ANALYSIS OF TIME-PERIODIC NAVIER-STOKES EQUATIONS IN A MOVING DOMAIN AND NUMERICAL COMPUTATIONS WITH RADIAL BASIS NEURAL NETWORKS. APPLICAION TO ARTIFICIAL HEARTS BLOOD FLOW

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Abstract

The dynamics of blood flow within an artificial heart (AH) chamber are governed by the time-periodic Navier-Stokes (NS) equations. These equations are coupled with a hyperbolic partial differential equation (PDE) that describes the dynamic behavior of the membrane (diaphragm). This coupled system of PDEs is subject to a moving boundary.

In this work, we decouple the system by assuming that the solution of the membrane equation is known within a well-defined Banach space. Subsequently, we solve only the NS problem and not the coupled fluid-structure problem. We conduct an analysis, specifically examining the existence and uniqueness of a time-periodic strong solution for the Navier-Stokes (NS) problem in dimensions $N = 2, 3$ within a moving domain. The moving domain Ω_t is supposed to be a $L^\infty(\mathbf{W}^{2,\hat{2}^*}) \cap H^1(\mathbf{H}^2) \cap H^2(\mathbf{L}^2)$ local perturbation of a $C^{1,1}$ reference domain Ω_0 where $\hat{2}^* = 2 + \frac{4}{\epsilon}$, $\epsilon > 0$ if $N = 2$ and $\hat{2}^* = 3 + \frac{\epsilon}{4-\epsilon}$, $\epsilon \in (0, 3]$ if $N = 3$. Typically, our moving domain Ω_t is of class $L^\infty(\mathbf{W}^{2,\hat{2}^*}) \cap H^1(\mathbf{H}^2) \cap H^2(\mathbf{L}^2)$, which is weaker that the C^3 in space regularity presented in the literature.

Subsequently, we proceed to numerically solve the problem in dimension $N = 2$ using radial basis neural network (RBNN) functions. To validate our computational framework, we compare our results against existing literature, particularly by solving benchmark-driven cavity problems up to a Reynolds number $Re = 10,000$. Finally, we solve numerically the time-periodic NS equations for different AH geometries. The results we obtain demonstrate the strong dependence of the blood flow behavior on the AH geometry and motivate the optimal shape design of AH, which we plan to address in future work.

Key words: Navier- Stokes equations, Radial basis neural network, Artificial heart

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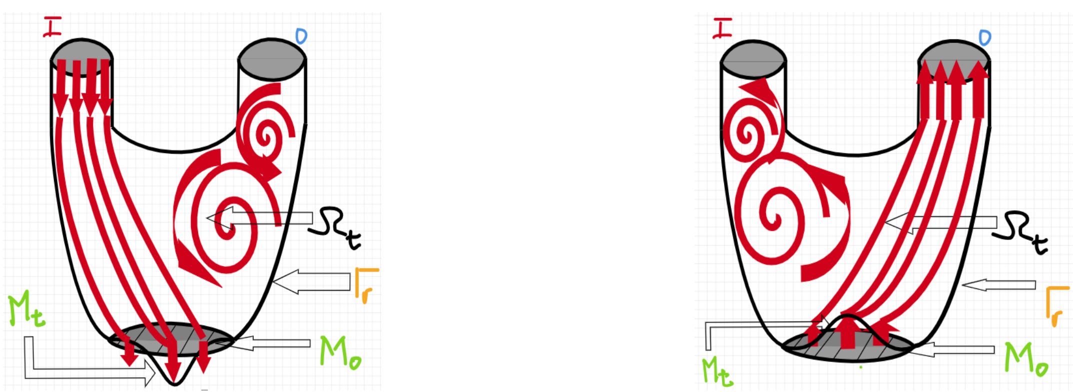
Chapter 1

Introduction

1.1 Motivation

In this thesis, we analyze the time-periodic Navier-Stokes (NS) equations in a moving domain, which describe the blood flow in an artificial heart. We also perform numerical computations using neural networks. The problem is periodic with a period of $4T$, where $T > 0$. Figure 1.1 illustrates the design of the artificial heart (AH) domain considered here during the first and second half-periods.



(a) First half period $[-T, T]$ (b) Second half period $[T, 3T]$

Figure 1.1: A schematic view of an artificial heart: Ω_t is the moving domain occupied by the blood, I is the inlet valve, O the outlet valve, Γ_r the rigid boundary, M_t the membrane moving up and down.

The Navier-Stokes (NS) equations are nonlinear partial differential equations (PDEs) that describe the motion of Newtonian fluids, such as gases and most liquids. These equations are essential for modeling phenomena such as weather patterns, airflow around airplane wings, and blood flow through vessels and organs. In this thesis, we use the NS equations to model blood flow in an artificial heart.

The NS equations hold significant interest in mathematics, particularly because the existence of a smooth solution in three dimensions (known as the NS millennium problem) remains unresolved. Specifically, if we assume that the external force in the NS equations is identically zero, and that for any smooth and divergence-free initial velocity, a certain growth condition hypothesis holds, and the initial kinetic energy is globally bounded, the challenge is to determine whether a global smooth solution exists such that the kinetic energy remains globally bounded at all times. This problem is one of the seven most important open problems in mathematics.

This thesis, proposed to me by my thesis supervisor Arian Novruzi, is a part of his project on the optimal shape design of artificial hearts. Here we consider only the analysis and numerical computations of the time-periodic NS equations, describing the blood flow in a moving AH domain.

1.2 Presentation of the mathematical model of artificial hearts

In this section, we present our mathematical model. The dynamics of blood in an AH are governed by the NS equations, coupled with the hyperbolic PDEs describing the motion of the membrane or diaphragm. An external compressor activates the motion of the membrane, which creates the flow of the blood. Figure 1.1 shows us the flow of the blood in an AH design during the first and the second half-periods.

We will present the complete mathematical model as introduced by Ahmed et al. [2]. We start by presenting the mathematical model of the membrane, which is a hyperbolic and time-periodic PDE with homogeneous Dirichlet boundary conditions. Next, we will present the equations of blood flow in the heart chamber, which are nonlinear time-periodic NS equations, subject to the incompressibility condition as well as non-homogeneous Dirichlet boundary conditions. This gives rise to the complete coupled system of nonlinear PDEs with moving boundaries, describing the complete fluid-structure system of the blood flowing in an AH.

1.2.1 Mathematical model of the membrane

The motion of the fluid in an AH is generated by the motion of the membrane. In this paragraph, we will describe the mathematical model of the membrane.

The mathematical model of the membrane can be derived by applying the D'Alembert principle of least action to an appropriate integral function. We denote by N the dimension of the space. We denote by ρ_m the mass density of the membrane, and $z = z(x_1, ..., x_{N-1}, t)$ the distance from the membrane at time t to the base of the heart chamber $B_{N-1}(\ell_0)$ where the membrane is attached. Here $B_{N-1}(\ell_0)$ represents the open ball in $\mathbb{R}^{N-1}$ of center $\mathbf{0}$ and with radius ℓ_0. The kinetic and potential energies of the membrane are respectively given by

$$K = \frac{1}{2}\rho_m(\partial_t z)^2 \text{ and } V = \frac{1}{2}\,{}^t(\nabla z)\cdot\sigma\cdot\nabla z,$$

where $\sigma = (\sigma_{ij})$ represents the stress (force per unit of area) tensor of the membrane assumed to be symmetric matrix with constant entries, ρ_m the mass density of the membrane and the symbol " $\cdot$ " stands for the standard matrix-vector product. We introduce the Lagrangian operator, which is here the difference between the kinetic energy and the potential energy. To fix the idea, in the case $N = 3$, the Lagrangian operator is given by

$$\begin{aligned} L(z, \partial_{x_1} z, \partial_{x_2} z, \partial_t z) &= K - V \\ &= \frac{1}{2}[\rho_m(\partial_t z)^2 - (\sigma_{11}(\partial_{x_1} z)^2 + 2\sigma_{12}\partial_{x_1} z\partial_{x_2} z + \sigma_{22}(\partial_{x_2} z)^2)]. \end{aligned}$$

The action is defined by the following integral

$$I := \int_{B_{N-1}(\ell_0)} L(z, \partial_{x_1} z, \partial_{x_2} z, \partial_t z)\, \mathrm{d}x_1\, \mathrm{d}x_2.$$

By the D'Alembert principle of least action, we have the following Euler- Lagrange equation which describes the dynamic of the membrane,

$$\rho_m \partial_{tt} z - lz = f, \tag{1.2.1}$$

where

$$lz := \sigma_{11}\partial_{x_1x_1} z + 2\sigma_{12}\partial_{x_1x_2} z + \sigma_{22}\partial_{x_2x_2} z,$$

and

$$f := p_m - p(x_1, x_2, z(x_1, x_2, t), t),$$

represents the difference between the internal pressure inside the compressor acting on the membrane and the pressure of the blood inside the heart chamber. We use the fact that the volume m_0 of the blood being sucked during the first half period $(-T, T)$ is equal to the volume of the blood being pumped out during the second half period $(T, 3T)$ to write

$$\begin{aligned} &-\rho \int_{-T}^{T} \Big(\int_{B_{N-1}(\ell_0)} \partial_t z(x_1, ..., x_{N-1}, t)\, \mathrm{d}x_1 ... \mathrm{d}x_{N-1} \Big) \mathrm{d}t \\ &= \rho \int_{T}^{3T} \Big(\int_{B_{N-1}(\ell_0)} \partial_t z(x_1, ..., x_{N-1}, t)\, \mathrm{d}x_1 ... \mathrm{d}x_{N-1} \Big) \mathrm{d}t \\ &= m_0, \end{aligned} \tag{1.2.2}$$

which is equivalent to write after integration with respect to t and using the $4T$ time-periodicity of z

$$\int_{B_{N-1}(\ell_0)} (z(x_1, ..., x_{N-1}, 3T) - z(x_1, ..., x_{N-1}, T))\, \mathrm{d}x_1 ... \mathrm{d}x_{N-1} = \frac{m_0}{\rho}. \tag{1.2.3}$$

Now, by taking into consideration the boundary conditions, the periodicity and the equation (1.2.3), we have the following mathematical model of the membrane

$$\begin{cases} \rho_m(t)\partial_{tt} z(\cdot,t) - lz(\cdot,t) = p_m(\cdot,t) - p(\cdot, z(\cdot,t), t) \text{ in } B_{N-1}(\ell_0), & (1.2.4)\\ z = 0 \text{ on } \partial B_{N-1}(\ell_0), & (1.2.5)\\ z(\cdot,t) = z(\cdot, t+4T) \text{ in } B_{N-1}(\ell_0), & (1.2.6)\\ \partial_t z(\cdot,t) = \partial_t z(\cdot,t+4T) \text{ in } B_{N-1}(\ell_0), & (1.2.7)\\ \displaystyle\int_{M_0} (z(\cdot,3T) - z(\cdot,T))\, \mathrm{d}x_1 ... \mathrm{d}x_{N-1} = \frac{m_0}{\rho}, & (1.2.8) \end{cases}$$

where ρ is the mass density of the fluid and $4T$ is the period.

1.2.2 Mathematical model of the blood flow

We denote by Ω_t the moving domain of AH, which is an open and bounded set of $\mathbb{R}^N$. The vector $\mathbf{u} \in \mathbb{R}^N$ represents the velocity of the fluid and p is a scalar function representing the pressure. From the Newton fundamental principle of fluid dynamics associated with the incompressibility condition (divergence-free), and taking into consideration the time-periodicity, we know that the fluid motion is governed by the following time-periodic NS equations

$$\begin{cases} \rho \partial_t \mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \rho\vec{g} \text{ in } \Omega_t \times \{t\}, & (1.2.9)\\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega_t\times\{t\}, & (1.2.10)\\ \mathbf{u}(\cdot,t) = \mathbf{u}(\cdot,t+4T) \text{ in } \Omega_t,\ \forall\, t\in\mathbb{R}, & (1.2.11) \end{cases}$$

where $\vec{g}$ is the gravity vector (acceleration), ρ the mass density of the fluid (blood) and μ its kinematic viscosity. The boundary of our moving domain $\partial\Omega_t$, consists of four parts described as

$$\partial\Omega_t := M_t \cup \Gamma_r \cup \Gamma_i \cup \Gamma_o,$$

where M_t is the flexible membrane boundary, moving with time, Γ_r is the rigid membrane boundary on which the velocity vanishes, Γ_i is the inlet valve which opens during the diastole phase (general rest of the heart), i.e., in $[-T,T)$ to receive the blood inside the chamber, and closes during the systole phase (contraction phase of the membrane), i.e., in $[T,3T)$. On the other hand, Γ_o is the outlet valve which opens during the systole phase and closes during the diastole. We define the boundary conditions on each part as:

$$\begin{cases} \mathbf{u}(x,t) = 0, \quad (x,t)\in\Gamma_r\times\mathbb{R}, & (1.2.12)\\ \mathbf{u}(x,t) = k_i(t)\mathbf{u}_i(x), \quad (x,t)\in\Gamma_i\times\mathbb{R}, & (1.2.13)\\ \mathbf{u}(x,t) = k_o(t)\mathbf{u}_o(x), \quad (x,t)\in\Gamma_o\times\mathbb{R}, & (1.2.14)\\ \mathbf{u}(x,t) = (0,...,0,\partial_t z(x_1,...,x_{N-1},t)), \quad (x,t)\in M_t\times\{t\}, & (1.2.15) \end{cases}$$

where z is the solution to the membrane dynamic equations (1.2.4) − (1.2.8), $\mathbf{u}_i$ and $\mathbf{u}_o$ are the Poiseuille flow conditions, which are given parabolic velocity profiles. The periodic scalar function k_i resp. k_o, defined on $[-T,T)$ resp. $[T,3T)$ is the flux condition on Γ_i

resp. Γ_o and they are chosen such that the free divergence condition (1.2.10) holds. The expression for $\mathbf{u}_\alpha$ and k_α, $\alpha \in \{i, o\}$, will be precise later in Chapter 3, where we carry out the analysis of the fluid system.

1.2.3 Presentation of the mathematical problem and the main result

In this thesis, we assume that we know z, the solution of the mathematical model of the membrane (1.2.4) – (1.2.8) in a well defined Banach space to be defined later in Chapter 3, and we consider the NS problem (1.2.18) – (1.2.24). The objective is to prove the existence and uniqueness of a time-periodic solution to the NS problem (1.2.18)–(1.2.24) in dimension $N = 2, 3$ and numerically simulate the flow using a neural network method.

Since the membrane is moving with time, Ω_t represents the domain inside the heart chamber at time t, and M_t represents the position of the membrane at time t, which is given by

$$M_t := \{(x_1, ..., x_{N-1}, z(x_1, ..., x_{N-1}, t)) \in \mathbb{R}^N, \quad (x_1, ..., x_{N-1}) \in B_{N-1}(\ell_0),\ t \in \mathbb{R}\}, \tag{1.2.16}$$

and which is moving around the position M_0 given by

$$M_0 := B_{N-1}(\ell_0) \times \{0\}. \tag{1.2.17}$$

The mathematical problem we are considering reads

$$\begin{cases} \rho\partial_t \mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \rho\vec{g} \text{ in } \Omega_t \times \{t\}, \quad t \in \mathbb{R}, & (1.2.18)\\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega_t \times \{t\}, \quad t \in \mathbb{R}, & (1.2.19)\\ \mathbf{u}(\cdot, t) = \mathbf{u}(\cdot, t + 4T) \text{ in } \Omega_t, \quad t \in \mathbb{R}, & (1.2.20)\\ \mathbf{u}(x, t) = 0, \quad (x, t) \in \Gamma_r \times \mathbb{R}, & (1.2.21)\\ \mathbf{u}(x, t) = k_i(t)\mathbf{u}_i(x), \quad (x, t) \in \Gamma_i \times \mathbb{R}, & (1.2.22)\\ \mathbf{u}(x, t) = k_o(t)\mathbf{u}_o(x), \quad (x, t) \in \Gamma_o \times \mathbb{R}, & (1.2.23)\\ \mathbf{u}(x, t) = (0, ..., 0, \partial_t z(x_1, ..., x_{N-1}, t)), \quad (x, t) \in M_t \times \{t\}, \quad t \in \mathbb{R}. & (1.2.24) \end{cases}$$

There are a number of works in the literature dealing with the time-periodic NS equations in a moving domain. Hiroko [44] initiated the existence theory for time-periodic incompressible NS equations in a periodic moving domain. He proved the existence of solution in dimension $N = 2, 3$, with the assumption that the moving domain is C^3. Miyakawa and Teramoto [75] proved the existence of a global weak time-periodic solution in a domain whose boundary moves smoothly. They used a C^∞ diffeomorphism to set the problem on a fixed domain. Next, they proved the existence of a global weak time-periodic solution of the newly defined problem by a Faedo-Galerkin approach. Salvi [86] proved the existence of a time-periodic weak solution to NS equations on a moving domain using the elliptic regularization in a case of a C^3 moving boundary. Farwig et al. [31] proved an existence and uniqueness result of a time-periodic solution in the case of C^3 moving domains and dimension $N \geq 2$. In the case when the analysis is related directly to an application, Casanova [14] proved the existence of a time-periodic solution to the time-dependent NS equations coupled with a forth order equation of a beam, in a moving domain which turns out to have regularity H^4 in dimension $N = 2$. Knobloch and Krechetnikov [52] presented a list of problems leading to time-periodic NS problems in moving domain, and discussed related dynamics and challenges.

In this thesis, we prove the existence and uniqueness of a strong time-periodic solution $(\mathbf{u}, p)$ to the NS equations (1.2.18) − (1.2.24) with $(\mathbf{u}, p) \in L^2(\mathbf{H}^2) \cap H^1(\mathbf{L}^2) \times L^2(H^1)$ in dimension $N = 2, 3$ using the implicit function theorem. We weakened the standard C^3 regularity of the moving domain Ω_t as presented in the literature. Here we assume that the moving domain Ω_t is a $L^\infty(\mathbf{W}^{2,\hat{2}^*}) \cap H^1(\mathbf{H}^2) \cap H^2(\mathbf{L}^2)$ local perturbation of a $C^{1,1}$ reference domain Ω_0, where $\hat{2}^* = 2 + \frac{4}{\epsilon}$ with $\epsilon > 0$ if $N = 2$ and $\hat{2}^* = 3 + \frac{\epsilon}{4-\epsilon}$ with $\epsilon \in (0, 3]$ if $N = 3$. Therefore, our moving domain is of class $L^\infty(\mathbf{W}^{2,\hat{2}^*}) \cap H^1(\mathbf{H}^2) \cap H^2(\mathbf{L}^2)$ which is weaker that C^3. To the best of our knowledge, this is a new result in comparison with the C^3 in space regularity of Ω_t in the literature. We also present numerical simulations in Python of the blood flow in an AH domain using radial basis neural network (RBNN) functions. We use the benchmark lid-driven cavity problem on a NS flow up to $Re = 10,000$ to validate our numerical method.

This thesis is organized as follows: In Chapter 2, we will provide the mathematical background. We will start by presenting useful fundamental vector spaces and theorems for the analysis. We will also present background information on the analysis of Stokes and NS equations on a fixed domain because after transformation, the analysis of our problem will be done on a fixed domain.

Next, in Chapter 3, we will assume knowledge of z, the solution of the membrane equation in a well-defined Banach space, and we will solve the fluid system (1.2.18) − (1.2.24), which is a time-periodic NS equation in a variable domain Ω_t. More precisely, we will prove the existence and uniqueness of a strong time-periodic solution using the implicit function theorem.

In Chapter 4, we will present our numerical method. Since we have chosen a neural network method, we will start by including a general background information on neural networks method for solving PDEs. Next, we will introduce the RBNN method for solving PDEs. Additionally, we will present our numerical results using the RBNN method on the augmented Lagrangian formulation. We will test our method on several benchmark problems such as the lid-driven cavity problem for both the Stokes flow and a Navier-Stokes flow up to $Re = 10,000$. Finally, we solve the time-periodic NS equations on four artificial heart domains and visually observe a strong dependence of vortices development on domain geometry. This observation aligns with the findings reported in [97].

Chapter 2

Mathematical background

This chapter provides background on the analysis of weak solutions to Stokes and NS equations, namely, existence and uniqueness results, as discussed by Leray [60], Lions [63, 64, 65], Lions and Magenes [66], Serrin [88, 89, 90, 91], Ladyzhenskaya [54], Raviart and Girault [85], Temam [98], Farwig [29]. This provides a comprehensive foundation for understanding the concepts involved.

This background is instructive for our problem. Indeed, though our problem is NS equations in a moving domain, the analysis is made on a fixed domain using a transformation. This background deals with the standard analysis of Stokes and NS on a fixed domain.

We will present results and useful tools regarding the existence and uniqueness of the linear, nonlinear, steady, and unsteady cases of the NS equations on a fixed domain. We start this chapter by defining fundamental spaces for the study of NS equations. Most importantly, functional spaces with free divergence are emphasized. Next, we will examine the steady-state Stokes equation, discussing the existence and uniqueness of a weak solution. Then, we address the steady-state nonlinear NS equation, providing a general result of the existence of a weak solution for all $N \geq 2$, and uniqueness only for $N \leq 4$. Finally, we discuss the time-dependent Navier-Stokes equations, covering both the linear and the nonlinear cases. For the linear case, we present a general result of existence and uniqueness, and for the nonlinear case, known as the full NS equation, we state a result of existence for $N \leq 4$ and uniqueness for $N = 2$. In this review, we follow the reference by Temam [98].

2.1 Some functional spaces

Let $p,\ q \in [1, \infty]$, $s > 0$, where $\mathbb{R}$ is the set of real numbers. We denote by $\lfloor s \rfloor$ the integer part of s. Let $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. We denote by $N \in \mathbb{N}^*$ the dimension of the space, where $\mathbb{N}^*$ is the set of positive integers. In our case $N = 2,\ 3$. Let $X_t \subset \mathbb{R}^N$ be a smooth open and bounded set, depending on a variable $t \in \mathbb{R}$, such that for every $t \in \mathbb{R}$ and for every open set $\omega \subset \mathbb{R}^N$, if $\omega \subsetneq X_t$, i.e. $\omega \subset \overline{\omega} \subset X_t$, then $\omega \subsetneq X_{t+h}$ for all $|h|$ small and

$$\lim_{|h|\to 0} |X_{t+h} \setminus X_t| = 0, \tag{2.1.1}$$

where here $|\cdot|$ denote the Lebesgue measure in $\mathbb{R}^N$.

We denote by $L^p(X_t)$ the classical L^p space in X_t, and by $L^p_0(X_t)$ the class of L^p space in X_t with zero mean. More precisely

$$L^p_0(X_t) := \Big\{ f \in L^p(X_t), \quad \int_{X_t} f(x)\, \mathrm{d}x = 0 \Big\}. \tag{2.1.2}$$

We denote by $W^{k,p}(X_t)$ the classical Sobolev space of order k in X_t defined as

$$W^{k,p}(X_t) := \Big\{ u \in L^p(X_t), \quad D^\alpha u \in L^p(X_t),\ \forall\, \alpha \in \mathbb{N}^N,\ |\alpha| \le k \Big\} \tag{2.1.3}$$

endowed with the norm

$$\|u\|_{W^{k,p}(X_t)} := \begin{cases} \Big(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p(X_t)} \Big)^{\frac{1}{p}} \text{ if } p \neq \infty & (2.1.4) \\ \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty(X_t)} \text{ if } p = \infty. & (2.1.5) \end{cases}$$

For $k = 0$, the space $W^{k,p}(X_t)$ becomes the usual $L^p(X_t)$ space. For $s > 0$ and $p \in [1, +\infty)$, we introduce $W^{s,p}(X_t)$ the classical Besov space in X_t defined as

$$W^{s,p}(X_t) := \Big\{ u \in W^{\lfloor s \rfloor,p}(X_t), \quad \sum_{|\alpha|=\lfloor s \rfloor} \int_{X_t}\int_{X_t} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+p(s-\lfloor s \rfloor)}}\, \mathrm{d}x\, \mathrm{d}y < \infty \Big\} \tag{2.1.6}$$

endowed with the norm

$$\|u\|_{W^{s,p}(X_t)} := \Big(\|u\|^p_{W^{\lfloor s \rfloor,p}(X_t)} + \sum_{|\alpha|=\lfloor s \rfloor} \int_{X_t}\int_{X_t} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+p(s-\lfloor s \rfloor)}}\, \mathrm{d}x\, \mathrm{d}y \Big)^{\frac{1}{p}}. \tag{2.1.7}$$

We denote by $\mathcal{D}(X_t)$ the set of C^∞ functions with compact support in X_t and its topological dual $\mathcal{D}'(X_t)$ represents the usual space of distribution in X_t. We introduce the space $W^{s,p}_0(X_t) := \overline{\mathcal{D}(X_t)}^{\|\cdot\|_{W^{s,p}(X_t)}}$ i.e., the closure of the space $\mathcal{D}(X_t)$ with the norm of $W^{s,p}(X_t)$. For $s > 0$, we introduce the space $W^{-s,p}(X_t) := (W^{s,p}_0(X_t))'$ which is the topological dual of $W^{s,p}_0(X_t)$. For $p = 2$ we write $H^s(X_t)$ instead of $W^{s,2}(X_t)$.

In our case, t represents the time variable and the problem is periodic with period $4T$. For a given Banach space, we will use the under-script "T" to denote the $4T$ time-periodicity. Precisely, we consider the following time-dependent Sobolev's spaces.

$$L^q_T(W^{s,p}(X_t)) := \Big\{ u(\cdot,\tau) \in W^{s,p}(X_t), \ \ \textit{for a.e. } \tau \in \mathbb{R},$$
$$u(\cdot,\tau) = u(\cdot,\tau + 4T) \quad \textit{for a.e. } \tau \in \mathbb{R} \textit{ and}$$

$$\|u\|_{L^q_T(W^{s,p}(X_t))} := \Big(\int_{-T}^{3T} \|u(\cdot,\tau)\|^q_{W^{s,p}(X_t)} d\tau\Big)^{\frac{1}{q}} < \infty, \textit{if } q \in [1,\infty) \textit{ and,}$$

$$\|u\|_{L^\infty_T(W^{s,p}(X_t))} := \sup\{\|u(\cdot,\tau)\|_{W^{s,p}(X_t)},\ \tau \in (-T,3T)\} < \infty\Big\}, \tag{2.1.8}$$

endowed with the norm $\|\cdot\|_{L^q_T(W^{s,p}(X_t))}$ defined in (2.1.8). For $k \in \mathbb{N}$, $q \in [1,\infty]$, $p \in [1,\infty)$ and $s > 0$, we can also introduce the following space

$$W^{k,q}_T(W^{s,p}(X_t)) := \Big\{u \in L^q_T(W^{s,p}(X_t)),\ \partial^i_\tau u(x,\tau) \in L^q(W^{s,p}(X_t)),\ \ i = 0,...,k\Big\}, \tag{2.1.9}$$

endowed with the norm

$$\|u\|_{W^{k,q}_T(W^{s,p}(X_t))} := \sum_{i=0}^k \|\partial^i_\tau u\|_{L^q(W^{s,p}(X_t))}. \tag{2.1.10}$$

Since the domain X_t is moving with time, we need to give a sense to the i^{th} time derivative $\partial^i_\tau u$. Indeed, it is understood in the weak sense. Precisely, $\partial_\tau u$ is defined as the element of $L^q_T(W^{s,p}(X_t))$ such that, for all $\omega \subsetneq X_t$, and $t \in \mathbb{R}$, we have

$$\langle \partial_\tau u(\cdot,t), \xi\rangle_{W^{s,p}(\omega)\times(W^{s,p}(\omega))'} = \frac{\mathrm{d}}{\mathrm{d}t}\langle u(\cdot,t),\xi\rangle_{W^{s,p}(\omega)\times(W^{s,p}(\omega))'},\ \forall\, \xi \in (W^{s,p}(\omega))', \tag{2.1.11}$$

where $t \mapsto \langle u(\cdot,t),\xi\rangle_{W^{s,p}(\omega)\times(W^{s,p}(\omega))'}$ is a real valued function and $(W^{s,p}(\omega))'$ is the dual space of $W^{s,p}(\omega)$. By induction, we define higher order time derivatives $\partial^i_\tau u,\ i \geq 2$.

Similarly, we define the space

$$W^{k,q}_T(W^{s,p}_0(X_t)) := \Big\{u \in L^q_T(W^{s,p}_0(X_t)),\ \ \partial^i_\tau u(x,\tau) \in L^q(W^{s,p}_0(X_t)), i = 0,...,k\Big\}, \tag{2.1.12}$$

endowed with the norm

$$\|u\|_{W^{k,q}_T(W^{s,p}_0(X_t))} := \sum_{i=0}^k \|\partial^i_\tau u\|_{L^q(W^{s,p}_0(X_t))}. \tag{2.1.13}$$

Finally, we define the space

$$\begin{aligned} C^0_T(W^{s,p}(X_t)) := \Big\{u \in L^\infty_T(W^{s,p}(X_t)),\quad & \lim_{h\to 0^+}(\|u(\cdot,t+h) - u(\cdot,t)\|_{W^{s,p}(X_{t+h}\cap X_t)} \\ & + \|u(\cdot,t+h)\|_{W^{s,p}(X_{t+h}\setminus X_t)} \\ & + \|u(\cdot,t)\|_{W^{s,p}(X_t\setminus X_{t+h})}) = 0\Big\}. \end{aligned} \tag{2.1.14}$$

Vector spaces and vector functions will be written in bold font to distinguish them from scalar spaces and scalar functions. Namely,

$$\mathbf{L}^p(X_t) = L^p(X_t;\mathbb{R}^N) \text{ and } \mathbf{L}^p_0(X_t) = L^p_0(X_t;\mathbb{R}^N), \tag{2.1.15}$$

$$\boldsymbol{\mathcal{D}}(X_t) = \mathcal{D}(X_t;\mathbb{R}^N) \text{ and } \boldsymbol{\mathcal{D}}'(X_t) = \mathcal{D}'(X_t;\mathbb{R}^N) \tag{2.1.16}$$

$$\mathbf{W}^{s,p}(X_t) = W^{s,p}(X_t;\mathbb{R}^N) \text{ with } \|\mathbf{u}\|_{\mathbf{W}^{s,p}(X_t)} = \sum_{i=1}^{N} \|u_i\|_{W^{s,p}(X_t)}, \tag{2.1.17}$$

$$\mathbf{W}_0^{s,p}(X_t) = W_0^{s,p}(X_t;\mathbb{R}^N) \text{ with } \|\mathbf{u}\|_{\mathbf{W}_0^{s,p}(X_t)} = \sum_{i=1}^{N} \|u_i\|_{W_0^{s,p}(X_t)}. \tag{2.1.18}$$

Similarly, for the time dependent vector Sobolev spaces we have

$$\begin{aligned} W_T^{k,q}(\mathbf{W}^{s,p}(X_t)) &= W_T^{k,q}(W^{s,p}(X_t;\mathbb{R}^N)), \\ W_T^{k,q}(\mathbf{W}_0^{s,p}(X_t)) &= W_T^{k,q}(W_0^{s,p}(X_t;\mathbb{R}^N)), \\ C_T^0(\mathbf{W}^{s,p}(X_t)) &= C_T^0(W^{s,p}(X_t;\mathbb{R}^N)), \\ C_T^0(\mathbf{W}_0^{s,p}(X_t)) &= C_T^0(W_0^{s,p}(X_t;\mathbb{R}^N)). \end{aligned}$$

Note that because of the time-periodicity, we have $X_t = X_{t+4T}$, $\forall\, t \in \mathbb{R}$.

The implicit function theorem will be applied in the context of Banach spaces and it involves the concept of Fréchet differentiation. More precisely, let X, Y and Z be three Banach spaces and U resp. V be a nonempty and open set of X resp. Y. We denote by $\mathcal{B}(X,Y)$ the set of linear and continuous function from X to Y.

Definition 2.1.1 (Fréchet derivative)**.**

A function $f : U \to Y$ is said to be Fréchet differentiable (or simply differentiable) at a point $x_0 \in U$ if there exists a linear and continuous map, denote here by $\mathrm{D}f(x_0) \in \mathcal{B}(X,Y)$ such that

$$\lim_{y \to 0,\, y \in X} \frac{\|f(x_0+y) - f(x_0) - Df(x_0)(y)\|_Y}{\|y\|_X} = 0. \tag{2.1.19}$$

The function f is said to be differentiable in U if it is differentiable at every point of U. Furthermore, if the map $x \mapsto Df(x)$ from U to $\mathcal{B}(X,Y)$ is continuous, then f is said to be of Fréchet C^1 (or of class C^1) in U.

Remark 2.1.2. *If $f : U \to Y$ is differentiable, then $Df(x)$ exists for all $x \in U$. Moreover, if the map $Df : x \in U \mapsto Df(x) \in \mathcal{B}(X,Y)$ is differentiable, its derivative is called the second derivative of f and denoted by $D^2 f : U \to \mathcal{B}(X, \mathcal{B}(X,Y))$. Inductively we can define higher order derivatives of all orders.*

Definition 2.1.3 (Fréchet C^k function)**.**

A function $f : U \to Y$ is said to be Fréchet C^k (or of class C^k), where $k \geq 1$, is a positive integer, if the k^{th} order (Fréchet) derivative of f exists and is continuous.

The function f is said to be Fréchet C^∞ (or of class C^∞) if it is of class C^k for all positive integer $k \in \mathbb{N}$.

Now we can state the implicit function theorem for Banach spaces, which is crucial in proving our existence result of our NS problem.

Theorem 2.1.4 (Implicit function theorem)**.**

Let $f \in C^k(U \times V; Z)$, $(k \geq 1)$ and $(x_0, y_0) \in U \times V$ such that $f(x_0, y_0) = z_0 \in Z$. If the partial Fréchet derivative of f with respect to y at (x_0, y_0) denoted by $D_y f(x_0, y_0)$ from Y to Z is a topological isomorphism (ie a linear homeomorphism or a linear bi-continuous map), then there exists a unique C^k map φ from a sufficiently small neighborhood $U_{x_0} \subset U$ of x_0 to V such that

$$\varphi(x_0) = y_0 \text{ and} \tag{2.1.20}$$

$$\forall\, (x, y) \in U_{x_0} \times V, \quad f(x, y) = z_0 \Longleftrightarrow y = \varphi(x). \tag{2.1.21}$$

Proof: Lang [56, pp 19-20], [17, Theorem 3, 4, pp 138-139] ∎

Let Ω_0 be a fixed (time-independent) open and bounded set with regular boundary, contained in $\mathbb{R}^N$. The following Sobolev embedding theorem, is crucial for the analysis of the NS problem in a fix domain and we will heavily rely on it. We choose to present it in the context of vector functions and spaces.

Theorem 2.1.5 (Sobolev's embeddings)**.**

If Ω_0 is open and Lipschitz, and $p \in (0, +\infty)$ we have:

$$\text{if } s < \frac{N}{p}, \text{ then } \mathbf{W}^{s,p}(\Omega_0) \hookrightarrow \mathbf{L}^q(\Omega_0), \ \forall q \in [p, \hat{p}] \text{ with } \frac{1}{\hat{p}} = \frac{1}{p} - \frac{s}{N} \tag{2.1.22a}$$

$$\text{if } s - \frac{N}{p} = 0, \text{ then } \mathbf{W}^{s,p}(\Omega_0) \hookrightarrow \mathbf{L}^q(\Omega_0), \ \forall q \in [p, +\infty) \tag{2.1.22b}$$

$$\text{if } 0 < s - \frac{N}{p} \notin \mathbb{N}, \text{ then } \mathbf{W}^{s,p}(\Omega_0) \hookrightarrow \mathbf{C}^{m,\sigma}(\overline{\Omega}_0) \tag{2.1.22c}$$

where $\sigma \in [0, \hat{\sigma}]$ and $m = [s - \frac{N}{p}]$ and $\hat{\sigma} = s - \frac{N}{p} - m \in (0, 1)$.

Moreover, if Ω_0 is bounded then the embedding (2.1.22a) resp. (2.1.22b) resp. (2.1.22c) are compact for $q \in [p, \hat{p})$, resp. $q \in [p, +\infty)$ resp. $\sigma \in [0, \hat{\sigma})$.

Proof: Adams and Fournier [1, Theorem 4.12, pp 86-87], Evans [28, Thereom 6, pp 284], Brezis [11, Corollary 9.14 and Corollary 9.15 pp 284-285], Gilbarg and Trudinger [36, Corollary 7.11, pp 151]. ∎

Now we will provide some helpful tools for the analysis of a weak solution to the Stokes and NS equations. Further details can be found in [98, 85, 37]. We begin by introducing the space

$$\mathbf{E}(\Omega_0) := \{\mathbf{u} \in \mathbf{L}^2(\Omega_0), \quad \nabla \cdot \mathbf{u} \in L^2(\Omega_0)\}, \tag{2.1.23}$$

which is a Hilbert space endowed with the following scalar product and norm:

$$(\mathbf{u}, \mathbf{v})_{\mathbf{E}(\Omega_0)} := (\mathbf{u}, \mathbf{u})_{\mathbf{L}^2(\Omega_0)} + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_{L^2(\Omega_0)} \text{ and } \|\mathbf{u}\|_{\mathbf{E}(\Omega_0)} := [(\mathbf{u}, \mathbf{u})_{\mathbf{E}(\Omega_0)}]^{1/2}. \tag{2.1.24}$$

For $p \in [1,\infty)$, we also introduce the space $L^p_{loc}(\Omega_0)$ of locally p-integrable functions on Ω_0 as

$$L^p_{loc}(\Omega_0) := \{f : \Omega_0 \mapsto \mathbb{R} \text{ mesurable}, \int_K |f(x)|^p \, \mathrm{d}x < \infty, \ \forall\, K \subset \Omega_0, \ K \text{ compact}\}. \quad (2.1.25)$$

Let us suppose that Ω_0 is open, bounded and Lipschitz. We have the following results.

Theorem 2.1.6. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega}_0)$ is dense in $\boldsymbol{E}(\Omega_0)$.*

Proof: Temam [98, Theorem 1.1, pp 6]. ∎

We have the following trace theorem.

Theorem 2.1.7 (The trace theorem in $\mathbf{H}^1(\Omega_0)$)**.**

There exists a linear and continuous map $\boldsymbol{\gamma}_0 \in \mathcal{B}(\mathbf{H}^1(\Omega_0), \mathbf{L}^2(\partial\Omega_0))$ called the trace operator such that

$$\boldsymbol{\gamma}_0 \mathbf{u} = \mathbf{u}|_{\partial\Omega_0}, \ \forall\, \mathbf{u} \in \mathbf{H}^1(\Omega_0).$$

We note that $\mathbf{H}^{1/2}(\partial\Omega_0) = \boldsymbol{\gamma}_0(\mathbf{H}^1(\Omega_0))$ and $\mathbf{H}^1_0(\Omega_0)$ is the kernel of $\boldsymbol{\gamma}_0$. The space $\mathbf{H}^{1/2}(\partial\Omega_0)$ is dense in $\mathbf{L}^2(\partial\Omega_0)$ and moreover, there exists a linear an continuous operator $\mathbf{l}_0 \in \mathcal{B}(\mathbf{H}^{1/2}(\partial\Omega_0), \mathbf{H}^1(\Omega_0))$ called the lifting operator and such that $\boldsymbol{\gamma}_0 \circ \mathbf{l}_0 = \mathbf{id}$, where $\mathbf{id}$ is the identity operator in $\mathbf{H}^{1/2}(\partial\Omega_0)$.

Proof: Lions and Magenes [66, Theorem 8.3, pp 39] ∎

More generally, we have the following total trace theorem.

Theorem 2.1.8 (The total trace Theorem in $\mathbf{W}^{k,p}(\Omega_0)$)**.**

Let $k \in \mathbb{N}$, $p \in [1,+\infty)$ and Ω_0 open and bounded domain of class C^k. Then there exists a linear and bounded operator

$$\begin{aligned} \boldsymbol{\Gamma} : \mathbf{W}^{k,p}(\Omega_0) &\longrightarrow \mathbf{L}^p(\partial\Omega_0) \\ \mathbf{u} &\mapsto \boldsymbol{\Gamma}(\mathbf{u}) := (\boldsymbol{\gamma}_0(\mathbf{u}), ..., \boldsymbol{\gamma}_{k-1}(\mathbf{u})) \end{aligned} \quad (2.1.26)$$

where $\boldsymbol{\gamma}_j(\mathbf{u}) = (\gamma_j(u_1), ..., \gamma_j(u_N))$, $j = 0, ..., k-1$, with

$$\gamma_j(u_i) := \frac{\partial^j u_i}{\partial \boldsymbol{\nu}^j}|_{\partial\Omega_0} = \sum_{\|\boldsymbol{\alpha}\|=j} \frac{j!}{\boldsymbol{\alpha}!} D^{\boldsymbol{\alpha}} u_i|_{\partial\Omega_0} \cdot \boldsymbol{\nu}^{\boldsymbol{\alpha}}, \ \forall\, \boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N, \ \forall\, \mathbf{u} \in \mathbf{C}^k(\overline{\Omega}_0) \quad (2.1.27)$$

called the j^{th} order normal derivative of u_i, with $\|\boldsymbol{\alpha}\| := \sum_{l=1}^N |\alpha_l|$ and $\boldsymbol{\nu}^{\boldsymbol{\alpha}} := (\nu_1^{\alpha_1}, ..., \nu_N^{\alpha_N})$, where $\boldsymbol{\nu} := (\nu_1, ..., \nu_N)$ represents the unit outward normal vector.

Moreover, if $\mathbf{u} \in \mathbf{W}^{k,p}(\Omega_0)$ then

$$\boldsymbol{\Gamma}(\mathbf{u}) = \mathbf{0} \text{ if and only if } \mathbf{u} \in \mathbf{W}^{k,p}_0(\Omega_0)$$

$$\boldsymbol{\Gamma}(\mathbf{W}^{k,p}(\Omega_0)) = \prod_{j=0}^{k-1} \mathbf{W}^{k-j-\frac{1}{p},p}(\partial\Omega_0), \quad p \in (1,+\infty)$$

$$\mathbf{\Gamma}(\mathbf{W}^{k,1}(\Omega_0)) = \prod_{j=0}^{k-2} \mathbf{W}^{k-j-1,1}(\partial\Omega_0) \times \mathbf{L}^1(\partial\Omega_0).$$

Proof: Domenico and Provenzano [27, Theorem 3.4, pp 8], Lions and Magenes [66, Theorem 8.3, pp 39]. ∎

The following theorem, is crucial because it utilizes the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega}_0)$ in $\mathbf{E}(\Omega_0)$, as given by Theorem 2.1.6, to comprehend the space $H^{-\frac{1}{2}}(\partial\Omega_0) := (H_0^{\frac{1}{2}}(\partial\Omega_0))'$ as the trace of $\mathbf{E}(\Omega_0)$. This result will be beneficial for the weak analysis of the Stokes problem.

Theorem 2.1.9. *There exists a linear and continuous operator $\tilde{\gamma} \in \mathcal{B}(\mathbf{E}(\Omega_0), H^{-1/2}(\partial\Omega_0))$ such that*

$$\tilde{\gamma}(\mathbf{u}) = \mathbf{u} \cdot \boldsymbol{\nu}|_{\partial\Omega_0}, \ \forall \ \mathbf{u} \in \boldsymbol{\mathcal{D}}(\overline{\Omega}_0) \tag{2.1.28}$$

and we have the following generalized Stokes formula

$$(\mathbf{u}, \nabla w)_{\mathbf{L}^2(\Omega_0)} + (\nabla \cdot \mathbf{u}, w)_{\mathbf{L}^2(\Omega_0)} = \langle \tilde{\gamma}\mathbf{u}, \gamma_0 w \rangle, \ \forall \mathbf{u} \in \mathbf{E}(\Omega_0), \ w \in H^1(\Omega_0). \tag{2.1.29}$$

Moreover the operator $\tilde{\gamma}$ maps $\mathbf{E}(\Omega_0)$ onto $H^{-1/2}(\partial\Omega_0)$.

Proof: Temam [98, Theorem 1.2, pp 9] ∎

We recall $\boldsymbol{\mathcal{D}}'(\Omega_0) := \mathcal{D}'(\Omega_0; \mathbb{R}^N)$ is the space a vector distribution on Ω_0 (see (2.1.16)) and we introduce the following functional spaces with free divergence.

$$\begin{aligned} \boldsymbol{\mathcal{D}}_\sigma(\Omega_0) &= \mathcal{D}(\Omega_0; \mathbb{R}^N) \cap \{\nabla \cdot \mathbf{u} = 0 \text{ in } \boldsymbol{\mathcal{D}}'(\Omega_0)\}, && (2.1.30)\\ \mathbf{L}^2_\sigma(\Omega_0) &= L^2(\Omega_0; \mathbb{R}^N) \cap \{\nabla \cdot \mathbf{u} = 0 \text{ in } \boldsymbol{\mathcal{D}}'(\Omega_0)\}, && (2.1.31)\\ \mathbf{H}^1_{0,\sigma}(\Omega_0) &= H^1_0(\Omega_0; \mathbb{R}^N) \cap \{\nabla \cdot \mathbf{u} = 0 \text{ in } \boldsymbol{\mathcal{D}}'(\Omega_0)\}, && (2.1.32)\\ \mathbf{H}^2(\Omega_0) \cap \mathbf{H}^1_{0,\sigma}(\Omega_0) &= H^2(\Omega_0; \mathbb{R}^N) \cap \mathbf{H}^1_{0,\sigma}(\Omega_0), && (2.1.33) \end{aligned}$$

endow it with the inner products

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_{\mathbf{L}^2_\sigma(\Omega_0)} &= \int_{\Omega_0} \mathbf{u} \cdot \mathbf{v} \, dx,\\ (\mathbf{u}, \mathbf{v})_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} &= \int_{\Omega_0} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx := \sum_{i=1}^{N} \int_{\Omega_0} \nabla u_i \cdot \nabla v_i \, \mathrm{d}x,\\ (\mathbf{u}, \mathbf{v})_{\mathbf{H}^2(\Omega_0) \cap \mathbf{H}^1_{0,\sigma}(\Omega_0)} &= \int_{\Omega_0} \Delta \mathbf{u} \cdot \Delta \mathbf{v} \, dx. \end{aligned}$$

More generally, the underscript σ before a Banach space X will always denote the associated space with free divergence, i.e.,

$$X_\sigma := X \cap \{\nabla \cdot \mathbf{u} = 0 \text{ in } \boldsymbol{\mathcal{D}}'(\Omega_0)\}. \tag{2.1.34}$$

The following results are also very useful for the analysis of the Stokes problem in the spaces of free divergence.

Proposition 2.1.10. *If* $\mathbf{f} \in \boldsymbol{\mathcal{D}}'(\Omega_0)$ *then the assertions* (i) *and* (ii) *are equivalent:*

$$(i)\ \ \mathbf{f} = \nabla p \text{ for some scalar function } p \in \mathcal{D}'(\Omega_0),$$
$$(ii)\ \ \langle \mathbf{f}, \mathbf{v} \rangle = 0\ \ \forall\, \mathbf{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega_0),$$

where here $\langle \cdot, \cdot \rangle$ *represents the distribution pairing between* $\boldsymbol{\mathcal{D}'}(\Omega_0)$ *and* $\boldsymbol{\mathcal{D}}(\Omega_0)$.

Proof: Temam [98, Proposition 1.1, pp 14] ∎

Proposition 2.1.11.

(i) *If a scalar distribution* $p \in \mathcal{D}'(\Omega_0)$ *satisfies* $\nabla p \in \mathbf{L}^2(\Omega_0)$, *then*

$$p \in L^2(\Omega_0) \text{ and } \|p\|_{L^2(\Omega_0)} \leq C(\Omega_0)\|\nabla p\|_{\mathbf{L}^2(\Omega_0)}$$

(ii) *If a scalar distribution* $p \in \mathcal{D}'(\Omega_0)$ *satisfies* $\nabla p \in \mathbf{H}^{-1}(\Omega_0)$ *then*

$$p \in L^2(\Omega_0) \text{ and } \|p\|_{L^2(\Omega_0)} \leq C(\Omega_0)\|\nabla p\|_{\mathbf{H}^{-1}(\Omega_0)}.$$

Proof: Temam [98, Proposition 1.2, pp 14] ∎

The following theorem gives the Helmholtz decomposition of $\mathbf{L}^2$ functions which is highly used in our case. Precisely, the differential of the function $\mathbf{F}$ we use to set the implicit function theorem depends a time-periodic Stokes problem. However, as the right hand side of that problem will not be in a space of free divergence, we will need the following Helmholtz decomposition theorem.

Theorem 2.1.12 (Helmholtz decomposition of $\mathbf{L}^2$).

Let Ω_0 *be a Lipschitz, open and bounded set of* $\mathbb{R}^N$, *then we have the following Helmholtz decomposition*

$$\mathbf{L}^2(\Omega_0) = \mathbf{L}^2_\sigma(\Omega_0) \oplus \mathbf{H}_1 \oplus \mathbf{H}_2 \tag{2.1.35}$$

where

$$\mathbf{H}_1 = \{\mathbf{u} \in \mathbf{L}^2(\Omega_0),\ \mathbf{u} = \nabla p,\ \text{for some } p \in H^1(\Omega_0) \text{ and } \Delta p = 0\} \tag{2.1.36}$$
$$\mathbf{H}_2 = \{\mathbf{u} \in \mathbf{L}^2(\Omega_0),\ \mathbf{u} = \nabla p,\ \text{for some } p \in H^1_0(\Omega_0)\}. \tag{2.1.37}$$

Proof: Temam [98, Theorem 1.5, pp 16] ∎

Remark 2.1.13. *We note that the equation* (2.1.35) *in Theorem* 2.1.12 *above represents the Helmholtz decomposition of* $\mathbf{L}^2$ *spaces. It states that every function* $\mathbf{f} \in \mathbf{L}^2(\Omega_0)$ *can be decomposed as* $\mathbf{f} = \mathbf{f}_\sigma + \nabla q$, *where* $\mathbf{f}_\sigma \in \mathbf{L}^2_\sigma(\Omega_0)$ *is a free divergence function and* $q \in H^1(\Omega_0)$. *A more detailed proof of the Helmholtz decomposition can also be found in Raviart and Girault [85].*

We will need the following definition of a Gelfand triple for the analysis of weak solution to Stokes and NS equations.

Definition 2.1.14 (Gelfand triple)**.**

Let $(\mathbf{H}, \langle\cdot,\cdot\rangle_{\mathbf{H}})$ *be a separable Hilbert space and* $\mathbf{V}$ *a reflexive Banach space such that* $\mathbf{V}$ *is dense in* $\mathbf{H}$ *and the canonical embedding* $\mathbf{V} \hookrightarrow \mathbf{H}$ *is continuous. If* $\mathbf{V}'$ *resp.* $\mathbf{H}'$ *denote the topological dual of* $\mathbf{V}$ *resp.* $\mathbf{H}$ *such that*

$$\mathbf{V} \hookrightarrow \mathbf{H} \equiv \mathbf{H}' \hookrightarrow \mathbf{V}' \tag{2.1.38}$$

then the triplet $(\mathbf{V}, \mathbf{H}, \mathbf{V}')$ *is called a Gelfand triple.*

The Gelfand triple is very useful in the analysis of parabolic PDEs such as the time-dependent NS equations. When the solutions are understood in a distribution sense, most of the time we need to give a sense to the initial or time-periodic condition. Proposition 2.1.15 is very important because it uses the Gelfand triple to set the problem in a time continuous Banach space.

Proposition 2.1.15.

Let $\mathbf{V}$, $\mathbf{H}$ *and* $\mathbf{V}'$ *be three Hilbert spaces such that* $(\mathbf{V}, \mathbf{H}, \mathbf{V}')$ *is a Gelfand triple. If* $\mathbf{u} \in L^2_T(\mathbf{V})$ *and* $\mathbf{u}' \in L^2_T(\mathbf{V}')$ *then after modification on a set of zero measure, we have* $\mathbf{u} \in C^0_T(\mathbf{H})$ *and we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|^2_{\mathbf{H}} = 2\langle \mathbf{u}', \mathbf{u}\rangle_{\mathbf{V}'\times\mathbf{V}}. \tag{2.1.39}$$

Proof: Temam [98, Lemma 1.2, pp 260] ∎

When $\mathbf{V} = \mathbf{H}$, we have the following corollary.

Corollary 2.1.16.

If $\mathbf{u} \in H^1_T(\mathbf{V})$*, then after possibly modification on a set of zero measure, we have* $\mathbf{u} \in C^0_T(\mathbf{V})$ *and we have*

$$\mathbf{u}(\cdot, t) = \mathbf{u}(\cdot, s) + \int_s^t \mathbf{u}'(\cdot, \tau)\,\mathrm{d}\tau, \quad \textit{for all } -T \le s \le t \le 3T. \tag{2.1.40}$$

Moreover, we have

$$\|\mathbf{u}\|_{C^0_T(\mathbf{V})} \le C(T)\|\mathbf{u}\|_{H^1_T(\mathbf{V})}, \tag{2.1.41}$$

where $C(T)$ *is a constant depending only on* T*.*

Proof: Evans [28, Theorem 2, pp 302] ∎

The following proposition, concerns the application of the Gelfand triple when $(\mathbf{V}, \mathbf{H}) = (\mathbf{H}^1_0(\Omega_0), \mathbf{L}^2(\Omega_0))$. It will be highly useful for the analysis of weak solutions to Stokes and NS equations.

Proposition 2.1.17.

If $\mathbf{u} \in L^2_T(\mathbf{H}^1_0(\Omega_0))$ *and* $\mathbf{u}' \in L^2_T(\mathbf{H}^{-1}(\Omega_0))$ *then after possibly modification on a set of zero measure we have* $\mathbf{u} \in C^0_T(\mathbf{L}^2(\Omega_0))$*. Moreover, the map* $t \mapsto \|\mathbf{u}(\cdot,t)\|_{\mathbf{L}^2(\Omega_0)}$ *is absolutely continuous and we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}^2(\Omega_0)} = 2\langle \mathbf{u}'(\cdot,t),\ \mathbf{u}(\cdot,t)\rangle,\ a.e\ t \in [-T,3T]. \tag{2.1.42}$$

Furthermore, we have the estimation

$$\|\mathbf{u}(\cdot,t)\|_{C^0_T(\mathbf{L}^2(\Omega_0))} \leq C(T)\Big[\|\mathbf{u}\|_{L^2_T(\mathbf{H}^1_0(\Omega_0))} + \|\mathbf{u}'\|_{L^2_T(\mathbf{H}^{-1}(\Omega_0))}\Big], \tag{2.1.43}$$

where the constant $C(T)$ *depends only on* T*.*

Proof: Evans [28, Theorem 3, pp 303]. ∎

More generally, when $(\mathbf{V},\mathbf{H}) = (\mathbf{H}^{m+2}(\Omega_0),\mathbf{H}^{m+1}(\Omega_0))$, for all positive (non-negative) integers $m \in \mathbb{N}$, Proposition 2.1.18 presents another application of the Gelfand triple. However, a direct proof is given in [28] using mollifiers by convolution. This application is very important as we will use it in our problem, particularly in the case where $m = 0$.

Proposition 2.1.18.

