A non-commutative $*$-algebra of Borel functions

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Abstract

To the pair (E, σ), where E is a countable Borel equivalence relation on a standard Borel space $(X, \mathscr{A})$ and σ a normalized Borel $\mathbb{T}$-valued 2-cocycle on E, we associate a sequentially weakly closed Borel $*$-algebra $\mathcal{B}^*_r(E, \sigma)$, contained in the bounded linear operators on $\ell^2(E)$.

Associated to $\mathcal{B}^*_r(E, \sigma)$ is a natural (Borel) Cartan subalgebra (Definition 6.4.10) $L(\mathcal{B}_o(X))$ isomorphic to the bounded Borel functions on X. Then $L(\mathcal{B}_o(X))$ and its normalizer (the set of the unitaries $u \in \mathcal{B}^*_r(E, \sigma)$ such that $u^* f u \in L(\mathcal{B}_o(X))$, $f \in L(\mathcal{B}_o(X))$) countably generates the Borel $*$-algebra $\mathcal{B}^*_r(E, \sigma)$.

In this thesis, we study $\mathcal{B}^*_r(E, \sigma)$ and in particular prove that:

i) If E is smooth, then $\mathcal{B}^*_r(E, \sigma)$ is a type I Borel $*$-algebra (Definition 6.3.10).

ii) If E is a hyperfinite, then $\mathcal{B}^*_r(E, \sigma)$ is a Borel AF-algebra (Definition 7.5.1).

iii) Generalizing Kumjian's definition, we define a Borel twist Γ over E and its associated sequentially closed Borel $*$-algebra $\mathcal{B}^*_r(\Gamma)$.

iv) Let a Borel Cartan pair $(\mathcal{B}, \mathcal{B}_0)$ denote a sequentially closed Borel $*$-algebra $\mathcal{B}$ with a Borel Cartan subalgebra $\mathcal{B}_0$, where $\mathcal{B}$ is countably $\mathcal{B}_0$-generated. Generalizing Feldman-Moore's result, we prove that any pair $(\mathcal{B}, \mathcal{B}_0)$ can be realized uniquely as a pair $(\mathcal{B}^*_r(E, \sigma), L(\mathcal{B}_o(X)))$. Moreover, we show that the pair $(\mathcal{B}^*_r(E), L(\mathcal{B}_o(X)))$ is a complete invariant of the countable Borel equivalence relation E.

v) We prove a Krieger type theorem, by showing that two aperiodic hyperfinite countable equivalence relations are isomorphic if and only if their associated Borel $*$-algebras $\mathcal{B}_r^*(E_1)$ and $\mathcal{B}_r^*(E_2)$ are isomorphic.

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Dedication

Je dédie cette thèse à mes fils Xavier et Yohan et à mon défunt frère Daniel.

Contents

Chapter 1

Introduction

The interplay between operator algebras and dynamical systems has been very important since the early work of Murray and von Neumann. Several of the first examples of von Neumann factors were obtained by the so-called group measure space construction $W^*(X, \mu, G)$ associated to the measurable action of a discrete countable group G on a measured space (X, μ). In his fundamental work on orbit equivalence, H. Dye in [Dye1] and [Dye2] studied measurable dynamical systems of the form (X, μ, T), with T a finite measure-preserving ergodic transformation and proved that any two such dynamical systems are orbit equivalent. In a sequence of papers (in particular [Kr1] and [Kr2]), W. Krieger studied and classified up to orbit equivalence nonsingular ergodic transformations T on a Lebesgue space (X, μ). He associated to (X, μ, T) an ergodic flow and proved that two systems are orbit equivalent if and only if their associated flow are conjugate and if and only if their associated von Neumann factor are isomorphic.

Recall that a C*-subalgebra (resp. W^*-subalgebra) B of A is regular if the normalizer $\{a \in A; aBa^* \subset B \text{ and } a^*Ba \subset B\}$ generates A as a C*-algebra (resp. as a W^*-algebra). In 1977, Feldman and Moore in [FM1] and [FM2] studied von Neumann algebras associated to countable Borel equivalence relations E on a measurable space

$(X, \mathscr{A}, \mu)$. The abelian algebra $L^\infty(X, \mu)$ is a natural Cartan subalgebra of the von Neumann algebra associated to E. Then a Cartan subalgebra B of a C*-algebra (resp. von Neumann algebra) A is a regular maximal abelian subalgebra of A, containing an approximate unit of A and with a faithful (resp. normal) conditional expectation from A to B.

The main result of [FM2] implies that two countable Borel equivalence relations are measurewise orbit equivalent if and only if there is an isomorphism of the corresponding von Neumann algebras keeping $L^\infty(X, \mu)$ globally fixed. In fact, Feldman and Moore associate a von Neumann algebra $W^*(E, \sigma)$ to a measurable twisted groupoid (E, σ) where E is as above and σ is a 2-cocycle on E with values in $\mathbb{T}$. They prove that any pair (M, A) of a von Neumann algebra M acting on a separable Hilbert space and a Cartan subalgebra A of M can be uniquely realized as $(W^*(E, \sigma), L^\infty(X, \mu))$.

The topological analogue of Krieger's result was studied by Giordano, Putnam and Skau in [GPS1]. They classified up to strong orbit equivalence minimal homeomorphisms ϕ on the Cantor set X; denoted (X, ϕ). They proved that two Cantor minimal systems are strongly orbit equivalent if and only if their associated C*-algebras (the cross-product) $C^*(X, \phi)$ are isomorphic. As in the measurable setting, the abelian subalgebra $C(X)$ is a natural Cartan subalgebra of the cross-product $C^*(X, \phi)$. Moreover, any two Cantor systems (X, ϕ) and (X, ψ) are flip conjugate (i.e. ϕ is conjugate to ψ or ψ^{-1}) if and only if there is an isomorphism of the corresponding cross-products, keeping $C(X)$ globally fixed.

Any freely acting topological dynamical system (X, ϕ) can be viewed as a topological groupoid $E_\phi \subset X \times X$. In, for example, [Re], [Re1] and [Re2], J. Renault analyzes topological groupoids $\mathscr{G}$ using the reduced C^*-algebras $C_r^*(\mathscr{G})$ associated to $\mathscr{G}$. In the case of Cantor minimal systems, the pair of C^*-algebras $(C^*(X, \phi), C(X))$ corresponds to the pair $(C_r^*(E_\phi), C(E_\phi^{(0)}))$.

In [Re], J. Renault gives a detailed analysis of the C*-algebra $C_r^*(E, \sigma)$ associated to a topological twisted groupoid (E, σ) where E is a principal r-discrete groupoid and σ is 2-cocycle on E with values in $\mathbb{T}$. In this setting, the abelian subalgebra $C_0(E^{(0)})$ is still a Cartan subalgebra of $C^*_r(E, \sigma)$, in a stronger sense (see 6.6.1). Then Renault proves that any pair (A, B) of a separable C*-algebra A and a (strong) Cartan subalgebra B can be uniquely realized as a pair $(C^*_r(E, \sigma), C_0(E^{(0)}))$. Renault's result is a topological analogue to Feldman and Moore's result, but with a stronger notion of Cartan subalgebra.

In [Ku], A. Kumjian introduces the notion of a principal twist Γ over E and studies the C*-algebra $C^*_r(\Gamma, E)$ associated to it. Topological twisted groupoid (E, σ) is a special case of principal twist (Γ, E). In this setting, the abelian subalgebra $C_0(E^{(0)})$ is a natural diagonal subalgebra of $C^*_r(\Gamma, E)$. A diagonal subalgebra is a Cartan subalgebra whose pure states can be extended uniquely to the whole algebra. Kumjian proves that to any pair (A, B) composed of a separable C*-algebra A and a diagonal subalgebra B can be uniquely realized as a pair $(C^*_r(\Gamma, E), C_0(E^{(0)}))$. Kumjian improved the result of Renault in [Re], but principal twists are not general enough to recover Cartan pairs (A, B).

In 2008, Renault in [Re2] showed that any Cartan pair (A, B) as above can be realized uniquely as $(C^*_r(\Gamma, E), C_0(E^{(0)}))$ where Γ is a twist over a topologically principal groupoid E. This is the algebraic topological analogue of Feldman and Moore's result. It is interesting to parallel the results of Renault and of Feldman and Moore in terms of groupoid and operator algebra: Renault's result in [Re] is the topological groupoid analogue, while his second in [Re2] is the C*-algebra analogue.

While von Neumann algebras are in correspondence with measurable dynamical systems and C*-algebras with topological dynamical systems, the link between operator algebras and Borel dynamical systems has not yet been so studied.

In Borel dynamical systems, we study countable groups G of Borel automorphisms on a standard (typically uncountable) Borel space $(X, \mathscr{A})$. To any such group of Borel automorphism, we associate a countable Borel equivalence relation

$E_G \subset X \times X$. In [FM1], Feldman and Moore prove that any countable Borel equivalence relation E of $(X, \mathscr{A})$ is induced by a countable group of Borel automorphisms.

Smooth countable Borel equivalence relations E on $(X, \mathscr{A})$, can be classified by studying their quotient spaces X/E. Note that finite Borel equivalence relations (i.e., the equivalence class of any $x \in X$ is finite) are smooth. In 1982, B. Weiss in [W] proved that any hyperfinite Borel equivalence relation E (i.e., increasing union of finite Borel subequivalence relations E_n) is induced by a Borel automorphism ϕ of $(X, \mathscr{A})$. In 1994, R. Dougherty, S. Jackson and A.S. Kechris classified in [DJKe] aperiodic non-smooth hyperfinite countable Borel equivalence relations up to orbit equivalence by the cardinality of their non-atomic invariant ergodic probability measures. In 2007, B. Miller and C. Rosendal in [MRo] showed that the full group $[E]$ of E (as an abstract group) is a complete invariant of orbit equivalence of countable Borel equivalence relations E whose equivalence classes are all of cardinality of at least 3.

The interplay between Borel dynamical systems and operator algebras started in the late 60s. In a sequence of papers [Dav1], [Dav2] and [Dav3], E.B. Davies studied sequentially weakly closed C*-algebras and called them Σ^*-algebras. In particular in [Dav3], he studied the Σ^*-algebra $\mathscr{B}(X, G)$ associated to a Borel G-space and showed that if G acts freely on X, then there is a natural one-to-one correspondence between the set of non-atomic invariant ergodic probability measures on X and the set of normalized sequentially normal extremal traces of $\mathscr{B}(X, G)$. One cannot fail to foresee how this results will be linked with the classification result of [DJKe]. Moreover from Davies's work, we can deduce the existence of a faithful sequentially normal conditional expectation from $\mathscr{B}(X, G)$ to the bounded Borel functions $\mathcal{B}_o(X)$ and that the normalizer of the maximal abelian subalgebra of $\mathcal{B}_o(X)$ generates $\mathscr{B}(X, G)$ as a Σ^*-algebra.

In 1986, D. Sullivan, B. Weiss and J.D.M. Wright in [SulWWr] studied actions of a countable group G of homeomorphisms on a Polish space X. They proved that any two generically free and generically ergodic (i.e., any nonatomic open invariant subset of X is meagre) such actions are generically orbit equivalent. They also defined and studied the Σ^*-algebra $M^*(E_G)$ associated to the Borel equivalence relation E_G induced by the action of G. Moreover from their work, we can deduce the existence of a sequentially normal conditional expectation $\Delta : M^*(E_G) \longrightarrow \mathcal{B}_o(X)$ and that the subalgebra of bounded Borel function $\mathcal{B}_o(X)$ is maximal abelian in $M^*(E_G)$. In fact, the main focus of [SulWWr] is on the monotone complete AW*-algebra $M^*(E_G)/J$ where J is the two-sided ideal $\{f \in M^*(E_G) : \text{supp}(\Delta(ff^*)) \text{ is meagre}\}$ and $\mathcal{D}$. We denote this algebra by $M^*(\mathcal{D}, G)$ where $\mathcal{D}$ is the Dixmier algebra $\mathcal{B}_o(X)/J$. This algebra $M^*(\mathcal{D}, G)$ is a type III AW*-factor. The first examples were constructed independently by Takenouchi and Dyer, answering a long standing problem of Kaplansky. In [SulWWr], the authors prove that any two type III AW*-factor $M^*(\mathcal{D}, G)$, associated to a generically free and generically ergodic action of any countable group G are isomorphic.

Let X be a compact separable metric space with no isolated points and G be a countable discrete group of homeomorphisms of X. In 1990, J.D.M. Wright in [Wr4] studied the Borel closure $\mathscr{A}^\sigma$ of the reduced crossed-product $C(X) \times_r G$ acting on the Hilbert space $\ell^2(G) \otimes \ell^2(X)$. He proves in the Borel $*$-algebra setting, that the subalgebra of bounded Borel functions $\mathcal{B}_o(X)$ is a Cartan subalgebra. The study of the AW*-algebra $\mathscr{A}^\sigma/\mathscr{I}$, where $\mathscr{I} = \{z \in \mathscr{A}^\sigma; \text{supp}(\Delta(ff^*)) \text{ is meagre}\}$, is then the focus of [Wr4].

The papers of [Dav3], [SulWWr] and [Wr4] lead us to use Borel $*$-algebras to further develop the interplay between Borel dynamical systems and operator algebras.

The first goal of this thesis is to associate a (weakly sequentially closed) Borel $*$-algebra to a countable Borel equivalence relation E and a $\mathbb{T}$-valued normalized 2-cocycle σ of E. In Chapter 7, we present two constructions of the Borel $*$-algebra $\mathcal{B}^*_r(E, \sigma)$. The first one follows Feldman and Moore's construction of [FM2], but in a measure-free context. In the second construction, we follow Kumjian's construction in [Ku] by viewing (E, σ) as a Borel twist. In both approaches, the main difficulty is to construct a faithful family of sequentially normal representations. Notice that, contrary to the topological case, any Borel twist can be realized by a $\mathbb{T}$-valued 2-cocycle on E.

We then show that the abelian subalgebra $\mathcal{B}_o(X)$ is a (Borel) Cartan subalgebra of $\mathcal{B}^*_r(E,\sigma)$. Moreover the (Borel) Cartan subalgebras are (Borel) diagonal subalgebras. Thus, as in the measurable setting, we only have one case to consider in the Borel setting (unlike the three levels of [Re], [Ku] and [Re2] in the topological setting).

In Chapter 7, we prove the analogue of Feldman and Moore's result for Borel $*$-algebras; more precisely, if $\mathcal{B}$ is a (sequentially weakly closed) Borel $*$-algebra, and $\mathcal{B}_0$ is an abelian (standard) Borel $*$-subalgebra, and if $\mathcal{B}$ is countable $\mathcal{B}_0$-generated then the pair $(\mathcal{B},\mathcal{B}_0)$ can be realized uniquely as a pair $(\mathcal{B}^*_r(E,\sigma),\mathcal{B}_o(X))$.

In this chapter, we also study how a second countable Hausdorff principal étale groupoid and its associated C*-algebra $C^*_r(E,\sigma)$ embeds in $\mathcal{B}^*_r(E,\sigma)$. We show that for any smooth countable Borel equivalence relation E, its associated Borel $*$-algebra $\mathcal{B}^*_r(E,\sigma)$ is of type I. We give a definition of approximatively finite Borel (BAF) $*$-algebra, and show that if E is hyperfinite, then $\mathcal{B}^*_r(E,\sigma)$ is a BAF $*$-algebra.

As in [Dav3], we obtain a natural one-to-one correspondence between the set of non-atomic invariant ergodic probability measures on X and the set of normalized sequentially normal extremal traces of $\mathcal{B}^*_r(E,\sigma)$. We combine this result with the Theorem 9.1 of [DJKe] to obtain in Chapter 8 a Borel analogue of Krieger's result (that we call a Borel-Krieger type Theorem), i.e., two aperiodic hyperfinite countable Borel equivalence relations are orbit equivalent if and only if their associated Borel $*$-algebras are isomorphic.

In Chapter 2, we recall the basic definitions of topological, Borel and measurable spaces. In Chapter 3, we present topological, Borel, and measured groupoids. In Chapter 4, we take a deeper look at the theory of countable Borel equivalence relations. In Chapter 5, we present the cohomology of countable Borel equivalence relations and the Borel twist of countable Borel equivalence relations. In Chapter

6, we present operator algebras, starting with C*-algebras, von Neumann algebras and more importantly for our cause, Borel $*$-algebras. We also surveys results in the measured and topological setting of [FM2], [Re], [Ku] and [Re2] for Cartan pairs of C*-algebras.

Chapter 2

Spaces

In this chapter we recall definitions and terminology for topological, Borel and measurable spaces. We follow the notations and definitions of [Ke].

2.1 Topological Spaces

Let X be a space. If $\mathcal{T}$ is a collection of subsets of X which contains the empty set and X and is closed under arbitrary unions and under finite intersections, then the pair $(X, \mathcal{T})$ is a **topological space**. For any set in $A \subset X$, let $\mathcal{T}|_A = \{U \cap A \text{ with } U \in \mathcal{T}\}$. The pair $(A, \mathcal{T}|_A)$ is called the **topological space restricted to** A. When $A \in \mathcal{T}$, the pair $(A, \mathcal{T}|_A)$ is also known as a **topological subspace** of $(X, \mathcal{T})$.

Let $\mathcal{E}$ be a collection of subsets of X. The smallest topology on X which contains $\mathcal{E}$ is denoted $\mathcal{T}_{\mathcal{E}}$ and the elements of $\mathcal{E}$ form a subbasis, or a set of **generators** of $(X, \mathcal{T}_{\mathcal{E}})$. When $\mathcal{E}$ is a countable collection of clopen subsets of X, then $(X, \mathcal{T}_{\mathcal{E}})$ is **zero-dimensional**.

For $i \in I$, let $(X_i, \mathcal{T}_i)$ be topological spaces. Let $X = \prod_{i \in I} X_i$ be the product space and $\pi_i : X \to X_i$ be the canonical projections. Let $\mathcal{E} = \{\pi_i^{-1}(A) : A \in \mathcal{T}_i, i \in I\}$ then $\mathcal{T}_{\mathcal{E}}$ is the **product topology** on X. It is the smallest topology which makes the canonical projection continuous (see definition below). For two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ we simply use the notation $(X \times Y, \mathcal{T} \times \mathcal{S})$ for the product space, with the product topology.

Definition 2.1.1 *A subset D of a topological space $(X, \mathcal{T})$ is* ***dense*** *if for all $U \in \mathcal{T}$, $U \neq \emptyset$, then $D \cap U \neq \emptyset$. If X contains a countable dense subset, then $(X, \mathcal{T})$ is* ***separable****.*

Definition 2.1.2 *A topological space $(X, \mathcal{T})$ is* ***Hausdorff*** *if every two distinct points of X have disjoint open neighborhoods.*

Definition 2.1.3 *Let $d : X \times X \to$ be a distance on X. The pair (X, d) is a* ***metric space****. The topology $\mathcal{T}_d$ generated by the open balls $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$, for any $\varepsilon > 0$, is called the* ***topology induced by*** *d. The metric space (X, d) is* ***complete*** *if every Cauchy sequence in X has a limit point in X.*

Definition 2.1.4 *Let $(X, \mathcal{T})$ be a topological space. If the topology $\mathcal{T}$ can be induced by a distance d, then $(X, \mathcal{T})$ is* ***metrizable****. We say that the distance d is* ***compatible*** *with the topology $\mathcal{T}$.*

Hence by definition, metrizable spaces are Hausdorff.

Definition 2.1.5 *We call $(X, \mathcal{T})$ a* ***Polish space****, if it is separable and it admits a compatible distance d such that (X, d) is complete.*

The strictly positive integers are denoted $\mathbb{N}^*$. For any space X, we denote by $\mathscr{P}(X)$ the collection of all subsets of X.

Example 2.1.6 *For each $k \in \mathbb{N}^*$, let $(A_k, \mathcal{T}_k)$ be a topological space with A_k countable and $\mathcal{T}_k = \mathscr{P}(A_k)$. Moreover, assume that the number of A_k with more than one point is infinite. If $d_k : A_k \times A_k \to \{0, 1\}$ is the distance defined by $d(x, y) = 0$ if and only if $x = y$, then this distance is compatible with the topology $\mathcal{T}_k$. The space $K = \prod_{k \in \mathbb{N}^*} A_k$*

with the product topology $\mathcal{T} = \prod_{k\in\mathbb{N}^*} \mathcal{T}_k$ is Polish. A compatible distance on K for the topology $\mathcal{T}$ is

$$d(x,y) = \sum_{k\in\mathbb{N}^*} \frac{d(x_i,y_i)}{2^k}.$$

Let $\varepsilon > 0$; then the balls $B(x,\varepsilon)$ are open and closed. Thus $(K,\mathcal{T})$ is totally disconnected with no isolated points. Moreover, K it is compact if every A_k is finite.

Definition 2.1.7 *Let $(X,\mathcal{T})$ be a topological space. If $(X,\mathcal{T})$ is totally disconnected, metrizable, compact, and without isolated points, then $(X,\mathcal{T})$ is a* ***Cantor space****.*

Example 2.1.8 *From the previous example, if all the A_k are finite then, the topological space $(K,\mathcal{T})$ is a Cantor space.*

Definition 2.1.9 *Let $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ be topological spaces. A map $f : X \to Y$ is* ***continuous*** *if for every open set V inside Y then the set $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is an open set in X. A map $f : X \to Y$ is* ***open*** *if for every open set U inside X then the set $f(U) = \{y \in Y : \exists\, x \in U \text{ such that } f(x) = y\}$ is an open set in Y.*

Definition 2.1.10 *Let $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ be topological spaces. If there is a continuous map $\phi : X \to Y$ which is bijective and open then the two topological spaces $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ are* ***homeomorphic****.*

Theorem 2.1.11 [Cantor] *The Cantor space is unique up to homeomorphism.*

2.2 Borel Spaces

Let X be a space. A collection of subsets $\mathscr{A}$ of X which contain the empty set and is closed under complements and arbitrary countable unions is called a **σ-algebra**. A **measurable** space is a pair $(X,\mathscr{A})$ where X is a space and $\mathscr{A}$ is a σ-algebra. For any set $A \subset X$, the **measurable space restricted to** A is the pair $(A,\mathscr{A}|_A)$ where $\mathscr{A}|_A$ is the restricted σ-algebra.

Let $\mathcal{E}$ be a collection of subsets of X. The smallest σ-algebra on X containing $\mathcal{E}$ is denoted $\mathscr{A}_{\mathcal{E}}$ and the elements of $\mathcal{E}$ are called the **generators** of $(X, \mathscr{A}_{\mathcal{E}})$.

Let $(X_i, \mathscr{A}_i)$ be measurable spaces indexed by $i \in I$. Let $X = \prod_{i \in I} X_i$ and $\pi_i : X \to X_i$ be the canonical projections. If $\mathcal{E} = \{\pi_i^{-1}(A) : A \in \mathscr{A}_i, i \in I\}$, then $\mathscr{A}_{\mathcal{E}}$ is the **product σ-algebra** on X. It is the smallest σ-algebra which makes the canonical projection measurable maps (see definition below). For two measurable spaces $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ we simply use the notation $(X_1 \times X_2, \mathscr{A}_1 \times \mathscr{A}_2)$ for the measurable product space. In that case, the sigma algebra $\mathscr{A}_1 \times \mathscr{A}_2$ is equivalent to the smallest σ-algebra generated by $\mathcal{P} = \{A \times B : A \in \mathscr{A}_1 \text{ and } B \in \mathscr{A}_2\}$.

Definition 2.2.1 *Let $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ be measurable spaces. Let $f : X_1 \to X_2$ be a map. The map f is called a* ***measurable map*** *if for all $A \in \mathscr{A}_2$ then $f^{-1}(A) \in \mathscr{A}_1$. The measurable map f is called a* ***measurable isomorphism*** *if it is bijective and if f^{-1} is a measurable map. If there is a measurable isomorphism between two measurable spaces, then the measurable spaces are* ***measurably isomorphic****.*

Definition 2.2.2 *Let $(X, \mathscr{A})$ be a measurable space. A measurable isomorphism f from X to X is called a* ***measurable automorphism****.*

Let f be a map defined between two subsets A and B of two spaces X and Y respectively. The set $\text{graph}(f) = \{(x, y) \in X \times Y : f(x) = y)\}$ is the **graph** of f.

Definition 2.2.3 *Let $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ be measurable spaces. Let $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$. A* ***partial measurable map*** *$f : A_1 \to A_2$ is a measurable map between the measurable spaces $(A_1, \mathscr{A}|_{A_1})$ and $(A_2, \mathscr{A}|_{A_2})$.*

Definition 2.2.4 *Let $(X, \mathcal{T})$ be a topological space. If $\mathscr{A} = \mathscr{A}_{\mathcal{T}}$, then $(X, \mathscr{A})$ is called a* ***Borel space*** *and the elements of $\mathscr{A}$ are called the Borel subsets of X.*

When the measurable spaces are Borel spaces, then we will use the word Borel instead of measurable for the corresponding definitions.

Definition 2.2.5 *Let $(X, \mathscr{A})$ be a Borel space. If there is a Polish topology $\mathcal{T}$ on X such that $\mathscr{A} = \mathscr{A}_{\mathcal{T}}$ then $(X, \mathscr{A})$ is called a* ***standard Borel space****.*

Example 2.2.6 *Let $X_n = \{x_1, ..., x_n\}$ (n distinct points) and $\mathcal{T}_n$ be the discrete topology on X_n. Then $\mathscr{A}_{\mathcal{T}_n}$ is the collection of all subsets of X_n and $(X_n, \mathscr{A}_{\mathcal{T}_n})$ are standard Borel spaces.*

Example 2.2.7 *Let $\mathcal{T}$ be the discrete topology on $\mathbb{N}$. Then $(\mathbb{N}, \mathscr{A}_{\mathcal{T}})$ is a standard Borel spaces.*

Example 2.2.8 *Let $\mathcal{E} = \{B(x, \varepsilon) \subset \mathbb{R} : x, \varepsilon \in \mathbb{Q} \text{ with } \varepsilon > 0\}$ be the open balls centred on rational points with rational radiuses (with the usual metric on $\mathbb{R}$). Then $(\mathbb{R}, \mathscr{A}_{\mathcal{E}})$ is a standard Borel space.*

The following are standard results.

Theorem 2.2.9 *Let $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ be standard Borel spaces. Let $f : Y_1 \to Y_2$ be a map for $Y_1 \in \mathscr{A}_1$ and $Y_2 \in \mathscr{A}_2$. The following are equivalent:*

i) f is a Borel map and

ii) graph(f) is a Borel set in $(X_1 \times X_2, \mathscr{A}_1 \times \mathscr{A}_2)$.

Moreover, a ***Borel graph*** *is a subset $U \in \mathscr{A}_1 \times \mathscr{A}_2$ such that if $(x, y) \in U$ and $(x, y') \in U$ then $y = y'$. Clearly, if f is a Borel map, then graph(f) is a Borel graph and for any Borel graph U, you can define a Borel map f_U where if $(x, y) \in U$ then $f_U(x) = y$.*

The cardinality of a set A is denoted $|A|$. We use the following notation: $|\emptyset| = 0$, $|\{x_1, x_2, ..., x_n\}| = n$ (here $i \neq j$ iff $x_i \neq x_j$), $|\mathbb{N}| = \omega$ (countably infinite) and $|\mathbb{R}| = c$.

Theorem 2.2.10 *Let $(X, \mathscr{A})$ be a standard Borel space. Then $|X| \in \{\omega, c, 1, 2, ..., n, ...\}$.*

Theorem 2.2.11 *Let $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ be standard Borel spaces. Then $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ are Borel isomorphic if and only if $|X_1| = |X_2|$.*

From this theorem, any standard Borel space is Borel isomorphic to one of our previous examples of standard Borel spaces.

2.3 Measured Spaces

Definition 2.3.1 *Let $(X, \mathscr{A})$ be a standard Borel space. Let μ be a measure on $(X, \mathscr{A})$. The triple $(X, \mathscr{A}, \mu)$ is called a measured space.*

2.4 Functions

We end this chapter by introducing important function spaces and some standard results used throughout this thesis.

Definition 2.4.1 *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. The set of* ***continuous functions with compact support****, denoted $C_c(X)$, consists of all continuous complex-valued functions f such that there exists a compact set $K \subset X$ with the property that for every $x \notin K$, then $f(x) = 0$.*

The functions in $C_c(X)$ are always bounded with respect to the sup-norm.

Definition 2.4.2 *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. The set of* ***continuous functions vanishing at infinity****, denoted $C_0(X)$, consists of all continuous complex-valued functions f such that for any $\varepsilon > 0$, there exists a compact set $K \subset X$ with the property that for every $x \notin K$, then $|f(x)| < \varepsilon$.*

Then $C_c(X)$ is dense in $C_0(X)$. Moreover $C_0(X)$ is a Banach space with respect to the sup-norm.

Definition 2.4.3 *Let* $(X, \mathscr{A})$ *be a measurable space. The set of* ***bounded measurable functions*** *(with respect to the sup-norm), denoted* $\mathcal{M}_0(X)$*, consists of all complex-valued measurable functions* f*, such that there exists* $M < \infty$ *such that* $|f(x)| < M$ *for all* $x \in X$*. When* $\mathscr{A}$ *are Borel sets, then the bounded measurable functions are called the* ***bounded Borel functions****, denoted* $\mathcal{B}_o(X)$*.*

Theorem 2.4.4 *Let* $(X, \mathscr{A})$ *be a measurable space. Let* $\{f_n\}$ *be a bounded sequence of bounded measurable functions. If the sequence* $\{f_n\}$ *converges pointwise to* f*, i.e., for every* $x \in X$ *the sequence of complex numbers* $\{f_n(x)\}$ *converges to* $f(x)$*, then* f *is a bounded measurable function.*

The proof of this theorem can be found in [Ru] (Corollaries p.14). In particular, $\mathcal{M}_0(X)$ is closed under bounded pointwise limit.

Theorem 2.4.5 *Let* $(X, \mathcal{T})$ *be a Polish space and* $(X, \mathscr{A})$ *be its corresponding standard Borel space. The class of real-valued bounded Borel functions of* $\mathcal{B}_o(X)$ *is the smallest class of real-valued functions containing the real-valued continuous functions with compact support of* $C_c(X)$*, which is closed under taking bounded pointwise limits of monotone (increasing or decreasing) sequences of functions (i.e., if* $\{f_n\}$ *are in the class, with* $|f_n| < M$ *for some* $M < \infty$*, and* $f_n \nearrow f$ *or* $f_n \searrow f$ *pointwise, then* f *is in the class).*

The proof of this theorem can be found in [Ped8] (Proposition 6.2.9).

Chapter 3

Groupoids

We begin with an introductory example. This example motivates the definition of a more general object called groupoid. We follow loosely the example given by [Re4] in the beginning of Section 2.3.1.

Let G be a group of automorphisms of a space X. The **action** $\alpha : X \times G \to X$ given by $\alpha((x, g)) = g(x)$ for $g \in G$ and $x \in X$ turns X into a **G-space**. If the action is free, i.e., $\alpha((x, g)) = x$ if and only if $g = \mathrm{id}_X$ (the identity on X), then X is a **free G-space**.

We define

$$\mathscr{G} = \mathscr{G}(X, G) = \{(x, g, y) \in X \times G \times X : x = g(y)\}$$

and endow $\mathscr{G}$ with the following algebraic structure:

1) a product map $m : \mathscr{G}^{(2)} \to \mathscr{G}$ where

$$\mathscr{G}^{(2)} = \{((x, g, y), (y, h, z)) \in \mathscr{G} \times \mathscr{G}\} \subset \mathscr{G} \times \mathscr{G}$$

is the set of composable pairs such that $m((x, g, y), (y, h, z)) = (x, gh, z)$,

2) an inverse map $\mathscr{G} \to \mathscr{G}$ such that $(x, g, y)^{-1} = (y, g^{-1}, x)$,

3) a range map and source map, $r : \mathscr{G} \to X$ and $s : \mathscr{G} \to X$ given by $r(x, g, y) = x$ and $s(x, g, y) = y$, and

4) an inclusion map $i : X \to \mathscr{G}$ given by $i(x) = (x, \mathrm{id}_X, x)$.

The resulting object is called a **transformation group groupoid**.

If X has a topology, a Borel structure or a measurable structure, then we want G to be a group of homeomorphisms, Borel automorphisms or measurable automorphisms. We also want to endow the transformation group groupoid with a corresponding structure to make the partially defined product map and inverse map continuous, Borel or measurable.

We recall definitions and terminology in the study of topological groupoids, Borel groupoids and measurable groupoids.

3.1 Groupoids

In this section, we follow mainly the notations of [Re].

Definition 3.1.1 *([Re], Definition 1.1) A* ***groupoid*** *is a set $\mathscr{G}$ endowed with a product map $(g_1, g_2) \mapsto g_1 g_2 : \mathscr{G}^{(2)} \to \mathscr{G}$ where $\mathscr{G}^{(2)} \subseteq \mathscr{G} \times \mathscr{G}$ is called* ***set of composable pairs****, and an inverse map $g \mapsto g^{-1} : \mathscr{G} \to \mathscr{G}$ such that it satisfy the following properties:*

i) $(g^{-1})^{-1} = g$ for all $g \in \mathscr{G}$.

ii) If (g_1, g_2) and (g_2, g_3) are in $\mathscr{G}^{(2)}$, then $(g_1 g_2, g_3)$ and $(g_1, g_2 g_3)$ are in $\mathscr{G}^{(2)}$ with $(g_1 g_2) g_3 = g_1 (g_2 g_3)$.

iii) $(g^{-1}, g) \in \mathscr{G}^{(2)}$ for all $g \in \mathscr{G}$, and if $(g, g_1) \in \mathscr{G}^{(2)}$, then $g^{-1}(g g_1) = g_1$.

iv) $(g, g^{-1}) \in \mathscr{G}^{(2)}$ for all $g \in \mathscr{G}$, and if $(g_1, g) \in \mathscr{G}^{(2)}$, then $(g_1 g) g^{-1} = g_1$.

Definition 3.1.2 *Let $\mathscr{G}$ a groupoid. The maps $r : \mathscr{G} \to \mathscr{G}$ and $s : \mathscr{G} \to \mathscr{G}$ defined by*

$$r(g) = gg^{-1} \text{ and } s(g) = g^{-1}g$$

*are called the **range map** and **source map** respectively.*

Remark 3.1.3 *Let $\mathscr{G}$ be a groupoid and let g and h be elements of $\mathscr{G}$. The pair (g, h) is in $\mathscr{G}^{(2)}$ if and only if $r(h) = s(g)$.*

Definition 3.1.4 *Let $\mathscr{G}$ a groupoid. The **set of units** of $\mathscr{G}$, denoted $\mathscr{G}^{(0)}$, is defined by*

$$\mathscr{G}^{(0)} = r(\mathscr{G}).$$

Definition 3.1.5 *Let $\mathscr{G}$ be a groupoid. Let x and y be in $\mathscr{G}^{(0)}$. The **fibers** of the range and source maps are denoted $\mathscr{G}^x = r^{-1}(x)$ and $\mathscr{G}_y = s^{-1}(y)$ respectively.*

Remark 3.1.6 *For all $g \in \mathscr{G}$, $r(g)g = g$ and $gs(g) = g$. The set of units could have been defined using $s(\mathscr{G})$.*

Definition 3.1.7 *Let $\mathscr{G}$ be a groupoid and $x \in \mathscr{G}^{(0)}$. The **isotropy group of** x, denoted $\mathscr{G}^x_x$, is defined by*

$$\mathscr{G}^x_x = \mathscr{G}^x \cap \mathscr{G}_x.$$

*If $\mathscr{G}^x_x$ is the trivial group, then we say that the point x has **trivial isotropy**.*

Definition 3.1.8 *Let $\mathscr{G}$ be a groupoid. The **isotropy bundle** of $\mathscr{G}$, denoted $\mathscr{G}'$, is defined by*

$$\mathscr{G}' = \{g \in \mathscr{G} : r(g) = s(g)\}.$$

Remark 3.1.9 *Clearly $\mathscr{G}^{(0)} \subseteq \mathscr{G}'$ and $\mathscr{G}' = \coprod_{x \in \mathscr{G}^{(0)}} \mathscr{G}^x_x$.*

Example 3.1.10 *If G is a group with neutral element e, then $G^{(2)} = G \times G$, $G^{(0)} = \{e\}$ and $G' = G$.*

Example 3.1.11 *Let E be an equivalence relation on a space X (i.e., $E \subset X \times X$). We can equip the equivalence relation with a natural groupoid structure. Let w, x, y and z be any points in X.*

1) Product Map: $((w,x),(y,z)) \in E^{(2)}$ *if and only if* $x = y$ *and then* $(w,x)(x,z) = (w,z)$.

