A 3-D FEM METHOD TO PREDICT CURRENT DENSITIES INDUCED IN CONDUCTING OBJECTS EXPOSED TO ELF MAGNETIC FIELDS

by

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To Mom and Dad

Abstract

This thesis studies a numerical method to predict the current densities induced in conducting objects exposed to Extremely Low Frequency (ELF) magnetic fields. The Three Dimensional Finite Element Method (3-D FEM) was chosen to accomplish this task. The 3-D FEM numerical formulation is derived and presented as is the resulting algorithm written in FORTRAN code. The results are then presented and compared to other verified solutions to verify the validity of the formulation and the code. These results show that the 3-D FEM formulation works well for very simple objects. But, due to the fact that the developed algorithm requires much computer memory and that the governing coupled equations to solve produce matrices which are not always well conditioned, complex objects cannot yet be analyzed using this method.

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Chapter 1 Introduction

Humans and other living organisms are exposed to extremely low frequency (ELF) electromagnetic (EM) fields every day. These fields are produced by many sources such as high voltage transmission lines and power stations, computers and other electrical office equipment, electrical appliances at home, induction furnaces and arc-welders in electrosteel industries, and many others. There is much concern by the scientific community as well as the public about the possible biological effects these fields might have on humans. "In recent years, biomedical researchers have demonstrated low-frequency electromagnetic fields can disrupt the body's natural immune system, modify the production of hormones and help promote tumor growth."[1]. It has been known for about 100 years that magnetophosphenes phenomenon (luminous impressions due to magnetic field excitation of the retina) exist in electrosteel industries and other similar industries[2]. In order to determine the physical effects these fields may have on living organisms, it is necessary to determine the way in which the external fields couple to the whole body and determine the current density levels induced inside the exposed biological body[3,4]. This thesis presents a numerical approach to predict the induced currents inside a human body exposed to ELF magnetic fields.

1.1 Background

Kaune and Phillips[5] published a review paper in 1985 on the research that has been done on the interaction of ELF electromagnetic fields with intact organisms since 1967. This paper dealt mainly with electric field effects. In 1987, Tenforde and Kaune[6] published a more complete review dealing with the biological effects of magnetic fields. It is clear from these review papers that electric field coupling to biological bodies has received much more attention then magnetic field coupling.

It appears that Barnes et Al.[7] were the first to publish a paper dealing with ELF electromagnetic field coupling to Humans. Computer simulations were used to study the transmission line field coupling to linemen performing maintenance on these lines. The human body was modeled as an ellipsoid with perfectly conducting boundaries and the inside was consisted of distributed resistance. The human body has since been modeled for numerical calculations as an ellipsoid[8,9], a collection of ellipsoids[10], a sphere[11], a collection of spheres[12] or cylinders[13], a grounded semi-ellipsoid[14,15], and a axially symmetrical model[16,17].

Spiegel[18] was the first to use a large number of small blocks to realistically approximate the shape and properties of a human and a baboon. The method of moments was used to calculate the current densities induced by electric fields. His results were not in good agreement with measurements made by Deno[4,19] and Kaune[20,21]. This may be due to the several approximations made in his analysis. The conclusions that the current densities and the conductivities of the biological objects are independent from one another may find its roots in his approximations[5]. However, Spiegel did show that it was impossible to measure the induced currents due to electromagnetic fields in quadrupeds (baboons) and to extrapolate this data to the induced currents in bipeds (humans) exposed to these same fields as was thought possible by other researchers[14,21,22]. Nevertheless, he did show that a more realistic model of man may be used to perform theoretical analysis.

Prior to 1976, most studies neglected the magnetic field coupling to bodies compared to electric field effects. This may be due to the large difference in magnitude of the two field levels under high voltage transmission lines and reserved sponsorship by utility companies[8]. However, Spiegel[11] showed that induced currents due to transmission line electric or magnetic fields are comparable. He showed that size of an exposed body was a major factor in determining the amount of magnetic field coupling to humans and other animals. "Beneath an EHV transmission line, electric field coupling dominates for biological objects smaller than man, but magnetic field coupling is on the same order of magnitude as electric field coupling for a man and large animals." In his next paper[8] he presented calculations of induced current densities in a homogeneous prolate spheroid model of man caused by magnetic field coupling. Two important facts are presented. The first fact is that "a lineman wearing a metallized suit or working in an enclosed conductive shield for protection from the electric field is not similarly protected from the magnetic field". There still doesn't exist a practical way to shield humans against ELF Magnetic fields[6]. The second fact is that the induced current density levels in the body are a "much better indicator of acute electrical hazards such as loss of muscular control or heart fibrillation than the total current". In 1982 Hart and Marino[15] modelled a man as a semi-ellipsoid on a ground plane. They used image theory to find analytical solutions of induced current densities in the human body due to transmission line electric and magnetic fields.

Late in 1982, Lovsund, Oberg, and Nillson[2] published results of measurements of ELF magnetic field levels present in electrosteel and welding industries in Sweden. The measurements were performed in conditions comparable to a realistic working environment. In these industries, the voltage levels are very low compared to the high current levels used, thus the effect of the electric fields may be neglected. The magnetic flux levels measured for welders and furnaces operating at 50 Hz were in the range 0.1 to 10 mT. Induction heaters operating at 50 to 1 000 Hz had levels up to 70 mT. In 1983 Caola, Deno, and Dymek[23] measured the magnetic flux levels in typical homes situated near 500 kV transmission lines. They found these levels to be in the order of 25 μT. They also found that houses offered no shielding protection against these magnetic fields.

Deford and Gandhi[24] used an impedance method to numerically calculate the specific absorption rate (SAR: mass normalized rate of energy absorption in Watts/kg) of biological bodies exposed to low frequency electromagnetic fields. The authors calculated the SAR induced on a thin cross-section of a torso of a man exposed to a vertically oriented 13.56 MHz magnetic field. The method has been shown to work in the MF-HF range (below 3 MHz).

Sullivan, Borup, and Gandhi[25] used the Finite-Difference Time-Domain Method to calculate induced currents. This method permits a very fine modeling of man (50000 blocks) but is only suitable for the frequency range of 100-450 MHz. Unfortunately, calculations in the ELF range would require too many iterations; the resulting errors would be too large.

1.2 Objective

Relatively little research has addressed the ELF magnetic field coupling to humans or other living organisms. It is believed that no papers concerned with finding the induced current densities in biological bodies exposed to ELF magnetic fields have presented results relating the induced field and current distributions with the nonhomogeneous electrical properties of the body being studied. These papers use homogeneous models to represent man. Kaune and Phillips[5] reported that it is a difficult and very time consuming task to make measurements in saline models.

The objective of this research is to develop a numerical formulation which predicts current densities induced in biological objects, homogeneous or heterogeneous, exposed to ELF magnetic fields more easily and more accurately than the formulations presently available. A three dimensional (3-D) Finite Element Method (FEM) is chosen to accomplish this goal. The reasons are explained below.

The conductivity of a biological body is very inhomogeneous. However, it is a known fact that the ELF magnetic field will be unperturbed by the presence of these bodies[6,10,24,26], because they are nonmagnetic (their permeability everywhere is equal to that of free space). Thus, the magnetic field is relatively unchanged whether biological bodies are present or not. This problem may therefore be identified as a boundary value problem. The boundary values are the known magnetic fields and the unknowns are the induced current densities inside the body.

The FEM is particularly well suited to solve boundary value problems of heterogeneous regions of interest. The biological body being studied is first divided into subregions having homogeneous conductivities (each region is allowed to have a different conductivity). A set of partial differential equations is then formulated for a generic subregion (FEM) using Maxwell's equations in the local form. Galerkin's method is then applied to the resulting equations to form a system of equations which is then appropriately solved.

1.3 Thesis Organization

The theoretical development, the corresponding FORTRAN code, and results are presented in this thesis. The validity of this numerical method is shown by comparing the results with known analytical solutions, proven numerical results, and published measurements.

The analytical solution of the induced current densities inside homogeneous cylinders exposed to magnetic fields oriented in a parallel and perpendicular fashion to the longitudinal axis are developed in chapter 2.

The 2-D FEM variational formulation to calculate induced current densities inside nonhomogeneous cylinders exposed to magnetic fields oriented parallel to the longitudinal axis is presented in chapter 3.

Chapter 4 deals with the development of the 3-D FEM formulation used to calculate the induced current density inside an arbitrarily shaped heterogeneous biological body exposed to an ELF magnetic field oriented in any direction.

Chapter 5 shows the algorithm of the FORTRAN code developed for this thesis subject. It shows how to input the necessary data to model an object and how to run the program. It also describes a typical output format.

The 3D FEM formulation results are then presented, compared to known analytical and numerical solutions, and analyzed for different cases in Chapter 6.

The conclusions and contributions of this thesis, as well as a brief indication of direction for further research is presented in chapter 7.

Chapter 2 Analytical Formulation

Very few published results about low frequency magnetically induced current densities inside saline filled objects have appeared in literature so far. Thus, it was found necessary to compare the 2-D and 3-D FEM formulation results with analytical solutions to verify the validity of the code developed in this thesis. Analytical solutions of current densities induced in saline filled cylinders which are exposed to magnetic fields of two orientations were developed by McLeod, et Al.[27] The two cases studied by these authors were a so called horizontal orientation (HO) and vertical orientation (VO). As can be seen in figures 2.1 and 2.2, the HO case refers to Helmholtz aiding coils, producing a spatially uniform magnetic field between them, being in the longitudinal plane. The VO case refers to the Helmholtz coils being vertical to the plane. The magnetic field in the HO case is parallel to the axis of symmetry of the cylinder while in the VO case, the magnetic field is perpendicular to the axis of symmetry of the cylinder. The cylinders under test in both cases are homogeneous.

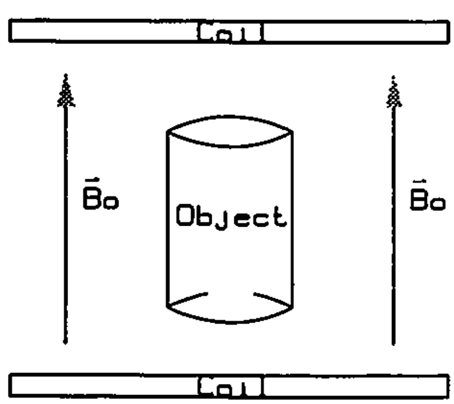


Figure 2.1: Geometry of HO case for the analytical formulation.

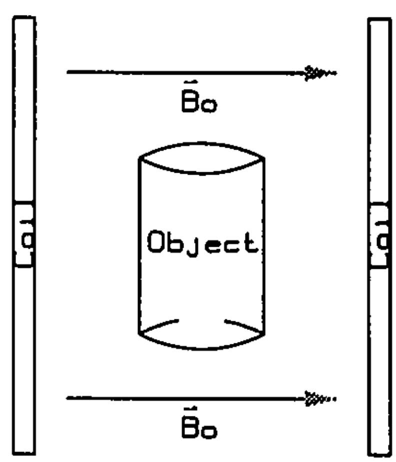


Figure 2.2: Geometry of VO case for the analytical formulation.

2.1 Horizontal Orientation (HO)

The object being studied is a cylinder with the radius a, height 2h, and conductivity σ. The cylindrical coordinates (r, ϕ, z) are therefore used in deriving the analytical solutions of the fields and current densities induced in the cylinder. Figure 2.3 shows the geometry of the problem.

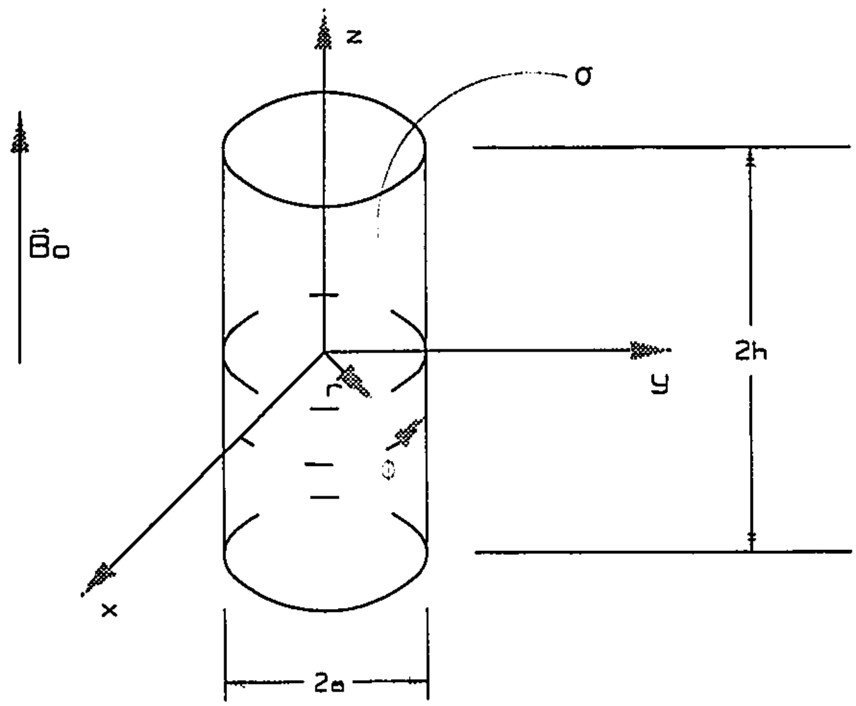


Figure 2.3: Cylindrical coordinates for the HO case for the analytical formulation.

The starting point to derive the solution are Maxwell's equations. Because only one frequency component is of interest, the phasor notation will be used.

$$\nabla \times \vec{E} = -j\omega\vec{B} \tag{2.1}$$

$$\nabla \times \vec{H} = j\omega\vec{D} + \vec{J} \tag{2.2}$$

$$\nabla \cdot \vec{D} = \rho \tag{2.3}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.4}$$

The constitutive relations will also be used.

$$\vec{D} = \varepsilon \vec{E} \tag{2.5}$$

$$\vec{B} = \mu \vec{H} \tag{2.6}$$

Where $\varepsilon = \varepsilon_r \varepsilon_o$ and $\mu = \mu_r \mu_o$

It is also known that in a conductive medium

$$\vec{J} = \sigma \vec{E} \tag{2.7}$$

For very low frequencies, such as 60 Hz, the displacement current is negligible compared to the conduction current. That is:

$$|\vec{J}| = |\sigma \vec{E}| \gg |j\omega \vec{D}| \tag{2.8}$$

So (2.2) can now be simplified to:

$$\nabla \times \vec{H} = \vec{J} \tag{2.9}$$

Substituting (2.6) and (2.7) into (2.1) yields:

$$\nabla \times \vec{J} = -j\omega\mu\sigma \vec{H} \tag{2.10}$$

The curl of the left and right hand side of (2.10) leads to:

$$\nabla \times \nabla \times \vec{J} = -j\omega\mu\sigma \vec{J} \tag{2.11}$$

The geometry of the problem is symmetrical about the z axis, thus the operator $\partial/\partial\phi = 0$. Assuming that the currents induced are small enough to produce negligible EMF, the magnetic field inside the cylinder has only a z component: $\vec{H}_i = \hat{z} H_i$. Also, J_r and J_z must be zero as the displacement current is assumed to be zero and the current continuity principle must hold.

Thus,

$$\begin{aligned}
\nabla \times \nabla \times \vec{J} = \hat{r}&\left\{\left[\frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial(rJ_\phi)}{\partial r} - \frac{1}{r}\frac{\partial J_r}{\partial\phi}\right)\right] - \left[\frac{\partial}{\partial z}\left(\frac{\partial J_r}{\partial z} - \frac{\partial J_z}{\partial r}\right)\right]\right\} \\
+\hat{\phi}&\left\{\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial J_z}{\partial\phi} - \frac{\partial J_\phi}{\partial z}\right)\right] - \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rJ_\phi)}{\partial r} - \frac{1}{r}\frac{\partial J_r}{\partial\phi}\right)\right]\right\} \\
+\hat{z}&\left\{\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial J_r}{\partial z} - \frac{\partial J_z}{\partial r}\right)\right)\right] - \left[\frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial J_z}{\partial\phi} - \frac{\partial J_\phi}{\partial z}\right)\right]\right\}
\end{aligned}$$

reduces to

$$\nabla \times \nabla \times \vec{J} = -\hat{\phi}\left(\frac{\partial^2 J_\phi}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rJ_\phi) \right) \right) \tag{2.12}$$

Substituting (2.12) into (2.11)

$$\frac{\partial^2 J_\phi}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rJ_\phi) \right) - j\omega\mu\sigma J_\phi = 0 \tag{2.13}$$

Making $\partial J_\phi / \partial z = 0$ to be consistent, reduces the equation to solve to

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rJ_\phi) \right) - j\omega\mu\sigma J_\phi = 0 \tag{2.14}$$

It can easily be seen that this is a Bessel equation.

The solution to (2.14) is

$$J_\phi = AJ_1(kr) + BN_1(kr) \tag{2.15}$$

where $k = \sqrt{-j\omega\mu\sigma}$, $J_1(kr)$ is the Bessel function of the first kind, and $N_1(kr)$ is the Newmann function of the first kind.

The current must be finite at $r = 0$, thus B must be set to zero due to the nature of the Newmann function.

$$J_\phi = AJ_1(kr) \tag{2.16}$$

From (2.10)

$$H_z = \frac{jAkJ_0(kr)}{\omega\mu\sigma} \tag{2.17}$$

and at $r = a$ the tangential magnetic fields are matched:

$$H_z|_{r=a} = H_0 \tag{2.18}$$

yielding

$$A = \frac{kH_0}{J_0(ka)} \tag{2.19}$$

Thus,

$$J_\phi = kH_0\frac{J_1(kr)}{J_0(ka)} \tag{2.20}$$

2.2 Vertical Orientation (VO)

When the magnetic field is perpendicular to the cylinder axis the solution is considerably more complicated. It is much simpler to analyze a parallelipiped. Assuming again that the induced current flow is small enough to produce negligible EMF, the solution of $\vec{J}$ and $\vec{H}$ can be found for a cylinder by approximating this cylinder with rectangular parallelipipeds. The cartesian coordinate system is therefore be used to analyze the problem. Figures 2.4 and 2.5 show the geometry of the problem.

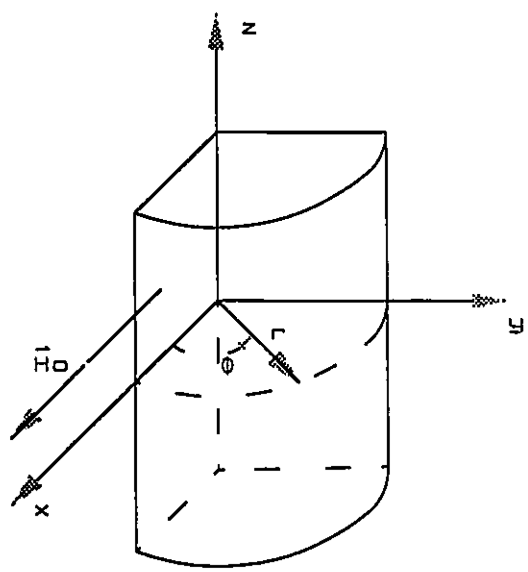


Figure 2.4: A quarter of the cylinder used in the VO case for the analytical formulation.

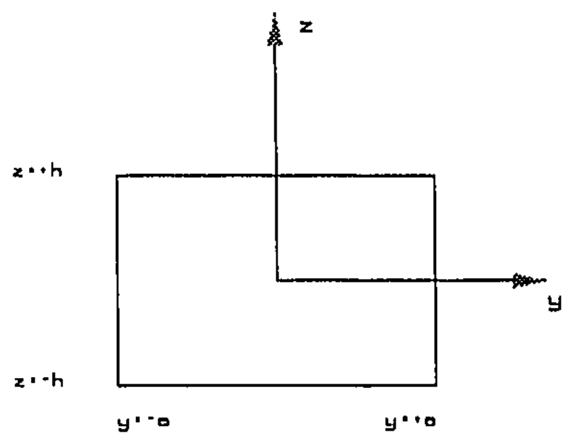


Figure 2.5: Rectangular parallelipiped used in the VO case for the analytical formulation.

B.R. McLeod, A.A. Pilla, and M.W. Sampsel developed solutions for induced currents and magnetic fields in one of their publications entitled "Electromagnetic Fields Induced by Helmholtz Aiding Coils Inside Saline-Filled Boundaries"[27].

The assumptions made were:

a) $|\vec{J}| \gg |j\omega\vec{D}|$ (conducting currents are much larger than displacement currents

b) $\partial/\partial x = 0$ the boundaries are uniform in the direction of the applied field

c) Back EMF is negligible

Using assumption a), the equation to solve is:

$$\nabla \times \nabla \times \vec{H} = \nabla \times \vec{J} = -j\omega\mu\sigma\vec{H} = \nabla(\nabla \cdot \vec{H}) - \nabla^2\vec{H} \tag{2.21}$$

or simply

$$\nabla^2\vec{H} - j\omega\mu\sigma\vec{H} = 0 \tag{2.22}$$

From assumptions b) and c) (2.22) may be written as

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} - j\omega\mu\sigma H_x = 0 \tag{2.23}$$

The authors used the expanded Fourrier Series, superposition, and matched the tangential magnetic fields at the boundary to arrive at the following solution for the resulting magnetic field:

$$\begin{aligned} H_x = &\sum_{m_{odd}} \frac{4H_0}{m\pi}\left[\pm\cos\left(\frac{m\pi}{2h}z\right)\right]\left[\frac{\cosh(\alpha y)}{\cosh(\alpha a)}\right] \\ &+ \sum_{n_{odd}} \frac{4H_0}{n\pi}\left[\pm\cos\left(\frac{n\pi}{2a}y\right)\right]\left[\frac{\cosh(\xi z)}{\cosh(\xi h)}\right] \end{aligned} \tag{2.24}$$

where

$$\alpha = \left[\left(\frac{m\pi}{2h}\right)^2 + j\omega\mu\sigma\right]^{\frac{1}{2}} \quad , \quad \xi = \left[\left(\frac{n\pi}{2a}\right)^2 + j\omega\mu\sigma\right]^{\frac{1}{2}} \quad \text{, and} \quad m,n = 1,3,5,\ldots$$

$$\pm\cos(\quad) = \text{ alternating sign cos series}$$

for an applied magnetic field of $\vec{H} = H_o\hat{x}$

The current density can be found from (2.9): $\nabla \times \vec{H} = \vec{J}$ or

$$\nabla \times \vec{H} = \vec{J} = \frac{\partial H_x}{\partial z}\hat{y} - \frac{\partial H_x}{\partial y}\hat{z} \tag{2.25}$$

Thus,

$$J_y = \sum_{n_{odd}} \frac{4H_0\xi}{n\pi}\left(\pm\cos\left(\frac{n\pi}{2a}y\right)\right)\left(\frac{\sinh(\xi z)}{\cosh(\xi h)}\right) - \sum_{m_{odd}} \frac{2H_0}{h}\left(\pm\sin\left(\frac{m\pi}{2h}z\right)\right)\left(\frac{\cosh(\alpha y)}{\cosh(\alpha a)}\right) \tag{2.26}$$

and

$$J_z = \sum_{n_{odd}} \frac{2H_0}{a}\left(\pm\sin\left(\frac{n\pi}{2a}y\right)\right)\left(\frac{\cosh(\xi z)}{\cosh(\xi h)}\right) - \sum_{m_{odd}} \frac{4H_0\alpha}{m\pi}\left(\pm\cos\left(\frac{m\pi}{2h}z\right)\right)\left(\frac{\sinh(\alpha y)}{\cosh(\alpha a)}\right) \tag{2.27}$$

If

$$\frac{\omega\mu\sigma}{(m\pi/2h)^2} < 0.05 \quad \text{and} \quad \frac{\omega\mu\sigma}{(n\pi/2a)^2} < 0.05$$

binomial expansion may be used on α and ξ and the higher order terms dropped, these terms are now reduced to:

$$\alpha \cong \frac{m\pi}{2h} + j\frac{\omega\mu\sigma h}{m\pi} \quad \text{and} \quad \xi \cong \frac{n\pi}{2a} + j\frac{\omega\mu\sigma a}{n\pi}$$

The resulting expressions for current densities are therefore:

$$J_y \cong j\frac{\omega\mu\sigma H_0(2a)}{\pi^2}\sum_{n_{odd}}\frac{4}{n^2}\left(\pm\cos\left(\frac{n\pi}{2a}y\right)\right)\left(\frac{\sinh\left(\frac{n\pi}{2a}z\right)}{\cosh\left(\frac{n\pi}{2a}h\right)}\right) \tag{2.28}$$

and

$$J_z \cong -j\frac{\omega\mu\sigma H_0(2h)}{\pi^2}\sum_{m_{odd}}\frac{4}{m^2}\left(\pm\cos\left(\frac{m\pi}{2h}z\right)\right)\left(\frac{\sinh\left(\frac{m\pi}{2h}y\right)}{\cosh\left(\frac{m\pi}{2h}a\right)}\right) \tag{2.29}$$

A cylinder may be approximated by a series of rectangular slabs of variable widths. Although $\partial/\partial x \neq 0$; the boundaries are variable in the direction of applied magnetic field, the authors have checked and published the results validating this method by a comparison with experimental results.

Chapter 3 2-D FEM Formulation

The 2-D FEM program is used to calculate the current densities induced in a multi-layered cylinder for the Horizontal Orientation case described in chapter 2. The 2-D FEM program is executed for a homogeneous cylinder and compared to the analytical solution described in chapter 2. This program is then executed for a multi-layered cylinder; the 3-D FEM results are compared to the 2-D program. Thus the purpose of the 2-D FEM algorithm is to verify the 3-D FEM formulation for accuracy in determining the induced current densities in a heterogeneous conducting object.

3.1 Requires Steps for the 2-D FEM Method

The steps to follow in order to formulate a 2-D FEM solution are listed below:

1) Determine the equation or system of equations to be solved
2) Determine the Newmann and Dirichlet boundary conditions
3) Determine appropriate functionals
4) Determine the type of finite element to be used to partition the problem
5) Formulate approximate solutions to the true solutions
6) Substitute these approximations into the functionals
7) Find the first variational on the functional and set it equal to zero. This is the stationary point of the functional where the solutions of the unknowns are found
8) Assemble a global matrix equation to include the effect of all elements defining the problem
9) Solve the system of equations to find the unknowns

3.2 Equation to be Solved

Figure 3.1 shows the geometry of the object being studied.

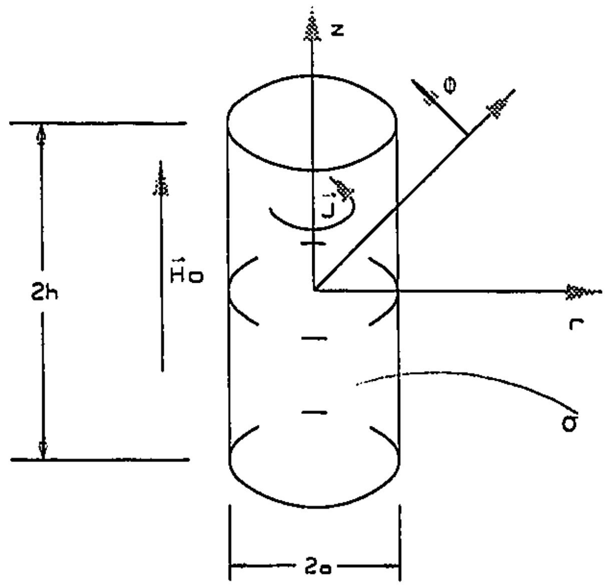


Figure 3.1: Cylinder with magnetic field parallel to the axis of symmetry for the 2-D FEM formulation.

The cylinder has a finite length $2h$, a radius a, and a conductivity σ. The magnetic field is parallel to the axis of symmetry (z) and its intensity is H_o. The only currents that can be induced are J_ϕ.

Figure 3.2 shows a cross section of figure 3.1

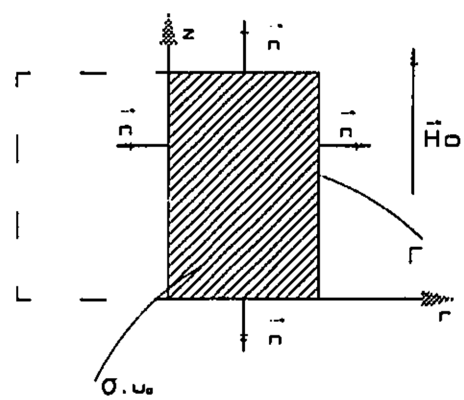


Figure 3.2: Normal vectors of a cross section of half a cylinder for the 2-D FEM formulation.

Because of the symmetry, this problem may be solved as a two dimensional problem. Using equations (2.1) to (2.8) (the displacement current is negligible compared to conduction current and there are no sources in the body) and taking the curl of equation (2.2)

$$\nabla \times \nabla \times \vec{H} = \nabla \times \vec{J} \tag{3.1}$$

substituting (2.1) and (2.7) into (3.1)

$$\nabla \times \nabla \times \vec{H} = -j\omega\sigma\vec{B} \tag{3.2}$$

using the identity

$$\nabla \times \nabla \times \vec{H} = \nabla(\nabla \cdot \vec{H}) - \nabla^2\vec{H}$$

and the fact that the magnetic flux density, and therefore the magnetic field, because of the nature of the problem: $\mu = \mu_o$ everywhere, must have zero divergence everywhere results in the following equation to be solved:

$$-\nabla^2\vec{H} + j\omega\mu\sigma\vec{H} = 0 \tag{3.3}$$

where the boundary conditions are:

a) (non-homogeneous Dirichlet boundary condition) the tangential magnetic field is continuous across the boundary:

$$H_t|_{r=a} = H_o \tag{3.4}$$

b) (homogeneous Newmann boundary condition) the normal component of the magnetic field is continuous across the boundary:

$$\left.\frac{\partial\vec{H}}{\partial\vec{n}}\right|_{r=a} = 0 \tag{3.5}$$

3.3 Defining the Functional

The idea is to find a functional of the unknown variable $\vec{H}$, find its first variational and equal it to zero to determine the stationary point. The symbol for the functional of $\vec{H}$ is $f(\vec{H})$.

An article titled "Computation of Electromagnetic Fields" by A. Wexler[28] demonstrates that an equation of the form:

$$Lu = f \tag{3.6}$$

will have the corresponding functional

$$f(u) = < Lu, u > - < u, f > - < f, u > \tag{3.7}$$

where the operator $< u, v >$ is defined as:

$$< u, v > = \int_{\Omega} u v^{*} d\Omega \tag{3.8}$$

In this case, L, u, and f are defined as:

$$L = -\nabla^2 + j\omega\mu\sigma \quad , \quad u = \overrightarrow{H} \quad , \quad f = 0 \quad , \quad \Omega = \text{Volume}$$

and because this is an axisymmetrical problem, the cylindrical coordinates are used. Therefore $d\Omega$ is defined as $d\Omega = 2\pi r dr dz$

Thus, the functional of $\overrightarrow{H}$ may now be written as:

$$\begin{aligned} f(\overrightarrow{H}) &= < Lu, u > - < u, f > - < f, u > \\ &= \int_{\Omega} \overrightarrow{H}^{*} (-\nabla^2 \overrightarrow{H} + j\omega\mu\sigma\overrightarrow{H}) d\Omega \\ &= -\int_{\Omega} (\overrightarrow{H}^{*})(\nabla^2 \overrightarrow{H}) d\Omega + \int_{\Omega} (\overrightarrow{H}^{*})(j\omega\mu\sigma\overrightarrow{H}) d\Omega \end{aligned} \tag{3.9}$$

Green's identity demonstrated in M.R. Spiegel's "Mathematical Handbook of Formulas and Tables"[29] is as follows:

$$\int_{S} \frac{\partial g}{\partial n} dS = \int_{\Omega} (f\nabla^2 g + \nabla f \cdot \nabla g) d\Omega$$

and rearranging the terms in the above equation, Green's Identity can also be written as:

$$\int_{\Omega} f(\nabla^2 g) d\Omega = \int_{S} f \frac{\partial g}{\partial n} dS - \int_{\Omega} (\nabla f \cdot \nabla g) d\Omega \tag{3.10}$$

Now, applying Green's identity, equation (3.9) can now be written as

$$f(\overrightarrow{H}) = \int_{\Omega} (\nabla \overrightarrow{H}^{*})(\nabla \overrightarrow{H}) d\Omega - \int_{S} (\overrightarrow{H}^{*}) \frac{\partial H}{\partial n} dS + \int_{\Omega} (\overrightarrow{H}^{*})(j\omega\mu\sigma\overrightarrow{H}) d\Omega \tag{3.11}$$

but, from the homogeneous Newmann boundary conditions stated in (3.5), the second term in the above equation is equal to zero and the functional may now be written as:

$$f(\vec{H}) = \int_\Omega (\nabla \vec{H}^*)(\nabla \vec{H}) d\Omega + \int_\Omega (\vec{H}^*)(j\omega\mu\sigma\vec{H}) d\Omega \qquad (3.12)$$

where: $H|_{r=a} = H_o$, and $\vec{H}$ at any other point inside the body under study is unknown.

The functional defined in equation (3.12) is complex. If it is possible, it is preferable to choose the simplest functional possible. The following functional is simpler and is very similar in form to the developed functional. It has the same boundary conditions.

$$f(\vec{H}) = \int_\Omega \frac{1}{\mu} (\nabla \vec{H})^2 d\Omega + \int_\Omega j\omega\sigma \vec{H}^2 d\Omega \qquad (3.13)$$

Because it is only a guess that (3.13) is an appropriate functional, it is necessary to validate this assumption. The functional is verified by taking the first variational and setting it to zero. If it is appropriate, the terms contained in the original equation to be solved will be present. Thus, taking the first variational of (3.13) and setting it to zero yields:

$$\delta f(\vec{H}) = \int_\Omega \frac{1}{\mu} 2(\nabla \vec{H}) \delta(\nabla \vec{H}) d\Omega + \int_\Omega j\omega\sigma 2\vec{H}\delta\vec{H} d\Omega = 0$$

which is equivalent to:

$$\delta f(\vec{H}) = \int_\Omega \frac{1}{\mu} (\nabla \vec{H}) \nabla \delta\vec{H} d\Omega + \int_\Omega j\omega\sigma \vec{H}\delta\vec{H} d\Omega = 0 \qquad (3.14)$$

Green's Lemma which is demonstrated in O.C. Zienkiewicz and K. Morgan's book titled "Finite Element and Approximations"[30] is stated as:

$$\int_\Omega \alpha(\nabla \vec{B}) d\Omega = -\int_\Omega (\nabla\alpha)\vec{B} d\Omega + \int_S \alpha\vec{B} \cdot \vec{n} dS \qquad (3.15)$$

Making $\alpha = \frac{1}{\mu}\nabla\vec{H}$ and $\vec{B} = \partial\vec{H}$ and applying Green's Lemma to equation (3.14) yields the following equation:

$$\delta f(\vec{H}) = -\int_\Omega \frac{1}{\mu} (\nabla^2 \vec{H}) \delta\vec{H} d\Omega + \int_S \frac{1}{\mu} \nabla\vec{H} \cdot \vec{n}\delta\vec{H} dS + \int_\Omega j\omega\sigma\vec{H}\delta\vec{H} d\Omega = 0 \qquad (3.16)$$

By combining the first and the third terms and applying the scalar triple product rule to the second term, this equation can now be rewritten as:

$$\int_{\Omega}\left(-\frac{1}{\mu}(\nabla^2\vec{H})+j\omega\sigma\vec{H}\right)\delta\vec{H}\,d\Omega+\int_{S}\frac{1}{\mu}\left(\frac{\partial\vec{H}}{\partial n}\right)\delta\vec{H}\,dS=0 \qquad (3.17)$$

Equation (3.17) can be decomposed into two parts:

a) the original equation to be solved

$$-\nabla^2\vec{H}+j\omega\sigma\mu\vec{H}=0 \qquad (3.18)$$

b) the homogeneous Newmann boundary conditions

$$\left.\frac{\partial\vec{H}}{\partial n}\right|_{r\,=\,\text{boundary}}=0 \qquad (3.19)$$

Therefore, the functional chosen is appropriate. The equation to be solved is satisfied as well as the Newmann Boundary conditions. The non homogeneous Dirichlet boundary conditions are satisfied by forcing the tangential magnetic field intensity on the surface of the cylinder to H_o.

3.4 A Look at One Element

The finite element used to solve this problem is a triangle like the one shown in figure 3.3. The approximation used to define $\vec{H}$ and the FEM matrix form of the equation to be solved for one element will be explained.

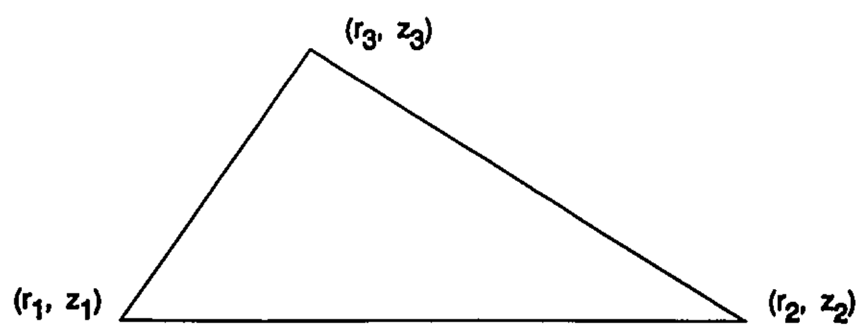


Figure 3.3: Typical triangular finite element for the 2-D FEM formulation.

The functional $f(\vec{H})$ is found in equation (3.13). Because of the geometry of the object being studied (i.e. a cylinder), the cylindrical coordinate system is used in the analysis. Thus,

$$d\Omega = r\,dr\,d\theta\,dz \tag{3.20}$$

$$dS = r\,d\theta\,dl \tag{3.21}$$

and

$$\int_0^{2\pi} d\theta = 2\pi \tag{3.22}$$

Combining (3.20) and (3.22) and substituting into (3.13) yields:

$$f(\vec{H}) = 2\pi\left\{\frac{1}{\mu}\int_\Omega (\nabla\vec{H})^2 r\,dr\,dz + j\omega\sigma\int_\Omega \vec{H}^2 r\,dr\,dz\right\} \tag{3.23}$$

Linear trial functions for these triangular finite elements were chosen to solve this problem as is shown in P.P. Sylvester and R.L. Ferrari's book titled "Finite Elements for Electrical Engineers"[31]. Thus, let H_z be of the form:

$$H_z = a + br + cz \tag{3.24}$$

H_{zi} at node i may be written in a matrix form of equation:

$$H_z = \begin{pmatrix} H_{z1} \\ H_{z2} \\ H_{z3} \end{pmatrix} = \begin{pmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.25}$$

isolating a, b, and c yields:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} H_{z1} \\ H_{z2} \\ H_{z3} \end{pmatrix} \tag{3.26}$$

Equation (3.24) can be rewritten in matrix form:

$$H_z = \begin{pmatrix} 1 & r & z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.27}$$

Substituting (3.26) in (3.27), H_z may now be written as a function of H_{zi} at nodes $i = 1, 2, 3$ as:

$$H_z = (1 \quad r \quad z) \cdot \begin{pmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} H_{z1} \\ H_{z2} \\ H_{z3} \end{pmatrix} \tag{3.28}$$

Which is equivalent to:

$$H_z = \sum_{i=1}^{3} H_{zi} \alpha_i(r, z) \tag{3.29}$$

where

$$\alpha_1 = \frac{1}{2S}\{(r_2 z_3 - r_3 z_2) + (z_2 - z_3)r + (r_3 - r_2)z\} \tag{3.29a}$$

$$\alpha_2 = \frac{1}{2S}\{(r_3 z_1 - r_1 z_3) + (z_3 - z_1)r + (r_1 - r_3)z\} \tag{3.29b}$$

$$\alpha_3 = \frac{1}{2S}\{(r_1 z_2 - r_2 z_1) + (z_1 - z_2)r + (r_2 - r_1)z\} \tag{3.29c}$$

where $2S$ is defined as twice the area of the triangle element which is equal to:

$$2S = \text{Determinant of the matrix } \begin{pmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{pmatrix} \tag{3.29d}$$

A quick calculation will reveal that:

$$\alpha_i(r_j, z_j) = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases} \tag{3.29e}$$

Equation (3.29) is now substituted into (3.23) to yield the following equation.

$$f(\overrightarrow{H}) = 2\pi \left\{ \frac{1}{\mu} \int_{\Omega} \left(\sum_{n=1}^{3} H_n \nabla \alpha_n \right)^2 r\,dr\,dz + j\omega\sigma \int_{\Omega} \left(\sum_{n=1}^{3} H_n \alpha_n \right)^2 r\,dr\,dz \right\} \tag{3.30}$$

Each term of this equation is expanded below.

a) the first term:

$$\frac{1}{\mu}\int_{\Omega}(\nabla\vec{H})^2 r\,dr\,dz = \frac{1}{\mu}\int_{\Omega}\left(\sum_{n=1}^{3} H_n\nabla\alpha_n\right)^2 r\,dr\,dz$$

$$= \frac{1}{\mu}\sum_{n=1}^{3}\sum_{m=1}^{3} H_n\left\{\int_{\Omega}\nabla\alpha_n \cdot \nabla\alpha_m r\,dr\,dz\right\}H_m \qquad (3.31)$$

Let S_{nm} be defined as

$$S_{nm} = \frac{1}{\mu}\int_{\Omega}\nabla\alpha_n \cdot \nabla\alpha_m r\,dr\,dz \qquad (3.32)$$

Substituting (3.29a) to (3.29d) into (3.32) and applying the operators to equation (3.32) will produce the following results:

$$S_{11} = \frac{(z_2 - z_3)^2 + (r_3 - r_2)^2}{4S\mu} \cdot \bar{r} \qquad (3.32a)$$

$$S_{12} = \frac{(z_2 - z_3)(z_3 - z_1) + (r_3 - r_2)(r_1 - r_3)}{4S\mu} \cdot \bar{r} \qquad (3.32b)$$

$$S_{13} = \frac{(z_2 - z_3)(z_1 - z_2) + (r_3 - r_2)(r_2 - r_1)}{4S\mu} \cdot \bar{r} \qquad (3.32c)$$

$$S_{21} = S_{12} \qquad (3.32d)$$

$$S_{22} = \frac{(z_3 - z_1)^2 + (r_1 - r_3)^2}{4S\mu} \cdot \bar{r} \qquad (3.32e)$$

$$S_{23} = \frac{(z_3 - z_1)(z_1 - z_2) + (r_1 - r_3)(r_2 - r_1)}{4S\mu} \cdot \bar{r} \qquad (3.32f)$$

$$S_{31} = S_{13} \qquad (3.32g)$$

$$S_{32} = S_{23} \qquad (3.32h)$$

$$S_{33} = \frac{(z_1 - z_2)^2 + (r_2 - r_1)^2}{4S\mu} \cdot \bar{r} \qquad (3.32i)$$

where

$$\bar{r} = \frac{(r_1 + r_2 + r_3)}{3} \qquad (3.32j)$$

Equation (3.31) can now be written as:

$$\frac{1}{\mu}\int_{\Omega}(\nabla\vec{H})^2 r\,dr\,dz = \sum_{n=1}^{3}\sum_{m=1}^{3}(H_n)(S_{nm})(H_m) \tag{3.33}$$

The matrix form of (3.33) is written as:

$$\frac{1}{\mu}\int_{\Omega}(\nabla\vec{H})^2 r\,dr\,dz = [H]^T[S][H] \tag{3.34}$$

where

$$[S]=\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \text{ and } [H]=\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}$$

and the S_{nm} are defined in (3.32a) to (3.32i).

b) the second term:

$$j\omega\sigma\int_{\Omega}\vec{H}^2 r\,dr\,dz = j\omega\sigma\int_{\Omega}\left(\sum_{n=1}^{3}H_n\alpha_n\right)^2 r\,dr\,dz$$

$$= j\omega\sigma\sum_{n=1}^{3}\sum_{m=1}^{3}H_n\left\{\int_{\Omega}\alpha_n\cdot\alpha_m r\,dr\,dz\right\}H_m \tag{3.35}$$

In the same manner, T_{nm} is defined as

$$T_{nm} = j\omega\sigma\int_{\Omega}(\alpha_n\cdot\alpha_m)r\,dr\,dz \tag{3.36}$$

and again going through the same procedure outlined previously for S_{nm}

$$T_{11}=\frac{j\omega\sigma 2S}{120}(6r_1+2r_2+2r_3) \tag{3.36a}$$

$$T_{12}=\frac{j\omega\sigma 2S}{120}(2r_1+2r_2+r_3) \tag{3.36b}$$

$$T_{13}=\frac{j\omega\sigma 2S}{120}(2r_1+r_2+2r_3) \tag{3.36c}$$

$$T_{21}=\frac{j\omega\sigma 2S}{120}(2r_1+2r_2+r_3) \tag{3.36d}$$

$$T_{22}=\frac{j\omega\sigma 2S}{120}(2r_1+6r_2+2r_3) \tag{3.36e}$$

$$T_{23} = \frac{j\omega\sigma 2S}{120}(r_1 + 2r_2 + 2r_3) \qquad (3.36f)$$

$$T_{31} = \frac{j\omega\sigma 2S}{120}(2r_1 + r_2 + 2r_3) \qquad (3.36g)$$

$$T_{32} = \frac{j\omega\sigma 2S}{120}(r_1 + 2r_2 + 2r_3) \qquad (3.36h)$$

$$T_{33} = \frac{j\omega\sigma 2S}{120}(2r_1 + 2r_2 + 6r_3) \qquad (3.36i)$$

Equation (3.35) can now be written as:

$$j\omega\sigma\int_{\Omega}\overline{H}^2 r\,dr\,dz = \sum_{n=1}^{3}\sum_{m=1}^{3}(H_n)(T_{nm})(H_m) \qquad (3.37)$$

The matrix form of (3.37) is written as:

$$j\omega\sigma\int_{\Omega}\overline{H}^2 r\,dr\,dz = [H]^T[T][H] \qquad (3.38)$$

where

$$[T] = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \text{ and } [H] = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}$$

and the T_{nm} are defined in (3.36a) to (3.36i).

The functional can now be represented in matrix form by substituting equations (3.34) and (3.38) into equation (3.23) yielding

$$f(H) = 2\pi\{[H]^T[S][H] + [H]^T[T][H]\} \qquad (3.39)$$

In order to find the unknowns (H inside the body), the first variational of the functional is found and made equal to zero ($\delta f(H) = 0$).

$$\delta f(H) = \delta(2\pi\{[H]^T[S][H] + [H]^T[T][H]\}) = 0 \qquad (3.40)$$

This is in fact minimizing the functional, or at least, finding its stationary point. Solving (3.40) yields the following equation

$$[S][H] + [T][H] = 0 \qquad (3.41)$$

which may also be written as:

$$\{[S] + [T]\}[H] = 0 \qquad (3.42)$$

3.5 Global Matrix Assembly

The object being studied is modelled by being discretized into many triangles (finite elements). As is described in section 3.4, each triangle is locally characterized by three nodes and a certain conductivity. Globally, each triangle is numbered and every node making up the object is also numbered. The $[S]$ and $[T]$ matrices are then found for every element. They are then assembled into a global (G) final system of equations:

$$[GST]\cdot[GH]=0 \tag{3.43}$$

where $[GST]=[GS]+[GT]$
and $[GS]$, $[GT]$, and $[GH]$ are the global $[S]$, $[T]$, and $[H]$ matrices respectively.

The following example will illustrate how the global matrix is assembled. Two elements shown in figure 3.4 will suffice to explain the assembly procedure.

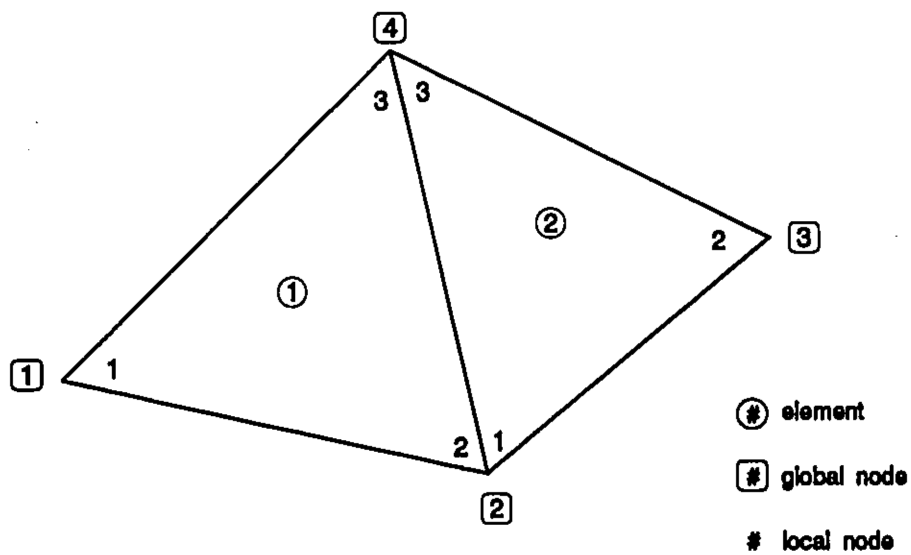


Figure 3.4: Global Matrix Assembly for the 2-D FEM formulation.

The circled numbers indicate the element number. The numbers in the square box indicate the global node numbering and the others indicate the local node numbering for each triangular element.

The $[ST]$ matrix is defined as $[ST]=[S]+[T]$. This $[ST]$ matrix for each element is written below using the local node numbering scheme.

For element 1

$$[ST]^{(1)}=\begin{pmatrix} ST_{11}^{(1)} & ST_{12}^{(1)} & ST_{13}^{(1)} \\ ST_{21}^{(1)} & ST_{22}^{(1)} & ST_{23}^{(1)} \\ ST_{31}^{(1)} & ST_{32}^{(1)} & ST_{33}^{(1)} \end{pmatrix} \quad (3.44a)$$

For element 2

$$[ST]^{(2)}=\begin{pmatrix} ST_{11}^{(2)} & ST_{12}^{(2)} & ST_{13}^{(2)} \\ ST_{21}^{(2)} & ST_{22}^{(2)} & ST_{23}^{(2)} \\ ST_{31}^{(2)} & ST_{32}^{(2)} & ST_{33}^{(2)} \end{pmatrix} \quad (3.45a)$$

The same $[ST]$ matrices may be expressed in terms of global node numbering scheme as described below:

$$[ST]^{(1)}=\begin{pmatrix} ST_{11}^{(1)} & ST_{12}^{(1)} & ST_{14}^{(1)} \\ ST_{21}^{(1)} & ST_{22}^{(1)} & ST_{24}^{(1)} \\ ST_{41}^{(1)} & ST_{42}^{(1)} & ST_{44}^{(1)} \end{pmatrix} \quad (3.44b)$$

$$[ST]^{(2)}=\begin{pmatrix} ST_{22}^{(2)} & ST_{23}^{(2)} & ST_{24}^{(2)} \\ ST_{32}^{(2)} & ST_{33}^{(2)} & ST_{34}^{(2)} \\ ST_{42}^{(2)} & ST_{43}^{(2)} & ST_{44}^{(2)} \end{pmatrix} \quad (3.45b)$$

Now $[GST]=[ST]^{(1)}+[ST]^{(2)}$.

The size of the global matrix is the square of the number of nodes used to describe the object. In this case, the size of the matrix will be 4X4.

The final assembly is quite simple. The terms $ST_{ij}^{(e)}$ for each element (e) is placed in the $(i,j)^{th}$ position of the $[GST]$ matrix. If a term is already present, the new term is added to it.

The global matrix is written below:

$$[GST]=\begin{pmatrix} ST_{11}^{(1)} & ST_{12}^{(1)} & 0 & ST_{14}^{(1)} \\ ST_{21}^{(1)} & ST_{22}^{(1)}+ST_{22}^{(2)} & ST_{23}^{(2)} & ST_{24}^{(1)}+ST_{24}^{(2)} \\ 0 & ST_{32}^{(2)} & ST_{33}^{(2)} & ST_{34}^{(2)} \\ ST_{41}^{(1)} & ST_{42}^{(1)}+ST_{42}^{(2)} & ST_{43}^{(2)} & ST_{44}^{(1)}+ST_{44}^{(2)} \end{pmatrix} \quad (3.46)$$

and now rewriting equation (3.46) in terms of local node numbering, and substituting in (3.43), the system of equations may now be written as:

$$\begin{pmatrix} ST_{11}^{(1)} & ST_{12}^{(1)} & 0 & ST_{13}^{(1)} \\ ST_{21}^{(1)} & ST_{22}^{(1)}+ST_{11}^{(2)} & ST_{12}^{(2)} & ST_{23}^{(1)}+ST_{13}^{(2)} \\ 0 & ST_{21}^{(2)} & ST_{22}^{(2)} & ST_{23}^{(2)} \\ ST_{31}^{(1)} & ST_{32}^{(1)}+ST_{31}^{(2)} & ST_{32}^{(2)} & ST_{33}^{(1)}+ST_{33}^{(2)} \end{pmatrix} \cdot \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{pmatrix} = 0 \tag{3.47}$$

Up to now, the non-homogeneous Dirichlet conditions have not been taken into account. It is at this point that they are dealt with. For the sake of this demonstration, let node 3 be a Dirichlet node. That is to say, H_3 is known and equal to the impinging tangential magnetic field outside the object. Rearranging equation (3.47) so that the source terms are now on the right hand side yields:

$$\begin{pmatrix} ST_{11}^{(1)} & ST_{12}^{(1)} & ST_{13}^{(1)} \\ ST_{21}^{(1)} & ST_{22}^{(1)}+ST_{11}^{(2)} & ST_{23}^{(1)}+ST_{13}^{(2)} \\ 0 & ST_{21}^{(2)} & ST_{23}^{(2)} \\ ST_{31}^{(1)} & ST_{32}^{(1)}+ST_{31}^{(2)} & ST_{33}^{(1)}+ST_{33}^{(2)} \end{pmatrix} \cdot \begin{pmatrix} H_1 \\ H_2 \\ H_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -ST_{12}^{(2)} \cdot H_3 \\ -ST_{22}^{(2)} \cdot H_3 \\ -ST_{32}^{(2)} \cdot H_3 \end{pmatrix} \tag{3.48}$$

Essentially, the terms in the third column were transferred to the right hand side. Equation (3.48) is now a system of four equations and three unknowns. Thus, one dependent equation. Row three of the left and right hand side of these equations are eliminated. The final system of equations to be solved, including boundary conditions, is:

$$\begin{pmatrix} ST_{11}^{(1)} & ST_{12}^{(1)} & ST_{13}^{(1)} \\ ST_{21}^{(1)} & ST_{22}^{(1)}+ST_{11}^{(2)} & ST_{23}^{(1)}+ST_{13}^{(2)} \\ ST_{31}^{(1)} & ST_{32}^{(1)}+ST_{31}^{(2)} & ST_{33}^{(1)}+ST_{33}^{(2)} \end{pmatrix} \cdot \begin{pmatrix} H_1 \\ H_2 \\ H_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -ST_{12}^{(2)} \cdot H_3 \\ -ST_{32}^{(2)} \cdot H_3 \end{pmatrix} \tag{3.49}$$

3.6 Current Density Formulation

The previous section has shown how the magnetic field strength was found for every node. It is now possible to proceed to calculate the current density at every node. From equation (2.2) the conducting current density is determined by taking the curl of $\overrightarrow{H}$. It has also been seen in section 3.2 that the only current density which can exist is J_ϕ. Taking the curl of $\overrightarrow{H}$ gives:

$$J_\phi = \nabla \times \overrightarrow{H} = \hat{\phi}\left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right] \tag{3.50}$$

But H_r is zero outside the object and thus must also be zero inside because of the boundary condition (3.5) which states that the normal component of the magnetic field must be continuous across the boundary. Equation (3.50) thus reduces to:

$$J_\phi = \nabla \times \overrightarrow{H} = -\hat{\phi}\left[\frac{\partial H_z}{\partial r}\right] \tag{3.51}$$

Numerically, $\vec{J}$ is found using the different finite differences formulae depending on the positions of the element under consideration with respect to the other elements forming the object. (3.51a), (3.51b), and (3.51c) respectively show the equation to find $\vec{J}$ using the backward, central, and forward difference formulae:

$$\vec{J} = \nabla \times \overrightarrow{H} = \hat{\phi}\left(\frac{H_z(r) - H_z(r-h)}{h}\right) \tag{3.51a}$$

$$\vec{J} = \nabla \times \overrightarrow{H} = \hat{\phi}\left(\frac{H_z(r-h) - H_z(r+h)}{2h}\right) \tag{3.51b}$$

$$\vec{J} = \nabla \times \overrightarrow{H} = \hat{\phi}\left(\frac{H_z(r+h) - H_z(r)}{h}\right) \tag{3.51c}$$

where r is the position of the node and h is the step size between nodes along the r axis.