If $\mathbf{u} \in L^2_T(\mathbf{H}^{m+2}(\Omega_0))$ *and* $\mathbf{u}' \in L^2_T(\mathbf{H}^{m}(\Omega_0))$*, then after possibly modification on a set of zero measure, we have* $\mathbf{u} \in C^0_T(\mathbf{H}^{m+1}(\Omega_0))$*. Furthermore, we have the estimation*

$$\|\mathbf{u}\|_{C^0_T(\mathbf{H}^{m+1}(\Omega_0))} \leq C(T,\Omega_0)\Big[\|\mathbf{u}\|_{L^2_T(\mathbf{H}^{m+2}(\Omega_0))} + \|\mathbf{u}'\|_{L^2_T(\mathbf{H}^{m}(\Omega_0))}\Big], \tag{2.1.44}$$

where the constant $C(T,\Omega_0)$ *depends only on* T *and* Ω_0*.*

Proof: Evans [28, Theorem 4, pp 304]. ∎

The Proposition 2.1.18 will be very useful in proving the existence of a time-periodic solution for NS in an artificial heart domain. Indeed, we aim to prove the $\mathbf{H}^2$ regularity of the solution, and we may need to give a sense of the periodicity using the continuity. More precisely, for $m = 0$ we will use in our case the following result.

Proposition 2.1.19.

If $\mathbf{u} \in L^2_T(\mathbf{H}^2(\Omega_0) \cap \mathbf{H}^1_{0,\sigma}(\Omega_0))$ *and* $\mathbf{u}' \in L^2_T(\mathbf{L}^2_\sigma(\Omega_0))$*, then* $\mathbf{u} \in C^0_T(\mathbf{H}^1_{0,\sigma}(\Omega_0))$ *up to a set of zero measure. Moreover, we have the following equality which is understood in the scalar distribution sense on* $(-T,3T)$*,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2_{\mathbf{L}^2_\sigma(\Omega_0)} = 2(\mathbf{u}',\mathbf{u})_{\mathbf{L}^2(\Omega_0)\times\mathbf{L}^2(\Omega_0)}. \tag{2.1.45}$$

Furthermore, we have the estimation

$$\|\mathbf{u}\|_{C^0_T(\mathbf{H}^1(\Omega_0))} \leq C(T,\Omega_0)\Big[\|\mathbf{u}\|_{L^2_T(\mathbf{H}^2(\Omega_0))} + \|\mathbf{u}'\|_{L^2_T(\mathbf{L}^2(\Omega_0))}\Big], \tag{2.1.46}$$

where the constant $C(T,\Omega_0)$ *depends only on* T *and* Ω_0*.*

Proof: Evans [28, Theorem 4, pp 304]. ∎

The following definition will be helpful in defining the notion of a very weak solution for the divergence problem, which we will heavily use. For more details, see Schumacher [87].

Definition 2.1.20 (Mackenhoupt weight)**.**

Let $q \in (1, +\infty)$. A Mackenhoupt weight function w associated to q is a positive and locally integrable function which satisfies

$$\sup_{Q \subset \mathbb{R}^N} \Big(\frac{1}{|Q|}\int_{|Q|} w(x)\,\mathrm{d}x\Big)\Big(\frac{1}{|Q|}\int_{Q} (w(x)^{\frac{-1}{q-1}})\,\mathrm{d}x\Big)^{q-1} < +\infty, \tag{2.1.47}$$

where Q is a cube in $\mathbb{R}^N$.

We denote by A_q the class of Mackenhoupt weight function associated to q. With this definition, we can introduce similarly the weighted Lebesgue space and Sobolev's spaces. Namely, we define

$$L^q_w(\Omega_0) := \{u \in L^1_{\mathrm{loc}}(\bar{\Omega}_0), \quad \|u\|_{q,w} := \Big(\int_{\Omega_0} |u(x)|^q\, w\,\mathrm{d}x\Big)^{\frac{1}{q}} < +\infty\}. \tag{2.1.48}$$

Its topological dual is defined as $(L^q_w(\Omega_0))' := L^{q'}_{w'}(\Omega_0)$ where q' is the conjugate of q ($\frac{1}{q} + \frac{1}{q'} = 1$) and $w' := w^{\frac{-1}{q-1}} \in A_{q'}$. For $m \in \mathbb{N}$, we can also define the weighted Sobolev's space as

$$W^{m,q}_w(\Omega_0) := \{u \in L^q_w(\Omega_0), \quad \|u\|_{m,q,w} := \sum_{|\alpha| \le m} \|D^\alpha u\|_{q,w} < +\infty\}. \tag{2.1.49}$$

We denote by $W^{m,q}_{w,0}(\Omega_0)$ the closure of $\mathcal{D}(\Omega_0)$ in $W^{m,q}_w(\Omega_0)$. We also define the space $W^{-m,q'}_{w'}(\Omega_0)$ to be the topological dual of the space $W^{m,q}_{w,0}$. Similarly, we denote by $W^{-m,q}_{w,0}(\Omega_0)$ the closure of $\mathcal{D}(\Omega_0)$ in $W^{-m,q}_w(\Omega_0)$, and we define the space $W^{-m,q}_{w,0}(\Omega_0)$ as the topological dual of $(W^{m,q'}_{w'}(\Omega_0))'$ (see [87]). We will use similar definition in bold for vector functionals and their spaces. Precisely, we will need to define the following functional space to prove the existence of solution to our problem in Chapter 3.

$$\begin{aligned}
\mathbf{Y}^{2,q'}_{w'}(\Omega_0) &:= \{\mathbf{u} \in \mathbf{W}^{2,q'}_{w'}(\Omega_0), \quad \mathbf{u}|_{\partial\Omega_0} = 0\},\\
\mathbf{Y}^{-2,q}_{w}(\Omega_0) &:= (\mathbf{Y}^{2,q'}_{w'}(\Omega_0))'\\
\mathbf{Y}^{2,q'}_{w',\sigma}(\Omega_0) &:= \{\boldsymbol{\phi} \in \mathbf{Y}^{2,q'}_{w'}(\Omega_0), \quad \nabla\cdot\boldsymbol{\phi} = 0\}\\
\mathbf{Y}^{-2,q}_{w,\sigma}(\Omega_0) &:= (\mathbf{Y}^{2,q'}_{w',\sigma}(\Omega_0))'.
\end{aligned}$$

2.2 Analysis of weak solution to Stokes and Navier-Stokes equations

2.2.1 The Stokes equations

In this section, we review the analysis of the time-independent Stokes equations. This review helps us to understand the use of free divergence spaces in the analysis of the Stokes problem. Note that the differential of the functional $\mathbf{F}$ used in Chapter 3 to apply the implicit function theorem defines a time-periodic Stokes problem, and we analyze it to demonstrate that this differential defines an isomorphism. We will also make use of the free divergence spaces in a similar way. We also note that the analysis of our time-periodic Stokes problem follows the line of [98, 5], which is a Faedo-Galerkin approach, and the method works for both initial value or time-periodic Stokes problems. We will also solve this problem numerically as a benchmark problem in the context of a lid-driven cavity.

Here, we present results on the existence and uniqueness of the associated weak variational formulation. Let $\mathbf{f} \in \mathbf{L}^2(\Omega_0)$ and denote by $\mathbf{u}$ a vector function representing the velocity of the fluid and p a scalar function representing the pressure of the fluid, both defined in Ω_0. Let ν denote the kinematic viscosity of the fluid, which is a (strictly) positive constant ($\nu > 0$). The steady-state Stokes problem reads:

Find $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega_0) \times H^1(\Omega_0)$ such that:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega_0, & (2.2.1a) \\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega_0, & (2.2.1b) \\ \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega_0. & (2.2.1c) \end{cases}$$

We want to know in which functional space to search the unique solution $(\mathbf{u}, p)$ of equations (2.2.1a) – (2.2.1c) and in which sense should we understand them. Because of (2.2.1b) and (2.2.1c), it makes sense to search the solution $\mathbf{u}$ in the space $\mathbf{H}^1_{0,\sigma}(\Omega_0)$ defined in equation (2.1.32). Then if we take the $\mathbf{L}^2-$ scalar product in (2.2.1a) with a test function $\mathbf{v}$ still in $\mathbf{H}^1_{0,\sigma}(\Omega_0)$, we obtain:

$$\nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} = (\mathbf{f}, \mathbf{v})_{\mathbf{L}^2(\Omega_0)}, \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{0,\sigma}(\Omega_0). \tag{2.2.2}$$

Now we can write what we call the variational formulation of the steady state Stokes problem:

Find $\mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$ such that:

$$\nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} = (\mathbf{f}, \mathbf{v})_{\mathbf{L}^2_\sigma(\Omega_0)}, \; \forall\, \mathbf{v} \in \mathbf{H}^1_0(\Omega_0). \tag{2.2.3a}$$

We can see that the problem (2.2.3*a*) is equivalent to the problem (2.2.1a) – (2.2.1c) in some sense. In fact, if $\mathbf{u}$ satisfies (2.2.3a) then $\mathbf{u} \in \mathbf{H}^1_0(\Omega_0)$. We can only expect (2.2.1a) – (2.2.1c) to be satisfied in a weaker sense. Since $\mathbf{u} \in \mathbf{H}^1_0(\Omega_0)$, then the trace $\gamma_0(\mathbf{u})$ is in $\mathbf{H}^{\frac{1}{2}}(\partial\Omega_0)$. Moreover, $\mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$ implies $\nabla \cdot \mathbf{u} = 0$ in $\boldsymbol{\mathcal{D}}'(\Omega_0)$. From the equation (2.2.3a) we have

$$\langle -\nu\Delta\mathbf{u} - \mathbf{f}, \mathbf{v}\rangle = 0, \; \forall\, \mathbf{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega_0). \tag{2.2.4}$$

By Proposition 2.1.10 and Proposition 2.1.11 there exists $p \in L^2(\Omega_0)$, such that:

$$-\nu\Delta\mathbf{u} - \mathbf{f} = -\nabla p. \tag{2.2.5}$$

Moreover, we have the following Lemma:

Lemma 2.2.1.

Let Ω_0 be a bounded and Lipschitz domain. Then the following conditions are equivalent:

(i) $\mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$ and satisfies the variational formulation (2.2.3a).

(ii) $\mathbf{u} \in \mathbf{H}^1_0(\Omega_0)$ and satisfies the equations (2.2.1a) − (2.2.1c) in the following sense:

There exists $p \in L^2(\Omega_0)$ such that:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \boldsymbol{\mathcal{D}}'(\Omega_0) & (2.2.6a)\\ \nabla\cdot\mathbf{u} = 0 \text{ in } \boldsymbol{\mathcal{D}}'(\Omega_0) & (2.2.6b)\\ \gamma_0(\mathbf{u}) = \mathbf{0} \text{ in } \mathbf{H}^{\frac{1}{2}}(\partial\Omega_0) & (2.2.6c)\end{cases}$$

Proof: [98, Lemma 2.1, pp 22] ∎

The following theorem gives the existence and uniqueness of the variational formulation of the steady state Stokes problem (2.2.1a) − (2.2.1c).

Theorem 2.2.2 (Existence and Uniquenes)**.**

If Ω_0 is a Lipschitz domain and $\mathbf{f} \in \mathbf{L}^2(\Omega_0)$ then the problem (2.2.3a) has a unique solution $\mathbf{u}$ (the result is also valid if $\mathbf{f} \in \mathbf{H}^{-1}(\Omega_0)$). Moreover, there exists p in the space $L^2_{loc}(\Omega_0)$ defined in (2.1.25) such that (2.2.6a) − (2.2.6b) are satisfied.

If Ω_0 is a bounded and Lipschitz domain, then $p \in L^2(\Omega_0)$ and (2.2.6a)−(2.2.6c) are satisfied by $\mathbf{u}$ and p.

Proof: Temam [98, Theorem 2.1, pp 23]. ∎

The following result concerns the non-homogeneous Stokes problem. This result is important because our problem involves a NS equation with non-homogeneous boundary conditions.

Theorem 2.2.3 (Existence and uniqueness for the non-homogeneous case)**.**

If Ω_0 is a bounded and Lipschitz domain, if $\mathbf{f} \in \mathbf{H}^{-1}(\Omega_0)$, $g \in L^2(\Omega_0)$ and $\boldsymbol{\phi} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega_0)$ and under the following general flux condition

$$\int_{\Omega_0} g\,\mathrm{d}x = \int_{\partial\Omega_0} \boldsymbol{\phi}\cdot\boldsymbol{\nu}\,\mathrm{d}s, \tag{2.2.7}$$

then there exists $\mathbf{u} \in \mathbf{H}^1(\Omega_0)$ *and* $p \in L^2(\Omega_0)$ *solution to the following non-homogeneous Stokes problem*

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \ \text{in}\ \boldsymbol{\mathcal{D}}'(\Omega_0), & (2.2.8a)\\ \nabla\cdot\mathbf{u} = g \ \text{in}\ \boldsymbol{\mathcal{D}}'(\Omega_0), & (2.2.8b)\\ \gamma_0(\mathbf{u}) = \boldsymbol{\phi} \ \text{on}\ \mathbf{H}^{\frac{1}{2}}(\partial\Omega_0), & (2.2.8c)\end{cases}$$

The vector function $\mathbf{u}$ *is unique, whereas* p *is unique up to an additive constant (ie.,* $p \in L^2_{\mathbb{R}}(\Omega_0) := L^2(\Omega_0)/\mathbb{R}$ *which is isomorphic to* $L^2_0(\Omega_0)$*).*

Proof: Temam [98, Theorem 2.4, pp 31] ∎

Now we will introduce the steady-state and nonlinear NS problem. First, we will provide the variational formulation of the homogeneous problem. Next, we will discuss the properties of the trilinear form arising from the nonlinear term in the variational formulation. Finally, we will present results on the existence and uniqueness of a weak solution to the associated problem.

2.2.2 The steady state Navier-Stokes equation

In this paragraph, we review the steady-state and nonlinear NS equations. Although our problem is time-dependent, it makes sense to review this background to see classical results of existence of nonlinear NS and understand the difficulty of such a problem in dimension $N \leq 4$. Moreover we utilize this problem heavily for numerical evaluation of our code in the context of the driven cavity problem up to $Re = 10,000$.

This problem is closer to our problem than the previous steady state Stokes problem. Here, we encounter the appearance of the nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$. We will demonstrate how to address this term and how it impacts the space where we search the solution $(\mathbf{u}, p)$. We assume Ω_0 to be an open, bounded, and Lipschitz set in $\mathbb{R}^N$. If $\mathbf{f} \in \mathbf{L}^2(\Omega_0)$, then the homogeneous and steady-state and nonlinear NS problem reads:

Find a vector function $\mathbf{u}$ and a scalar function p such that:

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \ \text{in}\ \Omega_0, & (2.2.9a)\\ \nabla\cdot\mathbf{u} = 0 \ \text{in}\ \Omega_0, & (2.2.9b)\\ \mathbf{u} = \mathbf{0} \ \text{on}\ \partial\Omega_0. & (2.2.9c)\end{cases}$$

For the same reason as in the Stokes problem (2.2.1a) – (2.2.1c), it makes sense to search for the solution $\mathbf{u}$ in $\mathbf{H}^1_{0,\sigma}(\Omega_0)$. Then, if we take again the scalar product in (2.2.9a) (p being smooth) with a test function $\mathbf{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega_0)$, and after integrating by parts, we obtain:

$$\nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} + \mathbf{b}(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{\mathbf{L}^2(\Omega_0)},\ \forall\, \mathbf{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega_0) \tag{2.2.10}$$

where $\mathbf{b}$ is the trilinear form defined by:

$$\mathbf{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) := ((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w})_{\mathbf{L}^2(\Omega_0)} = \sum_{i,j=1}^{N} \int_{\Omega_0} u_i \frac{\partial v_j}{\partial x_i} w_j \,\mathrm{d}x. \tag{2.2.11}$$

Conversely, if $\mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$ and satisfies (2.2.10) for all $\mathbf{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega_0)$, we know by Proposition 2.1.10 that there exists a distribution $p \in \mathcal{D}'(\Omega_0)$ such that (2.2.9a) and (2.2.9b) are satisfied in the weak sense, and (2.2.9c) is satisfied in the trace sense $\boldsymbol{\gamma}_0$.

Note that in (2.2.11), if $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are in $\mathbf{H}^1_{0,\sigma}(\Omega_0)$, then the trilinear form is not a priori well-defined in dimension $N > 4$. Indeed, if $\mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$, then by Sobolev's embedding (2.1.22a), $u_i \in L^{\frac{2N}{N-2}}(\Omega_0)$. If $\mathbf{v} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$, then $v_j \in H^1_0(\Omega_0)$ and therefore, $\frac{\partial v_j}{\partial x_i} \in L^2(\Omega_0)$. If $\mathbf{w} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$, the problem we encounter is that $\mathbf{w}$ is not in $\mathbf{L}^N(\Omega_0)$ and therefore the integral in (2.2.11) in not well-defined.

For the trilinear form to be well-defined in any dimension $N \geq 2$, it is enough to have w_j in $L^N(\Omega_0)$. So, let us introduce the space $\mathbf{H}^1_{0,\sigma}(\Omega_0) \cap \mathbf{L}^N(\Omega_0)$, endowed with the norm:

$$\|u\|_{\mathbf{H}^1_{0,\sigma}(\Omega_0) \cap \mathbf{L}^N(\Omega_0)} := \|\mathbf{u}\|_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} + \|\mathbf{u}\|_{\mathbf{L}^N(\Omega_0)}. \tag{2.2.12}$$

The space $\mathbf{H}^1_{0,\sigma}(\Omega_0) \cap \mathbf{L}^N(\Omega_0)$ is a subspace of $\mathbf{H}^1_{0,\sigma}(\Omega_0)$ and there are equal only for $N = 2,\ 3$ or 4. The trilinear form $\mathbf{b}$ has the following properties:

Lemma 2.2.4. *For Ω_0 Lipschitz domain, bounded or unbounded, the trilinear form $\mathbf{b}$ is continuous on $\mathbf{H}^1_0(\Omega_0) \times \mathbf{H}^1_0(\Omega_0) \times \mathbf{H}^1_0(\Omega_0) \cap \mathbf{L}^N(\Omega_0)$ for all $N \geq 2$.*

Proof: Temam [98, Lemma 1.1, pp 161] ∎

Remark 2.2.5. *in dimension $N = 2, 3, 4$, because of the Sobolev's embedding $\mathbf{H}^1(\Omega_0) \hookrightarrow \mathbf{L}^N(\Omega_0)$ we have $\mathbf{H}^1(\Omega_0) \cap \mathbf{L}^N(\Omega_0) = \mathbf{H}^1(\Omega_0)$. Therefore, under the hypothesis of Lemma 2.2.4, we can conclude that the trilinear form $\mathbf{b}$ is continuous on $\mathbf{H}^1_0(\Omega_0) \times \mathbf{H}^1_0(\Omega_0) \times \mathbf{H}^1_0(\Omega_0)$ for $N = 2, 3, 4$.*

More precisely, we have the following lemma.

Lemma 2.2.6. *If Ω_0 is open and Lipschitz, and $N \geq 2$ then the trilinear form $\mathbf{b}$ is continuous in $\mathbf{H}^1_0(\Omega_0) \times \mathbf{H}^1_0(\Omega_0) \times \mathbf{H}^1_0(\Omega_0) \cap \mathbf{L}^N(\Omega_0)$ and in $\mathbf{H}^1_{0,\sigma}(\Omega_0) \times \mathbf{H}^1_{0,\sigma} \times \mathbf{H}^1_{0,\sigma}(\Omega_0) \cap \mathbf{L}^N(\Omega_0)$.*

Furthermore, if Ω_0 is bounded and $N \leq 4$, then the trilinear form $\mathbf{b}$ is continuous on $\mathbf{H}^1_0(\Omega_0) \times \mathbf{H}^1_0(\Omega_0) \times \mathbf{H}^1_0(\Omega_0)$ and in $\mathbf{H}^1_{0,\sigma}(\Omega_0) \times \mathbf{H}^1_{0,\sigma}(\Omega_0) \times \mathbf{H}^1_{0,\sigma}(\Omega_0)$.

Proof: Temam [98, Lemma 1.1, pp 161 and Lemma 1.2, pp 162] ∎

Now, we give some properties of the nonlinear term $\mathbf{b}$ useful in the analysis on the nonlinear NS problem.

Lemma 2.2.7. *If Ω_0 is open and Lipschitz and $N \geq 2$, then we have:*

$$\mathbf{b}(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0,\ \forall\, \mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega_0),\ \mathbf{v} \in \mathbf{H}^1_0(\Omega_0) \cap \mathbf{L}^N(\Omega_0) \tag{2.2.13a}$$

$$\mathbf{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -\mathbf{b}(\mathbf{u}, \mathbf{w}, \mathbf{v}),\ \forall\, \mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega_0),\ \mathbf{v}, \mathbf{w} \in \mathbf{H}^1_0(\Omega_0) \cap \mathbf{L}^N(\Omega_0) \tag{2.2.13b}$$

Proof: Temam [98, Lemma 1.3, pp 163] ∎

Now, we want to provide the equivalent weak variational formulation of the nonlinear NS problem (2.2.9a) − (2.2.9c). We assume Ω_0 to be open, bounded, and Lipschitz, and $N \geq 2$. If $\mathbf{f} \in \mathbf{H}^{-1}(\Omega_0)$, then its weak variational formulation reads:

Find a vector function $\mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$ such that:

$$\nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} + \mathbf{b}(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}^{-1}(\Omega_0) \times \mathbf{H}^1_0(\Omega_0)}, \; \forall \, \mathbf{v} \in \mathbf{H}^1_{0,\sigma}(\Omega_0) \cap \mathbf{L}^N(\Omega_0). \quad (2.2.14a)$$

It is easy to see that the new problem (2.2.14a) is equivalent to (2.2.9a)−(2.2.9c), understood in the sense mentioned in the above lemma, referred to as Lemma 2.2.1.

Theorem 2.2.8 (Existence)**.**

Let Ω_0 be open, bounded and Lipschitz set of $\mathbb{R}^N$ and $N \geq 2$. If $\mathbf{f} \in \mathbf{H}^{-1}(\Omega_0)$ then the variational problem (2.2.14a) has at least one solution $\mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$ and there exists a distribution $p \in L^1_{loc}(\Omega_0)$ such that the problem (2.2.9a) − (2.2.9c) are satisfied.

Proof: Temam [98, Theorem 1.2, pp 164] ∎

For the uniqueness, we only have this restricted result.

Theorem 2.2.9 (Uniqueness)**.**

If $N \leq 4$ and if the kinetic viscosity coefficient ν is large enough (or $\mathbf{f}$ small enough) so that

$$\nu^2 > C(N) \|\mathbf{f}\|_{\mathbf{H}^1_{0,\sigma}(\Omega_0)'} \quad (2.2.15)$$

then the problem (2.2.14a) has a unique solution.

Proof: Temam [98, Theorem 1.3, pp 167] ∎

Finally, we will address the time-dependent NS problem, which is closely related to our case. We will present results on the existence and uniqueness of the variational formulation for both the linear and nonlinear cases.

2.2.3 The evolution Navier-Stokes equations

In this paragraph, we delve into the review of the time-dependent NS equations, which is the nature of our problem. The distinction between this equation and ours lies in the fact that our problem is time-periodic and involves moving boundaries. Though this problem is not periodic, its analysis involves crucial concepts (Gelfand triplet and its applications, use of functions with free divergence) which are helpful in our case and we will use them similarly.

Here, we begin with the linear case, followed by the nonlinear case, which is more relevant to our specific problem.

Let Ω_0 be a bounded and Lipschitz domain in $\mathbb{R}^N$, and let $T > 0$ be fixed. Here, Q denotes the cylinder $\Omega_0 \times [0, T]$. For the linear case, the problem is formulated as follows:

Find a vector function $\mathbf{u} : Q \to \mathbb{R}^N$ and a scalar function $p : Q \to \mathbb{R}$ such that:

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in } Q, & (2.2.16)\\ \nabla\cdot\mathbf{u} = 0 \text{ in } Q, & (2.2.17)\\ \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega_0 \times [0,T], & (2.2.18)\\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \text{ in } \Omega_0, & (2.2.19)\end{cases}$$

where $\mathbf{f}$ and $\mathbf{u}_0$ are given functions defined on Q and Ω_0 respectively. If we suppose that $\mathbf{u}$ and p are classical solutions to (2.2.16) – (2.2.19), we can see that by taking the scalar product with a test function $\mathbf{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega_0)$ in (2.2.16), after integrating by parts we get

$$(\partial_t \mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Omega_0)} + \nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} = (\mathbf{f}, \mathbf{v})_{\mathbf{L}^2(\Omega_0)},\ \forall\, \mathbf{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega_0). \tag{2.2.20}$$

By density, equation (2.2.20) can hold for all $\mathbf{v} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$.

If X is a given Banach space, and $a, b \in \mathbb{R}$ such that $a < b$, and $p \in [1,\infty]$ we introduce the following space

$$L^p(a,b;X) := \{f : [a,b] \to X \text{mesurable}, \quad \int_a^b \|f(t)\|_X^p \,\mathrm{d}t < \infty \text{ if } p \in [1,\infty), \text{ or } \sup_{t\in[a,b]} \|f(t)\|_X < \infty \text{ if } p = \infty\} \tag{2.2.21}$$

endow with the norm

$$\|f\|_{L^p(a,b;X)} := \begin{cases} \left(\int_a^b \|f(t)\|_X^p \,\mathrm{d}t\right)^{\frac{1}{p}} \text{ if } p \in [1,\infty) & (2.2.22)\\ \sup_{t\in[a,b]} \|f(t)\|_X \text{ if } p = \infty. & (2.2.23)\end{cases}$$

For $p \in (1,\infty)$, if p^* is the conjugate of p, *i.e.*, $\frac{1}{p} + \frac{1}{p^*} = 1$ and X' the topological dual of X, then we can define the dual space of $L^p(a,b;X)$ as

$$(L^p(a,b;X))' := L^{p^*}(a,b;X'). \tag{2.2.24}$$

Moreover, if $f \in L^{p^*}(a,b;X')$ and $g \in L^p(a,b;X)$, then by the Hölder inequality, the map $t \mapsto \langle f(t), g(t)\rangle_{X'\times X}$ is integrable because $t \mapsto \|f(t)\|_{X'} \in L^{p^*}((a,b))$ and $t \mapsto \|g(t)\|_X \in L^p((a,b))$. Therefore, we define the duality pairing between $L^{p^*}(a,b;X')$ and $L^p(a,b;X)$ as

$$\langle f, g\rangle_{L^{p^*}(a,b;X')\times L^p(a,b;X)} := \int_a^b \langle f(t), g(t)\rangle_{X'\times X} \,\mathrm{d}t. \tag{2.2.25}$$

Similarly if $\mathbf{X} := X \times ... \times X = X^N$ is a vector Banach space, and $p \in [1,\infty]$, we define in bold font the following vector Banach space

$$\mathbf{L}^p(a,b;\mathbf{X}) := L^p(a,b;X)^N, \tag{2.2.26}$$

endow with the norm

$$\|\mathbf{f}\|_{\mathbf{L}^p(a,b;\mathbf{X})} := \sum_{i=1}^{N} \|f_i\|_{L^p(a,b;X)}. \tag{2.2.27}$$

If the function f resp. $\mathbf{f}$ is periodic of period $4T$ as it will often be the case in this thesis, and if $a = -T$ and $b = 3T$, we will use the notation $L_T(X)$ instead of $L^p(-T, 3T; X)$ resp. $\mathbf{L}^p_T(\mathbf{X})$ instead of $L^p(-T, 3T; \mathbf{X})$.

We will need the following lemma. This lemma is very useful in the analysis of the time-dependent NS equations.

Lemma 2.2.10. *Let $\mathbf{X}$ be a given (vector) Banach space, and $\mathbf{X}'$ its topological dual. Let $\mathbf{u}$ and $\mathbf{g}$ be two functions in $\mathbf{L}^1(a, b; \mathbf{X})$. The following conditions are equivalent:*

(i) $\mathbf{u}' = \mathbf{g}$ *a.e. i.e:*

$$\mathbf{u}(\cdot, t) = \xi + \int_0^t \mathbf{g}(\cdot, s)\mathrm{d}s, \quad \textit{for } \xi \in \mathbf{X}, \ \textit{a.e. } t \in [a, b] \tag{2.2.28a}$$

(ii) $\forall \phi \in \mathcal{D}((a, b))$,

$$\int_a^b \mathbf{u}(\cdot, t)\phi'(t)\mathrm{d}t = -\int_a^b \mathbf{g}(\cdot, t)\phi(t)\mathrm{d}t \quad \left(\phi' = \frac{\mathrm{d}\phi}{\mathrm{d}t}\right) \tag{2.2.28b}$$

(iii) $\forall \boldsymbol{\eta} \in \mathbf{X}'$,

$$\frac{d}{dt}\langle \mathbf{u}, \boldsymbol{\eta} \rangle_{\mathbf{X}\times\mathbf{X}'} = \langle \mathbf{g}, \boldsymbol{\eta} \rangle_{\mathbf{X}\times\mathbf{X}'} \ \textit{in } \mathcal{D}'((a, b)). \tag{2.2.28c}$$

Furthermore, if one of the conditions $(i)-(iii)$ is satisfied, then $\mathbf{u}$ in particular is a.e. equal to a continuous function from $[a, b]$ into $\mathbf{X}$.

Proof: Temam [98, Lemma 1.1, pp 250] ∎

Note that according to Lemma 2.2.10, in the case $\mathbf{X} = \mathbf{L}^2(\Omega_0)$, and by identifying $\mathbf{L}^2(\Omega_0)$ with its topological dual, the following remark is highly used in the analysis of the time-dependent NS equation.

Remark 2.2.11.

If $\mathbf{u} \in \mathbf{W}^{1,1}(a, b, \mathbf{L}^2(\Omega_0))$ then we have

$$(\partial_t \mathbf{u}, \boldsymbol{\eta})_{\mathbf{L}^2(\Omega_0)} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}, \boldsymbol{\eta})_{\mathbf{L}^2(\Omega_0)}, \quad \forall\, \boldsymbol{\eta} \in \mathbf{L}^2(\Omega_0). \tag{2.2.29}$$

By Lemma 2.2.10 and Remark 2.2.11, we have:

$$(\partial_t \mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Omega_0)} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Omega_0)}, \ \ \forall\, \mathbf{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega_0). \tag{2.2.30}$$

This leads to the following weak variational formulation of the problem (2.2.16) − (2.2.19):

For $\mathbf{f} \in \mathbf{L}^2(0,T;(\mathbf{H}^1_{0,\sigma}(\Omega_0))')$, for $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega_0)$, find $\mathbf{u} \in \mathbf{L}^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0))$ such that:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u},\mathbf{v})_{\mathbf{L}^2(\Omega_0)} + \nu(\mathbf{u},\mathbf{v})_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} = \langle \mathbf{f},\mathbf{v}\rangle_{(\mathbf{H}^1_{0,\sigma}(\Omega_0))'\times\mathbf{H}^1_{0,\sigma}(\Omega_0)}\ \forall\, \mathbf{v} \in \mathbf{H}^1_{0,\sigma}(\Omega_0), & (2.2.31a)\\ \mathbf{u}(\cdot,0) = \mathbf{u}_0. & (2.2.31b) \end{cases}$$

We need to give a meaning of equation (2.2.31b). If we write equation (2.2.31a) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{u},\mathbf{v}\rangle = \langle \mathbf{f} - \nu\mathbf{A}\mathbf{u},\mathbf{v}\rangle,\ \forall\, \mathbf{v} \in \mathbf{H}^1_{0,\sigma}(\Omega_0) \tag{2.2.32}$$

where $\langle\cdot,\cdot\rangle$ is the duality between $\mathbf{H}^1_{0,\sigma}(\Omega_0)'$ and $\mathbf{H}^1_{0,\sigma}(\Omega_0)$ and $\mathbf{A}$ is a linear and continuous map from $\mathbf{H}^1_{0,\sigma}(\Omega_0)$ to $\mathbf{H}^1_{0,\sigma}(\Omega_0)'$ defined by

$$\langle \mathbf{A}\mathbf{u},\mathbf{v}\rangle := (\mathbf{u},\mathbf{v})_{\mathbf{H}^1_{0,\sigma}(\Omega_0)},\ \forall\, \mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega_0),\ \forall\, \mathbf{v} \in \mathbf{H}^1_{0,\sigma}(\Omega_0), \tag{2.2.33}$$

then according to Lemma 2.2.10,

$$\mathbf{u}' = \mathbf{f} - \nu\mathbf{A}\mathbf{u} \in \mathbf{L}^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)'),\ \text{and } \mathbf{u} \in C^0([0,T];\mathbf{H}^1_{0,\sigma}(\Omega_0)'). \tag{2.2.34}$$

We can also use the fact that $(\mathbf{H}^1_{0,\sigma}(\Omega_0), \mathbf{L}^2_\sigma(\Omega_0), \mathbf{H}^1_{0,\sigma}(\Omega_0)')$ is a Gelfand triple (see Definition 2.1.14) and use Proposition 2.1.15 to conclude that $\mathbf{u} \in C^0([0,T], \mathbf{L}^2_\sigma(\Omega_0))$ after modification on a set of zero measure. This result gives a sense to equation (2.2.31*b*). An alternative formulation of the weak problem (2.2.31a) – (2.2.31b) reads:

For $\mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)')$, for $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega_0)$, find $\mathbf{u} \in \mathbf{L}^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0))$ such that:

$$\mathbf{u}' \in \mathbf{L}^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)') \tag{2.2.35a}$$

$$\mathbf{u}' + \nu\mathbf{A}\mathbf{u} = \mathbf{f} \text{ on } (0,T) \tag{2.2.35b}$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}_0. \tag{2.2.35c}$$

Using Lemma 2.2.10, we can see that the two variational formulations are equivalent.

Theorem 2.2.12 (Existence and Uniqueness)**.**

For $\mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)')$*, for* $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega_0)$*, there exists a unique* $\mathbf{u}$ *satisfying* (2.2.35a) – (2.2.35c)*. Moreover,* $\mathbf{u} \in C^0([0,T];\mathbf{L}^2_\sigma(\Omega_0))$ *.*

Proof: Temam [98, Theorem 1.1, 254]. ∎

Now we consider the time-dependent and nonlinear case, known as the full NS equations. Here we are going to present results on the existence and uniqueness of a weak solution in dimension $N \leq 4$ for the homogeneous boundary condition and when an initial condition is given. The trilinear form $\mathbf{b}$ can therefore be defined on $\mathbf{H}^1_{0,\sigma}(\Omega_0) \times \mathbf{H}^1_{0,\sigma}(\Omega_0) \times \mathbf{H}^1_{0,\sigma}(\Omega_0)$ and it is continuous. According to equation (2.2.13a) of Lemma 2.2.7, we have:

$$\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0,\ \forall\, \mathbf{u} \in \mathbf{H}^1_{0,\sigma}(\Omega_0),\ \forall\, \mathbf{v} \in \mathbf{H}^1_0(\Omega_0). \tag{2.2.36}$$

For $\mathbf{u},\ \mathbf{v} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$, we denote by $\mathbf{B}(\mathbf{u},\mathbf{v})$ the element of $\mathbf{H}^1_{0,\sigma}(\Omega_0)'$ defined by:

$$\langle \mathbf{B}(\mathbf{u},\mathbf{v}), \mathbf{w}\rangle = \mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}),\ \forall\ \mathbf{w} \in \mathbf{H}^1_{0,\sigma}(\Omega_0) \tag{2.2.37}$$

and we defined $\mathbf{B}(\mathbf{u}) := \mathbf{B}(\mathbf{u},\mathbf{u}) \in \mathbf{H}^1_{0,\sigma}(\Omega_0)'$. The classical formulation of the full Navier-Stokes equation reads:

Find a vector function $\mathbf{u}:\ Q \to \mathbb{R}^N$ and a scalar function $p:\ Q \to \mathbb{R}$ such that:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in } Q, & (2.2.38a)\\ \nabla\cdot\mathbf{u} = 0 \text{ in } Q, & (2.2.38b)\\ \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega_0 \times (0,T), & (2.2.38c)\\ \mathbf{u}(x,0) = \mathbf{u}_0 \text{ in } \Omega_0, & (2.2.38d) \end{cases}$$

where the function $\mathbf{f}$ and $\mathbf{u}_0$ are given and defined in Q and Ω_0 respectively. If $\mathbf{u}$ and p are classical solution, (let us say $\mathbf{u} \in C^2(\overline{Q})$ and $p \in C^1(\overline{Q})$) then $\mathbf{u} \in L^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0))$ and satisfy:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u},\mathbf{v})_{\mathbf{L}^2(\Omega_0)} + \nu(\mathbf{u},\mathbf{v})_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} + \mathbf{b}(\mathbf{u},\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v})_{\mathbf{L}^2(\Omega_0)},\ \forall\ \mathbf{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega_0). \tag{2.2.39}$$

Again by density, equation (2.2.39) holds for all $\mathbf{v}$ in $\mathbf{H}^1_{0,\sigma}(\Omega_0)$ and the weak formulation reads:

For $\mathbf{f} \in L^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)')$ and $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega_0)$, find $\mathbf{u} \in L^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0))$ such that:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u},\mathbf{v})_{\mathbf{L}^2_\sigma(\Omega_0)} + \nu(\mathbf{u},\mathbf{v})_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} + \mathbf{b}(\mathbf{u},\mathbf{u},\mathbf{v}) & \\ \qquad = \langle \mathbf{f},\mathbf{v}\rangle_{(\mathbf{H}^1_{0,\sigma}(\Omega_0))'\times\mathbf{H}^1_{0,\sigma}(\Omega_0)},\quad \forall\ \mathbf{v} \in \mathbf{H}^1_{0,\sigma}(\Omega_0) & (2.2.40a)\\ \mathbf{u}(0) = \mathbf{u}_0. & (2.2.40b) \end{cases}$$

We have the following lemma

Lemma 2.2.13.

We suppose that $N \leq 4$. If $\mathbf{u} \in \mathbf{L}^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0))$, then the function $\mathbf{Bu}$ defined by:

$$\langle \mathbf{Bu}(t),\mathbf{v}\rangle := \mathbf{b}(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}),\ \forall\ \mathbf{v} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)\ a.e.\ t \in [0,T] \tag{2.2.41}$$

belongs to $L^1(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)')$.

Proof: Temam [98, Lemma 3.1, pp 281]. ∎

As in the linear case, we give an equivalent variational formulation of the problem (2.2.38a)–(2.2.38d).

For $\mathbf{f} \in L^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)')$ and $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega_0)$, find $\mathbf{u} \in L^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0))$ such that:

$$\begin{cases} \mathbf{u}' \in L^1(0,T;\mathbf{H}^1_0(\Omega_0)'), & (2.2.42a)\\ \mathbf{u}' + \nu\mathbf{Au} + \mathbf{Bu} = \mathbf{f} \text{ on } [0,T], & (2.2.42b)\\ \mathbf{u}(0) = \mathbf{u}_0. & (2.2.42c) \end{cases}$$

The existence of a weak solution is ensured by the following theorem.

Theorem 2.2.14 (Existence of the full Navier-Stokes equation)**.**

We suppose that $N \leq 4$*. If* $\mathbf{f} \in L^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)')$ *and* $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega_0)$*, then there exists at least one function* $\mathbf{u} \in L^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0))$ *which satisfies* (2.2.42a) − (2.2.42c)*. Moreover,* $\mathbf{u} \in C^0(0,T;\mathbf{L}^2_\sigma(\Omega_0))$ *and satisfies the following strong energy inequality*

$$\|\mathbf{u}(\cdot,t)\|_{C^0_T(\mathbf{L}^2(\Omega_0))} \leq C(T)\Big[\|\mathbf{u}\|_{L^2_T(\mathbf{H}^1_0(\Omega_0))} + \|\mathbf{u}'\|_{L^2_T(\mathbf{H}^1_0(\Omega_0)')}\Big], \tag{2.2.43}$$

Proof: Temam [98, Theorem 3.1, pp 282] ∎

We have the following results on the uniqueness and regularities in dimension two and three. The set Ω_0 is still open bounded and Lipschitz. Proofs are founds in [98, 85].

Theorem 2.2.15 (Uniqueness in dimension $N = 2$)**.**

If $N = 2$*, then the solution* $\mathbf{u}$ *of the problem* (2.2.42a) − (2.2.42c) *given by the existence Theorem* 2.2.14 *is unique. Moreover,* $\mathbf{u}$ *is almost everywhere equal to a continuous function in* $C^0([0,T];\mathbf{L}^2_\sigma(\Omega_0))$ *and*

$$\mathbf{u}(t) \longrightarrow \mathbf{u}_0 \text{ in } \mathbf{L}^2_\sigma(\Omega_0), \text{ when } t \to 0. \tag{2.2.44}$$

Proof: Temam [98, Theorem 3.2, pp 294] ∎

This result gives uniqueness of Navier-Stokes in cases $N = 3$.

Theorem 2.2.16 (Uniqueness in dimension $N \geq 2$)**.**

If $N \geq 2$*, then there exists at most one solution* $\mathbf{u}$ *of* (2.2.42a) − (2.2.42c) *given by the existence theorem, Theorem* 2.2.14 *satisfying:*

$$\mathbf{u} \in L^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)) \cap C^0(0,T;\mathbf{L}^2_\sigma(\Omega_0)), \tag{2.2.45}$$

$$\mathbf{u} \in L^s(0,T;\mathbf{L}^r(\Omega_0)), \tag{2.2.46}$$

with $\frac{2}{s} + \frac{N}{r} \leq 1$ *when* Ω_0 *is bounded and with* $\frac{2}{s} + \frac{N}{r} = 1$ *when* Ω_0 *is unbounded.*

Proof: Temam [98, Theorem 3.4, pp 297 for $N = 3$, $s = 8$, $r = 4$; Remark 3.6, pp 298 for $N \geq 2$] ∎

We also have the following results which give uniqueness and regularity.

Theorem 2.2.17 (Uniqueness and regularity in dimension $N = 2$)**.**

We suppose $N = 2$*. If* $\mathbf{f}$ *and* $\mathbf{f}'$ *belong to* $L^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)')$*,* $\mathbf{f}(0,\cdot) \in \mathbf{L}^2_\sigma(\Omega_0)$ *and* $\mathbf{u}_0 \in \mathbf{H}^2(\Omega_0) \cap \mathbf{H}^1_{0,\sigma}(\Omega_0)$*, then the unique solution* $\mathbf{u}$ *of the problem* (2.2.42a) − (2.2.42c) *given by Theorem* 2.2.15 *satisfies*

$$\mathbf{u}' \in L^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)) \cap L^\infty(0,T;\mathbf{L}^2_\sigma(\Omega_0)). \tag{2.2.47}$$

Moreover, if $\partial\Omega_0$ *is* C^2 *and* $\mathbf{f} \in L^\infty(0,T;\mathbf{L}^2_\sigma(\Omega_0))$*, then the function* $\mathbf{u}$ *satisfies*

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}^2(\Omega_0)). \tag{2.2.48}$$

Proof: Temam [98, Theorem 3.5, pp 299 and Theorem 3.6, pp 301] ∎

Theorem 2.2.18 (Uniqueness and regularity in dimension $N = 3$)**.**

We suppose $N = 3$. If $\mathbf{f}$ and $\mathbf{u}_0$ satisfy

$$\mathbf{u}_0 \in \mathbf{H}^2(\Omega_0) \cap \mathbf{H}^1_{0,\sigma}(\Omega_0), \tag{2.2.49}$$

$$\mathbf{f} \in L^\infty(0,T;\mathbf{L}^2_\sigma(\Omega_0)),\ \mathbf{f}' \in L^1(0,T;\mathbf{L}^2_\sigma(\Omega_0)). \tag{2.2.50}$$

If the following condition which is satisfied when the kinematic viscosity coefficient ν is large enough or when $\mathbf{f}$ and $\mathbf{u}_0$ are small enough holds, namely

$$\frac{d_2}{\nu} + (1+d_1^2)\Big(\|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega_0)} + \frac{Td_2}{\nu}\Big)^{\frac{1}{2}} \exp\Big(\int_0^T \|\mathbf{f}'(t)\|_{\mathbf{L}^2(\Omega_0)}\,\mathrm{d}t\Big) < \frac{\nu^3}{c^2}, \tag{2.2.51}$$

where the positive constants d_1 and d_2 are given by

$$d_1 = \|\mathbf{f}(0)\|_{\mathbf{L}^2_\sigma(\Omega_0)} + \nu c_0 \|\mathbf{u}_0\|_{\mathbf{H}^2(\Omega_0)} + c_1\|\mathbf{u}_0\|^2_{\mathbf{H}^2(\Omega_0)}, \tag{2.2.52}$$

$$d_2 = \|\mathbf{f}\|^2_{L^\infty(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)')}, \tag{2.2.53}$$

and where c_0, c_1 and c are positive constants such that

$$\|\mathbf{u}\|_{\mathbf{L}^4(\Omega_0)} \le c_0 \|\mathbf{u}\|^{\frac{1}{4}}_{\mathbf{L}^2(\Omega_0)} \|\mathbf{u}\|^{\frac{3}{4}}_{\mathbf{H}^1_0(\Omega_0)},\ \forall\, \mathbf{u} \in \mathbf{H}^1_0(\Omega_0), \tag{2.2.54}$$

$$|\mathbf{b}(\mathbf{u},\mathbf{u},\mathbf{v})| \le c_1 \|\mathbf{u}\|^2_{\mathbf{L}^4(\Omega_0)} \|\mathbf{v}\|_{\mathbf{H}^1_{0,\sigma}(\Omega_0)},\ \forall\, \mathbf{u},\ \mathbf{v} \in \mathbf{H}^1_{0,\sigma}(\Omega_0), \tag{2.2.55}$$

$$|\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w})| \le c \|\mathbf{u}\|_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} \|\mathbf{v}\|_{\mathbf{H}^1_{0,\sigma}(\Omega_0)} \|\mathbf{w}\|_{\mathbf{H}^1_{0,\sigma}(\Omega_0)},\ \forall\, \mathbf{u},\ \mathbf{v},\ \mathbf{w} \in \mathbf{H}^1_{0,\sigma}(\Omega_0), \tag{2.2.56}$$

then there exists a unique solution of the problem (2.2.42a) – (2.2.42c) given by Theorem 2.2.14 which satisfies

$$\mathbf{u}' \in L^2(0,T;\mathbf{H}^1_{0,\sigma}(\Omega_0)) \cap L^\infty(0,T;\mathbf{L}^2_\sigma(\Omega_0)). \tag{2.2.57}$$

Moreover, if the boundary $\partial\Omega_0$ is C^∞, then

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}^2(\Omega_0)). \tag{2.2.58}$$

Proof: Temam [98, Theorem 3.7, pp 303 and Theorem 3.8, pp 306] ∎

Chapter 3

Analysis of the fluids dynamics system of artificial heart

In this chapter, we delve into the analysis of a fluid dynamics system characterized by a time-periodic NS equation within a moving domain. Our primary focus is on establishing the existence and uniqueness of a time-periodic strong solution through the application of the implicit function theorem.

3.1 The problem and main result

Let $t \in \mathbb{R}$ and Ω_t be a bounded and Lipschitz domain (a non-empty and simply connected open set) in $\mathbb{R}^N$, where $N = 2, 3$, depending on t. Its boundary is denoted as

$$\partial\Omega_t = \Gamma_r \cup \Gamma_i \cup \Gamma_o \cup M_t. \tag{3.1.1}$$

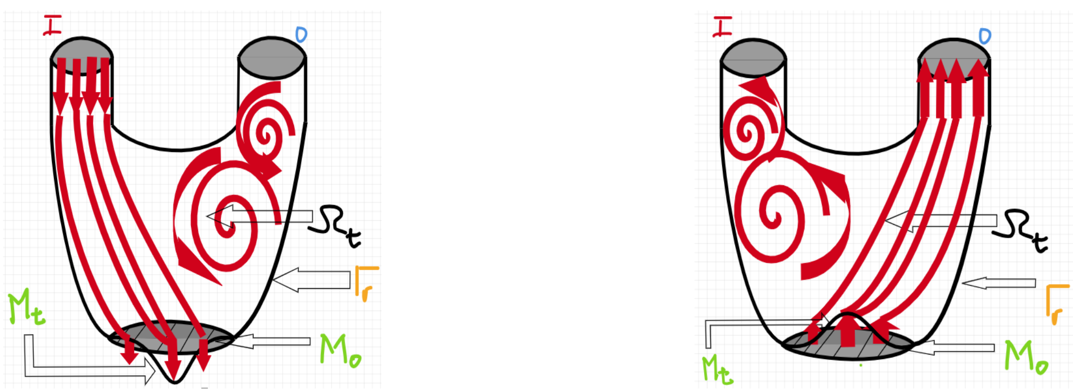
The domain Ω_t represents the moving blood chamber of the artificial heart. The boundaries Γ_α, where $\alpha \in \{r, i, o\}$, respectively represent the rigid boundary, and the inlet and outlet, which are valves that can open or close to allow blood to enter or exit the heart chamber. Finally, we have the boundary M_t, representing the position of the moving membrane. Its motion is responsible for the blood flow.

A periodic pressure generated by an external compressor activates the membrane M_t, causing it to move up and down periodically with a period of $4T$, where $T > 0$. The motion of the membrane, in turn, activates the blood flow inside the heart chamber. More precisely, the artificial heart operates as follows: At time $t = -T$, the membrane is at its upper position M_{-T}, and its velocity is zero. The valve at Γ_o closes, and the valve at Γ_i opens. During the time $(-T, T)$, the membrane moves progressively downward, pulling the blood in from the inlet, which fills the heart chamber. At a certain time, let us say $t = 0$ for instance, we assume the membrane to be flat, i.e., in the horizontal position (see assumption (3.1.4) and (3.1.14)), and its velocity to have reached its minimum value (most negative value). After reaching its minimum value, the velocity starts increasing, and the membrane continues to move downward until the time $t = T$, when the membrane reaches its lower position M_T with zero velocity, and the valve at Γ_i closes, while the valve at Γ_o opens. During the time $(T, 3T)$, the membrane moves progressively upward, pushing the blood out through Γ_o. At a certain time again, let us say at time $t = 2T$, the membrane reaches its flat position (see (3.1.14)) with its maximum velocity (most positive value). Next, the velocity starts decreasing, and the membrane continues to move upward until the time $t = 3T$, when the membrane reaches its upper position with $M_{3T} = M_{-T}$ and with zero velocity, and the periodic cycle continues.

Let us denote by $B_{N-1}(\ell_0)$ the open ball in $\mathbb{R}^{N-1}$, centered at the origin with radius ℓ_0 and for $\alpha \in \{i, o\}$, we denote by $B_{N-1}(C_\alpha, r_\alpha)$ the open ball in $\mathbb{R}^{N-1}$ centered at C_α with radius r_α. We denote by h_α the high at which Γ_α is formed. Throughout this work we make the following assumptions:

$$\begin{cases} 1)\ \Gamma_i := B_{N-1}(C_i, r_i) \times \{h_i\},\ \Gamma_o := B_{N-1}(C_o, r_o) \times \{h_o\},\ M_0 := B_{N-1}(\ell_0) \times \{0\}, & (3.1.2) \\ 2)\ \Gamma_r \text{ near } \Gamma_i, \text{ resp. } \Gamma_o, \text{ forms a right angle with base } \Gamma_i, \text{ resp. } \Gamma_o, & (3.1.3) \\ 3)\ M_t := \{(x', z(x', t)),\ x' := (x_1, \dots, x_{N-1}) \in B_{N-1}(\ell_0),\ t \in \mathbb{R}\}, \text{ with} & \\ \qquad z : B_{N-1}(\ell_0) \times \mathbb{R} \mapsto \mathbb{R}, M_0 = M_{2T} := \{(x', 0),\ x' \in B_{N-1}(\ell_0)\}, & (3.1.4) \\ 4) M_0 \cup \Gamma_r, \text{ is } C^{1,1}, \text{ and} & (3.1.5) \\ 5) \text{We denote by } \Omega_0 \text{ the domain that } \Gamma_i \cup \Gamma_o \cup \Gamma_r \cup M_0 \text{ encloses.} & (3.1.6) \end{cases}$$

More precisely, we have the following picture:



(a) First half period $[-T, T]$

(b) Second half period $[T, 3T)$

Figure 3.1: A schematic view of an artificial heart: Ω_t is the moving domain occupied by the blood, I is the inlet valve, O the outlet valve, Γ_r the rigid boundary, M_t the membrane moving up and down.

Note that there is no discontinuities in the fluid domain. We assume that the membrane M_t moves across the boundary M_0. With these notations introduced, we can now proceed to introduce the mathematical model of the fluid, which is the blood here. All through this thesis, we assume the blood to be an incompressible Newtonian fluid with constant viscosity coefficient and constant density (see Ahmed et al. [2], Tavoularis et al. [97], Casanova [14]).