2) Inverse Map: $(x,y)^{-1} = (y,x)$.

Let $(x,y) \in E$. Since $r(x,y) = (x,y)(y,x) = (x,x)$ and $s(x,y) = (y,x)(x,y) = (y,y)$, then the unit space of E is its diagonal, i.e., $E^{(0)} = \{(x,x) : x \in X\} \cong X$. Moreover $E' \cong X$.

Example 3.1.12 *Let (X,G) be a G-space as in the introduction of this chapter and $\mathscr{G}(X,G)$ be a the corresponding transformation group groupoid. Since $r((x,g,y)) = (x,g,y)(y,g^{-1},x) = (x,id_X,x)$, then $\mathscr{G}^{(0)} = \{(x,id_X,x) : x \in X\} \cong X$. If $\mathscr{G}$ is a free transformation group groupoid then $\mathscr{G}' \cong X$.*

Definition 3.1.13 *Let $\mathscr{G}$ be a groupoid. If $\mathscr{G}' = \mathscr{G}^{(0)}$, then $\mathscr{G}$ is a* ***principal*** *groupoid.*

Remark 3.1.14 *There is a bijective correspondence between principal groupoids and equivalence relations. Indeed, a groupoid $\mathscr{G}$ induces a canonical equivalence relation on $\mathscr{G}^{(0)} \times \mathscr{G}^{(0)}$ given by*

$$E_{\mathscr{G}} = \{(r(g),s(g)); g \in \mathscr{G}\}.$$

If $\mathscr{G}$ is principal, then the homomorphism $\Phi : \mathscr{G} \to E_{\mathscr{G}}$, given by $\Phi(g) = (r(g),s(g))$ is bijective.

3.2 Topological Groupoids

Now we introduce the notion of a topological groupoid.

Definition 3.2.1 *[Re2] Let $\mathscr{G}$ be a groupoid. Suppose $(\mathscr{G}, \mathcal{T})$ is a topological space. If the structure product and inverse maps are continuous and the source and range maps are continuous and open, where $\mathscr{G}^{(2)} \subseteq \mathscr{G} \times \mathscr{G}$ has the restricted topology of $(\mathscr{G} \times \mathscr{G}, \mathcal{T} \times \mathcal{T})$ and $\mathscr{G}^{(0)} \subseteq \mathscr{G}$ has the restricted topology of $(\mathscr{G}, \mathcal{T})$, then $\mathscr{G}$ is a* ***topological groupoid****.*

Definition 3.2.2 *Let $\mathscr{G}$ be a locally compact Hausdorff groupoid. A continuous* ***Haar system*** *is a family of nonnegative measures $\{\lambda^x\}_{x \in \mathscr{G}^{(0)}}$ with the following properties;*

1) supp$(\lambda^x) = \mathscr{G}^x$, $x \in \mathscr{G}^{(0)}$,

2) for a function $f : \mathscr{G} \to \mathbb{C}$ continuous and with compact support, the function

$$x \mapsto \lambda^x(f)$$

on $\mathscr{G}^{(0)}$ is continuous and with compact support, and

3) is invariant in the sense that

$$\int f(gg_1)d\lambda^{s(g)}(g_1) = \int f(g_1)d\lambda^{r(g)}(g_1), \quad \text{for every } g \in \mathscr{G}.$$

From now on, unless otherwise specified, we will assume that a topological groupoid $\mathscr{G}$ is locally compact, Hausdorff, second countable and that it possesses a Haar system.

Definition 3.2.3 *[Re2] Let $\mathscr{G}$ be a topological groupoid. If the range and source maps are local homeomorphisms, then $\mathscr{G}$ is* ***étale****.*

Remark 3.2.4 *[Re2] When $\mathscr{G}$ is an étale groupoid and $x \in \mathscr{G}^{(0)}$, the counting measures $\{\lambda^x\}$ on the set $\mathscr{G}^x$ form a canonical Haar system.*

Definition 3.2.5 *[Ku] Let $\mathscr{G}$ be an étale groupoid. If $\mathscr{G}$ is principal, then it is referred as a* ***topological equivalence relation****.*

Definition 3.2.6 *[Re2] Let $\mathscr{G}$ be an étale groupoid. If the interior of $\mathscr{G}'$ is $\mathscr{G}^{(0)}$ then $\mathscr{G}$ is* ***topologically principal****.*

Proposition 3.2.7 *[Re2] Let $\mathscr{G}$ be an étale groupoid. If $\mathscr{G}^{(0)}$ has the Baire property, then the following are equivalent:*

1) $\mathscr{G}$ is essentially principal.

2) The set of points in $\mathscr{G}^{(0)}$ with trivial isotropy is dense.

Definition 3.2.8 *[Re3] A* ***twist*** *over a groupoid $\mathscr{G}$ is a groupoid extension*

$$\mathbb{T} \times X \to \Gamma \to \mathscr{G}$$

where Γ and $\mathscr{G}$ are topological groupoids, $\mathbb{T}$ is the circle group, X is a space and, at the level of unit spaces, $X \to \Gamma^{(0)} \to \mathscr{G}^{(0)}$ are identification maps. In the topological setting, we require that all maps to be continuous and the identification maps to be homeomorphisms. So one says that Γ is twist over $\mathscr{G}$ or that the pair $(\Gamma, \mathscr{G})$ is a ***twisted groupoid****.*

Here for a complex number $z \in \mathbb{C}$, we denote by $\overline{z}$ its complex conjugate.

Example 3.2.9 *Let $\mathscr{G}$ be a topological equivalence relation and $\sigma : \mathscr{G}^{(2)} \to \mathbb{T}$ be a continuous 2-cocycle, so that it satisfies $\sigma(g_1, g_2)\sigma(g_1g_2, g_3) = \sigma(g_2, g_3)\sigma(g_1, g_2g_3)$ for all $(g_1, g_2), (g_2, g_3) \in \mathscr{G}^{(2)}$. Let $\Gamma = \mathbb{T} \times \mathscr{G}$ endowed with the following operations;*

1) Product Map*: $(s, g_1)(t, g_2) = (st\sigma(g_1, g_2), g_1g_2)$,*

2) Inverse Map*: $(s, g)^{-1} = (\overline{s}\overline{\sigma(g, g^{-1})}, g^{-1})$ and*

3) $\mathbb{T}$-action*: $s(t, g) = (st, g)$,*

with $s, t \in \mathbb{T}$ and $g, g_1, g_2 \in \mathscr{G}$, then Γ is a twist over $\mathscr{G}$.

The previous example was studied extensively by Renault in [Re] but was not called a twist. It is in fact a particular case of a twisted groupoid of the form $(\Gamma, \mathscr{G})$ with $\mathscr{G}$ étale, second countable locally compact Hausdorff and principal. This type of twisted groupoids was studied by Kumjian in [Ku]. Later Renault in [Re2] extended the study to a twisted groupoid of the form $(\Gamma, \mathscr{G})$ with $\mathscr{G}$ étale, second countable locally compact Hausdorff and topologically principal. In Section 6.6, we study the C^*-algebra associated to a twisted groupoid and their algebraic structure. The results on the algebraic structure of these C^*-algebras motivate the definition of twisted groupoids.

3.3 Borel Groupoids

Now we introduce definitions and results for groupoids with Borel a structure.

Definition 3.3.1 *[AnaRe] Let $\mathscr{G}$ be a groupoid. Suppose $(\mathscr{G}, \mathscr{A})$ is a standard Borel space. If the product, inverse, source and range maps are Borel, where $\mathscr{G}^{(2)} \subseteq \mathscr{G} \times \mathscr{G}$ has the restricted Borel structure of $(\mathscr{G} \times \mathscr{G}, \mathscr{A} \times \mathscr{A})$ and $\mathscr{G}^{(0)} \subseteq \mathscr{G}$ has the restricted Borel structure of $(\mathscr{G}, \mathscr{A})$, then $\mathscr{G}$ is a **Borel groupoid**.*

Definition 3.3.2 *[AnaRe] Let $\mathscr{G}$ be a Borel groupoid. A **Borel Haar system** is a family of nonnegative sigma-finite measures $\{\lambda^x\}_{x \in \mathscr{G}^{(0)}}$ with the following properties;*

1) *$supp(\lambda^x) = \mathscr{G}^x$, $x \in \mathscr{G}^{(0)}$,*

2) *for every nonnegative Borel function $f : \mathscr{G} \to \mathbb{C}$, the function $\lambda(f) : \mathscr{G}^{(0)} \to \mathbb{C}$, defined by $x \mapsto \lambda^x(f)$, is Borel,*

3) *is invariant in the sense that for every $g \in \mathscr{G}$, then*

$$\int f(gg_1) d\lambda^{s(g)}(g_1) = \int f(g_1) d\lambda^{r(g)}(g_1)$$

4) and is proper in the sense that there is a positive Borel function $f : \mathscr{G} \to \mathbb{C}$ such that $\lambda(f)$ is the constant function one.

Definition 3.3.3 *[AnaRe] A Borel groupoid $\mathscr{G}$ is **r-discrete** if for all $x \in \mathscr{G}^{(0)}$, $\mathscr{G}^x$ is countable. The counting measures on $\mathscr{G}^x$ form a natural Borel Haar system for $\mathscr{G}$.*

Remark 3.3.4 *As in Remark* 3.1.14, *there is a bijective correspondence between principal (and r-discrete) Borel groupoids and (countable) Borel equivalence relations.*

We will study countable Borel equivalence relations in depth in Chapter 4.

Definition 3.3.5 *Let Γ be a Borel groupoid and $\mathbb{T}$ be the circle group. Let $k_1, k_2 \in \mathbb{T}$ and $(\tau_1, \tau_2) \in \Gamma^{(2)}$. If Γ is a free $\mathbb{T}$-space such that*

i) $\mathscr{G} \cong \Gamma/\mathbb{T}$ is a countable Borel equivalence relation, where the canonical quotient map $q : \Gamma \to \mathscr{G}$ is a groupoid homomorphism,

ii) $(k_1\tau_1, k_2\tau_2) \in \Gamma^{(2)}$,

iii) $(k_1\tau_1)(k_2\tau_2) = (k_1k_2)(\tau_1\tau_2)$ and

iv) $\Gamma' \cong \mathbb{T} \times \Gamma^{(0)}$

*then $(\Gamma, \mathscr{G})$ is called a **Borel twist** or we say that Γ is a twist over $\mathscr{G}$.*

As in the topological setting (see Definition 3.2.8), this definition can be generalized by not requiring that $\mathscr{G}$ is a countable Borel equivalence relation. We study Borel twists in Section 5.3.

Definition 3.3.6 *Two Borel twists are isomorphic, if there is a groupoid isomorphism $\Phi : \Gamma_1 \to \Gamma_2$, such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathbb{T} \times X & \xrightarrow{\iota_1} & \Gamma_1 & \xrightarrow{q_1} & \mathscr{G} \\ \downarrow{\scriptstyle id_{\mathbb{T}\times X}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle id_{\mathscr{G}}} \\ \mathbb{T} \times X & \xrightarrow{\iota_2} & \Gamma_2 & \xrightarrow{q_2} & \mathscr{G} \end{array}$$

Following Kumjian in [Ku], we introduce the following definition.

Definition 3.3.7 *Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$. The set of isomorphism classes of Borel twists Γ such that $\Gamma/\mathbb{T} \cong E$ is denoted by $Tw(E)$.*

3.4 Countable Measured Equivalence Relations

We now extend the definitions to groupoids with a measurable structure.

Definition 3.4.1 *[AnaRe] Let $\mathscr{G}$ be a Borel groupoid with a Borel Haar system $\lambda = \{\lambda^x\}_{x \in \mathscr{G}^{(0)}}$. Let μ be a measure on $\mathscr{G}^{(0)}$. The measure μ is* ***quasi-invariant*** *with respect to $\mathscr{G}$ and λ if the measure $\mu \circ \lambda$ is such that $\mu \circ \lambda(\mathcal{G}) = 0$ if and only if $\mu \circ \lambda(\mathcal{G}^{-1}) = 0$ for all Borel subset $\mathcal{G}$ of $\mathscr{G}$.*

Definition 3.4.2 *[AnaRe] If $\mathscr{G}$ is a Borel groupoid with a be Borel Haar system λ and a quasi-invariant measure μ on $\mathscr{G}^{(0)}$. Then the triple $(\mathscr{G}, \lambda, \mu)$ is called a* ***measured groupoid****.*

We now introduce an important countable Borel equivalence relation, the so-called tail equivalence relation.

Definition 3.4.3 *Let $X = \prod_{k \in \mathbb{N}^*} \{0, 1\}$ be the Cantor space. The equivalence relation given by*

$$E_0 = \{(x, y) \in X \times X; \exists N > 0 \text{ such that } x_n = y_n, \forall n \geq N\}$$

is a countable Borel equivalence relation, called ***tail equivalence****.*

Example 3.4.4 *Let E_0 be tail equivalence on X with the canonical Borel Haar system λ of counting measures. For any $a \in [0, 1]$, let d_a be the measure on $\{0, 1\}$ such that $d_a(0) = \frac{1}{1+a}$ and $d_a(1) = \frac{a}{1+a}$ and define a measure on X by $\mu_a = \prod_{k \in \mathbb{N}^*} d_a$. Then (E, λ, μ_a) is a measured Borel groupoid.*

Measured groupoids coming from countable Borel equivalence relations with a measure on the unit space and a measurable 2-cocycle were studied by Feldman and Moore in [FM1] and [FM2]. In particular in [FM2], they study von Neumann algebras associated to these measure groupoids. We will present their construction and their results in Section 6.5.

Chapter 4

Borel Equivalence Relations

One of the goal of this thesis was to associate to countable Borel equivalence relations canonical operators algebras and to study their properties. In the measured case, the most general construction was obtained in [FM2] by Feldman and Moore. To a measured countable equivalence relation E on a Lebesgue space (X, μ) and a 2-cocycle σ on E with value in the circle group $\mathbb{T}$, they associate a von Neumann algebra $W^*(E, \sigma)$ generalizing Murray-von Neumann group measure space construction.

In this chapter, we recall the notion and first properties of countable Borel equivalence relations. We then present examples of hyperfinite Borel equivalence relations using the notion of tail equivalence on a Bratteli diagram.

4.1 Countable Borel Equivalence Relation

In this section, we present definitions and results related to countable Borel equivalence relations (Borel principal r-discrete groupoids). We follow mainly [DJKe].

Recall that an equivalence relation E on a set X is a subset of $X \times X$ such that for all $x, y, z \in X$,

i) E is reflexive; $\forall_{x \in X} \left((x, x) \in E\right)$,

ii) E is symmetric; $(x, y) \in E \implies (y, x) \in E$ and

iii) E is transitive; $(x, y) \in E$ and $(y, z) \in E \implies (x, z) \in E$.

For $x \in X$, let $[x]_E$ or simply $[x]$ denote the **equivalence class of** x. For two equivalent elements x and y in X, we use the notation xEy, $x \sim_E y$ or $x \sim y$. We will denote by

$$q : X \longrightarrow X/E$$

the **canonical quotient map**.

E is a **countable** equivalence relation if $[x]_E$ is countable for all $x \in X$.

Let $(X, \mathscr{A})$ be a standard Borel space. Then an equivalence relation E on $(X, \mathscr{A})$ is a **Borel equivalence relation** if $E \subset X \times X$ belongs to the σ-algebra $\mathscr{A} \times \mathscr{A}$. Notice that $(E, \mathscr{A} \times \mathscr{A}|_E)$ is a standard Borel space.

We endow X/E with the Borel σ-algebra $\mathscr{A}/E = q(\mathscr{A})$.

Definition 4.1.1 *Let E be a Borel equivalence relation on $(X, \mathscr{A})$. Then E is* ***finite*** *if every equivalence class is finite,* ***uniformly finite of order*** *n if every equivalence class is of cardinality at most n and* ***aperiodic*** *if every equivalence class is infinite.*

Example 4.1.2 *Let G be a countable group of Borel automorphisms of $(X, \mathscr{A})$ and let*

$$E_G = \{(x, y) \in X \times X; \exists_{g \in G}(g(x) = y)\}$$

be the orbit equivalence relation. Then E_G is a countable Borel equivalence relation. Moreover, if G is infinite and acts freely on $(X, \mathscr{A})$, then E_G is aperiodic.

Conversely in [FM1], Theorem 1, Feldman and Moore prove.

Theorem 4.1.3 *Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$ with X with X uncountable. Then there is a countable group G of Borel automorphisms such that $E = E_G$. Moreover, G can be chosen such that*

$$xEy \iff (\exists_{g \in G}(g^2 = 1 \wedge g(x) = y)).$$

Remark 4.1.4 *In [Ad], S. Adams showed that in Theorem* 4.1.3*, the group G cannot be chosen to act freely in general.*

Definition 4.1.5 *Two Borel equivalence relations E_1 and E_2 on standard Borel spaces $(X, \mathscr{A})$ and $(Y, \mathcal{C})$, are **isomorphic**, noted $E_1 \cong E_2$, if there is a Borel isomorphism $f : X \longrightarrow Y$ such that $xE_1y \Leftrightarrow f(x)E_2f(y)$.*

Remark 4.1.6 *Recall that if E is a Borel equivalence relation on a standard Borel space $(X, \mathscr{A})$, then $(E, \mathscr{A} \times \mathscr{A}|_E)$ is a standard Borel space. Keeping the notation of Definition* 4.1.5*, note that if $f : X \to Y$ is a Borel isomorphism such that $xE_1y \Leftrightarrow f(x)E_2f(y)$, then $f \times f : X \times X \to Y \times Y$ is a Borel isomorphism and a bijection from E_1 to E_2, thus the spaces E_1 and E_2 are Borel isomorphic.*

Definition 4.1.7 *Let E be a Borel equivalence relation on $(X, \mathscr{A})$. Let Y be a Borel subset of X. The **restriction** of E to Y, noted $E|_Y$, is*

$$E|_Y = E \cap (Y \times Y).$$

Definition 4.1.8 *A subset A of X is E**-invariant**, or simply **invariant** when no confusion arises, if $x \in A$ and xEy, then $y \in A$.*

Definition 4.1.9 *Let E be a Borel equivalence relation on $(X, \mathscr{A})$ and A be a subset of X. The E-**saturation** of A, denoted $[A]_E$, is defined by*

$$[A]_E = \{x \in X | \exists_{y \in A}(xEy)\}.$$

The subset $[A]_E$ is the smallest E**-invariant** set containing A.

Theorem 4.1.10 *Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$. Let $\kappa : X \longrightarrow \mathbb{N}^*$ be the the map which associate to each $x \in X$, the cardinality of its equivalence class $[x]$. Then κ is a Borel function.*

Proof: By Theorem 4.1.3, there exists a countable group of Borel automorphisms $G = \{g_k; k \in \mathbb{N}\}$ such that $E = \cup_{k\in\mathbb{N}} \text{graph}(g_k)$. Let $N = \{g_{k_1}, ..., g_{k_n}\}$ be any subset of G of n elements. The subset

$$J(N) = \{x \in X : g_{k_i}(x) \neq g_{k_j}(x) \text{ for all } i \neq j\}$$

is a Borel subset of X. For $n \in \mathbb{N}^*$, let

$$X(n) = \{x \in X : |[x]_E| = n\} = \kappa^{-1}(n).$$

To complete the proof it suffices to show that $X(n)$ is Borel. The set

$$J_n = \bigcup_{N \subset G} J(N)$$

is also a Borel subset of X. Moreover since $J_n = \{x \in X : |[x]_E| \geq n\}$, then $J_{n+1} \setminus J_n = X(n)$ is Borel. ∎

Definition 4.1.11 *The* ***pseudo-group*** *of an equivalence relation E, denoted $[[E]]$, is the set of partial Borel bijections $f : A \longrightarrow B$ between Borel subsets A and B of X, such that $f(x)Ex$ for all $x \in A$. The* ***full group*** *of an equivalence relation E, denoted $[E]$, is the set of Borel automorphisms $f : X \longrightarrow X$ such that $f(x)Ex$ for all $x \in X$.*

By Theorem 1.1 of [MRo] the full group, as an abstract group, is an invariant of isomorphism for the countable Borel equivalence relations provided that the cardinality of any class is at least 2.

Theorem 4.1.12 *[MRo] Let E_1 and E_2 be two countable Borel equivalence relations on $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ respectively. Suppose $|[x_k]_{E_k}| > 2$ for all $x_k \in X_k$, $k = 1, 2$. Then $E_1 \cong E_2$ if and only if $[E_1] \cong [E_2]$.*

With this theorem we can easily construct an invariant for all countable Borel equivalence relations. Recall, for $n \in \mathbb{N}^*$,

$$X(n) = \{x \in X : |[x]_E| = n\}.$$

The sets $X(n)$ are E-invariant Borel subsets of X. Let $c(n)$ be the cardinality of $X(n)$ and $Z = X \setminus (X(1) \cup X(2))$.

Corollary 4.1.13 *Let E_1 and E_2 be two countable Borel equivalence relations on $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ respectively. For $i = 1, 2$, let $c_i(n)$, $X_i(n)$ and Z_i be defined as above but for E_i. Then E_1 and E_2 are isomorphic if and only if $c_1(1) = c_2(1)$, $c_1(2) = c_2(2)$ and $[E_1|_{Z_1}] \cong [E_2|_{Z_2}]$.*

Definition 4.1.14 *Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$. Two Borel subsets A and B are Hopf equivalent ($A \sim B$) if there exists a partial Borel bijection $f \in [[E]]$ such that $f(A) = B$.*

We look now at the notions of subequivalence relation and embeddable relation.

Definition 4.1.15 *Let E and E' be two countable Borel equivalence relations on $(X, \mathscr{A})$. If $E' \subseteq E$, then E' is a **subequivalence relation** of E.*

Definition 4.1.16 *Let E_1 and E_2 be two countable Borel equivalence relations. The Borel equivalence relation E_1 is **embeddable** into E_2, denoted $E_1 \sqsubseteq E_2$, if there is an injective Borel map $f : X_1 \longrightarrow X_2$ such that*

$$(x, y) \in E_1 \iff (f(x), f(y)) \in E_2.$$

*If $E_1 \sqsubseteq E_2$ and $E_2 \sqsubseteq E_1$, then we call E_1 and E_2 **bi-embeddable**, denoted $E_1 \approx E_2$.*

Remark 4.1.17 *The following are equivalent:*

1) E_1 is embeddable into E_2.

2) There exists a Borel subset Y_2 of X_2 such that $E_1 \cong E_2|_{Y_2}$.

4.2 Invariant and Ergodic Measures on E

In this section we recall the notion of an invariant and an ergodic measure on a countable Borel equivalence relation, introduced in [FM1].

Definition 4.2.1 *[FM1] Let μ be a measure on X. Let λ^x be the counting measure on the set $\{(x,y) \in E; y \in X\}$ and λ_x be the counting measure on the set $\{(y,x) \in E; y \in X\}$, then the measures on E $\int \lambda^x d\mu(x)$ and $\int \lambda_x d\mu(x)$ are denoted by ν and ν^{-1}.*

Theorem 4.2.2 *[FM1] Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$, μ a probability measure on $(X, \mathscr{A})$ and G be a countable Borel group of Borel automorphisms such that $E = E_G$. The following are equivalent;*

1) $\mu = \mu g$ for all $g \in G$, where $\mu g(A) = \mu(g^{-1}(A))$ for all $A \in \mathscr{A}$.

2) If $f \in [E]$, then $\mu = \mu f$, where $\mu f(A) = \mu(f^{-1}(A))$ for all $A \in \mathscr{A}$.

3) If $f \in [[E]]$, with $f : A \longrightarrow B$, then $\mu(A) = \mu(B)$.

4) $\nu = \nu^{-1}$.

Definition 4.2.3 *Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$. A probability measure μ on $(X, \mathscr{A})$ is E-**invariant**, if it satisfies one of the equivalent conditions of the previous theorem. If $\mu(A) = 0$ or $\mu(A) = 1$ for all invariant Borel subsets $A \in \mathscr{A}$, then μ is E-**ergodic**.*

Notation 4.2.4 *Let E be a countable Borel equivalence relation. Then let $\mathcal{I}_E$ denote the set of all E-invariant probability measures, $\mathcal{E}_E$ denote the set of all E-ergodic probability measures and $\mathcal{IE}_E = \mathcal{I}_E \cap \mathcal{E}_E$. If we are considering only non-atomic measures, then let $\mathcal{I}_E^\circ$, $\mathcal{E}_E^\circ$ and $\mathcal{IE}_E^\circ$ denote the corresponding sets.*

4.3 Compressible Equivalence Relations

In this section we define compressible countable Borel equivalence relations and present some of their properties. We follow [DJKe]. As above, E denotes a countable Borel equivalence relation on $(X, \mathscr{A})$.

Definition 4.3.1 *A Borel subset A is E-**full** if it has non-empty intersection with every equivalence class of E, i.e. if $A \cap [x]_E \neq \emptyset$ for all $x \in X$.*

Definition 4.3.2 *A Borel equivalence relation E is **compressible** if there is a non-empty Borel subset A of X such that $A \sim X$ and $X \setminus A$ is E-full.*

Definition 4.3.3 *A subset A of X is E-**compressible** if $E|_A$ is compressible.*

Definition 4.3.4 *Let A and B be Borel subsets of X. Then $A \preceq B$ if there is a Borel subset $C \subseteq B$ such that $A \sim C$. Thus we get, by the usual Schroeder-Bernstein argument, that $A \sim B$ if and only if $(A \preceq B)$ and $(B \preceq A)$.*

Theorem 4.3.5 *E is not compressible if and only if E admits an E-invariant probability measure.*

Definition 4.3.6 *A Borel subset A of X is E-**paradoxical** if there is there are disjoint Borel subsets B and C of A such that $A \sim B$ and $A \sim C$.*

Theorem 4.3.7 *Then the following are equivalent:*

1) E is compressible,

2) There is a sequence $\{A_n\}$ of pairwise disjoint full Borel sets with $A_n \sim A_m$ for all n, m and

3) X is E-paradoxical (with B and C subsets of $\bigcup A_n$).

4.4 Smooth Borel Equivalence Relations

In this section we introduce the class of smooth countable Borel equivalence relations. All definitions and results can be found in [KeM]. As in the preceding sections, E denotes a countable Borel equivalence relation on $(X, \mathscr{A})$.

Definition 4.4.1 *Let E be a countable Borel equivalence relation. Then E is* ***smooth****, if there is a Borel surjection $\phi : X \longrightarrow Y$, where $(Y, \mathcal{C})$ is a standard Borel space, such that*

$$\forall_{x,y\in X}(xEy \Longleftrightarrow \phi(x) = \phi(y)).$$

Remark 4.4.2

1) This definition is equivalent to the existence of a ***Borel separating family****, i.e. a sequence $\{B_n\}_{n\in\mathbb{N}}$ of Borel subsets of X such that*

$$\forall_{x,y\in X}(xEy \Longleftrightarrow \forall_{n\in\mathbb{N}}(x \in B_n \Leftrightarrow y \in B_n)).$$

2) For a countable Borel equivalence relation E, recall that a ***Borel transversal*** *for E is a Borel set $B \subseteq X$ which intersects each class of E at exactly one point. Then E is smooth if and only if E as a* ***Borel selector****, i.e. a Borel function $f : X \longrightarrow X$ whose image is a Borel transversal for E.*

For the next lemma we need the uniformization theorem whose proof can be found in [Ke] Theorem 18.10.

Theorem 4.4.3 *Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be standard Borel spaces and let $P \subseteq X \times Y$ be a Borel subset. If every section $P_x = \{(z, y) \in P; z = x\}$ is countable, then there exist $P^* \subseteq P$ such that*

$$\forall_{x\in X}\left(\exists_{y\in X}((x, y) \in P) \Longleftrightarrow \exists!_{y\in X}((x, y) \in P^*)\right).$$

P^* is called a uniformization of P. Moreover $P = \cup_{n\in\mathbb{N}} P_n$, where P_n are a Borel graphs of $X \times Y$.

Lemma 4.4.4 *If E is a smooth countable Borel equivalence relation, then E admits a Borel transversal. If E is uniformly finite of order n, then there is a finite set of Borel selectors $\{f_i\}_{i=1}^n$ for E, whose images partition X. If E is aperiodic, then there is a countable set of Borel selectors $\{f_i\}_{i\in\mathbb{N}}$ for E, whose images partition X.*

Proof: Let $\phi : X \longrightarrow Y$ be a map witnessing that E is smooth, and define $A \subseteq Y \times X$ by

$$(y, x) \in A \Longleftrightarrow \phi(x) = y.$$

By the uniformization theorem (4.4.3), it follows that A has a uniformization $f : Y \longrightarrow X$ and that $f \circ \phi$ is the desired Borel selector. ∎

Theorem 4.4.5 *Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$. Then E is smooth if and only if X/E is a standard Borel space.*

Proof:

Let $\phi : X \longrightarrow Y$ be a Borel surjection verifying that E is smooth. Let $q : X \longrightarrow X/E$ be the quotient map and let $\Phi = \phi \circ q^{-1}$, where $\Phi([x]) = \phi(x)$ for all $x \in X$, be the induced bijection (well-defined) from X/E to Y. Since ϕ is countable to one, it sends Borel sets to Borel sets. Thus an E-invariant set B is Borel if and only if $\Phi(q(B)) = \phi(B)$ is Borel.

Conversely, since X/E is a standard Borel space there exists a a countable family $\{B_n\}_{n\in\mathbb{N}}$ of sets which separate points of X/E. Define $A_n = p^{-1}(B_n)$, then $\{A_n\}_{n\in\mathbb{N}}$ is a Borel separating family for E. ∎

A direct consequence of Theorem 4.4.5 is that a finite Borel equivalence relation is smooth.

If $\phi : X \longrightarrow X$ is a (Borel) automorphism, let

$$E_\phi = \{(x, y) \in X \times X; \exists_{k\in\mathbb{Z}} \phi^k(x) = y\}$$

denote the countable (Borel) equivalence relation induced by ϕ.

Theorem 4.4.6 *Let E be a countable Borel equivalence relation. If E is smooth, then there exists a Borel automorphism ϕ such that $E = E_\phi$.*

Proof: For $n \in \mathbb{N}^* \cup \{\infty\}$, define $A_n = \{x \in X : |[x]_E| = n\}$ and $E_n = E|_{A_n}$. Then A_n are E-invariant Borel sets. Thus E is a disjoint union of the E_n for $n \in \mathbb{N}^* \cup \{\infty\}$.

For $n \in \mathbb{N}^*$, partition A_n with Borel transversals $f_k : A_n \longrightarrow B_k$ for $k = 1, ..., n$ such that $A_n = \bigsqcup_{k=1}^n B_k$. Define the partial Borel map $\phi_n : A_n \longrightarrow A_n$ by

$$\phi_n(x) = \begin{cases} f_1(x), & \text{if } x \in B_n \\ f_{k+1}(x), & \text{if } x \in B_k,\ k < n \end{cases}.$$

For $n = \infty$, partition A_∞ with Borel transversals $f_k : A_\infty \longrightarrow B_k$ for $k \in \mathbb{Z}$ such that $A_\infty = \bigsqcup_{k\in\mathbb{Z}}^n B_k$. Define the partial Borel map $\phi_\infty : A_\infty \longrightarrow A_\infty$ by

$$\phi_\infty(x) = \begin{cases} f_{k+1}(x), & \text{if } x \in B_k \end{cases}.$$

Finally with $\phi(x) = \phi_n(x)$ if $x \in A_n$ for $n \in \mathbb{N}^* \cup \{\infty\}$, it follows that $E = E_\phi$. ∎

Corollary 4.4.7 *For $i = 1, 2$, let E_i be two smooth countable Borel equivalence relations. Then $E_1 \cong E_2$ if and only if the cardinalities of the standard Borel space A_k^i/E_i are equal, where $A_k^i = \{x \in X : |[x]_{E_i}| = k\}$, for $k \in \{\infty, 1, 2, ...\}$.*

Recall that if E is a countable Borel equivalence relation on $(X, \mathscr{A})$, $\mathcal{IE}_E$ (resp. $\mathcal{IE}_E^\circ$) denotes the set of E-invariant, ergordic (resp. non-atomic) probability measures on $(X, \mathscr{A})$.

Theorem 4.4.8 *Let E be a countable Borel equivalence relation. If $\mathcal{IE}_E^\circ$ is not empty, then E is not smooth.*

Proof: Suppose E is smooth and $\mu \in \mathcal{IE}_E$. Then there is a sequence $\mathscr{C} = \{A_i\}_{i\in\mathbb{N}}$ of Borel sets separating the points in X/E (and $\mathscr{C}$ is closed under taking complement). Let $q : X \longrightarrow X/E$ be the quotient map and define $\nu(A) = \mu(q^{-1}(A))$, where A is a Borel set of X/E. Then by ergodicity of μ, $\nu(A)$ is equal to zero or one. Let $\mathscr{C}_1 = \{A \in \mathscr{C}; \nu(A) = 1\}$ and

$$B = \bigcap_{A\in\mathscr{C}_1} A.$$

Then $\nu(B) = 1$.

Claim : B is a singleton; this implies that μ is atomic ($\mathcal{IE}_E^\circ = \emptyset$).

If not, suppose $x, y \in B$ and $x \neq y$, then there is is a set $D \in \mathscr{C}$ such that $x \in D$ and $y \notin D$. If $\nu(D) = 0$ then $x \notin B$ and if $\nu(D) = 1$ then $y \notin B$ which leads to a contradiction. ∎

The next theorem will give another formulation of non-smoothness. Recall that E_0 denotes tail equivalence (see Definition 3.4.3) and that E_1 is embeddable in E_2 is denoted by $E_1 \sqsubseteq E_2$ (see Definition 4.1.16).

Theorem 4.4.9 *Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$. Then E is non-smooth if and only if $E_0 \sqsubseteq E$.*

4.5 Hyperfinite Borel Equivalence Relations

In this section we introduce the class of hyperfinite countable Borel equivalence relations.

Definition 4.5.1 *A countable Borel equivalence relation E is **hyperfinite** if $E = \bigcup_{n\in\mathbb{N}^*} E_n$, where the E_n's are an increasing sequence (i.e., $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$) of finite Borel equivalence relations.*

The next theorem shows, as in the smooth case, that countable hyperfinite Borel equivalence relations can be written E_ϕ for some Borel automorphism $\phi : X \longrightarrow X$.

Theorem 4.5.2 *[W] Let E be a countable Borel equivalence on a standard Borel space $(X, \mathscr{A})$. The following conditions are equivalent:*

1) $E = E_\phi = \{(x, y) \in X \times X | \exists_{k\in\mathbb{Z}}(\phi^k(x) = y)\}$, where $\phi : X \longrightarrow X$ is a Borel automorphism.

2) $E = \bigcup_{n\in\mathbb{N}^*} E_n$, where the $(E_n)_{n\geqslant 1}$ forms an increasing sequence of uniformly finite Borel equivalence relations E_n of order n.

3) $E = \bigcup_{n\in\mathbb{N}^*} E_n$, where the $(E_n)_{n\geqslant 1}$ forms an increasing sequence of finite Borel equivalence relations.

By Theorem 4.4.6, a smooth countable Borel equivalence relation is hyperfinite. In the next section we give examples of countable non-smooth hyperfinite Borel equivalence relations.

4.6 Bratteli Diagram and Tail Equivalence

Let us first recall the definition of a Bratteli diagram. They were introduced by O. Bratteli in [Bra] to classify approximatively finite (AF) C^*-algebras.