The backward and forward differences are used only in conjunction with nodes which lie on the boundary of the object. The central difference formula is used for all other nodes inside the object.

For example, the situation depicted in figure 3.5 is considered.

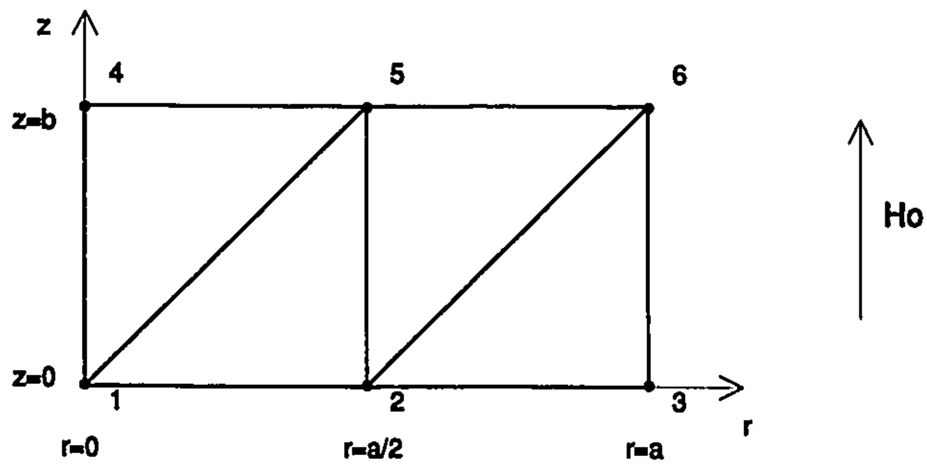


Figure 3.5: Finite Differences Scheme to calculate $\overline{J}$ for the 2-D FEM formulation.

This figure represents a disk of radius a and height b with a conductivity σ. This disk is submerged in a magnetic field $\overrightarrow{H} = H_o\hat{z}$. The magnetic field at each node is found as described in the previous section. J_ϕ is then found at each node using the finite differences formulae (3.51a), (3.51b), and (3.51c) substituting $h = a/2$. The forward difference formula is used to find J_ϕ at nodes 1 and 4. For nodes 2 and 5 the central differences formula is used. J_ϕ is found using the backward difference formula at the remaining nodes 3 and 6.

It has been shown how the 2-D FEM formulation has been developed. This algorithm will be used to compare the results of the 3-D FEM algorithm.

Chapter 4 3-D FEM Formulation

The 3-D FEM formulation allows for the complete analysis of a non symmetric body of finite dimensions. Low frequency approximations are used. Thus the only restriction applicable is that the body should be small compared to the wavelength of the magnetic field. The approximation made in the analysis is that the displacement currents are negligible compared to the conduction currents.

The method used to find the induced current densities is the finite element method which uses the weighted residuals approach with Galerkin's option. This could also be called Method of Moments with Galerkin's approach with sub-sectional basis. This, in principle, yields the same system of equations as the FEM method using the variational approach. The method is based on a similar method developed by S.J. Salon and J.P. Peng[32]. The code was written by the author of this thesis and is original.

Maxwell's equations are used as a starting point. Then the definitions of the vector magnetic potential $\vec{A}$ and scalar potential ϕ are used to formulate the problem in terms of these quantities. The boundary conditions are easily solved once $\vec{A}$ and ϕ are found everywhere inside the body being studied. It is then an easy task to determine the current densities inside this body.

The finite element used for this analysis is the parallelipiped shown in figure 4.1. The body under study can thus be partitioned into parallelipipeds of different electrical properties; in this case conductivities. The human body can thus be modelled this way.

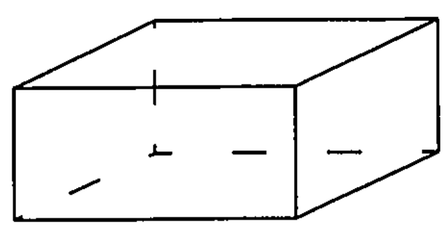


Figure 4.1: A typical finite element parallelipiped for the 3-D FEM formulation.

4.1 Formulation of the Problem

As was stated before, the formulation starts with Maxwell's equation in local form. The solution is then formulated using the vector magnetic potential and the scalar potential. The current densities can then be calculated.

Maxwell's equations in local form are:

$$\nabla \times \vec{E} = -j\omega\vec{B} \tag{2.1}$$

$$\nabla \times \vec{H} = j\omega\vec{D} + \vec{J} \tag{2.2}$$

$$\nabla \cdot \vec{D} = \rho \tag{2.3}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.4}$$

and the constitutive relations are:

$$\vec{D} = \varepsilon\vec{E} \tag{2.5}$$

$$\vec{B} = \mu\vec{H} \tag{2.6}$$

$$\vec{J} = \sigma\vec{E} \tag{2.7}$$

again at low frequencies, the displacement current is negligible compared to the conduction current

$$|j\omega\vec{D}| \ll |\vec{J}| \Rightarrow \nabla \times \vec{H} = \vec{J} \tag{2.8}$$

also, since there are no sources in the body:

$$\rho = 0 \Rightarrow \nabla \cdot \vec{D} = 0 \tag{4.1}$$

The vector magnetic potential $\vec{A}$ and the scalar potential ϕ are defined below:

$$\vec{B} = \nabla \times \vec{A} \tag{4.2}$$

$$\vec{E} = -j\omega\vec{A} - \nabla\phi \tag{4.3}$$

Since there are two unknowns to solve for ($\vec{A}$ and ϕ) two equations are needed. The two equations will be the current continuity Equation and the diffusion equation.

The current continuity equation is found by taking the divergence of (2.8). Thus

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} \tag{4.4}$$

and substituting (2.7) and (4.3) into (4.4) yields the following equation

$$\nabla \cdot (\sigma(j\omega\vec{A} + \nabla\phi)) = 0 \tag{4.5}$$

The diffusion equation is found in the following manner. Substituting (2.6) into (4.2) yields

$$\frac{1}{\mu}\nabla \times \vec{A} = \vec{H} \tag{4.6}$$

Taking the curl on both sides of this equation

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \vec{A}\right) = \nabla \times \vec{H} = \vec{J} \tag{4.7}$$

and substituting (2.7) and (4.3) yields

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \vec{A}\right) = \sigma(-j\omega\vec{A} - \nabla\phi) \tag{4.8}$$

Recapitulating, the coupled equations to solve are

$$\nabla \times \frac{1}{\mu}\nabla \times \vec{A} + \sigma(j\omega\vec{A} + \nabla\phi) = 0 \tag{4.9}$$

and

$$\nabla \cdot (\sigma(j\omega\vec{A} + \nabla\phi)) = 0 \tag{4.10}$$

4.2 FEM Formulation

The two coupled equations to solve are (4.9) and (4.10). The task now is to formulate these two equations into a Finite Element formulation. The following will explain the steps needed to accomplish this task.

Let $\hat{A}$ and $\hat{\phi}$ be approximate solutions to (4.9) and (4.10) and let $\overline{W}$ and W be a vector weighting function and a scalar weighting function respectively. Next, the method of weighted residuals is applied to the resulting equations. This method is in fact a scalar product between the weighting functions ($\overline{W}$ and W) and the approximate solutions ($\hat{A}$ and ϕ). The definition of a scalar product is:

$$<\vec{u},\vec{v}>=\int_V \vec{u}\cdot\vec{v}^{*}dV$$

if $\vec{v}$ is real, this reduces to

$$\langle\vec{u},\vec{v}\rangle=\int_V \vec{u}\cdot\vec{v}dV \tag{4.11}$$

The first step is to take the scalar products of both sides of equations (4.9) and (4.10). The result is:

$$\left\langle\nabla\times\frac{1}{\mu}\nabla\times\hat{A}+\sigma(j\omega\hat{A}+\nabla\hat{\phi}),\overline{W}\right\rangle=\int_R\left(\nabla\times\frac{1}{\mu}\nabla\times\hat{A}+\sigma(j\omega\hat{A}+\nabla\hat{\phi})\right)\cdot\overline{W}dR \tag{4.12}$$

and

$$\langle\nabla\cdot(\sigma(j\omega\hat{A}+\nabla\hat{\phi})),W\rangle=\int_R(\nabla\cdot(j\omega\hat{A}+\nabla\hat{\phi})))WdR \tag{4.13}$$

The resulting equations are:

$$\int_R\overline{W}\cdot\left(\nabla\times\frac{1}{\mu}\nabla\times\hat{A}+\sigma(j\omega\hat{A}+\nabla\hat{\phi})\right)dR=R_1 \tag{4.14}$$

and

$$\int_R W(\nabla\cdot(j\omega\hat{A}+\nabla\hat{\phi})))dR=R_2 \tag{4.15}$$

It was stated earlier that the body under study can be divided into many elements with each element having a different electrical property: in this case, each element has its own conductivity σ. If the body is modelled by using NB elements and if each element is allowed to have a different conductivity σ, the coupled equations (4.14) and (4.15) may now be written as follows:

$$\sum_{k=1}^{NB}\left(\int_{R^k}\left(\overline{W}^k\cdot\nabla\times\frac{1}{\mu}\nabla\times\hat{A}^k\right)dR^k+\int_{R^k}(\sigma^k\overline{W}^k\cdot(j\omega\hat{A}^k+\nabla\hat{\phi}^k))dR^k\right)=\sum_{k=1}^{NB}R_1^k \tag{4.16}$$

and

$$\sum_{k=1}^{NB}\left(\int_{R^k}(W^k\nabla\cdot(\sigma^k(j\omega\hat{A}^k+\nabla\phi^k)))dR^k\right)=\sum_{k=1}^{NB}R_2^k \tag{4.17}$$

Forcing $\hat{A}$ and $\hat{\phi}$ to converge to the exact solution is accomplished by forcing $\sum_{k=1}^{NB} R_1^k$ and $\sum_{k=1}^{NB} R_2^k$ to zero.

The divergence theorem is:

$$\int_V \nabla \cdot \vec{c}\, dV = \int_S \vec{c} \cdot \vec{n}\, dS \tag{4.18}$$

It, and vector identities are applied to (4.16) and (4.17) which results in the following equations:

$$\sum_k \left(\frac{1}{\mu}\int_{R^k} (\nabla \times \overline{W}^k) \cdot (\nabla \times \hat{A})dR^k - \int_{S^k}\left(\overline{W}^k \times \left(\frac{1}{\mu}\nabla \times \hat{A}^k\right)\right) \cdot \hat{n}dS^k\right.$$
$$\left. + j\omega\sigma^k \int_{R^k} \overline{W}^k \cdot \hat{A}dR^k + \sigma^k \int_{R^k} \overline{W}^k \cdot \nabla\hat{\phi}dR^k\right) = 0 \tag{4.19}$$

and

$$\sum_k \left(\int_{R^k} (\nabla W^k \cdot \sigma^k (j\omega\hat{A}^k + \nabla\hat{\phi}^k))dR^k - \int_{S^k} W^k(\sigma^k(j\omega\hat{A}^k + \nabla\hat{\phi}^k)) \cdot \hat{n}dS^k\right) = 0 \tag{4.20}$$

but, form (2.6) and (4.2): $\hat{H} = \frac{1}{\mu}\nabla \times \hat{A}$ and from (2.7) and (4.3): $J_n = -\sigma(j\omega\hat{A} + \nabla\hat{\phi}) \cdot \vec{n}$, the coupled equations can now be written as:

$$\sum_k \left(\frac{1}{\mu}\int_{R^k} (\nabla \times \overline{W}^k) \cdot (\nabla \times \hat{A}^k)dR^k + j\omega\sigma^k \int_{R^k} \overline{W}^k \cdot \hat{A}^k dR^k\right.$$
$$\left. + \sigma^k \int_{R^k} \overline{W}^k \cdot \nabla\hat{\phi}^k dR^k\right) = \sum_k \int_{S^k} (\overline{W}^k \times \hat{H}^k) \cdot \hat{n}dS^k \tag{4.21}$$

and

$$\sum_k \left(\int_{R^k} \nabla W^k \cdot \sigma^k (j\omega\hat{A}^k + \nabla\hat{\phi}^k)dR^k\right) = \sum_k \int_{S^k} W^k J_n^k dS^k \tag{4.22}$$

Two terms in these equations are of particular interest. One of the terms is used to ensure the continuity of currents and the other ensures that the boundary conditions are satisfied.

The term

$$\sum_k \int_{S^k} W^k J_n^k dS^k$$

ensures the current continuation criteria. An examination of this term, leads to the conclusion that this term must be zero. The reasons are as follows:

a) if the surface of the element of interest lies on the surface of the body, the current normal component must be zero,

b) if the surface of the element of interest does not lie on the surface of the body, but inside the body, the contribution of both elements sharing this surface is such that the above term becomes zero on this surface.

Thus, to ensure the current continuation criteria:

$$\sum_k \int_{S^k} W^k J_n^k dS^k = 0 \tag{4.23}$$

The term

$$\sum_k \int_{S^k} (\overline{W}^k \times \hat{H}^k) \cdot \vec{n} dS^k$$

ensures the boundary conditions. The same argument as above applies here as for the preceding term for an element inside the body. The magnetic field must be continuous across the boundary because the permittivity of the body is the same as free space permittivity. But, when the element is on the surface of the body, the magnetic field intensity cannot be zero, it must be the same as the outside magnetic field intensity. Thus,

$$\sum_k \int_{S^k} (\overline{W}^k \times \hat{H}^k) \cdot \vec{n} dS^k \quad \begin{cases} = 0 \text{ for elements inside the body} \\ \neq 0 \text{ for elements on the surface of the body} \end{cases} \tag{4.24}$$

The coupled equations to solve can now be written as:

$$\sum_k \left(\frac{1}{\mu} \int_{R^k} (\nabla \times \overline{W}^k) \cdot (\nabla \times \hat{A}^k) dR^k + j\omega\sigma^k \int_{R^k} \overline{W}^k \cdot \hat{A}^k dR^k + \sigma^k \int_{R^k} \overline{W}^k \cdot \nabla\hat{\phi}^k dR^k \right)$$

$$= \sum_k \int_{S^k} (\overline{W}^k \times \hat{H}^k) \cdot \vec{n} dS^k \quad \begin{cases} = 0 \text{ for elements inside the body} \\ \neq 0 \text{ for elements on the surface of the body} \end{cases} \tag{4.25}$$

and

$$\sum_k \left(\int_{R^k} \nabla W^k \cdot \sigma^k (j\omega\hat{A}^k + \nabla\hat{\phi}^k) dR^k \right) = 0 \tag{4.26}$$

4.3 A Look at One Element

The previous section describes the coupled equations to solve. This section will provide a more detailed look at the parameters used for one element. The finite element used is a parallelipiped. Each surface of the parallelipiped is parallel to one of the axis in the cartesian coordinate system. The node numbering scheme is shown below in figure 4.2.

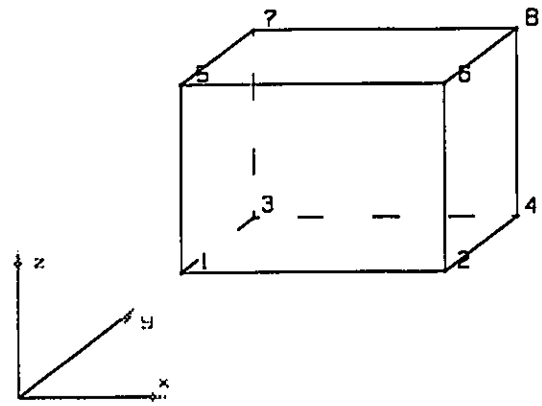


Figure 4.2: Node numbering scheme for an element for the 3-D FEM formulation.

The table 4.1 illustrates the correlation between the local node numbering a the local coordinates.

Node	Coordinates
1	(x_1, y_1, z_1)
2	(x_2, y_2, z_2)
3	(x_3, y_3, z_3)
4	(x_4, y_4, z_4)
5	(x_5, y_5, z_5)
6	(x_6, y_6, z_6)
7	(x_7, y_7, z_7)
8	(x_8, y_8, z_8)

Table 4.1: Coordinates of the basic finite element nodes for the 3-D FEM formulation.

The approximate values for the vector magnetic potential and the scalar potential, $\hat{A}$ and $\hat{\phi}$, are defined through a linear interpolation function α. The weighting functions are functions of this exact same interpolation functions, thus the name Galerkin's approach.

The linear interpolative function for element k (one of the elements defining the body studied) is:

$$[\alpha]=\begin{cases}[0] & \text{for } (x,y,z)\notin R^k \\ [\alpha_1,\alpha_2,\ldots,\alpha_8] & \text{for } (x,y,z)\in R^k\end{cases} \tag{4.27}$$

where

$$\alpha_i(x_j,y_j,z_j)=\begin{cases}0 \text{ for } i\neq j \\ 1 \text{ for } i=j\end{cases}$$

Therefore,

$$[\alpha]^T=\begin{bmatrix}\alpha_1\\ \cdot\\ \cdot\\ \cdot\\ \alpha_8\end{bmatrix} \tag{4.28}$$

where

$$\alpha_1=\frac{-(x-x_8)(y-y_8)(z-z_8)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \tag{4.28a}$$

$$\alpha_2=\frac{(x-x_1)(y-y_8)(z-z_8)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \tag{4.28b}$$

$$\alpha_3=\frac{(x-x_8)(y-y_1)(z-z_8)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \tag{4.28c}$$

$$\alpha_4=\frac{-(x-x_1)(y-y_1)(z-z_8)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \tag{4.28d}$$

$$\alpha_5=\frac{(x-x_8)(y-y_8)(z-z_1)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \tag{4.28e}$$

$$\alpha_6=\frac{-(x-x_1)(y-y_8)(z-z_1)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \tag{4.28f}$$

$$\alpha_7=\frac{-(x-x_8)(y-y_1)(z-z_1)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \tag{4.28g}$$

$$\alpha_8=\frac{(x-x_1)(y-y_1)(z-z_1)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \tag{4.28h}$$

The unknowns $\hat{A}$ and $\hat{\phi}$ are defined as:

$$\hat{\phi} = \sum_{i=1}^{8} \alpha_i \hat{\varphi}_i = [\alpha][\hat{\phi}] \tag{4.29}$$

$$\hat{A} = \sum_{i=1}^{8} \alpha_i \hat{A}_{xi}\hat{x} + \alpha_i \hat{A}_{yi}\hat{y} + \alpha_i \hat{A}_{zi}\hat{z}$$

$$= ([\alpha]\hat{x} \quad [\alpha]\hat{y} \quad [\alpha]\hat{z}) \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix} \tag{4.30}$$

and the weighting functions are defined as:

$$W = [\alpha]^T \tag{4.31}$$

$$\overline{W} = \begin{pmatrix} [\alpha]^T\hat{x} \\ [\alpha]^T\hat{y} \\ [\alpha]^T\hat{z} \end{pmatrix} \tag{4.32}$$

where ^{T} denotes the transpose of the matrix.

The equations (4.21) and (4.22) for element k can be written as:

$$\frac{1}{\mu}\int_{R^k} (\nabla \times \overline{W}^k) \cdot (\nabla \times \hat{A}^k) dR^k + j\omega\sigma^k \int_{R^k} \overline{W}^k \hat{A}^k dR^k$$

$$+\sigma^k \int_{R^k} \overline{W}^k \cdot \nabla\hat{\phi}^k dR^k = \int_{S^k} (\overline{W}^k \times \hat{H}^k) \cdot \vec{n} dS^k \tag{4.33}$$

and

$$\int_{R^k} \nabla W^k \cdot \sigma^k (j\omega\hat{A} + \nabla\hat{\phi}) dR^k = 0 \tag{4.34}$$

Equations (4.33) and (4.34) must be written in a matrix form of system of linear equations. A first look at these equations reveals that this matrix has the form:

$$[S] \cdot \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \\ [\hat{\phi}] \end{pmatrix} = \begin{pmatrix} [F_1] \\ [F_2] \\ [F_3] \\ [0] \end{pmatrix} \tag{4.35}$$

The form that each term of equations (4.33) and (4.34) will take and their location in the above $[S]$ matrix will be shown. The calculations leading to these results are found in Appendix 4.

$$\frac{1}{\mu}\int_{R^k}(\nabla\times\overline{W}^k)\cdot(\nabla\times\hat{A}^k)dR^k=\begin{pmatrix}[a_{11}] & [a_{12}] & [a_{13}]\\ [a_{21}] & [a_{22}] & [a_{23}]\\ [a_{31}] & [a_{32}] & [a_{33}]\end{pmatrix}\cdot\begin{pmatrix}[\hat{A}_x]\\ [\hat{A}_y]\\ [\hat{A}_z]\end{pmatrix} \tag{4.33a}$$

$$j\omega\sigma^k\int_{R^k}\overline{W}^k\hat{A}dR^k=\begin{pmatrix}[b_{11}] & [0] & [0]\\ [0] & [b_{22}] & [0]\\ [0] & [0] & [b_{33}]\end{pmatrix}\cdot\begin{pmatrix}[\hat{A}_x]\\ [\hat{A}_y]\\ [\hat{A}_z]\end{pmatrix} \tag{4.33b}$$

$$\sigma^k\int_{R^k}\overline{W}^k\cdot\nabla\hat{\phi}^k dR^k=\begin{pmatrix}[c_1]\\ [c_2]\\ [c_3]\end{pmatrix}\cdot[\hat{\phi}] \tag{4.33c}$$

$$\int_{S^k}(\overline{W}^k\times\hat{H}^k)\cdot\hat{n}dS^k=\begin{pmatrix}[F_1]\\ [F_2]\\ [F_3]\end{pmatrix} \tag{4.33d}$$

$$j\omega\sigma^k\int_{R^k}\nabla W^k\cdot\hat{A}^k dR^k=([d_1]\quad[d_2]\quad[d_3])\cdot\begin{pmatrix}[\hat{A}_x]\\ [\hat{A}_y]\\ [\hat{A}_z]\end{pmatrix} \tag{4.34a}$$

$$\sigma^k\int_{R^k}\nabla W^k\cdot\nabla\hat{\phi}^k dR^k=[e_1]\cdot[\hat{\phi}] \tag{4.34b}$$

It is to be noted that the size of each of the sub-matrices (i.e.: $[a_{11}]$, ..., $[b_{11}]$, ... $[e_1]$) is 8×8. Using the above equations, (4.33a) thru (4.34b), a final matrix equation may now be written as:

$$\begin{pmatrix}[a_{11}]+[b_{11}] & [a_{12}] & [a_{13}] & [c_1]\\ [a_{21}] & [a_{22}]+[b_{22}] & [a_{23}] & [c_2]\\ [a_{31}] & [a_{32}] & [a_{33}]+[b_{33}] & [c_3]\\ [d_1] & [d_2] & [d_3] & [e_1]\end{pmatrix}\cdot\begin{pmatrix}[\hat{A}_x]\\ [\hat{A}_y]\\ [\hat{A}_z]\\ [\hat{\phi}]\end{pmatrix}=\begin{pmatrix}[F_1]\\ [F_2]\\ [F_3]\\ [0]\end{pmatrix} \tag{4.36}$$

Thus, for one element there are eight nodes and 4x8=32 unknowns.

4.4 Global Matrix Assembly

The global matrix assembly is performed in much the same fashion as for the 2-D FEM formulation. The procedure is somewhat complicated by the fact that four unknowns per node (A_x, A_y, A_z, ϕ) are dealt with as opposed to only one in the 2-D FEM formulation.

The following example will demonstrate how the assembly process is performed when more than one unknown per node is present. Figure 4.3 shows an object which is represented by two elements. Each element has three nodes and there are two unknowns per node (A, ϕ). This is the simplest geometry which can be used to explain the assembly procedure. Using two parallelipipeds with four unknowns per node would have resulted in a (48X48) global matrix which is very impractical to write on paper and would have only clouded the issue. For this reason the geometry in figure 4.3 was chosen for this demonstration.

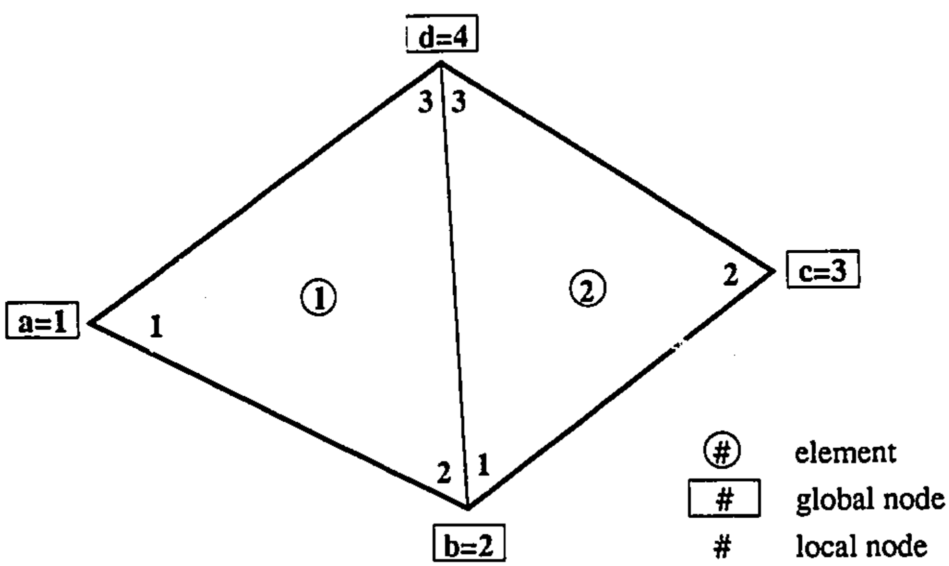


Figure 4.3: Global Matrix Assembly for the 3-D FEM formulation.

A fictional coupled set of equations similar to that of equations (4.33) and (4.34) will be used for this demonstration. They are as follows:

$$\Gamma_1(\hat{A}^k)+\Gamma_2(\hat{\phi}^k)=\Gamma_3(\hat{H}^k) \tag{4.37}$$

and

$$\Gamma_4(\hat{A}^k)+\Gamma_5(\hat{\phi}^k)=0 \tag{4.38}$$

where $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, and F_5 are arbitrary operators. The resulting matrix equation is:

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.39}$$

if the object has only one element defining it.

To get a clearer picture of what is going on when more than one unknown is present, the matrices are divided into sub-matrices.

$$\begin{pmatrix} \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} & \begin{pmatrix} S_{14} & S_{15} & S_{16} \\ S_{24} & S_{25} & S_{26} \\ S_{34} & S_{35} & S_{36} \end{pmatrix} \\ \begin{pmatrix} S_{41} & S_{42} & S_{43} \\ S_{51} & S_{52} & S_{53} \\ S_{61} & S_{62} & S_{63} \end{pmatrix} & \begin{pmatrix} S_{44} & S_{45} & S_{46} \\ S_{54} & S_{55} & S_{56} \\ S_{64} & S_{65} & S_{66} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \\ \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \tag{4.40}$$

These sub-matrices are labelled:

$$[s]^{[11]}=\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \tag{4.41a}$$

$$[s]^{[12]}=\begin{pmatrix} S_{14} & S_{15} & S_{16} \\ S_{24} & S_{25} & S_{26} \\ S_{34} & S_{35} & S_{36} \end{pmatrix} \tag{4.41b}$$

$$[s]^{[21]}=\begin{pmatrix} S_{41} & S_{42} & S_{43} \\ S_{51} & S_{52} & S_{53} \\ S_{61} & S_{62} & S_{63} \end{pmatrix} \tag{4.41c}$$

$$[s]^{[22]}=\begin{pmatrix} S_{44} & S_{45} & S_{46} \\ S_{54} & S_{55} & S_{56} \\ S_{64} & S_{65} & S_{66} \end{pmatrix} \tag{4.41d}$$

The $[S]$ matrix can thus be represented as:

$$\begin{pmatrix} [S]^{[11]} & [S]^{[12]} \\ [S]^{[21]} & [S]^{[22]} \end{pmatrix} \tag{4.41}$$

Using this scheme, the $[S]$ and solution matrices for each element is represented in equation (4.42) and (4.43). The superscript $^{(e)}$ denotes the element these terms belong to.

element (1)

$$\begin{pmatrix} \begin{pmatrix} S_{11}^{(1)} & S_{12}^{(1)} & S_{13}^{(1)} \\ S_{21}^{(1)} & S_{22}^{(1)} & S_{23}^{(1)} \\ S_{31}^{(1)} & S_{32}^{(1)} & S_{33}^{(1)} \end{pmatrix} & \begin{pmatrix} S_{14}^{(1)} & S_{15}^{(1)} & S_{16}^{(1)} \\ S_{24}^{(1)} & S_{25}^{(1)} & S_{26}^{(1)} \\ S_{34}^{(1)} & S_{35}^{(1)} & S_{36}^{(1)} \end{pmatrix} \\ \begin{pmatrix} S_{41}^{(1)} & S_{42}^{(1)} & S_{43}^{(1)} \\ S_{51}^{(1)} & S_{52}^{(1)} & S_{53}^{(1)} \\ S_{61}^{(1)} & S_{62}^{(1)} & S_{63}^{(1)} \end{pmatrix} & \begin{pmatrix} S_{44}^{(1)} & S_{45}^{(1)} & S_{46}^{(1)} \\ S_{54}^{(1)} & S_{55}^{(1)} & S_{56}^{(1)} \\ S_{64}^{(1)} & S_{65}^{(1)} & S_{66}^{(1)} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \\ A_3^{(1)} \end{pmatrix} \\ \begin{pmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \\ \phi_3^{(1)} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \tag{4.42}$$

element (2)

$$\begin{pmatrix} \begin{pmatrix} S_{11}^{(2)} & S_{12}^{(2)} & S_{13}^{(2)} \\ S_{21}^{(2)} & S_{22}^{(2)} & S_{23}^{(2)} \\ S_{31}^{(2)} & S_{32}^{(2)} & S_{33}^{(2)} \end{pmatrix} & \begin{pmatrix} S_{14}^{(2)} & S_{15}^{(2)} & S_{16}^{(2)} \\ S_{24}^{(2)} & S_{25}^{(2)} & S_{26}^{(2)} \\ S_{34}^{(2)} & S_{35}^{(2)} & S_{36}^{(2)} \end{pmatrix} \\ \begin{pmatrix} S_{41}^{(2)} & S_{42}^{(2)} & S_{43}^{(2)} \\ S_{51}^{(2)} & S_{52}^{(2)} & S_{53}^{(2)} \\ S_{61}^{(2)} & S_{62}^{(2)} & S_{63}^{(2)} \end{pmatrix} & \begin{pmatrix} S_{44}^{(2)} & S_{45}^{(2)} & S_{46}^{(2)} \\ S_{54}^{(2)} & S_{55}^{(2)} & S_{56}^{(2)} \\ S_{64}^{(2)} & S_{65}^{(2)} & S_{66}^{(2)} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} A_1^{(2)} \\ A_2^{(2)} \\ A_3^{(2)} \end{pmatrix} \\ \begin{pmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \\ \phi_3^{(2)} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \tag{4.43}$$

The coefficients in the $[S]$ matrix for element $^{(e)}$ are now defined in terms of the sub-matrices coefficients. Let

$$
\begin{array}{llllll}
S_{11}^{(e)}=s_{11}^{(e)[11]} & S_{12}^{(e)}=s_{12}^{(e)[11]} & S_{13}^{(e)}=s_{13}^{(e)[11]} & S_{14}^{(e)}=s_{11}^{(e)[12]} & S_{15}^{(e)}=s_{12}^{(e)[12]} & S_{16}^{(e)}=s_{13}^{(e)[12]} \\
S_{21}^{(e)}=s_{21}^{(e)[11]} & S_{22}^{(e)}=s_{22}^{(e)[11]} & S_{23}^{(e)}=s_{23}^{(e)[11]} & S_{24}^{(e)}=s_{21}^{(e)[12]} & S_{25}^{(e)}=s_{22}^{(e)[12]} & S_{26}^{(e)}=s_{23}^{(e)[12]} \\
S_{31}^{(e)}=s_{31}^{(e)[11]} & S_{32}^{(e)}=s_{32}^{(e)[11]} & S_{33}^{(e)}=s_{33}^{(e)[11]} & S_{34}^{(e)}=s_{31}^{(e)[12]} & S_{35}^{(e)}=s_{32}^{(e)[12]} & S_{36}^{(e)}=s_{33}^{(e)[12]} \\
S_{41}^{(e)}=s_{11}^{(e)[21]} & S_{42}^{(e)}=s_{12}^{(e)[21]} & S_{43}^{(e)}=s_{13}^{(e)[21]} & S_{44}^{(e)}=s_{11}^{(e)[22]} & S_{45}^{(e)}=s_{12}^{(e)[22]} & S_{46}^{(e)}=s_{13}^{(e)[22]} \\
S_{51}^{(e)}=s_{21}^{(e)[21]} & S_{52}^{(e)}=s_{22}^{(e)[21]} & S_{53}^{(e)}=s_{23}^{(e)[21]} & S_{54}^{(e)}=s_{21}^{(e)[22]} & S_{55}^{(e)}=s_{22}^{(e)[22]} & S_{56}^{(e)}=s_{23}^{(e)[22]} \\
S_{61}^{(e)}=s_{31}^{(e)[21]} & S_{62}^{(e)}=s_{32}^{(e)[21]} & S_{63}^{(e)}=s_{33}^{(e)[21]} & S_{64}^{(e)}=s_{31}^{(e)[22]} & S_{65}^{(e)}=s_{32}^{(e)[22]} & S_{66}^{(e)}=s_{33}^{(e)[22]}
\end{array}
\tag{4.44}
$$

Equations (4.42) and (4.43) are rewritten in terms of the coefficients defined in (4.44).

element (1)

$$
\begin{pmatrix}
\begin{pmatrix} s_{11}^{(1)[11]} & s_{12}^{(1)[11]} & s_{13}^{(1)[11]} \\ s_{21}^{(1)[11]} & s_{22}^{(1)[11]} & s_{23}^{(1)[11]} \\ s_{31}^{(1)[11]} & s_{32}^{(1)[11]} & s_{33}^{(1)[11]} \end{pmatrix} &
\begin{pmatrix} s_{11}^{(1)[12]} & s_{12}^{(1)[12]} & s_{13}^{(1)[12]} \\ s_{21}^{(1)[12]} & s_{22}^{(1)[12]} & s_{23}^{(1)[12]} \\ s_{31}^{(1)[12]} & s_{32}^{(1)[12]} & s_{33}^{(1)[12]} \end{pmatrix} \\
\begin{pmatrix} s_{11}^{(1)[21]} & s_{12}^{(1)[21]} & s_{13}^{(1)[21]} \\ s_{21}^{(1)[21]} & s_{22}^{(1)[21]} & s_{23}^{(1)[21]} \\ s_{31}^{(1)[21]} & s_{32}^{(1)[21]} & s_{33}^{(1)[21]} \end{pmatrix} &
\begin{pmatrix} s_{11}^{(1)[22]} & s_{12}^{(1)[22]} & s_{13}^{(1)[22]} \\ s_{21}^{(1)[22]} & s_{22}^{(1)[22]} & s_{23}^{(1)[22]} \\ s_{31}^{(1)[22]} & s_{32}^{(1)[22]} & s_{33}^{(1)[22]} \end{pmatrix}
\end{pmatrix}
\cdot
\begin{pmatrix}
\begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \\ A_3^{(1)} \end{pmatrix} \\
\begin{pmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \\ \phi_3^{(1)} \end{pmatrix}
\end{pmatrix}
=
\begin{pmatrix}
\begin{pmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{pmatrix}
\tag{4.45}
$$

element (2)

$$
\begin{pmatrix}
\begin{pmatrix} s_{11}^{(2)[11]} & s_{12}^{(2)[11]} & s_{13}^{(2)[11]} \\ s_{21}^{(2)[11]} & s_{22}^{(2)[11]} & s_{23}^{(2)[11]} \\ s_{31}^{(2)[11]} & s_{32}^{(2)[11]} & s_{33}^{(2)[11]} \end{pmatrix} &
\begin{pmatrix} s_{11}^{(2)[12]} & s_{12}^{(2)[12]} & s_{13}^{(2)[12]} \\ s_{21}^{(2)[12]} & s_{22}^{(2)[12]} & s_{23}^{(2)[12]} \\ s_{31}^{(2)[12]} & s_{32}^{(2)[12]} & s_{33}^{(2)[12]} \end{pmatrix} \\
\begin{pmatrix} s_{11}^{(2)[21]} & s_{12}^{(2)[21]} & s_{13}^{(2)[21]} \\ s_{21}^{(2)[21]} & s_{22}^{(2)[21]} & s_{23}^{(2)[21]} \\ s_{31}^{(2)[21]} & s_{32}^{(2)[21]} & s_{33}^{(2)[21]} \end{pmatrix} &
\begin{pmatrix} s_{11}^{(2)[22]} & s_{12}^{(2)[22]} & s_{13}^{(2)[22]} \\ s_{21}^{(2)[22]} & s_{22}^{(2)[22]} & s_{23}^{(2)[22]} \\ s_{31}^{(2)[22]} & s_{32}^{(2)[22]} & s_{33}^{(2)[22]} \end{pmatrix}
\end{pmatrix}
\cdot
\begin{pmatrix}
\begin{pmatrix} A_1^{(2)} \\ A_2^{(2)} \\ A_3^{(2)} \end{pmatrix} \\
\begin{pmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \\ \phi_3^{(2)} \end{pmatrix}
\end{pmatrix}
=
\begin{pmatrix}
\begin{pmatrix} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{pmatrix}
\tag{4.46}
$$

Equations (4.47) and (4.48) are the same as (4.45) and (4.46) but are written using the global node numbers as expressed by the subscripts. For element 1 the local nodes 1, 2, and 3 are represented by the global nodes a=1, b=2, and d=4 respectively. The local nodes 1, 2, and 3 are represented by the global nodes b=2, c=3, and d=4 respectively for element 2.

element (1)

$$\begin{pmatrix} \begin{pmatrix} s_{aa}^{(1)[11]} & s_{ab}^{(1)[11]} & s_{ad}^{(1)[11]} \\ s_{ba}^{(1)[11]} & s_{bb}^{(1)[11]} & s_{bd}^{(1)[11]} \\ s_{da}^{(1)[11]} & s_{db}^{(1)[11]} & s_{dd}^{(1)[11]} \end{pmatrix} & \begin{pmatrix} s_{aa}^{(1)[12]} & s_{ab}^{(1)[12]} & s_{ad}^{(1)[12]} \\ s_{ba}^{(1)[12]} & s_{bb}^{(1)[12]} & s_{bd}^{(1)[12]} \\ s_{da}^{(1)[12]} & s_{db}^{(1)[12]} & s_{dd}^{(1)[12]} \end{pmatrix} \\ \begin{pmatrix} s_{aa}^{(1)[21]} & s_{ab}^{(1)[21]} & s_{ad}^{(1)[21]} \\ s_{ba}^{(1)[21]} & s_{bb}^{(1)[21]} & s_{bd}^{(1)[21]} \\ s_{da}^{(1)[21]} & s_{db}^{(1)[21]} & s_{dd}^{(1)[21]} \end{pmatrix} & \begin{pmatrix} s_{aa}^{(1)[22]} & s_{ab}^{(1)[22]} & s_{ad}^{(1)[22]} \\ s_{ba}^{(1)[22]} & s_{bb}^{(1)[22]} & s_{bd}^{(1)[22]} \\ s_{da}^{(1)[22]} & s_{db}^{(1)[22]} & s_{dd}^{(1)[22]} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} A_a^{(1)} \\ A_b^{(1)} \\ A_d^{(1)} \end{pmatrix} \\ \begin{pmatrix} \phi_a^{(1)} \\ \phi_b^{(1)} \\ \phi_d^{(1)} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} F_a^{(1)} \\ F_b^{(1)} \\ F_d^{(1)} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \quad (4.47)$$

element 2

$$\begin{pmatrix} \begin{pmatrix} s_{bb}^{(2)[11]} & s_{bc}^{(2)[11]} & s_{bd}^{(2)[11]} \\ s_{cb}^{(2)[11]} & s_{cc}^{(2)[11]} & s_{cd}^{(2)[11]} \\ s_{db}^{(2)[11]} & s_{dc}^{(2)[11]} & s_{dd}^{(2)[11]} \end{pmatrix} & \begin{pmatrix} s_{bb}^{(2)[12]} & s_{bc}^{(2)[12]} & s_{bd}^{(2)[12]} \\ s_{cb}^{(2)[12]} & s_{cc}^{(2)[12]} & s_{cd}^{(2)[12]} \\ s_{db}^{(2)[12]} & s_{dc}^{(2)[12]} & s_{dd}^{(2)[12]} \end{pmatrix} \\ \begin{pmatrix} s_{bb}^{(2)[21]} & s_{bc}^{(2)[21]} & s_{bd}^{(2)[21]} \\ s_{cb}^{(2)[21]} & s_{cc}^{(2)[21]} & s_{cd}^{(2)[21]} \\ s_{db}^{(2)[21]} & s_{dc}^{(2)[21]} & s_{dd}^{(2)[21]} \end{pmatrix} & \begin{pmatrix} s_{bb}^{(2)[22]} & s_{bc}^{(2)[22]} & s_{bd}^{(2)[22]} \\ s_{cb}^{(2)[22]} & s_{cc}^{(2)[22]} & s_{cd}^{(2)[22]} \\ s_{db}^{(2)[22]} & s_{dc}^{(2)[22]} & s_{dd}^{(2)[22]} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} A_b^{(2)} \\ A_c^{(2)} \\ A_d^{(2)} \end{pmatrix} \\ \begin{pmatrix} \phi_b^{(2)} \\ \phi_c^{(2)} \\ \phi_d^{(2)} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} F_b^{(2)} \\ F_c^{(2)} \\ F_d^{(2)} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \quad (4.48)$$

Each sub-matrix, as represented in (4.41), in each matrix representing an element have the size NE X NE = 3 X 3 where NE is the number of nodes describing one element. At this stage, the sub-matrices are expanded to a new size of NN X NN = 4 X 4 where NN is the total number of nodes describing the object. Each coefficient $s_{ij}^{(e)[kl]}$ is placed in the $(i,j)^{th}$ position of the expanded sub-matrices $[kl]$ for each element $^{(e)}$. The $[S]$ matrices may now be written as:

element (1)

$$\left(\begin{pmatrix} S_{aa}^{(1)[11]} & S_{ab}^{(1)[11]} & 0 & S_{ad}^{(1)[11]} \\ S_{ba}^{(1)[11]} & S_{bb}^{(1)[11]} & 0 & S_{bd}^{(1)[11]} \\ 0 & 0 & 0 & 0 \\ S_{da}^{(1)[11]} & S_{db}^{(1)[11]} & 0 & S_{dd}^{(1)[11]} \end{pmatrix} \begin{pmatrix} S_{aa}^{(1)[12]} & S_{ab}^{(1)[12]} & 0 & S_{ad}^{(1)[12]} \\ S_{ba}^{(1)[12]} & S_{bb}^{(1)[12]} & 0 & S_{bd}^{(1)[12]} \\ 0 & 0 & 0 & 0 \\ S_{da}^{(1)[12]} & S_{db}^{(1)[12]} & 0 & S_{dd}^{(1)[12]} \end{pmatrix} \atop \begin{pmatrix} S_{aa}^{(1)[21]} & S_{ab}^{(1)[21]} & 0 & S_{ad}^{(1)[21]} \\ S_{ba}^{(1)[21]} & S_{bb}^{(1)[21]} & 0 & S_{bd}^{(1)[21]} \\ 0 & 0 & 0 & 0 \\ S_{da}^{(1)[21]} & S_{db}^{(1)[21]} & 0 & S_{dd}^{(1)[21]} \end{pmatrix} \begin{pmatrix} S_{aa}^{(1)[22]} & S_{ab}^{(1)[22]} & 0 & S_{ad}^{(1)[22]} \\ S_{ba}^{(1)[22]} & S_{bb}^{(1)[22]} & 0 & S_{bd}^{(1)[22]} \\ 0 & 0 & 0 & 0 \\ S_{da}^{(1)[22]} & S_{db}^{(1)[22]} & 0 & S_{dd}^{(1)[22]} \end{pmatrix}\right) \cdot \left(\begin{pmatrix} A_a^{(1)} \\ A_b^{(1)} \\ 0 \\ A_d^{(1)} \end{pmatrix} \atop \begin{pmatrix} \phi_a^{(1)} \\ \phi_b^{(1)} \\ 0 \\ \phi_d^{(1)} \end{pmatrix}\right) = \left(\begin{pmatrix} F_a^{(1)} \\ F_b^{(1)} \\ 0 \\ F_d^{(1)} \end{pmatrix} \atop \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) \qquad (4.49)$$

element (2)

$$\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{bb}^{(2)[11]} & S_{bc}^{(2)[11]} & S_{bd}^{(2)[11]} \\ 0 & S_{cb}^{(2)[11]} & S_{cc}^{(2)[11]} & S_{cd}^{(2)[11]} \\ 0 & S_{db}^{(2)[11]} & S_{dc}^{(2)[11]} & S_{dd}^{(2)[11]} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{bb}^{(2)[12]} & S_{bc}^{(2)[12]} & S_{bd}^{(2)[12]} \\ 0 & S_{cb}^{(2)[12]} & S_{cc}^{(2)[12]} & S_{cd}^{(2)[12]} \\ 0 & S_{db}^{(2)[12]} & S_{dc}^{(2)[12]} & S_{dd}^{(2)[12]} \end{pmatrix} \atop \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{bb}^{(2)[21]} & S_{bc}^{(2)[21]} & S_{bd}^{(2)[21]} \\ 0 & S_{cb}^{(2)[21]} & S_{cc}^{(2)[21]} & S_{cd}^{(2)[21]} \\ 0 & S_{db}^{(2)[21]} & S_{dc}^{(2)[21]} & S_{dd}^{(2)[21]} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{bb}^{(2)[22]} & S_{bc}^{(2)[22]} & S_{bd}^{(2)[22]} \\ 0 & S_{cb}^{(2)[22]} & S_{cc}^{(2)[22]} & S_{cd}^{(2)[22]} \\ 0 & S_{db}^{(2)[22]} & S_{dc}^{(2)[22]} & S_{dd}^{(2)[22]} \end{pmatrix}\right) \cdot \left(\begin{pmatrix} 0 \\ A_b^{(2)} \\ A_c^{(2)} \\ A_d^{(2)} \end{pmatrix} \atop \begin{pmatrix} 0 \\ \phi_b^{(2)} \\ \phi_c^{(2)} \\ \phi_d^{(2)} \end{pmatrix}\right) = \left(\begin{pmatrix} 0 \\ F_b^{(2)} \\ F_c^{(2)} \\ F_d^{(2)} \end{pmatrix} \atop \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) \qquad (4.50)$$

The $[S]$ and solution matrices must now be written using the local node numbers. However, the $[\hat{A},\hat{\phi}]$ matrices representing the unknowns keep the global node numbering scheme. Substituting the coefficients of equation (4.44) into (4.49) and (4.50) yields the following element equations:

element (1)

$$\begin{pmatrix} \begin{pmatrix} S_{11}^{(1)} & S_{12}^{(1)} & 0 & S_{13}^{(1)} \\ S_{21}^{(1)} & S_{22}^{(1)} & 0 & S_{23}^{(1)} \\ 0 & 0 & 0 & 0 \\ S_{31}^{(1)} & S_{32}^{(1)} & 0 & S_{33}^{(1)} \end{pmatrix} & \begin{pmatrix} S_{14}^{(1)} & S_{15}^{(1)} & 0 & S_{16}^{(1)} \\ S_{24}^{(1)} & S_{25}^{(1)} & 0 & S_{26}^{(1)} \\ 0 & 0 & 0 & 0 \\ S_{34}^{(1)} & S_{35}^{(1)} & 0 & S_{36}^{(1)} \end{pmatrix} \\ \begin{pmatrix} S_{41}^{(1)} & S_{42}^{(1)} & 0 & S_{43}^{(1)} \\ S_{51}^{(1)} & S_{52}^{(1)} & 0 & S_{53}^{(1)} \\ 0 & 0 & 0 & 0 \\ S_{61}^{(1)} & S_{62}^{(1)} & 0 & S_{63}^{(1)} \end{pmatrix} & \begin{pmatrix} S_{44}^{(1)} & S_{45}^{(1)} & 0 & S_{46}^{(1)} \\ S_{54}^{(1)} & S_{55}^{(1)} & 0 & S_{56}^{(1)} \\ 0 & 0 & 0 & 0 \\ S_{64}^{(1)} & S_{65}^{(1)} & 0 & S_{66}^{(1)} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} A_a^{(1)} \\ A_b^{(1)} \\ 0 \\ A_d^{(1)} \end{pmatrix} \\ \begin{pmatrix} \phi_a^{(1)} \\ \phi_b^{(1)} \\ 0 \\ \phi_d^{(1)} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} F_1^{(1)} \\ F_2^{(1)} \\ 0 \\ F_3^{(1)} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \qquad (4.51)$$

element (2)

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{11}^{(2)} & S_{12}^{(2)} & S_{13}^{(2)} \\ 0 & S_{21}^{(2)} & S_{22}^{(2)} & S_{23}^{(2)} \\ 0 & S_{31}^{(2)} & S_{32}^{(2)} & S_{33}^{(2)} \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{14}^{(2)} & S_{15}^{(2)} & S_{16}^{(2)} \\ 0 & S_{24}^{(2)} & S_{25}^{(2)} & S_{26}^{(2)} \\ 0 & S_{34}^{(2)} & S_{35}^{(2)} & S_{36}^{(2)} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{41}^{(2)} & S_{42}^{(2)} & S_{43}^{(2)} \\ 0 & S_{51}^{(2)} & S_{52}^{(2)} & S_{53}^{(2)} \\ 0 & S_{61}^{(2)} & S_{62}^{(2)} & S_{63}^{(2)} \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{44}^{(2)} & S_{45}^{(2)} & S_{46}^{(2)} \\ 0 & S_{54}^{(2)} & S_{55}^{(2)} & S_{56}^{(2)} \\ 0 & S_{64}^{(2)} & S_{65}^{(2)} & S_{66}^{(2)} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} 0 \\ A_b^{(2)} \\ A_c^{(2)} \\ A_d^{(2)} \end{pmatrix} \\ \begin{pmatrix} 0 \\ \phi_b^{(2)} \\ \phi_c^{(2)} \\ \phi_d^{(2)} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 \\ F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \qquad (4.52)$$

The global $[GS]$and $[GF]$ matrix assembly is performed by simply adding the two $[S]$ element matrices to the $[GS]$ matrix and adding the two element solution matrices to the $[GF]$ matrix respectively.

$$[GS]=[S]^{(1)}+[S]^{(2)}=$$

$$\begin{pmatrix}\begin{pmatrix}S_{11}^{(1)} & S_{12}^{(1)} & 0 & S_{13}^{(1)}\\ S_{21}^{(1)} & S_{22}^{(1)}+S_{11}^{(2)} & S_{12}^{(2)} & S_{23}^{(1)}+S_{13}^{(2)}\\ 0 & S_{21}^{(2)} & S_{22}^{(2)} & S_{23}^{(2)}\\ S_{31}^{(1)} & S_{32}^{(1)}+S_{31}^{(2)} & S_{32}^{(2)} & S_{33}^{(1)}+S_{33}^{(2)}\end{pmatrix} & \begin{pmatrix}S_{14}^{(1)} & S_{15}^{(1)} & 0 & S_{16}^{(1)}\\ S_{24}^{(1)} & S_{25}^{(1)}+S_{14}^{(2)} & S_{15}^{(2)} & S_{26}^{(1)}+S_{16}^{(2)}\\ 0 & S_{24}^{(2)} & S_{25}^{(2)} & S_{26}^{(2)}\\ S_{34}^{(1)} & S_{35}^{(1)}+S_{34}^{(2)} & S_{35}^{(2)} & S_{36}^{(1)}+S_{36}^{(2)}\end{pmatrix}\\ \begin{pmatrix}S_{41}^{(1)} & S_{42}^{(1)} & 0 & S_{43}^{(1)}\\ S_{51}^{(1)} & S_{52}^{(1)}+S_{41}^{(2)} & S_{42}^{(2)} & S_{53}^{(1)}+S_{43}^{(2)}\\ 0 & S_{51}^{(2)} & S_{52}^{(2)} & S_{53}^{(2)}\\ S_{61}^{(1)} & S_{62}^{(1)}+S_{61}^{(2)} & S_{62}^{(2)} & S_{63}^{(1)}+S_{63}^{(2)}\end{pmatrix} & \begin{pmatrix}S_{44}^{(1)} & S_{45}^{(1)} & 0 & S_{46}^{(1)}\\ S_{54}^{(1)} & S_{55}^{(1)}+S_{44}^{(2)} & S_{45}^{(2)} & S_{56}^{(1)}+S_{46}^{(2)}\\ 0 & S_{54}^{(2)} & S_{55}^{(2)} & S_{56}^{(2)}\\ S_{64}^{(1)} & S_{65}^{(1)}+S_{64}^{(2)} & S_{65}^{(2)} & S_{66}^{(1)}+S_{66}^{(2)}\end{pmatrix}\end{pmatrix} \qquad (4.53)$$

$$[GF]=[F]^{(1)}+[F]^{(2)}=$$

$$\begin{pmatrix}\begin{pmatrix}F_1^{(1)}\\ F_2^{(1)}+F_1^{(2)}\\ F_2^{(2)}\\ F_3^{(1)}+F_3^{(2)}\end{pmatrix}\\ \begin{pmatrix}0\\0\\0\\0\end{pmatrix}\end{pmatrix} \qquad (4.54)$$

The following final system of equation, $[GS]\cdot\begin{pmatrix}[GA]\\ [G\phi]\end{pmatrix}=[GF]$, to solve for A and ϕ at each node shown in figure 4.4, is written as:

$$\begin{pmatrix}S_{11}^{(1)} & S_{12}^{(1)} & 0 & S_{13}^{(1)} & S_{14}^{(1)} & S_{15}^{(1)} & 0 & S_{16}^{(1)}\\ S_{21}^{(1)} & S_{22}^{(1)}+S_{11}^{(2)} & S_{12}^{(2)} & S_{23}^{(1)}+S_{13}^{(2)} & S_{24}^{(1)} & S_{25}^{(1)}+S_{14}^{(2)} & S_{15}^{(2)} & S_{26}^{(1)}+S_{16}^{(2)}\\ 0 & S_{21}^{(2)} & S_{22}^{(2)} & S_{23}^{(2)} & 0 & S_{24}^{(2)} & S_{25}^{(2)} & S_{26}^{(2)}\\ S_{31}^{(1)} & S_{32}^{(1)}+S_{31}^{(2)} & S_{32}^{(2)} & S_{33}^{(1)}+S_{33}^{(2)} & S_{34}^{(1)} & S_{35}^{(1)}+S_{34}^{(2)} & S_{35}^{(2)} & S_{36}^{(1)}+S_{36}^{(2)}\\ S_{41}^{(1)} & S_{42}^{(1)} & 0 & S_{43}^{(1)} & S_{44}^{(1)} & S_{45}^{(1)} & 0 & S_{46}^{(1)}\\ S_{51}^{(1)} & S_{52}^{(1)}+S_{41}^{(2)} & S_{42}^{(2)} & S_{53}^{(1)}+S_{43}^{(2)} & S_{54}^{(1)} & S_{55}^{(1)}+S_{44}^{(2)} & S_{45}^{(2)} & S_{56}^{(1)}+S_{46}^{(2)}\\ 0 & S_{51}^{(2)} & S_{52}^{(-)} & S_{53}^{(2)} & 0 & S_{54}^{(2)} & S_{55}^{(2)} & S_{56}^{(2)}\\ S_{61}^{(1)} & S_{62}^{(1)}+S_{61}^{(2)} & S_{62}^{(2)} & S_{63}^{(1)}+S_{63}^{(2)} & S_{64}^{(1)} & S_{65}^{(1)}+S_{64}^{(2)} & S_{65}^{(2)} & S_{66}^{(1)}+S_{66}^{(2)}\end{pmatrix}\cdot\begin{pmatrix}A_a\\ A_b\\ A_c\\ A_d\\ \phi_a\\ \phi_b\\ \phi_c\\ \phi_d\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}F_1^{(1)}\\ F_2^{(1)}+F_1^{(2)}\\ F_2^{(2)}\\ F_3^{(1)}+F_3^{(2)}\end{pmatrix}\\ \begin{pmatrix}0\\0\\0\\0\end{pmatrix}\end{pmatrix} \qquad (4.55)$$

As was stated earlier, the 3-D FEM formulation uses parallelipipeds (6 sides, 8 nodes) as elements and each node has four unknowns (A_x, A_y, A_z, ϕ). Thus the final system of equations for the complete 3-D FEM formulation is:

$$[GS]\cdot\begin{pmatrix}[A_x]\\ [A_y]\\ [A_z]\\ [\phi]\end{pmatrix}=\begin{pmatrix}[GF_1]\\ [GF_2]\\ [GF_3]\\ [0]\end{pmatrix} \tag{4.56}$$

The object under study will be modelled by a certain number of adjacent elements, each having their own electrical properties (σ), in which there will be NN nodes in total. Thus for NN nodes, each with four unknowns, the size of the global $[GS]$ matrix will be $(4\times NN, 4\times NN)$. The $[GA_x]$, $[GA_y]$, $[GA_z]$, $[\phi]$ matrices will be $(NN\times 1)$ in size as will be the $[GF_1]$, $[GF_2]$, $[GF_3]$, and $[0]$ matrices.