For all $x \in \mathbb{R}^N$, we will use the notation $x' = (x_1, ..., x_{N-1})$ such that $x = (x', x_N)$. Let $z = z(x', t)$, where $x' \in B_{N-1}(\ell_0)$ and $t \in \mathbb{R}$, be the function defining the motion of the membrane M_t. Let $\mathbf{u} = \mathbf{u}(x,t)$ and $p = p(x,t)$ represent the blood velocity and pressure, respectively, where $x = (x_1, ..., x_N) \in \Omega_t \subset \mathbb{R}^N$. Then, $\mathbf{u}$ and p satisfy:

$$\begin{cases} \rho\partial_t\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \rho\vec{g} \quad \text{in } \Omega_t \times \{t\}, \quad t \in \mathbb{R}, & \text{(3.1.7a)} \\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega_t \times \{t\}, \quad t \in \mathbb{R}, & \text{(3.1.7b)} \\ \mathbf{u}(x,-T) = \mathbf{u}(x,3T), \quad x \in \Omega_{-T}, & \text{(3.1.7c)} \\ \mathbf{u}(x,t) = 0, \quad (x,t) \in \Gamma_r \times \mathbb{R}, & \text{(3.1.7d)} \\ \mathbf{u}(x,t) = k_i(t)\mathbf{u}_i(x), \quad (x,t) \in \Gamma_i \times \mathbb{R}, & \text{(3.1.7e)} \\ \mathbf{u}(x,t) = k_o(t)\mathbf{u}_o(x), \quad (x,t) \in \Gamma_o \times \mathbb{R}, & \text{(3.1.7f)} \\ \mathbf{u}(x,t) = (0,...,0,\partial_t z(x',t)), \quad (x,t) \in M_t \times \mathbb{R}. & \text{(3.1.7g)} \end{cases}$$

Here ρ is the blood density, μ the blood viscosity and $\vec{g}$ is the gravity vector and we can suppose that $\vec{g} = (0, ..., 0, g_N)$. The functions $\mathbf{u}_i$ and $\mathbf{u}_o$ are given "parabolic velocity profiles" on Γ_i and Γ_o. Namely we have

$$\mathbf{u}_i(x) = \frac{d^2(x, \partial\Omega_t\backslash\Gamma_i)}{\int_{\Gamma_i} d^2(x, \partial\Omega_t\backslash\Gamma_i)\,\mathrm{d}x}\boldsymbol{\nu}_i, \quad x \in \Gamma_i, \tag{3.1.8}$$

$$\mathbf{u}_o(x) = \frac{d^2(x, \partial\Omega_t\backslash\Gamma_o)}{\int_{\Gamma_o} d^2(x, \partial\Omega_t\backslash\Gamma_o)\,\mathrm{d}x}\boldsymbol{\nu}_o, \quad x \in \Gamma_o, \tag{3.1.9}$$

where $\boldsymbol{\nu}_i$, resp. $\boldsymbol{\nu}_o$, is the unit exterior normal vecor on Γ_i, resp. Γ_o. The function k_i resp. k_o is equal to zero in $[T, 3T) + 4T\mathbb{Z}$ resp. $[-T, T) + 4T\mathbb{Z}$. Moreover, k_i and k_o are chosen so that the divergence free condition (3.1.7b) is satisfied, and for a given set $A \subset \mathbb{R}^N$, and $x \in \mathbb{R}^N$, $d(x, A)$ represents the distance from x to A. Integrating (3.1.7b) in Ω_t gives

$$k_i(t)\mathbb{1}_{[-T,T)}(t) + k_o(t)\mathbb{1}_{[T,3T)}(t) + \int_{M_t} (0, ..., 0, \partial_t z)\cdot\boldsymbol{\nu}_t \,\mathrm{d}\sigma = 0, \ \forall\, t \in [-T, 3T], \tag{3.1.10}$$

where for a given subset $A \subset \mathbb{R}$, the notation $\mathbb{1}_A$ represents the characteristic function of A and $\boldsymbol{\nu}_t$ is the normal exterior unit vector on M_t defined as

$$\boldsymbol{\nu}_t = (\nabla_{x'} z, -1)(1 + |\nabla_{x'} z(x', t)|^2)^{-1/2}.$$

Therefore,

$$k_i(t) = \mathbb{1}^T_{[-T,T)}(t)\int_{B_{N-1}(\ell_0)} \partial_t z(x', t)\,\mathrm{d}x',$$

$$k_o(t) = \mathbb{1}^T_{[T,3T)}(t)\int_{B_{N-1}(\ell_0)} \partial_t z(x', t)\,\mathrm{d}x',$$

where $\mathbb{1}^T_{[-T,T)}$ resp. $\mathbb{1}^T_{[T,3T)}$ represents the characteristics function of the set $[-T, T) + 4T\mathbb{Z}$ resp. $[T, 3T) + 4T\mathbb{Z}$. We can introduce the following vector function which represents the boundary data.

$$\mathbf{g}(x,t) = (\mathbf{u}_i(x)k_i(t) + \mathbf{u}_o(x)k_o(t))\mathbb{1}_{\Gamma_i\cup\Gamma_o}(x) + (0, ..., 0, \partial_t z(x', t))\mathbb{1}_{M_t}(x)$$

$$
\begin{aligned}
&= \Big(\mathbf{u}_i(x)\mathbb{1}^T_{[-T,T)}(t) \\
&+ \mathbf{u}_o(x)\mathbb{1}^T_{[T,3T)}(t)\Big)\mathbb{1}_{\Gamma_i\cup\Gamma_o}(x)\int_{B_{N-1}(\ell_0)}\partial_t z(x',t)\mathrm{d}x' \\
&+ (0,...,0,\partial_t z(x',t))\mathbb{1}_{M_t}(x),
\end{aligned} \tag{3.1.11}
$$

where the characteristic function of M_t, which depends on z is defined as

$$
\mathbb{1}_{M_t}(x) = \begin{cases} 1 \text{ if } x \in \{(x', z(x',t)),\ x' \in B_{N-1}(\ell_0)\}, \\ 0 \text{ else.} \end{cases} \tag{3.1.12}
$$

Therefore, the equations (3.1.7a) – (3.1.7g) can be written as

$$
\begin{cases}
\rho\partial_t\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \rho\vec{g}, \quad \text{in } \Omega_t\times\{t\}, \quad t\in\mathbb{R}, & (3.1.13a)\\
\nabla\cdot\mathbf{u} = 0 \text{ in } \Omega_t\times\{t\}, \quad t\in\mathbb{R}, & (3.1.13b)\\
\mathbf{u}(x,-T) = \mathbf{u}(x,3T), \quad x\in\Omega_{-T}, & (3.1.13c)\\
\mathbf{u}(x,t) = \mathbf{g}(x,t), \quad (x,t)\in\partial\Omega_t\times\mathbb{R}. & (3.1.13d)
\end{cases}
$$

In the following, we will assume that the periodic motion of the membrane z is known. We will prove the existence of a unique solution $(\mathbf{u},p)$ to (3.1.13a) – (3.1.13d) for $\|z\|_{Z_T} \ll 1$.

We start by introducing the following specific Banach spaces. We recall that the under-script “T” on a given Banach space will always denote the $4T$ time-periodicity.

$$
\begin{aligned}
Z_T &:= L^\infty_T(W_{00}^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(B_{N-1}(\ell_0))) \cap H^1_T(H^{3/2}_{00}(B_{N-1}(\ell_0))) \cap H^2_T(L^2(B_{N-1}(\ell_0))) \\
&\cap \Big\{\partial_t z \le 0 \ \ in \ \ (-T,T), \quad \partial_t z \ge 0 \ \ in \ \ (T,3T), \quad z(\cdot,0) = z(\cdot,2T) = 0, \\
&\qquad \partial_t z(\cdot,(2k+1)T) = 0, \ \ \forall\, k\in\mathbb{Z}, \ \int_{B_{N-1}(\ell_0)}(z(\cdot,3T) - z(\cdot,T))\,\mathrm{d}x' = \frac{m_0}{\rho}\Big\}, && (3.1.14)\\
\mathbf{Z}_T &:= L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(\Omega_0)) \cap H^1_T(\mathbf{H}^2(\Omega_0)) \cap H^2_T(\mathbf{L}^2(\Omega_0)) \\
&\cap \{\mathbf{z} \ / \ \forall\, t\in(-T,3T), \text{ supp } (\mathbf{z}(\cdot,t))\cap\partial\Omega_0 \subset M_0\} \\
&\cap \{\mathbf{z} \ / \ \partial_t\mathbf{z}(\cdot,(2k+1)T) = 0, \ \forall\, k\in\mathbb{Z}\}, && (3.1.15)\\
\mathbf{U}_T &:= L^2_T(\mathbf{H}^2(\Omega_t)) \cap H^1_T(\mathbf{L}^2(\Omega_t)), && (3.1.16)\\
\mathbf{V}_T &:= L^2_T(\mathbf{H}^2(\Omega_t)\cap\mathbf{H}^1_0(\Omega_t)) \cap H^1_T(\mathbf{L}^2(\Omega_t)), && (3.1.17)\\
P_T &:= L^2_T(H^1(\Omega_t)\cap L^2_0(\Omega_t)), && (3.1.18)\\
\mathbf{W}_T &:= L^2_T(\mathbf{H}^2(\Omega_0)\cap\mathbf{H}^1_0(\Omega_0)) \cap H^1_T(\mathbf{L}^2(\Omega_0)), && (3.1.19)\\
\mathbf{W}^{\mathbf{h}}_T &:= L^2_T(\mathbf{H}^2(\Omega_0)) \cap H^1_T(\mathbf{L}^2(\Omega_0)), && (3.1.20)\\
Q_T &:= L^2_T(H^1(\Omega_0)\cap L^2_0(\Omega_0)), && (3.1.21)\\
\mathbf{R}_{\mathbf{w}} &:= L^2_T(\mathbf{L}^2(\Omega_0)), && (3.1.22)\\
R_q &:= Q_T \cap H^1_T(H^1(\Omega_0)'), && (3.1.23)
\end{aligned}
$$

where the spaces $W_{00}^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(B_{N-1}(\ell_0))$ resp. $H_{00}^{3/2}(B_{N-1}(\ell_0))$ are functions in $W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(B_{N-1}(\ell_0))$ resp. $H^{3/2}(B_{N-1}(\ell_0))$ which have zero extension in $W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(\mathbb{R}^{N-1})$ resp. $H^{3/2}(\mathbb{R}^{N-1})$ and

$$N=2: \begin{cases} \hat{2}=2+\epsilon, \\ \hat{2}^*=2+\frac{4}{\epsilon}, \quad \epsilon>0, \end{cases} \qquad N=3: \begin{cases} \hat{2}=6-\epsilon, \\ \hat{2}^*=3+\frac{\epsilon}{4-\epsilon}, \quad \epsilon\in(0,3] \end{cases}. \tag{3.1.24}$$

Note that the assumptions placed on the description of the space Z_T essentially serve to describe the motion of the membrane, i.e., moving downward during the first half period $(-T,T)$ and upward during the second half period, taking the horizontal position at the times $t=0$ and $2T$ and having zero velocity at the upper and lower position. The regularity we assumed on z, i.e., $z\in Z_T$, is sufficient to construct its extension on the space $\mathbf{Z}_T$. Therefore, the local perturbation will be of class $\mathbf{Z}_T$, and since our reference domain Ω_0 is essentially $C^{1,1}$, except near the inlet and outlet, we will have a $\mathbf{Z}_T$ moving domain Ω_t which is weaker than the C^3 assumption presented in the literature ([14, 31, 44, 75]). Because of the $L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(B_{N-1}(\ell_0)))$ regularity of z, that extension defines a $L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(\Omega_0))$-diffeomorphism. The space $\mathbf{U}_T\times P_T$ is where we prove the existence result of a strong solution to our original time-periodic NS problem. However, the analysis of the original NS problem involves a series of transformations. So, we begin by transforming the original non-homogeneous NS problem onto a homogeneous NS problem where the solution is sought in $\mathbf{V}_T\times P_T$. We then use the diffeomorphism we built to set the homogeneous NS problem on a fixed domain, and the space $\mathbf{W}_T\times Q_T$ is where we find the solution using the implicit function theorem. The extension of the boundary data on the fixed domain will then be in the space $\mathbf{W}^{\mathbf{h}}_T$. The range of the map on which we apply the implicit function theorem is $\mathbf{R}_{\mathbf{w}}\times R_q$.

Remark 3.1.1 (Motivation for the choice of $\hat{2}$ and $\hat{2}^*$)**.**

We have chosen $\hat{2}$ and $\hat{2}^*$ such that they satisfy the following properties:

- $1>\frac{N}{\hat{2}^*}$ *so that the Sobolev's embedding* $W^{1,\hat{2}^*}\hookrightarrow C^0$ *holds.*
- *The Sobolev's embeddings* $\mathbf{H}^1(\Omega_0)\hookrightarrow\mathbf{L}^{\hat{2}}(\Omega_0)$ *and* $\mathbf{H}^1(\Omega_0)\hookrightarrow\mathbf{L}^{\hat{2}^*}(\Omega_0)$ *hold for* $N=2,\ 3$.
- *The number* $\hat{2},\ \hat{2}^*$ *satisfy* $\frac{1}{\hat{2}}+\frac{1}{\hat{2}^*}=\frac{1}{2}$, *such that we can apply the generalized Holder's inequality. Precisely, if* $f\in L^{\hat{2}}(\Omega_0)$ *and* $g\in L^{\hat{2}^*}(\Omega_0)$ *then the product* $fg\in L^2(\Omega_0)$ *with the estimation* $\|fg\|_{L^2(\Omega_0)}\leq\|f\|_{L^{\hat{2}}(\Omega_0)}\|g\|_{L^{\hat{2}^*}(\Omega_0)}$.

Indeed, for $N=2$, *since* $1=\frac{N}{2}$, *we have the Sobolev's embedding* $\mathbf{H}^1(\Omega_0)\hookrightarrow\mathbf{L}^q(\Omega_0)$ *for all* $q\geq 2$ *and for* $N=3$, *since* $1<\frac{N}{2}$ *we have the Sobolev's embedding* $\mathbf{H}^1(\Omega_0)\hookrightarrow\mathbf{L}^q(\Omega_0)$ *for all* $q\in[2,6]$. *We can take* $\hat{2}=6-\epsilon$ *and denote by* $\hat{2}^*$ *the number such that* $\frac{1}{\hat{2}^*}+\frac{1}{\hat{2}}=\frac{1}{2}$, *i.e* $\hat{2}^*=3+\frac{\epsilon}{4-\epsilon}$. *Note that for* $\epsilon\in(0,3]$, *we have* $\hat{2},\ \hat{2}^*\in(3,6]$ *which implies* $H^1(\Omega_0)\hookrightarrow\mathbf{L}^{\hat{2}^*}(\Omega_0)$ *and* $H^1(\Omega_0)\hookrightarrow\mathbf{L}^{\hat{2}}(\Omega_0)$.

We will assume knowledge of the solution to the membrane dynamic equation z in the Banach space Z_T, defined in (3.1.14) and we will prove the existence and the uniqueness of a time-periodic solution $(\mathbf{u}, p)$ to the problem (3.1.13b) $-$ (3.1.13d) in the space $\mathbf{U}_T \times P_T$, defined in (3.1.16) and (3.1.18). The main result of this chapter reads.

Theorem 3.1.2 (Main result)**.**

Assume the hypotheses (3.1.2)-(3.1.6) *hold. For* $\|z\|_{Z_T} \ll 1$*, we consider its extension by zero in* $\mathbb{R}^{N-1}$ *still denoted* z*, we set* $\partial\Omega_t = \{(x', x_N + z(x', t)),\ (x', x_N) \in \partial\Omega_0\}$*. Then* $\partial\Omega_t$ *encloses an open set* Ω_t*, which is* $4T$ *periodic and there exists a unique time-periodic strong solution* $(\mathbf{u}, p) \in \mathbf{U}_T \times P_T$ *of the Navier-Stokes problem* (3.1.13a) $-$ (3.1.13d)*. Furthermore,* $\mathbf{u} \in C^0_T(\mathbf{H}^1(\Omega_t))$*.*

The technique we will use is the implicit function theorem. More precisely, we will transform the NS equations (3.1.13a)-(3.1.13d) in a fixed domain. Next, we prove that the differential of the new NS equations on a fixed domain depends on a time-periodic Stokes problem which defines an isomorphism. This allows to conclude with existence of the solution by using the implicit function theorem.

There are a number of works in the literature dealing with the time-periodic NS equations in a moving domain. Hiroko [44] initiated the existence theory for time-periodic incompressible NS equations in a periodic moving domain. He proved the existence of solution in dimension $N = 2, 3$, with the assumptions that the moving domain is C^3. It was followed by Miyakawa and Teramoto [75], who proved the existence of a global weak solution in a domain whose boundary moves smoothly. He used a C^∞ diffeomorphism to set the problem on a fixed domain. Next, he proved the existence of a weak solution of the newly defined problem by a Faedo-Galerkin approach. Salvi [86] proved the existence of a periodic weak solution to NS equations on a moving domain using the elliptic regularization in a case of a C^3 moving boundary. Farwig et al. [31] prove an existence and uniqueness result in the case of C^3 moving domains and dimension $N \geq 2$. In the case when the analysis is related directly to an application, Casanova [14] proved the existence of a solution to the time-dependent NS equation coupled with a forth order equation of a beam, in a moving domain which turns out to have regularity H^4 and in dimension $N = 2$.

In our case, with the assumptions (3.1.2) $-$ (3.1.6) on the reference domain Ω_0, we have a $\mathbf{Z}_T$ regularity on the moving domain Ω_t which is weaker than the result available in the literature. We prove a result of existence and uniqueness of a time-periodic strong solution in the moving domain Ω_t.

The outline of this chapter is as follows. We begin by presenting auxiliary results that are useful for proving the (Fréchet) C^1 regularity of a map we will define later and which we will use for the implicit function theorem. Next, we set the problem within a fixed domain. Following this, we establish the existence of the transformation and boundary data in the appropriate Banach spaces. Finally, we apply the implicit function theorem to the newly defined problem on a fixed domain, proving both existence and uniqueness.

3.2 Some auxiliary results

In this section we present a few classical functional analysis results that we will use. Mainly, the following propositions show results about the C^1 regularity of a product. It will be very useful when showing the C^1 regularity of the map $\mathbf{F}$ (see (3.3.25)). Though the results seems classic, we did not find any explicit reference. This is why we prove it here.

Proposition 3.2.1. *Let $f : Z \times W \to R_f$ be a function such that $f(z,w) = a(z)b(w)$, with $a : Z \to R_a$ and $b : W \to R_b$ two C^1 functions, where Z, W, R_a, R_b, R_f are Banach spaces. We assume that there exists $C > 0$ such that for all $(\tilde{a}, \tilde{b}) \in R_a \times R_b$ we have*

$$\tilde{a}\tilde{b} \in R_f \text{ and } \|\tilde{a}\tilde{b}\|_{R_f} \leq C\|\tilde{a}\|_{R_a}\|\tilde{b}\|_{R_b}. \tag{3.2.1}$$

Then $f \in C^1(Z \times W; R_f)$.

Proof: For $z, h, \alpha \in Z$ and $w, k, \beta \in W$, we set

$$\begin{cases} \Delta a(z,h) := a(z+h) - a(z), & \Delta b(w,k) := b(w+k) - b(w), \\ \delta a(z,h) := a(z+h) - a(z) - a'(z;h), & \delta b(w,k) := b(w+k) - b(w) - b'(w;k), \\ \Delta a'(z,h;\alpha) := a'(z+h;\alpha) - a'(z;\alpha), & \Delta b'(w,k;\beta) := b'(w+k;\beta) - b'(w;\beta), \end{cases}$$

where $a'(z;h)$ resp. $b'(w;k)$ represents the Fréchet derivative of a resp. b in the direction h resp. k. We also set

$$\begin{cases} \Delta f(z,w,h,k) := f(z+h,w+k) - f(z,w) \\ \qquad = \Delta a(z,h)b(w+k) + a(z)\Delta b(w,k), & (3.2.2) \\ D_{(z,w)}f(z,w;h,k) := a'(z;h)b(w) + a(z)b'(w;k) & (3.2.3) \\ \delta f(z,w;h,k) := \Delta f(z,w,h,k) - D_{(z,w)}f(z,w;h,k) \\ \qquad = \delta a(z,h)b(w+k) + a(z)\delta b(w,k) + a'(z;h)\Delta b(w,k) & (3.2.4) \\ \Delta D_{(z,w)}f(z,w,h,k;\alpha,\beta) := D_{(z,w)}f(z+h,w+k;\alpha,\beta) - D_{(z,w)}f(z,w;\alpha,\beta) \\ \qquad = \Delta a'(z,h;\alpha)b(w+k) + a'(z;\alpha)\Delta b(w,k) \\ \qquad + \Delta a(z,h)b'(w+k;\beta) + a(z)\Delta b'(w,k;\beta). & (3.2.5) \end{cases}$$

Claim: $f \in C^0(Z \times W; R_f)$.

Indeed, from (3.2.1) and (3.2.2) we have

$$\begin{aligned} \|\Delta f(z,w,h,k)\|_{R_f} &\leq \|\Delta a(z,h)b(w+k)\|_{R_f} + \|a(z)\Delta b(w,k)\|_{R_f} \\ &\leq C\|\Delta a(z,h)\|_{R_a}\|b(w+k)\|_{R_b} + C\||a(z)\|_{R_a}\|\Delta b(w,k)\|_{R_b} \\ &\longrightarrow 0 \text{ as } \|h\|_Z \to 0 \text{ and } \|k\|_W \to 0, \end{aligned} \tag{3.2.6}$$

because the maps $z \mapsto a(z)$ and $w \mapsto b(w)$ are continue.

Claim: f is (Fréchet) differentiable at (z,w) at its Fréchet derivative reads

$$D_{(z,w)}f(z,w): \ (h,k) \in Z \times W \mapsto D_{(z,w)}f(z,w;h,k) \in R_f. \tag{3.2.7}$$

Indeed, using (3.2.1) and (3.2.4) we get

$$\begin{aligned}\|\delta f(z,w;h,k)\|_{R_f} &\leq \|\delta a(z,h)b(w+k)\|_{R_f} + \|a(z)\delta b(w,k)\|_{R_f} + \|a'(z;h)\Delta b(w,k)\|_{R_f}\\ &\leq C\Big(\|\delta a(z,h)\|_{R_a}\|b(w+k)\|_{R_b} + \|a(z)\|_{R_a}\|\delta b(w,k)\|_{R_b}\\ &+ \|a'(z;h)\|_{R_a}\|\Delta b(w,k)\|_{R_b}\Big)\\ &\leq o(\|h\|_Z) + o(\|k\|_W)\\ &\leq o(\|(h,k)\|_{Z\times W}), \end{aligned} \tag{3.2.8}$$

because the maps $z \mapsto a(z)$ and $z \mapsto b(z)$ are differentiable. Here for $x \in \{h, k, (h,k)\}$ and $X \in \{Z, W, Z\times W\}$, $o(\|x\|_X)$ is a real function satisfying $\frac{o(\|x\|_X)}{\|x\|_X} \to 0$ as $\|x\|_X \to 0$.

Claim: The map $D_{(z,w)}f: \ (z,w) \in Z\times W \mapsto D_{(z,w)}f(z,w) \in \mathcal{B}(Z\times W; R_f)$ is continuous.

Indeed, using again (3.2.1) and (3.2.5) and the C^1 regularity of a and b, we get

$$\begin{aligned}\|\Delta D_{(z,w)}f(z,w,h,k;\alpha,\beta)\|_{R_f} &\leq \|\Delta a'(z,h;\alpha)b(w+k)\|_{R_f} + \|a'(z;\alpha)\Delta b(w,k)\|_{R_f}\\ &+ \|\Delta a(z,h)b'(w+k;\beta)\|_{R_f} + \|a(z)\Delta b'(w,k;\beta)\|_{R_f}\\ &\leq C\Big(\|\Delta a'(z,h;\alpha)\|_{R_a}\|b(w+k)\|_{R_b} + \|a'(z;\alpha)\|_{R_a}\|\Delta b(w,k)\|_{R_b}\\ &+ \|\Delta a(z,h)\|_{R_a}\|b'(w+k;\beta)\|_{R_b} + \|a(z)\|_{R_a}\|\Delta b'(w,k;\beta)\|_{R_b}\Big)\\ &\leq C\Big(\|\Delta a'(z,h;\alpha)\|_{R_a}\|b(w+k)\|_{R_b}\\ &+ o_k(1)\|\alpha\|_Z + o_h(1)\|\beta\|_W + \|a(z)\|_{R_a}\|\Delta b'(w,k;\beta)\|_{R_b}\Big). \end{aligned} \tag{3.2.9}$$

Taking the sup over $\{(\alpha,\beta) \in Z\times W, \quad \|(\alpha,\beta)\|_{Z\times W} \leq 1\}$, by the continuity of the maps $z \in Z \mapsto a'(z;\cdot) \in \mathcal{B}(Z, R_a)$ and $w \in W \mapsto b(w;\cdot) \in \mathcal{B}(W, R_b)$ we get

$$\begin{aligned}\|\Delta D_{(z,w)}f(z,w,h,k;\cdot,\cdot)\|_{\mathcal{B}(Z\times W;R_a)} &\leq C\Big(o_h(1)\|b(w+k)\|_{R_b} + o_k(1) + o_h(1) + o_k(1)\|a(z)\|_{R_a}\Big)\\ &= o_h(1) + o_k(1). \end{aligned} \tag{3.2.10}$$

where for $x \in \{h,k\}$ and $X \in \{Z, W\}$ we have $o_x(1) \to 0$ as $\|x\|_X \to 0$. ∎

The following proposition is a generalization of Proposition 3.2.1.

Proposition 3.2.2. *Let $p \geq 2$ an integer and Z, R_a and R_a^i, $i \in \{1,...,p\}$ a collection of Banach spaces. Let $a: \ Z \to R_a$ a function such that $a(z) = \prod_{i=1}^p a_i(z)$ with $a_i \in C^1(Z; R_a^i)$. We assume that there exists a constant $C > 0$ such that for all $\tilde{a}_i \in R_a^i$ we have*

$$\prod_{i=1}^p \tilde{a}_i \in R_a \ \text{and} \ \Big\|\prod_{i=1}^p \tilde{a}_i\Big\|_{R_a} \leq C\prod_{i=1}^p \|\tilde{a}_i\|_{R_a^i}. \tag{3.2.11}$$

Then $a \in C^1(Z; R_a)$.

Proof: Let $z, h, \alpha \in Z$. We set

$$a'(z;h) := \sum_{i=1}^{p} a_i'(z;h) \prod_{j=1, j\neq i}^{p} a_j(z), \tag{3.2.12}$$

$$\Delta a(z,h) := a(z+h) - a(z) = \prod_{i=1}^{p} a_i(z+h) - \prod_{i=1}^{p} a_i(z)$$

$$= \sum_{i=1}^{p} \Delta a_i(z,h) \prod_{j=1}^{i-1} a_j(z) \prod_{j=i+1}^{p} a_j(z+h), \tag{3.2.13}$$

$$\delta a(z,h) := \Delta a(z,h) - a'(z;h)$$

$$= \sum_{i=1}^{p} \prod_{j=1}^{i-1} a_j(z) \Big(\Delta a_i(z,h) \prod_{j=i+1}^{p} a_j(z+h) - a_i'(z;h) \prod_{j=i+1}^{p} a_i(z) \Big)$$

$$= \sum_{i=1}^{p} \prod_{j=1}^{i-1} a_j(z) \Big(\delta a(z,h) \prod_{j=i+1}^{p} a_j(z+h) + a_i'(z;h) \Big(\prod_{j=i+1}^{p} a_j(z+h) - \prod_{j=i+1}^{p} a_i(z) \Big) \Big)$$

$$= \sum_{i=1}^{p} \prod_{j=1}^{i-1} a_j(z) \Big(\delta a(z,h) \prod_{j=i+1}^{p} a_j(z+h)$$

$$+ a_i'(z;h) \sum_{j=i+1}^{p} \Delta a_j(z,h) \prod_{k=i+1}^{j-1} a_j(z) \prod_{k=j+1}^{p} a_j(z+h) \Big), \tag{3.2.14}$$

$$\Delta a'(z,h;\alpha) := a'(z+h;\alpha) - a'(z;\alpha)$$

$$= \sum_{i=1}^{p} a_i'(z+h;\alpha) \prod_{j=1, j\neq i}^{p} a_j(z+h) - \sum_{i=1}^{p} a_i'(z;\alpha) \prod_{j=1, j\neq i}^{p} a_j(z)$$

$$= \sum_{i=1}^{p} \Big(\Delta a_i'(z,h;\alpha) \prod_{j=1, j\neq i}^{p} a_j(z+h) + a_i'(z;\alpha) \Big(\prod_{j=1, j\neq i}^{p} a_j(z+h) - \prod_{j=1, j\neq i}^{p} a_j(z) \Big) \Big)$$

$$= \sum_{i=1}^{p} \Big(\Delta a_i'(z,h;\alpha) \prod_{j=1, j\neq i}^{p} a_j(z+h)$$

$$+ a_i'(z;\alpha) \sum_{j=1, j\neq i}^{p} \Delta a_j(z,h) \prod_{j=1, k\neq i}^{j-1} a_j(z) \prod_{k=j+1, k\neq i}^{p} a_j(z+h) \Big). \tag{3.2.15}$$

Claim: $a \in C^0(Z; R_a)$.

Indeed, from the continuity of the maps $z \mapsto a_i,\ i \in \{1, ..., p\}$ and (3.2.11) and (3.2.13), we have

$$\|\Delta a(z,h)\|_{R_a} \leq C \sum_{i=1}^{p} \|\Delta a_i(z,h)\|_{R_a^i} \prod_{j=1}^{i-1} \|a_j(z)\|_{R_a^j} \prod_{j=i+1}^{p} \|a_j(z+h)\|_{R_a^j} \longrightarrow 0 \text{ as } \|h\|_Z \to 0. \tag{3.2.16}$$

Claim: The map $z \mapsto a(z)$ is Fréchet differentiable and its derivative reads

$$z \in Z \mapsto a'(z;\cdot) \in \mathcal{B}(Z, R_a). \tag{3.2.17}$$

Indeed, from (3.2.11) and (3.2.14) and using the fact that for $i \in \{1,...,p\}$. $a_i \in C^1(Z; R_a^i)$ we get

$$\|\delta a(z,h)\|_{R_a} \leq C \sum_{i=1}^{p} \prod_{j=1}^{i-1} \|a_j(z)\|_{R_a^j} \Big(o(\|h\|_Z) \prod_{j=i+1}^{p} \|a_j(z+h)\|_{R_a^j}$$
$$+ \|h\|_Z \sum_{j=i+1}^{p} o_h(1) \prod_{k=i+1}^{j-1} \|a_j(z)\|_{R_a^j} \prod_{k=j+1}^{p} \|a_j(z+h)\|_{R_a^j} \Big) \leq o(\|h\|_Z). \tag{3.2.18}$$

Claim: The map $z \in Z \mapsto a'(z;\cdot) \in \mathcal{B}(Z, R_a)$ is continuous.

Indeed, from (3.2.11) and (3.2.15) and using the fact that for all $i \in \{1,...,p\}$, $z \mapsto a_i'(z)$ is continue, we have

$$\|\Delta a'(z,h;\alpha)\|_{R_a} \leq C \sum_{i=1}^{p} \Big(\|\Delta a_i'(z,h;\alpha)\|_{R_a^i} \prod_{j=1, j\neq i}^{p} \|a_j(z+h)\|_{R_a^j}$$
$$+ \|\alpha\|_Z \sum_{j=1, j\neq i}^{p} o_h(1) \prod_{j=1, k\neq i}^{j-1} \|a_j(z)\|_{R_a^j} \prod_{k=j+1, k\neq i}^{p} \|a_j(z+h)\|_{R_a^j} \Big).$$

Taking the sup over $\{\alpha \in Z, \quad \|\alpha\|_Z \leq 1\}$, we get

$$\|\Delta a'(z,h;\cdot)\|_{\mathcal{B}(Z,R_a)} \leq C \sum_{i=1}^{p} \Big(o_h(1) \prod_{j=1, j\neq i}^{p} \|a_j(z+h)\|_{R_a^j}$$
$$+ \sum_{j=1, j\neq i}^{p} o_h(1) \prod_{j=1, k\neq i}^{j-1} \|a_j(z)\|_{R_a^j} \prod_{k=j+1, k\neq i}^{p} \|a_j(z+h)\|_{R_a^j} \Big)$$
$$= o_h(1) \longrightarrow 0 \text{ as } \|h\|_Z \to 0.$$

∎

3.3 Setting the problem in a fixed domain

To proceed with the solution of (3.1.13a) − (3.1.13d) in the moving space-time domain $\Omega_t \times \{t\}$, where $t \in (-T, 3T)$, we make a series of transformations so that the problem becomes more approachable from a functional viewpoint. First, we perform the change of variable $\mathbf{v} = \mathbf{u} - \mathbf{g}$, i.e., $\mathbf{u} = \mathbf{v} + \mathbf{g}$, such that $\mathbf{v}$ satisfies the homogeneous boundary condition on M_t. Then, we transform the problem into a fixed domain, where the final existence result will be established.

For this reason, we assume that $\mathbf{g}$ has an extension denoted by the same letter, $\mathbf{g} \in \mathbf{U}_T$. Since, according to (3.1.11), $\mathbf{g}$ depends on z, its extension will also depend on z in a way that will become clearer later. With $\mathbf{u} = \mathbf{v} + \mathbf{g}$, we transform (3.1.13a) – (3.1.13d) into the following problem:

Find $(\mathbf{v}, p) \in \mathbf{V}_T \times P_T$ solving

$$\begin{cases} \rho\Big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{g}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{g}\Big) - \mu\Delta\mathbf{v} + \nabla p \\ \qquad\qquad\qquad\qquad = \mathbf{f} \quad in \quad \Omega_t \times \{t\},\ t \in (-T, 3T), & (3.3.1a) \\ \nabla\cdot\mathbf{v} = -\nabla\cdot\mathbf{g} \quad on \quad \Omega_t \times \{t\},\ t \in (-T, 3T), & (3.3.1b) \\ \mathbf{v}(\cdot, -T) = \mathbf{v}(\cdot, 3T), & (3.3.1c) \\ \mathbf{v} = 0, \quad on \quad \partial\Omega_t \times \{t\},\ t \in (-T, 3T), & (3.3.1d) \end{cases}$$

where the space $\mathbf{V}_T$ resp. P_T is defined in (3.1.17) resp. (3.1.18), and

$$\mathbf{f} := \rho(\vec{g} - \partial_t \mathbf{g} - (\mathbf{g}\cdot\nabla)\mathbf{g}) + \mu\Delta\mathbf{g}. \tag{3.3.2}$$

3.3.1 Transformation to a fixed domain

Before we transform (3.3.1a)-(3.3.1d) in a fixed domain, let us first present some preliminary calculations that relates the derivatives of functions in a variable domain with those in a fixed domain.

Assume $\mathbf{T}(t) = (T_1(t), ..., T_N(t)) : \bar{\Omega}_0 \mapsto \bar{\Omega}_t$ is a $4T$ periodic $L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(\Omega_0))-$ diffeomorphism such that $\mathbf{T}(M_0) = M_t$ and $\mathbf{T}(\Gamma_\alpha) = \Gamma_\alpha, \quad \alpha \in \{i, o, r\}$. The existence of such a diffeomorphism is proven later in Lemma 3.3.2. Let $\mathbf{T}^{-1} = (T_1^{-1}, ..., T_N^{-1}) : \bar{\Omega}_t \mapsto \bar{\Omega}_0$ be its inverse. For a given smooth scalar function ψ in Ω_t we set $\varphi = \psi \circ \mathbf{T}$, or equivalently $\psi = \varphi \circ \mathbf{T}^{-1}$. We will write the formulas that connect the derivatives of φ and ψ.

First, let us precise the notations for the gradient and the derivatives of scalar and vector functions. If h is a scalar function and $\mathbf{h} = [h_i]$ is a column vector function, then ∇h is a column vector and $[\nabla\mathbf{h}]$ is a matrix. For example, for $N = 3$ we have

$$\nabla h = [\partial_j h]_j = \begin{bmatrix} \partial_1 h \\ \partial_2 h \\ \partial_3 h \end{bmatrix}, \quad [\nabla\mathbf{h}] = [\partial_j h_i]_{ij} = \begin{bmatrix} \partial_1 h_1 & \partial_2 h_1 & \partial_3 h_1 \\ \partial_1 h_2 & \partial_2 h_2 & \partial_3 h_2 \\ \partial_1 h_3 & \partial_2 h_2 & \partial_3 h_3 \end{bmatrix}. \tag{3.3.3}$$

Note that here we are considering the standard derivative which applies for functions $f : \mathbb{R}^n \to \mathbb{R}^m$, with m, n positive integers, in a Cartesian coordinate system. Whereas the Fréchet derivative generalizes the standards derivative to functions defined on Banach spaces as in Definition 2.1.1. Most of the time, we will write "Fréchet derivative" to differentiate with the standard derivative or simply write "derivative" when there is no confusion.

Next, using the relations $\mathbf{T}^{-1} \circ \mathbf{T}(x) = x$, for all $x \in \overline{\Omega}_0$ and $\mathbf{T} \circ \mathbf{T}^{-1}(x) = x$, for all $x \in \overline{\Omega}_t$, we find that the gradient of $\mathbf{T}$ and $\mathbf{T}^{-1}$ are related by

$$[\nabla\mathbf{T}^{-1}] \circ \mathbf{T} \cdot [\nabla\mathbf{T}] = I. \tag{3.3.4}$$

Indeed, if we set $y = \mathbf{T}(x)$, then $\mathbf{T}^{-1} \circ \mathbf{T}(x) = \mathbf{T}^{-1}(y) = x$ implies

$$\begin{aligned}
\delta_{ij} = \partial_{x_j} x_i = \partial_{x_j}(\mathbf{T}_i^{-1}(y)) &= \sum_{k=1}^{N} \partial_{y_k}(\mathbf{T}_i^{-1}(y))\partial_{x_j}(\mathbf{T}_k(x)) \\
&= \sum_{k=1}^{N} ((\partial_{y_k}\mathbf{T}_i^{-1}) \circ \mathbf{T})(\partial_{x_j}\mathbf{T}_k) \\
&= \sum_{k=1}^{N} ([\nabla \mathbf{T}^{-1}] \circ \mathbf{T})_{ik}[\nabla \mathbf{T}]_{kj} \\
&= (([\nabla \mathbf{T}^{-1}] \circ \mathbf{T}) \cdot [\nabla \mathbf{T}]\Big)_{ij}.
\end{aligned}$$

The equation (3.3.4) is equivalent to

$$[\nabla \mathbf{T}^{-1}] \circ \mathbf{T} = [\nabla \mathbf{T}]^{-1}. \tag{3.3.5}$$

For the functions φ and ψ, as $\psi \circ \mathbf{T} = \varphi$ we get

$${}^t[\nabla \mathbf{T}] \cdot [\nabla \psi] \circ \mathbf{T} = [\nabla \varphi], \tag{3.3.6}$$

hence

$$[\nabla \psi] \circ \mathbf{T} = {}^t[\nabla \mathbf{T}]^{-1} \cdot [\nabla \varphi]. \tag{3.3.7}$$

Taking $\partial_i \psi$ instead of ψ and $({}^t[\nabla \mathbf{T}]^{-1} \cdot [\nabla \varphi])_i$ instead of ${}^t[\nabla \mathbf{T}]^{-1} \cdot [\nabla \varphi]$, using again the formula above (3.3.7) we have

$${}^t[\nabla \mathbf{T}] \cdot [\nabla \partial_i \psi] \circ \mathbf{T} = [\nabla({}^t[\nabla \mathbf{T}]^{-1} \cdot [\nabla \varphi])_i], \tag{3.3.8}$$

hence

$$[\nabla \partial_i \psi] \circ \mathbf{T} = {}^t[\nabla \mathbf{T}]^{-1} \cdot \left[\nabla({}^t\,[\nabla \mathbf{T}]^{-1} \cdot [\nabla \varphi])_i\right]. \tag{3.3.9}$$

Taking the transpose of (3.3.9) for all i gives

$$\begin{aligned}
[D^2\psi] \circ \mathbf{T} &= {}^t\,[\nabla \partial_i \psi] \circ \mathbf{T} \\
&= {}^t\left[\nabla({}^t\,[\nabla \mathbf{T}]^{-1} \cdot [\nabla \varphi])\right] \cdot [\nabla \mathbf{T}]^{-1}.
\end{aligned} \tag{3.3.10}$$

Assuming furthermore that $\mathbf{T} = \mathbf{T}(x,t)$, $\psi = \psi(x,t)$ and $\varphi(x,t) := \psi \circ \mathbf{T} = \psi(\mathbf{T}(x,t),t)$, we have

$$\partial_t \varphi = (\partial_t \psi) \circ \mathbf{T} + \partial_t \mathbf{T} \cdot ([\nabla \psi] \circ \mathbf{T}), \tag{3.3.11}$$

hence

$$(\partial_t \psi) \circ \mathbf{T} = \partial_t \varphi - \partial_t \mathbf{T} \cdot ({}^t[\nabla \mathbf{T}]^{-1} \cdot [\nabla \varphi]). \tag{3.3.12}$$

If $\mathbf{v} = [v_i]$, p and $\mathbf{g}$ are given smooth function in $\Omega_t \times \{t\}$, as in (3.3.1a) – (3.3.1d), then we define $\mathbf{w} = [w_i]$, $\mathbf{h}$ and q with $\mathbf{w} := \mathbf{v} \circ \mathbf{T}$, $\mathbf{h} := \mathbf{g} \circ \mathbf{T}$ and $q := p \circ \mathbf{T}$. Moreover, the time derivative of $\mathbf{v}$ is given by

$$(\partial_t \mathbf{v}) \circ \mathbf{T} := [\partial_t v_i] \circ \mathbf{T}$$

$$= \partial_t \mathbf{w} - \partial_t \mathbf{T} \cdot {}^t[\nabla \mathbf{T}]^{-1} \cdot [\nabla \mathbf{w}]. \tag{3.3.13}$$

To compute the gradient of $\mathbf{v}$ in terms of the gradient of $\mathbf{w}$ we first use equation (3.3.7) at each component. Precisely, for all j we have

$$[\nabla v_j] \circ \mathbf{T} = {}^t[\nabla \mathbf{T}]^{-1} \cdot [\nabla w_j]. \tag{3.3.14}$$

Taking the transpose in equation (3.3.14) we get

$$[\nabla \mathbf{v}] \circ \mathbf{T} = [\nabla \mathbf{w}] \cdot [\nabla \mathbf{T}]^{-1}. \tag{3.3.15}$$

The terms with first order derivatives in space of $\mathbf{v}$ and p are given by

$$\begin{aligned}((\mathbf{v}\cdot\nabla)\mathbf{g}) \circ \mathbf{T} &= \left[\sum (v_j \circ \mathbf{T})(\partial_j\ g_i) \circ \mathbf{T}\right] \\ &= [\nabla \mathbf{h}] \cdot [\nabla \mathbf{T}]^{-1} \cdot \mathbf{w}, \end{aligned} \tag{3.3.16}$$

$$\begin{aligned}((\mathbf{g}\cdot\nabla)\mathbf{v}) \circ \mathbf{T} &= \left[\sum (g_j \circ \mathbf{T})(\partial_j\ v_i) \circ \mathbf{T}\right] \\ &= [\nabla \mathbf{w}] \cdot [\nabla \mathbf{T}]^{-1} \cdot \mathbf{h}, \end{aligned} \tag{3.3.17}$$

$$(\nabla p) \circ \mathbf{T} = {}^t\, [\nabla \mathbf{T}]^{-1} \cdot \nabla q, \tag{3.3.18}$$

$$(\nabla \cdot \mathbf{v}) \circ \mathbf{T} = \mathrm{tr}([\nabla \mathbf{v}] \circ \mathbf{T}) = \mathrm{tr}([\nabla \mathbf{w}] \cdot [\nabla \mathbf{T}]^{-1}), \tag{3.3.19}$$

and the Laplacian of $\mathbf{v}$ is given by

$$\begin{aligned}(\Delta \mathbf{v}) \circ \mathbf{T} &= [\mathrm{tr}(\left[D^2 v_i\right] \circ \mathbf{T})] \\ &= \mathrm{tr}({}^t\left[\nabla({}^t\, [\nabla \mathbf{T}]^{-1} \cdot [\nabla w_i])\right] \cdot [\nabla \mathbf{T}]^{-1}) \\ &=: \mathrm{tr}({}^t\left[\nabla({}^t\, [\nabla \mathbf{T}]^{-1} \cdot {}^t\, [\nabla \mathbf{w}])\right] \cdot [\nabla \mathbf{T}]^{-1}). \end{aligned} \tag{3.3.20}$$

Now, assume that $(\mathbf{v}, p) \in \mathbf{V}_T \times P_T$ solves the problem (3.3.1a) – (3.3.1d), and assume that $\mathbf{g}$ is the boundary condition of the problem (3.1.13a) – (3.1.13d) given by the equation (3.1.11), then we define the function $\mathbf{h}$ in $\Omega_0 \times \mathbb{R}$ by

$$\mathbf{h}(x,t) := (\mathbf{u}_i(x)k_i(t) + \mathbf{u}_o(x)k_o(t))\mathbb{1}_{\Gamma_i \cup \Gamma_o}(x) + (0, ..., 0, \partial_t z(x', t))\mathbb{1}_{M_0}(x), \tag{3.3.21}$$

where the characteristic function of M_0 is defined by

$$\mathbb{1}_{M_0}(x) := \begin{cases} 1 \text{ if } x \in \{(x', 0),\ x' \in B_{N-1}(\ell_0)\}, \\ 0 \text{ else.} \end{cases} \tag{3.3.22}$$

Define $\mathbf{w} = [w_i]$ and q with $\mathbf{w} := \mathbf{v} \circ \mathbf{T}$, $q := p \circ \mathbf{T}$. Then $(\mathbf{w}, q)$ is solution the problem

Find $(\mathbf{w}, q) \in \mathbf{W}_T \times Q_T$, solving

$$\begin{cases} \quad \rho(\partial_t \mathbf{w} - \partial_t \mathbf{T} \cdot {}^t[\nabla \mathbf{T}]^{-1} \cdot [\nabla \mathbf{w}]) \\ +\rho\Big([\nabla \mathbf{h}] \cdot [\nabla \mathbf{T}]^{-1} \cdot \mathbf{w} + [\nabla \mathbf{w}] \cdot [\nabla \mathbf{T}]^{-1} \cdot \mathbf{h} + [\nabla \mathbf{w}] \cdot [\nabla \mathbf{T}]^{-1} \cdot \mathbf{w}\Big) \\ -\mu \mathrm{tr}({}^t\left[\nabla({}^t\, [\nabla \mathbf{T}]^{-1} \cdot [\nabla \mathbf{w}])\right] \cdot [\nabla \mathbf{T}]^{-1}) \\ +{}^t\, [\nabla \mathbf{T}]^{-1} \cdot \nabla q \\ -\mathbf{e} = 0, \quad \text{in} \quad \Omega_0 \times (-T, 3T), & (3.3.23a) \\ \mathrm{tr}([\nabla \mathbf{w}] \cdot [\nabla \mathbf{T}]^{-1}) + \mathrm{tr}([\nabla \mathbf{h}] \cdot [\nabla \mathbf{T}]^{-1}) = 0, \text{ in } \Omega_0 \times (-T, 3T), & (3.3.23b) \end{cases}$$

where the space $\mathbf{W}_T$ resp. Q_T is defined in (3.1.19) resp. (3.1.21) and

$$\begin{aligned}\mathbf{e} = {}& \rho(\vec{g} - \partial_t \mathbf{h} + \partial_t \mathbf{T} \cdot {}^t[\nabla \mathbf{T}]^{-1} \cdot [\nabla \mathbf{h}] - [\nabla \mathbf{h}] \cdot [\nabla \mathbf{T}]^{-1} \cdot \mathbf{h}) \\ &+ \mu \mathrm{tr}({}^t\Big[\nabla({}^t\,[\nabla \mathbf{T}]^{-1} \cdot [\nabla \mathbf{h}])\Big] \cdot [\nabla \mathbf{T}]^{-1}),\end{aligned} \tag{3.3.24}$$

and $\mathbf{T} = \mathbf{I} + \mathbf{z}$, where $\mathbf{z}$ is an element of the space $\mathbf{Z}_T$ given by (3.1.15) and is the extension of $(0, ..., 0, z)$ given by Lemma 3.3.5.

We will find $(\mathbf{w}, q)$ solution to the problem (3.3.23a)$-$(3.3.23b) using the implicit function theorem associated to the map

$$\begin{aligned}\mathbf{F} : Z_T \times \mathbf{W}_T \times Q_T &\longrightarrow \mathbf{R}_{\mathbf{w}} \times R_q, \\ (z, \mathbf{w}, q) &\mapsto (F_1(z, \mathbf{w}, q), F_2(z, \mathbf{w})),\end{aligned} \tag{3.3.25}$$

where

$$\begin{aligned}F_1(z, \mathbf{w}, q) := {}& \rho(\partial_t \mathbf{w} - \partial_t \mathbf{T} \cdot {}^t[\nabla \mathbf{T}]^{-1} \cdot [\nabla \mathbf{w}]) \\ &+ \rho\Big([\nabla \mathbf{h}] \cdot [\nabla \mathbf{T}]^{-1} \cdot \mathbf{w} + [\nabla \mathbf{w}] \cdot [\nabla \mathbf{T}]^{-1} \cdot \mathbf{h} + [\nabla \mathbf{w}] \cdot [\nabla \mathbf{T}]^{-1} \cdot \mathbf{w}\Big) \\ &- \mu \mathrm{tr}({}^t\Big[\nabla({}^t\,[\nabla \mathbf{T}]^{-1} \cdot {}^t\,[\nabla \mathbf{w}])\Big] \cdot [\nabla \mathbf{T}]^{-1}) \\ &+ {}^t\,[\nabla \mathbf{T}]^{-1} \cdot \nabla q \\ &- \mathbf{e},\end{aligned} \tag{3.3.26}$$

$$F_2(z, \mathbf{w}) := (\mathrm{tr}([\nabla \mathbf{w}] \cdot [\nabla \mathbf{T}]^{-1}) + \mathrm{tr}([\nabla \mathbf{h}] \cdot [\nabla \mathbf{T}]^{-1}))|\nabla \mathbf{T}|. \tag{3.3.27}$$

We recall the spaces $\mathbf{R}_{\mathbf{w}} = L^2_T(\mathbf{L}^2(\Omega_0))$ and $R_q = Q_T \cap H^1_T(H^1(\Omega_0)')$ (see (3.1.22), (3.1.23)).

To set the implicit function theorem, we will firstly show that the function $\mathbf{F}$ given by (3.3.25) is well-defined. Secondly, we will show that $\mathbf{F}$ is Fréchet C^1 near a point $((z_0, \mathbf{w}_0), q_0)$, defined later and which is such that $\mathbf{F}((z_0, \mathbf{w}_0), q_0) = (\mathbf{0}, 0)$. Finally, we will show the differential of $\mathbf{F}$ with respect to $(z, \mathbf{w})$ at $((z_0, \mathbf{w}_0), q_0)$ called $D_{(z,\mathbf{w})}\mathbf{F}((z_0, \mathbf{w}_0), z_0)$, is an isomorphism.

Since the analysis is made in the fixed space $(\mathbf{W}_T, Q_T)$, we need the following Proposition 3.3.1 which shows the link between the variable space $(\mathbf{V}_T, P_T)$ and the fixed space $(\mathbf{W}_T, Q_T)$. Precisely, they are equal up to a diffeomorphism.

Proposition 3.3.1 (Link between the spaces $(\mathbf{V}_T, P_T)$ and $(\mathbf{W}_T, Q_T)$)**.**

Let $\mathbf{z} \in \mathbf{Z}_T$. Assume $\mathbf{T} = \mathbf{I} + \mathbf{z}$ is a $L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(\Omega_0))-$diffeomorphism from B_0 to itself, where $B_0 \subset \mathbb{R}^N$ is a ball large enough to contain Ω_0. Then

$$\mathbf{W}_T = \mathbf{V}_T \circ \mathbf{T}, \tag{3.3.28}$$

$$Q_T = P_T \circ \mathbf{T}, \tag{3.3.29}$$

where the sign " $\circ$ " here stands for the the composition. For example,

$$\mathbf{V}_T \circ \mathbf{T} := \{\mathbf{v} \circ \mathbf{T}, \quad \mathbf{v} \in \mathbf{V}_T\}, \tag{3.3.30}$$

where for all $\mathbf{v} \in \mathbf{V}_T$, the composition $\mathbf{v} \circ \mathbf{T}$ is defined as

$$\mathbf{v} \circ \mathbf{T}(x,t) := \mathbf{v}(\mathbf{T}(x,t),t), \quad \forall\, (x,t) \in \Omega_0 \times \mathbb{R}. \tag{3.3.31}$$

Proof: Claim:

$$L^2_T(\mathbf{L}^2(\Omega_0)) = L^2_T(\mathbf{L}^2(\Omega_t)) \circ \mathbf{T}, \tag{3.3.32}$$
$$L^2_T(\mathbf{H}^1(\Omega_0)) = L^2_T(\mathbf{H}^1(\Omega_t)) \circ \mathbf{T}, \tag{3.3.33}$$
$$L^2_T(\mathbf{H}^1_0(\Omega_0)) = L^2_T(\mathbf{H}^1_0(\Omega_t)) \circ \mathbf{T}, \tag{3.3.34}$$
$$L^2_T(\mathbf{H}^2(\Omega_0)) = L^2_T(\mathbf{H}^2(\Omega_t)) \circ \mathbf{T}, \tag{3.3.35}$$
$$L^2_T(\mathbf{H}^2(\Omega_0) \cap \mathbf{H}^1_0(\Omega_0)) = L^2_T((\mathbf{H}^2(\Omega_t) \cap \mathbf{H}^1_0(\Omega_t))) \circ \mathbf{T}, \tag{3.3.36}$$
$$L^2_T(H^1(\Omega_0)) = L^2_T(H^1(\Omega_t)) \circ \mathbf{T}. \tag{3.3.37}$$

We will only prove (3.3.32), (3.3.33), (3.3.34), (3.3.35) and (3.3.36) for vector functions because (3.3.37) can be carried out with the same calculations as in (3.3.33) for scalar functions. Since $\mathbf{T}$ is an $L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(\Omega_0))-$ diffeomorphism for $\|z\|_{Z_T} \ll 1$, then due to the injection $\mathbf{W}^{2,\hat{2}^*}(\Omega_0) \hookrightarrow \mathbf{W}^{1,\infty}(\Omega_0)$, we have $\mathbf{T},\ \mathbf{T}^{-1} \in L^\infty_T(\mathbf{W}^{1,\infty}(\Omega_0))$.