Definition 4.6.1 *(See for example [GPS3]) A* ***Bratteli diagram*** *is an infinite directed graph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set and $\mathcal{E}$ is the edge set. They are disjoint unions of non-empty finite sets;*

$$\mathcal{V} = \bigsqcup_{i=1}^{\infty} V_i \qquad \textit{and} \qquad \mathcal{E} = \bigsqcup_{i=1}^{\infty} E_i.$$

The graph has the following property:

1) *An edge $e \in E_i$ goes form a vertex V_i, noted $s(e)$, to a vertex in V_{i+1}, noted $r(e)$. The map s is called the* ***source map*** *and r is called the* ***range map****.*

2) *The graph has no* ***sink****;*

$$\forall_{v \in \mathcal{V}} \left(s^{-1}(v) \neq \emptyset \right).$$

To a Bratteli diagram $\mathcal{D}$ we can associate its path space, denoted $\Omega_{\mathcal{D}}$, as follows: Start with a **source** $v \in \mathcal{V}$, i.e. $r^{-1}(v) = \emptyset$ and define

$$\Omega_v = \{(e_i)_{i=n}^{\infty} : e_i \in E_i,\, s(e_n) = v,\, s(e_{i+1}) = r(e_i),\, i \geqslant n\}.$$

For each $n \geqslant 1$, we endow E_n with its discrete topology and the product space $\prod_{i \geqslant n} E_i$ with the corresponding product topology. Then $\Omega_v \subset \prod_{i \geqslant n} E_i$ with the relative topology is compact metrizable and zero dimensional. Then $\Omega_{\mathcal{D}}$ is the disjoint union of the Ω_v and equipped with the topological sum topology is a locally compact, metrizable and zero dimensional space and the **clopen cylinder sets**

$$U_{(e_n,\ldots,e_m)} = \{x \in \Omega_{\mathcal{D}} : x_n = e_n, \ldots, x_m = e_m\}$$

form a basis for the topology of Ω_v.

Definition 4.6.2 ***Tail equivalence*** *on the Bratteli diagram $\mathcal{D}$ is the equivalence relation $E_{\mathcal{D}}$ defined on $\Omega_{\mathcal{D}}$ by*

$$E_{\mathcal{D}} = \{(x, y) \in \Omega_{\mathcal{D}} \times \Omega_{\mathcal{D}}; \exists N > 0 \text{ such that } x_n = y_n,\ \forall n \geqslant N\}.$$

For each $n \geqslant 1$, let $E_{\mathcal{D}}^{(n)}$ be the finite subequivalence relation of $E_{\mathcal{D}}$ defined by

$$E_{\mathcal{D}}^{(n)} = \{(x, y) \in \Omega_{\mathcal{D}} \times \Omega_{\mathcal{D}}; x_i = y_i,\ i \geqslant n\}$$

(i.e., two paths in $\Omega_{\mathcal{D}}$ are $E_{\mathcal{D}}^{(n)}$-equivalent if they agree from level n).

By construction, $(E_{\mathcal{D}}^{(n)})_{n\geqslant 1}$ forms an increasing sequence of subequivalence relations of $E_{\mathcal{D}}$ and

$$E_{\mathcal{D}} = \bigcup_{i=1}^{\infty} E_{\mathcal{D}}^{(n)}.$$

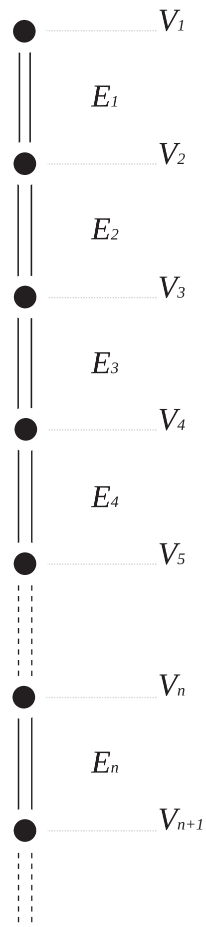
For $n \geqslant 1$, $E_{\mathcal{D}}^{(n)}$ is a compact, open subset of the product space $\Omega_{\mathcal{D}} \times \Omega_{\mathcal{D}}$, and therefore Borel. Hence, $E_{\mathcal{D}}$ is a hyperfinite countable Borel equivalence relation on $\Omega_{\mathcal{D}}$.

Using Bratteli diagrams and tail equivalence, we now construct examples of hyperfinite countable Borel equivalence relations.

We start with the Bratteli diagram of the UHF C^*-algebra 2^∞. Tail equivalence on this diagram is the tail equivalence E_0 for the Definition 3.4.3.

Example 4.6.3 *Let $\mathcal{D}_1$ be the Bratteli diagram:*

Figure 4.1: $\mathcal{D}_1$



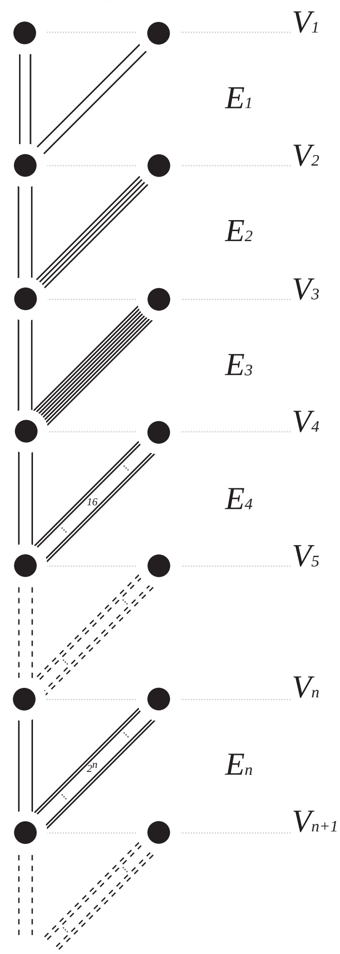
For each $i \in \mathbb{N}^*$, $V_i = \{v^i\}$ and $E_i = \{e_0^i, e_1^i\}$. The range of all edges in E_i is v^{i+1}, and the source for the edges in E_i is v^i. The product measure

$$\mu = \bigotimes_{i=1}^{\infty} \left(\frac{1}{2}\delta_{e_0^i} + \frac{1}{2}\delta_{e_1^i} \right)$$

is the only non-atomic ergodic $E_{\mathcal{D}_1}$-invariant probability measure. Moreover $E_{\mathcal{D}_1}$ is isomorphic to E_0.

Example 4.6.4 *Let $\mathcal{D}_0$ be the Bratteli diagram:*

Figure 4.2: $\mathcal{D}_0$



For each $i \in \mathbb{N}^*$, $V_i = \{v_0^i, v_1^i\}$ and $E_i = \{e_0^i, e_1^i, a_1^i, ..., a_{2^i}^i\}$. The range of all edges in E_i is v_0^{i+1}, the source of the edges e_j^i, $j = 0$ or 1, is v_0^i and the source of the edges a_k^i, $k = 1, ..., 2^i$, is v_1^i. In this example, the map

$$\phi : \Omega_{\mathcal{D}_0} \longrightarrow \Omega_{\mathcal{D}_0},$$

defined by

$$\phi((a_k^i e_j^{i+1}, ...)) = (a_{k+j2^i}^{i+1}, ...)$$

and

$$\phi((e_j^1, ...)) = (a_j^1, ...)$$

for $j = 0$ or 1 is a Borel injection, belonging to $[[E_{\mathcal{D}_0}]]$.

If A_i is the subset of $\Omega_{\mathcal{D}_0}$ of all paths beginning with an edge a_k^i, $k = 1, ..., 2^i$;

$$A = \bigsqcup_{i=1}^{\infty} A_i \text{ and } B = \{(e_j^i); i \geqslant 1, j = \text{ 0 or 1}\},$$

then

$$\phi(\Omega_{\mathcal{D}_0}) = A$$

and for $i \geqslant 1$,

$$\phi^i(B) = A_i.$$

Lemma 4.6.5 *The Borel space $\Omega_{\mathcal{D}_0}$ does not carry any $E_{\mathcal{D}_0}$-invariant measure.*

Proof: If μ were such an $E_{\mathcal{D}_0}$-invariant probability measure, then $1 = \mu(\Omega_{\mathcal{D}_0}) = \mu(A) + \mu(B)$ and $1 = \mu(\Omega_{\mathcal{D}_0}) = \mu(\phi(\Omega_{\mathcal{D}_0})) = \mu(A)$, then $\mu(B) = 0$ and $\mu(B) = \mu(\phi^i(B)) = \mu(A_i)$. Thus $\mu(A) = 0$ and $\mu(\Omega_{\mathcal{D}_0}) = 0$ which is a contradiction. ∎

Then, since $\Omega_{\mathcal{D}_0} \sim A$, and A^c full, then

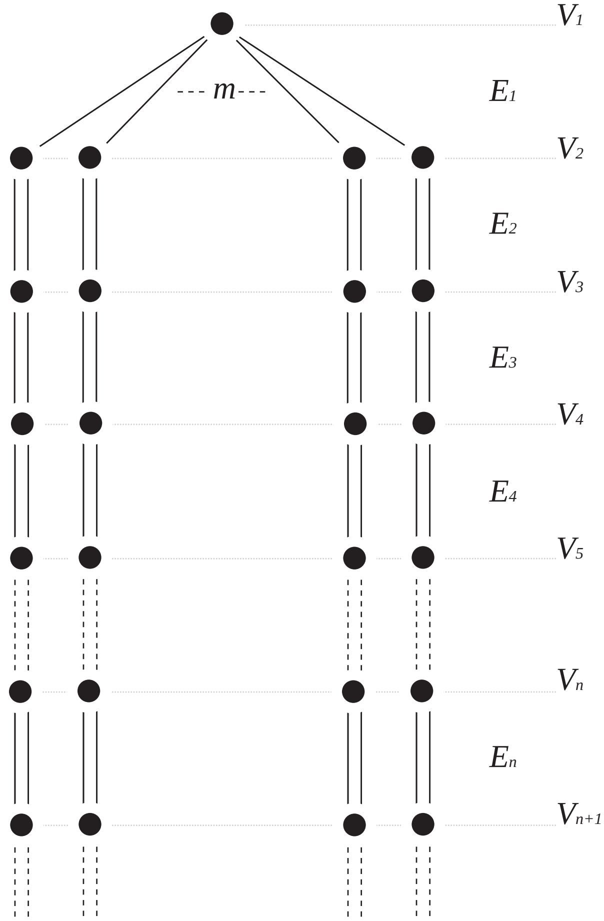
Fact: The tail equivalence $E_{\mathcal{D}_0}$ on $\mathcal{D}_0$ is a compressible countable Borel equivalence relation.

The sets $(A_i)_{i\geqslant 1}$ form a sequence of pairwise disjoints, $E_{\mathcal{D}_0}$-invariant, full Borel sets (with $A_i \sim A_j$ for all $i, j > 0$). As $E_{\mathcal{D}_0}|_B$ is isomorphic to E_0, then

Fact: $E_{\mathcal{D}_0}$ is non-smooth.

Example 4.6.6 *For $m > 1$, let $\mathcal{D}_m$ be the Bratteli diagram:*

Figure 4.3: $\mathcal{D}_m$



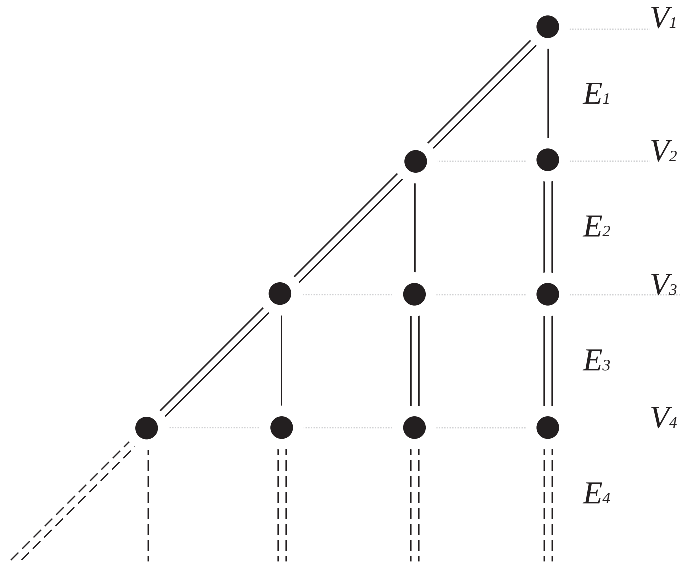
Let $V_1 = \{v^1\}$ and $E_1 = \{e_1^1, ..., e_m^1\}$. For each $i \in \mathbb{N}^*$, $i > 1$, $V_i = \{v_1^i, ..., v_m^i\}$ and $E_i = \{e^i_{(0,1)}, e^i_{(1,1)}, e^1_{(0,2)}, e^i_{(1,2)}, ..., e^i_{(0,m)}, e^i_{(1,m)}\}$. The vertex v^1 is the source of all edges in E_1 and the range of the edge e_k^1 is v_k^2, for $k = 1, ..., m$. For $i > 1$, the source of the edges $e^i_{(0,k)}$ and $e^i_{(1,k)}$ is v^i_k and the range of the edges $e^i_{(0,k)}$ and $e^i_{(1,k)}$ is v_k^{i+1}. For $k \in \{1, ..., m\}$, the product measures

$$\mu_k = \delta_{e_k^1} \otimes \left(\bigotimes_{i=2}^{\infty} \left(\frac{1}{2} \delta_{e^i_{(0,k)}} + \frac{1}{2} \delta_{e^i_{(1,k)}} \right) \right)$$

are the m measures in $\mathcal{IE}^\circ_{E_{\mathcal{D}_m}}$.

Example 4.6.7 *Let $\mathcal{D}_\omega$ be the Bratteli diagram:*

Figure 4.4: $\mathcal{D}_\omega$



Here $V_1 = \{v^1\}$ and $E_1 = \{b_0^1, b_1^1, a^1\}$. For each $i \in \mathbb{N}^*$, $i > 1$,

$$V_i = \{v^i, w_1^i, ..., w_{i-1}^i\} \text{ and } E_i = \{b_0^i, b_1^i, a^i, e^i_{(0,1)}, e^i_{(1,1)}, ..., e^i_{(0,i-1)}, e^i_{(1,i-1)}\}.$$

The source of b_0^i and b_1^i is v^i and their range is v^{i+1}. The source of a^i is v^i and its range is w_i^{i+1}. For $k = 1, ..., i-1$, the source of $e^i_{(0,k)}$ and $e^i_{(1,k)}$ is w_k^i and their range is w_k^{i+1}. The measure

$$\mu_1 = \bigotimes_{i=1}^{\infty} \left(\frac{1}{2}\delta_{b_0^i} + \frac{1}{2}\delta_{b_1^i} \right)$$

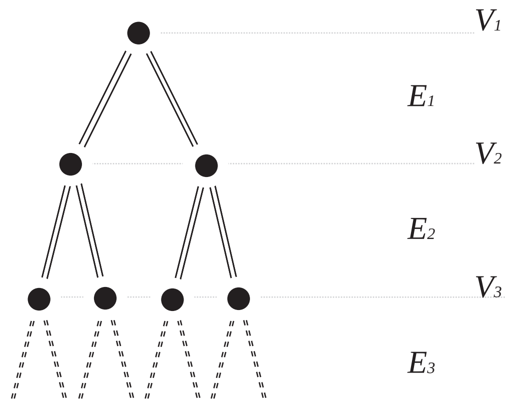
together with the measures μ_k, for $k > 2$, given by

$$\mu_k = \left(\bigotimes_{i=1}^{k} \left(\frac{1}{2}\delta_{b_0^i} + \frac{1}{2}\delta_{b_1^i} \right) \right) \otimes (\delta_{a^k}) \otimes \left(\bigotimes_{i=k+1}^{\infty} \left(\frac{1}{2}\delta_{e^i_{(0,k)}} + \frac{1}{2}\delta_{e^i_{(1,k)}} \right) \right)$$

are the measures in $\mathcal{IE}^\circ_{E_{\mathcal{D}_\omega}}$.

Example 4.6.8 *Let $\mathcal{D}_c$ be the following Bratteli diagram;*

Figure 4.5: $\mathcal{D}_c$



That example can be described as follows: Let W_i be the set of all binary words of length i. For each $i > 1$, let $\tau : W_i \longrightarrow W_{i-1}$ be the truncation of the last digit. Let W_ω be the set of (countable) infinite binary words and $\tau_i : W_\omega \longrightarrow W_i$ be the truncation that keeps the first n digits of the word. Let $V_1 = \{v^1\}$ and $E_1 = \bigcup_{k \in W_1} \{a^1_k, b^1_k\}$. For $i \in \mathbb{N}^*$, $i > 1$,

$$V_i = \bigcup_{k \in W_{i-1}} \{v^i_k\} \text{ and } E_i = \bigcup_{k \in W_i} \{a^i_k, b^i_k\}.$$

The vertex v^1 is the source of all edges in E_1. For any $k \in W_i$, the vertex $v^i_{\tau(k)}$ is the source of the (four) edges a^i_k and b^i_k. The range for the (two) edges a^i_k and b^i_k is v^{i+1}_k. Let $k \in W_\omega$, the measures

$$\mu_k = \bigotimes_{i=1}^{\infty} \left(\frac{1}{2} \delta_{a^i_{\tau_i(k)}} + \frac{1}{2} \delta_{b^i_{\tau_i(k)}} \right)$$

are the measures in $\mathcal{IE}^\circ_{E_{\mathcal{D}_c}}$.

Fact: For any $\mathcal{D}_k$, $k \in \{c, \omega, 0, 1, 2, 3, ...\}$, $E_{\mathcal{D}_k}$ is an aperiodic, non-smooth and hyperfinite countable Borel equivalence relation.

4.7 Classification of Hyperfinite Borel Equivalence Relations

The goal of this section is to survey the classification of all hyperfinite Borel equivalence relations up to isomorphism (orbit equivalence) on the standard Borel space $(X, \mathscr{A})$. We will first give the classification of the smooth hyperfinite Borel equivalence relations. The classification of the non-smooth Borel equivalence relation is based (almost entirely) on the result of Theorem 9.1 of [DJKe].

Let us recall the smooth case (see Section 4.4), for $n \in \mathbb{N}^* \cup \{\infty\}$, let $X(n)$ be the points in X with orbits of cardinality n and let $E_n = E|_{X(n)}$. Then E is a disjoint union of the E_n. Now since E_n is smooth, the quotient space $X(n)/E_n$ is a standard Borel space and let q_n be its cardinality. Then (from Corollary 4.4.7), two smooth Borel equivalence relations E and E' are isomorphic if and only if $q_n = q'_n$ for all $n \in \mathbb{N}^* \cup \{\infty\}$.

In the non-smooth case we can assume that the Borel equivalence relation is aperiodic. If not, let $A = X(\infty)$ be the points in X with orbits of infinite cardinality and decompose $E = E|_{A^C} \sqcup E|_A$ (periodic and aperiodic blocks). The restriction of E to its periodic part is smooth; this was handle above. We now present the classification up to isomorphism of countable, aperiodic, non-smooth and hyperfinite Borel equivalence relations.

Theorem 4.7.1 *[DJKe] Let E_1 and E_2 be two countable, aperiodic, non-smooth and hyperfinite Borel equivalence relations. Then*

$$E_1 \cong E_2 \iff |\mathcal{IE}^\circ_{E_1}| = |\mathcal{IE}^\circ_{E_2}|.$$

Corollary 4.7.2 *Let E be a countable, aperiodic, non-smooth and hyperfinite Borel equivalence relation. Then there exists unique $k \in \{w, c, 0, 1, 2, ...\}$ such that*

$$E \cong E_{\mathcal{D}_k}.$$

Chapter 5

Cohomology

5.1 Cocycles and coboundaries

In this section we present definitions and results on cohomology of countable Borel equivalence relations. We follow closely the definitions and results of [FM1], but they are given in a purely Borel context (with no reference to a measure on X). In this section $(\mathbb{A}, +)$ is an abelian Polish group with neutral element denoted 0, and E is a countable Borel equivalence relation on $(X, \mathscr{A})$.

Definition 5.1.1 *For $n \geqslant 1$, let $E^{(n)}$ be the Borel subset of $\prod_{k=1}^{n+1}(X, \mathscr{A})$ given by*

$$(x_1, ..., x_{n+1}) \in E^{(n)} \Longleftrightarrow \forall_{i,j=1,...,n+1}(x_i E x_j).$$

Clearly, $E^{(1)} = E$.

Definition 5.1.2 *Let $n \geqslant 1$. A Borel function $f : E^{(n)} \longrightarrow \mathbb{A}$ is a n-**cochain** if there exists $k \in \{1, ..., n\}$, such that $x_k = x_{k+1}$ entails $f(x_1, ..., x_{n+1}) = 0$. The n-cochains, denoted $C^n(E, \mathbb{A})$, form an abelian group.*

Definition 5.1.3 *For $n = 0$, $E^{(0)} = X$ and $C^0(E, \mathbb{A})$ is the set of Borel functions from X to $\mathbb{A}$.*

Definition 5.1.4 *For any $n \geqslant 1$, the maps $\delta_n : C^{n-1}(E, \mathbb{A}) \longrightarrow C^n(E, \mathbb{A})$ are defined by*

$$(\delta_n f)(x_1, ..., x_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} f(x_1, ..., \hat{x}_k, ..., x_{n+1}),$$

where

$$(x_1, ..., \hat{x}_k, ..., x_{n+1}) \in E^{(n-1)}$$

denote the element of $E^{(n-1)}$ obtained by removing the k^{th} variable of $(x_1, ..., x_{n+1})$.

The following computation shows that $\delta_{n+1} \circ \delta_n = 0$:
For $n \geqslant 1$,

$$\delta_{n+1}(\delta_n f)(x_1, ..., x_{n+2}) = \sum_{k=1}^{n+2} (-1)^{k+1} (\delta_n f)(x_1, ..., \hat{x}_k, ..., x_{n+2})$$

which expands and simplifies to

$$\sum_{k=1}^{n+2} \left(\sum_{k'=1}^{k-1} (-1)^{k+k'} f(x_1, ..., \hat{x}_{k'}, ..., \hat{x}_k, ..., x_{n+2}) + \sum_{k'=k+1}^{n+2} (-1)^{k+k'+1} f(x_1, ..., \hat{x}_k, ..., \hat{x}_{k'}, ..., x_{n+2}) \right),$$

(when $k = 1$ (resp. $k = n+2$) the first (resp. second) inside sum is empty) and for any $1 \leqslant i < j \leqslant n+2$, the term $f(x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+2})$ appears exactly twice in $\delta_{n+1}(\delta_n f)(x_1, ..., x_{n+2})$, one with $k' = i$ and $k = j$ leading to $(-1)^{i+j} f(x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+2})$ and one with $k = i$ and $k' = j$ leading to $(-1)^{i+j+1} f(x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+2})$ whose sum is 0, this implies that $\delta_{n+1}(\delta_n f)(x_1, ..., x_{n+2}) = 0$ for any $f \in C^n(E, \mathbb{A})$. Thus we get a long exact sequence:

$$0 \xrightarrow{\delta_0} C(X, \mathbb{A}) \xrightarrow{\delta_1} C(E, \mathbb{A}) \xrightarrow{\delta_2} C^2(E, \mathbb{A}) \xrightarrow{\delta_3} ...$$

where δ_0 is the inclusion map.

Definition 5.1.5 *For $n \geqslant 0$, a Borel n-**cocycle** is an element of $\ker(\delta_{n+1})$. The set of all n-cocycles will be denoted $Z^n(E, \mathbb{A})$. A Borel n-**coboundary** is an element of $im(\delta_n)$. The set of all n-coboundaries will be denoted $B^n(E, \mathbb{A})$. The n-**cohomology group** is $Z^n(E, \mathbb{A})/B^n(E, \mathbb{A})$, denoted $H^n(E, \mathbb{A})$.*

By definition for $n \geqslant 2$, two cocycles f and g in $Z^n(E, \mathbb{A})$ are **cohomologous** if $f - g \in B^n(E, \mathbb{A})$.

Theorem 5.1.6 *Let E be a smooth countable Borel equivalence relation, and $\mathbb{A}$ be an abelian Polish group. Then for all $n \geqslant 1$, $H^n(E, \mathbb{A}) = 0$.*

Proof: As E is smooth, let us fix a Borel selector $f : X \longrightarrow X$ of E. Then, let us associate to any n-cocycle $\sigma \in Z^n(E, \mathbb{A})$, the $(n-1)$-cochain $c : E^{(n-1)} \longrightarrow \mathbb{A}$ given by

$$c(x_1, ..., x_n) = \sigma(f(x_1), x_1, ..., x_n).$$

As

$$(\delta_{n+1}\sigma)(x_0, x_1, ..., x_{n+1}) = \sum_{k=0}^{n+1} (-1)^k \sigma(x_0, ..., \hat{x}_k, ..., x_{n+1}) = 0,$$

it follows that

$$\begin{aligned} \sigma(x_1, ..., x_{n+1}) &= - \sum_{k=1}^{n+1} (-1)^k \sigma(x_0, ..., \hat{x}_k, ..., x_{n+1}) \\ &= \sum_{k=1}^{n+1} (-1)^{k+1} c(x_1, ..., \hat{x}_k, ..., x_{n+1}) \\ &= (\delta_n c)(x_1, ..., x_{n+1}), \end{aligned}$$

with $x_0 = f(x_1)$. Thus $\sigma \in B^n(E, \mathbb{A})$. ∎

The next theorem follows a proof, in the topological setting, of H. Matui [Mat], which also applies in the Borel setting.

Theorem 5.1.7 *Let $E = \cup_{i \geqslant 1} E_i$ be a hyperfinite countable Borel equivalence relation, where $(E_i)_{i \geqslant 1}$ is an increasing sequence of finite equivalence relations. Let $\mathbb{A}$ be an abelian Polish group. For all $n \geqslant 2$, $H^n(E, \mathbb{A}) = 0$.*

Proof: Let $n \geqslant 2$. Let $\sigma \in \ker(\delta^{n+1})$. We construct inductively a sequence $(\tau_i)_{i\geqslant 1}$, $\tau_i \in C^{n-1}(E, \mathbb{A})$, such that

$$\delta^n(\tau_i)(\xi) = \sigma(\xi),\ \forall \xi \in E_i^{(n)} \ \text{ and } \ \tau_{i+1}(\eta) = \tau_i(\eta),\ \forall \eta \in E_i^{(n-1)}.$$

Then if we define $\tau \in C^{n-1}(E, \mathbb{A})$ by

$$\tau(\eta) = \lim_{i\to\infty} \tau_i(\eta),\ \forall \eta \in E^{(n-1)},$$

then we get $\delta^n(\tau) = \sigma$. Hence $H^n(E, \mathbb{A}) = 0$.

For τ_1, since $H^n(E_1, \mathbb{A}) = 0$ there exists $\tau_1 \in C^{n-1}(E, \mathbb{A})$ such that

$$\delta^n(\tau_1)(\xi) = \sigma(\xi),\ \forall \xi \in E_1^{(n)}.$$

Suppose τ_i is constructed. Since $H^n(E_{i+1}, \mathbb{A}) = 0$, then there exists $f \in C^{n-1}(E, \mathbb{A})$, with $\mathrm{supp}(f) \subseteq E_{i+1}^{(n-1)}$, such that

$$\delta^n(f)(\xi) = \sigma(\xi),\ \forall \xi \in E_{i+1}^{(n)}.$$

Hence we have

$$\delta^n(f - \tau_i)(\xi) = 0,\ \forall \xi \in E_i^{(n)}.$$

The restriction of $f - \tau_i$ to $E_i^{(n-1)}$ is in $H^{n-1}(E_i, \mathbb{A})$. Since $H^{n-1}(E_i, \mathbb{A}) = 0$, then there exists $g \in C^{n-2}(E, \mathbb{A})$, with $\mathrm{supp}(f) \subseteq E_i^{(n-2)}$, such that

$$\delta^{n-1}(g)(\eta) = (f - \tau_i)(\eta),\ \forall \eta \in E_i^{(n-1)}.$$

Set $\tau_{i+1} = f - \delta^{n-1}(g)$. Then

$$\delta^n(\tau_{i+1})(\xi) = \delta^{(n)}(f)(\xi) = \sigma(\xi),$$

for all $\xi \in E_{i+1}^{(n)}$, and

$$\tau_{i+1}(\eta) = f(\eta) - \delta^{n-1}(g)(\eta) = f(\eta) - (f - \tau_i)(\eta) = \tau_i(\eta),$$

for all $\eta \in E_i^{(n-1)}$, which completes the proof. ∎

5.2 2-cocycles

We identify by $\mathbb{T}$ the set of complex numbers of modulus 1. The $\mathbb{T}$-valued Borel 2-cocycles play an important role in this thesis. Thus we review the definitions and give some results in that specific context. In this section E denotes a countable Borel equivalence relation on $(X, \mathscr{A})$.

Definition 5.2.1 *A $\mathbb{T}$-valued Borel* ***2-cocycle*** *$\sigma \in Z^2(E, \mathbb{T})$ of E is a Borel function $\sigma : E^{(2)} \to \mathbb{T}$ which satisfies the following property*

$$\sigma(x, y, z)\sigma^{-1}(w, y, z)\sigma(w, x, z)\sigma^{-1}(w, x, y) = 1$$

or

$$\sigma(w, y, z)\sigma(w, x, y) = \sigma(x, y, z)\sigma(w, x, z)$$

for all $w \sim_E x \sim_E y \sim_E z$.

If there is a $c \in C(E, \mathbb{A})$, such that a Borel 2-cocycle σ can be written as

$$\sigma(x, y, z) = c(x, y)c^{-1}(x, z)c(y, z)$$

is a Borel ***2-coboundary****.*

Definition 5.2.2 *Two Borel 2-cocycles σ_1 and σ_2 are* ***cohomologous*** *if there exists $c \in B^2(E, \mathbb{T})$ such that*

$$\sigma_1(x, y, z) = \sigma_2(x, y, z)c(x, y)c^{-1}(x, z)c(y, z)$$

for all $x \sim_E y \sim_E z$.

From now on a 2-cocycle on E will always refer to a $\mathbb{T}$-valued Borel 2-cocycle.

Definition 5.2.3 *A 2-cocycle σ is* ***normalized*** *if it is equal to 1 whenever two or three variables are identical.*

Hence a normalized 2-cocycle satisfies the extra condition that $\sigma(x,y,x) = 1$ for all $(x,y) \in E$.

Lemma 5.2.4 *Every 2-cocycle is cohomologous to a normalized one.*

Proof: Let σ be a 2-cocycle. We have

$$\sigma(a,c,d)\sigma(a,b,c) = \sigma(b,c,d)\sigma(a,b,d);$$

then if $a = c = x$ and $b = d = x$, we obtain

$$\begin{aligned} \sigma(x,x,y)\sigma(x,y,x) &= \sigma(y,x,y)\sigma(x,y,y) \\ \sigma(x,y,x) &= \sigma(y,x,y). \end{aligned}$$

If $\sqrt{\bullet} : \mathbb{T} \longrightarrow \mathbb{T}$ denotes the Borel map defined by $\sqrt{e^{2\pi i\theta}} = e^{\pi i\theta}$ for $\theta \in [0,1[$, then the function $g : E \longrightarrow \mathbb{T}$ defined by

$$g(x,y) = \sqrt{\sigma(x,y,x)}$$

is Borel and such that $g(x,y) = g(y,x)$ and $g(x,x) = 1$ for all $(x,y) \in E$ and $x \in X$. If $\tau : E^{(2)} \longrightarrow \mathbb{T}$ denotes the Borel function defined by

$$\tau = \sigma(\delta g)^{-1},$$

then for all $x \in X$ and $(x,y) \in E$ we have:

1) $\tau(x,x,x) = 1$,

2) $\tau(x,x,y) = \sigma(x,x,y)(g(x,y)g^{-1}(x,y)g(x,x))^{-1} = 1$,

3) $\tau(x,y,y) = \sigma(x,y,y)(g(y,y)g^{-1}(x,y)g(x,y))^{-1} = 1$ and

4) $$\begin{aligned} \tau(x,y,x) &= \sigma(x,y,x)(g(y,x)g^{-1}(x,x)g(x,y))^{-1} \\ &= \sigma(x,y,x)(g(y,x)g(x,y))^{-1} \\ &= \sigma(x,y,x)(g(x,y)^2)^{-1} \\ &= 1. \end{aligned}$$

Hence τ is a normalized 2-cocycle. ∎

Lemma 5.2.5 *A normalized 2-cocycle is skew symmetric, i.e. permuting any two variables inverses (conjugates) the value of the 2-cocycle.*

Proof: Let $\tau : E^{(2)} \longrightarrow \mathbb{T}$ be a normalized 2-cocycle. We have

$$\tau(a, c, d)\tau(a, b, c) = \tau(b, c, d)\tau(a, b, d);$$

then if $a = x$, $b = d = y$ and $c = z$, we obtain

$$\begin{aligned} \tau(x, z, y)\tau(x, y, z) &= \tau(y, z, y)\tau(x, y, y) \\ \tau(x, z, y) &= 1 \\ \tau(x, z, y) &= \tau^{-1}(x, y, z). \end{aligned}$$

Similarly if $a = c = x$, $b = y$ and $d = z$, then $\tau(x, y, z) = \tau^{-1}(y, x, z)$. If $a = d = x$, $b = z$ and $c = y$, then $\tau(z, y, x) = \tau(x, z, y) = \tau^{-1}(x, y, z)$. ∎

5.3 The Borel Twist

In this section, for a countable Borel equivalence relation E on $(X, \mathscr{A})$, we establish the link between isomorphism classes of Borel twists $Tw(E)$ (see Definitions 3.3.5 and 3.3.7) and the 2-cohomology group $H^2(E, \mathbb{T})$. We first associate a Borel twist to a 2-cocycle, defined on a countable Borel equivalence relation, and then show that any Borel twist arises this way.

Let E be a countable Borel equivalence relation and σ be a normalized 2-cocycle on E with values in the torus $\mathbb{T}$. Let Γ be the set of pairs of the form $(a, (x, y)) \in \mathbb{T} \times E$. We then define a multiplication on Γ by

$$(a, (x, w))(b, (y, z)) = (ab\sigma(x, y, z), (x, z)), \text{ if } w = y,$$

an involution by

$$(a, (x, y))^{-1} = (\overline{a}, (y, x)),$$

and a free Borel $\mathbb{T}$-action by

$$b(a, (x, y)) = (ab, (x, y)),$$

for all $(w, x, y, z) \in E^{(3)}$ and $a, b \in \mathbb{T}$. Then Γ is a Borel twist, that we will (sometimes) refer to as E **twisted by** σ.

In the Borel case, we will show that any Borel twist of a countable equivalence relation is induced by a 2-cocycle of E. The proof follows closely the proof of the topological analogue by Kumjian in [Ku] and uses the following result of Kallman [Kall].