4.5 Current Density Formulation

The $[S]$ matrix for each element was calculated and then assembled into a global matrix $[GS]$. The global system of equations was solved and the unknowns $\vec{A}$ and ϕ at each node defining the object was determined. The remaining step left is to determine the current densities in the object. These current densities are found in the center of each element. Since $\vec{A}$ and ϕ at each node of each element is now known, it is an easy task to calculate the current density $\vec{J}$ inside each element. The conduction current density is defined by equations (2.7) and (4.3) as:

$$\vec{J}=\sigma\vec{E}=\sigma(-j\omega\vec{A}-\nabla\phi) \tag{4.57}$$

Making use of the previous numerical equivalence of ϕ and $\vec{A}$ in (4.29) and (4.30):

$$\phi=[\alpha]\cdot[\phi] \qquad \vec{A}=([\alpha]\hat{x}\quad[\alpha]\hat{y}\quad[\alpha]\hat{z})\cdot\begin{pmatrix}[\hat{A}_x]\\ [\hat{A}_y]\\ [\hat{A}_z]\end{pmatrix}$$

the current density can now be written as:

$$\vec{J} = -j\omega\sigma([\alpha]\hat{x} \quad [\alpha]\hat{y} \quad [\alpha]\hat{z}) \cdot \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix} - \sigma\nabla([\alpha] \cdot [\phi]) \tag{4.58}$$

The current density (eq. (4.58)) can be written in such a way that the contributions of $\vec{A}$ and ϕ at each node of each element is more apparent. This equivalent equation is:

$$\vec{J} = -j\omega\sigma([\alpha][\hat{A}x]\hat{x} + [\alpha][\hat{A}y]\hat{y} + [\alpha][\hat{A}z]\hat{z})$$

$$-\sigma\left(\left[\frac{\partial\alpha}{\partial x}\right][\hat{\phi}]\hat{x} + \left[\frac{\partial\alpha}{\partial y}\right][\hat{\phi}]\hat{y} + \left[\frac{\partial\alpha}{\partial z}\right][\hat{\phi}]\hat{z}\right)$$

which is equal to:

$$\begin{aligned} \vec{J} = & -\sigma\sum_{i=1}^{8}\left(\frac{\partial\alpha_i}{\partial x}\phi_i + j\omega\alpha_i A_{xi}\right)\hat{x} \\ & -\sigma\sum_{i=1}^{8}\left(\frac{\partial\alpha_i}{\partial y}\phi_i + j\omega\alpha_i A_{yi}\right)\hat{y} \\ & -\sigma\sum_{i=1}^{8}\left(\frac{\partial\alpha_i}{\partial z}\phi_i + j\omega\alpha_i A_{zi}\right)\hat{z} \end{aligned} \tag{4.59}$$

The current density is calculated in the center of the element at $(x = x_\zeta, y = y_\zeta, z = z_\zeta)$ as shown in figure 4.4.

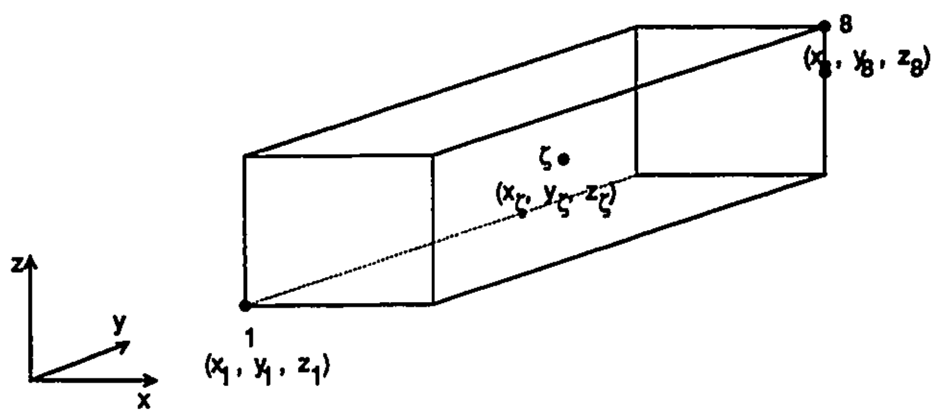


Figure 4.4: Nodes for Current Density Calculation for the 3-D FEM formulation.

The current density will be written in a matrix form. $[\alpha]$, $\left[\frac{\partial\alpha}{\partial x}\right]$, $\left[\frac{\partial\alpha}{\partial y}\right]$, and $\left[\frac{\partial\alpha}{\partial z}\right]$ must first be calculated. These partial results are:

$$[\alpha]^T = \frac{1}{\text{VOL}} \begin{pmatrix} -(x-x_8)(y-y_8)(z-z_8) \\ (x-x_1)(y-y_8)(z-z_8) \\ (x-x_8)(y-y_1)(z-z_8) \\ -(x-x_1)(y-y_1)(z-z_8) \\ (x-x_8)(y-y_8)(z-z_1) \\ -(x-x_1)(y-y_8)(z-z_1) \\ -(x-x_8)(y-y_1)(z-z_1) \\ (x-x_1)(y-y_1)(z-z_1) \end{pmatrix} \tag{4.60}$$

$$\left[\frac{\partial\alpha}{\partial x}\right]^T = \frac{1}{\text{VOL}} \begin{pmatrix} -(y-y_8)(z-z_8) \\ (y-y_8)(z-z_8) \\ (y-y_1)(z-z_8) \\ -(y-y_1)(z-z_8) \\ (y-y_8)(z-z_1) \\ -(y-y_8)(z-z_1) \\ -(y-y_1)(z-z_1) \\ (y-y_1)(z-z_1) \end{pmatrix} \tag{4.61a}$$

$$\left[\frac{\partial\alpha}{\partial y}\right]^T = \frac{1}{\text{VOL}} \begin{pmatrix} -(x-x_8)(z-z_8) \\ (x-x_1)(z-z_8) \\ (x-x_8)(z-z_8) \\ -(x-x_1)(z-z_8) \\ (x-x_8)(z-z_1) \\ -(x-x_1)(z-z_1) \\ -(x-x_8)(z-z_1) \\ (x-x_1)(z-z_1) \end{pmatrix} \tag{4.61b}$$

$$\left[\frac{\partial\alpha}{\partial z}\right]^T=\frac{1}{\text{VOL}}\begin{pmatrix}-(x-x_8)(y-y_8)\\(x-x_1)(y-y_8)\\(x-x_8)(y-y_1)\\-(x-x_1)(y-y_1)\\(x-x_8)(y-y_8)\\-(x-x_1)(y-y_8)\\-(x-x_8)(y-y_1)\\(x-x_1)(y-y_1)\end{pmatrix} \tag{4.61c}$$

where

$$\text{VOL}=(x_8-x_1)(y_8-y_1)(z_8-z_1)$$

$(x_\zeta, y_\zeta, z_\zeta)$ is in the center of the element. These terms equal to:

$$x_\zeta=\frac{x_8+x_1}{2}\qquad y_\zeta=\frac{y_8+y_1}{2}\qquad z_\zeta=\frac{z_8+z_1}{2} \tag{4.62}$$

Rewriting (4.60), (4.61a), (4.61b), and (4.61c) at the center position of the element, $(x_\zeta, y_\zeta, z_\zeta)$, yields these partial results:

$$[\alpha]^T_{(x,y,z)=(x_\zeta,y_\zeta,z_\zeta)}=\frac{(x_8-x_1)(y_8-y_1)(z_8-z_1)}{8\text{VOL}}\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}=\frac{1}{8}\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix} \tag{4.63}$$

$$\left[\frac{\partial\alpha}{\partial x}\right]^T_{(x,y,z)=(x_\zeta,y_\zeta,z_\zeta)}=\frac{(y_8-y_1)(z_8-z_1)}{4\text{VOL}}\begin{pmatrix}-1\\1\\-1\\1\\-1\\1\\-1\\1\end{pmatrix}=\frac{1}{4(x_8-x_1)}\begin{pmatrix}-1\\1\\-1\\1\\-1\\1\\-1\\1\end{pmatrix} \tag{4.64a}$$

$$\left[\frac{\partial \alpha}{\partial y}\right]^T_{(x,y,z)=(x_\zeta,y_\zeta,z_\zeta)} = \frac{1}{4(y_8-y_1)}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \tag{4.64b}$$

$$\left[\frac{\partial \alpha}{\partial z}\right]^T_{(x,y,z)=(x_\zeta,y_\zeta,z_\zeta)} = \frac{1}{4(z_8-z_1)}\begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.64c}$$

Substituting (4.63), (4.64a), (4.64b), and (4.64c) into equation (4.59) and defining $DX = x_8 - x_1$, $DY = y_8 - y_1$, and $DZ = z_8 - z_1$ yields the following numerical equation defining the current density in the middle of an element.

$$\vec{J} = \sigma\begin{pmatrix} \begin{pmatrix} \frac{1}{4DX}(\phi_1-\phi_2+\phi_3-\phi_4+\phi_5-\phi_6+\phi_7-\phi_8) \\ -j\frac{\omega}{8}(A_{x1}+A_{x2}+A_{x3}+A_{x4}+A_{x5}+A_{x6}+A_{x7}+A_{x8}) \end{pmatrix}\hat{x} \\ \begin{pmatrix} \frac{1}{4DY}(\phi_1+\phi_2-\phi_3-\phi_4+\phi_5+\phi_6-\phi_7-\phi_8) \\ -j\frac{\omega}{8}(A_{y1}+A_{y2}+A_{y3}+A_{y4}+A_{y5}+A_{y6}+A_{y7}+A_{y8}) \end{pmatrix}\hat{y} \\ \begin{pmatrix} \frac{1}{4DZ}(\phi_1+\phi_2+\phi_3+\phi_4-\phi_5-\phi_6-\phi_7-\phi_8) \\ -j\frac{\omega}{8}(A_{z1}+A_{z2}+A_{z3}+A_{z4}+A_{z5}+A_{z6}+A_{z7}+A_{z8}) \end{pmatrix}\hat{z} \end{pmatrix} \tag{4.65}$$

Chapter 5 3-D FEM Algorithm

The 3DFEM.F program that executes the simulations has been written mostly in FORTRAN with one subroutine being written in the C programming language. A description of the FORTRAN code developed by the author will be discussed. The input and output file formats will be explained as well as the minor modifications to be made to the main program in order to calculate the current density solution everywhere inside a particular object being studied. An example of how to compile, link, and execute the program in the ULTRIX environment on the DEC Micro-VAX work station can be found in Appendix 4.

FORTRAN is an ideal programming language to execute calculations having complex variables such as the solution in the thesis. The original program used the IMSL math library subroutine ZGECO and ZGESL to solve the complex matrix equation. These routines do not take advantage of the fact that the [GS] matrix set up by the program, as for all of the FEM solutions, is sparse. In this case, the matrix has on average, 10% to 20 % non-zero elements depending on the geometry of the object being studied. As demonstrated in Chapter 4, the final matrices used to solve for the vector magnetic potential $\vec{A}$ and the scalar electric potential ϕ at each node grows exponentially for each additional element needed to model the object. In order to be able to solve a larger systems of equations, thus more elements to model the object being studied, a sparse matrix equation solver had to be found or developed. SPARSE 1.3a written by K.S. Kundert and A. Sangiovanni-Vincetteli at UCLA is such a subroutine[33]. Because this routine has been written in C, it is necessary to choose an environment that will allow compilation of FORTRAN and C coded routines and permit linking them together to form an executable file. The ULTRIX environment on the DEC Micro-VAX work stations will permit this

The flowchart in figure 5.1 below describes the algorithm of the program. Comparing the flowchart to the program, it is seen that each box of the flowchart corresponds to a subroutine of the program. This eases the reading and understanding of the program. Before running the program, the user must create an input file FEM3D.IN and bring about some changes to the main program

FEM3D.F. The problem is to calculate the current density induced in an object exposed to extra low frequency magnetic fields. The user must model the object with brick shaped elements. One may use less elements for a coarse estimate or more elements for a better calculation. Once the number of nodes and bricks needed to model the object is now known, the first 5 lines of the program starting with the word PARAMETER may now be modified. The variables TNODE and TBRICK represent the total number of nodes and bricks respectively used to model the object.

The magnetic field is represented by the variables HXR, HXI, HYR, HYI, HZR, and HZI such that $\vec{H} = (HXR + jHXI)\vec{x} + (HYR + jHYI)\vec{y} + (HZR + jHZI)\vec{z}$. The FREQUENCY variable is self explanatory and its value is in Hz. These are the only lines to be modified for every simulation executed for a different model or magnetic field used.

The input file FEM3D.IN describes the modelled object. This file has two main parts. The first describes the global node numbering scheme. On each line, the user must enter the node numbers and their corresponding x, y, and z coordinates as shown below:

Node i	X coordinate of node i	Y coordinate of node i	Z coordinate of node i

The nodes must be ordered starting at 1 and in ascending order.

The second part of the input file gives the correspondence between the global nodes and the local node numbering scheme for each brick shaped element. Also included is the conductivity of each element. Each line has 10 entries describing the element. These entries are:

element number	local node 1	local node 2	local node 3	local node 4	local node 5	local node 6	local node 7	local node 8	conductivity (S/m)

The elements must be also be ordered starting at 1 and in ascending order. For local node i, the user must enter the number of the global node which occupies the same position as this local node. An example of the steps needed to be applied in order to solve a problem is given in Appendix 4 and will clarify what the input file format should be.

Once the input file has been created and the program modified, the FORTRAN program is compiled and then linked to the compiled C coded program SPARSE 1.3a. This file may now be executed.

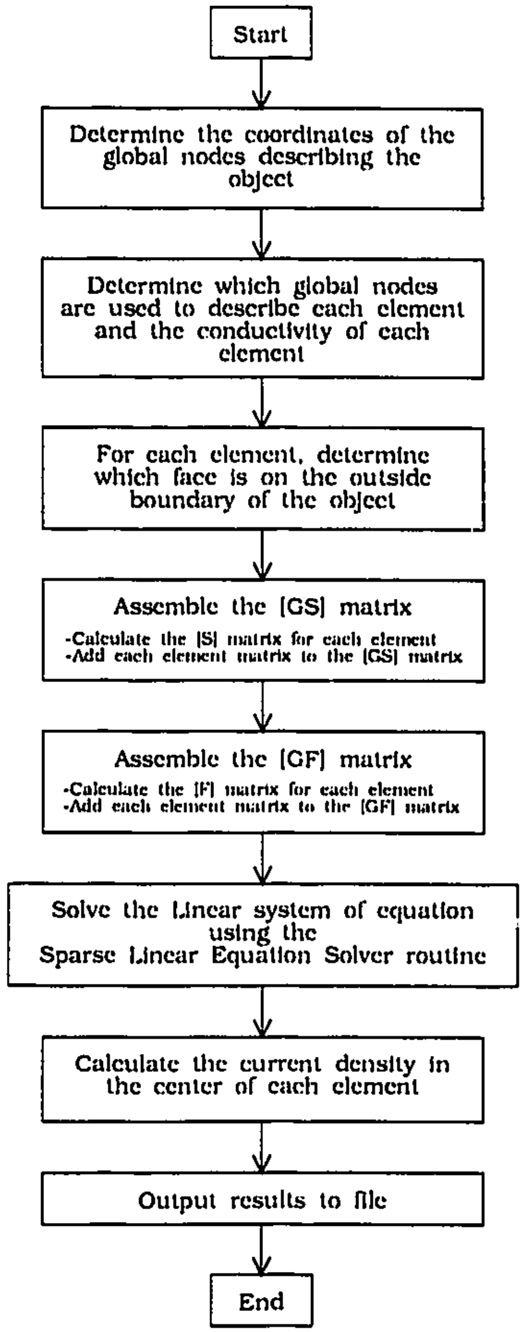


Figure 5.1: Algorithm for 3DFEM.F program

The first action taken by the program is reading the input files created in the manner discussed above. This sets up the local and global node numbering scheme and the element numbering scheme as well as the conductivity of each element (boxes 1 and 2). The next step (box 3) is to determine which elements are on the surface of the object being modelled and which faces of these elements are actually part of the outside boundary of this object. This is accomplished quite easily. Each element has 6 faces and each face has 4 nodes attached to them. The program takes one element at a time and compares it to all other elements describing the object. The 4 nodes describing each face of the element in question are compared to the 4 nodes of each surface of each element to see if there is a match. If there is a match, the program concludes that this face is inside the object, otherwise the program concludes that the surface is on the boundary. A variable FACE(face#,element#) has been created to record the status of each face of each element. FACE(face#,element#) will take on a value of 1 if a face is on the surface of the object and a value of 0 otherwise. This information will be used in the assembly of the [GF] matrix (box 6) to ensure that the boundary conditions are satisfied in equation (4.25).

The next action taken by the program is to assemble a global [GS] matrix. For each element, the program calculates the [S] element matrix as show in Chapter 4 at equation (4.36) or in Appendix 5 . This matrix is then added to the global [GS] matrix in the same manner as shown in the example of section 4.4. These terms are stored in the [GS] matrix in such a manner that the SPARSE 1.3a subroutine will be able to interpret it. It also takes advantage of the sparseness of the matrix by not storing the zero terms of the matrix, thus saving valuable memory space.

The [GF] matrix is then assembled (box 5). Again, the [F] element matrix is calculated for each element and then added to the global [GF] matrix. Unlike the [GS] matrix, all elements, zeroes and all, are stored in the [GF] matrix. The linear equation is now complete. The SPARSE 1.3a subroutine inputs the [GS] and [GF] matrix and solves for the unknowns $\vec{A}$ and ϕ at every node. A detailed explanation of the subroutine SPARSE 1.3a and the manner it is to be used is presented in the user's manual entitled "Sparse User's Guide - A Sparse Linear Equation Solver"[33] . Thus, if the reader wants to know more about this subroutine, the manual above should be read.

The current density in the center of each brick shaped element can now easily be calculated (box 7). Section 4.5 shows how the conduction current density equation (4.57) can be numerically formulated (4.65). This formulation is straight forward and is used by the program. The results are then printed to an output file.

The output file created states the magnetic field in which the object is immersed, information about the matrix it has operated on, and the values of the current densities in each element. The condition number helps in determining how ill-conditioned the matrix equation is. This helps to estimate the error in the solution. Round off error represents the errors that occur due to factorization of the matrix. The current densities of each element are than tabulated in the order that the elements were numbered. For each element, the number of the element and the corresponding x, y, z coordinates of the center of that element is given. This is then followed by the angle in degrees. The output format is shown in Appendix 4.

Chapter 6 Results and Analysis

This chapter presents the 3-D FEM formulation results and compares them to analytical solutions, previously published theoretical solutions, and validated numerical solutions. The following cases were chosen to verify the validity of the 3-D FEM formulation. The first cases are cylinders submerged in a 60 Hz magnetic field oriented parallel to the longitudinal axis of the cylinders. The objects used in these cases are: a) three homogeneous cylinders each having a different conductivity, b) a layered cylinder with the outside layer having a higher conductivity than the inside layer, and c) a square and a rectangular homogeneous cylinder. The last two sub-cases are cylinders submerged in a 60 Hz magnetic field oriented perpendicular to the longitudinal axis of the cylinders. The two cylinders used have different lengths. The above cases are judged sufficient to show whether the 3-D FEM formulation will yield satisfactory results when predicting the induced current densities inside a) variously shaped homogeneous objects each having a different conductivity value close to what is found in the human body, b) objects which are not homogeneous, and c) objects where the magnetic field orientation is arbitrary.

6.1 Homogeneous Long Cylinder with $\vec{H}$ Parallel to the Longitudinal Axis

This section compares the induced current density solutions, for a homogeneous long cylinder immersed in a magnetic field, between the analytical, 2-D FEM and 3-D FEM formulation. The magnetic field's orientation is parallel to the longitudinal axis of the cylinder as shown in figure 6.1a. Figure 6.1b shows how the cylinder is modelled for the 2-D FEM formulation and figure 6.1c demonstrates how the same cylinder is divided into 177 elements (416 nodes) for the 3-D FEM solution.

Graphs 6.1, 6.2, 6.3, and 6.4 compare the results obtained by all three methods. The dimensions of the cylinders used for all three cases are the same but the conductivity differs.

One will observe that for each different cylinder conductivity, the three methods yield almost identical results. The 2-D FEM solution diverges slightly at the center and the boundary of the cylinder. This is due to the fact that the forward and backward differences respectively are used to calculate the current density as opposed to the central difference used for all other points in the cylinder. As expected, the relationship between, the induced current densities in the cylinder and the cylinder's conductivity, is linear. This shows that the 3-D FEM formulation is valid for finding induced current density solutions inside homogeneous conducting objects having a conductivity value close to that found in the human body.

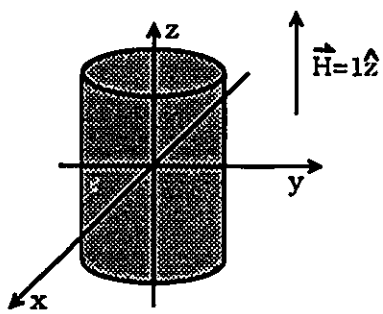


Figure 6.1a: Homogeneous long cylinder with $\overline{H}$ parallel to the longitudinal axis

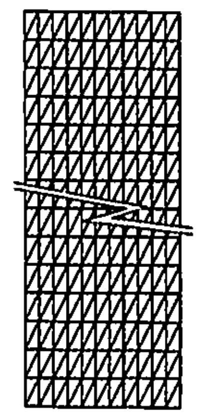


Figure 6.1b: Triangularization for 2-D FEM solution for the cylinder in figure 6.1a

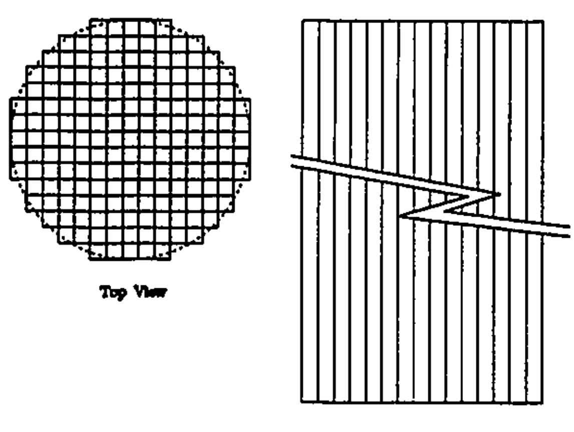
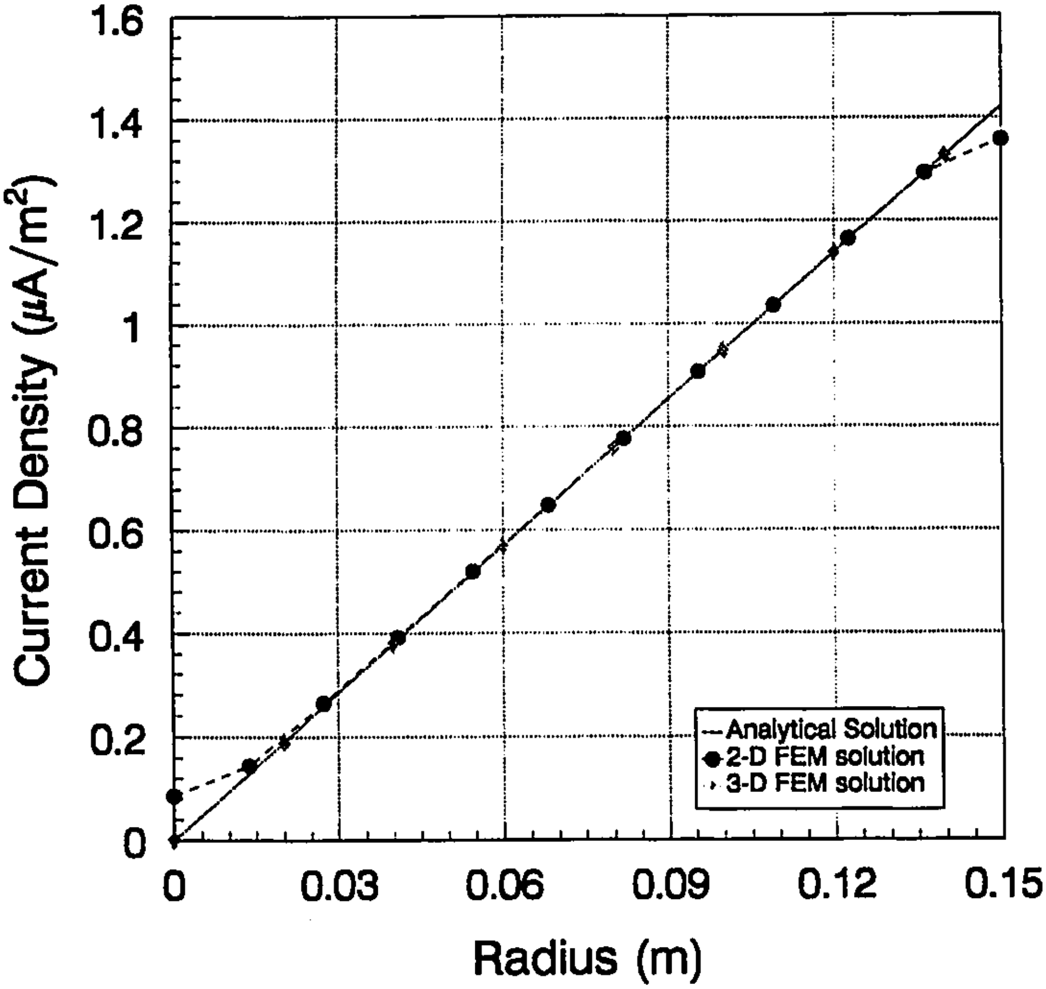


Figure 6.1c: Division of cylinder into elements for 3-D FEM Solution for the cylinder in figure 6.1a

Homogeneous Long Cylinder with $\vec{H}$ Parallel to the Longitudinal Axis

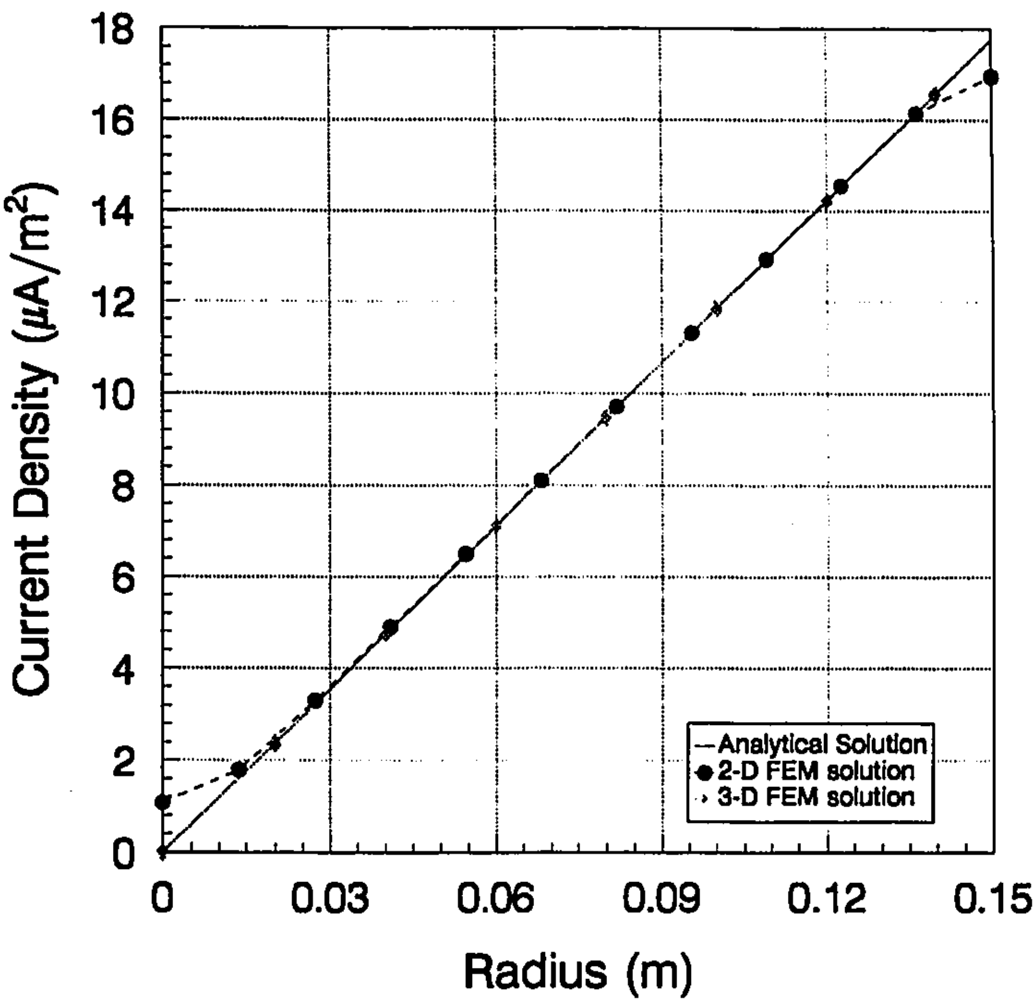
radius = 0.15 m, height = 1.8 m, σ=0.04 S/m



Graph 6.1: Homogeneous long cylinder with $\vec{H}$ parallel to the longitudinal axis - σ = 0.04 S/m

Homogeneous Long Cylinder with $\vec{H}$ parallel to the longitudinal axis

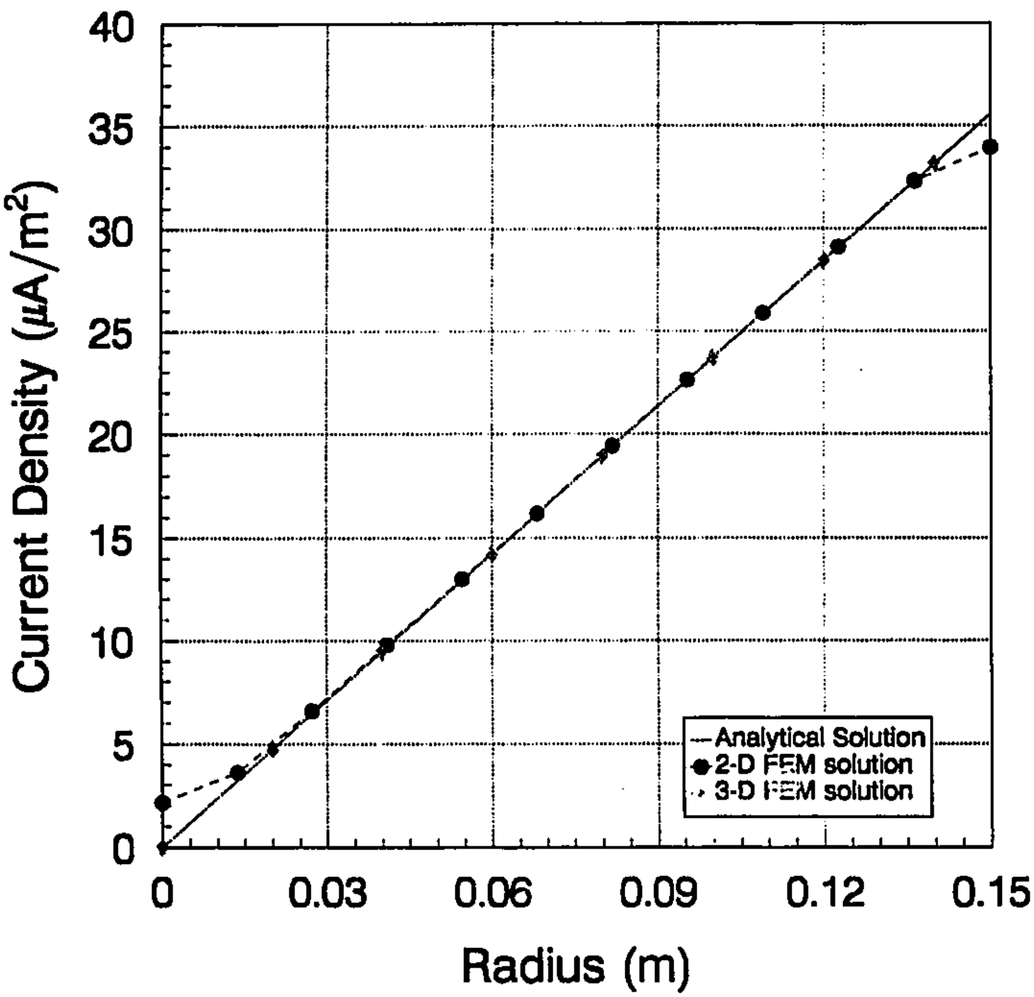
radius = 0.15 m, height = 1.8 m, σ=0.50 S/m



Graph 6.2: Homogeneous long cylinder with $\vec{H}$ parallel to the longitudinal axis - σ = 0.50 S/m

Homogeneous Long Cylinder with $\overrightarrow{H}$ parallel to the longitudinal axis

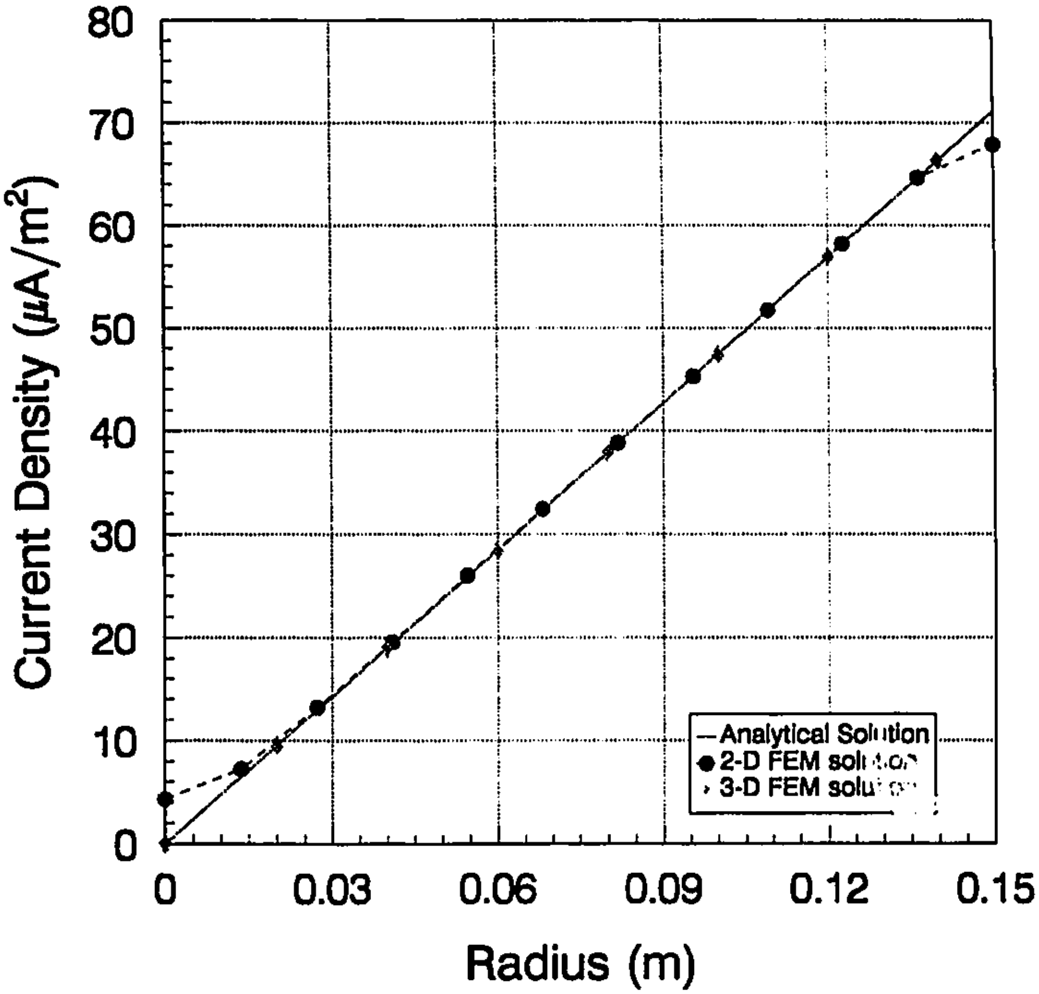
radius = 0.15 m, height = 1.8 m, σ=1.00 S/m



Graph 6.3: Homogeneous long cylinder with $\overrightarrow{H}$ parallel to the longitudinal axis - σ = 1.00 S/m

Homogeneous Long Cylinder with $\vec{H}$ parallel to the longitudinal axis

radius = 0.15 m, height = 1.8 m, σ=2.00 S/m



Graph 6.4: Homogeneous long cylinder with $\vec{H}$ parallel to the longitudinal axis - σ = 2.00 S/m

6.2 Layered Long Cylinder with $\vec{H}$ Parallel to the Longitudinal Axis

The case being studied in this section is very similar to the previous one except that the cylinder used is a layered cylinder having two different conductivities as depicted in figure 6.2a. The outside layer of the cylinder has a higher conductivity than the inside one. Figures 6.2b and 6.2c show how the cylinder is modelled for the 2-D FEM and 3-D FEM (177 elements - 416 nodes) formulation respectively.

Graph 6.3 compares the results of the two numerical solutions. The 2-D FEM solution is taken as the correct solution and the 3-D FEM solution is compared to it.

The 3-D FEM formulation yield similar results to the 2-D FEM solution. It shows a clear discontinuity in the induced current densities where the cylinder conductivity abruptly changes. One can observe that the calculated current densities are identical in the inside layers of the cylinder and slightly different in the outside layer. In order to get a reasonable number of data points along the different positions along the radius, very long elements had to be used because of the computer memory limitations. This accounts for most of the errors in the 3-D FEM solution. The finite element method generally yields its best results when the dimensions of the elements are close. If more memory was available, more elements could have been used to model the cylinder, thus making the shapes of the element closer to cubes rather than long rectangular parallelipipeds.

These results show the 3-D FEM formulation is valid for finding current density solutions inside heterogeneous conducting objects.

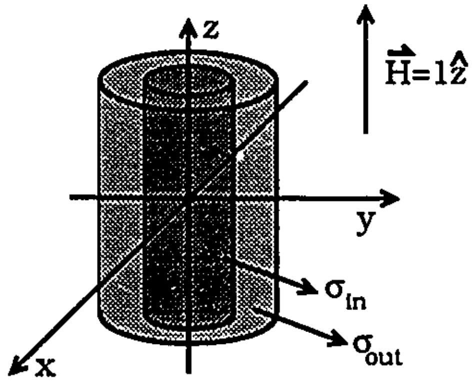


Figure 6.2a: Layered long cylinder with $\vec{H}$ parallel to the longitudinal axis

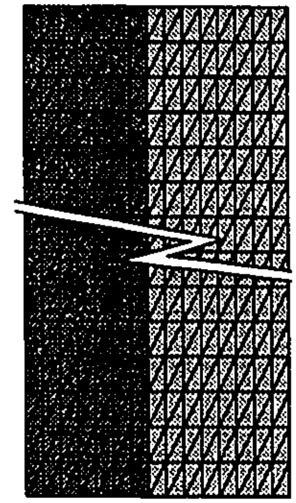
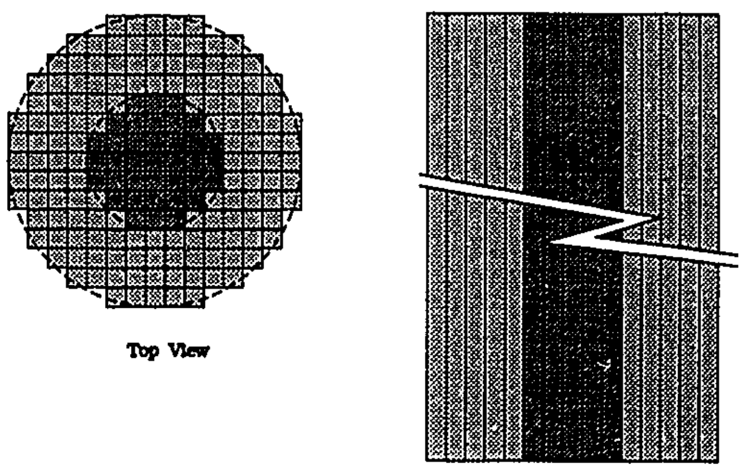


Figure 6.2b: Triangularization for 2-D FEM solution for the cylinder in figure 6.2a



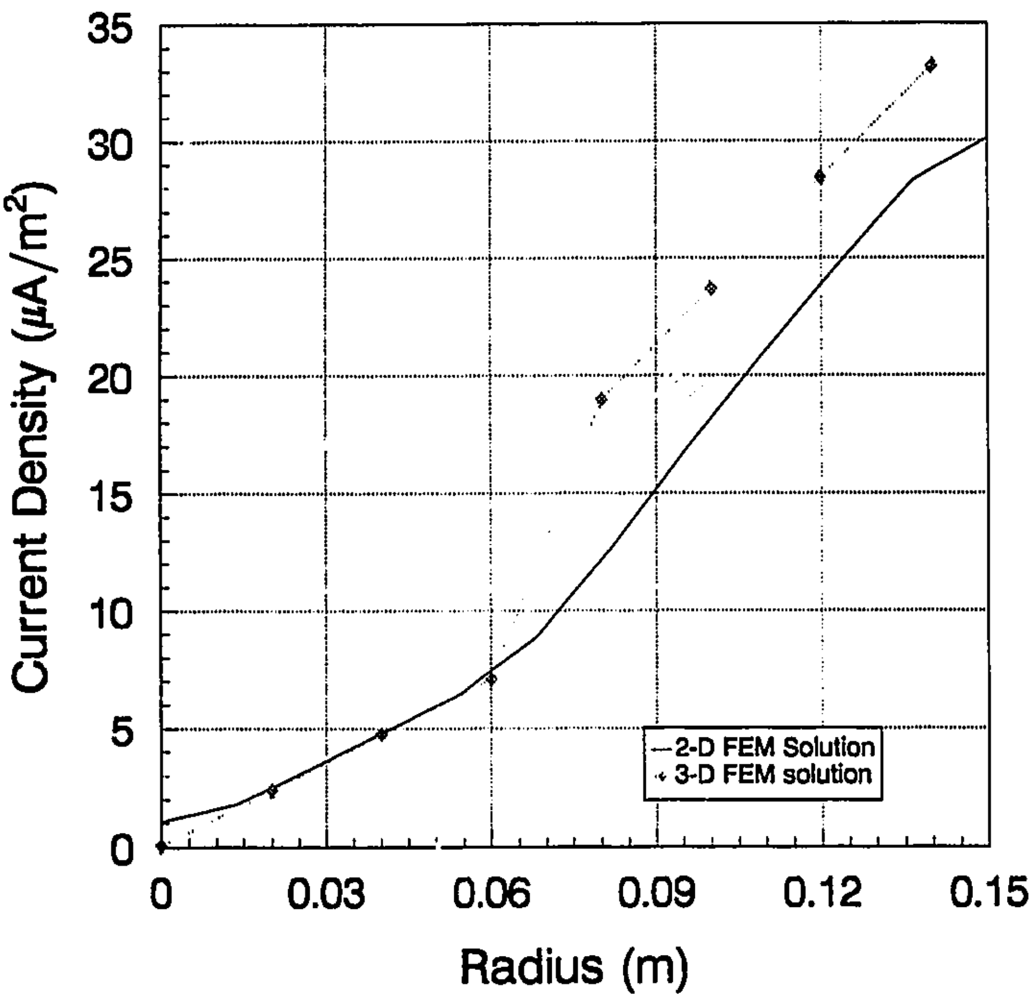
Top View

Side View

Figure 6.2c: Division of cylinder into elements for 3-D FEM Solution for the cylinder in figure 6.2a

Layered Long Cylinder with $\vec{H}$ Parallel to the Longitudinal Axis

rad_{in}=0.07 m, $\sigma_{(in)}$=0.5 S/m, rad_{out}=0.15 m, $\sigma_{(out)}$=1.0 S/m
height = 1.8 m



Graph 6.5: Layered long cylinder with $\vec{H}$ parallel to the longitudinal axis

6.3 Homogeneous Square and Rectangular Cylinders with $\vec{H}$ Parallel to the Longitudinal Axis

The results presented here show the 3-D FEM solution compared to the analytical solution for a square and rectangular cylinder. The dimensions of the width and height are 0.3m by 0.3m for the square cylinder and 0.3m by 0.6m for the rectangular cylinder. The analytical solution supposes that the cylinder is of infinite length. This is not possible for the 3-D FEM solution, thus it was given an arbitrary length of 1m. The current densities are then calculated at the middle points along the length and compared. It is to be noted that the 3-D FEM results for cylinders having longer lengths were computed. In order to ensure that the 1m length chosen was long enough to compare its results with analytical solutions.

Figure 6.3a shows the geometry of the homogeneous square cylinder and the magnetic field orientation relative to it. Figure 6.3b displays the model discretization for the 3-D FEM solution into 105 elements (256 nodes). All elements have the same dimensions. The results for the square cylinder are presented in figure 6.4.

Graph 6.6 shows the geometry of the homogeneous rectangular cylinder. The discretization for this model is similar to that of the square cylinder: the number of elements making it up are the same but their height is double that of the elements making up the square cylinder. Graphs 6.7 and 6.8 present the current density results at the center of the cylinder along the height and width respectively.

The 3-D FEM solution is in good agreement with the analytical solution for the case of the square cylinder. It is believed that the slight discrepancy at the boundary is due to the coarseness of the object discretization.

At first glance, graphs 6.7 and 6.8 seem to show that the 3-D FEM solution is in agreement to a lesser degree for the rectangular cylinder. Even though the coarseness of the discretization will cause some errors, the 3-D FEM solution still displays the correct behavior of the current density. It shows that the current density is higher in the narrower part of the cylinders as compared to the thicker part - which was expected.

These results show that the 3-D FEM program is reasonable to calculate and predict the behavior of the induced current densities inside objects having different shapes.

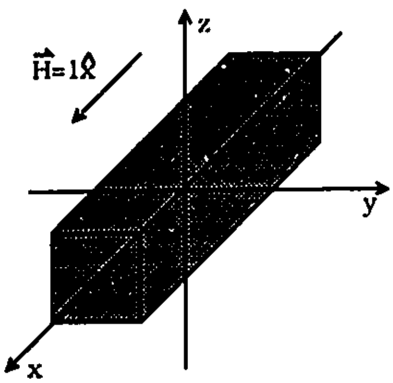


Figure 6.3a: Homogeneous square cylinder with $\overline{H}$ parallel to the longitudinal axis

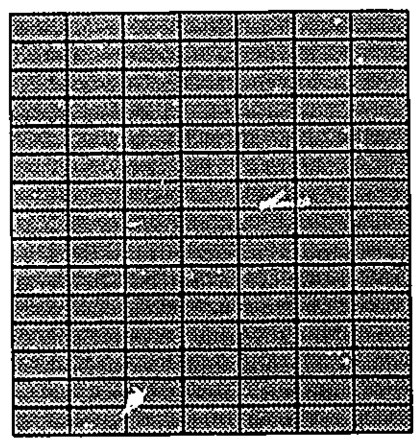
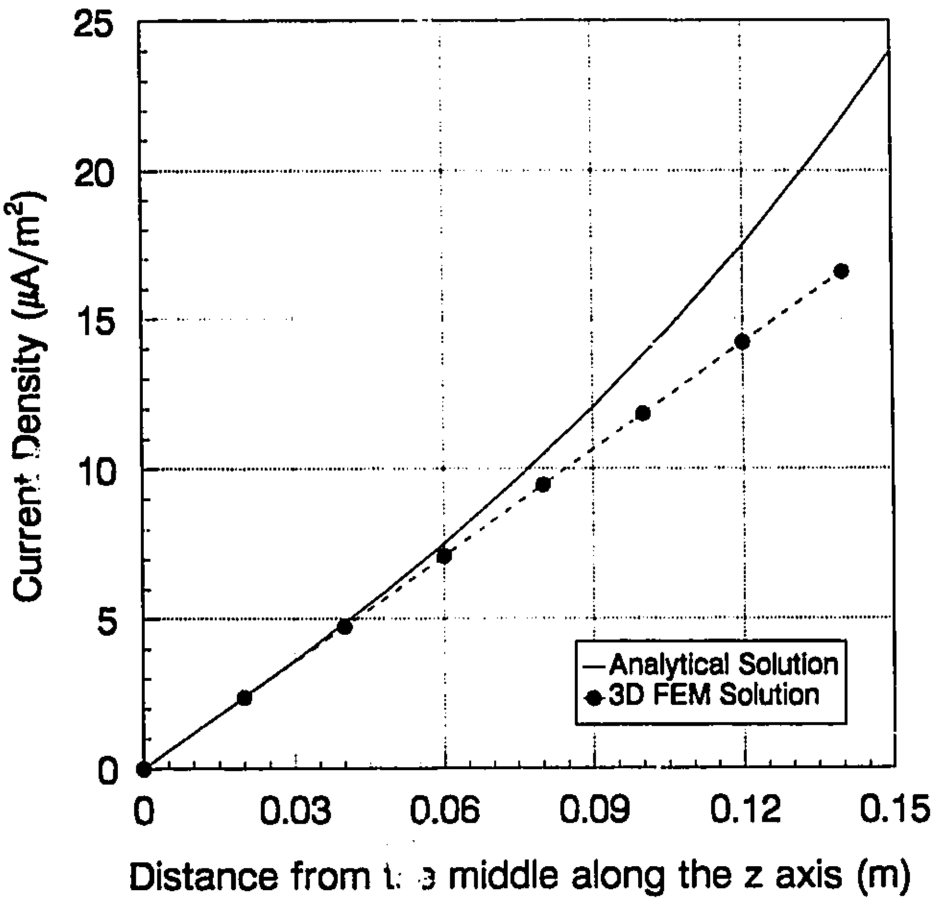


Figure 6.3b: Division of cylinder into elements for 3-D FEM Solution for the cylinder in figure 6.3a

Homogeneous Square Cylinder with $\vec{H}$ Parallel to the Longitudinal Axis

Width = 0.3m, Height = 0.3m, σ=0.5 S/m

J_y @$y=0$, for $0 \le z \le 0.15$m



Graph 6.6: Homogeneous square cylinder with $\vec{H}$ parallel to the longitudinal axis

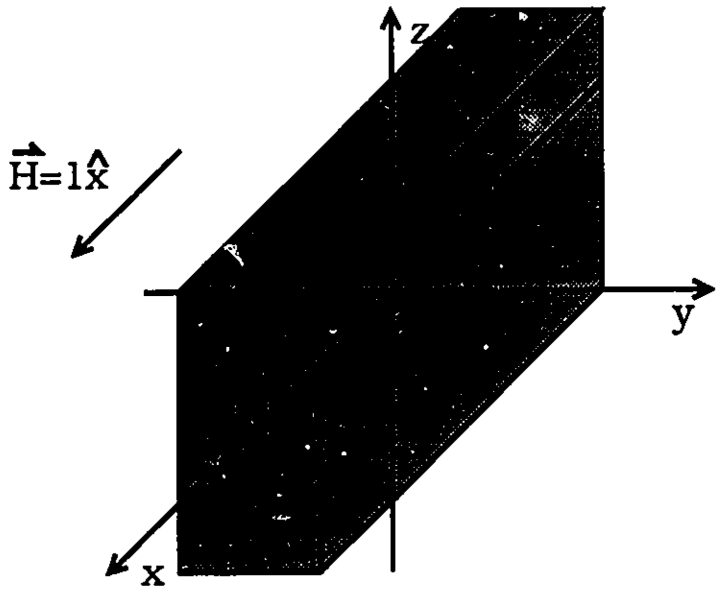
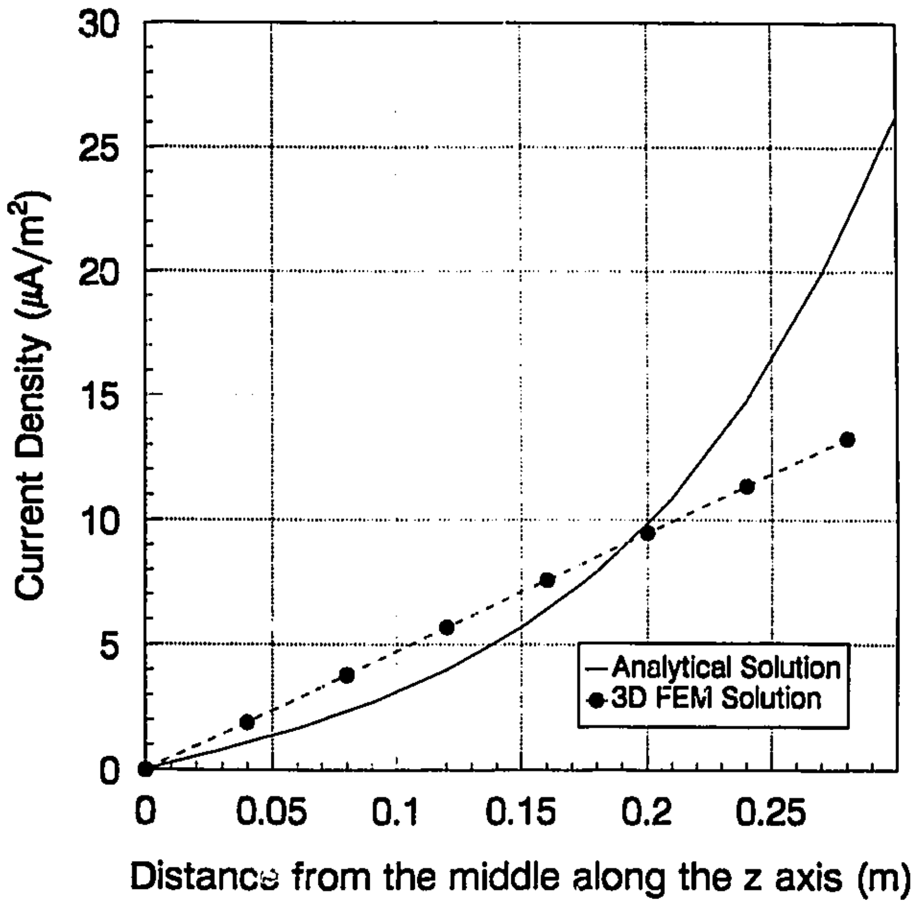


Figure 6.4: Homogeneous rectangular cylinder with $\vec{H}$ parallel to the longitudinal axis

Homogeneous Rectangular Cylinder with $\vec{H}$ Parallel to the Longitudinal Axis

Width = 0.3m, Height = 0.6m, σ=0.5 S/m

J_y @$y=0$, for $0 \leq z \leq 0.30$m

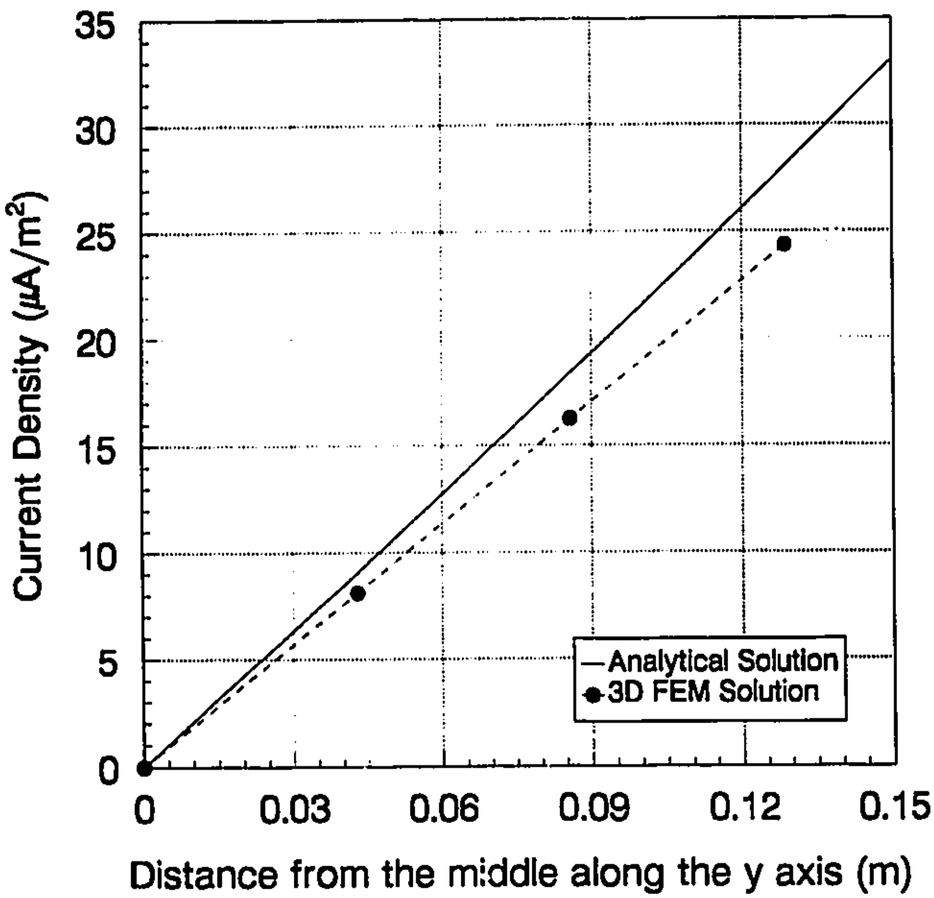


Graph 6.7: J_y in the homogeneous rectangular cylinder with $\vec{H}$ parallel to the longitudinal axis

Homogeneous Rectangular Cylinder with $\overrightarrow{H}$ Parallel to the Longitudinal Axis

Width = 0.3m, Height = 0.6m, σ=0.5 S/m

J_z @ $y = 0$, for $0 \le y \le 0.15$m



Graph 6.8: J_z in the homogeneous rectangular cylinder with $\overrightarrow{H}$ parallel to the long axis

6.4 Homogeneous Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

This section presents the results of the 3-D FEM solution compared to the analytical solution for homogeneous cylinders of different lengths submerged in a magnetic field which is oriented perpendicular to the longitudinal axis of the cylinder.

