UNIVERSITE D'OTTAWA
Ecole des Etudes Supérieures

UNIVERSITY OF OTTAWA
School of Graduate Studies

Title of thesis: A STUDY OF SENSITIVITY REDUCTION DESIGN APPROACH

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Degree: M.A. Sc. (Electrical Engineering)

Date: September 25th, 1970

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A STUDY OF SENSITIVITY REDUCTION DESIGN APPROACH.

by

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Submitted in partial fulfillment
of the requirements for the degree of
Master of Applied Science

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University of Ottawa
Ottawa, Ontario
July, 1970

ABSTRACT

This thesis is concerned with the reduction of sensitivity in Linear Optimal Control Systems. The classical concept of Sensitivity and the important results, in context of modern Control Theory have been summarized. The potentials of adding the sensitivity terms in the quadratic performance index for the reduction of the Plant Sensitivity to small parameter perturbations have been investigated. Some design criteria and theoretical results in the role of weighting matrices, that are significant in the Sensitivity Reduction Design approaches have also been presented.

ACKNOWLEDGEMENTS.

The author wishes to express his appreciation and thanks to Professor M. Krieger, for his guidance, suggestions, and encouraging discussions, throughout the entire period of research.

The author is particularly thankful to Professor G.S. Glinski, Chairman, Department of Electrical Engineering, for his interest in the work, and the facilities he provided, during the period the research was completed. Author's sincere appreciation and thanks are due to Dr. J. Kruus, former Professor in the department, who introduced the author to the subject of Optimal Control Theory, and Sensitivity.

Gratitude is also expressed to the colleagues, and graduate students in the Department of Electrical Engineering, in particular, B.S. Rao, and G. Tremblay.

The author is grateful to the National Research Council, for the financial assistance received through the Research Grants Nos, N.R.C. A2580, and N.R.C. A5157.

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CHAPTER I

INTRODUCTION

The study of sensitivity of dynamic system in recent years has become increasingly important. Its applications are no longer confined in the field of Control theory, but has spread to areas such as Network and System theory, Reliability, and Circuit design.

In the field of Control theory, despite the rapid development of the theory of Optimal Control Systems, its applications to practical system design seems to be stagnant at some points. This is mainly because of too many gross idealizations that are made. The difference between the mathematical model and the actual plant is not as small as is often thought. Various factors such as aging, noise, not only need to be considered, but also to be counteracted.

In mathematical language, engineers are interested in analyzing the effects of disturbances or parameter variation on the solution of differential equations. These disturbances may have widely differing character, they may be small or large, momentary or permanent, they may be related to initial conditions etc. These disturbances often lead to the violation of the given constraints on states, and control variables, resulting in the degradation of the system perfomance.

The property of adaptive system demands that system be able to monitor its parameters as disturbance changes, while satisfying an optimal control strategy. If adaptive systems are ever to become a reality, then the sensitivty problem from the Optimal Control point of view first needs to be resolved.

The primary objective of this thesis was to investigate the potentials of adding the sensitivity term into the performance index to be optimized in order to achieve a reduction in sensitivity to plant parameter deviations. The secondary objective is to present the practical and theoretical problems that are encountered, and that need to be solved to achieve further simplicity in the sensitivity reduction techniques.

Chapter II introduces certain basic classical concepts of sensitivity. A relation between two classical definitions is established for linear systems, and a proof for the same relationship is extended to regulators.

Chapter III is devoted to sensitivity in context of Optimal Control systems. It presents some of the basic relationship of sensitivity theory, and provides a new derivation of the relationship between the closed and open-loop sensitivity.

In chapter IV, the sensitivity reduction technique is posed as an overall optimization problem. The technique for improving the performance sensitivity is discussed. The detailed formulation of the trajectory sensitivity reduction problem is presented. The technique is also extended to include the case of large parameter deviations. In the concluding section of this chapter some important design considerations for comparison of the proposed augmented system with the original system are discussed.

Chapter V presents an illustrative example which highlights the practical implications of the points discussed in the preceeding chapter. The concluding section is devoted to the designers dilemma that exists in the choice of the weighting matrix.

Chapter VI basically presents some important theoritical results regarding the weighting matrix. These results are extracted from the current literature, and may serve as very important design tools.

It should be noted that because of the nature of this thesis, the conclusions and suggestions for further research are made at the end of each chapter.

CHAPTER II

THE SENSITIVITY PROBLEM

The objective of this chapter is to introduce two basic (classical) concepts of sensitivity, and to present the relationship between the two. In the later part some discussions of the basic results will be presented.

II.1 CLASSICAL SENSITIVITY.

Two notions of sensitivity considered classical are:-

a) The dependence of solutions of differential equation on parameters. This is a well covered subject in the theory of differential equations.

b) The Bode's sensitivity function, which relates the sensitivity to transfer function representation.

For notational simplicity, the discussion will be restricted to a single parameter variation. However the translation to a set of parameters is straight forward.

a) Sensitivity of Solutions of Differential Equations.

Consider the vector differential equation

$$\dot{x} = f[x, t, u, \mu] \qquad x(t_0) = x_0 \tag{I}$$

where $x^T = [x_1, x_2, \ldots\ldots\ldots x_n]$ is the state vector, $u^T = [u_1, u_2, \ldots\ldots\ldots\ldots u_m]$ is the control vector, μ is a time invariant parameter, and t is the independent variable (time).

The vector $\frac{\partial x}{\partial \mu}(t) = \sigma(t)$ (2)

is called the sensitivity vector, or simply sensitivity, and $\frac{\partial x_i}{\partial \mu} = \sigma_i$ is called sensitivity variable(sometimes referred to in the technical literature as sensitivity coefficient or parameter coefficient).

Under mild assumptions of continuous differentiability, and uniform Lipschitz conditions*, the solution of (1) exists uniquely and $\sigma_i(t)$ satisfies the linear time varying differential equation

$$\dot{\sigma}_i = \sum_{j=1}^{n} f_x^i(t,x,u,\mu)\,\sigma_j + f_\mu^i\,(t,x,u,\mu) \qquad \sigma_i(t_0) = 0 \tag{3}$$

where $f_x = \frac{\partial f}{\partial x}$. Solutions of these equations contain all the information about small parameter deviations, because

$$\Delta x(t) \cong \sigma(t)\,\Delta\mu \text{ for small } \Delta\mu \tag{4}$$

For the linear system, equations (I) and (2) have the form

$$\dot{x} = A(t,\mu)x + B(t,\mu)u\,, \qquad x(t_0) = x_0 \tag{5}$$

$$\dot{\sigma} = A(t,\mu)\,\sigma + A_\mu(t,\mu)\,x + B_\mu(t,\mu)\,u, \qquad \sigma(t_0) = 0 \tag{6}$$

where A_μ is the matrix of the partial derivatives $\frac{\partial a_{ij}(t,\mu)}{\partial \mu}$. For and only for linear systems, we can write a differential equation for the difference caused by large parameter deviations.

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*If in the closed interval (a, b) the function f(x) has a derivative $f'(x)$ on (a, b) with an upper bound M, then for any n such that $n > 2M\,(b-a)^2/\eta$ the sum:

$S_n = \sum_{1-1}^{n} f(x)\,\Delta x$: $\Delta x = \frac{b-a}{n}$, $x = a + \frac{i(b-a)}{n}$ approaximates the intregal $I = \int_a^b f(x)\,dx$ within η, so that $|S_n - I| < \eta$, provided M is such

Let

$$\Delta x(t) = x^1(t) - x(t)$$

where x(t) is the solution of (5) and $x^1(t)$ is the solution of

$$\dot{x}^1 = A_1(t)\, x^1 + B_1(t)\, u$$

It can be easily shown that

$$\Delta\dot{x} = A_1(t)\,\Delta x + \Delta A(t)\, x + \Delta B(t)\, u; \qquad \Delta x(t_0) = 0 \tag{7}$$

where

$$\Delta A = A_1 - A \text{ and } \Delta B = B_1 - B$$

Unlike equation (6) which determines $\Delta x(t)$ for any sufficiently small $\Delta\mu$, the change of matrices to A_1 and B_1 must be known in advance for (7).

However the functions $\sigma(t)$ and $\Delta x(t)$ are not measures of sensitivity. To obtain a measure, one must choose an approoapriate norm of these functions of time(that is reducing to a number expressing the "size" of the function) such as

$$\left(\int_{t_0}^{t_1} \sigma^2(t)\, dt \right), \quad \max_{t_0 < t < t_1} |\sigma(t)|, \qquad |\sigma(t_1)|$$

The first of these measures is a root mean square, while the second is concerned only with the maximum, and the third is merely the magnitude of the sensitivity at a particular time. The choice depends of course on the application, and on the mathematical tool to be employed. The survey of literature shows that the Lipschitz condition

$$|f(x_2) - f(x_1)| < M\, |x_2 - x_1| \text{ is satisfied.}$$

that far more complex measures can be defined, but the question is always whether they can be of any use. A measure is essential for the purpose of precise comparison of two systems: however it hardly uncovers the entire story, and might be misleading unless used with caution.

b) Bode's Sensitivity Function. (Sensitivity of Linear Time-invariant Feedback Systems)

Feedback is employed in Control systems from two points of view, that is to achieve:-

I) monitored, instead of pre-programmed control.

2) reduction of sensitivity to parameter changes, and noise disturbances, and also to obtain stability improvement, increase of input impedence, decrease of output impedence.

In this section the sensitivity of the closed -loop transfer function with respect to deviations in the plant transfer function is derived, and a brief discussion of this is function is included.

Consider the the single loop, time invariant linear feedback system of Fig.1. P is the plant to be controlled, with the variable parameter μ, with the output y, and the control input u. G and H are compensating networks. The transfer function from

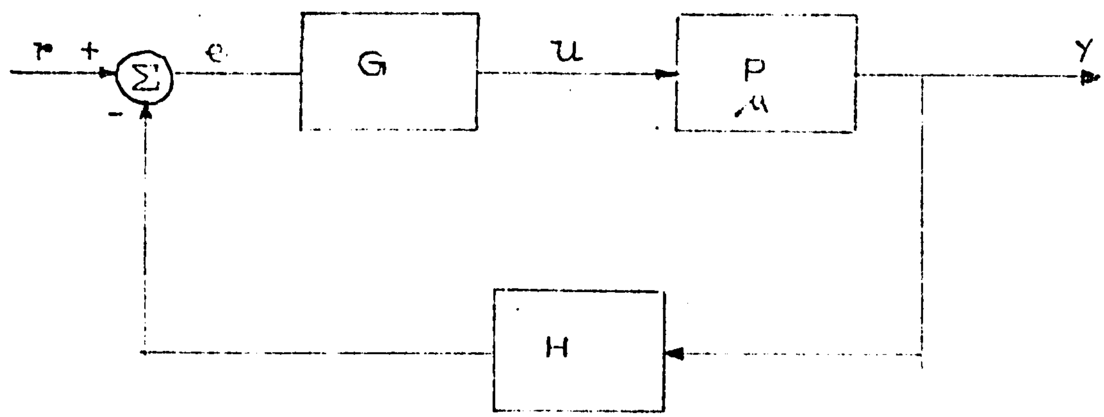


Figure.1.

the reference input r to the output y is given by

$$T(s,\mu) = \frac{Y(s)}{R(s)} = \frac{G(s)\,P(s,\mu)}{1 + G(s)\,P(s,\mu)\,H(s)} \tag{8}$$

It is evident that if the magnitude of $G(s)\,P(s,\mu)\,H(s)$ (the loop transmission) is large compared to unity, then the closed loop system is approaximately independent (insensitive) of the plant P. The classical sensitivity function of a transfer function $P(s,\mu)$ with respect to a parameter μ is defined by

$$S_{\mu}^{P} = \frac{\mu}{P(s,\mu)} \frac{\partial P(s,\mu)}{\partial \mu} \tag{9}$$

When this definition is applied to the system of Fig 1, then the transfer function T(s) with respect to deviations in $P(s,\mu)$ is then easily found to be

$$S_{P}^{T}(s) = \frac{P(s)}{T(s)} \frac{\partial T(s)}{\partial P(s)}$$

$$= \frac{1}{1 + G(s)\,P(s)\,H(s)} \tag{10}$$

The expression $1 + G(s)\,P(s)\,H(s)$ represents the so called " return difference ". From (10) we conclude the classical result that for the feedback to reduce sensitivity, the magnitude of the return difference must be larger than unity. The sensitivity of T with respect to parameter μ is given by

$$S_{\mu}^{T} = \frac{dT/T}{d\mu/\mu} = S_{P}^{T}\, S_{\mu}^{P}$$

A point to note that the sensitivity function is a function of s or jω and is therefore not a measure of sensitivity. Ameasure of sensitivity is for example the real number

$$\int_{-\infty}^{\infty} S_{P}^{T}(-j\omega)\, S_{P}^{T}(j\omega)\, d\omega$$

In linear feedback systems, it is of some interest to study the sensitivity of the closed-loop system's characteristic roots with respect to open-loop parameters. The difficulty lies in evaluating how various sensitivities can be used in a sensitivity control (as distinghuised from sensitivity analysis) design procedure. The effect of parameter deviation on the stability of a transfer function T has been also studied via the sensitivity function.[1]

II. 2 Relation between classical Sensitivities.

Cruz and Perkins[2] found an interesting relation between the sensitivity variable $\sigma(t) = \frac{\partial y}{\partial \mu}(t)$, where y(t) is the output of the feedback system of Fig.1, and the sensitivity function S_{P}^{T} in (10). This relation enables one to interpret an extension of equatin (10) to multi-variable systems (where r, e, u, y of Fig.1 are vectors) and to regulators (where $r \equiv 0$)

Consider an open-loop system with transfer function

$$\frac{Y_o(s)}{R(s)} = T_o(s,\mu) = G_o(s,\mu)\, P(s,\mu)$$

which is equivalent to the system of Fig.1 characterized by equation (8) in the sense that

$$T_o(s,\mu) = G_o(s)\, P(s,\mu).$$

That is in the absence of initial conditions and disturbances, the output $y_o(t)$ of the open-loop system is identical to $y_c(t)$ of the feedback system for all $r(t)$. Cruz and Perkins [2] then showed that if $P(s,\mu)$ changes from "nominal" value to $P_1 = P(s,\mu_1)$, then the Laplace transform of the differences

$$\Delta y_c = y_{c_1} - y_c \quad \text{and} \quad \Delta y_o = y_{o_1} - y_o$$

are related by

$$\Delta Y_c(s) = (1 + P_1(s)\, G(s)\, H(s))^{-1}\, \Delta Y_o(s) \tag{11}$$

This expression is valid for multivariable systems where P, G, H are transfer matrices and the unity in (11) is the unit matrix and reciprocal is matrix inversion. The existence of the inverse in (11) follows from the assumption that the feedback is physically meaningful.

For differential parameter deviations, equation (11) becomes

$$\Sigma_c(s) = (1 + P\, G\, H)^{-1}\, \Sigma_o(s) \tag{12}$$

where

$$\Sigma(s) = L\{\sigma(t)\} = L\left\{\frac{\partial y}{\partial \mu}(t)\right\}.$$

Obviously, for the single variable case, $(1 + PGH)^{-1}$ is the sensitivity function.

The relation (12) is also valid for regulators. Consider Fig. 1

with $r=0$. The output $y_c(t)$ is generated by initial conditions in the plant P, and through the networks H and G a control u(t) is generated. The same output $y_o(t) \equiv y_c(t)$ can also be generated in an open-loop manner by applying $u_o(t) \equiv u_c(t)$ to the identical plant with the same initial conditions. Then it can be shown as below that $\sigma_c(t)$ and $\sigma_o(t)$ are related by the equation (12).

