Analysis of Box–Girder Bridges By Macro–Element Substructural Method

By

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ABSTRACT

An efficient macro–element substructure method for static analysis of straight box–girder bridge structures is successfully developed. The structure is considered as an assemblage of macro–element substructures. All of these macro–element substructures are identical in geometry. Each substructure is derived by applying the principle of static condensation in solving its equilibrium equations. Stiffness and load matrices of each substructure are assembled by rectangular flat shell elements which are composed of 8 degrees of freedom membrane stress and 12 degrees of freedom plate bending elements.

This special purpose macro–element substructure method provides considerable efficiency in practical designs of straight box–girder bridges. A computer program is developed in which the proposed procedures are implemented, and several numerical examples are presented which compare the proposed method with other numerical methods. Good agreement is found in all these examples.

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NOTATIONS

A	Length of the plate in plate examples
[A]	Relationship matrix in plate bending element between nodal displacements and coefficients in deflection function w.
a	Half length of the small element, that is the shell element
B	Width of the plate in plate examples
[B]	Geometry property matrix of small element
b	Half width of the small element, that is the shell element
d	Thickness of beam
D	$D = \frac{Et^3}{12(1-\nu^2)}$
[D]	Material property matrix of small element
E	Modulus of elasticity.
[F]	Load matrix of small element
h	Height of beam
[K]	Stiffness matrix of small element
[Kb]	Stiffness matrix of plate bending element
[Kp]	Stiffness matrix of membrane stress element
[N]	Shape function of small element
P	Concentrated load
q	Uniformly distributed load
[S]	Membrane stress matrix of small element

t Thickness of small element

$[T]$ Transformation matrix of coordinates

u Displacement in x direction of small element

v Displacement in y direction of small element

w Displacement in z direction (deflection) of small element

θ_x Rotation in x direction of small element

θ_y Rotation in y direction of small element

θ_z Rotation in z direction of small element

ν Poisson's ratio

σ_x Stress in x direction of small element

σ_y Stress in y direction of small element

σ_z Stress in z direction of small element

ε_x Strain in x direction of small element

ε_y Strain in y direction of small element

ε_z Strain in z direction of small element

Chapter 1

INTRODUCTION

1.1 General

In recent years, there has been a general tendency to adopt box-girder construction in highway bridges (Fig. 1.1). In the practical bridge design, how to choose the most suitable type of bridge mainly depends on economic and aesthetic [1] considerations. Because of their favorable torsional stiffness and beautiful appearance, box-girder sections have been widely used as economic and aesthetic solutions for overpasses, undercrossings, separation structures and viaducts in today's modern highway systems.

It is important to develop sound analytical techniques to analyze box-girder bridges. Straight box-girder bridges can be analyzed by using the efficient numerical method - computer program presented in this thesis. Generally speaking, an efficient computer program relies on a good computing strategy and efficient numerical techniques. Based on these principles, a macro-element substructure has been developed to analyze straight box-girder bridges. Since the behavior of box-shaped cross-sections are closely related to other thin-walled cross-sections, this program can also be used to solve other thin-walled cross-section structures, as well as thin plate bending and membrane problems which are two special cases of flat shell structures. Several numerical examples are presented in Chapter 4 which involved different types of structures. The computer program is limited to isotropic materials. Any arbitrary bridge cross-section may be modelled as an assemblage of rectangular flat shell elements (Fig.1.3). A direct stiffness method is used in the analysis.

1.2 General Philosophy of Macro–Element

Straight box-girder bridge structural systems can be regarded as an assemblage of macro-element substructures which are interconnected at their exterior common boundaries. Figure 1.2 shows macro-element substructures I, II and III for single cell and double cell box-girder structures. The substructures in each case have the same geometric properties so that the derivation of stiffness and force matrices is necessary only for one typical macro-element substructure.

Any arbitrary straight box–girder bridge section can be modelled in different ways. Typical modelling of different box–girder sections is shown in Figure 1.3. Each macro–element substructure is assembled by flat shell elements in both directions as shown in Figure 1.4. In Figure 1.4, nodes $1, 2, \ldots,$ and 14 are within the sub–structure and separated from adjacent sub–structures. These nodes are referred to as free nodes or internal nodes which are condensed before assembling the macro–element substructure into structure. Nodes $15, 16, \ldots,$ and 28 are on the boundary of the sub–structure and connected to adjacent sub-structures. These nodes are referred to as boundary nodes or common nodes which will be solved at structural level. After assembling small flat shell elements into a macro–element substructure, all internal degrees of freedom are condensed by applying standard process of static condensation. Thus only degrees of freedom associated with external boundaries in two transversal planes are retained for subsequent resolution in which each substructure is already viewed as a single large element. Unlike the classic finite element method, the macro–element substructure method divide the process of solving equilibrium equations into two stages, and in this way, the advantage of repetition of box–girder bridges is reflected.

Flat shell elements, used to develop macro-elements, are the simplest shell elements which are combinations of plate bending and membrane stress elements. General elastic theory is used in deriving the stiffness matrix of flat shell elements. In other words, straight box–girder structures are modelled as a number of macro–element substructures which are assembled by flat shell elements.

All of the input data files of this computer program are much simpler than those of the conventional finite element programs. This is because: only one of the data file of macro-element substructures is needed to be input and, all others can be generated from this typical substructure data file. This provides an alternative solution to tedious mesh and data preparation work of finite element. However, the same level of accuracy as provided by the corresponding finite element method can be obtained. This is a more efficient and practical way to analyze straight box-girder bridge structures.

1.3 Objective and Scope

The objective of the research is to develop a macro-element substructure with which straight box-girder bridges can be effectively modelled and analyzed. This method of macro-element substructure has been derived from established finite element method and provides a simpler and more effective procedure in the design of box-girder bridges.

The computer program is devised to be as general as possible and capable of treating any kinds of straight box-girder bridges under different load conditions. For the purpose of practical use, full attention is given to computing economy and a relatively sophisticated program is developed in order to achieve this purpose. All the displacements, moments and stresses at required positions can be obtained from this program.

This special purpose computer program takes advantage of the repetition and special structural characteristics of straight box-girder section structures. Applications of the program have been successfully made in static analysis of rectangular plate bending, folded plate, and box-girder bridge structures.

1.4 Review of Previous Research

A continuous research program on box-girder bridges has been conducted at the University of Ottawa, including studies on the structural behavior of box-girder bridges. The results of this research program have been presented in detail in a number of research papers. Two published papers are "finite strip analysis of continuous structures" [35] and "analysis of continuous, haunched box-girder bridge by finite strips" [36]. Many analytical models and methods have been developed for linear elastic analysis of box-girder bridges. Typical numerical solution methods are finite strip, finite element, boundary element, etc.

The review of the current research is based on the analysis of box-girder bridges by using the method of macro-element substructure which is developed by applying static condensation principle. Thus reviews of development of static condensation principle and research of box-girder bridges are briefly presented in the following paragraphs.

Gutkowski [9, 10] and Wah [12] have demonstrated the macro–approach for the analysis of plate system. Gutkowski's Finite Panel Method Approach [9] is for the analysis of continuous rectangular, isotropic plate system. Wah[12] partitions a quadrilateral plate into two triangles which are joined by writing continuity (compatibility) equations. His solution and approach for solving elastic quadrilateral plates by a combination of analytical procedures are based on least squares.

GangaRao [13, 14] has developed a macro–approach to model beams and plates. A macro flexibility approach for studying plate behavior was introduced by Parsons, GangaRao and Peterson [11]. Deflected shapes of macro–elements of rectangular shapes have been obtained by a shape function that satisfies all four boundary conditions and the bi–harmonic equation. The convergence of the macro–approach has been verified.

A finite element scheme for static and free vibration analysis of box–girder bridges with orthogonal boundaries and arbitrary combinations of straight and horizontally curved sections is presented by Fam and Turkstra [15]. In that paper, a variety of special purpose elements for different types of box beams is developed to suit the behavior characteristics of thin box sections.

Zhang and Lyons [16] applied thin–walled box beam element to the analysis of multi–box bridges in practical design. By comparing with other numerical methods, this method provides a versatile and accurate analytical tool for the practical design of box-beam bridges.

A summary of the development and application of the finite strip method in the field of bridge engineering was presented by Loo and Cusens [22]. They also developed a flexibility approach to extend the finite strip method to the analysis of indeterminate bridge structures. Murray [21] also did the elastic analysis of box-girders, stiffened plates and other folded plate structures by using finite strip method and stiffness method.

In 1979, macro-element approach was applied to curved box-girder bridge analysis by Jirousek, Bouberguig and Saygun. [8]. Their paper entitled "A macro-element analysis of prestressed curved box-girder bridges" was published in 1988. An efficient macro-element formulation for static analysis of curved box-girder bridges has been developed for a variety of cross-sections. A substructure oriented finite element program of considerable efficiency has been developed for practical applications.

Scordelis, Chan, Ketchum and Walt [7] presented three new computer programs which incorporate automatic prestressing and several other new features into three existing computer programs for the analysis of box-girder bridges. These computer programs make it possible to perform detailed linear elastic analysis of a wide range of prestressed concrete box-girder bridges in an expeditious and accurate manner with a minimum of simplifications and assumptions.

Based on an extended Vlasov's thin-walled beam theory, Razaqpur and Li [37] developed a box-girder finite element which includes extension, biaxial bending, torsion, torsional warping, distortion, distortional warping and shear lag effects for multi-branch multi-cell box girder structures. The exact shape functions are used to eliminate the need for dividing the box into many elements in the longitudinal direction. Favorable results are obtained.

There is the necessity to develop an appropriate computer program for the analysis of straight box-girder bridges by taking advantage of its structural characteristics. This stems from the fact that previous works do not specifically address this necessity.

There are two major advantages in this approach compared to the classical finite element method. Data preparation work is greatly reduced because the element mesh and data preparation is required only for one typical macro-element substructure. The computation efficiency of the solution will also be greatly enhanced. The stiffness matrix of a typical macro-element substructure is obtained by using the static condensation principle. Once the typical macro-element substructure is derived, the overall stiffness matrix of the structure can be assembled by using the same stiffness matrix of a typical macro-element substructure. Therefore, the computer memory and C.P.U. time are greatly reduced in the solution.

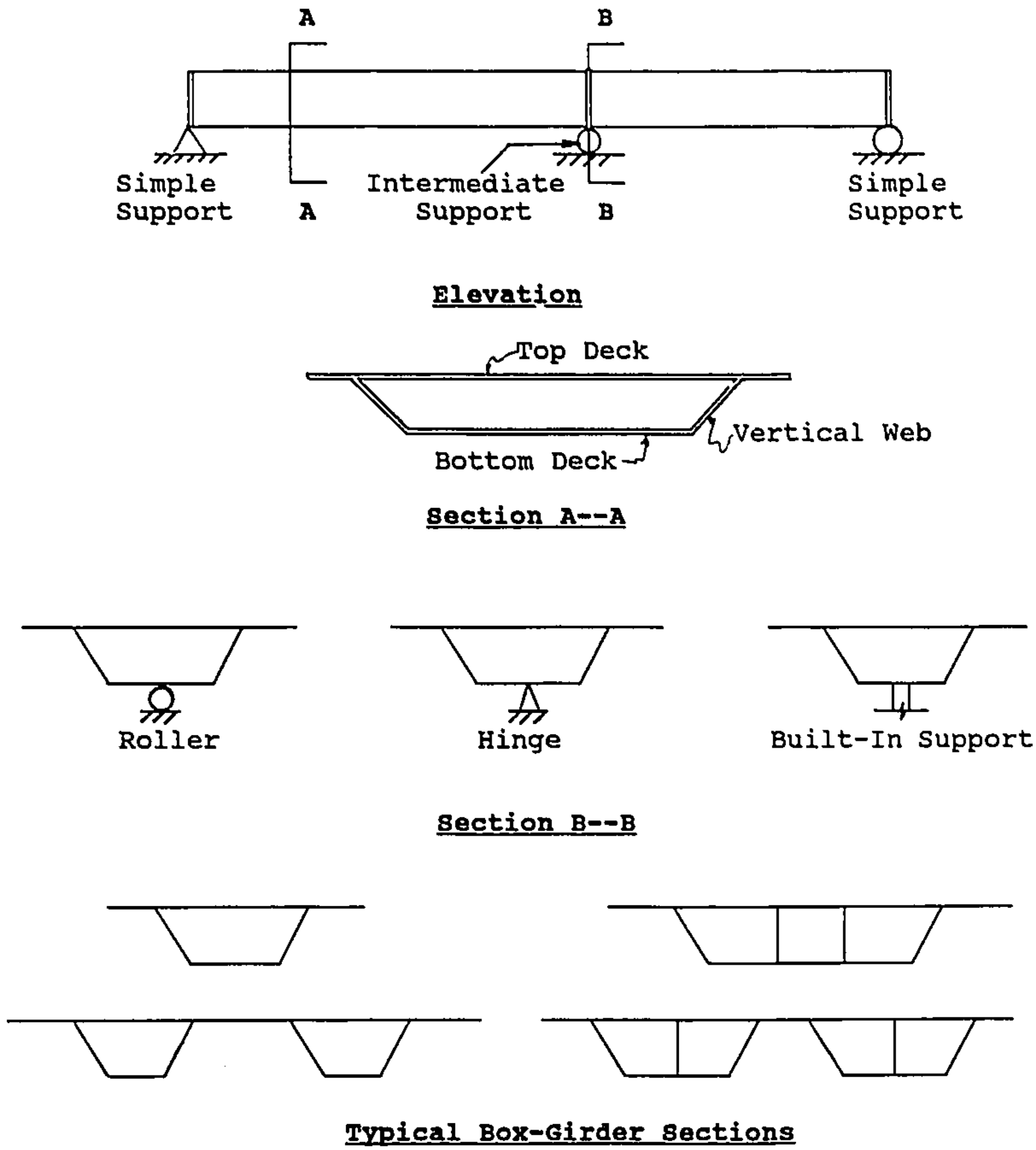
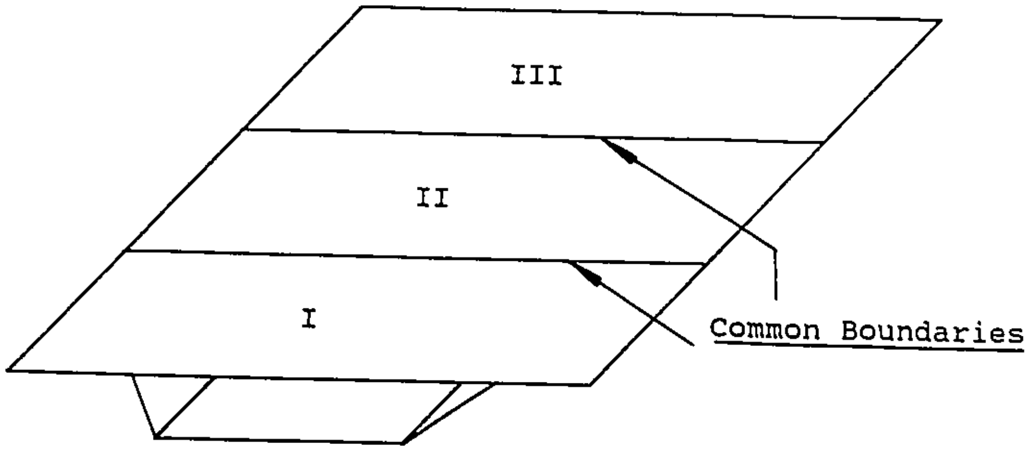
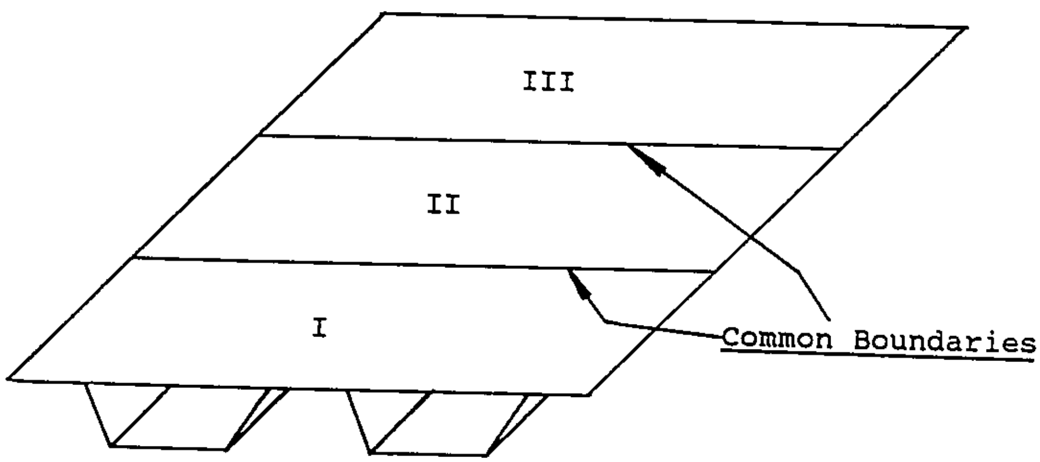


Figure 1.1: Box–girder/Folded plate structure



(a) single box-girder



(b) twin box-girder

Figure 1.2: Substructure in box girder bridge

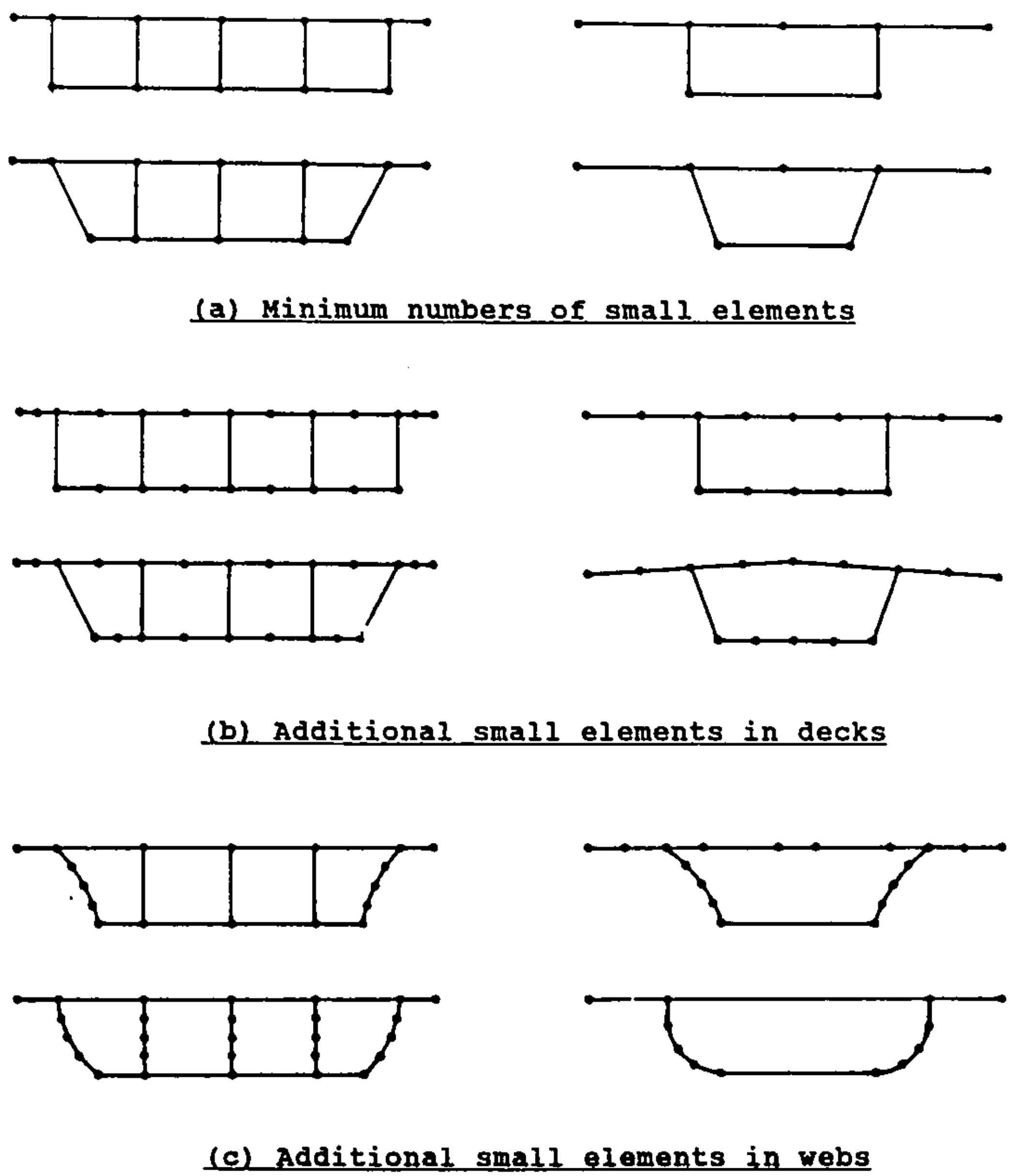


Figure 1.3: Modelling of different cross section in box-girder bridge

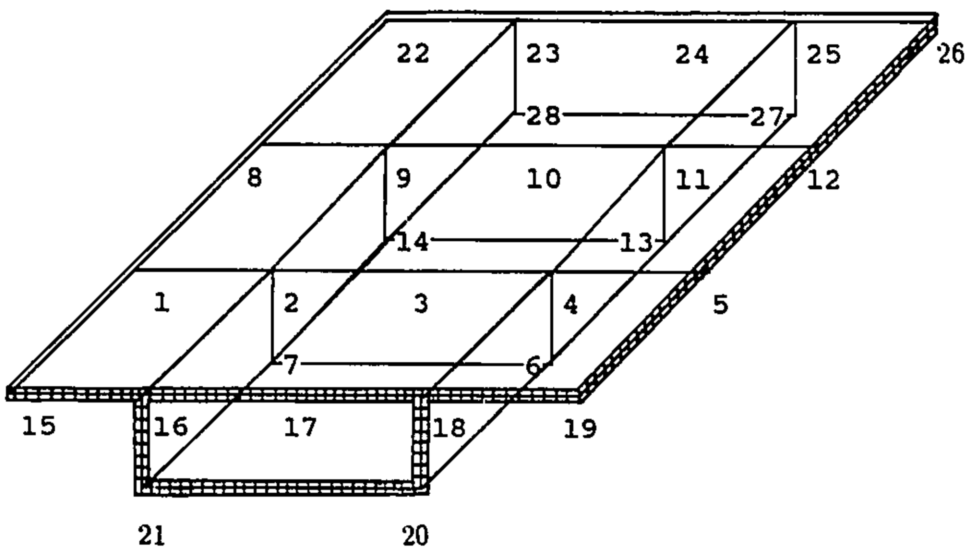


Figure 1.4: Macro–element substructure

Chapter 2

THEORY OF FLAT–SHELL ELEMENT

2.1 Introduction

The macro–element used in this investigation is made up of flat shell elements Macro–elements extend over the full width of the structure and are interconnected at their adjacent boundaries. Rectangular flat shell elements are used in each macro–element. The flat shell element is the combination of a plate bending element and a membrane element. Only the same type of element for both membrane and plate bending can be combined. Because of the particular geometric properties of straight box–girder bridges, the rectangular bending element is adopted here and the corresponding rectangular membrane element is developed as well. In the next few sections, membrane and plate bending elements will be described as well as the flat shell element. Once the flat shell element is derived, the macro-element substructure can be easily formed by using static condensation method which will be introduced in the next chapter.

Other than nodal loads, the uniform surface loads and concentrated loads may be applied anywhere on the surface of shell elements.

2.2 Membrane element

2.2.1 Introduction

An eight degrees of freedom rectangular membrane element in Figure 2.1 is similar to the membrane element which was fully described by Yang[2]. The element has length $2a$, width $2b$ and constant thickness t. Two degrees of freedom, displacements u and v in the x and y directions respectively, are assumed at each of the four corners. There are four pairs of nodal forces of Fx and Fy and four pairs of nodal displacements of u and v.

2.2.2 Shape function

The basic assumptions in the derivation of shape functions are as follows:

1. The boundary lines of each element remain straight after deformation.

2. The displacement surface is expressed in a second-order equation.

3. The strain ϵ_x (or ϵ_y) is independent of x (or y) but varies linearly with y (or x). The shearing strain γ_{xy} varies linearly with both x and y.