There are no closed form analytical solutions developed for this problem, but McLeod, et Al[27] developed and validated with measurements an approximate analytical solution (See Chapter 2). The cylinders are approximated by a series of rectangular slabs of variable widths and fixed heights. One would use equations (2.33) and (2.34) to calculate the y and z component of the current density at $x = x_1$, in the cylinder presented in figure 6.5a. The variables h and a would be substituted with the value which represents half the height of the cylinder and the value representing half the width of the cylinder at $x = x_1$.

The short cylinder and the orientation of the magnetic field in which it is submerged is presented in figure 6.5a and the manner in which it has been modelled (185 elements using 312 nodes) for the 3-D FEM solution is presented in figure 6.5b. Graphs 6.9a, 6.9b, 6.9c, and 6.9d compare the analytical and 3-D FEM solutions for the z component of the current density (J_z) along the y axis for different values of x. Graphs 6.10a, 6.10b, 6.10c, and 6.10d compare the two solutions for the y component of the current density (J_y) along the z axis for different values of x.

The situation of the longer cylinder is the same as that of the shorter one and is presented in figures 6.6a and 6.6b. The same number of elements were used to model both cylinders. Therefore, the latter case uses longer elements. Graphs 6.11a, 6.11b, 6.11c, and 6.11d present the J_z solution along the y axis at different positions along x and graphs 6.12a, 6.12b, 6.12c, and 6.12d present the J_y solution along the z axis at different positions along x.

A study of the results reveals that the 3-D FEM solutions are in very good agreement with the analytical solutions for positions inside the cylinder from the center to a radius equalling approximately three quarters of the total cylinder radius. It is unfortunate that the 3-D FEM solutions seem to diverge for positions in the outside quarter of the cylinders. There are too few elements to model the cylinder appropriately to reflect the current density behavior near the outside boundaries of the cylinders. This is due the fact that the developed algorithm requires much computer memory and that the governing coupled equations to solve produce matrices which are not always well conditioned. Nevertheless, a comparison between the 3-D FEM solution for the short cylinder and the 3-D FEM solution for the longer cylinder shows that as the ratio of the height over the radius gets larger, the ratio of the current density in the narrow area of the cylinder over the current density in the larger area of the cylinder also gets larger. And, this is in complete agreement with the analytical solutions.

These results demonstrate the 3-D FEM formulation shows promise for predicting the induced current densities inside conducting objects submerged in a magnetic field being oriented in any direction relative to the object.

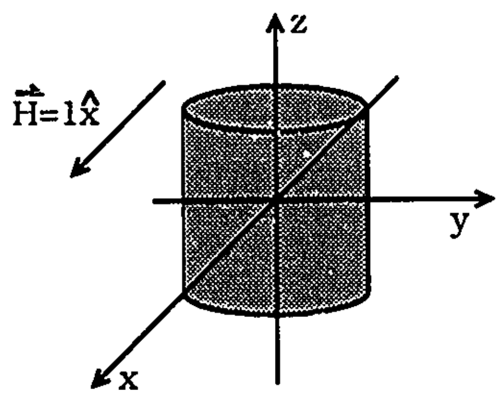
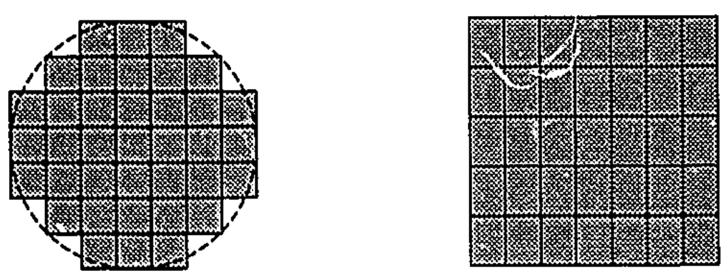


Figure 6.5a: Homogeneous short cylinder with $\vec{H}$ perpendicular to the longitudinal axis

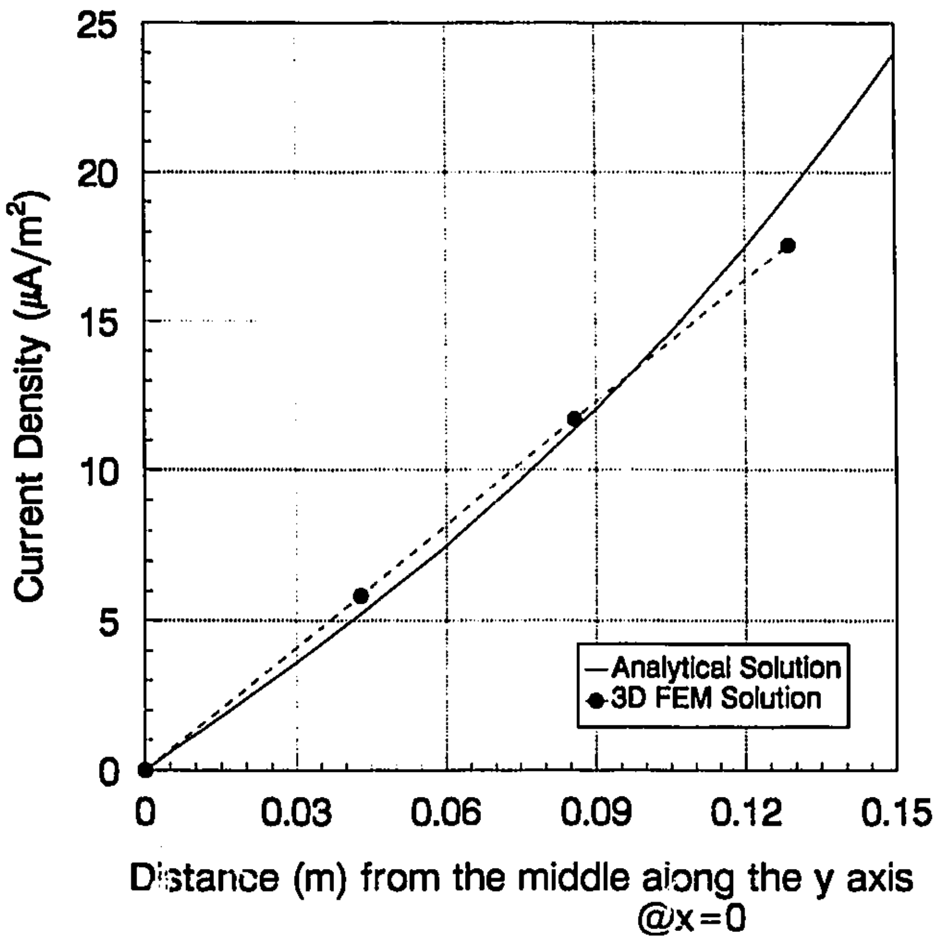


Top View / Side View

Figure 6.5b: Division of Cylinder in figure 6.5a into elements for 3-D FEM Solution

Homogeneous Short Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

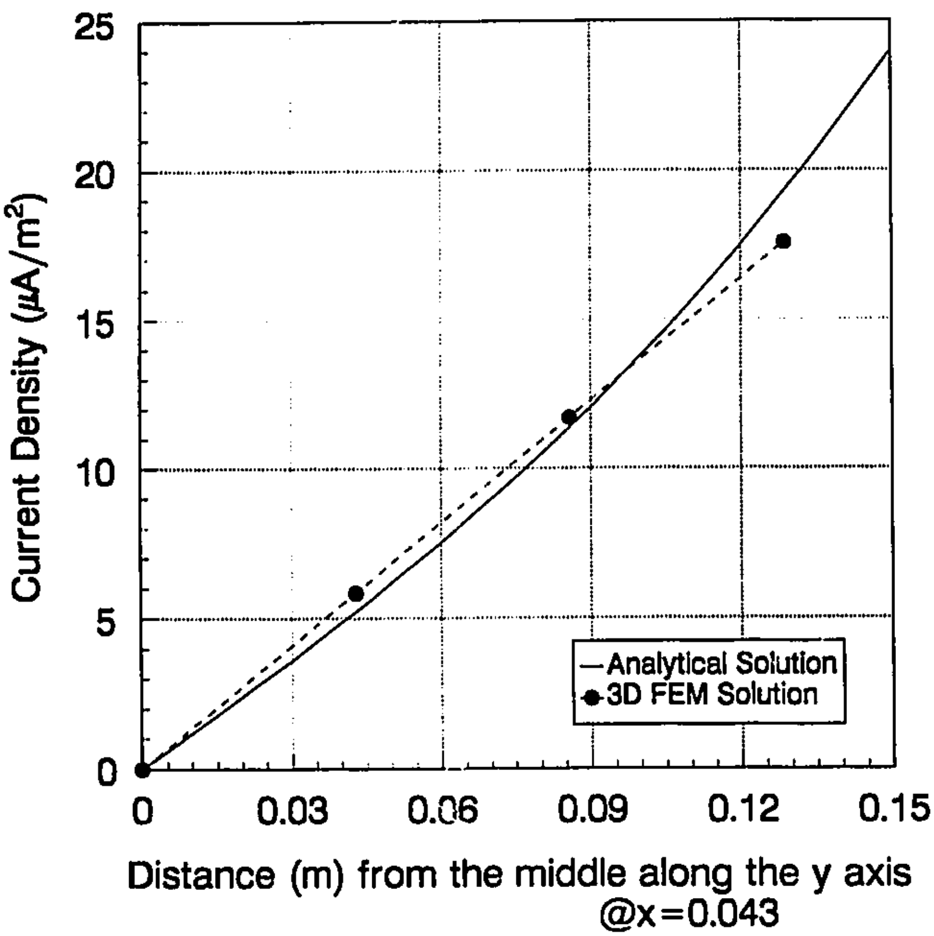
Radius = 0.15m, Height = 0.3m, σ=0.5 S/m
J_z @$z = 0$m, $x = 0$m, for $0 \le y \le 0.15$m



Graph 6.9a: J_z @ x=0m in the homogeneous short cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Short Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

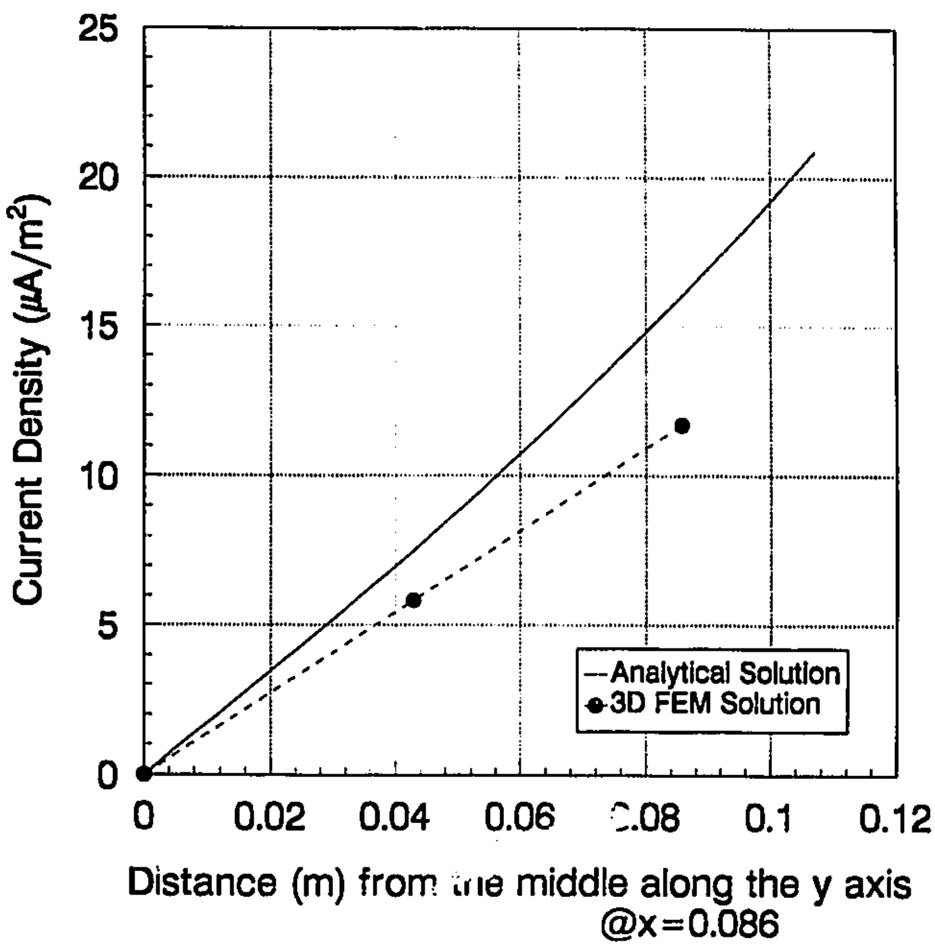
Radius = 0.15m, Height = 0.3m, σ=0.5 S/m
J_z @$z = 0$m, $x = 0.043$m, for $0 \le y \le 0.15$m



Graph 6.9b: J_z @ x=0.043m in the homogeneous short cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Short Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.3m, σ=0.5 S/m
J_z @$z = 0$m, $x = 0.086$m, for $0 \le y \le 0.11$m

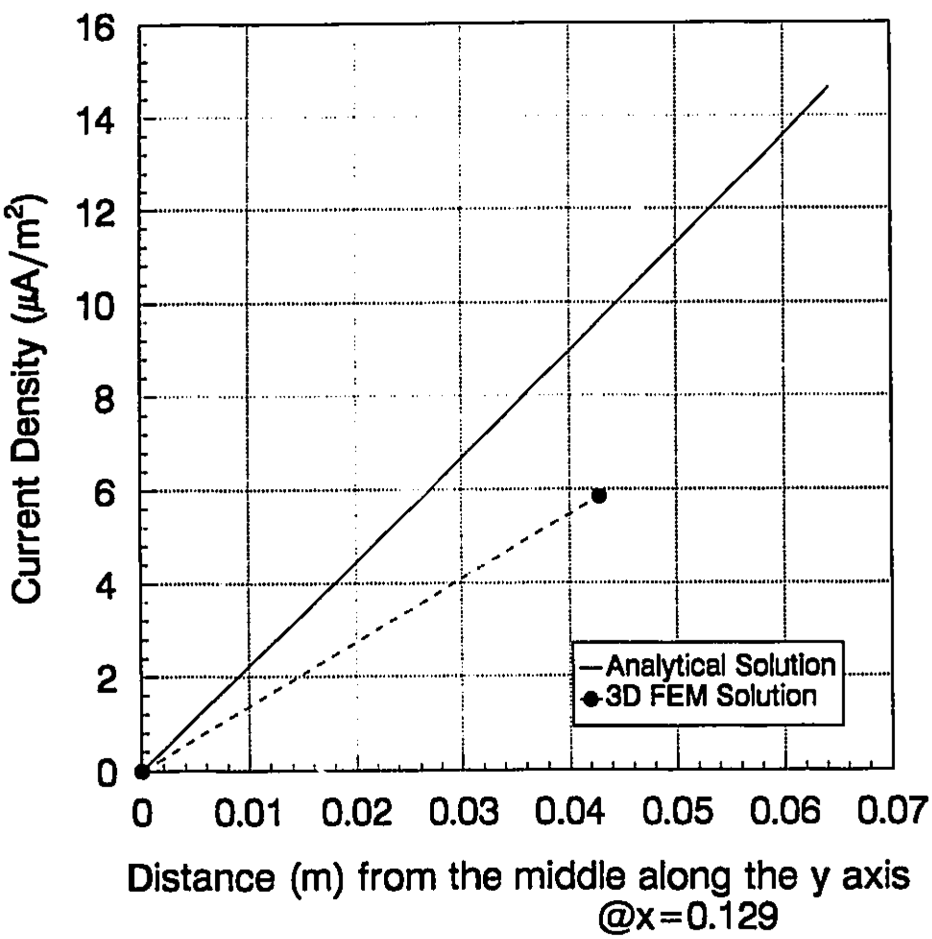


Graph 6.9c: J_z @ x=0.086m in the homogeneous short cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Short Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.3m, σ=0.5 S/m

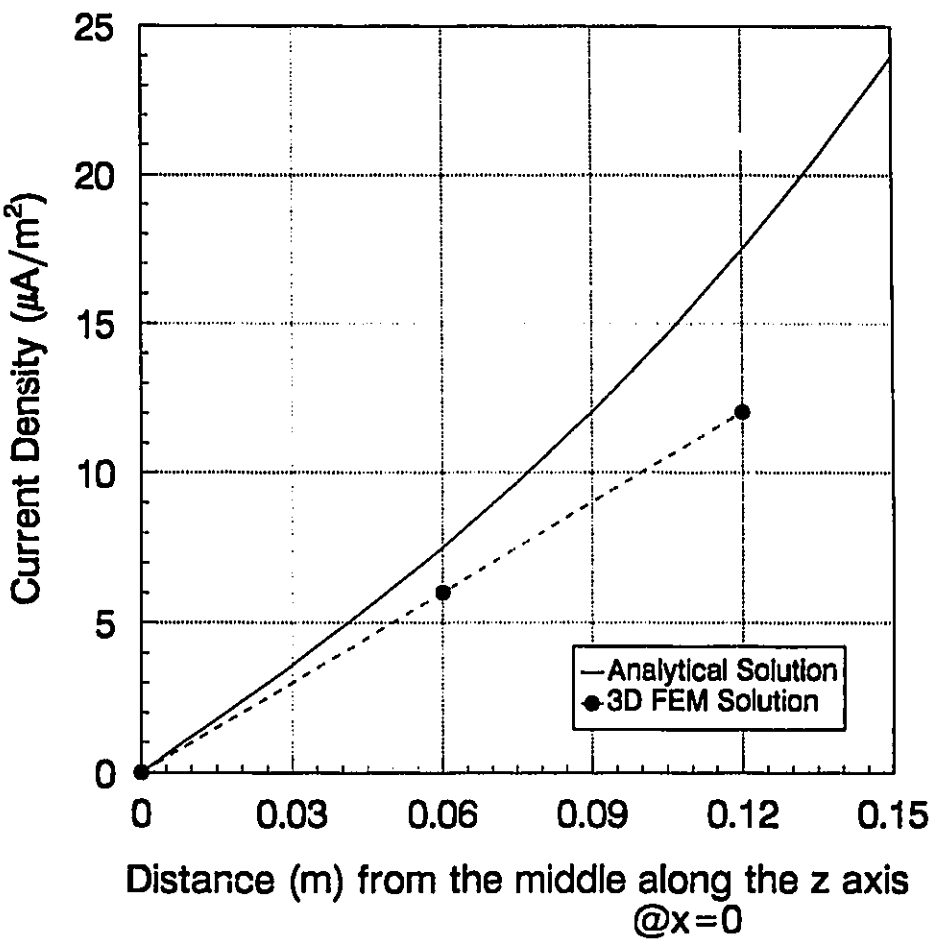
J_z @$z = 0$m, $x = 0.129$m, for $0 \le y \le 0.06$m



Graph 6.9d: J_z @ x=0.129m in the homogeneous short cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Short Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.3m, σ=0.5 S/m
J_y @$y = 0$m, $x = 0$m, for $0 \le z \le 0.15$m

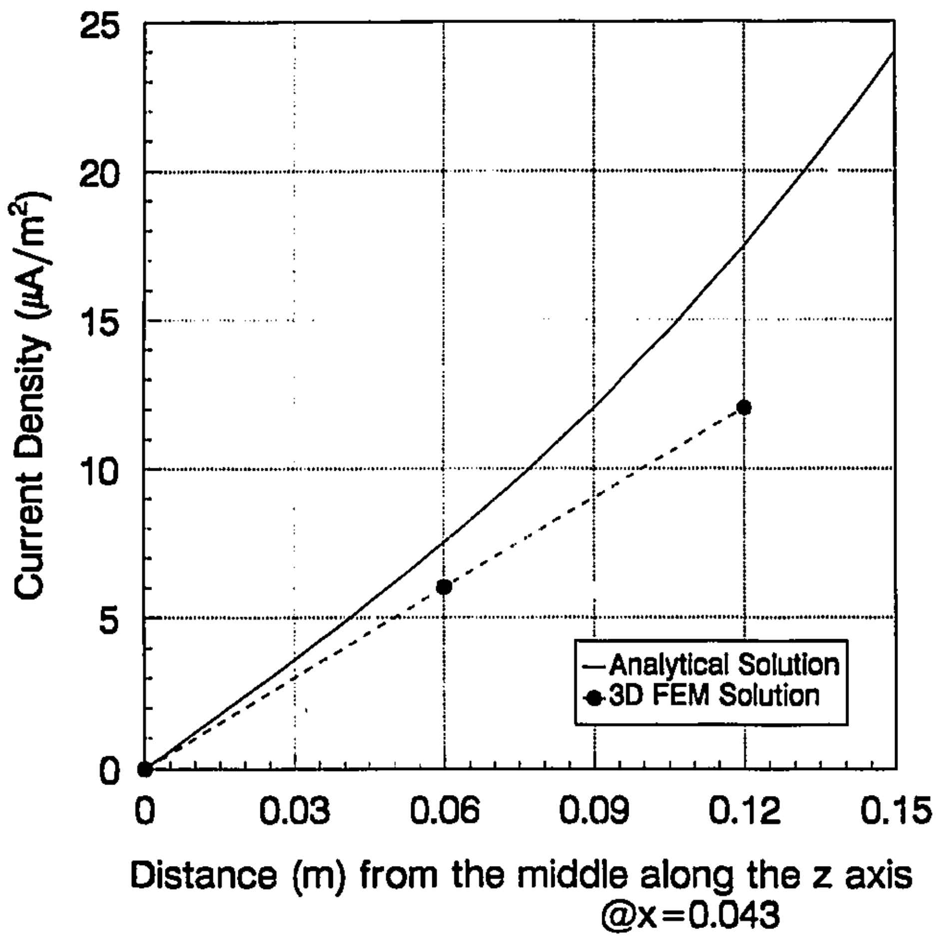


Graph 6.10a: J_y @ x=0.0m in the homogeneous short cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Short Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.3m, σ=0.5 S/m

J_y @$y = 0$m, $x = 0.043$m, for $0 \le z \le 0.15$m

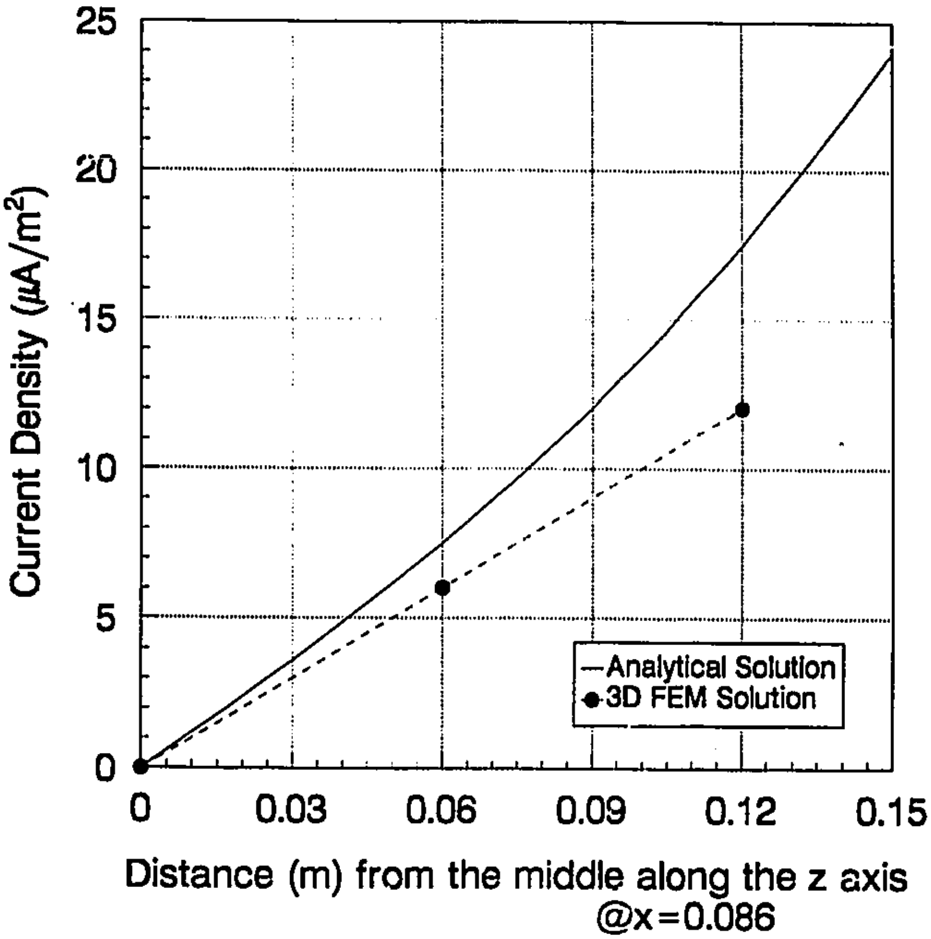


Graph 6.10b: J_y @ x=0.043m in the homogeneous short cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Short Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.3m, σ=0.5 S/m

J_y @$y = 0$m, $x = 0.086$m, for $0 \le z \le 0.15$m

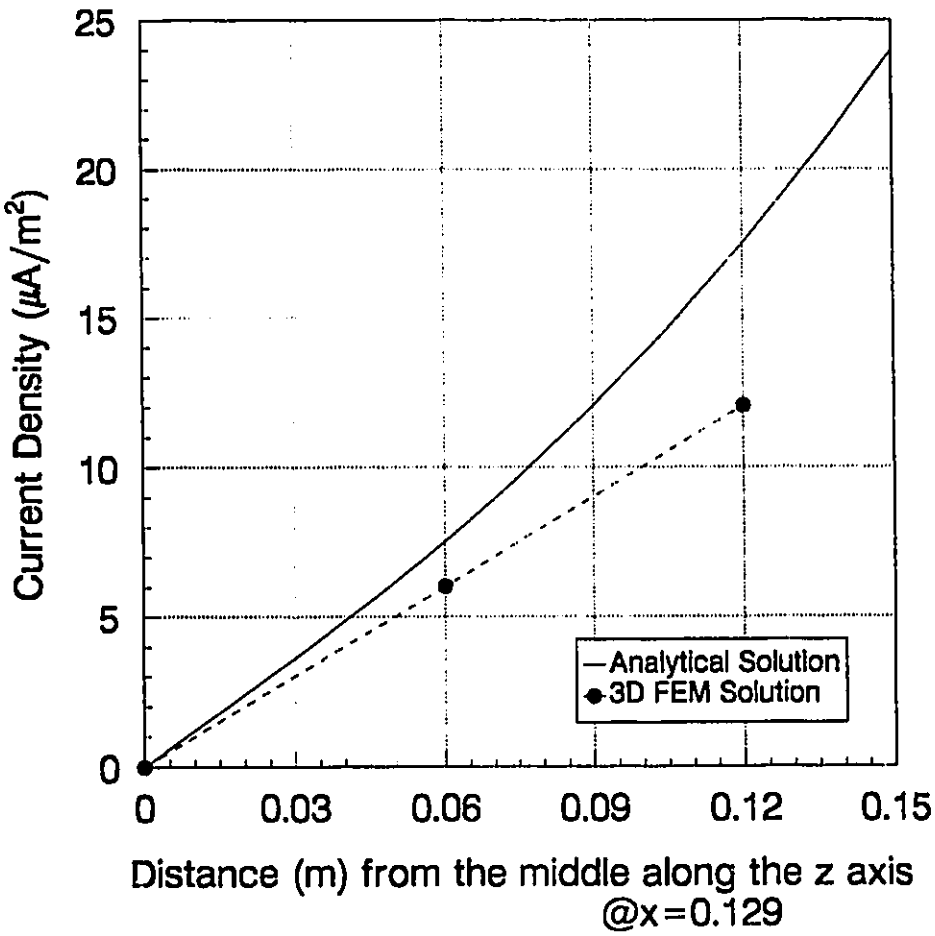


Graph 6.10c: J_y @ x=0.086m in the homogeneous short cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Short Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.3m, σ=0.5 S/m

J_y @$y = 0$m, $x = 0.129$m, for $0 \le z \le 0.15$m



Graph 6.10d: J_y @ x=0.129m in the homogeneous short cylinder with $\vec{H}$ perpendicular to the longitudinal axis

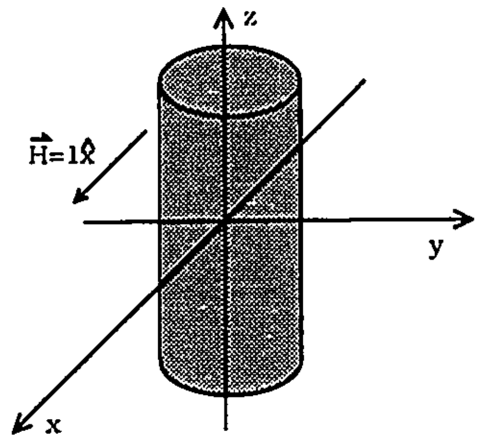


Figure 6.6a: Homogeneous long cylinder with $\overline{H}$ perpendicular to the longitudinal axis

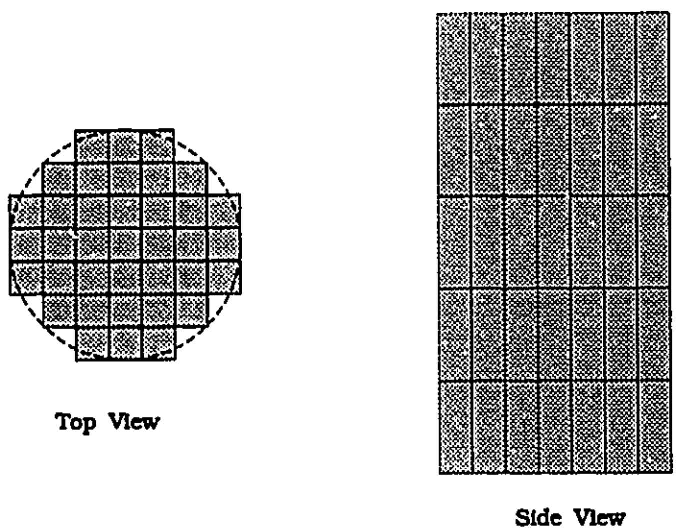
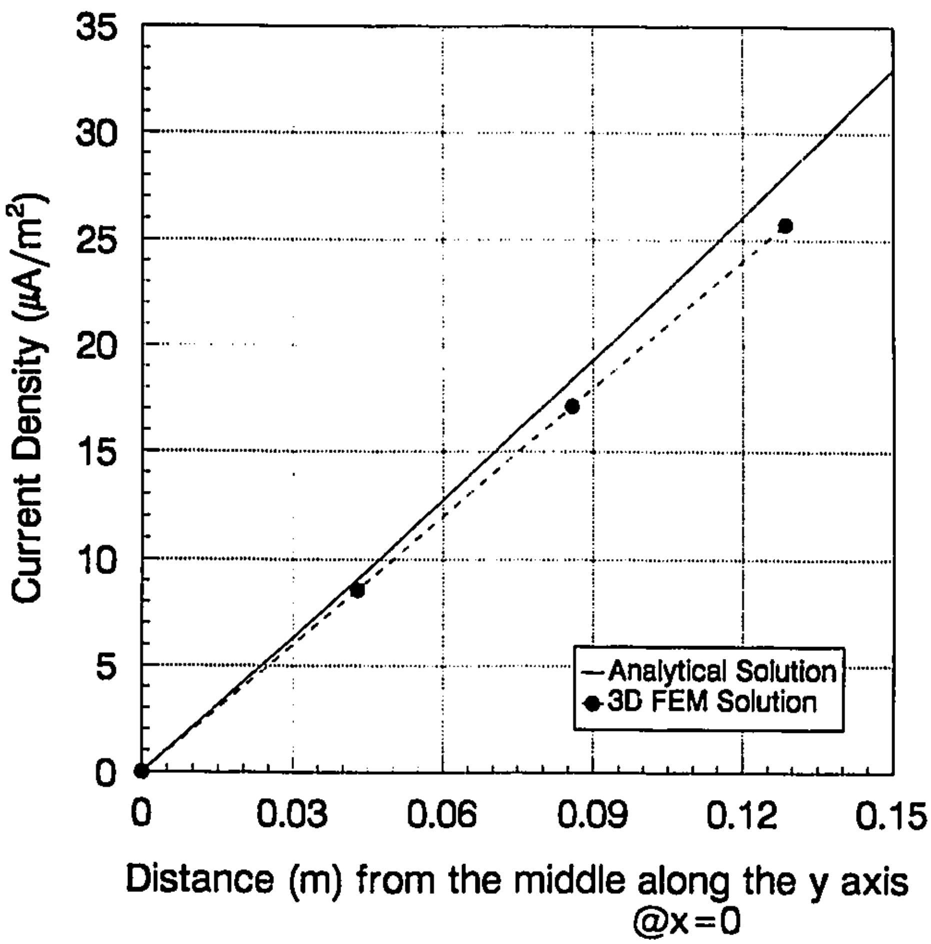


Figure 6.6b: Division of cylinder in figure 6.6a into elements for 3-D FEM solution

Homogeneous Long Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.6m, σ=0.5 S/m

J_z @$z = 0$m, $x = 0$m, for $0 \le y \le 0.15$m

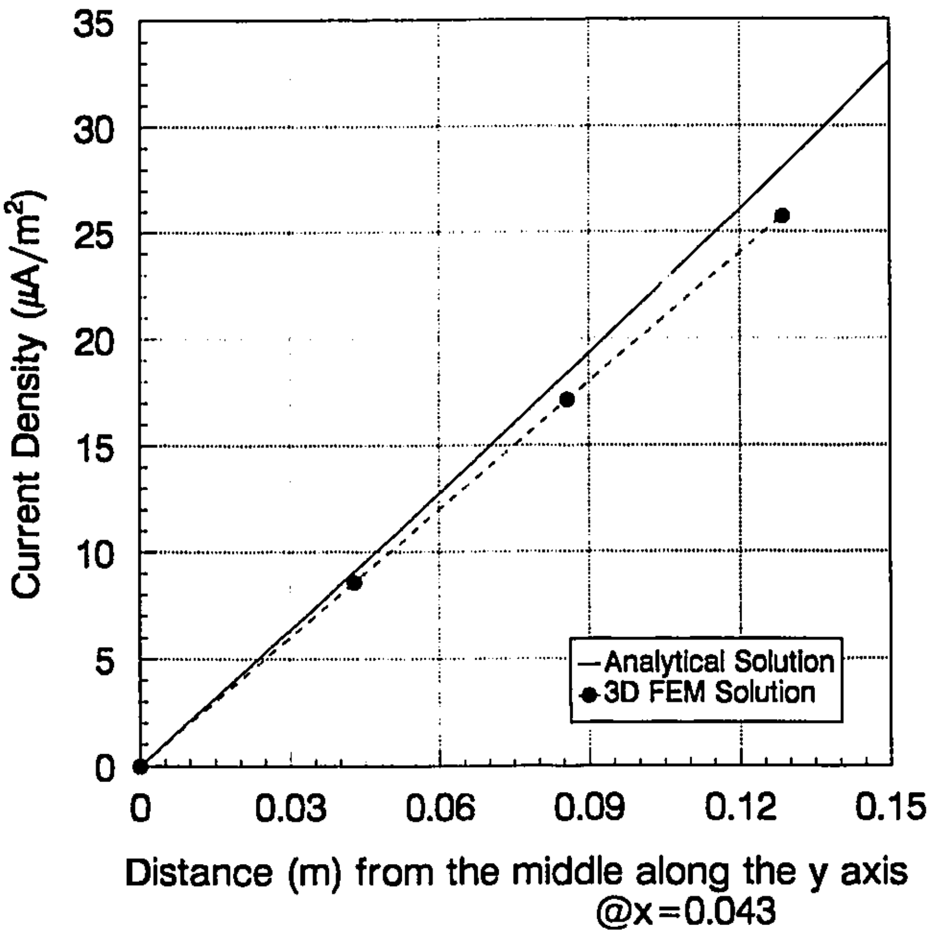


Graph 6.11a: J_z @ x=0.0m in the homogeneous long cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Long Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.6m, σ=0.5 S/m

J_z @$z = 0$m, $x = 0.043$m, for $0 \le y \le 0.15$m

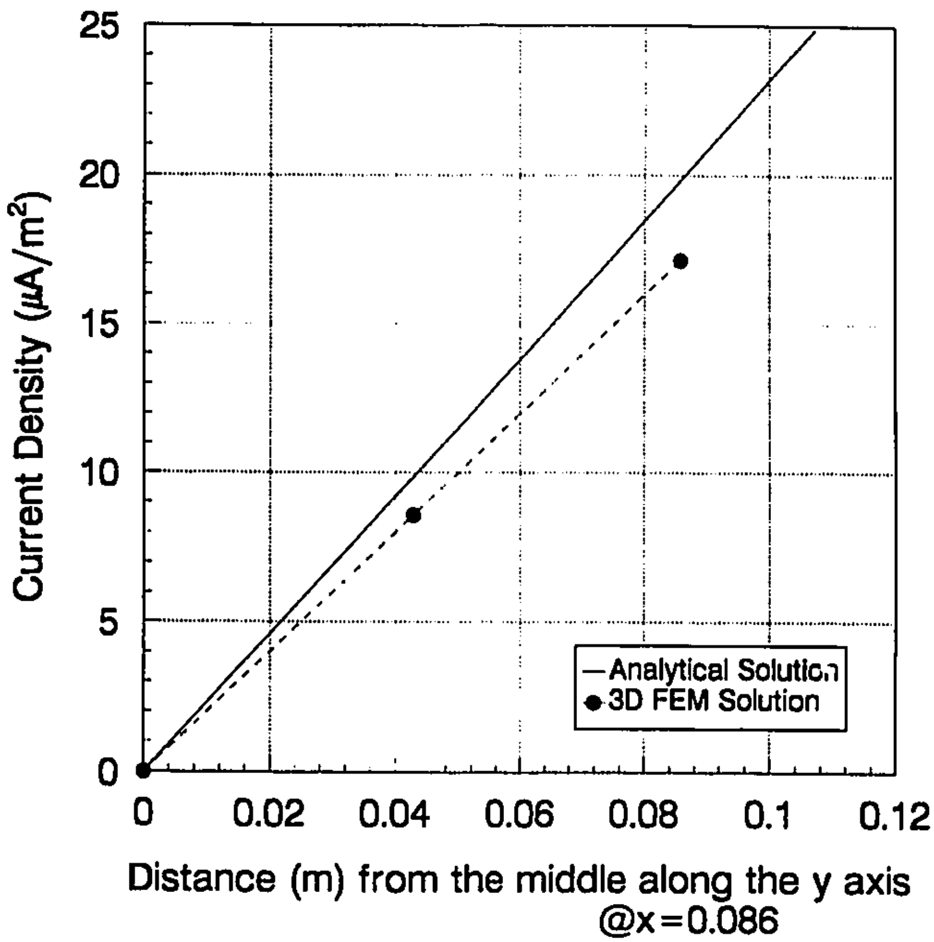


Graph 6.11b: J_z @ x=0.043m in the homogeneous long cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Long Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.6m, σ=0.5 S/m

J_z @$z = 0$m, $x = 0.086$m, for $0 \le y \le 0.11$m

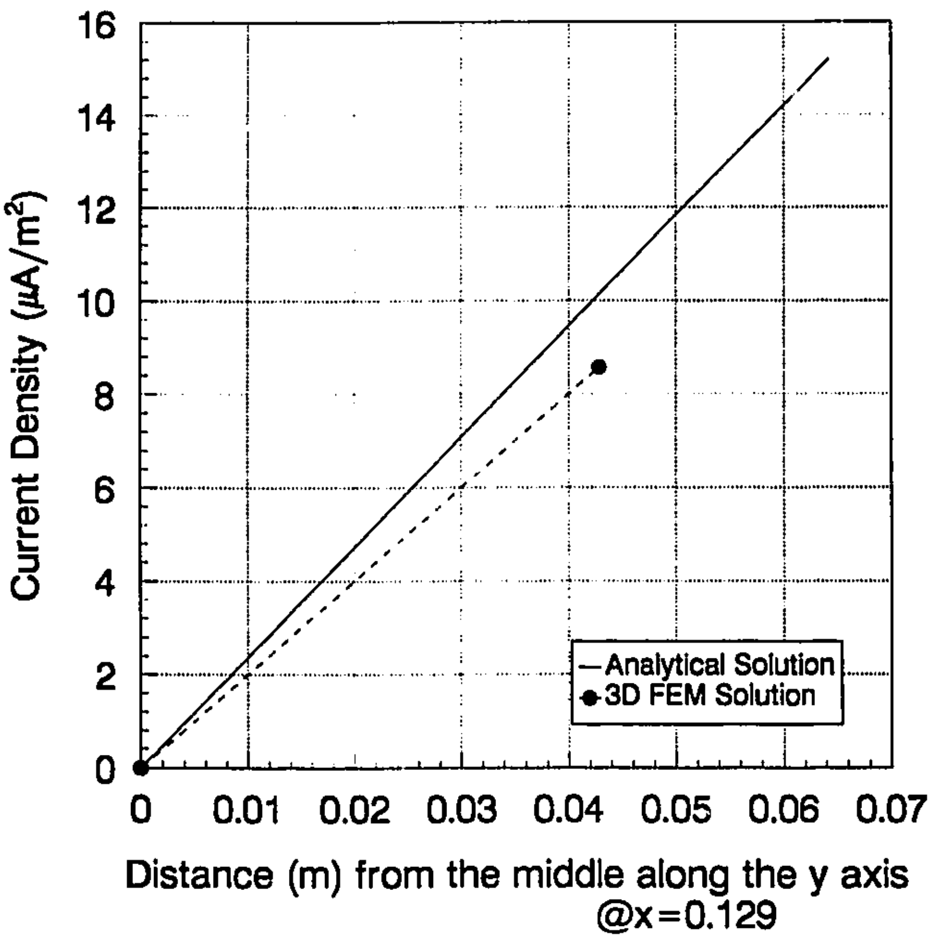


Graph 6.11c: J_z @ x=0.086m in the homogeneous long cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Long Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.6m, σ=0.5 S/m

J_z @$z = 0$m, $x = 0.129$m, for $0 \le y \le 0.06$m

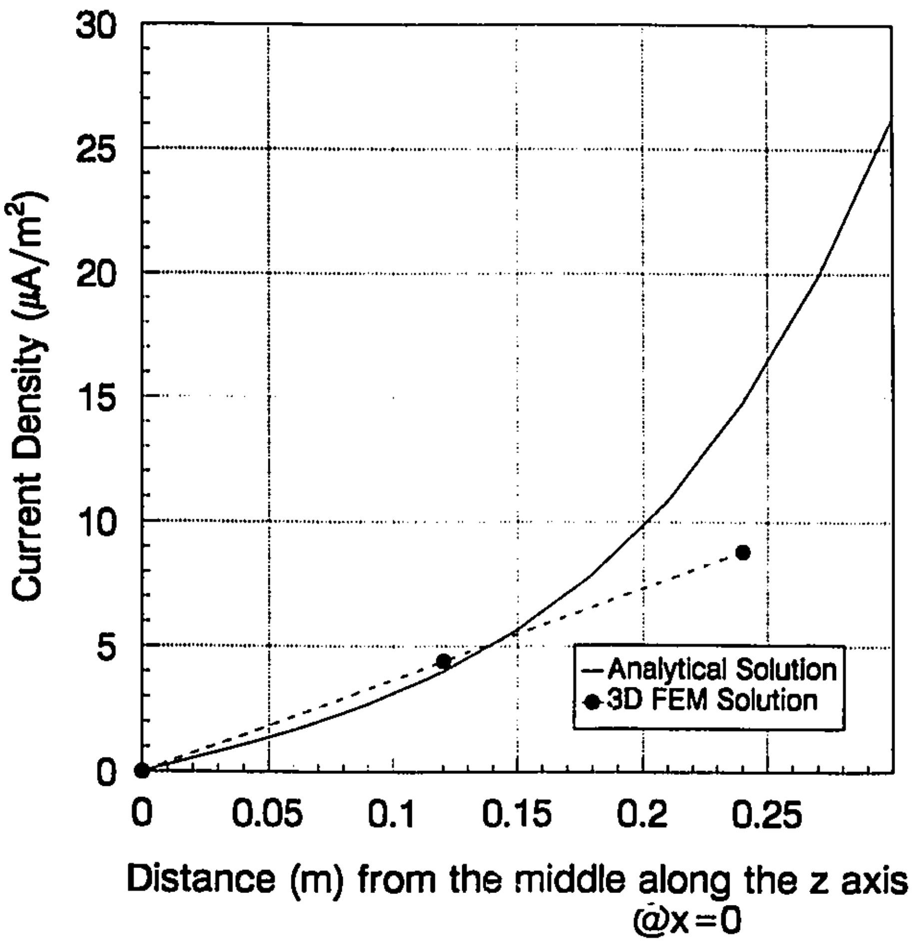


Graph 6.11d: J_z @ x=0.129m in the homogeneous long cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Long Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.6m, σ=0.5 S/m

J_y @$y = 0$m, $x = 0$m, for $0 \le z \le 0.30$m

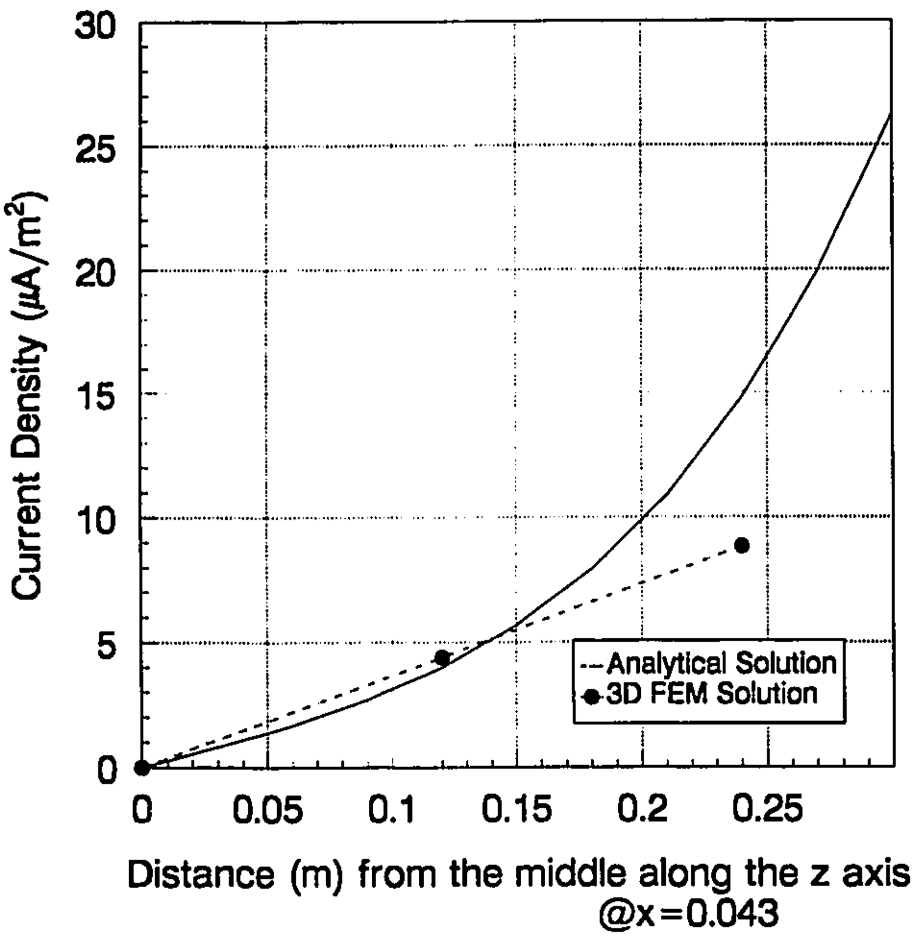


Graph 6.12a: J_y @ x=0.0m in the homogeneous long cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Long Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.6m, σ=0.5 S/m

J_y @$y = 0$m, $x = 0.043$m, for $0 \leq z \leq 0.30$m

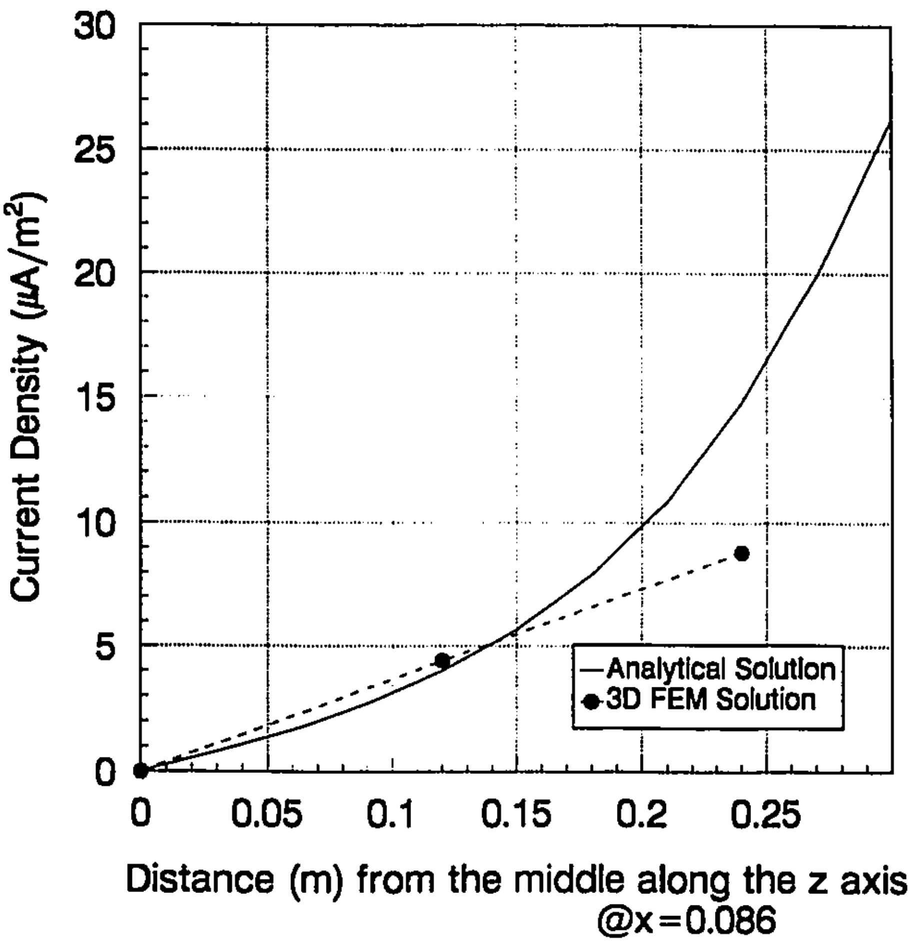


Graph 6.12b: J_y @ x=0.043m in the homogeneous long cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Long Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.6m, σ=0.5 S/m

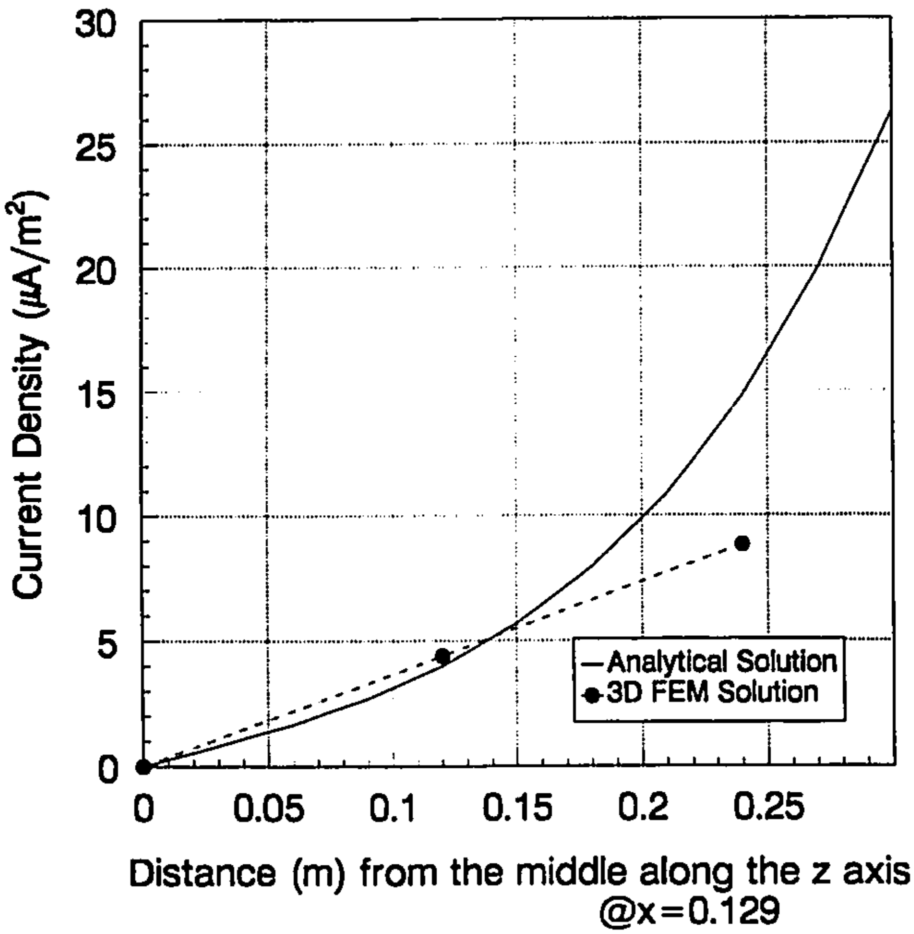
J_y @$y = 0$m, $x = 0.086$m, for $0 \le z \le 0.30$m



Graph 6.12c: J_y @ x=0.086m in the homogeneous long cylinder with $\vec{H}$ perpendicular to the longitudinal axis

Homogeneous Long Cylinder with $\vec{H}$ Perpendicular to the Longitudinal Axis

Radius = 0.15m, Height = 0.6m, σ=0.5 S/m
J_y @$y = 0$m, $x = 0.129$m, for $0 \le z \le 0.30$m



Graph 6.12d: J_y @ x=0.129m in the homogeneous long cylinder with $\vec{H}$ perpendicular to the longitudinal axis

6.5 Conclusion

The 3-D FEM results were presented and compared to proven solutions in this chapter. These results have shown that the 3-D FEM formulation is valid for a homogeneous or heterogeneous conducting object submerged in an arbitrarily oriented low frequency magnetic field. The results have also shown that the formulation has severe limitations due to the conditioning of the matrices in the final system of equations to solve. A condition number of 1 (one) indicates that a matrix is well conditioned. The reciprocal of the condition number[34] of the matrix described in the "Sparse User's Guide - A Sparse Linear Equation Solver"[33] was calculated as 1.3×10^{-6} for the situation in figure 6.5a (Homogeneous short cylinder with $\vec{H}$ perpendicular to the longitudinal axis). Thus, this would indicate that uncertainty in these results are unavoidable.