To show this consider linear plant

$$\dot{x} = A(\mu)\, x + B(\mu)\, u\,, \qquad x(0) = x_0$$
$$y = C(\mu)\, x$$

The matrix of transfer function from control inputs u to the outputs y is given by

$$P(s,\mu) = C(\mu)\Phi(s,\mu)\, B(s,\mu)$$

where $\Phi(s,\mu) = \left[sI - A(\mu)\right]^{-1}$

The output y is given by

$$Y(s) = P(s,\mu)\, U(s) + C(\mu)\, \Phi\,(s,\mu)\, x_0 \qquad (13)$$

Suppose the loop is now closed by a feedback network F, then

$$U(s) = -\,F(s)\, Y(s) \qquad (14)$$

Then the closed-loop output $y_c(t)$ is given by

$$Y_c(s) = -P(s,\mu)\;F(s)\;Y_c(s) + C(\mu)\Phi(s,\mu)\, x_0 \qquad (15)$$

Let the closed loop sensitivity be denoted by

$$\Sigma_c(s) = L\{\sigma_c\} = L\left\{\frac{\partial y_c}{\partial \mu}(t)\right\}$$

Then differentiating (15) with respect to μ

$$\Sigma_c(s) = -P(s,\mu)\, F(s)\, \Sigma_c(s) - P_\mu(s,\mu)\, F(s)\, Y_c(s) + \left[C(\mu)\Phi(s,\mu)\right]_\mu x_0 \tag{16}$$

Solving (15) and (16) for $Y_c(s)$ and $\Sigma_c(s)$, we have

$$Y_c(s) = \left[I + P(s,\mu)\, F(s)\right]^{-1} C(\mu)\, \Phi(s,\mu)\, x_0 \tag{17}$$

$$\Sigma_c(s) = \left[I + P(s,\mu)\, F(s)\right]^{-1} \Big\{ -P_\mu(s,\mu)\, F(s,\mu) \left[I + P(s,\mu)\, F(s)\right]^{-1} C(\mu)\Phi(s,\mu)\, x_0 + \left[C(\mu)\Phi(s,\mu)\right]_\mu x_0 \Big\}. \tag{18}$$

Supposing the system is operated open-loop as indicated in (13). Then the open-loop sensitivity is given by

$$\Sigma_o(s) = P_\mu(s,\mu)\, U(s) + \left[C(\mu)\Phi(s,\mu)\right]_\mu x_0 \tag{19}$$

Assume now that the open-loop control is identical to the control of the closed-loop system. The open-loop control is given by (14) and (19) becomes

$$\Sigma_o(s) = -P_\mu(s,\mu)\, F(s) \left[I + P(s,\mu)\, F(s)\right]^{-1} C(\mu)\Phi(s,\mu)\, x_0 + \left[C(\mu)\Phi(s,\mu)\right]_\mu x_o \tag{20}$$

Comparing (20) with (18) we see that

$$\Sigma_c(s) = \left[I + P(s,\mu)\, F(s)\right]^{-1} \Sigma_o(s) \tag{21}$$

which is the relation (12) with $G(s)\, H(s) = F(s)$.

The reduction of sensitivity by feedback when the magnitude of return difference is larger than unity, can now be given a new and precise meaning. The feedback system can be considered less sensitive than the equivalent open-loop system, if some measure of $\sigma_c(t)$ is less than that of $\sigma_o(t)$.

In particular, let us consider

$$\int_0^{t_1} \sigma_c^T(t)\, Z\, \sigma_c(t)\, dt < \int_0^{t_1} \sigma_o^T(t)\, Z\, \sigma_o(t)\, dt \qquad (22)$$

where Z is a positive definite matrix, σ^T is a row vector, and $\sigma^T Z \sigma$ is a positive definite quadratic form. Applying Parseval's theorem to (22) and using (12), it follws that if the Hermetian matrix

$$\left[I + P(-j\omega)G(-j\omega)H(-j\omega)\right]^T Z \left[I + P(j\omega)G(j\omega)H(j\omega)\right] - Z \qquad (23)$$

is positive semi-definite for all ω, then (22) holds for all t_1 (provided the intregal is finite).

For the scalar case (23) reduces to familiar:-

$$\left| I + P(j\omega)G(j\omega)H(j\omega) \right| = \text{magnitude of return difference}$$

greater or equal to unity for all ω.

If a paticular t_1 is considered in (22) then it is sufficient that matrix (23) be positive semi-definite only for $\omega = K\frac{2\pi}{t_1}$, $K = 0, \pm 1, \pm 2, \ldots$ instead for all ω.[2]

CHAPTER III

SENSITIVITY IN OPTIMAL CONTROL SYETEMS

In this chapter, we shall discuss the sensitivity problem in context of Optimal control theory, as formulated below. Furthermore, we shall present a few basic results, followed by a relevant discussion.

III. 1: Optimal Control Problem Formulation.

1) The equations of motion of the dynamical system satisfy the vector differential equation

$$\dot{x}(t) = f[x(t), u(t)] \tag{1}$$

x(t) being an n-dimensional state vector, and u(t) is an r-vector representing control input. The vector valued function f depends on x and u.

2) The control u(t) is constrained to have values in a set Ω in Euclidian space E^r.

$$\text{that is } u(t) \in \Omega, \quad \text{for all } t \tag{2}$$

3) The initial and final states of the system are constrained in the usual manner,

$$x(t_1) \in M_1 \tag{3}$$

$$x(t_2) \in M_2 \tag{4}$$

M_1, M_2 being sets in n-dimensional space.

4) The preformance index or cost functional is given by

$$J(u) = \int_{t_1}^{t_2} L(\dot{x}(t), u(t))\, dt \tag{5}$$

The performance index ascertains how the state transfer is realized, and the choice of the cost functional poses a difficult engineering problem. The optimization problem is to find a u(t) satisfying (1) and (2) and transferring the states from $x(t_1) \in M_1$ to $x(t_2) \in M_2$, while minimizing the 'cost' J(u).

The sensitivity problem arises, when we realize that the function f appearing in (1) is, in general an idealization of the actual behaviour of a physical system. The most common difficulty is, that f actually depends upon certain parameters, say $\mu_1, \mu_2, \ldots\ldots, \mu_m$, whose values are either not known exactly or may change, during the life span of the system. Hence for more realistic representation of the system, one must replace (1) by

$$\dot{x}(t) = f\left[x(t), u(t), \mu\right] \tag{1'}$$

As a consequence of dependence of f on parameter μ, one is inclined to expect that the optimal control will be a function of parameters μ. If such control is possible to derive and implement(in context of the problem formulated above), then it may be called an 'Optimal Adaptive Control'.*

The dependence of the system dynamics on a set of parameters makes the cost, the solution x(t), the optimal control $u^*(t)$, and the final states dependent on these parameters.

* For definition of the term see reference 27.

Depending upon the paticular problem at hand, the engineer may take interst in

1) the dependence of J on μ - normally regarded as the 'Performance sensitivity problem'.

2) dependence of x on μ, regarded as the "Trajectory sensitivity problem'.

III. 2 Basic Results:-

The first basic result[3] can be expressed as follows:-

Let $u = u^*$ be the optimal control for the system

$$x = f\left[x(t), u(t), t, \mu\right] \tag{6}$$

with respect to the performance index,

$$J = \int_{t_1}^{t_2} L(x(t), u(t), \mu, t)\, dt \tag{7}$$

where $\mu = \mu^*$, μ^* denoting the nominal value of the vector parameter μ. Then the first variation δJ of the performance functional, caused by a variation $\delta\mu$ of the parameter μ, is the same whether an open or closed loop implementation of the control u(t) is used.

This result tends to suggest, that some of the benifits, that a feedback implementation normally provides, are not achieved in optimal systems.

We shall summarize a derivation of Pagurek's result[3] due to Kokotovic and Sannuti[4].

Let the initial and target manifolds be M_1 and M_2 respectively, that is

$$x(t_1) \in M_1 \tag{8a}$$

$$x(t_2) \in M_2 \tag{8b}$$

Let H be the Hamiltonian,

$$H = -L + p^T f \tag{9}$$

the costate p(t) satisfies

$$p(t) = -\nabla_x H \tag{10}$$

where

$$\nabla_x H = \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \vdots \\ \frac{\partial H}{\partial x_n} \end{bmatrix}$$

For the functional (7), system equation (6), and the condition (10), the first variation δJ is

$$\delta J = -\int_{t_1}^{t_2} \left\{ (\nabla_u H)^T \delta u + (\nabla_\mu H)^T \delta\mu \right\} dt + (p^T \delta x)\Big|_{t_1}^{t_2} \tag{11}$$

Now a point to note is that (11) holds whether the control is implemented in open or closed-loop form because δu can depend on δx or not.
For $\mu = \mu^*$

$$\nabla_u H = 0 \tag{12}$$

and hence (11) simplifies to

$$\delta J = -\int_{t_1}^{t_2} (\nabla_\mu H)^T \delta\mu \, dt + (p^T \delta x)\Big|_{t_1}^{t_2} \tag{13}$$

Denoting the open and closed loop quantities by the subscripts $_o$, and $_c$ respectively, we have from (13)

$$\delta J_o - \delta J_c = \left\{ p^T(\delta x_o - \delta x_c) \right\} \Big|_{t_1}^{t_2} \tag{14}$$

The expression (14) is the generalized form of Pagurek's result. In certain special cases, for instance, if M_1 is a point, M_2 is the whole space of x, that is

$$\delta x_o(t_1) - \delta x_c(t_1) = 0$$

and $p(t_2) = 0$; then it reduces to the original result

$$\delta J_o - \delta J_c = 0$$

The second basic result that we are going to consider, is a direct consequence of a result by Kalman[5]. This is more restrictive in a sense, since it was originally derived for single-input, linear time invariant control systems.

Consider a completely controllable plant of nth order:-

$$\dot{x}(t) = A\,x(t) + B\,u(t) \tag{15}$$

let

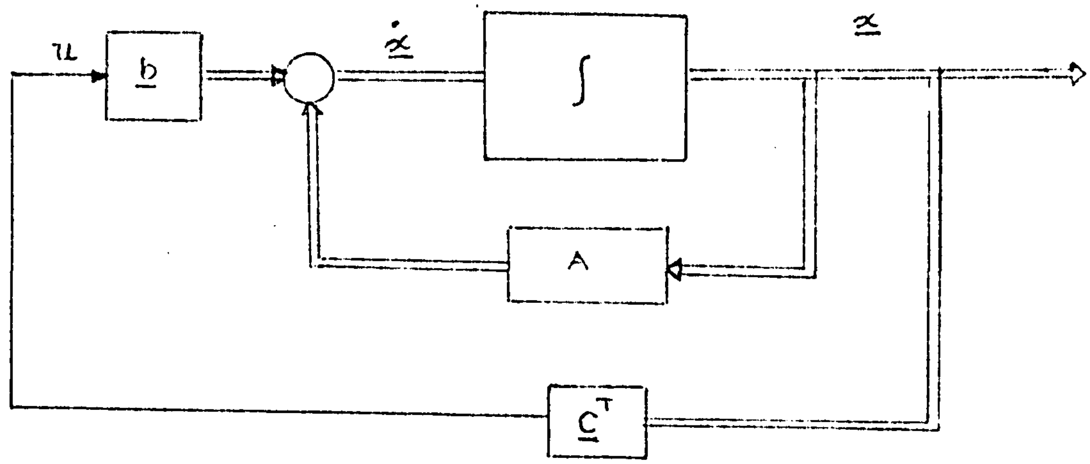
$$u(t) = C^T x(t) \tag{16}$$

be the optimal control law for the performance index

$$J(u) = \int_0^{\infty} \left[x^T(t)\,Q\,x(t) + u^2(t) \right] dt \tag{17}$$

where Q is positive definite; then the following condition is satisfied by the controller C;

$$\left| 1 - C^T (j\omega I - A)^{-1} b \right| \geqslant 1 \tag{18}$$



$$A = \begin{bmatrix} 0 & 1 & 0 \ldots\ldots & 0 \\ 0 & 0 & 1 \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \cdot & & & \cdot \\ -a_0 & -a_1 & \ldots\ldots & -a_{n-1} \end{bmatrix} : \qquad B = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

Fig. 2. Single-input plant, and Optimal Feedback Control.

where the equality may hold for some but not all ω. Kalman also proved an inverse theorem which states; that if a stable control law C satisfies (18), then it is optimal with respect to a performance index of the type (17), for some Q.

The expression (18) simply means that the return difference of the optimal control system has a magnitude not less than 1 for all frequencies.

We shall now establish the meaning of the expression (18) on the trajectory sensitivity of optimal systems. As is well known, for completely controllable systems, we can assume A to be of companion form, and b to have all entries zero but the last $b_n = 1$. We shall prime the actual quantities in order to distinguish them from the nominal quantities:

$$e_c(t) = x_c'(t) - x(t) \tag{19}$$

and

$$e_o(t) = x_o'(t) - x(t) \tag{20}$$

where $x_c'(t)$, and $x_o'(t)$ denote the actual closed-loop, and open-loop state, and $x(t)$ the nominal optimal state. The actual plant matrix A' is expressed as

$$A' = A + ba^T \tag{21}$$

where a is defined by

$$a_i = (A')_{ni} - (A)_{ni} \qquad i = 1, 2, \ldots\ldots n \tag{22}$$

We shall now derive the relationship between the actual state $x'(t)$ and the nominal optimal state $x(t)$, for both the open and the closed-loop case.

Open-loop case:

$$\text{nominal} \quad \dot{x}(t) = A\,x(t) + b\,u(t)$$

Laplace transform, and further rearrangement give

$$(sI - A)\,x(s) = b\,u(s) \tag{23}$$

$$\text{actual;} \quad x_o'(t) = A'\,x_o'(t) + b\,u(t)$$
$$= (A + ba^T)\,x_o'(t) + b\,u(t)$$

Similarly, we have

$$\left[(sI - A) - ba^T\right] x_o'(s) = b\,u(s) \tag{24}$$

From (23) and (24), we have

$$\left[(sI - A) - ba^T\right] x_o'(s) = (sI - A)\,x(s)$$

that is

$$x_o'(s) = \left[(sI - A) - ba^T\right]^{-1} (sI - A)\,x(s)$$

$$x_o'(s) = \left[I - (sI - A)^{-1} ba^T\right]^{-1} x(s)$$

Let

$$(sI - A)^{-1} b = c\,; \quad \text{then}$$

$$x_o'(s) = \left[I - ca^T\right]^{-1} x(s)$$

Using the matrix inversion lemma;

$$\left[I_n + gh^T\right]^{-1} = \left[I_n - \frac{gh^T}{1 + h^T g}\right]$$

we have

$$x_o'(s) = \left[I + \frac{c\,a^T}{1 - a^T c}\right] x(s)$$

$$x_o'(s) = \left[I + \frac{(sI - A)^{-1} ba^T}{1 - a^T(sI - A)^{-1}b} \right] x(s) \tag{25}$$

<u>Closed-loop case</u> ;

nominal; $\dot{x}(t) = A\,x(t) + b\,u(t)$

$$= A\,x(t) + b\,C^T x(t)$$

Laplace transform gives

$$\left[sI - A - b\,C^T \right] x(s) = 0 \tag{26}$$

actual;

$$x_c'(t) = (A + ba^T)\,x_c'(t) + b\,C^T x_c'(t)$$

Similarly

$$\left[sI - A - ba^T - b\,C^T \right] x_c'(s) = 0 \tag{27}$$

From (26) and (27), we have

$$\left[(sI - A - b\,C^T) - ba^T \right] x_c'(s) = (sI - A - b\,C^T)\,x(s)$$

that is

$$x_c'(s) = \left[(sI - A - b\,C^T) - ba^T \right]^{-1} (sI - A - b\,C^T)\,x(s)$$

$$= \left[I - (sI - A - b\,C^T)^{-1} ba^T \right]^{-1} x(s)$$

Let

$$\left[(sI - A) - b\,C^T \right]^{-1} b = d$$

therefore

$$d = \left[I - (sI - A)^{-1} b\,C^T \right]^{-1} (sI - A)^{-1} b$$

$$= \left[I + \frac{(sI - A)^{-1} b\,C^T}{1 - C^T (sI - A)^{-1} b} \right] (sI - A)^{-1} b.$$

From the preceeding page, it follows;

$$x_c'(s) = \left[I - d\, a^T \right]^{-1} x(s)$$

using the same inversion lemma;

$$x_c'(s) = \left[I + \frac{d\, a^T}{1 - a^T d} \right] x(s)$$

that is

$$x_c'(s) = \left[I + \frac{\left[I + \frac{(sI - A)^{-1} b\, C^T}{1 - C^T (sI - A)^{-1} b} \right] (sI - A)^{-1} b\, a^T}{1 - a^T \left[I + \frac{(sI - A)^{-1} b\, C^T}{1 - C^T (sI - A)^{-1} b} \right] (sI - A)^{-1} b} \right] x(s)$$

$$x_c'(s) = \left[I + \rho(s) \frac{(sI - A)^{-1} b\, a^T}{1 - \rho(s) a^T (sI - A)^{-1} b} \right] x(s) \qquad (28)$$

where

$$\rho(s) = \frac{1}{1 - C^T (sI - A)^{-1} b}$$

Thus we have;

$$x_c'(s) = M_c(s)\, x(s) \qquad (28')$$

$$x_o'(s) = M_o(s)\, x(s) \qquad (25')$$

where matrices $M_c(s)$ and $M_o(s)$ are given by

$$M_c(s) = I + \rho(s) \frac{(sI - A)^{-1} b\, a^T}{i - \rho(s)\, a^T (sI - A)^{-1} b}$$

and

$$M_o(s) = I + \frac{(sI - A)^{-1} b\, a^T}{1 - a^T (sI - A)^{-1} b}$$

The scalar $\rho(s)$ is given by

$$\rho(s) = \frac{1}{1 - C^T (sI - A)^{-1} b} \tag{28}$$

That is the inverse of the 'return difference'.