4. Static equivalent can be used in finding element stress resultants which act at the corners of elements.

Based on the above listed assumptions, displacement function u and v can be assumed in the following simple form:

$$u(x,y) = c_1 + c_2 x + c_3 y + c_4 xy \qquad (2.1)$$

$$v(x,y) = c_5 + c_6 x + c_7 y + c_8 xy \qquad (2.2)$$

where eight constants c can be uniquely determined by using the following eight nodal displacement conditions:

$$u = u_1 \quad and \quad v = v_1 \quad at(0,0)$$

$$u = u_2 \quad and \quad v = v_2 \quad at(0,2b)$$

$$u = u_3 \quad and \quad v = v_3 \quad at(2a,0)$$

$$u = u_4 \quad and \quad v = v_4 \quad at(2a,2b)$$

Substituting the above nodal displacement conditions into displacement functions, the coefficients c_i ($i = 1, ...8$) are obtained. Back substitution of the coefficients into the displacement equations(2.1) and(2.2) gives.

$$\left\{ \begin{array}{c} u(x,y) \\ v(x,y) \end{array} \right\} = \left[\ N \ \right] \{\delta\} \tag{2.3}$$

where [N] is the shape function,

$$[N] = \left[\begin{array}{cccccccc} (1-\frac{x}{2a})(1-\frac{y}{2b}) & 0 & (1-\frac{x}{2a})\frac{y}{2b} & 0 & (1-\frac{y}{2b})\frac{x}{2a} & 0 & \frac{xy}{4ab} & 0 \\ 0 & \frac{x}{2a}(1-\frac{y}{2b}) & 0 & (1-\frac{x}{2a})\frac{y}{2b} & 0 & (1-\frac{y}{2b})\frac{x}{2a} & 0 & \frac{xy}{4ab} \end{array} \right] \tag{2.4}$$

2.2.3 Strain–displacement relations

The strain–displacement relations are:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}$$

Substituting the displacement equation 2.3 into the above strain displacement relations gives,

$$\{\varepsilon\} = [B]\{\delta\} \tag{2.5}$$

where

$$[\mathbf{B}] = \left[\begin{array}{cccccccc} -\frac{1}{2a}(1-\frac{y}{2b}) & 0 & -\frac{y}{4ab} & 0 & \frac{1}{2a}(1-\frac{y}{2b}) & 0 & \frac{y}{4ab} & 0 \\ 0 & -\frac{1}{2b}(1-\frac{x}{2a}) & 0 & \frac{1}{2b}(1-\frac{x}{2a}) & 0 & -\frac{x}{4ab} & 0 & \frac{x}{4ab} \\ -\frac{1}{2b}(1-\frac{x}{2a}) & -\frac{1}{2a}(1-\frac{y}{2b}) & \frac{1}{2b}(1-\frac{x}{2a}) & -\frac{y}{4ab} & -\frac{x}{4ab} & \frac{1}{2a}(1-\frac{y}{2b}) & \frac{x}{4ab} & \frac{y}{4ab} \end{array}\right] \tag{2.6}$$

It is seen that the strains ε_x are constant in the x direction and vary linearly in the y direction. The shear strains γ_{xy} vary linearly with both the x and y coordinates.

2.2.4 Strain–stress relations

The stress–strain relations are:

$$\left\{\begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array}\right\} = \left[\begin{array}{ccc} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{array}\right] \left\{\begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array}\right\} \tag{2.7}$$

In our particular problem, plane stress state should be considered. For isotropic material, we have

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{2.8}$$

Symbolically,

$$\{\sigma\} = [\mathbf{D}]\{\epsilon\} \tag{2.9}$$

where [**D**] is the material property matrix.

2.2.5 Stiffness matrix

Using energy method, stiffness matrix can be obtained as shown below:

$$\{\mathbf{F}\} = [\mathbf{Kp}]\{\delta\} \tag{2.10}$$

where [**Kp**] is stiffness matrix of membrane element and,

$$[\mathbf{Kp}] = t \int\int_{Area} [\mathbf{B}]^T [\mathbf{D}] [\mathbf{B}] \; dxdy \tag{2.11}$$

After integration, the following stiffness matrix for plane stress state is obtained.

$$[\mathbf{Kp}] = \begin{bmatrix} c_1 & & & & & & & \\ c_2 & c_3 & & & & & & \\ c_7 & c_5 & c_1 & & & & & \\ -c_5 & c_8 & -c_2 & c_3 & & symmetry & & \\ c_4 & -c_5 & -\frac{c_1}{2} & c_2 & c_1 & & & \\ c_5 & c_6 & c_2 & -\frac{c_3}{2} & -c_2 & c_3 & & \\ -\frac{c_1}{2} & -c_2 & c_4 & c_5 & c_7 & -c_5 & c_1 & \\ -c_2 & -\frac{c_3}{2} & -c_5 & c_6 & c_5 & c_8 & c_2 & c_3 \end{bmatrix} \quad (2.12)$$

where

$$c_1 = \left(\frac{b}{3a} + \frac{1-\nu}{6}\frac{a}{b}\right)\frac{Et}{1-\nu^2}$$

$$c_2 = \left(\frac{\nu}{4} + \frac{1-\nu}{8}\right)\frac{Et}{1-\nu^2}$$

$$c_3 = \left(\frac{a}{3b} + \frac{1-\nu}{6}\frac{b}{a}\right)\frac{Et}{1-\nu^2}$$

$$c_4 = \left(-\frac{b}{3a} + \frac{1-\nu}{12}\frac{a}{b}\right)\frac{Et}{1-\nu^2}$$

$$c_5 = \left(\frac{\nu}{4} - \frac{1-\nu}{8}\right)\frac{Et}{1-\nu^2}$$

$$c_6 = \left(\frac{a}{6b} - \frac{1-\nu}{6}\frac{b}{a}\right)\frac{Et}{1-\nu^2}$$

$$c_7 = (\frac{b}{6a} - \frac{1-\nu}{6}\frac{a}{b})\frac{Et}{1-\nu^2}$$

$$c_8 = (-\frac{a}{3b} + \frac{1-\nu}{12}\frac{b}{a})\frac{Et}{1-\nu^2} \qquad (2.13)$$

The derivation of the stiffness coefficients k_{ij} $(i, j = 1, ...8)$ can be illustrated by the following examples.

The stiffness coefficient k_{16} which is defined as the amount of force F_{xi} produced by a unit displacement v_k is obtained by substituting the first column in matrix [**B**] of Eq 2.6, and material matrix [**D**] of Eq 2.8 into Eq 2.11 respectively:

$$k_{16} = t\int_0^{2a}\int_0^{2b}\left[\begin{array}{ccc} -(\frac{1}{2a})(1-\frac{y}{2b}) & 0 & -(\frac{1}{2b})(1-\frac{x}{2a}) \end{array}\right]$$

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\left\{\begin{array}{c} 0 \\ -\frac{x}{4ab} \\ \frac{1}{2a}(1-\frac{y}{2b}) \end{array}\right\} dxdy$$

$$= \frac{Et}{1-\nu^2}\int_0^{2a}\int_0^{2b}\left[\begin{array}{ccc} -(\frac{1}{2a})(1-\frac{y}{2b}) & -(\frac{\nu}{2a})(1-\frac{y}{2b}) & -(\frac{1-\nu}{4b})(1-\frac{x}{2a}) \end{array}\right]$$

$$\left\{\begin{array}{c} 0 \\ -\frac{x}{4ab} \\ \frac{1}{2a}(1-\frac{y}{2b}) \end{array}\right\} dxdy$$

$$= \frac{Et}{1-\nu^2}\int_0^{2a}\int_0^{2b}\left[(\frac{\nu x}{8a^2b})(1-\frac{y}{2b}) - (\frac{1-\nu}{8ab})(1-\frac{x}{2a})(1-\frac{y}{2b})\right] dxdy$$

$$= \frac{Et}{1-\nu^2}\left[\frac{\nu}{8a^2b}\frac{4a^2}{2}(2b - \frac{4b^2}{4b}) - \frac{1-\nu}{8ab}(ab)\right]$$

$$= \frac{Et}{1-\nu^2}\left[\frac{\nu}{4} - \frac{1-\nu}{8}\right] = c_5$$

2.2.6 Stress matrix

After computing displacements $u_1, v_1, \ldots, u_4, v_4$, membrane stresses of each element can be obtained by following the steps shown below. From the stress strain relation equations,

$$\begin{aligned} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} &= \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \\ &= [\mathbf{D}]\{\varepsilon\} \\ &= [\mathbf{D}][\mathbf{B}]\{\delta\} \\ &= [\mathbf{S}]\{\delta\} \end{aligned} \quad (2.14)$$

That is,

$$\{\mathbf{F}i\} = [\mathbf{S}i]\{\delta i\} \quad (2.15)$$

Substituting matrices [**D**] and [**B**] into the equation(2.15), the stress matrix [Si] at node i can be obtained.

$$[\mathbf{S}i] = \frac{E}{1-\nu^2} \begin{bmatrix} -\frac{1}{2a}(1-\frac{y}{2b}) & (-\frac{\nu}{2b})(1-\frac{x}{2a}) & -\frac{y}{4ab} & (\frac{\nu}{2b})(1-\frac{x}{2a}) & (\frac{1}{2a})(1-\frac{y}{2b}) & -\frac{\nu x}{4ab} & \frac{y}{4ab} & \frac{\nu x}{4ab} \\ (-\frac{\nu}{2a})(1-\frac{y}{2b}) & (-\frac{1}{2b})(1-\frac{x}{2a}) & (-\frac{\nu y}{4ab}) & (\frac{1}{2b})(1-\frac{x}{2a}) & (\frac{\nu}{2a})(1-\frac{y}{2b}) & -\frac{x}{4ab} & \frac{\nu y}{4ab} & \frac{x}{4ab} \\ (-\frac{1-\nu}{4b})(1-\frac{x}{2a}) & (-\frac{1-\nu}{4a})(1-\frac{y}{2b}) & (\frac{1-\nu}{4b})(1-\frac{x}{2a}) & (-\frac{1-\nu}{8ab})y & (-\frac{1-\nu}{8ab})x & (\frac{1-\nu}{4a})(1-\frac{y}{2b}) & (\frac{1-\nu}{8ab})x & (\frac{1-\nu}{8ab})y \end{bmatrix} \tag{2.16}$$

Substituting the nodal coordinates $(0,0)$, $(0,2b)$, $(2a,0)$, and $(2a,2b)$, the total membrane stress matrix [S] of element can be obtained as,

$$[\mathrm{S}] = \frac{E}{1-\nu^2}\begin{bmatrix} -\frac{1}{2a} & -\frac{\nu}{2b} & 0 & \frac{\nu}{2b} & -\frac{1}{2a} & 0 & 0 & 0 \\ -\frac{\nu}{2a} & -\frac{1}{2b} & 0 & \frac{1}{2b} & -\frac{\nu}{2a} & 0 & 0 & 0 \\ -\frac{1-\nu}{4b} & -\frac{1-\nu}{4a} & \frac{1-\nu}{4b} & 0 & 0 & \frac{1-\nu}{4a} & 0 & 0 \\ 0 & -\frac{\nu}{2b} & -\frac{1}{2a} & \frac{\nu}{2b} & 0 & 0 & \frac{1}{2a} & 0 \\ 0 & -\frac{1}{2b} & -\frac{\nu}{2a} & \frac{1}{2b} & 0 & 0 & \frac{\nu}{2a} & 0 \\ -\frac{1-\nu}{4b} & 0 & \frac{1-\nu}{4b} & -\frac{1-\nu}{4a} & 0 & 0 & 0 & \frac{1-\nu}{4a} \\ -\frac{1}{2a} & 0 & 0 & 0 & \frac{1}{2a} & \frac{\nu}{2b} & 0 & \frac{\nu}{2b} \\ -\frac{\nu}{2a} & 0 & 0 & 0 & \frac{\nu}{2a} & \frac{1}{2b} & 0 & \frac{1}{2b} \\ 0 & -\frac{1-\nu}{4a} & 0 & 0 & -\frac{1-\nu}{4b} & \frac{1-\nu}{4a} & \frac{1-\nu}{4b} & 0 \\ 0 & 0 & -\frac{1}{2a} & 0 & 0 & -\frac{\nu}{2b} & \frac{1}{2a} & \frac{1}{2b} \\ 0 & 0 & -\frac{\nu}{2a} & 0 & 0 & -\frac{1}{2b} & \frac{\nu}{2a} & \frac{1}{2b} \\ 0 & 0 & 0 & -\frac{1-\nu}{4a} & -\frac{1-\nu}{4b} & 0 & \frac{1-\nu}{4b} & \frac{1-\nu}{4a} \end{bmatrix} \tag{2.17}$$

2.3 Plate bending element

2.3.1 Introduction

The simplest type of plate bending element, the rectangular non-conforming element, is used here. Mathematical derivations of this element is well documented by O. C. Zienkiewicz [3, 4]. Therefore, only the fundamental steps and results are described in this thesis.

2.3.2 Shape function

The deflection of plate bending element can be entirely described by one quantity - the deflection w of middle plane of the plate. If complete slope continuity is required, it will bring much more difficulty in computation. Based on the continuity of w and partial continuity of rotation, a twelve degrees of freedom non-conforming rectangular plate bending element is taken as in Figure 2.2.

The deflection function $w(x, y)$ is expressed in terms of a twelve-parameter polynomial;

$$\begin{aligned} w(x,y) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 \\ &\quad \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 \alpha_{11} x^3 y + \alpha_{12} xy^3 \end{aligned} \tag{2.18}$$

Similar to the membrane element case, the relation between deflection $w(x, y)$ and nodal displacement $\{\delta\}$ can be expressed as,

$$w(x,y) = [\mathrm{N}]\{\delta\} \tag{2.19}$$

where [N] is the shape function.

2.3.3 Strain–displacement relations

The generalized 'strain' can be defined as:

$$\begin{aligned} \varepsilon_x &= -\frac{\partial^2 w}{\partial x^2} \\ \varepsilon_y &= -\frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= 2\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{2.20}$$

Substituting the displacement equation 2.18 into the above equation, the similar relationships to membrane element between 'strain' and displacement can be obtained as,

$$\{\varepsilon\} = [\mathbf{B}]\{\delta\} \tag{2.21}$$

2.3.4 Stress–strain relations

Similar to the membrane element, stress–strain relations are:

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = \left[\begin{array}{ccc} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{array} \right] \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\} \tag{2.22}$$

For isotropic bending plate,

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = \frac{Et^3}{12(1-\nu^2)} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{array} \right] \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\} \tag{2.23}$$

Symbolically:

$$\{\sigma\} = [\mathbf{D}]\{\epsilon\} \tag{2.24}$$

where $[\mathbf{D}]$ is the material property matrix.

2.3.5 Stiffness matrix

Using energy method, stiffness matrix can be obtained as shown below:

$$\{\mathbf{F}\} = [\mathbf{Kb}]\{\delta\} \tag{2.25}$$

where

$$\{\mathbf{F}\} = \begin{Bmatrix} F_i \\ F_j \\ F_k \\ F_l \end{Bmatrix}$$

and

$$\{\delta\} = \begin{Bmatrix} \delta_i \\ \delta_j \\ \delta_k \\ \delta_l \end{Bmatrix}$$

$[\mathbf{Kb}]$ is stiffness matrix,

$$\begin{aligned} [\mathbf{Kb}] &= \int\int_{Area} [\mathbf{B}]^T [\mathbf{D}] [\mathbf{B}]\, dx dy \\ &= \frac{1}{60ab} [\mathbf{L}] \{d_{11} [\mathbf{K_1}] + d_{22} [\mathbf{K_2}] + \end{aligned}$$

$$d_{12}[\mathrm{K}_3] + d_{33}[\mathrm{K}_4]\}[\mathrm{L}] \qquad (2.26)$$

where $[\mathrm{K}_1]$, $[\mathrm{K}_2]$, $[\mathrm{K}_3]$, $[\mathrm{K}_4]$ and $[\mathrm{L}]$ are the same as shown by Zienkiewicz [3, 4].

2.3.6 Load matrix

Three load types are considered. They are nodal loads acting at element nodes, concentrated loads acting anywhere on the elements and uniformly distributed surface loads of shell element.

1. The load matrix due to uniform surface load q is given by,

$$\left\{\begin{array}{c} F_i \\ F_j \\ F_k \\ F_l \end{array}\right\} = 4qab \left\{\begin{array}{c} \frac{1}{4} \\ -\frac{b}{12} \\ \frac{a}{12} \\ \frac{1}{4} \\ \frac{b}{12} \\ \frac{a}{12} \\ \frac{1}{4} \\ -\frac{b}{12} \\ -\frac{a}{12} \\ \frac{1}{4} \\ \frac{b}{12} \\ -\frac{a}{12} \end{array}\right\}$$

where

$$\{\mathbf{F_i}\} = \begin{Bmatrix} F_{wi} \\ F_{\theta_{xi}} \\ F_{\theta_{yi}} \end{Bmatrix}$$

2. The load matrix due to a concentrated load p acting at the point (x_0, y_0) of the local coordinate system of the element can be derived from the following equation.

$$\begin{aligned} \{\mathbf{F}\} &= p\{[\mathbf{A}]^{-1}\}^T \int\int_{Area} [\mathbf{N}(x_0, y_0)]^T \, dx dy \\ &= p\{[\mathbf{A}]^{-1}\}^T \begin{Bmatrix} 1 \\ x_0 \\ y_0 \\ {x_0}^2 \\ x_0 y_0 \\ {y_0}^2 \\ {x_0}^3 \\ {x_0}^2 y_0 \\ x_0 {y_0}^2 \\ {y_0}^3 \\ {x_0}^3 y_0 \\ x_0 {y_0}^3 \end{Bmatrix} \end{aligned} \tag{2.27}$$

where matrix [**A**] relates the coefficients α_i ($i = 1, ..., 12$) in the displacements function to the nodal displacements, that is:

$$\{\delta\} = [\mathbf{A}]\{\alpha\} \tag{2.28}$$

and

$$
[\mathrm{A}] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2b & 0 & 0 & 4b^2 & 0 & 0 & 0 & 8b^3 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -4b & 0 & 0 & 0 & -12b^2 & 0 & 0 \\
0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 4b^2 & 0 & 0 & 8b^3 \\
1 & 2a & 0 & 4a^2 & 0 & 0 & 8a^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -2a & 0 & 0 & -4a^2 & 0 & 0 & -8a^3 & 0 \\
0 & 1 & 0 & 4a & 0 & 0 & 12a^2 & 0 & 0 & 0 & 0 & 0 \\
1 & 2a & 2b & 4a^2 & 4ab & 4b^2 & 8a^3 & 8a^2b & 8ab^2 & 8b^3 & 16a^3b & 16ab^3 \\
0 & 0 & -1 & 0 & -2a & -4b & 0 & -4a^2 & -8ab & -12b^2 & -8a^3 & -24ab^2 \\
0 & 1 & 0 & 4a & 2b & 0 & 12a^2 & 8ab & 4b^2 & 0 & -24ba^2 & 8b^3
\end{bmatrix} \tag{2.29}
$$

2.4 Flat shell element

2.4.1 Introduction

Generally speaking, the shell element will be subjected to both bending and membrane forces. The flat shell element, the simplest but the most efficient shell element, is obtained by combining membrane and plate bending elements.

2.4.2 Stiffness of flat shell element in local coordinates

Consider a flat shell element shown in Figure 2.3. Its properties are easily obtained by combining the corresponding membrane element and plate bending element [3, 4, 5, 6]. The displacements prescribed for membrane forces do not affect bending deformation and vice versa. The rotation θ_z does not enter as parameter into definition of deformation in either mode. Because of the assembly of element, rotation θ_z needs to be taken into account and associated with fictitious couple M_z.

The stiffness matrix of flat shell element at each node can be made up from the following submatrices:

$$[Kf] = \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{[Kp]}} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & 0 \\ 0 & 0 & & [Kb] & & 0 \\ 0 & 0 & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (2.30)$$

where $[Kp]$ and $[Kb]$ are the corresponding stiffness matrices for membrane(2×2) and bending(3×3) respectively. The following tables show how the above equations work.

Table a: membrane stiffness $[Kp]$ (8×8, equation 2.12)

$Kp(1,1)$	$Kp(1,2)$	$Kp(1,3)$	$Kp(1,4)$
$Kp(2,1)$	$Kp(2,2)$	$Kp(2,3)$	$Kp(2,4)$
$Kp(3,1)$	$Kp(3,2)$	$Kp(3,3)$	$Kp(3,4)$
$Kp(4,1)$	$Kp(4,2)$	$Kp(4,3)$	$Kp(4,4)$

where

$$Kp(i,j) = \begin{bmatrix} k(1,1) & k(1,2) \\ k(2,1) & k(2,2) \end{bmatrix} \tag{2.31}$$

Table b: plate bending stiffness $[Kb]$ (12×12, equation 2.26)

$Kb(1,1)$	$Kb(1,2)$	$Kb(1,3)$	$Kb(1,4)$
$Kb(2,1)$	$Kb(2,2)$	$Kb(2,3)$	$Kb(2,4)$
$Kb(3,1)$	$Kb(3,2)$	$Kb(3,3)$	$Kb(3,4)$
$Kb(4,1)$	$Kb(4,2)$	$Kb(4,3)$	$Kb(4,4)$

where

$$Kb(i,j) = \begin{bmatrix} k'(1,1) & k'(1,2) & k'(1,3) \\ k'(2,1) & k'(2,2) & k'(2,3) \\ k'(3,1) & k'(3,2) & k'(3,3) \end{bmatrix} \tag{2.32}$$

Table c in next page is a combination of tables a and b, which represents the flat shell element stiffness.

Table c: flat shell stiffness $[Kf]$ (24×24)

$Kp(1,1)$	0	$Kp(1,2)$	0	$Kp(1,3)$	0	$Kp(1,4)$	0
0	$Kb(1,1)$	0	$Kb(1,2)$	0	$Kb(1,3)$	0	$Kb(1,4)$
0	0	0	0	0	0	0	0
$Kp(2,1)$	0	$Kp(2,2)$	0	$Kp(2,3)$	0	$Kp(2,4)$	0
0	$Kb(2,1)$	0	$Kb(2,2)$	0	$Kb(2,3)$	0	$Kb(2,4)$
0	0	0	0	0	0	0	0
$Kp(3,1)$	0	$Kp(3,2)$	0	$Kp(3,3)$	0	$Kp(3,4)$	0
0	$Kb(3,1)$	0	$Kb(3,2)$	0	$Kb(3,3)$	0	$Kb(3,4)$
0	0	0	0	0	0	0	0
$Kp(4,1)$	0	$Kp(4,2)$	0	$Kp(4,3)$	0	$Kp(4,4)$	0
0	$Kb(4,1)$	0	$Kb(4,2)$	0	$Kb(4,3)$	0	$Kb(4,4)$
0	0	0	0	0	0	0	0

where

$$\begin{Bmatrix} Kp(i,j) & \\ & Kb(i,j) \end{Bmatrix} = \begin{bmatrix} k(1,1) & k(1,2) & 0 & 0 & 0 & 0 \\ k(2,1) & k(2,2) & 0 & 0 & 0 & 0 \\ 0 & 0 & k'(1,1) & k'(1,1) & k'(1,3) & 0 \\ 0 & 0 & k'(2,1) & k'(2,2) & k'(2,3) & 0 \\ 0 & 0 & k'(3,1) & k'(3,2) & k'(3,3) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.33}$$

2.5 Transformation of coordinates

To assemble individual elements into a macro–element and macro–elements into a structure, it is convenient to establish a global coordinate system.

Flat shell element in local coordinates has already been discussed in the previous section. Displacement components can be transformed from global coordinates xyz to local coordinates $x'y'z'$ (Figure 2.4) by a suitable transformation matrix $[\mathbf{T}]$ as,

$$[\delta'] = [\mathbf{T}]\,[\delta] \tag{2.34}$$

because the same amount of work must be done in the two coordinate systems,

$$[\mathbf{F}]^T [\delta] = [\mathbf{F}']^T [\delta'] \tag{2.35}$$

simplifying ,

$$[\mathbf{F}]^T [\delta] = [\mathbf{F}']^T [\mathbf{T}]\,[\delta] \tag{2.36}$$

or

$$[\mathbf{F}] = [\mathbf{T}]^T \left[\mathbf{F}'\right] \tag{2.37}$$

The transformed "stiffness" can be easily derived as

$$[\mathbf{K}] = [\mathbf{T}]^T [\mathbf{K}f]\,[\mathbf{T}] \tag{2.38}$$

where $[\mathbf{K}f]$ is stiffness matrix in local coordinates as shown in equation

(2.30), and [T] is,

$$[\mathbf{T}] = \begin{bmatrix} \mathbf{L} & 0 & 0 & 0 \\ 0 & \mathbf{L} & 0 & 0 \\ 0 & 0 & \mathbf{L} & 0 \\ 0 & 0 & 0 & \mathbf{L} \end{bmatrix} \tag{2.39}$$

$$[\mathbf{L}] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \tag{2.40}$$

$$[\lambda] = \begin{bmatrix} \lambda_{x'x} & \lambda_{x'y} & \lambda_{x'z} \\ \lambda_{y'x} & \lambda_{y'y} & \lambda_{y'z} \\ \lambda_{z'x} & \lambda_{z'y} & \lambda_{z'z} \end{bmatrix} \tag{2.41}$$

where $\lambda_{x'x}$ = cosine of angle between x and x' axis, etc.