Proposition 5.3.1 *([Kall], Proposition 7.1) Let G be a locally compact group with a countable basis, and let Γ be a standard Borel G-space such that Γ/G is countably separated. Then Γ/G is standard, and there is a Borel cross-section*

$$h : \Gamma/G \longrightarrow \Gamma.$$

Theorem 5.3.2 *Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$. For any twist Γ in $Tw(E)$, there exists a 2-cocycle σ of E such that $\Gamma = (E, \sigma)$. Moreover if $(E, \sigma_1) \cong (E, \sigma_2)$, then σ_1 and σ_2 are cohomologous.*

Proof: The construction above associates to E and σ a Borel twist. Moreover, straightforward computations show that Borel twists associated to cohomologous (normalized) 2-cocycles of a Borel equivalence relation are isomorphic. Hence, we only have to show that each Borel twist is associated to a normalized 2-cocycle. Let Γ be a Borel twist with $E \cong \Gamma/\mathbb{T}$. By 5.3.1, there exists a Borel cross-section

$$s : E \longrightarrow \Gamma.$$

Let $\sigma_s : E^{(2)} \longrightarrow \mathbb{T}$ denote the Borel function such that

$$\sigma_s(x, y, z)s(x, z) = s(x, y)s(y, z)$$

for all $(x, y, z) \in E^{(2)}$. Since

$$\begin{aligned} s(w, x)s(x, y)s(y, z) &= \sigma_s(w, x, y)s(w, y)s(y, z) \\ &= \sigma_s(w, x, y)\sigma_s(w, y, z)s(w, z) \end{aligned}$$

and

$$\begin{aligned} s(w, x)s(x, y)s(y, z) &= s(w, x)\sigma_s(x, y, z)s(x, z) \\ &= \sigma_s(x, y, z)s(w, x)s(x, z) \\ &= \sigma_s(x, y, z)\sigma_s(w, x, z)s(w, z) \end{aligned}$$

then

$$\sigma_s(w, x, y)\sigma_s(w, y, z) = \sigma_s(x, y, z)\sigma_s(w, x, z),$$

and therefore $\sigma_s \in Z^2(E, \mathbb{T})$. If

$$t : E \longrightarrow \Gamma$$

is an other Borel cross-section and $\sigma_t : E^{(2)} \longrightarrow \mathbb{T}$ such that

$$\sigma_t(x,y,z)t(x,z) = t(x,y)t(y,z),$$

then consider the Borel function $k : E \to \mathbb{T}$ such that for $(x,y) \in E$

$$t(x,y) = k(x,y)s(x,y).$$

Since

$$\begin{aligned}
\sigma_t(x,y,z)t(x,z) &= t(x,y)t(y,z) \\
\sigma_t(x,y,z)k(x,z)s(x,z) &= k(x,y)s(x,y)k(y,z)s(y,z) \\
\sigma_t(x,y,z)k(x,z)s(x,z) &= k(x,y)k(y,z)\sigma_s(x,y,z)s(x,z)
\end{aligned}$$

then

$$\sigma_t(x,y,z) = k(x,y)\overline{k(x,z)}k(y,z)\sigma_s(x,y,z),$$

thus σ_s and σ_t are cohomologous. ∎

This result does differ from the result of Kumjian in the topological setting: only topological twists with a continuous cross-section are induced by continuous 2-cocycles. Some topological twists do not have a continuous cross-section, while all Borel twists do have Borel cross-sections.

Remark 5.3.3 *We can equip $Tw(E)$ with a group structure (as in Remark 2 and Proposition 3 of [Ku]) which makes the bijection between $Tw(E)$ and $H^2(E,\mathbb{T})$ of Theorem 5.3.2 a group isomorphism.*

Chapter 6

Borel $*$-algebras

The first three sections of this chapter introduce the facts on the theory of C^*-algebras, von Neumann algebras and Borel $*$-algebras that we will use in the rest of the thesis, and we review in the fourth one the definitions of Cartan subalgebras. In the last section, we review AF-algebras.

Recall that we want to associate to a countable Borel equivalence relation a Borel $*$-algebra. In Section 5, we present the measurable-von Neumann algebra case, following Feldman and Moore's results in [FM1] and [FM2], and in Section 6 the topological-C^*-algebra case, following Renault in [Re], [Re1] and [Re2] and Kumjian in [Ku].

The construction of the Borel $*$-algebra associated to a countable Borel equivalence relation is presented in Chapter 7.

Throughout this chapter, $\mathcal{H}$ will denote a Hilbert space and $\mathcal{B}(\mathcal{H})$ the involutive algebra of bounded linear operator on $\mathcal{H}$.

6.1 C^*-algebras

Definition 6.1.1 *A C^***-algebra** $\mathcal{A}$ is an involutive Banach algebra whose norm satisfies, for all $f \in \mathcal{A}$, $\|f^* f\| = \|f\|^2$.*

Example 6.1.2

1) *$\mathcal{B}(\mathcal{H})$, with the operator norm, is a C^*-algebra. Note that if $\mathcal{H}$ is infinite dimensional, then $\mathcal{B}(\mathcal{H})$ is not separable.*

2) *Any norm-closed involutive subalgebra of $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. Such an algebra is called a **concrete** C^***-algebra**.*

3) *If X be a locally compact Hausdorff space, then $C_0(X)$, the continuous functions vanishing at infinity on X (see Definition 2.4.2), equipped with the sup-norm is an abelian C^*-algebra.*

In fact, any C^*-algebra has a concrete representation. To concretise this last sentence, we recall the definition of a representation and in particular of the GNS-representation of a C^*-algebra.

Definition 6.1.3 *A **representation** of C^*-algebra $\mathcal{A}$ is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. The representation $(\pi, \mathcal{H})$ is **faithful** when it is injective.*

Remark 6.1.4 *If $(\pi, \mathcal{H})$ is a representation of a C^*-algebra $\mathcal{A}$, then $\|\pi(a)\| \leqslant \|a\|$ for all $a \in \mathcal{A}$.*

Definition 6.1.5 *Let $\mathcal{A}$ be a C^*-algebra. A bounded positive linear form of norm one $\phi : \mathcal{A} \longrightarrow \mathbb{C}$ is called a **state**.*

Notation 6.1.6 *The set of all states on $\mathcal{A}$ will be denoted $S(\mathcal{A})$. Then $S(\mathcal{A})$ is a convex set. The extreme points in $S(\mathcal{A})$ are called the **pure states** and are denoted $P(\mathcal{A})$.*

Example 6.1.7 *By Riesz representation theorem, the states on $C_0(X)$ correspond to the probability measures on X and the pure states to the Dirac measures on X.*

We know present the GNS-representation, following [Ped], Theorem 3.3.3.

Theorem 6.1.8 *Let $\mathcal{A}$ be a C^*-algebra and $\phi \in S(\mathcal{A})$. There exist a representation $(\pi_\phi, \mathcal{H}_\phi)$ of $\mathcal{A}$ and a vector of norm one, noted ξ_ϕ, such that $\phi(A) = \langle \xi_\phi, \pi_\phi(A)\xi_\phi \rangle$ for all $A \in \mathcal{A}$ and $\pi_\phi(\mathcal{A})\xi_\phi$ is dense in $\mathcal{H}_\phi$.*

Definition 6.1.9 *Let $\mathcal{A}$ be a C^*-algebra and $\phi \in S(\mathcal{A})$. The representation $(\pi_\phi, \mathcal{H}_\phi)$ of Theorem* 6.1.8 *is called the* ***Gelfand-Naimark-Segal (GNS) representation induced by*** *ϕ.*

Definition 6.1.10 *([Ped], 3.7.6 and 4.3.7) Let $\mathcal{A}$ be a C^*-algebra. Let $(\pi_\phi, \mathcal{H}_\phi)$ be the GNS representation induced by $\phi \in S(\mathcal{A})$. The* ***universal representation*** *of $\mathcal{A}$ is defined by*

$$(\pi_u, \mathcal{H}_u) = \bigoplus_{\phi \in S(\mathcal{A})} (\pi_\phi, \mathcal{H}_\phi).$$

The ***atomic representation*** *of $\mathcal{A}$ is defined by*

$$(\pi_a, \mathcal{H}_a) = \bigoplus_{\phi \in P(\mathcal{A})} (\pi_\phi, \mathcal{H}_\phi).$$

Theorem 6.1.11 *([Ped], 4.3.11) Let $\mathcal{A}$ be a C^*-algebra. The atomic representation is faithful.*

Corollary 6.1.12 *([Ped], Theorem 1.1.7) If $\mathcal{A}$ is a commutative C^*-algebra, then $\mathcal{A}$ is isomorphic to $C_0(P(\mathcal{A}))$.*

We end this section with two useful theorems for C^*-algebras.

Theorem 6.1.13 *([An], Theorem 3.2) Let $\mathcal{A}_0$ be a C^*-subalgebra of a C^*-algebra $\mathcal{A}$ and ϕ be a pure state of $\mathcal{A}_0$. Then there exists a unique pure state Φ of $\mathcal{A}$ such that*

$\Phi|_{\mathcal{A}_0} = \phi$ *if and only if for each* $x \in \mathcal{A}$ *and* $\varepsilon > 0$ *there is* $b \in \mathcal{A}_0^+$, *with* $\|b\| = \phi(b) = 1$, *and* $y \in \mathcal{A}_0$ *such that*

$$\|bxb - y\| < \varepsilon.$$

Definition 6.1.14 *([An], Definition 3.3) Let* $\mathcal{A}_0$ *be a* C^**-subalgebra of a* C^**-algebra* $\mathcal{A}$*. Then* $\mathcal{A}_0$ *has the* ***extension property relative to*** $\mathcal{A}$ *if every pure state of* $\mathcal{A}_0$ *can be extended uniquely to a pure state of* $\mathcal{A}$*.*

6.2 von Neumann algebras

Throughout this section, $\mathcal{H}$ denotes a Hilbert space and $\langle \bullet | \bullet \rangle$ its inner product. Moreover $\{x_i\} \subset \mathcal{B}(\mathcal{H})$ will denote a bounded net of operators ($\|x_i\| < k < \infty$ for all i) and $\{x_n\} \subset \mathcal{B}(\mathcal{H})$ is a bounded sequence. This section follows closely Sections 2.1 and 2.2 of [Ped].

Definition 6.2.1 *A net* $\{x_i\}$ *is* ***weakly convergent*** *to* $x \in \mathcal{B}(\mathcal{H})$, *denoted* $x_i \xrightarrow{w} x$, *if for every vectors* $\xi, \eta \in \mathcal{H}$ *the net* $\{\langle x_i \xi | \eta \rangle\}$ *converges to* $\langle x\xi | \eta \rangle$.

Definition 6.2.2 *A net* $\{x_i\}$ *is* ***strongly convergent*** *to* $x \in \mathcal{B}(\mathcal{H})$, *denoted* $x_i \xrightarrow{s} x$, *if for each vector* $\xi \in \mathcal{H}$ *the net* $\{\langle x_i \xi | x_i \xi \rangle\}$ *converges to* $\langle x\xi | x\xi \rangle$.

Notation 6.2.3 *Let* $\mathcal{M}$ *be a subset of* $\mathcal{B}(\mathcal{H})$*. We will denote by* $\mathcal{M}^w$ *(resp.* $\mathcal{M}^s$*) its weak closure (resp. strong closure).*

Remark 6.2.4 *As any strongly convergent net in* $\mathcal{B}(\mathcal{H})$ *is weakly convergent, then for any* $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{M}^s \subseteq \mathcal{M}^w$.

Definition 6.2.5 *Let* $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$*. The* ***commutant*** *of* $\mathcal{M}$ *is the set*

$$\mathcal{M}' = \{x \in \mathcal{B}(\mathcal{H}) : \forall_{f \in \mathcal{M}}\, xf = fx\}.$$

Theorem 6.2.6 *([Ped], Theorem 2.2.2) Let $\mathcal{W}$ be a C^*-algebra containing the identity of $\mathcal{B}(\mathcal{H})$. The following are equivalent:*

$$a)\ \mathcal{W} = \mathcal{W}''. \qquad b)\ \mathcal{W} = \mathcal{W}^s. \qquad c)\ \mathcal{W} = \mathcal{W}^w.$$

Definition 6.2.7 *Let $\mathcal{W}$ be a C^*-algebra containing the identity of $\mathcal{B}(\mathcal{H})$. If $\mathcal{W} = \mathcal{W}''$, then $\mathcal{W}$ is called a **von Neumann algebra**.*

Example 6.2.8

1) *$\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.*

2) *Let (X, μ) be a measured space. Any $f \in L^\infty(X, \mu)$ defines a multiplication operator $M_f \in \mathcal{B}(L^2(X, \mu))$, by*

$$(M_f \xi)(x) = f(x)\xi(x), \quad \xi \in L^2(X, \mu), x \in X.$$

Then $\{M_f; f \in L^\infty(X, \mu)\}$ is an abelian von Neumann algebra of $\mathcal{B}(L^2(X, \mu))$.

6.3 Borel $*$-algebra

In this section, we introduce Borel $*$-algebra. They will be used, in Chapter 7, to study countable Borel equivalence relations. From a topological space $(X, \mathcal{T})$, recall from Theorem 2.4.5 that any bounded Borel function on X is a pointwise limit of a sequence of continuous functions. Borel $*$-algebras can be seen as a possible non-commutative analogue of this result. The definitions and results are taken from Pedersen's book [Ped].

Definition 6.3.1 *([Ped], 4.5.2) Let $\mathcal{H}$ be an Hilbert space and $\mathcal{B}(\mathcal{H})_{sa}$ be the set of self-adjoint elements of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})_{sa}$, $x \in \mathcal{B}(\mathcal{H})_{sa}$ and $\{x_n\}$ be elements in $\mathcal{B}(\mathcal{H})_{sa}$. Then*

1) $\mathscr{B}^m_{\mathcal{H}}(\mathcal{M})$ *will denote the smallest subset of* $\mathcal{B}(\mathcal{H})_{sa}$ *containing* $\mathcal{M}$ *and such that if* $\{x_n\}$ *is a norm bounded, monotone (increasing or decreasing) sequence of* $\mathscr{B}^m_{\mathcal{H}}(\mathcal{M})$ *converging strongly to* x*, then* $x \in \mathscr{B}^m_{\mathcal{H}}(\mathcal{M})$.

2) $\mathscr{B}^s_{\mathcal{H}}(\mathcal{M})$ *will denote the smallest subset of* $\mathcal{B}(\mathcal{H})_{sa}$ *containing* $\mathcal{M}$ *and such that if* $\{x_n\}$ *is a norm bounded sequence of* $\mathscr{B}^s_{\mathcal{H}}(\mathcal{M})$ *converging strongly to* x*, then* $x \in \mathscr{B}^s_{\mathcal{H}}(\mathcal{M})$.

3) $\mathscr{B}^w_{\mathcal{H}}(\mathcal{M})$ *will denote the smallest subset of* $\mathcal{B}(\mathcal{H})_{sa}$ *containing* $\mathcal{M}$ *and such that if* $\{x_n\}$ *is a norm bounded sequence of* $\mathscr{B}^w_{\mathcal{H}}(\mathcal{M})$ *converging weakly to* x*, then* $x \in \mathscr{B}^w_{\mathcal{H}}(\mathcal{M})$.

Remark 6.3.2 *By definition, it follows that*

$$\mathscr{B}^m_{\mathcal{H}}(\mathcal{M}) \subseteq \mathscr{B}^s_{\mathcal{H}}(\mathcal{M}) \subseteq \mathscr{B}^w_{\mathcal{H}}(\mathcal{M}).$$

Definition 6.3.3 *([Ped], 4.5.14) Let* $\mathcal{A}$ *be a* C^**-subalgebra of* $\mathcal{B}(\mathcal{H})$.
The ***monotone Borel envelope*** *of* $\mathcal{A}$ *in* $\mathcal{B}(\mathcal{H})$*, denoted* $\mathscr{B}^m_{\mathcal{H}}(\mathcal{A})$*, is*

$$\mathscr{B}^m_{\mathcal{H}}(\mathcal{A}) = \mathscr{B}^m_{\mathcal{H}}(\mathcal{A}_{sa}) + i\mathscr{B}^m_{\mathcal{H}}(\mathcal{A}_{sa}).$$

The ***strong Borel envelope*** *of* $\mathcal{A}$ *in* $\mathcal{B}(\mathcal{H})$*, denoted* $\mathscr{B}^s_{\mathcal{H}}(\mathcal{A})$*, is*

$$\mathscr{B}^s_{\mathcal{H}}(\mathcal{A}) = \mathscr{B}^s_{\mathcal{H}}(\mathcal{A}_{sa}) + i\mathscr{B}^s_{\mathcal{H}}(\mathcal{A}_{sa}).$$

The ***weak Borel envelope*** *of* $\mathcal{A}$ *in* $\mathcal{B}(\mathcal{H})$*, denoted* $\mathscr{B}^w_{\mathcal{H}}(\mathcal{A})$*, is*

$$\mathscr{B}^w_{\mathcal{H}}(\mathcal{A}) = \mathscr{B}^w_{\mathcal{H}}(\mathcal{A}_{sa}) + i\mathscr{B}^w_{\mathcal{H}}(\mathcal{A}_{sa}).$$

These envelopes are all C^*-algebras.

Theorem 6.3.4 *([Ped], Theorem 4.5.4) Let* $\mathcal{A}$ *be a* C^**-algebra of* $\mathcal{B}(\mathcal{H})$*. Then* $\mathscr{B}^m_{\mathcal{H}}(\mathcal{A})$ *is the self-adjoint part of a* C^**-algebra.*

Definition 6.3.5 *([Ped], 4.5.5) Let $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be a concrete C^*-algebra. Then $\mathcal{B}$ is a **Borel $*$-algebra** if*

$$\mathcal{B}_{sa} = \mathscr{B}^m_{\mathcal{H}}(\mathcal{B}_{sa}).$$

Definition 6.3.6 *([Dav1]) Let $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be a concrete C^*-algebra. Then $\mathcal{B}$ is a **Σ^*-algebra** if*

$$\mathcal{B}_{sa} = \mathscr{B}^w_{\mathcal{H}}(\mathcal{B}_{sa}).$$

The Σ^*-algebras were first introduced and studied by Davies in [Dav1], [Dav2] and [Dav3]. By definition, if $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ is a concrete C^*-algebra and if $\mathcal{B}_{sa} = \mathscr{B}^s_{\mathcal{H}}(\mathcal{B}_{sa})$ or $\mathcal{B}_{sa} = \mathscr{B}^w_{\mathcal{H}}(\mathcal{B}_{sa})$, then $\mathcal{B}$ is a Borel $*$-algebra. In particular, a Σ^*-algebra is a Borel $*$-algebra. Let $\mathcal{W} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Since $\mathcal{W}_{sa} \subseteq \mathscr{B}^m_{\mathcal{H}}(\mathcal{W}_{sa}) \subseteq \mathcal{W}^w_{sa}$, then any von Neumann algebra is a Borel $*$-algebra. The converse is not always true (see Example 6.3.8).

Theorem 6.3.7 *([Ped], 4.5.5) If $\mathcal{H}$ is a separable Hilbert space, then any Borel $*$-algebra in $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.*

Example 6.3.8

Let $(X, \mathscr{A})$ be the uncountable standard Borel space. Let $\mathcal{H} = \ell^2(X)$ denote the Hilbert space of complex valued functions such that

$$\langle \xi | \xi \rangle = \sum_{x \in X} |\xi(x)|^2 < \infty.$$

If $\xi \in \ell^2(X)$, then its support $\operatorname{supp}(\xi) = \{x \in X : \xi(x) \neq 0\}$ is countable. Thus any element in $\ell^2(X)$ is a bounded Borel functions. For $x \in X$, let $\mathcal{X}_x$ denote the characteristic function of the singleton $\{x\}$. Then $\{\mathcal{X}_x : x \in X\}$ forms an orthonormal basis of $\ell^2(X)$, therefore $\ell^2(X)$ is not separable.

To a bounded Borel function $f \in \mathcal{B}_o(X)$, we associate the multiplication operator M_f given by

$$(M_f \xi)(x) = f(x)\xi(x),\ x \in X,\ \xi \in \ell^2(X).$$

Then $\|M_f\| = \|f\|_\infty = \sup_{x\in X} |f(x)|$.

We will frequently identify a function $f \in \mathcal{B}_o(X)$ with its associated multiplication operator $M_f \in \mathcal{B}(\ell^2(X))$.

Let $\{f_n\} \subset \mathcal{B}_o(X)$ be a bounded sequence of Borel function converging weakly to f. As $\langle f_n \mathcal{X}_x | \mathcal{X}_x \rangle = f_n(x)$, then $\langle f \mathcal{X}_x | \mathcal{X}_x \rangle = f(x)$ which means that $\{f_n\}$ converges pointwise to f, thus $f \in \mathcal{B}_o(X)$. Hence $\mathcal{B}_o(X)$ is a Borel $*$-algebra in $\mathcal{B}(\ell^2(X))$. Since $\mathcal{B}_o(X)'' = \mathrm{Bnd}(X)$, the bounded complex-valued functions of X, then $\mathcal{B}_o(X)$ is not a von Neumann algebra.

Thus $\mathcal{B}_o(X)$ is an example of Borel $*$-algebra which is also a Σ^*-algebra. This is also the case of a larger class of Borel $*$-algebra.

Theorem 6.3.9 *Let $\{X_k : k = 1, 2, ..., \infty\}$ be a sequence of standard Borel spaces and $\mathcal{H}_k$ be the Hilbert space of dimension k. Let*

$$B = \prod_{k=1}^{\infty} B_0(X_k, \mathcal{B}(\mathcal{H}_k))$$

where $B_0(X_k, \mathcal{B}(\mathcal{H}_k))$ is the $*$-algebra of bounded Borel functions of the form $f : X_k \to \mathcal{B}(\mathcal{H}_k)$. Then B is sequentially weakly closed, thus a Borel $*$-algebra.

Recall that if X is a standard Borel space, $|X|$ denotes its cardinal.

Definition 6.3.10 *Any Borel $*$-algebra of the form*

$$B = \prod_{k=1}^{\infty} B_0(X_k, \mathcal{B}(\mathcal{H}_k))$$

is called a ***type I*** *Borel $*$-algebra. It follows that the ordered sequence of cardinals $\{|X_k| : k = 1, 2, ..., \infty\}$ is an invariant of class of isomorphism of type I Borel $*$-algebras.*

The previous theorem and definition follow Theorem 6.3.4, Corollary 6.3.5 and paragraph 6.3.5 of [Ped].

For a locally compact Hausdorff space $(X, \mathcal{T})$, we identify $C_0(X)$ with $\{M_f \in \mathcal{B}(\ell^2(X)); f \in C_0(X)\}$. Then, from Theorem 2.4.5, the monotone Borel envelope of $C_0(X)$ is $\mathcal{B}_o(X)$. Moreover, since $\mathcal{B}_o(X)$ is closed under pointwise limits of bounded sequences, then the weak Borel envelope of $C_0(X)$ is also $\mathcal{B}_o(X)$. Thus in the abelian case there is no distinction between the three Borel envelopes. In this thesis, we use the weak Borel envelope for our purposes (to associate a Borel $*$-algebra to a countable Borel equivalence relation).

Definition 6.3.11 *Let $\mathcal{A}$ be a C^*-algebra in $\mathcal{B}(\mathcal{H})$. The **Borel envelope** of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, denoted $\mathscr{B}_{\mathcal{H}}(\mathcal{A})$, is chosen to be the weak Borel envelope.*

We now turn our attention on representations of Borel $*$-algebras.

Definition 6.3.12 *Let $\mathcal{K}$ be a Hilbert space and $\mathcal{B}$ be a Borel $*$-subalgebra of $\mathcal{B}(\mathcal{K})$. A **σ-representation** of $\mathcal{B}$ is a representation $(\pi, \mathcal{H})$ such that if $\{x_n\}$ is a bounded sequence of self-adjoint elements of $\mathcal{B}$ then*

$$x_n \xrightarrow{w} x \text{ if and only if } \pi(x_n) \xrightarrow{w} \pi(x).$$

By definition, if ϕ is a σ-normal state, the GNS-representation $(\pi_\phi, \mathcal{H}_\phi)$ is a σ-normal representation. Moreover if F is a separating family of σ-normal states of a Borel $*$-algebra $\mathcal{B}$, then the representation

$$\bigoplus_{\phi \in F} (\pi_\phi, \mathcal{H}_\phi)$$

is a faithful σ-representation of $\mathcal{B}$ (see paragraph 4.5.5 of [Ped]).

The Borel envelope of a C^*-algebra depends on the Hilbert space on which it is acting. For example, $\mathscr{B}_{\ell^2(X)}(C_0(X)) = \mathcal{B}_o(X)$, but $\mathscr{B}_{L^2(X,\mu)}(C_0(X)) = L^\infty(X, \mu)$. But if the Hilbert spaces are spatially isomorphic, then the Borel envelopes are isomorphic.

Gert K. Pedersen in his book [Ped] and some of his articles [Ped$k$], $k = 1, ..., 7$, for example, studies monotone Borel envelopes of C^*-algebras taken in their universal representation. Thus we make a definition, by sticking the word universal in front, when we are making reference to this Borel envelope.

Definition 6.3.13 *([Ped], paragraph 4.5.6) Let $\mathcal{A}$ be a C^*-algebra and $(\pi_u, \mathcal{H}_u)$ its universal representation. The **universal enveloping Borel $*$-algebra** of $\mathcal{A}$ is given by*

$$\mathscr{B}_u(\mathcal{A}) = \mathscr{B}^m_{\mathcal{H}_u}(\pi_u(\mathcal{A}_{sa})) + i\mathscr{B}^m_{\mathcal{H}_u}(\pi_u(\mathcal{A}_{sa})).$$

The universal enveloping Borel $*$-algebra associated to $\mathcal{A}$ could be defined using the atomic representation $(\pi_a, \mathcal{H}_a)$. This is a direct consequence of the following two lemmas.

Lemma 6.3.14 *([Ped], 4.5.9) Let $\mathcal{A}$ be a C^*-algebra. Any representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ can be uniquely extended to a σ-representation $(\pi'', \mathcal{H})$ of $\mathscr{B}_u(\mathcal{A})$ such that $\pi''(\mathscr{B}_u(\mathcal{A})_{sa}) = \mathscr{B}_{\mathcal{H}}(\pi(\mathcal{A}_{sa}))$.*

Lemma 6.3.15 *([Ped], 4.5.13) The σ-representation $(\pi''_a, \mathcal{H}_a)$ of $\mathscr{B}_u(\mathcal{A})$ is faithful.*

Thus for any C^*-algebra $\mathcal{A}$, then

$$\mathscr{B}_u(\mathcal{A}) \cong \mathscr{B}^m_{\mathcal{H}_a}(\pi_a(\mathcal{A}_{sa})) + i\mathscr{B}^m_{\mathcal{H}_a}(\pi_a(\mathcal{A}_{sa})).$$

There is also an interesting non-commutative generalization of the notion standard Borel spaces, given by Pedersen in [Ped], to Borel $*$-algebras motivated by the example following the definition.

Definition 6.3.16 *([Ped], 4.6.1) We say that a Borel $*$-algebra $\mathcal{B}$ is **standard** if there is a separable C^*-algebra $\mathcal{A}$ with universal enveloping Borel $*$-algebra $\mathscr{B}_u(\mathcal{A})$, and a central projection $z \in \mathscr{B}_u(\mathcal{A})$, such that $\mathcal{B}$ is isomorphic (as a C^*-algebra) to $z\mathscr{B}_u(\mathcal{A})$.*

Example 6.3.17 *([Ped], 4.6.9) The Borel $*$-algebra of bounded Borel functions $\mathcal{B}_o(X)$, over a Borel space $(X, \mathscr{A})$, is standard if and only if $(X, \mathscr{A})$ is standard.*

We finish this section with a definition and a technical result used later in the thesis.

Definition 6.3.18 *([Ped], paragraph 4.5.5) Let $\mathcal{B}$ be a Borel $*$-algebra of $\mathcal{B}(\mathcal{H})$. Then $\mathcal{B}$ is **countably generated** if there is a sequence $\{x_n\} \subset \mathcal{B}$ such that no proper Borel $*$-subalgebra of $\mathcal{B}$ contains that sequence.*

For a C^*-algebra $\mathcal{A}$, recall that if $x \in \mathcal{A}$, then $|x| = \sqrt{x^*x} \in \mathcal{A}$.

Theorem 6.3.19 *([Ped], Proposition 4.5.17) Let $\{x_n\}$ be a sequence in a Borel $*$-algebra $\mathcal{B}$. If $\sum |x_n|$ and $\sum |x_n^*|$ are convergent (and thus belong to $\mathcal{B}$), then $\sum x_n$ is convergent and belongs to $\mathcal{B}$.*

6.4 Cartan Subalgebras

In the constructions of operator algebras associated to dynamical systems, the so-called Cartan subalgebras play a crucial role. In this section, we introduce them in the measurable (mainly following [FM1]), topological (mainly following [Re2]) and Borel context.

Definition 6.4.1 *Let $\mathcal{A}$ be a C^*-algebra. An abelian C^*-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ is **maximal abelian** if*

$$\mathcal{A}_0' \cap \mathcal{A} = \mathcal{A}_0.$$

Definition 6.4.2 *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{A}_0$ be a C^*-subalgebra.*

*a) The map $\Delta : \mathcal{A} \longrightarrow \mathcal{A}_0$ is a **conditional expectation** if it satisfies the following properties,*

i) Δ is a linear and surjective,

ii) $\Delta^2(f) = \Delta(\Delta(f)) = \Delta(f)$, *iii)* $\Delta(f) \geqslant 0$ *for all* $f \geqslant 0$*, and*

iv) $\Delta(afb) = a\Delta(f)b$

for all a *and* b *in* $\mathcal{A}_0$ *and* f *in* $\mathcal{A}$.

b) A conditional expectation Δ *is* ***faithful*** *if* $\Delta(ff^*) = 0$*, then* $f = 0$ *for all* $f \in \mathcal{A}$.

c) A conditional expectation Δ *is* ***normal*** *if for all bounded nets* $\{f_k\}_{k\in K}$ *in* $\mathcal{A}$ *converging weakly to* f, $\{\Delta(f_k)\}_{k\in K}$ *is a bounded net converging weakly to* $\Delta(f)$.

d) A conditional expectation Δ *is* ***sequentially normal*** *if for all bounded sequences* $\{f_n\}_{n=1}^{\infty}$ *in* $\mathcal{A}$ *converging weakly to* f, $\{\Delta(f_n)\}_{n=1}^{\infty}$ *is a bounded sequence converging weakly to* $\Delta(f)$.

Definition 6.4.3

a) ([Ku], p.970) Let $\mathcal{A}$ *be a* C^**-algebra and* $\mathcal{A}_0$ *be a* C^**-subalgebra. The set of all operators* a *in* $\mathcal{A}$ *such* $a\mathcal{A}_0a^* \subset \mathcal{A}_0$ *and* $a^*\mathcal{A}_0a \subset \mathcal{A}_0$ *is called the* ***normalizer*** *of* $\mathcal{A}_0$*, denoted* $\tilde{N}_{\mathcal{A}}(\mathcal{A}_0)$.

b) Let $\mathcal{B}$ *be a Borel* $*$*-algebra and* $\mathcal{B}_0$ *be a Borel* $*$*-subalgebra. The* ***(Borel) normalizer*** *of* $\mathcal{B}_0$ *is*

$$N_{\mathcal{B}}(\mathcal{B}_0) = \{u \in \mathcal{U}(\mathcal{B}); u\mathcal{B}_0u^* = \mathcal{B}_0\}.$$

Remark 6.4.4 *Keeping the notations of the previous definition, we have:*

1) $\mathcal{A}_0 \subset \tilde{N}_{\mathcal{A}}(\mathcal{A}_0)$ *and* $\mathcal{B}_0 \subset N_{\mathcal{B}}(\mathcal{B}_0)$.

2) If $\mathcal{A}$ *is unital, then* $\tilde{N}_{\mathcal{A}}(\mathcal{A}_0) \cap \mathcal{U}(\mathcal{A}) = N_{\mathcal{A}}(\mathcal{A}_0)$.

We now define regular subalgebras for C^*-algebras, von Neumann algebras and Borel $*$-algebras.

Definition 6.4.5 *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{A}_0$ be a C^*-subalgebra. If $\tilde{N}_{\mathcal{A}}(\mathcal{A}_0)$ is norm dense in $\mathcal{A}$, then $\mathcal{A}_0$ is called **regular**.*

Definition 6.4.6 *Let $\mathcal{W}_0$ be a von Neumann subalgebra of a von Neumann algebra $\mathcal{W}$. If $N_{\mathcal{W}}(\mathcal{W}_0)$ is weakly dense in $\mathcal{W}$, then $\mathcal{W}_0$ is called **(von Neumann) regular**.*

Definition 6.4.7 *Let $\mathcal{B}_0$ be a Borel $*$-subalgebra of $\mathcal{B}$ a Borel $*$-algebra. If the Borel envelope of $N_{\mathcal{B}}(\mathcal{B}_0)$ is $\mathcal{B}$, then $\mathcal{B}_0$ is called **(Borel) regular**.*

We now define Cartan subalgebras for C^*-algebras, von Neumann algebras and Borel $*$-algebras.

Definition 6.4.8 *([Re2]) Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{A}_0$ be a C^*-subalgebra. If $\mathcal{A}_0$ is a regular, maximal abelian C^*-subalgebra of $\mathcal{A}$ and if there exists a faithful conditional expectation $\Delta : \mathcal{A} \longrightarrow \mathcal{A}_0$, then $\mathcal{A}_0$ is called a **Cartan** subalgebra.*

Definition 6.4.9 *([FM2]) Let $\mathcal{W}$ be a von Neumann algebra. If $\mathcal{W}_0$ is a (von Neumann) regular, maximal abelian and if there exists a faithful normal conditional expectation Δ from $\mathcal{W}$ to $\mathcal{W}_0$, then $\mathcal{W}_0$ is called a **(von Neumann) Cartan** subalgebra.*

Definition 6.4.10 *Let $\mathcal{B}$ be a Borel $*$-algebra and $\mathcal{B}_0$ be an Borel $*$-subalgebra. If $\mathcal{B}_0$ is (Borel) regular, maximal abelian and if there exists a faithful sequentially normal conditional expectation Δ from $\mathcal{B}$ to $\mathcal{B}_0$, then $\mathcal{B}_0$ is called a **(Borel) Cartan** subalgebra.*

Clearly, the von Neumann and Borel definitions of regular (or Cartan) are the same for subalgebras of $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ separable.

Definition 6.4.11 *([Ku], p.971) Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{A}_0$ be a Cartan subalgebra. If $\mathcal{A}_0$ has the extension property relative to $\mathcal{A}$ (*6.1.14*), then we say that $\mathcal{A}_0$ is a **diagonal** subalgebra.*

The next two definitions are the analogue in the Borel setting.

Definition 6.4.12 *Let $\mathcal{B}_0$ be a Borel $*$-subalgebra of a Borel $*$-algebra $\mathcal{B}$. Then $\mathcal{B}_0$ has the (Borel)* ***extension property relative to*** *$\mathcal{B}$ if every sequentially normal pure state of $\mathcal{B}_0$ can be extended uniquely to a sequentially normal pure state of $\mathcal{B}$.*

Definition 6.4.13 *Let $\mathcal{B}$ be a Borel $*$-algebra and $\mathcal{B}_0$ be a Cartan subalgebra. If $\mathcal{B}_0$ has the extension property relative to $\mathcal{B}$, then $\mathcal{B}_0$ is a (Borel)* ***diagonal*** *subalgebra.*

It is known in the topological setting that some Cartan subalgebras are not diagonal (see [Ku] or [Re2]). The next proposition shows that in the Borel case the notions of Borel diagonal subalgebra and Borel Cartan subalgebra coincide.

Proposition 6.4.14 *Let $\mathcal{B}$ be an unital Borel $*$-algebra, $\mathcal{B}_0$ be an abelian Borel $*$-subalgebra of $\mathcal{B}$. If $\mathcal{B}_0 \cong \mathcal{B}_o(X)$ is (Borel) Cartan, then it is diagonal.*

Proof: Without loss of generality we will identify $\mathcal{B}_0$ with $\mathcal{B}_o(X)$. Since $\mathcal{B}_o(X)$ is Cartan in $\mathcal{B}$, then we denote by Δ the sequentially normal conditional expectation from $\mathcal{B}$ to $\mathcal{B}_o(X)$. For $x \in X$, recall that the Dirac measures ϕ_x are the sequentially normal pure states of $\mathcal{B}_o(X)$ (see for example [Dav1]). We want to show that for any $x \in X$ the corresponding sequentially normal pure states ϕ_x can be extended uniquely to $\mathcal{B}$. Since $\Phi_x : \mathcal{B} \to \mathbb{C}$ given by $\Phi_x(f) = \phi_x(\Delta(f))$, for $f \in \mathcal{B}$, is a sequentially normal pure state which extends ϕ_x, it will be the unique extension of ϕ_x. For $x \in X$ fixed, recall from Theorem 6.1.13 that showing that ϕ_x as a unique extension is equivalent to show that for any $f \in \mathcal{B}$ and $\varepsilon > 0$, there are $b \in \mathcal{B}_o(X)^+$, with $\|b\| = \phi_x(b) = 1$, and $y \in \mathcal{B}_o(X)$ such that

$$\|bfb - y\| < \varepsilon.$$

Set $b = \mathcal{X}_x \in \mathcal{B}_o(X)$ the characteristic function of the point $x \in X$ (clearly $\|b\| = \phi_x(b) = 1$). For any $a \in \mathcal{B}_o(X)$, set $\lambda_x = a(x)I$, where I is the identity, then $ab = ba = \lambda_x b = b\lambda_x$.