Now the closed-loop error is given by

$$e_c(s) = x_c'(s) - x(s)$$

$$= \left[\frac{\rho(s)\,(sI - A)^{-1} b\, a^T}{1 - \rho(s)\, a^T (sI - A)^{-1} b}\right] x(s)$$

and the open-loop error is given by

$$e_o(s) = x_o'(s) - x(s) = \left[\frac{(sI - A)^{-1} b\, a^T}{1 - a^T (sI - A)^{-1} b}\right] x(s)$$

The relationship between the closed-loop error $e_c(s)$ and the open-loop error $e_o(s)$ is then obtained as

$$e_c(s) = \frac{1 - a^T (sI - A)^{-1} b}{1 - \rho(s)\, a^T (sI - A)^{-1} b}\, \rho(s)\, e_o(s) \tag{29}$$

For infinitesimally small variations a , (29) reduces to

$$e_c(s) = \rho(s)\, e_o(s) \tag{30}$$

If Kalman's result, as expressed in (18) is now taken into account, then we see (bearing in mind $\rho(s) = (1 - C^T(j\omega I - A)^{-1}b)^{-1}$) that

$$\left|(e_c(j\omega))_i\right| \leqslant \left|(e_o(j\omega))_i\right| \quad \text{for } i = 1, 2, \ldots. \tag{31}$$

The subscript i stands for the ith component of the vector $e(j\omega)$. The result (31) which is a frequency domain characterisation can be shown by the application of Parseval's theorem to be the sufficient condition for the inequality (expressed in time domain)

$$\int_0^\infty \left[(e_c(t))_i\right]^2 dt < \int_0^\infty \left[(e_o(t))_i\right]^2 dt, \quad i = 1, 2. \ldots. \tag{32}$$

to be true.

The basic result on the trajectory sensitivity as expressed by (32) summed up as follows:- ' If a single input linear system is optimal with respect to a performance index of the type ' intregal of a quadratic form in the state plus u^2', then the closed-loop implementation is better than the open-loop implementation in the sense that the intregal of error square in each of the state components is smaller for the closed-loop case'.

III. 3 Discussion.

The basic results expressed by (14) and (32) have in common the comparison between the open and the closed-loop solutions. This is a common characteristic used in modern sensitivity analysis, and owes its origin to a well known result obtained by Cruz and Perkins[6]. In their approach of sensitivity analysis, they defined sensitivity of a system, as the operator, which when applied to the open-loop error, gives the closed loop error. For instance, for the single input system described by (15), they showed that (for infinitesimal variations)

$$e_c(s) = S(s)\, e_o(s) \tag{33}$$

where

$$S(s) = \left[I-(sI-A)^{-1} bC^T \right]^{-1} \tag{34}$$

The matrix S(s) is called the 'Sensitivity matrix'. The concept of sensitivity matrix is of some benifit to the designers. A sensitivity index can be defined as an integral of a quadratic form in the error vector. If the inequality criterion

$$\int_0^{\infty} (e_c^T(t)\; W\; e_c^T(t))\; dt < \int_c^{\infty} e_o^T(t)\; W\; e_o(t)\; dt \tag{35}$$

is satisfied, then the control signal should be implemented in the closed-loop form, and the designer has to choose some appropriate positive-definite matrix W to satisfy the criterion (35).

Actually, for the optimal closed-loop control be better than the open-loop control, in the sense of (35), it is sufficient to show that

$$e_c^T(-j\omega)\; W\; e_c(j\omega) - e_o^T(j\omega)\; W\; e_o(j\omega) \leqslant 0 \tag{35'}$$

According to (30)

$$e_c^T(-j\omega)\; W\; e_c(-j\omega) = \left|\rho(j\omega)\right|^2 e_o^T(-j\omega)\; W\; e_o(j\omega)$$

In view of (28'') and (18) we have

$$\left|\rho(j\omega)\right|^2 \leqslant 1$$

and therefore, an optimal single input linear system satisfies (35') and (35). It may be worthwhile to mention that J.P.Herner[7] derived an interesting relationship among S(s), $\rho(s)$ and $e_o(s)$

$$S(s)\, e_o(s) = \rho(s)\, e_o(s)$$

that is $\rho(s)$ is an eigen value of $S(s)$ corresponding to the eigen vector $e_o(s)$.

The basic results that we have so far considered, points out two fundamental properties of the optimal linear systems; -

(i) If variations of the performance index are our main concern, then both the open and closed-loop implementation of controls are equivalent.

(ii) If variations in the state are of prime significance, then in a certain sense a closed-loop system is better than an open-loop system.

These allow the designer the choice of implementation of the control signal, however they are not directly suitable for design purposes, as will be discussed in the following chapters.

CHAPTER IV

SENSITIVITY REDUCTION AS AN AUGMENTED OPTIMIZATION PROBLEM.

In this chapter we shall present the details of performance and trajectory sensitivity in the open-loop non-linear optimal system, and the trajectory sensitivity in the linear optimal regulators. The basic idea is to introduce the sensitivity terms in the performance index, and sensitivity reduction is posed as an overall optimization problem. A discussion on several aspects of this method that must be considered prior to design will also be presented.

IV. 1 Open-loop Non-linear Optimal Systems.

We shall denote the system optimized without the sensitivity terms in the performance index by S_1, and the system optimized with sensitivity terms in the performance index by S_2. These will be described fully below. Also for notational simplicity we shall consider only a single scalar parameter μ subject to perturbations.

The system S_1 :

is given by

$$\dot{x} = f(x, u, \mu); \qquad x(t_0) = x_0 \tag{1}$$

x being n-dimensional state, u the m-dimensional control, and with the performance index

$$I(\mu, \mu_o) = F(x(t_1)) + \int_{t_0}^{t_1} L(x(t), u(t))\, dt \qquad (2)$$

The performance functional I is a function of a nominal parameter μ_o, as well as the actual value μ, because the control is computed on the basis of μ_o while the actual trajectory depends on the actual value of μ. Therefore $I(\mu, \mu_o)$. The functions F and L are scalar.

Now due to optimality

$$\frac{\partial I^*}{\partial \mu_o} = 0$$

I^* being the optimal value of I.

Now $I(\mu, \mu_o)$ as defined above has the following properties:-

(i) If the nominal parameter is not specified, I becomes a function of whatever control is used, $I(\mu, u)$. If the actual value is known and used in computing the optimal control $u^*(t)$ then, of course, $I^*(\mu, \mu) = I^*(\mu)$ represents the best that can be achieved if μ is known. Thus

$$I^*(\mu, \mu_o) \geqslant I^*(\mu)$$

(ii) The control which minimises the cost functional is not necessarily the same for all values of μ_o, unless it is proved, or assumed so.

(iii) We shall assume that for each value of the parameter μ, the control $u^*(t, \mu)$ which minimizes $I(\mu, \mu) = I(\mu)$ is unique.

Supposing we are concerned about the steep rise of $I^*(\mu, \mu_o)$ with μ, I^* being the optimal value of I, then we are naturally inclined to consider the minimization of the functional

$$J = I(\mu, \mu_o) + \lambda \left(\frac{\partial I}{\partial \mu}\right)^{2K} \tag{3}$$

where λ is a positive integer.

More explicitly, this can be described as follows:-

We have

$$\frac{\partial I}{\partial \mu} = F_x^T(x(t_1))\,\sigma(t1) + \int_{t_o}^{t_1} L_x^T(x, u)\,\sigma(t)\,dt$$

where $\sigma(t) = \frac{\partial x(t)}{\partial \mu}$ satisfies

$$\dot{\sigma}(t) = f_x(x, u, \mu)\sigma + f_\mu(x, u, \mu); \qquad \sigma(t_o) = 0 \tag{4}$$

In the above F_x, L_x, and f_μ are vectors of partial derivatives

$F_x^T = \left[\frac{\partial F}{\partial x_1}, \ldots\ldots\ldots\ldots \frac{\partial F}{\partial x_n}\right]$, and f_x is given by the matrix,

$$f_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}. & \cdots\cdots\cdots\cdots & \frac{\partial f_n}{\partial x_n} \\ . & & . \\ . & \cdots\cdots\cdots\cdots & . \\ \frac{\partial f_n}{x_1}. & \cdots\cdots\cdots\cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Supposing, we let

$$\dot{\sigma}^o = L_x^T(x, u)\sigma \qquad \sigma^o(t_o) = 0$$

then $\left(\frac{\partial I}{\partial \mu}\right)$ reduces to

$$\frac{\partial I}{\partial \mu} = F_x^T \sigma(t_1) + \sigma^o(t_1)$$

Now we are in a position to introduce the system S_2.

The system S_2:

is then the system

$$\dot{x} = f(x,u,\mu) \qquad x(t_0) = x_0$$
$$\dot{\sigma}^o = L_x^T(x,u)\sigma \qquad \sigma^o(t_0) = 0$$
$$\dot{\sigma} = f_x(x,u,\mu)\sigma + f_\mu(x,u,\mu): \quad \sigma(t_0) = 0$$

where the performance index

$$J = F(x(t_1)) + \int_{t_0}^{t_1} L(x,u)\,dt + \left[F_x^T(x(t_1))\sigma(t_1) + \sigma^o(t_1)\right]^{2K}$$

is to be minimized over the same class of admissible controls u(t) as in (2). This is a standard Bolza-type optimization problem.

Let us denote the optimal value of I for S_1 by I_1^* and that for J in (3) for S_2 by I_2^*. It is obvious that

$$I_2^* \geqslant I_1^*$$

also

$$\left.\frac{\partial I_2^*}{\partial \mu}\right|_{\mu=\mu_0} \geqslant \left.\frac{\partial I_1^*}{\partial \mu}\right|_{\mu=\mu_0}$$

, because if the inequality is reversed, then u_1^* applied to S_2, yields a lower value of J, than that given by u_2^*. In general these are strict inequalities.

If we compute I^* for several values of parameter μ, then we shall be able to discuss the relative merits of the optimization of the systems S_1 and S_2. Let us consider a purely hypothetical example as shown in Fig 3.

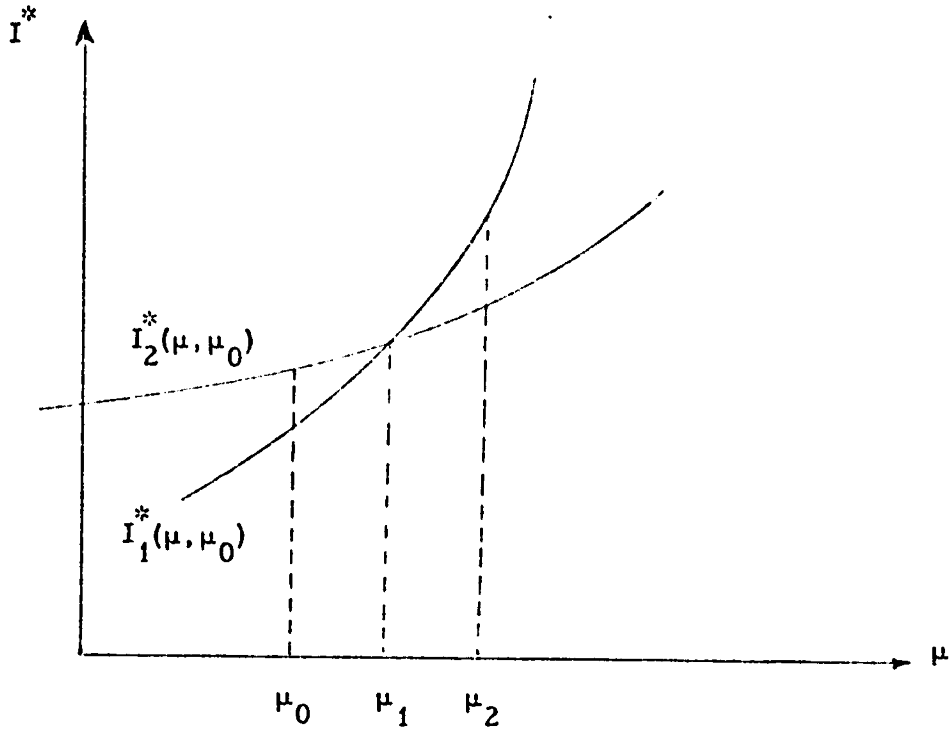


Fig. 3. Comparison of $I_1^*(\mu, \mu_0)$ and $I_2^*(\mu, \mu_0)$.

We see from the Figure 3, that at a sacrifice of performance and at the cost of a more complex computation, the sensitivity of the system S_2 as measured by $\left.\frac{\partial I}{\partial \mu}\right|_{\mu=\mu_0}$, will be lower than that in the system S_1.

As evident from Fig 3, only when the deviation $\Delta\mu$ is such that $\mu = \mu_0 + \Delta\mu$ is greater than μ_1, then

$$I_2^*(\mu, \mu_0) < I_1^*(\mu, \mu_0)$$

However such an improvement might be obtained in the original optimization of I, with far less complexity in computation by choosing a larger value for the nominal parameter μ_0, because obviously

$$I_1^*(\mu_2, \mu_2) < I_2^*(\mu_2, \mu_0)$$

Thus several serious doubts may arise with regards to the justification of optimization of the system S_2, with the sensitivity terms in the performance index. Thus for a particular problem or restricted class of problems, considerable investigation is needed to determine, if any worthwhile benifits can be claimed from the use of a performance index of the type as in equation (3)

In addition to these doubts, it is only in a few cases that the minimum value of I, and $\frac{\partial I^*}{\partial \mu}$ are of practical interest. Usually the performance index is a weighted sum of several components, and the designer must bear in mind that it is the proper balance among these components and their individual sensitivity which is important. Moreover, even the few cases, where I^* and $\frac{\partial I^*}{\partial \mu}$ are of interest, are further reduced in number because the performance index of the type (3) is not applicable to cases. such as the minimum fuel minimum time, and minimum energy, where L and F are independent of $\dot{x}$. In these cases and in the open loop operation I depends only on μ, and not on μ_o, and when $\mu \neq \mu_o$ the terminal condition, generally, will not be met. This may explain the scarcity of papers in the literature in this particular aspect of the sensitivity problem.

One question may arise is whether the trajectory sensitivity, that is the deviations of the optimal response can be controlled in open loop manner. The performance index may be reformulated, and it is sensible to consider

$$J = I + \lambda\, P(\sigma(t)) \qquad \lambda > 0 \tag{5}$$

where $P(\sigma(t))$ is some non-negative functional of $\sigma(t)$, for example

$$P(\sigma(t)) = \sigma^T(t_1)\, G\sigma(t_1) + \int_{t_0}^{t_1} \sigma^T(t)\, S\sigma(t)\, dt$$

where G, and S are positive, semidefinite symmetric matrices. Again a compromise between the performance and sensitivity can be expected. The performance index (5) is suitable for problems like minimum time and minimum fuel, where x(t) does not appear explicitly, and the on-off nature of the optimal control is then preserved in S_2.

It may be noted that both performance indices (3) and (5) are special cases of the general problem of minimizing

$$J = F_2(x(t_1), \sigma(t_1)) + \int_{t_0}^{t_1} L_2(x, \sigma, u, \mu)\, dt$$

with

$$\dot{x} = f(x, u, \mu) \qquad x(t_0) = x_0$$

$$\dot{\sigma} = f_x(x, u, \mu)\sigma + f_\mu(x, u, \mu) \qquad \sigma(t_0) = 0$$

By viewing the variables $\sigma(s)$ as additional state variables, this problem is basically a standard optimization problem, except the fact, that: when the control $u^*(t)$ is converted to an equivalent closed loop control $u^*(x)$, then the variable $\sigma(t)$ given by equn (4) no longer represents $\frac{\partial x}{\partial \mu}$ of the closed loop system. Although the dimension of the problem is increased, a compromise between performance and sensitivity can be expected. For a closed loop solution the additional difficulty arises in that, the closed loop sensitivity $\frac{\partial x}{\partial \mu}$ is unknown in advance.