2.5.1 Local direction cosines

In the case of a straight box girder bridge, The global x axis (Figure 2.5) can be always taken as to parallel to the length of the bridge. And each local coordinate is selected in the way that its x' axis is parallel to the global x axis (Figure 2.5). Under this condition, all the relevant direction cosines can be easily obtained from the element $i\ j\ k\ l$ as shown in Figure 2.6. Direction cosines of x' are,

$$\begin{aligned} \lambda_{x'x} &= 1 \\ \lambda_{x'y} &= 0 \\ \lambda_{x'z} &= 0 \end{aligned}$$

Direction cosines of y' are,

$$
\begin{aligned}
\lambda_{y'x} &= 0 \\
\lambda_{y'y} &= \frac{y_j - y_i}{\sqrt{(z_j - z_i)^2 + (y_j - y_i)^2}} \\
\lambda_{y'z} &= \frac{z_j - z_i}{\sqrt{(z_j - z_i)^2 + (y_j - y_i)^2}}
\end{aligned}
$$

Direction cosines of z' are,

$$
\begin{aligned}
\lambda_{z'x} &= 0 \\
\lambda_{z'y} &= -\frac{z_j - z_i}{\sqrt{(z_j - z_i)^2 + (y_j - y_i)^2}} \\
\lambda_{z'z} &= \frac{y_j - y_i}{\sqrt{(z_j - z_i)^2 + (y_j - y_i)^2}}
\end{aligned}
$$

The above geometrical relations can be obtained only by considering the sectional plane passing vertically through ij.

Once the stiffness matrix has been transformed into the global coordinate system, it is possible to assemble the overall stiffness matrix for the whole structure in the conventional manner.

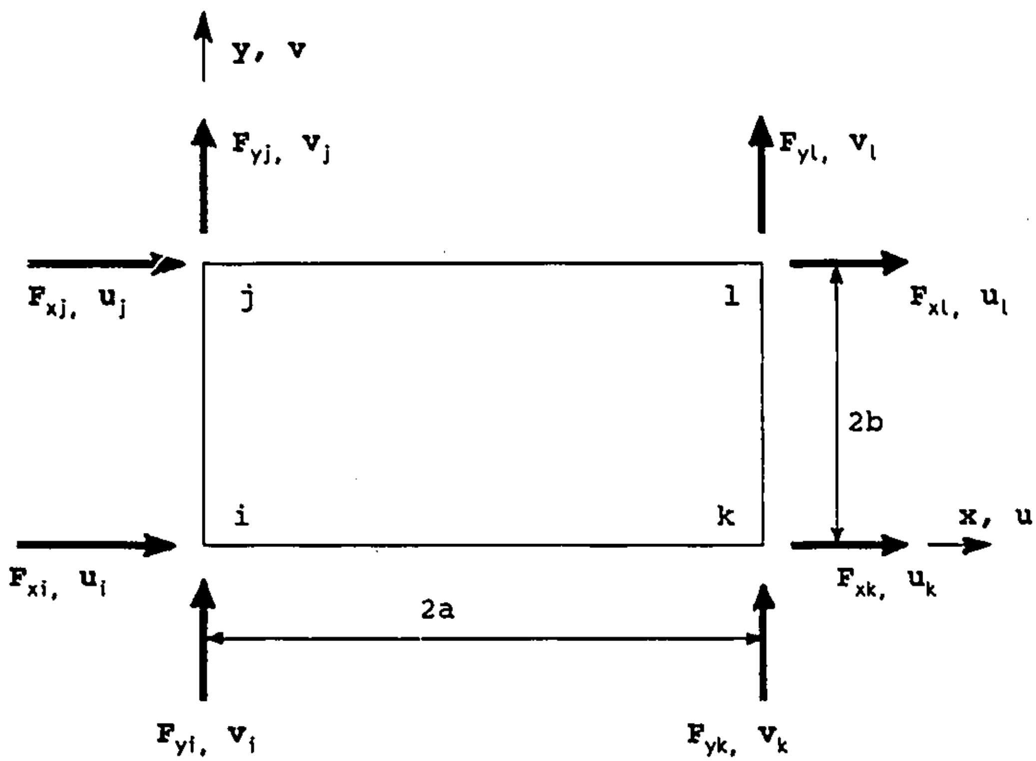
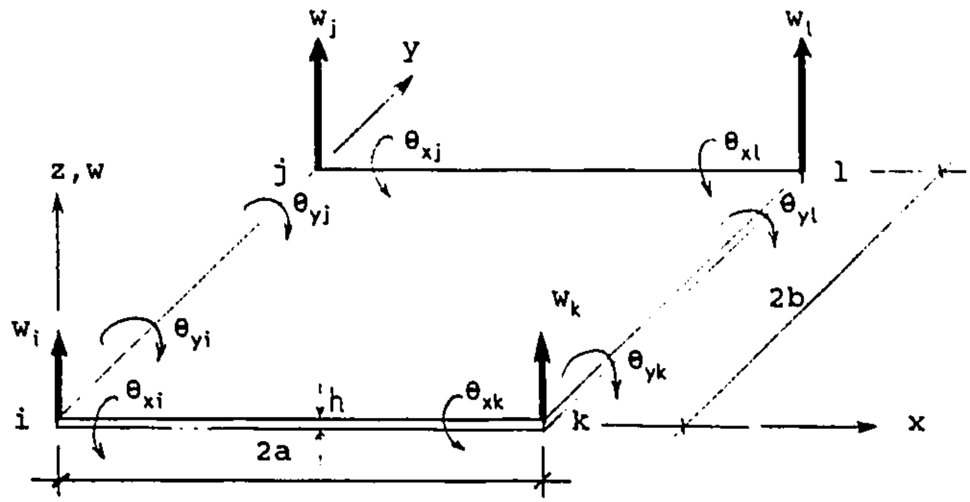


Figure 2.1: Eight degrees of freedom rectangular element



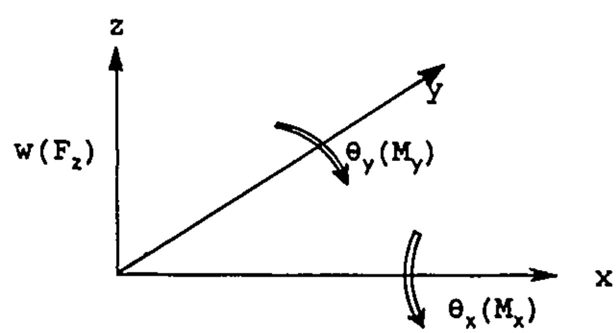
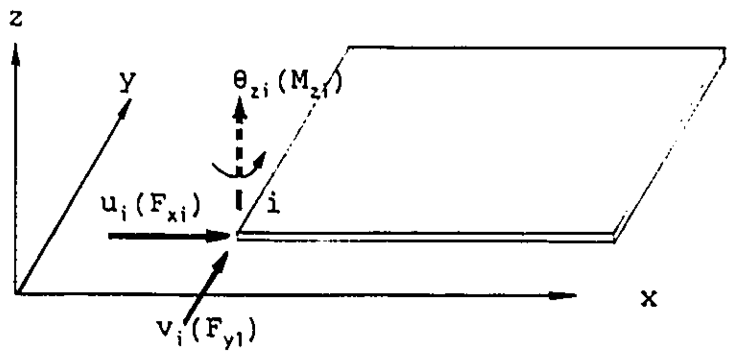
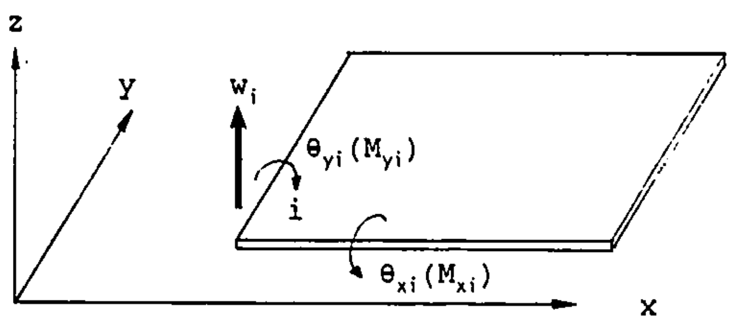


Figure 2.2: Twelve degrees of freedom non–conforming rectangular plate bending element



"membrane forces" and deformation



"bending forces" and deformation

Figure 2.3: A flat shell element subjected to membrane and bending action

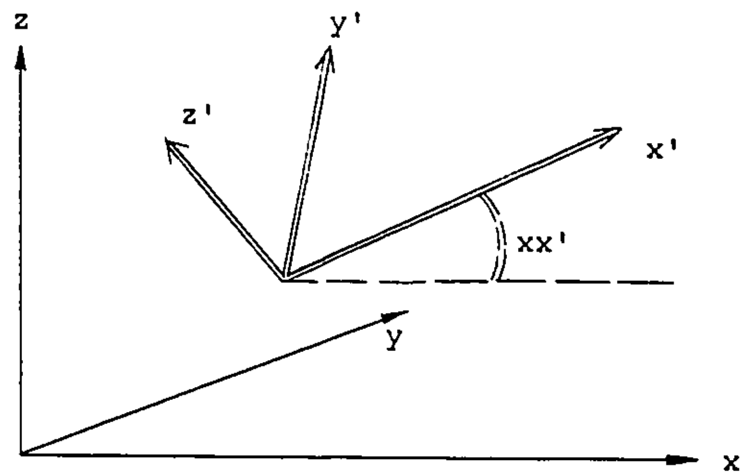


Figure 2.4: Local and global coordinates (xyz denotes common global system and $x'y'z'$ denotes local coordinate system)

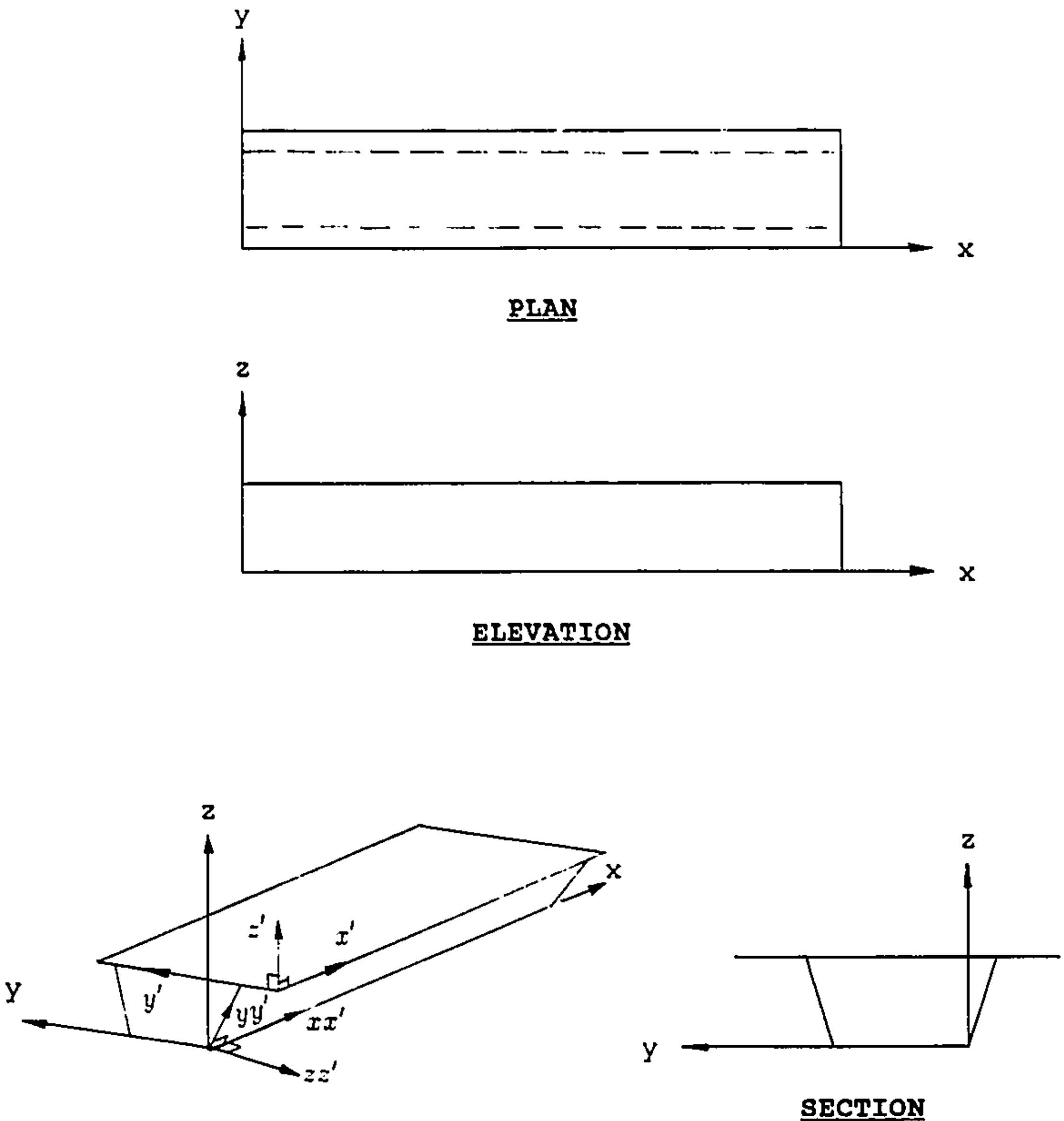


Figure 2.5: Coordinate systems in a stright box–girder bridge (xyz is a global system, $x'y'z'$ and $xx'yy'zz'$ are local systems)

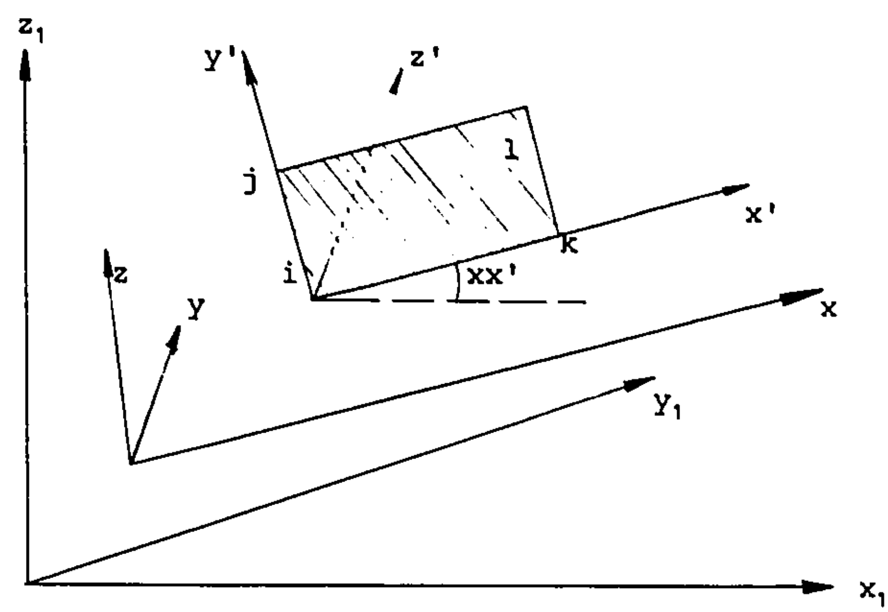


Figure 2.6: Local and global coordinates of a typical rectangular element (Instead of taking an arbitrary global coordinate $x_1y_1z_1$, take a special global coordinate xyz whose x axis is parallel to the length of the element, along which its local axis x' lies.)

Chapter 3

MACRO–ELEMENT

3.1 Introduction

Macro–element substructure is obtained from the assemblage of flat shell elements by applying static condensation method. In this chapter, the principle of static condensation method and the computer program will be discussed. The static condensation method which is used to develop macro–element substructure is an exact process and not an approximation. The same accurate result as the corresponding finite elements used in the formulation of the substructure can be obtained.

3.2 The General Principle of condensation

The macro-approach being presented herein models the substructure with its boundary conditions. The nodes of each substructure are divided into two parts. One of them consists of those nodes which are modeled as free boundaries and in most cases they lie within the internal domain of a substructure. The other one consists of those nodes lying on the boundary of the substructure. The former is referred to as centroidal nodes and the latter, boundary nodes. By forcing deflection and slope compatibility of boundary nodes between adjacent elements (common edge), it is possible to determine deflection, slope and moment at any nodes in the substructure. This allows internal degrees of freedom to be eliminated first and the substructure be formed. In this case, equilibrium equations of the structure are only associated with common boundary conditions between substructures. By solving the equilibrium equations, common boundary nodal displacements are obtained first. The displacements and stresses of internal nodes can be obtained simply by applying boundary displacements. The procedure outlined above is the alternative way to solve the problem and its application is shown below.

The equilibrium equation is of the form:

$$[\mathsf{S}]\{\delta\} = \{\mathbf{P}\} \tag{3.1}$$

Equation 3.1 represents stiffness and force relationship of a substructure. So the right side represents loads applied to nodes by adjacent elements.

Alternatively, equation 3.1 can be written as

$$\begin{bmatrix} S_c & S_{cb} \\ S_{bc} & S_b \end{bmatrix} \begin{Bmatrix} \delta_c \\ \delta_b \end{Bmatrix} = \begin{Bmatrix} P_c \\ P_b \end{Bmatrix} \tag{3.2}$$

Where parameters with subscript c represent internal values and those with subscript b, boundary values.

δ, $\mathbf{S}$ and $\mathbf{P}$ matrices can be partitioned in the following forms,

$$\{\delta\} = \begin{Bmatrix} \delta_c \\ \delta_b \end{Bmatrix} \tag{3.3}$$

$$[\mathbf{S}] = \begin{bmatrix} S_c & S_{cb} \\ S_{bc} & S_b \end{bmatrix} \tag{3.4}$$

$$\{\mathbf{P}\} = \begin{Bmatrix} P_c \\ P_b \end{Bmatrix} \tag{3.5}$$

Since the equilibrium equations for internal nodes only contain the displacement measures and have nothing to do with boundary conditions of the structure, the internal displacements can be expressed in terms of the boundary displacements of the macro–element substructure.

Using the upper part of equation 3.2, internal displacements $\{\delta_\mathbf{c}\}$ can be solved for,

$$\{\delta_\mathbf{c}\} = -[\mathbf{S_c}]^{-1}([\mathbf{S_{cb}}]\{\delta_\mathbf{b}\} - \{\mathbf{P_c}\}) \tag{3.6}$$

Substituting $\{\delta_\mathbf{c}\}$ from equation 3.6 into the lower partition of equation 3.2 gives:

$$[\mathbf{S}]\,\{\delta_\mathbf{b}\} = \{\mathbf{P}\} \tag{3.7}$$

in which

$$[S] = -[S_b] - [S_{bc}][S_c]^{-1}[S_{cb}]$$

$$\{P\} = -\{P_b\} - [S_{bc}][S_c]^{-1}\{P_c\}$$

The above procedure outlines the principle of static condensation. The condensed macro-element is now treated like any other element and [S] and $\{P\}$ in equation 3.7 can then be assembled into the structure in the usual way. After solving for $\{\delta_b\}$ from the structural equilibrium equations, $\{\delta_c\}$ can be solved by using equation 3.6. This is a recovery process. $\{\delta_c\}$ is needed for stresses calculation. Equation 3.6 serves as a constraint relation between $\{\delta_c\}$ and $\{\delta_b\}$. The typical macro–element substructure as shown in Figure 1.4 can be treated like a conventional element in finite element method.

When $\{\delta_c\}$ only contains a single degree of freedom, it is obvious that equation 3.7 represents the Gauss elimination method of solving equilibrium equations. The process ends only when the internal degrees of freedom $\{\delta_c\}$ have been eliminated. Elimination of the remaining degrees of freedom $\{\delta_b\}$ is done on the structural level, after all macro–element substructures have been assembled. Thus condensation is simply the first step in solving the structural equations $[S]\{\delta\} = \{P\}$. The same solution vector $\{\delta\}$ would result if all degrees of freedom were carried into $\{\delta\}$. Then there would be more structural equations. However, the effort required to solve them would be no greater if the degrees of freedom in $\{\delta\}$ were analyzed so that all $\{\delta_c\}$ were processed first - but this is accomplished when the $\{\delta_c\}$ are condensed before assembly.

3.3 Macro–element substructure

In order to describe the main features of macro–element substructure modelling technique, the following example is included.

This is a twin–cell prestressed concrete bridge as shown in Figure 3.1. This example demonstrates the usefulness of the macro–element substructure in reducing the data preparation for a three span bridge. Based on different problems, the size of macro-element substructure is different, for example, it can be as large as a whole span of the structure or it can be a small part of a structural span. Let us take span 1 as a substructure. For spans which are structurally identical and equal in length, only one substructure needs to be defined, since the same substructure can be used to generate data files for all identical spans. As indicated in Figure 3.1(a), the model of one typical span can be used for all three identical spans. For a very long multi–span bridge its advantage is obvious.

A typical three dimensional macro–element substructure grid layout for a typical span is shown in Figure 3.1(c).

3.4 Computer program organization

Program organization may be summarized as follows:

1. Input data file.
2. Sum small element load matrix into macro-element substructural load matrix Po.
3. Sum small element stiffness matrix into macro-element substructural stiffness S_e.
4. Condense the internal degrees of freedom for both matrices Po and S_e.
5. Sum macro–element substructural load matrix Po into structural load matrix P.
6. Sum macro–element substructural stiffness matrix S_e. into structural stiffness matrix K_s.
7. Apply boundary conditions of the structure.
8. Solve equilibrium equations of the structure.
9. Compute condensed internal nodal displacements in each substructure.
10. Compute small element stresses.

Alternatively, reference can be made to the main flow charts from figures 3.2 to 3.5.