Nevertheless, the results show that the solutions are good within one order of magnitude and give the user a tool to get a reasonable idea of the magnitude and the behavior of the current densities to be present in a conducting object submerged in a magnetic field.

Chapter 7 Conclusion

The objective of this thesis was to develop a numerical formulation which would predict induced current densities in biological objects, or objects having similar conductivities, exposed to ELF magnetic fields more easily and with more accuracy than the formulations presently available. That is to say that the task of modelling an object being studied and using the program developed in this thesis should be as simple as possible, and the results yielded by this formulation should be more accurate.

A FORTRAN program was developed based on the Three Dimensional Finite Element Method (3-D FEM) numerical formulation. The program is easy to use: only a few lines need to be modified before it is compiled and executed, and the input file format required by the program is simple.

The 3-D FEM program was tested for several cases to verify its validity. It has been demonstrated that the formulation is valid but that there are severe limitations due to the conditioning of the matrices derived from the governing coupled equations to solve. The cases of the homogeneous and layered long cylinders immersed 60 Hz magnetic fields oriented parallel to the longitudinal axes have shown that the program is valid for predicting the induced current densities inside simple homogeneous and heterogeneous objects having conductivity values close to that found in the human body. The following cases also have the 60 Hz magnetic field orientated parallel to the longitudinal axes of the objects but this time, the objects have different shapes: they are homogeneous square and rectangular cylinders. The corresponding results show that the formulation is valid for simple objects having different shapes. These results also start to show the limitations of the formulation. The last cases presented are used to show that the formulation and the program are valid for objects being immersed in arbitrarily oriented 60 Hz magnetic fields. The objects used are round homogeneous short and long cylinders immersed in magnetic fields oriented perpendicular to their longitudinal axes. It is seen that the formulation is still valid, but the limitations for this formulation are clearly shown by these results.

The results have shown that the program works well for very simple object under study. These objects may be homogeneous ones having different conductivities or may be heterogeneous. The magnetic fields in which these objects are immersed may be arbitrarily oriented. Unfortunately, at the present time, it is not possible to use this program to determine the induced current densities in complicated biological objects due to their complexity which would require a large number of elements to model them and the ill-conditioning of the matrices found in the matrices found in the final system of equations to solve.

The 3-D FEM used to developed the program uses linear interpolative functions in its formulation. It would interesting study the effect of using higher order interpolative functions to replace the linear ones. If further work is to be continued in this area, some means of dealing with limitations imposed by the ill-conditioning of the matrices will have to be studied. It is worth investigating a formulation which includes $\vec{E}$ and $\vec{H}$ instead of $\vec{A}$ and ϕ and determining if the matrices are better conditioned and yield better solutions. Of course, the number of unknowns per node would be increased to six from four which would imply a larger system of equations to solve and more computer memory would be required. Also, a new 3-D finite element called the edge-element[35] could be used to model conducting objects and compare those results to the ones presented in this thesis.

Appendix 1 Program Listing for the 2-D FEM Solution for the Homogeneous Cylinder

```
      PROGRAM CFEM_CYL1
C     PROGRAM CFEM_CYL1.FOR
C     OUTPUT DATA FOUND IN CFEM_CYL1.DAT
      IMPLICIT NONE
      PARAMETER   Ho=1.0D0
      PARAMETER   SIGMA=0.04D0
      PARAMETER   FREQ=60.0D0
      PARAMETER   NNHS=12
      PARAMETER   NNVS=91
      PARAMETER   RADIUS=0.15D0
      PARAMETER   HEIGHT=1.80D0
      PARAMETER   EPSILON=1.0D-10
C*********************************************************************
C     FEM_CYL will calculate the CURRENT distribution in a cylinder
C     with finite conductivity, free space permitivity and
C     permeability, due to a time varying magnetic field parallel
C     to the cylinder axix.
C     Because this is an axisymmetrical problem.
C     the cylindrical coordinate sytem are used.
C*********************************************************************
C     ASIZE       - actual dimension size of the arrays used
C     H           - matrix with S,T elements
C     B           - vector with boundary conditions
C     NOTR        - total number of triangles
C     NOPTL       - total number of nodes
C     NDIR        - number of dirichlet nodes
C     NNHS        - number of nodes on horizontal sections
C     NNVS        - number of nodes on vertical sections
C     R(i)        - R coordinate of node (i)
C     Z(i)        - Z coordinate of node (i)
C     P(i)        - dirichlet potential of node (i)
C     IV(t,i)     - node # on grid for triangle (t), node (i)
C     NOU         - number of unknowns
C     I1,I2,I3 - I1=IV(triangle being analyzed, node 1)
C     PI          - 3.141526...
C     R1,R2,R3 - R coordinates of triangle being analyzed
C     Z1,Z2,Z3 - Z coordinates of triangle being analyzed
C     S           - S shape element matrix of triangle being analyzed
C     T           - T shape element matrix of triangle being analyzed
C     MUE_NOT  - permeability of free space = 4*PI*1x10-7
C     SIGMA       - conductivity of cylinder in mho/m
C     RAD_FREQ - radial frequency of H field
C     Ho          - strength of H field tangential to axis of symmetry
C     MR(i)       - R coordinate of point i in the cylinder
C     MZ(i)       - Z coordinate of point i in the cylinder
C     MB(i)       - Magnetic field strength of the point i in the
C                   cylinder
C**********************************************************************
```

```
      INTEGER      NOTR,NOPTL,NDIR
      INTEGER      ASIZE, I, K
      INTEGER      NOLE, TNOE
      INTEGER      IV((NNHS-1)*(NNVS-1)*2,3)
      INTEGER      ITR, L, LC, NOU
      INTEGER      I1,I2,I3
      REAL*8       R(NNHS*NNVS), Z(NNHS*NNVS), P(NNVS)
      REAL*8       PI, MR(NNHS*NNVS),MZ(NNHS*NNVS)
      REAL*8       R1, R2, R3, Z1, Z2, Z3
      REAL*8       S(3,3), T(3,3)
      REAL*8       MUE_NOT, RAD_FREQ
      COMPLEX*16   H((NNHS-1)*NNVS,(NNHS-1)*NNVS), B((NNHS-1)*NNVS)
      COMPLEX*16   DCMPLX, MB(NNHS*NNVS)
      INTEGER      JOB,PIV((NNHS-1)*(NNVS))
      REAL*8       RCOND
      COMPLEX*16   WORK((NNHS-1)*NNVS)
      INTEGER      NODE(NNVS,NNHS)
      REAL*8       DR(NNHS),DZ(NNVS)
      COMPLEX*16   J(NNHS)
      COMMON /A/ R1, R2, R3, Z1, Z2, Z3
      COMMON /B/ I1, I2, I3, ITR
      COMMON /C/ S, MUE_NOT
      COMMON /D/ T, RAD_FREQ
      COMMON /E/ PI
      OPEN (UNIT=1, FILE='M4A.DAT',STATUS='NEW')

1     FORMAT(1X,' ')
2     FORMAT(1X,' CONDITION OF THE MATRIX IS: ',E15.7)
C     INITIALIZE STARTING VALUES
      PI=4.0D0*DATAN(1.0D0)
      MUE_NOT  =4.0D0*PI*1.0D-7
      RAD_FREQ =2.0D0*PI*FREQ
      NOTR     =(NNHS-1)*(NNVS-1)*2
      NOPTL    =NNHS*NNVS
      NDIR     =NNVS
      NOU      =NOPTL-NDIR
      ASIZE    =NOU
C-----------------------------------------------------------------------
C     'GRID' will initialize R,Z,P,IV
      CALL GRID (R,Z,P,IV,NDIR,Ho,NOTR,NOPTL,NNHS,NNVS,
     &           NODE,DR,DZ,HEIGHT,RADIUS)
C-----------------------------------------------------------------------
C     assemble the H and B matrix for the triangle elements

      DO ITR=1,NOTR
C       calculate I1,I2,I3,R1,R2,R3,Z1,Z2,Z3
        CALL CAL_IRZ(NOTR,NOPTL,R,Z,IV)
C     calculate S matrix
        CALL S_MAT
C       calculate T matrix
        CALL T_MAT(SIGMA)

        DO I=1,3
          L=IV(ITR,I)-NDIR
          IF(IV(ITR,I).GT.NDIR) THEN
            DO  K=1,3
              IF(IV(ITR,K).GT.NDIR) THEN
                LC=IV(ITR,K)-NDIR
                H(L,LC)=H(L,LC)+DCMPLX(S(I,K),T(I,K))
```

```
             ELSE
              B(L)=B(L)-P(IV(ITR,K))*DCMPLX(S(I,K),T(I,K))
            ENDIF
          ENDDO
        ENDIF
      ENDDO
    ENDDO
C-----------------------------------------------------------------------
C     Solve the system of equations [H] * x = [B]
C     Results are placed in B
      CALL ZGECO(H,ASIZE,ASIZE,PIV,RCOND,WORK)
      JOB=0
      CALL ZGESL(H,ASIZE,ASIZE,PIV,B,JOB)
C-----------------------------------------------------------------------
C     A_GRID writes a data file containing the R and Z coordinates
C     with the corresponding magnetic field matrix 'H'.
      CALL A_GRID(NOPTL,NDIR,NNHS,NNVS,NOU,R,Z,P,B,MR,MZ,MB,
     &            HEIGHT,EPSILON)
C     CURRENT will find the current distribution in the center of the
C     cylinder denoted J(COUNT).
      CALL CURRENT(NNHS,NNVS,NOPTL,MR,MZ,MB,J,RADIUS)
      WRITE(1,1)
      WRITE(1,2) RCOND
      CLOSE (1)
      CLOSE (2)
      STOP
      END

C-----------------------------------------------------------------------
C*********************** SUBROUTINE SECTION ***************************
C-----------------------------------------------------------------------
      SUBROUTINE GRID (R,Z,P,IV,NDIR,Ho,NOTR,NOPTL,NNHS,NNVS,
     &                 NODE,DR,DZ,HEIGHT,RADIUS)

      IMPLICIT NONE

      INTEGER NOTR,NOPTL
      INTEGER NNHS,NNVS,NDIR
      INTEGER I,J,M,N,K
      INTEGER IV(NOTR,3), NODE(NNVS,NNHS)
      REAL*8  Ho,HEIGHT,RADIUS
      REAL*8  DR(NNHS),DZ(NNVS),R(NOPTL),Z(NOPTL),P(NDIR)

C     calculate R coordinate of vertical grid line
      DO I=1,NNHS
        DR(I)=0.0D0+(I-1)*(RADIUS/(NNHS-1))
      ENDDO
C     calculate Z coordinate of horizontal grid line
      DO I=1,NNVS
        DZ(I)=HEIGHT - (I-1)*(HEIGHT/(NNVS-1))
      ENDDO
C     assign node number and potential
      M=NDIR+1
      N=1
      DO K=1,NNHS
        DO I=1,NNVS
          IF((K.EQ.NNHS).AND.(I.GE.1).AND.(I.LE.NNVS)) THEN
            NODE(I,K)=N
            R(N)=DR(K)
```

```
          Z(N)=DZ(I)
          P(N)=Ho
          N=N+1
        ELSE
          NODE(I,K)=M
          R(M)=DR(K)
          Z(M)=DZ(I)
          M=M+1
        ENDIF
      ENDDO
    ENDDO
C   number the triangles and assign nodes to them (triangle elements)
    N=0
    DO J=1,NNHS-1
      DO I=1,NNVS-1
        DO M=1,2
          N=N+1
          IV(N,1)=NODE(I,J)
          IV(N,2)=NODE(I+1,J+M-1)
          IV(N,3)=NODE(I-M+2,J+1)
        ENDDO
      ENDDO
    ENDDO
    RETURN
    END
C-----------------------------------------------------------------------
    SUBROUTINE CAL_IRZ(NOTR,NOPTL,R,Z,IV)
    IMPLICIT NONE
    COMMON /A/ R1,R2,R3,Z1,Z2,Z3
    COMMON /B/ I1,I2,I3,ITR
    INTEGER NOTR,NOPTL
    INTEGER I1,I2,I3,IV(NOTR,3),ITR
    REAL*8  R1,R2,R3,Z1,Z2,Z3,R(NOPTL),Z(NOPTL)
    I1=IV(ITR,1)
    I2=IV(ITR,2)
    I3=IV(ITR,3)
    R1=R(I1)
    R2=R(I2)
    R3=R(I3)
    Z1=Z(I1)
    Z2=Z(I2)
    Z3=Z(I3)

    RETURN
    END
C-----------------------------------------------------------------------
    SUBROUTINE S_MAT
    IMPLICIT NONE
    COMMON /A/ R1,R2,R3,Z1,Z2,Z3
    COMMON /C/ S,MUE_NOT
    REAL*8 R1,R2,R3,Z1,Z2,Z3,S(3,3),MUE_NOT
    REAL*8 DT,AREA,R_AVG,S_CONST
    REAL*8 X1,X2,X3,Y1,Y2,Y3
    DT(X1,Y1,X2,Y2,X3,Y3)=X1*Y2-X2*Y1+X2*Y3-X3*Y2+X3*Y1-X1*Y3
    AREA=DABS(DT(R1,Z1,R2,Z2,R3,Z3)/2.0D0)
    R_AVG=(R1+R2+R3)/3.0D0
    S_CONST=R_AVG/(4.0D0*AREA*MUE_NOT)
```

```
      S(1,1)=((Z2-Z3)**2+(R3-R2)**2)*S_CONST
      S(1,2)=((Z2-Z3)*(Z3-Z1)+(R3-R2)*(R1-R3))*S_CONST
      S(1,3)=((Z2-Z3)*(Z1-Z2)+(R3-R2)*(R2-R1))*S_CONST
      S(2,1)=S(1,2)
      S(2,2)=((Z3-Z1)**2+(R1-R3)**2)*S_CONST
      S(2,3)=((Z3-Z1)*(Z1-Z2)+(R1-R3)*(R2-R1))*S_CONST
      S(3,1)=S(1,3)
      S(3,2)=S(2,3)
      S(3,3)=((Z1-Z2)**2+(R2-R1)**2)*S_CONST
      RETURN
      END
C-----------------------------------------------------------------------
      SUBROUTINE T_MAT(SIGMA)
      IMPLICIT NONE
      COMMON /A/ R1,R2,R3,Z1,Z2,Z3
      COMMON /D/ T,RAD_FREQ
      REAL*8 SIGMA
      REAL*8 R1,R2,R3,Z1,Z2,Z3
      REAL*8 RAD_FREQ, T(3,3)
      REAL*8 DT,T_CONST,AREA
      REAL*8 X1,X2,X3,Y1,Y2,Y3
      DT(X1,Y1,X2,Y2,X3,Y3)=X1*Y2-X2*Y1+X2*Y3-X3*Y2+X3*Y1-X1*Y3
      AREA=DABS(DT(R1,Z1,R2,Z2,R3,Z3)/2.0D0)
      T_CONST=(2.0D0 * AREA * RAD_FREQ * SIGMA) / 120.0D0

      T(1,1)=(6.0D0*R1 + 2.0D0*R2 + 2.0D0*R3)*T_CONST
      T(1,2)=(2.0D0*R1 + 2.0D0*R2 + 1.0D0*R3)*T_CONST
      T(1,3)=(2.0D0*R1 + 1.0D0*R2 + 2.0D0*R3)*T_CONST
      T(2,1)=(2.0D0*R1 + 2.0D0*R2 + 1.0D0*R3)*T_CONST
      T(2,2)=(2.0D0*R1 + 6.0D0*R2 + 2.0D0*R3)*T_CONST
      T(2,3)=(1.0D0*R1 + 2.0D0*R2 + 2.0D0*R3)*T_CONST
      T(3,1)=(2.0D0*R1 + 1.0D0*R2 + 2.0D0*R3)*T_CONST
      T(3,2)=(1.0D0*R1 + 2.0D0*R2 + 2.0D0*R3)*T_CONST
      T(3,3)=(2.0D0*R1 + 2.0D0*R2 + 6.0D0*R3)*T_CONST
      RETURN
      END
C-----------------------------------------------------------------------
      SUBROUTINE A_GRID(NOPTL,NDIR,NNHS,NNVS,NOU,R,Z,P,B,MR,MZ,MB,
     &                  HEIGHT,EPSLN)

      IMPLICIT  NONE

      INTEGER     NOPTL,NDIR,NNHS,NNVS,NOU
      INTEGER     J, I, K
      REAL*8      R(NOPTL), Z(NOPTL), P(NDIR), MR(NOPTL),MZ(NOPTL)
      REAL*8      HEIGHT,EPSLN
      COMPLEX*16 B(NOU),TEMP, MB(NOPTL)
      DO I=1,NDIR
        K=I*NNHS
        MR(K)=R(I)
        MZ(K)=Z(I)
        MB(K)=DCMPLX(P(I),0.0D0)
      ENDDO

      DO J=1,(NNHS-1)
        DO I=1,NDIR
          K=J+(I-1)*NNHS
          MR(K)=R(I+J*NDIR)
          MZ(K)=Z(I+J*NDIR)
```

```
          MB(K)=B(I+(J-1)*NDIR)
        ENDDO
      ENDDO
      RETURN
      END
C     ----------------------------------------------------------
      SUBROUTINE CURRENT(NNHS,NNVS,NOPTL,MR,MZ,MB,J,RADIUS)
      IMPLICIT NONE
      COMMON /D/ T, RAD_FREQ
      COMMON /E/ PI
      INTEGER     NNHS,NNVS,NOPTL
      INTEGER     COUNT,I,L,K
      REAL*8      RAD_FREQ, T(3,3)
      REAL*8      MR(NOPTL),MZ(NOPTL),P,R, PI
      REAL*8      RADIUS,DIV,MJ,AJ
      COMPLEX*16 MB(NOPTL),JTEMP,J(NOPTL)
C     CALCULATE THE CURRENT DISTRIBUTION
      COUNT=1
      DO L=1,NNVS
        DO I=1,NNHS
          IF (I.EQ.1) THEN
            DIV=MR(I+1+(L-1)*NNHS)-MR(I+(L-1)*NNHS)
            JTEMP=MB(I+(L-1)*NNHS)-MB(I+1+(L-1)*NNHS)
           ELSE IF (I.LT.NNHS) THEN
              DIV=MR(I+1+(L-1)*NNHS)-MR(I-1+(L-1)*NNHS)
              JTEMP=MB(I-1+(L-1)*NNHS)-MB(I+1+(L-1)*NNHS)
             ELSE
                DIV=MR(I+(L-1)*NNHS)-MR(I-1+(L-1)*NNHS)
                JTEMP=MB(I-1+(L-1)*NNHS)-MB(I+(L-1)*NNHS)
          ENDIF
          J(COUNT)=JTEMP/DCMPLX(DIV,0.0D0)
          COUNT=COUNT+1
        ENDDO
      ENDDO
1     FORMAT(1X,'CFEM_2D_CYL1.DAT OUTPUT DATA FILE')
2     FORMAT(1X,'CURRENT DENSITY DISTRIBUTION')
3     FORMAT(1X,' ')
4     FORMAT(7X,'Z',13X,'R',9X,'MAG(J)',8X,'ANG(J)')
5     FORMAT(1X,D11.4,3X,D11.4,3X,D11.4,3X,D11.4)
      WRITE(1,1)
      WRITE(1,2)
      WRITE(1,3)
      WRITE(1,4)
      DO I=1,NOPTL
        MJ=CDABS(J(I))
        IF (MJ.EQ.0.0D0) THEN
          AJ=0.0D0
         ELSE
          AJ=DATAN2D(DIMAG(J(I)),DREAL(J(I)))
        ENDIF
        WRITE(1,5) MZ(I),MR(I),MJ,AJ
      ENDDO
      WRITE(*,*) 'DATA IS FOUND IN "M4A.DAT"'
      RETURN
      END
```

Appendix 2 Program Listing for the 2-D FEM Solution for the Layered Cylinder

```
      PROGRAM CFEM_CYL2
C     PROGRAM CFEM_CYL2.FOR
C     OUTPUT DATA FOUND IN CFEM_2D_CYL2.DAT
      IMPLICIT NONE
      PARAMETER   Ho=1.0D0
      PARAMETER   SIGMA1=0.5D0
      PARAMETER   SIGMA2=1.0D0
      PARAMETER   DRAD=0.070D0
      PARAMETER   FREQ=60.0D0
      PARAMETER   NNHS=16
      PARAMETER   NNVS=101
      PARAMETER   RADIUS=0.15D0
      PARAMETER   HEIGHT=1.80D0
      PARAMETER   EPSILON=1.0D-10
C**********************************************************************
C     CFEM_CYL2 will calculate the CURRENT distribution in a cylinder
C     with two different layers of finite conductivity, and homogenous
C     free space permitivity and permeability, due to a time
C     varying magnetic field parallel to the cylinder axix.
C     Because this is an axisymmetrical problem.
C     the cylindrical coordinate sytem will be used.
C**********************************************************************
C     ASIZE      - actual dimension size of the arrays used
C     H          - matrix with S,T elements
C     B          - vector with boundary conditions
C     NOTR       - total number of triangles
C     NOPTL      - total number of nodes
C     NDIR       - number of dirichlet nodes
C     NNHS       - number of nodes on horizontal sections
C     NNVS       - number of nodes on vertical sections
C     R(i)       - R coordinate of node (i)
C     Z(i)       - Z coordinate of node (i)
C     P(i)       - dirichlet potential of node (i)
C     IV(t,i)    - node # on grid for triangle (t), node (i)
C     NOU        - number of unknowns
C     I1,I2,I3 - I1=IV(triangle being analyzed, node 1)
C     PI         - 3.141526...
C     R1,R2,R3 - R coordinates of triangle being analyzed
C     Z1,Z2,Z3 - Z coordinates of triangle being analyzed
C     S          - S shape element matrix of triangle being analyzed
C     T          - T shape element matrix of triangle being analyzed
C     MUE_NOT    - permeability of free space = 4*PI*1x10-7
C     SIGMA1     - conductivity of inside cylinder in mho/m
C     SIGMA2     - conductivity of outside cylinder in mho/m
C     DRAD       - Radius of conductivity discontinuity
C     RAD_FREQ - radial frequency of H field
C     Ho         - strength of H field tangential to axis of symmetry
C     MR(i)      - R coordinate of point i in the cylinder
C     MZ(i)      - Z coordinate of point i in the cylinder
C     MB(i)      - Magnetic field strength of the point i in the
C                  cylinder
C**********************************************************************
```

```
      INTEGER       NOTR,NOPTL,NDIR
      INTEGER       ASIZE, I, K
      INTEGER       NOLE, TNOE
      INTEGER       IV((NNHS-1)*(NNVS-1)*2,3)
      INTEGER       ITR, L, LC, NOU
      INTEGER       I1,I2,I3
      REAL*8        R(NNHS*NNVS), Z(NNHS*NNVS), P(NNVS)
      REAL*8        PI, MR(NNHS*NNVS),MZ(NNHS*NNVS)
      REAL*8        R1, R2, R3, Z1, Z2, Z3
      REAL*8        S(3,3), T(3,3)
      REAL*8        MUE_NOT, RAD_FREQ
      COMPLEX*16    H((NNHS-1)*NNVS,(NNHS-1)*NNVS), B((NNHS-1)*NNVS)
      COMPLEX*16    DCMPLX, MB(NNHS*NNVS)
      INTEGER       JOB,PIV((NNHS-1)*(NNVS))
      REAL*8        RCOND
      COMPLEX*16    WORK((NNHS-1)*NNVS)
      INTEGER       NODE(NNVS,NNHS)
      REAL*8        DR(NNHS),DZ(NNVS)
      COMPLEX*16    J(NNHS)
      COMMON /A/ R1, R2, R3, Z1, Z2, Z3
      COMMON /B/ I1, I2, I3, ITR
      COMMON /C/ S, MUE_NOT
      COMMON /D/ T, RAD_FREQ
      COMMON /E/ PI
      OPEN (UNIT=1, FILE='CFEM_2D_CYL2.DAT',STATUS='NEW')

1     FORMAT(1X,' ')
2     FORMAT(1X,' CONDITION OF THE MATRIX IS: ',E15.7)
C     INITIALIZE STARTING VALUES
      PI=4.0D0*DATAN(1.0D0)
      MUE_NOT  =4.0D0*PI*1.0D-7
      RAD_FREQ =2.0D0*PI*FREQ
      NOTR     =(NNHS-1)*(NNVS-1)*2
      NOPTL    =NNHS*NNVS
      NDIR     =NNVS
      NOU      =NOPTL-NDIR
      ASIZE    =NOU
C-----------------------------------------------------------------------
C     'GRID' will initialize R,Z,P,IV
      CALL GRID (R,Z,P,IV,NDIR,Ho,NOTR,NOPTL,NNHS,NNVS,
     &           NODE,DR,DZ,HEIGHT,RADIUS)
C-----------------------------------------------------------------------
C     assemble the H and B matrix for the triangle elements

      DO ITR=1,NOTR
C       calculate I1,I2,I3,R1,R2,R3,Z1,Z2,Z3
        CALL CAL_IRZ(NOTR,NOPTL,R,Z,IV)
C       calculate S matrix
        CALL S_MAT
C       calculate T matrix
        CALL T_MAT(SIGMA1,SIGMA2,DRAD)

        DO I=1,3
          L=IV(ITR,I)-NDIR
          IF(IV(ITR,I).GT.NDIR) THEN
            DO  K=1,3
              IF(IV(ITR,K).GT.NDIR) THEN
                LC=IV(ITR,K)-NDIR
                H(L,LC)=H(L,LC)+DCMPLX(S(I,K),T(I,K))
```

```
                ELSE
                 B(L)=B(L)-P(IV(ITR,K))*DCMPLX(S(I,K),T(I,K))
               ENDIF
             ENDDO
           ENDIF
         ENDDO
       ENDDO
C-----------------------------------------------------------------------
C      Solve the system of equations [H] * x = [B]
C      Results are placed in B
       CALL ZGECO(H,ASIZE,ASIZE,PIV,RCOND,WORK)
       JOB=0
       CALL ZGESL(H,ASIZE,ASIZE,PIV,B,JOB)
C-----------------------------------------------------------------------
C      A_GRID writes a data file containing the R and Z coordinates
C      with the corresponding magnetic field matrix 'H'.
       CALL A_GRID(NOPTL,NDIR,NNHS,NNVS,NOU,R,Z,P,B,MR,MZ,MB,
      &            HEIGHT,EPSILON)
C      CURRENT will find the current distribution in the center of the
C      cylinder denoted J(COUNT).
       CALL CURRENT(NNHS,NNVS,NOPTL,MR,MZ,MB,J,RADIUS)
       WRITE(1,1)
       WRITE(1,2) RCOND
       CLOSE (1)
       CLOSE (2)
       STOP
       END

C-----------------------------------------------------------------------
C*********************** SUBROUTINE SECTION ****************************
C-----------------------------------------------------------------------
       SUBROUTINE GRID (R,Z,P,IV,NDIR,Ho,NOTR,NOPTL,NNHS,NNVS,
      &                 NODE,DR,DZ,HEIGHT,RADIUS)

       IMPLICIT NONE

       INTEGER NOTR,NOPTL
       INTEGER NNHS,NNVS,NDIR
       INTEGER I,J,M,N,K
       INTEGER IV(NOTR,3), NODE(NNVS,NNHS)
       REAL*8  Ho,HEIGHT,RADIUS
       REAL*8  DR(NNHS),DZ(NNVS),R(NOPTL),Z(NOPTL),P(NDIR)

C      calculate R coordinate of vertical grid line
       DO I=1,NNHS
         DR(I)=0.0D0+(I-1)*(RADIUS/(NNHS-1))
       ENDDO
C      calculate Z coordinate of horizontal grid line
       DO I=1,NNVS
         DZ(I)=HEIGHT - (I-1)*(HEIGHT/(NNVS-1))
       ENDDO
C      assign node number and potential
       M=NDIR+1
       N=1
       DO K=1,NNHS
         DO I=1,NNVS
           IF((K.EQ.NNHS).AND.(I.GE.1).AND.(I.LE.NNVS)) THEN
             NODE(I,K)=N
             R(N)=DR(K)
```

```
          Z(N)=DZ(I)
          P(N)=Ho
          N=N+1
        ELSE
          NODE(I,K)=M
          R(M)=DR(K)
          Z(M)=DZ(I)
          M=M+1
        ENDIF
      ENDDO
    ENDDO
C   number the triangles and assign nodes to them (triangle elements)
    N=0
    DO J=1,NNHS-1
      DO I=1,NNVS-1
        DO M=1,2
          N=N+1
          IV(N,1)=NODE(I,J)
          IV(N,2)=NODE(I+1,J+M-1)
          IV(N,3)=NODE(I-M+2,J+1)
        ENDDO
      ENDDO
    ENDDO
    RETURN
    END
C-----------------------------------------------------------------------
    SUBROUTINE CAL_IRZ(NOTR,NOPTL,R,Z,IV)
    IMPLICIT NONE
    COMMON /A/ R1,R2,R3,Z1,Z2,Z3
    COMMON /B/ I1,I2,I3,ITR
    INTEGER NOTR,NOPTL
    INTEGER I1,I2,I3,IV(NOTR,3),ITR
    REAL*8  R1,R2,R3,Z1,Z2,Z3,R(NOPTL),Z(NOPTL)
    I1=IV(ITR,1)
    I2=IV(ITR,2)
    I3=IV(ITR,3)
    R1=R(I1)
    R2=R(I2)
    R3=R(I3)
    Z1=Z(I1)
    Z2=Z(I2)
    Z3=Z(I3)

    RETURN
    END
C-----------------------------------------------------------------------
    SUBROUTINE S_MAT
    IMPLICIT NONE
    COMMON /A/ R1,R2,R3,Z1,Z2,Z3
    COMMON /C/ S,MUE_NOT
    REAL*8 R1,R2,R3,Z1,Z2,Z3,S(3,3),MUE_NOT
    REAL*8 DT,AREA,R_AVG,S_CONST
    REAL*8 X1,X2,X3,Y1,Y2,Y3
    DT(X1,Y1,X2,Y2,X3,Y3)=X1*Y2-X2*Y1+X2*Y3-X3*Y2+X3*Y1-X1*Y3
    AREA=DABS(DT(R1,Z1,R2,Z2,R3,Z3)/2.0D0)
    R_AVG=(R1+R2+R3)/3.0D0
    S_CONST=R_AVG/(4.0D0*AREA*MUE_NOT)
```

```
      S(1,1)=((Z2-Z3)**2+(R3-R2)**2)*S_CONST
      S(1,2)=((Z2-Z3)*(Z3-Z1)+(R3-R2)*(R1-R3))*S_CONST
      S(1,3)=((Z2-Z3)*(Z1-Z2)+(R3-R2)*(R2-R1))*S_CONST
      S(2,1)=S(1,2)
      S(2,2)=((Z3-Z1)**2+(R1-R3)**2)*S_CONST
      S(2,3)=((Z3-Z1)*(Z1-Z2)+(R1-R3)*(R2-R1))*S_CONST
      S(3,1)=S(1,3)
      S(3,2)=S(2,3)
      S(3,3)=((Z1-Z2)**2+(R2-R1)**2)*S_CONST
      RETURN
      END
C-----------------------------------------------------------------------
      SUBROUTINE T_MAT(SIGMA1,SIGMA2,DRAD)
      IMPLICIT NONE
      COMMON /A/ R1,R2,R3,Z1,Z2,Z3
      COMMON /D/ T,RAD_FREQ
      REAL*8 SIGMA1,SIGMA2,DRAD,SIG
      REAL*8 R1,R2,R3,Z1,Z2,Z3
      REAL*8 RAD_FREQ, T(3,3)
      REAL*8 DT,T_CONST,AREA
      REAL*8 X1,X2,X3,Y1,Y2,Y3
      DT(X1,Y1,X2,Y2,X3,Y3)=X1*Y2-X2*Y1+X2*Y3-X3*Y2+X3*Y1-X1*Y3
      AREA=DABS(DT(R1,Z1,R2,Z2,R3,Z3)/2.0D0)
      IF ((R1.LE.DRAD).AND.(R2.LE.DRAD).AND.(R3.LE.DRAD)) THEN
        SIG=SIGMA1
       ELSE
        SIG=SIGMA2
      ENDIF
      T_CONST=(2.0D0 * AREA * RAD_FREQ * SIG) / 120.0D0

      T(1,1)=(6.0D0*R1 + 2.0D0*R2 + 2.0D0*R3)*T_CONST
      T(1,2)=(2.0D0*R1 + 2.0D0*R2 + 1.0D0*R3)*T_CONST
      T(1,3)=(2.0D0*R1 + 1.0D0*R2 + 2.0D0*R3)*T_CONST
      T(2,1)=(2.0D0*R1 + 2.0D0*R2 + 1.0D0*R3)*T_CONST
      T(2,2)=(2.0D0*R1 + 6.0D0*R2 + 2.0D0*R3)*T_CONST
      T(2,3)=(1.0D0*R1 + 2.0D0*R2 + 2.0D0*R3)*T_CONST
      T(3,1)=(2.0D0*R1 + 1.0D0*R2 + 2.0D0*R3)*T_CONST
      T(3,2)=(1.0D0*R1 + 2.0D0*R2 + 2.0D0*R3)*T_CONST
      T(3,3)=(2.0D0*R1 + 2.0D0*R2 + 6.0D0*R3)*T_CONST
      RETURN
      END
C-----------------------------------------------------------------------
      SUBROUTINE A_GRID(NOPTL,NDIR,NNHS,NNVS,NOU,R,Z,P,B,MR,MZ,MB,
     &                  HEIGHT,EPSLN)

      IMPLICIT  NONE

      INTEGER     NOPTL,NDIR,NNHS,NNVS,NOU
      INTEGER     J, I, K
      REAL*8      R(NOPTL), Z(NOPTL), P(NDIR), MR(NOPTL),MZ(NOPTL)
      REAL*8      HEIGHT,EPSLN
      COMPLEX*16 B(NOU),TEMP, MB(NOPTL)
      DO I=1,NDIR
        K=I*NNHS
        MR(K)=R(I)
        MZ(K)=Z(I)
        MB(K)=DCMPLX(P(I),0.0D0)
      ENDDO
```

```
      DO J=1,(NNHS-1)
        DO I=1,NDIR
          K=J+(I-1)*NNHS
          MR(K)=R(I+J*NDIR)
          MZ(K)=Z(I+J*NDIR)
          MB(K)=B(I+(J-1)*NDIR)
        ENDDO
      ENDDO
      RETURN
      END
C     ------------------------------------------------------------
      SUBROUTINE CURRENT(NNHS,NNVS,NOPTL,MR,MZ,MB,J,RADIUS)
      IMPLICIT NONE
      COMMON /D/ T, RAD_FREQ
      COMMON /E/ PI
      INTEGER     NNHS,NNVS,NOPTL
      INTEGER     COUNT,I,L,K
      REAL*8      RAD_FREQ, T(3,3)
      REAL*8      MR(NOPTL),MZ(NOPTL),P,R, PI
      REAL*8      RADIUS,DIV,MJ,AJ
      COMPLEX*16 MB(NOPTL),JTEMP,J(NOPTL)
C     CALCULATE THE CURRENT DISTRIBUTION
      COUNT=1
      DO L=1,NNVS
        DO I=1,NNHS
          IF (I.EQ.1) THEN
            DIV=MR(I+1+(L-1)*NNHS)-MR(I+(L-1)*NNHS)
            JTEMP=MB(I+(L-1)*NNHS)-MB(I+1+(L-1)*NNHS)
           ELSE IF (I.LT.NNHS) THEN
              DIV=MR(I+1+(L-1)*NNHS)-MR(I-1+(L-1)*NNHS)
              JTEMP=MB(I-1+(L-1)*NNHS)-MB(I+1+(L-1)*NNHS)
             ELSE
                DIV=MR(I+(L-1)*NNHS)-MR(I-1+(L-1)*NNHS)
                JTEMP=MB(I-1+(L-1)*NNHS)-MB(I+(L-1)*NNHS)
          ENDIF
          J(COUNT)=JTEMP/DCMPLX(DIV,0.0D0)
          COUNT=COUNT+1
        ENDDO
      ENDDO
1     FORMAT(1X,'CFEM_2D_CYL1.DAT OUTPUT DATA FILE')
2     FORMAT(1X,'CURRENT DENSITY DISTRIBUTION')
3     FORMAT(1X,' ')
4     FORMAT(7X,'Z',13X,'R',9X,'MAG(J)',8X,'ANG(J)')
5     FORMAT(1X,D11.4,3X,D11.4,3X,D11.4,3X,D11.4)
      WRITE(1,1)
      WRITE(1,2)
      WRITE(1,3)
      WRITE(1,4)

      DO I=1,NOPTL
        MJ=CDABS(J(I))
        IF (MJ.EQ.0.0D0) THEN
          AJ=0.0D0
         ELSE
          AJ=DATAN2D(DIMAG(J(I)),DREAL(J(I)))
        ENDIF
        WRITE(1,5) MZ(I),MR(I),MJ,AJ
      ENDDO
      WRITE(*,*) 'DATA IS FOUND IN CFEM_2D_CYL2.DAT'
```

```
RETURN
END
```