IV. 2: Closed loop Linear Optimal Regulators.

In this section, we consider the so called state regulator problem. Basically, the solution of the state regulator problem for linear plants, leads to an Optimal feed back system with the property that the components of the state vector x(t) are kept near zero, without excessive expenditure of the ccontrol energy. By and large, the quadratic performance index is used for computational convenience, and the linearity property of the feed back system. The minimum value of the performance index under such formulation is of no interest, and hence for the purpose of sensitivity, it is sensible to consider a performance index of the type(5) rather than (3), because the term $\left(\frac{\partial I}{\partial \mu}\right)^{2K}$ is not only of little interest, but the addition of this term will destroy the linearity property of the closed loop system.

a) Time invariant system:-

Let us consider the linear time invariant plant

$$\dot{x} = A(\mu)\, x + B(\mu)\, u : \qquad x(0) = x_0 \tag{6}$$

and the performance index

$$I = 1/2\int_0^{\infty}\left[x^T Q x + u^T R u\right]dt \tag{7}$$

where Q is a positive semi definite symmetric matrix, and R is a symmetric positive definite matrix. x is an n-vector representing the states, and control u is unconstrained. The matrix A and B may depend on the time t, and the time invariant parameter μ. The optimal control is given by

$$u^*(x) = -K x \tag{8}$$

where the matrix K, known as the 'feedback gain matrix' is given by

$$K = R^{-1}B^T P \tag{9}$$

and P is the steady-state solution(at $t = \infty$) of the non linear first order matrix differential equation

$$\dot{P}(t) = P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t) + Q : \; P(0) = 0 \tag{10}$$

This equation is of the Riccati type, and is often referred to as 'Riccati Equation'.

The closed loop optimal system is therefore given by

$$\dot{x} = (A(\mu) - B(\mu) K) x \qquad x(0) = x_0 \tag{11}$$

and the closed-loop sensitivity $\sigma = \frac{\partial x}{\partial \mu}$ by

$$\dot{\sigma} = (A(\mu) - B(\mu) K)\sigma + (\dot{A}_{\mu}(\mu) - B_{\mu}(\mu) K) x : \sigma(0) = 0 \tag{12}$$

If the system (6) is completely controllable and the matrix Q is such that $x^T(t)\, Q\, x(t) \not\equiv 0$ for solutions x(t) of $\dot{x} = A x$ for all initial conditions (a complete observability condition), then a solution of the Riccati

Equation (10) settles down to a unique steady state solution $P(\infty)$ for any positive semi-definite and symmetrical initial condition matrix $P(0)$, cn (10), and the closed -loop system (11) is asymptotically stable even if $\dot{x} = A\,x$, is not stable. This is a brief summary from Athans and Falb.

The optimal system described so far will be referred to as S_1 in contrast with the system S_2 where sensitivity terms will be added to the performance index (7).

Consider now the plant (6) minimized with respect to

$$J = 1/2 \int_0^{\infty} \left[x^T Q\, x + u^T R\, u + 2\, x^T W \rho + \rho^T S \rho \right] dt \qquad (13)$$

where W and S are symmetric matrices and $\rho(t)$ is an approaximation of the closed-loop sensitivity. The problem of equation for $\rho(t)$ arises. The open-loop sensitivity is given by

$$\dot{\sigma}_o = A\,\sigma_o + A_\mu x + B_\mu u\,, \qquad \sigma(0) = 0 \qquad (14)$$

and may be quite different from the closed-loop sensitivity $\sigma(t)$ which is the important one, but unfortunately this is unknown until the problem is solved. The closed-loop sensitivity may be so different from the open-loop sensitivity that the addition of quadratic sensitivity term involving ρ, to form the augmented performance index may be detrimental rather than benificial. It is therefore essential that an equation for $\rho(t)$ must be assumed, which should approaximate the closed-loop sensitivity. Taking a clue from (12), we assume a linear equation of the form

$$\dot{\rho} = \widetilde{A}\rho + \widetilde{B}\,x \qquad \rho(0) = 0 \qquad (15)$$

where A, B are as yet undetermined matrices. The problem of minimizing the performance index (13) subject to the constraints (6) and (15) may be considered as an augmented optimization problem, with the augmented state

$$Z = \begin{bmatrix} x \\ \hdashline \rho \end{bmatrix} \tag{16}$$

which satisfies

$$\dot{Z} = \bar{A}\, Z + \bar{B}\, u \qquad Z(0) = \begin{bmatrix} x_0 \\ \hdashline 0 \end{bmatrix} \tag{17}$$

and minimizes

$$J = 1/2 \int_0^{\infty} \left[Z^T Q\, Z + u^T R\, u \right] dt \tag{18}$$

where the augmented matrices are of the form

$$\bar{A} = \begin{bmatrix} A & 0 \\ \tilde{B} & \tilde{A} \end{bmatrix} \; : \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \; : \quad \bar{Q} = \begin{bmatrix} Q & W \\ W & S \end{bmatrix} \tag{19}$$

and where $\bar{Q}$ is positive semi-definite (S must therefore be a positive semi-definite matrix)

The optimal control is therefore given by

$$u^*(Z) = u^*(x,\rho) = -\bar{K}\, Z = -[K_1 \,\vdots\, K_2]\, Z = -K_1 x - K_2 \rho \tag{20}$$

and the closed-loop system by

$$\left.\begin{aligned} \dot{x} &= (A - BK_1)\, x - BK_2 \rho & x(0) &= x_0 \\ \dot{\rho} &= \tilde{A}\rho + \tilde{B}\, x & \rho(0) &= 0 \end{aligned}\right\} \tag{21}$$

Note that in (21) is not the exact closed-loop sensitivity $\sigma = \frac{\partial x}{\partial \mu}$ which is obtained by differentiating the first equation in (21) and is now given by

$$\dot{\sigma} = (A - BK_1)\sigma + (A_\mu - B_\mu K_1)x - B_\mu K_2 \rho - BK_2 \frac{\partial \rho}{\partial \mu} : \sigma(0) = 0 \quad (22)$$

where

$$\frac{d}{dt}\left(\frac{\partial \rho}{\partial \mu}\right) = \tilde{A}\left(\frac{\partial \rho}{\partial \mu}\right) + \tilde{B}\sigma \qquad \rho(0) = 0 \quad (23)$$

The closed-loop sensitivity $\sigma(t)$ is therefore, in general, the solution of 4n coupled differential equations for $x(t)$, $\rho(t)$, $\sigma(t)$ and $\frac{\partial \rho}{\partial \mu}(t)$. Comparing (22) with (15), we can see that in order that $\rho(t)$ closely approaximates $\sigma(t)$, we should have

$$\tilde{A} = A - B\tilde{K}_1 - B_\mu \tilde{K}_2 \quad \text{and} \quad \tilde{B} = A_\mu - B_\mu \tilde{K}_1 \quad (24)$$

where assumed $\tilde{K}_1$ and $\tilde{K}_2$ are close to the actual K_1 and K_2 of (20). The last term, $BK_2\frac{\partial \rho}{\partial \mu}$, is neglected in (15). Thus, if $\tilde{K}_1 \cong K_1$, and $\tilde{K}_2 \cong K_2$ and $BK_2\frac{\partial \rho}{\partial \mu}$ is negligible, then $\rho(t)$ is given by

$$\dot{\rho} = (A - B\tilde{K}_1 - B_\mu \tilde{K}_2)\rho + (A_\mu - B_\mu \tilde{K}_1)\, x : \quad \rho(0) = 0 \quad (25)$$

and is a good approaximation to $\sigma(t)$ given by (15).

If the first order approaximation given by (25) fails because $BK_2\frac{\partial \rho}{\partial \mu}$ is not negligible, then this term must be added to (25) and another vector equation for a vector $\gamma(t)$ to approaximate the equation for $\frac{\partial \rho}{\partial \mu}$ must be added to (21). This will still leave $\rho(t)$ only an approaximation, a second order one, to $\sigma(t)$. Since the optimal control for the second order approaximation is of the form

$$u = -K_1 x - K_2 \rho - K_3 \gamma$$

then the closed loop optimal system is given by the equations (26) to (28)

$$\dot{x} = (A - BK_1)\,x - BK_2\rho - BK_3\gamma \tag{26}$$

$$\dot{\rho} = (A - B\tilde{K}_1 - B_\mu\tilde{K}_2)\rho + (A_\mu - B_\mu\tilde{K}_1)\,x - (B\tilde{K}_2 + B_\mu\tilde{K}_3)\,\gamma \tag{27}$$

$$\dot{\gamma} = (A - B\tilde{K}_1 - B_\mu\tilde{K}_2)\,\gamma + (A_\mu - B_\mu\tilde{K}_1)\rho \tag{28}$$

Differentiating these equations with respect to μ, and keeping in mind that the matrices A and B in (27) and (28) depend only on the nominal parameter μ_o, and not on μ, we have

$$\dot{\sigma} = (A - BK_1)\sigma + (A_\mu - B_\mu K_1)x - B_\mu K_2\rho - BK_2\frac{\partial\rho}{\partial\mu} - B_\mu K_3\gamma - BK_3\frac{\partial\gamma}{\partial\mu} \tag{29}$$

$$\frac{d}{dt}\left(\frac{\partial\rho}{\partial\mu}\right) = (A - B\tilde{K}_1 - B_\mu\tilde{K}_2)\frac{\partial\rho}{\partial\mu} + (A_\mu - B_\mu\tilde{K}_1)\sigma - (B\tilde{K}_2 + B_\mu\tilde{K}_3)\,\frac{\partial\gamma}{\partial\mu} \tag{30}$$

$$\frac{d}{dt}\left(\frac{\partial\gamma}{\partial\mu}\right) = (A - B\tilde{K}_1 - B_\mu\tilde{K}_2)\,\frac{\partial\gamma}{\partial\mu} + (A_\mu - B_\mu\tilde{K}_1)\,\frac{\partial\rho}{\partial\mu} \tag{31}$$

By comparison of (27) with (29) and of (28) with (30), it is evident that when $\frac{\partial\gamma}{\partial\mu} = \frac{\partial^3 x}{\partial\mu^3}$ is very small, then the last term in (29) and (30) may be negligible. If in addition $\tilde{K}_1 \cong K_1$, $\tilde{K}_2 \cong K_2$, and $\tilde{K}_3 \cong K_3$, then $\rho(t) = \sigma(t)$ and $\gamma(t) = \frac{\partial\rho}{\mu}(t)$.

If one obtains $\bar{K} = [K_1 \vdots K_2]$ by solving the set of quadratic algebraic equations

$$PA + A^TP - PBR^{-1}B^TP + Q = 0 \tag{32}$$

$$K = R^{-1}B^TP$$

then one can substitute $K_1 = K_1$ and $K_2 = K_2$ in (24) and (19) and there is no need to assume the values of these matrices. This is applicable to low order systems. It should be emphasised at this point, that one should not underestimate the difficulty in isolating the proper solution of these algebraic equations. For higher systems, where a numerical solution by digital computer is desired, it is easier to solve the Riccati differential equation (10) with the augmented matrices than the above set of algebraic equations. One may also substitute $\tilde{K}_1 = K_1$, and $\tilde{K}_2 = K_2$ into the Riccati equation as well, but unfortunately this will transform it from a relatively easy initial value problem to a complicated two-point boundary value problem:

$$\frac{d}{dt}\bar{P}(t) = \bar{P}(t)\bar{A}(\bar{K}) + \bar{A}^T(\bar{K})\bar{P}(t) - \bar{P}(t)\bar{B}R^{-1}\bar{B}^T\bar{P}(t) + \bar{Q} : \bar{P}(0) = 0 \tag{33}$$

$$\frac{d}{dt}\bar{K} = 0 \qquad \bar{K}(0) = \bar{K} = R^{-1}\bar{B}^T\bar{P}(\infty) = [K_1 \vdots K_2] \tag{34}$$

where

$$\bar{A}(\bar{K}) = \left[\begin{array}{c:c} A & 0 \\ \hdashline A_\mu - B_\mu K_1 & A - BK_1 - B_\mu K_2 \end{array}\right] \tag{35}$$

It may also be feasible to substitute $\bar{K}(t) = R^{-1}\bar{B}^T\bar{P}(t)$ into (33) and then solve the resulting initial value problem for $\bar{P}(\infty)$. What one should take great care to note here is, that since A and B contain elements of P, the equation (33) is then no longer of the Riccati type. At this exploratory stage, the stability properties of this equation are not established. The equation may be unstable, it may have more than one stable critical point.

In brief, the optimization problem of S_2 is computationally identical to the standard optimization of S_1, except the fact that the values of $\tilde{K}_1$ and $\tilde{K}_2$ must be assumed and the dimension of the problem is generally doubled. Since the vector $\rho(t)$ does not exist physically, the equation (15) has to be simulated. Hence in addition to the linear memory-less state-feedback as in S_1, the controller in the augmented system S_2 will have to include the simulation of the linear dynamical system as shown in Fig. 4. The matrix transfer function of the dynamic controller is given by

$$U(s) = -\left[K_1 + K_2(sI - A)^{-1}B \right] X(s) \tag{36}$$

where $\tilde{A}$ and $\tilde{B}$ are given by (24).

b) Time-variant system.

We shall briefly sketch the formulation of the problem for the time varying case, where A and B are time varying, or the integration in the performance index is over a finite time interval. The performance index in the standard system S_1 to be minimized is

$$J = 1/2\, x^T(t_1)Fx(t_1) + 1/2\int_{t_c}^{t_1} \left[x^T Q\, x + u^T R\, u \right] dt \tag{37}$$

The control u(t) as usual, is assumed unconstrained. The term $x^T(t_1)\, F\, x(t_1)$ is often called the terminal cost and its purpose is to gurantee that the error at the terminal time t_1 is small. It may be pointed out here that F = 0 in the time invariant case, because the terminal cost at $t_1 \to \infty$ does not make much engineering sense, since the engineer is always interested in the response of a system in finite time. One may question why then let $t_1 \to \infty$. It is done for the

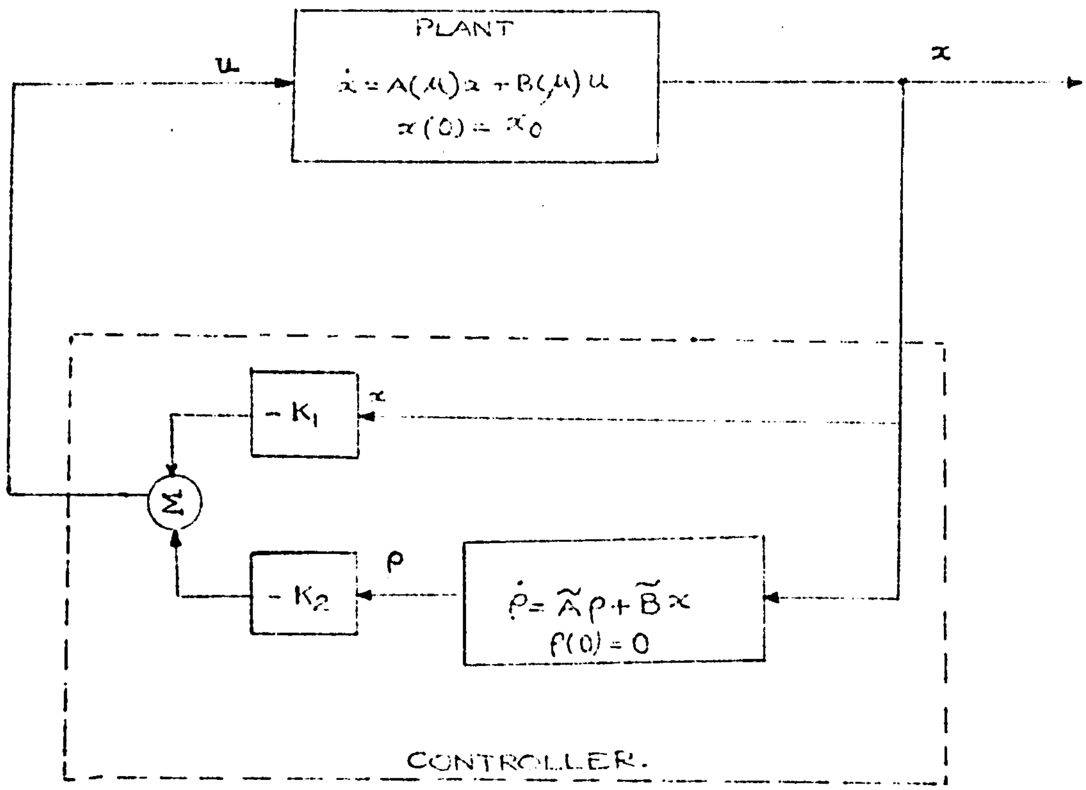


Fig. 4. Structure of the system S_2

following reasons:

1) to gurantee that the state stays near zero after an initial transient interval.

2) to avoid the (somewhat arbitrary) specification of a large terminal time t_1.