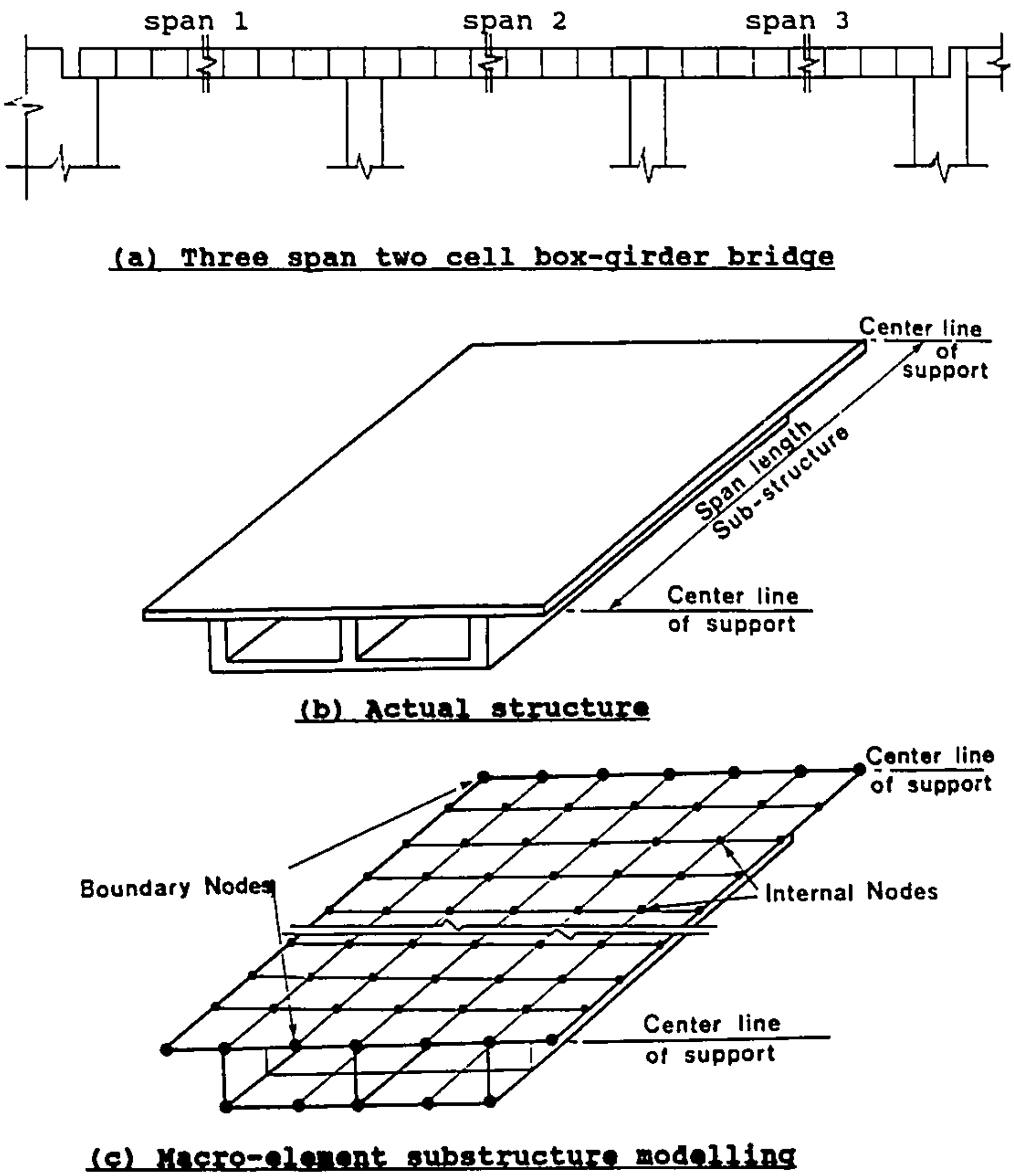


Figure 3.1: Macro–element substructure layout

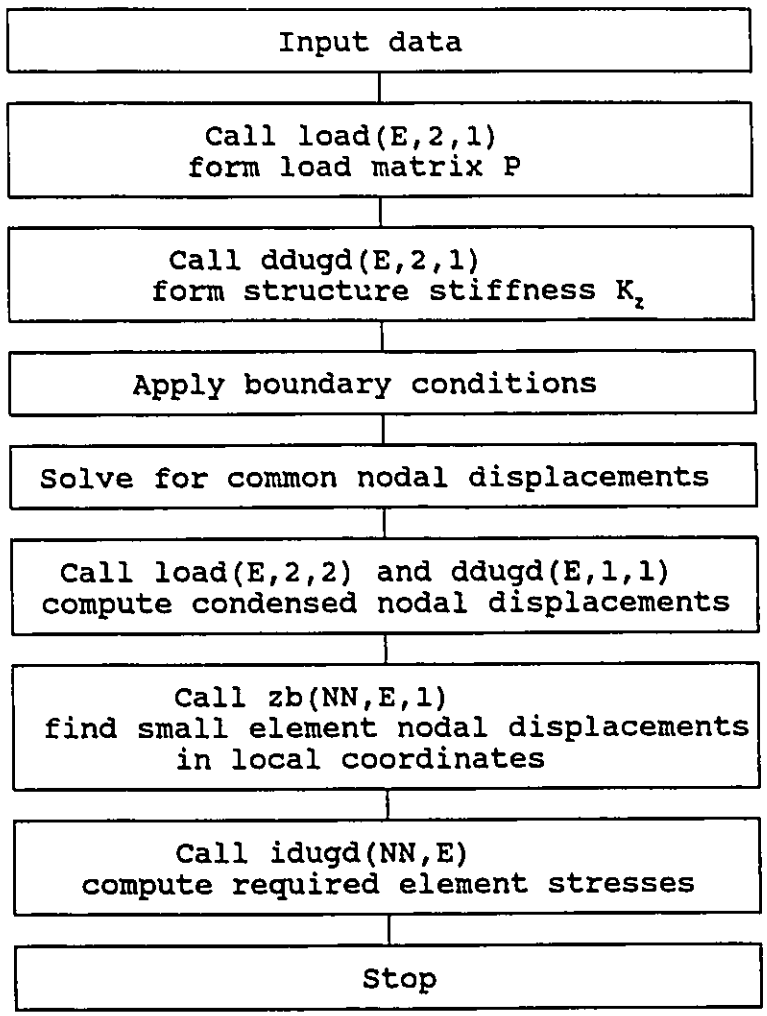


Figure 3.2: Flow chart of main program

Subroutine ddugd(E,LLE,LEN)

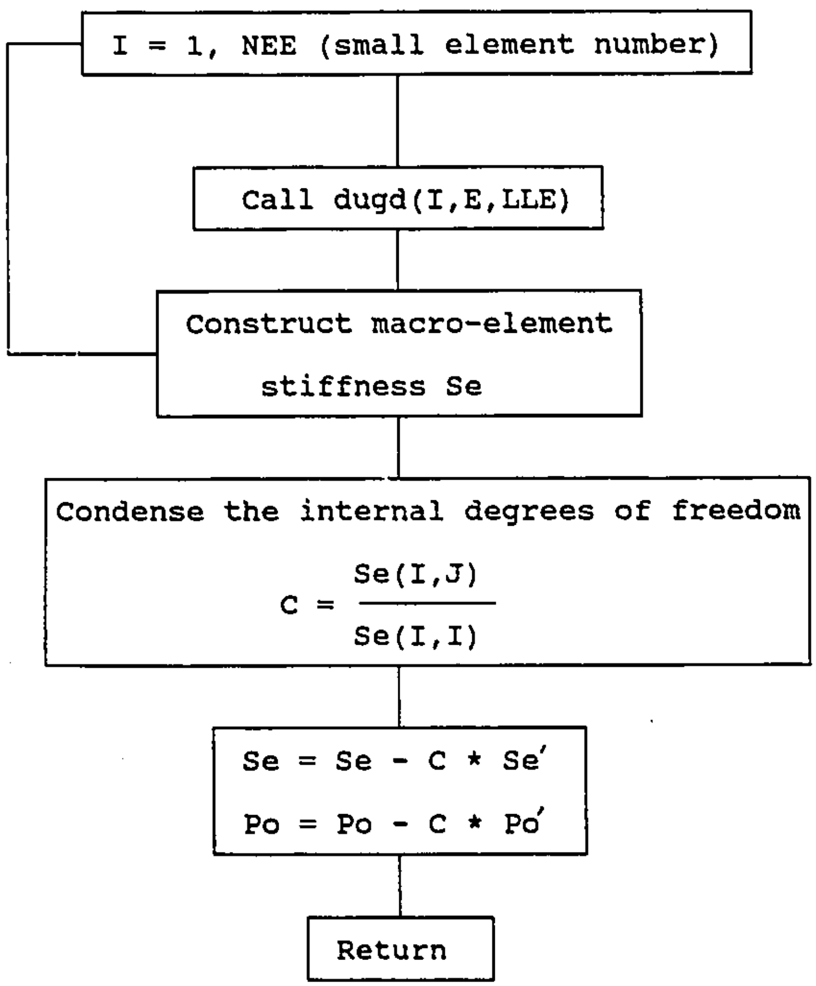


Figure 3.3: Flow chart of subroutine DDUGD

Subroutine load(E,LLE,LEN)

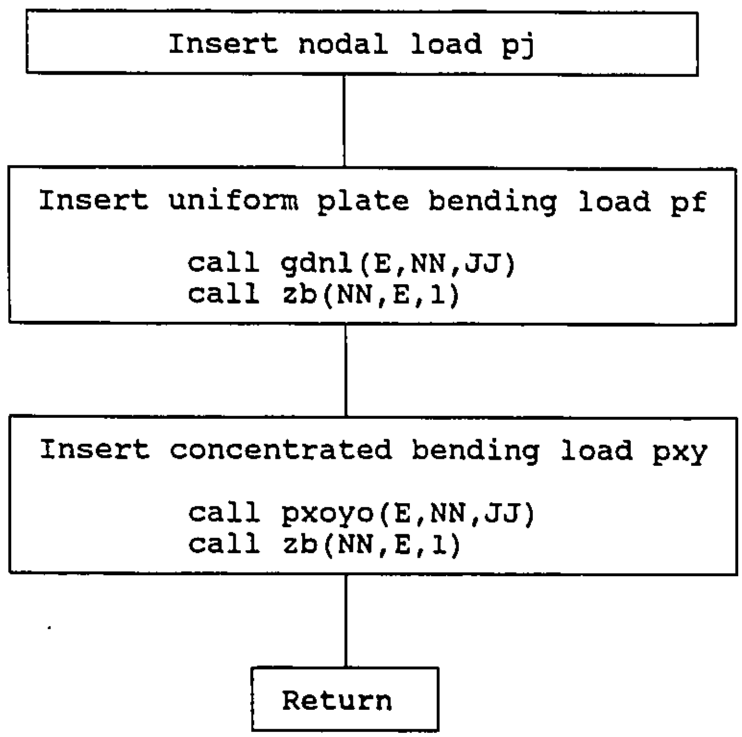


Figure 3.4: Flow chart of subroutine LOAD

Chapter 4

NUMERICAL EXAMPLES

4.1 General

In this chapter several numerical examples are provided and tested to demonstrate the practical application of the concept presented in Chapters 1 and 3. For convenience, this special purpose substructure oriented program will be named MESP (Macro–Element Substructure Program). Unless otherwise stated, all units in these numerical examples are in S. I. units. The MESP in FORTRAN program is given in Appendix A.

4.2 Membrane Stress

The first example is the application of MESP to the in-plane tension stress analysis. The second example involves the analysis of a cantilever beam, using MESP, and then comparing the results with analytical solutions. It is obvious from the examples that the accuracy of results is improved by increasing numbers of elements.

The in-plane tension application is demonstrated by using Yang's square plate [2] with parabolic tension forces at both sides as shown in Figure 4.1. Due to double symmetry, only one-quarter of the plate needs to be analyzed. Two macro-element substructure models are adopted. First, one-quarter of the plate is modelled as one macro-element substructure in which 4 and 16 small elements are used. One-quarter of the plate is then modelled as four macro-element substructures in which 16 small elements are taken. The comparison of the extension with analytical solutions and Yang's results are shown in Table 4.1. The stresses are also compared in Table 4.2. From the comparisons in Table 4.1 and 4.2, the same accurate results as Yang's finite element method are obtained from the macro-element substructural method.

The second example involves a cantilever beam with a unit point load at the end and,

Height	$h = 2.0$ or 3.0
Length	$A = 10h = 20$ or 30
Thickness	$t = 1.0$
Modulus of elasticity	$E = 1000000$
Poisson's Ratio	$\nu = 0.3$

The tip deflection and the maximum bending stress obtained from MESP are compared with the results from beam theory in Table 4.3.

Table 4.4 shows the results of a cantilever beam analyzed by using different numbers of elements. The results show rapid convergency as the number of elements increases.

4.3 Plate bending

The MESP is applied to a clamped square plate with two types of loading; a uniformly distributed and a concentrated load acting at the center of the plate. The parameters of the plate are as follows:

Length	$A = 4.0$
Height	$B = 4.0$
Thickness	$t = 0.05$
Modulus of elasticity	$E = 1000000$
Poisson's ratio	$\nu = 0.3$

With the double symmetrical nature of the plate, only one quarter of the plate is analyzed.

Below are results published by Timoshenko [32] for a clamped plate.

(i) For a unit uniformly distributed load q.

center deflection	$w = -0.00126\frac{A^4}{D}q = -0.0282$
edge moment	$M_e = -0.0513A^2q = -0.8208$
center moment	$M_c = 0.0231A^2q = 0.3696$

(ii) For a concentrated load p

center deflection	$w = -0.0056\frac{A^2}{D}p = -0.00783$
edge moment	$M_e = -0.1257p = -0.1257$

A comparison of the results obtained from MESP with Timoshenko's is shown in Tables 4.5. and 4.6 respectively.

4.4 Thin–walled box section

As a first example, a straight single cell box-girder bridge model with cantilever slab is examined. The model is shown in Figure 4.2. The load considered is an asymmetrical point load of 224 lbf applied at midspan and producing torsional effects.

The symmetry structural model is analyzed by modeling its half span with five macro-element substructures. Each macro-element is formed by 48 small elements which are simple flat shell elements. Figure 4.3 is the deformed shape of the cross section at midspan. Figure 4.4 shows the longitudinal in-plane stresses at midspan. The longitudinal and transverse bending stresses at midspan are shown in Figures 4.5 and 4.6 respectively. Detailed analysis of the problem can be found in Refs [8, 23, 24, 25]. Due to different w· ·s of defining boundary conditions and the accuracy of the different method employed, the results vary considerably in different references.

Figure 4.7 shows the model of example 2 which is a two-span three-cell box-girder bridge with diaphragms provided at all support sections to ensure stability and to reduce stress concentration around the supporting columns. The bridge was previously analyzed by Scordelis [20] and Rockey [33]. The rigid diaphragms at intermediate supports are treated as point restraints. Scordelis studied the bridge by using folded plate theory, finite element and finite segment approach while Rockey used a finite strip approach. In both cases, fairly good results were obtained.

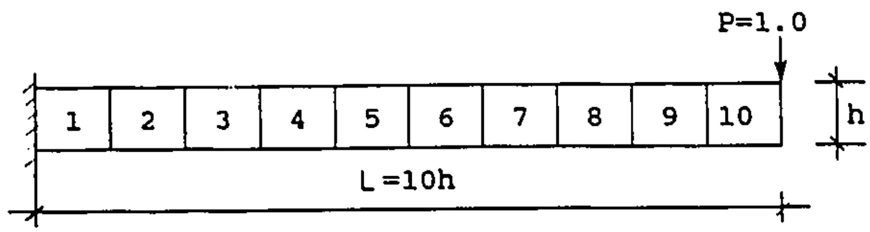
The two span bridge is analyzed by modelling each span with ten macro-element substructures as shown in Figure 4.8. Each macro-element is formed by 32 small elements which are simple flat shell elements. Figures 4.9 and 4.10 show the longitudinal in-plane stresses at loaded and intermediate support sections respectively. Results from MESP compare fairly well with those by Scordelis [20] and Rockey [33]. A slightly large torsional rotation is produced because of different ways of handling rigid diaphragms. This can be improved if using more elements and refined boundary conditions.

Point	Results from Ref [34]	One macro-element				Four macro-elements	
		Small element number in each macro-element					
		4		16		16	
		Yang	MESP	Yang	MESP	Yang	MESP
1	1.4764	1.4769	1.4730	1.4757	1.4760	1.4769	1.479
2						1.4550	1.457
3	1.3895			1.3892	1.3890	1.3899	1.394
4						1.2842	1.286
5	1.1428	1.149	1.149	1.1442	1.1440	1.1425	1.143
6						0.9712	0.9696
7	0.7786			0.7873	0.7873	0.7795	0.7790
8						0.5972	0.5788
9	0.3647	0.4946	0.4966	0.4159	0.4161	0.3862	0.3858

Table 4.1: The comparation of in-plane extension ($\times 10^{-3}$ in)

Point	σ_x			σ_y		
	Results of Ref[34]	Yang	MESP	Results of Ref[34]	Yang	MESP
A	909.6	854.1	854.0	78.7	102.4	102.3
B	456.2	479.3	479.6	18.8	32.9	32.9
C	975.0	960.1	960,0	217.8	224.6	224.6
D	852.6	837.9	838.0	171.6	109.3	177.1
E	609.4	597.3	597.2	91.9	95.5	95.4
F	249.2	271.4	271.6	14.9	22.1	22.0
G	993.5	989.7	989.5	310.2	309.3	309.8
H	962.5	958.5	962.2	292.3	293.6	295.4
I	900.0	896.3	900.1	261.4	262.8	264.2
J	806.9	803.1	803.2	213.8	218.9	218.5
K	683.1	679.2	676.9	157.4	163.5	161.64
L	528.1	525.4	523.1	97.0	101.1	100.1
M	342.6	341.9	432.1	41.9	43.7	43.6
N	126.6	139.9	139.5	5.6	8.1	8.2

Table 4.2: The comparation of in-plane stresses (psi)



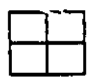
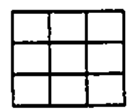
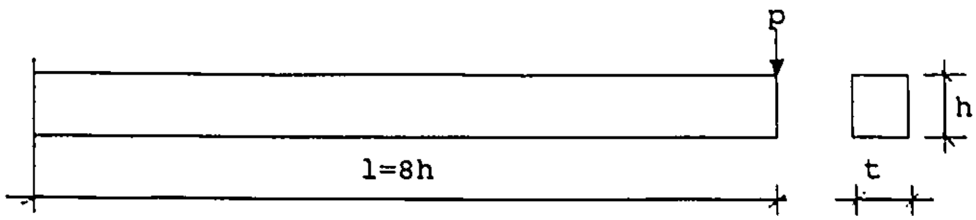
Macro-element	 h=2	 h=3
Tip deflection from beam theory $f=\frac{PL^3}{3EI}$	0.004	0.004
MESP results	0.003581	0.003812
Max. stress from beam theory $\sigma=\frac{60P}{h}$	30	20
MESP results	27.93	20.11

Table 4.3: The comparison of the cantilever beam



E, $\nu=0$

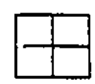
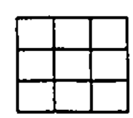
Macro-element numbers	Tip deflection multiplier $\frac{p}{Et}$	Max. bending stress multiplier $\frac{p}{th}$	Macro-element type
2	688.5	14.0	
4	1377.0	30.0	
8	1836.0	41.33	
10	1913.0	43.33	
10	1994.0	45.89	
Beam theory [33]	2066.8	48.0	

Table 4.4: The analysis of the cantilever beam with different element meshes

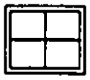
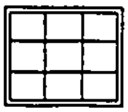
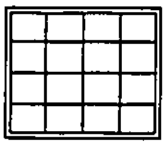
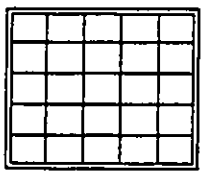
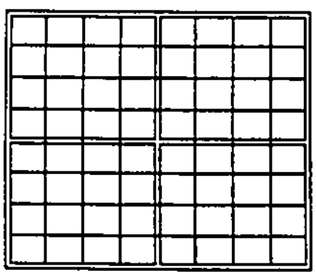
5 element types	Deflection w	Error %	M_e	Error %	M_c	Error %
	-0.03138	11.28	-0.7617	7.2	0.4445	20.27
	-0.0298	5.67	-0.7929	3.4	0.3993	8.4
	-0.02916	3.4	-0.8045	2.0	0.3834	4.1
	-0.02886	2.34	-0.8101	1.3	0.3781	2.3
	-0.02869	1.74	-0.8133	0.9	0.3745	1.3
Results from Timoshenko [32]	w = -0.0282		M_e = -0.8208		M_c = 0.3696	

Table 4.5: The comparation of the plate bending results under uniform load

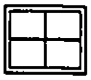
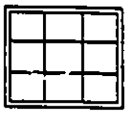
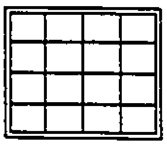
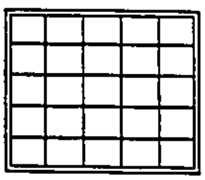
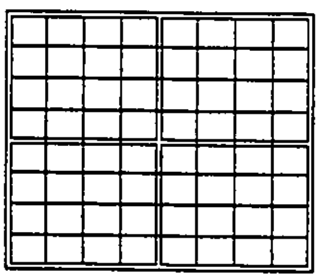
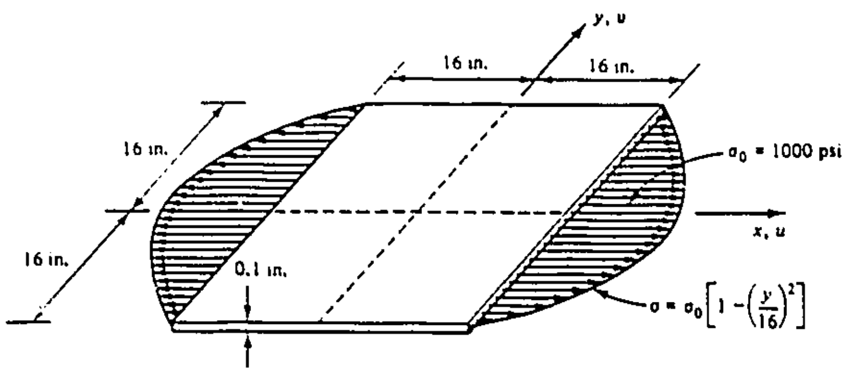
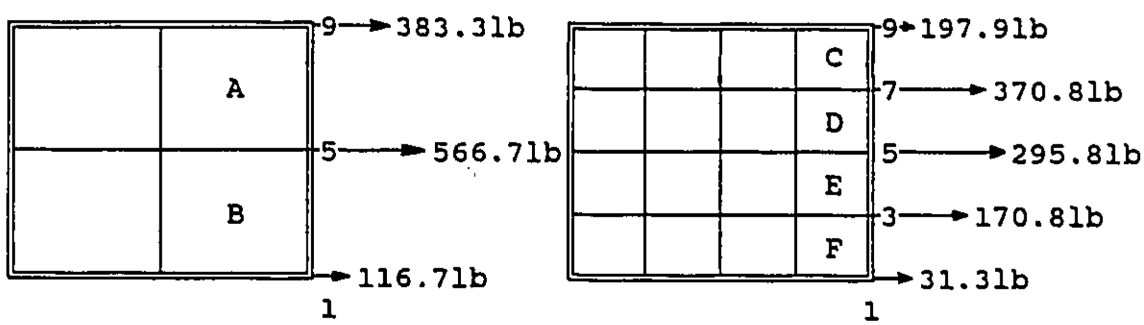
5 element types	Deflection w	Error %	M_e	Error %
	-0.008574	9.5	-0.1178	6.3
	-0.008261	5.5	-0.1214	3.42
	-0.008111	3.6	-0.1233	2.0
	-0.00803	2.6	-0.1214	1.3
	-0.007981	2.0	-0.1245	1.0
Results from Timoshenko [32]	w = -0.00783		M_e = -0.1257	

Table 4.6: The comparation of the plate bending results under concentrated load

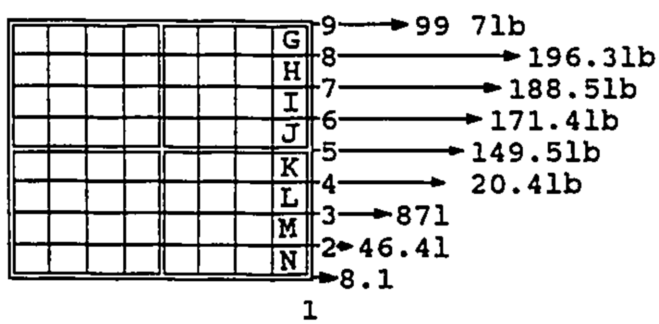


(a) Parabolically loaded square plate



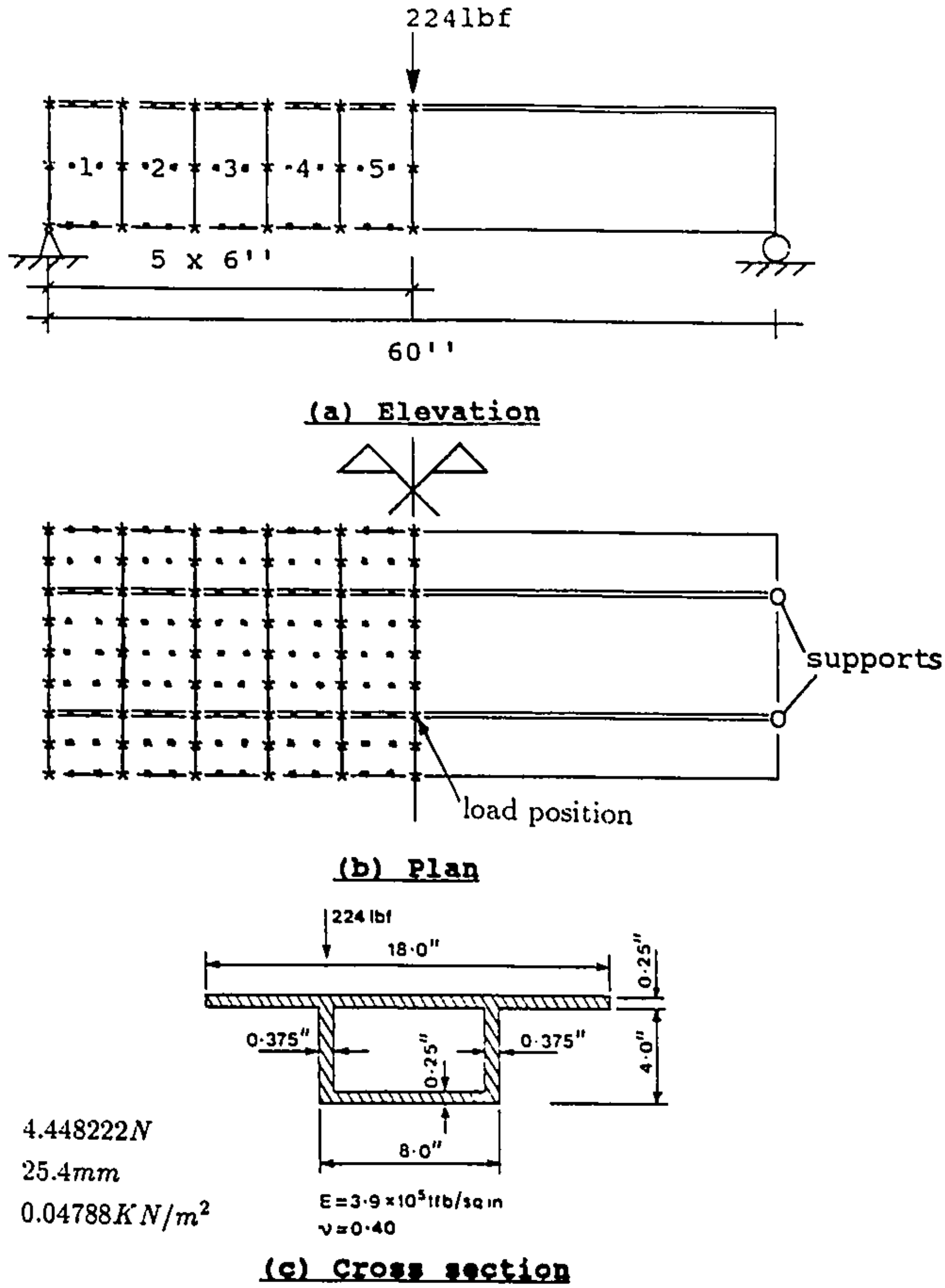
(b) One macro-element model

$$
\begin{aligned}
1lbf &= 4.448222N \\
1in &= 25.4mm \\
1lbf/ft^2 &= 0.04788KN/m^2 \\
1lbf/in^2 &= 6.894757KN/m^2
\end{aligned}
$$