Claim: The element bfb is in $\mathcal{B}_o(X)$.

Since $\mathcal{B}_o(X)$ is maximal abelian, we have

$$\begin{aligned}(bfb)a &= bf(ba)\\ &= bf(b\lambda_x)\\ &= (bfb)\lambda_x\\ &= \lambda_x(bfb)\\ &= (\lambda_x b)fb\\ &= a(bfb),\end{aligned}$$

hence $bfb \in \mathcal{B}_o(X)$. Since $b \in \mathcal{B}_o(X)$, then $\Delta(bfb) = b\Delta(f)b$ and $\Delta(bfb) = bfb$. Thus $bfb = b\Delta(f)b$ and

$$\|bfb - b\Delta(f)b\| = 0 < \varepsilon.$$

Hence by Theorem 6.1.13 (choosing $y = b\Delta(f)b$), the (sequentially normal) pure state ϕ_x can be extended uniquely to a (sequentially normal pure) state of $\mathcal{B}$. Since the choice of x (thus the choice of the sequentially normal pure state of $\mathcal{B}_0$) is arbitrary, the Borel $*$-subalgebra $\mathcal{B}_0$ has the extension property. Thus $\mathcal{B}_0$ is diagonal in $\mathcal{B}$. ∎

Definition 6.4.15 *Let $\mathcal{B}$ be a Borel $*$-algebra and $\mathcal{B}_0$ be a Borel $*$-subalgebra. The **Weyl group**, denoted $W_{\mathcal{B}}(\mathcal{B}_0)$, is defined by*

$$W_{\mathcal{B}}(\mathcal{B}_0) = N_{\mathcal{B}}(\mathcal{B}_0)/U(\mathcal{B}_0).$$

Definition 6.4.16 *Let $\mathcal{B}_0$ be a Borel $*$-subalgebra of a Borel $*$-algebra $\mathcal{B}$. The Borel $*$-algebra $\mathcal{B}$ is **countably $\mathcal{B}_0$-generated** if $\mathcal{B}$ is generated by $\mathcal{B}_0$ and $N_{\mathcal{B}}(\mathcal{B}_0)$ and if $W_{\mathcal{B}}(\mathcal{B}_0)$ is countably generated.*

In particular, when $(X, \mathscr{A})$ is a standard Borel space, $\mathcal{B}_o(X)$ is countably generated, hence if $\mathcal{B}_0 \cong \mathcal{B}_o(X)$ and the Borel $*$-algebra is $\mathcal{B}$ is countably $\mathcal{B}_0$-generated, then

$\mathcal{B}$ is countably generated. We will see later in Chapter 7, that the Borel $*$-algebra $\mathcal{B}$ associated to a Borel twist is always $\mathcal{B}_0$-countably generated by a distinguished diagonal Borel $*$-subalgebra $\mathcal{B}_0$.

6.5 The von Neumann Algebra $W^*(E, \sigma, \mu)$

Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$, σ a Borel 2-cocycle of E and μ be a probability measure on X. In this section we present the construction of the von Neumann algebra $W^*(E, \sigma, \mu)$, given by Feldman and Moore [FM2]. It generalizes Krieger's result in [Kr1].

We denote by $\mathcal{B}(E)$, the set of **Borel functions** $f : E \to \mathbb{C}$, and let

$$\|f\|_\infty = \sup_{(x,y)\in E} |f(x, y)|$$

be its sup-norm. Then $\mathcal{B}_o(E)$ will denote the set of **bounded Borel functions**.

Definition 6.5.1 *[FM2] Let $f \in \mathcal{B}(E)$. The **band** of f, denoted band(f), is the smallest number $n \in \mathbb{N} \cup \{\infty\}$ such that*

$$\forall_{x\in X} \left(|\{y \in X; f(x, y) \neq 0\}| + |\{y \in X; f(y, x) \neq 0\}| \leq n\right).$$

*Then $\mathcal{B}_c(E)$ will denote the set of **band-limited bounded Borel functions**, i.e.,*

$$\mathcal{B}_c(E) = \{f \in \mathcal{B}_o(E); band(f) < \infty\}.$$

The set of band-limited bounded Borel functions is a complex vector space. Now if σ is a normalized 2-cocycle on E, then $\mathcal{B}_c(E)$ becomes a $*$-algebra with the following operations: for f and $g \in \mathcal{B}_c(E)$ and $(x, y) \in E$:

1. multiplication: $(f \cdot g)(x, y) = \sum_{z\in[x]} f(x, z)g(z, y)\sigma(x, z, y)$.
2. involution: $f^*(x, y) = \overline{f(y, x)}$.

Definition 6.5.2 *The $*$-algebra associated to E and to the normalized 2-cocycle σ is denoted $\mathcal{B}_c(E,\sigma)$.*

Notation 6.5.3 *There is a (Borel) bijection between $\mathcal{B}_o(X)$ and the elements of $\mathcal{B}_c(E,\sigma)$ whose support is on the diagonal of E. Thus we will also use the notation $\mathcal{B}_o(X)$ for this $*$-subalgebra.*

Recall that to a measure μ on X, we associate two measures ν and ν^{-1} on E (Definition 4.2.1). Then let

$$\mathcal{H}_\mu = L^2(E,\nu^{-1})$$

denote the Hilbert space of square ν^{-1}-measurable functions on E. Hence a Borel function $\xi \in \mathcal{B}(E)$ belongs to $\mathcal{H}_\mu$ if

$$\int_X \sum_{y\sim x} |\xi(y,x)|^2 \, d\mu(x) < \infty.$$

The let $(L_\mu, \mathcal{H}_\mu)$ denote the $*$-representation of $\mathcal{B}_c(E,\sigma)$, defined by

$$\left(L_\mu(f)\xi\right)(x,y) = \sum_{z\sim x} f(x,z)\xi(z,y)\sigma(x,z,y)$$

for all $f \in \mathcal{B}_c(E,\sigma)$ and $\xi \in \mathcal{H}_\mu$.

Definition 6.5.4 *The von Neumann algebra associated to (E,σ) and μ is*

$$W^*(E,\sigma,\mu) = L_\mu(\mathcal{B}_c(E,\sigma))'' \subset \mathcal{B}(\mathcal{H}_\mu).$$

The abelian von Neumann subalgebra $L_\mu(\mathcal{B}_o(X))$ is denoted $L^\infty(X,\mu)$.

Theorem 6.5.5 *([FM2], Proposition 2.9) Let E be a countable Borel equivalence relation on $(X,\mathscr{A})$, σ a normalized 2-cocycle and μ be a quasi-invariant ergodic probability measure on X. Then the subalgebra $L^\infty(X,\mu)$ of $W^*(E,\sigma,\mu)$ has the following properties:*

1) $L^\infty(X,\mu)$ is maximal abelian.

2) $L^\infty(X,\mu)$ is regular.

3) There exists a faithful normal conditional expectation from $W^*(E,\sigma,\mu)$ onto $L^\infty(X,\mu)$.

This theorem admits a converse (Theorem 6.5.7).

Definition 6.5.6 *Let $\mathcal{A}_0$ (resp. $\mathcal{B}_0$) be a C^*-subalgebra of $\mathcal{A}$ (resp. $\mathcal{B}$). The pairs $(\mathcal{A},\mathcal{A}_0)$ and $(\mathcal{B},\mathcal{B}_0)$ are **isomorphic as pairs** if there exists an isomorphism $\Phi : \mathcal{A}\to\mathcal{B}$ of C^*-algebras such that $\Phi(\mathcal{A}_0)=\mathcal{B}_0$, denoted by $(\mathcal{A},\mathcal{A}_0)\cong(\mathcal{B},\mathcal{B}_0)$.*

Theorem 6.5.7 *([FM2], Theorem 1.) Let $\mathcal{W}_0$ be a Cartan subalgebra of a von Neumann algebra $\mathcal{W}$ acting on a separable Hilbert space. Then there exist a countable Borel equivalence relation E on a measurable space (X,μ), a σ a normalized $\mathbb{T}$-valued 2-cocycle and an isomorphism of pairs such that*

$$(\mathcal{W},\mathcal{W}_0)\cong(W^*(E,\sigma,\mu),L^\infty(X,\mu)).$$

The countable measured equivalence relation E on (X,μ) is unique up to isomorphism and the 2-cocycle σ on E is unique up to coboundary.

6.6 C^*-Algebras Associated to Twisted Groupoids.

In this section, we present the C^*-algebra version of the results discussed in Section 6.5. They were obtained in several papers by Kumjian and Renault, more precisely [Ku], [Re] and [Re2].

We begin this section by presenting how a C^*-algebra is associated to a topological twisted groupoid $(\Gamma,\mathscr{G})$ (Definition 3.2.8). This construction can be found in many articles like [Ku] or [Re2].

A function $f:\Gamma\to\mathbb{C}$ is equivariant if $\lambda f(\gamma)=f(\lambda\gamma)$ for all $\lambda\in\mathbb{T}$ and for all $\gamma\in\Gamma$. Let $x\in\mathscr{G}^{(0)}=X$ and let $q:\Gamma\to\mathscr{G}$ be the quotient map.

Let $C_c(\Gamma, \mathscr{G})$ denote the vector space of equivariant functions with compact support. For $g \in \mathscr{G}$ we denote by $\hat{g}$ an element in $q^{-1}(g)$. Equipped with the following operations, for $f, h \in C_c(\Gamma, \mathscr{G})$,

1) multiplication

$$(f \cdot h)(\gamma) = \sum_{g \in \mathscr{G}^{d(\gamma)}} f(\gamma \hat{g}) h(\hat{g}^{-1})$$

2) involution

$$f^*(\gamma) = \overline{f(\gamma^{-1})},$$

$C_c(\Gamma, \mathscr{G})$ becomes a $*$-algebra. Notice that the multiplication is well-defined since the functions f and g are equivariant.

The set of equivariant functions $\xi : \Gamma \to \mathbb{C}$ such that

$$\sum_{g \in \mathscr{G}_x} |\xi(\hat{g})|^2 < \infty,$$

is a Hilbert space, denoted $L^2(\Gamma, \mathscr{G}, \delta_x)$. Let $\xi \in L^2(\Gamma, \mathscr{G}, \delta_x)$.

For any $x \in X$, let π_x be the $*$-representation of $C_c(\Gamma, \mathscr{G})$ into $\mathcal{B}(L^2(\Gamma, \mathscr{G}, \delta_x))$, defined by

$$(\pi_x(f)\xi)(\gamma) = \sum_{\hat{g} \in \mathscr{G}^x} f(\gamma \hat{g}) \xi(\hat{g}^{-1}).$$

The reduced C*-algebra associated to $(\Gamma, \mathscr{G})$, denoted $C_r^*(\Gamma, \mathscr{G})$, is the completion of $C_c(\Gamma, \mathscr{G})$ with the norm

$$\|f\| = \sup_{x \in X} \|\pi_x(f)\|.$$

The first topological result which parallels the work of Feldman and Moore was obtained by Renault. Here the topological twist comes from a topological countable equivalence relation with a continuous 2-cocycle.

Theorem 6.6.1 *([Re], Proposition 4.13) Let $(\Gamma, \mathscr{G})$ be a twisted groupoid. Suppose Γ is a twist over an equivalence relation $\mathscr{G}$ given by a continuous 2-cocycle. Let $\mathcal{A} = C_r^*(\Gamma, \mathscr{G})$ and $\mathcal{A}_0 = C_0(\mathscr{G}^{(0)})$. Then the pair $(\mathcal{A}, \mathcal{A}_0)$ has the following properties:*

1) $\mathcal{A}_0$ *is maximal abelian,*

2) $\mathcal{A}_0$ *is regular,*

3) there exists a faithful conditional expectation from $\mathcal{A}$ *onto* $\mathcal{A}_0$*,*

4) each element of the ample semigroup (in this case, the set of partial of homeomorphism of the unit space of $\mathscr{G}$*, see [Re] p.105) acts relatively freely on the unit space (see [Re], 1.2.14) and*

5) let $\mathcal{N}_{\mathcal{A}}(\mathcal{A}_0) = \{a \in U_p(\mathcal{A}) : r(a), d(a) \in \mathcal{A}_0$ *and* $a(\mathcal{A}_0 d(a))a^* = \mathcal{A}_0 r(a)\}$ *where* $U_p(\mathcal{A})$ *are the partial isometry of* $\mathcal{A}$ *and* $\mathcal{P}(\mathcal{A}_0)$ *the projections in* $\mathcal{A}_0$*, then the exact sequence of semigroups (see Remark p.105 in [Re])*

$$\mathcal{P}(\mathcal{A}_0) \to U_p(\mathcal{A}_0) \to \mathcal{N}_{\mathcal{A}}(\mathcal{A}_0) \xrightarrow{s} [[\mathscr{G}]] \to \mathcal{P}(\mathcal{A}_0)$$

splits, i.e., there exists a continuous section k *for* s *such that* $k(se) = k(s)e$*,* $k(es) = ek(s)$ *and* $k(e) = e$*, for every* $e \in \mathcal{P}(\mathcal{A}_0)$ *and* $s \in [[\mathscr{G}]]$*.*

The result of Renault gives a pair of C*-algebras with very specific algebraic properties. This theorem has a converse.

Theorem 6.6.2 *([Re], Theorem 4.15) Let* $(\mathcal{A}, \mathcal{A}_0)$ *be a pair of C*-algebra with the algebraic properties of the previous theorem. Then there exists a twisted groupoid* $(\Gamma, \mathscr{G})$ *where* Γ *is a twist over an equivalence relation* $\mathscr{G}$ *given by a continuous 2-cocycle, such that*

$$(\mathcal{A}, \mathcal{A}_0) \cong (C_r^*(\Gamma, \mathscr{G}), C_0(\mathscr{G}^{(0)})).$$

The twisted groupoid $(\Gamma, \mathscr{G})$ *is unique up to isomorphism.*

In their respective settings, Renault's theorems and Feldman and Moore's theorems have the same starting point: an equivalence relation with a 2-cocycle. But the parallel is not perfect since Renault's pair of C*-algebras has more algebraic properties then Feldman and Moore's pair of von Neumann algebras. The next development in the topological setting to get closer to the algebraic setup of Feldman and Moore was obtained by A. Kumjian in [Ku].

The starting point of Kumjian's construction is a twisted groupoid $(\Gamma, \mathscr{G}))$, with $\mathscr{G}$ an equivalence relation (principal groupoid). Here the twist Γ over $\mathscr{G}$ is not necessarily arising from a continuous 2-cocycle of $\mathscr{G}$, making the class of the objects $(\Gamma, \mathscr{G})$ more general than in Renault's result.

Theorem 6.6.3 *([Ku], Section 2,* 9° *Theorem) Let* $(\Gamma, \mathscr{G})$ *be a twisted groupoid. Suppose* $\mathscr{G}$ *is an equivalence relation. Let* $\mathcal{A} = C^*_r(\Gamma, \mathscr{G})$ *and* $\mathcal{A}_0 = C_0(\mathscr{G}^{(0)})$*. Then the pair* $(\mathcal{A}, \mathcal{A}_0)$ *satisfies:*

1) $\mathcal{A}_0$ *is maximal abelian,*

2) $\mathcal{A}_0$ *is regular,*

3) there exists a faithful conditional expectation from $\mathcal{A}$ *onto* $\mathcal{A}_0$ *and*

4) Every pure state of $\mathcal{A}_0$ *can be extended uniquely to a pure state of* $\mathcal{A}$*.*

This theorem also has a converse.

Theorem 6.6.4 *([Ku], Section 1,* 1° *Theorem) Let* $(\mathcal{A}, \mathcal{A}_0)$ *be a pair of* C^**-algebras with the algebraic properties* 1) *to* 4) *of the previous theorem. Then there exists a twisted groupoid* $(\Gamma, \mathscr{G})$ *where* Γ *is a twist over an equivalence relation* $\mathscr{G}$*, such that*

$$(\mathcal{A}, \mathcal{A}_0) \cong (C^*_r(\Gamma, \mathscr{G}), C_0(\mathscr{G}^{(0)})).$$

The twisted groupoid $(\Gamma, \mathscr{G})$ *is unique up to isomorphism.*

The following results were obtained (again) by Renault in [Re2]. It characterizes pairs of C*-algebras arising from from twisted groupoids $(\Gamma, \mathscr{G})$ with $\mathscr{G}$ a topologically principal groupoid.

Theorem 6.6.5 *([Re2], Theorem 4.2 + Proposition 4.3 + Corollary 4.9) Let* $(\Gamma, \mathscr{G})$ *be a twisted groupoids. Let* $(\mathcal{A}, \mathcal{A}_0) = (C_r^*(\Gamma, \mathscr{G}), C_0(\mathscr{G}^{(0)}))$*, where* $\mathscr{G}$ *is topologically principal. Then the pair* $(\mathcal{A}, \mathcal{A}_0)$ *satisfies:*

1) $\mathcal{A}_0$ *is maximal abelian,*

2) $\mathcal{A}_0$ *is regular and*

3) there exists a faithful conditional expectation from $\mathcal{A}$ *onto* $\mathcal{A}_0$*.*

This result is the C*-algebraic equivalent of Feldman and Moore's result. Again this result has a converse.

Theorem 6.6.6 *([Re2], Theorem 5.9) Let* $(\mathcal{A}, \mathcal{A}_0)$ *be a pair of C*-algebras with the algebraic properties* 1) *to* 3) *of the previous theorem. Then there exists a twisted groupoid* $(\Gamma, \mathscr{G})$ *where* Γ *is a twist over a topologically principal groupoid* $\mathscr{G}$*, such that*

$$(\mathcal{A}, \mathcal{A}_0) \cong (C_r^*(\Gamma, \mathscr{G}), C_0(\mathscr{G}^{(0)})).$$

The twisted groupoid $(\Gamma, \mathscr{G})$ *is unique up to isomorphism.*

Thus in the topological setting we have three "levels" of results: the theorems from [Ku] generalized the theorems from [Re], which in turn were generalized by the theorems from [Re2]. As in the measurable setting, we will see in Chapter 7 that all three "levels" coincide in the Borel setting.

6.7 AF-Algebras

In this section, we introduce the class of approximatively finite C*-algebras (AF-algebras). Then we make the connection between AF-algebra, Bratteli diagram and the reduced C*-algebra associated with the tail equivalence relation on a Bratteli diagram. (See for example [Da], Chapter 3, for general properties of AF-algebras.)

Definition 6.7.1 *Let $\mathcal{A}$ be a C*-algebra. The following conditions are equivalent;*

i) For any $\varepsilon > 0$ and for any finite set of elements $\{a_1, ..., a_n\} \subset \mathcal{A}$, there exists a finite dimensional C-algebra $\mathcal{A}(\{a_1, ..., a_n\}, \varepsilon)$ containing a subset $\{b_1, ..., b_n\}$ such that $\|a_k - b_k\| < \varepsilon$ for all $k = 1, ..., n$.*

ii) There exists an increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq ...$ of finite dimensional algebras such that $\bigcup_{k \geq 1} \mathcal{A}_k$ is norm dense in $\mathcal{A}$.

If $\mathcal{A}$ satisfies one of the above (equivalent) conditions, then $\mathcal{A}$ is called an ***AF-algebra****.*

In [Bra], O. Bratteli associates to any Bratteli diagram $\mathcal{D}$ an AF-algebra. He also defines an equivalence between Bratteli diagrams, and proves

Proposition 6.7.2 *([Bra], Theorem 2.7) Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Bratteli diagrams.*

$$\mathcal{A}_{\mathcal{D}_1} \cong \mathcal{A}_{\mathcal{D}_2} \text{ if and only if } \mathcal{D}_1 \text{ and } \mathcal{D}_2 \text{ are equivalent.}$$

With this proposition, we can now associate to any AF-algebra $\mathcal{A}$ a unique, up to equivalence, Bratteli diagram, denoted $\mathcal{D}_{\mathcal{A}}$. (See Section 3 of [GPS], for example, for more details on AF-algebras and equivalency between Bratteli diagrams.) Now tail equivalence on $\mathcal{D}_{\mathcal{A}}$, $E_{\mathcal{D}_{\mathcal{A}}}$, is a topological equivalence relation. The reduced C*-algebra associated to $E_{\mathcal{D}_{\mathcal{A}}}$, $C^*_r(E_{\mathcal{D}_{\mathcal{A}}})$, allows us to reconstruct the original AF-algebras.

Theorem 6.7.3 *Let $\mathcal{A}$ be an AF-algebra and let $\mathcal{D}_A$ be its Bratteli diagram. Then*

$$\mathcal{A} \cong C^*_r(E_{\mathcal{D}_{\mathcal{A}}}).$$

The details of this result can be found in [ExRe], for example.

In this thesis, we focus on countable Borel equivalence relations. Thus tail equivalence on Bratteli diagrams, $E_{\mathcal{D}_{\mathcal{A}}}$, provides many examples of countable hyperfinite Borel equivalence relations to study. The next set of conditions, presented in [Ped], are results used to define in general the notion of a type I C*-algebra.

Definition 6.7.4 *Let $\mathcal{A}$ be a separable C^*-algebra. The following conditions are equivalent;*

i) A^{**} is (a von Neumann) algebra of type I.

ii) $\mathscr{B}_u(\mathcal{A})$ is (a Borel $*$-algebra) of type I.

iii) $\mathcal{A}$ has no factor representation of type II.

iv) $\mathcal{A}$ has no factor representation of type III.

Example 6.7.5 *Any AF-algebra*

$$\mathcal{A} \cong \bigoplus_{k=1}^{\infty} C_0(X_k, \mathcal{K}(\mathcal{H}_k)),$$

where $\{X_k\}$ is a sequence of zero-dimensional Hausdorff spaces and $C_0(X_k, \mathcal{K}(\mathcal{H}_k))$ are the continuous functions $f : X_k \to \mathcal{K}(\mathcal{H}_k)$ vanishing at infinity with $\mathcal{K}(\mathcal{H}_k)$ the compact operators in the Hilbert space $\mathcal{H}_k$ of dimension k, for $k = 1, 2, ..., \infty$, is an example of a smooth AF-algebra.

The Corollary 4.7.2 gives examples of Bratteli diagrams associated to non-smooth aperiodic and hyperfinite Borel equivalence relations.

Chapter 7

A Non-Commutative Algebra of Borel Functions

In this chapter we associate to a Borel twist (Γ, E) a (sequentially weakly closed) Borel $*$-algebra $\mathcal{B}_r^*(\Gamma)$. In this construction, that the algebra of bounded Borel functions $\mathcal{B}_o(X)$ is a natural abelian subalgebra of $\mathcal{B}_r^*(\Gamma)$. We then study the algebraic structure of the pair $(\mathcal{B}_r^*(\Gamma), \mathcal{B}_o(X))$ and we show that

1. $\mathcal{B}_o(X)$ is a (Borel) Cartan subalgebra of $\mathcal{B}_r^*(\Gamma)$,

2. $\mathcal{B}_r^*(\Gamma)$ is $\mathcal{B}_o(X)$-countably generated.

Moreover, any pair (A, B) of sequentially weakly closed Borel $*$-algebras satisfying conditions 1 and 2 can be realized uniquely as $(\mathcal{B}_r^*(\Gamma), \mathcal{B}_o(X))$ for a Borel twist $(\Gamma, E))$ up to isomorphism.

More precisely, the Borel twist Γ over E is part of the following diagram

$$\mathbb{T} \times X \longrightarrow \Gamma \underset{s}{\overset{q}{\rightleftarrows}} E$$

where $(X, \mathscr{A})$ is a standard Borel space, E a countable Borel equivalence relation on X such that $E^{(0)} = X$, $q : \Gamma \longrightarrow E$ is the quotient map and $s : E \longrightarrow \Gamma$ is a Borel cross-section. Recall from Theorem 5.3.2 that any twist Γ over E is induced by a Borel 2-cocycle $\sigma \in H^2(E, \mathbb{T})$.

7.1 Band Limited Bounded Borel Functions

Let E be a countable Borel equivalence relation over a standard Borel space $(X, \mathscr{A})$ and σ a $\mathbb{T}$-valued Borel 2-cocycle of E. Following closely section 2 of [FM2], we associate in this section a $*$-algebra $\mathcal{B}_c(E, \sigma)$ of Borel functions on (E, σ).

Notation 7.1.1 *We will denote by $\mathcal{B}(E)$ the complex vector space of* ***Borel functions*** *on E and by $\mathcal{B}_o(E)$ the subspace of* ***bounded Borel functions*** *on E. If $f \in \mathcal{B}(E)$, its sup-norm is given by:*

$$\|f\|_\infty = \sup_{(x,y)\in E}(|f(x,y)|).$$

Definition 7.1.2 *Let $f \in \mathcal{B}(E)$ and*

$$N_f = \{n \in \mathbb{N}; \forall_{x\in X}(|\{y \in X; f(x,y) \neq 0\}| + |\{y \in X; f(y,x) \neq 0\}| \leq n)\}.$$

Then the ***band*** *of f, denoted band(f), is defined by*

$$band(f) = \begin{cases} \infty & \text{if } N_f = \emptyset \\ \inf(N_f) & \text{if } N_f \neq \emptyset \end{cases}.$$

Definition 7.1.3 *([FM2], Definition 2.1) We denote by $\mathcal{B}_c(E)$ the linear subspace of $\mathcal{B}_o(E)$ of* ***band limited bounded Borel functions****:*

$$\mathcal{B}_c(E) = \{f \in \mathcal{B}_o(E); band(f) < \infty\}.$$

If $f, g \in \mathcal{B}_c(E)$ and $\sigma \in Z^2(E, \mathbb{T})$ is a $\mathbb{T}$-valued 2-cocycle of E, then the function $f \cdot g : E \to \mathbb{C}$ defined for $(x, y) \in E$, by

$$(f \cdot g)(x,y) = \sum_{z \in [x]} f(x,z)g(z,y)\sigma(x,z,y)$$

is in $\mathcal{B}_c(E)$.

With this product, and the involution given for $(x,y) \in E$, by

$$f^*(x,y) = \overline{f(y,x)\sigma(x,y,x)}$$

$\mathcal{B}_c(E)$ becomes a $*$-algebra that we denote by $\mathcal{B}_c(E,\sigma)$.

Proposition 7.1.4 *Let E be a countable Borel equivalence relation and σ_1 and σ_2 be two $\mathbb{T}$-valued 2-cocycles. If σ_1 and σ_2 are cohomologous, then $\mathcal{B}_c(E,\sigma_1)$ and $\mathcal{B}_c(E,\sigma_2)$ are isomorphic $*$-algebras.*

Proof: By assumption, there exists $c \in C^1(E,\mathbb{T})$ such that

$$\sigma_2(x,y,z) = \sigma_1(x,y,z)c(x,y)c^{-1}(x,z)c(y,z).$$

Let $\Phi : \mathcal{B}_c(E,\sigma_2) \longrightarrow \mathcal{B}_c(E,\sigma_1)$ be given for $f \in \mathcal{B}_c(E,\sigma_2)$ by

$$\Phi(f)(x,y) = f(x,y)c(x,y), \quad (x,y) \in E.$$

Let f and g be in $\mathcal{B}_c(E,\sigma_2)$. The map Φ is obviously linear. Now

$$\begin{aligned}
(\Phi(f)\cdot\Phi(g))(x,z) &= \sum_{y\in[x]_E} \Phi(f)(x,y)\Phi(g)(y,z)\sigma_1(x,y,z) \\
&= \sum_{y\in[x]_E} f(x,y)c(x,y)g(y,z)c(y,z)\sigma_1(x,y,z) \\
&= \sum_{y\in[x]_E} f(x,y)g(y,z)c(x,z)c(x,y)c^{-1}(x,z)c(y,z)\sigma_1(x,y,z) \\
&= c(x,z)\sum_{y\in[x]_E} f(x,y)g(y,z)\sigma_2(x,y,z) \\
&= (f\cdot g)(x,z)c(x,z) \\
&= \Phi(f\cdot g)(x,z)
\end{aligned}$$

and

$$\begin{aligned}(\Phi(f))^*(x,y) &= \overline{\Phi(f)(y,x)\sigma_1(x,y,x)} \\ &= \overline{f(y,x)c(y,x)\sigma_1(x,y,x)} \\ &= \overline{f(y,x)c(x,y)c^{-1}(x,x)c(y,x)\sigma_1(x,y,x)}c(x,y) \\ &= \overline{f(y,x)\sigma_2(x,y,x)}c(x,y) \\ &= \Phi(f^*)(x,y).\end{aligned}$$

∎

Consequently, by 5.2.4 we can and will without loss of generality assume that in the definition of $\mathcal{B}_c(E,\sigma)$ the 2-cocycle σ is normalized and therefore the involution is given for $f \in \mathcal{B}_c(E,\sigma)$ and $(x,y) \in E$ by

$$f^*(x,y) = \overline{f(y,x)}.$$

Recall that (E_1,σ_1) and (E_2,σ_2) are (Borel) isomorphic if there is a Borel bijection $\phi : X_1 \longrightarrow X_2$ such $(x,y) \in E_1$ if and only if $(\phi(x),\phi(y)) \in E_2$. For every $n \geqslant 0$, ϕ induces an isomorphism $\hat{\phi}$ from the n-cochains $C^n(E_2,\mathbb{T})$ to $C^n(E_1,\mathbb{T})$.

Proposition 7.1.5 *Let (E_1,σ_1) and (E_2,σ_2) be two countable Borel equivalence relations on $(X,\mathscr{A})$ with σ_1 and σ_2 two normalized Borel 2-cocycles. If ϕ is a Borel isomorphism between E_1 and E_2 and if $\hat{\phi}(\sigma_2)$ and σ_1 are cohomologuous then the two $*$-algebras $\mathcal{B}_c(E_1,\sigma_1)$ and $\mathcal{B}_c(E_2,\sigma_2)$ are isomorphic.*

Proof: By Proposition 7.1.4, we only have to show that the isomorphism $\phi : X_1 \to X_2$ induces an isomorphism from $\mathcal{B}_c(E_1,\hat{\phi}(\sigma_2))$ to $\mathcal{B}_c(E_2,\sigma_2)$. Recall that $\phi \times \phi : E_1 \to E_2$ is a Borel isomorphism. Let $\Phi : \mathcal{B}(E_2) \longrightarrow \mathcal{B}(E_1)$ be defined by

$$\Phi(f)(x,y) = f(\phi(x),\phi(y)),$$

for $f \in \mathcal{B}(E_2)$, $(x,y) \in E_1$. A straightforward computation shows that Φ induces an isomorphism from $\mathcal{B}_c(E_1, \hat{\phi}(\sigma_2))$ to $\mathcal{B}_c(E_2, \sigma_2)$. ∎

In the second part of this section, we embed the bounded Borel functions on $(X, \mathscr{A})$ and the elements of the pseudo-group $[[E]]$ of E in $\mathcal{B}_c(E,\sigma)$.

Example 7.1.6 *To a function $f \in \mathcal{B}_o(X)$, we associate the function $m_f \in \mathcal{B}_c(E,\sigma)$, defined by*

$$m_f(x,y) = \begin{cases} f(x), & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}.$$

Then m_f is an element of $\mathcal{B}_c(E,\sigma)$ such that band$(m_f) = 1$.

Example 7.1.7 *Let $\phi : A \to B$ be an element of the pseudo-group $[[E]]$ and let $u_\phi : E \to \mathbb{C}$ be the function given by*

$$(u_\phi)(x,y) = \begin{cases} 1, & \text{if } y \in A \text{ and } \phi(y) = x \\ 0, & \text{if not} \end{cases}.$$

Then u_ϕ is the characteristic function of the graph of ϕ^{-1} in $X \times X$, thus u_ϕ is a band one element of $\mathcal{B}_c(E,\sigma)$.

One checks easily that $u_\phi^* = u_{\phi^{-1}}$. We then have

$$\begin{aligned}(u_\phi \cdot u_\phi^*)(x,z) &= \sum_{y\sim x} u_\phi(x,y)u_\phi^*(y,z)\sigma(x,y,z) \\ &= \sum_{y\sim x} u_\phi(x,y)u_{\phi^{-1}}(y,z)\sigma(x,y,z) \\ &= \begin{cases} 1, & \text{if } x = z \text{ and } z \in B \\ 0, & \text{else} \end{cases} \\ &= m_{\mathcal{X}_B}(x,z),\end{aligned}$$

and

$$\begin{aligned}(u_\phi^* \cdot u_\phi)(x,z) &= \sum_{y\sim x} u_\phi^*(x,y)u_\phi(y,z)\sigma(x,y,z)\\ &= \sum_{y\sim x} u_{\phi^{-1}}(x,y)u_\phi(y,z)\sigma(x,y,z)\\ &= \begin{cases} 1, & \textit{if } x=z \textit{ and } z\in A\\ 0, & \textit{else}\end{cases}\\ &= m_{\mathcal{X}_A}(x,z).\end{aligned}$$

Then $\{m(f), u_\phi; f \in \mathcal{B}_o(X), \phi \in [[E]]\}$ is a set of generators of $\mathcal{B}_c(E,\sigma)$. As in Proposition 2.3 of [FM2], we have;

Proposition 7.1.8 *For all $f \in \mathcal{B}_c(E,\sigma)$, there exist $N > 0$, $f_k \in \mathcal{B}_o(X)$ and $\phi_k \in [[E]]$ for $1 \leqslant k \leqslant N$, such that*

$$f = \sum_{k=1}^{N} m_{f_k} \cdot u_{\phi_k}.$$

Proof: Let $F = \{(x,y) \in E; f(x,y) \neq 0\}$ be the support of f; $\pi_r : F \to X$ be the right projection (i.e. $\pi_r(x,y) = y$) and $\pi_l : F \to X$ be the left one (i.e. $\pi_l(x,y) = x$). As in the proof of Theorem 1 of [FM1], F can be partitioned

$$F = \bigsqcup_{i\in\mathbb{N}} F_i$$

where π_r is injective on every F_i.

As mentioned in Proposition 2.3 of [FM2], we can rearrange the partition using the following algorithm:

For $i \geqslant 0$

For $j \geqslant i+1$

The set $A_{i,j} = (F_j \cap \pi_r^{-1}(\pi_r(F_i)^c))$ is the subset of F_j whose right projection is disjoint from the right projection of F_i. Annex this set to F_i, i.e.

$$F_i = F_i \cup A_{i,j},$$

and then remove it from F_j, i.e.

$$F_j = F_j \setminus A_{i,j}.$$

end.

end.

This new partition of F is such that π_r is injective on every F_i and $\pi_r(F_j) \subseteq \pi_r(F_i)$ for all $j > i$. Since $f \in \mathcal{B}_c(E, \sigma)$, then there is at most a finite number of non-empty F_i, i.e. there exists an $s \in \mathbb{N}$ such that $F_k = \emptyset$ for all $k > s$ (in fact $s = \text{band}(f)$). Thus

$$F = \bigsqcup_{p=1}^{s} F_p.$$

Similarly F can be partitioned by

$$F = \bigsqcup_{q=1}^{t} F'_q,$$

where π_l is injective on every F'_q and $\pi_l(F'_j) \subseteq \pi_l(F'_i)$ for all $j > i$. With

$$F''_k = F_{p,q} = F_p \cap F'_q,$$

for $p = 1, ..., s$ and $q = 1, ..., t$, then

$$F = \bigsqcup_{k=1}^{st} F''_k$$

where π_r and π_l are injective on each F''_k. Let $\phi_k \in [[E]]$ be such that

$$F''_k = \text{graph}(\phi_k^{-1}),$$

and define

$$f_k(x) = \begin{cases} f(x, \phi_k^{-1}(x)), & \text{if } x \in \mathrm{dom}(\phi_k^{-1}) \\ 0, & \text{else} \end{cases}.$$

As

$$\begin{aligned}(m_{f_k} \cdot u_{\phi_k})(x,y) &= \sum_{z \sim x} m_{f_k}(x,z) u_{\phi_k}(z,y)\sigma(x,z,y) \\ &= f_k(x) u_{\phi_k}(x,y) \\ &= \begin{cases} f(x, \phi_k^{-1}(x)), & \text{if } \phi_k(y) = x \\ 0, & \text{else} \end{cases}\end{aligned}$$

we then have

$$f = \sum_{k=1}^{st} m_{f_k} \cdot u_{\phi_k}.$$

∎

As in [FM2], Proposition 2.4, we show that every element of $\mathcal{B}_c(E, \sigma)$ is linear combination of elements of the form $m_f \cdot u_\phi$ with $f \in \mathcal{B}_o(X)$ with $|f| = 1$ and $\phi \in [E]$, the full group of E. To achieve this goal, we need the following.