A natural extension of the linear optimal regulator problem given by (16) and (37) with a view towards reducing the sensitivity is to consider the performance index

$$I = 1/2\Big\{x^T(t_1)Fx(t_1) + 2\rho^T(t_1)Hx(t_1) + \rho^T(t_1)G\,\rho(t_1) + \int_{t_0}^{t_1}\left[x^T Q\, x + u^T R\, u + 2\rho^T W\, x + \rho^T S\rho\right] dt\Big\} \tag{38}$$

The problem is then to minimize (38) subject to

$$\dot{x} = A\,x + B\,u \qquad\qquad x(0) = x_0 \tag{39}$$

$$\dot{\rho} = \left[A - B\tilde{K}_1(t) - B_\mu \tilde{K}_2(t)\right]\rho + \left[A_\mu - B_\mu \tilde{K}_1(t)\right] x\,; \quad \rho(0) = 0 \tag{40}$$

where the terms between the brackets represent the augmented matrices $\tilde{A}$ and $\tilde{B}$ as given by (24). The matrices A, B, Q, R, W and S appearing in the equations representing the system and the performance index may be time varying. The optimal control is then given by

$$u^*(x, \rho, t) = - K_1(t)\, x - K_2(t)\,\rho \tag{41}$$

where

$$\bar{K}(t) = \left[K_1(t) \,\vdots\, K_2(t)\right] = R^{-1}\bar{B}^T\bar{P}(t) \tag{42}$$

$\bar{P}(t)$ is the solution for $t_0 \leq t \leq t_1$ of the Riccati equation

$$-\frac{d}{dt}\ \bar{P}(t) = \bar{P}(t)\bar{A} + \bar{A}^T\bar{P}(t) - \bar{P}(t)\bar{B}R^{-1}\bar{B}^T\bar{P}(t) + \bar{Q} : \bar{P}(t_1) = F \tag{43}$$

where

$$\bar{A} = \left[\begin{array}{c|c} A & 0 \\ \hline A_\mu - B_\mu\tilde{K}_1(t) & A - B\tilde{K}_1(t) - B_\mu\tilde{K}_2(t) \end{array}\right] \quad \bar{B} = \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] \tag{44}$$

$$\bar{Q} = \left[\begin{array}{c|c} Q & W \\ \hline W & S \end{array}\right] \quad : \quad \bar{F} = \left[\begin{array}{c|c} F & H \\ \hline H & G \end{array}\right] \tag{45}$$

c) Large Parameter-Deviation.

The theory developed under IV. 2a is for differential parameter variaton problem. It would be interesting to see, if the theory can be extended to plants, with large parameter deviations.

For linear plants, a differential equation for Δx

$$\Delta x = x_1 - x$$

where

$$\dot{x} = A(\mu_1)\, x + B(\mu_1)\, u$$

$$\dot{x}_1 = A(\mu_2)\, x_1 + B(\mu_2)\, u_2$$

can be written as shown below :-

Let

$$\Delta A = A(\mu_2) - A(\mu_1) = A_2 - A_1$$

$$\Delta B = B(\mu_2) - B(\mu_1) = B_2 - B_1$$

From above,

$$\dot{x}_1 - \dot{x} = A_2 x_1 - A_1 x + B_2 u_2 - B_1 u$$

that is

$$\Delta\dot{x} = A_2x_1 - A_2x + A_2x - A_1x + B_2u_2 - B_2u + B_2u - B_1u$$

$$\Delta\dot{x} = A_2(x_1 - x) + (A_2 - A_1)\,x + B_2(u_2 - u) + (B_2 - B_1)\,u$$

that is

$$\Delta\dot{x} = A_2\,\Delta x + \Delta A\,x + B_2\Delta u + \Delta B\,u : \text{ where } \Delta u = u_2 - u.$$

Just as in the case of differential parameter-variations, we can consider the minimization of

$$J = 1/2\int_0^\infty [x^T Q\,x + u^T R\,u + 2\,x^T W\rho + \rho^T S\rho]dt \tag{46}$$

subject to

$$\dot{x} = A_1x + B_1u \qquad x(0) = 0 \tag{47}$$

$$\dot{\rho} = (A_2 - B_2\widetilde{K}_1 - \Delta B\widetilde{K}_2)\rho + (\Delta A - \Delta B\widetilde{K}_1)\,x\,;\ \rho(0) = 0 \tag{48}$$

Note that the equation (48) is analogous to equation(25), bearing in mind that the current values are $A_2 = A$, $B_2 = B$, and the derivatives A_μ, B_μ are replaced by the deviations ΔA, and ΔB respectively. The equation (48) for $\rho(t)$ has to be so chosen that it approaximates the equation for $\Delta x(t)$. The result is

$$u^*(x,\rho) = -K_1x - K_2\rho \tag{49}$$

As shown above

$$\Delta\dot{x} = A_2\Delta x + \Delta A\,x + B_2\Delta u + \Delta B\,u$$

Also

$$\Delta u = -K_1\Delta x - K_2\Delta\rho$$

$$\Delta\dot{x} = A_2\Delta x + \Delta A\,x + B_2(-K_1\Delta x - K_2\Delta\rho) + \Delta B(-K_1x - K_2\rho)$$

$$\Delta\dot{x} = (A_2 - B_2K_1)\,\Delta x - \Delta BK_2\rho + (\Delta A - \Delta BK_1)\,x - B_2K_2\Delta\rho$$

The actual $\Delta x(t)$ is given by the additional equations

$$\Delta\dot{x} = (A_2 - B_2K_1)\,\Delta x - \Delta BK_2\rho + (\Delta A - \Delta BK_1)\,x - BK_2\Delta\rho : \quad \Delta x(0) = 0 \tag{50}$$

$$\Delta\dot{\rho} = (A_2 - B_2\tilde{K}_1 - \Delta B\tilde{K}_2)\,\Delta\rho + (\Delta A - \Delta B\tilde{K}_1)\,\Delta x : \qquad \Delta\rho(0) = 0 \tag{51}$$

Comparing the equation (48) for with (50) for Δx, we see that if $\tilde{K}_1$ and $\tilde{K}_2$ are chosen close enough to the actual resulting K_1 and K_2 in (49) then, if the effect of the term $B_2K_2\Delta\rho$ of (50) is negligible, $\rho(t)$ is an approaximation of $\Delta x(t)$.

It should be noted that the above equations are entirely analogous to those of section IV.2a (see equn 21) for the differential sensitivity and in fact, the same algorithm can be used with the augmented matrices

$$\bar{A} = \left[\begin{array}{c|c} A_1 & 0 \\ \hline \Delta A - \Delta B\tilde{K}_1 & A_2 - B_2\tilde{K}_1 - \Delta B\tilde{K}_2 \end{array}\right] \; : \; \bar{B} = \left[\begin{array}{c} B \\ \hline 0 \end{array}\right]$$

$$: \; \bar{Q} = \left[\begin{array}{c|c} Q & W \\ \hline W & S \end{array}\right] \tag{52}$$

for the sensitivity minimization.

d) Multiple Parameter- Deviations.

So far we have only considered a scalar parameter. The theory proposed however can be extended to take care of multiple parameter deviations. Let us consider the case where two parameters μ_1 and μ_2 are subject to deviations. The system equation therefore is given by

$$\dot{x} = A(\mu_1, \mu_2)\, x + B(\mu_1, \mu_2)\, u \;; \qquad x(0) = x_0$$

$$\rho_1 = \frac{\partial x}{\partial \mu_1} \qquad \rho_2 = \frac{\partial x}{\partial \mu_2}$$

From above it follws:

$$\dot{\rho}_1 = A_{\mu_1} x + A\,\rho_1 + B_{\mu_1} u + B\,u_{\mu_1}\,, \text{ and}$$

$$\dot{\rho}_2 = A_{\mu_2} x + A\,\rho_2 + B_{\mu_2} u + B\,u_{\mu_2}\,.$$

The control law will be approaximately given by

$$u \cong - \widetilde{K}_1 x - \widetilde{K}_2 \rho_1 - \widetilde{K}_3 \rho_2$$

$$u_{\mu_1} = - \widetilde{K}_1 \rho_1 \quad ; \quad u_{\mu_2} = - \widetilde{K}_1 \rho_2 \; :$$ neglecting higher order derivatives, that is

$$\dot{\rho}_1 = A_{\mu_1} x + A\,\rho_1 + B_{\mu_1} (-\widetilde{K}_1 x - \widetilde{K}_2 \rho_1 - \widetilde{K}_3 \rho_2) - B\,\widetilde{K}_1 \rho_1$$

$$\dot{\rho}_1 = (A - B\widetilde{K}_1 - B_{\mu_1} \widetilde{K}_2)\,\rho_1 - B_{\mu_1} \widetilde{K}_3 \rho_2 + (A_{\mu_1} - B_{\mu_1} \widetilde{K}_1)\,x$$

Similarly,

$$\dot{\rho}_2 = A\rho_2 + A_{\mu_2} x + B_{\mu_2} (-\widetilde{K}_1 x - \widetilde{K}_2 \rho_1 - \widetilde{K}_3 \rho_2) - B\widetilde{K}_1 \rho_2$$

$$\dot{\rho}_2 = (A - B\breve{K}_1 - B_{\mu_2}\tilde{K}_3)\rho_2 - B_{\mu_2}\tilde{K}_2\rho_1 + (A_{\mu_2} - B_{\mu_2}\tilde{K}_1)x$$

We can now consider the equations,

$$\dot{x} = A(\mu_1, \mu_2)x + B(\mu_1, \mu_2)u \qquad x(0) = 0$$

$$\dot{\rho}_1 = (A - B\tilde{K}_1 - B_{\mu_1}\tilde{K}_2)\rho_1 - B_{\mu_1}\tilde{K}_3\rho_2 + (A_{\mu_1} - B_{\mu_1}\tilde{K}_1)x : \quad \rho_1(0) = 0$$

$$\dot{\rho}_2 = (A - B\breve{K}_1 - B_{\mu_2}\tilde{K}_3)\rho_2 - B_{\mu_2}\tilde{K}_2\rho_1 + (A_{\mu_2} - B_{\mu_2}\tilde{K}_1)x : \quad \rho_2(0) = 0$$

and the performance index,

$$J = 1/2\int_0^\infty [x^T Q x + u^T R u + 2x^T W_1\rho_1 + 2x^T W_2\rho_2 + 2\rho_1^T V\rho_2$$
$$+ \rho_1^T S_1\rho_1 + \rho_2^T S_2\rho_2]\, dt$$

where $\breve{K}_1, \tilde{K}_2, \tilde{K}_3$ are assumed matrices close enough to the actual K_1, K_2, and K_3 of the minimizing control

$$u^* = -K_1x - K_2\rho_1 - K_3\rho_2$$

Again, we should bear in mind that ρ_1 and ρ_2 are only approaximations of the actual closed-loop sensitivities. We can introduce some further simplification, if in the above equations, we choose

$$B_{\mu_1}\tilde{K}_2 = B_{\mu_2}\tilde{K}_3.$$

Then the transfer functions from x to ρ_1 and to ρ_2 will all have the same denominator, thus the number of parameters will affect only the zeroes of the transfer functions. This suggests that in many cases one may use an approaximately equivalent single parameter, instead of considering multiple parameters. However, the difficulty in the task of determining such an equivalent single parameter should not be underestimated, and it may require some trial and error simulation.

IV. 3 Discussions and Comments.

The technique of minimizing the trajectory sensitivity by introducing the sensitivity terms in the performance index has been studied by several authors.[11 - 16] The idea was first introduced by Tuel[13], and later considerable investigation has been done by others, such as Kriendler, De Russo, Dougherty. In this section, we shall summarize the important design features of the techniques proposed by several authors.[8 - 17]

As has been pointed out earlier, to generate $\rho(t)$, for the control law given by (20), the equation (25) has to be simulated in one path of the feedback controller as shown in Fig. 4. Thus the controller of S_2 contains dynamics, generally of order n. This of course renders the system S_2, complicated compared to the system S_1.

Kalman[18] postulated, that matrices F, Q are required to be non-negative, along with certain controllability and observability conditions, to ensure existence and uniqueness of solution. These postulates, natural for the original system $\{A, B, Q\}$ may not be fulfilled in the augmented system $\{\bar{A}, \bar{B}, \bar{Q}\}$, and weaker conditions might hold.

The differential equation for $\rho(t)$ depends on the matrices $\tilde{K}_1(t)$ and $\tilde{K}_2(t)$. A very natural question is, then how does one obtain the matices $\tilde{K}_1(t)$ and $\tilde{K}_2(t)$.

One way is to substitute in (44), that is, in the equation for the augmented matrix A,

$$\tilde{K}_1(t) = K_1(t) \text{ and } \tilde{K}_2(t) = K_2(t)$$

The equation (43) is then no longer of Riccati type, because the augmented matrices $\bar{A}$ and $\bar{B}$ contain elements of $\bar{P}$. However, the resulting $\bar{K}$ is correct if the modified equation has a unique solution in the control interval. Having obtained $\tilde{K}_1$ and $\tilde{K}_2$, $\bar{K}$ can be checked by resolving (43) as a Riccati equation. Intuitively, it seems that since the variation $\frac{\partial \rho}{\partial \mu}$ was neglected in approximmation of $\rho(t)$ to $\sigma(t)$, substituting $\tilde{K}_1(t) = K_1(t)$ and $\tilde{K}_2(t) = K_2(t)$ will generally, not give the best results.

The other alternative approach is somewhat similar to that adopted by Merriam.[19] This approach avoids the uncertainty of the modified equation, and allows the provision of studying the effect of changing $\tilde{K}_1$ and $\tilde{K}_2$. The matrices weighting the sensitivity terms are usually left at the designer's disposal, and the final choice is made to satisfy several realistic design constraints. The values of these are gradually increased in trial and error fashion. Thus one can initially assume for $\tilde{K}_1$ the value of K in S_1 and then update $\tilde{K}_1$, and $\tilde{K}_2$ as the trial and error design proceeds. The essential thing is that $\sigma(t)$ closely approximates $\rho(t)$, especially when $\rho(t)$ decreases with the increasing values of the sensitivity weighting matrices. It does not seem difficult to devise an algorithm, for this procedure, but substantial amount of investigation is required before any generalized algorithm is attempted.

If the approximation of $\sigma(t)$ by $\rho(t)$ does not work, namely $\sigma(t)$ is larger than $\rho(t)$, then the neglected term $BK_2(t)\frac{\partial\rho}{\partial\mu}$ should be taken into account. This requires addition of the term $BK_2\gamma$ to the differential equation for $\rho(t)$, and addition of the vector differential equation for γ to that for x and ρ. Incorporation of each higher order term, it should be noted, increases the dimension of the problem and of the controller by n.

IV.4 Comparison of S_1 and S_2.

The problem of sensitivity reduction posed as an overall augmented optimization problem, unfortunately is not free of complications. Although the augmented problem is likely to reduce the sensitivity of the system, the crucial question is whether the additional trouble is worthwhile. Although there is no clear cut answer to it, it is only fair to point out, that the decision should be based on the selection of suitable basis for comparison of the system S_2 with system S_1.

It is always assumed that some or all the elements of the matrices entering the perfromance indices of the system S_1 and S_2, are at the disposal of the designer to achieve satisfactory response, of course, under certain control limitation. The fact that the minimization of the quadratic cost criteria is only a means to and end, changes the optimization procedure from synthesis to design, introduces a measure of subjectivity in the design evaluation, and makes comparison of S_1

and S_2 considerably difficult. The addition of sensitivity term into the performance index of the system S_1, affects besides the sensitivity, both the control u(t) and the response $x_2(t)$. It is by and large unsatisfactory to design S_2 by merely adding a sensitivity term to the performance index of S_1, and then just accepting the design, if the sensitivity of S_2 is lower than that of S_1. The basic question is, whether S_2 is significantly better than S_1 in satisfying often conflicting design goals and constraints as reduced sensitivity, speed of response, maximum overshoot, maximum control magnitude, and noise immunity. Considerable amount of numerical experimentation is nec essary for each case, before any practically meaningful answer can be found.

It should be noted that different basis for comparison may lead to different conclusions. If there is no limitations on the magnitude of the control and the feedback coefficients, a criterion for the usefulness of S_2 is provided by the question:-

Given a particular desirable response of S_1, is it possible to produce this response in S_2, with reduced sensitivity, perhaps at the expense of higher magnitude of control ?

In the next chapter an illustrative example will be presented which will clarify most of the points discussed under this section and bring out several design criteria worthy of careful consideration.

CHAPTER V

AN ILLUSTRATIVE EXAMPLE.