Let $\mathbf{w} \in L^2_T(\mathbf{L}^2(\Omega_0))$, then $\mathbf{v} := \mathbf{w} \circ \mathbf{T}^{-1} \in L^2_T(\mathbf{L}^2(\Omega_t))$. Indeed, we set $x = \mathbf{T}(y,\cdot)$ and we introduce the notation $|\nabla \mathbf{T}| := \det([\nabla \mathbf{T}])$ (the determinant of the Jacobian matrix) to have

$$\begin{aligned}\int_{-T}^{3T}\int_{\Omega_t} |\mathbf{v}(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_{-T}^{3T}\int_{\mathbf{T}(\Omega_0)} |\mathbf{w} \circ \mathbf{T}^{-1}(x,\cdot)|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{-T}^{3T}\int_{\Omega_0} |\mathbf{w}(y,t)|^2\, |\nabla \mathbf{T}| \,\mathrm{d}y\,\mathrm{d}t \\ &\leq \|\mathbf{T}\|_{L^\infty_T(\mathbf{W}^{1,\infty}(\Omega_0))} \int_{-T}^{3T}\int_{\Omega_0} |\mathbf{w}(y,t)|^2 \,\mathrm{d}y\,\mathrm{d}t < \infty. \end{aligned} \tag{3.3.38}$$

This show that $\mathbf{v} \in L^2_T(\mathbf{L}^2(\Omega_t))$ and therefore $\mathbf{w} = \mathbf{v} \circ \mathbf{T} \in L^2_T(\mathbf{L}^2(\Omega_t)) \circ \mathbf{T}$ and we have

$$L^2_T(\mathbf{L}^2(\Omega_0)) \subset L^2_T(\mathbf{L}^2(\Omega_t)) \circ \mathbf{T}. \tag{3.3.39}$$

Let $\mathbf{w} \in L^2_T(\mathbf{H}^1(\Omega_0))$, then as before we have $\mathbf{v} := \mathbf{w} \circ \mathbf{T}^{-1} \in L^2_T(\mathbf{H}^1(\Omega_t))$. Indeed, using (3.3.15) and we have

$$\begin{aligned}\int_{-T}^{3T}\int_{\Omega_t} |\nabla \mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_{-T}^{3T}\int_{\Omega_0} |[\nabla \mathbf{v}] \circ \mathbf{T}|^2 |\nabla \mathbf{T}| \,\mathrm{d}y\,\mathrm{d}t \\ &= \int_{-T}^{3T}\int_{\Omega_0} |[\nabla \mathbf{w}] \cdot [\nabla \mathbf{T}]^{-1}|^2 |\nabla \mathbf{T}| \,\mathrm{d}y\,\mathrm{d}t\end{aligned}$$

$$\leq \|\mathbf{T}\|^3_{L^\infty_T(\mathbf{W}^{1,\infty}(\Omega_0))} \int_{-T}^{3T} \int_{\Omega_0} |\nabla \mathbf{w}|^2 \,\mathrm{d}y\,\mathrm{d}t < +\infty. \tag{3.3.40}$$

From (3.3.38) and (3.3.40), we have $\mathbf{w} = \mathbf{v} \circ \mathbf{T} \in L^2_T(\mathbf{H}^1(\Omega_t)) \circ \mathbf{T}$ and we have

$$L^2_T(\mathbf{H}^1(\Omega_0)) \subset L^2_T(\mathbf{H}^1(\Omega_t)) \circ \mathbf{T}. \tag{3.3.41}$$

Moreover, if $\mathbf{w}|_{\partial\Omega_0} = 0$ then $\mathbf{v}|_{\partial\Omega_t} = 0$. Indeed, Let us suppose that $\mathbf{w}|_{\partial\Omega_0} = 0$ and let $x \in \partial\Omega_t$. Since $\mathbf{T}(\partial\Omega_0, t) = \partial\Omega_t$, there exists $y \in \partial\Omega_0$ such that $x = \mathbf{T}(y,t)$. So, since $\mathbf{w}|_{\partial\Omega_0} = 0$, we have

$$\mathbf{v}(x,t) = \mathbf{w} \circ \mathbf{T}^{-1}(x,t) = \mathbf{w}(y,t) = 0. \tag{3.3.42}$$

This show that if $\mathbf{w} \in \mathbf{H}^1_0(\Omega_0)$ then $\mathbf{w} = \mathbf{v} \circ \mathbf{T} \in \mathbf{H}^1_0(\Omega_t) \circ \mathbf{T}$, which implies that $\mathbf{H}^1_0(\Omega_0) \subset \mathbf{H}^1_0(\Omega_t) \circ \mathbf{T}$. Therefore, we have

$$L^2_T(\mathbf{H}^1_0(\Omega_0)) \subset L^2_T(\mathbf{H}^1_0(\Omega_t)) \circ \mathbf{T}. \tag{3.3.43}$$

Now, let $\mathbf{w} \in L^2_T(\mathbf{H}^2(\Omega_0))$, then $\mathbf{v} := \mathbf{w} \circ \mathbf{T}^{-1} \in L^2_T(\mathbf{H}^2(\Omega_t))$. Indeed, using (3.3.10) we have

$$\begin{aligned}
\int_{-T}^{3T} \int_{\Omega_t} |D^2\mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_{-T}^{3T} \int_{\Omega_0} |[D^2\mathbf{v}] \circ \mathbf{T}|^2 \,|\nabla\mathbf{T}|\,\mathrm{d}y\,\mathrm{d}t \\
&= \int_{-T}^{3T} \int_{\Omega_0} \left| {}^t\!\left[\nabla\!\left({}^t[\nabla\mathbf{T}]^{-1} \cdot {}^t[\nabla\mathbf{w}]\right)\right] \cdot [\nabla\mathbf{T}]^{-1} \right|^2 |\nabla\mathbf{T}|\,\mathrm{d}y\,\mathrm{d}t \\
&\leq C\|\mathbf{T}\|^3_{L^\infty_T(\mathbf{W}^{1,\infty}(\Omega_0))} \int_{-T}^{3T} \int_{\Omega_0} |D^2\mathbf{T}|^2|\nabla\mathbf{w}|^2 + |\nabla\mathbf{T}|^2|D^2\mathbf{w}|^2 \,\mathrm{d}y\,\mathrm{d}t \\
&\leq C\|\mathbf{T}\|^3_{L^\infty_T(\mathbf{W}^{1,\infty}(\Omega_0))} \int_{-T}^{3T} \Big(\|D^2\mathbf{T}\|^2_{\mathbf{L}^{\hat{2}^*}(\Omega_0)} \|\nabla\mathbf{w}\|^2_{\mathbf{L}^{\hat{2}}(\Omega_0)} + \\
&\qquad\qquad \|\nabla\mathbf{T}\|^2_{\mathbf{L}^\infty(\Omega_0)} \|D^2\mathbf{w}\|^2_{\mathbf{L}^2(\Omega_0)} \Big)\,\mathrm{d}t \\
&\leq C\|\mathbf{T}\|^4_{L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(\Omega_0))} \|\mathbf{w}\|^2_{L^2_T(\mathbf{H}^2(\Omega_0))} < +\infty.
\end{aligned} \tag{3.3.44}$$

The estimations (3.3.38), (3.3.40), (3.3.44) show that if $\mathbf{w} \in L^2_T(\mathbf{H}^2(\Omega_0))$ then $\mathbf{w} := \mathbf{v} \circ \mathbf{T} \in L^2_T(\mathbf{H}^2(\Omega_t)) \circ \mathbf{T}$. Therefore,

$$L^2_T(\mathbf{H}^2(\Omega_0)) \subset L^2_T(\mathbf{H}^2(\Omega_t)) \circ \mathbf{T}. \tag{3.3.45}$$

Hence

$$L^2_T(\mathbf{H}^2(\Omega_0) \cap \mathbf{H}^1_0(\Omega_0)) \subset L^2_T(\mathbf{H}^2(\Omega_t) \cap \mathbf{H}^1_0(\Omega_t)) \circ \mathbf{T}. \tag{3.3.46}$$

The opposite inclusions of (3.3.39), (3.3.41), (3.3.43), (3.3.45) and (3.3.46) are similar by using the change of variable $y = \mathbf{T}^{-1}(x, \cdot)$, which proves the claims (3.3.32), (3.3.33), (3.3.34), (3.3.35), (3.3.36)(3.3.37).

Now, let us show (3.3.28) and (3.3.29). We start by showing that $\mathbf{W}_T \subset \mathbf{V}_T \circ \mathbf{T}$ and $Q_T \subset P_T \circ \mathbf{T}$. Let $\mathbf{w} \in \mathbf{W}_T$ and $q \in Q_T$. We will show that

$$\mathbf{v} := \mathbf{w} \circ \mathbf{T}^{-1} \in \mathbf{V}_T, \tag{3.3.47}$$

$$p := q \circ \mathbf{T}^{-1} - \frac{1}{|\Omega_t|} \int_{\Omega_0} q \, |\nabla \mathbf{T}| \, \mathrm{d}y \in P_T. \tag{3.3.48}$$

Note that the function p defined in (3.3.48) has zero mean in Ω_t. Indeed,

$$\begin{aligned} \frac{1}{|\Omega_t|} \int_{\Omega_t} p \, \mathrm{d}x &= \frac{1}{|\Omega_t|} \int_{\Omega_t} q \circ \mathbf{T}^{-1} \, \mathrm{d}x - \frac{1}{|\Omega_t|} \int_{\Omega_0} q \, |\nabla \mathbf{T}| \, \mathrm{d}y \\ &= \frac{1}{|\Omega_t|} \int_{\Omega_0} q \, |\nabla \mathbf{T}| \, \mathrm{d}y - \frac{1}{|\Omega_t|} \int_{\Omega_0} q \, |\nabla \mathbf{T}| \, \mathrm{d}y = 0. \end{aligned} \tag{3.3.49}$$

Moreover, with (3.3.37) and (3.3.49), we have $p \in L^2_T(H^1(\Omega_0) \cap L^2_0(\Omega_t)) = P_T$, which proves (3.3.48). Therefore, if $q \in Q_T$ then $q = p \circ \mathbf{T} \in P_T \circ \mathbf{T}$ and we have

$$Q_T \subset P_T \circ \mathbf{T}. \tag{3.3.50}$$

Additionally, with (3.3.36) and (3.3.33), we have $\mathbf{v} \in L^2_T(\mathbf{H}^2(\Omega_t) \cap \mathbf{H}^1_0(\Omega_t))$. To complete the proof of (3.3.47), it remains to show that

$$\mathbf{v} \in H^1_T(\mathbf{L}^2(\Omega_t)). \tag{3.3.51}$$

Since $\mathbf{w} \in L^2_T(\mathbf{L}^2(\Omega_0))$, according to (3.3.32), we have $\mathbf{v} \in L^2_T(\mathbf{L}^2(\Omega_t))$.

Let $t \in (-T, 3T)$. We will show that $\partial_t \mathbf{v} \in L^2_T(\mathbf{L}^2(\Omega_t))$. Here we define $\partial_t \mathbf{v}$ as the element of $L^2_T(\mathbf{L}^2(\Omega_t))$ such that

$$\int_{\Omega_t} \partial_t \mathbf{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}x := -\int_{\Omega_t} \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} \, \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial\Omega_t} \mathbf{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \quad \forall \, \boldsymbol{\varphi} \in \mathcal{D}_T(\boldsymbol{\mathcal{D}}(\Omega_t)). \tag{3.3.52}$$

Note that this definition in compatible with the classical Reynolds control theorem (see Niven et al. [77, Theorem 2.1, pp 4]) when $\mathbf{v}$ have more regularity in time because since $\mathbf{v}(\cdot, t) \in \mathbf{H}^1_0(\Omega_t)$, in this case we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \mathbf{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}x &= \int_{\Omega_t} \partial_t (\mathbf{v} \cdot \boldsymbol{\varphi}) \, \mathrm{d}x + \int_{\partial\Omega_t} \mathbf{v} \cdot \boldsymbol{\varphi} (\mathbf{v} \cdot \boldsymbol{\nu}_t) \, \mathrm{d}s \\ &= \int_{\Omega_t} \partial_t (\mathbf{v} \cdot \boldsymbol{\varphi}) \, \mathrm{d}x \\ &= \int_{\Omega_t} \partial_t \mathbf{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \int_{\Omega_t} \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} \, \mathrm{d}x, \quad \forall \, \boldsymbol{\varphi} \in \mathcal{D}_T(\boldsymbol{\mathcal{D}}(\Omega_t)), \end{aligned} \tag{3.3.53}$$

since $\mathbf{v}(\cdot, t) \in \mathbf{H}^1_0(\Omega_t))$, where $\boldsymbol{\nu}_t$ is the unit outward vector to the moving boundary $\partial\Omega_t$. Hence, we recover the definition (3.3.52).

Let $\boldsymbol{\varphi} \in \mathcal{D}_T(\boldsymbol{\mathcal{D}}(\Omega_t))$. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \mathbf{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_0} \mathbf{w} \cdot \boldsymbol{\varphi} \circ \mathbf{T} |\nabla \mathbf{T}| \, \mathrm{d}x \\ &= \int_{\Omega_0} \partial_t \Big(\mathbf{w} \cdot \boldsymbol{\varphi} \circ \mathbf{T} |\nabla \mathbf{T}| \Big) \, \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega_0} \Big(\partial_t \mathbf{w} \cdot \boldsymbol{\varphi} \circ \mathbf{T} |\nabla \mathbf{T}| + \mathbf{w} \partial_t |\nabla \mathbf{T}| \cdot \boldsymbol{\varphi} \circ \mathbf{T}\Big)\, \mathrm{d}x + \int_{\Omega_0} \mathbf{w} \cdot \partial_t (\boldsymbol{\varphi} \circ \mathbf{T}) |\nabla \mathbf{T}|\, \mathrm{d}x \\
&= \int_{\Omega_t} \Big((\partial_t \mathbf{w}) \circ \mathbf{T}^{-1} + (\mathbf{w} \partial_t |\nabla \mathbf{T}|) \circ \mathbf{T}^{-1} |\nabla \mathbf{T}^{-1}|\Big) \cdot \boldsymbol{\varphi}\, \mathrm{d}x + \\
&\int_{\Omega_t} \mathbf{v} \cdot \partial_t \boldsymbol{\varphi}\, \mathrm{d}x + \int_{\Omega_t} \mathbf{v} \cdot (\partial_t \mathbf{z} \cdot [\nabla \boldsymbol{\varphi}])\, \mathrm{d}x \\
&= \int_{\Omega_t} \Big((\partial_t \mathbf{w}) \circ \mathbf{T}^{-1} + (\mathbf{w} \partial_t |\nabla \mathbf{T}|) \circ \mathbf{T}^{-1} |\nabla \mathbf{T}^{-1}|\Big) \cdot \boldsymbol{\varphi}\, \mathrm{d}x + \int_{\Omega_t} \mathbf{v} \cdot \partial_t \boldsymbol{\varphi}\, \mathrm{d}x \\
&=: \int_{\Omega_t} \partial_t \mathbf{v} \cdot \boldsymbol{\varphi}\, \mathrm{d}x + \int_{\Omega_t} \mathbf{v} \cdot \partial_t \boldsymbol{\varphi}\, \mathrm{d}x, \qquad (3.3.54)
\end{aligned}
$$

where we use the formula $\partial_t(\boldsymbol{\varphi} \circ \mathbf{T}) = (\partial_t \boldsymbol{\varphi}) \circ \mathbf{T} + \partial_t \mathbf{z} \cdot [\nabla \boldsymbol{\varphi}] \circ \mathbf{T}$ (See (3.3.12)) and the fact that $\partial_t \mathbf{z} = \mathbf{g} = 0$ in Ω_t. Let us show that

$$
\partial_t \mathbf{v} := (\partial_t \mathbf{w}) \circ \mathbf{T}^{-1} + (\mathbf{w} \partial_t |\nabla \mathbf{T}|) \circ \mathbf{T}^{-1} |\nabla \mathbf{T}^{-1}| \in L^2_T(\mathbf{L}^2(\Omega_t)). \qquad (3.3.55)
$$

For the first term of (3.3.55), since $\mathbf{w} \in H^1_T(\mathbf{L}^2(\Omega_0))$, then $\partial_t \mathbf{w} \in L^2_T(\mathbf{L}^2(\Omega_0))$. Therefore, using (3.3.32), we have $(\partial_t \mathbf{w}) \circ \mathbf{T}^{-1} \in L^2_T(\mathbf{L}^2(\Omega_t))$. For the second term of (3.3.55), since $\mathbf{T} \in H^1_T(\mathbf{H}^2(\Omega_0)) \hookrightarrow H^1_T(\mathbf{W}^{1,\hat{2}^*}(\Omega_0))$ and $\mathbf{T}^{-1} \in L^\infty_T(\mathbf{W}^{1,\infty}(B_0))$, we have

$$
\begin{aligned}
&\int_{-T}^{3T} \int_{\Omega_t} \Big|(\mathbf{w} \partial_t |\nabla \mathbf{T}|) \circ \mathbf{T}^{-1} |\nabla \mathbf{T}^{-1}|\Big|^2\, \mathrm{d}x\, \mathrm{d}t \\
&\leq \|\mathbf{T}^{-1}\|^2_{L^\infty_T(\mathbf{W}^{1,\infty}(B_0))} \int_{-T}^{3T} \int_{\Omega_0} |\mathbf{w}(\partial_t |\nabla \mathbf{T}|)|^2\, \mathrm{d}x\, \mathrm{d}t \\
&\leq \|\mathbf{T}^{-1}\|^2_{L^\infty_T(\mathbf{W}^{1,\infty}(B_0))} \int_{-T}^{3T} \|\mathbf{w}\|^2_{\mathbf{L}^{\hat{2}}(\Omega_0)} \|\partial_t |\nabla \mathbf{T}|\|^2_{\mathbf{L}^{\hat{2}^*}(\Omega_0)}\, \mathrm{d}t \\
&\leq \|\mathbf{T}^{-1}\|^2_{L^\infty_T(\mathbf{W}^{1,\infty}(B_0))} \|\mathbf{w}\|^2_{L^\infty_T(\mathbf{H}^1(\Omega_0))} \|\mathbf{T}\|^2_{H^1_T(\mathbf{H}^2(\Omega_0))} < +\infty, \qquad (3.3.56)
\end{aligned}
$$

which shows that $(\mathbf{w} \partial_t |\nabla \mathbf{T}|) \circ \mathbf{T}^{-1} |\nabla \mathbf{T}^{-1}| \in L^2_T(\mathbf{L}^2(\Omega_t))$. Therefore (3.3.55) holds. This proves (3.3.51) and hence (3.3.47).

So, if $\mathbf{w} \in \mathbf{W}_T$ then $\mathbf{v} := \mathbf{w} \circ \mathbf{T}^{-1} \in \mathbf{V}_T$ and $\mathbf{w} = \mathbf{v} \circ \mathbf{T} \in \mathbf{V}_T \circ \mathbf{T}$. Hence,

$$
\mathbf{W}_T \subset \mathbf{V}_T \circ \mathbf{T}. \qquad (3.3.57)
$$

Similarly, using the change of variable $y = \mathbf{T}^{-1}(x, \cdot)$, we can show that for $(\mathbf{v}, p) \in \mathbf{V}_T \times P_T$ we have

$$
\mathbf{w} := \mathbf{v} \circ \mathbf{T} \in \mathbf{W}_T, \qquad (3.3.58)
$$

$$
q := p \circ \mathbf{T} - \frac{1}{|\Omega_0|} \int_{\Omega_t} p\, |\nabla \mathbf{T}^{-1}|\, \mathrm{d}x \in Q_T, \qquad (3.3.59)
$$

which proves the inverse inclusions

$$
\mathbf{V}_T \circ \mathbf{T} \subset \mathbf{W}_T,
$$

$$P_T \circ \mathbf{T} \subset Q_T,$$

and completes the proof. ∎

3.3.2 Extension of the perturbation and of the boundary data

In this section, we construct the transformation $\mathbf{T}$ and the extension $\mathbf{h}$ describing the boundary condition based on the assumption $z \in Z_T$. Indeed, the function z is used to construct the transformation $\mathbf{T} = \mathbf{I} + \mathbf{z}$, where $\mathbf{z}$ is an appropriate extension of $(0, ..., 0, z)$, and the function $\mathbf{h}$ defines the boundary condition $\mathbf{g}$ for the problem (3.1.13a) – (3.1.13d). The regularity of z clearly affects the smoothness of the function $\mathbf{F}$ defined in (3.3.25). We will show that under the assumptions (3.1.2) – (3.1.6), and the regularity of z, we can construct a sufficiently smooth function $\mathbf{z}$, which allows the use of the implicit function theorem, and we will construct the extension $\mathbf{h}$.

Lemma 3.3.2 (Existence of an extension of z)**.**

Under the assumptions (3.1.2)-(3.1.6) on the fixed domain Ω_0, and given $z \in Z_T$, there exists a linear and continuous extension $\mathbf{z} \in \mathbf{Z}_T$ of $(0, ..., 0, z)$, where $\mathbf{Z}_T$ is the space defined in (3.1.15). Moreover, for z such that $\|z\|_{Z_T} \ll 1$, the operator $\mathbf{T} := \mathbf{I} + \mathbf{z}$ is a $L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(\Omega_0))-$ diffeomorphism from B_0 to itself, where $B_0 \subset \mathbb{R}^N$ is a large ball containing Ω_0. Furthermore, if $\partial\Omega_t = \Gamma_i \cup \Gamma_o \cup \Gamma_r \cup M_t$, with $M_t = \{(x, z(x,t)),\ t \in \mathbb{R}\}$, and Ω_t is the domain $\partial\Omega_t$ encloses then $\mathbf{T}(\Omega_0) = \Omega_t$ and $\mathbf{T}(\Gamma_\alpha) = \Gamma_\alpha$, $\alpha \in \{i, o, r\}$, $\mathbf{T}(M_0) = M_t$.

Proof:

Let $t \in (-T, 3T)$ and consider the extension by zero of $z = z(\cdot, t)$ in $\mathbb{R}^{N-1}$, still denoted by z. Following Demengel [25], we define the operator

$$\bar{z}(x,t) := \frac{\eta_m(x_N)}{x_N^{N-1}} \int_{(0,\, x_N)^{N-1}} z(x' + s', t)\, \mathrm{d}s', \forall\, x \in \mathbb{R}^N_+ := \mathbb{R}^N \cap \{x_N \geq 0\}, \tag{3.3.60}$$

where $\eta_m \in \mathcal{D}(\mathbb{R})$ is a cut-off function such that $\eta_m(0) = 1$ and $\eta_m(x_N) = 0$ for $|x_N| > \varepsilon$, where $0 < \varepsilon \ll 1$, i.e., supp $(\eta_m) \subsetneq [-\varepsilon, \varepsilon]$. We set

$$\mathbf{z}(x,t) = (0, .., 0, \bar{z}(x,t)),\ \forall\, x \in \mathbb{R}^N_+,\ \forall\, t \in \mathbb{R}, \tag{3.3.61}$$

and we will show that $\mathbf{z}$ is the required extension.

- Claim 1: $\bar{z} \in L^\infty_T(W^{2,\hat{2}^*}(\mathbb{R}^N_+)) \cap H^1_T(H^2(\mathbb{R}^N_+)) \cap H^2_T(L^2(\mathbb{R}^N_+))$.

Indeed, note that according to [25, Theorem 3.67, Exercise 3.12], the map

$$\begin{aligned} W^{k-\frac{1}{p},p}(\mathbb{R}^{N-1}) &\longrightarrow W^{k,p}(\mathbb{R}^N_+) \\ z &\mapsto \bar{z} \end{aligned} \tag{3.3.62}$$

where $\bar{z}$ is defined as in (3.3.60), is linear and continuous for all integer, $k \geq 0$ and $p \in (1,+\infty)$ and we have

$$\|\bar{z}\|_{W^{k,p}(\mathbb{R}^N_+)} \leq C\|z\|_{W^{k-\frac{1}{p},p}(\mathbb{R}^{N-1})}. \tag{3.3.63}$$

Since for *a.e.* $t \in (-T, 3T)$, we have

$$z(\cdot,t) \in W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(\mathbb{R}^{N-1}) \cap H^{3/2}(\mathbb{R}^{N-1}) \cap L^2(\mathbb{R}^{N-1}), \tag{3.3.64}$$

then according to (3.3.62) we have

$$\bar{z}(\cdot,t) \in W^{2,\hat{2}^*}(\mathbb{R}^N_+) \cap H^2(\mathbb{R}^N_+) \cap L^2(\mathbb{R}^N_+), \tag{3.3.65}$$

and because of (3.3.63) we have the estimations

$$\|\bar{z}(\cdot,t)\|_{W^{2,\hat{2}^*}(\mathbb{R}^N_+)} \leq C\|z(\cdot,t)\|_{W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(\mathbb{R}^{N-1})}, \tag{3.3.66}$$

$$\|\bar{z}(\cdot,t)\|_{H^2(\mathbb{R}^N_+)} \leq C\|z(\cdot,t)\|_{H^{3/2}(\mathbb{R}^{N-1})}, \tag{3.3.67}$$

$$\|\bar{z}(\cdot,t)\|_{L^2(\mathbb{R}^N_+)} \leq C\|z(\cdot,t)\|_{L^2(\mathbb{R}^{N-1})}, \tag{3.3.68}$$

which gives by taking the sup on (3.3.66), and raising to the square and integration over $(-T, 3T)$ (3.3.67), (3.3.68),

$$\bar{z} \in L^\infty_T(W^{2,\hat{2}^*}(\mathbb{R}^N_+)) \cap L^2_T(H^2(\mathbb{R}^N_+)) \cap L^2_T(L^2(\mathbb{R}^N_+)), \tag{3.3.69}$$

with the estimations

$$\|\bar{z}\|_{L^\infty_T(W^{2,\hat{2}^*}(\mathbb{R}^N_+))} \leq C\|z\|_{L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(\mathbb{R}^{N-1}))}, \tag{3.3.70}$$

$$\|\bar{z}\|_{L^2_T(H^2(\mathbb{R}^N_+))} \leq C\|z\|_{L^2_T(H^{3/2}(\mathbb{R}^{N-1}))}, \tag{3.3.71}$$

$$\|\bar{z}\|_{L^2_T(L^2(\mathbb{R}^N_+))} \leq C\|z\|_{L^2_T(L^2(\mathbb{R}^{N-1}))}. \tag{3.3.72}$$

Note that for all $t \in (-T, 3T)$ and for all $h > 0$, we have

$$\begin{aligned}\frac{\bar{z}(\cdot,t+h) - \bar{z}(\cdot,t)}{h} &= \frac{\eta_m(x_N)}{x_N^{N-1}} \int_{(0,x_N)^{N-1}} \frac{z(x'+s',t+h) - z(x'+s',t)}{h}\, \mathrm{d}s' \\ &\underset{h\to 0^+}{\longrightarrow} \frac{\eta_m(x_N)}{x_N^{N-1}} \int_{(0,x_N)^{N-1}} \partial_t z(x'+s',t)\, \mathrm{d}s' = \overline{\partial_t z}(\cdot,t).\end{aligned} \tag{3.3.73}$$

So, we have $\partial_t \bar{z} = \overline{\partial_t z}$. Similarly, we can show that $\partial_{tt}\bar{z} = \overline{\partial_{tt} z}$ by replacing $\bar{z}$ by $\partial_t \bar{z}$ and z by $\partial_t z$ in (3.3.73). Since

$$\partial_t z \in L^2_T(H^{3/2}(\mathbb{R}^{N-1})) \cap L^2_T(L^2(\mathbb{R}^{N-1})), \tag{3.3.74}$$

and

$$\partial_{tt} z \in L^2_T(L^2(\mathbb{R}^{N-1})), \tag{3.3.75}$$

on one side, we will use (3.3.71), (3.3.72) by replacing z by $\partial_t z$ and $\bar{z}$ by $\partial_t \bar{z} = \overline{\partial_t z}$ and on another side we use (3.3.72) by replacing z by $\partial_{tt} z$ and $\bar{z}$ by $\partial_{tt}\bar{z} = \overline{\partial_{tt} z}$. We get the claim 1 with the estimations

$$\|\bar{z}\|_{H^1_T(H^2(\mathbb{R}^N_+))} \leq C\|z\|_{H^1_T(H^{3/2}(\mathbb{R}^{N-1}))}, \tag{3.3.76}$$

$$\|\bar{z}\|_{H^2_T(L^2(\mathbb{R}^N_+))} \leq C\|z\|_{H^2_T(L^2(\mathbb{R}^{N-1}))}. \tag{3.3.77}$$

- Claim 2: $\mathbf{T} = \mathbf{I} + \mathbf{z}$ is a $L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(\Omega_0))-$ diffeomorphism for z small.

Indeed, in view of the estimations (3.3.70), (3.3.76), (3.3.77), and the equation (3.3.61), we have clearly $\mathbf{z} \in \mathbf{Z}_T$ (see (3.1.15)). Since for all t, $z(\cdot,t)$ and η_m are functions with compact support, then $\bar{z}(\cdot,t)$ has compact support and we can suppose that supp $(\bar{z}(\cdot,t)) \subset B_0$, where B_0 is a ball in $\mathbb{R}^N$ large enough to contains Ω_0. Therefore, we have supp $(\mathbf{z}(\cdot,t)) \subset B_0$ and we can choose z small enough such that $\mathbf{T}(B_0) = (\mathbf{I} + \mathbf{z}(\cdot,t))(B_0) = B_0$.

Note that since $\hat{2}^* > N$, then by the Sobolev's embedding, we have $\mathbf{T} \in L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(B_0)) \hookrightarrow L^\infty_T(\mathbf{W}^{1,\infty}(B_0))$ (see [1, Theorem 4.12]). Therefore we have $\mathbf{T} \in L^\infty_T(\mathbf{W}^{1,\infty}(B_0))$. Moreover, since for all $t \in \mathbb{R}$, the determinant of the Jacobian matrix is given by

$$\det([\nabla \mathbf{T}(\cdot,t)]) := |\nabla \mathbf{T}(\cdot,t)| = \left| \begin{bmatrix} I_{N-1} & \mathbf{0}_{N-1,1} \\ {}^t(\nabla_{x'}\bar{z}(\cdot,t)) & 1 + \partial_N \bar{z}(\cdot,t) \end{bmatrix} \right| = 1 + \partial_N \bar{z}(\cdot,t) \neq 0, \tag{3.3.78}$$

from the inverse function theorem for Lipschitz functions (see [20, Theorem 1]), $\mathbf{T}(\cdot,t)$ is a $\mathbf{W}^{1,\infty}(B_0)-$ diffeomorphism, which implies that,

$$\mathbf{T}^{-1}(\cdot,t) \in \mathbf{W}^{1,\infty}(B_0). \tag{3.3.79}$$

Moreover, since B_0 is bounded and independent of t, then

$$\mathbf{T}^{-1} \in L^\infty_T(\mathbf{L}^\infty(B_0)). \tag{3.3.80}$$

Since $\mathbf{T}^{-1} \in L^\infty_T(\mathbf{L}^\infty(B_0)) \hookrightarrow L^\infty_T(\mathbf{L}^{\hat{2}^*}(B_0))$, it remains to handle the space regularity of $\mathbf{T}^{-1}$.

Now, since $\mathbf{T} \in L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(B_0))$ we have $[\nabla \mathbf{T}] \in L^\infty_T(\mathbf{W}^{1,\hat{2}^*}(B_0))$. Therefore, from the equation $\mathbf{T}^{-1} \circ \mathbf{T}(x,t) = x \in B_0$, and using (3.3.5), we have

$$[\nabla \mathbf{T}^{-1}] = [\nabla \mathbf{T}]^{-1} \circ \mathbf{T}^{-1}. \tag{3.3.81}$$

We note that the matrix $[\nabla \mathbf{T}]^{-1} = [\mathbf{I} + \nabla \mathbf{z}]^{-1}$ explicitly reads

$$[\nabla \mathbf{T}]^{-1} = \begin{bmatrix} I_{N-1} & \mathbf{0}_{N-1,1} \\ -\frac{{}^t(\nabla_{x'}\bar{z})}{1+\partial_N \bar{z}} & \frac{1}{1+\partial_N \bar{z}} \end{bmatrix}. \tag{3.3.82}$$

Since $\bar{z} \in L^\infty_T(W^{2,\hat{2}^*}(B_0)) \hookrightarrow L^\infty_T(W^{1,\infty}(B_0))$, then for $z \in Z_T$ such that $\|z\|_{Z_T} \ll 1$ we have

$$1 + \partial_N \bar{z} \in L^\infty_T(L^\infty(B_0)) \text{ and } \frac{\partial_i \bar{z}}{1 + \partial_N \bar{z}} \in L^\infty_T(W^{1,\hat{2}^*}(B_0)). \tag{3.3.83}$$

Therefore, all the entries of the matrix $[\nabla \mathbf{T}]^{-1}$ defined in (3.3.82) belong to $L^\infty_T(W^{1,\hat{2}^*}(B_0))$ and we have $[\nabla \mathbf{T}^{-1}] \in L^\infty_T(\mathbf{W}^{1,\hat{2}^*}(B_0))$, which implies

$$\mathbf{T}^{-1} \in L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(B_0)). \tag{3.3.84}$$

So, $\mathbf{T}$ is a $L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(B_0))-$ diffeomorphism.

Finally, with the equation of $\bar{z}$ given by equation (3.3.60), we have for all $t \in [-T, 3T]$,

$$\bar{z}(\cdot,t) \equiv 0 \quad \text{in} \quad \Big\{x \in \mathbb{R}^N_+, \quad |x'| > \ell_0 + |x_N|\sqrt{N-1}\Big\} \cup \Big\{x \in \mathbb{R}^N_+, \quad |x_N| > \varepsilon\Big\}, \tag{3.3.85}$$

because supp $(z(\cdot,t)) \in B_{N-1}(\ell_0)$ and supp $(\eta_m) \in [\varepsilon, \varepsilon]$. Hence

$$\text{supp } (\bar{z}(\cdot,t)) \subset \Big\{x \in \mathbb{R}^N_+, \quad |x'| \leq \ell_0 + |x_N|\sqrt{N-1}\Big\} \cap \Big\{x \in \mathbb{R}^N_+, \quad |x_N| \leq \varepsilon\Big\}. \tag{3.3.86}$$

- Claim 3: $\mathbf{T}(\Gamma_\alpha) = \Gamma_\alpha$, $\alpha \in \{o, i, r\}$, $\mathbf{T}(M_0) = M_t$ and $\mathbf{T}(\Omega_0) = \Omega_t$,

Indeed, as Ω_0 is a $C^{1,1}$ domain, it is enough to show that

$$\text{supp } (\bar{z}(\cdot,t)) \cap \partial\Omega_0 \subset M_0. \tag{3.3.87}$$

Let $x = (x', x_N) \in$ supp $(\bar{z}(\cdot,t)) \cap \partial\Omega_0$. By contradiction, let us suppose that $x \notin M_0$. Then since $\bar{z}$ is the extension of z and supp $(z) = B_{N-1}(\ell_0)$, we have supp $(\bar{z}(\cdot,t)) \cap \Gamma_r = \emptyset$, and since $0 < \varepsilon \ll 1$, we have supp $(\bar{z}(\cdot,t)) \cap (\Gamma_i \cup \Gamma_o) = \emptyset$. Therefore, we have

$$x = (x', x_N) \in \text{supp } (\bar{z}(\cdot,t)) \cap (\Gamma_r \cup \Gamma_i \cup \Gamma_o) = \emptyset \tag{3.3.88}$$

which is a contradiction. Therefore, $x \in M_0$ and (3.3.87) holds. By taking the closure in (3.3.87), we get

$$\text{supp } (\mathbf{z}(\cdot,t)) \cap \partial\Omega_0 \subset M_0, \tag{3.3.89}$$

and since $\mathbf{T} = \mathbf{I} + \mathbf{z}$, this proves the Claim 3 and completes the proof of Lemma 3.3.2. ∎

Remark 3.3.3. *We remark that from the above Lemma 3.3.2 and the choice of the number $\hat{2}^*$, there exists a constant $C > 0$ such that for all $z \in Z_T$,*

$$\begin{aligned} &\mathbf{z} \in L^\infty_T(\mathbf{W}^{1,\infty}(\Omega_0)), \\ &\|\mathbf{z}\|_{L^\infty_T(\mathbf{W}^{1,\infty}(\Omega_0))} \leq C\|\mathbf{z}\|_{L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(\Omega_0))} \leq C\|z\|_{L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(B_{N-1}\ell_0))} \leq C\|z\|_{Z_T}. \end{aligned} \tag{3.3.90}$$

Lemma 3.3.4 (Existence of an extension of $\mathbf{h}$).

Under the assumptions (3.1.2)-(3.1.6) on the domain Ω_0, and given $z \in Z_T$, there exists an extension $\mathbf{h}$ of the function $\mathbf{h}$ defined in (3.3.21) with $\mathbf{h} \in \mathbf{W}^{\mathbf{h}}_T := L^2_T(\mathbf{H}^2(\Omega_0)) \cap H^1_T(\mathbf{L}^2(\Omega_0))$ (see (3.1.20)). Moreover, the map $z \in Z_T \mapsto \mathbf{h} \in \mathbf{W}^{\mathbf{h}}_T$ is linear and continuous.

Note that the extension $\mathbf{h}$ is different from the one defined in (3.3.21) due to the appearance of cut-off functions η_i and η_o and also due to the appearance of the new function $\bar{z}$ which itself extend z, but they are equal on the boundary. However, this extension called again $\mathbf{h}$ is smoother than the one we defined in (3.3.21).

Proof: Since the function $\mathbf{g}$ is given by the formula (3.1.11), then we will look for the extension of $\mathbf{h}$ in the form

$$\begin{aligned}\mathbf{h}(x,t) &= \eta_i(x)\mathbf{u}_i(x)k_i(t) + \eta_o(x)\mathbf{u}_o(x)k_o(t) + (0,...,0,\partial_t\bar{z}(x,t)) \\ &= \mathbf{u}_i(x)\mathbb{1}^T_{[-T,T)}(t)\eta_i(x)\int_{B_{N-1}(\ell_0)} \partial_t z(x',t)\,\mathrm{d}x' \\ &+ \mathbf{u}_o(x)\mathbb{1}^T_{[T,3T)}(t)\eta_o(x)\int_{B_{N-1}(\ell_0)} \partial_t z(x',t)\,\mathrm{d}x' \\ &+ (0,...,0,\partial_t\bar{z}(x,t)),\ \forall\, x=(x',x_N)\in\mathbb{R}^N,\ \forall\, t\in\mathbb{R}, \end{aligned} \tag{3.3.91}$$

where η_i, resp. η_o is a positive (i.e non-negative) cut-off function which worth one on Γ_i, resp. Γ_o, and zero outside of a small neighborhood of Γ_i, resp. Γ_o. The function z is the zero extension on $\mathbb{R}^{N-1}$ of the membrane equation (1.2.4) $-$ (1.2.8). Note that, from the expression of $\mathbf{h}$ defined in (3.3.91), we have

$$\mathbf{h}(x,t) = \mathbf{u}_i(x)k_i(t) + \mathbf{u}_o(x)k_o(t),\ \forall\,(x,t)\in\Gamma_i\cup\Gamma_o\times\mathbb{R}, \tag{3.3.92}$$

$$\mathbf{h}(x,t) = (0,...,0,\partial_t\bar{z}(x,t)) = (0,..,0,\partial_t z(x',t)),\ \forall\,(x,t)\in M_0\times\mathbb{R}. \tag{3.3.93}$$

This shows that the function defined in (3.3.91) is indeed the extension of the function defined in (3.3.21). It remains to show that $\mathbf{h}\in\mathbf{W}^{\mathbf{h}}_T$ and the map $z\in Z_T\mapsto\mathbf{h}\in\mathbf{W}^{\mathbf{h}}_T$ is linear and continuous.

By construction, we have $\mathbf{u}_\alpha\in\mathbf{H}^2(\Omega_0),\ \alpha\in\{i,o\}$. Moreover, since $z\in H^2_T(L^2(B_{N-1}(\ell_0)))$, its zero extension in $\mathbb{R}^{N-1}$, still denoted z, belongs to $H^2_T(L^2(\mathbb{R}^{N-1}))$. Therefore, $\partial_t z\in H^1_T(L^2(\mathbb{R}^{N-1}))$, which implies $t\mapsto\int_{B_{N-1}(\ell_0)}\partial_t z(y',t)\,\mathrm{d}y'\in H^1_T(\mathbb{R})$. Therefore, the second and the third line of the right-hand side of equation (3.3.91) belong to $H^1_T(\mathbf{H}^2(\Omega_0))$. Indeed, for all $\alpha\in\{i,o\}$ since $\eta_\alpha,\ \nabla\eta_\alpha,\ D^2\eta_\alpha$ are bounded, and since $k_\alpha(t)\in L^2(\mathbb{R})$ we have $\eta_\alpha(x)\mathbf{u}_\alpha(x)k_\alpha(t)\in L^2_T(\mathbf{H}^2(\Omega_0))$ with

$$\|\eta_\alpha\mathbf{u}_\alpha k_\alpha\|_{L^2_T(\mathbf{H}^2(\Omega_0))}\le C\|k_\alpha\|_{L^2_T(\mathbb{R})}\|\mathbf{u}_\alpha\|_{\mathbf{H}^2(\Omega_0)}\le C\|z\|_{Z_T}\|\mathbf{u}_\alpha\|_{\mathbf{H}^2(\Omega_0)}. \tag{3.3.94}$$

Now, we will show the $H^1_T(\mathbf{L}^2(\Omega_0))$ regularity of $\eta_\alpha\mathbf{u}_\alpha k_\alpha$. Note that

$$\begin{aligned}k_i'(t) &= \delta(-T)\int_{B_{N-1}(\ell_0)}\partial_t z(x',-T)\,\mathrm{d}x' - \delta(T)\int_{B_{N-1}(\ell_0)}\partial_t z(x',T)\,\mathrm{d}x' \\ &+ \mathbb{1}^T_{[-T,T)}(t)\int_{B_{N-1}(\ell_0)}\partial_{tt}z(x',t)\mathrm{d}x' \\ &= \mathbb{1}^T_{[-T,T)}(t)\int_{B_{N-1}(\ell_0)}\partial_{tt}z(x',t)\mathrm{d}x', \end{aligned} \tag{3.3.95}$$

because the membrane has zero velocity at the upper and lower position i.e., $\partial_t z(\cdot,(2k+1)T) = 0,\ \forall\ k \in \mathbb{Z}$. Here for arbitrarily $a \in \mathbb{R}$, $\delta(a) \in \mathcal{D}'(\mathbb{R})$ is the Dirac distribution centered at the point a and defined as

$$\langle \delta(a), \varphi\rangle := \varphi(a),\ \forall\ \varphi \in \mathcal{D}(\mathbb{R}). \tag{3.3.96}$$

Similarly

$$k_o'(t) = \mathbb{1}^T_{[T,3T)}(t) \int_{B_{N-1}(\ell_0)} \partial_{tt} z(x',t)\mathrm{d}x'. \tag{3.3.97}$$

Therefore, $k_\alpha'(t) \in L^2(\mathbb{R})$ because

$$\begin{aligned}\int_{\mathbb{R}} |k_\alpha'|^2\ \mathrm{d}t &\leq \int_{\mathbb{R}} \Big| \int_{B_{N-1}(\ell_0)} \partial_{tt} z(x',t)\ \mathrm{d}x' \Big|^2\ \mathrm{d}t \\ &\leq C \int_{\mathbb{R}} \int_{B_{N-1}(\ell_0)} |\partial_{tt} z(x',t)|^2\ \mathrm{d}x'\ \mathrm{d}t \leq C\|z\|^2_{H^2(L^2(B_{N-1}(\ell_0)))} < +\infty.\end{aligned} \tag{3.3.98}$$

This permits to have $\eta_\alpha(x)\mathbf{u}_\alpha(x)k_\alpha'(t) \in L^2_T(\mathbf{H}^2(\Omega_0))$ with

$$\|\eta_\alpha \mathbf{u}_\alpha k_\alpha'\|_{L^2_T(\mathbf{H}^2(\Omega_0))} \leq C\|k_\alpha'\|_{L^2_T(\mathbb{R})}\|\mathbf{u}_\alpha\|_{\mathbf{H}^2(\Omega_0)} \leq C\|z\|_{Z_T}\|\mathbf{u}_\alpha\|_{\mathbf{H}^2(\Omega_0)}. \tag{3.3.99}$$

Therefore,

$$\eta_\alpha(x)\mathbf{u}_\alpha(x)k_\alpha(t) \in H^1_T(\mathbf{H}^2(\Omega_0)) \hookrightarrow L^2_T(\mathbf{H}^2(\Omega_0)) \cap H^1_T(\mathbf{L}^2(\Omega_0)) := \mathbf{W}^{\mathbf{h}}_T \tag{3.3.100}$$

and from (3.3.94), (3.3.99), (3.3.100) we get

$$\|\eta_\alpha \mathbf{u}_\alpha k_\alpha\|_{\mathbf{W}^{\mathbf{h}}_T} \leq C\|\eta_\alpha \mathbf{u}_\alpha k_\alpha\|_{H^1_T(\mathbf{H}^2(\Omega_0))} \leq C\|k_\alpha\|_{H^1_T(\mathbb{R})}\|\mathbf{u}_\alpha\|_{\mathbf{H}^2(\Omega_0)} \leq C\|z\|_{Z_T}\|\mathbf{u}_\alpha\|_{\mathbf{H}^2(\Omega_0)}. \tag{3.3.101}$$

It remains to show that the fourth line in the right-hand side of equation (3.3.91) belongs to $\mathbf{W}^{\mathbf{h}}_T$. Note that the extension $\bar{z}$ satisfies $\partial_t \bar{z} \in L^2_T(H^2(\mathbb{R}^N_+)) \cap H^1_T(L^2(\mathbb{R}^N_+))$. This permits us to conclude that

$$(0,...,0,\partial_t \bar{z}) = \partial_t \mathbf{z} \in L^2_T(\mathbf{H}^2(\Omega_0)) \cap H^1_T(\mathbf{L}^2(\Omega_0)) = \mathbf{W}^{\mathbf{h}}_T, \tag{3.3.102}$$

with the estimation

$$\begin{aligned}\|\partial_t \mathbf{z}\|_{\mathbf{W}^{\mathbf{h}}_T} &\leq C\|\partial_t \bar{z}\|_{L^2_T(H^2(\mathbb{R}^N_+)) \cap H^1_T(L^2(\mathbb{R}^N_+))} \\ &\leq C\|\partial_t z\|_{L^2_T(H^{3/2}(\mathbb{R}^{N-1})) \cap H^1_T(L^2(\mathbb{R}^{N-1}))} \\ &\leq C\|z\|_{Z_T}.\end{aligned} \tag{3.3.103}$$

From the formula (3.3.91), the map $z \in Z_T \mapsto \mathbf{h} \in \mathbf{W}^{\mathbf{h}}_T$ is clearly linear by construction and because of the estimations (3.3.101) and (3.3.103) we have

$$\|\mathbf{h}\|_{\mathbf{W}^{\mathbf{h}}_T} \leq C\|z\|_T, \tag{3.3.104}$$

which proves the continuity. ∎

The following lemma shows that the map $\mathbf{F}$ is well defined. Precisely, we need to show that $F_1(z,\mathbf{w},q) \in \mathbf{R}_{\mathbf{w}}$ and $F_2(z,\mathbf{w}) \in R_q$, (see (3.1.22), (3.1.23)).

Lemma 3.3.5. *Assume the hypotheses (3.1.2)-(3.1.6) hold. If $(\mathbf{w}, q) \in \mathbf{W}_T \times Q_T$ where $\mathbf{W}_T$ resp. Q_T is given by (3.1.19) resp. (3.1.21) then $F_1(z, \mathbf{w}, q) \in \mathbf{R}_{\mathbf{w}}$ and $F_2(z, \mathbf{w}) \in R_q$.*

Proof: We note that the term F_2 depends on the determinant of the Jacobian matrix $|\nabla \mathbf{T}|$. In addition, the terms $[\nabla \mathbf{T}]$ and $[\nabla \mathbf{T}]^{-1}$ will sometimes appear in the estimations of both F_1 and F_2, but since for z small, $\mathbf{T}$ is a $L^\infty_T(\mathbf{W}^{1,\infty}(\Omega_0))-$diffeomorphism (see Lemma 3.3.2), they are all bounded in space and time. Therefore, we will estimate them with a generic constant $C = C(T) > 0$. Let us suppose that $(\mathbf{w}, q) \in \mathbf{W}_T \times Q_T$.

To simplify and make the following computations more explicit, it is worth noting that

$$[\nabla \mathbf{T}] = \begin{bmatrix} I_{N-1} & \mathbf{0}_{N-1,1} \\ {}^t(\nabla_{x'} \bar{z}) & 1 + \partial_N \bar{z} \end{bmatrix}, \quad [\nabla \mathbf{T}]^{-1} = \begin{bmatrix} I_{N-1} & \mathbf{0}_{N-1,1} \\ -\frac{{}^t(\nabla_{x'} \bar{z})}{1+\partial_N \bar{z}} & \frac{1}{1+\partial_N \bar{z}} \end{bmatrix}, \quad |\nabla \mathbf{T}| = 1 + \partial_N \bar{z}. \tag{3.3.105}$$

Indeed, we note that the proof is lengthy and technical. Essentially, it considers each term of $\mathbf{F}$ in the form of products $a(z)b(\mathbf{w})$ and carries out the computations in the spirit of Propositions 3.2.1 and 3.2.2. These propositions will be useful for demonstrating the C^1 regularity of $\mathbf{F}$.

(i) We start by proving that $F_1(z, \mathbf{w}, q) \in \mathbf{R}_{\mathbf{w}}$.

Indeed, note that every term of (3.3.26) is of the form $f(z, \mathbf{w}) = a(z)b(\mathbf{w})$, where a and b and two C^1 functions to be precised for each term of (3.3.26).

For the first term of the first line of (3.3.26), taking only the general term of $\partial_t \mathbf{w}$, we have

$$f(z, \mathbf{w}) = \partial_t \mathbf{w}, \quad a(z) = 1, \quad b(\mathbf{w}) = \partial_t w_i, \tag{3.3.106}$$

with the estimation

$$\begin{aligned} \|a(z)b(\mathbf{w})\|^2_{L^2_T(L^2(\Omega_0))} &= \int_{-T}^{3T} \int_{\Omega_0} |a(z)|^2 |b(\mathbf{w})|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \|a(z)\|^2_{L^\infty_T L^\infty(\Omega_0)} \|b(\mathbf{w})\|^2_{L^2_T(L^2(\Omega_0))} &(3.3.107) \\ &\leq C \|\mathbf{w}\|^2_{H^1_T(\mathbf{L}^2(\Omega_0))}. &(3.3.108) \end{aligned}$$

For the second term of the first line of (3.3.26) we have (we consider only the highest order terms in z; the estimation of the lower order terms is similar and simpler)

$$f(z, \mathbf{w}) := \partial_t \mathbf{T} \cdot {}^t[\nabla \mathbf{T}]^{-1} \cdot [\nabla \mathbf{w}], \quad a(z) := a_1(z)a_2(z) = \partial_t \bar{z} \frac{\partial_i \bar{z}}{1 + \partial_N \bar{z}}, \quad b(\mathbf{w}) := b_1(\mathbf{w}) = \partial_k w_j, \tag{3.3.109}$$

with the estimations

$$\begin{aligned} \|a(z)b(\mathbf{w})\|^2_{L^2_T(L^2(\Omega_0))} &= \int_{-T}^{3T} \int_{\Omega_0} |a(z)|^2 |b(\mathbf{w})|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \int_{-T}^{3T} \|a(z)\|^2_{L^\infty(\Omega_0)} \|b(\mathbf{w})\|^2_{L^2(\Omega_0)} \, \mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&\leq \|a(z)\|^2_{L^2_T(L^\infty(\Omega_0))}\|b(\mathbf{w})\|^2_{L^\infty_T(L^2(\Omega_0))}\\
&=: \|a(z)\|^2_{R_a}\|b(\mathbf{w})\|^2_{R_b} && (3.3.110)\\
&\leq C\|a_1(z)\|^2_{R^1_a}\|a_2(z)\|^2_{R^2_a}\|b(\mathbf{w})\|^2_{R_b} && (3.3.111)\\
&:= C\|\partial_t\bar z\|^2_{L^2_T(L^\infty(\Omega_0))}\left\|\frac{\partial_i\bar z}{1+\partial_N\bar z}\right\|^2_{L^\infty_T(L^\infty(\Omega_0))}\|\partial_k w_j\|^2_{L^\infty_T(L^2(\Omega_0))}\\
&\leq C\|z\|^2_{H^1_T(H^{3/2}(B_{N-1}(\ell_0)))}\|z\|^2_{L^\infty_T(W^{2-\frac{1}{\hat 2^*},\hat 2^*}(B_{N-1}(\ell_0)))}\\
&\qquad\qquad \|\mathbf{w}\|^2_{L^2_T(\mathbf{H}^2(\Omega_0))\cap H^1_T(\mathbf{L}^2(\Omega_0))}, && (3.3.112)
\end{aligned}
$$

where we used the embedding $W^{1,\hat 2^*}(\Omega_0)\hookrightarrow L^\infty(\Omega_0)$.

Here it is important to make the following remark (the proof of Lemma 3.3.5 will continue after the remark).

Remark 3.3.6. *First, observe that in the estimations (3.3.110), (3.3.111), we have used the notations of Propositions 3.2.1 and 3.2.2. The reason is that we will use these propositions to prove the C^1 regularity of $\mathbf{F}$. Note also that the estimation (3.3.110) holds if instead of $(a(z),b(\mathbf{w}))$ we take $(\tilde a,\tilde b)\in R_a\times R_b$. Furthermore, the estimation (3.3.111) holds separately for $\|a(z)\|_{R_a}$ resp. $\|b(\mathbf{w})\|_{R_b}$ in terms of the product of $\|a_i(z)\|_{R^i_a}$ resp. $\|b_i(\mathbf{w})\|_{R^i_b}$ i.e. $\|a(z)\|_{R_a}\leq C\prod_{i=1}^2\|a_i(z)\|_{R^i_a}$, and $\|b(\mathbf{w})\|_{R_b}\leq C\prod_{i=1}^1\|b_i(\mathbf{w})\|_{R^i_b}$. These inequalities hold also if we replace $a_i(z)$ resp. $b_i(\mathbf{w})$ by any $\tilde a_i\in R^i_a$ resp. $\tilde b_i\in R^i_b$.*

In other words, if the functions $z\mapsto a_i(z)$ and $\mathbf{w}\mapsto b_i(\mathbf{w})$ are C^1, and the function $z\mapsto a(z)$ and $\mathbf{w}\mapsto b(\mathbf{w})$ satisfy the condition of Proposition 3.2.2 and the function $z\mapsto f(z,\mathbf{w})=a(z)b(\mathbf{w})$ satisfies the condition of Proposition 3.2.1, therefore, f is C^1. This remark holds for all the estimations of the terms of $\mathbf{F}$ that we will present hereafter. For each of them, the functions a_i, b_i and the spaces R^i_a and R^i_b are clear from the context. We will highlight them only for the most complex terms.

For the first term of the second line of (3.3.26), we proceed similarly, taking again the dominant terms and we have

$$
f(z,\mathbf{w})=[\nabla\mathbf{h}]\cdot[\nabla\mathbf{T}]^{-1}\cdot\mathbf{w},\ a(z)=\partial_i h_j(\bar z)\frac{\partial_k\bar z}{1+\partial_N\bar z},\ b(\mathbf{w})=w_l. \tag{3.3.113}
$$

Since $\partial_i h_j\in H^1(\Omega_0)\hookrightarrow L^{\hat 2^*}(\Omega_0)$ and $w_l\in H^1(\Omega_0)\hookrightarrow L^{\hat 2}(\Omega_0))$, we have the estimations

$$
\begin{aligned}
\|a(z)b(\mathbf{w})\|^2_{L^2_T(L^2(\Omega_0))} &\leq \int_{-T}^{3T}\|a(z)\|^2_{L^{\hat 2^*}(\Omega_0)}\|b(\mathbf{w})\|^2_{L^{\hat 2}(\Omega_0)}\,\mathrm{d}t\\
&\leq \|a(z)\|^2_{L^2_T(L^{\hat 2^*}(\Omega_0))}\|b(\mathbf{w})\|^2_{L^\infty_T(L^{\hat 2}(\Omega_0))} && (3.3.114)\\
&\leq \|\partial_i h_j\|^2_{L^2_T(H^1(\Omega_0))}\left\|\frac{\partial_k\bar z}{1+\partial_N\bar z}\right\|^2_{L^\infty_T(L^\infty(\Omega_0))}\|w_l\|^2_{L^\infty_T(H^1(\Omega_0))}\\
&\leq C\|\mathbf{h}\|^2_{L^2(\mathbf{H}^2(\Omega_0))}\|\bar z\|^2_{L^\infty(W^{2,\hat 2^*}(\Omega_0))}\|w_l\|^2_{L^\infty_T(H^1(\Omega_0))}\\
&\leq C\|z\|^2_{L^2_T(H^{3/2}(B_{N-1}(\ell_0)))}\|z\|^2_{L^\infty_T(W^{2-\frac{1}{\hat 2^*},\hat 2^*}(B_{N-1}(\ell_0)))}
\end{aligned}
$$

$$\|\mathbf{w}\|^2_{L^2_T(\mathbf{H}^2(\Omega_0))\cap H^1_T(\mathbf{L}^2(\Omega_0))} \tag{3.3.115}$$

where we used the fact that the map $z \in L^2_T(H^{3/2}(B_{N-1}(\ell_0))) \mapsto \mathbf{h} \in L^2_T(\mathbf{H}^2(\Omega_0))$ is continuous.