Lemma 7.1.9 *Let $f \in \mathcal{B}_o(X)$. Then $f \in \mathcal{B}_o(X)$ is a linear combination of four Borel functions of modulus one.*

Proof: Without loss of generality, we can assume that f is real-valued and that $\|f\| \leqslant 1$. Then $f_\pm = \frac{1}{2}\left(f \pm i\sqrt{1-f^2}\right)$ is of modulus one and $f = f_+ + f_-$. ∎

Lemma 7.1.10 *Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$. For any $\phi : A \to B$ in $[[E]] \setminus [E]$, there exist two Borel automorphisms h_0 and h_1 in $[E]$ and* *two disjoints Borel sets A_0 and A_1 of X such that $A = A_0 \sqcup A_1$ and*

$$\phi(x) = \begin{cases} h_0(x) & \text{if } x \in A_0 \\ h_1(x) & \text{if } x \in A_1 \end{cases}.$$

Proof: Let $\phi \in [[E]]$, but $\phi \notin [E]$, with $\phi : A \to B$. We say that a point $x \in B \setminus A$ has a **negative orbit** if for all $k > 0$, then $\phi^{-k}(x) \in A \cap B$. We denote that (infinite) orbit $\cup_{k>0}\{\phi^{-k}(x)\}$ by $\mathcal{O}_\phi^-(x)$. We say that a point $x \in A \setminus B$ has a **positive orbit** if for all $k > 0$, then $\phi^k(x) \in A \cap B$. We denote that (infinite) orbit $\cup_{k>0}\{\phi^k(x)\}$ by $\mathcal{O}_\phi^+(x)$. We say that a point $x \in A \cap B$ has an **orbit** if for all $k \in \mathbb{Z}$, then $\phi^k(x) \in A \cap B$. We denote that orbit $\cup_{k\in\mathbb{Z}}\{\phi^k(x)\}$ by $\mathcal{O}_\phi(x)$. Notice that under this definition, any fixed point of ϕ has an orbit. Let $n > 0$. Define the following sets;

$$\begin{aligned}
\overline{B} &= \{x \in B \setminus A; \mathcal{O}_\phi^-(x) \subset A \cap B\}, \\
\overline{B}_n &= \phi^{-n}(\overline{B}), \\
\overline{A} &= \{x \in A \setminus B; \mathcal{O}_\phi^+(x) \subset A \cap B\}, \\
\overline{A}_n &= \phi^n(\overline{A}), \\
C &= \{x \in A \cap B; \mathcal{O}_\phi(x) \subset A \cap B\}, \\
D &= (A \cap B) \setminus \left(C \sqcup \left(\sqcup_{m>0}\overline{A}_m\right) \sqcup \left(\sqcup_{m>0}\overline{B}_m\right)\right), \\
A^n &= \{x \in A \setminus B; \phi^n(x) \in B \setminus A\}, \\
B^n &= \phi^n(A^n). \\
A_0 &= \overline{A} \sqcup C \sqcup D \sqcup (\sqcup_{m>0}A^m) \sqcup \left(\sqcup_{m>0}\overline{A}_{2m}\right) \sqcup \left(\sqcup_{m>0}\overline{B}_{2m-1}\right) \\
A_1 &= A \setminus A_0
\end{aligned}$$

The two following Borel automorphisms of $(X, \mathscr{A})$

$$h_0(x) = \begin{cases} x, & \text{if } x \in (A \cup B)^c \\ \phi^{-1}(x), & \text{if } x \in \overline{B} \sqcup \left(\sqcup_{m>0} \overline{A}_{2m-1}\right) \sqcup \left(\sqcup_{m>0} \overline{B}_{2m}\right) \\ \phi^{-n}(x), & \text{if } x \in B^n \\ \phi(x), & \text{else.} \end{cases}$$

and

$$h_1(x) = \begin{cases} x, & \text{if } x \in (A \cup B)^c \sqcup \overline{A} \sqcup \overline{B} \\ \phi^{-1}(x), & \text{if } x \in \left(\sqcup_{m>0} \overline{A}_{2m}\right) \sqcup \left(\sqcup_{m>0} \overline{B}_{2m-1}\right) \\ \phi^{-n}(x), & \text{if } x \in B^n \\ \phi(x), & \text{else.} \end{cases}$$

are elements of $[E]$ such that

$$\phi(x) = \begin{cases} h_0(x) & \text{if } x \in A_0 \\ h_1(x) & \text{if } x \in A_1 \end{cases}.$$

∎

Theorem 7.1.11 *The vector space $\mathcal{B}_c(E, \sigma)$ is generated by*

$$\mathcal{G} = \{m_f \cdot u_\phi; f \in \mathcal{B}_o(X), \|f\|_\infty = 1 \text{ and } \phi \in [E]\}.$$

Proof: Let $\phi : A \to B$ be an element of $[[E]] \setminus [E]$. By Lemma 7.1.10, there exist for $i = 0, 1$, $A_i \in \mathscr{A}$ and $h_i \in [E]$ such that $A = A_0 \sqcup A_1$ and, for $x \in A$,

$$\phi(x) = \begin{cases} h_0(x) & \text{if } x \in A_0 \\ h_1(x) & \text{if } x \in A_1 \end{cases}.$$

Then let us check that

$$u_\phi = m_{\mathcal{X}_{\phi(A_0)}} \cdot u_{h_0} + m_{\mathcal{X}_{\phi(A_1)}} \cdot u_{h_1}.$$

For $j = 0, 1$, we have

$$\begin{aligned}(m_{\mathcal{X}_{\phi(A_j)}} \cdot u_{h_j})(x,y) &= \sum_{z\sim x} m_{\mathcal{X}_{\phi(A_j)}}(x,z) u_{h_j}(z,y)\sigma(x,z,y) \\ &= \begin{cases} u_{h_j}(x,y), & \text{if } x \in \phi(A_j) \\ 0, & \text{else} \end{cases} \\ &= \begin{cases} \mathcal{X}_{\phi(A_j)}, & \text{if } h_j(y) = x \\ 0, & \text{else} \end{cases}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\Big(m_{\mathcal{X}_{\phi(A_0)}} \cdot u_{h_0} + m_{\mathcal{X}_{\phi(A_1)}} \cdot u_{h_1}\Big)(x,y) &= ... \\ ... &= \begin{cases} \mathcal{X}_{\phi(A_0)}(x), & \text{if } h_0(y) = x \\ 0, & \text{else} \end{cases} + \begin{cases} \mathcal{X}_{\phi(A_1)}(x), & \text{if } h_1(y) = x \\ 0, & \text{else} \end{cases} \\ &= \begin{cases} 1, & \text{if } x \in \phi(A) \text{ and } \phi(y) = x \\ 0, & \text{else} \end{cases} \\ &= u_\phi(x,y).\end{aligned}$$

Thus $u_\phi \in \text{span}\{\mathcal{G}\}$. ∎

7.2 The Borel $*$-Algebra of (E,σ)

In this section we associate a Borel $*$-algebra to (E,σ). In order to do this we first define a sequentially normal representation of $\mathcal{B}_c(E,\sigma)$. Throughout this section, μ denotes a probability measure on $(X,\mathscr{A})$ and

$$\nu^{-1} = \int \lambda_x d\mu(x)$$

is the associated measure on the standard Borel space $(E,\mathcal{A}\times\mathcal{A}|_E)$ (Definition 4.2.1).

Definition 7.2.1 *Let*

$$\mathcal{H}_\mu = L^2(E,\nu^{-1})$$

denote the Hilbert space of the ν^{-1}-square summable functions on E. Then the representation $(L_\mu,\mathcal{H}_\mu)$ of $\mathcal{B}_c(E,\sigma)$ is defined by

$$\left(L_\mu(f)\xi\right)(x,y) = \sum_{z\sim x} f(x,z)\xi(z,y)\sigma(x,z,y)$$

for all $f\in\mathcal{B}_c(E,\sigma)$ and $\xi\in\mathcal{H}_\mu$.

If $f\in\mathcal{B}_c(E,\sigma)$, with $M=\|f\|_\infty$ and $n=\text{band}(f)$, then by Theorem 7.1.11 we have

$$f=\sum_{k=1}^{n}\lambda_k m_{f_k}\cdot u_{\phi_k},$$

and therefore

$$\begin{aligned}
\|L_\mu(f)\xi\|_2^2 &= \int\sum_{y\sim x}\left|(L_\mu(f)\xi)(y,x)\right|^2 d\mu(x)\\
&= \int\sum_{y\sim x}\left|\sum_{k=1}^{n}(L_\mu(\lambda_k m_{f_k}\cdot u_{\phi_k})\xi)(y,x)\right|^2 d\mu(x)\\
&\leqslant \int\sum_{y\sim x}\sum_{k=1}^{n}|\lambda_k|^2|\xi(\phi_k^{-1}(y),x)|^2 d\mu(x)\\
&\leqslant M^2\sum_{k=1}^{n}\int\sum_{y\sim x}|\xi(\phi_k^{-1}(y),x)|^2 d\mu(x)
\end{aligned}$$

$$=M^2 n\|\xi\|_2^2,$$

This shows that $L_\mu(f) \in \mathcal{B}(\mathcal{H}_\mu)$, for any $f \in \mathcal{B}_c(E,\sigma)$.

Definition 7.2.2 *The representation given by*

$$(L, \ell^2(E)) = \bigoplus_{x\in X}(L_{\delta_x}, \mathcal{H}_{\delta_x}).$$

is called the (left) reduced representation of $\mathcal{B}_c(E,\sigma)$ and let $\|L(f)\|_r$ denote the operator norm of $L(f)$, for $f \in \mathcal{B}_c(E,\sigma)$.

Recall that the elements in $\ell^2(E)$ are (Borel) functions $\xi : E \longrightarrow \mathbb{C}$ with countable support on E such that

$$\sum_{(x,y)\in E} |\xi(x,y)|^2 < \infty.$$

Let us denote by

$$\left\{\xi_x^{(y)}\right\}_{(x,y)\in E}$$

the orthonormal basis for $\ell^2(E)$ given by $\xi_x^{(y)}$ the characteristic function of the point $(x,y) \in E$.

For $f \in \mathcal{B}_c(E,\sigma)$, $L(f)$ is a decomposable operator (Section 2.5 of [Dix]), i.e.,

$$L(f) \in \mathcal{D} = \prod_{x\in X} \mathcal{B}(\mathcal{H}_{\delta_x}).$$

Recall that, any operator in $a \in \mathcal{D}$ can be written as

$$a = \bigoplus_{x\in X} a^{(x)}$$

where

$$a^{(x)}(y,z) = \left\langle a\xi_z^{(x)} | \xi_y^{(x)} \right\rangle.$$

Then for $a \in \mathcal{D}$,

$$\|a\|_r = \sup_{x\in X} \|a^{(x)}\|_x$$

where $\|\cdot\|_x$ is the operator norm in $\mathcal{B}(\mathcal{H}_{\delta_x})$. Therefore,

$$L(f) = \bigoplus_{x \in X} L(f)^{(x)},$$

with $L(f)^{(x)}(y,z) = \left\langle L(f)\xi_y^{(x)} | \xi_x^{(x)} \right\rangle = f(y,z)\sigma(y,z,x)$. Thus the reduced representation is faithful of $\mathcal{B}_c(E,\sigma)$.

Remark that if $f \in \mathcal{B}(E)$, then its associated operator is not necessarily bounded (take the constant function $f(x,y) = 1$ for example). We introduce the following definition.

Definition 7.2.3 *Let E be a countable Borel equivalence relation and σ be a $\mathbb{T}$-valued Borel 2-cocycle. The set of* ***Borel operators*** *associated to (E,σ), denoted $M(E,\sigma)$, is the set*

$$M(E,\sigma) = L(\mathcal{B}_{op}(E,\sigma)) \subset \mathcal{D}$$

where $\mathcal{B}_{op}(E,\sigma) = \{f \in \mathcal{B}(E) | L(f) \in \mathcal{D}\}$.

The previous definition follows closely the construction in Section 2 of [SulWWr], where they associate to a countable Borel equivalence relation E a weakly sequentially closed algebra $M(E)$. Here the definition was adapted to include the Borel 2-cocycle σ.

Definition 7.2.4 *Let $a \in \mathcal{D}$. The* ***coordinate function*** *associated to the operator a, denoted $c(a)$, is a function $c(a) : E \longrightarrow \mathbb{C}$ defined by*

$$c(a)(x,y) = \left\langle a\xi_y^{(x)} | \xi_x^{(x)} \right\rangle,$$

where $\xi_z^{(x)}$ is the characteristic function of the point (z,x).

Remark 7.2.5 *For any Borel operators f and $\xi \in \ell^2(E)$, then*

$$(L(f)\xi)(x,z) = \sum_{y \sim x} f(x,y)\xi(y,z)\sigma(x,y,z),$$

and therefore

$$c(L(f))(x,y) = \left\langle L(f)\xi_y^{(x)} | \xi_x^{(x)} \right\rangle = f(x,y).$$

Thus for any $f \in \mathcal{B}_{op}(E,\sigma)$, then $c(L(f)) = f$.

Lemma 7.2.6

$$\mathcal{B}_{op}(E,\sigma) \subset \mathcal{B}_o(E).$$

Proof: Let $f \in \mathcal{B}_{op}(E,\sigma)$, then $\|L(f)\|_r < \infty$. For any $(x,y) \in E$, then

$$\begin{aligned} \|L(f)\|_r^2 &\geqslant \left\langle L(f)\xi_y^{(x)} | L(f)\xi_y^{(x)} \right\rangle \\ &= \sum_{w \sim y} |f(w,y)|^2 \\ &\geqslant |f(x,y)|^2. \end{aligned}$$

Thus $\|f\|_\infty \leqslant \|L(f)\|_r$. Hence $f \in \mathcal{B}_o(E)$. ∎

Theorem 7.2.7 *The set of Borel operators $M(E,\sigma)$ is a C^*-subalgebra of $\mathcal{B}(\ell^2(E))$ which is sequentially closed with respect to the weak operator topology, thus is a Borel $*$-algebra.*

Proof: Let f and g be in $\mathcal{B}_{op}(E,\sigma)$ and $\lambda \in \mathbb{C}$.

Since $f+g$ and λf are Borel functions and $(L, \ell^2(E))$ is a representation, then it follows immediately that $M(E,\sigma)$ is a complex vector space.

For the involution, the map $\theta : E \longrightarrow E$ defined by $\theta(x,y) = (y,x)$ is Borel isomorphism. Then $f^* = \overline{f \circ \theta}$ is a bounded Borel function. Since

$$\begin{aligned} \left\langle L(f^*)\xi_z^{(x)} | \xi_y^{(x)} \right\rangle &= (f^*)(y,z)\sigma(x,y,z) \\ &= f^*(y,z)\sigma(x,y,z) \\ &= \overline{f(z,y)}\sigma(x,y,z) \end{aligned}$$

$$= \left\langle \xi_z^{(x)} | L(f)\xi_y^{(x)} \right\rangle$$
$$= \left\langle L(f)^*\xi_z^{(x)} | \xi_y^{(x)} \right\rangle,$$

then $L(f^*) = L(f)^*$.

For multiplication, in general we don't know if $f \cdot g$ is a bounded function, but since $L(f)L(g)$ is a bounded operator, straightforward computation shows that

$$c(L(f)L(g)) = \left\langle L(f)L(g)\xi_y^{(x)} | \xi_x^{(x)} \right\rangle = \sum_{z\in[x]} f(x,z)g(z,y)\sigma(x,z,y) = (f\cdot g)(x,y),$$

which shows that $f \cdot g$ is a bounded function on E such that $L(f\cdot g) = L(g)L(f)$. It remains to show that $c(L(f\cdot g)) = f\cdot g$ is a Borel function. Let $G = \{\phi_i\}_{i=1}^{\infty}$ be a countable group of Borel automorphisms of $(X, \mathscr{A})$ such that $E = E_G$. Let

$$A_n = \{(x, \phi_n(x)) | x \in X\} \subset E$$

for $n \geqslant 1$. The A_n are not necessarily disjoint since the group G is not assumed to act freely. Let $D_1 = A_1$, and

$$D_n = \Delta_n \setminus D_{n-1}$$

for $n \geqslant 2$. The D_n are disjoints and

$$E = \bigsqcup_{n=1}^{\infty} D_n.$$

For $n \geqslant 1$, define $f_n(x,y) = \mathcal{X}_{D_n}(x,y)f(x,y)$ where $(x,y) \in E$, which is a bounded Borel function since it is the pointwise product of two bounded Borel functions. For $n \geqslant 1$ define the Borel functions,

$$a_n : E \longrightarrow E \text{ by } a_n(x,z) = (x, g_n(x)),$$

$$b_n : E \longrightarrow E \text{ by } b_n(x,z) = (g_n(x), z)$$

and

$$c_n : E \longrightarrow E^{(2)} \text{ by } c_n(x,z) = (x, g_n(x), z).$$

Now for $(x,z) \in E$,

$$\begin{aligned}(f\cdot g)(x,z) &= \sum_{y\in[x]} f(x,y)g(y,z)\sigma(x,y,z)\\ &= \sum_{n=1}^{\infty} f_n(x,\phi_n(x))g(\phi_n(x),z)\sigma(x,\phi_n(x),z)\\ &= \sum_{n=1}^{\infty}(f_n\circ a_n)(x,z)(g\circ b_n)(x,z)(\sigma\circ c_n)(x,z)\end{aligned}$$

which is the the pointwise limit of the Borel functions

$$\sum_{n=1}^{N} h_n,$$

where $h_n(x,z) = (f_n \circ a_n)(x,z)(g\circ b_n)(x,z)(\sigma\circ c_n)(x,z)$. Thus $f\cdot g$ is a bounded Borel function.

For weak sequential closure, let $\{f_n\}_{n=1}^{\infty}$ be a (bounded) sequence in $\mathcal{B}_{op}(E,\sigma)$ such that $L(f_n)$ converges weakly to a and let $f = c(a)$. Clearly for all $(x,y)\in E$ $f_n(x,y)$ converges to $f(x,y)$ making the sequence $\{f_n\}_{n=1}^{\infty}$ converging pointwise to f, thus f is a bounded Borel function. ∎

Remark 7.2.8 *The previous proof is an adaptation of Lemma 2.1 of [SulWWr]. We make the connection between the two constructions. Let (E,σ) be such that σ is trivial. If $f\in\mathcal{B}_{op}(E,\sigma)$ with*

$$L(f) = \bigoplus_{x\in X} L(f)^{(x)},$$

then for $x\sim y$ we have $L(f)^{(x)} = L(f)^{(y)}$. From $f\in\mathcal{B}_{op}(E,\sigma)$ construct a bounded operator $\mathcal{Q}(L(f))$ on

$$\ell^2(X) = \bigoplus_{[x]\in X/E} \ell^2([x]),$$

defined by

$$\mathcal{Q}(L(f)) = \bigoplus_{[x] \in X/E} L(f)^{(y)},$$

where $y \in [x]$. Explicitly, if $f \in \mathcal{B}_{op}(E, \sigma)$ and $\xi \in \ell^2(X)$ then

$$\left(\mathcal{Q}(L(f))\xi\right)(x) = \sum_{y \in [x]} f(x, y)\xi(y).$$

The resulting Borel $*$-algebra $\mathcal{Q}(M(E, \sigma))$ is then the construction defined in [SulWWr]. Then we get the following result.

Corollary 7.2.9 *Let (E, σ) be such that σ is trivial. Then*

$$M(E, \sigma) \cong \mathcal{Q}(M(E, \sigma)).$$

Proof: By construction $M(E, \sigma)$ is the amplification of $\mathcal{Q}(M(E, \sigma))$. ∎

We now define a Borel $*$-algebra, by following the construction of the reduced C*-algebra associated to a topological groupoid, like in [Re], [Mu] or [Pat2].

Definition 7.2.10 *The* ***reduced Borel $*$-algebra*** *associated to (E, σ), denoted $\mathcal{B}_r^*(E, \sigma)$, is the Borel envelope of $L(\mathcal{B}_c(E, \sigma))$ in $\mathcal{B}(\ell^2(E))$.*

Example 7.2.11 *(Decomposition of the Borel $*$-algebra $\mathcal{B}_r^*(E, \sigma)$ by E-invariant Borel subsets) If A be an E-invariant subset (see Definition* 4.1.8*), then*

$$\mathcal{B}_r^*(E, \sigma) = \mathcal{B}_r^*(E|_A, \sigma) \bigoplus \mathcal{B}_r^*(E|_{A^c}, \sigma).$$

Remark 7.2.12 *Since $\mathcal{B}_c(E, \sigma) \subset M(E, \sigma)$ and $M(E, \sigma)$ is a weakly sequentially closed Borel $*$-algebra, it follows that $\mathcal{B}_r^*(E, \sigma) \subseteq M(E, \sigma)$. We will show later that they coincide when E is hyperfinite. We do not know if they coincide in general.*

Proposition 7.2.13 *The subalgebra $L(\mathcal{B}_o(X))$ of $\mathcal{B}_r^*(E, \sigma)$ is maximal abelian.*

Proof: Let a and b be in $\mathcal{B}_o(X)$ and ξ in $\ell^2(E)$. We have

$$\begin{aligned}(L(a)L(b)\xi)(x,y) &= \sum_{w\in[x]} a(x,w)(L(b)\xi)(w,y)\sigma(x,w,y)\\ &= a(x,x)(L(b)\xi)(x,y)\\ &= a(x,x)\left(\sum_{z\in[x]} b(x,z)\xi(z,y)\sigma(x,z,y)\right)\\ &= a(x,x)b(x,x)\xi(x,y)\end{aligned}$$

and similary $(L(b)L(a)\xi)(x,y) = b(x,x)a(x,x)\xi(x,y)$. This shows that $L(\mathcal{B}_o(X))$ is abelian. Let A be an abelian subalgebra of $\mathcal{B}_r^*(E,\sigma)$ containing $L(\mathcal{B}_o(X))$. Let $\hat{f}\in A$ and $f = c(\hat{f})$. We need to show that $f\in\mathcal{B}_o(X)$. Suppose that there exists $(x,y)\in E$ with $x\neq y$ and $f(x,y)\neq 0$. Pick $a\in\mathcal{B}_o(X)$ such that $a(x,x)\neq a(y,y)$. Since $L(f)L(a) = L(a)L(f)$, the computations

$$\begin{aligned}(L(f)L(a)\xi_y^{(z)})(x,z) &= \sum_{w\in[x]} f(x,w)(L(a)\xi_y^{(z)})(w,z)\sigma(x,w,y)\\ &= \sum_{w\in[x]} f(x,w)\left(\sum_{w'\in[x]} a(w,w')\xi_y^{(z)}(w',z)\sigma(w,w',z)\right)\sigma(x,w,y)\\ &= \sum_{w\in[x]} f(x,w)a(w,y)\sigma(w,y,z)\sigma(x,w,y)\\ &= f(x,w)a(y,y)\end{aligned}$$

and

$$\begin{aligned}(L(a)L(f)\xi_y^{(z)})(x,z) &= \sum_{w\in[x]} a(x,w)(L(f)\xi_y^{(z)})(w,z)\sigma(x,w,y)\\ &= a(x,x)(L(f)\xi_y^{(z)})(x,z)\\ &= a(x,x)\left(\sum_{w'\in[x]} f(x,w')\xi_y^{(z)}(w',z)\sigma(x,w',z)\right)\\ &= a(x,x)f(x,w)\end{aligned}$$

show that $a(x,x) = a(y,y)$ which is a contradiction. Since the function f is bounded by Lemma 7.2.6, then $f \in \mathcal{B}_o(X)$ and $\hat{f} \in L(\mathcal{B}_o(X))$. ∎

Proposition 7.2.14 *The map $\Delta : \mathcal{B}_r^*(E,\sigma) \longrightarrow L(\mathcal{B}_o(X))$ defined by*

$$\Delta(\hat{f}) = L(c(\hat{f})_\Delta),$$

with $\hat{f} \in \mathcal{B}_r^*(E,\sigma)$ and $f = c(\hat{f})$ and where

$$c(\hat{f})_\Delta(x,y) = \begin{cases} c(\hat{f})(x,y) & \text{if } x = y \\ 0 & \text{else} \end{cases}$$

is a sequentially normal faithful conditional expectation.

Proof: Let $\hat{f} \in \mathcal{B}_r^*(E,\sigma)$. The coordinate function $c(\hat{f})$ (bounded, by Lemma 7.2.6) multiplied pointwise by the characteristic function of the diagonal Δ in $X \times X$ is a Borel function that can be viewed as an element of $\mathcal{B}_o(X)$. The operator $\Delta(\hat{f})$ is the representation of that function, i.e.

$$\Delta(\hat{f}) = L(\mathcal{X}_\Delta c(\hat{f})).$$

Clearly this is a linear map which is surjective. Let e be the identity element of $\mathcal{B}(\ell^2(E))$. Since $c(e) = \mathcal{X}_\Delta$, then $\Delta(e) = e$. The computation

$$\begin{aligned} \Delta(\Delta(\hat{f})) &= L(\mathcal{X}_\Delta c(L(\mathcal{X}_\Delta c(\hat{f})))) \\ &= L(\mathcal{X}_\Delta \mathcal{X}_\Delta c(\hat{f})) \\ &= L(\mathcal{X}_\Delta c(\hat{f})) \\ &= \Delta(\hat{f}) \end{aligned}$$

shows that Δ is a projection. Let $\hat{a}$ and $\hat{b}$ be in $L(\mathcal{B}_o(X))$. Let $a = c(\hat{a})$, $b = c(\hat{b})$ and $f = c(\hat{f})$. Let $(x,y) \in E$. The computations

$$(a \cdot f \cdot b)_\Delta(x,y) = \begin{cases} (a \cdot f \cdot b)(x,x) & \text{if } x = y \\ 0 & \text{else} \end{cases}$$

where

$$\begin{aligned}(a \cdot f \cdot b)(x,x) &= \sum_{w \in [x]} a(x,w)\,(f \cdot b)\,(w,x) \\ &= a(x,x)\,(f \cdot b)\,(x,x) \\ &= a(x,x)\left(\sum_{w' \in [x]} f(x,w')b(w',x)\right) \\ &= a(x,x)f(x,x)b(x,x)\end{aligned}$$

and

$$\begin{aligned}(a \cdot f_\Delta \cdot b)(x,y) &= \sum_{w \in [x]} a(x,w)\,(f_\Delta \cdot b)\,(w,y) \\ &= a(x,x)\,(f_\Delta \cdot b)\,(x,y) \\ &= a(x,x)\left(\sum_{w' \in [x]} f_\Delta(x,w')b(w',y)\right) \\ &= a(x,x)f_\Delta(x,y)b(y,y) \\ &= \begin{cases} a(x,x)f(x,x)b(x,x) & \text{if } x = y \\ 0 & \text{else} \end{cases}\end{aligned}$$

show that $\hat{a}\Delta(\hat{f})\hat{b} = \Delta(\hat{a}\hat{f}\hat{b})$.

Now if $\Delta(\hat{f}\hat{f}^*) = 0$ then $(f \cdot f^*)_\Delta = 0$. Since

$$\begin{aligned}(f \cdot f^*)(x,x) &= \sum_{w \in [x]} f(x,w)f^*(w,x) \\ &= \sum_{w \in [x]} |f(x,w)|^2\end{aligned}$$

then $f(x,w) = 0$ for all $(x,w) \in E$, thus $\hat{f} = 0$.

Finally if $\{\hat{f}_n\}_{n\in\mathbb{N}}$ is a bounded sequence of operators in $\mathcal{B}_r^*(E,\sigma)$ that converges weakly to $\hat{f}$, then $c(\hat{f}_n)_{n=1}^{\infty}$ is a bounded sequence that converges pointwise to $c(\hat{f})$. Thus $\{\mathcal{X}_\Delta c(\hat{f}_n)\}_{n=1}^{\infty}$ is a bounded sequence that converges pointwise to $\mathcal{X}_\Delta c(\hat{f})$. Since weak sequential convergence on $L(\mathcal{B}_o(X))$ is equivalent to pointwise convergence on $\mathcal{B}_o(X)$, it follows that Δ is a sequentially normal conditional expectation. ∎

We now give an explicit description of an element of the normalizer of $L(\mathcal{B}_o(X))$ in $\mathcal{B}_r^*(E,\sigma)$. Recall

$$N_{\mathcal{B}_r^*(E,\sigma)}(L(\mathcal{B}_o(X))) = \{u \in \mathcal{U}(\mathcal{B}_r^*(E,\sigma)); uL(\mathcal{B}_o(X))u^* = L(\mathcal{B}_o(X))\}.$$

Proposition 7.2.15 *Let $\hat{u} \in N_{\mathcal{B}_r^*(E,\sigma)}(L(\mathcal{B}_o(X)))$. There exists $f : X \to \mathbb{T}$ an (unitary) element of $\mathcal{B}_o(X)$ and a Borel automorphism ϕ of $(X, \mathscr{A})$ such that*

$$c(\hat{u}) = m_f \cdot u_\phi.$$

Proof: Let $u = c(\hat{u})$ and for any $\hat{a} \in L(\mathcal{B}_o(X))$, let $a = c(\hat{a})$. Then there exists a Borel injection $\phi : X \longrightarrow X$ such that

$$\hat{u}\hat{a}\hat{u}^* = L(a \circ \phi^{-1}).$$

Since

$$\begin{aligned}
(u \cdot a \cdot u^*)(x,x) &= \sum_{y\in[x]} u(x,y)\,(au^*)\,(y,x)\sigma(x,y,x) \\
&= \sum_{y\in[x]} u(x,y) \left(\sum_{z\in[x]} a(y,z)u^*(z,x)\sigma(y,z,x) \right) \\
&= \sum_{y\in[x]} u(x,y)a(y,y)u^*(y,x) \\
&= \sum_{y\in[x]} |u(x,y)|^2 a(y,y)
\end{aligned}$$

then

$$(a \circ \phi^{-1})(x,x) = \sum_{y \in [x]} |u(x,y)|^2 a(y,y).$$

If $a = \delta_{\phi^{-1}(x)}$ (the characteristic function of the singleton $\{\phi^{-1}(x)\}$, for $x \in X$), then

$$1 = \sum_{y \in [x]} |u(x,y)|^2 \delta_{\phi^{-1}(x)}(y,y).$$

Thus $\phi^{-1}(x) \in [x]$ and $|u(x,\phi^{-1}(x))|^2 = 1$ for all $x \in X$. Then

$$x \in X \mapsto f(x) = u(x,\phi^{-1}(x)) \in \mathbb{T}$$

is a Borel function. For any $(x,z) \in E$, $z \neq \phi^{-1}(x)$, we have for $a = \delta_z$,

$$0 = \sum_{y \in [x]} |u(x,y)|^2 \delta_z(y,y).$$

This implies that $u(x,z) = 0$. The conclusion of the proposition follows. ∎

As a corollary, we get a Borel analogue of part (3) of Proposition 2.10 of [FM2].

Corollary 7.2.16 *Let $\mathcal{B} = \mathcal{B}_r^*(E,\sigma)$ and $\mathcal{B}_0 = L(\mathcal{B}_o(X))$. The **Weyl group***

$$W_{\mathcal{B}}(\mathcal{B}_0) = N_{\mathcal{B}}(\mathcal{B}_0)/\mathcal{U}(\mathcal{B}_0)$$

is isomorphic to $[E]$.

Recall that for countable Borel equivalence relation that $[E]$ is a countably generated (as a group). Thus we get the following corollary.

Corollary 7.2.17 *$\mathcal{B}_r^*(E,\sigma)$ is countably $L(\mathcal{B}_o(X))$-generated.*

Proposition 7.2.18 *Let $\hat{u} \in N_{\mathcal{B}_r^*(E,\sigma)}(L(\mathcal{B}_o(X)))$ and $\hat{f} \in \mathcal{B}_r^*(E,\sigma)$. The sequentially normal conditional expectation Δ is such that $\Delta(\hat{u}\hat{f}\hat{u}^*) = \hat{u}\Delta(\hat{f})\hat{u}^*$.*

Proof: Let $f = c(\hat{f})$ and $u = c(\hat{u})$. Let $(x,y) \in E$. To prove the proposition, it is enough to show that the two functions $c(\Delta(\hat{u}\hat{f}\hat{u}^*))$ and $c(\hat{u}\Delta(\hat{f})\hat{u}^*)$ are equal, which follows from the following computations. For $(x,y) \in E$, we have:

$$c(\Delta(\hat{u}\hat{f}\hat{u}^*))(x,y) = \begin{cases} (u \cdot f \cdot u^*)(x,x) & \text{if } x = y \\ 0 & \text{else} \end{cases}$$

where

$$\begin{aligned}
(u \cdot f \cdot u^*)(x,x) &= \sum_{w \in [x]} u(x,w)\,(f \cdot u^*)\,(w,x) \\
&= u(x,\phi^{-1}(x))\,(f \cdot u^*)\,(\phi^{-1}(x),x) \\
&= u(x,\phi^{-1}(x)) \left(\sum_{w' \in [x]} f(\phi^{-1}(x),w')u^*(w',x)\sigma(\phi^{-1}(x),w',x) \right) \\
&= u(x,\phi^{-1}(x)) f(\phi^{-1}(x),\phi^{-1}(x)) \overline{u^*(x,\phi^{-1}(x))} \\
&= f(\phi^{-1}(x),\phi^{-1}(x)) \\
&= f_\Delta(\phi^{-1}(x),\phi^{-1}(x))
\end{aligned}$$

and

$$\begin{aligned}
c(\hat{u}\Delta(\hat{f})\hat{u}^*)(x,y) &= \sum_{w \in [x]} u(x,w)\,(f_\Delta u^*)\,(w,y)\sigma(x,w,y) \\
&= u(x,\phi^{-1}(x))\,(f_\Delta u^*)\,(w,y)\sigma(x,\phi^{-1}(x),y) \\
&= u(x,\phi^{-1}(x)) \left(\sum_{w' \in [x]} f_\Delta(\phi^{-1}(x),w')u^*(w',y)\sigma(\phi^{-1}(x),w',x) \right) \sigma(x,\phi^{-1}(x),y) \\
&= u(x,\phi^{-1}(x)) \left(f_\Delta(\phi^{-1}(x),\phi^{-1}(x)) u^*(\phi^{-1}(x),y) \right) \sigma(x,\phi^{-1}(x),y) \\
&= u(x,\phi^{-1}(x)) \left(f_\Delta(\phi^{-1}(x),\phi^{-1}(x)) \overline{u(y,\phi^{-1}(x))} \right) \sigma(x,\phi^{-1}(x),y) \\
&= \begin{cases} f_\Delta(\phi^{-1}(x),\phi^{-1}(x)) & \text{if } x = y \\ 0 & \text{else} \end{cases}.
\end{aligned}$$

∎

The Propositions 7.2.13, 7.2.14 and 7.2.15 can be summarized with the next Corollary.

Corollary 7.2.19 *The subalgebra $L(\mathcal{B}_o(X))$ of $\mathcal{B}_r^*(E, \sigma)$ is a Cartan subalgebra.*

Theorem 7.2.20 *The subalgebra $L(\mathcal{B}_o(X))$ has the extension property (see Definition* 6.4.12*) relative to $\mathcal{B}_r^*(E, \sigma)$.*

Proof: This follows directly from Proposition 6.4.14. Nevertheless we give a proof in this context. Let $\hat{f} \in \mathcal{B}_r^*(E, \sigma)$ and ϕ be a sequentially normal pure state of $L(\mathcal{B}_o(X))$. Since $L(\mathcal{B}_o(X)) \cong \mathcal{B}_o(X)$, then ϕ is a Dirac measure δ_x, for some $x \in X$. Let $\lambda = c(\hat{f})(x, x)$ and $\hat{g} = L(\lambda \mathcal{X}_x)$. Clearly $\hat{g} \in L(\mathcal{B}_o(X))$. Then $\hat{h} = L(\mathcal{X}_x)$ is positive, of norm one and $\hat{h}\hat{f}\hat{h} = \hat{g}$. ∎

The pair $(\mathcal{B}_r^*(E, \sigma), L(\mathcal{B}_o(X)))$ is a pair of a Borel $*$-algebra and a (Borel) Cartan subalgebra $L(\mathcal{B}_o(X))$. In Section 7.7, we will state and prove a generalization to the Borel case of Feldman and Moore converse statement.