V. 1 Design of S_1 and S_2. (for a first order plant)

The system S_1 is the first order plant

$$\dot{x} = ax + u : \quad a_o = -1 \quad : \quad x(0) = 1 \tag{1}$$

where a_o is the nominal value of a, and the performance index is

$$I = 1/2 \int_0^{\infty} (Qx^2 + u^2)\, dt \tag{2}$$

where Q is a positive real number. The minimizing control is easily found to be

$$u^* = -k\,x : \quad k = a + \sqrt{a^2 + Q} \tag{3}$$

The closed loop system is therefore given by

$$\dot{x} = (a - k)\, x : \qquad x(0) = 1 \tag{4}$$

hence

$$\dot{\sigma} = (a - k)\sigma + x \qquad \sigma(0) = 0 : \quad \sigma = \frac{\partial x}{\partial a} \tag{5}$$

that is the transfer functions are:

$$X(s) = \frac{x(0)}{s - a + k} \tag{6}$$

and

$$\Sigma(s) = \frac{x(0)}{(s - a + k)^2} \tag{7}$$

The system S_2 is the same plant (1), but the performance index

$$J = 1/2\int_0^\infty [Q(x(t))^2 + (u(t))^2 + S(\rho(t))^2]\, dt \tag{8}$$

with $Q>0$, and $S>0$, is to be minimized. $\rho(t)$ is given by

$$\dot{\rho} = (a_o - \widetilde{k}_1)\rho + x : \qquad \rho(0) = 0 \tag{9}$$

The minimizing control is given by

$$u^* = - k_1 x - k_2 \rho \tag{10}$$

where k_1 and k_2 are, as outlined in the preceeding chapter, obtained from the steady state solution $P = P(\infty)$ of the Riccati equation

$$\dot{P}(t) = P(t)A + A^T P(t) - P(t) B R^{-1} B^T P(t) + Q$$

In this example,

$$k = b^T P = [1 \quad 0] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = [p_{11} \quad p_{12}]$$

hence

$$k_1 = p_{11} \text{ and } k_2 = p_{12}$$

At steady state, $\dot{p} = 0$, and the matrix Riccati type differential equation reduces to a set of quadratic algebraic equations for the elements of P.

In the example the algebraic equations, after rearrangements reduce to

$$\left.\begin{aligned} k_1^2 - 2(ak_1 + k_2) - Q &= 0 \\ k_1 k_2 - [(a_o - \widetilde{k}_1 + a)k_2 + p_{22}] &= 0 \\ k_2^2 - 2(a_o - \widetilde{k}_1)p_{22} - S &= 0 \end{aligned}\right\} \tag{11}$$

The closed loop system is given by

$$\dot{x} = (a - k_1)x - k_2\rho \quad : \qquad x(0) = 1$$
$$\dot{\rho} = (a_o - \tilde{k}_1)\rho + x \quad : \qquad \rho(0) = 0 \tag{12}$$

and the closed loop sensitivity $\sigma(t) = \frac{\partial x}{\partial a}$ is given by the additional equations

$$\dot{\sigma} = (a - k_1)\sigma + x - k_2\frac{\partial \rho}{\partial a} \qquad \sigma(0) = 0 \tag{13}$$

$$\frac{d}{dt}\left(\frac{\partial \rho}{\partial a}\right) = (a_o - \tilde{k}_1)\frac{\partial \rho}{\partial a} + \sigma \qquad \frac{\partial \rho}{\partial a}(0) = 0 \tag{14}$$

If the quadratic algebraic equations are solved, rather than the Riccati differential equations, the assumed value in (9) can be taken as k_1. Then if we neglect the term $k_2\frac{\partial \rho}{\partial a}$ the equation (9) for $\rho(t)$ is identical to the equation (13) for $\sigma(t)$.

For $a = a_o = -1$, and $\tilde{k}_1 = k_1$, the equations (11) reduce to

$$k_1^2 + 2k_1 - 2k_2 - Q = 0$$
$$2k_2(1 + k_1) - p_{22} = 0 \tag{15}$$
$$k_2^2 + 2(1 + k_1)p_{22} - S = 0$$

The signs of the roots are given by the condition that the matrix $P = P(\infty)$ must be positive definite. This gives the conditions

$$k_1 > 0 \quad : \qquad k_1 p_{22} - k_2^2 > 0 \tag{16}$$

the equation for k_1 is given by,

$$k_1 = \frac{2a \pm\sqrt{4\,a^2 + 4Q}}{2} = a \pm\sqrt{a + Q} \qquad (16)$$

which is the same as the equation indicated by (3).

The system S_2 has the configuration shown in Fig. 5, with the transfer function

$$X(s) = \frac{(s - a_o + \tilde{k}_1)\, x(0)}{s^2 + (k_1 + \tilde{k}_1 - a - a_o)\, s + (k_1 - a)(\tilde{k}_1 - a_o) + k_2} \qquad (17)$$

The closed loop sensitivity $\sigma(t)$ has the following Laplace transform

$$\Sigma(s) = \left[\frac{s - a_o + \tilde{k}_1}{s^2 + (k_1 + \tilde{k}_1 - a - a_o)\, s + (k_1 - a)(\tilde{k}_1 - a_o) + k_2}\right]^2 x(0) \qquad (18)$$

Since $k_2 > 0$, for $\tilde{k}_1 = k_1$ and $a = a_o = -1$, it can be written as

$$X(s) = \frac{(s + 1 + k_1)\, x(0)}{(s + 1 + j\, k_2)\,(s + 1 - j\, k_2)} \qquad (19)$$

clearly, if $Q \gg S$, then $k_1 \gg k_2$, and

$$X(s) = \frac{x(0)}{(s + 1 + k_1)}$$

which is the same the Laplace transform of the response of S_1. Thus, when the sensitivity term S^2 has an appreciable effect, the response $x_2(t)$ is that of an underdamped second order system, while $x_1(t)$ is the response of the forst order system.

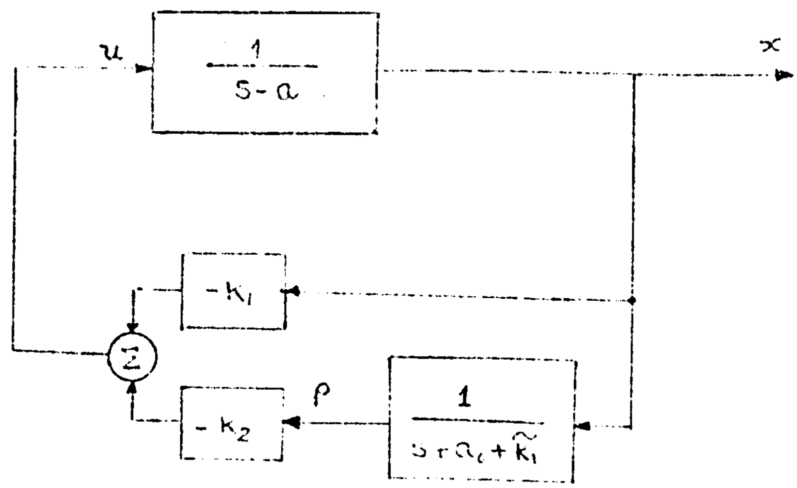


Fig. 5. Configuration of S_2.

V.2. Comparison of S_1 and S_2.

Since the responses of S_1 and S_2 are of a different nature, the two systems were compared on the basis of equal maximum magnitude of the control u(t). In the present example

$$\max_{0 \leq t < \infty} \left| u(t) \right| = \left| u(0) \right| = k_1$$

and hence S_1 and S_2 are compared on the basis of $k = k_1$.

The closed loop sensitivities are shown in Figure. 6. Figures. 7 and 8 show the responses of the systems S_1 and S_2 respectively. The control functions are shown in Fig. 9. Figures. 7 and 8 also show the responses when the parameter deviates by Δa, that is

$$a = a_o + \Delta a = -1 + 4$$

That $\Delta a = 4$ cannot be considered 'small' is evident from the fact that $\Delta x(t)$ for $\Delta a = -4$ are quite different.

It is seen that at no extra effort in terms of the maximum magnitude of the control, there is some improvement of $\sigma(t)$ in S_2, as well as the nominal response. For example, the 5% settling time of S_1 is 0.75 sec while that of S_2 is 0.47 sec. Another important point to note, that while the improvement in differential sensitivity is not spectacular, considerable improvement is achieved in large-parameter deviations: for instance at $a = a_o + \Delta a = 3$, the system S_1 is on the verge of instability while the response of S_2 is slower, but reasonably good and well behaved. It should be also noted that the approximate closed loop sensitivity $\rho(t)$ is reasonably close to the exact sensitivity $\sigma(t)$.

The stability margin with respect to Δa can be easily obtained for S_1 and S_2. From the characteristic equation for S_1

$$s - a_o - \Delta a + k = 0$$

the critical Δa is given by

$$\Delta a_{c1} = - a_o + k \tag{20}$$

The characteristic equation for S_2, for $\tilde{k}_1 = k_1$, is given by

$$s^2 + s(2k_1 - 2a_o - \Delta a) + (k_1 - a_o)(k_1 - a_o - \Delta a) + k_2 = 0$$

If $k_2 \geqslant (k_1 - a_o)^2$: then $\Delta a_{c2} = 2(k_1 - a_o)$.

If $k_2 \leqslant (k_1 - a_o)^2$: then

$$\Delta a_{c2} = - a_o + k_1 + \frac{k_2}{k_1 - a_o} \tag{21}$$

Since the first of equations (11) limits k_2 to

$k_2 \leqslant 1/2\,(k_1^2 - 2\,a_o k_1) = 1/2\,(k_1 - a_o)^2 - 1/2\,a_o^2$ which is less than $(k_1 - a_o)^2$: the only alternative (21) is valid. Hence on the basis of $k = k_1$, the system S_2 always has a larger margin of stability with respect to deviations in the parameter a than that for the system S_1. In this case, for example, for $a_o = -1$, $k_1 = 3$, $k_2 = 7.5$; we have

$$\Delta a_{c1} = 4 : \qquad \Delta a_{c2} = 5.875$$

The entire root locus of S_2 for the parameter a is shown in Fig. 10.

The value of $Q = 0$ was chosen in this case to obtain the largest possible k_2 consistent with $k_1 = 3$ to give the best results. It was found, that for

$$Q = 10 : \qquad S = 31.25 : \qquad k_1 = 3 : \qquad k_2 = 2.5$$

the 5% settling time is 0.59, and the peak of the graph for $\sigma(t)$ is 0.087, which is somewhat worse than the case shown in figures 6 and 8. The choice of Q = 0, as was found, is not necessarily always the best. For example $k_1 = 9$, Q = 0 gave a response with an overshoot larger than 5%. As was pointed out in the later section of the preceeding chapter, considerable amount of difficulty was encountered in the selection of proper Q to give desired response with significant reduction in the system sensitivity. Although no clear cut design procedure exists, with regards to the choice of weighting matrix, some knowledge has been gained about its role. The next chapter will be basically devoted to it.

It is obvious from Fig. 6, that for this particular example,

$$\int_0^\infty (\sigma_2(t))^2 \, dt < \int_0^\infty (\sigma_1(t))^2 \, dt \tag{22}$$

Just as a matter of interest we tried to see if the sufficient condition obtained by applying Parseval's theorem is valid for this example. By Parseval's theorem (22) becomes equivalent to

$$\int_0^\infty |\Sigma_2(j\omega)|^2 \, d\omega < \int_0^\infty |\Sigma_1(j\omega)|^2 \, d\omega \tag{23}$$

Let,

$$\frac{\Sigma_1(j\omega)}{\Sigma_2(j\omega)} = T(j\omega)$$

Hence a sufficient condition for (22) is

$$|T(j\omega)| > 1 \quad \text{for all} \quad \omega \geqslant 0 \tag{24}$$

From (7) and (18), for $k = k_1$,

$$T(j\omega) = \left[1 + \frac{k_2}{(j\omega + 1 + k_1)^2}\right]^2$$

It is clear that at some $\omega = \omega_1$, the plot of $T(j\omega)$ crosses the unit circle and remains within it for $\omega > \omega_1$, violating (24). The sufficient condition (23), we thus found out is too restrictive for this example.

V.3 Use of the term $x^T W \rho$ in the Performance index.

The addition of the term $S\rho^2$ to the performance index (2) of S_1, as we found out, resulted in an underdamped second order system S_2 with state x(t) overshooting zero. Supposing that such an overshoot of x(t) is not permissible. Some numerical experimentation was done with this particular problem, and the purpose of this section is to explain the possibility of achieving an overdamped responses in the system S_2, similar to those in S_1.

For the system S_2 to have real roots, (11) shows that k_2 must be negative. Also from (15) we find that since k_1 and p_{22} must be positive, for k_2 to be negative, the middle equation of (15) should be of the form

$$2k_2(1 + k_2) - p_{22} - W = 0 \qquad W < 0$$

This corresponds to the modified performance index

$$I = 1/2 \int_0^\infty \left[Q(x(t))^2 + (u(t))^2 + 2Wx\rho + S(\rho(t))^2 \right] dt$$

At first, we tried to adjust the performance index so that $\tilde{k}_1 = k_1 = 3$ and the response is overdamped with a 5% settling time of 0.75, comparable with results for S_1 presented in Figures. 6 and 7. Unfortunately, the sensitivity $\sigma_2(t)$ was larger than $\sigma_1(t)$, and the design of S_2 therefore could not be accepted. In the process of experimentation, it became clear that inorder to get an overdamped response in S_2, with reduced sensitivity, higher gains must be used. The graphs for such a case are shown in Figures. 11 and 12, where the state $x_1(t)$ and $x_2(t)$ for both systems have the same settling time. Evidently S_2 is less sensitive, at the price of a thirteen times larger maximum magnitude of the control.

From Fig. 12, it is seen that $\rho_2(t)$ is much smaller than $\sigma_2(t)$, which suggests that further improvement can be achieved using a higher order approximation of $\sigma_2(t)$, which means the addition of additional equations to (12). Increasing the gains does not solve all problems. During numerical experimentation it was found that the approximation of $\sigma(t)$ by $\rho(t)$ deteriorates, as gain is increased to reduce the sensitivity of the system S_2. Although the addition of the cross product term $2\,W\,x\,\rho$ does preserve the damped nature of the response, and allows us the basis of 5% settling time, for the comparison of two systems, it certainly is not above criticism. Extensive amount of computational investigation has to be done, before the introduction of this cross product term in the performance index , can be positively claimed as benificial.

V.4 Comments and Conclusions.

The results and conclusions of the study of the augmented optimization problem for the sensitivity reduction can be summarized as follows:-

The closed loop sensitivity can be introduced in the formulation of the problem only approximately. The simulation of the equation for the approximate closed loop sensitivity introduces dynamics into the usually static state feedback controller. This provides theoretical justification for using dynamic feedback even though all the state components are used by the controller.

The design of S_2 is more complicated than S_1 because of addition of n differential equations for ρ, containing unknown components of matrices $\widetilde{K}_1$ and $\widetilde{K}_2$, and also due to the addition of more weighting multipliers in the performance index. In higher order systems there might be severe complications, and these should not be under-rated.

In some cases the approximation $\rho \cong \sigma$ may break down. This requires the need to check the accuracy of $\rho(t)$, by solving for $\sigma(t)$. This includes solving 4n differential equations.

The addition of the sensitivity terms alters the response $x_1(t)$ of the system S_1. In general, a specified response of S_1, may not be attainable in S_2 with an appreciable sensitivity reduction. It may be attainable with difficulty, and at the cost of higher control magnitudes.