Similarly, for the second term of the second line of (3.3.26), and we have

$$f(z,\mathbf{w}) = [\nabla\mathbf{w}]\cdot[\nabla\mathbf{T}]^{-1}\cdot\mathbf{h}, \quad a(z) = h_k(\bar{z})\frac{\partial_l\bar{z}}{1+\partial_N\bar{z}}, \quad b(\mathbf{w}) = \partial_i w_j, \tag{3.3.116}$$

with the estimations

$$\begin{aligned}
\|a(z)b(\mathbf{w})\|^2_{L^2_T(L^2(\Omega_0))} &\leq \int_{-T}^{3T}\|a(z)\|^2_{L^\infty(\Omega_0)}\|b(\mathbf{w})\|^2_{L^2(\Omega_0)}\,\mathrm{d}t \\
&\leq \|a(z)\|^2_{L^2_T(L^\infty(\Omega_0))}\|b(\mathbf{w})\|^2_{L^\infty_T(L^2(\Omega_0))} \qquad (3.3.117)\\
&\leq C\|h_k\|^2_{L^2_T(L^\infty(\Omega_0))}\left\|\frac{\partial_l\bar{z}}{1+\partial_N\bar{z}}\right\|^2_{L^\infty_T(L^\infty(\Omega_0))}\|\partial_i w_j\|^2_{L^\infty_T(L^2(\Omega_0))}\\
&\leq C\|\mathbf{h}\|^2_{L^2_T(\mathbf{H}^2(\Omega_0))}\|\bar{z}\|^2_{L^\infty_T(W^{2,\hat{2}^*}(\Omega_0))}\|\mathbf{w}\|^2_{L^\infty_T(\mathbf{H}^2(\Omega_0))\cap H^1_T(\mathbf{L}^2(\Omega_0))}\\
&\leq C\|z\|^2_{L^2_T(H^{3/2}(B_{N-1}(\ell_0)))}\|z\|^2_{L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(B_{N-1}(\ell_0)))}\\
&\qquad\|\mathbf{w}\|^2_{L^\infty_T(\mathbf{H}^2(\Omega_0))\cap H^1_T(\mathbf{L}^2(\Omega_0))}. \qquad (3.3.118)
\end{aligned}$$

Similarly, for the third term of the second line of (3.3.26), we have

$$\begin{aligned}
f(z,\mathbf{w}) = [\nabla\mathbf{w}]\cdot[\nabla\mathbf{T}]^{-1}\cdot\mathbf{w}, \quad a(z) = a_1(z) := \frac{\partial_i\bar{z}}{1+\partial_N\bar{z}},\\
b_1(\mathbf{w}) := \partial_j w_k, \quad b_2(\mathbf{w}) := w_l, \quad b(\mathbf{w}) = b_1(\mathbf{w})b_2(\mathbf{w}).
\end{aligned} \tag{3.3.119}$$

Since $w_l \in H^2(\Omega_0) \hookrightarrow L^\infty(\Omega_0)$, we have the estimations

$$\begin{aligned}
\|a(z)b(\mathbf{w})\|^2_{L^2_T(L^2(\Omega_0))} &\leq \|a(z)\|^2_{L^\infty_T(L^\infty(\Omega_0))}\int_{-T}^{3T}\|b_1(\mathbf{w})\|^2_{L^2(\Omega_0)}\|b_2(\mathbf{w})\|^2_{L^\infty(\Omega_0)}\,\mathrm{d}t\\
&\leq \|a_1(z)\|^2_{L^\infty_T(L^\infty(\Omega_0))}\|b_1(\mathbf{w})\|^2_{L^\infty_T(L^2(\Omega_0))}\|b_2(\mathbf{w})\|^2_{L^2_T(H^2(\Omega_0))}\\
&=: \|a_1(z)\|^2_{R^1_a}\|b_1(\mathbf{w})\|^2_{R^1_b}\|b_2(\mathbf{w})\|^2_{R^2_b} \qquad (3.3.120)\\
&\leq C\|\bar{z}\|^2_{L^\infty_T(W^{2,\hat{2}^*}(\Omega_0))}\|\mathbf{w}\|^4_{L^\infty_T(\mathbf{H}^2(\Omega_0))\cap H^1_T(\mathbf{L}^2(\Omega_0))}\\
&\leq C\|z\|^2_{L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(\Omega_0))}\|\mathbf{w}\|^4_{L^\infty_T(\mathbf{H}^2(\Omega_0))\cap H^1_T(\mathbf{L}^2(\Omega_0))}. \qquad (3.3.121)
\end{aligned}$$

The third line of (3.3.26), after expanding the gradient, we have two dominant terms. One with second order space derivative on $\mathbf{w}$, and the other with the second order space on $\mathbf{T}$ (so on $\bar{z}$). For the term with the second order derivative on $\mathbf{w}$ we have the following general expression

$$f(z,\mathbf{w}) = \mathrm{tr}\Big({}^t\left[\left({}^t[\nabla\mathbf{T}]^{-1}\cdot\nabla\,{}^t[\nabla\mathbf{w}]\right)\right]\cdot[\nabla\mathbf{T}]^{-1}\Big),$$

$$a(z) = \frac{\partial_i \bar{z} \partial_j \bar{z}}{(1+\partial_N \bar{z})^p}, \qquad b(\mathbf{w}) = \partial_{k,l} w_m, \tag{3.3.122}$$

where p is a positive integer, with the estimations

$$\begin{aligned}
\|a(z)b(\mathbf{w})\|^2_{L^2_T(L^2(\Omega_0))} &\leq \|a(z)\|^2_{L^\infty_T(L^\infty(\Omega_0))} \|b(\mathbf{w})\|^2_{L^2_T(L^2(\Omega_0))} && (3.3.123)\\
&\leq C\|\bar{z}\|^4_{L^\infty_T(W^{1,\infty}(\Omega_0))} \|\mathbf{w}\|^2_{L^2_T(\mathbf{L}^2(\Omega_0))}\\
&\leq C\|\bar{z}\|^4_{L^\infty_T(W^{2,\hat{2}^*}(\Omega_0))} \|\mathbf{w}\|^2_{L^2_T(\mathbf{L}^2(\Omega_0))}\\
&\leq C\|z\|^4_{L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(B_{N-1}(\Omega_0)))} \|\mathbf{w}\|^2_{L^2_T(\mathbf{L}^2(\Omega_0))}, && (3.3.124)
\end{aligned}$$

where we used the embedding $W^{1,\hat{2}^*}(\Omega_0) \hookrightarrow L^\infty(\Omega_0)$. Similarly, for the term with the second order derivative on $\mathbf{T}$, we have

$$\begin{aligned}
f(z,\mathbf{w}) &= \mathrm{tr}\Big(\nabla\, {}^t\Big[\big({}^t[\nabla\mathbf{T}]^{-1} \cdot {}^t[\nabla\mathbf{w}]\big)\Big] \cdot [\nabla\mathbf{T}]^{-1}\Big),\\
a(z) &= \frac{\partial_i \bar{z} \partial_j \bar{z}}{(1+\partial_N \bar{z})^p} \partial_{k,l}\bar{z}, \qquad b(\mathbf{w}) = \partial_m w_n, && (3.3.125)
\end{aligned}$$

with the estimations

$$\begin{aligned}
\|a(z)b(\mathbf{w})\|^2_{L^2_T(L^2(\Omega_0))} &\leq \int_{-T}^{3T} \|a(z)\|^2_{L^{\hat{2}^*}(\Omega_0)} \|b(\mathbf{w})\|^2_{L^{\hat{2}}(\Omega_0)}\, \mathrm{d}t\\
&\leq \|a(z)\|^2_{L^\infty_T(L^{\hat{2}^*}(\Omega_0))} \|b(\mathbf{w})\|^2_{L^2_T(L^{\hat{2}}(\Omega_0))} && (3.3.126)\\
&\leq \|\partial_{k,l}\bar{z}\|^2_{L^\infty_T(L^{\hat{2}^*}(\Omega_0))} \left\| \frac{\partial_i \bar{z} \partial_j \bar{z}}{(1+\partial_N \bar{z})^p} \right\|^2_{L^\infty_T(L^\infty(\Omega_0))} \|\partial_m w_n\|^2_{L^2_T(H^1(\Omega_0))}\\
&\leq C\|\bar{z}\|^6_{L^\infty_T(W^{2,\hat{2}^*}(\Omega_0))} \|\mathbf{w}\|^2_{L^2_T(\mathbf{H}^2(\Omega_0))}\\
&\leq C\|z\|^6_{L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(B_{N-1}(\ell_0)))} \|\mathbf{w}\|^2_{L^2_T(\mathbf{H}^2(\Omega_0))}. && (3.3.127)
\end{aligned}$$

For the term of the forth line of (3.3.26) we have

$$f(z,q) = {}^t[\nabla\mathbf{T}]^{-1} \cdot \nabla q, \quad a(z) = \frac{\partial_i \bar{z}}{1+\partial_N \bar{z}}, \quad b(q) = \partial_j q, \tag{3.3.128}$$

with the estimations

$$\begin{aligned}
\|a(z)b(q)\|^2_{L^2_T(L^2(\Omega_0))} &\leq \|a(z)\|^2_{L^\infty_T(L^\infty(\Omega_0))} \|b(q)\|^2_{L^2_T(L^2(\Omega_0))} && (3.3.129)\\
&\leq C\|\bar{z}\|^2_{W^{2,\hat{2}^*}(\Omega_0)} \|q\|^2_{L^2_T(H^1(\Omega_0))}\\
&\leq C\|z\|^2_{L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(B_{N-1}(\ell_0)))} \|q\|^2_{L^2_T(H^1(\Omega_0))}. && (3.3.130)
\end{aligned}$$

We can show that each term of the fifth line of (3.3.26) is in $\mathbf{R_w} = L^2_T(\mathbf{L}^2(\Omega_0))$ by proceeding exactly as in (3.3.107), (3.3.108), (3.3.110), (3.3.111), (3.3.112), (3.3.114), (3.3.115), (3.3.117), (3.3.118), (3.3.120), (3.3.121), (3.3.123), (3.3.124), (3.3.126), (3.3.127), (3.3.129), (3.3.130), with $\mathbf{h}$ instead of $\mathbf{w}$, and noting the continuity of the map

$$z \in L^2_T(H^{3/2}(B_{N-1}(\ell_0))) \mapsto \mathbf{h} \in L^2_T(\mathbf{H}^2(\Omega_0)) \text{ with } \|\mathbf{h}\|_{L^2_T(\mathbf{H}^2(\Omega_0))} \le C\|z\|_{L^2_T(H^{3/2}(B_{N-1}(\ell_0)))}.$$

The estimations (3.3.108), (3.3.112), (3.3.115), (3.3.118), (3.3.121), (3.3.124), (3.3.127), (3.3.130) prove that $F_1 \in \mathbf{R_w}$ for $z \in Z_T$ small.

(ii) Now, we prove that $F_2(z, \mathbf{w}) \in R_q = L^2_T(H^1(\Omega_0) \cap L^2_0(\Omega_0)) \cap H^1_T(H^1(\Omega_0)')$.

Note that taking into account (3.3.105) we have

$$\begin{aligned} F_2(z,\mathbf{w}) &= \operatorname{tr}([\nabla(\mathbf{w}+\mathbf{h})]\cdot[\nabla\mathbf{T}]^{-1})|\nabla\mathbf{T}| \\ &= \Big(\partial_1(w_1+h_1) - \partial_N(w_1+h_1)\frac{\partial_1\bar z}{1+\partial_N\bar z} + \ldots + \\ &\qquad \partial_{N-1}(w_{N-1}+h_{N-1}) - \partial_N(w_{N-1}+h_{N-1})\frac{\partial_{N-1}\bar z}{1+\partial_N\bar z} + \frac{\partial_N(w_N+h_N)}{1+\partial_N\bar z}\Big)(1+\partial_N\bar z) \\ &= \nabla\cdot(\mathbf{w}+\mathbf{h}) + \sum_{i=1}^{N-1}\Big(\partial_i(w_i+h_i)\partial_N\bar z - \partial_N(w_i+h_i)\partial_i\bar z\Big) \end{aligned} \quad (3.3.131)$$

Firstly we will show that $F_2(z, \mathbf{w}) \in L^2_T(H^1(\Omega_0))$. Again here we will consider only the terms with highest order derivative. We consider the general terms

$$f(z,\mathbf{w}) = \partial_i w_j \partial_k \bar z, \text{ with } a(z) = \partial_k \bar z, \quad b(\mathbf{w}) = \partial_i w_j. \quad (3.3.132)$$

We have the estimations

$$\begin{aligned} \|a(z)b(\mathbf{w})\|^2_{L^2_T(H^1(\Omega_0))} &\le \int_{-T}^{3T} \|a(z)b(\mathbf{w})\|^2_{L^2(\Omega_0)} + \|a(z)\nabla b(\mathbf{w})\|^2_{\mathbf{L}^2(\Omega_0)} + \|\nabla a(z) b(\mathbf{w})\|^2_{\mathbf{L}^2(\Omega_0)}\,\mathrm{d}t \\ &\le \|a(z)\|^2_{L^\infty_T(L^\infty(\Omega_0))}\|b(\mathbf{w})\|^2_{L^2_T(L^2(\Omega_0))} + \|a(z)\|^2_{L^\infty_T(L^\infty(\Omega_0))}\|\nabla b(\mathbf{w})\|^2_{L^2_T(\mathbf{L}^2(\Omega_0))} \\ &\quad + \|\nabla a(z)\|^2_{L^\infty_T(\mathbf{L}^{\hat 2^*}(\Omega_0))}\|b(\mathbf{w})\|^2_{L^2_T(L^{\hat 2}(\Omega_0))} \\ &\le C\|a(z)\|^2_{L^\infty_T(W^{1,\hat 2^*}(\Omega_0))}\|b(\mathbf{w})\|^2_{L^2_T(H^1(\Omega_0))} \quad (3.3.133) \\ &\le C\|\bar z\|^2_{L^\infty_T(W^{2,\hat 2^*}(\Omega_0))}\|\mathbf{w}\|^2_{L^2_T(\mathbf{H}^2(\Omega_0))} \\ &\le C\|z\|^2_{L^\infty_T(W^{2-\frac{1}{\hat 2^*},\hat 2^*}(B_{N-1}(\ell_0)))}\|\mathbf{w}\|^2_{L^2_T(\mathbf{H}^2(\Omega_0))}, \quad (3.3.134) \end{aligned}$$

where we used the embedding $W^{1,\hat 2^*}(\Omega_0) \hookrightarrow L^\infty(\Omega_0)$.

Similarly, for the other term in h_i we have

$$f(z) = \partial_i h_j(\bar z)\partial_k \bar z, \text{ with } a(z) = \partial_k \bar z, \quad b(z) = \partial_i h_j(\bar z), \quad (3.3.135)$$

with the estimations

$$\|a(z)b(z)\|^2_{L^2_T(H^1(\Omega_0))} \le \|a(z)\|^2_{L^\infty_T(L^\infty(\Omega_0))}\|b(z)\|^2_{L^2_T(L^2(\Omega_0))} + \|a(z)\|^2_{L^\infty_T(L^\infty(\Omega_0))}\|\nabla b(z)\|^2_{L^2_T(\mathbf{L}^2(\Omega_0))}$$

$$
\begin{aligned}
&+ \|\nabla a(z)\|^2_{L^\infty_T(\mathbf{L}^{\hat{2}^*}(\Omega_0))}\|b(z)\|^2_{L^2_T(L^{\hat{2}}(\Omega_0))} \\
&\leq C\|a(z)\|^2_{L^\infty_T(W^{1,\hat{2}^*}(\Omega_0))}\|b(z)\|^2_{L^2_T(H^1(\Omega_0))} \qquad (3.3.136)\\
&\leq C\|\bar{z}\|^2_{L^\infty_T(W^{2,\hat{2}^*}(\Omega_0))}\|\mathbf{h}\|^2_{L^2_T(\mathbf{H}^2(\Omega_0))} \\
&\leq C\|z\|^2_{L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(B_{N-1}(\ell_0)))}\|z\|^2_{H^{3/2}(B_{N-1}(\ell_0))}. \qquad (3.3.137)
\end{aligned}
$$

Estimations (3.3.134) and (3.3.137) prove that $F_2(z,\mathbf{w}) \in L^2_T(H^1(\Omega_0))$. Moreover, if we set $\mathbf{v} := \mathbf{w}\circ\mathbf{T}^{-1}$ and $\mathbf{g} := \mathbf{h}\circ\mathbf{T}^{-1}$ then by the regularity of $\mathbf{w}$ and $\mathbf{h}$, we have $\mathbf{v}\in\mathbf{V}_T$ and $\mathbf{g}\in L^2_T(\mathbf{H}^2(\Omega_t))\cap H^1_T(\mathbf{L}^2(\Omega_t))$. Also, using the fact that the function k_i and k_o are constructed to satisfy equation (3.1.10), and the fact that $\mathbf{v}$ vanishes on the boundary, we have

$$
\begin{aligned}
\int_{\Omega_0} F_2(z,\mathbf{w})\,\mathrm{d}x &= \int_{\Omega_t}\nabla\cdot(\mathbf{v}+\mathbf{g})\,\mathrm{d}x = \int_{\partial\Omega_t}(\mathbf{v}+\mathbf{g})\cdot\boldsymbol{\nu}_t\,\mathrm{d}\sigma = \int_{\partial\Omega_t}\mathbf{g}\cdot\boldsymbol{\nu}_t\,\mathrm{d}\sigma \\
&= k_i(t)\int_{\Gamma_i}\mathbf{u}_i\cdot\boldsymbol{\nu}_i\,\mathrm{d}\sigma + k_o(t)\int_{\Gamma_o}\mathbf{u}_o\cdot\boldsymbol{\nu}_o\,\mathrm{d}\sigma + \int_{M_t}(0,...,0,\partial_t z(x',t))\cdot\boldsymbol{\nu}_t\,\mathrm{d}\sigma \\
&= k_i(t)+k_o(t)+\int_{M_t}(0,...,0,\partial_t z(x',t))\cdot\boldsymbol{\nu}_t\,\mathrm{d}\sigma = 0. \qquad (3.3.138)
\end{aligned}
$$

This shows that $F_2(z,\mathbf{w})\in L^2_T(H^1(\Omega_0)\cap L^2_0(\Omega_0)) = Q_T$.

Finally, we need to show that $F_2(z,\mathbf{w})\in H^1_T(H^1(\Omega_0)')$. Since we already have $F_2(z,\mathbf{w})\in L^2_T(H^1(\Omega_0))\hookrightarrow L^2_T(H^1(\Omega_0)')$, it remains to show that $\partial_t F_2(z,\mathbf{w})\in L^2_T(H^1(\Omega_0)')$.

We note that we cannot proceed with the calculation straightforward as before because of the presence of the term $\partial_t(\mathbf{w}+\mathbf{h})$ and the fact that the maximal regularity of $\mathbf{w}$ in t is $H^1_T(\mathbf{L}^2(\Omega_0))$. So, a priori the term $\partial_t\nabla\mathbf{w}$ have no sense.

Note that $\partial_t F_2(z,\mathbf{w})\in L^2_T(H^1(\Omega_0)')$ is equivalent to show the existence of an element in $L^2_T(H^1(\Omega_0)')$, denoted $\partial_t F_2(z,\mathbf{w})$ such that

$$
\int_{-T}^{3T}\langle\partial_t F_2(z,\mathbf{w}),\varphi\rangle\,\xi\,\mathrm{d}t = -\int_{-T}^{3T}\langle F_2(z,\mathbf{w}),\varphi\rangle\,\xi'\,\mathrm{d}t,\ \forall\,\varphi\in H^1(\Omega_0),\ \forall\,\xi\in\mathcal{D}(-T,3T). \qquad (3.3.139)
$$

We have

$$
\begin{aligned}
\int_{-T}^{3T}\langle F_2(z,\mathbf{w}),\varphi\rangle\,\xi'\,\mathrm{d}t = \int_{-T}^{3T}\Big(&-\sum_{\alpha\in\{i,o\}}\int_{\Gamma_\alpha}k'_\alpha(t)\mathbf{u}^N_\alpha(x)\varphi\,\mathrm{d}x + \int_{B_{N-1}(\ell_0)}\partial_{tt}z(x',t)\varphi(x',0)\,\mathrm{d}x' \\
&+\int_{\Omega_0}\partial_t(\mathbf{w}+\mathbf{h})\cdot\nabla\varphi\,\mathrm{d}x \\
&+\int_{\Omega_0}\sum_{i=1}^{N-1}\partial_t(w_i+h_i)(\partial_N\bar{z}\partial_i\varphi-\partial_i\bar{z}\partial_N\varphi)\,\mathrm{d}x
\end{aligned}
$$

$$+\int_{\Omega_0}\sum_{i=1}^{N-1}(w_i+h_i)(\partial_{t,N}\bar{z}\partial_i\varphi-\partial_{t,i}\bar{z}\partial_N\varphi)\,\mathrm{d}x\Big)\xi\,\mathrm{d}t$$
$$=:\int_{-T}^{3T}\langle\partial_t F_2(z,\mathbf{w}),\varphi\rangle\,\xi\,\mathrm{d}t. \tag{3.3.140}$$

In the calculations above, we used the facts that $\mathbf{w}\in\mathbf{H}_0^1(\Omega_0)$ and that the second order space derivatives of $\bar{z}$ cancel.

It remains to show that $\partial_t F_2(z,\mathbf{w})\in L_T^2(H^1(\Omega_0)')$. We will show that each of its term as defined in (3.3.140) belong to $L_T^2(H^1(\Omega_0)')$.

For the term defined on the boundary $\Gamma_\alpha,\ \alpha\in\{i,o\}$ we have

$$f(z,\mathbf{w})=k'_\alpha(t)\mathbf{u}_\alpha^N(x),\quad a(z)=k'_\alpha(t),\quad b(\mathbf{w})=\mathbf{u}_\alpha^N(x), \tag{3.3.141}$$

with the estimations,

$$\|a(z)b(\mathbf{w})\|^2_{L_T^2(H^1(\Omega_0)')}=\int_{-T}^{3T}\Big(\sup_{\|\varphi\|_{H^1(\Omega_0)}\le 1}\int_{\Gamma_\alpha}a(z)b(\mathbf{w})\varphi\,\mathrm{d}x\Big)^2\,\mathrm{d}t$$
$$\le C\|a(z)\|^2_{L_T^2(\mathbb{R})}\|b(\mathbf{w})\|^2_{L_T^\infty(L^\infty(\Gamma_\alpha))}$$
$$=:C\|a(z)\|^2_{R_a}\|b(\mathbf{w})\|^2_{R_b} \tag{3.3.142}$$
$$\le C\int_{-T}^{3T}\Big|\int_{B_{N-1}(\ell_0)}\partial_{tt}z(x',t)\,\mathrm{d}x'\Big|^2\,\mathrm{d}t\le C\|z\|^2_{L_T^2(H^2(B_{N-1}(\ell_0)))}, \tag{3.3.143}$$
$$\tag{3.3.144}$$

using (3.3.95) and (3.3.97)).

For the term with the integral in $B_{N-1}(\ell_0)$ we have

$$f(z,\mathbf{w})=\partial_{tt}z(x',t),\quad b(\mathbf{w})=1, \tag{3.3.145}$$

with the estimations

$$\|a(z)b(\mathbf{w})\|^2_{L_T^2(H^1(\Omega_0)')}\le C\|a(z)\|^2_{L_T^2(L^2(B_{N-1}(\ell_0)))}\|b(\mathbf{w})\|^2_{L_T^\infty(L^\infty(B_{N-1}(\ell_0)))} \tag{3.3.146}$$
$$\le C\|z\|^2_{L_T^2(H^2(B_{N-1}(\ell_0)))}. \tag{3.3.147}$$

Now, we will estimate the terms with integral in Ω_0, and we will see only the terms of highest order as follows. We have

$$f(z,\mathbf{w})=\partial_t w_i\partial_j\bar{z},\quad a(z)=\partial_j\bar{z},\quad b(\mathbf{w})=\partial_t w_i, \tag{3.3.148}$$

with the estimations

$$\|a(z)b(\mathbf{w})\|^2_{L_T^2(H^1(\Omega_0)')}:=\int_{-T}^{3T}\Big(\sup_{\|\varphi\|_{H^1}\le 1}\int_{\Omega_0}\partial_t w_i\partial_j\bar{z}\partial_k\varphi\,\mathrm{d}x\Big)^2\,\mathrm{d}t$$

$$\begin{aligned}
&\le \|a(z)\|^2_{L^\infty_T(L^\infty(\Omega_0))}\|b(\mathbf{w})\|^2_{L^2_T(L^2(\Omega_0))} \\
&=: \|a(z)\|^2_{R_a}\|b(\mathbf{w})\|^2_{R_b} && (3.3.149)\\
&\le C\|z\|^2_{L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(\Omega_0))}\|\mathbf{w}\|^2_{H^1_T(\mathbf{L}^2(\Omega_0))}. && (3.3.150)
\end{aligned}$$

Similarly, for the terms of the same type with w_i replace by h_i we have

$$a(z) = \partial_t h_i(\bar{z})\partial_j\bar{z}, \quad a_1(z) = \partial_j\bar{z}, \quad a_2(z) = \partial_t h_i(\bar{z}), \tag{3.3.151}$$

with the estimations

$$\begin{aligned}
\|a_1(z)a_2(z)\|^2_{L^2_T(H^1(\Omega_0)')} &\le \|a_1(z)\|^2_{L^\infty_T(L^\infty(\Omega_0))}\|a_2(z)\|^2_{L^2_T(L^2(\Omega_0))} \\
&=: \|a_1(z)\|^2_{R^1_a}\|a_2\|^2_{R^2_a} && (3.3.152)\\
&\le C\|z\|^2_{L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(\Omega_0))}\|\mathbf{h}\|^2_{H^1_T(\mathbf{L}^2(\Omega_0))} \\
&\le C\|z\|^2_{L^\infty_T(W^{2-\frac{1}{\hat{2}^*},\hat{2}^*}(\Omega_0))}\|z\|^2_{L^2_T(L^2(B_{N-1}(\ell_0)))}. && (3.3.153)
\end{aligned}$$

For the terms involving $w_i\partial_{t,j}\bar{z}$ we have

$$f(z,\mathbf{w}) = w_i\partial_{t,j}\bar{z}, \quad a(z) = \partial_{t,j}\bar{z}, \quad b(\mathbf{w}) = w_i, \tag{3.3.154}$$

with the estimations

$$\begin{aligned}
\|a(z)b(\mathbf{w})\|^2_{L^2_T(H^1(\Omega_0)')} &:= \int_{-T}^{3T}\Big(\sup_{\|\varphi\|_{H^1}\le 1}\int_{\Omega_0} w_i\partial_{t,j}\bar{z}\partial_k\varphi\,\mathrm{d}x\Big)^2\,\mathrm{d}t \\
&\le \|a(z)\|^2_{L^2_T(L^{\hat{2}^*}(\Omega_0))}\|b(\mathbf{w})\|^2_{L^\infty_T(L^{\hat{2}}(\Omega_0))} \\
&\le C\|a(z)\|^2_{L^2_T(H^1(\Omega_0))}\|b(\mathbf{w})\|^2_{L^\infty_T(H^1(\Omega_0))} && (3.3.155)\\
&\le C\|\bar{z}\|^2_{H^1_T(H^2(\Omega_0))}\|\mathbf{w}\|^2_{L^\infty_T(\mathbf{H}^1(\Omega_0))} \\
&\le C\|z\|^2_{H^1_T(H^{3/2}(B_{N-1}(\ell_0)))}\|\mathbf{w}\|^2_{L^2_T(\mathbf{H}^2(\Omega_0))\cap H^1_T(\mathbf{L}^2(\Omega_0))}, && (3.3.156)
\end{aligned}$$

since $H^1(\Omega_0) \hookrightarrow L^{\hat{2}^*}(\Omega_0)$ and $H^1(\Omega_0) \hookrightarrow L^{\hat{2}}(\Omega_0))$.

For the similar terms with $h_j\partial_{t,j}\bar{z}$ we have

$$f(z,\mathbf{w}) = h_i(\bar{z})\partial_{t,j}\bar{z}, \quad a_1(z) = \partial_{t,j}\bar{z}, \quad a_2(z) = h_i, \tag{3.3.157}$$

with the estimations

$$\begin{aligned}
\|a_1(z)a_2(z)\|^2_{L^2_T(H^1(\Omega_0)')} &\le \|a_1(z)\|^2_{L^2_T(L^{\hat{2}^*}(\Omega_0))}\|a_2(z)\|^2_{L^\infty_T(L^{\hat{2}}(\Omega_0))} \\
&\le C\|a_1(z)\|^2_{L^2_T(H^1(\Omega_0))}\|\mathbf{h}\|^2_{L^\infty_T(H^1(\Omega_0))} && (3.3.158)\\
&\|z\|^2_{H^1_T(H^{3/2}(B_{N-1}(\ell_0)))}\|\mathbf{h}\|^2_{L^2_T(\mathbf{H}^2(\Omega_0))\cap H^1_T(\mathbf{L}^2(\Omega_0))} \\
&\le C\|z\|^2_{H^1_T(H^{3/2}(B_{N-1}(\ell_0)))}\|z\|^2_{H^1_T(H^{3/2}(B_{N-1}(\ell_0)))\cap H^2_T(L^2(B_{N-1}(\ell_0)))}. && (3.3.159)
\end{aligned}$$

The estimations (3.3.134), (3.3.137), (3.3.143), (3.3.147), (3.3.150), (3.3.153), (3.3.156), (3.3.159) show that $\partial_t F_2(z, \mathbf{w}) \in L^2_T(H^1(\Omega_0)')$ which completes the proof. ∎

3.4 Analysis of the Navier-Stokes equation

In this section, we analyze the time-periodic NS equations (3.3.23a) − (3.3.23b). We will show the existence of a strong time-periodic solution in the space $\mathbf{W}_T \times Q_T$. We will show that $\mathbf{F}$ is Fréchet C^1 near a certain point $((z_0, \mathbf{w}_0), q_0)$ which is such that $\mathbf{F}((z_0, \mathbf{w}_0), q_0) = (\mathbf{0}, 0)$, and that its differential with respect to $(z, \mathbf{w})$ at $((z_0, \mathbf{w}_0), q_0)$ is an isomorphism.

3.4.1 Existence of a local strong time-periodic solution

The aim is to solve the problem (3.3.23a) − (3.3.23b). We will use the implicit function theorem, applied to the following map

$$\begin{aligned} \mathbf{F} : Z_T \times \mathbf{W}_T \times Q_T &\longrightarrow \mathbf{R}_{\mathbf{w}} \times R_q, \\ (z, \mathbf{w}, q) &\longmapsto (F_1(z, \mathbf{w}, q), F_2(z, \mathbf{w})), \end{aligned} \tag{3.4.1}$$

where $F_1(z, \mathbf{w}, q)$ resp. $F_2(z, \mathbf{w})$ is given by (3.3.26) resp. (3.3.27). Because of the Lemma 3.3.5, the map $\mathbf{F}$ is well defined.

To begin with, we choose to implement the implicit function theorem on the function $\mathbf{F}$ near the point $(z_0, \mathbf{w}_0, q_0) = (0, \mathbf{0}, q_{\vec{g}})$ where $q_{\vec{g}}$ is such that $\mathbf{F}(z_0, \mathbf{w}_0, q_{\vec{g}}) = (0, 0)$. Indeed, if $z_0 = 0$ then we have $\mathbf{T} = \mathbf{I}$ and $\mathbf{h} = \mathbf{0}$. Moreover, if $\mathbf{w}_0 = \mathbf{0}$ then $F_2(z_0, \mathbf{w}_0) = 0$ and $F_1(z_0, \mathbf{w}_0, q_0) = \nabla q_0 - \rho \vec{g}$. Since $\vec{g} = (0, ..., 0, g_N)$, we can take

$$q_{\vec{g}} = q_{\vec{g}}(x) = \rho \vec{g} \cdot x - \frac{\rho}{|\Omega_0|} \int_{\Omega_0} \vec{g} \cdot x \, \mathrm{d}x, \ \forall \, x \in \Omega_0. \tag{3.4.2}$$

We will show that $\mathbf{F}$ is C^1 near $(z_0, (\mathbf{w}_0, q_0))$ and that $D_{(\mathbf{w},q)}\mathbf{F}(z_0, (\mathbf{w}_0, q_0))$ is a topological isomorphism from $\mathbf{W}_T \times Q_T$ onto $\mathbf{R}_{\mathbf{w}} \times R_q$, depending on the Stokes problem. More precisely, we have the following result.

Proposition 3.4.1 (Differentiability).

The map $\mathbf{F}$ defined in (3.4.1) is C^1 in a neighborhood of $(0, (\mathbf{0}, q_0))$. Moreover, its Fréchet differential with respect to $(\mathbf{w}, q)$ at $(0, (\mathbf{0}, q_0))$ reads

$$\begin{aligned} D_{(\mathbf{w},q)}\mathbf{F}(0, (\mathbf{0}, q_0)) : \mathbf{W}_T \times Q_T &\longrightarrow \mathbf{R}_{\mathbf{w}} \times R_q \\ (\mathbf{w}, q) &\longmapsto (D_{(\mathbf{w},q)}F_1(0, (\mathbf{0}, q_0))(\mathbf{w}, q), D_{\mathbf{w}}F_2(0, \mathbf{0})(\mathbf{w})), \end{aligned} \tag{3.4.3}$$

with

$$\begin{cases} D_{(\mathbf{w},q)}F_1(0, (\mathbf{0}, q_0))(\mathbf{w}, q) = \rho \partial_t \mathbf{w} - \mu \Delta \mathbf{w} + \nabla q, & (3.4.4) \\ D_{\mathbf{w}}F_2(0, \mathbf{0})(\mathbf{w}) = \nabla \cdot \mathbf{w}. & (3.4.5) \end{cases}$$

Proof: The proof relies on Proposition 3.2.1 and Proposition 3.2.2 and the estimations of Lemma 3.3.5, precisely (3.3.108), (3.3.111), (3.3.112), (3.3.115), (3.3.118), (3.3.121), (3.3.124), (3.3.127), (3.3.130) for $F_1(z, \mathbf{w}, q)$ and (3.3.134), (3.3.137), (3.3.143), (3.3.147), (3.3.150), (3.3.153), (3.3.156), (3.3.159) for $F_2(z, \mathbf{w})$.

Indeed, based on Remark 3.3.6, each term $f(z, \mathbf{w}) = a(z)b(\mathbf{w})$ will be in $C^1(Z_T \times \mathbf{W}_T; R_f)$ if $a_i \in C^1(Z_T; R_a^i)$ and $b_i \in C^1(Z_T; R_b^i)$ under the assumptions (3.2.11), (3.2.1) of Proposition 3.2.2 and Proposition 3.2.1.

Moreover, all these factors, except terms of the form $a(z) = \frac{1}{(1+\partial_N \bar{z})^p}$, p positive integer are linear and continuous, so they are C^1. The factor $a(z) = \frac{1}{(1+\partial_N \bar{z})^p}$ is also $C^1(Z_T; R_a)$ with $R_a = L_T^\infty(W^{1,\hat{2}^*}(\Omega_0))$. The proof of this result can be carried out by straightforward calculations. A more elegant proof is by considering the map

$$\begin{aligned} m : Z_T \times L_T^\infty(W^{1,\hat{2}^*}(\Omega_0)) &\longrightarrow L_T^\infty(W^{1,\hat{2}^*}(\Omega_0)) \\ (z, w) &\mapsto (1 + \partial_N \bar{z})^p w - 1, \end{aligned} \tag{3.4.6}$$

where p is a positive integer and apply the implicit function theorem at the point $(z, w) = (0, 1)$.

Indeed, based on Proposition 3.2.2 and Proposition 3.2.1, the map m is C^1 and we have $m(0, 1) = 0$. Moreover, the (Fréchet) differential of m at $(0, 1)$ reads

$$D_w m(0, 1) : \tilde{w} \in L_T^\infty(W^{1,\hat{2}}(\Omega_0)) \mapsto \tilde{w} \in L_T^\infty(W^{1,\hat{2}^*}(\Omega_0)) \tag{3.4.7}$$

is a diffeomorphism. Therefore, by the implicit function theorem, there exists a unique C^1 function φ defined on a neighborhood U_0 of 0 such that

$$(1 + \partial_N \bar{z})^p \varphi(z) = 1, \qquad \text{i.e.} \qquad \varphi(z) = \frac{1}{(1 + \partial_N \bar{z})^p}, \ \forall \, z \in U_0, \tag{3.4.8}$$

which show that the map $z \mapsto \frac{1}{(1+\partial_N \bar{z})^p}$ is C^1 for z small.

Finally, we obtain (3.4.4) and (3.4.5) by differentiating $\mathbf{F}$ with respect to $(\mathbf{w}, q)$ at $(0, (\mathbf{0}, q_0))$. Precisely, if we denote by G_i, $i = 1, ..., 4$, the function defined by the i^{th} line of (3.3.26), then we have

$$\begin{cases} D_{\mathbf{w}} F_2(0, \mathbf{0})(\mathbf{w}) = \nabla \cdot \mathbf{w}, \forall \, \mathbf{w} \in \mathbf{W}_T, & (3.4.9) \\ D_{\mathbf{w}} G_1(0, \mathbf{0})(\mathbf{w}) = \rho \, \partial_t \mathbf{w}, \ \forall \, \mathbf{w} \in \mathbf{W}_T, & (3.4.10) \\ D_{\mathbf{w}} G_2(0, \mathbf{0})(\mathbf{w}) = 0, \ \forall \, \mathbf{w} \in \mathbf{W}_T, & (3.4.11) \\ D_{\mathbf{w}} G_3(0, \mathbf{0})(\mathbf{w}) = -\mu \Delta \mathbf{w}, \ \forall \, \mathbf{w} \in \mathbf{W}_T, & (3.4.12) \\ D_q G_4(0, 0)(q) \quad = \nabla q, \ \forall \, q \in Q_T. & (3.4.13) \end{cases}$$

From (3.4.10) − (3.4.13), we can deduce that

$$D_{(\mathbf{w},q)} F_1(0, (\mathbf{0}, 0))(\mathbf{w}, q) = \rho \partial_t \mathbf{w} - \mu \Delta \mathbf{w} + \nabla q. \tag{3.4.14}$$

∎

We need the following result in order to apply the implicit function theorem on the map $\mathbf{F}$.

Theorem 3.4.2 (Existence and uniqueness of the associated Stokes Problem)**.**

The map $(\mathbf{w}, q) \mapsto D_{(\mathbf{w},q)}\mathbf{F}(0,(\mathbf{0},q_0))(\mathbf{w},q)$ *defined by* (3.4.3) *is a topological isomorphism.*

Before we continue with the proof of this theorem which starts at page 67, we will present a few auxiliary results.

Note that the proof of Theorem 3.4.2 is based on classical results on homogeneous time-periodic Stokes equations for C^2 domains (see [3, 4, 23, 22]). As our domain is not C^2 and the implicit function theorem requires initially non homogeneous Stokes equations, we will need the following lemma.

Lemma 3.4.3 (Divergence)**.**

Assume the hypotheses (3.1.2)-(3.1.6) *on* Ω_0 *hold. If* $g \in R_q$, *(see* (3.1.23)*), then there exists* $\mathbf{w}_g \in \mathbf{W}_T$ *(see* (3.1.19)*) solving*

$$\nabla \cdot \mathbf{w}_g = g \quad in \ \ \Omega_0 \times (-T, 3T), \tag{3.4.15}$$

such that,

$$\|\mathbf{w}_g\|_{L^2_T(\mathbf{H}^2(\Omega_0))} \leq C\|g\|_{L^2_T(H^1(\Omega_0))}, \tag{3.4.16}$$

and

$$\|\mathbf{w}_g\|_{H^1_T(\mathbf{L}^2(\Omega_0))} \leq C\|g\|_{H^1_T(H^1(\Omega_0)')}, \tag{3.4.17}$$

with $C = C(T) > 0$.

The proof of this lemma starts at page 64.

Theorem 3.4.4 (Existence of a very weak solution for Stokes)**.**

Let Ω_0 *be bounded domain with* $C^{1,1}$ *boundary. Let* $\mathbf{f} \in \mathbf{Y}_w^{-2,q}(\Omega_0)$ *and* $g \in W_{w,0}^{-1,q}(\Omega_0)$ *such that* $\langle g, 1\rangle = 0$. *Then, there exists a unique very weak solution* $\mathbf{u} \in \mathbf{L}^2_w(\Omega_0)$ *to the Stokes problem*

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f} \ \ in \ \Omega_0, \ \ \nabla\cdot\mathbf{u} = g \ \ in \ \Omega_0, \tag{3.4.18}$$

in the sense that $\mathbf{u}$ *uniquely satisfies*

$$-\langle \mathbf{u}, \Delta\boldsymbol{\phi}\rangle = \langle \mathbf{f}, \boldsymbol{\phi}\rangle, \ \forall\, \boldsymbol{\phi} \in \mathbf{Y}^{2,q'}_{w'}(\Omega_0), \tag{3.4.19}$$

$$\langle \mathbf{u}, \nabla\phi\rangle = \langle g, \psi\rangle, \ \forall\, \psi \in W^{1,q'}_{w'}(\Omega_0), \tag{3.4.20}$$

where $\langle\cdot,\cdot\rangle$ *represents here the duality pairing. Furthermore, we have the estimation*

$$\|\mathbf{u}\|_{\mathbf{L}^q_w(\Omega_0} \leq C\Big(\|\mathbf{f}\|_{\mathbf{Y}_w^{-2,q}(\Omega_0)} + \|g\|_{W^{-1,q}_{w,0}(\Omega_0)}\Big). \tag{3.4.21}$$

Proof: Schumacher [87, Theorem 3.1, pp 130-131]. ∎

Remark 3.4.5.

If we take $q = 2$, $w = 1$, *then* $g \in (H^1(\Omega_0))'$ *with* $\langle g, 1\rangle = 0$ *and* $\mathbf{f} \in \mathbf{H}^{-2}(\Omega_0) := (\mathbf{H}_0^2(\Omega_0))'$. *Moreover, if we take* $\mathbf{f} = \mathbf{0}$, *then according Theorem 3.4.4, the divergence problem* $\nabla \cdot \mathbf{u} = g$ *has a very weak solution* $\mathbf{u} \in \mathbf{L}^2(\Omega_0)$ *with the estimation*

$$\|\mathbf{u}\|_{\mathbf{L}^2(\Omega_0)} \leq C\|g\|_{H^1(\Omega_0)'}. \tag{3.4.22}$$

Proof: (**Proof of Lemma** 3.4.3)

We first suppose that the whole boundary $\partial\Omega_0$ is $C^{1,1}$. This is essential to obtain $\mathbf{H}^2 \times H^1$ regularity for the Stokes problem. Let us consider the Stokes problem

$$\begin{cases} -\Delta\mathbf{w} + \nabla q = 0, \quad \text{in } \Omega_0 \times (-T, 3T), & (3.4.23)\\ \nabla \cdot \mathbf{w} = g, \quad \text{in } \Omega_0 \times (-T, 3T), & (3.4.24)\\ \mathbf{w} = \mathbf{0}, \quad \text{on } \partial\Omega_0 \times (-T, 3T). & (3.4.25)\end{cases}$$

Since $g \in L^2_T(H^1(\Omega_0) \cap L^2_0(\Omega_0))$ then *a.e.*, $t \in (-T, 3T)$, $g(\cdot, t) \in H^1(\Omega_0) \cap L^2_0(\Omega_0)$. According to Da Veiga [23], Solonnikov [92], Da Veiga [22], Amrouche and Girault [4, 3], the problem (3.4.23) − (3.4.25) has a unique solution $(\mathbf{w}(\cdot, t), q(\cdot, t)) \in \mathbf{H}^2(\Omega_0) \cap \mathbf{H}^1_0(\Omega_0) \times H^1(\Omega_0) \cap L^2_0(\Omega_0)$. Moreover, for *a.e* $t \in (-T, 3T)$, we have the estimation

$$\|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2(\Omega_0)} + \|q(\cdot, t)\|_{H^1(\Omega_0)} \leq C\|g(\cdot, t)\|_{H^1(\Omega_0)}. \tag{3.4.26}$$

Estimation (3.4.26) implies

$$\|\mathbf{w}(\cdot, t)\|_{\mathbf{H}^2(\Omega_0)} \leq C\|g(\cdot, t)\|_{H^1(\Omega_0)}. \tag{3.4.27}$$

Then by raising to the square (3.4.27) and integration over $(-T, 3T)$, we obtain $\mathbf{w} \in L^2(\mathbf{H}^2(\Omega_0) \cap \mathbf{H}^1_0(\Omega_0))$, and the estimation

$$\|\mathbf{w}\|_{L^2(\mathbf{H}^2(\Omega_0))} \leq C\|g\|_{L^2(H^1(\Omega_0))}. \tag{3.4.28}$$

Now, let us show the periodicity. It comes from the periodicity of g and the estimation (3.4.26). Indeed, set $\tilde{\mathbf{w}} := \mathbf{w}(\cdot, -T) - \mathbf{w}(\cdot, 3T)$ and $\tilde{q} = q(\cdot, -T) - q(\cdot, 3T)$. Then, $(\tilde{\mathbf{w}}, \tilde{q}) \in \mathbf{H}^2(\Omega_0) \cap \mathbf{H}^1_0(\Omega_0) \times H^1(\Omega_0) \cap L^2_0(\Omega_0)$ is the unique solution to the Stokes problem

$$\begin{cases} -\Delta\tilde{\mathbf{w}} + \nabla\tilde{q} = 0, \text{ in } \Omega_0, & (3.4.29)\\ \nabla \cdot \tilde{\mathbf{w}} = g(\cdot, -T) - g(\cdot, 3T) = 0, \text{ in } \Omega_0, & (3.4.30)\\ \tilde{\mathbf{w}} = \mathbf{0}, \text{ on } \partial\Omega_0, & (3.4.31)\end{cases}$$

and from (3.4.26) we have

$$\|\tilde{\mathbf{w}}\|_{\mathbf{H}^2(\Omega_0)} + \|\tilde{q}\|_{H^1(\Omega_0)} \leq 0 \implies (\tilde{\mathbf{w}}, \tilde{q}) = (\mathbf{0}, 0) \tag{3.4.32}$$

which shows the periodicity of $(\mathbf{w}, q)$.

Now, let us suppose that Ω_0 is as in our case, *i.e.* satisfying the assumptions (3.1.2)-(3.1.6). From classical regularity results of elliptic PDEs systems, we know that for *a.e.*

$t \in (-T, 3T)$, we have the existence result $\mathbf{w}(\cdot, t)$, resp. $p(\cdot, t)$, in $\mathbf{H}^2$, resp. H^1, in Ω_0 except near the corners. We will extend this solution to have the desired regularity even near the corners of Γ_i and Γ_o.

Let us consider the case of Γ_i, the case of Γ_o is similar. Without loss of generality, we take a local coordinate system on Γ_i such that $\Gamma_i = B_{N-1}(0, r_i) \times \{0\}$ and $\Omega_0 \subset \mathbb{R}^N_- := \{x \in \mathbb{R}^N, \quad x_N < 0\}$. Let $\hat{\Omega}_0 = \Omega_0 \cup \Gamma_i \cup \{x := (x', x_N) \in \mathbb{R}^N, \quad (x', -x_N) \in \Omega_0\}$. Note that the new domain $\hat{\Omega}_0$ is $C^{1,1}$ except near Γ_o and symmetric with respect to $x_N = 0$. We can extend $\mathbf{w}$, q and g in $\hat{\Omega}_0$ as follows. Let us denote by $\hat{\mathbf{w}} = (\hat{w}_1, ..., \hat{w}_N)$, $\hat{q}$ and $\hat{\mathbf{g}}$ the extension of $\mathbf{w} = (w_1, .., w_N)$, q and g respectively. We set

$$\hat{w}_N(x', x_N, t) = \begin{cases} w_N(x', x_N, t) & \text{if } x_N < 0, \\ -w_N(x', -x_N, t) & \text{if } x_N > 0, \end{cases}$$

and for all $i = 1, ..., N-1$ we set

$$\hat{w}_i(x', x_N, t) = \begin{cases} w_i(x', x_N, t) & \text{if } x_N < 0, \\ w_i(x', -x_N, t) & \text{if } x_N > 0, \end{cases}$$

and

$$\hat{q}(x', x_N, t) = \begin{cases} q(x', x_N, t) \text{ if } x_N < 0, \\ q(x', -x_N, t) \text{ if } x_N > 0, \end{cases}$$

$$\hat{g}(x', x_N, t) = \begin{cases} g(x', x_N, t) \text{ if } x_N < 0, \\ g(x', -x_N, t) \text{ if } x_N > 0. \end{cases}$$

Note that $\hat{g} \in L^2_T(H^1(\hat{\Omega}_0) \cap L^2_0(\hat{\Omega}_0))$ and $(\hat{\mathbf{w}}, \hat{q}) \in L^2_T(\mathbf{H}^1_0(\hat{\Omega}_0)) \times L^2_T(L^2_0(\hat{\Omega}_0))$ and $(\hat{\mathbf{w}}, \hat{q})$ solves weakly in $\hat{\Omega}_0$ the Stokes problem

$$\begin{cases} -\Delta \hat{\mathbf{w}} + \nabla \hat{q} = 0 \text{ in } \hat{\Omega}_0 \times (-T, 3T), & (3.4.33) \\ \nabla \cdot \hat{\mathbf{w}} = \hat{g} \text{ in } \hat{\Omega}_0 \times (-T, 3T), & (3.4.34) \\ \hat{\mathbf{w}} \quad = \mathbf{0} \text{ on } \partial\hat{\Omega}_0 \times (-T, 3T). & (3.4.35) \end{cases}$$

Indeed, in $\hat{\Omega}_0 \cap \{x_N < 0\} = \Omega_0$ we have $\hat{g} = g \in L^2_T(H^1(\Omega_0) \cap L^2_0(\Omega_0))$, $(\hat{\mathbf{w}}, \hat{q}) = (\mathbf{w}, q) \in L^2_T(\mathbf{H}^1_0(\Omega_0)) \times L^2_T(L^2(\Omega_0))$ and the problem is already solved in Ω_0.

Now, in $\hat{\Omega}_0 \cap \{x_N > 0\} \times (-T, 3T)$ we have

$$-\Delta \hat{w}_N(x, t) + \partial_{x_N} \hat{q}(x, t) = \Delta w_1(x', -x_N, t) - \partial_{x_N} q(x', -x_N, t) = 0,$$

and for all $i = 1, ..., N-1$

$$-\Delta \hat{w}_i(x, t) + \partial_{x_i} \hat{q}(x, t) = -\Delta w_i(x', -x_N, t) + \partial_{x_i} q(x', -x_N, t) = 0,$$

and

$$\partial_{x_1} \hat{w}_1(x, t) + ... + \partial_{x_N} \hat{w}_N(x, t) = \partial_{x_1} w_1(x', -x_{x_N}, t) + ... + \partial_{x_N} w_N(x', -x_N, t)$$

$$= g(x', -x_N, t).$$

Classical local regularity results for the Stokes system, see [4, 3, 23, 22], imply that the problem (3.4.33) – (3.4.35) has a unique solution $(\hat{\mathbf{w}}, \hat{q}) \in L^2_T(\mathbf{H}^2(\hat{\Omega}_0) \cap \mathbf{H}^1_0(\hat{\Omega}_0)) \times L^2_T(H^1(\hat{\Omega}_0) \cap L^2_0(\hat{\Omega}_0))$ except in a neighborhood of Γ_o, and the estimation (3.4.26) hold for $\hat{\Omega}_0$. Repeating the same argument on Γ_o permits to obtain the same regularity even near Γ_o.

Now, we assume $g \in H^1(H^1(\Omega_0)')$ then for $a.e,\ t \in (-T, 3T)$, $g(\cdot, t) \in H^1(\Omega_0)'$ and $\partial_t g(\cdot, t) \in H^1(\Omega_0)'$. Therefore, according to [87], $\mathbf{w}(\cdot, t) \in \mathbf{L}^2(\Omega_0)$ is the unique very weak solution to (3.4.23) – (3.4.25) with the estimation

$$\|\mathbf{w}(\cdot, t)\|_{\mathbf{L}^2(\Omega_0)} \leq C\|g(\cdot, t)\|_{H^1(\Omega_0)'}. \tag{3.4.36}$$

By raising to the square (3.4.36) and integrating it over $(-T, 3T)$ we get

$$\|\mathbf{w}\|_{L^2(\mathbf{L}^2(\Omega_0))} \leq C\|g\|_{L^2(H^1(\Omega_0)')}. \tag{3.4.37}$$

Now, let us consider the problem

$$\begin{cases} -\Delta(\partial_t \mathbf{w}) + \nabla(\partial_t q) = 0, \ \text{ in } \Omega_0 \times (-T, 3T) & (3.4.38) \\ \nabla \cdot (\partial_t \mathbf{w}) = \partial_t g, \ \text{ in } \Omega_0 \times (-T, 3T) & (3.4.39) \\ \partial_t \mathbf{w} = \mathbf{0} \ \text{ on } \partial\Omega_0 \times (-T, 3T). & (3.4.40) \end{cases}$$

Using again [87], the problem (3.4.38) – (3.4.40) has a unique very weak solution $\partial_t \mathbf{w} \in \mathbf{L}^2(\Omega_0)$ such that

$$\|\partial_t \mathbf{w}\|_{L^2(\mathbf{L}^2(\Omega_0))} \leq C\|\partial_t g\|_{L^2(H^1(\Omega_0)')}. \tag{3.4.41}$$

Now, we show that $\partial_t \mathbf{w}$ is actually the time derivative of $\mathbf{w}$. For all $h > 0$, we set

$$\begin{aligned} \delta_{\mathbf{w}} &:= \frac{\mathbf{w}(\cdot, t+h) - \mathbf{w}(\cdot, t)}{h} - \partial_t \mathbf{w}(\cdot, t), \\ \delta_q &:= \frac{q(\cdot, t+h) - q(\cdot, t)}{h} - \partial_t q(\cdot, t), \\ \delta_g &:= \frac{g(\cdot, t+h) - g(\cdot, t)}{h} - \partial_t g(\cdot, t). \end{aligned}$$

Then we have

$$\begin{cases} -\Delta\delta_{\mathbf{w}} + \nabla\delta_q = 0, \ \text{ in } \Omega_0 \times (-T, 3T), & (3.4.42) \\ \nabla \cdot \delta_{\mathbf{w}} = \delta_g, \ \text{ in } \Omega_0 \times (-T, 3T), & (3.4.43) \\ \delta_{\mathbf{w}} \ = \mathbf{0} \ \text{ on } \partial\Omega_0 \times (-T, 3T). & (3.4.44) \end{cases}$$

Therefore, since $\delta_g \in L^2_T(H^1(\Omega_0)')$, then $\delta_{\mathbf{w}}$ is the unique very weak solution to the problem (3.4.42) – (3.4.44), and as in (3.4.37), we have

$$\|\delta_{\mathbf{w}}\|_{L^2(\mathbf{L}^2(\Omega_0))} \leq C\|\delta_g\|_{L^2(H^1(\Omega_0)')} \underset{h \to 0^+}{\longrightarrow} 0, \tag{3.4.45}$$

which show that $\partial_t \mathbf{w}$ is the time derivative of $\mathbf{w}$ and $\mathbf{w} \in H^1_T(\mathbf{L}^2(\Omega_0))$ with the estimation

$$\begin{aligned}
\|\mathbf{w}\|_{H^1(\mathbf{L}^2(\Omega_0))} &= \|\mathbf{w}\|_{L^2(\mathbf{L}^2(\Omega_0))} + \|\partial_t \mathbf{w}\|_{L^2(\mathbf{L}^2(\Omega_0))} \\
&\leq C(\|g\|_{L^2(H^1(\Omega_0)')} + \|\partial_t g\|_{L^2(H^1(\Omega_0)')}) \\
&= C\|g\|_{H^1(H^1(\Omega_0)')},
\end{aligned}$$

which ends the proof. ∎

Now, we can start the proof of Theorem 3.4.2.

Proof: (**Proof of Thereom** 3.4.2)
Theorem 3.4.2, is basically a result of existence and uniqueness of the time-periodic non-homogeneous Stokes problem given by equations (3.4.4) − (3.4.5). This will be done in three major steps. Firstly, we prove the injectivity of the differential operator $D_{(\mathbf{w},q)}\mathbf{F}(0,(\mathbf{0},q_0))$. This is equivalent to prove the uniqueness of the Stokes Problem (3.4.4) − (3.4.5). Secondly, we prove its surjectivity, which is the existence. And finally, we prove the continuity of the map $D_{(\mathbf{w},q)}\mathbf{F}(0,(\mathbf{0},q_0))$ and its inverse. This will end the proof because the linearity comes directly from the linearity of the integrals and the differential operators involved.

- Injectivity.