7.3 Borel Envelope of $C_r^*(E,\sigma)$

An étale equivalence relation (E,σ) twisted by a continuous normalized 2-cocycle σ, can also be viewed as a countable Borel equivalence relation. In this section we study the C^*-algebras $C_r^*(E,\sigma)$ and $\mathcal{B}_r^*(E,\sigma)$ as subalgebras of $\mathcal{B}(\ell^2(E))$.

Theorem 7.3.1 *Let (E,σ) be an étale, Hausdorff, locally compact, second countable, principal groupoid twisted by a continuous normalized 2-cocycle σ. Then the Borel envelope $\mathscr{B}_{\ell^2(E)}(C_r^*(E,\sigma))$ of $C_r^*(E,\sigma)$ is equal to $\mathcal{B}_r^*(E,\sigma)$.*

Proof: Recall that $\mathcal{B}_r^*(E,\sigma)$ is the Borel envelope of $L(\mathcal{B}_c(E,\sigma))$ inside $\mathcal{B}(\ell^2(E))$ (see Definition 7.2.10). As any element of $C_c(E,\sigma)$ is also a band-limited Borel function in $\mathcal{B}_c(E,\sigma)$, then $C_r^*(E,\sigma) \subset \mathcal{B}_r^*(E,\sigma)$ and therefore $\mathscr{B}_{\ell^2(E)}(C_r^*(E,\sigma)) \subseteq \mathcal{B}_r^*(E,\sigma)$.

To prove the converse, it suffice to show that the generators $L(\mathcal{B}_c(E,\sigma))$ of $\mathcal{B}_r^*(E,\sigma)$ are inside $\mathcal{B} = \mathscr{B}_{\ell^2(E)}(C_r^*(E,\sigma))$. By Theorem 7.1.11, any element $f \in \mathcal{B}_c(E,\sigma)$ is finite linear combination of elements of the form $f \cdot u_\phi$, with $f \in \mathcal{B}_o(X)$ and $\phi \in [E]$. It is enough to show that $L(\mathcal{B}_o(X)) \subset \mathcal{B}$ and that $L(u_\phi) \in \mathcal{B}$ for all $\phi \in [E]$. As $C_0(X) \subset C_r^*(E,\sigma)$, then $\mathcal{B}_o(X) \subset \mathcal{B}$.

Now let $\phi \in [E]$. As E is étale, for any point $g = (x,y) \in E$ there exist an open subset U_g of E and a partial homeomorphism $\phi_g : D_g \to C_g$, C_g and D_g open subsets of X, such that $\mathrm{graph}(\phi_g) = U_g$. Thus $\cup_{g\in E} U_g$ is an open cover of E. As E is second countable, by Lindelöf's Theorem (see for example [Sch] Theorem 2 section I.74 p.72), $\cup_{g\in E} U_g$ as a countable sub-cover.

Let $\{\phi_k\}$ be a countable family of partial homeomorphisms of X, $\phi_k : D_k \to C_k$, such that $\{\mathrm{graph}(\phi_k)\}$ is a countable family of open bisections which covers E. Let $A_1 = \mathrm{graph}(\phi) \cap \mathrm{graph}(\phi_1)$. This is a Borel set. Let $\tilde{D}_1 = \{x \in D_1 : (x, \phi_1(x)) \in A_1\}$ and $\tilde{C}_1 = \phi_1(\tilde{D}_1)$, which are both Borel sets of X. Thus $\mathcal{X}_{\tilde{D}_1} \in \mathcal{B}_o(X)$ and $\mathcal{X}_{\tilde{D}_1} \cdot u_{\phi_1} \in \mathcal{B}$. Let $B_1 = X \setminus \tilde{D}_1$.

For $k > 1$, let $A_k = \mathrm{graph}(\phi) \cap \mathrm{graph}(\phi_k|_{B_{k-1}})$. This is a Borel set. Let $\tilde{D}_k = \{x \in D_k : (x, \phi_k(x)) \in A_k\}$ and $\tilde{C}_k = \phi_k(\tilde{D}_k)$, which are both Borel sets of X. Thus $\mathcal{X}_{\tilde{D}_k} \in \mathcal{B}_o(X)$ and $\mathcal{X}_{\tilde{D}_k} \cdot u_{\phi_k} \in \mathcal{B}$. Let $B_k = X \setminus \left(\cup_{j=1}^k \tilde{D}_j\right)$.

By construction some $\tilde{D}_k$ could be empty and are such that $X = \bigsqcup_{k=1}^\infty \tilde{D}_k$. Since

$$\begin{aligned}\sum_{k=1}^\infty |u_{\phi_k} \cdot \mathcal{X}_{\tilde{D}_k}| =& \sum_{k=1}^\infty \sqrt{(u_{\phi_k} \cdot \mathcal{X}_{\tilde{D}_k})^* \cdot (u_{\phi_k} \cdot \mathcal{X}_{\tilde{D}_k})} \\ =& \sum_{k=1}^\infty \sqrt{\mathcal{X}_{\tilde{D}_k} \cdot u_{\phi_k^{-1}} \cdot u_{\phi_k} \cdot \mathcal{X}_{\tilde{D}_k})} \\ =& \sum_{k=1}^\infty \mathcal{X}_{\tilde{D}_k} \\ =& I\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^\infty |(u_{\phi_k} \cdot \mathcal{X}_{\tilde{D}_k})^*| =& \sum_{k=1}^\infty \sqrt{(u_{\phi_k} \cdot \mathcal{X}_{\tilde{D}_k}) \cdot (U_{\phi_k} \cdot \mathcal{X}_{\tilde{D}_k})^*} \\ =& \sum_{k=1}^\infty \sqrt{u_{\phi_k} \cdot \mathcal{X}_{\tilde{D}_k} \cdot \mathcal{X}_{\tilde{D}_k} \cdot u_{\phi_k^{-1}}} \\ =& \sum_{k=1}^\infty \sqrt{u_{\phi_k} \cdot \mathcal{X}_{\tilde{D}_k} \cdot u_{\phi_k^{-1}}} \\ =& \sum_{k=1}^\infty \sqrt{\mathcal{X}_{\phi_k(\tilde{D}_k)}} \\ =& \sum_{k=1}^\infty \mathcal{X}_{\tilde{C}_k} \\ =& I,\end{aligned}$$

then by Theorem 6.3.19 the element

$$u = \sum_{k=1}^\infty u_{\phi_k} \cdot \mathcal{X}_{\tilde{D}_k}$$

converges to an element of $\mathcal{B}$. Since

$$(u_{\phi_k} \cdot \mathcal{X}_{\tilde{D}_k})(x,y) = \begin{cases} 1 & \text{if } y \in \tilde{D}_k \text{ and } \phi(y) = x \\ 0 & \text{else} \end{cases},$$

then $u = u_\phi$. The conclusions follows by definition. ∎

Remark 7.3.2

In [Re3], J. Renault construct the following Cartan pair of C^*-algebra (A,B) which does not have the extension property. The C*-algebra A, associated to the groupoid of germs of the pseudogroup generated by $T(x) = -x$ on $[-1,1]$ of germs $\Gamma = \{(\pm x, \pm 1, x) : x \in [-1,1]\}$ and the subalgebra B are

$$A = \left\{ f : [0,1] \to \mathbb{M}_2 | f \text{ continuous and } f(0) = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \right\}$$

and

$$B = \left\{ f : [0,1] \to \mathbb{D}_2 | f \text{ continuous and } f(0) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right\}$$

where $\mathbb{M}_2$ are the (2×2) matrices and $\mathbb{D}_2$ are the diagonal (2×2) matrices. The Borel envelope of the pair (A,B) inside $\mathcal{B}(\mathcal{H})$, where $\mathcal{H} = \oplus_{[0,1]}\mathbb{C}^2$, is then

$$\mathscr{B}_{\mathcal{H}}(A) = \left\{ f : [0,1] \to \mathbb{M}_2 | f \text{ Borel and } f(0) = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \right\}$$

and

$$\mathscr{B}_{\mathcal{H}}(B) = \left\{ f : [0,1] \to \mathbb{D}_2 | f \text{ Borel and } f(0) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right\}.$$

Then $\mathscr{B}_{\mathcal{H}}(B)$ is not maximal abelian in $\mathscr{B}_{\mathcal{H}}(A)$, as the Borel subalgebra

$$C = \left\{ f : [0,1] \to \mathbb{M}_2 | f \text{ Borel, } f(x) \in \mathbb{D}_2, x \in]0,1] \text{ and } f(0) = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \right\},$$

contains $\mathscr{B}_{\mathcal{H}}(B)$.

This example shows that Theorem 7.3.1 does not extend directly to Renault's generalized notion of a twist.

7.4 The Smooth Case

The goal of this section is to study Borel $*$-algebras constructed from smooth countable Borel equivalence relations. We will establish the correspondence between smooth countable Borel equivalence relations and type I Borel $*$-algebras.

Recall that a countable Borel equivalence relation E on $(X, \mathscr{A})$ is smooth if X/E is a standard Borel space (see Definition 4.4.1 and Theorem 4.4.5). Also a Borel $*$-algebra of type I can be written as

$$B = \prod_{k=1}^{\infty} B_0(X_k, \mathcal{B}(\mathcal{H}_k)),$$

where $B_0(X_k, \mathcal{B}(\mathcal{H}_k))$ is the Borel $*$-algebra of bounded Borel functions a standard Borel space f from X_k to $\mathcal{B}(\mathcal{H}_k)$ the bounded linear operators on the Hilbert space of dimension k (see Definition 6.3.10).

In general, straightforward computation shows that for any Borel twist (E, σ) the centre of $\mathcal{B}_{op}(E, \sigma)$ (Definition 7.2.3)

$$\mathcal{B}_{op}(E, \sigma) \cap \mathcal{B}_{op}(E, \sigma)'$$

can be identified with $\mathcal{B}_0(X/E)$ (as in the proof of Theorem 4.2 of [Dav3]). By combining Definition 4.4.1 and Example 6.3.17, E is smooth if and only if $\mathcal{B}_0(X/E)$ is a standard Borel $*$-algebra. Thus the centre of $\mathcal{B}_{op}(E, \sigma)$ is an abelian standard Borel $*$-algebra if and only if E is smooth.

Theorem 7.4.1 *Let E be a smooth countable Borel equivalence relation. Then $\mathcal{B}_r^*(E, \sigma)$ is a type I Borel $*$-algebra.*

Proof: For $n \in \mathbb{N}^* \cup \{\omega\}$, let $A_n = \{x \in X : |[x]| = n\}$. Define $\mathbb{M} = \{n \in \mathbb{N}^* \cup \{\omega\} : X_n \neq \emptyset\}$ and $\mathbb{M}^F = \mathbb{M} \setminus \{\omega\}$. For any $n \in \mathbb{M}$, the Borel subset A_n is E-invariant by Theorem 4.1.10. Moreover the space $(A_n, \mathscr{A}|_{A_n})$ is a standard Borel space and the quotient space $X_n = A_n/E$ is a standard Borel space with the quotient Borel structure, such that X/E is the disjoint union of the X_n. Since E is smooth, we can assume that σ is trivial by Theorem 5.1.7. Thus $\mathcal{B}_r^*(E,\sigma) \cong \mathcal{Q}(\mathcal{B}_r^*(E,\sigma))$ as defined in Remark 7.2.8. Thus any $f \in \mathcal{Q}(\mathcal{B}_r^*(E,\sigma))$ can be written as $f = \bigoplus_{X/E} f^{[x]}$. Then if $\mathcal{H}_n$ is the Hilbert space of dimension n, then $f^{[x]} \in \mathcal{B}(\mathcal{H}_n)$ if and only if $[x] \in X_n$. Thus we can rewrite the operator f as

$$f = \bigoplus_{n \in \mathbb{M}} \left(\bigoplus_{[x] \in X_n} f^{[x]} \right) = \bigoplus_{n \in \mathbb{M}} f_n,$$

where $f_n : X_n \to \mathcal{B}(\mathcal{H}_n)$ is a bounded Borel function defined by $f_n([x]) = f^{[x]}$.

Suppose first that E is aperiodic. In this case, any band-limited bounded Borel function is an element of the weakly sequentially closed Borel $*$-algebra $B_0(X_\omega, \mathcal{B}(\mathcal{H}_\omega))$, thus we have

$$\mathcal{Q}(L(\mathcal{B}_c(E,\sigma))) \subset \mathcal{Q}(\mathcal{B}_r^*(E,\sigma)) \subseteq B_0(X_\omega, \mathcal{B}(\mathcal{H}_\omega)).$$

Now we can find a compact Hausdorff topology on X_ω which generates the Borel structure of X_ω such that the Borel envelope of $C_c(X_\omega, \mathcal{K}(\mathcal{H}_\omega))$ (here every continuous function from X_ω to $\mathcal{K}(\mathcal{H}_\omega)$ has compact support) is $B_0(X_\omega, \mathcal{B}(\mathcal{H}_\omega))$ (see [Ped] paragraph 6.3.6). Since any function of compact support is a band-limited bounded Borel function then $C_c(X_\omega, \mathcal{K}(\mathcal{H}_\omega)) \subset \mathcal{Q}(L(\mathcal{B}_c(E,\sigma)))$, and by taking the Borel envelope of the following:

$$C_c(X_\omega, \mathcal{K}(\mathcal{H}_\omega)) \subset \mathcal{Q}(L(\mathcal{B}_c(E,\sigma))) \subset B_0(X_\omega, \mathcal{B}(\mathcal{H}_\omega))$$

the Borel $*$-algebra $\mathcal{Q}(\mathcal{B}_r^*(E,\sigma))$ is squeezed and equals to $B_0(X_\omega, \mathcal{B}(\mathcal{H}_\omega))$.

Now suppose E is finite. In this case, since the sequentially closed Borel $*$-algebra

$$\prod_{n \in \mathbb{M}^F} B_o(X_n, \mathcal{B}(\mathcal{H}_n))$$

contains any band-limited bounded Borel function, then we have

$$\mathcal{Q}(L(\mathcal{B}_c(E,\sigma))) \subset \mathcal{Q}(\mathcal{B}_r^*(E,\sigma)) \subseteq \prod_{n\in\mathbb{M}^F} B_o(X_n,\mathcal{B}(\mathcal{H}_n)).$$

Let $f \in \prod_{n\in\mathbb{M}^F} B_o(X_n,\mathcal{B}(\mathcal{H}_n))$. Let $B_n = \sqcup_{k=1}^n A_n$. For any $n \in \mathbb{M}^F$, the function $g_n = L(\mathcal{X}_{B_n})fL(\mathcal{X}_{B_n})$ is a band-limited bounded Borel function (actually band$(g_n) = n$). Then the sequence of operators $\{g_n\}_{n\in\mathbb{M}^F}$ is such that $g_n \overset{s}{\to} f$. thus $f \in \mathcal{Q}(\mathcal{B}_r^*(E,\sigma))$. This shows that

$$\mathcal{Q}(\mathcal{B}_r^*(E,\sigma)) = \prod_{n\in\mathbb{M}^F} B_o(X_n,\mathcal{B}(\mathcal{H}_n)).$$

Now let E be any smooth countable Borel equivalence relation. Let $A_p = X \setminus A_\omega$ be E-invariant Borel subset of X of periodic points (points in X with finite orbit). Then we can decompose $\mathcal{Q}(\mathcal{B}_r^*(E,\sigma))$ as the following direct sum

$$\mathcal{Q}(\mathcal{B}_r^*(E,\sigma)) = \mathcal{Q}(\mathcal{B}_r^*(E|_{A_p},\sigma_p)) \oplus \mathcal{Q}(\mathcal{B}_r^*(E|_{A_\omega},\sigma_\omega))$$

and by the first two cases the results is verified. ∎

Using the notation of the previous theorem for the X_k, $k = 1,2,...,\infty$, if B is a type I Borel $*$-algebra, then the centre of B is

$$B \cap B' = \mathcal{B}_0(\sqcup_{k=1}^\infty X_k).$$

It follows that a countable Borel equivalence relation E is smooth if and only if $\mathcal{B}_r^*(E,\sigma)$ is a type I Borel $*$-algebra. The next proposition shows how to associate to a type I Borel $*$-algebra B a smooth countable Borel equivalence relation E, such that $B \cong \mathcal{B}_r^*(E,\sigma)$.

Proposition 7.4.2 *Every type I Borel $*$-algebra is the Borel $*$-algebra associated to a smooth countable Borel equivalence relation.*

Proof: Let B be any Borel $*$-algebra of type I,

$$B = \prod_{k=1}^{\infty} B_0(X_k, \mathcal{B}(\mathcal{H}_k)).$$

For $n < \infty$, let $X_n^j = X_n$ and

$$Z_n = \bigsqcup_{j=0}^{n-1} X_n^j$$

be the standard Borel space of the disjoint union of n copies of X_n. For $n = \infty$, let $X_\infty^j = X_\infty$ and

$$Z_\infty = \bigsqcup_{j \in \mathbb{Z}} X_\infty^j$$

be the standard Borel space of the disjoint union of infinitely countably many copies of X_∞. The point $x \in X_n$ on the j-th copy X_n^j is denoted x^j. Let

$$Z = \bigsqcup_{k=1}^{\infty} Z_k$$

and let $\phi : Z \to Z$ be defined as

$$\phi(x) = \begin{cases} x^{j+1 \mod n} & \text{if } x \in Z_n \\ x^{j+1} & \text{if } x \in Z_\infty \end{cases}.$$

Then ϕ is a smooth Borel automorphism of Z such that $\mathcal{Q}(\mathcal{B}_r^*(E_\phi, \sigma)) = B$. ∎

7.5 The Hyperfinite Case

The goal of this section is to study the Borel $*$-algebras constructed from hyperfinite countable Borel equivalence relations (E,σ) (recall that σ can be assumed to be trivial for E hyperfinite). We give a (possible) definition for a Borel AF-algebras (BAF-algebra) and show that if the reduced Borel $*$-algebra $\mathcal{B}_r^*(E,\sigma)$ (Definition 7.2.10) is hyperfinite then $\mathcal{B}_r^*(E,\sigma)$ is a BAF-algebra. We will also established the equality between $\mathcal{B}_r^*(E,\sigma)$ and the Borel operators $M(E,\sigma)$ (Definition 7.2.3), when E is hyperfinite.

To characterize abstractly Borel $*$-algebras arising from hyperfinite countable Borel equivalence relations is an important problem.

Definition 7.5.1 *Let B be a Borel $*$-algebra inside $\mathcal{B}(\mathcal{H})$. If there exist a separable AF-algebra $A \subseteq B$ such that $\mathscr{B}_{\mathcal{H}}(A) = B$, then we say that B is a Borel approximatively finite $*$-algebra or a **BAF-algebra**.*

Of course if the Hilbert space $\mathcal{H}$ is separable and B is a BAF-algebra, then B is an AF von Neumann algebra and is equal to $A'' \subset \mathcal{B}(\mathcal{H})$.

Example 7.5.2 *The Borel envelope of*

$$\bigoplus_{k=1}^{\infty} C_0(X_k, \mathcal{K}(\mathcal{H}_k)),$$

(from the Example 6.7.5) is

$$\prod_{k=1}^{\infty} B_0(X_k, \mathcal{B}(\mathcal{H}_k)).$$

Thus any Borel $*$-algebra of type I is a BAF-algebra.

Example 7.5.3 *From the topological (also Borel) tail equivalence on $E_{\mathcal{D}_k}$, we can compute $C_r^*(E_{\mathcal{D}_k})$ which is an AF-algebra and by Theorem 7.3.1, the Borel envelope of $C_r^*(E_{\mathcal{D}_k})$ inside $\mathcal{B}(\ell^2(E))$ is $\mathcal{B}_r^*(E_{\mathcal{D}_k})$, thus a BAF-algebra.*

This result says that every reduced Borel $*$-algebra constructed from an hyperfinite Borel equivalence relation is a BAF-algebra. We do not know if every BAF-algebra arises this way.

Theorem 7.5.4 *Let E be a hyperfinite countable Borel equivalence relation on $(X, \mathscr{A})$. Then*

$$\mathcal{B}_r^*(E,\sigma) = M(E,\sigma).$$

Proof: We already know by definition that $\mathcal{B}_r^*(E,\sigma) \subseteq M(E,\sigma)$. To show that $M(E,\sigma) \subseteq \mathcal{B}_r^*(E,\sigma)$, we will show that for any $\hat{f} \in M(E,\sigma)$ there is a bounded sequence $\{\hat{f}_n\} \subset L(\mathcal{B}_c(E,\sigma))$ such that $\hat{f}_n \xrightarrow{w} \hat{f}$.

When E is hyperfinite, any 2-cocycle σ is cohomologuous to the trivial one, and E can be written as $E = \cup_{n\in\mathbb{N}^*} E_n$, with E_n uniformly finite and $E_n \subseteq E_{n+1}$.

Let $\hat{f} \in M(E,\sigma)$ and $f = c(\hat{f})$ be the corresponding coordinate function. Define

$$f_n(x,y) = \begin{cases} f(x,y) & \text{, if } (x,y) \in E_n \\ 0 & \text{, if } (x,y) \notin E_n \end{cases}.$$

As E_n is uniformly finite then f_n is in $\mathcal{B}_c(E,\sigma)$. The functions f_n are such that for any $(x,y) \in E$ there exists $N(x,y) > 0$ such that for all $m > N(x,y)$, then $f(x,y) = f_m(x,y)$ (in particular, the functions f_n converge pointwise to f).

Let $\hat{f}_n = L(f_n)$. The sequence $\{\hat{f}_n\}_{n\in\mathbb{N}^*}$ is a bounded sequence ($\|\hat{f}_n\| \leqslant \|\hat{f}\|$).

Since $\hat{f}$ is a bounded operator in $\mathcal{B}_r^*(E,\sigma)$, then for any $\xi \in \ell^2(E)$ such that $\|\xi\|_2^2 \leqslant 1$, we have $\|\hat{f}\xi\|_2^2 \leqslant \|\hat{f}\|^2$. In particular, for any element $\xi_x^{(y)}$ (the characteristic function of the point $(x,y) \in E$) of the orthonormal basis of $\ell^2(E)$, we have

$$\begin{aligned} \|\hat{f}\xi_x^{(y)}\|_2^2 &= \sum_{w\sim x} |(f \cdot \xi_x^{(y)})(w,y)|^2 \\ &= \sum_{w\sim x} |\sum_{z\sim y} f(w,z)\xi_x^{(y)}(z,y)|^2 \\ &= \sum_{w\sim x} |f(w,x)|^2 \leqslant \|\hat{f}\|^2. \end{aligned}$$

Then for any $\varepsilon > 0$, there is a finite subset $F_{(\varepsilon,x)} \subset [x]$ such that

$$\sum_{w\sim x} |f(w,x)|^2 \leqslant \sum_{w\in F_{(\varepsilon,x)}} |f(w,x)|^2 + \varepsilon.$$

Recall that any vector $\xi \in \ell^2(E)$ is of the form

$$\xi = \sum_{k\in I} \lambda_k \xi_{x_k}^{(y_k)},$$

where I is a countable index and $(x_k, y_k) \in E$. Since finite linear combinations of elements of the basis $\{\xi_x^{(y)}\}_{(x,y)\in E}$ form a dense subset of $\ell^2(E)$, then for any $\varepsilon > 0$, there exists a vector

$$\xi_0 = \sum_{k\in F} \lambda_k \xi_{x_k}^{(y_k)},$$

where $F \subset I$ is a finite index, such that

$$\|\xi - \xi_0\|_2^2 \leqslant \frac{\varepsilon}{8\|\hat{f}\|^2}.$$

With this we get

$$\begin{aligned}
\|(\hat{f} - \hat{f}_n)\xi_0\|_2^2 =& \|(\hat{f} - \hat{f}_n)(\sum_{k\in F} \lambda_k \xi_{x_k}^{(y_k)})\|_2^2 \\
=& \|\sum_{k\in F} \lambda_k (\hat{f} - \hat{f}_n) \xi_{x_k}^{(y_k)}\|_2^2 \\
\leqslant& \sum_{k\in F} \|\lambda_k (\hat{f} - \hat{f}_n) \xi_{x_k}^{(y_k)}\|_2^2 \\
=& \sum_{k\in F} \sum_{w\sim x_k} |\lambda_k|^2 |(f - f_n)(w, x_k)|^2 \quad (\bigstar)
\end{aligned}$$

Now for $\varepsilon_k = \frac{\varepsilon}{|F|2|\lambda_k|^2}$ (here $|F|$ is the cardinality of the set F), let $F_{(\varepsilon_k,x_k)}$ be the finite subset of $[x_k]$ such that

$$\sum_{w\sim x_k} |f(w,x_k)|^2 \leqslant \sum_{w\in F_{(\varepsilon_k,x_k)}} |f(w,x_k)|^2 + \varepsilon_k.$$

Thus the line $(\bigstar)$ equals to:

$$\sum_{k\in F} |\lambda_k|^2 \left(\sum_{w\in F_{(\varepsilon_k,x_k)}} |(f - f_n)(w, x_k)|^2 + \sum_{w\notin F_{(\varepsilon_k,x_k)}} |(f - f_n)(w, x_k)|^2 \right),$$

and since $(f - f_n)(w, x_k) = f(w, x_k)$ or $(f - f_n)(w, x_k) = 0$, the previous line is bounded by

$$\begin{aligned}
&\sum_{k\in F}|\lambda_k|^2\left(\sum_{w\in F_{(\varepsilon_k,x_k)}}|(f-f_n)(w,x_k)|^2+\sum_{w\notin F_{(\varepsilon_k,x_k)}}|f(w,x_k)|^2\right)\\
&\quad\leqslant\sum_{k\in F}|\lambda_k|^2\left(\sum_{w\in F_{(\varepsilon_k,x_k)}}|(f-f_n)(w,x_k)|^2+\frac{\varepsilon}{|F|2|\lambda_k|^2}\right)\\
&\quad=\sum_{k\in F}\left(\sum_{w\in F_{(\varepsilon_k,x_k)}}|\lambda_k|^2|(f-f_n)(w,x_k)|^2+\frac{\varepsilon}{2|F|}\right)\\
&\quad=\left(\sum_{k\in F}\sum_{w\in F_{(\varepsilon_k,x_k)}}|\lambda_k|^2|(f-f_n)(w,x_k)|^2\right)+\frac{\varepsilon}{2},
\end{aligned}$$

Now for every (w, x_k) there exists $N(w, x_k) \in \mathbb{N}^*$ such that for all $n \geqslant N(w, x_k)$ then $f(w, x_k) = f_n(w, x_k)$. Let $N = \max\{N(w, x_k)\}$. Now if $n \geqslant N$, then

$$\|(f - f_n)\xi_0\|_2^2 \leqslant \frac{\varepsilon}{2}.$$

Finally, since (with $n \geqslant N$)

$$\begin{aligned}
\|(f-f_n)\xi\|_2^2 =&\|(f-f_n)(\xi-\xi_0+\xi_0)\|_2^2\\
\leqslant&\|(f-f_n)(\xi-\xi_0)\|_2^2+\|(f-f_n)\xi_0\|_2^2\\
\leqslant&\|f-f_n\|^2\|\xi-\xi_0\|_2^2+\frac{\varepsilon}{2}\\
\leqslant&4\|f\|^2\left(\frac{\varepsilon}{8\|f\|^2}\right)+\frac{\varepsilon}{2}\\
=&\varepsilon,
\end{aligned}$$

then $f_n \xrightarrow{s} f$ which in turn implies weak convergence. ∎

Remark 7.5.5 *The above proof shows also that the strong Borel envelope of $L(\mathcal{B}_c(E, \sigma))$ is $M(E, \sigma)$.*

7.6 The Borel $*$-Algebra of a Borel Twist

This section generalize to the Borel case the work of A. Kumjian in [Ku]. We associate to a Borel twist Γ a Borel $*$-algebra $\mathcal{B}_r^*(\Gamma)$. Recall (see Section 5.3) that, the Borel twist Γ over E is given by the following diagram

$$\mathbb{T} \times X \longrightarrow \Gamma \underset{s}{\overset{q}{\rightleftarrows}} E$$

where $(X, \mathscr{A})$ is a standard Borel space, E a countable Borel equivalence relation on X such that $E^{(0)} = X$, $q : \Gamma \longrightarrow E$ is the quotient map and $s : E \longrightarrow \Gamma$ is a Borel cross-section. Moreover $q(\mathbb{T} \times X) = E^{(0)} = X$. Recall from Theorem 5.3.2, any twist Γ over E is induced by a Borel 2-cocycle $\sigma \in H^2(E, \mathbb{T})$.

Definition 7.6.1

1) *The **equivariant Borel functions** over Γ, denoted $\mathcal{B}(\Gamma)$, are defined as*

$$\mathcal{B}(\Gamma) = \{f : \Gamma \to \mathbb{C} \text{ Borel}; \forall_{\lambda \in \mathbb{T}} \forall_{\gamma \in \Gamma} (\lambda f(\gamma) = f(\lambda\gamma))\}.$$

2) *For $f \in \mathcal{B}(\Gamma)$, let*

$$\|f\|_\infty = \sup_{\gamma \in \Gamma} \{|f(\gamma)|\}.$$

*The **bounded equivariant Borel functions** over Γ, denoted $\mathcal{B}_o(\Gamma)$, are the functions $f \in \mathcal{B}(\Gamma)$ such that $\|f\|_\infty < \infty$.*

Remark 7.6.2 *If $\gamma \in supp(f) = \{\gamma \in \Gamma; f(\gamma) \neq 0\}$, then for all $\lambda \in \mathbb{T}$*

$$f(\lambda\gamma) = \lambda f(\gamma) \neq 0,$$

thus $\lambda\gamma \in supp(f)$.

Definition 7.6.3 *Let $f \in \mathcal{B}(\Gamma)$. The **band** of f, denoted band(f), is*

$$band(f) = \sup_{\mathbb{N} \cup \{\infty\}} \{\#\{g \in E; r(s(g)) = d(\gamma) \text{ and } f(\gamma s(g)) \neq 0\} +$$

$$\#\{g \in E; d(s(g)) = r(\gamma) \text{ and } f(s(g)\gamma) \neq 0\}\}.$$

The ***band limited bounded equivariant Borel functions*** *over* Γ *are defined as*

$$\mathcal{B}_c(\Gamma) = \{f \in \mathcal{B}_o(\Gamma); band(f) \text{ is finite}\}.$$

Definition 7.6.4 *The vector space* $\mathcal{B}_c(\Gamma)$ *becomes a* $*$*-algebra with the following operations:*

i) multiplication: $(f \cdot h)(\gamma) = \sum_{g \in E^{d(q(\gamma))}} f(\gamma s(g))h(s(g)^{-1}),$

ii) involution: $f^*(\gamma) = \overline{f(\gamma^{-1})}.$

Definition 7.6.5

1) *The* ***equivariant normalizer functions*** *of* Γ*, denoted* $N(\Gamma)$*, are functions* $u \in \mathcal{B}_o(\Gamma)$ *such that* $u^*u = uu^* = 1$ *and* $q(supp(u)) = graph(\phi^{-1})$ *for some* $\phi \in [E]$*. It follows that if* $u \in N(\Gamma)$ *then* $band(u) = 1$*, hence* $N(\Gamma) \subset \mathcal{B}_c(\Gamma)$*.*

2) *The* ***equivariant diagonal functions*** *of* Γ*, denoted* $D(\Gamma)$*, are functions* $f \in \mathcal{B}_o(\Gamma)$ *such that* $q(supp(u)) \subset X$*. It follows that if* $f \in D(\Gamma)$ *and* $f \neq 0$*, then* $band(f) = 1$*, hence* $D(\Gamma) \subset \mathcal{B}_c(\Gamma)$*.*

Proposition 7.6.6 *There is a natural isomorphism between* $\mathcal{B}_c(\Gamma)$ *to* $\mathcal{B}_c(E, \sigma)$*.*

Proof: Let $f \in \mathcal{B}_c(\Gamma)$, $f' \in \mathcal{B}_c(E, \sigma)$, $g \in E$ and $\gamma \in q^{-1}(g)$ with $\gamma = k_\gamma s(g) \in \Gamma$. Define $Q : \mathcal{B}_c(\Gamma) \to \mathcal{B}_c(E, \sigma)$ by

$$Q(f)(g) = f(s(g))$$

In this case $Q^{-1} : \mathcal{B}_c(E, \sigma) \to \mathcal{B}_c(\Gamma)$ is

$$Q^{-1}(f')(\gamma) = k_\gamma f'(g).$$

Then Q can be used to show that $\mathcal{B}_c(\Gamma)$ and $\mathcal{B}_c(E, \sigma)$ are isomorphic as $*$-algebra. ∎

For any $x \in X$, we consider the vector space,

$$L^2(\Gamma, E, \delta_x) = \{\xi \in \mathcal{B}_o(\Gamma) : Q(\xi) \in \mathcal{H}_{\delta_x}\}.$$

Then we have:

Lemma 7.6.7

1) Endowed with the inner product,

$$\langle \xi | \eta \rangle_x = \sum_{g \in E_x} \overline{\xi(s(g))} \eta(s(g)),$$

ξ and η in $L^2(\Gamma, E, \delta_x)$, then $L^2(\Gamma, E, \delta_x)$ is a Hilbert space.

2) $\{\xi_g\}_{g \in E_x}$, with $\xi_g = Q^{-1}(\mathcal{X}_g)$ where $\mathcal{X}_g \in \mathcal{B}_c(E, \sigma)$ is the characteristic function of the point $g \in E$, forms an orthogonal basis for $L^2(\Gamma, E, \delta_x)$.

3) Since $\xi_g \in \mathcal{B}_c(\Gamma)$, then for any $x \in X$, $\mathcal{B}_c(\Gamma)$ is a dense subspace in $L^2(\Gamma, E, \delta_x)$.

For all $x \in X$, we now define a representation L_{δ_x} of $\mathcal{B}_c(\Gamma)$ in $L^2(\Gamma, E, \delta_x)$ by,

$$(L_{\delta_x}(f)\xi)(\gamma) = \sum_{g \in E^x} f(\gamma s(g)) \xi(s(g)^{-1}).$$

Definition 7.6.8 *Let*

$$(L, \ell^2(\Gamma)) = \bigoplus_{x \in X} (L_{\delta_x}, L^2(\Gamma, E, \delta_x)).$$

The ***reduced Borel $*$-algebra associate to*** *Γ, denoted $\mathcal{B}_r^*(\Gamma)$, is the Borel envelope of $L(\mathcal{B}_c(\Gamma))$ in $\mathcal{B}(\ell^2(\Gamma))$.*

7.7 The Feldman-Moore Theorem for Borel Twist

Throughout this section, A denotes a (weakly sequentially closed) Borel $*$-algebra with a (Borel) Cartan subalgebra such that $B \cong \mathcal{B}_o(X)$, where X is a standard Borel space (X is also the space of sequentially normal pure states of B) and such that A is countably B-generated. We shall refer to this pair (A, B) as a Borel Cartan pair.

Let $N_A(B)$ the the normalizer of B in A. Recall that for any $u \in N_A(B)$ we can associate a Borel automorphism $\phi_u : X \to X$, such that $u^* f u = f \circ \phi_u$ for all $f \in B$. Let $\Delta : A \to B$ be the sequentially normal conditional expectation from A to B. For any $x \in X$, then the pure state $x \circ \Delta : A \to \mathbb{C}$ is the unique (by Proposition 6.4.14) extension of x to A. To simplify notation, we shall use again x for the corresponding pure state on A.

We first associate a Borel twist to the Borel Cartan pair (A, B).