Appendix 3 Program Listing for the 3-D FEM Solution

```
C       PROGRAM FEM3D.F
        PARAMETER   (TNODE=27,TBRICK=8,DIM3=TNODE*3)
        PARAMETER   (HXR=0.0D0,HXI=0.0D0)
        PARAMETER   (HYR=0.0D0,HYI=0.0D0)
        PARAMETER   (HZR=1.0D0,HZI=0.0D0)
        PARAMETER   (FREQUENCY=60.0D0)
C
C    INTEGER     TNB                 ! total number of brick elements
C    INTEGER     TNN                 ! total number of nodes
C    INTEGER     NODE(TBRICK,8)      ! [ NODE(TNB,8)]
C                                    ! NODE(BRICK,BRICK_NODE)
C    INTEGER     NDIM                ! available dimension size of all arrays
C    INTEGER     FACE(6,TBRICK)      ! [ FACE(6,TNB) ]
C                                    ! FACE(a,b): face a of brick b
C                                    ! FACE(a,b)=1 if on surface
C                                    ! FACE(a,b)=0 if not on surface
C    REAL*8      MUE                 ! permeability (H/m)
C    REAL*8      SIGMA(TBRICK)       ! [ SIGMA(TNB) ]
C                                    ! conductivity (siemens/m) of each element
C    REAL*8      FREQ                ! frequency in Hz
C    REAL*8      W                   ! radial frequency (rad/s)
C    REAL*8      PI                  ! PI~= 3.14159
C    REAL*8      XN(TNODE)           ! [ XN(TNN) ]
C                                    ! X coordinate of node (I)
C    REAL*8      YN(TNODE)           ! [ YN(TNN) ]
C                                    ! Y coordinate of node (I)
C    REAL*8      ZN(TNODE)           ! [ ZN(TNN) ]
C                                    ! Z coordinate of node (I)
C    INTEGER     AS                  ! assembled global S matrix
C    COMPLEX*16  AF(DIM3)            ! [ AF(NDIM) ]
C                                    ! assembled global F matrix
C    COMPLEX*16  SAF(DIM3)           ! solution matrix (new AF)
C    COMPLEX*16  HX                  ! X component of the magnetic field
C    COMPLEX*16  HY                  ! Y component of the magnetic field
C    COMPLEX*16  HZ                  ! Z component of the magnetic field
C    COMPLEX*16  JX(TBRICK),JY(TBRICK),JZ(TBRICK)
C
C    COMPLEX*16  DCMPLX
C    INTEGER I,JC
C
        IMPLICIT NONE
        INTEGER     TNB,TNN,NODE(TBRICK,8),NDIM,FACE(6,TBRICK),AS
        REAL*8      MUE,SIGMA(TBRICK),FREQ,W,PI
        REAL*8      XN(TNODE),YN(TNODE),ZN(TNODE)
        COMPLEX*16  AF(DIM3),SAF(DIM3),HX,HY,HZ
        COMPLEX*16  JX(TBRICK),JY(TBRICK),JZ(TBRICK)
        COMPLEX*16  DCMPLX
C
        COMMON /A/  MUE,W
        COMMON /B/  TNN
        COMMON /C/  TNB
        COMMON /D/  NDIM
        COMMON /E/  HX,HY,HZ
```

```
C
      OPEN(UNIT=1,FILE='fem3d.out',STATUS='NEW')
      OPEN(UNIT=2,FILE='fem3d.in',STATUS='OLD')
C
110   FORMAT (1X,'H=  ( ( ',D11.4,' ) + j ( ',D11.4,' ) ) x')
111   FORMAT (1X,'  + ( ( ',D11.4,' ) + j ( ',D11.4,' ) ) y')
112   FORMAT (1X,'  + ( ( ',D11.4,' ) + j ( ',D11.4,' ) ) z')
C
      WRITE(1,110) HXR,HXI
      WRITE(1,111) HYR,HYI
      WRITE(1,112) HZR,HZI
      WRITE(1,*) ' '
      PI  = 4.0D0*DATAN(1.0D0)
      MUE = PI*4.0D-7
      FREQ = FREQUENCY
      W   = FREQ*2*PI
      HX = DCMPLX(HXR,HXI)
      HY = DCMPLX(HYR,HYI)
      HZ = DCMPLX(HZR,HZI)
      TNB = TBRICK
      TNN = TNODE
      NDIM= DIM3
      READ(2,*)
      READ(2,*)
      READ(2,*)
      READ(2,*)
C*************** INPUT SUBROUTINES *********************
C     enter the nodes and their coordinates
      CALL NCOOR(XN,YN,ZN)
C     specify which node is with which brick element
      CALL NOBRIK(NODE,SIGMA)
C     specify which surfaces are on the boundary
      CALL BOUNDF(FACE,NODE)
C*********** CALCULATING SUBROUTINES *****************
C     assemble global AS matrix (NDIMxNDIM)
      CALL GLOBAS(XN,YN,ZN,NODE,SIGMA,AS)
C     assemble global AF matrix (NDIMx1)
      CALL GLOBAF(XN,YN,ZN,NODE,FACE,AF)
C     solve the system of equations
      CALL SOLVAS(AS,AF,SAF)
C     calculate the current densities for each element
      CALL CURDEN(NDIM,SIGMA,SAF,NODE,XN,YN,ZN,JX,JY,JZ)
C     CLOSE(1)
C     CLOSE(2)
      STOP
      END
C
C******************************************************
C****************  SUBROUTINE SECTION *****************
C******************************************************
C
C
C
      SUBROUTINE NCOOR(XN,YN,ZN)
C     enter the nodes and their coordinates
C
      COMMON /B/ TNN
C
```

```
      IMPLICIT NONE
      INTEGER I,J
      INTEGER TNN
      REAL*8  XN(TNN)
      REAL*8  YN(TNN)
      REAL*8  ZN(TNN)
C
      DO 300 I=1,TNN
        READ(2,*) J,XN(J),YN(J),ZN(J)
300   CONTINUE
      RETURN
      END
C
C
C
      SUBROUTINE NOBRIK(NODE,SIGMA)
C     specify which node is with which brick element
C
      COMMON /C/ TNB
C
      IMPLICIT NONE
      INTEGER I,J
      INTEGER TNB
      INTEGER NODE(TNB,8)
      REAL*8  SIGMA(TNB)
C
      DO 301 I=1,TNB
        READ(2,*) J,NODE(J,1),NODE(J,2),NODE(J,3),NODE(J,4),
     ^              NODE(J,5),NODE(J,6),NODE(J,7),NODE(J,8),SIGMA(J)
301   CONTINUE
      RETURN
      END
C
C
C
      SUBROUTINE BOUNDF(FACE,NODE)
C     specify which faces of which bricks are on the boundary
C
      COMMON /C/ TNB
C
      IMPLICIT NONE
      INTEGER I,J
      INTEGER TNB
      INTEGER NODE(TNB,8)
      INTEGER FACE(6,TNB)
C
      DO 302 I=1,TNB
        DO 303 J=1,6
          FACE(J,I)=1
303     CONTINUE
302   CONTINUE
      DO 305 I=1,TNB
        DO 306 J=1,TNB
          IF((NODE(I,1).EQ.NODE(J,5)).AND.(NODE(I,2).EQ.NODE(J,6)).AND.
     ^       (NODE(I,3).EQ.NODE(J,7)).AND.(NODE(I,4).EQ.NODE(J,8)))
     ^       FACE(1,I)=0
          IF((NODE(I,1).EQ.NODE(J,3)).AND.(NODE(I,2).EQ.NODE(J,4)).AND.
     ^       (NODE(I,5).EQ.NODE(J,7)).AND.(NODE(I,6).EQ.NODE(J,8)))
     ^       FACE(2,I)=0
```

```
          IF((NODE(I,1).EQ.NODE(J,2)).AND.(NODE(I,3).EQ.NODE(J,4)).AND.
     ^       (NODE(I,5).EQ.NODE(J,6)).AND.(NODE(I,7).EQ.NODE(J,8)))
     ^       FACE(3,I)=0
          IF((NODE(I,2).EQ.NODE(J,1)).AND.(NODE(I,4).EQ.NODE(J,3)).AND.
     ^       (NODE(I,6).EQ.NODE(J,5)).AND.(NODE(I,8).EQ.NODE(J,7)))
     ^       FACE(4,I)=0
          IF((NODE(I,3).EQ.NODE(J,1)).AND.(NODE(I,4).EQ.NODE(J,2)).AND.
     ^       (NODE(I,7).EQ.NODE(J,5)).AND.(NODE(I,8).EQ.NODE(J,6)))
     ^       FACE(5,I)=0
          IF((NODE(I,5).EQ.NODE(J,1)).AND.(NODE(I,6).EQ.NODE(J,2)).AND.
     ^       (NODE(I,7).EQ.NODE(J,3)).AND.(NODE(I,8).EQ.NODE(J,4)))
     ^       FACE(6,I)=0
306     CONTINUE
305   CONTINUE
      RETURN
      END
C
C
C
      SUBROUTINE GLOBAS(XN,YN,ZN,NODE,SIGMA,AS)
C
      COMMON /A/ MUE,W
      COMMON /B/ TNN
      COMMON /C/ TNB
      COMMON /D/ NDIM
C
C     INTEGER     AS          ! global S matrix
C     INTEGER     sfcreate    ! variable used in the sparse matrix subroutine
C     INTEGER     sfgetelement! variable used in the sparse matrix subroutine
C     INTEGER     element     ! variable used in the sparse matrix subroutine
C     INTEGER     BRICK       ! brick number
C     INTEGER     I,J,K,L     ! do loop variables
C     REAL*8      SIG         ! SIG=SIGMA(i) i is the current element
C     REAL*8      DX          ! Xj-Xi
C     REAL*8      DY          ! Yj-Yi
C     REAL*8      DZ          ! Zj-Zi
C     COMPLEX*16 S(24,24)
C
      IMPLICIT NONE
      INTEGER     TNN,TNB,NDIM,NODE(TNB,8),BRICK,I,J,K,L,AS
      INTEGER     sfcreate,sfgetelement,element,error,ROW,COLUMN
      REAL*8      SIGMA(TNB),SIG,MUE,W,XN(TNN),YN(TNN),ZN(TNN),DX,DY,DZ
      REAL*8      REAL_S,IMAG_S
      COMPLEX*16 S(24,24)
C
      COMMON /F/ DX,DY,DZ
C
C     create AS matrix
      AS=sfcreate(1,1,error)
      write(1,*) ' error from sfcreate=',error
C
C     clear AS matrix
      call sfclear(AS)
C
C     load matrix
      DO 309 BRICK=1,TNB
        SIG=SIGMA(BRICK)
        DX=XN(NODE(BRICK,8))-XN(NODE(BRICK,1))
```

```
      DY=YN(NODE(BRICK,8))-YN(NODE(BRICK,1))
      DZ=ZN(NODE(BRICK,8))-ZN(NODE(BRICK,1))
      CALL SMAT(S,SIG)
      DO 310 I=1,8,1
        DO 311 J=1,8,1
          DO 312 K=0,2,1
            DO 313 L=0,2,1
              ROW    = NODE(BRICK,I)+K*TNN
              COLUMN = NODE(BRICK,J)+L*TNN
              element=sfgetelement(AS,ROW,COLUMN)
              REAL_S=DREAL(S(I+K*8,J+L*8))
              IMAG_S=DIMAG(S(I+K*8,J+L*8))
              CALL sfadd1complex(element,REAL_S,IMAG_S)
313         CONTINUE
312       CONTINUE
311     CONTINUE
310   CONTINUE
309   CONTINUE
C
      RETURN
      END
C
C
C
      SUBROUTINE SMAT(S,SIG)
C
      COMMON /A/ MUE,W
      COMMON /F/ DX,DY,DZ
C
C     INTEGER    I,J
C     REAL*8     SR    ! temp real varialble for S(I,J) element
C     REAL*8     SI    ! temp imaginary varialble for S(I,J) element
C
      IMPLICIT NONE
      REAL*8 DX,DY,DZ,SIG,MUE,W,SR,SI
      COMPLEX*16 S(24,24),DCMPLX
C
        SR     = DX*((+4.0D0*DZ**2)+(4.0D0*DY**2))/(MUE*36.0D0*DY*DZ)
        SI     = (W*SIG*DX*DY*DZ)/27.0D0
      S(1,1)   = DCMPLX(SR,SI)
        SR     = DX*((+2.0D0*DZ**2)+(2.0D0*DY**2))/(MUE*36.0D0*DY*DZ)
        SI     = (W*SIG*DX*DY*DZ)/54.0D0
      S(1,2)   = DCMPLX(SR,SI)
        SR     = DX*((-4.0D0*DZ**2)+(2.0D0*DY**2))/(MUE*36.0D0*DY*DZ)
        SI     = (W*SIG*DX*DY*DZ)/54.0D0
      S(1,3)   = DCMPLX(SR,SI)
        SR     = DX*((-2.0D0*DZ**2)+(1.0D0*DY**2))/(MUE*36.0D0*DY*DZ)
        SI     = (W*SIG*DX*DY*DZ)/108.0D0
      S(1,4)   = DCMPLX(SR,SI)
        SR     = DX*((+2.0D0*DZ**2)-(4.0D0*DY**2))/(MUE*36.0D0*DY*DZ)
        SI     = (W*SIG*DX*DY*DZ)/54.0D0
      S(1,5)   = DCMPLX(SR,SI)
        SR     = DX*((+1.0D0*DZ**2)-(2.0D0*DY**2))/(MUE*36.0D0*DY*DZ)
        SI     = (W*SIG*DX*DY*DZ)/108.0D0
      S(1,6)   = DCMPLX(SR,SI)
        SR     = DX*((-2.0D0*DZ**2)-(2.0D0*DY**2))/(MUE*36.0D0*DY*DZ)
        SI     = (W*SIG*DX*DY*DZ)/108.0D0
      S(1,7)   = DCMPLX(SR,SI)
        SR     = DX*((-1.0D0*DZ**2)-(1.0D0*DY**2))/(MUE*36.0D0*DY*DZ)
```

```
  SI     = (W*SIG*DX*DY*DZ)/216.0D0
S(1,8)   = DCMPLX(SR,SI)
S(1,9)   = DCMPLX(-DZ/(MUE*12.0D0),0.0D0)
S(1,10)  = -S(1,9)
S(1,11)  = +S(1,9)
S(1,12)  = -S(1,9)
S(1,13)  = DCMPLX(-DZ/(MUE*24.0D0),0.0D0)
S(1,14)  = -S(1,13)
S(1,15)  = +S(1,13)
S(1,16)  = -S(1,13)
S(1,17)  = DCMPLX(-DY/(MUE*12.0D0),0.0D0)
S(1,18)  = -S(1,17)
S(1,19)  = DCMPLX(-DY/(MUE*24.0D0),0.0D0)
S(1,20)  = -S(1,19)
S(1,21)  = +S(1,17)
S(1,22)  = -S(1,17)
S(1,23)  = +S(1,19)
S(1,24)  = -S(1,19)
S(2,1)   = +S(1,2)
S(2,2)   = +S(1,1)
S(2,3)   = +S(1,4)
S(2,4)   = +S(1,3)
S(2,5)   = +S(1,6)
S(2,6)   = +S(1,5)
S(2,7)   = +S(1,8)
S(2,8)   = +S(1,7)
S(2,9)   = +S(1,9)
S(2,10)  = -S(1,9)
S(2,11)  = +S(1,9)
S(2,12)  = -S(1,9)
S(2,13)  = +S(1,13)
S(2,14)  = -S(1,13)
S(2,15)  = +S(1,13)
S(2,16)  = -S(1,13)
S(2,17)  = +S(1,17)
S(2,18)  = -S(1,17)
S(2,19)  = +S(1,19)
S(2,20)  = -S(1,19)
S(2,21)  = +S(1,17)
S(2,22)  = -S(1,17)
S(2,23)  = +S(1,19)
S(2,24)  = -S(1,19)
S(3,1)   = +S(1,3)
S(3,2)   = +S(1,4)
S(3,3)   = +S(1,1)
S(3,4)   = +S(1,2)
S(3,5)   = +S(1,7)
S(3,6)   = +S(1,8)
S(3,7)   = +S(1,5)
S(3,8)   = +S(1,6)
S(3,9)   = -S(1,9)
S(3,10)  = +S(1,9)
S(3,11)  = -S(1,9)
S(3,12)  = +S(1,9)
S(3,13)  = -S(1,13)
S(3,14)  = +S(1,13)
S(3,15)  = -S(1,13)
S(3,16)  = +S(1,13)
S(3,17)  = +S(1,19)
```

```
S(3,18) = -S(1,19)
S(3,19) = +S(1,17)
S(3,20) = -S(1,17)
S(3,21) = +S(1,19)
S(3,22) = -S(1,19)
S(3,23) = +S(1,17)
S(3,24) = -S(1,17)
S(4,1)  = +S(1,4)
S(4,2)  = +S(1,3)
S(4,3)  = +S(1,2)
S(4,4)  = +S(1,1)
S(4,5)  = +S(1,8)
S(4,6)  = +S(1,7)
S(4,7)  = +S(1,6)
S(4,8)  = +S(1,5)
S(4,9)  = -S(1,9)
S(4,10) = +S(1,9)
S(4,11) = -S(1,9)
S(4,12) = +S(1,9)
S(4,13) = -S(1,13)
S(4,14) = +S(1,13)
S(4,15) = -S(1,13)
S(4,16) = +S(1,13)
S(4,17) = +S(1,19)
S(4,18) = -S(1,19)
S(4,19) = +S(1,17)
S(4,20) = -S(1,17)
S(4,21) = +S(1,19)
S(4,22) = -S(1,19)
S(4,23) = +S(1,17)
S(4,24) = -S(1,17)
S(5,1)  = +S(1,5)
S(5,2)  = +S(1,6)
S(5,3)  = +S(1,7)
S(5,4)  = +S(1,8)
S(5,5)  = +S(1,1)
S(5,6)  = +S(1,2)
S(5,7)  = +S(1,3)
S(5,8)  = +S(1,4)
S(5,9)  = +S(1,13)
S(5,10) = -S(1,13)
S(5,11) = +S(1,13)
S(5,12) = -S(1,13)
S(5,13) = +S(1,9)
S(5,14) = -S(1,9)
S(5,15) = +S(1,9)
S(5,16) = -S(1,9)
S(5,17) = -S(1,17)
S(5,18) = +S(1,17)
S(5,19) = -S(1,19)
S(5,20) = +S(1,19)
S(5,21) = -S(1,17)
S(5,22) = +S(1,17)
S(5,23) = -S(1,19)
S(5,24) = +S(1,19)
S(6,1)  = +S(1,6)
S(6,2)  = +S(1,5)
S(6,3)  = +S(1,8)
S(6,4)  = +S(1,7)
```

```
S(6,5)   = +S(1,2)
S(6,6)   = +S(1,1)
S(6,7)   = +S(1,4)
S(6,8)   = +S(1,3)
S(6,9)   = +S(1,13)
S(6,10)  = -S(1,13)
S(6,11)  = +S(1,13)
S(6,12)  = -S(1,13)
S(6,13)  = +S(1,9)
S(6,14)  = -S(1,9)
S(6,15)  = +S(1,9)
S(6,16)  = -S(1,9)
S(6,17)  = -S(1,17)
S(6,18)  = +S(1,17)
S(6,19)  = -S(1,19)
S(6,20)  = +S(1,19)
S(6,21)  = -S(1,17)
S(6,22)  = +S(1,17)
S(6,23)  = -S(1,19)
S(6,24)  = +S(1,19)
S(7,1)   = +S(1,7)
S(7,2)   = +S(1,8)
S(7,3)   = +S(1,5)
S(7,4)   = +S(1,6)
S(7,5)   = +S(1,3)
S(7,6)   = +S(1,4)
S(7,7)   = +S(1,1)
S(7,8)   = +S(1,2)
S(7,9)   = -S(1,13)
S(7,10)  = +S(1,13)
S(7,11)  = -S(1,13)
S(7,12)  = +S(1,13)
S(7,13)  = -S(1,9)
S(7,14)  = +S(1,9)
S(7,15)  = -S(1,9)
S(7,16)  = +S(1,9)
S(7,17)  = -S(1,19)
S(7,18)  = +S(1,19)
S(7,19)  = -S(1,17)
S(7,20)  = +S(1,17)
S(7,21)  = -S(1,19)
S(7,22)  = +S(1,19)
S(7,23)  = -S(1,17)
S(7,24)  = +S(1,17)
S(8,1)   = +S(1,8)
S(8,2)   = +S(1,7)
S(8,3)   = +S(1,6)
S(8,4)   = +S(1,5)
S(8,5)   = +S(1,4)
S(8,6)   = +S(1,3)
S(8,7)   = +S(1,2)
S(8,8)   = +S(1,1)
S(8,9)   = -S(1,13)
S(8,10)  = +S(1,13)
S(8,11)  = -S(1,13)
S(8,12)  = +S(1,13)
S(8,13)  = -S(1,9)
S(8,14)  = +S(1,9)
S(8,15)  = -S(1,9)
```

```
S(8,16) = +S(1,9)
S(8,17) = -S(1,19)
S(8,18) = +S(1,19)
S(8,19) = -S(1,17)
S(8,20) = +S(1,17)
S(8,21) = -S(1,19)
S(8,22) = +S(1,19)
S(8,23) = -S(1,17)
S(8,24) = +S(1,17)
S(9,1)  = +S(1,9)
S(9,2)  = +S(1,9)
S(9,3)  = -S(1,9)
S(9,4)  = -S(1,9)
S(9,5)  = +S(1,13)
S(9,6)  = +S(1,13)
S(9,7)  = -S(1,13)
S(9,8)  = -S(1,13)
  SR    = DY*((+4.0D0*DZ**2)+(4.0D0*DX**2))/(36.0D0*MUE*DX*DZ)
  SI    = (W*SIG*DX*DY*DZ)/27.0D0
S(9,9)  = DCMPLX(SR,SI)
  SR    = DY*((-4.0D0*DZ**2)+(2.0D0*DX**2))/(36.0D0*MUE*DX*DZ)
  SI    = (W*SIG*DX*DY*DZ)/54.0D0
S(9,10) = DCMPLX(SR,SI)
  SR    = DY*((+2.0D0*DZ**2)+(2.0D0*DX**2))/(36.0D0*MUE*DX*DZ)
  SI    = (W*SIG*DX*DY*DZ)/54.0D0
S(9,11) = DCMPLX(SR,SI)
  SR    = DY*((-2.0D0*DZ**2)+(1.0D0*DX**2))/(36.0D0*MUE*DX*DZ)
  SI    = (W*SIG*DX*DY*DZ)/108.0D0
S(9,12) = DCMPLX(SR,SI)
  SR    = DY*((+2.0D0*DZ**2)-(4.0D0*DX**2))/(36.0D0*MUE*DX*DZ)
  SI    = (W*SIG*DX*DY*DZ)/54.0D0
S(9,13) = DCMPLX(SR,SI)
  SR    = DY*((-2.0D0*DZ**2)-(2.0D0*DX**2))/(36.0D0*MUE*DX*DZ)
  SI    = (W*SIG*DX*DY*DZ)/108.0D0
S(9,14) = DCMPLX(SR,SI)
  SR    = DY*((+1.0D0*DZ**2)-(2.0D0*DX**2))/(36.0D0*MUE*DX*DZ)
  SI    = (W*SIG*DX*DY*DZ)/108.0D0
S(9,15) = DCMPLX(SR,SI)
  SR    = DY*((-1.0D0*DZ**2)-(1.0D0*DX**2))/(36.0D0*MUE*DX*DZ)
  SI    = (W*SIG*DX*DY*DZ)/216.0D0
S(9,16) = DCMPLX(SR,SI)
S(9,17) = DCMPLX(-DX/(MUE*12.0D0),0.0D0)
S(9,18) = DCMPLX(-DX/(MUE*24.0D0),0.0D0)
S(9,19) = -S(9,17)
S(9,20) = -S(9,18)
S(9,21) = +S(9,17)
S(9,22) = +S(9,18)
S(9,23) = -S(9,17)
S(9,24) = -S(9,18)
S(10,1) = -S(1,9)
S(10,2) = -S(1,9)
S(10,3) = +S(1,9)
S(10,4) = +S(1,9)
S(10,5) = -S(1,13)
S(10,6) = -S(1,13)
S(10,7) = +S(1,13)
S(10,8) = +S(1,13)
S(10,9) = +S(9,10)
S(10,10)= +S(9,9)
```

```
S(10,11)= +S(9,12)
S(10,12)= +S(9,11)
S(10,13)= +S(9,14)
S(10,14)= +S(9,13)
S(10,15)= +S(9,16)
S(10,16)= +S(9,15)
S(10,17)= +S(9,18)
S(10,18)= +S(9,17)
S(10,19)= -S(9,18)
S(10,20)= -S(9,17)
S(10,21)= +S(9,18)
S(10,22)= +S(9,17)
S(10,23)= -S(9,18)
S(10,24)= -S(9,17)
S(11,1)  = +S(1,9)
S(11,2)  = +S(1,9)
S(11,3)  = -S(1,9)
S(11,4)  = -S(1,9)
S(11,5)  = +S(1,13)
S(11,6)  = +S(1,13)
S(11,7)  = -S(1,13)
S(11,8)  = -S(1,13)
S(11,9)  = +S(9,11)
S(11,10)= +S(9,12)
S(11,11)= +S(9,9)
S(11,12)= +S(9,10)
S(11,13)= +S(9,15)
S(11,14)= +S(9,16)
S(11,15)= +S(9,13)
S(11,16)= +S(9,14)
S(11,17)= +S(9,17)
S(11,18)= +S(9,18)
S(11,19)= -S(9,17)
S(11,20)= -S(9,18)
S(11,21)= +S(9,17)
S(11,22)= +S(9,18)
S(11,23)= -S(9,17)
S(11,24)= -S(9,18)
S(12,1)  = -S(1,9)
S(12,2)  = -S(1,9)
S(12,3)  = +S(1,9)
S(12,4)  = +S(1,9)
S(12,5)  = -S(1,13)
S(12,6)  = -S(1,13)
S(12,7)  = +S(1,13)
S(12,8)  = +S(1,13)
S(12,9)  = +S(9,12)
S(12,10)= +S(9,11)
S(12,11)= +S(9,10)
S(12,12)= +S(9,9)
S(12,13)= +S(9,16)
S(12,14)= +S(9,15)
S(12,15)= +S(9,14)
S(12,16)= +S(9,13)
```

```
S(12,17)= +S(9,18)
S(12,18)= +S(9,17)
S(12,19)= -S(9,18)
S(12,20)= -S(9,17)
S(12,21)= +S(9,18)
S(12,22)= +S(9,17)
S(12,23)= -S(9,18)
S(12,24)= -S(9,17)
S(13,1)  = +S(1,13)
S(13,2)  = +S(1,13)
S(13,3)  = -S(1,13)
S(13,4)  = -S(1,13)
S(13,5)  = +S(1,9)
S(13,6)  = +S(1,9)
S(13,7)  = -S(1,9)
S(13,8)  = -S(1,9)
S(13,9)  = +S(9,13)
S(13,10)= +S(9,14)
S(13,11)= +S(9,15)
S(13,12)= +S(9,16)
S(13,13)= +S(9,9)
S(13,14)= +S(9,10)
S(13,15)= +S(9,11)
S(13,16)= +S(9,12)
S(13,17)= -S(9,17)
S(13,18)= -S(9,18)
S(13,19)= +S(9,17)
S(13,20)= +S(9,18)
S(13,21)= -S(9,17)
S(13,22)= -S(9,18)
S(13,23)= +S(9,17)
S(13,24)= +S(9,18)
S(14,1)  = -S(1,13)
S(14,2)  = -S(1,13)
S(14,3)  = +S(1,13)
S(14,4)  = +S(1,13)
S(14,5)  = -S(1,9)
S(14,6)  = -S(1,9)
S(14,7)  = +S(1,9)
S(14,8)  = +S(1,9)
S(14,9)  = +S(9,14)
S(14,10)= +S(9,13)
S(14,11)= +S(9,16)
S(14,12)= +S(9,15)
S(14,13)= +S(9,10)
S(14,14)= +S(9,9)
S(14,15)= +S(9,12)
S(14,16)= +S(9,11)
S(14,17)= -S(9,18)
S(14,18)= -S(9,17)
S(14,19)= +S(9,18)
S(14,20)= +S(9,17)
S(14,21)= -S(9,18)
S(14,22)= -S(9,17)
S(14,23)= +S(9,18)
S(14,24)= +S(9,17)
S(15,1)  = +S(1,13)
```

```
S(15,2)  = +S(1,13)
S(15,3)  = -S(1,13)
S(15,4)  = -S(1,13)
S(15,5)  = +S(1,9)
S(15,6)  = +S(1,9)
S(15,7)  = -S(1,9)
S(15,8)  = -S(1,9)
S(15,9)  = +S(9,15)
S(15,10) = +S(9,16)
S(15,11) = +S(9,13)
S(15,12) = +S(9,14)
S(15,13) = +S(9,11)
S(15,14) = +S(9,12)
S(15,15) = +S(9,9)
S(15,16) = +S(9,10)
S(15,17) = -S(9,17)
S(15,18) = -S(9,18)
S(15,19) = +S(9,17)
S(15,20) = +S(9,18)
S(15,21) = -S(9,17)
S(15,22) = -S(9,18)
S(15,23) = +S(9,17)
S(15,24) = +S(9,18)
S(16,1)  = -S(1,13)
S(16,2)  = -S(1,13)
S(16,3)  = +S(1,13)
S(16,4)  = +S(1,13)
S(16,5)  = -S(1,9)
S(16,6)  = -S(1,9)
S(16,7)  = +S(1,9)
S(16,8)  = +S(1,9)
S(16,9)  = +S(9,16)
S(16,10) = +S(9,15)
S(16,11) = +S(9,14)
S(16,12) = +S(9,13)
S(16,13) = +S(9,12)
S(16,14) = +S(9,11)
S(16,15) = +S(9,10)
S(16,16) = +S(9,9)
S(16,17) = -S(9,18)
S(16,18) = -S(9,17)
S(16,19) = +S(9,18)
S(16,20) = +S(9,17)
S(16,21) = -S(9,18)
S(16,22) = -S(9,17)
S(16,23) = +S(9,18)
S(16,24) = +S(9,17)
S(17,1)  = +S(1,17)
S(17,2)  = +S(1,17)
S(17,3)  = +S(1,19)
S(17,4)  = +S(1,19)
S(17,5)  = -S(1,17)
S(17,6)  = -S(1,17)
S(17,7)  = -S(1,19)
S(17,8)  = -S(1,19)
S(17,9)  = +S(9,17)
S(17,10) = +S(9,18)
S(17,11) = +S(9,17)
S(17,12) = +S(9,18)
```

```
S(17,13)= -S(9,17)
S(17,14)= -S(9,18)
S(17,15)= -S(9,17)
S(17,16)= -S(9,18)
  SR    = DZ*((+4.0D0*DY**2)+(4.0D0*DX**2))/(36.0D0*MUE*DX*DY)
  SI    = (W*SIG*DX*DY*DZ)/27.0D0
S(17,17)= DCMPLX(SR,SI)
  SR    = DZ*((-4.0D0*DY**2)+(2.0D0*DX**2))/(36.0D0*MUE*DX*DY)
  SI    = (W*SIG*DX*DY*DZ)/54.0D0
S(17,18)= DCMPLX(SR,SI)
  SR    = DZ*((+2.0D0*DY**2)-(4.0D0*DX**2))/(36.0D0*MUE*DX*DY)
  SI    = (W*SIG*DX*DY*DZ)/54.0D0
S(17,19)= DCMPLX(SR,SI)
  SR    = DZ*((-2.0D0*DY**2)-(2.0D0*DX**2))/(36.0D0*MUE*DX*DY)
  SI    = (W*SIG*DX*DY*DZ)/108.0D0
S(17,20)= DCMPLX(SR,SI)
  SR    = DZ*((+2.0D0*DY**2)+(2.0D0*DX**2))/(36.0D0*MUE*DX*DY)
  SI    = (W*SIG*DX*DY*DZ)/54.0D0
S(17,21)= DCMPLX(SR,SI)
  SR    = DZ*((-2.0D0*DY**2)+(1.0D0*DX**2))/(36.0D0*MUE*DX*DY)
  SI    = (W*SIG*DX*DY*DZ)/108.0D0
S(17,22)= DCMPLX(SR,SI)
  SR    = DZ*((+1.0D0*DY**2)-(2.0D0*DX**2))/(36.0D0*MUE*DX*DY)
  SI    = (W*SIG*DX*DY*DZ)/108.0D0
S(17,23)= DCMPLX(SR,SI)
  SR    = DZ*((-1.0D0*DY**2)-(1.0D0*DX**2))/(36.0D0*MUE*DX*DY)
  SI    = (W*SIG*DX*DY*DZ)/216.0D0
S(17,24)= DCMPLX(SR,SI)
S(18,1) = -S(1,17)
S(18,2) = -S(1,17)
S(18,3) = -S(1,19)
S(18,4) = -S(1,19)
S(18,5) = +S(1,17)
S(18,6) = +S(1,17)
S(18,7) = +S(1,19)
S(18,8) = +S(1,19)
S(18,9) = +S(9,18)
S(18,10)= +S(9,17)
S(18,11)= +S(9,18)
S(18,12)= +S(9,17)
S(18,13)= -S(9,18)
S(18,14)= -S(9,17)
S(18,15)= -S(9,18)
S(18,16)= -S(9,17)
S(18,17)= +S(17,18)
S(18,18)= +S(17,17)
S(18,19)= +S(17,20)
S(18,20)= +S(17,19)
S(18,21)= +S(17,22)
S(18,22)= +S(17,21)
S(18,23)= +S(17,24)
S(18,24)= +S(17,23)
S(19,1) = +S(1,19)
S(19,2) = +S(1,19)
S(19,3) = +S(1,17)
S(19,4) = +S(1,17)
S(19,5) = -S(1,19)
S(19,6) = -S(1,19)
S(19,7) = -S(1,17)
```

```
S(19,8)  = -S(1,17)
S(19,9)  = -S(9,17)
S(19,10)= -S(9,18)
S(19,11)= -S(9,17)
S(19,12)= -S(9,18)
S(19,13)= +S(9,17)
S(19,14)= +S(9,18)
S(19,15)= +S(9,17)
S(19,16)= +S(9,18)
S(19,17)= +S(17,19)
S(19,18)= +S(17,20)
S(19,19)= +S(17,17)
S(19,20)= +S(17,18)
S(19,21)= +S(17,23)
S(19,22)= +S(17,24)
S(19,23)= +S(17,21)
S(19,24)= +S(17,22)
S(20,1)  = -S(1,19)
S(20,2)  = -S(1,19)
S(20,3)  = -S(1,17)
S(20,4)  = -S(1,17)
S(20,5)  = +S(1,19)
S(20,6)  = +S(1,19)
S(20,7)  = +S(1,17)
S(20,8)  = +S(1,17)
S(20,9)  = -S(9,18)
S(20,10)= -S(9,17)
S(20,11)= -S(9,18)
S(20,12)= -S(9,17)
S(20,13)= +S(9,18)
S(20,14)= +S(9,17)
S(20,15)= +S(9,18)
S(20,16)= +S(9,17)
S(20,17)= +S(17,20)
S(20,18)= +S(17,19)
S(20,19)= +S(17,18)
S(20,20)= +S(17,17)
S(20,21)= +S(17,24)
S(20,22)= +S(17,23)
S(20,23)= +S(17,22)
S(20,24)= +S(17,21)
S(21,1)  = +S(1,17)
S(21,2)  = +S(1,17)
S(21,3)  = +S(1,19)
S(21,4)  = +S(1,19)
S(21,5)  = -S(1,17)
S(21,6)  = -S(1,17)
S(21,7)  = -S(1,19)
S(21,8)  = -S(1,19)
S(21,9)  = +S(9,17)
S(21,10)= +S(9,18)
S(21,11)= +S(9,17)
S(21,12)= +S(9,18)
S(21,13)= -S(9,17)
S(21,14)= -S(9,18)
S(21,15)= -S(9,17)
S(21,16)= -S(9,18)
S(21,17)= +S(17,21)
S(21,18)= +S(17,22)
```

```
S(21,19)= +S(17,23)
S(21,20)= +S(17,24)
S(21,21)= +S(17,17)
S(21,22)= +S(17,18)
S(21,23)= +S(17,19)
S(21,24)= +S(17,20)
S(22,1)  = -S(1,17)
S(22,2)  = -S(1,17)
S(22,3)  = -S(1,19)
S(22,4)  = -S(1,19)
S(22,5)  = +S(1,17)
S(22,6)  = +S(1,17)
S(22,7)  = +S(1,19)
S(22,8)  = +S(1,19)
S(22,9)  = +S(9,18)
S(22,10)= +S(9,17)
S(22,11)= +S(9,18)
S(22,12)= +S(9,17)
S(22,13)= -S(9,18)
S(22,14)= -S(9,17)
S(22,15)= -S(9,18)
S(22,16)= -S(9,17)
S(22,17)= +S(17,22)
S(22,18)= +S(17,21)
S(22,19)= +S(17,24)
S(22,20)= +S(17,23)
S(22,21)= +S(17,18)
S(22,22)= +S(17,17)
S(22,23)= +S(17,20)
S(22,24)= +S(17,19)
S(23,1)  = +S(1,19)
S(23,2)  = +S(1,19)
S(23,3)  = +S(1,17)
S(23,4)  = +S(1,17)
S(23,5)  = -S(1,19)
S(23,6)  = -S(1,19)
S(23,7)  = -S(1,17)
S(23,8)  = -S(1,17)
S(23,9)  = -S(9,17)
S(23,10)= -S(9,18)
S(23,11)= -S(9,17)
S(23,12)= -S(9,18)
S(23,13)= +S(9,17)
S(23,14)= +S(9,18)
S(23,15)= +S(9,17)
S(23,16)= +S(9,18)
S(23,17)= +S(17,23)
S(23,18)= +S(17,24)
S(23,19)= +S(17,21)
S(23,20)= +S(17,22)
S(23,21)= +S(17,19)
S(23,22)= +S(17,20)
S(23,23)= +S(17,17)
S(23,24)= +S(17,18)
S(24,1)  = -S(1,19)
S(24,2)  = -S(1,19)
S(24,3)  = -S(1,17)
S(24,4)  = -S(1,17)
S(24,5)  = +S(1,19)
```

```
      S(24,6)  = +S(1,19)
      S(24,7)  = +S(1,17)
      S(24,8)  = +S(1,17)
      S(24,9)  = -S(9,18)
      S(24,10)= -S(9,17)
      S(24,11)= -S(9,18)
      S(24,12)= -S(9,17)
      S(24,13)= +S(9,18)
      S(24,14)= +S(9,17)
      S(24,15)= +S(9,18)
      S(24,16)= +S(9,17)
      S(24,17)= +S(17,24)
      S(24,18)= +S(17,23)
      S(24,19)= +S(17,22)
      S(24,20)= +S(17,21)
      S(24,21)= +S(17,20)
      S(24,22)= +S(17,19)
      S(24,23)= +S(17,18)
      S(24,24)= +S(17,17)
      RETURN
      END
C
C
C
      SUBROUTINE GLOBAF(XN,YN,ZN,NODE,FACE,AF)
C     assemble global AF matrix (NDIMx1)
C
      COMMON /B/ TNN
      COMMON /C/ TNB
      COMMON /D/ NDIM
      COMMON /E/ HX,HY,HZ
c
C     INTEGER    BRICK        ! brick number
C     INTEGER    I,K          ! do loop variables
C     INTEGER    FAC(6)       ! temp.FACE(6,b) variable for brick b
C     REAL*8     DX           ! Xj-Xi
C     REAL*8     DY           ! Yj-Yi
C     REAL*8     DZ           ! Zj-Zi
C     COMPLEX*16 F(24)        ! F matrix for each brick
C
      IMPLICIT NONE
      INTEGER   TNN,TNB,NDIM,NODE(TNB,8),BRICK,I,K,FACE(6,TNB),FAC(6)
      REAL*8    XN(TNN),YN(TNN),ZN(TNN),DX,DY,DZ
      COMPLEX*16 F(24),AF(NDIM),HX,HY,HZ
      COMMON /F/ DX,DY,DZ
C
C     equal all AF matrix elements to zero
      CALL ZERMAT(AF)
c
      DO 314 BRICK=1,TNB
        DX=XN(NODE(BRICK,8))-XN(NODE(BRICK,1))
        DY=YN(NODE(BRICK,8))-YN(NODE(BRICK,1))
        DZ=ZN(NODE(BRICK,8))-ZN(NODE(BRICK,1))
        FAC(1)=FACE(1,BRICK)
        FAC(2)=FACE(2,BRICK)
        FAC(3)=FACE(3,BRICK)
        FAC(4)=FACE(4,BRICK)
        FAC(5)=FACE(5,BRICK)
        FAC(6)=FACE(6,BRICK)
```

```
      CALL FMAT(F,FAC)
      DO 315 K=0,2,1
        DO 316 I=1,8,1
          AF(NODE(BRICK,I)+K*TNN)=AF(NODE(BRICK,I)+K*TNN)+F(I+K*8)
316     CONTINUE
315   CONTINUE
314   CONTINUE
      RETURN
      END
C
C
C
      SUBROUTINE ZERMAT(AF)
C     make AF matrices all zeroes
C
      COMMON /D/ NDIM
C
C     INTEGER    I            !DO LOOP variables
C
      IMPLICIT NONE
      INTEGER    I,NDIM
      COMPLEX*16 AF(NDIM)
      COMPLEX*16 DCMPLX
C
      DO 307 I=1,NDIM,1
        AF(I)=DCMPLX(0.0D0,0.0D0)
307   CONTINUE
      RETURN
      END
C
C
C
      SUBROUTINE FMAT(F,FAC)
C     create the F matrix for each element
C
      COMMON /E/ HX,HY,HZ
      COMMON /F/ DX,DY,DZ
C
C     INTEGER    I        ! counter
C
      IMPLICIT NONE
      INTEGER    I,FAC(6)
      REAL*8     DX,DY,DZ
      COMPLEX*16 F(24),HX,HY,HZ
C
      DO 317 I = 1,24,1
        F(I)=0.0D0
317   CONTINUE
      F(1)  = (-FAC(1)*DX*DY*HY + FAC(2)*DX*DZ*HZ)/4.0D0
      F(2)  = F(1)
      F(3)  = (-FAC(1)*DX*DY*HY - FAC(5)*DX*DZ*HZ)/4.0D0
      F(4)  = F(3)
      F(5)  = ( FAC(6)*DX*DY*HY + FAC(2)*DX*DZ*HZ)/4.0D0
      F(6)  = F(5)
      F(7)  = (-FAC(5)*DX*DZ*HZ + FAC(6)*DX*DY*HY)/4.0D0
      F(8)  = F(7)
      F(9)  = ( FAC(1)*DX*DY*HX - FAC(3)*DY*DZ*HZ)/4.0D0
      F(10)= ( FAC(1)*DX*DY*HX + FAC(4)*DY*DZ*HZ)/4.0D0
      F(11)= F(9)
```

```
      F(12)= F(10)
      F(13)= (-FAC(3)*DY*DZ*HZ - FAC(6)*DX*DY*HX)/4.0D0
      F(14)= ( FAC(4)*DY*DZ*HZ - FAC(6)*DX*DY*HX)/4.0D0
      F(15)= F(13)
      F(16)= F(14)
      F(17)= (-FAC(2)*DX*DZ*HX + FAC(3)*DY*DZ*HY)/4.0D0
      F(18)= (-FAC(2)*DX*DZ*HX - FAC(4)*DY*DZ*HY)/4.0D0
      F(19)= ( FAC(3)*DY*DZ*HY + FAC(5)*DX*DZ*HX)/4.0D0
      F(20)= (-FAC(4)*DY*DZ*HY + FAC(5)*DX*DZ*HX)/4.0D0
      F(21)= F(17)
      F(22)= F(18)
      F(23)= F(19)
      F(24)= F(20)
      RETURN
      END
C
C
C
      SUBROUTINE SOLVAS(AS,AF,SASAF)
C
      COMMON /D/ NDIM
C
      IMPLICIT NONE
      INTEGER NDIM,AS,error,sffactor
      INTEGER sffilestats
      COMPLEX*16 AF(ndim),SASAF(ndim)
      REAL*8  lbefore,lafter,round,growth,sfroundoff,sflargestelement
      REAL*8  norm,cond,sfnorm,sfcondition
C
C     compute L-infinity norm of AS
      norm=sfnorm(AS)
C
C     find largest element prior to factorization
      lbefore=sflargestelement(AS)
C
C     order and factor AS
C
      error=sffactor(AS)
C
C     find lower bound on largest element which occured in any sub-matrices
C     during factorization
C
      lafter=sflargestelement(AS)
C
C     compute growth
      growth=lafter/lbefore
C
C     determine roundoff
      round=sfroundoff(AS,lafter)
C
C     write stats to file
      error=sffilestats(AS,"stats","stats")
C
C     compute L-infinity condition number
      cond=sfcondition(AS,norm,error)
C
C     solve the system
      call sfsolve(AS,AF,SASAF)
C
```

```
      write(1,*) ' norm=',norm,' cond=',cond
      write(1,*) ' lbefore=',lbefore,' lafter=',lafter
      write(1,*) ' growth=',growth,' roundoff=',round
      write(1,*) ' estimated error=',round/cond
      write(1,*) ' '
      return
      end
C
C
C
      SUBROUTINE CURDEN(NDIM,SIGMA,SAF,NODE,XN,YN,ZN,JX,JY,JZ)
C
      COMMON /A/ MUE,W
      COMMON /B/ TNN
      COMMON /C/ TNB
C
      IMPLICIT NONE
      INTEGER     I
      INTEGER     NDIM
      INTEGER     TNB
      INTEGER     TNN
      INTEGER     NODE(TNB,8)
      REAL*8      XN(TNN),YN(TNN),ZN(TNN)
      REAL*8      MUE,W
      REAL*8      SIGMA(TNB)
      REAL*8      CX,CY,CZ,MJX,MJY,MJZ,AJX,AJY,AJZ
      COMPLEX*16 SAF(NDIM)
      COMPLEX*16 j,DCMPLX
      COMPLEX*16 JX2,JY2,JZ2
      COMPLEX*16 JX(TNB),JY(TNB),JZ(TNB)
C
      j=DCMPLX(0.0D0,1.0D0)
      DO 318 I=1,TNB,1
        JX2=(SAF(NODE(I,1))+SAF(NODE(I,2))+SAF(NODE(I,3))
     ^       +SAF(NODE(I,4))+SAF(NODE(I,5))+SAF(NODE(I,6))
     ^       +SAF(NODE(I,7))+SAF(NODE(I,8)))
     ^       *j*W/8.0D0
        JX(I)=SIGMA(I)*(-JX2)
        JY2=(SAF(NODE(I,1)+TNN)+SAF(NODE(I,2)+TNN)+SAF(NODE(I,3)+TNN)
     ^       +SAF(NODE(I,4)+TNN)+SAF(NODE(I,5)+TNN)+SAF(NODE(I,6)+TNN)
     ^       +SAF(NODE(I,7)+TNN)+SAF(NODE(I,8)+TNN))
     ^       *j*W/8.0D0
        JY(I)=SIGMA(I)*(-JY2)
        JZ2=(SAF(NODE(I,1)+2.D0*TNN)+SAF(NODE(I,2)+2.D0*TNN)
     ^       +SAF(NODE(I,3)+2.D0*TNN)+SAF(NODE(I,4)+2.D0*TNN)
     ^       +SAF(NODE(I,5)+2.D0*TNN)+SAF(NODE(I,6)+2.D0*TNN)
     ^       +SAF(NODE(I,7)+2.D0*TNN)+SAF(NODE(I,8)+2.D0*TNN))
     ^       *j*W/8.0D0
        JZ(I)=SIGMA(I)*(-JZ2)
318   CONTINUE
C
100   FORMAT(1X,'CURRENT DENSITIES FOR EACH ELEMENT')
101   FORMAT(1X,' ')
102   FORMAT(1X,'BLOCK',7X,'X',12X,'Y',12X,'Z',9X,
     ^'J: Current Density (A/m**2)')
103   FORMAT(56X,'( MAG J , ANG J )')
104   FORMAT(1X,I5,2X,D11.4,2X,D11.4,2X,D11.4,4X,
```

```
     ^'Jx=(',D11.4,',',D11.4,')')
105   FORMAT(49X,'Jy=(',D11.4,',',D11.4,')')
106   FORMAT(49X,'Jz=(',D11.4,',',D11.4,')')
C
      WRITE(1,100)
      WRITE(1,101)
      WRITE(1,102)
      WRITE(1,103)
      WRITE(1,101)
      DO 319 I=1,TNB
        CX=(XN(NODE(I,8))+XN(NODE(I,1)))/2.0D0
        CY=(YN(NODE(I,8))+YN(NODE(I,1)))/2.0D0
        CZ=(ZN(NODE(I,8))+ZN(NODE(I,1)))/2.0D0
        MJX=CDABS(JX(I))
        IF (JX(I).EQ.DCMPLX(0.0D0,0.0D0)) THEN
          AJX=0.0D0
         ELSE
          AJX=DATAN2D(DIMAG(JX(I)),DREAL(JX(I)))
        ENDIF
        MJY=CDABS(JY(I))
        IF (JY(I).EQ.DCMPLX(0.0D0,0.0D0)) THEN
          AJY=0.0D0
         ELSE
          AJY=DATAN2D(DIMAG(JY(I)),DREAL(JY(I)))
        ENDIF
        MJZ=CDABS(JZ(I))
        IF (JZ(I).EQ.DCMPLX(0.0D0,0.0D0)) THEN
          AJZ=0.0D0
         ELSE
          AJZ=DATAN2D(DIMAG(JZ(I)),DREAL(JZ(I)))
        ENDIF
      WRITE(1,104) I,CX,CY,CZ,MJX,AJX
      WRITE(1,105) MJY,AJY
      WRITE(1,106) MJZ,AJZ
      WRITE(1,101)
319   CONTINUE
      RETURN
      END
```

Appendix 4 FEM3D.F Program Demonstration in the DEC ULTRIX Environment

This section demonstrates how to use the **fem3d.f** program. It shows how an object being studied is to be modelled and how the input data file to the **fem3d.f** program must be written. It then shows how the user must modify the PARAMETER lines in the program before compilation and linkage. A sample output file is then presented.

fem3d.f is written in FORTRAN and the subroutine used to solve the sparse linear system of equations resulting from the formulation of the solution is written in C. The ULTRIX environment on the DEC Micro-VAX work stations can compile programs written in both languages, link them together, and create a single executable file. This appendix shows the steps necessary to run the program correctly with the help of an example. The flowchart in figure A4.c presents these steps.

The object modeled is a cube of dimensions 0.2m × 0.2m × 0.2m and the magnetic field chosen is $\vec{H} = H_0\hat{z}$ where $H_o = 1$. Figure A4.a sets up the problem. The conductivity of the cube is chosen to be 0.1 S/m. This cube is modelled by 8 cube shaped elements of equal dimensions. In this figure, the circled numbers identify the elements and the squared ones identify the global node numbering scheme.

The global node numbering scheme for both elements and the nodes are the same. The first node is identified as node 1 and is located by first identifying the nodes located at the lowest value z coordinates, then identifying which of these nodes are located at the lowest y coordinate, and finally, of these nodes, which one is located at the lowest value x coordinate. The next node, and subsequent nodes, is found in the same manner but excluding the node that has been numbered. The local node numbering scheme is found the same way as for the global scheme but is done for each element. Figure A4.b shows the global and local node numbering scheme for element 6 in figure A4.a.

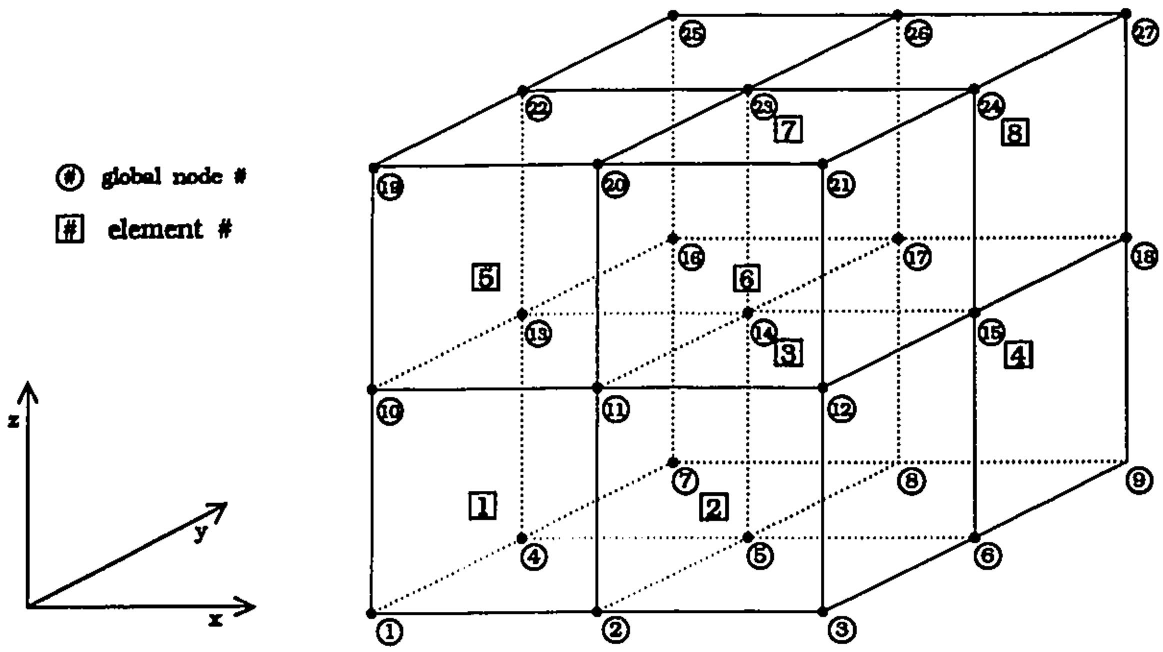


Figure A4.a: Example object for the FEM3D.F program demonstration

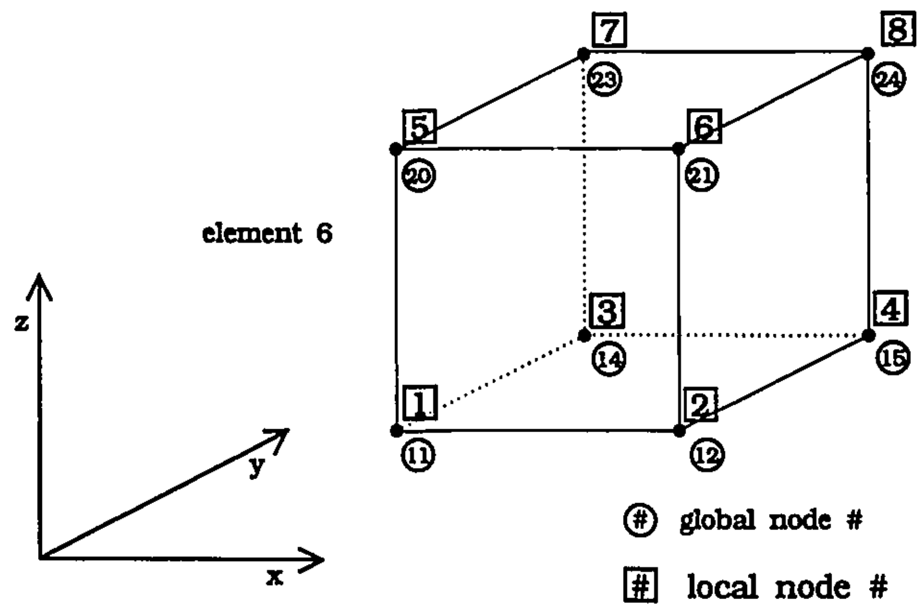


Figure A4.b: Example element for the FEM3D.F program demonstration

As indicated by the user flowchart, the input file must be created. It is named **fem3d.in**. The first four lines may be used by the user for identification purposes. It is suggested to include the name of the original file name, the total number of elements and nodes modelling the object. The input file formats is described in chapter 5 and is written below:

```
The input file is: INPUT1.IN
Total number of blocks is:  8
Total number of nodes is : 27

  1   -1.0D-01    -1.0D-01    -1.0D-01
  2    0.0D+00    -1.0D-01    -1.0D-01
  3   +1.0D-01    -1.0D-01    -1.0D-01
  4   -1.0D-01     0.0D+00    -1.0D-01
  5    0.0D+00     0.0D+00    -1.0D-01
  6   +1.0D-01     0.0D+00    -1.0D-01
  7   -1.0D-01    +1.0D-01    -1.0D-01
  8    0.0D+00    +1.0D-01    -1.0D-01
  9   +1.0D-01    +1.0D-01    -1.0D-01
 10   -1.0D-01    -1.0D-01     0.0D+00
 11    0.0D+00    -1.0D-01     0.0D+00
 12   +1.0D-01    -1.0D-01     0.0D+00
 13   -1.0D-01     0.0D+00     0.0D+00
 14    0.0D+00     0.0D+00     0.0D+00
 15   +1.0D-01     0.0D+00     0.0D+00
 16   -1.0D-01    +1.0D-01     0.0D+00
 17    0.0D+00    +1.0D-01     0.0D+00
 18   +1.0D-01    +1.0D-01     0.0D+00
 19   -1.0D-01    -1.0D-01    +1.0D-01
 20    0.0D+00    -1.0D-01    +1.0D-01
 21   +1.0D-01    -1.0D-01    +1.0D-01
 22   -1.0D-01     0.0D+00    +1.0D-01
 23    0.0D+00     0.0D+00    +1.0D-01
 24   +1.0D-01     0.0D+00    +1.0D-01
 25   -1.0D-01    +1.0D-01    +1.0D-01
 26    0.0D+00    +1.0D-01    +1.0D-01
 27   +1.0D-01    +1.0D-01    +1.0D-01
  1     1     2     4     5    10    11    13    14    1.0D-01
  2     2     3     5     6    11    12    14    15    1.0D-01
  3     4     5     7     8    13    14    16    17    1.0D-01
  4     5     6     8     9    14    15    17    18    1.0D-01
  5    10    11    13    14    19    20    22    23    1.0D-01
  6    11    12    14    15    20    21    23    24    1.0D-01
  7    13    14    16    17    22    23    25    26    1.0D-01
  8    14    15    17    18    23    24    26    27    1.0D-01
```

The parameter lines of the **fem3d.f** program must be modified accordingly. For this problem, these lines should be written as:

```
      PARAMETER   (TNODE=27,TBRICK=8,DIM3=TNODE*3)
      PARAMETER   (HXR=0.0D0,HXI=0.0D0)
      PARAMETER   (HYR=0.0D0,HYI=0.0D0)
      PARAMETER   (HZR=1.0D0,HZI=0.0D0)
      PARAMETER   (FREQUENCY=60.0D0)
```

fem3d.f is ready to be compiled with the FORTRAN compiler via the command:

```
f77 -c fem3d.f
```

The next step is to link the compiled versions of the FORTRAN and C coded routines together and create an executable file. The command is:

```
f77 fem3d.o sparse.a -o fem3d
```

The program is in executable form and is run by simply entering the word:

```
fem3d
```

The output file **fem3d.out** is created and its contents are shown below:

```
H=  ( (  0.0000D+00 ) + j (  0.0000D+00 ) ) x
  + ( (  0.0000D+00 ) + j (  0.0000D+00 ) ) y
  + ( (  0.1000D+01 ) + j (  0.0000D+00 ) ) z

 error from sfcreate= xxxxx.xx
 norm=xxxxx.xx                            cond=xxxxx.xx
 lbefore=xxxxx.xx                         lafter=xxxxx.xx
 growth=xxxxx.xx                          roundoff=xxxxx.xx
 estimated error=xxxxx.xx

CURRENT DENSITIES FOR EACH ELEMENT

BLOCK       X            Y            Z           J: Current Density (A/m**2)
           (m)          (m)          (m)                ( MAG J , ANG J )

    1  -0.5000D-01  -0.5000D-01  -0.5000D-01     Jx=( 0.1184D-05,-0.9000D+02)
                                                 Jy=( 0.1184D-05, 0.9000D+02)
                                                 Jz=( 0.6707D-16,-0.7693D-03)

    2   0.5000D-01  -0.5000D-01  -0.5000D-01     Jx=( 0.1184D-05,-0.9000D+02)
                                                 Jy=( 0.1184D-05,-0.9000D+02)
                                                 Jz=( 0.9992D-16, 0.1800D+03)

    3  -0.5000D-01   0.5000D-01  -0.5000D-01     Jx=( 0.1184D-05, 0.9000D+02)
                                                 Jy=( 0.1184D-05, 0.9000D+02)
                                                 Jz=( 0.1708D-16, 0.1846D-02)

    4   0.5000D-01   0.5000D-01  -0.5000D-01     Jx=( 0.1184D-05, 0.9000D+02)
                                                 Jy=( 0.1184D-05,-0.9000D+02)
                                                 Jz=( 0.9495D-16,-0.4775D-03)

    5  -0.5000D-01  -0.5000D-01   0.5000D-01     Jx=( 0.1184D-05,-0.9000D+02)
                                                 Jy=( 0.1184D-05, 0.9000D+02)
                                                 Jz=( 0.6045D-16, 0.6623D-03)

    6   0.5000D-01  -0.5000D-01   0.5000D-01     Jx=( 0.1184D-05,-0.9000D+02)
                                                 Jy=( 0.1184D-05,-0.9000D+02)
                                                 Jz=( 0.1065D-15,-0.1800D+03)

    7  -0.5000D-01   0.5000D-01   0.5000D-01     Jx=( 0.1184D-05, 0.9000D+02)
                                                 Jy=( 0.1184D-05, 0.9000D+02)
                                                 Jz=( 0.1046D-16, 0.2819D-03)

    8   0.5000D-01   0.5000D-01   0.5000D-01     Jx=( 0.1184D-05, 0.9000D+02)
                                                 Jy=( 0.1184D-05,-0.9000D+02)
                                                 Jz=( 0.8834D-16, 0.5043D-03)
```

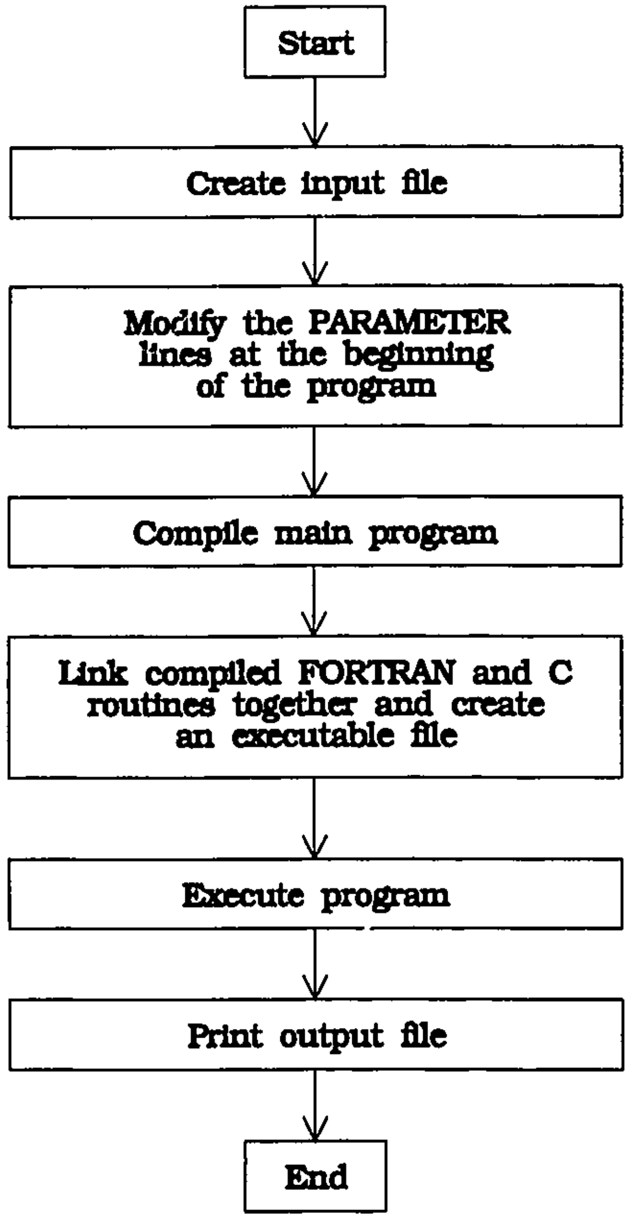


Figure A4.c: User flowchart for the FEM3D.F program demonstration

Appendix 5 Calculations of the [S] and [F] Matrix Elements

The coupled equations to solve for element k in the 3-D FEM formulation are found in Chapter 4 as equations (4.25) and (4.26). It is understood that the following equation developments are for element k and thus the k superscripts have been omitted in this appendix.

$$\frac{1}{\mu}\int_R (\nabla \times \overline{W}) \cdot (\nabla \times \hat{A}) dR + j\omega\sigma \int_R \overline{W}\hat{A}\, dR$$

$$+\sigma \int_R \overline{W} \cdot \nabla\hat{\phi} dR = \int_S (\overline{W} \times \hat{H}) \cdot \vec{n} dS \tag{4.25}$$

and

$$\int_R \nabla W \cdot \sigma(j\omega\hat{A} + \nabla\hat{\phi}) dR = 0 \tag{4.26}$$

The unknowns $\hat{A}$ and $\hat{\phi}$ are defined as:

$$\hat{\phi} = \sum_{i=1}^{8} \alpha_i \hat{\phi}_i = \lfloor\alpha\rfloor \{\hat{\phi}\} \tag{4.29}$$

$$\hat{A} = \sum_{i=1}^{8} \alpha_i \hat{A}_{xi}\hat{x} + \alpha_i \hat{A}_{yi}\hat{y} + \alpha_i \hat{A}_{zi}\hat{z}$$

$$= (\lfloor\alpha\rfloor\hat{x} \quad \lfloor\alpha\rfloor\hat{y} \quad \lfloor\alpha\rfloor\hat{z}) \begin{pmatrix} \{\hat{A}_x\} \\ \{\hat{A}_y\} \\ \{\hat{A}_z\} \end{pmatrix} \tag{4.30}$$

and the weighting functions are defined as:

$$W = \lfloor\alpha\rfloor^T \tag{4.31}$$

$$\overline{W} = \begin{pmatrix} \lfloor\alpha\rfloor^T\hat{x} \\ \lfloor\alpha\rfloor^T\hat{y} \\ \lfloor\alpha\rfloor^T\hat{z} \end{pmatrix} \tag{4.32}$$

where ^{T} denotes the transpose of the matrix.