Let $(\mathbf{w}, q)$ in $\mathbf{W}_T \times Q_T$, solution to the homogeneous Stokes problem

$$\begin{cases} \rho\partial_t \mathbf{w} - \mu\Delta\mathbf{w} + \nabla q = 0, \quad \text{in } \Omega_0 & (3.4.46) \\ \nabla\cdot\mathbf{w} = 0, \quad \text{in } \Omega_0. & (3.4.47) \end{cases}$$

We will show that $(\mathbf{w}, q) = (\mathbf{0}, 0)$. By multiplying equation (3.4.46) and (3.4.47) by test functions $\boldsymbol{\zeta} \in \mathbf{H}^1_0(\Omega_0)$ and $\chi \in L^2(\Omega_0)$ and after integrating by parts, we get the following equivalent problem

Find $(\mathbf{w}, q) \in \mathbf{W}_T \times Q_T$ such that

$$\begin{cases} \displaystyle\int_{\Omega_0} \rho\, \partial_t \mathbf{w}\cdot\boldsymbol{\zeta} + \mu\nabla\mathbf{w} : \nabla\boldsymbol{\zeta} - q\nabla\cdot\boldsymbol{\zeta}\mathrm{d}y = 0,\ \forall\, \boldsymbol{\zeta} \in \mathbf{H}^1_0(\Omega_0), & (3.4.48a) \\ \displaystyle\int_{\Omega_0} \chi\nabla\cdot\mathbf{w}\ \mathrm{d}y = 0,\ \forall\, \chi \in L^2(\Omega_0),. & (3.4.48b) \end{cases}$$

Since equations $(3.4.48a)$ and $(3.4.48b)$ are true for arbitrary $\boldsymbol{\zeta} \in \mathbf{H}^1_0(\Omega_0)$ and $\chi \in L^2(\Omega_0)$, in particular, for $\boldsymbol{\zeta} = \mathbf{w}(t)$, and $\chi = q$ the equations $(3.4.48a) - (3.4.48b)$ imply

$$\int_{\Omega_0} \rho\, \partial_t \mathbf{w}\cdot\mathbf{w} + \mu|\nabla\mathbf{w}|^2\ \mathrm{d}y = 0 \tag{3.4.49}$$

for almost every $t \in [-T, 3T]$. Therefore according to Proposition 2.1.17, we have

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}(\cdot,t)\|^2_{\mathbf{L}^2(\Omega_0)} + 2\mu\ \|\mathbf{w}(\cdot,t)\|^2_{\mathbf{H}^1_0(\Omega_0)} = 0, \tag{3.4.50}$$

i.e

$$\rho \int_{-T}^{3T} \frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{w}(\cdot,t)\|^2_{\mathbf{L}^2(\Omega_0)} \,\mathrm{d}t + 2\mu \int_{-T}^{3T} \|\mathbf{w}(\cdot,t)\|^2_{\mathbf{H}^1_0(\Omega_0)} = 2\mu \|\mathbf{w}\|^2_{L^2(\mathbf{H}^1_0(\Omega_0))} = 0 \qquad (3.4.51)$$

because $\mathbf{w}(\cdot,-T) = \mathbf{w}(\cdot,3T)$. So, $\mathbf{w} = \mathbf{0}$.

Now, using again equation (3.4.48a), we have for all $t \in [-T,3T]$

$$\int_{\Omega_0} q \nabla \cdot \boldsymbol{\zeta} \,\mathrm{d}y = 0,\ \forall \boldsymbol{\zeta} \in \mathbf{H}^1_0(\Omega_0) \implies \int_{\Omega_0} \boldsymbol{\zeta} \nabla q \,\mathrm{d}y = 0,\ \forall \boldsymbol{\zeta} \in H^1_0(\Omega_0).$$

Therefore $\nabla q \equiv 0$ in $\mathbf{H}^{-1}(\Omega_0)$, i.e, $q = c \in \mathbb{R}$. As $q \in L^2_0(\Omega_0)$, we get $q = 0$ which proves the injectivity and therefore the uniqueness of the solution to the problem (3.4.4) – (3.4.5). The next step is the surjectivity, which is the existence.

- Surjectivity

Let $(\mathbf{f}, g) \in L^2_T(\mathbf{L}^2(\Omega_0)) \times L^2_T(H^1(\Omega_0) \cap L^2_0(\Omega_0)) \cap H^1_T(H^1(\Omega_0)')$. We search $(\mathbf{w}, q) \in \mathbf{W}_T \times Q_T$, solution to the following Stokes problem

$$\begin{cases} \rho \partial_t \mathbf{w} - \mu \Delta \mathbf{w} + \nabla q = \mathbf{f}, \text{ in } \Omega_0 & (3.4.52) \\ \nabla \cdot \mathbf{w} = g \text{ in } \Omega_0. & (3.4.53) \end{cases}$$

By applying integrals over Ω_0 for a.e. $t \in (-T,3T)$, and taking test functions in $\mathbf{H}^1_0(\Omega_0) \times L^2(\Omega_0)$, this problem is equivalent to

Search $(\mathbf{w}, q) \in \mathbf{W}_T \times Q_T$ / a.e. $t \in (-T,3T)$,

$$\begin{cases} \rho \int_{\Omega_0} \partial_t \mathbf{w}(\cdot,t) \cdot \boldsymbol{\zeta} \,\mathrm{d}y + \mu \int_{\Omega_0} \nabla \mathbf{w}(\cdot,t) : \nabla \boldsymbol{\zeta} \,\mathrm{d}y - \int_{\Omega_0} q(\cdot,t)\, \nabla \cdot \boldsymbol{\zeta} \,\mathrm{d}y = \int_{\Omega_0} \mathbf{f}(\cdot,t) \cdot \boldsymbol{\zeta} \,\mathrm{d}y, \\ \qquad \forall\, \boldsymbol{\zeta} \in \mathbf{H}^1_0(\Omega_0), & (3.4.54a) \\ \int_{\Omega_0} \chi\, \nabla \cdot \mathbf{w}(\cdot,t) \,\mathrm{d}y = \int_{\Omega_0} g(\cdot,t)\, \chi \,\mathrm{d}y,\ \forall\, \chi \in L^2(\Omega_0), & (3.4.54b) \end{cases}$$

Note that the problem (3.4.54a)–(3.4.54b) consists in looking for a strong solution to a weak formulation. So, since by hypothesis $(\mathbf{w}, q) \in L^2_T(\mathbf{H}^2(\Omega_0)) \times L^2_T(H^1(\Omega_0))$, by integrating by part, we recover (3.4.52) – (3.4.53), such that the equivalence holds.

We will solve the problem (3.4.54a) – (3.4.54b) in several steps. The first consists in setting again an equivalent version of (3.4.54a) – (3.4.54b) in the space of divergence free functions, using an appropriate change of variables. The second consists in solving the equivalent Stokes problem by a Faedo approach.

Step 1: Transformation to a divergence free problem.

Since $g \in L^2_T(H^1(\Omega_0) \cap L^2_0(\Omega_0)) \cap H^1_T(H^1(\Omega_0)')$ then by the Lemma 3.4.3, there exists a unique $\mathbf{w}_g \in \mathbf{W}_T$ such that

$$\nabla \cdot \mathbf{w}_g = g \quad \text{in } \Omega_0 \times (-T,3T). \qquad (3.4.55)$$

Next, we search a solution to (3.4.54a) – (3.4.54b) on the form $\mathbf{w} = \boldsymbol{\gamma} + \mathbf{w}_g$, where $\boldsymbol{\gamma}$ is such that $(\boldsymbol{\gamma}, q)$ solves the Stokes problem

Find $(\boldsymbol{\gamma}, q) \in \mathbf{W}_T \times Q_T$ such that for *a.e.* $t \in (-T, 3T)$

$$\begin{cases} \rho \displaystyle\int_{\Omega_0} \partial_t \boldsymbol{\gamma}(\cdot,t) \cdot \boldsymbol{\zeta} \, \mathrm{d}y + \mu \int_{\Omega_0} \nabla \boldsymbol{\gamma}(\cdot,t) : \nabla \boldsymbol{\zeta} \, \mathrm{d}y - \int_{\Omega_0} q(\cdot,t) \, \nabla \cdot \boldsymbol{\zeta} \mathrm{d}y = \int_{\Omega_0} \mathbf{r}(\cdot,t) \cdot \boldsymbol{\zeta} \, \mathrm{d}y, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \boldsymbol{\zeta} \in \mathbf{H}^1_0(\Omega_0), & \text{(3.4.56a)} \\ \displaystyle\int_{\Omega_0} \chi \, \nabla \cdot \boldsymbol{\gamma}(\cdot,t) \, \mathrm{d}y = 0, \ \forall\, \chi \in L^2(\Omega_0), & \text{(3.4.56b)} \end{cases}$$

where

$$\mathbf{r} := \mathbf{f} - \rho \partial_t \mathbf{w}_g + \mu \Delta \mathbf{w}_g \in L^2(\mathbf{L}^2(\Omega_0)) \tag{3.4.57}$$

and

$$\int_{\Omega_0} \mathbf{r}(\cdot,t) \cdot \boldsymbol{\zeta} \, \mathrm{d}y = \int_{\Omega_0} (\mathbf{f}(\cdot,t) - \rho \, \partial_t \mathbf{w}_g(\cdot,t)) \cdot \boldsymbol{\zeta} \, \mathrm{d}y - \mu \int_{\Omega_0} \nabla \mathbf{w}_g(\cdot,t) : \nabla \boldsymbol{\zeta} \, \mathrm{d}y.$$

Proposition 3.4.6. *Let* $\mathbf{f} \in L^2_T(\mathbf{L}^2(\Omega_0))$ *and we set* $\mathbf{r} \in L^2(\mathbf{L}^2(\Omega_0))$ *given as in* (3.4.57). *Then* $(\boldsymbol{\gamma}, q) \in \mathbf{W}_T \times Q_T$ *and solves the problem* (3.4.56a) – (3.4.56b) *if and only if* $(\mathbf{w}, q) \in \mathbf{W}_T \times Q_T$ *and solves the problem* (3.4.54a) – (3.4.54b).

The proof of Proposition 3.4.6 is straightforward, so we place it at the end of this chapter to maintain focus.

Step 2: Formulation in the divergent free space.

From (3.4.56b) clearly we have $\nabla \cdot \boldsymbol{\gamma} = 0$ and then the problem (3.4.56a) – (3.4.56b) can be written equivalently in the following form:

Find $\boldsymbol{\gamma} \in \mathbf{W}_{T,\sigma} := \{\mathbf{w} \in \mathbf{W}_T, \quad \nabla \cdot \mathbf{w} = 0\}$ such that

$$\rho \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_0} \boldsymbol{\gamma}(\cdot,t) \cdot \boldsymbol{\zeta} \, \mathrm{d}y + \mu \int_{\Omega_0} \nabla \boldsymbol{\gamma}(\cdot,t) : \nabla \boldsymbol{\zeta} \mathrm{d}y = \int_{\Omega_0} \mathbf{r}(\cdot,t) \cdot \boldsymbol{\zeta} \mathrm{d}y, \ \forall\, \boldsymbol{\zeta} \in \mathbf{H}^1_{0,\sigma}(\Omega_0). \tag{3.4.58a}$$

Indeed, if $(\boldsymbol{\gamma}, q) \in \mathbf{W}_T \times Q_T$ solves (3.4.56a) – (3.4.56b) then by (3.4.56b) we have $\nabla \cdot \boldsymbol{\gamma} = 0$ a.e. in $(-T, 3T)$ and therefore $\boldsymbol{\gamma} \in \mathbf{W}_{T,\sigma}$. Furthermore, as from Proposition 2.1.19 we have

$$\int_{\Omega_0} \partial_t \boldsymbol{\gamma} \cdot \boldsymbol{\zeta} \, \mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_0} \boldsymbol{\gamma} \cdot \boldsymbol{\zeta} \, \mathrm{d}y, \ \forall\, \boldsymbol{\zeta} \in \mathbf{H}^1_{0,\sigma}(\Omega_0), \tag{3.4.59}$$

taking $\boldsymbol{\zeta} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$ in (3.4.56a) implies (3.4.58a).

Inversely, if $\boldsymbol{\gamma} \in \mathbf{W}_{T,\sigma}$ and solves (3.4.58a) then $\boldsymbol{\gamma} \in \mathbf{W}_T$ and satisfies equation (3.4.56b). Using again (3.4.59) we have

$$\rho \int_{\Omega_0} \partial_t \boldsymbol{\gamma}(\cdot,t) \cdot \boldsymbol{\zeta} + \mu \int_{\Omega_0} \nabla \boldsymbol{\gamma}(\cdot,t) : \nabla \boldsymbol{\zeta} \, \mathrm{d}y - \int_{\Omega_0} \mathbf{r}(\cdot,t) \cdot \boldsymbol{\zeta} \, \mathrm{d}y = 0, \ \forall\, \boldsymbol{\zeta} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega_0) \tag{3.4.60}$$

i.e.

$$\langle \rho\partial_t\boldsymbol{\gamma} - \mu\Delta\boldsymbol{\gamma} - \mathbf{r}, \boldsymbol{\zeta} \rangle = 0, \quad \forall\, \boldsymbol{\zeta} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega_0). \tag{3.4.61}$$

Since Ω_0 is open, bounded and Lipschitz, according to the Proposition 2.1.10, there exists a distribution $q = q(\cdot,t) \in \mathcal{D}'(\Omega_0)$ such that

$$\rho\, \partial_t\boldsymbol{\gamma}(\cdot,t) - \mu\Delta\boldsymbol{\gamma}(\cdot,t) + \nabla q(\cdot,t) - \mathbf{r}(\cdot,t) = 0 \quad \text{in } \boldsymbol{\mathcal{D}}'_\sigma(\Omega_0), \text{ a.e. } t \in (-T, 3T). \tag{3.4.62}$$

But since

$$\partial_t\boldsymbol{\gamma},\ \Delta\boldsymbol{\gamma},\ \mathbf{r} \in L^2(\mathbf{L}^2(\Omega_0)), \tag{3.4.63}$$

(3.4.62) in combination with Lemma 2.1.11 implies

$$q \in L^2_T(H^1(\Omega_0) \cap L^2_0(\Omega)) = Q_T, \tag{3.4.64}$$

and $(\boldsymbol{\gamma}, q)$ solves (3.4.56a) – (3.4.56b) which completes the proof of the claim.

Step 3: Solution of (3.4.58a)*:*

Before we solve (3.4.58a), without loss of generality, we can suppose that $\mathbf{r} \in L^2(\mathbf{L}^2_\sigma(\Omega_0))$ because since $\mathbf{r}(t) \in \mathbf{L}^2(\Omega_0)$, for almost every $t \in (-T, 3T)$, according to Proposition 2.1.12 and Remark 2.1.13, we have the following Helmoltz decomposition

$$\mathbf{r}(\cdot,t) = \mathbf{r}_\sigma(\cdot,t) + \nabla q_{\mathbf{r}}(\cdot,t), \text{ in } \mathbf{L}^2(\Omega_0), \text{ for a.e. } t \in (-T, 3T), \tag{3.4.65}$$

with $\mathbf{r}_\sigma(\cdot,t) \in \mathbf{L}^2_\sigma(\Omega_0)$ and $q_{\mathbf{r}}(\cdot,t) \in H^1(\Omega_0) \cap L^2_0(\Omega_0)$. And since $\mathbf{r} \in L^2(\mathbf{L}^2(\Omega_0))$, then $\mathbf{r}_\sigma \in L^2(\mathbf{L}^2_\sigma(\Omega_0))$ and $q_{\mathbf{r}} \in L^2(H^1(\Omega_0) \cap L^2_0(\Omega_0))$. Then for a.e. $t \in (-T, 3T)$ and $\boldsymbol{\zeta} \in \mathbf{H}^1_{0,\sigma}(\Omega_0)$ we have

$$\int_{\Omega_0} \mathbf{r} \cdot \boldsymbol{\zeta}\, \mathrm{d}y = \int_{\Omega_0} (\mathbf{r}_\sigma + \nabla q_{\mathbf{r}}) \cdot \boldsymbol{\zeta}\, \mathrm{d}y = \int_{\Omega_0} \mathbf{r}_\sigma \cdot \boldsymbol{\zeta}\, \mathrm{d}y, \tag{3.4.66}$$

since $\nabla \cdot \boldsymbol{\zeta} = 0$ in Ω_0 and $\boldsymbol{\zeta} = 0$ on $\partial\Omega_0$.

The rest of the proof continues as in [5, 53, 98]. More precisely, let $-\Delta : \mathbf{H}^1_{0,\sigma}(\Omega_0) \mapsto \mathbf{H}^{-1}_\sigma(\Omega_0) = (\mathbf{H}^1_{0,\sigma}(\Omega_0))'$ with $-\Delta\mathbf{u} = \mathbf{f}$. Then $-\Delta$ defines an isomorphism from $\mathbf{H}^1_{0,\sigma}(\Omega_0)$ onto $\mathbf{H}^{-1}_\sigma(\Omega_0)$. If we consider $(-\Delta)^{-1} : \mathbf{L}^2_\sigma(\Omega_0) \mapsto \mathbf{L}^2_\sigma(\Omega_0)$, $(-\Delta)^{-1}\mathbf{f} = \mathbf{u}$ if $-\Delta\mathbf{u} = \mathbf{f}$ then $(-\Delta)^{-1}$ is compact and self-adjoint. Therefore it has a countable sequence of eigenfunctions and eigenvectors denoted by $(\boldsymbol{\varphi}_n, \mu_n)$, with $\mu_n > 0$ and $\mu_n \to 0$ as $n \to \infty$. It follows that $-\Delta\boldsymbol{\varphi}_n = \lambda_n\boldsymbol{\varphi}_n$, $\lambda_n = \mu_n^{-1} \to \infty$. The system $(\boldsymbol{\varphi}_n)$ can be chosen orthonormal in $\mathbf{L}^2_\sigma(\Omega_0)$ and orthogonal in $\mathbf{H}^1_{0,\sigma}(\Omega_0)$. It turns out that

$$\mathbf{L}^2_\sigma(\Omega_0) = \{\mathbf{u} = \sum_{n=1}^{\infty} c_n\boldsymbol{\varphi}_n, \quad \sum_{n=1}^{\infty} |c_n|^2 < \infty\}, \tag{3.4.67}$$

$$\mathbf{H}^1_{0,\sigma}(\Omega_0) = \{\mathbf{u} = \sum_{n=1}^{\infty} c_n\boldsymbol{\varphi}_n, \quad \sum_{n=1}^{\infty} \lambda_n|c_n|^2 < \infty\}, \tag{3.4.68}$$

$$\mathbf{H}^2_0(\Omega_0) \cap \mathbf{H}^1_{0,\sigma}(\Omega_0) = \{\mathbf{u} = \sum_{n=1}^{\infty} c_n\boldsymbol{\varphi}_n, \quad \sum_{n=1}^{\infty} \lambda_n^2|c_n|^2 < \infty\}. \tag{3.4.69}$$

So, let $\mathbf{r} \in L^2_T(\mathbf{L}^2_\sigma(\Omega_0))$, so $\mathbf{r}(t) = \sum_{n=1}^\infty r_n(t)\boldsymbol{\varphi}_n$ with $\sum_{n=1}^\infty \int_{-T}^{3T} |r_n(t)|^2 < \infty$. We look for $\boldsymbol{\gamma}(t) = \sum_{n=1}^\infty \gamma_n(t)\boldsymbol{\varphi}_n \in \mathbf{W}_{T,\sigma}$ such that $\rho\partial_t\boldsymbol{\gamma} - \mu\Delta\boldsymbol{\gamma} = \mathbf{r}$. Hence necessarily

$$\rho\gamma_n'(t) + \mu\lambda_n\gamma_n = r_n(t), \quad so \tag{3.4.70}$$

$$\gamma_n(t) = \frac{1}{\rho}\int_{-T}^{t} e^{\lambda_n \frac{\mu}{\rho}(s-t)} r_n(s)ds + \frac{1}{\rho}\cdot\frac{1}{e^{4\lambda_n\frac{\mu}{\rho}T}-1}\int_{-T}^{3T} e^{\lambda_n\frac{\mu}{\rho}(s-t)} r_n(s)ds. \tag{3.4.71}$$

Note that $\gamma_n(-T) = \gamma_n(3T)$. Multiplying (3.4.70) by $\boldsymbol{\gamma}_n$ and integrating in $(-T, 3T)$ gives

$$\mu\lambda_n \int_{-T}^{3T} |\gamma_n(t)|^2 dt = \int_{-T}^{3T} r_n(t)\gamma_n(t)dt. \tag{3.4.72}$$

Since the sequence λ_n tends to ∞, we can suppose that $\lambda_n > \frac{1}{\mu}$, $\forall\, n$. Then we have

$$r_n(t)\gamma_n(t) \le |r_n(t)\gamma_n(t)| \le |r_n(t)\sqrt{\mu\lambda_n}\gamma_n(t)| \le \frac{1}{2}|r_n(t)|^2 + \frac{\mu\lambda_n}{2}|\gamma_n(t)|^2 \tag{3.4.73}$$

i.e.

$$\mu\lambda_n\int_{-T}^{3T}|\gamma_n(t)|^2\,\mathrm{d}t = \int_{-T}^{3T} r_n(t)\gamma_n(t)\,\mathrm{d}t \le \frac{1}{2}\int_{-T}^{3T}|r_n(t)|^2\,\mathrm{d}t + \frac{\mu\lambda_n}{2}\int_{-T}^{3T}|\gamma_n(t)|^2\,\mathrm{d}t. \tag{3.4.74}$$

Therefore

$$\frac{\mu\lambda_n}{2}\int_{-T}^{3T}|\gamma_n(t)|^2\,\mathrm{d}t \le \frac{1}{2}\int_{-T}^{3T}|r_n(t)|^2\,\mathrm{d}t. \tag{3.4.75}$$

Taking the sum over n in the above inequality (3.4.75) we get

$$\mu\int_{-T}^{3T}\sum_{n=1}^{\infty}\lambda_n|\gamma_n(t)|^2dt \le \sum_{n=1}^{\infty}\int_{-T}^{3T}|r_n(t)|^2dt,$$

so

$$\boldsymbol{\gamma} \in L^2(\mathbf{H}^1_{0,\sigma}(\Omega_0)). \tag{3.4.76}$$

Similarly, multiplying (3.4.70) by $\boldsymbol{\gamma}_n'$ and itegrating in $(-T, 3T)$ gives

$$\rho\int_{-T}^{3T}|\gamma_n'(t)|^2dt = \int_{-T}^{3T} r_n(t)\gamma_n'(t)dt, \quad so$$

$$\rho\int_{-T}^{3T}\sum_{n=1}^{\infty}|\gamma_n'(t)|^2dt \le \sum_{n=1}^{\infty}\int_{-T}^{3T}|r_n(t)|^2dt,$$

so

$$\boldsymbol{\gamma}' \in L^2(\mathbf{L}^2_\sigma(\Omega_0)) \tag{3.4.77}$$

and therefore

$$\boldsymbol{\gamma} \in H^1(\mathbf{L}^2_\sigma(\Omega_0)). \tag{3.4.78}$$

Note that (3.4.70) implies also

$$\mu^2\lambda_n^2|\gamma_n|^2 \quad\leq\quad 2(|r_n|^2+\rho^2|\gamma_n'|^2). \tag{3.4.79}$$

This inequality gives

$$\mu^2\int_{-T}^{3T}\sum_{n=1}^{\infty}\lambda_n^2|\gamma_n|^2dt \quad\leq\quad 2\int_{-T}^{3T}\sum_{n=1}^{\infty}|r_n|^2dt+2\rho^2\int_{-T}^{3T}\sum_{n=1}^{\infty}|\gamma_n'|^2dt<\infty,$$

which shows that

$$\boldsymbol{\gamma}\in L^2(\mathbf{H}^2(\Omega_0)\cap\mathbf{H}^1_{0,\sigma}(\Omega_0))\cap H^1(\mathbf{L}^2_\sigma(\Omega_0)). \tag{3.4.80}$$

Therefore, according to Proposition 2.1.19 we have up to a modification on a set of zero measure

$$\boldsymbol{\gamma}\in C([-T,3T];\mathbf{H}^1_{0,\sigma}(\Omega_0)). \tag{3.4.81}$$

As $\gamma_n(-T)=\gamma_n(3T)$ implies $\boldsymbol{\gamma}(-T)=\boldsymbol{\gamma}(3T)$, we have the periodicity. From (3.4.80) we get $\boldsymbol{\gamma}\in\mathbf{W}_{T,\sigma}$ and solve (3.4.58a).

So, for all $(\mathbf{f},g)\in\mathbf{R_w}\times R_q$ there exists a solution $(\mathbf{w},q)\in\mathbf{W}_T\cap C^0_T(\mathbf{H}^1(\Omega_0))\times Q_T$ to the Stokes problem (3.4.54a)−(3.4.54b), obtained by constructing q as in (3.4.64) and by setting $\mathbf{w}=\boldsymbol{\gamma}+\mathbf{w}_g\in\mathbf{W}_T$, where $(\boldsymbol{\gamma},q)\in\mathbf{W}_T\times Q_T$ solves the Stokes problem (3.4.56a)−(3.4.56b) and $\mathbf{w}_g$ solution to the divergence problem given by Lemma 3.4.3 and $\mathbf{r}$ being the right hand side defined in (3.4.57).

- Bi-continuity.

For all $(\mathbf{w},q)\in\mathbf{W}_T\times Q_T$ we have

$$\|D_{(\mathbf{w},q)}F_1(0,(\mathbf{0},q_0))\|_{\mathbf{R_w}}=\|\partial_t\mathbf{w}-\mu\Delta\mathbf{w}+\nabla q\|_{\mathbf{R_w}}\leq C\Big(\|\mathbf{w}\|_{\mathbf{W}_T}+\|q\|_{Q_T}\Big), \tag{3.4.82}$$

and

$$\|D_{\mathbf{w}}F_2(0,0)(\mathbf{w})\|_{R_q}=\|\nabla\cdot\mathbf{w}\|_{R_q}\leq C\|\mathbf{w}\|_{\mathbf{W}_T}. \tag{3.4.83}$$

On the other-hand, let $(\mathbf{f},g)\in L^2_T(\mathbf{L}^2(\Omega_0)\times L^2_T(H^1(\Omega_0)\cap L^2_0(\Omega_0)))$ such that $(\mathbf{w},q)$ solves the non-homogeneous Stokes problem (3.4.52) − (3.4.53), as given by Proposition 3.4.6. Since $\nabla\cdot\mathbf{w}=g$, according to Lemma 3.4.3 we have the estimation

$$\|\mathbf{w}\|_{\mathbf{W}_T}\leq C\|g\|_{R_q} \tag{3.4.84}$$

and

$$\begin{aligned}\|\nabla q\|_{L^2_T(\mathbf{L}^2(\Omega_0))}\leq\|q\|_{L^2_T(H^1(\Omega_0))}&\leq\|\mathbf{f}-\partial_t\mathbf{w}+\mu\Delta\mathbf{w}\|_{L^2_T(\mathbf{L}^2(\Omega_0))}\\&\leq C(\|\mathbf{f}\|_{\mathbf{R_w}}+\|\mathbf{w}\|_{\mathbf{W}_T})\\&\leq C(\|\mathbf{f}\|_{\mathbf{R_w}}+\|g\|_{R_q}),\end{aligned} \tag{3.4.85}$$

which show the isomorphism of the map $D_{(\mathbf{w},q)}\mathbf{F}(0,(\mathbf{0},q_0))$ and ends the proof of Proposition 3.4.2. ∎

Theorem 3.4.7 (Existence and uniquesness of $(\mathbf{w}, q)$)**.**

The problem (3.3.23a) – (3.3.23b) *has a unique time-periodic strong solution* $(\mathbf{w}, q) \in \mathbf{W}_T \times Q_T$ *for* $\|z\|_{Z_T} \ll 1$. *Moreover,* $\mathbf{w} \in C^0_T(\mathbf{H}^1(\Omega_0))$.

Proof:

We have shown in Proposition 3.4.1 that the map $\mathbf{F}$ is C^1 in a neighborhood of $(0, (\mathbf{0}, q_0))$, with $\mathbf{F}(0, (\mathbf{0}, q_0)) = (\mathbf{0}, 0)$. Moreover, from Theorem 3.4.2, we have shown that $(\mathbf{w}, q) \mapsto D_{(\mathbf{w},q)}\mathbf{F}(0, (\mathbf{0}, q_0))(\mathbf{w}, q)$ is a topological isomorphism. Therefore by the implicit function theorem, there exists a unique C^1 map φ_0 from a small neighborhood $N_0 \subset Z_T$ of 0 to $\mathbf{W}_T \times Q_T$ such that

$$\mathbf{F}(z, \varphi_0(z)) = (\mathbf{0}, 0), \ \forall\, z \in N_0. \tag{3.4.86}$$

It turns out that for a given $z \in N_0$, if $\varphi_0(z) = (\mathbf{w}_z, q_z)$, then $(\mathbf{w}_z, q_z) \in \mathbf{W}_T \times Q_T$ is the unique local solution to the periodic NS problem (3.3.23a) – (3.3.23b). Since $\mathbf{w}_z \in \mathbf{W}_T \hookrightarrow L^2_T(\mathbf{H}^2(\Omega_0)) \cap H^1_T(\mathbf{L}^2(\Omega_0))$, by Proposition 2.1.18, we have $\mathbf{w}_z \in C^0_T(\mathbf{H}^1(\Omega_0))$. ∎

Theorem 3.4.8 (Existence and uniqueness of $(\mathbf{u}, p)$)**.**

The problem (3.1.13a)-(3.1.13d) *has a unique time-periodic strong solution* $(\mathbf{u}, p) \in \mathbf{U}_T \cap C^0_T(\mathbf{H}^1(\Omega_t)) \times P_T$ *for* $\|z\|_{Z_T} \ll 1$.

Proof:

For z small, we still denote by z its zero extension in $\mathbb{R}^{N-1}$. We construct $\bar{z}$ as in equation (3.3.60) and we denote by $\mathbf{z} \in \mathbf{Z}_T$ the extension of $(0, ..., 0, \bar{z})$ in $\mathbf{Z}_T$. We set $\mathbf{T} = \mathbf{I} + \mathbf{z}$. According to Lemma 3.3.2, $\mathbf{T}$ is a diffeomorphism of class $L^\infty_T(\mathbf{W}^{2,\hat{2}^*}(\Omega_0))$. According to Lemma 3.3.4, we know that the extension of $\mathbf{h}$ is in $\mathbf{W}^{\mathbf{h}}_T := L^2_T(\mathbf{H}^2(\Omega_0)) \cap H^1_T(\mathbf{L}^2(\Omega_0))$ and is given by the formula (3.3.91). Therefore, we set

$$\mathbf{g} = \mathbf{g}(x, t) := \mathbf{h} \circ \mathbf{T}^{-1} = \mathbf{h}(\mathbf{T}^{-1}(x, t), t). \tag{3.4.87}$$

Let $(\mathbf{w}, q) \in \mathbf{W}_T \cap C^0_T(\mathbf{H}^1(\Omega_0)) \times Q_T$ be the unique solution of (3.3.23a) – (3.3.23b) given by Theorem 3.4.7. Therefore, $\mathbf{F}(z, \mathbf{w}, q) = (\mathbf{0}, 0)$. We set

$$\mathbf{v} = \mathbf{v}(x, t) := \mathbf{w} \circ \mathbf{T}^{-1} = \mathbf{w}(\mathbf{T}^{-1}(x, t), t), \tag{3.4.88}$$

$$\mathbf{u} = \mathbf{v} + \mathbf{g}, \tag{3.4.89}$$

$$p = p(x, t) := q \circ \mathbf{T}^{-1} - \frac{1}{|\Omega_t|} \int_{\Omega_0} q\, |\nabla \mathbf{T}|\, \mathrm{d}y. \tag{3.4.90}$$

Note that such p has zero mean in Ω_t. Since $(\mathbf{w}, q) \in \mathbf{W}_T \times Q_T$, then, according Proposition 3.3.1, we have

$$\mathbf{v} \in \mathbf{V}_T := L^2_T(\mathbf{H}^2(\Omega_t) \cap \mathbf{H}^1_0(\Omega_t)) \cap H^1_T(\mathbf{L}^2(\Omega_t)), \tag{3.4.91}$$

$$p \in P_T := L^2_T(H^1(\Omega_t) \cap L^2_0(\Omega_t)), \tag{3.4.92}$$

$$\mathbf{g} \in \mathbf{U}_T = L^2_T(\mathbf{H}^2(\Omega_t)) \cap H^1_T(\mathbf{L}^2(\Omega_0)). \tag{3.4.93}$$

It remains to show that $\mathbf{v} \in C^0_T(\mathbf{H}^1(\Omega_t))$ and $\mathbf{g} \in C^0_T(\mathbf{H}^1(\Omega_t))$. We will prove only the case of $\mathbf{v}$ since the case of $\mathbf{g}$ is similar. Without loss of generality we prove it at $t = 0$. We recall that the space $C^0_T(\mathbf{H}^1(\Omega_t))$ is defined in (2.1.14). The proof relies on the property $\mathbf{w} \in C^0_T(\mathbf{H}^1(\Omega_0))$. We set

$$\delta(t) := \|\mathbf{v}(\cdot,t) - \mathbf{v}(\cdot,0)\|_{\mathbf{H}^1(\Omega_t\cap\Omega_0)} + \|\mathbf{v}(\cdot,t)\|_{\mathbf{H}^1(\Omega_t\setminus\Omega_0)} + \|\mathbf{v}(\cdot,0)\|_{\mathbf{H}^1(\Omega_0\setminus\Omega_t)}, \tag{3.4.94}$$

$$=: \delta_1(t) + \delta_2(t) + \delta_3(t). \tag{3.4.95}$$

We will show that

$$\lim_{t\to 0} \delta(t) = 0. \tag{3.4.96}$$

We will prove this only for the H^1 semi-norm because the proof for the L^2 norm is similar and easier. Let us first consider the term $\delta_2(t)$. By contradiction, let us assume that

$$\lim_{t\to 0} \delta_2(t) = C_2 > 0, \text{ which implies } \lim_{t\to 0} \delta_2^2(t) = C_2^2 > 0. \tag{3.4.97}$$

Therefore, using (3.3.15), we have

$$\begin{aligned}
\delta_2^2(t) &= \int_{\Omega_t\setminus\Omega_0} |\nabla\mathbf{v}(x,t)|^2 \, \mathrm{d}x \\
&= \int_{\mathbf{T}^{-1}(\Omega_t\setminus\Omega_0)} |[\nabla\mathbf{v}(x,t)] \circ \mathbf{T}(\cdot,t)|^2 \, |\nabla\mathbf{T}(\cdot,t)| \, \mathrm{d}x \\
&= \int_{\mathbf{T}^{-1}(\Omega_t\setminus\Omega_0)} |[\nabla\mathbf{w}(x,t)] \cdot [\nabla\mathbf{T}(x,t)]^{-1}|^2 |\nabla\mathbf{T}(x,t)| \, \mathrm{d}x \\
&\leq 2\int_{\mathbf{T}^{-1}(\Omega_t\setminus\Omega_0)} |[\nabla(\mathbf{w}(x,t) - \mathbf{w}(x,0))] \cdot [\nabla\mathbf{T}(x,t)]^{-1}|^2 |\nabla\mathbf{T}(x,t)| \, \mathrm{d}x \\
&+ 2\int_{\mathbf{T}^{-1}(\Omega_t\setminus\Omega_0)} |[\nabla\mathbf{w}(x,0)] \cdot [\nabla\mathbf{T}(x,t)]^{-1}|^2 |\nabla\mathbf{T}(x,t)| \, \mathrm{d}x.
\end{aligned} \tag{3.4.98}$$

By passing the limit as $t \to 0$, in (3.4.98), using the fact that $\mathbf{w} \in C^0_T(\mathbf{H}^1(\Omega_0))$ and applying the Lebesgue dominated convergence theorem, we get

$$0 < C_2^2 \leq 2\int_{\Omega_0} \lim_{t\to 0} \mathbb{1}_{\mathbf{T}^{-1}(\Omega_t\setminus\Omega_0)}(x) |\nabla\mathbf{w}(x,0)|^2 \, \mathrm{d}x = 0, \tag{3.4.99}$$

since

$$\lim_{t\to 0} |\Omega_t \setminus \Omega_0| = 0, \tag{3.4.100}$$

which is a contradiction. Therefore, $C_2 = 0$.

Similarly, we prove that

$$\lim_{t\to 0} \delta_3(t) = 0. \tag{3.4.101}$$

For δ_1 we after changing the variable and using (3.3.15), we have

$$\delta_1(t)^2 = \int_{\Omega_t\cap\Omega_0} |\nabla(\mathbf{v}(x,t) - \mathbf{v}(\cdot,0))|^2 \, \mathrm{d}x$$

$$
\begin{aligned}
&\le \int_{\Omega_0} |[\nabla(\mathbf{w}(x,t)] \cdot [\nabla \mathbf{T}(x,t)]^{-1} |\nabla \mathbf{T}(x,t)| - \mathbf{w}(x,0))|^2 \, \mathrm{d}x \\
&\le C \int_{\Omega_0} |[\nabla \mathbf{w}(x,t)] - [\nabla \mathbf{w}(x,0)]|^2 \, \mathrm{d}x \\
&+ C \int_{\Omega_0} [\nabla \mathbf{w}(x,0)] \cdot \Big([\nabla \mathbf{w}(x,t)]^{-1} |\nabla \mathbf{T}(x,t)| - I \Big) \, \mathrm{d}x \\
&\le C \Big(\|\mathbf{w}(\cdot,t) - \mathbf{w}(\cdot,0)\|^2_{\mathbf{H}^1(\Omega_0)} + \|\mathbf{w}(\cdot,0)\|^2_{\mathbf{H}^1(\Omega_0)} \|[\nabla \mathbf{T}(\cdot,t)]^{-1} |\nabla \mathbf{T}(\cdot,t)| - I\|^2_{\mathbf{L}^{\hat{2}^*}(\Omega_0)} \Big) \\
&\longrightarrow 0 \text{ as } t \to 0, \qquad (3.4.102)
\end{aligned}
$$

because $\mathbf{w} \in C^0(\mathbf{H}^1(\Omega_0))$, $|\nabla \mathbf{T}| \in L^\infty_T(\mathbf{L}^\infty(\Omega_0))$ and $\mathbf{T} \in H^1_T(\mathbf{H}^2(\Omega_0)) \hookrightarrow C^0_T(\mathbf{H}^2(\Omega_0)) \hookrightarrow C^0_T(\mathbf{W}^{1,\hat{2}^*}(\Omega_0))$.

The same way, we prove that $\mathbf{g} \in C^0(\mathbf{H}^1(\Omega_t))$.

Since $(\mathbf{w}, q)$ solves (3.3.23a) – (3.3.23b), then by doing the change of variable in the opposite sense as in section 3.3.1, we have $(\mathbf{v}, p)$ solve the problem (3.3.1a) – (3.3.1d). This proves that $(\mathbf{u}, p) \in \mathbf{U}_T \cap C^0_T(\mathbf{H}^1(\Omega_t)) \times P_T$ and solves the problem (3.1.13a)-(3.1.13d).

Note that the uniqueness of $(\mathbf{u}, p)$ is assured by the uniqueness of $(\mathbf{w}, q)$. Indeed, for $i \in \{1,2\}$, if $(\mathbf{u}^i, p^i) \in \mathbf{U}_T \times P_T$ are two solutions of (3.1.13a)-(3.1.13d), then we set $(\mathbf{w}^i, q^i) = (\mathbf{u}^i \circ \mathbf{T} - \mathbf{g} \circ \mathbf{T}, p^i \circ \mathbf{T} - \frac{1}{|\Omega_0|} \int_{\Omega_t} p^i \, |\nabla \mathbf{T}^{-1}| \, \mathrm{d}x)$. Therefore, $(\mathbf{w}^i, q^i) \in \mathbf{W}_T \times Q_T$ and solve the problem (3.3.23a) – (3.3.23b) and moreover, we have $\mathbf{F}(z, \mathbf{w}^i, q^i) = (\mathbf{0}, 0)$. Therefore $(\mathbf{w}^1, q^1) = (\mathbf{w}^2, q^2)$ which implies $(\mathbf{u}^1, p^1) = (\mathbf{u}^2, p^2)$. ∎

Proof: (**Proof of Proposition** 3.4.6)
$\Longrightarrow$) Let us suppose that $(\boldsymbol{\gamma}, q) \in \mathbf{W}_T \times Q_T$ and solves the problem (3.4.56*a*) – (3.4.56*b*). Then $\mathbf{w} = \boldsymbol{\gamma} + \mathbf{w}_g \in \mathbf{W}_T$. Therefore, for *a.e* $t \in (-T, 3T)$, equation (3.4.56a) with $\boldsymbol{\gamma} = \mathbf{w} - \mathbf{w}_g$ becomes

$$
\begin{aligned}
\rho \int_{\Omega_0} \partial_t \mathbf{w}(\cdot,t) \cdot \boldsymbol{\zeta} \, \mathrm{d}y + \mu \int_{\Omega_0} \nabla \mathbf{w}(\cdot,t) : \nabla \boldsymbol{\zeta} \, \mathrm{d}y - \int_{\Omega_0} q(\cdot,t) \, \nabla \cdot \boldsymbol{\zeta} \mathrm{d}y \\
= \int_{\Omega_0} \mathbf{r}(\cdot,t) \cdot \boldsymbol{\zeta} \, \mathrm{d}y + \rho \int_{\Omega_0} \partial_t \mathbf{w}_g(\cdot,t) \cdot \boldsymbol{\zeta} \, \mathrm{d}y + \mu \int_{\Omega_0} \nabla \mathbf{w}_g(\cdot,t) : \nabla \boldsymbol{\zeta} \, \mathrm{d}y \\
= \int_{\Omega_0} \mathbf{f}(\cdot,t) \cdot \boldsymbol{\zeta} \, \mathrm{d}y, \ \forall \, \boldsymbol{\zeta} \in \mathbf{H}^1_0(\Omega_0)
\end{aligned}
$$

which proves equation (3.4.54a). Similarly, equation (3.4.56b) with $\boldsymbol{\gamma} = \mathbf{w} - \mathbf{w}_g$ becomes

$$
\begin{aligned}
0 = \int_{\Omega_0} \chi \, \nabla \cdot \boldsymbol{\gamma}(\cdot,t) \, \mathrm{d}y &= \int_{\Omega_0} \chi \, \nabla \cdot (\mathbf{w}(\cdot,t) - \mathbf{w}_g(\cdot,t)) \, \mathrm{d}y \\
&= \int_{\Omega_0} \chi \, \nabla \cdot \mathbf{w}(\cdot,t) \, \mathrm{d}y - \int_{\Omega_0} \chi \, \nabla \cdot g(\cdot,t) \, \mathrm{d}y, \ \forall \, \chi \, \in L^2(\Omega_0)
\end{aligned}
$$

which proves equation (3.4.54b).

$\Longleftarrow$) On the other-hand, if $(\mathbf{w}, q) \in \mathbf{W}_T \times Q_T$ and solves the problem (3.4.54a) − (3.4.54b), then $\boldsymbol{\gamma} = \mathbf{w} - \mathbf{w}_g \in \mathbf{W}_T$ and by similar calculations, we find that $(\boldsymbol{\gamma}, q)$ solves that problem (3.4.56a) − (3.4.56b). ∎

Chapter 4

Numerical solution of the time-periodic Navier-Stokes equations using neural networks

Numerical methods for solving PDEs were effectively utilized in the mid-1940s by John Von Neumann. Among the successful methods are the finite difference method (FDM), the finite element method (FEM), and the finite volume method (FVM) (see Courant et al. [21]).

FDM involves discretizing the domain and using divided differences to create a finite difference scheme that approximates the original PDEs. It offers advantages such as mathematical ease of understanding, flexibility in implementing divided differences (forward, backward, or centered), suitability for both linear and nonlinear PDEs, and the fact that finite difference numerical schemes often result in algebraic equations that can sometimes be solved directly (particularly for forward approximations). Additionally, error analysis in this method, including truncation and approximation errors, is well-developed in the literature (see LeVeque [62], Dimov et al. [26]).

FEM offers flexibility in modeling PDEs with complex (polygonal) geometries, making it widely used in science and engineering for mechanical and fluid dynamics problems (see Strand and Fix [94], Ciarlet [18], Johnson [49], Ciarlet et al. [19], Hughes [46], Brenner [10]). It involves partitioning the domain into sets of non-overlapping polygonal finite elements, then finding piece-wise approximation functions and the subspace of approximation where the finite element numerical scheme is defined. Error analysis for FEM is also well-developed.

FVM presents advantages such as ease in handling complex geometries using unstructured meshes. The method is well-suited for approximating problems or functions with large gradients or jump discontinuities (see Godlewski and Raviart [38], Tadmor [95], LeVeque [61]). It ensures the conservation of mass, momentum, and energy within each control volume and across the entire domain. Error analysis for FVM is also well-developed.

In the last 15 years, deep learning, in the form of deep neural networks (NN), has been effectively used in diverse domains (see LeCun et al. [58], Lu et al. [67], Lagaris et al. [55, 55]), such as computer vision and natural language processing and it has been quite extensively used in the pass few years in the field of scientific computing. In addition, solving PDEs with deep learning has emerged. In particular, we can replace traditional discretization methods such as FDM, FDM, and FVM with a NN method for solving PDEs. Although the error analysis is not yet well-developed, there are experimental results in the literature showing great success (see Lu et al. [67], Raissi et al. [83, 84], Lee and Kang [59], Wang and Mendel [99], Yentis and Zaghloul [101]) using a neural network method called physics informed neural networks (PINNs). The idea of the method is as follows: given a PDE, we suppose that the solution is approximated with a NN function, which is a composition of linear and nonlinear transformations (depending on the unknown parameters) of the inputs (space and time variables) followed by a nonlinear transformation through an activation function. Then, our new problem consists of minimizing the residual of the PDE with respect to the parameters. The network with the optimized parameters provides an approximation of the solution to the PDE.

Compared with traditional mesh-based approaches such as FEM, FDM, and FVM, the NN method is mesh-free. It takes advantage of automatic differentiation (see Raissi et al. [83]). With automatic differentiation, we can directly differentiate functions and avoid truncation errors which appear when we approach the derivatives with finite difference schemes. It also breaks the curse of dimensionality (see Poggio et al. [81], Grohs et al. [39]) since it deals with scanning points i.e., randomly distributed points scatter inside the domain and on the boundary. Another general advantage of this approach is that it solves forward and inverse PDEs with just slight modifications to the code (see Raissi et al. [83, 84], Tartakovsky et al. [96], He et al. [43], Chen et al. [16]). Additionally, an extension of this method applies to solving integro-differential PDEs, fractional PDEs, and stochastic PDEs (see Zhang et al. [102], Yang et al. [100], Nabian and Meidani [76], Zhang et al. [103]). There are many types of existing NNs such as convolutional NNs and recurrent NNs, but the simplest is the feedforward neural network (FNN), also called multi-layer Perceptron (MLP).

In this thesis, we are going to use a special case of FNN architecture, called radial basis neural network (RBNN). A RBNN function is a special case of FNN function with only one hidden layer. The reason behind this choice is that as a function of the parameters, a RBNN function is a convex function and the optimization is more accurate than the traditional FNN function with multiple hidden layers (see [57, 24, 69, 32, 9, 47]). We will apply the RBNN methods to solve the time-independent Stokes, Steady state NS problem in the context of lid-driven cavity problem. We will also use it to solve our time-dependent NS equations.

For the Stokes problem, we first consider its equivalent augmented Lagrangian formulation. This approach transforms the Stokes problem into a minimization problem in terms of the parameters of the NN-approximated solution, which we solve using Uzawa's algorithm (see [33, 35, 6, 7, 93, 78, 85, 37]). In the case of the time dependent Navier-Stokes problem, we approach the nonlinear term using the backward differential formula of order 2 (BDF2) coupled with a second order Runge-Kutta as described by Bermejo and Saavedra [8], which transforms the time-dependent NS equations into a time-dependent Stokes equations. Like before, we solve the Stokes problem using again Uzawa's algorithm. Note also that, since our method use integrals, we use the Gauss-Legendre quadrature numerical method available in the python NumPy library. This approximation is of degree $2n-1$, where n here is the given number of points, meaning that we have an exact approximation of the integral for all polynomial of degree less or equal to $2n-1$. We took Gauss points as training points to compute the integrals. This method provides very good accuracy of integrals.

To test the efficiency of our method, we solve some benchmark problems. We start by solving what we choose to call here "toy" problem. Indeed, the "toy" problem is a simple time-periodic NS problem. Here the advantage is the fact that the exact solution is known (see Pedergnana et al. [79]). Therefore, we can compare the result (neural network solution) given by our code with the exact solution and compute the error of accuracy. Next, we solve the lid-driven cavity problem for Stokes and NS flow up to $Re = 10,000$ and compare it with some results in the literature. Finally, we apply this method for our time-periodic NS problem in the context of AH and present numerical results.

4.1 General formulation of the feed-forward neural network method

In this section, we provide an overview of feedforward neural networks (FFNNs) with multiple hidden layers and describe their use in approximating general Dirichlet partial differential equations (PDEs). While our simulations utilize a neural network model similar to a single hidden layer FFNN, known as the radial basis neural network (RBNN), the underlying approximation method remains the same which is basically by minimization of a certain loss function. Notably, due to the convex nature of the RBNN with respect to its parameters, it often outperforms traditional FFNNs with multiple hidden layers (see [57, 24, 69, 32, 9, 47]).

We begin by presenting a general description of the FFNN method for approximating arbitrary functions. This is followed by an illustration of how FFNNs can be applied to solve general PDEs, as outlined in [55, 41, 72, 73]. This discussion serves as a foundational background, covering essential concepts before delving into the specifics of our approach. Additionally, examples of neural network methods applied to classic PDEs are included to provide practical insights.

4.1.1 Neural networks and approximation of functions

Let $x \in \mathbb{R}^N$, and $d \geq 1$ an integer. A single-layer FNN with input x and output $y \in \mathbb{R}^m$ is given by the following construction

$$x \in \mathbb{R}^N \quad \longrightarrow \quad y = W^1 \cdot \sigma(W^0 \cdot x + b^0) + b^1 \in \mathbb{R}^m \tag{4.1.1}$$

where the couples $(W^0,\ b^0) \in \mathbb{R}^{d\times N} \times \mathbb{R}^d$ and $(W^1,\ b^1) \in \mathbb{R}^{m\times d} \times \mathbb{R}^m$ are the parameters of the network called weights and biases, and σ is the nonlinear activation function. More precisely, our single hidden layer has d neurons and at the k^{th} neuron, we have the linear and nonlinear transformations on the input x as:

$$\sigma(W_k^0 \cdot x + b_k^0) = \sigma\Big(\sum_{j=1}^{N} W_{kj}^0 x_j + b_k^0\Big), \ \ \forall\, k \in \{1, ..., d\} \tag{4.1.2}$$

where $W_k^0 := [W_{k,1}^0, ..., W_{k,N}^0]$ is the k^{th} row of the weight matrix W^0. The parameters W_{kj}^0 is the weight from the input unit x_j to the k^{th} neuron and b_k^0 is the bias associated to the k^{th} neuron. It can be illustrates with the Figure 4.1.

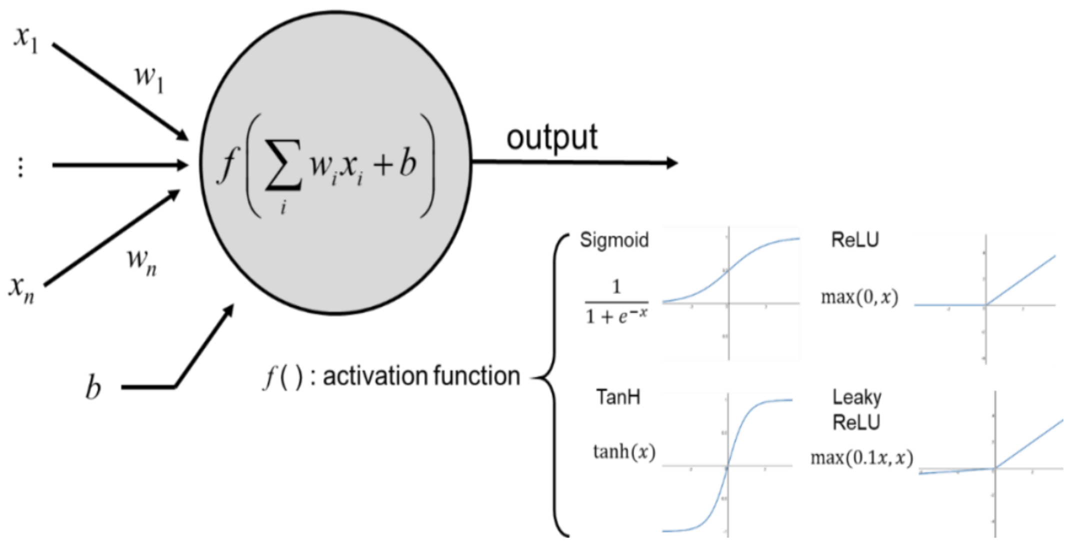


Figure 4.1: Artificial neuron.

Some common activation functions read:

$$\text{ReLU} : x \longmapsto \max(0, x),$$
$$\text{Sigmoid} : x \longmapsto \frac{1}{1+\exp(-x)},$$
$$\tanh : x \longmapsto \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)}.$$

Note that in our case, we use the radial Gaussian activation function $r \mapsto \exp(-(\varepsilon r)^2)$, where ε is called the shape parameter. Unlike the aforementioned activation functions, the Gaussian possesses the advantage of preserving the local properties of its derivatives, making it more suitable for solving PDEs. Moreover, the gradient vanishing problem, which is common in other activation functions, does not occur here, particularly near the origin. In practice, the Gaussian activation function performs better than all the other activation functions mentioned above (see Jiang et al. [47]).

To construct a FNN with L hidden layers, ($L \geq 2$, integer), we need to define at each layer the transformation

$$\Sigma^{i-1} := x^{i-1} \in \mathbb{R}^{N_{i-1}} \longmapsto y^i := \sigma(W^{i-1} \cdot x^{i-1} + b^{i-1}) \in \mathbb{R}^{N_i}, \quad \forall\, i \in \{1, ..., L\}, \tag{4.1.3}$$

where x^{i-1} resp. y^i is the inputs resp. outputs of the i^{th} hidden layer, N_{i-1} is the number of neurons of the i^{th} hidden layer with $N_0 = N$. The parameters $(W^{i-1}, b^{i-1}) \in \mathbb{R}^{N_i \times N_{i-1}} \times \mathbb{R}^{N_i}$ are the weights and biases of the i^{th} hidden layer. Namely, given $x := x^0 \in \mathbb{R}^N$ as the main input, the output of our multi-layers FNN reads

$$y = W^L \cdot (\Sigma^{L-1} \circ ... \circ \Sigma^0(x)) + b^L, \tag{4.1.4}$$

where the output $y := N(x, P)$ depends on the main input x and the parameter

$$P := \{W^i, b^i,\ i = 0, ..., L\} \in \Theta,$$

where Θ denote the set of parameters. In a general way, a NN with parameters P is a function of the variable x in $\mathbb{R}^N$ and will be denoted by $N(x, P)$. The Figure 4.2 represents a multi-layer FNN with three hidden layers and four neurons each.