Notation 7.7.1 *Let (A, B) be a Borel Cartan pair. Let us denote by*

1) $\Gamma(B) = \{[u, x] \in A^*; u \in N_A(B), x \in X\}$, *where* $[u, x](a) = x(u^* a)$, $a \in A$,

2) $F(B) = \{[u, x] \in A^*; u \in N_A(B), x \in X, \phi_u(x) = x\}$,

3) $E(B) = \{q([u, x]) = (\phi_u(x), x) \in X \times X; [u, x] \in \Gamma(B)\}$.

Lemma 7.7.2 *Let (A, B) be a Borel Cartan pair. If $[u, x] \in \Gamma(B)$, then $x(u) \neq 0$ if and only if $\phi_u(x) = x$.*

Proof: If $\phi_u(x) \neq x$, then

$$x(u) = x(\mathcal{X}_x u) = x(u u^* \mathcal{X}_x u) = x(u \mathcal{X}_{\phi_u(x)}) = x(u) x(\mathcal{X}_{\phi_u(x)}) = 0.$$

If $\phi_u(x) = x$, then $u^* \mathcal{X}_x u = \mathcal{X}_x$, hence $\mathcal{X}_x u = u \mathcal{X}_x$. Let us show first that $u \mathcal{X}_x = x(u) \mathcal{X}_x$. Let $f, g \in B$. Then $g \mathcal{X}_x = f \mathcal{X}_x$ if and only if $g(x) = f(x)$. We have

$$(u \mathcal{X}_x) f = u f \mathcal{X}_x$$

$$= uf\mathcal{X}_x u^* u$$
$$= (f \circ \phi_u^{-1})\mathcal{X}_{\phi_u^{-1}(x)} u$$
$$= f\mathcal{X}_x u$$
$$= f(u\mathcal{X}_x)$$

and B is maximal abelian, thus $u\mathcal{X}_x$ is in B. If $y \neq x$ then $y(x(u)\mathcal{X}_x) = y(u\mathcal{X}_x) = 0$ and $x(u\mathcal{X}_x) = x(x(u)\mathcal{X}_x) = x(u)$. This shows that

$$u\mathcal{X}_x = x(u)\mathcal{X}_x$$
$$\mathcal{X}_x u^* = \mathcal{X}_x x(u^*).$$

Since $x(u^*\mathcal{X}_x u) = x(\mathcal{X}_x) = 1$ and $x(u^*\mathcal{X}_x u) = x(x(u^*)\mathcal{X}_x x(u)) = x(u^*)x(u)$, then $\overline{x(u)}x(u) = 1$, which implies that $x(u) \in \mathbb{T}$. ∎

Proposition 7.7.3 *Let (A, B) be a Borel Cartan pair. Then $\Gamma(B)$ can be endowed with a Borel twist structure over $E(B)$.*

Proof: Equip $\Gamma(B)$ with the following operations:

1) (Product) $[v, y][u, x] = [vu, x]$ if and only if $\phi_u(x) = y$.

2) (Inverse) $[u, x]^* = [u^*, \phi_u(x)]$.

The the range and the source maps are given by:

$$r([u, x]) = [u, x][u, x]^* = [1, \phi_u(x)]$$

and

$$d([u, x]) = [u, x]^*[u, x] = [1, x],$$

and the object $\Gamma(B)^{(0)}$ can be identified by X. Then Γ becomes a free $\mathbb{T}$-space under the action given, for $\lambda \in \mathbb{T}$, by

$$\lambda[u, x] = [\overline{\lambda}u, x].$$

Since A is countably B-generated, let U be a countable group of unitaries in $N_A(B)$ generating A. Define

$$E = \bigcup_{u \in U} \operatorname{graph}(\phi_u^{-1}) \subset X \times X,$$

since ϕ_u are Borel automorphisms and U is countable, then $E = E(B)$ is a countable Borel equivalence relation. Let $q : \Gamma(B) \to E(B)$ be defined by

$$q([u, x]) = (\phi_u(x), x).$$

The map q is $\mathbb{T}$-invariant, since $\phi_u = \phi_{\lambda u}$. Moreover $q([u, x]) = (x, x)$, a unit of $E(B)$, if and only if $[u, x] \in F(B)$. Recall by construction that $x(a) = x(\mathcal{X}_x a \mathcal{X}_x)$. Let $[u, x]$ be such that $\phi_u(x) = x$, then we want to show that $[u, x] = [x(u), x]$.

By Lemma 7.7.2, if $\phi_u(x) = x$, then $x(u)\mathcal{X}_x = u\mathcal{X}_x$. Now this implies that for any $a \in A$, then

$$\begin{aligned} [u, x](a) &= x(u^* a) \\ &= x(\mathcal{X}_x u^* a \mathcal{X}_x) \\ &= x(\mathcal{X}_x x(u^*) a \mathcal{X}_x) \\ &= x(x(u^*) a) \\ &= [x(u), x](a) \end{aligned}$$

which shows that $[u, x] = [x(u), x] = x(u^*)[1, x]$. Thus an element of $F(B)$ can be written as an element of $\mathbb{T} \times X$.

Now if $[u, x]$ and $[v, z]$ are two elements of Γ such that $q([u, x]) = q([v, z])$, then $z = x$ and $\phi_u(x) = \phi_v(x)$. Moreover $[v^*, \phi_v(x)][u, x] = [v^* u, x]$, with $\phi_{v^* u}(x) = x$, then $[v^* u, x] = x(u^* v)[1, x]$ and $[u, x] = x(u^* v)[v, x]$. In particular, $[u, x] = [v, x]$ if and only if $x(v^* u) = 1$. It follows that $\Gamma(B)$ is a Borel twist. ∎

The following proposition is a Borel analogue of Proposition 4.15 of [Re2]. It shows that for a Borel twist (Γ, E) the Borel twist associated to the Borel Cartan pair $(\mathcal{B}_r^*(\Gamma), L(\mathcal{B}_o(X)))$ is again (Γ, E).

Proposition 7.7.4 *Keeping the notation of Proposition* 7.7.3*, let* (Γ, E) *be a Borel twist. Let* $A = \mathcal{B}_r^*(\Gamma)$ *and* $B = L(\mathcal{B}_o(X))$*. We have an isomorphism of Borel twist:*

$$\begin{array}{ccccc} F(B) & \xrightarrow{\hat{\iota}} & \Gamma(B) & \overset{\hat{q}}{\underset{\hat{s}}{\rightleftarrows}} & E(B) \\ \downarrow & & \downarrow \Phi & & \downarrow \\ \mathbb{T} \times X & \xrightarrow{\iota} & \Gamma & \overset{q}{\underset{s}{\rightleftarrows}} & E \end{array}$$

Proof: We verified already in the previous proposition that $E(B) = E$.

Let $x \in X$, $\gamma \in \Gamma$, with $\gamma = k_\gamma s(q(\gamma)))$ and $k_\gamma \in \mathbb{T}$, such that $d(\gamma) = x$, $r(\gamma) = y$ and $\phi \in [E]$, with $\phi(x) = y$. In general for any $\phi \in [E]$, let $u_\phi \in N(\Gamma)$ be such that

$$u_\phi(\gamma) = \begin{cases} k_\gamma & \text{if } \phi(d(\gamma)) = r(\gamma) \\ 0 & \text{else} \end{cases}.$$

Let $u \in N_A(B)$. Let $\Phi : \Gamma(B) \to \Gamma$ be defined by

$$\Phi([u, x]) = u(\hat{s}(\phi_u(x), x))\hat{s}(\phi_u(x), x).$$

The map Φ is a groupoid homomorphism: For $[u, x]$,

$$\begin{aligned} \Phi([u, x])^* &= (u(s((\phi_u(x), x)))s((\phi_u(x), x)))^* \\ &= \overline{(u(s((\phi_u(x), x)))}s((\phi_u(x), x))^{-1} \\ &= u^*(s((\phi_u(x), x))^{-1})s((\phi_u(x), x))^{-1} \\ &= u^*(s((x, \phi_u(x))))s((x, \phi_u(x))) \\ &= \Phi([u^*, \phi_u(x)]), \end{aligned}$$

and now with $[v, y]$, $\phi_u(x) = y$ and $\phi_v(y) = z$,

$$\Phi([vu, x]) = (vu)(s((\phi_{uv}(x), x)))s((\phi_{uv}(x), x))$$

$$
\begin{aligned}
&= \sum_{w \in [x]_E} v(s((z,x))s((w,x)))u(s((w,z)))s((z,x)) \\
&= v(s((z,x))s((x,y)))u(s((y,x)))s((\phi_{uv}(x),x)) \\
&= v(s((z,x))s((x,y)))u(s((y,x)))k^{-1}s((z,y))s((y,x)) \\
&= v(k^{-1}s((z,x))s((x,y)))s((z,y))u(s((y,x)))s((y,x)) \\
&= v(s((z,y)))s((z,y))u(s((y,x)))s((y,x)) \\
&= \Phi([v,y])\Phi([u,x]).
\end{aligned}
$$

Moreover $\Phi([u,x]) \in \mathbb{T} \times X$ if and only if $[u,x] \in F(B)$.

Let $\Psi : \Gamma \to \Gamma(B)$ be defined by

$$\Psi(\gamma) = [u_\phi(\gamma)u_\phi, x],$$

where $\phi \in [E]$ is chosen such that $(\phi(x), x) = (r(\gamma), d(\gamma))$. The map Ψ does not depend on the choice of ϕ and is a groupoid homomorphism. Moreover, we get $(\Psi \circ \Phi)([u,x]) = [u,x]$ and $(\Phi \circ \Psi)(\gamma) = \gamma$, since

$$
\begin{aligned}
(\Psi \circ \Phi)([u,x]) &= \Psi(u(s(\phi_u(x),x))s((\phi_u(x),x))) \\
&= [u_\phi(u(s(\phi_u(x),x))s((\phi_u(x),x)))u_\phi, x] \\
&= [u(s(\phi_u(x),x))u_\phi(s((\phi_u(x),x)))u_\phi, x] \\
&= [u(s(\phi_u(x),x))u_\phi, x] \\
&= [u,x]
\end{aligned}
$$

and

$$
\begin{aligned}
(\Phi \circ \Psi)(\gamma) &= \Phi([u_\phi(\gamma)u_\phi, x]) \\
&= u_\phi(\gamma)u_\phi(s((\phi(x),x)))s((\phi(x),x)) \\
&= k_\gamma u_\phi(s((\phi(x),x)))s((\phi(x),x)) \\
&= k_\gamma((\phi(x),x))
\end{aligned}
$$

$$= \gamma.$$

∎

Theorem 7.7.5 *Let A be a Borel $*$-algebra with a Cartan subalgebra $B \cong \mathcal{B}_o(X)$, where X is a standard Borel space (X is also the space of sequentially normal pure states of B), such that A is countably B-generated. Then there exists (E, σ), E unique up to isomorphism and σ unique up to a 2-coboundary, such that $A \cong \mathcal{B}_r^*(E, \sigma)$ and where the isomorphism sends B onto $L(\mathcal{B}_o(X))$.*

Proof:

We divide the proof in four steps:

1. By Proposition 7.7.3, we associate to the Borel Cartan pair (A, B) a Borel twist $\Gamma = \Gamma(B)$ over the countable Borel equivalence relation $E = E(B)$.

2. To any operator, we associate an equivariant (bounded) Borel function over Γ.

3. Show that (A, B) and $(\mathcal{B}_r^*(\Gamma), L(\mathcal{B}_o(X)))$ can be realized as algebras of operators acting in spatially equivalent Hilbert spaces.

4. By Theorem 5.3.2, there exists a normalized 2-cocycle σ such that $\Gamma \cong (E, \sigma)$.

The steps **2.** and **3.** need to be verified.

2. For any $a \in A$, we denote by $\mathcal{L}(a) : \Gamma \to \mathbb{C}$ the equivariant complex valued function defined by

$$\mathcal{L}(a)([v, x]) = [v, x](a).$$

By Lemma 7.7.2, if $\mathcal{L}(N_A(B)) \subseteq N(\Gamma)$ and $\mathcal{L}(B) \subseteq D(\Gamma)$. Let us show that $\mathcal{L}(N_A(B)) = N(\Gamma)$.

We want to show that for any element $\tilde{u} \in N(\Gamma)$, there is $u \in N_A(B)$ such that $\mathcal{L}(u) = \tilde{u}$. By hypothesis on A, let $\{u_{\phi_n}\}_{n\in\mathbb{N}}$ of $N_A(B)$ be a countable subset which together with B generated A. Let E be the countable Borel equivalence relation

$$E = \bigcup_{n\in\mathbb{N}} \operatorname{graph}(\phi_n^{-1}).$$

We can assume that $u_{\phi_0} = 1$. Now set

$$A_0 = \{x \in X; \phi(x) = x\}$$

and $B_0 = A_0$ and for $k > 0$ define inductively

$$A_n = \{x \in X \setminus B_{n-1}; \phi(x) = \phi_n(x)\} \text{ and } B_n = \bigcup_{i=0}^{n} A_i.$$

Then $\{A_n\}_{n\in\mathbb{N}}$ forms a Borel partition of the space X. Let $h_n \in B$ be defined by

$$x(h_n) = \tilde{u}([u_{\phi_n}, x])\mathcal{X}_{A_n}(x),\ x \in X,$$

and

$$g_n = u_{\phi_n} h_n \in A.$$

By construction, the elements

$$a_N = \sum_{n=0}^{N} |g_n| = \sum_{n=0}^{N} \sqrt{g_n^* g_n} = \sum_{n=0}^{N} \sqrt{h_n^* u_{\phi_n}^* u_{\phi_n} h_n} = \sum_{n=0}^{N} \mathcal{X}_{A_n} = \mathcal{X}_{\left(\bigcup_{n=0}^{N} A_n\right)} = \mathcal{X}_{B_{N-1}}$$

and

$$b_N = \sum_{n=0}^{N} |g_n^*| = \sum_{n=0}^{N} \sqrt{g_n g_n^*} = \sum_{n=0}^{N} \sqrt{u_{\phi_n} h_n h_n^* u_{\phi_n}^*} = \sum_{n=0}^{N} \sqrt{u_{\phi_n} \mathcal{X}_{A_n} u_{\phi_n}^*} = \sum_{n=0}^{N} \sqrt{\mathcal{X}_{\phi_n(A_n)}}$$
$$= \sum_{n=0}^{N} \mathcal{X}_{\phi(A_n)} = \mathcal{X}_{\phi\left(\bigcup_{n=0}^{N} A_n\right)} = \mathcal{X}_{\phi(B_{N-1})}$$

are such that $a_N \nearrow 1$ and $b_N \nearrow 1$ strongly, thus by Theorem 6.3.19, $\sum_{n=0}^{N} g_n$ is convergent to an element $u = \sum_{n=0}^{\infty} g_n$ in A and, the element $u \in N_A(B)$ is such that $\mathcal{L}(u) = \tilde{u}$.

The map $\mathcal{L}$ is clearly linear. Moreover $\mathcal{L}(a)^* = \mathcal{L}(a^*)$, for any $a \in A$, since

$$\begin{aligned}
\mathcal{L}(a)^*([v,x]) &= \overline{\mathcal{L}(a)([v^*, \phi_v(x)])} \\
&= \phi_v(x)(va) \\
&= \phi_v(x)(a^*v^*) \\
&= \phi_v(x)(vv^*a^*v^*) \\
&= x(v^*a^*) \\
&= \mathcal{L}(a^*)([v,x]).
\end{aligned}$$

Let $[v,x] \in \Gamma$, $b \in B$, $u \in N_A(B)$ and $a \in A$. First $\mathcal{L}(ab) = \mathcal{L}(a) \cdot \mathcal{L}(b)$, since

$$\mathcal{L}(ab)([v,x]) = x(v^*ab)$$

and

$$\begin{aligned}
(\mathcal{L}(a) \cdot \mathcal{L}(b))([v,x]) &= \sum_{g \in E^x} \mathcal{L}(a)([v,x]c(g))\mathcal{L}(b)(c(g)^{-1}) \\
&= \mathcal{L}(a)([kv,x])kx(b) \\
&= x(\overline{k}v^*a)kx(b) \\
&= x(v^*ab).
\end{aligned}$$

We also have $\mathcal{L}(au) = \mathcal{L}(a) \cdot \mathcal{L}(u)$, since

$$\mathcal{L}(au)([v,x]) = x(v^*au)$$

and

$$\begin{aligned}
(\mathcal{L}(a) \cdot \mathcal{L}(u))([v,x]) &= \sum_{g \in E^x} \mathcal{L}(a)([v,x]c(g))\mathcal{L}(u)(c(g)^{-1}) \\
&= \mathcal{L}(a)([kvu^*, \phi_u(x)])k\mathcal{L}(u)([u,x]) \\
&= \phi_u(x)(\overline{k}uv^*a)k \\
&= x(u^*uv^*au)
\end{aligned}$$

$$= x(v^*au).$$

If A_c denote the linear span of $\{B, N_A(B)\}$, then by restriction the map L is a $*$-homomorphism between A_c and $\mathcal{B}_c(\Gamma)$.

3. Since the conditional expectation $\Delta : A \to B$ is faithful and $x : A \to \mathbb{C}$ is a sequentially normal pure state then, by using the GNS construction, the representation

$$(\pi, \mathcal{H}) = \bigoplus_{x \in X} (\pi_x, \mathcal{H}_x)$$

is a faithful sequentially normal representation of A. Thus for all $a \in A$ we have

$$\|a\| = \sup_{x \in X} \{ \sup_{\substack{d \in \mathcal{H}_x \\ x(d^*d) \leq 1}} \{ \|ad\|_x = x(d^*a^*ad) \} \}.$$

Thus we identify A with $\pi(A) \subset \mathcal{B}(\mathcal{H})$.

Recall that by Lemma 7.6.7 $\{\xi_g\}_{g \in E_x}$ is an orthonormal basis for the Hilbert space $L^2(\Gamma, E, \delta_x)$. For any element ξ_g of the basis $\{\xi_g\}_{g \in E_x}$, by definition, there is an element $[u_g, x] \in \Gamma$ such that $\xi_g([u_g, x]) = 1$, thus by taking $\mathcal{X}_x \in B$ and $u_g \in N_A(B)$, then $d_g = u_g \mathcal{X}_x$, is such that $\mathcal{L}(d_g) = \xi_g$. Thus for any vector

$$\xi = \sum_{g \in E_x} \lambda_g \xi_g \ \in L^2(\Gamma, E, \delta_x),$$

we can associate a vector

$$d = \sum_{g \in E_x} \lambda_g d_g \ \in \mathcal{H}_x,$$

via the same map $\mathcal{L}$, which is a linear surjective isometry from $\mathcal{H}_x$ to $L^2(\Gamma, E, \delta_x)$. Since $\{\xi_g\}_{g \in E_x}$ is an orthonormal basis for $\mathcal{H}_x$, then $\{\xi_g\}_{g \in E_x}$ is an orthonormal basis for $L^2(\Gamma, E, \delta_x)$.

For any $x \in X$, we get a map

$$\mathcal{L}_x : \mathcal{H}_x \to L^2(\Gamma, E, \delta_x)$$

such that $\forall d \in \mathcal{H}_x$,

$$\mathcal{L}_x(\pi_x(a)d) = L_{\delta_x}(\mathcal{L}(a))\mathcal{L}_x(d).$$

The matrix representation for both operators $\pi_x(a)$ in A and $L_{\delta_x}(\mathcal{L}(f))$ in $\mathcal{B}(L^2(\Gamma, E, \delta_x))$ are the identical, since

$$\begin{aligned}\pi_x(f)_{i,j} &= x(d_i^* f d_j) \\ &= \mathcal{L}(d_i^* f d_j)([1, x]) \\ &= (\mathcal{L}(d_i^*) \cdot \mathcal{L}(f d_j))([1, x]) \\ &= \langle \mathcal{L}(d_i) | \mathcal{L}(f) \cdot \mathcal{L}(d_j) \rangle_x \\ &= \langle \xi_i | \mathcal{L}(f) \xi_j \rangle_x \\ &= L_x(\mathcal{L}(f))_{i,j}.\end{aligned}$$

Thus $\|f\| = \|\mathcal{L}(f)\|$. The map $\mathcal{L}_x : d \mapsto \xi$ is an unitary between the Hilbert spaces $\mathcal{H}_x$ and $L^2(\Gamma, E, \delta_x)$ which make them spatially isomorphic. The map $\mathcal{L}$ is an unitary between the Hilbert bundle spaces $\mathcal{H}$ and $\ell^2(\Gamma)$ which make them spatially isomorphic.

Since A_c and $L(\mathcal{B}_c(\Gamma))$ are isomorphic $*$-algebras acting in spatially equivalent Hilbert spaces then their corresponding Borel closures are isomorphic, i.e.,

$$A \cong \mathcal{B}_r^*(\Gamma).$$

Remark that the uniqueness of (E, σ), up to isomorphism, follows from Proposition 7.7.4.

∎

Remark 7.7.6 *The previous proof is the Borel analogue of Theorem 1° of Section 3 of [Ku], where the unicity part was adapted from [Re2], and some ideas came from the lines of the proof in* [Re1] *(Theorem p.439).*

7.8 Invariants of Borel Equivalence Relations

In this section, we present two results on invariants of countable Borel equivalence relation. The first one is a particular case of Theorem 1.1 of [MRo]. Our second result uses isomorphism of Cartan pairs between Borel $*$-algebras.

Theorem 7.8.1 *([MRo], Theorem 1.1) Let E_1 and E_2 be two countable Borel equivalence relations on $(X, \mathscr{A})$ such that $|[x]_{E_i}| \geqslant 3$, for all $x \in X$ and $i = 1, 2$. The following are equivalent.*

1) E_1 is isomorphic to E_2, and

2) $[E_1]$ is isomorphic to $[E_2]$.

The cardinality condition of the previous theorem cannot be removed. Here is an example given by [M].

Example 7.8.2 *Let $\phi : X \to X$ and $\psi : X \to X$ be two Borel automorphisms such that $\phi \circ \phi = I_X$ and $\psi \circ \psi = I_X$ (i.e., there is at most two points in the orbit of any points). Denote by $A_2, B_2 \in \mathcal{A}$ the points with exactly two points in their orbits for ϕ and ψ respectively. Suppose A_2 and B_2 are infinite with different cardinalities. Then $[E_\phi]$ and $[E_\psi]$ are both isomorphic to the unique abelian group, with the cardinality of the continuum, whose elements are of order at most 2, but the standard Borel spaces X/E_ϕ and X/E_ψ are not isomorphic, as A_2/E_ϕ and B_2/E_ψ have different cardinalities.*

As a consequence of Theorem 7.7.5, we have the following invariant:

Corollary 7.8.3 *Let E_1 and E_2 be two countable Borel equivalence relations on $(X, \mathscr{A})$. The following are equivalent:*

1) E_1 is isomorphic to E_2, and

2) $(\mathcal{B}_r^*(E_1), \mathcal{B}_o(X))$ *is isomorphic to* $(\mathcal{B}_r^*(E_2), \mathcal{B}_o(X))$.

Proof: This follows by the unicity of the (in this case trivial) Borel twist associated to a Borel Cartan pair of Theorem 7.7.5. ∎

Chapter 8

A Borel Krieger Type Theorem

Let $(X, \mathscr{A})$ be a standard Borel space and μ a measure on X. Recall that a Borel automorhism $T : X \to X$ is a **nonsingular transformation** (here we use the terminology transformation for an automorphism) if for any $B \in \mathscr{A}$,

$$\mu(T^{-1}(B)) = 0 \text{ iff } \mu(B) = 0.$$

T is an **ergodic transformation** if for any $B \in \mathscr{A}$,

$$T^{-1}(B) = B \text{ implies } \mu(B) = 0 \text{ or } \mu(B^c) = 0.$$

In [Kr1], W. Krieger associates to any nonsingular transformation T acting ergodically on a Lebesgue space (X, μ) an ergodic $\mathbb{R}$-flow, the so-called associated flow. If $W^*(X, \mu, T)$ denotes the von Neumann algebra associated to the ergodic dynamical system (X, μ, T), then Krieger proves:

Theorem. *Let (X_i, μ_i, T_i), $i = 1, 2$, be two nonsingular ergodic transformations. The following statements are equivalent:*

1) (X_1, μ_1, T_1) and (X_2, μ_2, T_2) are measurably orbit equivalent.

2) $W^*(X_1, \mu_1, T_1)$ and $W^*(X_2, \mu_2, T_2)$ are isomorphic.

3) The associated flows of (X_1, μ_1, T_1) and (X_2, μ_2, T_2) are conjugate.

As any ergodic nonsingular dynamical system (X, μ, T) is orbit equivalent to an ergodic hyperfinite countable equivalence relation E_T on (X, μ), Krieger's theorem can be stated as follows:

Theorem. *Let (X_i, μ_i, T_i), $i = 1, 2$, be two nonsingular ergodic transformations. The following statements are equivalent:*

1) E_{T_1} and E_{T_2} are orbit equivalent.

2) $W^*(X_1, E_{T_1})$ and $W^*(X_2, E_{T_2})$ are isomorphic.

3) The associated flows of (X_1, μ_1, E_{T_1}) and (X_2, μ_2, E_{T_2}) are conjugate.

In Section 8.4, we will prove a Krieger type theorem in the Borel setting. To get this result, we first make, in Section 8.1, the connection between invariant (ergodic) probability measures on E and (extremal) normalized sequentially normal traces of $\mathcal{B}_r^*(E, \sigma)$. This correspondence established, we recall and use, in Section 8.2, the Dougherty-Jackson-Kechris classification theorem of aperiodic non-smooth hyperfinite countable Borel equivalence relations to get a classification of their Borel $*$-algebras. The classification of smooth countable Borel equivalence relations classification is covered in Section 8.3.

8.1 Traces on $\mathcal{B}_r^*(E,\sigma)$

Let E be a countable Borel equivalence relation on $(X,\mathscr{A})$ and σ be a normalized $\mathbb{T}$-valued 2-cocycle on E.

If Δ denotes the sequentially normal conditional expectation from $\mathcal{B}_r^*(E,\sigma)$ onto $L(\mathcal{B}_o(X))$, defined in Proposition 7.2.14, and if μ is an E-invariant measure on $(X,\mathscr{A})$, we verify in Lemma 8.1.2 that

$$\tau_\mu(\hat{f}) = \int \Delta(\hat{f})d\mu, \quad \hat{f} \in \mathcal{B}_r^*(E,\sigma),$$

defines a sequentially normal trace on $\mathcal{B}_r^*(E,\sigma)$ and we show in Proposition 8.1.3, that $\mu \mapsto \tau_\mu$ induces a bijective correspondence between E-invariant probability measures on $(X,\mathscr{A})$ and normalized sequentially normal traces on $\mathcal{B}_r^*(E,\sigma)$.

Definition 8.1.1 *Let $\mathcal{B}$ be a Borel $*$-algebra. A **normalized sequentially normal trace** τ is a sequentially normal state of $\mathcal{B}$ such that $\tau(x^*x) = \tau(xx^*)$, for all $x \in \mathcal{B}$. The set of all normalized sequentially normal traces of $\mathcal{B}$ is denoted $\mathcal{T}(\mathcal{B})$. The extreme points of $\mathcal{T}(\mathcal{B})$ are called the (normalized sequentially normal) **characters**. The set of all normalized sequentially normal characters of $\mathcal{B}$ is denoted $\partial\mathcal{T}(\mathcal{B})$.*

Lemma 8.1.2 *Let E be a countable Borel equivalence relation on $(X,\mathscr{A})$ and σ be a normalized $\mathbb{T}$-valued 2-cocycle on E. If μ is an E-invariant measure on $(X,\mathscr{A})$ and Δ denotes the sequentially normal conditional expectation from $\mathcal{B}_r^*(E,\sigma)$ onto $L(\mathcal{B}_o(X))$, then τ_μ belongs to $\mathcal{T}(\mathcal{B}_r^*(E,\sigma))$.*

Proof: As Δ and integration with respect to μ are linear and preserve positivity, τ_μ is a positive linear form. Moreover, as Δ is sequentially normal, $\Delta(I) = 1$ and as μ is a probability measure, τ_μ is a sequentially normal state. Let us check that τ_μ is a trace. Let $\hat{f} \in \mathcal{B}_r^*(E,\sigma)$ and $f = c(\hat{f})$ be the coordinate function of $\hat{f}$. By Proposition 7.2.14, we have

$$\Delta(\hat{f}\hat{f}^*) = L((f \cdot f^*)_\Delta)$$

and

$$(f \cdot f^*)(x,x) = \sum_{z\in[x]} f(x,z)f^*(z,x).$$

Hence

$$\tau_\mu(\hat{f}\hat{f}^*) = \int \sum_{z\in[x]} |f(x,z)|^2 d\mu(x) = \nu(|f|^2),$$

and similarly

$$\tau_\mu(\hat{f}^*\hat{f}) = \int \sum_{z\in[x]} |f(z,x)|^2 d\mu(x) = \nu^{-1}(|f|^2).$$

As μ is E-invariant, $\nu = \nu^{-1}$ and therefore τ_μ is a trace. ∎

Proposition 8.1.3 *Let E be a countable Borel equivalence relation on $(X, \mathscr{A})$ and σ be a normalized $\mathbb{T}$-valued 2-cocycle on E. If $\mathcal{I}_E$ denotes the set of E-invariant probability measures on $(X, \mathscr{A})$, then the convex map*

$$\mu \mapsto \tau_\mu$$

is a bijective correspondence between $\mathcal{I}_E$ and $\mathcal{T}(\mathcal{B}_r^*(E,\sigma))$.

Proof: Let $\tau \in \mathcal{T}(\mathcal{B}_r^*(E,\sigma))$, and μ be defined by

$$\mu_\tau(A) = \tau(L(\mathcal{X}_A)),\ A \in \mathscr{A}.$$

As τ is a normalized normal state, μ is a probability measure on $\mathscr{A}$.

Let $\phi \in [[E]]$ be a partial Borel transformation from A to B and let u_ϕ be the partial isometry of $\mathcal{B}_r^*(E,\sigma)$ such that $u_\phi^* u_\phi = L(\mathcal{X}_A)$ and $u_\phi u_\phi^* = L(\mathcal{X}_B)$. Then

$$\mu_\tau(A) = \tau(L(\mathcal{X}_A)) = \tau(u_\phi u_\phi^*) = \tau(u_\phi^* u_\phi) = \tau(L(\mathcal{X}_B)) = \mu_\tau(B).$$

This shows that $\mu_\tau \in \mathcal{I}_E$.

As $\tau_{\mu_\tau} = \tau$ and $\mu_{\tau_\mu} = \mu$, and as the correspondance $\mu \in \mathcal{I}_E \mapsto \tau \in \mathcal{T}(\mathcal{B}_r^*(E,\sigma))$ preserves convex combinations, the proposition is proved. ∎

As the set of E-invariant and ergodic probability measures of $(X,\mathscr{A})$ are the extreme points of $\mathcal{I}_E$, we have:

Corollary 8.1.4 *Let E be a countable Borel equivalence relation on $(X,\mathscr{A})$ and σ be a normalized $\mathbb{T}-$valued 2-cocycle of E. There is a bijection between the set $\mathcal{I}\mathcal{E}_E$, E-invariant and E-ergodic probability measures on $(X,\mathscr{A})$, and the set of normalized sequentially normal characters of $\mathcal{B}_r^*(E,\sigma)$.*

Remark 8.1.5 *The Proposition* 8.1.3 *and Corollary* 8.1.4 *are just Theorem 4.2 and Corollary 4.3 of the article of Davies [Dav2] adapted to countable Borel equivalence relations.*

8.2 The Non-Smooth Classification

For $i = 1, 2$, let $\phi_i : X \to X$ be a Borel automorphism of $(X, \mathscr{A})$ and E_i be its associated countable hyperfinite Borel equivalence relation (as defined in Theorem 4.5.2). We say that ϕ_1 and ϕ_2 are Borel orbit equivalent if $E_1 \cong E_2$ (see Definition 4.1.5).

In this section, we will moreover assume that ϕ_1 and ϕ_2 are aperiodic non-smooth Borel automorphisms, and therefore E_1 and E_2 are non-smooth aperiodic hyperfinite countable Borel equivalence relations. Recall from Theorem 4.7.1 that two aperiodic countable non-smooth hyperfinite Borel equivalence relations are isomorphic if and only if the cardinalities of their sets of non-atomic, invariant and ergodic probability measures are equal.

Theorem 8.2.1 *Let E_1 and E_2 be two non-smooth aperiodic hyperfinite countable Borel equivalence relations. Then $E_1 \cong E_2$ if and only if $\mathcal{B}^*_r(E_1) \cong \mathcal{B}^*_r(E_2)$.*

Proof: That $E_1 \cong E_2$ implies $\mathcal{B}^*_r(E_1) \cong \mathcal{B}^*_r(E_2)$ is true in general. For the converse, if $\mathcal{B}^*_r(E_1) \cong \mathcal{B}^*_r(E_2)$, then the cardinality of $\partial\mathcal{T}(\mathcal{B}^*_r(E_1))$ and $\partial\mathcal{T}(\mathcal{B}^*_r(E_2))$ are the same which implies by Lemma 8.1.4 that the cardinality of $\mathcal{IE}^\circ_{E_1}$ and $\mathcal{IE}^\circ_{E_2}$ are also the same. Thus by Theorem 4.7.1, $E_1 \cong E_2$. ∎

8.3 The Smooth Classification

In this section, ϕ_1 and ϕ_2 are smooth Borel automorphisms, i.e., E_1 and E_2 are smooth. Recall as a consequence of Theorem 7.4.1 that a countable Borel equivalence relation E is smooth if and only if $\mathcal{B}_r^*(E,\sigma)$ is a type I Borel $*$-algebra (see Definition 6.3.10).

Theorem 8.3.1 *Let E_1 and E_2 be two smooth countable Borel equivalence relations on $(X,\mathscr{A})$. Then $E_1 \cong E_2$ if and only if $\mathcal{B}_r^*(E_1) \cong \mathcal{B}_r^*(E_2)$.*

Proof: That $E_1 \cong E_2$ implies $\mathcal{B}_r^*(E_1) \cong \mathcal{B}_r^*(E_2)$ is true in general. To prove the converse statement, keeping the notation of Theorem 7.4.1, note that as E_1 and E_2 are smooth countable Borel equivalence relations, we have, for $i = 1, 2$,

$$\mathcal{B}_r^*(E_i) = \prod_{k=1}^{\infty} B_0(X_k^i, \mathcal{B}(\mathcal{H}_k)).$$

Moreover, for $k \in 1, 2, ..., \infty$, we have by 7.4.1, that $X_k^i = A_k^i / E_i$, where $A_k^i = \{x \in X : |[x]_{E_i}| = k\}$ are E_i-invariant Borel subset of X. As $\mathcal{B}_r^*(E_1) \cong \mathcal{B}_r^*(E_2)$, then $|X_k^1| = |X_k^2|$ for all $k \in \{1, 2, ..., \infty\}$ (see Definition 6.3.10). Thus by Corollary 4.4.7, we have that $E_1 \cong E_2$. ∎

8.4 The Borel Krieger Type Theorem

In this section, we use the results of the Sections 8.2 and 8.3 to state a Borel Krieger type Theorem:

Theorem 8.4.1 *Let E_1 and E_2 be two aperiodic hyperfinite countable Borel equivalence relations. Then $E_1 \cong E_2$ if and only if $\mathcal{B}_r^*(E_1) \cong \mathcal{B}_r^*(E_2)$.*

Proof: That $E_1 \cong E_2$ implies $\mathcal{B}_r^*(E_1) \cong \mathcal{B}_r^*(E_2)$ is true in general.

If E_1 and E_2 are smooth countable Borel equivalence relations, then by Theorem 8.3.1, $\mathcal{B}_r^*(E_1) \cong \mathcal{B}_r^*(E_2)$ if and only if $E_1 \cong E_2$.

If E_1 and E_2 are aperiodic non-smooth hyperfinite countable Borel equivalence relations, then by Theorem 8.2.1, $\mathcal{B}_r^*(E_1) \cong \mathcal{B}_r^*(E_2)$ if and only if $E_1 \cong E_2$. ∎

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