The linear interpolative function for element k is:

$$[\alpha]=\begin{cases}[0] & \text{for } (x,y,z)\notin R^k\\ [\alpha_1,\alpha_2,\ldots,\alpha_8] & \text{for } (x,y,z)\in R^k\end{cases} \tag{4.27}$$

where

$$\alpha_i(x_j,y_j,z_j)=\begin{cases}0 \text{ for } i\neq j\\ 1 \text{ for } i=j\end{cases}$$

Therefore,

$$[\alpha]^T=\begin{bmatrix}\alpha_1\\ \cdot\\ \cdot\\ \cdot\\ \alpha_8\end{bmatrix} \tag{107}$$

where

$$\alpha_1=\frac{-(x-x_8)(y-y_8)(z-z_8)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \qquad \alpha_2=\frac{(x-x_1)(y-y_8)(z-z_8)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)}$$

$$\alpha_3=\frac{(x-x_8)(y-y_1)(z-z_8)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \qquad \alpha_4=\frac{-(x-x_1)(y-y_1)(z-z_8)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)}$$

$$\alpha_5=\frac{(x-x_8)(y-y_8)(z-z_1)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \qquad \alpha_6=\frac{-(x-x_1)(y-y_8)(z-z_1)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)}$$

$$\alpha_7=\frac{-(x-x_8)(y-y_1)(z-z_1)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)} \qquad \alpha_8=\frac{(x-x_1)(y-y_1)(z-z_1)}{(x_8-x_1)(y_8-y_1)(z_8-z_1)}$$

The terms in equations (4.25) and (4.26) are developed below.

$$\frac{1}{\mu}\int_R(\nabla\times\overline{W})\cdot(\nabla\times\hat{A})\,dR \tag{A5.1}$$

$$(\nabla\times\overline{W})=\nabla\times\begin{bmatrix}[\alpha]^T\hat{x}\\ [\alpha]^T\hat{y}\\ [\alpha]^T\hat{z}\end{bmatrix}=\begin{bmatrix}\nabla\times[\alpha]^T\hat{x}\\ \nabla\times[\alpha]^T\hat{y}\\ \nabla\times[\alpha]^T\hat{z}\end{bmatrix}=\begin{bmatrix}\hat{y}\frac{\partial[\alpha]^T}{\partial z}-\hat{z}\frac{\partial[\alpha]^T}{\partial y}\\ \hat{z}\frac{\partial[\alpha]^T}{\partial x}-\hat{x}\frac{\partial[\alpha]^T}{\partial z}\\ \hat{x}\frac{\partial[\alpha]^T}{\partial y}-\hat{y}\frac{\partial[\alpha]^T}{\partial x}\end{bmatrix} \quad (A5.2)$$

$$(\nabla\times\hat{A})=\nabla\times[[\alpha]\hat{x}\quad[\alpha]\hat{y}\quad[\alpha]\hat{z}]\cdot\begin{bmatrix}[\hat{A}_x]\\ [\hat{A}_y]\\ [\hat{A}_z]\end{bmatrix}$$

$$=[\nabla\times[\alpha]\hat{x}\quad\nabla\times[\alpha]\hat{y}\quad\nabla\times[\alpha]\hat{z}]\cdot\begin{bmatrix}[\hat{A}_x]\\ [\hat{A}_y]\\ [\hat{A}_z]\end{bmatrix}$$

$$=\left[\hat{y}\frac{\partial[\alpha]}{\partial z}-\hat{z}\frac{\partial[\alpha]}{\partial y}\quad\hat{z}\frac{\partial[\alpha]}{\partial x}-\hat{x}\frac{\partial[\alpha]}{\partial z}\quad\hat{x}\frac{\partial[\alpha]}{\partial y}-\hat{y}\frac{\partial[\alpha]}{\partial x}\right]\cdot\begin{bmatrix}[\hat{A}_x]\\ [\hat{A}_y]\\ [\hat{A}_z]\end{bmatrix} \quad (A5.3)$$

Substituting (A5.2) and (A5.3) in (A5.1) yields:

$$\frac{1}{\mu}\int_R(\nabla\times\overline{W})\cdot(\nabla\times\hat{A})dR$$

$$=\frac{1}{\mu}\int_R\begin{pmatrix}[a'_{11}] & [a'_{12}] & [a'_{13}]\\ [a'_{21}] & [a'_{22}] & [a'_{23}]\\ [a'_{31}] & [a'_{32}] & [a'_{33}]\end{pmatrix}\cdot\begin{pmatrix}[\hat{A}_x]\\ [\hat{A}_y]\\ [\hat{A}_z]\end{pmatrix}dR$$

where

$$[a_{11}']=\frac{\partial[\alpha]^T}{\partial y}\frac{\partial[\alpha]}{\partial y}+\frac{\partial[\alpha]^T}{\partial z}\frac{\partial[\alpha]}{\partial z}\quad[a_{12}']=-\frac{\partial[\alpha]^T}{\partial y}\frac{\partial[\alpha]}{\partial x}\quad[a_{13}']=-\frac{\partial[\alpha]^T}{\partial z}\frac{\partial[\alpha]}{\partial x}$$

$$[a_{21}']=-\frac{\partial[\alpha]^T}{\partial x}\frac{\partial[\alpha]}{\partial y}\quad[a_{22}']=\frac{\partial[\alpha]^T}{\partial x}\frac{\partial[\alpha]}{\partial x}+\frac{\partial[\alpha]^T}{\partial z}\frac{\partial[\alpha]}{\partial z}\quad[a_{23}']=-\frac{\partial[\alpha]^T}{\partial z}\frac{\partial[\alpha]}{\partial y}$$

$$[a_{31}']=-\frac{\partial[\alpha]^T}{\partial x}\frac{\partial[\alpha]}{\partial z}\quad[a_{32}']=-\frac{\partial[\alpha]^T}{\partial y}\frac{\partial[\alpha]}{\partial z}\quad[a_{33}']=\frac{\partial[\alpha]^T}{\partial x}\frac{\partial[\alpha]}{\partial x}+\frac{\partial[\alpha]^T}{\partial y}\frac{\partial[\alpha]}{\partial y}$$

$$=\frac{1}{\mu}\int_R \begin{pmatrix} [a'_{11}] & [a'_{12}] & [a'_{13}] \\ [a'_{21}] & [a'_{22}] & [a'_{23}] \\ [a'_{31}] & [a'_{32}] & [a'_{33}] \end{pmatrix} dR \cdot \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\mu}\int_R [a'_{11}]dR & \frac{1}{\mu}\int_R [a'_{12}]dR & \frac{1}{\mu}\int_R [a'_{13}]dR \\ \frac{1}{\mu}\int_R [a'_{21}]dR & \frac{1}{\mu}\int_R [a'_{22}]dR & \frac{1}{\mu}\int_R [a'_{23}]dR \\ \frac{1}{\mu}\int_R [a'_{31}]dR & \frac{1}{\mu}\int_R [a'_{32}]dR & \frac{1}{\mu}\int_R [a'_{33}]dR \end{pmatrix} \cdot \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix}$$

$$= \begin{pmatrix} [a_{11}] & [a_{12}] & [a_{13}] \\ [a_{21}] & [a_{22}] & [a_{23}] \\ [a_{31}] & [a_{32}] & [a_{33}] \end{pmatrix} \cdot \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix} dR \qquad (A5.4)$$

where

$$[a_{ij}] = \frac{1}{\mu}\int_R [a_{ij}']dR$$

$$j\omega\sigma \int_R \overline{W} \hat{A}\, dR \qquad (A5.5)$$

$$\overline{W} \cdot \hat{A} = \begin{pmatrix} [\alpha]^T \hat{x} \\ [\alpha]^T \hat{y} \\ [\alpha]^T \hat{z} \end{pmatrix} \cdot \begin{pmatrix} [\alpha]\hat{x} & [\alpha]\hat{y} & [\alpha]\hat{z} \end{pmatrix} \cdot \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix}$$

$$= \begin{pmatrix} [\alpha]^T[\alpha] & [0] & [0] \\ [0] & [\alpha]^T[\alpha] & [0] \\ [0] & [0] & [\alpha]^T[\alpha] \end{pmatrix} \cdot \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix} \qquad (A5.6)$$

Substituting (A5.6) in (A5.5) yields:

$$j\omega\sigma\int_R \overline{W}\cdot\hat{A}\,dR = j\omega\sigma\int_R \begin{pmatrix} [\alpha]^T[\alpha] & [0] & [0] \\ [0] & [\alpha]^T[\alpha] & [0] \\ [0] & [0] & [\alpha]^T[\alpha] \end{pmatrix}\cdot\begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix} dR$$

$$= \begin{pmatrix} [b_{11}] & [0] & [0] \\ [0] & [b_{22}] & [0] \\ [0] & [0] & [b_{33}] \end{pmatrix}\cdot\begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix} \tag{A5.7}$$

where

$$[b_{ij}] = j\omega\sigma\int_R [\alpha]^T[\alpha]\,dR$$

$$\sigma\int_R \overline{W}\cdot\nabla\hat{\phi}\,dR \tag{A5.8}$$

$$\nabla\hat{\phi} = \nabla([\alpha][\hat{\phi}]) = \nabla[\alpha][\hat{\phi}]$$

$$= \left[\frac{\partial[\alpha]}{\partial x}\hat{x} + \frac{\partial[\alpha]}{\partial y}\hat{y} + \frac{\partial[\alpha]}{\partial z}\hat{z}\right][\hat{\phi}] \tag{A5.9}$$

$$\overline{W}\cdot\nabla\hat{\phi} = \begin{pmatrix} [\alpha]^T\hat{x} \\ [\alpha]^T\hat{y} \\ [\alpha]^T\hat{z} \end{pmatrix}\cdot\left[\frac{\partial[\alpha]}{\partial x}\hat{x} + \frac{\partial[\alpha]}{\partial y}\hat{y} + \frac{\partial[\alpha]}{\partial z}\hat{z}\right][\hat{\phi}]$$

$$= \begin{pmatrix} [\alpha]^T\frac{\partial[\alpha]}{\partial x} \\ [\alpha]^T\frac{\partial[\alpha]}{\partial y} \\ [\alpha]^T\frac{\partial[\alpha]}{\partial z} \end{pmatrix}\cdot[\hat{\phi}] \tag{A5.10}$$

Substituting (A5.10) in (A5.8) yields:

$$\sigma\int_r \overline{W}\cdot\nabla\hat{\phi}dR = \sigma\int_R \begin{pmatrix} [\alpha]^T\dfrac{\partial[\alpha]}{\partial x} \\ [\alpha]^T\dfrac{\partial[\alpha]}{\partial y} \\ [\alpha]^T\dfrac{\partial[\alpha]}{\partial z} \end{pmatrix}\cdot[\hat{\phi}]dR$$

$$= \sigma\int_R \begin{pmatrix} [\alpha]^T\dfrac{\partial[\alpha]}{\partial x} \\ [\alpha]^T\dfrac{\partial[\alpha]}{\partial y} \\ [\alpha]^T\dfrac{\partial[\alpha]}{\partial z} \end{pmatrix} dR[\hat{\phi}]$$

$$= \begin{pmatrix} [c_1] \\ [c_2] \\ [c_3] \end{pmatrix}\cdot[\hat{\phi}] \qquad (A5.11)$$

where

$$[c_1] = \sigma\int_R [\alpha]^T\frac{\partial[\alpha]}{\partial x}dR$$

$$[c_2] = \sigma\int_R [\alpha]^T\frac{\partial[\alpha]}{\partial y}dR$$

$$[c_3] = \sigma\int_R [\alpha]^T\frac{\partial[\alpha]}{\partial z}dR$$

$$\int_S (\overline{W}\times\hat{H})\cdot\hat{n}dS \qquad (A5.12)$$

$$\overline{W}\times\hat{H} = \begin{pmatrix} [\alpha]^T\hat{x} \\ [\alpha]^T\hat{y} \\ [\alpha]^T\hat{z} \end{pmatrix}\times(H_x\hat{x}+H_y\hat{y}+H_z\hat{z}) \qquad (A5.13)$$

$$(\overline{W}\times\hat{H})\cdot\hat{n}=\begin{pmatrix}[\alpha]^T(H_y\hat{z}-H_z\hat{y})\\ [\alpha]^T(H_z\hat{x}-H_x\hat{z})\\ [\alpha]^T(H_x\hat{y}-H_y\hat{x})\end{pmatrix}\cdot(n_x\hat{x}+n_y\hat{y}+n_z\hat{z})$$

$$=\begin{pmatrix}[\alpha]^T(H_yn_z-H_zn_y)\\ [\alpha]^T(H_zn_x-H_xn_z)\\ [\alpha]^T(H_xn_y-H_yn_x)\end{pmatrix} \qquad (A5.14)$$

Substituting (A5.14) in (A5.12) yields:

$$\int_S(\overline{W}\times\hat{H})\cdot\hat{n}dS=\begin{pmatrix}[F_1]\\ [F_2]\\ [F_3]\end{pmatrix} \qquad (A5.15)$$

where

$$[f_1]=\int_S[\alpha]^T(H_yn_z-H_zn_y)dS$$

$$[f_2]=\int_S[\alpha]^T(H_zn_x-H_xn_z)dS$$

$$[f_3]=\int_S[\alpha]^T(H_xn_y-H_yn_x)dS$$

$$\boxed{\int_R\nabla W\cdot\sigma(j\omega\hat{A}+\nabla\hat{\phi})=0 \qquad (A5.16)}$$

$$\int_R\nabla W\cdot\sigma(j\omega\hat{A}+\nabla\hat{\phi})=j\omega\sigma\int_R\nabla W\cdot\hat{A}dR+\sigma\int_R\nabla W\cdot\nabla\phi dR$$

$$\boxed{j\omega\sigma\int_R\nabla W\cdot\hat{A}dR \qquad (A5.17)}$$

$$\nabla W=\nabla[\alpha]^T=\left[\frac{\partial[\alpha]^T}{\partial x}\hat{x}+\frac{\partial[\alpha]^T}{\partial y}\hat{y}+\frac{\partial[\alpha]^T}{\partial z}\hat{z}\right]$$

$$\nabla W \cdot \hat{A} = \left[\frac{\partial[\alpha]^T}{\partial x}\hat{x} + \frac{\partial[\alpha]^T}{\partial y}\hat{y} + \frac{\partial[\alpha]^T}{\partial z}\hat{z} \right] \cdot ([\alpha]\hat{x} \quad [\alpha]\hat{y} \quad [\alpha]\hat{z}) \cdot \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix}$$

$$= \left(\frac{\partial[\alpha]^T}{\partial x}[\alpha] \quad \frac{\partial[\alpha]^T}{\partial y}[\alpha] \quad \frac{\partial[\alpha]^T}{\partial z}[\alpha] \right) \cdot \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix} \qquad (A5.18)$$

Substituting (A5.18) in (A5.17) yields:

$$j\omega\sigma \int_R \nabla W \cdot \hat{A}\, dR = j\omega\sigma \int_R \left(\frac{\partial[\alpha]^T}{\partial x}[\alpha] \quad \frac{\partial[\alpha]^T}{\partial y}[\alpha] \quad \frac{\partial[\alpha]^T}{\partial z}[\alpha] \right) \cdot \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix} dR$$

$$= j\omega\sigma \int_R \left(\frac{\partial[\alpha]^T}{\partial x}[\alpha] \quad \frac{\partial[\alpha]^T}{\partial y}[\alpha] \quad \frac{\partial[\alpha]^T}{\partial z}[\alpha] \right) dR \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix}$$

$$= ([d_1] \quad [d_2] \quad [d_3]) \cdot \begin{pmatrix} [\hat{A}_x] \\ [\hat{A}_y] \\ [\hat{A}_z] \end{pmatrix} \qquad (A5.19)$$

where

$$[d_1] = j\omega\sigma \int_R \frac{\partial[\alpha]^T}{\partial x}[\alpha] dR$$

$$[d_2] = j\omega\sigma \int_R \frac{\partial[\alpha]^T}{\partial y}[\alpha] dR$$

$$[d_3] = j\omega\sigma \int_R \frac{\partial[\alpha]^T}{\partial z}[\alpha] dR$$

$$\sigma \int_R \nabla W \cdot \nabla \phi\, dR \qquad (A5.20)$$

$$\nabla W = \nabla[\alpha]^T = \left[\frac{\partial[\alpha]^T}{\partial x}\hat{x} + \frac{\partial[\alpha]^T}{\partial y}\hat{y} + \frac{\partial[\alpha]^T}{\partial z}\hat{z} \right] \qquad (A5.21)$$

$$\nabla\phi = \nabla([\alpha][\hat{\phi}]) = \nabla[\alpha][\hat{\phi}]$$

$$= \left[\frac{\partial[\alpha]^T}{\partial x}\hat{x} + \frac{\partial[\alpha]^T}{\partial y}\hat{y} + \frac{\partial[\alpha]^T}{\partial z}\hat{z} \right][\hat{\phi}] \quad (A5.22)$$

$$\nabla W \cdot \nabla\hat{\phi} = \left[\frac{\partial[\alpha]^T}{\partial x}\frac{\partial[\alpha]}{\partial x} + \frac{\partial[\alpha]^T}{\partial y}\frac{\partial[\alpha]}{\partial y} + \frac{\partial[\alpha]^T}{\partial z}\frac{\partial[\alpha]}{\partial z} \right] \cdot [\hat{\phi}] \quad (A5.23)$$

Substituting (A5.23) in (A5.20) yields:

$$\sigma \int_R (\nabla W \cdot \nabla\hat{\phi}) = \sigma \int_R \left[\frac{\partial[\alpha]^T}{\partial x}\frac{\partial[\alpha]}{\partial x} + \frac{\partial[\alpha]^T}{\partial y}\frac{\partial[\alpha]}{\partial y} + \frac{\partial[\alpha]^T}{\partial z}\frac{\partial[\alpha]}{\partial z} \right] \cdot [\hat{\phi}]dR$$

$$= \sigma \int_R \left[\frac{\partial[\alpha]^T}{\partial x}\frac{\partial[\alpha]}{\partial x} + \frac{\partial[\alpha]^T}{\partial y}\frac{\partial[\alpha]}{\partial y} + \frac{\partial[\alpha]^T}{\partial z}\frac{\partial[\alpha]}{\partial z} \right] dR\,[\hat{\phi}]$$

$$= [e_1][\hat{\phi}] \quad (A5.24)$$

where

$$[e_1] = \sigma \int_R \left[\frac{\partial[\alpha]^T}{\partial x}\frac{\partial[\alpha]}{\partial x} + \frac{\partial[\alpha]^T}{\partial y}\frac{\partial[\alpha]}{\partial y} + \frac{\partial[\alpha]^T}{\partial z}\frac{\partial[\alpha]}{\partial z} \right] dR$$

The submatrices $[a_{11}], \ldots, [a_{33}]$, $[b_{11}], \ldots, [b_{33}]$, $[c_1], \ldots, [c_3]$, $[d_1], \ldots, [d_3]$, and $[e_1]$ are all 8x8 matrices. For example $[a_{11}]$ represents an 8x8 matrix such that:

$$[a_{11}] = \begin{pmatrix} (a_{11})_{11} & (a_{11})_{12} & \cdot & \cdot & \cdot & \cdot & (a_{11})_{17} & (a_{11})_{18} \\ (a_{11})_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (a_{11})_{28} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (a_{11})_{71} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (a_{11})_{78} \\ (a_{11})_{81} & (a_{11})_{82} & \cdot & \cdot & \cdot & \cdot & (a_{11})_{87} & (a_{11})_{88} \end{pmatrix} \quad (A5.25)$$

The $[S]$ matrix is formed by the developed submatrices above to form a 32×32 matrix such that:

$$[S] = \begin{pmatrix} [a_{11}]+[b_{11}] & [a_{12}] & [a_{13}] & [c_1] \\ [a_{21}] & [a_{22}]+[b_{22}] & [a_{23}] & [c_2] \\ [a_{31}] & [a_{32}] & [a_{33}]+[b_{33}] & [c_3] \\ [d_1] & [d_2] & [d_3] & [e_1] \end{pmatrix} \tag{A5.26}$$

Using the formulas above to calculate the values of the elements of the submatrices, the values of the elements of the $[S]$ matrix are listed below:

Let

$$\Delta x = (x_8 - x_1) \qquad \Delta y = (y_8 - y_1) \qquad \Delta z = (z_8 - z_1)$$

$$\Delta V = \Delta x \Delta y \Delta z$$

$$s_{1,1} = (a_{11})_{1,1} + (b_{11})_{1,1} = \frac{\Delta x}{36\mu\Delta y\Delta z}(4(\Delta z)^2 + 4(\Delta y)^2) + \frac{j\omega\sigma\Delta V}{27}$$

$$s_{1,2} = (a_{11})_{1,2} + (b_{11})_{1,2} = \frac{\Delta x}{36\mu\Delta y\Delta z}(2(\Delta z)^2 + 2(\Delta y)^2) + \frac{j\omega\sigma\Delta V}{54}$$

$$s_{1,3} = (a_{11})_{1,3} + (b_{11})_{1,3} = \frac{\Delta x}{36\mu\Delta y\Delta z}(-4(\Delta z)^2 + 2(\Delta y)^2) + \frac{j\omega\sigma\Delta V}{54}$$

$$s_{1,4} = (a_{11})_{1,4} + (b_{11})_{1,4} = \frac{\Delta x}{36\mu\Delta y\Delta z}(-2(\Delta z)^2 + (\Delta y)^2) + \frac{j\omega\sigma\Delta V}{108}$$

$$s_{1,5} = (a_{11})_{1,5} + (b_{11})_{1,5} = \frac{\Delta x}{36\mu\Delta y\Delta z}(2(\Delta z)^2 - 4(\Delta y)^2) + \frac{j\omega\sigma\Delta V}{54}$$

$$s_{1,6} = (a_{11})_{1,6} + (b_{11})_{1,6} = \frac{\Delta x}{36\mu\Delta y\Delta z}((\Delta z)^2 - 2(\Delta y)^2) + \frac{j\omega\sigma\Delta V}{108}$$

$$s_{1,7} = (a_{11})_{1,7} + (b_{11})_{1,7} = \frac{\Delta x}{36\mu\Delta y\Delta z}(-2(\Delta z)^2 - 2(\Delta y)^2) + \frac{j\omega\sigma\Delta V}{108}$$

$$s_{1,8} = (a_{11})_{1,8} + (b_{11})_{1,8} = \frac{\Delta x}{36\mu\Delta y\Delta z}(-(\Delta z)^2 - (\Delta y)^2) + \frac{j\omega\sigma\Delta V}{216}$$

$$s_{1,9} = (a_{12})_{1,1} = -\frac{\Delta z}{12\mu} \qquad s_{1,10} = (a_{12})_{1,2} = -s_{1,9}$$

$$s_{1,11} = (a_{12})_{1,3} = s_{1,9} \qquad s_{1,12} = (a_{12})_{1,4} = -s_{1,9}$$

$$s_{1,13} = (a_{12})_{1,5} = -\frac{\Delta z}{24\mu} \qquad s_{1,14} = (a_{12})_{1,6} = -s_{1,13}$$

$$s_{1,15} = (a_{12})_{1,7} = s_{1,13} \qquad s_{1,16} = (a_{12})_{1,8} = -s_{1,13}$$

$$s_{1,17} = (a_{13})_{1,1} = -\frac{\Delta y}{12\mu} \qquad s_{1,18} = (a_{13})_{1,2} = -s_{1,17}$$

$$s_{1,19}=(a_{13})_{1,3}=-\frac{\Delta y}{24\mu}$$

$$s_{1,20}=(a_{13})_{1,4}=-s_{1,19}$$

$$s_{1,21}=(a_{13})_{1,5}=s_{1,17}$$

$$s_{1,22}=(a_{13})_{1,6}=-s_{1,17}$$

$$s_{1,23}=(a_{13})_{1,7}=s_{1,19}$$

$$s_{1,24}=(a_{13})_{1,8}=-s_{1,19}$$

$$s_{1,25}=(c_1)_{1,1}=-\frac{\sigma\Delta y\Delta z}{18}$$

$$s_{1,26}=(c_1)_{1,2}=-s_{1,25}$$

$$s_{1,27}=(c_1)_{1,3}=-\frac{\sigma\Delta y\Delta z}{36}$$

$$s_{1,28}=(c_1)_{1,4}=-s_{1,27}$$

$$s_{1,29}=(c_1)_{1,5}=s_{1,27}$$

$$s_{1,30}=(c_1)_{1,6}=-s_{1,27}$$

$$s_{1,31}=(c_1)_{1,7}=-\frac{\sigma\Delta y\Delta z}{72}$$

$$s_{1,32}=(c_1)_{1,8}=-s_{1,31}$$

$$s_{2,1}=(a_{11})_{2,1}+(b_{11})_{2,1}=s_{1,2}$$

$$s_{2,2}=(a_{11})_{2,2}+(b_{11})_{2,2}=s_{1,1}$$

$$s_{2,3}=(a_{11})_{2,3}+(b_{11})_{2,3}=s_{1,4}$$

$$s_{2,4}=(a_{11})_{2,4}+(b_{11})_{2,4}=s_{1,3}$$

$$s_{2,5}=(a_{11})_{2,5}+(b_{11})_{2,5}=s_{1,6}$$

$$s_{2,6}=(a_{11})_{2,6}+(b_{11})_{2,6}=s_{1,5}$$

$$s_{2,7}=(a_{11})_{2,7}+(b_{11})_{2,7}=s_{1,8}$$

$$s_{2,8}=(a_{11})_{2,8}+(b_{11})_{2,8}=s_{1,7}$$

$$s_{2,9}=(a_{12})_{2,1}=s_{1,9}$$

$$s_{2,10}=(a_{12})_{2,2}=-s_{1,9}$$

$$s_{2,11}=(a_{12})_{2,3}=s_{1,9}$$

$$s_{2,12}=(a_{12})_{2,4}=-s_{1,9}$$

$$s_{2,13}=(a_{12})_{2,5}=s_{1,13}$$

$$s_{2,14}=(a_{12})_{2,6}=-s_{1,13}$$

$$s_{2,15}=(a_{12})_{2,7}=s_{1,13}$$

$$s_{2,16}=(a_{12})_{2,8}=-s_{1,13}$$

$$s_{2,17}=(a_{13})_{2,1}=s_{1,17}$$

$$s_{2,18}=(a_{13})_{2,2}=-s_{1,17}$$

$$s_{2,19}=(a_{13})_{2,3}=s_{1,19}$$

$$s_{2,20}=(a_{13})_{2,4}=-s_{1,19}$$

$$s_{2,21}=(a_{13})_{2,5}=s_{1,17}$$

$$s_{2,22}=(a_{13})_{2,6}=-s_{1,17}$$

$$s_{2,23}=(a_{13})_{2,7}=s_{1,19}$$

$$s_{2,24}=(a_{13})_{2,8}=-s_{1,19}$$

$$s_{2,25}=(c_1)_{2,1}=s_{1,25}$$

$$s_{2,26}=(c_1)_{2,2}=-s_{1,25}$$

$$s_{2,27}=(c_1)_{2,3}=s_{1,27}$$

$$s_{2,28}=(c_1)_{2,4}=-s_{1,27}$$

$$s_{2,29}=(c_1)_{2,5}=s_{1,27}$$

$$s_{2,30}=(c_1)_{2,6}=-s_{1,27}$$

$$s_{2,31}=(c_1)_{2,7}=s_{1,31}$$

$$s_{2,32}=(c_1)_{2,8}=-s_{1,31}$$

$$s_{3,1}=(a_{11})_{3,1}+(b_{11})_{3,1}=s_{1,3}$$

$$s_{3,2}=(a_{11})_{3,2}+(b_{11})_{3,2}=s_{1,4}$$

$$s_{3,3}=(a_{11})_{3,3}+(b_{11})_{3,3}=s_{1,1}$$

$$s_{3,4}=(a_{11})_{3,4}+(b_{11})_{3,4}=s_{1,2}$$

$$s_{3,5}=(a_{11})_{3,5}+(b_{11})_{3,5}=s_{1,7}$$

$$s_{3,6}=(a_{11})_{3,6}+(b_{11})_{3,6}=s_{1,8}$$

$$s_{3,7}=(a_{11})_{3,7}+(b_{11})_{3,7}=s_{1,5}$$

$$s_{3,8}=(a_{11})_{3,8}+(b_{11})_{3,8}=s_{1,6}$$

$s_{3,9} = (a_{12})_{3,1} = -s_{1,9}$ $s_{3,10} = (a_{12})_{3,2} = s_{1,9}$

$s_{3,11} = (a_{12})_{3,3} = -s_{1,9}$ $s_{3,12} = (a_{12})_{3,4} = s_{1,9}$

$s_{3,13} = (a_{12})_{3,5} = -s_{1,13}$ $s_{3,14} = (a_{12})_{3,6} = s_{1,13}$

$s_{3,15} = (a_{12})_{3,7} = -s_{1,13}$ $s_{3,16} = (a_{12})_{3,8} = s_{1,13}$

$s_{3,17} = (a_{13})_{3,1} = s_{1,19}$ $s_{3,18} = (a_{13})_{3,2} = -s_{1,19}$

$s_{3,19} = (a_{13})_{3,3} = s_{1,17}$ $s_{3,20} = (a_{13})_{3,4} = -s_{1,17}$

$s_{3,21} = (a_{13})_{3,5} = s_{1,19}$ $s_{3,22} = (a_{13})_{3,6} = -s_{1,19}$

$s_{3,23} = (a_{13})_{3,7} = s_{1,17}$ $s_{3,24} = (a_{13})_{3,8} = -s_{1,17}$

$s_{3,25} = (c_1)_{3,1} = s_{1,27}$ $s_{3,26} = (c_1)_{3,2} = -s_{1,27}$

$s_{3,27} = (c_1)_{3,3} = s_{1,25}$ $s_{3,28} = (c_1)_{3,4} = -s_{1,25}$

$s_{3,29} = (c_1)_{3,5} = s_{1,31}$ $s_{3,30} = (c_1)_{3,6} = -s_{1,31}$

$s_{3,31} = (c_1)_{3,7} = s_{1,27}$ $s_{3,32} = (c_1)_{3,8} = -s_{1,27}$

$s_{4,1} = (a_{11})_{4,1} + (b_{11})_{4,1} = s_{1,4}$ $s_{4,2} = (a_{11})_{4,2} + (b_{11})_{4,2} = s_{1,3}$

$s_{4,3} = (a_{11})_{4,3} + (b_{11})_{4,3} = s_{1,2}$ $s_{4,4} = (a_{11})_{4,4} + (b_{11})_{4,4} = s_{1,1}$

$s_{4,5} = (a_{11})_{4,5} + (b_{11})_{4,5} = s_{1,8}$ $s_{4,6} = (a_{11})_{4,6} + (b_{11})_{4,6} = s_{1,7}$

$s_{4,7} = (a_{11})_{4,7} + (b_{11})_{4,7} = s_{1,6}$ $s_{4,8} = (a_{11})_{4,8} + (b_{11})_{4,8} = s_{1,5}$

$s_{4,9} = (a_{12})_{4,1} = -s_{1,9}$ $s_{4,10} = (a_{12})_{4,2} = s_{1,9}$

$s_{4,11} = (a_{12})_{4,3} = -s_{1,9}$ $s_{4,12} = (a_{12})_{4,4} = s_{1,9}$

$s_{4,13} = (a_{12})_{4,5} = -s_{1,13}$ $s_{4,14} = (a_{12})_{4,6} = s_{1,13}$

$s_{4,15} = (a_{12})_{4,7} = -s_{1,13}$ $s_{4,16} = (a_{12})_{4,8} = s_{1,13}$

$s_{4,17} = (a_{13})_{4,1} = s_{1,19}$ $s_{4,18} = (a_{13})_{4,2} = -s_{1,19}$

$s_{4,19} = (a_{13})_{4,3} = s_{1,17}$ $s_{4,20} = (a_{13})_{4,4} = -s_{1,17}$

$s_{4,21} = (a_{13})_{4,5} = s_{1,19}$ $s_{4,22} = (a_{13})_{4,6} = -s_{1,19}$

$s_{4,23} = (a_{13})_{4,7} = s_{1,17}$ $s_{4,24} = (a_{13})_{4,8} = -s_{1,17}$

$s_{4,25} = (c_1)_{4,1} = s_{1,27}$ $s_{4,26} = (c_1)_{4,2} = -s_{1,27}$

$s_{4,27} = (c_1)_{4,3} = s_{1,25}$ $s_{4,28} = (c_1)_{4,4} = -s_{1,25}$

$s_{4,29} = (c_1)_{4,5} = s_{1,31}$ $s_{4,30} = (c_1)_{4,6} = -s_{1,31}$

$s_{4,31} = (c_1)_{4,7} = s_{1,27}$ $s_{4,32} = (c_1)_{4,8} = -s_{1,27}$

$s_{5,1} = (a_{11})_{5,1} + (b_{11})_{5,1} = s_{1,5}$ $s_{5,2} = (a_{11})_{5,2} + (b_{11})_{5,2} = s_{1,6}$

$s_{5,3}=(a_{11})_{5,3}+(b_{11})_{5,3}=s_{1,7}$ $s_{5,4}=(a_{11})_{5,4}+(b_{11})_{5,4}=s_{1,8}$

$s_{5,5}=(a_{11})_{5,5}+(b_{11})_{5,5}=s_{1,1}$ $s_{5,6}=(a_{11})_{5,6}+(b_{11})_{5,6}=s_{1,2}$

$s_{5,7}=(a_{11})_{5,7}+(b_{11})_{5,7}=s_{1,3}$ $s_{5,8}=(a_{11})_{5,8}+(b_{11})_{5,8}=s_{1,4}$

$s_{5,9}=(a_{12})_{5,1}=s_{1,13}$ $s_{5,10}=(a_{12})_{5,2}=-s_{1,13}$

$s_{5,11}=(a_{12})_{5,3}=s_{1,13}$ $s_{5,12}=(a_{12})_{5,4}=-s_{1,13}$

$s_{5,13}=(a_{12})_{5,5}=s_{1,9}$ $s_{5,14}=(a_{12})_{5,6}=-s_{1,9}$

$s_{5,15}=(a_{12})_{5,7}=s_{1,9}$ $s_{5,16}=(a_{12})_{5,8}=-s_{1,9}$

$s_{5,17}=(a_{13})_{5,1}=-s_{1,17}$ $s_{5,18}=(a_{13})_{5,2}=s_{1,17}$

$s_{5,19}=(a_{13})_{5,3}=-s_{1,19}$ $s_{5,20}=(a_{13})_{5,4}=s_{1,19}$

$s_{5,21}=(a_{13})_{5,5}=-s_{1,17}$ $s_{5,22}=(a_{13})_{5,6}=s_{1,17}$

$s_{5,23}=(a_{13})_{5,7}=-s_{1,19}$ $s_{5,24}=(a_{13})_{5,8}=s_{1,19}$

$s_{5,25}=(c_1)_{5,1}=s_{1,27}$ $s_{5,26}=(c_1)_{5,2}=-s_{1,27}$

$s_{5,27}=(c_1)_{5,3}=s_{1,31}$ $s_{5,28}=(c_1)_{5,4}=-s_{1,31}$

$s_{5,29}=(c_1)_{5,5}=s_{1,25}$ $s_{5,30}=(c_1)_{5,6}=-s_{1,25}$

$s_{5,31}=(c_1)_{5,7}=s_{1,27}$ $s_{5,32}=(c_1)_{5,8}=-s_{1,27}$

$s_{6,1}=(a_{11})_{6,1}+(b_{11})_{6,1}=s_{1,6}$ $s_{6,2}=(a_{11})_{6,2}+(b_{11})_{6,2}=s_{1,5}$

$s_{6,3}=(a_{11})_{6,3}+(b_{11})_{6,3}=s_{1,8}$ $s_{6,4}=(a_{11})_{6,4}+(b_{11})_{6,4}=s_{1,7}$

$s_{6,5}=(a_{11})_{6,5}+(b_{11})_{6,5}=s_{1,2}$ $s_{6,6}=(a_{11})_{6,6}+(b_{11})_{6,6}=s_{1,1}$

$s_{6,7}=(a_{11})_{6,7}+(b_{11})_{6,7}=s_{1,4}$ $s_{6,8}=(a_{11})_{6,8}+(b_{11})_{6,8}=s_{1,3}$

$s_{6,9}=(a_{12})_{6,1}=s_{1,13}$ $s_{6,10}=(a_{12})_{6,2}=-s_{1,13}$

$s_{6,11}=(a_{12})_{6,3}=s_{1,13}$ $s_{6,12}=(a_{12})_{6,4}=-s_{1,13}$

$s_{6,13}=(a_{12})_{6,5}=s_{1,9}$ $s_{6,14}=(a_{12})_{6,6}=-s_{1,9}$

$s_{6,15}=(a_{12})_{6,7}=s_{1,9}$ $s_{6,16}=(a_{12})_{6,8}=-s_{1,9}$

$s_{6,17}=(a_{13})_{6,1}=-s_{1,17}$ $s_{6,18}=(a_{13})_{6,2}=s_{1,17}$

$s_{6,19}=(a_{13})_{6,3}=-s_{1,19}$ $s_{6,20}=(a_{13})_{6,4}=s_{1,19}$

$s_{6,21}=(a_{13})_{6,5}=-s_{1,17}$ $s_{6,22}=(a_{13})_{6,6}=s_{1,17}$

$s_{6,23}=(a_{13})_{6,7}=-s_{1,19}$ $s_{6,24}=(a_{13})_{6,8}=s_{1,19}$

$s_{6,25}=(c_1)_{6,1}=s_{1,27}$ $s_{6,26}=(c_1)_{6,2}=-s_{1,27}$

$s_{6,27}=(c_1)_{6,3}=s_{1,31}$ $s_{6,28}=(c_1)_{6,4}=-s_{1,31}$

$s_{6,29}=(c_1)_{6,5}=s_{1,25}$ $s_{6,30}=(c_1)_{6,6}=-s_{1,25}$

$s_{6,31}=(c_1)_{6,7}=s_{1,27}$ $s_{6,32}=(c_1)_{6,8}=-s_{1,27}$

$s_{7,1}=(a_{11})_{7,1}+(b_{11})_{7,1}=s_{1,7}$ $s_{7,2}=(a_{11})_{7,2}+(b_{11})_{7,2}=s_{1,8}$

$s_{7,3}=(a_{11})_{7,3}+(b_{11})_{7,3}=s_{1,5}$ $s_{7,4}=(a_{11})_{7,4}+(b_{11})_{7,4}=s_{1,6}$

$s_{7,5}=(a_{11})_{7,5}+(b_{11})_{7,5}=s_{1,3}$ $s_{7,6}=(a_{11})_{7,6}+(b_{11})_{7,6}=s_{1,4}$

$s_{7,7}=(a_{11})_{7,7}+(b_{11})_{7,7}=s_{1,1}$ $s_{7,8}=(a_{11})_{7,8}+(b_{11})_{7,8}=s_{1,2}$

$s_{7,9}=(a_{12})_{7,1}=-s_{1,13}$ $s_{7,10}=(a_{12})_{7,2}=s_{1,13}$

$s_{7,11}=(a_{12})_{7,3}=-s_{1,13}$ $s_{7,12}=(a_{12})_{7,4}=s_{1,13}$

$s_{7,13}=(a_{12})_{7,5}=-s_{1,9}$ $s_{7,14}=(a_{12})_{7,6}=s_{1,9}$

$s_{7,15}=(a_{12})_{7,7}=-s_{1,9}$ $s_{7,16}=(a_{12})_{7,8}=s_{1,9}$

$s_{7,17}=(a_{13})_{7,1}=-s_{1,19}$ $s_{7,18}=(a_{13})_{7,2}=s_{1,19}$

$s_{7,19}=(a_{13})_{7,3}=-s_{1,17}$ $s_{7,20}=(a_{13})_{7,4}=s_{1,17}$

$s_{7,21}=(a_{13})_{7,5}=-s_{1,19}$ $s_{7,22}=(a_{13})_{7,6}=s_{1,19}$

$s_{7,23}=(a_{13})_{7,7}=-s_{1,17}$ $s_{7,24}=(a_{13})_{7,8}=s_{1,17}$

$s_{7,25}=(c_1)_{7,1}=s_{1,31}$ $s_{7,26}=(c_1)_{7,2}=-s_{1,31}$

$s_{7,27}=(c_1)_{7,3}=s_{1,27}$ $s_{7,28}=(c_1)_{7,4}=-s_{1,27}$

$s_{7,29}=(c_1)_{7,5}=s_{1,27}$ $s_{7,30}=(c_1)_{7,6}=-s_{1,27}$

$s_{7,31}=(c_1)_{7,7}=s_{1,25}$ $s_{7,32}=(c_1)_{7,8}=-s_{1,25}$

$s_{8,1}=(a_{11})_{8,1}+(b_{11})_{8,1}=s_{1,8}$ $s_{8,2}=(a_{11})_{8,2}+(b_{11})_{8,2}=s_{1,7}$

$s_{8,3}=(a_{11})_{8,3}+(b_{11})_{8,3}=s_{1,6}$ $s_{8,4}=(a_{11})_{8,4}+(b_{11})_{8,4}=s_{1,5}$

$s_{8,5}=(a_{11})_{8,5}+(b_{11})_{8,5}=s_{1,4}$ $s_{8,6}=(a_{11})_{8,6}+(b_{11})_{8,6}=s_{1,3}$

$s_{8,7}=(a_{11})_{8,7}+(b_{11})_{8,7}=s_{1,2}$ $s_{8,8}=(a_{11})_{8,8}+(b_{11})_{8,8}=s_{1,1}$

$s_{8,9}=(a_{12})_{8,1}=-s_{1,13}$ $s_{8,10}=(a_{12})_{8,2}=s_{1,13}$

$s_{8,11}=(a_{12})_{8,3}=-s_{1,13}$ $s_{8,12}=(a_{12})_{8,4}=s_{1,13}$

$s_{8,13}=(a_{12})_{8,5}=-s_{1,9}$ $s_{8,14}=(a_{12})_{8,6}=s_{1,9}$

$s_{8,15}=(a_{12})_{8,7}=-s_{1,9}$ $s_{8,16}=(a_{12})_{8,8}=s_{1,9}$

$s_{8,17}=(a_{13})_{8,1}=-s_{1,19}$ $s_{8,18}=(a_{13})_{8,2}=s_{1,19}$

$s_{8,19}=(a_{13})_{8,3}=-s_{1,17}$ $s_{8,20}=(a_{13})_{8,4}=s_{1,17}$

$s_{8,21}=(a_{13})_{8,5}=-s_{1,19}$ $s_{8,22}=(a_{13})_{8,6}=s_{1,19}$

$s_{8,23} = (a_{13})_{8,7} = -s_{1,17}$ $\qquad$ $s_{8,24} = (a_{13})_{8,8} = s_{1,17}$

$s_{8,25} = (c_1)_{8,1} = s_{1,31}$ $\qquad$ $s_{8,26} = (c_1)_{8,2} = -s_{1,31}$

$s_{8,27} = (c_1)_{8,3} = s_{1,27}$ $\qquad$ $s_{8,28} = (c_1)_{8,4} = -s_{1,27}$

$s_{8,29} = (c_1)_{8,5} = s_{1,27}$ $\qquad$ $s_{8,30} = (c_1)_{8,6} = -s_{1,27}$

$s_{8,31} = (c_1)_{8,7} = s_{1,25}$ $\qquad$ $s_{8,32} = (c_1)_{8,8} = -s_{1,25}$

$s_{9,1} = (a_{21})_{1,1} = s_{1,9}$ $\qquad$ $s_{9,2} = (a_{21})_{1,2} = s_{1,9}$

$s_{9,3} = (a_{21})_{1,3} = -s_{1,9}$ $\qquad$ $s_{9,4} = (a_{21})_{1,4} = -s_{1,9}$

$s_{9,5} = (a_{21})_{1,5} = s_{1,13}$ $\qquad$ $s_{9,6} = (a_{21})_{1,6} = s_{1,13}$

$s_{9,7} = (a_{21})_{1,7} = -s_{1,13}$ $\qquad$ $s_{9,8} = (a_{21})_{1,8} = -s_{1,13}$

$$s_{9,9} = (a_{22})_{1,1} + (b_{22})_{1,1} = \frac{\Delta y}{36\mu\Delta x\Delta z}(4(\Delta z)^2 + 4(\Delta x)^2) + \frac{j\omega\sigma\Delta V}{27}$$

$$s_{9,10} = (a_{22})_{1,2} + (b_{22})_{1,2} = \frac{\Delta y}{36\mu\Delta x\Delta z}(-4(\Delta z)^2 + 2(\Delta x)^2) + \frac{j\omega\sigma\Delta V}{54}$$

$$s_{9,11} = (a_{22})_{1,3} + (b_{22})_{1,3} = \frac{\Delta y}{36\mu\Delta x\Delta z}(2(\Delta z)^2 + 2(\Delta x)^2) + \frac{j\omega\sigma\Delta V}{54}$$

$$s_{9,12} = (a_{22})_{1,4} + (b_{22})_{1,4} = \frac{\Delta y}{36\mu\Delta x\Delta z}(-2(\Delta z)^2 + (\Delta x)^2) + \frac{j\omega\sigma\Delta V}{108}$$

$$s_{9,13} = (a_{22})_{1,5} + (b_{22})_{1,5} = \frac{\Delta y}{36\mu\Delta x\Delta z}(2(\Delta z)^2 - 4(\Delta x)^2) + \frac{j\omega\sigma\Delta V}{54}$$

$$s_{9,14} = (a_{22})_{1,6} + (b_{22})_{1,6} = \frac{\Delta y}{36\mu\Delta x\Delta z}(-2(\Delta z)^2 - 2(\Delta x)^2) + \frac{j\omega\sigma\Delta V}{108}$$

$$s_{9,15} = (a_{22})_{1,7} + (b_{22})_{1,7} = \frac{\Delta y}{36\mu\Delta x\Delta z}((\Delta z)^2 - 2(\Delta x)^2) + \frac{j\omega\sigma\Delta V}{108}$$

$$s_{9,16} = (a_{22})_{1,8} + (b_{22})_{1,8} = \frac{\Delta y}{36\mu\Delta x\Delta z}(-(\Delta z)^2 - (\Delta x)^2) + \frac{j\omega\sigma\Delta V}{216}$$

$s_{9,17} = (a_{23})_{1,1} = -\dfrac{\Delta x}{12\mu}$ $\qquad$ $s_{9,18} = (a_{23})_{1,2} = -\dfrac{\Delta x}{24\mu}$

$s_{9,19} = (a_{23})_{1,3} = -s_{9,17}$ $\qquad$ $s_{9,20} = (a_{23})_{1,4} = -s_{9,18}$

$s_{9,21} = (a_{23})_{1,5} = s_{9,17}$ $\qquad$ $s_{9,22} = (a_{23})_{1,6} = s_{9,18}$

$s_{9,23} = (a_{23})_{1,7} = -s_{9,17}$ $\qquad$ $s_{9,24} = (a_{23})_{1,8} = -s_{9,18}$

$s_{9,25} = (c_2)_{1,1} = -\dfrac{\sigma\Delta x\Delta z}{18}$ $\qquad$ $s_{9,26} = (c_2)_{1,2} = -\dfrac{\sigma\Delta x\Delta z}{36}$

$s_{9,27} = (c_2)_{1,3} = -s_{9,25}$ $\qquad$ $s_{9,28} = (c_2)_{1,4} = -s_{9,26}$

$s_{9,29} = (c_2)_{1,5} = s_{9,26}$ $\qquad$ $s_{9,30} = (c_2)_{1,6} = -\dfrac{\sigma\Delta x\Delta z}{72}$

$s_{9,31} = (c_2)_{1,7} = -s_{9,26}$ $\qquad$ $s_{9,32} = (c_2)_{1,8} = -s_{9,30}$

$s_{10,1} = (a_{21})_{2,1} = -s_{1,9}$

$s_{10,2} = (a_{21})_{2,2} = -s_{1,9}$

$s_{10,3} = (a_{21})_{2,3} = s_{1,9}$

$s_{10,4} = (a_{21})_{2,4} = s_{1,9}$

$s_{10,5} = (a_{21})_{2,5} = -s_{1,13}$

$s_{10,6} = (a_{21})_{2,6} = -s_{1,13}$

$s_{10,7} = (a_{21})_{2,7} = s_{1,13}$

$s_{10,8} = (a_{21})_{2,8} = s_{1,13}$

$s_{10,9} = (a_{22})_{2,1} + (b_{22})_{2,1} = s_{9,10}$

$s_{10,10} = (a_{22})_{2,2} + (b_{22})_{2,2} = s_{9,9}$

$s_{10,11} = (a_{22})_{2,3} + (b_{22})_{2,3} = s_{9,12}$

$s_{10,12} = (a_{22})_{2,4} + (b_{22})_{2,4} = s_{9,11}$

$s_{10,13} = (a_{22})_{2,5} + (b_{22})_{2,5} = s_{9,14}$

$s_{10,14} = (a_{22})_{2,6} + (b_{22})_{2,6} = s_{9,13}$

$s_{10,15} = (a_{22})_{2,7} + (b_{22})_{2,7} = s_{9,16}$

$s_{10,16} = (a_{22})_{2,8} + (b_{22})_{2,8} = s_{9,15}$

$s_{10,17} = (a_{23})_{2,1} = s_{9,18}$

$s_{10,18} = (a_{23})_{2,2} = s_{9,17}$

$s_{10,19} = (a_{23})_{2,3} = -s_{9,18}$

$s_{10,20} = (a_{23})_{2,4} = -s_{9,17}$

$s_{10,21} = (a_{23})_{2,5} = s_{9,18}$

$s_{10,22} = (a_{23})_{2,6} = s_{9,17}$

$s_{10,23} = (a_{23})_{2,7} = -s_{9,18}$

$s_{10,24} = (a_{23})_{2,8} = -s_{9,17}$

$s_{10,25} = (c_2)_{2,1} = s_{9,26}$

$s_{10,26} = (c_2)_{2,2} = s_{9,25}$

$s_{10,27} = (c_2)_{2,3} = -s_{9,26}$

$s_{10,28} = (c_2)_{2,4} = -s_{9,25}$

$s_{10,29} = (c_2)_{2,5} = s_{9,30}$

$s_{10,30} = (c_2)_{2,6} = s_{9,26}$

$s_{10,31} = (c_2)_{2,7} = -s_{9,30}$

$s_{10,32} = (c_2)_{2,8} = -s_{9,26}$

$s_{11,1} = (a_{21})_{3,1} = s_{1,9}$

$s_{11,2} = (a_{21})_{3,2} = s_{1,9}$

$s_{11,3} = (a_{21})_{3,3} = -s_{1,9}$

$s_{11,4} = (a_{21})_{3,4} = -s_{1,9}$

$s_{11,5} = (a_{21})_{3,5} = s_{1,13}$

$s_{11,6} = (a_{21})_{3,6} = s_{1,13}$

$s_{11,7} = (a_{21})_{3,7} = -s_{1,13}$

$s_{11,8} = (a_{21})_{3,8} = -s_{1,13}$

$s_{11,9} = (a_{22})_{3,1} + (b_{22})_{3,1} = s_{9,11}$

$s_{11,10} = (a_{22})_{3,2} + (b_{22})_{3,2} = s_{9,12}$

$s_{11,11} = (a_{22})_{3,3} + (b_{22})_{3,3} = s_{9,9}$

$s_{11,12} = (a_{22})_{3,4} + (b_{22})_{3,4} = s_{9,10}$

$s_{11,13} = (a_{22})_{3,5} + (b_{22})_{3,5} = s_{9,15}$

$s_{11,14} = (a_{22})_{3,6} + (b_{22})_{3,6} = s_{9,16}$

$s_{11,15} = (a_{22})_{3,7} + (b_{22})_{3,7} = s_{9,13}$

$s_{11,16} = (a_{22})_{3,8} + (b_{22})_{3,8} = s_{9,14}$

$s_{11,17} = (a_{23})_{3,1} = s_{9,17}$

$s_{11,18} = (a_{23})_{3,2} = s_{9,18}$

$s_{11,19} = (a_{23})_{3,3} = -s_{9,17}$

$s_{11,20} = (a_{23})_{3,4} = -s_{9,18}$

$s_{11,21} = (a_{23})_{3,5} = s_{9,17}$

$s_{11,22} = (a_{23})_{3,6} = s_{9,18}$

$s_{11,23} = (a_{23})_{3,7} = -s_{9,17}$

$s_{11,24} = (a_{23})_{3,8} = -s_{9,18}$

$s_{11,25} = (c_2)_{3,1} = s_{9,25}$

$s_{11,26} = (c_2)_{3,2} = s_{9,26}$

$s_{11,27} = (c_2)_{3,3} = -s_{9,25}$ $s_{11,28} = (c_2)_{3,4} = -s_{9,26}$

$s_{11,29} = (c_2)_{3,5} = s_{9,26}$ $s_{11,30} = (c_2)_{3,6} = s_{9,30}$

$s_{11,31} = (c_2)_{3,7} = -s_{9,26}$ $s_{11,32} = (c_2)_{3,8} = -s_{9,30}$

$s_{12,1} = (a_{21})_{4,1} = -s_{1,9}$ $s_{12,2} = (a_{21})_{4,2} = -s_{1,9}$

$s_{12,3} = (a_{21})_{4,3} = s_{1,9}$ $s_{12,4} = (a_{21})_{4,4} = s_{1,9}$

$s_{12,5} = (a_{21})_{4,5} = -s_{1,13}$ $s_{12,6} = (a_{21})_{4,6} = -s_{1,13}$

$s_{12,7} = (a_{21})_{4,7} = s_{1,13}$ $s_{12,8} = (a_{21})_{4,8} = s_{1,13}$

$s_{12,9} = (a_{22})_{4,1} + (b_{22})_{4,1} = s_{9,12}$ $s_{12,10} = (a_{22})_{4,2} + (b_{22})_{4,2} = s_{9,11}$

$s_{12,11} = (a_{22})_{4,3} + (b_{22})_{4,3} = s_{9,10}$ $s_{12,12} = (a_{22})_{4,4} + (b_{22})_{4,4} = s_{9,9}$

$s_{12,13} = (a_{22})_{4,5} + (b_{22})_{4,5} = s_{9,16}$ $s_{12,14} = (a_{22})_{4,6} + (b_{22})_{4,6} = s_{9,15}$

$s_{12,15} = (a_{22})_{4,7} + (b_{22})_{4,7} = s_{9,14}$ $s_{12,16} = (a_{22})_{4,8} + (b_{22})_{4,8} = s_{9,13}$

$s_{12,17} = (a_{23})_{4,1} = s_{9,18}$ $s_{12,18} = (a_{23})_{4,2} = s_{9,17}$

$s_{12,19} = (a_{23})_{4,3} = -s_{9,18}$ $s_{12,20} = (a_{23})_{4,4} = -s_{9,17}$

$s_{12,21} = (a_{23})_{4,5} = s_{9,18}$ $s_{12,22} = (a_{23})_{4,6} = s_{9,17}$

$s_{12,23} = (a_{23})_{4,7} = -s_{9,18}$ $s_{12,24} = (a_{23})_{4,8} = -s_{9,17}$

$s_{12,25} = (c_2)_{4,1} = s_{9,26}$ $s_{12,26} = (c_2)_{4,2} = s_{9,25}$

$s_{12,27} = (c_2)_{4,3} = -s_{9,26}$ $s_{12,28} = (c_2)_{4,4} = -s_{9,25}$

$s_{12,29} = (c_2)_{4,5} = s_{9,30}$ $s_{12,30} = (c_2)_{4,6} = s_{9,26}$

$s_{12,31} = (c_2)_{4,7} = -s_{9,30}$ $s_{12,32} = (c_2)_{4,8} = -s_{9,26}$

$s_{13,1} = (a_{21})_{5,1} = s_{1,13}$ $s_{13,2} = (a_{21})_{5,2} = s_{1,13}$

$s_{13,3} = (a_{21})_{5,3} = -s_{1,13}$ $s_{13,4} = (a_{21})_{5,4} = -s_{1,13}$

$s_{13,5} = (a_{21})_{5,5} = s_{1,9}$ $s_{13,6} = (a_{21})_{5,6} = s_{1,9}$

$s_{13,7} = (a_{21})_{5,7} = -s_{1,9}$ $s_{13,8} = (a_{21})_{5,8} = -s_{1,9}$

$s_{13,9} = (a_{22})_{5,1} + (b_{22})_{5,1} = s_{9,13}$ $s_{13,10} = (a_{22})_{5,2} + (b_{22})_{5,2} = s_{9,14}$

$s_{13,11} = (a_{22})_{5,3} + (b_{22})_{5,3} = s_{9,15}$ $s_{13,12} = (a_{22})_{5,4} + (b_{22})_{5,4} = s_{9,16}$

$s_{13,13} = (a_{22})_{5,5} + (b_{22})_{5,5} = s_{9,9}$ $s_{13,14} = (a_{22})_{5,6} + (b_{22})_{5,6} = s_{9,10}$

$s_{13,15} = (a_{22})_{5,7} + (b_{22})_{5,7} = s_{9,11}$ $s_{13,16} = (a_{22})_{5,8} + (b_{22})_{5,8} = s_{9,12}$

$s_{13,17} = (a_{23})_{5,1} = -s_{9,17}$ $s_{13,18} = (a_{23})_{5,2} = -s_{9,18}$

$s_{13,19} = (a_{23})_{5,3} = s_{9,17}$ $s_{13,20} = (a_{23})_{5,4} = s_{9,18}$

$s_{13,21} = (a_{23})_{5,5} = -s_{9,17}$

$s_{13,22} = (a_{23})_{5,6} = -s_{9,18}$

$s_{13,23} = (a_{23})_{5,7} = s_{9,17}$

$s_{13,24} = (a_{23})_{5,8} = s_{9,18}$

$s_{13,25} = (c_2)_{5,1} = s_{9,26}$

$s_{13,26} = (c_2)_{5,2} = s_{9,30}$

$s_{13,27} = (c_2)_{5,3} = -s_{9,26}$

$s_{13,28} = (c_2)_{5,4} = -s_{9,30}$

$s_{13,29} = (c_2)_{5,5} = s_{9,25}$

$s_{13,30} = (c_2)_{5,6} = s_{9,26}$

$s_{13,31} = (c_2)_{5,7} = -s_{9,25}$

$s_{13,32} = (c_2)_{5,8} = -s_{9,26}$

$s_{14,1} = (a_{21})_{6,1} = -s_{1,13}$

$s_{14,2} = (a_{21})_{6,2} = -s_{1,13}$

$s_{14,3} = (a_{21})_{6,3} = s_{1,13}$

$s_{14,4} = (a_{21})_{6,4} = s_{1,13}$

$s_{14,5} = (a_{21})_{6,5} = -s_{1,9}$

$s_{14,6} = (a_{21})_{6,6} = -s_{1,9}$

$s_{14,7} = (a_{21})_{6,7} = s_{1,9}$

$s_{14,8} = (a_{21})_{6,8} = s_{1,9}$

$s_{14,9} = (a_{22})_{6,1} + (b_{22})_{6,1} = s_{9,14}$

$s_{14,10} = (a_{22})_{6,2} + (b_{22})_{6,2} = s_{9,13}$

$s_{14,11} = (a_{22})_{6,3} + (b_{22})_{6,3} = s_{9,16}$

$s_{14,12} = (a_{22})_{6,4} + (b_{22})_{6,4} = s_{9,15}$

$s_{14,13} = (a_{22})_{6,5} + (b_{22})_{6,5} = s_{9,10}$

$s_{14,14} = (a_{22})_{6,6} + (b_{22})_{6,6} = s_{9,}$[illegible]