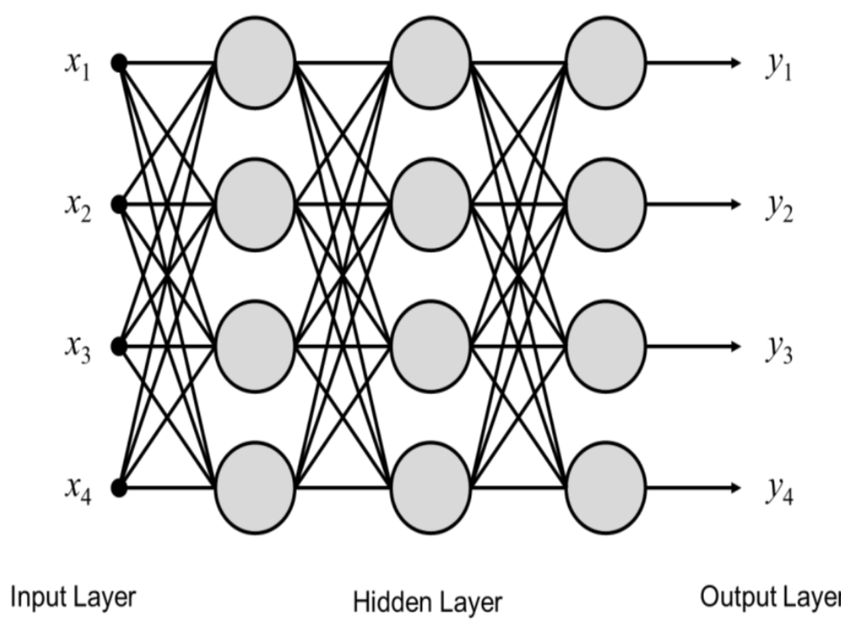


Figure 4.2: Multi-Layer feed-forward neural network.

When we approximate a function using the NN method, the parameter $P \in \Theta$ of the network is optimized through a minimization algorithm. For instance let us assume that we want to approximate a general functional $x \mapsto f(x)$ defined in a domain $\Omega \subset \mathbb{R}^N$ with a multi-layer FNN. The method consists in minimizing the following *Loss* functional.

$$P \mapsto Loss(P) = \|f(x) - N(x,P)\|_{MSE} := \frac{1}{|\hat{\Omega}|} \sum_{x \in \hat{\Omega}} |f(x) - N(x,P)|^2, \tag{4.1.5}$$

where $\hat{\Omega}$ is the set of discrete training points randomly distributed in Ω and on the boundary and $|\hat{\Omega}|$ the number of its elements and $\|\cdot\|_{MSE}$ is the mean squared error defined as in equation (4.1.5). Many other norms could be used to define the *Loss* function such as the the absolute norm or the $L^2 - norm$. We then need to numerically solve the following minimization problem

Search $\hat{P} \in \Theta$ such that:

$$Loss(\hat{P}) = \min_{P \in \Theta} Loss(P), \tag{4.1.6a}$$

where Θ is the set of parameters. Since the optimization problem (4.1.6a) in non convex, we may need many trials to find optimal parameters. The function $x \mapsto N(x, \hat{P})$, where $\hat{P}$ is the optimal parameter, will represents what we called the NN approximation of the function f.

4.1.2 The neural network method for solving Dirichlet PDEs

In this section, we give a description on how to use the neural network method to solve a general initial and Dirichlet boundary value problem.

For $K = 2$ and $T > 0$, we suppose Ω is a bounded domain of $\mathbb{R}^N$ with regular boundary $\partial\Omega$. Let us consider the following nonlinear Dirichlet problem:

Find a smooth function $u : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ solving:

$$\begin{cases} f(x,t,u,\partial_t u, \partial^k_{x_i} u) = 0 \text{ in } \Omega \times (0,T), & (4.1.7a) \\ u = g \text{ on } \partial\Omega \times (0,T), & (4.1.7b) \\ u(\cdot,0) = \varphi \text{ in } \Omega, & (4.1.7c) \end{cases}$$

where for simplicity of presentation, $\partial^k_{x_i} u$ denotes a vector of all first and second order derivatives of u, with $k = 1, ...K, i = 1, ..., N$. The function f is a nonlinear scalar function such that the problem (4.1.7a) − (4.1.7c) is well-posed and g resp. φ represents the Dirichlet boundary condition resp. initial condition.

We present a method for finding an NN approximation to the problem (4.1.7a) − (4.1.7c) by minimizing an appropriate *Loss* function which depends on the parameters of the NN. Let us denote by $\hat{u}$ the NN solution of (4.1.7a) − (4.1.7c) we are looking for. We have

$$\hat{u} = N(x,t,\hat{P}). \tag{4.1.8}$$

The goal is to find the parameter $\hat{P} \in \Theta$, so that $\hat{u}$ approximates (4.1.7a) − (4.1.7c) as well as possible. In practice, there are two methods to consider.

- First method

The first method consists in including the BC and the IC into the *Loss* function defined by

$$Loss(P) = \frac{1}{N_f}\sum_{j=1}^{N_f}\Big|f(x_f^j, t_f^j, N(x_f^j, t_f^j, P), \partial_t^k N(x_f^j, t_f^j, P), \partial_{x_i}^k N(x_f^j, t_f^j, P))\Big|^2 + \frac{1}{N_{BC}}\sum_{j=1}^{N_{BC}}\Big|N(x_{BC}^j, t_{BC}^j, P) - g(x_{BC}^j, t_{BC}^j)\Big|^2 + \frac{1}{N_{IC}}\sum_{j=1}^{N_{IC}}\Big|N(x_{IC}^j, 0, P) - \varphi(x_{IC}^j)\Big|^2. \tag{4.1.9}$$

Here, $(x_f^j, t_f^j)_j$ represents the N_f training data randomly distributed inside the space-time domain $\Omega \times (0, T)$, $(x_{BC}^j, t_{BC}^j)_j$ are the N_{BC} training data on the boundary $\partial\Omega \times (0, T)$ and $(x_{IC}^j)_j$ are the N_{IC} training data chosen inside the domain Ω to enforce the initial condition. The first method consist in finding the optimal parameter by solving the following optimization problem:

Find $\hat{P} \in \Theta$ such that:

$$Loss(\hat{P}) = \min_{P\in\Theta} Loss(P). \tag{4.1.10a}$$

- Second method

The second method consists in including the boundary and initial condition directly inside the network when it is possible. More precisely, we consider the following new network

$$N_1(x, t, P) := A(x, t) + F(x, t, N(x, t, P)), \tag{4.1.11}$$

where $A(x, t)$ satisfies only the boundary and initial condition and does not depend on the parameter P. The network $N_1(x, t, P)$ is build with the particularity to satisfy automatically the boundary and initial condition. The term $F(x, t, N(x, t, P))$ is constructed such as to have no contribution for the boundary and initial condition. In this case, the *Loss* function to minimize reads

$$Loss(P) = \frac{1}{N_f}\sum_{j=1}^{N_f}\frac{1}{N_f}\Big|f(x_f^j, t_f^j, N_1(x_f^j, t_f^j, P), \partial_t^k N_1(x_f^j, t_f^j, P), \partial_{x_i}^k N_1(x_f^j, t_f^j, P))\Big|^2. \tag{4.1.12}$$

So, the approximation $\hat{u} = N_1(x, t, \hat{P})$ is identified by the parameters $\hat{P}$ which solve the following problem. Find $\hat{P} \in \Theta$ such that:

$$Loss(\hat{P}) = \min_{P\in\Theta} Loss(P). \tag{4.1.13a}$$

In this case, the approximated solution of our PDE (4.1.7a) − (4.1.7c) is given by

$$\hat{u} = N_1(x, t, \hat{P}). \tag{4.1.14}$$

We can consider the minimization problems ($4.1.10a$) and ($4.1.13a$) as the training procedure of the NN. It involves the computation of the gradient of the *Loss* with respect to the parameters.

4.1.2.1 Some examples of using neural network to solve initial value problems

Here we give an illustration of the NN method on simple PDEs. We start with the first method and then the second method follows.

(i) Let us consider the following initial value problem:

Find u, a C^1 function solving:

$$\begin{cases} u'(x) = f(x,u) \text{ in } [0,1], & (4.1.15a) \\ u(0) = a, & (4.1.15b) \end{cases}$$

where $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is a nonlinear scalar function. If we apply the first method, we have the following *Loss* function

$$Loss(P) = \frac{1}{N_f} \sum_{j=1}^{N_f} \left| \partial_x N(x_f^j, P) - f(x_f^j, N(x_f^j, P)) \right|^2 + |N(0,P) - a|^2. \tag{4.1.16}$$

The solution is given by $\hat{u} = N(x, \hat{P})$, where $\hat{P}$ is obtained by solving the following minimization problem:

Search $\hat{P} \in \Theta$ such that:

$$Loss(\hat{P}) = \min_{P \in \Theta} Loss(P). \tag{4.1.17a}$$

Now we give the formulation using the second method. For the Problem (4.1.15a) – (4.1.15b), we introduce the new network N_1 as

$$N_1(x,P) = a + xN(x,P), \tag{4.1.18}$$

with

$$\partial_x(N_1(x,P)) = N(x,P) + x\partial_x N(x,P).$$

Here $A(x) = a$ satisfies the BC and $F(x, N(x,p)) = xN(x,P)$. The network is trained only inside the domain because the boundary condition is imposed on the new network. The *Loss* function reads

$$Loss(P) = \frac{1}{N_f} \sum_{j=1}^{N_f} \left| \partial_x N_1(x_f^j, P) - f(x_f^j, N_1(x_f^j, P)) \right|^2, \tag{4.1.19}$$

and the solution is $\hat{u} = N_1(x, \hat{P})$, where $\hat{P}$ solves

Find $\hat{P} \in \Theta$ such that:

$$Loss(\hat{P}) = \min_{P \in \Theta} Loss(P) \tag{4.1.20a}$$

(ii) Another example is the following second order an nonlinear ODE:

Find u, a C^2 function solving:

$$\begin{cases} u''(x) = f(x,u,u') \text{ in } [0,1] & (4.1.21a)\\ u(0) = a & (4.1.21b)\\ u'(0) = b. & (4.1.21c)\end{cases}$$

The *Loss* function is given by

$$Loss(P) = \frac{1}{N_f}\sum_{j=1}^{N_f}\left|\partial^2_{x^2}N(x_f^j,P) - f(x_f^j, N(x_f^j,P), \partial_x N(x_f^j,P))\right|^2 + |N(0,P)-a|^2 + |\partial_x N(0,P)-b|^2. \quad (4.1.22)$$

and the solution is given by $\hat{u} = N(x,\hat{P})$, where $\hat{P}$ solves

Search $\hat{P} \in \Theta$ such that:

$$Loss(\hat{P}) = \min_{P\in\Theta} Loss(P). \quad (4.1.23a)$$

For the problem (4.1.21a) − (4.1.21c), the new network reads

$$N_1(x,P) = a + bx + x^2 N(x,P).$$

If the BC were $u(0) = a$ and $u(1) = b$ then we would choose:

$$N_1(x,P) = a(1-x) + bx + x(1-x)N(x,P).$$

The *Loss* function reads

$$Loss(P) = \frac{1}{N_f}\sum_{j=1}^{N_f}\left|(\partial^2_x N_1)(x_f^j,P) - f(x_f^j, N_1(x_f^j,P), (\partial_x N_1)(x_f^j,P))\right|^2 \quad (4.1.24)$$

and the solution is still $\hat{u} = N_1(x,\hat{P})$, where $\hat{P}$ solves

Find $\hat{P} \in \Theta$ such that:

$$Loss(\hat{P}) = \min_{P\in\Theta} Loss(P). \quad (4.1.25a)$$

4.2 General formulation of the radial basis neural network functions

Here, we introduce the formulation of a special type of FNN architecture, called a radial basis neural network (RBNN). A RBNN is a specific case of an FNN with only one hidden layer. The advantage of this network is that, as a function of its weight parameters w_j, a RBNN is a convex function, making the optimization more accurate than the traditional FNNs with multiple hidden layers (see [57, 24, 69, 32, 9, 47]).

The concept of radial basis neural network functions (RBNNF) was first proposed by Hardy [42] using multi-variate scattered nodes interpolation and was gradually developed by Micchelli [74] and Powell [82] and his collaborators at the university of Cambridge. It was initially used by Kansa [50, 51] to solve PDEs. So far, many authors are using RBNNFs to solve PDEs (see [57, 24, 69, 32, 9, 47]). The idea of the method is as follow.

For a given input $x \in \mathbb{R}^N,\ N \in \mathbb{N}$ and a given radial activation function φ, the output of our network with n neurons ($n \in \mathbb{N}\backslash\{0\}$) reads

$$N(x, P) = \sum_{j=1}^{n} w_j \varphi(\|x - c^j\|), \tag{4.2.1}$$

where $P = {}^t[w_1, \cdots, w_n]$ is a vector representing the set of parameters and $c = {}^t[c^1, \cdots, c^n]$ a vector of vectors representing the different centers. We recall that here $\|\cdot\|$ represents the Euclidean norm in $\mathbb{R}^N$.

Let f be a scalar function known only on distinct collocation points $\{x^i,\ i = 1, ..., m\}$, with $m \in \mathbb{N} \setminus \{0\}$. The function f can be approximated using RBNN functions by solving the linear system

$$AP = b, \tag{4.2.2}$$

where $b = {}^t[f(x^1), ..., f(x^m)],\ P = {}^t[w_1, ..., w_n]$, the unknown set of parameters and $A = [a_{i,j}],\ i = 1., ..., m,\ j = 1, ..., n$ is the interpolation matrix defined by $a_{i,j} = \varphi(\|x^i - c^j\|)$. If $n = m$ and the matrix A is non-singular, then the solution of the linear system (4.2.2) is given by $\hat{P} = A^{-1}b$. As in the previous section, if we denote by $\hat{f}$ the approximation of f by RBNN functions then we have

$$\hat{f} = N(x, \hat{P}). \tag{4.2.3}$$

The singularity of the interpolation matrix A highly depends on the choice of the radial activation function and its parameters and how the training points are scattered in the domain. In practice, when $n = m$ we take $c^j = x^j$. In this case, the matrix A becomes symmetric. And since it is a non negative matrix, therefore, A is non-singular if it is positive definite, that is ${}^t zAz > 0,\ \forall z \in \mathbb{R}^n \setminus \{0\}$ (see [48, 32]). This implies that all the eigenvalues are strictly positive.

Fornberg and Flyer [32], Bochner [9] have proven that for the Gaussian radial function *i.e*, $\varphi(r) = \exp(-(\varepsilon r)^2)$, the interpolation matrix is positive definite. The number ε is called the shape parameter. Table 4.3 represents some examples of radial functions which were proven in [32] to be non-singular for any number of points scattered in the domain in the case $n = m$ and $a_{i,j} = \varphi(\|x^i - x^j\|)$.

Type of basis function	Radial function $\phi(r)$
Piecewise smooth RBFs	
Polyharmonic spline (PHS)	$r^m, m = 1, 3, 5, \ldots$
	$r^m \log(r), m = 2, 4, 6, \ldots$
Compact support ('Wendland')	$(1 - \varepsilon r)_+^m p(\varepsilon r)$, p certain polynomials
Infinitely smooth RBFs	
Gaussian (GA)	$e^{-(\varepsilon r)^2}$
Multiquadric (MQ)	$\sqrt{1 + (\varepsilon r)^2}$
Inverse quadratic (IQ)	$1/(1 + (\varepsilon r)^2)$
Inverse multiquadric (IMQ)	$1/\sqrt{1 + (\varepsilon r)^2}$
Bessel (BE) ($d = 1, 2, \ldots$)	$J_{d/2-1}(\varepsilon r)/(\varepsilon r)^{d/2-1}$

Table 4.3: Some examples of RBFs

The unknown parameter P in (4.2.2) is also found by minimizing the following *Loss* function

$$Loss(P) := \frac{1}{m}\sum_{i=1}^{m}(N(x^i, P) - f(x^i))^2 = \frac{1}{m}\sum_{i=1}^{m}\Big(\sum_{j=1}^{n} w_j \varphi(\|x^i - c^j\|) - f(x^i)\Big)^2. \quad (4.2.4)$$

RBNNFs is also used to solve PDEs by minimizing the associated loss function. Precisely, let us consider the following general scalar-valued PDEs

$$\begin{cases} L(u) = f & \text{in } \Omega \\ B(u) = g & \text{on } \partial\Omega, \end{cases} \quad (4.2.5)$$

where L and B are differential operators. The idea is to approximate the solution of (4.2.5) with a NN function of the form (4.2.1), next the new problem will be to minimize the loss function

$$Loss(P) = \frac{1}{N_L}\sum_{i=1}^{N_L}(L(N(x_L^i, P)) - f(x_L^i))^2 + \frac{1}{N_B}\sum_{i=1}^{N_B}(B(N(x_B^i, P)) - g(x_B^i))^2 \quad (4.2.6)$$

where for $\alpha \in \{L, B\}$, x_α^i is the collocation points inside the domain, resp. on the boundary and N_α is the number of collocation points inside the domain resp. on the boundary. This corresponds to the first method described in Section 4.1.2 for the FNN.

Whenever it is possible, we look for the neural solution of (4.2.5) in the following new form

$$N_1(x, P) = A(x)N(x, P) + g(x), \quad (4.2.7)$$

where $x \mapsto A(x)$ is a function which vanishes on the boundary and $x \mapsto g(x)$ enforces the boundary condition. This corresponds to the second method described in Section 4.1.2 for the FNN. Therefore, the boundary condition becomes strongly imposed. In this case, we consider the following *Loss* function

$$Loss(P) = \frac{1}{N_L}\sum_{i=1}^{N_L}(L(N_1(x_L^i, P)) - f(x_L^i))^2. \tag{4.2.8}$$

If $\hat{P}$ is the optimal parameter, then the solution $\hat{u}$ to the PDEs (4.2.5) is given by $\hat{u} = N_1(x, \hat{P})$.

4.3 Application of the radial basis neural network functions to solve the Stokes problem: The augmented Lagrangian approach

Let Ω be a bounded domain of $\mathbb{R}^N$, $(N = 2)$ with smooth boundary $\partial\Omega$. We consider the following generalized Stokes problem

$$\begin{cases} \alpha\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega, & (4.3.1)\\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega, & (4.3.2)\\ \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega, & (4.3.3)\end{cases}$$

where the unknown $\mathbf{u}$ and p represent respectively the velocity field and the pressure of an incompressible fluid. We know according to [87, 98, 30] that if $(\mathbf{f}, \mathbf{g}) \in \mathbf{H}^{-1}(\Omega_0) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega_0)$, under the general flux condition

$$\int_{\partial\Omega_0} \mathbf{g}\cdot\boldsymbol{\nu}\, \mathrm{d}s = 0, \tag{4.3.4}$$

where Ω_0 is open, bounded and $C^{1,1}$, then there exists a unique weak solution $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega_0) \times L_0^2(\Omega_0)$ to the problem (4.3.1) − (4.3.3) for every $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$ satisfying a certain condition (see Theorem 2.2.3).

For the Stokes problem (4.3.1) − (4.3.3), the difficulty is on the numerical implementation of both the incompressibility condition (4.3.2) and the boundary condition (4.3.3). When we use neural network, it is possible to impose only one of those conditions on the network but not both at the same time. For instance, in dimension $N = 2$, looking the velocity $\mathbf{u} = (u_1, u_2)$ as the rotational of a scalar stream function φ (*i.e* $u_1 = \partial_{x_2}\varphi$ and $u_1 = -\partial_{x_1}\varphi$) permits to impose the free divergence condition. On the other-hand, using the new network N_1 as described in equation (4.2.7) permits to impose the boundary condition. However, if the geometry of the domain is complex, the function $x \mapsto A(x)$ can be very difficult to find. Moreover, it is more difficult to find a network on which both the free divergence and the boundary condition are imposed. The augmented Lagrangian formulation appears to be a way out.

Firstly, we imposing the boundary condition by using the network $\mathbf{N}_1$ as

$$\mathbf{N}_1(x, P) = A(x)\mathbf{N}(x, P) + \mathbf{g}(x), \tag{4.3.5}$$

where A is as before a scalar function which vanished on the boundary, $\mathbf{g}$ is the boundary condition and $\mathbf{N}$ is the RBNN vector function. In practice, in the case where the geometry of the domain is complex as it is in the case for an AH domain, the function $A(x)$ is obtained by training a network to have a characteristic function of the domain.

Secondly, we introduce the Lagrangian functional associated to the problem (4.3.1) – (4.3.3) as

$$\mathcal{L}(\mathbf{u}, p) = \int_\Omega \frac{\alpha}{2}|\mathbf{u}|^2 + \frac{\mu}{2}|\nabla\mathbf{u}|^2 - p\nabla\cdot\mathbf{u} - \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x. \tag{4.3.6}$$

We know according to [35, 34] that the solution to the Stokes problem (4.3.1) – (4.3.3) is also solution to the following saddle point problem

Find $(\mathbf{u}, p)$ such that

$$\mathcal{L}(\mathbf{u}, q) \leq \mathcal{L}(\mathbf{u}, p) \leq \mathcal{L}(\mathbf{v}, p) \quad \forall\,(\mathbf{v}, q) \tag{4.3.7}$$

which implies that

$$\min_{\mathbf{v}}\max_{q}\mathcal{L}(\mathbf{v}, q) = \max_{q}\min_{\mathbf{v}}\mathcal{L}(\mathbf{v}, q) = \mathcal{L}(\mathbf{u}, p). \tag{4.3.8}$$

Now, following [34, 35], we introduce the augmented Lagrangian as

$$\mathcal{L}_r(\mathbf{u}, p) := \int_\Omega \frac{\alpha}{2}|\mathbf{u}|^2 + \frac{\mu}{2}|\nabla\mathbf{u}|^2 - p\nabla\cdot\mathbf{u} - \mathbf{f}\cdot\mathbf{u} + \frac{r}{2}(\nabla\cdot\mathbf{u})^2\,\mathrm{d}x, \tag{4.3.9}$$

which characterizes the problem

Find $(\mathbf{u}, p)$ such that:

$$\begin{cases} \alpha\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p - r\nabla(\nabla\cdot u) = \mathbf{f} \text{ in } \Omega & (4.3.10) \\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega & (4.3.11) \\ \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega. & (4.3.12) \end{cases}$$

In fact, the problem (4.3.1) – (4.3.3) and (4.3.10) – (4.3.12) are equivalent. The advantage of the augmented Lagrangian formulation is on the use of the penalty term $r\nabla(\nabla\cdot\mathbf{u})$ which help to implement well the free divergence condition, especially using Uzawa's algorithm (see [33, 35, 6, 7, 93, 78, 85, 37]). In practice, when we use Uzawa's algorithm, r is chosen in term of ρ as given by equation (4.3.15) and Figure 4.4.

Description of the Uzawa algorithm

- We initialize the pressure as $p^0 = p(x, P^0)$, where P^0 is the given initial parameter.

- We suppose known the pressure at the n^{th} iteration $p^n = p(x, P^n)$, and we compute the network representing the velocity $\mathbf{u}^n = \mathbf{u}(x, P^n)$ by minimizing the augmented Lagrangian, i.e., solving

$$\mathcal{L}_r(\mathbf{u}^n, p^n) = \min_{\mathbf{v}}\{\mathcal{L}_r(\mathbf{v}, p^n)\}. \tag{4.3.13}$$

  So, for p^n fixed, we are looking for the parameter P^n such that $\mathbf{u}^n := \mathbf{u}(\cdot, P^n)$ minimizes the augmented Lagrangian.

- Finally, we update the pressure using the formula

$$p^{n+1} = p^n - \rho \nabla \cdot \mathbf{u}^n. \tag{4.3.14}$$

- Repeat the process until the algorithm converges, meaning the Lagrangian no longer updates with each iteration.

The convergence of the Uzawa algorithm highly depends on the choice of the penalty r and the choice of ρ. In general, according to [35, 34], the algorithm converges under the condition

$$\rho \in (0, 2(r + \frac{1}{\mu})). \tag{4.3.15}$$

One experimental result they found is that for a given r sufficiently large, we should take ρ slightly larger that r as described the following Figure 4.4.

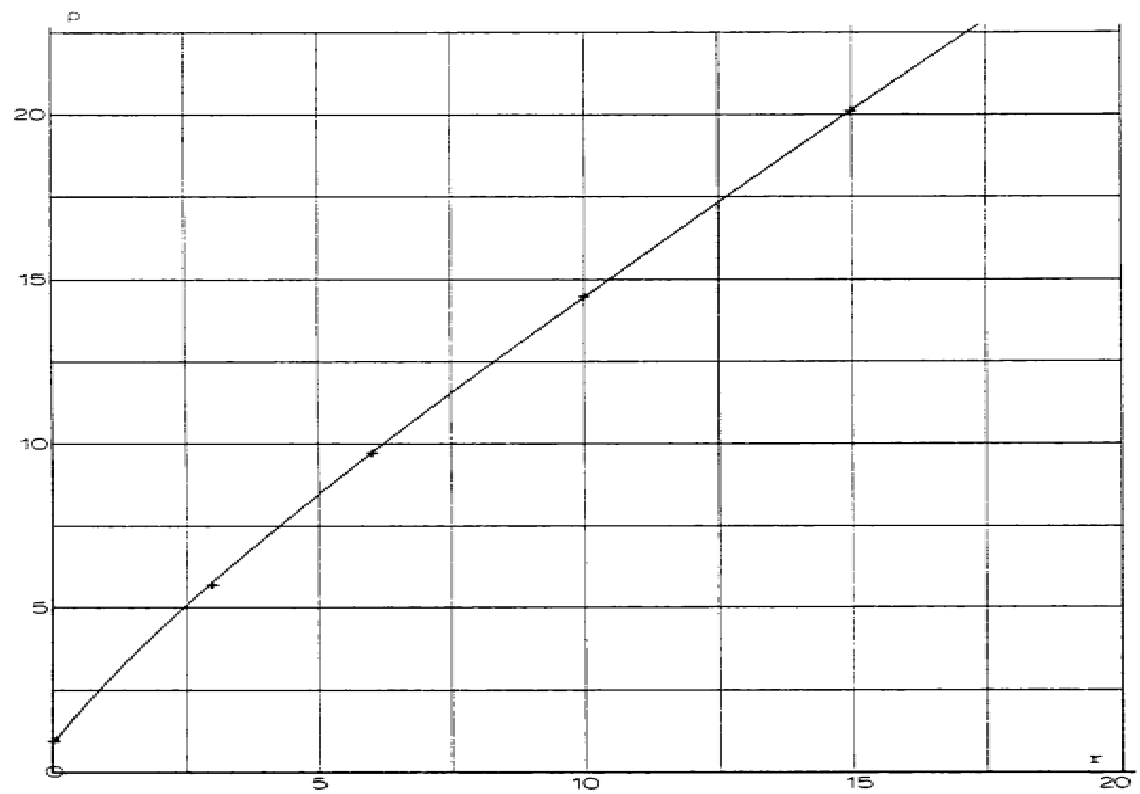


Figure 4.4: Agreement between r and ρ
taken from the reference Fortin and Thomasset [35]

4.4 Application of radial basis neural network on a Navier-Stokes problem: The augmented Lagrangian formulation

In this section, we consider the time-dependent and nonlinear NS problem (4.4.1)−(4.4.3). Our goal is to solve it numerically using radial basis neural network (RBNN) functions, which typically is a Galerkin method. We opt for the RBNN instead of the traditional feedforward neural network (FNN) because the RBNN function as a function parameter is convex and this enables standard minimization algorithms (e.g., gradient descent, Krylov, L-BFGS) to perform more efficiently. Indeed, we have tried classical FNN methods as descibed in Section 4.1, but our algorithm did not provide good results.

To achieve this, we discretize the time derivative term $\partial_t \mathbf{u}$ using the backward differentiation formula of second order (BDF2), as introduced in Section (4.4.3), combined with a second-order Runge-Kutta method following Bermejo and Saavedra [8]. Additionally, we apply a Crank-Nicolson scheme to the diffusion and augmented terms. In the subsequent sections, we provide a detailed explanation of the methodology. At each time step, we solve a problem of the type (4.3.1) − (4.3.3), as previously described for the Stokes problem.

4.4.1 Lagrangian approach

For $T > 0$, we consider the following incompressible NS equation for a Newtonian fluid

$$\begin{cases} \rho(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f} \text{ in } \Omega\times(0,T), & (4.4.1)\\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\times(0,T), & (4.4.2)\\ \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega\times(0,T), & (4.4.3)\end{cases}$$

where ρ is the density of the fluid and μ the dynamic viscosity of the fluid. We can suppose that we have initial or periodic condition. Note that in Chapter 3 we proved the existence and uniqueness of solution to this kind of problem in the periodic case. For the initial case, the analysis can be found in [98, 85, 37, 75]. Here we focus only on the implementation of our numerical method.

We will first change the problem into its equivalent dimensionless formulation. Let us make the following change of variable

$$x^* = \frac{1}{D}x,\ t^* = \frac{tU}{D},\ \mathbf{u}^* = \frac{\mathbf{u}}{U},\ p^* = \frac{p}{\rho U^2} \tag{4.4.4}$$

where U and D represent respectively the velocity of the fluid flow (m/s) and the characteristic length (m). Therefore, the new differential operators for the dimensionless variables read

$$\nabla = \frac{1}{D}\nabla^*,\ \Delta = \frac{1}{D^2}\Delta^*,\ \partial_t = \frac{U}{D}\partial_{t^*}. \tag{4.4.5}$$

By multiplying equation (4.4.1) – (4.4.3) by the factor $\frac{D}{\rho U^2}$ and setting $Re = \frac{\rho D U}{\mu}$, we can rewrite the equation (4.4.1) – (4.4.3) in terms of the new (dimensionless) variables and operators as

$$\begin{cases} \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{1}{Re}\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega\times(0,T), & (4.4.6)\\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\times(0,T), & (4.4.7)\\ \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega\times(0,T), & (4.4.8)\end{cases}$$

where all the stars on the variables and the operators were dropped to ease the notations. The dimensionless number Re represents the Reynolds number which is also defined as the ratio between the inertia and the viscus forces. Here we denote the new dimensionless boundary condition and source term by $\mathbf{g}^* = \mathbf{g}/U$ and $\mathbf{f}^* = \frac{D}{U^2}\mathbf{f}$.

Let us suppose that $\mathbf{u} = \mathbf{u}(X(x,s,t),t)$ where the map $t \mapsto X(x,s,t)$ represents the trajectory of a fluid particle that at $t = s$ is at the position $x \in \mathbb{R}^N$. Therefore, $t \mapsto X(x,s,t)$ solves the following Cauchy problem

$$\begin{cases} \frac{\mathrm{d}X(x,s,t)}{\mathrm{d}t} = \mathbf{u}(X(x,s,t),t) & (4.4.9)\\ X(x,s,s) = x. & (4.4.10)\end{cases}$$

We know that if $\mathbf{u} \in C^0(\mathbf{C}^{0,1}(\mathbb{R}^N))$ then Cauchy problem (4.4.9) − (4.4.10) has a unique solution given by the following formula

$$X(x,s,t) = x + \int_s^t \mathbf{u}(X(x,s,\tau),\tau)\,\mathrm{d}\tau. \tag{4.4.11}$$

By the chain rule, the time derivative $\frac{D\mathbf{u}}{Dt}$ of the map $t \mapsto \mathbf{u}(X(x,s,t),t)$ reads

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}. \tag{4.4.12}$$

Note that the term $\frac{D\mathbf{u}}{Dt}$ includes the nonlinear term of the NSE (4.4.6) − (4.4.8). The expression (4.4.12) is also called material derivative of the velocity $\mathbf{u}$. Therefore, the equation (4.4.6) − (4.4.8) becomes

$$\begin{cases} \dfrac{D\mathbf{u}}{Dt} - \dfrac{1}{Re}\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega\times(0,T) & (4.4.13)\\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\times(0,T) & (4.4.14)\\ \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega\times(0,T), & (4.4.15)\end{cases}$$

which represents the Lagrangian formulation of (4.4.6) − (4.4.8).

4.4.2 Implementation of the backward differential formula of order one

In this paragraph, we implement the first order semi-discrete Lagrangian scheme by applying a backward differential formula of order 1 (BDF1) for the problem (4.4.13) − (4.4.15), following [8].

We will discretize the time interval $[0,T]$ in $(t_n)_{n=0,\dots,K}$, with $t_0 = 0$ and $t_K = T$, where $K \in \mathbb{N}\setminus\{0\}$. We denote by $\Delta t := t_n - t_{n-1}$ the uniform time step. We remark that $x \mapsto X(x,s,t)$ has a group property in the sense that

$$\forall\, t_1,t_2 \in [0,T],\ X(x,s,t_2) = X(\cdot,t_1,t_2)\circ X(x,s,t_1) = X(X(x,s,t_1),t_1,t_2), \tag{4.4.16}$$

meaning that the trajectory of a particle from s to t_2 is equal to the trajectory form s to some arbitrarily time t_1 plus the trajectory from time t_1 to t_2. As in [8], we will use the notation

$$X^{k,l}(x) := X(x,t_l,t_k),\ \forall\, k,l \in \mathbb{N}, \tag{4.4.17}$$

Taking as initial time $s = t_n > 0$, we approach the material derivative $\frac{D\mathbf{u}}{Dt}$ with the following differentiation formula of order 1.

$$\frac{D\mathbf{u}}{Dt}(x,t_n) = \frac{\mathbf{u}(x,t_n) - \mathbf{u}(X^{n-1,n}(x),t_{n-1})}{\Delta t} + o(\Delta t) \tag{4.4.18}$$

$$= \frac{\mathbf{u}^n - \mathbf{u}^{n-1}\circ X^{n-1,n}}{\Delta t} + o(\Delta t), \tag{4.4.19}$$

where we used the notation $\mathbf{u}^n := \mathbf{u}(\cdot, t_n)$ and $\mathbf{u}^{n-1} \circ X^{n-1,n} := \mathbf{u}(X^{n-1,n}(x), t_{n-1})$. We recall that $X^{n-1,n} = X(x, t_n, t_{n-1})$, is the solution at time t_{n-1} to the ODE

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(x, t_n, t) = \mathbf{u}(X(x, t_n, t), t), \quad \text{in } [t_{n-1}, t_n] & (4.4.20) \\ X(x, t_n, t_n) = x, & (4.4.21) \end{cases}$$

which we solve at each iteration by a second order Runge-Kutta. Precisely, the solution $X^{n-1,n}$ is approximated by

$$X^{n-1,n} \approx x - \frac{\Delta t}{2}(\mathbf{u}(x, t_n) + \mathbf{u}(x - \Delta t\, \mathbf{u}(x, t_n), t_{n-1})). \tag{4.4.22}$$

This permits us to write the following BDF1 semi-discrete Lagrangian scheme for the NS equations, for all integer $n \geq 1$

$$\begin{cases} \dfrac{\mathbf{u}^n - \mathbf{u}^{n-1} \circ X^{n-1,n}}{\Delta t} - \dfrac{1}{Re} \Delta \mathbf{u}^n + \nabla p^n - \mathbf{f}^n = 0 \text{ in } \Omega & (4.4.23) \\ \nabla \cdot \mathbf{u}^n = 0 \ \text{ in } \Omega & (4.4.24) \\ \mathbf{u}^{n-1} \text{ known}, & (4.4.25) \end{cases}$$

where we use the notation $\mathbf{f}^n := \mathbf{f}(\cdot, t_n)$.

The above semi-discrete Lagrangian scheme initially proposed by Pironneau [80] gives an approximation of the weak solution $(\mathbf{u}, p)$ of order $O(\Delta t)$. It is very important to precise that at each time n, since $\mathbf{u}^{n-1}$ is known, equation (4.4.23) – (4.4.25) can be rewrite as the following Stokes equation

$$\begin{cases} \dfrac{1}{\Delta t} \mathbf{u}^n - \dfrac{1}{Re} \Delta \mathbf{u}^n + \nabla p^n = \mathbf{F}^n & (4.4.26) \\ \nabla \cdot \mathbf{u}^n = 0 & (4.4.27) \\ \mathbf{u}^n = \mathbf{g}^n. & (4.4.28) \end{cases}$$

where $\mathbf{F}^n := \mathbf{f}^n + \frac{1}{\Delta t} \mathbf{u}^{n-1} \circ X^{n-1,n}$. We remark that equation (4.4.26) – (4.4.28) is of the Stokes form (4.3.1) – (4.3.3) which we described in the previous section with $\alpha = \frac{1}{\Delta t}$ and $\mu = \frac{1}{Re}$. Therefore, in the case of the nonlinear NS equation, we solve by applying the same Uzawa's algorithm at each time step $n \geq 1$.

The BDF1 is simple to implement and produce reasonable results for Stokes and NS equation with moderate Re. But as the Re increases, we needed to applied the BDF2 described in the next paragraph (4.4.3), which produce a more accurate results.

4.4.3 Implementation of the of the backward differential formula of order two

In paragraph, we will implement a second order semi-discrete Lagrangian scheme by applying a backward differential formula of order 2 (BDF2) as in [8].

Taking again as initial time $s = t_n > 0$, we approach the material derivative $\frac{D\mathbf{u}}{Dt}$ with the following differentiation formula of order 2.

$$\frac{D\mathbf{u}}{Dt}(x, t_n) = \frac{3\mathbf{u}(x, t_n) - 4\mathbf{u}(X^{n-1,n}(x), t_{n-1}) + \mathbf{u}(X^{n-2,n}(x), t_{n-2})}{2\Delta t} + o(\Delta t^2) \quad (4.4.29)$$

$$= \frac{3\mathbf{u}^n - 4\mathbf{u}^{n-1} \circ X^{n-1,n} + \mathbf{u}^{n-2} \circ X^{n-2,n}}{2\Delta t} + o(\Delta t^2). \quad (4.4.30)$$

For all integer $n \geq 2$, we numerically solve the following BDF2

$$\begin{cases} \dfrac{3\mathbf{u}^n - 4\mathbf{u}^{n-1} \circ X^{n-1,n} + \mathbf{u}^{n-2} \circ X^{n-2,n}}{2\Delta t} - \dfrac{1}{Re}\Delta \mathbf{u}^n + \nabla p^n - \mathbf{f}^n = 0 \text{ in } \Omega & (4.4.31) \\ \nabla \cdot \mathbf{u}^n = 0 \text{ in } \Omega & (4.4.32) \\ \mathbf{u}^{n-1},\ \mathbf{u}^{n-2} \text{ known.} & (4.4.33) \end{cases}$$

According to [8], the above scheme gives an approximation of a weak solution $(\mathbf{u}, p)$ of the order $O(\Delta t^2)$. Here at each time step $n \geq 2$ we solve successively $X^{n-1,n}$ and $X^{n-2,n}$ using a second order Runge-Kutta. Indeed, $X^{n-1,n}$ is given by equation (4.4.22) and $X^{n-2,n}$ is approximated by

$$X^{n-2,n} \approx X^{n-1,n} - \frac{\Delta t}{2}(\mathbf{u}(X^{n-1,n}, t_{n-1}) + \mathbf{u}(X^{n-1,n} - \Delta t\ \mathbf{u}(X^{n-1,n}, t_{n-1}), t_{n-2}), \quad (4.4.34)$$

The problem (4.4.31) – (4.4.33) can be rewrite into the following model Stokes problem (4.3.1) – (4.3.3) we described earlier

$$\begin{cases} \dfrac{3}{2\Delta t}\mathbf{u}^n - \dfrac{1}{Re}\Delta \mathbf{u}^n + \nabla p^n = \mathbf{H}^n \text{ in } \Omega & (4.4.35) \\ \nabla \cdot \mathbf{u}^n = 0 \text{ in } \Omega & (4.4.36) \\ \mathbf{u}^{n-1}, \mathbf{u}^{n-2} \text{ known,} & (4.4.37) \end{cases}$$

with $\mathbf{H}^n := \mathbf{f}^n + \frac{4}{2\Delta t}\mathbf{u}^{n-1} \circ X^{n-1,n} + \frac{1}{2\Delta t}\mathbf{u}^{n-2} \circ X^{n-2,n}$. We solve again at each time step $n \geq 2$ the Stokes problem (4.4.35) – (4.4.37) using Uzawa's algorithm.

4.4.4 Implementation of the augmented Lagrangian method and Crank-Nicolson scheme

In this paragraph, we apply the augmented Lagrangian method to the Stokes problem (4.4.35) – (4.4.36). We recall that the boundary condition is automatically imposed in the network and the augmented Lagrangian method will help us to solve well the free divergence problem. We also add the Crank-Nicolson (CN) scheme on both the diffusion and the augmented term by averaging them at two consecutive time steps. It is important to mention that adding a CN scheme was really important in the sense that it has helped improving the accuracy for the lid-driven cavity benchmark problem for NS flow. Indeed, without implementation of the CN scheme, the result were reasonable up to $Re = 1,000$ and with the CN scheme implemented, we are able to go until $Re = 10,000$, while maintaining a good agreement with the results from the literature.

For all interger $n \geq 2$, the equivalent augmented Lagrangian formulation associated to the problem (4.4.35) – (4.4.37) reads

$$\begin{cases} \dfrac{3}{2\Delta t}\mathbf{u}^n - \dfrac{1}{Re}\Delta \mathbf{u}^n + \nabla p^n - \mathbf{H}^n - r\nabla(\nabla \cdot \mathbf{u}^n) = 0 \text{ in } \Omega & (4.4.38) \\ \nabla \cdot \mathbf{u}^n = 0 \text{ in } \Omega & (4.4.39) \\ \mathbf{u}^{n-1},\ \mathbf{u}^{n-2} \ \text{ known}, & (4.4.40) \end{cases}$$

where $r\nabla(\nabla \cdot \mathbf{u}^n)$ is called the augmented term. We apply the CN scheme (see [68, 70]) by averaging both the diffusion and augmented term at time t_{n-1} and t_n and get the following scheme

$$\begin{cases} \dfrac{3}{2\Delta t}\mathbf{u}^n - \dfrac{1}{2Re}(\Delta \mathbf{u}^n + \Delta \mathbf{u}^{n-1}) + \nabla p^n - \mathbf{H}^n - \dfrac{r}{2}\nabla(\nabla \cdot \mathbf{u}^n + \nabla \cdot \mathbf{u}^{n-1}) = 0 \text{ in } \Omega & (4.4.41) \\ \nabla \cdot \mathbf{u}^n = 0 \text{ in } \Omega & (4.4.42) \\ \mathbf{u}^{n-1},\ \mathbf{u}^{n-2} \ \text{ known}, & (4.4.43) \end{cases}$$

which can be put in the form of the model Stokes problem (4.3.10) – (4.3.12)

$$\begin{cases} \dfrac{3}{2\Delta t}\mathbf{u}^n - \dfrac{1}{2Re}\Delta \mathbf{u}^n + \nabla p^n - \dfrac{r}{2}\nabla(\nabla \cdot \mathbf{u}^n) = \mathbf{G}^n \text{ in } \Omega & (4.4.44) \\ \nabla \cdot \mathbf{u}^n = 0 \text{ in } \Omega & (4.4.45) \\ \mathbf{u}^{n-1}, \mathbf{u}^{n-2} \text{ known} & (4.4.46) \end{cases}$$

with $\mathbf{G}^n := \mathbf{H}^n + \frac{1}{2Re}\Delta \mathbf{u}^{n-1} + \frac{r}{2}\nabla(\nabla \cdot \mathbf{u}^{n-1})$. At each time step, we solves again the problem (4.4.44) – (4.4.46) using Uzawa's algorithm with the following full expression of the augmented Lagrangian

$$\begin{aligned} \mathcal{L}_r(\mathbf{u}^n, p^n) = & \frac{1}{2\Delta t}\int_\Omega \frac{3}{2}|\mathbf{u}^n|^2 - 4\mathbf{u}^{n-1} \circ X^{n-1,n} \cdot \mathbf{u}^n + \mathbf{u}^{n-2} \circ X^{n-2,n} \cdot \mathbf{u}^n \, \mathrm{d}x \\ & + \frac{1}{2Re}\int_\Omega \frac{1}{2}|\nabla \mathbf{u}^n|^2 + \nabla \mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n \, \mathrm{d}x \\ & - \int_\Omega p^n \nabla \cdot \mathbf{u}^n \, \mathrm{d}x \\ & + \int_\Omega \mathbf{f}^n \cdot \mathbf{u}^n \, \mathrm{d}x \\ & + \frac{r}{2}\int_\Omega \frac{1}{2}(\nabla \cdot \mathbf{u}^n)^2 + (\nabla \cdot \mathbf{u}^{n-1})(\nabla \cdot \mathbf{u}^n) \, \mathrm{d}x. \end{aligned} \quad (4.4.47)$$

4.5 Some numerical results

In this section, we present some numerical simulations in 2D. The aims is to solve numerically our time-periodic NS problem on fixed domain, with time dependent velocity profile given on the inlet, outlet and membrane.

To reach this target, our approach is to gradually solve numerically several benchmark problems for the Stokes, and Navier-Stokes flow. Indeed, we start by presenting our result for a particular time-periodic NS problem called here "toy" problem. For this problem, the exact solution can be found in [79]. Next, we solve the lid-driven cavity problem for the Stokes flow and compare it with the existing literature by Brian and Housam [12]. Additionally, we solve the lid-driven cavity problem on a NS flow with Reynold number (Re) up to $10,000$. We proceed by solving the Steady state NS on some artificial heart domains. Here the velocity profile on the inlet, outlet and membrane is time independent. Finally, we solve, the time-periodic NS problem on several fixed AH domains, with a given time dependent velocity profile on the inlet, outlet and membrane.

4.5.1 A time-dependent "toy" problem

In this section we consider the time-periodic and nonlinear NS problem (4.4.6) − (4.4.8), where $\mathbf{f} = \mathbf{0}$ and the exact solution is given explicitly in [79] and simple. Moreover, the solution is the same regardless the value of Re. This is why we call it here "toy" problem. Therefore, here we solve it for $Re = 1$. We use a RBNN with a Gaussian non linear radial function $\exp(-(\varepsilon r)^2)$ with the shape parameter $\varepsilon = 19.19$. Here we took $N_G = 50X50 = 2500$ Gauss points inside the domain to compute the integrals and we took $N_{hat} = 50X50 = 2500$ radial basis hat functions (or neurons).

The objective is to capture the exact solution, evaluate the error since the exact solution is known. We take $\Omega := (-1,1)^2$ and the period is $T = \frac{\pi}{2}$. The exact solution $\mathbf{u}_{ex} = (u_{1ex}, u_{2ex})$ is given by

$$u_{1ex}(x,y,t) := x\sin(4t) + y(\cos(4t)+2) \tag{4.5.1}$$

$$u_{2ex}(x,y,t) := x(\cos(4t)-2) - y\sin(4t) \tag{4.5.2}$$

$$p_{ex}(x,y,t) := \frac{3}{2}(x^2+y^2) + 4xy\sin(4t) - 2(x^2-y^2)\cos(4t). \tag{4.5.3}$$

We imposed the boundary condition on the network by taking

$$\mathbf{N}_1(x,y,t,P) = \mathbf{N}(x,y,t,P)(x-1)(x+1)(y-1)(y+1) + \mathbf{g}(x,y,t). \tag{4.5.4}$$

The boundary condition $\mathbf{g}$ is constructed by perturbing the exact solution with a cosine function, such that it matches the exact solution only on the boundary. Specifically, we took

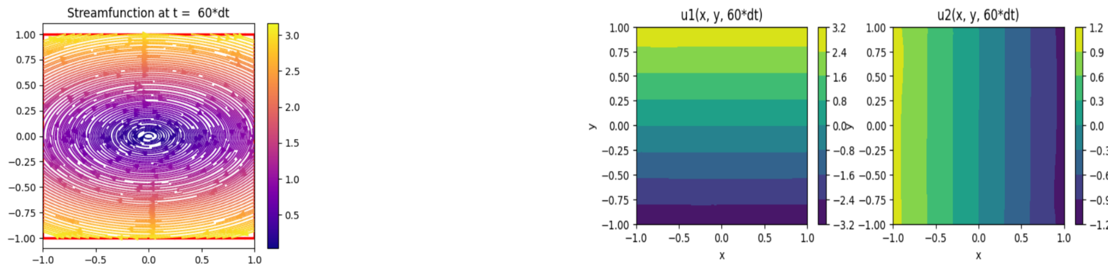
$$\begin{aligned}\mathbf{g}(x,y,t) &= \cos\Big(k\frac{2\pi}{T}(x-1)(x+1)(y-1)(y+1)\Big)\mathbf{u}_{ex}(x,y,t)\\ &= \cos(8(x-1)(x+1)(y-1)(y+1))\mathbf{u}_{ex}(x,y,t),\end{aligned} \tag{4.5.5}$$

with $k = 2$ and we note that

$$\mathbf{g} = \mathbf{u}_{ex} \text{ on } \partial\Omega_0 \times (0,T) \tag{4.5.6}$$

and in $(0,T)$, the function $\mathbf{g}$ has 2 oscillations around the exact solution and in $(0,2T) = (0,\pi)$, it has 4 oscillations.

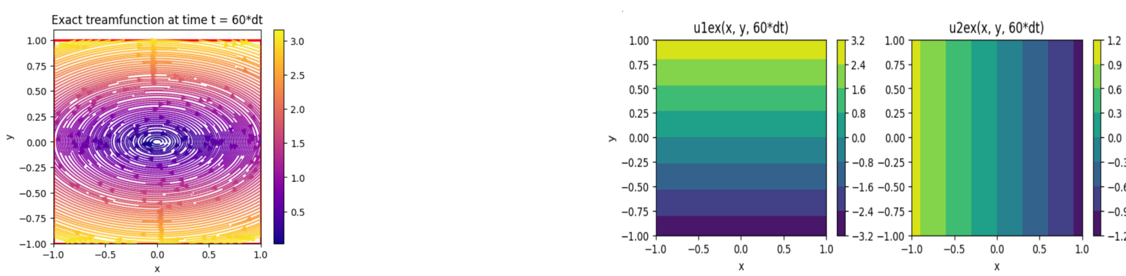
We use the BDF2 augmented Lagrangian scheme associated with CN. The following Figure 4.5 represents the neural network solution of the "toy" problem at time $t = \pi$.



(a) (b)

Figure 4.5: Approximated solution of the time-periodic Navier-Stokes equations ("toy" problem) at time $t = \pi$ i.e., $\mathbf{u}(x, y, \pi)$, using RBNNFs. a) Stream function, b) approximated velocity $(u_1(x, y, \pi), u_2(x, y, \pi))$

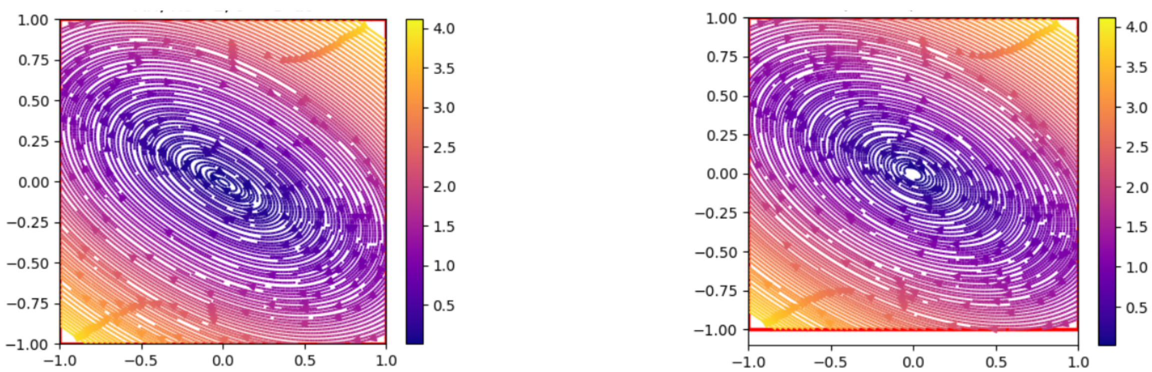
The following Figure 4.6 represents the exact solution of the "toy" problem.



(a) (b)

Figure 4.6: Exact solution of the time-periodic Navier-Stokes equations ("toy" problem) at time $t = \pi$ i.e., $\mathbf{u}_{ex}(x, y, \pi)$. a) Exact Stream function, b) Exact velocity $(u_{1ex}(x, y, \pi), u_{2ex}(x, y, \pi))$

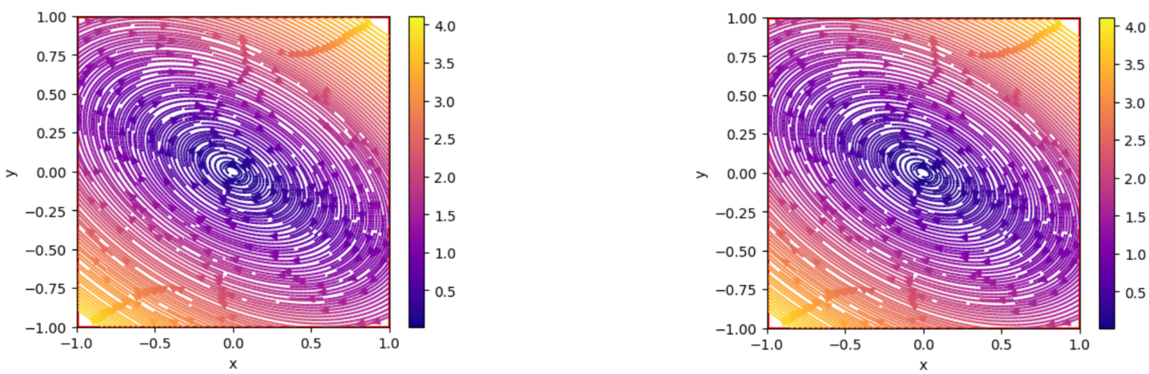
On the following Figure 4.7, we can see that our approximated solution is time-periodic. The period here is $T = \frac{\pi}{2} = 30\Delta t$. We visually compare $\mathbf{u}(x, y, 5\Delta t)$ and $\mathbf{u}(x, y, 35\Delta t)$ and we can notice that they are similar.



(a) $\mathbf{u}(x, y, 5\Delta t)$ (b) $\mathbf{u}(x, y, 35\Delta t)$

Figure 4.7: Time-periodicity of the approximated solution using RBNNFs

We can see from the analytic expression of the exact solution that it is time-periodic of period $T = \frac{\pi}{2} = 30\Delta t$. The following Figure 4.8 permits to see the periodicity of the exact solution.



(a) $\mathbf{u}_{ex}(x, y, 5\Delta t)$ (b) $\mathbf{u}_{ex}(x, y, 35\Delta t)$

Figure 4.8: Time-periodicity of the exact solution to the "toy" problem

We compute L^2 relative error of the solution, which is the $L^2 - norm$ of the difference over the $L^2 - norm$ of the exact solution. For this problem the error is at the order of 10^{-2}. Precisely the error is 0.003056923626 for u_1 and 0.009981707670 for u_2.

4.5.2 Stationary Stokes and Navier stokes flow on the lid-driven cavity problem

The lid-driven cavity is a classic problem in computational fluid dynamic. It usually serves as a benchmark for validating numerical methods. The cavity has a square shape fill with a fluid. At the top boundary of the cavity, we apply a tangential velocity in order to drive the fluid flow inside the cavity. On the three remaining boundaries (left, right and down), we apply a no-slip condition, i.e. zero velocity. Figure 4.9 represents the dimensionless 2D driven cavity we are considering.