(c) Four macro-element model

Figure 4.1: Analysis of in–plane stresses



$$1lbf = 4.448222N$$
$$1in = 25.4mm$$
$$1lbf/ft^2 = 0.04788KN/m^2$$

* ———— Common nodes with degrees of freedom used in assembly of macro-element

• ———— Internal nodes with degrees of freedom condensed at the macro-element level

Figure 4.2: Macro–element modelling of a straight single cell box–girder bridge model

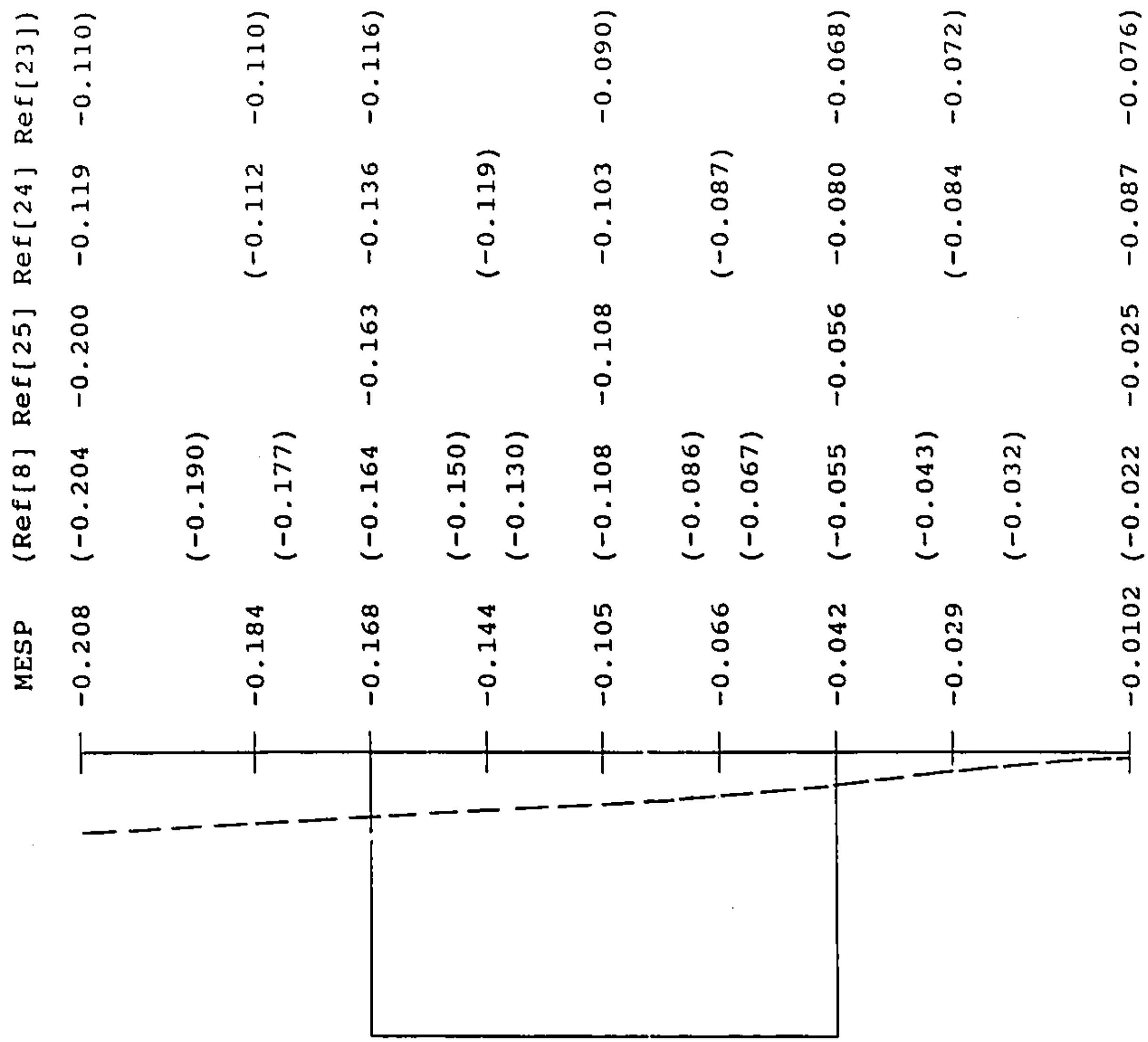


Figure 4.3: Distribution of displacements at center cross section for the box-girder bridge model in figure 4.2 with an eccentric point load $p = 224$ lbf at midspan (in)

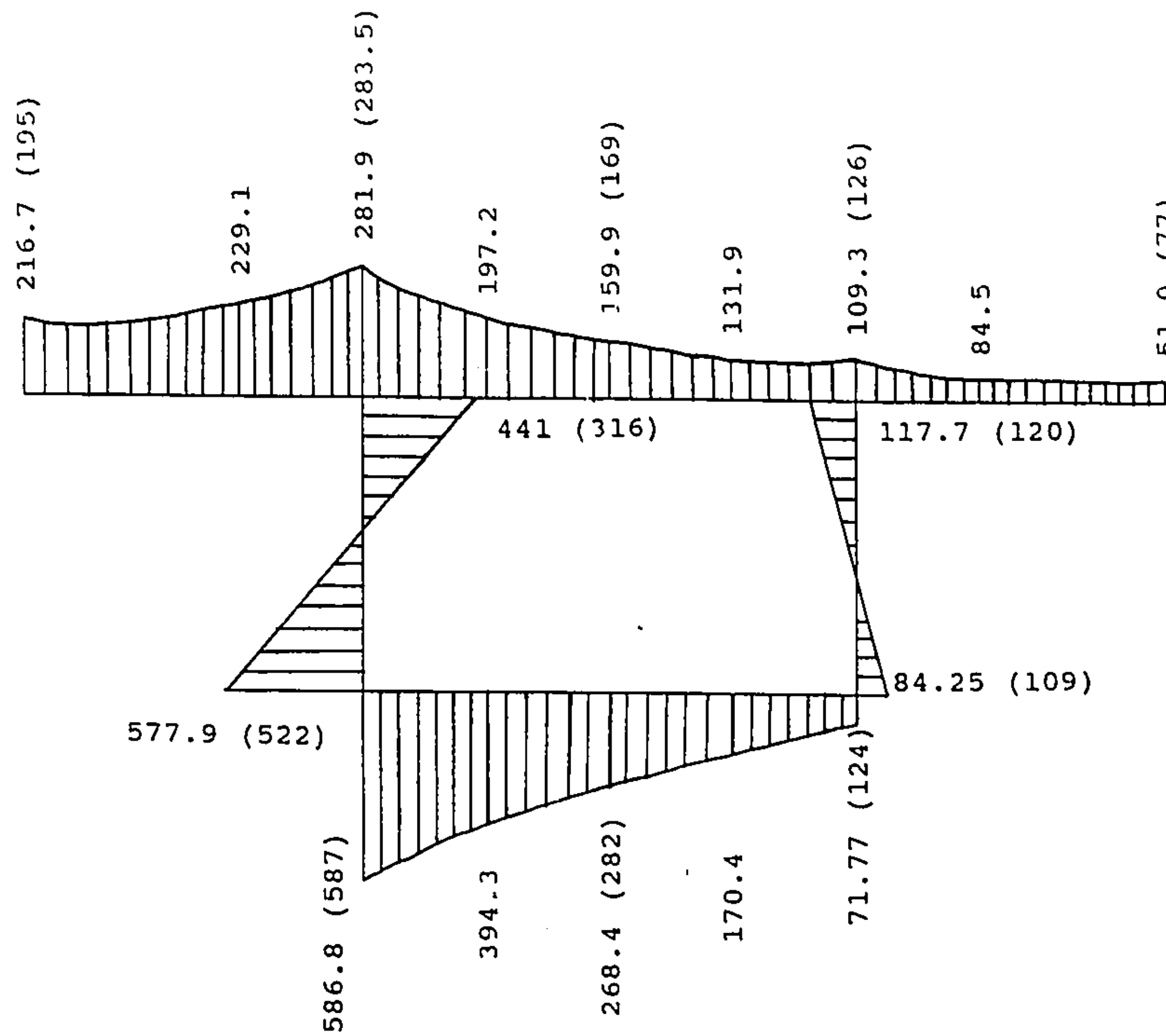
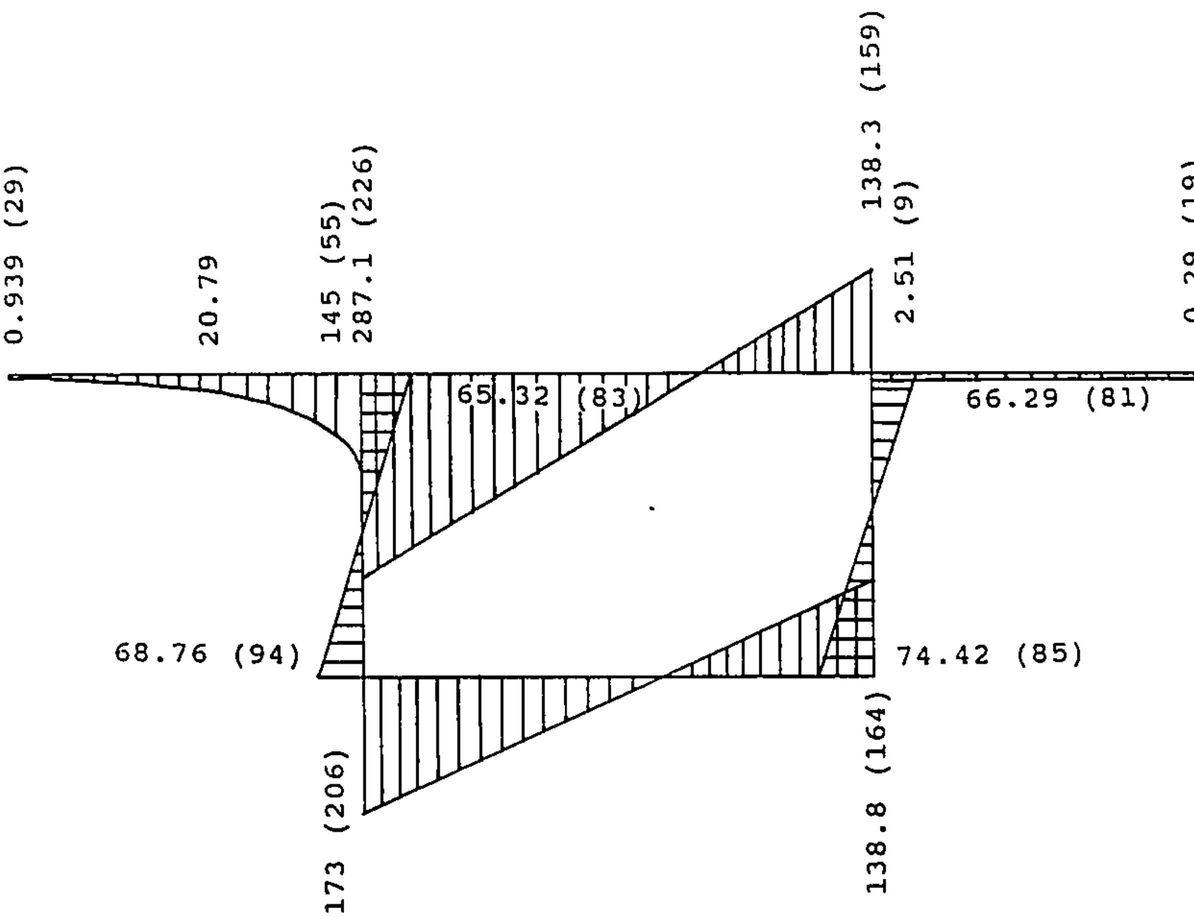


Figure 4.4: Distribution of longitudinal in-plane stresses at center cross section for the box-girder bridge model in figure 4.2 with an eccentric point load $p = 224$ lbf at midspan (lbf/sq in)



Values in ( ) is from Ref [8]

Figure 4.5: Distribution of longitudinal bending stresses at center cross section for the box-girder bridge model in figure 4.2 with an eccentric point load $p = 224$ lbf at midspan (lbf/sq in)

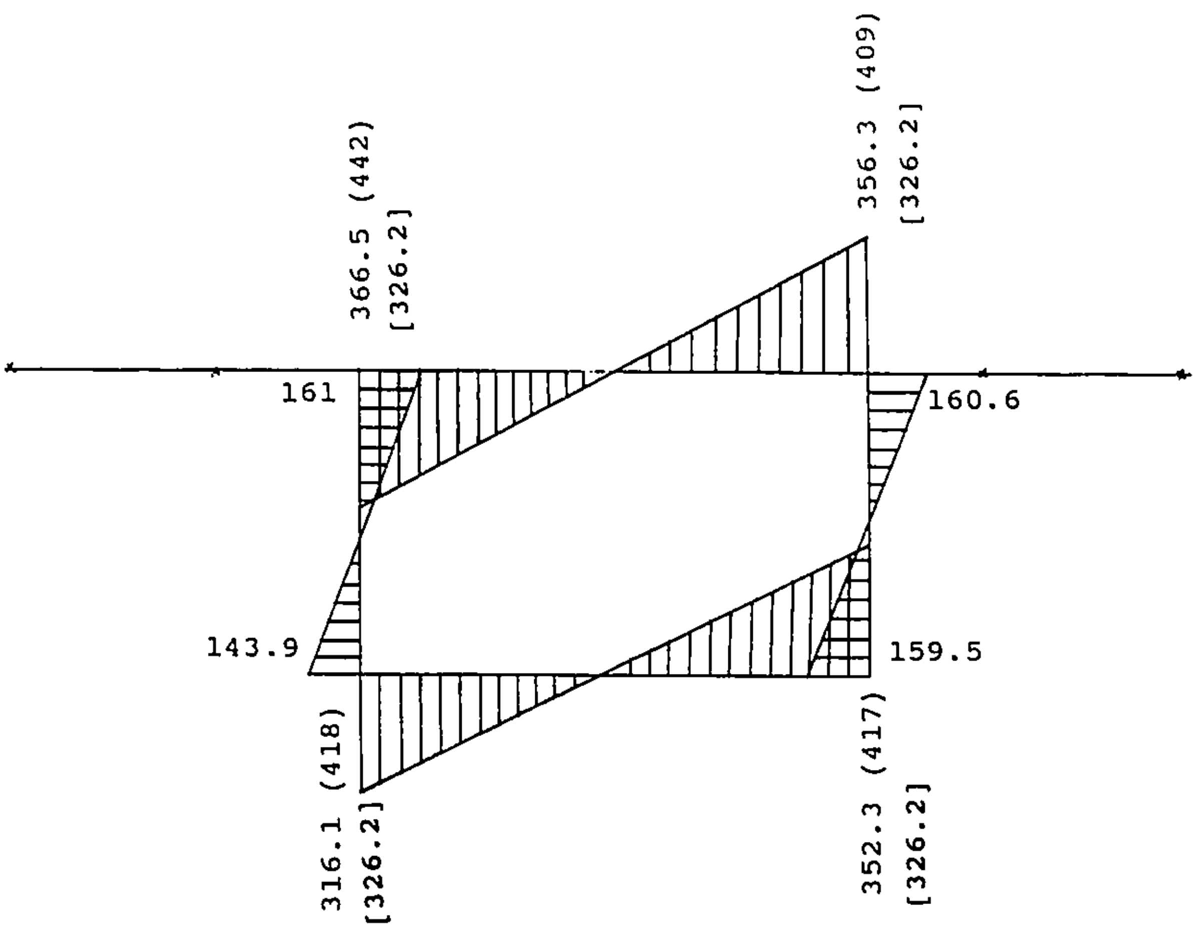
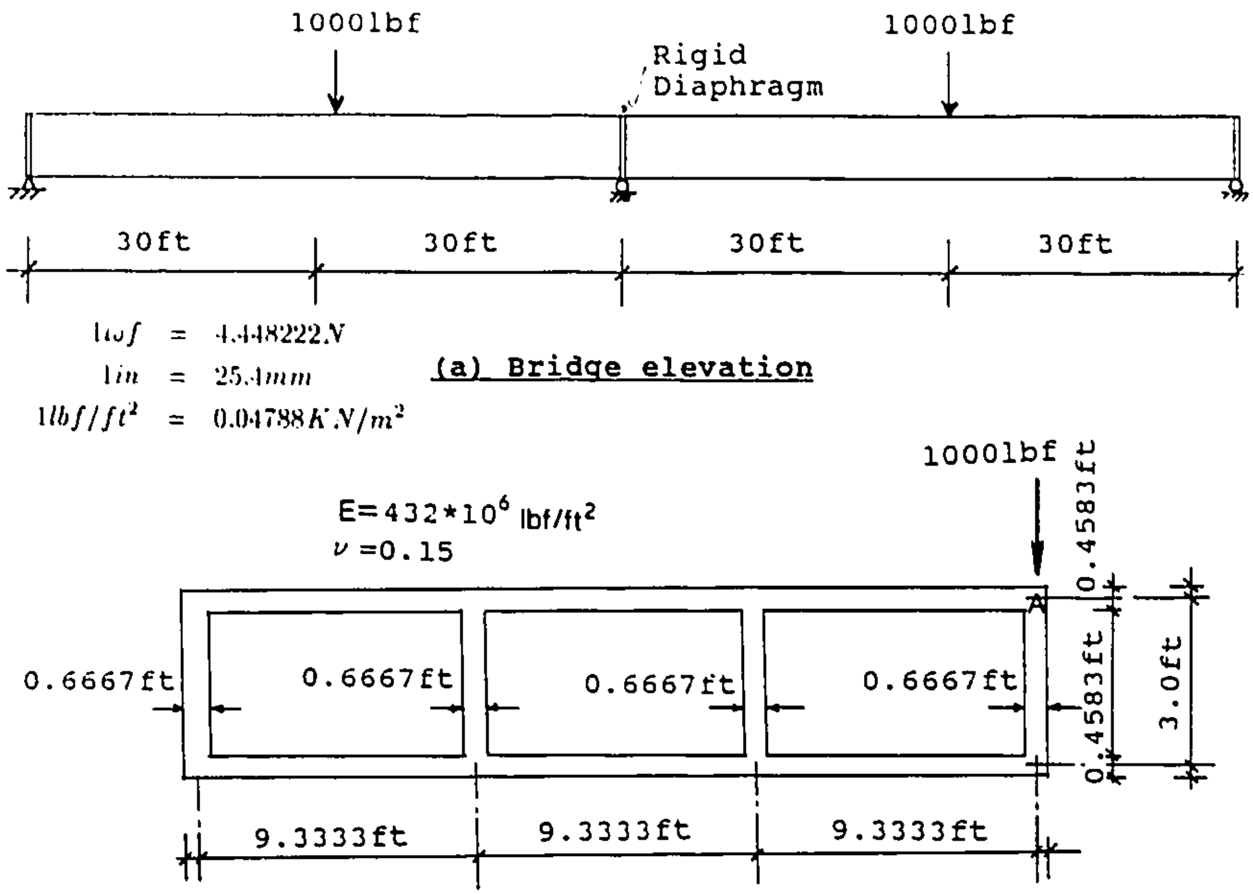
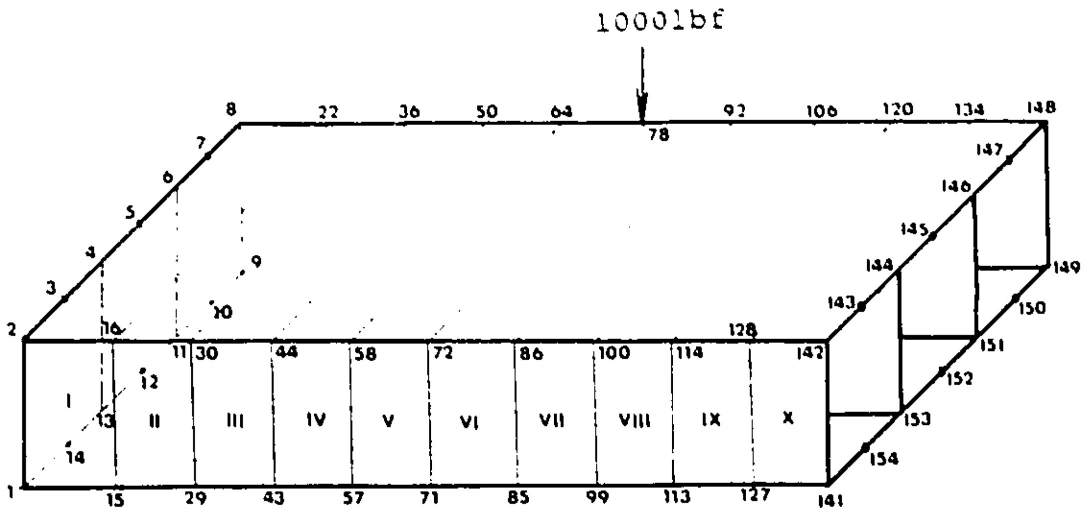


Figure 4.6: Distribution of transverse bending stresses at center cross section for the box-girder bridge model in figure 4.2 with an eccentric point load $p = 224$ lbf at midspan (lbf/sq in)