$s_{14,15} = (a_{22})_{6,7} + (b_{22})_{6,7} = s_{9,12}$

$s_{14,16} = (a_{22})_{6,8} + (b_{22})_{6,8} = s_{9,11}$

$s_{14,17} = (a_{23})_{6,1} = -s_{9,18}$

$s_{14,18} = (a_{23})_{6,2} = -s_{9,17}$

$s_{14,19} = (a_{23})_{6,3} = s_{9,18}$

$s_{14,20} = (a_{23})_{6,4} = s_{9,17}$

$s_{14,21} = (a_{23})_{6,5} = -s_{9,18}$

$s_{14,22} = (a_{23})_{6,6} = -s_{9,17}$

$s_{14,23} = (a_{23})_{6,7} = s_{9,18}$

$s_{14,24} = (a_{23})_{6,8} = s_{9,17}$

$s_{14,25} = (c_2)_{6,1} = s_{9,30}$

$s_{14,26} = (c_2)_{6,2} = s_{9,26}$

$s_{14,27} = (c_2)_{6,3} = -s_{9,30}$

$s_{14,28} = (c_2)_{6,4} = -s_{9,26}$

$s_{14,29} = (c_2)_{6,5} = s_{9,26}$

$s_{14,30} = (c_2)_{6,6} = s_{9,25}$

$s_{14,31} = (c_2)_{6,7} = -s_{9,26}$

$s_{14,32} = (c_2)_{6,8} = -s_{9,25}$

$s_{15,1} = (a_{21})_{7,1} = s_{1,13}$

$s_{15,2} = (a_{21})_{7,2} = s_{1,13}$

$s_{15,3} = (a_{21})_{7,3} = -s_{1,13}$

$s_{15,4} = (a_{21})_{7,4} = -s_{1,13}$

$s_{15,5} = (a_{21})_{7,5} = s_{1,9}$

$s_{15,6} = (a_{21})_{7,6} = s_{1,9}$

$s_{15,7} = (a_{21})_{7,7} = -s_{1,9}$

$s_{15,8} = (a_{21})_{7,8} = -s_{1,9}$

$s_{15,9} = (a_{22})_{7,1} + (b_{22})_{7,1} = s_{9,15}$

$s_{15,10} = (a_{22})_{7,2} + (b_{22})_{7,2} = s_{9,16}$

$s_{15,11} = (a_{22})_{7,3} + (b_{22})_{7,3} = s_{9,13}$

$s_{15,12} = (a_{22})_{7,4} + (b_{22})_{7,4} = s_{9,14}$

$s_{15,13} = (a_{22})_{7,5} + (b_{22})_{7,5} = s_{9,11}$

$s_{15,14} = (a_{22})_{7,6} + (b_{22})_{7,6} = s_{9,12}$

$s_{15,15} = (a_{22})_{7,7} + (b_{22})_{7,7} = s_{9,9}$

$s_{15,16} = (a_{22})_{7,8} + (b_{22})_{7,8} = s_{9,10}$

$s_{15,17} = (a_{23})_{7,1} = -s_{9,17}$

$s_{15,18} = (a_{23})_{7,2} = -s_{9,18}$

$s_{15,19} = (a_{23})_{7,3} = s_{9,17}$

$s_{15,20} = (a_{23})_{7,4} = s_{9,18}$

$s_{15,21} = (a_{23})_{7,5} = -s_{9,17}$

$s_{15,22} = (a_{23})_{7,6} = -s_{9,18}$

$s_{15,23} = (a_{23})_{7,7} = s_{9,17}$

$s_{15,24} = (a_{23})_{7,8} = s_{9,18}$

$s_{15,25} = (c_2)_{7,1} = s_{9,26}$

$s_{15,26} = (c_2)_{7,2} = s_{9,30}$

$s_{15,27} = (c_2)_{7,3} = -s_{9,26}$

$s_{15,28} = (c_2)_{7,4} = -s_{9,30}$

$s_{15,29} = (c_2)_{7,5} = s_{9,25}$

$s_{15,30} = (c_2)_{7,6} = s_{9,26}$

$s_{15,31} = (c_2)_{7,7} = -s_{9,25}$

$s_{15,32} = (c_2)_{7,8} = -s_{9,26}$

$s_{16,1} = (a_{21})_{8,1} = -s_{1,13}$

$s_{16,2} = (a_{21})_{8,2} = -s_{1,13}$

$s_{16,3} = (a_{21})_{8,3} = s_{1,13}$

$s_{16,4} = (a_{21})_{8,4} = s_{1,13}$

$s_{16,5} = (a_{21})_{8,5} = -s_{1,9}$

$s_{16,6} = (a_{21})_{8,6} = -s_{1,9}$

$s_{16,7} = (a_{21})_{8,7} = s_{1,9}$

$s_{16,8} = (a_{21})_{8,8} = s_{1,9}$

$s_{16,9} = (a_{22})_{8,1} + (b_{22})_{8,1} = s_{9,16}$

$s_{16,10} = (a_{22})_{8,2} + (b_{22})_{8,2} = s_{9,15}$

$s_{16,11} = (a_{22})_{8,3} + (b_{22})_{8,3} = s_{9,14}$

$s_{16,12} = (a_{22})_{8,4} + (b_{22})_{8,4} = s_{9,13}$

$s_{16,13} = (a_{22})_{8,5} + (b_{22})_{8,5} = s_{9,12}$

$s_{16,14} = (a_{22})_{8,6} + (b_{22})_{8,6} = s_{9,11}$

$s_{16,15} = (a_{22})_{8,7} + (b_{22})_{8,7} = s_{9,10}$

$s_{16,16} = (a_{22})_{8,8} + (b_{22})_{8,8} = s_{9,9}$

$s_{16,17} = (a_{23})_{8,1} = -s_{9,18}$

$s_{16,18} = (a_{23})_{8,2} = -s_{9,17}$

$s_{16,19} = (a_{23})_{8,3} = s_{9,18}$

$s_{16,20} = (a_{23})_{8,4} = s_{9,17}$

$s_{16,21} = (a_{23})_{8,5} = -s_{9,18}$

$s_{16,22} = (a_{23})_{8,6} = -s_{9,17}$

$s_{16,23} = (a_{23})_{8,7} = s_{9,18}$

$s_{16,24} = (a_{23})_{8,8} = s_{9,17}$

$s_{16,25} = (c_2)_{8,1} = s_{9,30}$

$s_{16,26} = (c_2)_{8,2} = s_{9,26}$

$s_{16,27} = (c_2)_{8,3} = -s_{9,30}$

$s_{16,28} = (c_2)_{8,4} = -s_{9,26}$

$s_{16,29} = (c_2)_{8,5} = s_{9,26}$

$s_{16,30} = (c_2)_{8,6} = s_{9,25}$

$s_{16,31} = (c_2)_{8,7} = -s_{9,26}$

$s_{16,32} = (c_2)_{8,8} = -s_{9,25}$

$s_{17,1} = (a_{31})_{1,1} = s_{1,17}$

$s_{17,2} = (a_{31})_{1,2} = s_{1,17}$

$s_{17,3} = (a_{31})_{1,3} = s_{1,19}$

$s_{17,4} = (a_{31})_{1,4} = s_{1,19}$

$s_{17,5} = (a_{31})_{1,5} = -s_{1,17}$

$s_{17,6} = (a_{31})_{1,6} = -s_{1,17}$

$s_{17,7} = (a_{31})_{1,7} = -s_{1,19}$

$s_{17,8} = (a_{31})_{1,8} = -s_{1,19}$

$s_{17,9}=(a_{32})_{1,1}=s_{9,17}$

$s_{17,10}=(a_{32})_{1,2}=s_{9,18}$

$s_{17,11}=(a_{32})_{1,3}=s_{9,17}$

$s_{17,12}=(a_{32})_{1,4}=s_{9,18}$

$s_{17,13}=(a_{32})_{1,5}=-s_{9,17}$

$s_{17,14}=(a_{32})_{1,6}=-s_{9,18}$

$s_{17,15}=(a_{32})_{1,7}=-s_{9,17}$

$s_{17,16}=(a_{32})_{1,8}=-s_{9,18}$

$$s_{17,17}=(a_{33})_{1,1}+(b_{33})_{1,1}=\frac{\Delta z}{36\mu\Delta x\Delta y}(4(\Delta y)^2+4(\Delta x)^2)+\frac{j\omega\sigma\Delta V}{27}$$

$$s_{17,18}=(a_{33})_{1,2}+(b_{33})_{1,2}=\frac{\Delta z}{36\mu\Delta x\Delta y}(-4(\Delta y)^2+2(\Delta x)^2)+\frac{j\omega\sigma\Delta V}{54}$$

$$s_{17,19}=(a_{33})_{1,3}+(b_{33})_{1,3}=\frac{\Delta z}{36\mu\Delta x\Delta y}(2(\Delta y)^2-4(\Delta x)^2)+\frac{j\omega\sigma\Delta V}{54}$$

$$s_{17,20}=(a_{33})_{1,4}+(b_{33})_{1,4}=\frac{\Delta z}{36\mu\Delta x\Delta y}(-2(\Delta y)^2-2(\Delta x)^2)+\frac{j\omega\sigma\Delta V}{108}$$

$$s_{17,21}=(a_{33})_{1,5}+(b_{33})_{1,5}=\frac{\Delta z}{36\mu\Delta x\Delta y}(2(\Delta y)^2+2(\Delta x)^2)+\frac{j\omega\sigma\Delta V}{54}$$

$$s_{17,22}=(a_{33})_{1,6}+(b_{33})_{1,6}=\frac{\Delta z}{36\mu\Delta x\Delta y}(-2(\Delta y)^2+(\Delta x)^2)+\frac{j\omega\sigma\Delta V}{108}$$

$$s_{17,23}=(a_{33})_{1,7}+(b_{33})_{1,7}=\frac{\Delta z}{36\mu\Delta x\Delta y}((\Delta y)^2-2(\Delta x)^2)+\frac{j\omega\sigma\Delta V}{108}$$

$$s_{17,24}=(a_{33})_{1,8}+(b_{33})_{1,8}=\frac{\Delta z}{36\mu\Delta x\Delta y}(-(\Delta y)^2-(\Delta x)^2)+\frac{j\omega\sigma\Delta V}{216}$$

$s_{17,25}=(c_3)_{1,1}=-\dfrac{\sigma\Delta x\Delta y}{18}$

$s_{17,26}=(c_3)_{1,2}=-\dfrac{\sigma\Delta x\Delta y}{36}$

$s_{17,27}=(c_3)_{1,3}=s_{17,26}$

$s_{17,28}=(c_3)_{1,4}=-\dfrac{\sigma\Delta x\Delta y}{72}$

$s_{17,29}=(c_3)_{1,5}=-s_{17,25}$

$s_{17,30}=(c_3)_{1,6}=-s_{17,26}$

$s_{17,31}=(c_3)_{1,7}=-s_{17,26}$

$s_{17,32}=(c_3)_{1,8}=-s_{17,28}$

$s_{18,1}=(a_{31})_{2,1}=-s_{1,17}$

$s_{18,2}=(a_{31})_{2,2}=-s_{1,17}$

$s_{18,3}=(a_{31})_{2,3}=-s_{1,19}$

$s_{18,4}=(a_{31})_{2,4}=-s_{1,19}$

$s_{18,5}=(a_{31})_{2,5}=s_{1,17}$

$s_{18,6}=(a_{31})_{2,6}=s_{1,17}$

$s_{18,7}=(a_{31})_{2,7}=s_{1,19}$

$s_{18,8}=(a_{31})_{2,8}=s_{1,19}$

$s_{18,9}=(a_{32})_{2,1}=s_{9,18}$

$s_{18,10}=(a_{32})_{2,2}=s_{9,17}$

$s_{18,11}=(a_{32})_{2,3}=s_{9,18}$

$s_{18,12}=(a_{32})_{2,4}=s_{9,17}$

$s_{18,13}=(a_{32})_{2,5}=-s_{9,18}$

$s_{18,14}=(a_{32})_{2,6}=-s_{9,17}$

$s_{18,15}=(a_{32})_{2,7}=-s_{9,18}$

$s_{18,16}=(a_{32})_{2,8}=-s_{9,17}$

$s_{18,17}=(a_{33})_{2,1}+(b_{33})_{2,1}=s_{17,18}$

$s_{18,18}=(a_{33})_{2,2}+(b_{33})_{2,2}=s_{17,17}$

$s_{18,19}=(a_{33})_{2,3}+(b_{33})_{2,3}=s_{17,20}$ $s_{18,20}=(a_{33})_{2,4}+(b_{33})_{2,4}=s_{17,19}$

$s_{18,21}=(a_{33})_{2,5}+(b_{33})_{2,5}=s_{17,22}$ $s_{18,22}=(a_{33})_{2,6}+(b_{33})_{2,6}=s_{17,21}$

$s_{18,23}=(a_{33})_{2,7}+(b_{33})_{2,7}=s_{17,24}$ $s_{18,24}=(a_{33})_{2,8}+(b_{33})_{2,8}=s_{17,23}$

$s_{18,25}=(c_3)_{2,1}=s_{17,16}$ $s_{18,26}=(c_3)_{2,2}=s_{17,25}$

$s_{18,27}=(c_3)_{2,3}=s_{17,28}$ $s_{18,28}=(c_3)_{2,4}=s_{17,26}$

$s_{18,29}=(c_3)_{2,5}=-s_{17,26}$ $s_{18,30}=(c_3)_{2,6}=-s_{17,25}$

$s_{18,31}=(c_3)_{2,7}=-s_{17,28}$ $s_{18,32}=(c_3)_{2,8}=-s_{17,26}$

$s_{19,1}=(a_{31})_{3,1}=s_{1,19}$ $s_{19,2}=(a_{31})_{3,2}=s_{1,19}$

$s_{19,3}=(a_{31})_{3,3}=s_{1,17}$ $s_{19,4}=(a_{31})_{3,4}=s_{1,17}$

$s_{19,5}=(a_{31})_{3,5}=-s_{1,19}$ $s_{19,6}=(a_{31})_{3,6}=-s_{1,19}$

$s_{19,7}=(a_{31})_{3,7}=-s_{1,17}$ $s_{19,8}=(a_{31})_{3,8}=-s_{1,17}$

$s_{19,9}=(a_{32})_{3,1}=-s_{9,17}$ $s_{19,10}=(a_{32})_{3,2}=-s_{9,18}$

$s_{19,11}=(a_{32})_{3,3}=-s_{9,17}$ $s_{19,12}=(a_{32})_{3,4}=-s_{9,18}$

$s_{19,13}=(a_{32})_{3,5}=s_{9,17}$ $s_{19,14}=(a_{32})_{3,6}=s_{9,18}$

$s_{19,15}=(a_{32})_{3,7}=s_{9,17}$ $s_{19,16}=(a_{32})_{3,8}=s_{9,18}$

$s_{19,17}=(a_{33})_{3,1}+(b_{33})_{3,1}=s_{17,19}$ $s_{19,18}=(a_{33})_{3,2}+(b_{33})_{3,2}=s_{17,20}$

$s_{19,19}=(a_{33})_{3,3}+(b_{33})_{3,3}=s_{17,17}$ $s_{19,20}=(a_{33})_{3,4}+(b_{33})_{3,4}=s_{17,18}$

$s_{19,21}=(a_{33})_{3,5}+(b_{33})_{3,5}=s_{17,23}$ $s_{19,22}=(a_{33})_{3,6}+(b_{33})_{3,6}=s_{17,24}$

$s_{19,23}=(a_{33})_{3,7}+(b_{33})_{3,7}=s_{17,21}$ $s_{19,24}=(a_{33})_{3,8}+(b_{33})_{3,8}=s_{17,22}$

$s_{19,25}=(c_3)_{3,1}=s_{17,26}$ $s_{19,26}=(c_3)_{3,2}=s_{17,28}$

$s_{19,27}=(c_3)_{3,3}=s_{17,25}$ $s_{19,28}=(c_3)_{3,4}=s_{17,26}$

$s_{19,29}=(c_3)_{3,5}=-s_{17,26}$ $s_{19,30}=(c_3)_{3,6}=-s_{17,28}$

$s_{19,31}=(c_3)_{3,7}=-s_{17,25}$ $s_{19,32}=(c_3)_{3,8}=-s_{17,26}$

$s_{20,1}=(a_{31})_{4,1}=-s_{1,19}$ $s_{20,2}=(a_{31})_{4,2}=-s_{1,19}$

$s_{20,3}=(a_{31})_{4,3}=-s_{1,17}$ $s_{20,4}=(a_{31})_{4,4}=-s_{1,17}$

$s_{20,5}=(a_{31})_{4,5}=s_{1,19}$ $s_{20,6}=(a_{31})_{4,6}=s_{1,19}$

$s_{20,7}=(a_{31})_{4,7}=s_{1,17}$ $s_{20,8}=(a_{31})_{4,8}=s_{1,17}$

$s_{20,9}=(a_{32})_{4,1}=-s_{9,18}$ $s_{20,10}=(a_{32})_{4,2}=-s_{9,17}$

$s_{20,11}=(a_{32})_{4,3}=-s_{9,18}$ $s_{20,12}=(a_{32})_{4,4}=-s_{9,17}$

$s_{20,13} = (a_{32})_{4,5} = s_{9,18}$

$s_{20,14} = (a_{32})_{4,6} = s_{9,17}$

$s_{20,15} = (a_{32})_{4,7} = s_{9,18}$

$s_{20,16} = (a_{32})_{4,8} = s_{9,17}$

$s_{20,17} = (a_{33})_{4,1} + (b_{33})_{4,1} = s_{17,20}$

$s_{20,18} = (a_{33})_{4,2} + (b_{33})_{4,2} = s_{17,19}$

$s_{20,19} = (a_{33})_{4,3} + (b_{33})_{4,3} = s_{17,18}$

$s_{20,20} = (a_{33})_{4,4} + (b_{33})_{4,4} = s_{17,17}$

$s_{20,21} = (a_{33})_{4,5} + (b_{33})_{4,5} = s_{17,24}$

$s_{20,22} = (a_{33})_{4,6} + (b_{33})_{4,6} = s_{17,23}$

$s_{20,23} = (a_{33})_{4,7} + (b_{33})_{4,7} = s_{17,22}$

$s_{20,24} = (a_{33})_{4,8} + (b_{33})_{4,8} = s_{17,21}$

$s_{20,25} = (c_3)_{4,1} = s_{17,28}$

$s_{20,26} = (c_3)_{4,2} = s_{17,26}$

$s_{20,27} = (c_3)_{4,3} = s_{17,26}$

$s_{20,28} = (c_3)_{4,4} = s_{17,25}$

$s_{20,29} = (c_3)_{4,5} = -s_{17,28}$

$s_{20,30} = (c_3)_{4,6} = -s_{17,26}$

$s_{20,31} = (c_3)_{4,7} = -s_{17,26}$

$s_{20,32} = (c_3)_{4,8} = -s_{17,25}$

$s_{21,1} = (a_{31})_{5,1} = s_{1,17}$

$s_{21,2} = (a_{31})_{5,2} = s_{1,17}$

$s_{21,3} = (a_{31})_{5,3} = s_{1,19}$

$s_{21,4} = (a_{31})_{5,4} = s_{1,19}$

$s_{21,5} = (a_{31})_{5,5} = -s_{1,17}$

$s_{21,6} = (a_{31})_{5,6} = -s_{1,17}$

$s_{21,7} = (a_{31})_{5,7} = -s_{1,19}$

$s_{21,8} = (a_{31})_{5,8} = -s_{1,19}$

$s_{21,9} = (a_{32})_{5,1} = s_{9,17}$

$s_{21,10} = (a_{32})_{5,2} = s_{9,18}$

$s_{21,11} = (a_{32})_{5,3} = s_{9,17}$

$s_{21,12} = (a_{32})_{5,4} = s_{9,18}$

$s_{21,13} = (a_{32})_{5,5} = -s_{9,17}$

$s_{21,14} = (a_{32})_{5,6} = -s_{9,18}$

$s_{21,15} = (a_{32})_{5,7} = -s_{9,17}$

$s_{21,16} = (a_{32})_{5,8} = -s_{9,18}$

$s_{21,17} = (a_{33})_{5,1} + (b_{33})_{5,1} = s_{17,21}$

$s_{21,18} = (a_{33})_{5,2} + (b_{33})_{5,2} = s_{17,22}$

$s_{21,19} = (a_{33})_{5,3} + (b_{33})_{5,3} = s_{17,23}$

$s_{21,20} = (a_{33})_{5,4} + (b_{33})_{5,4} = s_{17,24}$

$s_{21,21} = (a_{33})_{5,5} + (b_{33})_{5,5} = s_{17,17}$

$s_{21,22} = (a_{33})_{5,6} + (b_{33})_{5,6} = s_{17,18}$

$s_{21,23} = (a_{33})_{5,7} + (b_{33})_{5,7} = s_{17,19}$

$s_{21,24} = (a_{33})_{5,8} + (b_{33})_{5,8} = s_{17,20}$

$s_{21,25} = (c_3)_{5,1} = s_{17,25}$

$s_{21,26} = (c_3)_{5,2} = s_{17,26}$

$s_{21,27} = (c_3)_{5,3} = s_{17,26}$

$s_{21,28} = (c_3)_{5,4} = s_{17,28}$

$s_{21,29} = (c_3)_{5,5} = -s_{17,25}$

$s_{21,30} = (c_3)_{5,6} = -s_{17,26}$

$s_{21,31} = (c_3)_{5,7} = -s_{17,26}$

$s_{21,32} = (c_3)_{5,8} = -s_{17,28}$

$s_{22,1} = (a_{31})_{6,1} = -s_{1,17}$

$s_{22,2} = (a_{31})_{6,2} = -s_{1,17}$

$s_{22,3} = (a_{31})_{6,3} = -s_{1,19}$

$s_{22,4} = (a_{31})_{6,4} = -s_{1,19}$

$s_{22,5} = (a_{31})_{6,5} = s_{1,17}$

$s_{22,6} = (a_{31})_{6,6} = s_{1,17}$

$s_{22,7} = (a_{31})_{6,7} = s_{1,19}$ $s_{22,8} = (a_{31})_{6,8} = s_{1,19}$

$s_{22,9} = (a_{32})_{6,1} = s_{9,18}$ $s_{22,10} = (a_{32})_{6,2} = s_{9,17}$

$s_{22,11} = (a_{32})_{6,3} = s_{9,18}$ $s_{22,12} = (a_{32})_{6,4} = s_{9,17}$

$s_{22,13} = (a_{32})_{6,5} = -s_{9,18}$ $s_{22,14} = (a_{32})_{6,6} = -s_{9,17}$

$s_{22,15} = (a_{32})_{6,7} = -s_{9,18}$ $s_{22,16} = (a_{32})_{6,8} = -s_{9,17}$

$s_{22,17} = (a_{33})_{6,1} + (b_{33})_{6,1} = s_{17,22}$ $s_{22,18} = (a_{33})_{6,2} + (b_{33})_{6,2} = s_{17,21}$

$s_{22,19} = (a_{33})_{6,3} + (b_{33})_{6,3} = s_{17,24}$ $s_{22,20} = (a_{33})_{6,4} + (b_{33})_{6,4} = s_{17,23}$

$s_{22,21} = (a_{33})_{6,5} + (b_{33})_{6,5} = s_{17,18}$ $s_{22,22} = (a_{33})_{6,6} + (b_{33})_{6,6} = s_{17,17}$

$s_{22,23} = (a_{33})_{6,7} + (b_{33})_{6,7} = s_{17,20}$ $s_{22,24} = (a_{33})_{6,8} + (b_{33})_{6,8} = s_{17,19}$

$s_{22,25} = (c_3)_{6,1} = s_{17,26}$ $s_{22,26} = (c_3)_{6,2} = s_{17,25}$

$s_{22,27} = (c_3)_{6,3} = s_{17,28}$ $s_{22,28} = (c_3)_{6,4} = s_{17,26}$

$s_{22,29} = (c_3)_{6,5} = -s_{17,26}$ $s_{22,30} = (c_3)_{6,6} = -s_{17,25}$

$s_{22,31} = (c_3)_{6,7} = -s_{17,28}$ $s_{22,32} = (c_3)_{6,8} = -s_{17,26}$

$s_{23,1} = (a_{31})_{7,1} = s_{1,19}$ $s_{23,2} = (a_{31})_{7,2} = s_{1,19}$

$s_{23,3} = (a_{31})_{7,3} = s_{1,17}$ $s_{23,4} = (a_{31})_{7,4} = s_{1,17}$

$s_{23,5} = (a_{31})_{7,5} = -s_{1,19}$ $s_{23,6} = (a_{31})_{7,6} = -s_{1,19}$

$s_{23,7} = (a_{31})_{7,7} = -s_{1,17}$ $s_{23,8} = (a_{31})_{7,8} = -s_{1,17}$

$s_{23,9} = (a_{32})_{7,1} = -s_{9,17}$ $s_{23,10} = (a_{32})_{7,2} = -s_{9,18}$

$s_{23,11} = (a_{32})_{7,3} = -s_{9,17}$ $s_{23,12} = (a_{32})_{7,4} = -s_{9,18}$

$s_{23,13} = (a_{32})_{7,5} = s_{9,17}$ $s_{23,14} = (a_{32})_{7,6} = s_{9,18}$

$s_{23,15} = (a_{32})_{7,7} = s_{9,17}$ $s_{23,16} = (a_{32})_{7,8} = s_{9,18}$

$s_{23,17} = (a_{33})_{7,1} + (b_{33})_{7,1} = s_{17,23}$ $s_{23,18} = (a_{33})_{7,2} + (b_{33})_{7,2} = s_{17,24}$

$s_{23,19} = (a_{33})_{7,3} + (b_{33})_{7,3} = s_{17,21}$ $s_{23,20} = (a_{33})_{7,4} + (b_{33})_{7,4} = s_{17,22}$

$s_{23,21} = (a_{33})_{7,5} + (b_{33})_{7,5} = s_{17,19}$ $s_{23,22} = (a_{33})_{7,6} + (b_{33})_{7,6} = s_{17,20}$

$s_{23,23} = (a_{33})_{7,7} + (b_{33})_{7,7} = s_{17,17}$ $s_{23,24} = (a_{33})_{7,8} + (b_{33})_{7,8} = s_{17,18}$

$s_{23,9} = (a_{32})_{7,1} = -s_{9,17}$ $s_{23,10} = (a_{32})_{7,2} = -s_{9,18}$

$s_{23,11} = (a_{32})_{7,3} = -s_{9,17}$ $s_{23,12} = (a_{32})_{7,4} = -s_{9,18}$

$s_{23,13} = (a_{32})_{7,5} = s_{9,17}$ $s_{23,14} = (a_{32})_{7,6} = s_{9,18}$

$s_{23,15} = (a_{32})_{7,7} = s_{9,17}$ $s_{23,16} = (a_{32})_{7,8} = s_{9,18}$

$s_{23,17} = (a_{33})_{7,1} + (b_{33})_{7,1} = s_{17,23}$ $s_{23,18} = (a_{33})_{7,2} + (b_{33})_{7,2} = s_{17,24}$

$s_{23,19} = (a_{33})_{7,3} + (b_{33})_{7,3} = s_{17,21}$ $s_{23,20} = (a_{33})_{7,4} + (b_{33})_{7,4} = s_{17,22}$

$s_{23,21} = (a_{33})_{7,5} + (b_{33})_{7,5} = s_{17,19}$ $s_{23,22} = (a_{33})_{7,6} + (b_{33})_{7,6} = s_{17,20}$

$s_{23,23} = (a_{33})_{7,7} + (b_{33})_{7,7} = s_{17,17}$ $s_{23,24} = (a_{33})_{7,8} + (b_{33})_{7,8} = s_{17,18}$

$s_{23,25} = (c_3)_{7,1} = s_{17,26}$ $s_{23,26} = (c_3)_{7,2} = s_{17,28}$

$s_{23,27} = (c_3)_{7,3} = s_{17,25}$ $s_{23,28} = (c_3)_{7,4} = s_{17,26}$

$s_{23,29} = (c_3)_{7,5} = -s_{17,26}$ $s_{23,30} = (c_3)_{7,6} = -s_{17,28}$

$s_{23,31} = (c_3)_{7,7} = -s_{17,25}$ $s_{23,32} = (c_3)_{7,8} = -s_{17,26}$

$s_{24,1} = (a_{31})_{8,1} = -s_{1,19}$ $s_{24,2} = (a_{31})_{8,2} = -s_{1,19}$

$s_{24,3} = (a_{31})_{8,3} = -s_{1,17}$ $s_{24,4} = (a_{31})_{8,4} = -s_{1,17}$

$s_{24,5} = (a_{31})_{8,5} = s_{1,19}$ $s_{24,6} = (a_{31})_{8,6} = s_{1,19}$

$s_{24,7} = (a_{31})_{8,7} = s_{1,17}$ $s_{24,8} = (a_{31})_{8,8} = s_{1,17}$

$s_{24,9} = (a_{32})_{8,1} = -s_{9,18}$ $s_{24,10} = (a_{32})_{8,2} = -s_{9,17}$

$s_{24,11} = (a_{32})_{8,3} = -s_{9,18}$ $s_{24,12} = (a_{32})_{8,4} = -s_{9,17}$

$s_{24,13} = (a_{32})_{8,5} = s_{9,18}$ $s_{24,14} = (a_{32})_{8,6} = s_{9,17}$

$s_{24,15} = (a_{32})_{8,7} = s_{9,18}$ $s_{24,16} = (a_{32})_{8,8} = s_{9,17}$

$s_{24,17} = (a_{33})_{8,1} + (b_{33})_{8,1} = s_{17,24}$ $s_{24,18} = (a_{33})_{8,2} + (b_{33})_{8,2} = s_{17,23}$

$s_{24,19} = (a_{33})_{8,3} + (b_{33})_{8,3} = s_{17,22}$ $s_{24,20} = (a_{33})_{8,4} + (b_{33})_{8,4} = s_{17,21}$

$s_{24,21} = (a_{33})_{8,5} + (b_{33})_{8,5} = s_{17,20}$ $s_{24,22} = (a_{33})_{8,6} + (b_{33})_{8,6} = s_{17,19}$

$s_{24,23} = (a_{33})_{8,7} + (b_{33})_{8,7} = s_{17,18}$ $s_{24,24} = (a_{33})_{8,8} + (b_{33})_{8,8} = s_{17,17}$

$s_{24,25} = (c_3)_{8,1} = s_{17,28}$ $s_{24,26} = (c_3)_{8,2} = s_{17,26}$

$s_{24,27} = (c_3)_{8,3} = s_{17,26}$ $s_{24,28} = (c_3)_{8,4} = s_{17,25}$

$s_{24,29} = (c_3)_{8,5} = -s_{17,28}$ $s_{24,30} = (c_3)_{8,6} = -s_{17,26}$

$s_{24,31} = (c_3)_{8,7} = -s_{17,26}$ $s_{24,32} = (c_3)_{8,8} = -s_{17,25}$

$s_{25,1} = (d_1)_{1,1} = -\frac{j\omega\sigma\Delta y\Delta z}{18}$ $s_{25,2} = (d_1)_{1,2} = s_{25,1}$

$s_{25,3} = (d_1)_{1,3} = -\frac{j\omega\sigma\Delta y\Delta z}{36}$ $s_{25,4} = (d_1)_{1,4} = s_{25,3}$

$s_{25,5} = (d_1)_{1,5} = s_{25,3}$ $s_{25,6} = (d_1)_{1,6} = s_{25,3}$

$s_{25,7} = (d_1)_{1,7} = -\frac{j\omega\sigma\Delta y\Delta z}{72}$ $s_{25,8} = (d_1)_{1,8} = s_{25,7}$

$$s_{25,9} = (d_2)_{1,1} = -\frac{j\omega\sigma\Delta x\Delta z}{18} \qquad s_{25,10} = (d_2)_{1,2} = -\frac{j\omega\sigma\Delta x\Delta z}{36}$$

$$s_{25,11} = (d_2)_{1,3} = s_{25,9} \qquad s_{25,12} = (d_2)_{1,4} = s_{25,10}$$

$$s_{25,13} = (d_2)_{1,5} = s_{25,10} \qquad s_{25,14} = (d_2)_{1,6} = -\frac{j\omega\sigma\Delta x\Delta z}{72}$$

$$s_{25,15} = (d_2)_{1,7} = s_{25,10} \qquad s_{25,16} = (d_2)_{1,8} = s_{25,14}$$

$$s_{25,17} = (d_3)_{1,1} = -\frac{j\omega\sigma\Delta x\Delta y}{18} \qquad s_{25,18} = (d_3)_{1,2} = -\frac{j\omega\sigma\Delta x\Delta y}{36}$$

$$s_{25,19} = (d_3)_{1,3} = s_{25,18} \qquad s_{25,20} = (d_3)_{1,4} = -\frac{j\omega\sigma\Delta x\Delta y}{72}$$

$$s_{25,21} = (d_3)_{1,5} = s_{25,17} \qquad s_{25,22} = (d_3)_{1,6} = s_{25,18}$$

$$s_{25,23} = (d_3)_{1,7} = s_{25,18} \qquad s_{25,24} = (d_3)_{1,8} = s_{25,20}$$

$$s_{25,25} = (e_1)_{1,1} = \frac{\sigma}{36}\left(\frac{4\Delta y\Delta z}{\Delta x} + \frac{4\Delta x\Delta z}{\Delta y} + \frac{4\Delta x\Delta y}{\Delta z}\right)$$

$$s_{25,26} = (e_1)_{1,2} = \frac{\sigma}{36}\left(\frac{-4\Delta y\Delta z}{\Delta x} + \frac{2\Delta x\Delta z}{\Delta y} + \frac{2\Delta x\Delta y}{\Delta z}\right)$$

$$s_{25,27} = (e_1)_{1,3} = \frac{\sigma}{36}\left(\frac{2\Delta y\Delta z}{\Delta x} + \frac{-4\Delta x\Delta z}{\Delta y} + \frac{2\Delta x\Delta y}{\Delta z}\right)$$

$$s_{25,28} = (e_1)_{1,4} = \frac{\sigma}{36}\left(\frac{-2\Delta y\Delta z}{\Delta x} + \frac{-2\Delta x\Delta z}{\Delta y} + \frac{\Delta x\Delta y}{\Delta z}\right)$$

$$s_{25,29} = (e_1)_{1,5} = \frac{\sigma}{36}\left(\frac{2\Delta y\Delta z}{\Delta x} + \frac{2\Delta x\Delta z}{\Delta y} + \frac{-4\Delta x\Delta y}{\Delta z}\right)$$

$$s_{25,30} = (e_1)_{1,6} = \frac{\sigma}{36}\left(\frac{-2\Delta y\Delta z}{\Delta x} + \frac{\Delta x\Delta z}{\Delta y} + \frac{-2\Delta x\Delta y}{\Delta z}\right)$$

$$s_{25,31} = (e_1)_{1,7} = \frac{\sigma}{36}\left(\frac{\Delta y\Delta z}{\Delta x} + \frac{-2\Delta x\Delta z}{\Delta y} + \frac{-2\Delta x\Delta y}{\Delta z}\right)$$

$$s_{25,32} = (e_1)_{1,8} = \frac{\sigma}{36}\left(\frac{-\Delta y\Delta z}{\Delta x} + \frac{-\Delta x\Delta z}{\Delta y} + \frac{-\Delta x\Delta y}{\Delta z}\right)$$

$$s_{26,1} = (d_1)_{2,1} = -s_{25,1} \qquad s_{26,2} = (d_1)_{2,2} = -s_{25,1}$$

$$s_{26,3} = (d_1)_{2,3} = -s_{25,3} \qquad s_{26,4} = (d_1)_{2,4} = -s_{25,3}$$

$$s_{26,5} = (d_1)_{2,5} = -s_{25,3} \qquad s_{26,6} = (d_1)_{2,6} = -s_{25,3}$$

$$s_{26,7} = (d_1)_{2,7} = -s_{25,7} \qquad s_{26,8} = (d_1)_{2,8} = -s_{25,7}$$

$$s_{26,9} = (d_2)_{2,1} = s_{25,10} \qquad s_{26,10} = (d_2)_{2,2} = s_{25,9}$$

$$s_{26,11} = (d_2)_{2,3} = s_{25,10} \qquad s_{26,12} = (d_2)_{2,4} = s_{25,9}$$

$$s_{26,13} = (d_2)_{2,5} = s_{25,14} \qquad s_{26,14} = (d_2)_{2,6} = s_{25,10}$$

$$s_{26,15} = (d_2)_{2,7} = s_{25,14} \qquad s_{26,16} = (d_2)_{2,8} = s_{25,10}$$

$s_{26,17} = (d_3)_{2,1} = s_{25,18}$

$s_{26,18} = (d_3)_{2,2} = s_{25,17}$

$s_{26,19} = (d_3)_{2,3} = s_{25,20}$

$s_{26,20} = (d_3)_{2,4} = s_{25,18}$

$s_{26,21} = (d_3)_{2,5} = s_{25,18}$

$s_{26,22} = (d_3)_{2,6} = s_{25,17}$

$s_{26,23} = (d_3)_{2,7} = s_{25,20}$

$s_{26,24} = (d_3)_{2,8} = s_{25,18}$

$s_{26,25} = (e_1)_{2,1} = s_{25,26}$

$s_{26,26} = (e_1)_{2,2} = s_{25,25}$

$s_{26,27} = (e_1)_{2,3} = s_{25,28}$

$s_{26,28} = (e_1)_{2,4} = s_{25,27}$

$s_{26,29} = (e_1)_{2,5} = s_{25,30}$

$s_{26,30} = (e_1)_{2,6} = s_{25,29}$

$s_{26,31} = (e_1)_{2,7} = s_{25,32}$

$s_{26,32} = (e_1)_{2,8} = s_{25,31}$

$s_{27,1} = (d_1)_{3,1} = s_{25,3}$

$s_{27,2} = (d_1)_{3,2} = s_{25,3}$

$s_{27,3} = (d_1)_{3,3} = s_{25,1}$

$s_{27,4} = (d_1)_{3,4} = s_{25,1}$

$s_{27,5} = (d_1)_{3,5} = s_{25,7}$

$s_{27,6} = (d_1)_{3,6} = s_{25,7}$

$s_{27,7} = (d_1)_{3,7} = s_{25,3}$

$s_{27,8} = (d_1)_{3,8} = s_{25,3}$

$s_{27,9} = (d_2)_{3,1} = -s_{25,9}$

$s_{27,10} = (d_2)_{3,2} = -s_{25,10}$

$s_{27,11} = (d_2)_{3,3} = -s_{25,9}$

$s_{27,12} = (d_2)_{3,4} = -s_{25,10}$

$s_{27,13} = (d_2)_{3,5} = -s_{25,10}$

$s_{27,14} = (d_2)_{3,6} = -s_{25,14}$

$s_{27,15} = (d_2)_{3,7} = -s_{25,10}$

$s_{27,16} = (d_2)_{3,8} = -s_{25,14}$

$s_{27,17} = (d_3)_{3,1} = s_{25,18}$

$s_{27,18} = (d_3)_{3,2} = s_{25,20}$

$s_{27,19} = (d_3)_{3,3} = s_{25,17}$

$s_{27,20} = (d_3)_{3,4} = s_{25,18}$

$s_{27,21} = (d_3)_{3,5} = s_{25,18}$

$s_{27,22} = (d_3)_{3,6} = s_{25,20}$

$s_{27,23} = (d_3)_{3,7} = s_{25,17}$

$s_{27,24} = (d_3)_{3,8} = s_{25,18}$

$s_{27,25} = (e_1)_{3,1} = s_{25,27}$

$s_{27,26} = (e_1)_{3,2} = s_{25,28}$

$s_{27,27} = (e_1)_{3,3} = s_{25,25}$

$s_{27,28} = (e_1)_{3,4} = s_{25,26}$

$s_{27,29} = (e_1)_{3,5} = s_{25,31}$

$s_{27,30} = (e_1)_{3,6} = s_{25,32}$

$s_{27,31} = (e_1)_{3,7} = s_{25,29}$

$s_{27,32} = (e_1)_{3,8} = s_{25,30}$

$s_{28,1} = (d_1)_{4,1} = -s_{25,3}$

$s_{28,2} = (d_1)_{4,2} = -s_{25,3}$

$s_{28,3} = (d_1)_{4,3} = -s_{25,1}$

$s_{28,4} = (d_1)_{4,4} = -s_{25,1}$

$s_{28,5} = (d_1)_{4,5} = -s_{25,7}$

$s_{28,6} = (d_1)_{4,6} = -s_{25,7}$

$s_{28,7} = (d_1)_{4,7} = -s_{25,3}$

$s_{28,8} = (d_1)_{4,8} = -s_{25,3}$

$s_{28,9} = (d_2)_{4,1} = -s_{25,10}$

$s_{28,10} = (d_2)_{4,2} = -s_{25,9}$

$s_{28,11} = (d_2)_{4,3} = -s_{25,10}$ $s_{28,12} = (d_2)_{4,4} = -s_{25,9}$

$s_{28,13} = (d_2)_{4,5} = -s_{25,14}$ $s_{28,14} = (d_2)_{4,6} = -s_{25,10}$

$s_{28,15} = (d_2)_{4,7} = -s_{25,14}$ $s_{28,16} = (d_2)_{4,8} = -s_{25,10}$

$s_{28,17} = (d_3)_{4,1} = s_{25,20}$ $s_{28,18} = (d_3)_{4,2} = s_{25,18}$

$s_{28,19} = (d_3)_{4,3} = s_{25,18}$ $s_{28,20} = (d_3)_{4,4} = s_{25,17}$

$s_{28,21} = (d_3)_{4,5} = s_{25,20}$ $s_{28,22} = (d_3)_{4,6} = s_{25,18}$

$s_{28,23} = (d_3)_{4,7} = s_{25,18}$ $s_{28,24} = (d_3)_{4,8} = s_{25,17}$

$s_{28,25} = (e_1)_{4,1} = s_{25,28}$ $s_{28,26} = (e_1)_{4,2} = s_{25,27}$

$s_{28,27} = (e_1)_{4,3} = s_{25,26}$ $s_{28,28} = (e_1)_{4,4} = s_{25,25}$

$s_{28,29} = (e_1)_{4,5} = s_{25,32}$ $s_{28,30} = (e_1)_{4,6} = s_{25,31}$

$s_{28,31} = (e_1)_{4,7} = s_{25,30}$ $s_{28,32} = (e_1)_{4,8} = s_{25,29}$

$s_{29,1} = (d_1)_{5,1} = s_{25,3}$ $s_{29,2} = (d_1)_{5,2} = s_{25,3}$

$s_{29,3} = (d_1)_{5,3} = s_{25,7}$ $s_{29,4} = (d_1)_{5,4} = s_{25,7}$

$s_{29,5} = (d_1)_{5,5} = s_{25,1}$ $s_{29,6} = (d_1)_{5,6} = s_{25,1}$

$s_{29,7} = (d_1)_{5,7} = s_{25,3}$ $s_{29,8} = (d_1)_{5,8} = s_{25,3}$

$s_{29,9} = (d_2)_{5,1} = s_{25,10}$ $s_{29,10} = (d_2)_{5,2} = s_{25,14}$

$s_{29,11} = (d_2)_{5,3} = s_{25,10}$ $s_{29,12} = (d_2)_{5,4} = s_{25,14}$

$s_{29,13} = (d_2)_{5,5} = s_{25,9}$ $s_{29,14} = (d_2)_{5,6} = s_{25,10}$

$s_{29,15} = (d_2)_{5,7} = s_{25,9}$ $s_{29,16} = (d_2)_{5,8} = s_{25,10}$

$s_{29,17} = (d_3)_{5,1} = -s_{25,17}$ $s_{29,18} = (d_3)_{5,2} = -s_{25,18}$

$s_{29,19} = (d_3)_{5,3} = -s_{25,18}$ $s_{29,20} = (d_3)_{5,4} = -s_{25,20}$

$s_{29,21} = (d_3)_{5,5} = -s_{25,17}$ $s_{29,22} = (d_3)_{5,6} = -s_{25,18}$

$s_{29,23} = (d_3)_{5,7} = -s_{25,18}$ $s_{29,24} = (d_3)_{5,8} = -s_{25,20}$

$s_{29,25} = (e_1)_{5,1} = s_{25,29}$ $s_{29,26} = (e_1)_{5,2} = s_{25,30}$

$s_{29,27} = (e_1)_{5,3} = s_{25,31}$ $s_{29,28} = (e_1)_{5,4} = s_{25,32}$

$s_{29,29} = (e_1)_{5,5} = s_{25,25}$ $s_{29,30} = (e_1)_{5,6} = s_{25,26}$

$s_{29,31} = (e_1)_{5,7} = s_{25,27}$ $s_{29,32} = (e_1)_{5,8} = s_{25,28}$

$s_{30,1} = (d_1)_{6,1} = -s_{25,3}$ $s_{30,2} = (d_1)_{6,2} = -s_{25,3}$

$s_{30,3} = (d_1)_{6,3} = -s_{25,7}$ $s_{30,4} = (d_1)_{6,4} = -s_{25,7}$

$s_{30,5} = (d_1)_{6,5} = -s_{25,1}$ $s_{30,6} = (d_1)_{6,6} = -s_{25,1}$

$s_{30,7} = (d_1)_{6,7} = -s_{25,3}$ $s_{30,8} = (d_1)_{6,8} = -s_{25,3}$

$s_{30,9} = (d_2)_{6,1} = s_{25,14}$ $s_{30,10} = (d_2)_{6,2} = s_{25,10}$

$s_{30,11} = (d_2)_{6,3} = s_{25,14}$ $s_{30,12} = (d_2)_{6,4} = s_{25,10}$

$s_{30,13} = (d_2)_{6,5} = s_{25,10}$ $s_{30,14} = (d_2)_{6,6} = s_{25,9}$

$s_{30,15} = (d_2)_{6,7} = s_{25,10}$ $s_{30,16} = (d_2)_{6,8} = s_{25,9}$

$s_{30,17} = (d_3)_{6,1} = -s_{25,18}$ $s_{30,18} = (d_3)_{6,2} = -s_{25,17}$

$s_{30,19} = (d_3)_{6,3} = -s_{25,20}$ $s_{30,20} = (d_3)_{6,4} = -s_{25,18}$

$s_{30,21} = (d_3)_{6,5} = -s_{25,18}$ $s_{30,22} = (d_3)_{6,6} = -s_{25,17}$

$s_{30,23} = (d_3)_{6,7} = -s_{25,20}$ $s_{30,24} = (d_3)_{6,8} = -s_{25,18}$

$s_{30,25} = (e_1)_{6,1} = s_{25,30}$ $s_{30,26} = (e_1)_{6,2} = s_{25,29}$

$s_{30,27} = (e_1)_{6,3} = s_{25,32}$ $s_{30,28} = (e_1)_{6,4} = s_{25,31}$

$s_{30,29} = (e_1)_{6,5} = s_{25,26}$ $s_{30,30} = (e_1)_{6,6} = s_{25,25}$

$s_{30,31} = (e_1)_{6,7} = s_{25,28}$ $s_{30,32} = (e_1)_{6,8} = s_{25,27}$

$s_{31,1} = (d_1)_{7,1} = s_{25,7}$ $s_{31,2} = (d_1)_{7,2} = s_{25,7}$

$s_{31,3} = (d_1)_{7,3} = s_{25,3}$ $s_{31,4} = (d_1)_{7,4} = s_{25,3}$

$s_{31,5} = (d_1)_{7,5} = s_{25,3}$ $s_{31,6} = (d_1)_{7,6} = s_{25,3}$

$s_{31,7} = (d_1)_{7,7} = s_{25,1}$ $s_{31,8} = (d_1)_{7,8} = s_{25,1}$

$s_{31,9} = (d_2)_{7,1} = -s_{25,10}$ $s_{31,10} = (d_2)_{7,2} = -s_{25,14}$

$s_{31,11} = (d_2)_{7,3} = -s_{25,10}$ $s_{31,12} = (d_2)_{7,4} = -s_{25,14}$

$s_{31,13} = (d_2)_{7,5} = -s_{25,9}$ $s_{31,14} = (d_2)_{7,6} = -s_{25,10}$

$s_{31,15} = (d_2)_{7,7} = -s_{25,9}$ $s_{31,16} = (d_2)_{7,8} = -s_{25,10}$

$s_{31,17} = (d_3)_{7,1} = -s_{25,18}$ $s_{31,18} = (d_3)_{7,2} = -s_{25,20}$

$s_{31,19} = (d_3)_{7,3} = -s_{25,17}$ $s_{31,20} = (d_3)_{7,4} = -s_{25,18}$

$s_{31,21} = (d_3)_{7,5} = -s_{25,18}$ $s_{31,22} = (d_3)_{7,6} = -s_{25,20}$

$s_{31,23} = (d_3)_{7,7} = -s_{25,17}$ $s_{31,24} = (d_3)_{7,8} = -s_{25,18}$

$s_{31,25} = (e_1)_{7,1} = s_{25,31}$ $s_{31,26} = (e_1)_{7,2} = s_{25,32}$

$s_{31,27} = (e_1)_{7,3} = s_{25,29}$ $s_{31,28} = (e_1)_{7,4} = s_{25,30}$

$s_{31,29} = (e_1)_{7,5} = s_{25,27}$ $s_{31,30} = (e_1)_{7,6} = s_{25,28}$

$s_{31,31} = (e_1)_{7,7} = s_{25,25}$

$s_{31,32} = (e_1)_{7,8} = s_{25,26}$

$s_{32,1} = (d_1)_{8,1} = -s_{25,7}$

$s_{32,2} = (d_1)_{8,2} = -s_{25,7}$

$s_{32,3} = (d_1)_{8,3} = -s_{25,3}$

$s_{32,4} = (d_1)_{8,4} = -s_{25,3}$

$s_{32,5} = (d_1)_{8,5} = -s_{25,3}$

$s_{32,6} = (d_1)_{8,6} = -s_{25,3}$

$s_{32,7} = (d_1)_{8,7} = -s_{25,1}$

$s_{32,8} = (d_1)_{8,8} = -s_{25,1}$

$s_{32,9} = (d_2)_{8,1} = -s_{25,14}$

$s_{32,10} = (d_2)_{8,2} = -s_{25,10}$

$s_{32,11} = (d_2)_{8,3} = -s_{25,14}$

$s_{32,12} = (d_2)_{8,4} = -s_{25,10}$

$s_{32,13} = (d_2)_{8,5} = -s_{25,10}$

$s_{32,14} = (d_2)_{8,6} = -s_{25,9}$

$s_{32,15} = (d_2)_{8,7} = -s_{25,10}$

$s_{32,16} = (d_2)_{8,8} = -s_{25,9}$

$s_{32,17} = (d_3)_{8,1} = -s_{25,20}$

$s_{32,18} = (d_3)_{8,2} = -s_{25,18}$

$s_{32,19} = (d_3)_{8,3} = -s_{25,18}$

$s_{32,20} = (d_3)_{8,4} = -s_{25,17}$

$s_{32,21} = (d_3)_{8,5} = -s_{25,20}$

$s_{32,22} = (d_3)_{8,6} = -s_{25,18}$

$s_{32,23} = (d_3)_{8,7} = -s_{25,18}$

$s_{32,24} = (d_3)_{8,8} = -s_{25,17}$

$s_{32,25} = (e_1)_{8,1} = s_{25,32}$

$s_{32,26} = (e_1)_{8,2} = s_{25,31}$

$s_{32,27} = (e_1)_{8,3} = s_{25,30}$

$s_{32,28} = (e_1)_{8,4} = s_{25,29}$

$s_{32,29} = (e_1)_{8,5} = s_{25,28}$

$s_{32,30} = (e_1)_{8,6} = s_{25,27}$

$s_{32,31} = (e_1)_{8,7} = s_{25,26}$

$s_{32,32} = (e_1)_{8,8} = s_{25,25}$

The [F] matrix is a column matrix of the form:

$$[F] = \begin{pmatrix} [F_1] \\ [F_2] \\ [F_3] \\ [0] \end{pmatrix} \qquad (A5.27)$$

it has 32 elements in all. The values of the elements of the [F] matrix are listed below.

$$f_1 = (F_1)_1 = \frac{1}{4}(-Face_1\ \Delta x\Delta y II_y + Face_2\ \Delta x\Delta z II_z)$$

$$f_2 = (F_1)_2 = f_1$$

$$f_3 = (F_1)_3 = \frac{1}{4}(-Face_1\ \Delta x\Delta y II_y - Face_5\ \Delta x\Delta z II_z)$$

$$f_4 = (F_1)_4 = f_3$$

$$f_5 = (F_1)_5 = \frac{1}{4}(Face_6\ \Delta x\Delta y II_y + Face_2\ \Delta x\Delta z II_z)$$

$$f_6 = (F_1)_6 = f_5$$

$$f_7 = (F_1)_7 = \frac{1}{4}(-Face_5\ \Delta x\Delta z II_z + Face_6\ \Delta x\Delta y II_y)$$

$$f_8 = (F_1)_8 = f_7$$

$$f_9 = (F_2)_1 = \frac{1}{4}(Face_1\ \Delta x\Delta y II_x - Face_3\ \Delta y\Delta z II_z)$$

$$f_{10} = (F_2)_2 = \frac{1}{4}(Face_1\ \Delta x\Delta y II_x + Face_4\ \Delta y\Delta z II_y)$$

$$f_{11} = (F_2)_3 = f_9$$

$$f_{12} = (F_2)_4 = f_{10}$$

$$f_{13} = (F_2)_5 = \frac{1}{4}(-Face_3\ \Delta y\Delta z II_z - Face_6\ \Delta x\Delta y II_x)$$

$$f_{14} = (F_2)_6 = \frac{1}{4}(Face_4\ \Delta y\Delta z II_z - Face_6\ \Delta x\Delta y II_x)$$

$$f_{15} = (F_2)_7 = f_{13}$$

$$f_{16} = (F_2)_8 = f_{14}$$

$$f_{17} = (F_3)_1 = \frac{1}{4}(-Face_2\ \Delta x\Delta z II_x + Face_3\ \Delta y\Delta z II_y)$$

$$f_{18} = (F_3)_2 = \frac{1}{4}(-Face_2\ \Delta x\Delta z II_x - Face_4\ \Delta y\Delta z II_y)$$

$$f_{19} = (F_3)_3 = \frac{1}{4}(Face_3\ \Delta y\Delta z II_y Face_5\ \Delta x\Delta z II_x)$$

$$f_{20} = (F_3)_4 = \frac{1}{4}(-Face_4\ \Delta y \Delta z H_y Face_5\ \Delta x \Delta z H_x)$$

$$f_{21} = (F_3)_5 = f_{17}$$

$$f_{22} = (F_3)_6 = f_{18}$$

$$f_{23} = (F_3)_7 = f_{19}$$

$$f_{24} = (F_3)_8 = f_{20}$$

$$f_{25} = 0\ \ \ldots\ \ f_{32} = 0$$

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