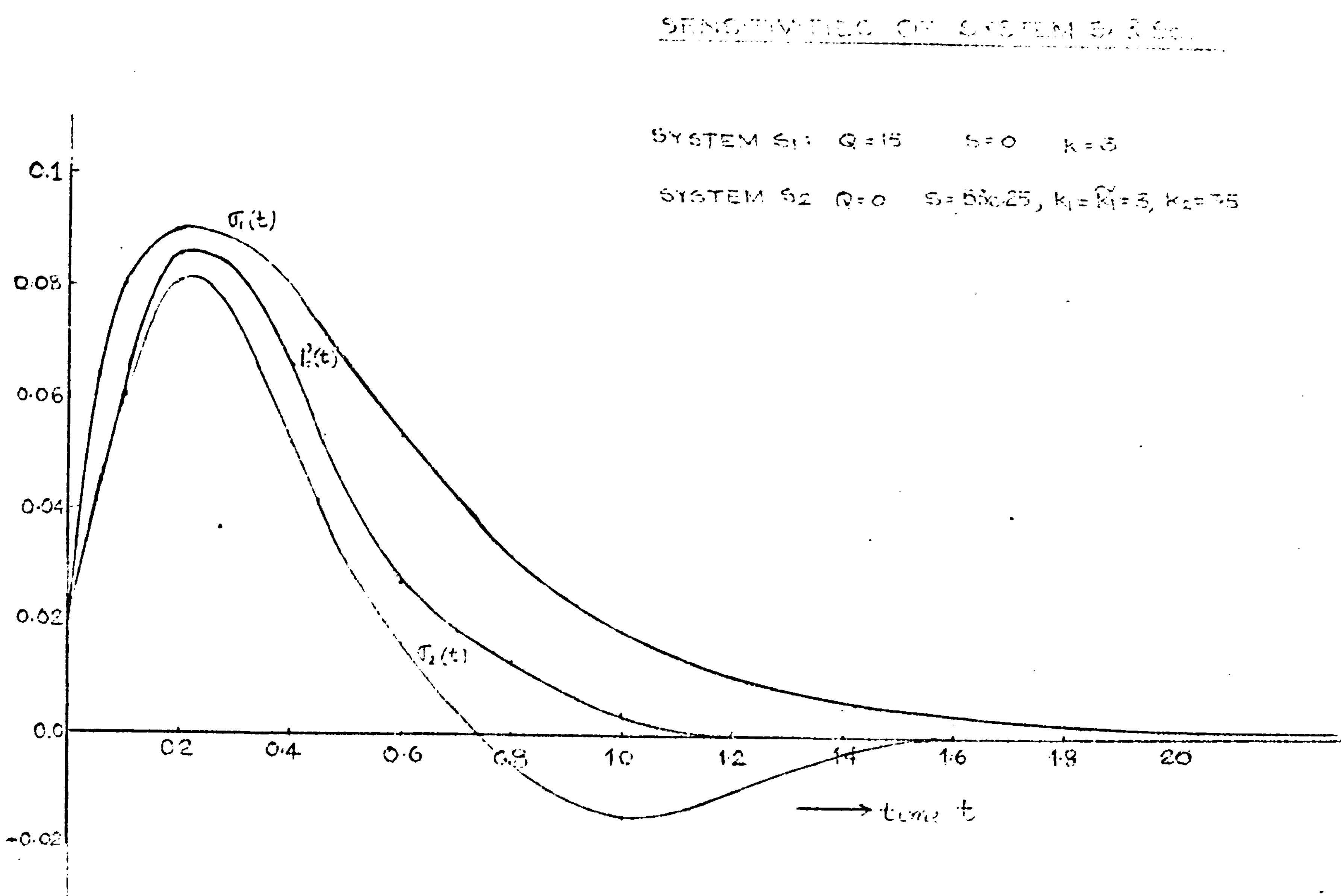
Although the reduction of sensitivity in S_2 enumerated in the example may not seem as much, one should bear in mind, that large sensitivity reduction achievable by linear feedback systems is associated with tremendous amount of increase in the loop gain (sometimes hundred folds), where as in the example the maximum magnitude of the control was limited. Also this basis for comparison of sensitivity between two systems was very severe.

It is by and large true to say, that the practical importance of the sensitivity problem suggests further investigation to determine situations where S_2 is superior to S_1. Intuitively, it seems that by employing ' observer's principle ' the system S_2 may prove to be advantageous where the entire state is not available for measurement. It may also be feasible, in certain multi-input plants to achieve parametric invariance, or reducing the terminal sensitivity

The present investigation can hardly be regarded as complete, and the scope for considerable amount of theoritical and experimental work exists. Devoloping a systematic approach for the design of the system S_2 (and that of S_1 as well) involves systematizing the selction of the weighting matrices in the performance index, and this will be of considerable benifit to the designers. Other areas of investigation may include the study of the margin of stability of S_1 and S_2 with respect to large parameter deviations, and extension of the augmented problem to the general non-linear case.

GRAPHICAL RESULTS FOR THE EXAMPLE COMPARING THE STANDARD SYSTEM WITH THE AUGMENTED SYSTEM AND THE ROLE OF THE WEIGHTING MATRIX.

List of graphs.



SYSTEM S1 : Q = 15 S = 0 k = 3
SYSTEM S2 Q = 0
σ1(t)
σ2(t)
0.1
0.08
0.06
0.04
0.02
0.0
-0.02
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
time t

Figure. 6

RESPONSES OF SYSTEM S1 FOR SEVERAL VALUES OF a

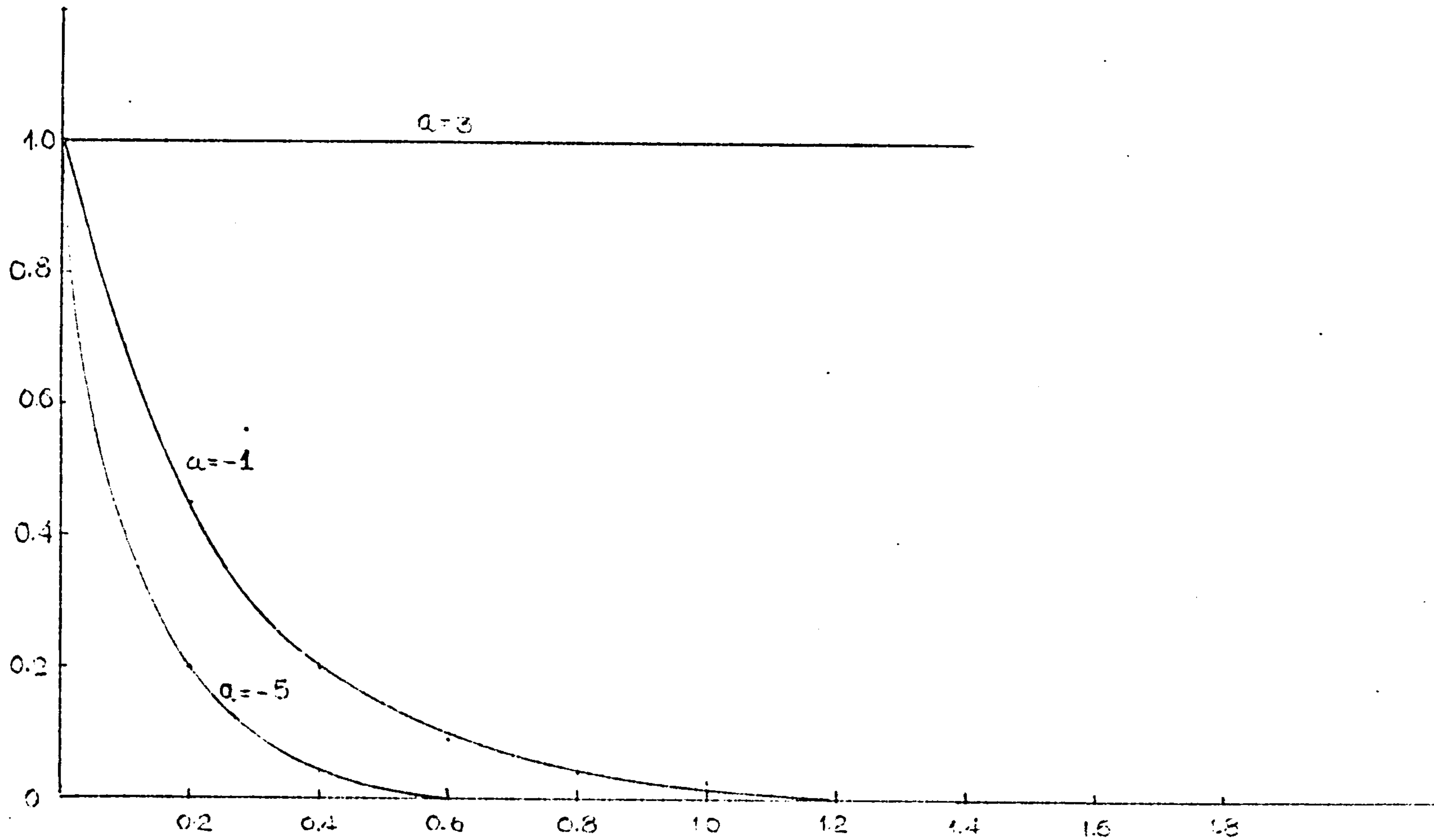


Figure. 7

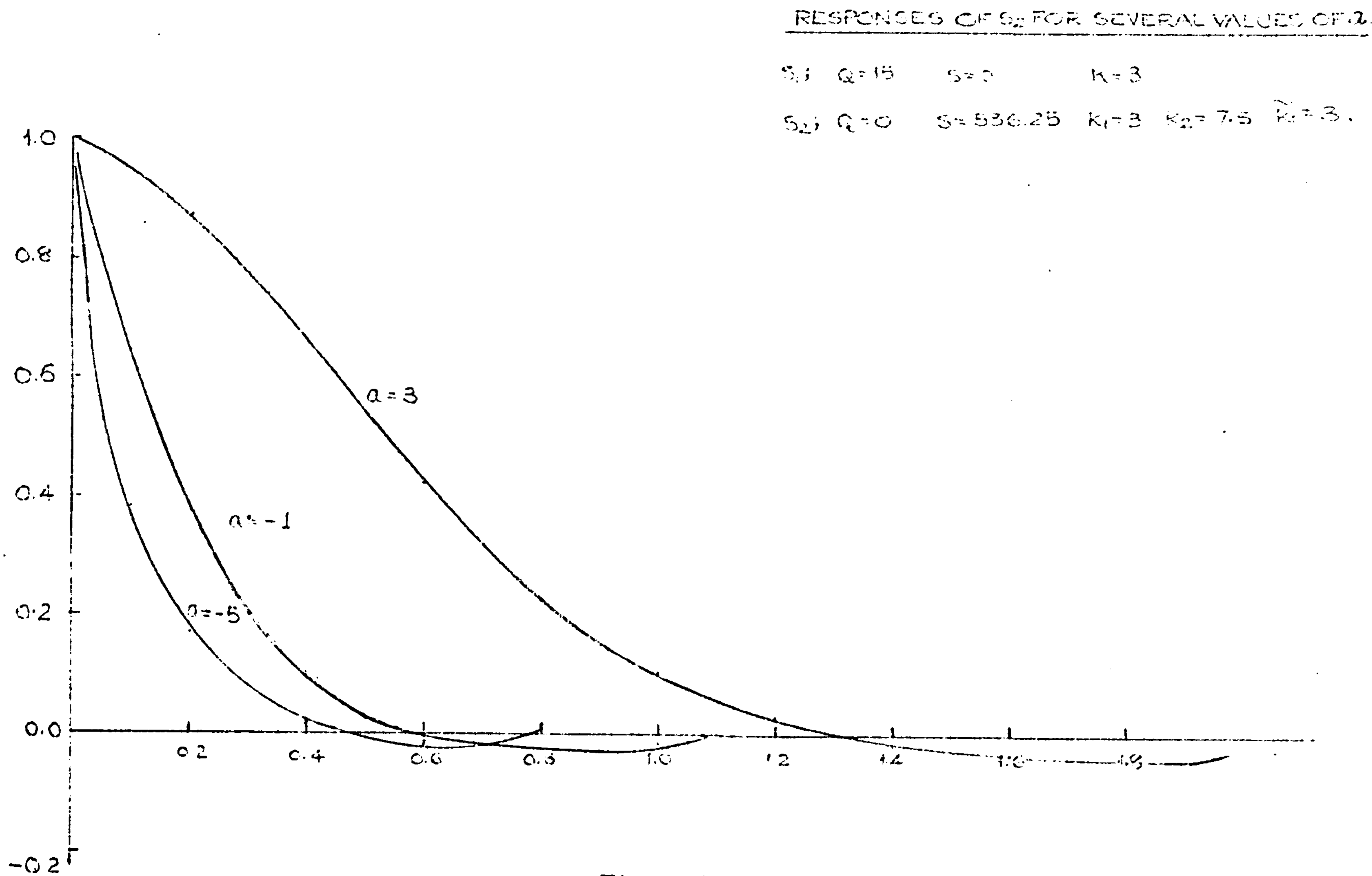

RESPONSES OF S_2 FOR SEVERAL VALUES OF a.
S_1; Q=15 S=0 K=3
S_2; Q=0 S=536.25 K_1=3 K_2=7.5
a=3
a=-1
a=-5
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0.2
0.4
0.6
1.0
1.2
1.4

Figure.8

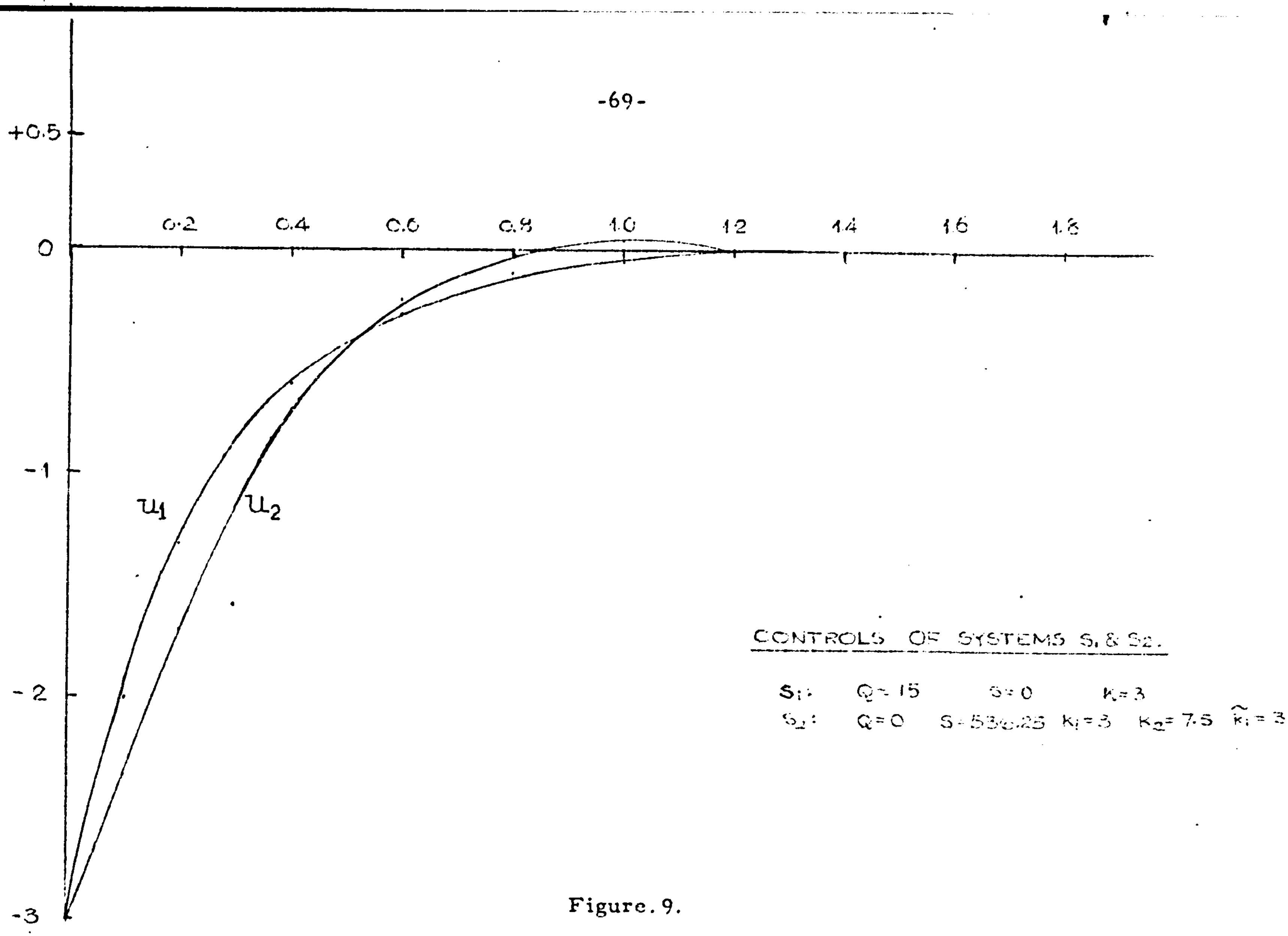
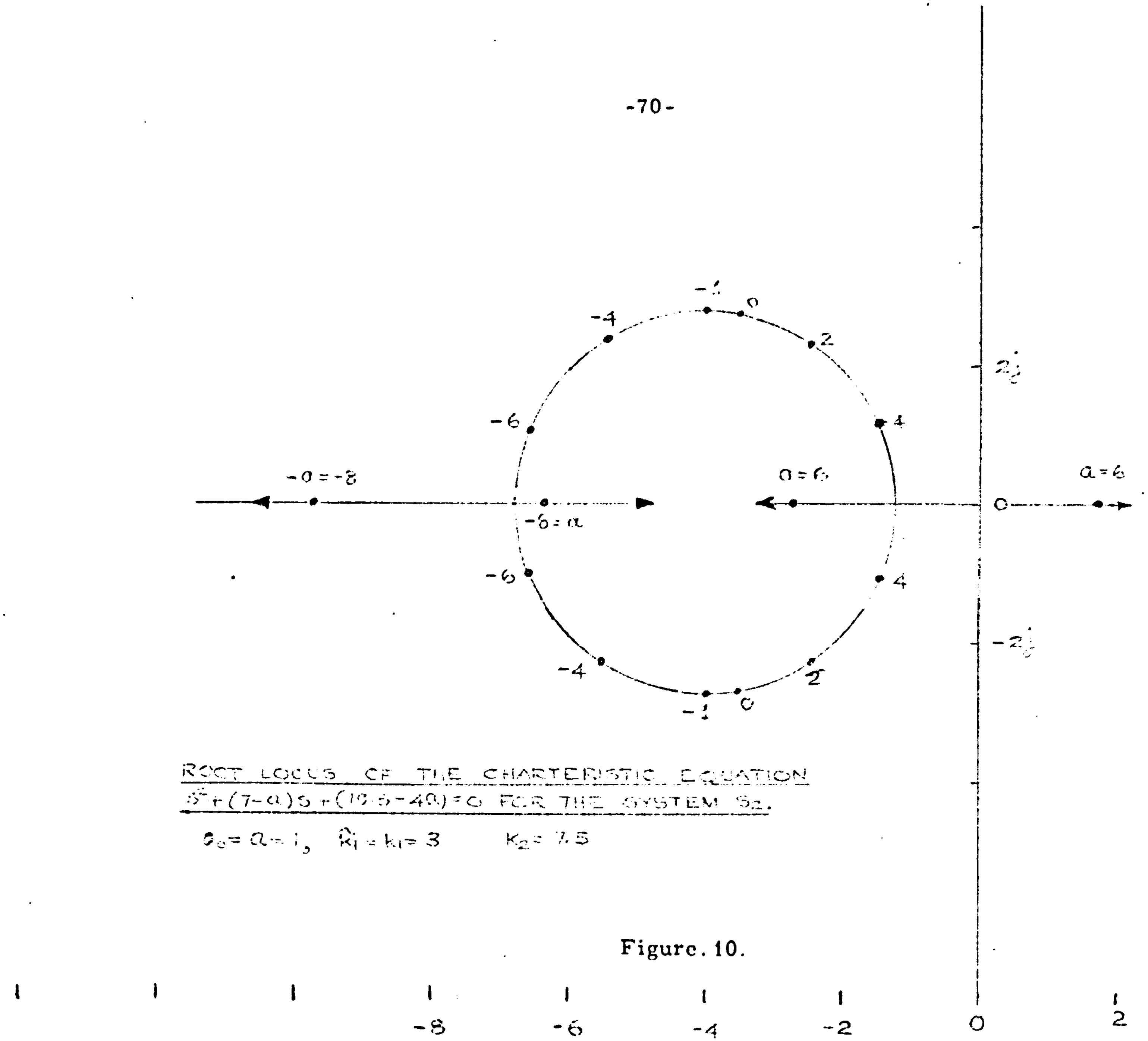