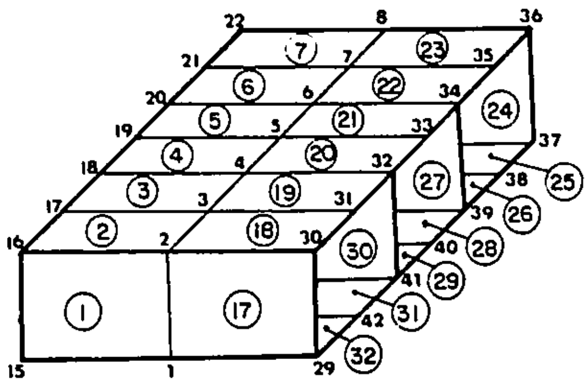


(b) Cross-section and load position

Figure 4.7: Two span three cell box-girder bridge



10 macro—element substructural modelling of the half bridge



typical macro—element substructure with 32 flat—shell elements

Figure 4.8: Macro-element idealization of the box-girder bridge in Figure 4.7

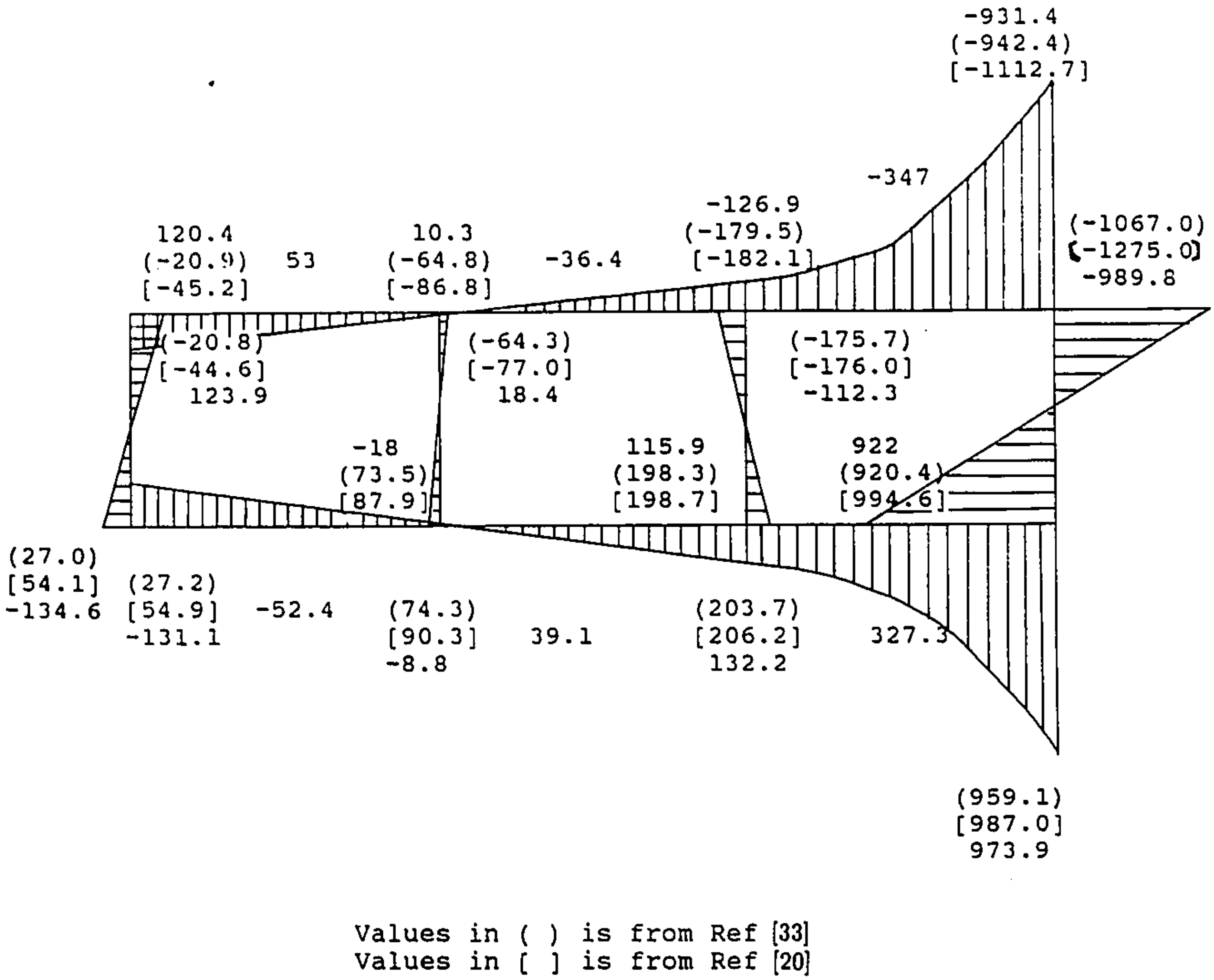


Figure 4.9: Longitudinal in-plane stresses at loaded section for the two span box-girder bridge in figure 4.7 (lbf/ft^2)

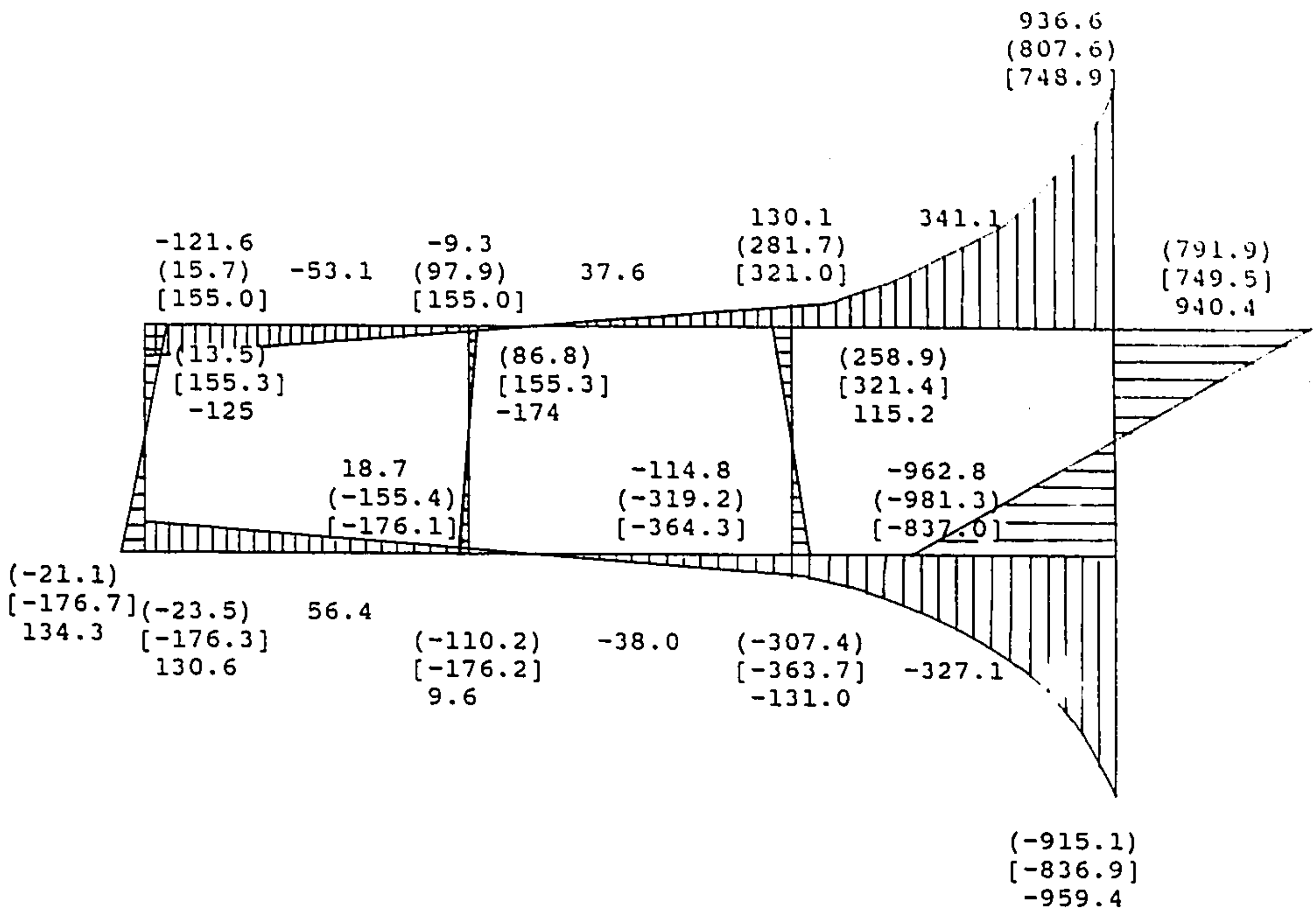


Figure 4.10: Longitudinal in-plane stresses at intermediate section for the two span box–girder bridge in figure 4.7 (lbf/ft^2)

Chapter 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The macro-element substructure approach presented in this thesis has been successfully used to analyze straight box-girder bridge structures with different types of cross-sections. Based on this approach, a general computer program-MESP has been developed for different straight box-girder bridge structures and other related structural problems. The following three types of analysis can be carried out using the MESP computer program:

1. In-plane stress, deep beam and thin-plate bending problems.

2. Folded plate structures.

3. Straight single-cell or multi-cell box-girder bridges

The accuracy of this method has been tested on a variety of examples. MESP program results have been compared with exact or other numerical solutions in different numerical examples in chapter 4.

The main advantage of this macro-element substructure approach over the conventional finite element method is the significant reduction of data preparation time. Unlike the classical finite element method which deals with the whole structure, the macro-element approach considers only one typical macro-element substructure. This produces a fairly small substructure grid as compared to that of the whole structure. More importantly, only one typical data set preparation is required. Based on this, data files for other substructures are automatically generated. This significantly reduces the work of the engineer without compromising accuracy; The computing efficiency is obvious because the whole structure is only the repetition of one typical macro-element substructure and computer memory and C.P.U. time in computer are greatly reduced, showing therefore, the economical advantage of the macro-element substructure approach.

5.2 Recommendations

This efficient macro-element substructure method as developed in this thesis should be continued and further developed to address following:

(1). Static analysis of curved box-girder bridges:

The macro-element substructure method presented here is applicable only to straight box-girder bridges. With the development of highway systems, curved box-girder bridges play more and more important roles in modern motorway and urban highway interchange systems. Relatively simple solutions for curved box-girder bridges are only available in special cases. Approximate analysis can be made by conventional finite element method. A more accurate element approach is necessary and further research development is required. Jirousek, Bouberguig and Saygun [8] introduced a macro-element formulation for static analysis of prestressed curved box-girder bridges and special purpose elements were developed. But it is not a good solution for engineers because of its complicated calculation. There is the necessity to do more studies on simplifying the analysis of curved box-girder bridge structures.

(2). Dynamic analysis of box-girder bridge structures:

This macro-element substructure method developed for static analysis of box-girder bridges can be extended to include dynamic analysis. Method [25] indicates that, with the simplified dynamic condensation method, dynamic effects of a substructure's internal degrees of freedoms can be taken into account in the dynamic analysis of a multi-substructure system. More accurate results were obtained both in eigen-pairs from free vibration analysis and in the higher-mode responses from dynamic response analysis compared with the static condensation method. The simplified method is nearly as simple in form as the static method, and thus it is not difficult to implement in multi-substructure system. Much more research on dynamic behavior analysis of box-girder structures is under progress.