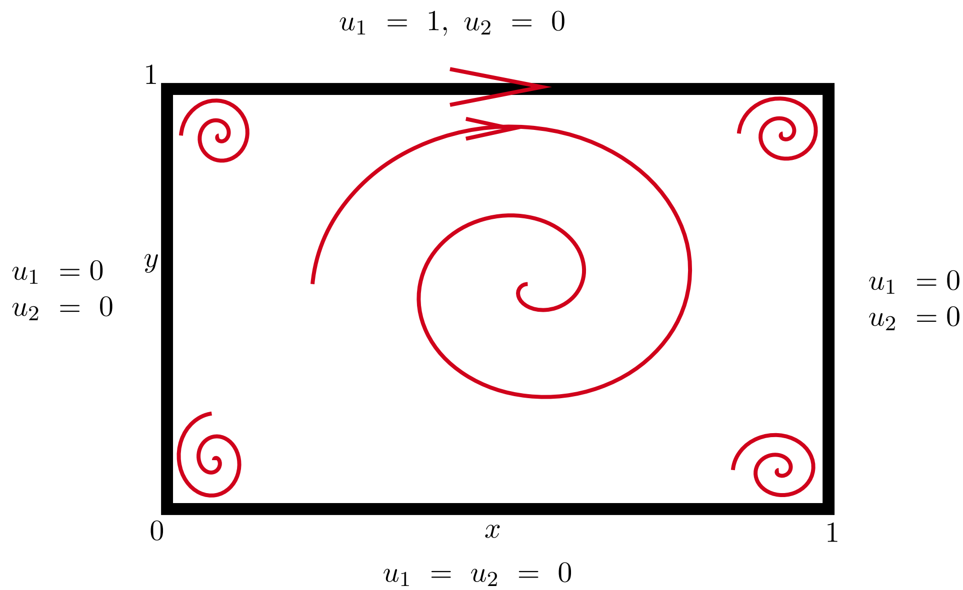


Figure 4.9: The dimensionless 2D lid-driven cavity.

4.5.2.1 Case of a stationary Stokes flow in a driven cavity

Let us consider the motion of a fluid in the above lid-driven cavity along the following Stokes flow

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = 0 \text{ in } \Omega, & (4.5.7) \\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega. & (4.5.8) \end{cases}$$

Here we impose the boundary condition on the network as

$$\mathbf{u}(x, y, P) := \mathbf{N}_1(x, y, P) = \mathbf{N}(x, y, P)xy(1-x)(1-y) + \mathbf{u}_b(x, y), \tag{4.5.9}$$

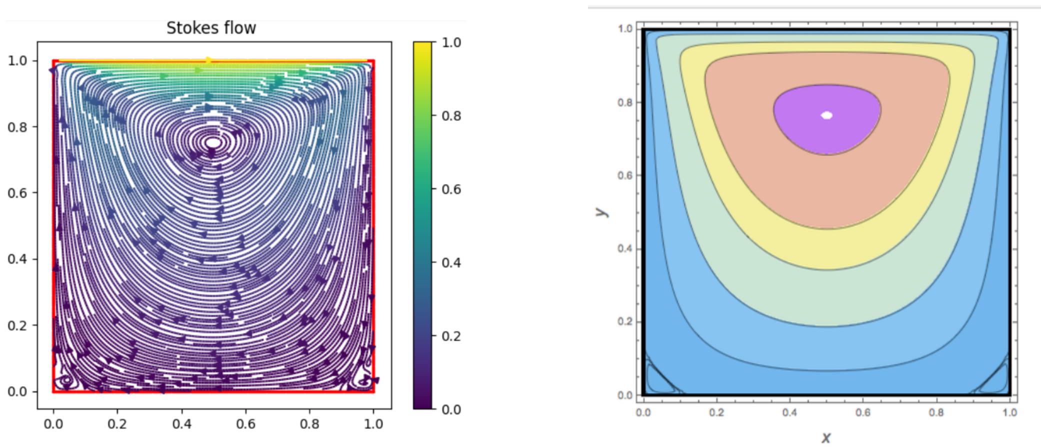
where $\mathbf{N}(x, y, P) := (N_1(x, y, P), N_2(x, y, P))$ is a vector network, P the parameter to optimize and $\mathbf{u}_b = (u_{1b}, u_{2b})$ is a vector function which satisfies the boundary condition. Here we took

$$u_{1b}(x, y) = xy(1-x)\Big(\frac{1}{\sqrt{x^2 + (1-y)^2}} + \frac{1}{\sqrt{(1-x)^2 + (1-y)^2}}\Big), \qquad u_{2b}(x, y) = 0. \tag{4.5.10}$$

More precisely, we have

$$u_1(x, y, P) = N(x, y, P)xy(1-x)(1-y) + u_{1b}(x, y) \tag{4.5.11}$$
$$u_2(x, y, P) = N_2(x, y, P)xy(1-x)(1-y). \tag{4.5.12}$$

Applying Uzawa's algorithm with $\rho = r = 10$. Figure 4.10*a* represents our neural network solution for the driven cavity problem on a Stokes flow and Figure 4.10*b* represents the result from the literature [12]. Here we took $N_G = 80X80 = 6400$ Gauss points inside the domain to compute the integrals and we took $N_{hat} = 50X50 = 2500$ radial basis hat functions. We used the Gaussian RBF $\exp(-(\varepsilon r)^2)$ with small support. We took $\varepsilon = 49$.



(a) (b)

Figure 4.10: Stream function of the lid-driven cavity problem on a Stokes flow. a) Neural network solution, b) Solution given by the reference Brian and Housam [12]

4.5.2.2 Case of a stationary nonlinear Navier-Stokes flow in a lid-driven cavity

Here we consider the driven cavity problem on a steady state nonlinear NS flow. Precisely, we solve the problem

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{1}{Re}\Delta\mathbf{u} + \nabla p = 0 \text{ in } \Omega, & (4.5.13) \\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega. & (4.5.14) \end{cases}$$

We can solve this problem the same way we solved the time-dependent NS equation (4.4.6)−(4.4.8) and obtain the solution of (4.5.13) − (4.5.14) as the limit when the time step Δt goes to 0 of the solution to time dependent NS equation. Indeed, since the boundary condition is time independent, the solution is also time independent (see [98, 85] for the existence of steady state NS). Therefore, the material derivative defined in (4.4.12) becomes

$$\frac{D\mathbf{u}}{Dt} = (\mathbf{u}\cdot\nabla)\mathbf{u}. \tag{4.5.15}$$

We recall that the idea of the Lagrangian method is to approach the material derivative (4.4.12) with a BDF. Meaning that, as the time step Δt goes to zero, the BDF (one or two) scheme converges to the material derivative. Therefore, in this case, the BDF scheme will converge to the right side of equation (4.5.15), allowing us to apply the same technique.

Since the boundary conditions are the same with the Stokes problem, we look for the neural network solution in the form describes in equation (4.5.9). This case is more interesting because the problem becomes difficult as the Re increases. We will solve it up to $Re = 10,000$ and compare our result with an existing literature, see [71, 40, 45]. Here we use the BDF2 augmented Lagrangian scheme associated with the CN scheme. We apply Uzawa's algorithm at each time step and we take $r = \rho = 10$. We took $N_G = 81 \times 81 = 6561$ Gauss points inside the domain to compute the integrals and we took $N_{hat} = 81 \times 81 = 6561$ radial basis hat functions. We used the Gaussian radial function $\exp(-(\varepsilon r)^2)$ with small support. We took $\varepsilon = 49$.

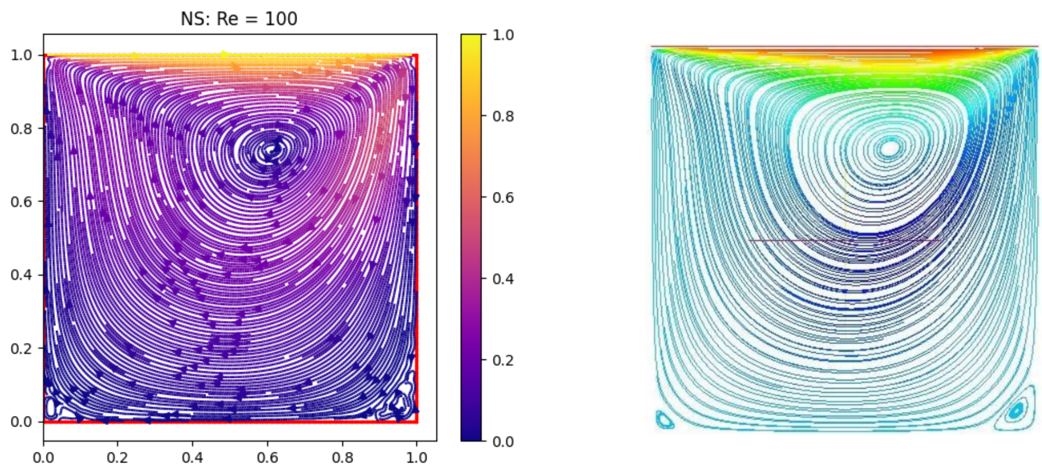


Figure 4.11: Driven cavity problem on a steady state Navier-Stokes flow at Re = 100., a) Neural network result, b) Result from McKay [71]

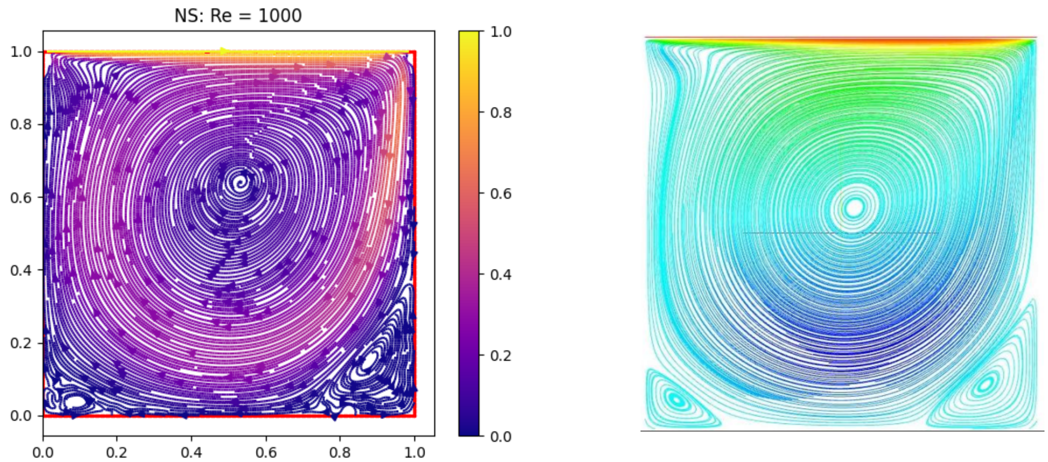
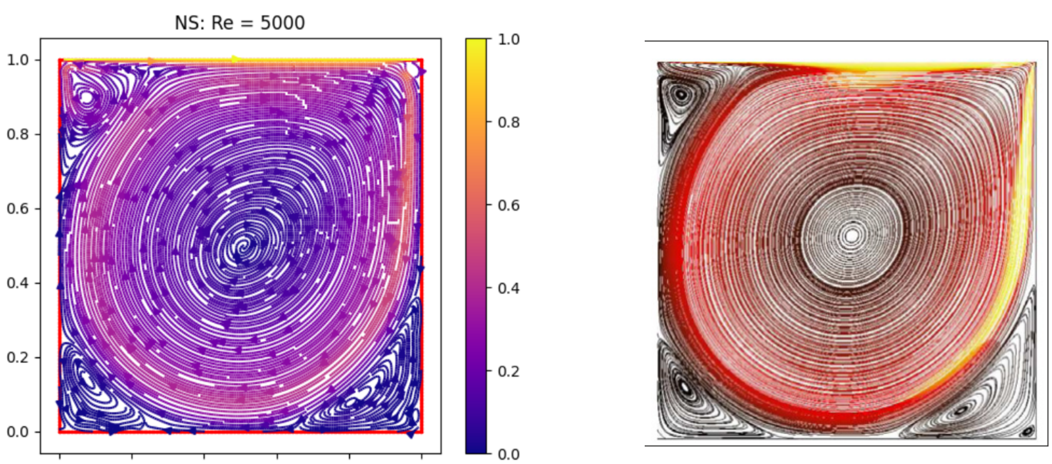


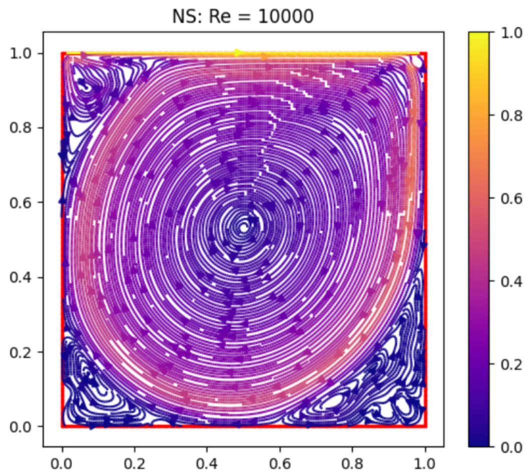
Figure 4.12

Figure 4.13: Driven cavity problem on a steady state Navier-Stokes flow at Re = 1000. a) Neural network result, b) Result from McKay [71]



(a) (b)

Figure 4.14: Driven cavity problem on a steady state Navier-Stokes flow at Re = 5000. a) Neural network result, b) Result from Hachem et al. [40]



(a)

(b)

Figure 4.15: Driven cavity problem on a steady state Navier-Stokes flow at Re = 10000. a) Neural network result, b) Result from Hachem et al. [40]

As a conclusion for the driven cavity problem for Stokes flow and NS flow at Reynolds numbers up to $Re = 10,000$, our code successfully captures the general flow patterns. However, its precision is less accurate, particularly near the corners of the cavity. This limitation may stem from the influence of the shape parameter ε. Indeed, the RBNN responds rather sensitively to ε. Meaning that accuracy of the solution is limited by the ill-conditioning issue of the interpolation matrix caused by the value of the shape parameter ε (see Bustamante et al. [13, Introduction], Chen and Leng [15, Section 2]). Moving forward, we will apply the same method to a more realistic scenario which is modeling blood flow in an AH domain. Given that the Reynolds number of blood is approximately $Re \approx 298$, we expect to have a reasonable general representation of the flow with our method.

4.5.3 Navier-Stokes problem on an artificial heart domain

In this section, we consider the time-periodic NS flow in an AH domain in dimension 2, with velocity profile given in the inlet, outlet and membrane. This case is supposed to simulate the problem we considered in the chapter 3 of this thesis. We note that this work is in the context of a project proposed to me by my thesis supervisor regarding the shape optimization of artificial hearts. The optimal shape can be found through the analysis of shape optimization and its numerical simulation, which we kept for future works. Here we do some “naive” shape optimization perturbing the rigid boundary Γ_r. Precisely, we propose four admissible shapes and we will numerically choose the optimal shape amount them, which would be the one presenting less vortices inside the heart chamber. Figure 4.16 gives us the admissible set of shapes we are considering. We choose to name them D_1, D_2, D_3, and D_4.

Let us consider the following problem where Ω_0 can represent one of the four AH admissible domains

$$\begin{cases} \rho\partial_t \mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \rho\vec{g} \quad \text{in } \Omega_0 \times (0,4T), \quad t\in\mathbb{R}, & (4.5.16a)\\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega_0\times(0,4T), & (4.5.16b)\\ \mathbf{u}(x,0) = \mathbf{u}(x,4T), \quad x\in\Omega_0, & (4.5.16c)\\ \mathbf{u}(x,t) = 0, \quad (x,t)\in\Gamma_r\times\mathbb{R}, & (4.5.16d)\\ \mathbf{u}(x,t) = k_i(t)\mathbf{u}_i(x), \quad (x,t)\in\Gamma_i\times\mathbb{R}, & (4.5.16e)\\ \mathbf{u}(x,t) = k_o(t)\mathbf{u}_o(x), \quad (x,t)\in\Gamma_o\times\mathbb{R}, & (4.5.16f)\\ \mathbf{u}(x,t) = k_m(t)\mathbf{u}_m(x), \quad (x,t)\in M_t\times\mathbb{R}. & (4.5.16g)\end{cases}$$

For a human blood, we have $\rho \approx 1.043g/cm^3$ and $\mu \approx 0.0035cm^2/S$. Therefore, the Re is around $Re \approx \frac{\rho}{\mu} = 298$, since we took into consideration the dimensions of a human heart. We suppose that the inlet is $\Gamma_i = (a_1, a_2)\times\{b_2\}$, the outlet is $\Gamma_o = (a_3, a_4)\times\{b_2\}$ and the membrane is $M_0 = (c_1, c_2)\times\{b_1\}$. We apply the following velocity profiles

$$\mathbf{u}_i(x) = (x-a_1)(x-a_2)\mathbb{1}_{\Gamma_i}(x,y)\ {}^t[0,1] \tag{4.5.17}$$

$$\mathbf{u}_o(x) = (x-a_3)(x-a_4)\mathbb{1}_{\Gamma_o}(x,y)\ {}^t[0,1] \tag{4.5.18}$$

$$\mathbf{u}_m(x) = (x-c_1)(x-c_2)\mathbb{1}_{M_0}(x,y)\ {}^t[0,1], \tag{4.5.19}$$

where we took

$$k_i(t) = \frac{\alpha}{2}\Big(\sin\Big(\frac{\pi(t-T_0)}{T}\Big) + \Big|\sin\Big(\frac{\pi(t-T_0)}{T}\Big)\Big|\Big), \tag{4.5.20}$$

$$k_o(t) = -\frac{\alpha}{2}\Big(\sin\Big(\frac{\pi(t-T_1)}{T}\Big) + \Big|\sin\Big(\frac{\pi(t-T_1)}{T}\Big)\Big|\Big), \tag{4.5.21}$$

$$k_m(t) = \frac{\beta}{2}\Big(\sin\Big(\frac{\pi(t-T_0)}{T}\Big), \tag{4.5.22}$$

where $T_0 = 0$, $T_1 = T_0 + 2T$ and $2T$ is the length of the first half period. Knowing that the stroke volume of the blood is around $SV = 66cm^3/beat$ and the depth of a human heart is around $d = 6cm$, the constant α and β are determined such that the volume of blood sucked during each half period has a constant value $SV/2 = 33cm^3$. Therefore, α and β are such that

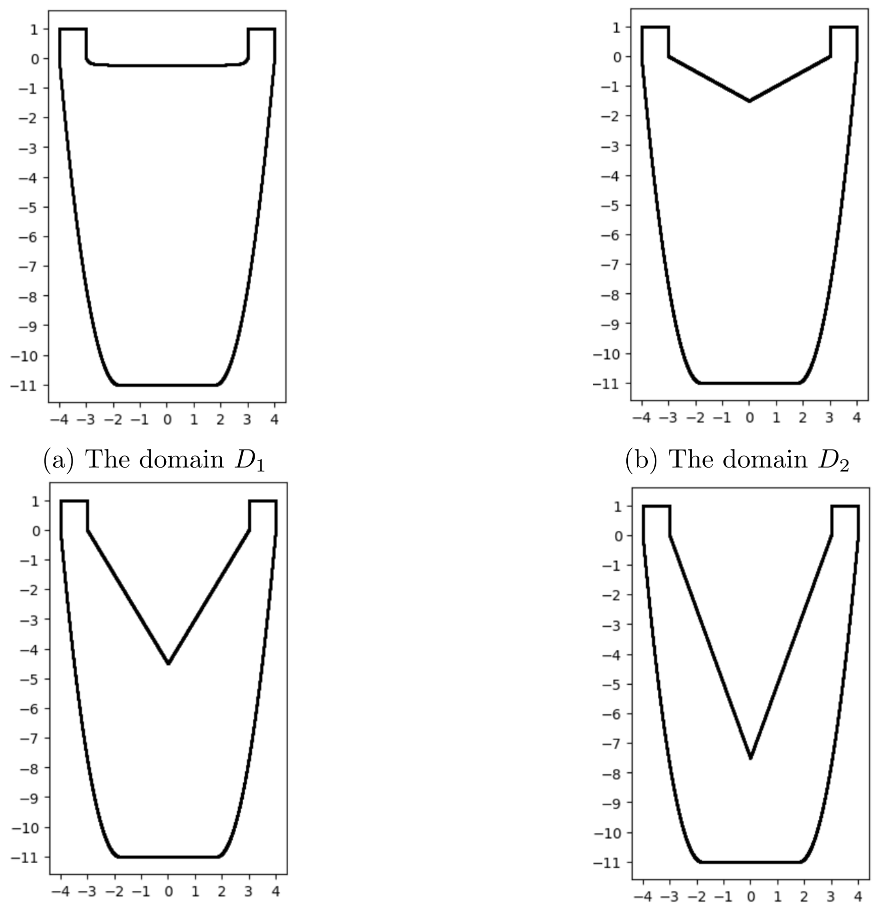
$$-d\int_{a_1}^{a_2} \alpha(x-a_1)(x-a_2)\,\mathrm{dx} = \mathrm{SV}/2 = 33, \tag{4.5.23}$$

$$-d\int_{c_1}^{c_2} \beta(x-c_1)(x-c_2)\,\mathrm{dx} = \mathrm{SV}/2 = 33. \tag{4.5.24}$$

Therefore,

$$\alpha = \frac{33}{(a_2-a_1)^3} \text{ and } \beta = \frac{33}{(c_1-c_2)^3}. \tag{4.5.25}$$

Here the time is discretized as $t_0 = 0$ and $t_n = t_{n-1} + \Delta t$, $\forall\, n \in \mathbb{N}\backslash\{0\}$. We took $\Delta t = \frac{2T}{N}$ with $N = 10$. Since each beat of heart last around $\frac{6}{7} \approx 0.857\ S$, we took $2T = \frac{6}{14} \approx 0.429$. This implies that in all those simulations $\Delta t \approx 0.0429S$. Here we use the BDF2 coupled with a CN scheme. We took $N_G = 80X80 = 6561$ Gauss points inside the domain to compute the integrals and we took $N_{hat} = 30X30 = 900$ radial basis hat functions. We used the Gaussian radial function $\exp(-(\varepsilon r)^2)$ with $\varepsilon = 2.24$.



(a) The domain D_1

(b) The domain D_2

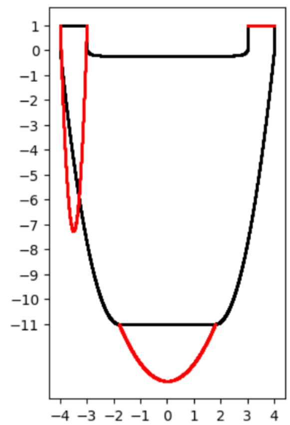
(c) The domain D_3

(d) The domain D_4

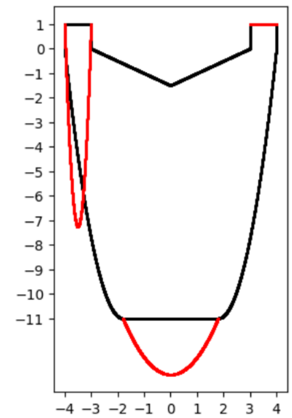
Figure 4.16: The admissible sets of AH domains.

4.5.3.1 Steady state Navier-Stokes problem on an artificial heart domain

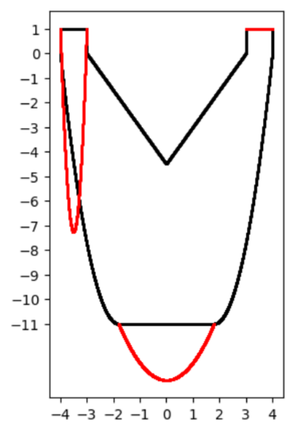
In this section, we solve the steady state NS in all the four "toy" design AH domains. We recall that here as before, the steady state NS equation is solved as the limit when the time t goes to infinity (Δt goes to zero) of the time-dependent NS equation. We fixed the velocity profile at a particular instance. We have chosen the instance of the first half period where the inlet is open, the outlet is closed and the amplitude of the velocity is maximal. Figure 4.17 illustrate the various domains and the velocity profiles on the inlet, outlet and membrane.



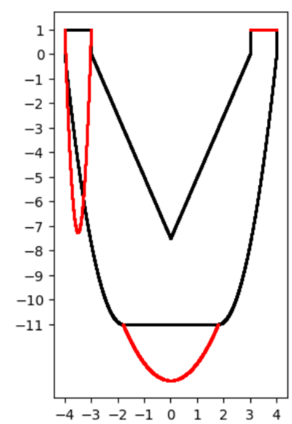
(a) The domain D_1 and the velocity profiles



(b) The domain D_2 and the velocity profiles



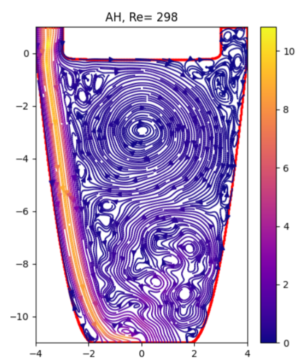
(c) The domain D_3 and the velocity profiles



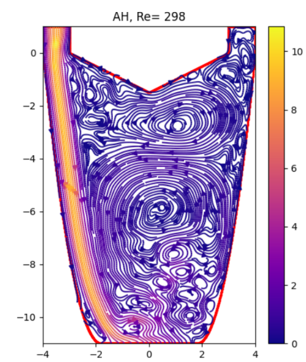
(d) The domain D_4 and the velocity profiles

Figure 4.17: Four “toy” design of AH domains and the velocity profiles on the inlet, outlet and membrane.

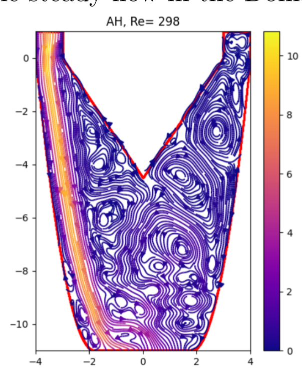
Figure 4.18 represents the motion of the blood in our four domains at that particular instance. We remark that the flow in D_1 presents a big vortice at the center of the chamber and another vortice close to the membrane. In the domain D_2 the big vortice diminishes and the second vortice which was close to the membrane increase a little bit. In domain D_3 there is no big vortice anymore. The flow presents only small vortices in the heart chamber. In the domain D_4, still there is no big vortice and the size of the vortices are smaller. So, when we perturb the upper rigid boundary of the heart chamber, the number and size of vortices diminishes and they are located in an area where their motions are limited. Finally, we remark that amount the four admissible domains, the domain D_4 seems to have less vortices inside the heart chamber.



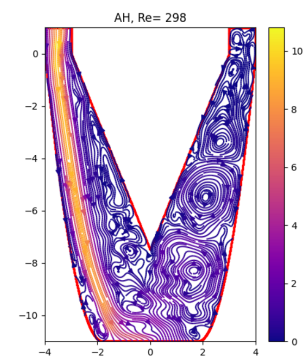
(a) The steady flow in the Domain D_1



(b) The steady flow in the Domain D_2



(c) The steady flow in the Domain D_3



(d) The steady flow in the Domain D_4

Figure 4.18: The steady state NS flow in an AH domain

4.5.3.2 Time periodic Navier-Stokes problem on an artificial heart domain

Here, the boundary conditions vary with time due to the non-static nature of the system. We present a complete simulation of blood flow within an artificial heart. For each domain, we visualize the flow at different time intervals. Our primary focus is to visually identify, among the four domains, the one with the least vortices inside the heart chamber, as vortices contribute to clot formation, which can damage the blood (see [97]).

Since the problem exhibits periodic behavior, we also anticipate observing a periodic pattern in the flow. We note that our analysis here relies on visual inspection of the flow dynamics and their periodicity. The time step is defined as $\Delta t = \frac{2T}{10}$, with the period given by $4T = 20\Delta t$.

Figure 4.19 below illustrates the blood motion within the artificial heart domain D_1 at various time points.

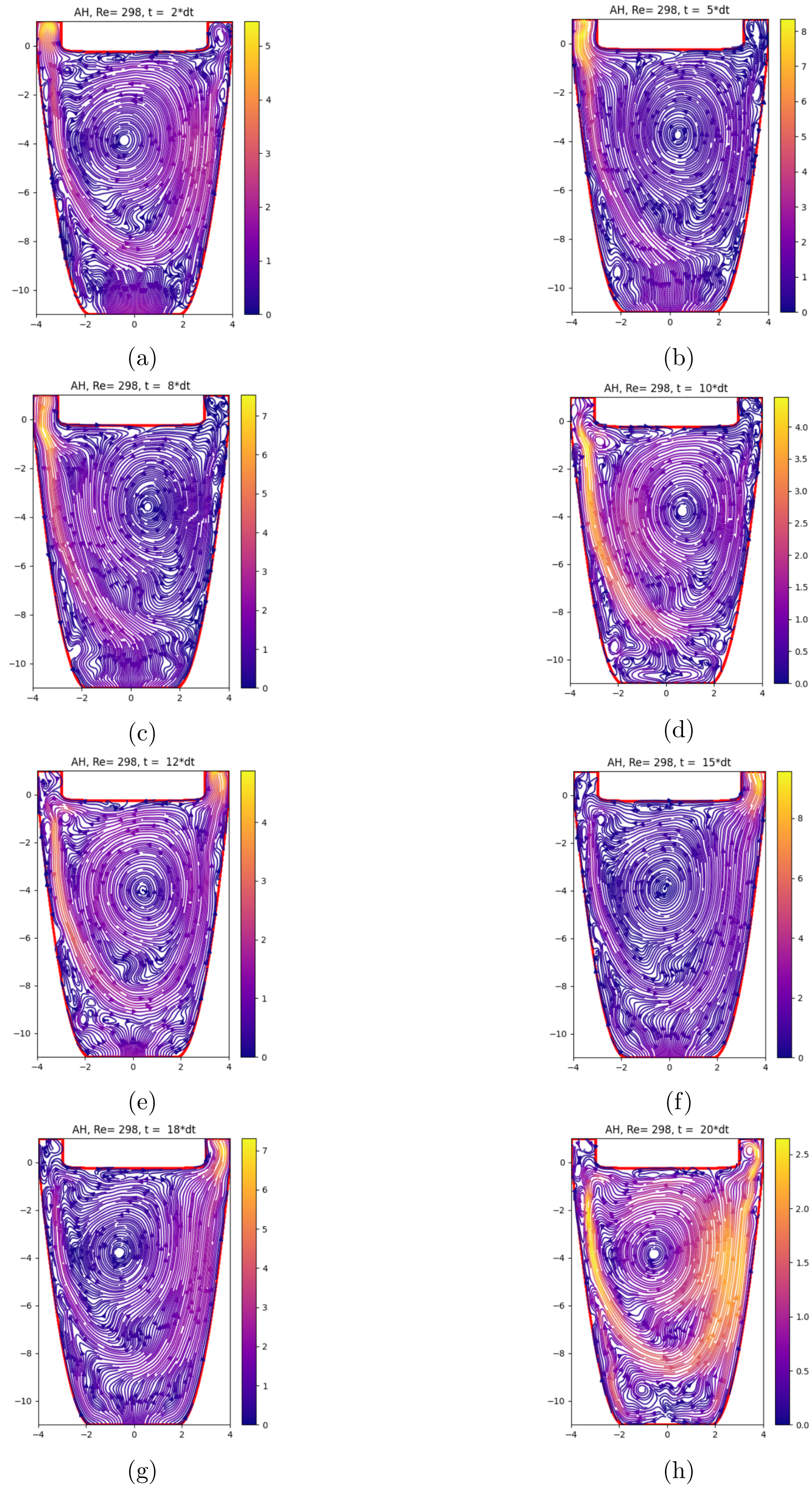


Figure 4.19: The flow of the blood in the domain D_1, the period is $4T = 20\ \Delta t$

Figure 4.20 shows the periodic behavior of the flow inside the AH domain D_1.

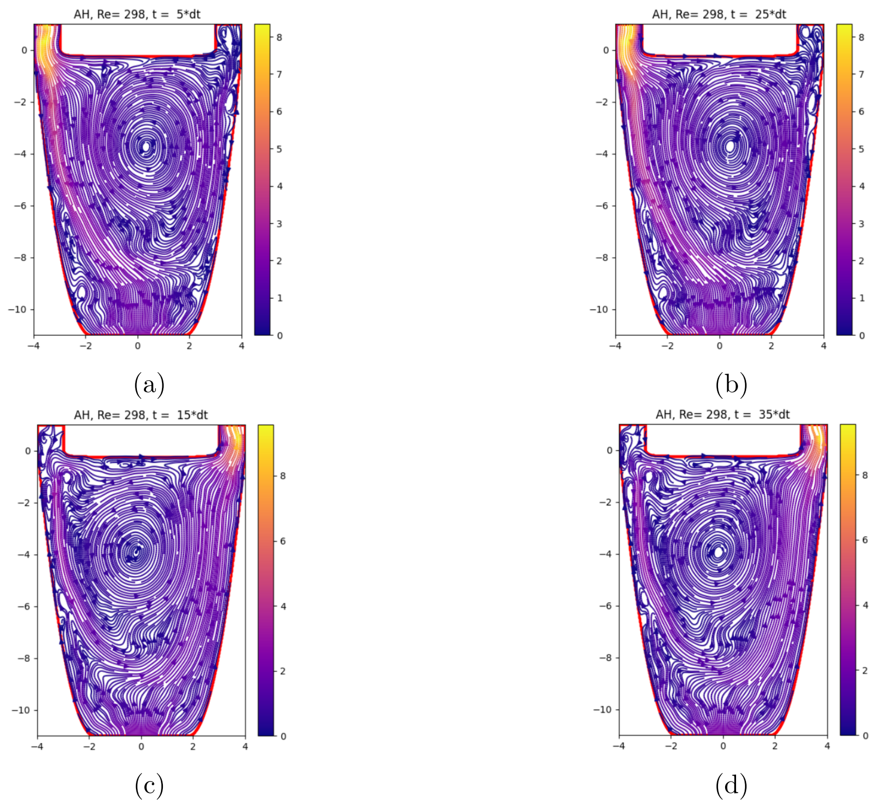
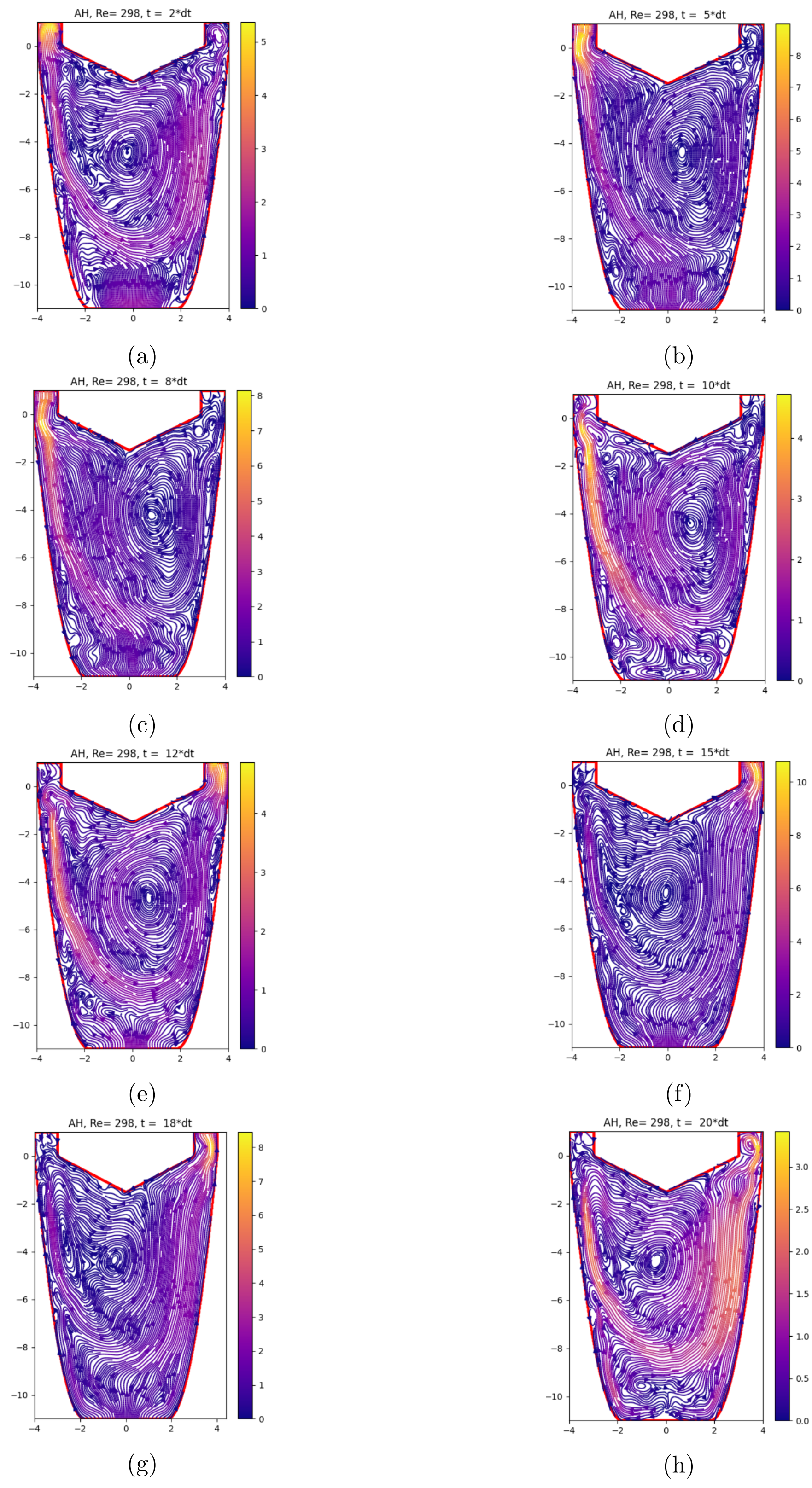


Figure 4.20: The periodic behavior of the flow in D_1, the period is $4T = 20\ \Delta t$

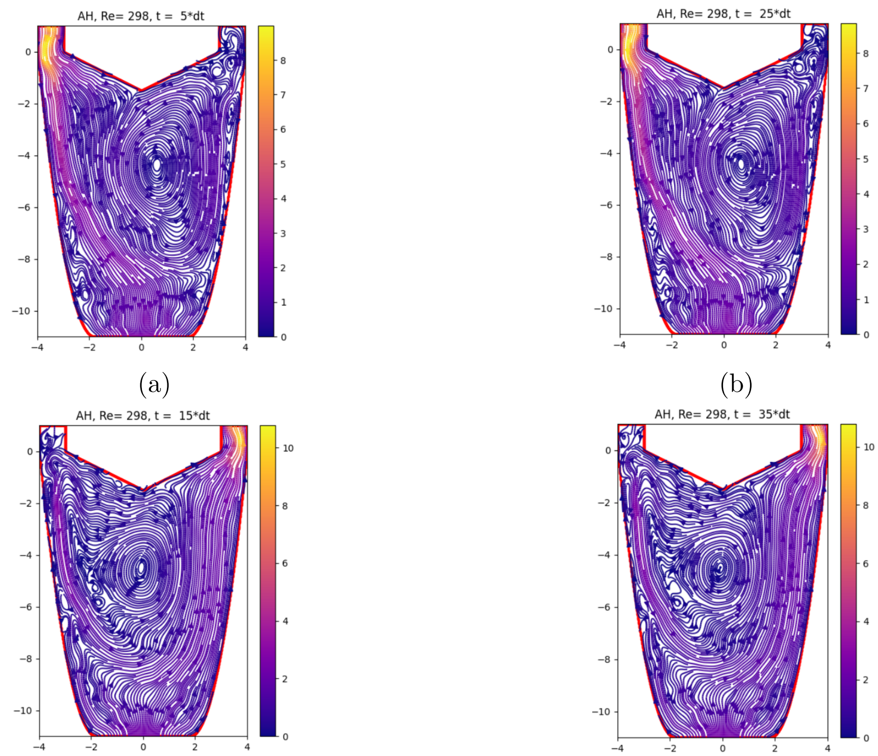
Figure 4.21 presents the flow of the blood inside the AH domain D_2.



(a) (b) (c) (d) (e) (f) (g) (h)

Figure 4.21: The flow of the blood in the domain D_2, the period is $4T = 20\ \Delta t$

Figure 4.22 presents the periodic behavior inside the AH domain D_2.



(a) (b) (c) (d)

Figure 4.22: The periodic behavior of the flow in D_2, the period is $4T = 20\ \Delta t$

Figure 4.23 presents the motion of the blood inside the AH domain D_3.

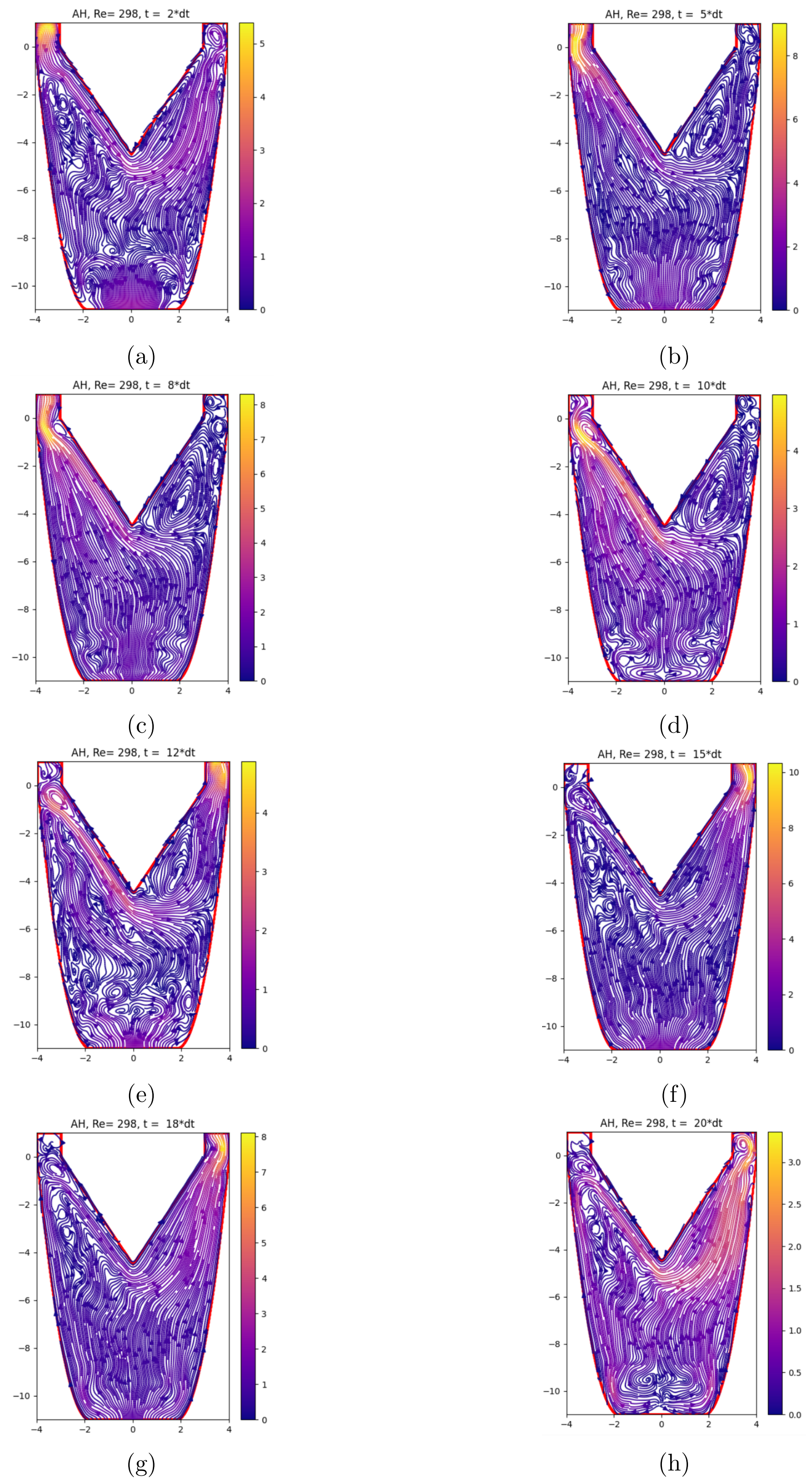
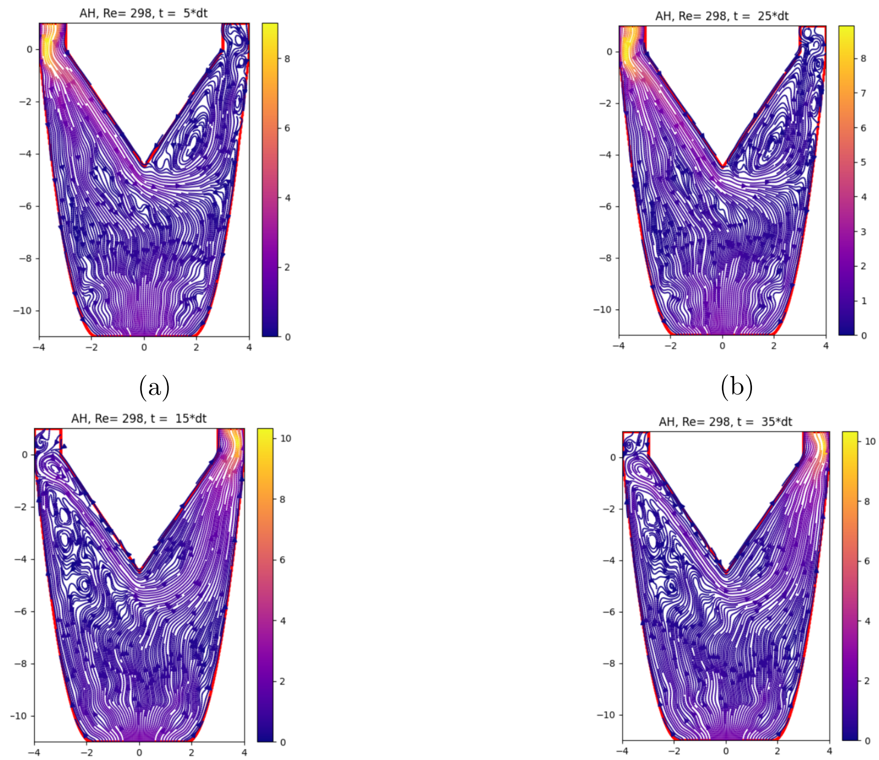


Figure 4.23: The flow of the blood in the domain D_3, the period is $4T = 20\ \Delta t$

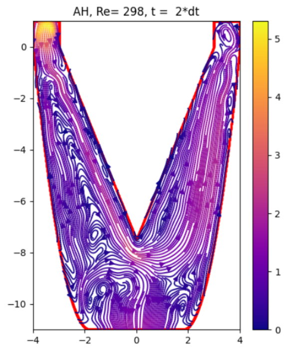
Figure 4.24 presents the periodic behavior inside the the AH domain D_3.



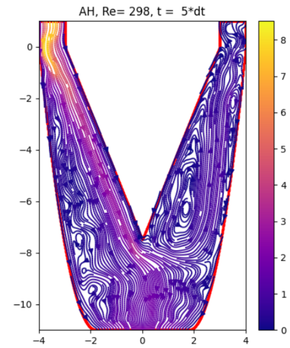
(a) (b) (c) (d)

Figure 4.24: The periodic behavior of the flow in D_3, the period is $4T = 20\ \Delta t$

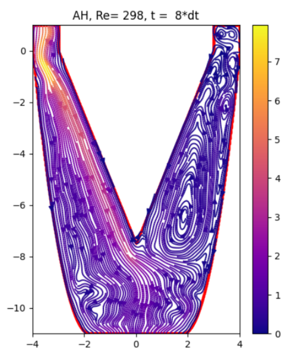
Figure 4.25 presents the motion of the blood inside the AH domain D_4.



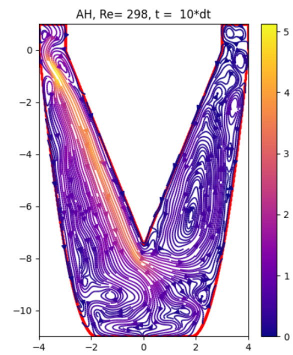
(a)



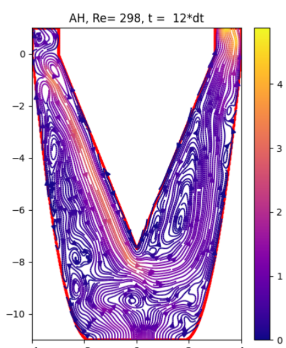
(b)



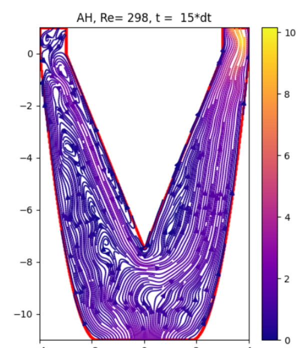
(c)



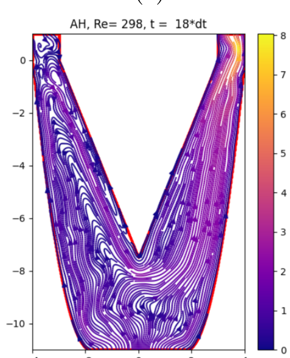
(d)



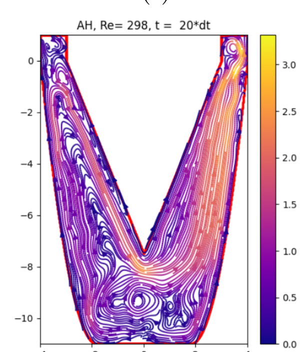
(e)



(f)



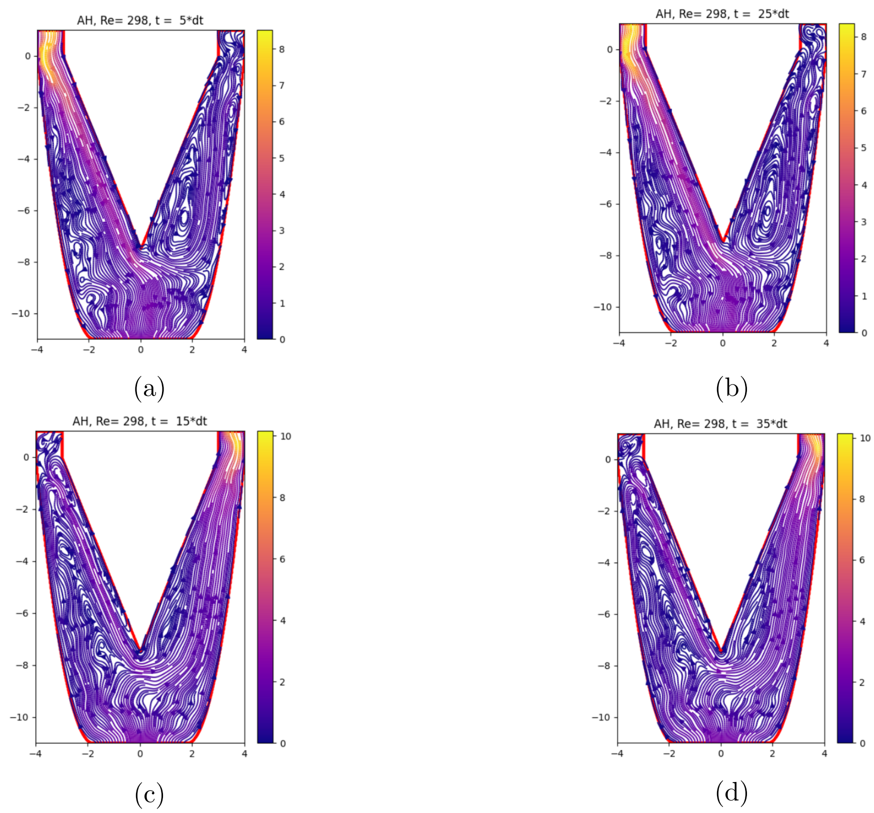
(g)



(h)

Figure 4.25: The flow of the blood in the domain D_4, the period is $4T = 20\ \Delta t$

Figure 4.26 presents the periodic behavior inside the AH domain D_4.



(a) (b) (c) (d)

Figure 4.26: The periodic behavior of the flow in D_4, the period is $4T = 20\ \Delta t$

After observation of our numerical results, we remark that the flow seems reasonable and periodic in all the four domains we have considered. We also remark that in all those domains, the domain D_4 seems to present less vortices inside the heart chamber.

Chapter 5

Conclusion

We are at the end of this PhD thesis. The goal was to carry on the analysis and numerical computation of blood flow in an AH design. To reach this goal, we needed to split it into two main objectives.

The first objective was the analysis of a time-periodic Navier-Stokes flow in a moving domain, with application to the blood flow in artificial hearts. More precisely, we aimed to find results regarding the existence and uniqueness of a time-periodic strong solution of the NS problem (1.2.18) − (1.2.24) in a moving domain using the implicit function theorem. This goal was reached in Chapter 3. Mainly, with the assumptions (3.1.2) -(3.1.6) on the reference domain Ω_0 which is essentially $C^{1,1}$ and with the Z_T regularity we imposed on z, solution to the membrane equation (1.2.4) − (1.2.8), our moving domain Ω_t is essentially a $L^\infty(\mathbf{W}^{2,\hat{2}^*}) \cap H^1(\mathbf{H}^2) \cap H^2(\mathbf{L}^2)$ perturbation of Ω_0. Note that this regularity of the moving domain is weaker than the standard assumption available in the literature, i.e., C^3 in space. We were able to prove the existence and uniqueness of a strong time-periodic solution to our nonlinear NS problem on the moving domain Ω_t.

The second objective was to numerically simulate the motion of the blood in a given artificial heart domain using a neural networks approach. We achieved that goal in Chapter 4. We used a special kind of neural neural with only one hidden layer called radial basis neural networks and we used the augmented Lagrangian formulation of the NS equations with a backward differentiation scheme in time of order 2 as described in [8] and we coupled it with a Crank-Nicolson scheme. Before using our code to solve the time-periodic NS equation in an artificial heart domain, we tested it on several benchmark problems. We used our code to solve the lid-driven cavity problem on a Stokes flow and compared it with the existing literature [12]. We also solved a particular time-periodic NS problem called the "toy" problem. The advantage of examining this problem is that the exact solution is given in [79]. Our code solved this problem with the L^2-relative error less than 10^{-2}. Additionally, we solved the lid-driven cavity problem on a Stokes and Navier-Stokes flow up to Reynolds number $10,000$, and we compared our results with the reference [12, 40]. Next, we applied our code to solve our time-periodic Navier-Stokes problem with a velocity profile on the inlet, outlet, and membrane. Finally, we did a “naive” numerical shape optimization by considering four artificial heart shapes D_1, D_2, D_3, and D_4. We solved the problem in each domain and selected the best shape among them.

This thesis is a part of a bigger project led by my thesis supervisor Arian Novruzi, which consists in finding the optimal shape design of an artificial heart. As future work, we planned to finish that project which involves the analysis of shape optimization. The problem is to minimize the energy associated with the system (1.2.18) − (1.2.24). Precisely, given an admissible rigid boundary $\Gamma := \Gamma_r$, if $(\mathbf{u}, p) := (\mathbf{u}(\Gamma), p(\Gamma))$ is solution to NS on the domain $\Omega_t := \Omega_t(\Gamma)$, the problem is to optimize the following shape functional (see Tavoularis et al. [97])

$$E(\Gamma) := \frac{1}{2}\int_{-T}^{T}\Big[\int_{\Omega_t}\Big(w_s|\nabla\mathbf{u}|^2 + w_\nu|\nabla\times\mathbf{u}|^2\Big)^2\,\mathrm{d}x + w_m p_m^2\Big]\,\mathrm{d}t, \tag{5.0.1}$$

where w_s, w_ν, and w_m are positive weights. The term with w_s minimizes the shear stress and contributes to reducing blood damage. The term with w_ν minimizes the vorticity and helps to prevent blood clot formation, and the term with w_m contributes to minimizing pump operating energy.

Also, knowing that the motion of the blood in an artificial heart design is activated by the motion of the membrane z, solving the equation of the membrane is a topic we are interested in exploring in the future.

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