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Appendix A

Macro–Element Substructure Program

```
C     ***********************************************************
C     * MACRO-ELEMENT SUBSTRUCTURAL METHOD IN BOX-GIRDER STRUCTURES *
C     ***********************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER E,DH,ZC,AA,H,Z
      REAL*8 MU,K,KZ(800,200)
      DIMENSION K(24,24),Y(100,2),ZY(100,2),T(24,24)
     *,JM(60,60),PJ(100,3),PO(450),FO(24)
     *,SE(400,400),JMM(150,4),D(100,6)
     *,P(1100),WY(24),ZC(140),SK(24,20),AA(100),F1(24),BD(300)
     *,BTO(50,4),PF(100,3),PXY(100,5),F2(24),MACRO(20)
      COMMON/L1/A,B,TO
      COMMON/L3/EO,MU
      COMMON/L2/KE,KS
      COMMON/L4/ND
      COMMON/L11/BTO
      COMMON/L5/NJE,NCJE,NEE
      COMMON/L6/T,Y,ZY
      COMMON/L7/K,SE,SK
      COMMON/L8/NPJ,NPF,NPXY
      COMMON/L12/PJ,PO,P
      COMMON/L9/JM,JMM
      COMMON/L10/FO,PF,PXY
C     ******************************************************
C     * DATA INPUT                                         *
C     ******************************************************
      READ(5,*)NJ,NE,NZ,NPJ,NPF,NPXY,ND,NP
      WRITE(6,5)NJ,NE,NZ,NPJ,NPF,NPXY,ND,NP
5     FORMAT(/,5X,'MACRO-ELEMENT METHOD',//,3X,'(1).INPUT DATA:',//,
     *5X,'NJ=',I3,2X,'NE=',I3,2X,'NZ=',I3,2X,'NPJ=',I3,2X,'NPF=',
     *I3,2X,'NPXY=',I3,//,5X,'ND=',I3,5X,'NP=',I3)
      NJ3=NJ*6
      READ(5,*)EO,MU
      WRITE(6,10)EO,MU
10    FORMAT(/,5X,'EO=',E14.5,2X,'MU=',F8.4)
      READ(5,*)(ZC(I),I=1,NZ)
      WRITE(6,15)(ZC(I),I=1,NZ)
15    FORMAT(/,5X,'ZC(I)=',16(/,15X,6I5))
      IF(ND.EQ.0) GO TO 19
      READ(5,*)((BTO(I,J),J=1,4),I=1,ND)
      WRITE(6,17)((BTO(I,J),J=1,4),I=1,ND)
17    FORMAT(/,5X,'DEMENSION OF ELEMENT',//, 2X,'SMALL NUMBER',
     *'A1=',5X,'B1=',5X,'T1=',//,5X,32(/,1X,4F10.5))
19    READ(5,*)A,B,TO
      WRITE(6,25)A,B,TO
25    FORMAT(/,5X,'GENERAL DIMENSION INFORMATION',
     */,5X,'A=',F10.4,2X,'B=',F10.4,2X,'T=',F10.4)
      IF(NP.EQ.0) GO TO 6
      READ(5,*)(MACRO(J),J=1,NP)
      WRITE(6,7)(MACRO(J),J=1,NP)
7     FORMAT(/,5X,'MACRO ELEMENT FOR PRINT',//,5X,5I5)
6     READ(5,*)NEE,NJE,NCJE
      WRITE(6,30)NEE,NJE,NCJE
30    FORMAT(/,5X,'NEE=',I3,2X,'NJE=',I3,2X,'NCJE=',I3)
      NJE3=6*NJE
```

```
      NCJE3=6*NCJE
      DO 45 MK=1,NE
      READ(5,*)(JM(MK,J),J=1,NJE)
      WRITE(6,65)MK,(JM(MK,J),J=1,NJE)
65    FORMAT(/,5X,'ELEMENT',I3,2X,//,'JM(I,J)=',50(16I4,/))
45    CONTINUE
      DO 71 J=1,NEE
      READ(5,*)(JMM(J,N),N=1,4)
      WRITE(6,75)(JMM(J,N),N=1,4)
75    FORMAT(/,5X,'JMM(J,K)=',50(9I5,/))
71    CONTINUE
      DO 77 J=1,NEE
      DO 77 M=1,2
      READ(5,*)Y(J,M),ZY(J,M)
83    IF(J.EQ.1) GO TO 84
      GO TO 89
84    IF(M.EQ.1) GO TO 85
      GO TO 89
85    WRITE(6,88)
88    FORMAT(/,5X,'THE COORDINATES OF NODES I & J IN THE ELEMENT',
     *//,3X,'  ELEMENT    Y(J,M)      Z(J,M)',//]
89    WRITE(6,80)J,Y(J,M),ZY(J,M)
80    FORMAT(5X,I5,2F10.3)
77    CONTINUE
      IF(NPJ.EQ.0) GO TO 98
      READ(5,*)((PJ(I,J),J=1,3),I=1,NPJ)
      WRITE(6,95)((PJ(I,J),J=1,3),I=1,NPJ)
95    FORMAT(/,5X,'PJ(I,J)=    '/,32(/,5X,1F12.3,2X,2F9.0))
98    IF(NPF.EQ.0) GO TO 105
      READ(5,*)((PF(I,J),J=1,3),I=1,NPF)
      WRITE(6,99)((PF(I,J),J=1,3),I=1,NPF)
99    FORMAT(/,5X,'PF(I,J)=',32(/,15X,1F10.7,2X,1F9.0,2X,1F9.0))
105   IF(NPXY.EQ.0) GO TO 107
      READ(5,*)((PXY(I,J),J=1,5),I=1,NPXY)
      WRITE(6,109)((PXY(I,J),J=1,5),I=1,NPXY)
109   FORMAT(/,5X,'PXY(I,J)=',32(/,15X,3(F10.7,2X),2(F9.0,2X)))
107   READ(5,*)NDIFF
      WRITE(6,91)NDIFF
91    FORMAT(/,5X,'NDIFF=',I4,2X)
C     ********************************************************
C     *  INSERT LOAD VECTOR                                  *
C     * CONSTRUCT TOTAL STIFFNESS MATRIX OF THE STRUCTURE    *
C     ********************************************************
      NBAND=6*(NDIFF+1)
      DO 100 I=1,NJ3
      DO 100 J=1,NBAND
      P(I)=0.
100   KZ(I,J)=0.
      JH=6*NJE+6*NCJE
      DO 129 E=1,NE
      DO 140 M=1,JH
140   PO(M)=0.
      CALL LOAD(E,2,1)
      CALL DDUGD(E,2,1)
      DO 120 I=1,NJE
```

```
      DO 120 II=1,6
      H=6*(I-1)+II
      DH=6*(JM(E,I)-1)+II
      DO 130 J=1,NJE
      DO 130 JJ=1,6
      L=6*(J-1)+JJ
      LL=6*(JM(E,J)-1)+JJ
      NDL=LL-DH+1
      IF(NDL.LT.0)GO TO 130
      KZ(DH,NDL)=KZ(DH,NDL)+SE(H,L)
130   CONTINUE
120   CONTINUE
      DO 150 IK=1,NJE
      AA(IK)=JM(E,IK)
      DO 150 J=1,6
150   P(6*AA(IK)-(6-J))=P(6*AA(IK)-(6-J))+PO(6*(IK-1)+J)
129   CONTINUE
      WRITE(6,169)(P(II),II=1,NJ3)
169   FORMAT(/,5X,'LOAD MATRIX OF THE STRUCTURE BEFORE APPLY B.C.
     * ',//,'NX=',11X,'NY=',11X,'Q=',10X,
     *'MX=',10X,'MY=',10X,'MZ=',//,40(/,6E13.3))
      DO 300 I=1,NZ
      Z=ZC(I)
      KZ(Z,1)=1
      DO 330 J=2,NBAND
330   KZ(Z,J)=0.
      IF(Z.GT.NBAND) JO=NBAND
      IF(Z.LE.NBAND) JO=Z
      DO 333 JZ=2,JO
      IZ=Z-JZ+1
333   KZ(IZ,JZ)=0.
      P(Z)=0.
300   CONTINUE
      JH=NJ3-1
      DO 350 IK=1,JH
      KKD=IK+NBAND-1
      IF(NJ3.GT.KKD) IM=KKD
      IF(NJ3.LE.KKD) IM=NJ3
      DO 400 I=IK+1,IM
      L=I-IK+1
      C=KZ(IK,L)/KZ(IK,1)
      LD=NBAND-L+1
      DO 450 J=1,LD
      M=J+I-IK
450   KZ(I,J)=KZ(I,J)-C*KZ(IK,M)
400   P(I)=P(I)-C*P(IK)
350   CONTINUE
      P(NJ3)=P(NJ3)/KZ(NJ3,1)
      DO 500 LK=1,NJ3-1
      I=NJ3-LK
      IF(NBAND.GT.NJ3-I+1)  JO=NJ3-I+1
      IF(NBAND.LE.NJ3-I+1)  JO=NBAND
      DO 550 J=2,JO
      H=J+I-1
550   P(I)=P(I)-KZ(I,J)*P(H)
```

```
500    P(I)=P(I)/KZ(I,1)
       WRITE(6,570)
570    FORMAT(//,3X,'(2).THE OUTPUT:',///,5X,'NODAL DISPLACEMENTS',
      *//,'NODE',4X,'U=',10X,'V=',10X,'W=',8X,
      *'CETA(X)=',5X,'CETA(Y)=',5X,'CETA(Z)=')
       DO 595 I=1,NJ
       DO 560 J=1,6
560    D(I,J)=P(6*I-6+J)
       WRITE(6,580)I,(D(I,J),.' 1,6)
580    FORMAT(/,I3,6E12.4)
595    CONTINUE
       IF(NP.EQ.0) GO TO 990
       DO 990 E=1,NE
       DO 11 LW=1,NP
       LO=MACRO(LW)
       IF(E.EQ.LO) GO TO 13
11     CONTINUE
       GO TO 990
13     DO 610 I=1,NJE
       DO 610 M=1,NJ
       MJ=JM(E,I)
       IF(MJ.EQ.M) GOTO 620
       GO TO 610
620    DO 640 N=1,6
640    BD((I-1)*6+N)=D(M,N)
610    CONTINUE
       IF(E.GT.1) GO TO 613
613    DO 618 M=1,JH
618    PO(M)=0.
       CALL LOAD(E,2,2)
       CALL DDUGD(E,1,1)
       JH=NJE3+NCJE3
       DO 645 MK=1,NJE3
       J=NCJE3+MK
       PO(J)=BD(MK)
645    CONTINUE
       MT=NCJE3+1
       DO 660 I=1,NCJE3
       P(I)=0.
       DO 666 J=NCJE3+1,JH
666    P(I)=P(I)+SE(I,J)*PO(J)
660    PO(I)=PO(I)-P(I)
       PO(NCJE3)=PO(NCJE3)/SE(NCJE3,NCJE3)
       DO 650 LK=1,NCJE3-1
       I=NCJE3-LK
       DO 656 J=I+1,NCJE3
656    PO(I)=PO(I)-SE(I,J)*PO(J)
650    PO(I)=PO(I)/SE(I,I)
       DO 950 NN=1,NEE
       DO 710 I=1,4
       DO 710 J=1,6
       N=6*(I-1)+J
       MM=6*(JMM(NN,I)-1)+J
710    WY(N)=PO(MM)
       DO 837 I=1,24
```

```
      DO 837 J=1,24
837   T(I,J)=0.0
838   CALL ZB(NN,E,1)
      IF(E.GT.1) GO TO 836
      IF(NN.GT.1) GO TO 836
836   DO 881 I=1,24
      F2(I)=0.
      DO 888 M=1,24
      F2(I)=F2(I)+T(I,M)*WY(M)
888   CONTINUE
881   CONTINUE
      DO 783 I=1,20
783   WY(I)=0
      DO 780 I=1,4
      DO 780 J=1,5
      IQ=6*(I-1)+J
      IP=5*(I-1)+J
780   WY(IP)=F2(IQ)
      DO 869 L=1,24
      DO 869 M=1,20
869   SK(L,M)=0
      CALL IDUGD(NN,E)
868   DO 860 LI=1,24
      F1(LI)=0.
      DO 860 J=1,20
      F1(LI)=F1(LI)+SK(LI,J)*WY(J)
860   CONTINUE
      WRITE(6,863)E,NN
863   FORMAT(/,5X,'E=',I2,5X,'INTERNAL FORCES IN SMALL ELEMENT'
     *,I2,//,9X,'SIGMA-X',6X,'SIGMA-Y',6X,
     *'TAU-XY',9X,'MX',11X,'MY',9X,'MXY',//)
      DO 867 LI=1,4
      LL=6*LI-5
      JJ=6*LI
      WRITE(6,853)JMM(NN,LI),(F1(J),J=LL,JJ)
853   FORMAT('NODE',I2,6E13.4,/)
867   CONTINUE
      WRITE(6,9001)
9001  FORMAT(9X,'BENDING  STRESS ',6X,//
     *,9X,'SIGMA-MX',5X,'SIGMA-MY',5X,'TAU-MXY',/)
      IF(ND.EQ.0) GO TO 8301
      DO 8303 I=1,ND
      IAT=INT(BTO(I,1))
      IF(IAT.EQ.NN) GO TO 8304
      GO TO 8303
8304  TOX=BTO(I,4)
      GO TO 8306
8303  CONTINUE
8301  TOX=TO
8306  CONTINUE
      DO 24 N=1,4
      LL=6*N-2
      JJ=6*N
      DO 26 NH=LL,JJ
26    F1(NH)=F1(NH)*6/(TOX*TOX)
```

```
      WRITE(6,28)JMM(NN,N),(F1(J),J=LL,JJ)
28    FORMAT('NODE',I2,3E13.4,/)
24    CONTINUE
950   CONTINUE
990   CONTINUE
      STOP
      END
C     ****************************************************************
      SUBROUTINE DDUGD(E,LLE,LEN)
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER E,DH,H
      REAL*8 MU,K,KZ(800,200)
      DIMENSION K(24,24),Y(100,2),ZY(100,2),T(24,24)
     *,JM(60,60),PJ(100,3),PO(450),P(1100)
     *,SE(400,400),JMM(150,4),BTO(50,4),SK(24,20)
      COMMON/L1/A,B,TO
      COMMON/L3/EO,MU
      COMMON/L2/KE,KS
      COMMON/L4/ND
      COMMON/L11/BTO
      COMMON/L5/NJE,NCJE,NEE
      COMMON/L6/T,Y,ZY
      COMMON/L7/K,SE,SK
      COMMON/L12/PJ,PO,P
      COMMON/L9/JM,JMM
      NJE3=6*NJE
      NCJE3=6*NCJE
      II=NCJE3+NJE3
      DO 1100 I=1,II
      DO 1100 J=1,II
1100  SE(I,J)=0.
      DO 1030 I=1,NEE
      CALL DUGD(I,E,LLE)
      DO 1048 L=1,4
      DO 1048 LL=1,6
      H=6*(L-1)+LL
      DH=6*(JMM(I,L)-1)+LL
      DO 1049 M=1,4
      DO 1049 MM=1,6
      N=6*(M-1)+MM
      NN=6*(JMM(I,M)-1)+MM
      SE(DH,NN)=SE(DH,NN)+K(H,N)
1049  CONTINUE
1048  CONTINUE
1030  CONTINUE
1090  DO 1111 KJ=1,NCJE3
      DO 1101 KI=KJ+1,II
      IF(SE(KJ,KI).EQ.0) GO TO 1101
      C=SE(KJ,KI)/SE(KJ,KJ)
      DO 1150 J=KI,II
1150  SE(KI,J)=SE(KI,J)-C*SE(KJ,J)
1130  PO(KI)=PO(KI)-C*PO(KJ)
1101  CONTINUE
1111  CONTINUE
      IF (LLE.EQ.1)GO TO 1190
```

```
      DO 1180 I=1,NJE3
      IJ=I+NCJE3
      DO 1180 J=1,NJE3
      JI=J+NCJE3
      SE(I,J)=SE(IJ,JI)
1180   CONTINUE
      IF(LEN.EQ.2)GO TO 1190
      DO 1110 I=1,NJE3
      IJ=I+NCJE3
1110  PO(I)=PO(IJ)
1190  RETURN
      END
C     **************************************************************
      SUBROUTINE DUGD(LF,E,LLE)
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER E
      REAL*8 MU,K,KE(12,12),KS(24,24)
      DIMENSION K(24,24),Y(100,2),ZY(100,2),T(24,24)
     *,SK(24,20),SE(400,400),BTO(50,4)
      COMMON/L1/A,B,TO
      COMMON/L3/EO,MU
      COMMON/L2/KE,KS
      COMMON/L4/ND
      COMMON/L6/T,Y,ZY
      COMMON/L11/BTO
      COMMON/L7/K,SE,SK
      DO 1200 L=1,24
      DO 1200 J=1,24
1200  KS(L,J)=0.0
      CALL JDUGD(LF,E,LLE)
      DO 1208 I=1,24
      DO 1208 J=1,24
1208  T(I,J)=0.0
      CALL ZB(LF,E,LLE)
      DO 1210 I=1,24
      DO 1210 J=1,24
      K(I,J)=0.
      DO 1210 N=1,24
      DO 1210 M=1,24
      K(I,J)=K(I,J)+T(N,I)*KS(N,M)*T(M,J)
1210  CONTINUE
1220  RETURN
      END
C     **************************************************************
      SUBROUTINE JDUGD(LF,E,LLE)
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER E
      REAL*8 MU,KE(12,12),K,KS(24,24)
      DIMENSION BTO(50,4)
      COMMON/L1/A,B,TO
      COMMON/L3/EO,MU
      COMMON/L2/KE,KS
      COMMON/L4/ND
      COMMON/L11/BTO
      IF(ND.EQ.0) GO TO 1301
```

```
      DO 1303 I=1,ND                                                    MAC03860
      IAT=INT(BTO(I,1))                                                 MAC03870
      IF(IAT.EQ.LF) GO TO 1304                                          MAC03880
      GO TO 1303                                                        MAC03890
 1304 AX=BTO(I,2)                                                       MAC03900
      BX=BTO(I,3)                                                       MAC03910
      TOX=BTO(I,4)                                                      MAC03920
      GO TO 1306                                                        MAC03930
 1303 CONTINUE                                                          MAC03940
 1301 AX=A                                                              MAC03950
      BX=B                                                              MAC03960
      TOX=TO                                                            MAC03970
 1306 P1=AX/BX                                                          MAC03980
      P2=BX/AX                                                          MAC03990
      ETV=EO*TOX/(1.0-MU*MU)                                            MAC04000
      C1=(P2/3+(1-MU)*P1/6)*ETV                                         MAC04010
      C2=(MU/4+(1-MU)/8)*ETV                                            MAC04020
      C3=(P1/3+(1-MU)*P2/6)*ETV                                         MAC04030
      C4=(-P2/3+(1-MU)*P1/12)*ETV                                       MAC04040
      C5=(MU/4-(1-MU)/8)*ETV                                            MAC04050
      C6=(P1/6-(1-MU)*P2/6)*ETV                                         MAC04060
      C7=(P2/6-(1-MU)*P1/6)*ETV                                         MAC04070
      C8=(-P1/3+(1-MU)*P2/12)*ETV                                       MAC04080
      DO 1300 I=1,24                                                    MAC04090
      DO 1300 J=1,24                                                    MAC04100
 1300 KS(I,J)=0.                                                        MAC04110
      KS(1,1)=C1                                                        MAC04120
      KS(2,1)=C2                                                        MAC04130
      KS(2,2)=C3                                                        MAC04140
      KS(7,1)=C7                                                        MAC04150
      KS(7,2)=C5                                                        MAC04160
      KS(7,7)=C1                                                        MAC04170
      KS(8,1)=-C5                                                       MAC04180
      KS(8,2)=C8                                                        MAC04190
      KS(8,7)=-C2                                                       MAC04200
      KS(8,8)=C3                                                        MAC04210
      KS(13,1)=C4                                                       MAC04220
      KS(13,2)=-C5                                                      MAC04230
      KS(13,7)=-C1/2                                                    MAC04240
      KS(13,8)=C2                                                       MAC04250
      KS(13,13)=C1                                                      MAC04260
      KS(14,1)=C5                                                       MAC04270
      KS(14,2)=C6                                                       MAC04280
      KS(14,7)=C2                                                       MAC04290
      KS(14,8)=-C3/2                                                    MAC04300
      KS(14,13)=-C2                                                     MAC04310
      KS(14,14)=C3                                                      MAC04320
      KS(19,1)=-C1/2                                                    MAC04330
      KS(19,2)=-C2                                                      MAC04340
      KS(19,7)=C4                                                       MAC04350
      KS(19,8)=C5                                                       MAC04360
      KS(19,13)=C7                                                      MAC04370
      KS(19,14)=-C5                                                     MAC04380
      KS(19,19)=C1                                                      MAC04390
      KS(20,1)=-C2                                                      MAC04400
```

```
      KS(20,2)=-C3/2                                                    MAC04410
      KS(20,7)=-C5                                                      MAC04420
      KS(20,8)=C6                                                       MAC04430
      KS(20,13)=C5                                                      MAC04440
      KS(20,14)=C8                                                      MAC04450
      KS(20,19)=C2                                                      MAC04460
      KS(20,20)=C3                                                      MAC04470
      CALL BDUGD(LF,E,LLE)                                              MAC04480
      KS(3,3)=KE(1,1)                                                   MAC04490
      KS(4,3)=KE(2,1)                                                   MAC04500
      KS(4,4)=KE(2,2)                                                   MAC04510
      KS(5,3)=KE(3,1)                                                   MAC04520
      KS(5,4)=KE(3,2)                                                   MAC04530
      KS(5,5)=KE(3,3)                                                   MAC04540
      KS(9,3)=KE(4,1)                                                   MAC04550
      KS(9,4)=KE(4,2)                                                   MAC04560
      KS(9,5)=KE(4,3)                                                   MAC04570
      KS(9,9)=KE(4,4)                                                   MAC04580
      KS(10,3)=KE(5,1)                                                  MAC04590
      KS(10,4)=KE(5,2)                                                  MAC04600
      KS(10,5)=KE(5,3)                                                  MAC04610
      KS(10,9)=KE(5,4)                                                  MAC04620
      KS(10,10)=KE(5,5)                                                 MAC04630
      KS(11,3)=KE(6,1)                                                  MAC04640
      KS(11,4)=KE(6,2)                                                  MAC04650
      KS(11,5)=KE(6,3)                                                  MAC04660
      KS(11,9)=KE(6,4)                                                  MAC04670
      KS(11,10)=KE(6,5)                                                 MAC04680
      KS(11,11)=KE(6,6)                                                 MAC04690
      KS(15,3)=KE(7,1)                                                  MAC04700
      KS(15,4)=KE(7,2)                                                  MACC4710
      KS(15,5)=KE(7,3)                                                  MAC04720
      KS(15,9)=KE(7,4)                                                  MAC04730
      KS(15,10)=KE(7,5)                                                 MAC04740
      KS(15,11)=KE(7,6)                                                 MAC04750
      KS(15,15)=KE(7,7)                                                 MAC04760
      KS(16,3)=KE(8,1)                                                  MAC04770
      KS(16,4)=KE(8,2)                                                  MAC04780
      KS(16,5)=KE(8,3)                                                  MAC04790
      KS(16,9)=KE(8,4)                                                  MAC04800
      KS(16,10)=KE(8,5)                                                 MAC04810
      KS(16,11)=KE(8,6)                                                 MAC04820
      KS(16,15)=KE(8,7)                                                 MAC04830
      KS(16,16)=KE(8,8)                                                 MAC04840
      KS(17,3)=KE(9,1)                                                  MAC04850
      KS(17,4)=KE(9,2)                                                  MAC04860
      KS(17,5)=KE(9,3)                                                  MAC04870
      KS(17,9)=KE(9,4)                                                  MAC04880
      KS(17,10)=KE(9,5)                                                 MAC04890
      KS(17,11)=KE(9,6)                                                 MAC04900
      KS(17,15)=KE(9,7)                                                 MAC04910
      KS(17,16)=KE(9,8)                                                 MAC04920
      KS(17,17)=KE(9,9)                                                 MAC04930
      KS(21,3)=KE(10,1)                                                 MAC04940
      KS(21,4)=KE(10,2)                                                 MAC04950
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      KS(21,5)=KE(10,3)                                                 MAC04960
      KS(21,9)=KE(10,4)                                                 MAC04970
      KS(21,10)=KE(10,5)                                                MAC04980
      KS(21,11)=KE(10,6)                                                MAC04990
      KS(21,15)=KE(10,7)                                                MAC05000
      KS(21,16)=KE(10,8)                                                MAC05010
      KS(21,17)=KE(10,9)                                                MAC05020
      KS(21,21)=KE(10,10)                                               MAC05030
      KS(22,3)=KE(11,1)                                                 MAC05040
      KS(22,4)=KE(11,2)                                                 MAC05050
      KS(22,5)=KE(11,3)                                                 MAC05060
      KS(22,9)=KE(11,4)                                                 MAC05070
      KS(22,10)=KE(11,5)                                                MAC05080
      KS(22,11)=KE(11,6)                                                MAC05090
      KS(22,15)=KE(11,7)                                                MAC05100
      KS(22,16)=KE(11,8)                                                MAC05110
      KS(22,17)=KE(11,9)                                                MAC05120
      KS(22,21)=KE(11,10)                                               MAC05130
      KS(22,22)=KE(11,11)                                               MAC05140
      KS(23,3)=KE(12,1)                                                 MAC05150
      KS(23,4)=KE(12,2)                                                 MAC05160
      KS(23,5)=KE(12,3)                                                 MAC05170
      KS(23,9)=KE(12,4)                                                 MAC05180
      KS(23,10)=KE(12,5)                                                MAC05190
      KS(23,11)=KE(12,6)                                                MAC05200
      KS(23,15)=KE(12,7)                                                MAC05210
      KS(23,16)=KE(12,8)                                                MAC05220
      KS(23,17)=KE(12,9)                                                MAC05230
      KS(23,21)=KE(12,10)                                               MAC05240
      KS(23,22)=KE(12,11)                                               MAC05250
      KS(23,23)=KE(12,12)                                               MAC05260
      DO 1540 I=1,24                                                    MAC05270
      DO 1540 J=1,I                                                     MAC05280
 1540 KS(J,I)=KS(I,J)                                                   MAC05290
      IF(LLE.EQ.1) GO TO 1550                                           MAC05300
      IF(E.GT.1) GO TO 1550                                             MAC05310
      IF(LF.GT.1) GO TO 1550                                            MAC05320
 1550 RETURN                                                            MAC05330
      END                                                               MAC05340
C     ******************************************************************MAC05350
      SUBROUTINE ZB(LF,E,LLE)                                           MAC05360
      IMPLICIT REAL*8(A-H,O-Z)                                          MAC05370
      INTEGER E                                                         MAC05380
      DIMENSION Y(100,2),ZY(100,2),T(24,24),YY(3,3),YL(6,6)             MAC05390
      COMMON/L6/T,Y,ZY                                                  MAC05400
      Y1=Y(LF,1)                                                        MAC05410
      Y2=Y(LF,2)                                                        MAC05420
      Z1=ZY(LF,1)                                                       MAC05430
      Z2=ZY(LF,2)                                                       MAC05440
      XYZ=SQRT((Y2-Y1)*(Y2-Y1)+(Z2-Z1)*(Z2-Z1))                         MAC05450
      XAX=1.                                                            MAC05460
      XAY=0.                                                            MAC05470
      XAZ=0.                                                            MAC05480
      YAX=0.                                                            MAC05490
      YAY=(Y2-Y1)/XYZ                                                   MAC05500
```

```
      YAZ=(Z2-Z1)/XYZ
      ZAX=0.
      ZAY=-(Z2-Z1)/XYZ
      ZAZ=(Y2-Y1)/XYZ
      DO 1610 I=1,3
      DO 1610 J=1,3
1610  YY(I,J)=0
      YY(1,1)=XAX
      YY(1,2)=XAY
      YY(1,3)=XAZ
      YY(2,1)=YAX
      YY(2,2)=YAY
      YY(2,3)=YAZ
      YY(3,1)=ZAX
      YY(3,2)=ZAY
      YY(3,3)=ZAZ
1612  DO 1615 I=1,6
      DO 1615 J=1,6
1615  YL(I,J)=0.
      YL(1,1)=YY(1,1)
      YL(1,2)=YY(1,2)
      YL(1,3)=YY(1,3)
      YL(2,1)=YY(2,1)
      YL(2,2)=YY(2,2)
      YL(2,3)=YY(2,3)
      YL(3,1)=YY(3,1)
      YL(3,2)=YY(3,2)
      YL(3,3)=YY(3,3)
      YL(4,4)=YY(1,1)
      YL(4,5)=YY(1,2)
      YL(4,6)=YY(1,3)
      YL(5,4)=YY(2,1)
      YL(5,5)=YY(2,2)
      YL(5,6)=YY(2,3)
      YL(6,4)=YY(3,1)
      YL(6,5)=YY(3,2)
      YL(6,6)=YY(3,3)
1622  DO 1630 I=1,24
      DO 1630 J=1,24
1630  T(I,J)=0.0
      DO 1640 I=1,4
      DO 1640 J=1,6
      DO 1640 MK=1,6
1640  T(6*(I-1)+J,6*(I-1)+MK)=YL(J,MK)
1642  RETURN
      END
C     ******************************************************************
      SUBROUTINE BDUGD(IE,E,LLE)
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER E
      REAL*8 MU,KE(12,12),K1,K2,K3,K4,K5,KL,KS(24,24)
      DIMENSION K1(12,12),K2(12,12),KL(12,12),K3(12,12)
     *,K4(12,12),K5(12,12),BTO(50,4)
      COMMON/L1/A,B,TO
      COMMON/L3/EO,MU
```

```
      COMMON/L2/KE,KS                                                   MAC06060
      COMMON/L4/ND                                                      MAC06070
      COMMON/L11/BTO                                                    MAC06080
      IF(ND.EQ.0) GO TO 1651                                            MAC06090
      DO 1653 I=1,ND                                                    MAC06100
1656  IAT=INT(BTO(I,1))                                                 MAC06110
      IF(IAT.EQ.IE) GO TO 1652                                          MAC06120
      GO TO 1653                                                        MAC06130
1652  AX=BTO(I,2)                                                       MAC06140
      BX=BTO(I,3)                                                       MAC06150
      TOX=BTO(I,4)                                                      MAC06160
      GO TO 1655                                                        MAC06170
1653  CONTINUE                                                          MAC06180
1651  AX=A                                                              MAC06190
      BX=B                                                              MAC06200
      TOX=TO                                                            MAC06210
1655  DO 1650 I=1,12                                                    MAC06220
      DO 1650 J=1,12                                                    MAC06230
1650  K1(I,J)=0                                                         MAC06240
      K1(1,1)=60                                                        MAC06250
      K1(3,1)=30                                                        MAC06260
      K1(3,3)=20                                                        MAC06270
      K1(4,1)=30                                                        MAC06280
      K1(4,3)=15                                                        MAC06290
      K1(4,4)=60                                                        MAC06300
      K1(6,1)=15                                                        MAC06310
      K1(6,3)=10                                                        MAC06320
      K1(6,4)=30                                                        MAC06330
      K1(6,6)=20                                                        MAC06340
      K1(7,1)=-60                                                       MAC06350
      K1(7,3)=-30                                                       MAC06360
      K1(7,4)=-30                                                       MAC06370
      K1(7,6)=-15                                                       MAC06380
      K1(7,7)=60                                                        MAC06390
      K1(9,1)=30                                                        MAC06400
      K1(9,3)=10                                                        MAC06410
      K1(9,4)=15                                                        MAC06420
      K1(9,6)=5                                                         MAC06430
      K1(9,7)=-30                                                       MAC06440
      K1(9,9)=20                                                        MAC06450
      K1(10,1)=-30                                                      MAC06460
      K1(10,3)=-15                                                      MAC06470
      K1(10,4)=-60                                                      MAC06480
      K1(10,6)=-30                                                      MAC06490
      K1(10,7)=30                                                       MAC06500
      K1(10,9)=-15                                                      MAC06510
      K1(10,10)=60                                                      MAC06520
      K1(12,1)=15                                                       MAC06530
      K1(12,3)=5                                                        MAC06540
      K1(12,4)=30                                                       MAC06550
      K1(12,6)=10                                                       MAC06560
      K1(12,7)=-15                                                      MAC06570
      K1(12,9)=10                                                       MAC06580
      K1(12,10)=-30                                                     MAC06590
      K1(12,12)=20                                                      MAC06600
```

```
      DO 1660 IA=1,12
      DO 1660 JA=1,IA
1660  K1(JA,IA)=K1(IA,JA)
      DO 1685 I=1,12
      DO 1685 J=1,12
1685  K2(I,J)=0.
      K2(1,1)=60
      K2(2,1)=-30
      K2(2,2)=20
      K2(4,1)=-60
      K2(4,2)=30
      K2(4,4)=60
      K2(5,1)=-30
      K2(5,2)=10
      K2(5,4)=30
      K2(5,5)=20
      K2(7,1)=30
      K2(7,2)=-15
      K2(7,4)=-30
      K2(7,5)=-15
      K2(7,7)=60
      K2(8,1)=-15
      K2(8,2)=10
      K2(8,4)=15
      K2(8,5)=5
      K2(8,7)=-30
      K2(8,8)=20
      K2(10,1)=-30
      K2(10,2)=15
      K2(10,4)=30
      K2(10,5)=15
      K2(10,7)=-60
      K2(10,8)=30
      K2(10,10)=60
      K2(11,1)=-15
      K2(11,2)=5
      K2(11,4)=15
      K2(11,5)=10
      K2(11,7)=-30
      K2(11,8)=10
      K2(11,10)=30
      K2(11,11)=20
      DO 1690 IA=1,12
      DO 1690 JA=1,IA
1690  K2(JA,IA)=K2(IA,JA)
1691  DO 1700 I=1,12
      DO 1700 J=1,12
1700  K3(I,J)=0
      K3(1,1)=30
      K3(2,1)=-15
      K3(3,1)=15
      K3(3,2)=-15
      K3(4,1)=-30
      K3(4,3)=-15
      K3(4,4)=30
```

```
      K3(5,4)=15
      K3(6,1)=-15
      K3(6,4)=15
      K3(6,5)=15
      K3(7,1)=-30
      K3(7,2)=15
      K3(7,4)=30
      K3(7,7)=30
      K3(8,1)=15
      K3(8,7)=-15
      K3(9,7)=-15
      K3(9,8)=15
      K3(10,1)=30
      K3(10,4)=-30
      K3(10,5)=-15
      K3(10,7)=-30
      K3(10,9)=15
      K3(10,10)=30
      K3(11,4)=-15
      K3(11,10)=15
      K3(12,7)=15
      K3(12,10)=-15
      K3(12,11)=-15
      DO 1730 IA=1,12
      DO 1730 JA=1,IA
1730  K3(JA,IA)=K3(IA,JA)
1771  DO 1780 I=1,12
      DO 1780 J=1,12
1780  K4(I,J)=0.
      K4(1,1)=84
      K4(2,1)=-6
      K4(2,2)=8
      K4(3,1)=6
      K4(3,3)=8
      K4(4,1)=-84
      K4(4,2)=6
      K4(4,3)=-6
      K4(4,4)=84
      K4(5,1)=-6
      K4(5,2)=-2
      K4(5,4)=6
      K4(5,5)=8
      K4(6,1)=-6
      K4(6,3)=-8
      K4(6,4)=6
      K4(6,6)=8
      K4(7,1)=-84
      K4(7,2)=6
      K4(7,3)=-6
      K4(7,4)=84
      K4(7,5)=6
      K4(7,6)=6
      K4(7,7)=84
      K4(8,1)=6
      K4(8,2)=-8
```

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      K4(8,4)=-6                                                         MAC07710
      K4(8,5)=2                                                          MAC07720
      K4(8,7)=-6                                                         MAC07730
      K4(8,8)=8                                                          MAC07740
      K4(9,1)=6                                                          MAC07750
      K4(9,3)=-2                                                         MAC07760
      K4(9,4)=-6                                                         MAC07770
      K4(9,6)=2                                                          MAC07780
      K4(9,7)=-6                                                         MAC07790
      K4(9,9)=8                                                          MAC07800
      K4(10,1)=84                                                        MAC07810
      K4(10,2)=-6                                                        MAC07820
      K4(10,3)=6                                                         MAC07830
      K4(10,4)=-84                                                       MAC07840
      K4(10,5)=-6                                                        MAC07850
      K4(10,6)=-6                                                        MAC07860
      K4(10,7)=-84                                                       MAC07870
      K4(10,8)=6                                                         MAC07880
      K4(10,9)=6                                                         MAC07890
      K4(10,10)=84                                                       MAC07900
      K4(11,1)=6                                                         MAC07910
      K4(11,2)=2                                                         MAC07920
      K4(11,4)=-6                                                        MAC07930
      K4(11,5)=-8                                                        MAC07940
      K4(11,7)=-6                                                        MAC07950
      K4(11,8)=-2                                                        MAC07960
      K4(11,10)=6                                                        MAC07970
      K4(11,11)=8                                                        MAC07980
      K4(12,1)=-6                                                        MAC07990
      K4(12,3)=2                                                         MAC08000
      K4(12,4)=6                                                         MAC08010
      K4(12,6)=-2                                                        MAC08020
      K4(12,7)=6                                                         MAC08030
      K4(12,9)=-8                                                        MAC08040
      K4(12,10)=-6                                                       MAC08050
      K4(12,12)=8                                                        MAC08060
      DO 1790 I=1,12                                                     MAC08070
      DO 1790 J=1,I                                                      MAC08080
1790  K4(J,I)=K4(I,J)                                                    MAC08090
1791  DO 1800 I=1,12                                                     MAC08100
      DO 1800 J=1,12                                                     MAC08110
1800  KL(I,J)=0.                                                         MAC08120
      KL(1,1)=1                                                          MAC08130
      KL(2,2)=2*BX                                                       MAC08140
      KL(3,3)=2*AX                                                       MAC08150
      KL(4,4)=1                                                          MAC08160
      KL(5,5)=2*BX                                                       MAC08170
      KL(6,6)=2*AX                                                       MAC08180
      KL(7,7)=1                                                          MAC08190
      KL(8,8)=2*BX                                                       MAC08200
      KL(9,9)=2*AX                                                       MAC08210
      KL(10,10)=1                                                        MAC08220
      KL(11,11)=2*BX                                                     MAC08230
      KL(12,12)=2*AX                                                     MAC08240
1801  DO 1850 I=1,12                                                     MAC08250
```

```
      DO 1850 J=1,12
      K5(I,J)=0
      Q1=(AX/BX)*(AX/BX)
      Q2=1/Q1
      TV=EO*TOX*TOX*TOX/(12*(1.0-MU*MU))
      DX=TV
      DY=TV
      DXY=TV*(1.0-MU)/2
      D1=TV*MU
      DO 1850 IJ=1,12
      K5(I,J)=K5(I,J)+KL(I,IJ)*(DX*Q2*K1(IJ,J)+DY*Q1*K2(IJ,J)+
     *D1*K3(IJ,J)+DXY*K4(IJ,J))/(60*AX*BX)
 1850 CONTINUE
      DO .888 I=1,12
      DO 1888 J=1,12
      KE(I,J)=0
      DO 1888 IJ=1,12
      KE(I,J)=KE(I,J)+K5(I,IJ)*KL(IJ,J)
 1888 CONTINUE
 1858 RETURN
      END
C     *****************************************************************
      SUBROUTINE LOAD(E,LLE,LEN)
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER E,A1(4)
      REAL*8 MU,K,KS
      DIMENSION K(24,24),Y(100,2),ZY(100,2),T(24,24)
     *,JM(60,60),PJ(100,3),PO(450),FO(24)
     *,SE(400,400),JMM(150,4),PE(24),P(600),SK(24,20)
     *,BTO(50,4),PF(100,3),PXY(100,5)
      COMMON/L1/A,B,TO
      COMMON/L3/EO,MU
      COMMON/L4/ND
      COMMON/L11/BTO
      COMMON/L5/NJE,NCJE,NEE
      COMMON/L6/T,Y,ZY
      COMMON/L7/K,SE,SK
      COMMON/L8/NPJ,NPF,NPXY
      COMMON/L12/PJ,PO,P
      COMMON/L9/JM,JMM
      COMMON/L10/FO,PF,PXY
      NJE3=6*NJE
      NCJE3=6*NCJE
      MA=NJE3+NCJE3
      DO 1901 JI=1,MA
 1901 PO(JI)=0.
      IF(NPJ.GT.0) GO TO 1900
      GO TO 1940
 1900 DO 1950 I=1,NPJ
      MAC=INT(PJ(I,3))
      IF(MAC.EQ.E) GOTO 1951
      GOTO 1950
 1951 J=INT(PJ(I,2))
      PO(J)=PJ(I,1)
 1950 CONTINUE
```

```
1940 IF (NPF.GT.0) GO TO 1945                                          MAC08810
     GO TO 1990                                                        MAC08820
1945 DO 1960 JJ=1,NPF                                                  MAC08830
     ME=INT(PF(JJ,3))                                                  MAC08840
     IF(ME.EQ.E) GO TO 1946                                            MAC08850
     GO TO 1960                                                        MAC08860
1946 NN=INT(PF(JJ,2))                                                  MAC08870
     CALL GDNL(E,NN,JJ)                                                MAC08880
     DO 1941 I=1,24                                                    MAC08890
     DO 1941 J=1,24                                                    MAC08900
1941 T(I,J)=0.0                                                        MAC08910
     CALL ZB(NN,E,1)                                                   MAC08920
     DO 1964 J=1,24                                                    MAC08930
     PE(J)=0.                                                          MAC08940
     DO 1964 M=1,24                                                    MAC08950
     PE(J)=PE(J)+T(M,J)*FO(M)                                          MAC08960
1964 CONTINUE                                                          MAC08970
1965 DO 1969 IK=1,4                                                    MAC08980
     A1(IK)=JMM(NN,IK)                                                 MAC08990
     DO 1969 J=1,6                                                     MAC09000
1969 PO(6*A1(IK)-(6-J))=PO(6*A1(IK)-(6-J))+PE(6*(IK-1)+J)              MAC09010
1960 CONTINUE                                                          MAC09020
1990 IF (NPXY.GT.0) GO TO 2260                                         MAC09030
     GO TO 2200                                                        MAC09040
2260 DO 2290 JJ=1,NPXY                                                 MAC09050
     ME=INT(PXY(JJ,5))                                                 MAC09060
     IF(ME.EQ.E) GO TO 2201                                            MAC09070
     GO TO 2290                                                        MAC09080
2201 NN=INT(PXY(JJ,4))                                                 MAC09090
     CALL PXOYO(E,NN,JJ)                                               MAC09100
     DO 2272 I=1,24                                                    MAC09110
     DO 2272 J=1,24                                                    MAC09120
2272 T(I,J)=0.0                                                        MAC09130
     CALL ZB(NN,E,1)                                                   MAC09140
     DO 2264 J=1,24                                                    MAC09150
     PE(J)=0.                                                          MAC09160
     DO 2264 M=1,24                                                    MAC09170
     PE(J)=PE(J)+T(M,J)*FO(M)                                          MAC09180
2264 CONTINUE                                                          MAC09190
2265 DO 2269 IK=1,4                                                    MAC09200
     A1(IK)=JMM(NN,IK)                                                 MAC09210
     DO 2269 J=1,6                                                     MAC09220
2269 PO(6*A1(IK)-(6-J))=PO(6*A1(IK)-(6-J))+PE(6*(IK-1)+J)              MAC09230
2290 CONTINUE                                                          MAC09240
2200 CONTINUE                                                          MAC09250
2373 RETURN                                                            MAC09260
     END                                                               MAC09270
C    *****************************************************************MAC09280
     SUBROUTINE GDNL(E,IE,JJ)                                          MAC09290
     IMPLICIT REAL*8(A-H,O-Z)                                          MAC09300
     INTEGER E                                                         MAC09310
     DIMENSION FO(24),BTO(50,4),PF(100,3),PXY(100,5)                   MAC09320
     COMMON/L1/A,B,TO                                                  MAC09330
     COMMON/L4/ND                                                      MAC09340
     COMMON/L11/BTO                                                    MAC09350
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      COMMON/L10/FO,PF,PXY                                              MAC09360
      FF=PF(JJ,1)                                                       MAC09370
      IF(ND.EQ.0) GO TO 2401                                            MAC09380
      DO 2403 I=1,ND                                                    MAC09390
2406  IAT=INT(BTO(I,1))                                                 MAC09400
      IF(IAT.EQ.IE) GO TO 2402                                          MAC09410
      GO TO 2403                                                        MAC09420
2402  AX=BTO(I,2)                                                       MAC09430
      BX=BTO(I,3)                                                       MAC09440
      TOX=BTO(I,4)                                                      MAC09450
      GO TO 2405                                                        MAC09460
2403  CONTINUE                                                          MAC09470
2401  AX=A                                                              MAC09480
      BX=B                                                              MAC09490
      TOX=TO                                                            MAC09500
2405  PAB=4*FF*AX*BX                                                    MAC09510
      DO 2400 MN=1,24                                                   MAC09520
2400  FO(MN)=0.                                                         MAC09530
      FO(3)=PAB/4                                                       MAC09540
      FO(4)=-PAB*BX/12                                                  MAC09550
      FO(5)=PAB*AX/12                                                   MAC09560
      FO(9)=FO(3)                                                       MAC09570
      FO(10)=-FO(4)                                                     MAC09580
      FO(11)=FO(5)                                                      MAC09590
      FO(15)=FO(3)                                                      MAC09600
      FO(16)=FO(4)                                                      MAC09610
      FO(17)=-FO(5)                                                     MAC09620
      FO(21)=FO(3)                                                      MAC09630
      FO(22)=FO(10)                                                     MAC09640
      FO(23)=FO(17)                                                     MAC09650
2480  RETURN                                                            MAC09660
      END                                                               MAC09670
C     *****************************************************************MAC09680
      SUBROUTINE PXOYO(E,IE,JJ)                                         MAC09690
      IMPLICIT REAL*8(A-H,O-Z)                                          MAC09700
      INTEGER E                                                         MAC09710
      DIMENSION FO(24),C(200,200),XYP(12),FOO(12)                       MAC09720
     *,BTO(50,4),PF(100,3),PXY(100,5)                                   MAC09730
      COMMON/L1/A,B,TO                                                  MAC09740
      COMMON/L4/ND                                                      MAC09750
      COMMON/L11/BTO                                                    MAC09760
      COMMON/L10/FO,PF,PXY                                              MAC09770
      PAB=PXY(JJ,1)                                                     MAC09780
      IF(ND.EQ.0) GO TO 2501                                            MAC09790
2506  IAT=INT(BTO(I,1))                                                 MAC09800
      IF(IAT.EQ.IE) GO TO 2502                                          MAC09810
      GO TO 2503                                                        MAC09820
2502  AX=BTO(I,2)                                                       MAC09830
      BX=BTO(I,3)                                                       MAC09840
      TOX=BTO(I,4)                                                      MAC09850
      GO TO 2505                                                        MAC09860
2503  CONTINUE                                                          MAC09870
2501  AX=A                                                              MAC09880
      BX=B                                                              MAC09890
      TOX=TO                                                            MAC09900
```

```
2505  DO 2510 LI=1,12
      FOO(LI)=0
      DO 2510 LJ=1,12
2510  C(LI,LJ)=0
      C(1,1)=1.
      C(2,3)=-1.
      C(3,2)=1.
      C(4,1)=1.
      C(4,3)=2*BX
      C(4,6)=4*BX*BX
      C(4,10)=8*BX*BX*BX
      C(5,3)=-1
      C(5,6)=-4*BX
      C(5,10)=-3*C(4,6)
      C(6,2)=1.
      C(6,5)=2*BX
      C(6,9)=4*BX*BX
      C(6,12)=C(4,10)
      C(7,1)=1.
      C(7,2)=2*AX
      C(7,4)=4*AX*AX
      C(7,7)=8*AX*AX*AX
      C(8,3)=-1.
      C(8,5)=-2*AX
      C(8,8)=-C(7,4)
      C(8,11)=-C(7,7)
      C(9,2)=1.
      C(9,4)=4*AX
      C(9,7)=12*AX*AX
      C(10,1)=1
      C(10,2)=2*AX
      C(10,3)=2*BX
      C(10,4)=4*AX*AX
      C(10,5)=4*AX*BX
      C(10,6)=4*BX*BX
      C(10,7)=8*AX*AX*AX
      C(10,8)=8*AX*BX*AX
      C(10,9)=8*BX*BX*AX
      C(10,10)=8*BX*BX*BX
      C(10,11)=16*AX*AX*AX*BX
      C(10,12)=16*BX*BX*BX*AX
      C(11,3)=-1
      C(11,5)=-2*AX
      C(11,6)=-4*BX
      C(11,8)=-4*AX*AX
      C(11,9)=-8*AX*BX
      C(11,10)=-12*BX*BX
      C(11,11)=-8*AX*AX*AX
      C(11,12)=-24*AX*BX*BX
      C(12,2)=1.0
      C(12,4)=4*AX
      C(12,5)=2*BX
      C(12,7)=12*AX*AX
      C(12,8)=8*AX*BX
      C(12,9)=4*BX*BX
```

```
      C(12,11)=24*AX*AX*BX
      C(12,12)=8*BX*BX*BX
      CALL MATINV(C,12)
      DO 2520 LM=1,12
2520  XYP(LM)=0
      XX=PXY(JJ,2)
      YY=PXY(JJ,3)
      XYP(1)=1.
      XYP(2)=XX
      XYP(3)=YY
      XYP(4)=XX*XX
      XYP(5)=XX*YY
      XYP(6)=YY*YY
      XYP(7)=XX**3
      XYP(8)=XX*XX*YY
      XYP(9)=YY*YY*XX
      XYP(10)=YY**3
      XYP(11)=(XX**3)*YY
      XYP(12)=XX*(YY**3)
      DO 2517 K=1,12
      DO 2517 J=1,12
      FOO(K)=FOO(K)+C(J,K)*XYP(J)
2517  CONTINUE
      DO 2550 LMN=1,24
2550  FO(LMN)=0
      FO(3)=FOO(1)*PAB
      FO(4)=FOO(2)*PAB
      FO(5)=FOO(3)*PAB
      FO(9)=FOO(4)*PAB
      FO(10)=FOO(5)*PAB
      FO(11)=FOO(6)*PAB
      FO(15)=FOO(7)*PAB
      FO(16)=FOO(8)*PAB
      FO(17)=FOO(9)*PAB
      FO(21)=FOO(10)*PAB
      FO(22)=FOO(11)*PAB
      FO(23)=FOO(12)*PAB
2580  RETURN
      END
C     ****************************************************************
      SUBROUTINE MATINV(A,N)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION INDEX(200,2),A(200,200)
      DO 2608 I=1,N
2608  INDEX(I,1)=0
      II=0
2609  AMAX=-1.
      DO 2610 I=1,N
      IF(INDEX(I,1))2610,2611,2610
2611  DO 2612 J=1,N
      IF(INDEX(J,1))2612,2613,2612
2613  TEMP=DABS(A(I,J))
      IF(TEMP-AMAX)2612,2612,2614
2614  IROW=I
      ICOL=J
```

```
      AMAX=TEMP
 2612 CONTINUE
 2610 CONTINUE
      IF(AMAX)2699,2615,2616
 2616 INDEX(ICOL,1)=IROW
      IF(IROW-ICOL)2619,2618,2619
 2619 DO 2620 J=1,N
      TEMP=A(IROW,J)
      A(IROW,J)=A(ICOL,J)
 2620 A(ICOL,J)=TEMP
      II=II+1
      INDEX(II,2)=ICOL
 2618 PIVOT=A(ICOL,ICOL)
      A(ICOL,ICOL)=1.
      PIVOT=1./PIVOT
      DO 2621 J=1,N
 2621 A(ICOL,J)=A(ICOL,J)*PIVOT
      DO 2622 I=1,N
      IF(I-ICOL)2623,2622,2623
 2623 TEMP=A(I,ICOL)
      A(I,ICOL)=0.
      DO 2624 J=1,N
 2624 A(I,J)=A(I,J)-A(ICOL,J)*TEMP
 2622 CONTINUE
      GO TO 2609
 2625 ICOL=INDEX(II,2)
      IROW=INDEX(ICOL,1)
      DO 2626 I=1,N
      TEMP=A(I,IROW)
      A(I,IROW)=A(I,ICOL)
 2626 A(I,ICOL)=TEMP
      II=II-1
 2699 IF(II)2625,2627,2625
 2615 CONTINUE
 2627 CONTINUE
      RETURN
      END
C     ******************************************************************
      SUBROUTINE IDUGD(LE,E)
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER E
      REAL*8 MU,K
      DIMENSION SK(24,20),BTO(50,4),K(24,24),SE(400,400)
      COMMON/L1/A,B,TO
      COMMON/L3/EO,MU
      COMMON/L4/ND
      COMMON/L11/BTO
      COMMON/L7/K,SE,SK
      IF(ND.EQ.0) GO TO 2709
      DO 2703 I=1,ND
 2706 IAT=INT(BTO(I,1))
      IF(IAT.EQ.LE) GO TO 2702
      GO TO 2703
 2702 AX=BTO(I,2)
      BX=BTO(I,3)
```

```
      TOX=BTO(I,4)                                                      MAC11560
      GO TO 2705                                                        MAC11570
2703  CONTINUE                                                          MAC11580
2709  AX=A                                                              MAC11590
      BX=B                                                              MAC11600
      TOX=TO                                                            MAC11610
2705  ETV=EO/(1.-MU*MU)                                                 MAC11620
      DO 2710 I=1,24                                                    MAC11630
      DO 2710 J=1,20                                                    MAC11640
2710  SK(I,J)=0.                                                        MAC11650
      SK(1,1)=(-1/(2*AX))*ETV                                           MAC11660
      SK(1,2)=-MU/(2*BX)*ETV                                            MAC11670
      SK(1,7)=MU/(2*BX)*ETV                                             MAC11680
      SK(1,11)=1/(2*AX)*ETV                                             MAC11690
      SK(2,1)=-MU/(2*AX)*ETV                                            MAC11700
      SK(2,2)=-1/(2*BX)*ETV                                             MAC11710
      SK(2,7)=1/(2*BX)*ETV                                              MAC11720
      SK(2,11)=MU/(2*AX)*ETV                                            MAC11730
      SK(3,1)=-(1-MU)/(4*BX)*ETV                                        MAC11740
      SK(3,2)=-(1-MU)/(4*AX)*ETV                                        MAC11750
      SK(3,6)=(1-MU)/(4*BX)*ETV                                         MAC11760
      SK(3,12)=(1-MU)/(4*AX)*ETV                                        MAC11770
      SK(7,2)=-MU/(2*BX)*ETV                                            MAC11780
      SK(7,6)=(-1/(2*AX))*ETV                                           MAC11790
      SK(7,7)=MU/(2*BX)*ETV                                             MAC11800
      SK(7,16)=(1/(2*AX))*ETV                                           MAC11810
      SK(8,2)=-1/(2*BX)*ETV                                             MAC11820
      SK(8,6)=-MU/(2*AX)*ETV                                            MAC11830
      SK(8,7)=1/(2*BX)*ETV                                              MAC11840
      SK(8,16)=MU/(2*AX)*ETV                                            MAC11850
      SK(9,1)=-(1-MU)/(4*BX)*ETV                                        MAC11860
      SK(9,6)=(1-MU)/(4*BX)*ETV                                         MAC11870
      SK(9,7)=-(1-MU)/(4*AX)*ETV                                        MAC11880
      SK(9,17)=(1-MU)/(4*AX)*ETV                                        MAC11890
      SK(13,1)=(-1/(2*AX))*ETV                                          MAC11900
      SK(13,11)=(1/(2*AX))*ETV                                          MAC11910
      SK(13,12)=(-MU/(2*BX))*ETV                                        MAC11920
      SK(13,17)=(MU/(2*BX))*ETV                                         MAC11930
      SK(14,1)=(-MU/(2*AX))*ETV                                         MAC11940
      SK(14,11)=(MU/(2*AX))*ETV                                         MAC11950
      SK(14,12)=(-1/(2*BX))*ETV                                         MAC11960
      SK(14,17)=(1/(2*BX))*ETV                                          MAC11970
      SK(15,2)=-(1-MU)/(4*AX)*ETV                                       MAC11980
      SK(15,11)=-(1-MU)/(4*BX)*ETV                                      MAC11990
      SK(15,12)=(1-MU)/(4*AX)*ETV                                       MAC12000
      SK(15,16)=(1-MU)/(4*BX)*ETV                                       MAC12010
      SK(19,6)=(-1/(2*AX))*ETV                                          MAC12020
      SK(19,12)=-MU/(2*BX)*ETV                                          MAC12030
      SK(19,16)=1/(2*AX)*ETV                                            MAC12040
      SK(19,17)=MU/(2*BX)*ETV                                           MAC12050
      SK(20,6)=-MU/(2*AX)*ETV                                           MAC12060
      SK(20,12)=-1/(2*BX)*ETV                                           MAC12070
      SK(20,16)=MU/(2*AX)*ETV                                           MAC12080
      SK(20,17)=1/(2*BX)*ETV                                            MAC12090
      SK(21,7)=-(1-MU)/(4*AX)*ETV                                       MAC12100
```

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      SK(21,11)=-(1-MU)/(4*BX)*ETV                                      MAC12110
      SK(21,16)=(1-MU)/(4*BX)*ETV                                       MAC12120
      SK(21,17)=(1-MU)/(4*AX)*ETV                                       MAC12130
      Q1=AX/BX                                                          MAC12140
      Q2=BX/AX                                                          MAC12150
      Q3=4*AX*BX                                                        MAC12160
      EV=EO*TOX*TOX*TOX/(12*(1.-MU*MU))                                 MAC12170
      DX=EV/Q3                                                          MAC12180
      DY=EV/Q3                                                          MAC12190
      D1=EV*MU/Q3                                                       MAC12200
      DXY=EV*(1-MU)/(2*Q3)                                              MAC12210
      SK(4,3)=6*(Q2*DX+Q1*D1)                                           MAC12220
      SK(4,4)=-8*AX*D1                                                  MAC12230
      SK(4,5)=8*BX*DX                                                   MAC12240
      SK(4,8)=-6*Q1*D1                                                  MAC12250
      SK(4,9)=-4*AX*D1                                                  MAC12260
      SK(4,13)=-6*Q2*DX                                                 MAC12270
      SK(4,15)=4*BX*DX                                                  MAC12280
      SK(5,3)=6*(Q1*DY+Q2*D1)                                           MAC12290
      SK(5,4)=-8*AX*DY                                                  MAC12300
      SK(5,5)=8*BX*D1                                                   MAC12310
      SK(5,8)=-6*Q1*DY                                                  MAC12320
      SK(5,9)=-4*AX*DY                                                  MAC12330
      SK(5,13)=-6*Q2*D1                                                 MAC12340
      SK(5,15)=4*BX*D1                                                  MAC12350
      SK(6,3)=-2*DXY                                                    MAC12360
      SK(6,4)=4*BX*DXY                                                  MAC12370
      SK(6,5)=-4*AX*DXY                                                 MAC12380
      SK(6,8)=2*DXY                                                     MAC12390
      SK(6,10)=4*AX*DXY                                                 MAC12400
      SK(6,13)=2*DXY                                                    MAC12410
      SK(6,14)=-4*BX*DXY                                                MAC12420
      SK(6,18)=-2*DXY                                                   MAC12430
      SK(10,3)=-6*Q1*D1                                                 MAC12440
      SK(10,4)=4*AX*D1                                                  MAC12450
      SK(10,8)=6*(Q1*DY+Q2*D1)                                          MAC12460
      SK(10,9)=8*AX*D1                                                  MAC12470
      SK(10,10)=8*BX*DX                                                 MAC12480
      SK(10,18)=-6*Q2*DX                                                MAC12490
      SK(10,20)=4*BX*DX                                                 MAC12500
      SK(11,3)=-6*Q1*DY                                                 MAC12510
      SK(11,4)=4*AX*DY                                                  MAC12520
      SK(11,8)=6*(Q1*DY+Q2*D1)                                          MAC12530
      SK(11,9)=8*AX*DY                                                  MAC12540
      SK(11,10)=8*BX*D1                                                 MAC12550
      SK(11,18)=-6*Q2*D1                                                MAC12560
      SK(11,20)=4*BX*D1                                                 MAC12570
      SK(12,3)=-2*DXY                                                   MAC12580
      SK(12,5)=4*AX*DXY                                                 MAC12590
      SK(12,8)=2*DXY                                                    MAC12600
      SK(12,9)=4*BX*DXY                                                 MAC12610
      SK(12,10)=4*AX*DXY                                                MAC12620
      SK(12,13)=2*DXY                                                   MAC12630
      SK(12,18)=-2*DXY                                                  MAC12640
      SK(12,19)=-4*BX*DXY                                               MAC12650
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      SK(16,3)=-6*Q2*DX                                                 MAC12660
      SK(16,5)=-4*BX*DX                                                 MAC12670
      SK(16,13)=6*(Q2*DX+Q1*D1)                                         MAC12680
      SK(16,14)=-8*AX*D1                                                MAC12690
      SK(16,15)=-8*BX*DX                                                MAC12700
      SK(16,18)=-6*Q1*D1                                                MAC12710
      SK(16,19)=-4*AX*D1                                                MAC12720
      SK(17,3)=-6*Q2*D1                                                 MAC12730
      SK(17,5)=-4*BX*D1                                                 MAC12740
      SK(17,13)=6*(Q1*DY+Q2*D1)                                         MAC12750
      SK(17,14)=-8*AX*DY                                                MAC12760
      SK(17,15)=-8*BX*D1                                                MAC12770
      SK(17,18)=-6*Q1*DY                                                MAC12780
      SK(17,19)=-4*AX*DY                                                MAC12790
      SK(18,3)=-2*DXY                                                   MAC12800
      SK(18,4)=4*BX*DXY                                                 MAC12810
      SK(18,8)=2*DXY                                                    MAC12820
      SK(18,13)=2*DXY                                                   MAC12830
      SK(18,14)=-4*BX*DXY                                               MAC12840
      SK(18,15)=-4*AX*DXY                                               MAC12850
      SK(18,18)=-2*DXY                                                  MAC12860
      SK(18,20)=4*AX*DXY                                                MAC12870
      SK(22,8)=-6*Q2*DX                                                 MAC12880
      SK(22,10)=-4*BX*DX                                                MAC12890
      SK(22,13)=-6*Q1*D1                                                MAC12900
      SK(22,14)=4*AX*D1                                                 MAC12910
      SK(22,18)=6*(Q2*DX+Q1*D1)                                         MAC12920
      SK(22,19)=8*AX*D1                                                 MAC12930
      SK(22,20)=-8*BX*DX                                                MAC12940
      SK(23,8)=-6*Q2*D1                                                 MAC12950
      SK(23,10)=-4*BX*D1                                                MAC12960
      SK(23,13)=-6*Q1*DY                                                MAC12970
      SK(23,14)=4*AX*DY                                                 MAC12980
      SK(23,18)=6*(Q1*DY+Q2*D1)                                         MAC12990
      SK(23,19)=8*AX*DY                                                 MAC13000
      SK(23,20)=-8*BX*D1                                                MAC13010
      SK(24,3)=-2*DXY                                                   MAC13020
      SK(24,8)=2*DXY                                                    MAC13030
      SK(24,9)=4*BX*DXY                                                 MAC13040
      SK(24,13)=2*DXY                                                   MAC13050
      SK(24,15)=-4*AX*DXY                                               MAC13060
      SK(24,18)=-2*DXY                                                  MAC13070
      SK(24,19)=-4*BX*DXY                                               MAC13080
      SK(24,20)=4*AX*DXY                                                MAC13090
 2799 RETURN                                                            MAC13100
      END                                                               MAC13110
```