+0.5
0
-1
-2
-3
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
u_1
u_2
CONTROLS OF SYSTEMS S_1 & S_2.
S_1: Q = 15 S = 0 k_1 = 3
S_2: Q = 0 S = 536.25 k_1 = 3 k_2 = 7.5 $\tilde{k}_1$ = 3.

Figure. 9.



ROOT LOCUS OF THE CHARTERISTIC EQUATION
$s^2+(7-a)s+(19.5-4a)=0$ FOR THE SYSTEM S_2.

$s_0 = a = 1, \hat{k}_1 = k_1 = 3 \qquad k_2 = 7.5$

Figure. 10.

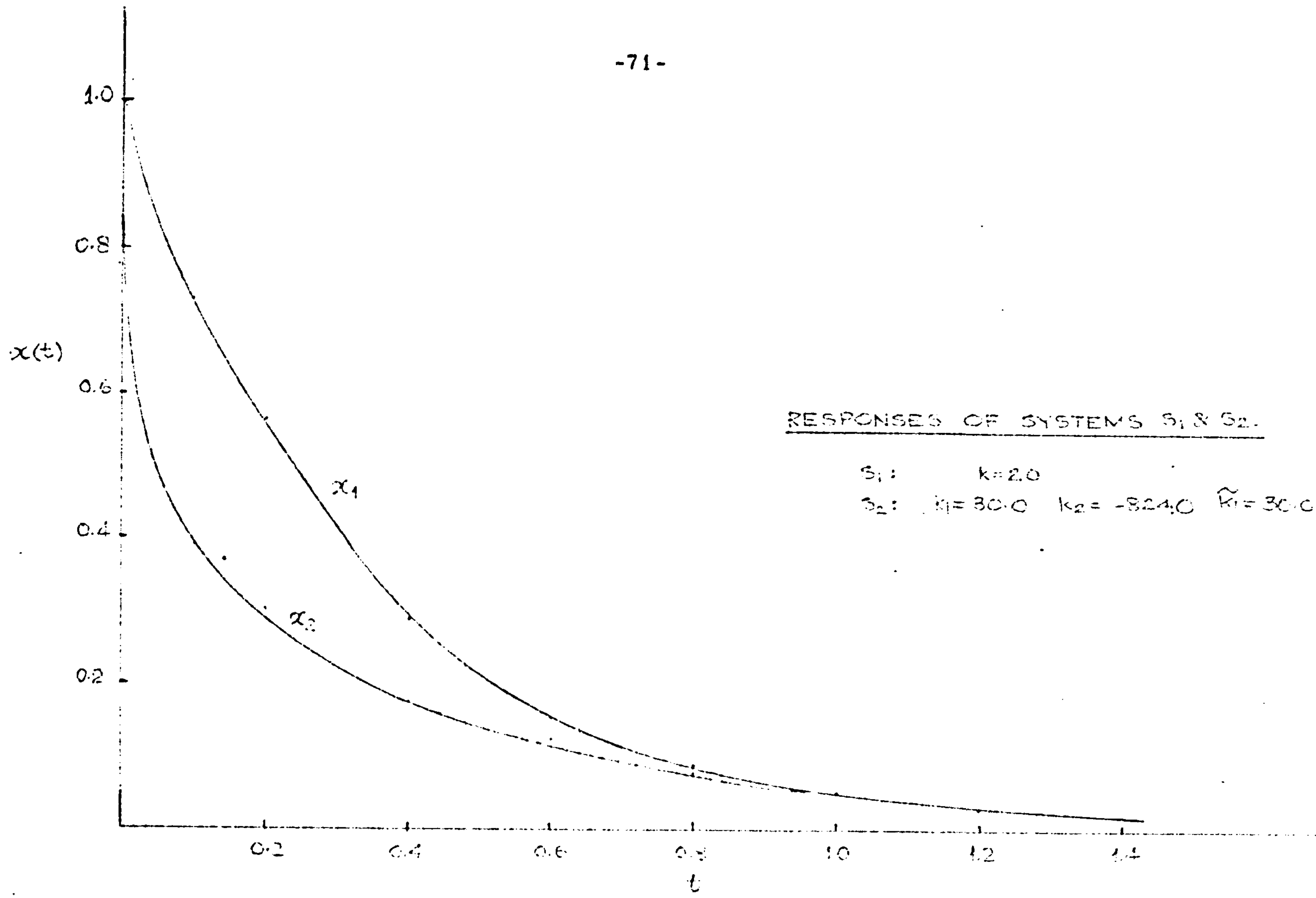

1.0
0.8
0.6
0.4
0.2
x(t)
x1
x2
0.2
0.4
0.6
0.8
1.0
1.2
1.4
t
RESPONSES OF SYSTEMS S1 & S2.
S1: k=2.0
S2: k1=30.0 k2= -824.0 k̃1=30.0

Figure. 11.

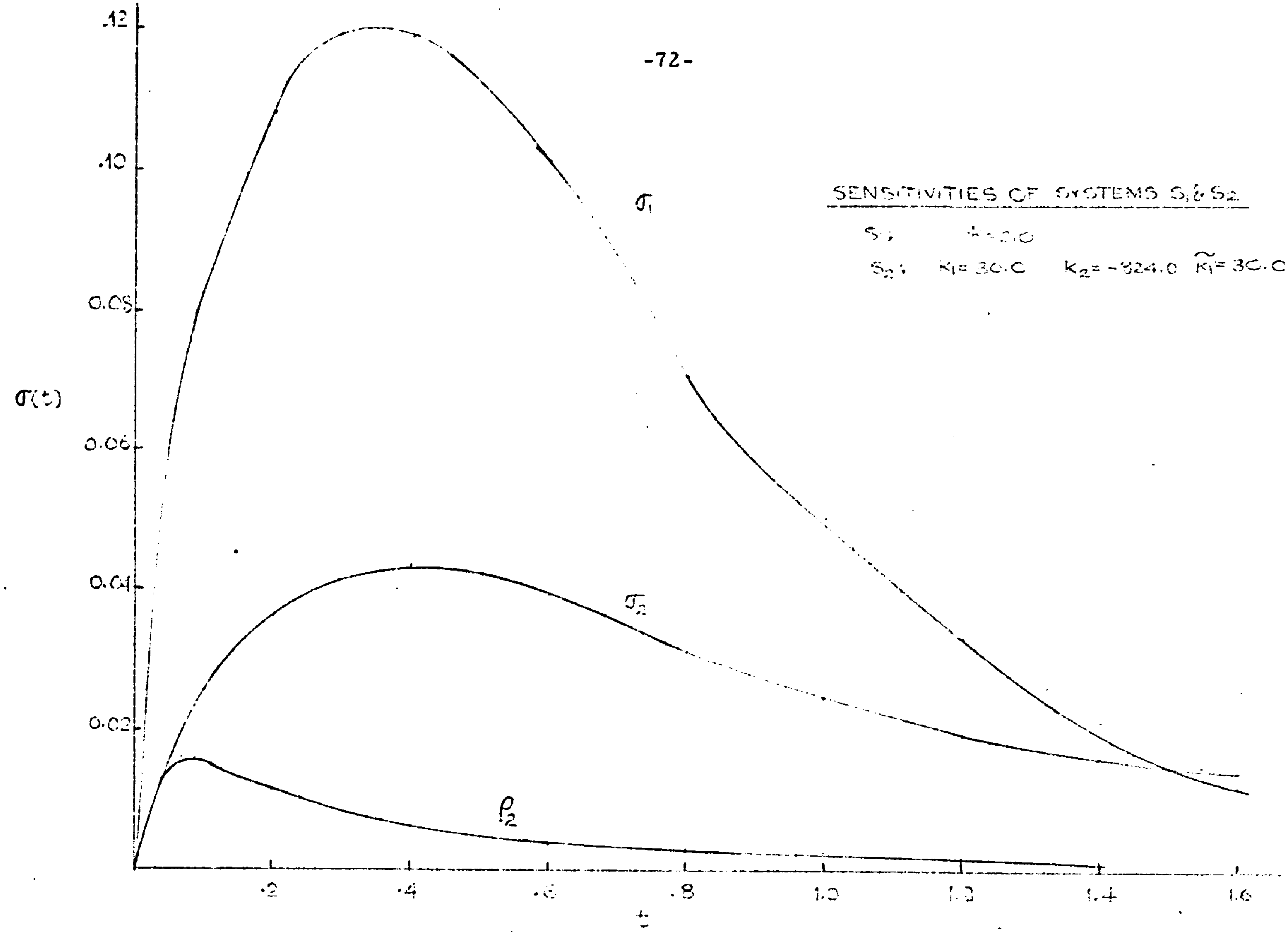

SENSITIVITIES OF SYSTEMS S1 & S2
S2: K1= 30.0 K2= -824.0 K̃1= 30.0
σ1
σ2
ρ2
σ(t)
.12
.10
0.08
0.06
0.04
0.02
.2
.4
.6
.8
1.0
1.4
1.6
t

Figure. 12.

CHAPTER VI

ON EQUIVALENT QUADRATIC WEIGHTING MATRIX.

In the earlier chapters, it has been borne out that the optimal system synthesis virtually becomes a system design, under realistic performance constraints, because of the role played by the weighting matrix in the performance index. The purpose of this chapter is to present some important theoritical findings on the equivalent weighting matrix. Most of the contents have been extracted from several recent publications in the literature.

VI. 1 Introduction.

Consider a linear time-invariant plant given by

$$\dot{x} = A x + B u : \qquad x(0) = x_0 \tag{1}$$

where x is an n-vector state, and u is a r-vector control to be determined to minimize the performance index

$$I = 1/2 \int_0^\infty (x^T Q x + u^T R u) \, dt. \tag{2}$$

Q is a symmetric non-negative definite matrix.

It is well known that the optimal control law is given by

$$u = - K x : \quad K = R^{-1} B^T P_\infty \tag{3}$$

where P_∞ is the steady state solution of the Matrix Riccati Equation

$$\dot{P} = P A + A^T P - P B R^{-1} B^T P + Q : \quad P(0) = 0 \tag{4}$$

To ensure the existence of a unique, symmetric and positive definite

P , and the stability of the closed loop optimal ststem when A has characteristic roots with non negative real parts, it is assumed as a sufficient but not a necessary condition that the olant given by (1) is completely controllable and that even though Q may be only non negative, $x^T Q x \neq 0$ for any solution of x(t) of $\dot{x} = A x$, $x(0) = 0$. This is due to Kalman.[18] A symmetric non negative Q satisfying this condition and a positive definite R will be called admissible.

It has been pointed out by several authors that the pair [Q, R] that determines the feedback gain matrix K via the Riccati equation is not unique. Two pairs, [Q, R] and $[Q_e, R_e]$, that yield the same matrix K will be considered equivalent. In particular, since the

$$1/2\,[n(n+1) + r(r+1)]$$

elements of Q, R determine only rn elements of K which shows that

$$1/2\,n(n+1) + 1/2\,r(r+1) - nr = 1/2[(n-r)^2 + n + r]$$

elements are redundant, it is convenient to reduce the number of elements in the equivalent $[Q_e, R_e]$ to a minimum, or to parameterize $[Q_e, R_e]$ by a minimum of parameters.

The study of equivalent [Q, R] has both theoretical and practical significance. Kalman[5] originally started the theoretical investigation in studying the asymptotic properties of linear optimal systems. In practice, the minimization of (2) serves as a design tool, for the determination of satisfactory control law (3). A suitable [Q, R] is determined purely by trial and error, and the common practice is to try only diagonal Q, R with as many zero elements as possible.

A designer has little feel for the role of cross-product terms in the performance index. A common problem is to know whether a non-diagonal admissible [Q, R] gives a better control, well suited to meet the design specifications. It is certainly very convenient to know that for every non-diagonal admissible [Q, R] there exists an equivalent diagonal [Q^*, R] : if there is, a search over only diagonal admissible [Q, R] is fully justified.

Unfortunately the knowledge about the problems as posed above is not clear-cut. In the multi-input case, a diagonal equivalent [Q, R] does not exist since it is shown that [Q_e, R_e] must depend on rn parameters.

VI. 2 Some theoretical developments.

In this section we shall briefly summarize the theoretical developments that have appeared currently in the literature, and are proving to be of importance for design purposes. We shall duly refer to the approapriate papers for detailed proof or analysis, as it is hard to provide a broad coverage of the developments.

The single-input case: diagonal Q_e.

When u is a scalar, we replace B in (1) by the vector b, $u^T R u$ in (2) by u^2, K in (3) by k^T and (4) becomes

$$\dot{P} = PA + A^T P - Pbb^T P + Q, \qquad P(0) = 0 \tag{5}$$

The solution P is an equilibrium point of (5), hence must satisfy the matrix quadratic equation

$$0 = PA + A^T P - Pbb^T P + Q \tag{6}$$

If we denote a solution of (6) with Q replaced by an equivalent Q_e, P by P_e, then we have

$$0 = P_e A + A^T P_e - P_e b b^T P_e + Q_e \tag{7}$$

In order to be equivalent to Q, Q_e and P_e must satisfy

$$P_e b = P b = k \tag{8}$$

Let

$$P - P_e = M \tag{9}$$

substracting (7) from (6) and using (9), we have

$$MA + A^T M + Q - Q_e = 0 \tag{10}$$

$$Mb = 0 \tag{11}$$

We need to eliminate the symmetric matrix M between (10) and (11) and solve for Q_e in terms of Q. Such a Q_e clearly satisfies (7) but it may not be an admissible Q. It is interesting that the admissibility of Q_e is not be needed, as the following result shows. This is due to Karl Hedric.[21]

<u>Lemma.</u> Given an admissible Q leading for $t \to \infty$, to the solution P_∞ of the Riccati equation (5), and given a symmetric matrix Q' leading to a solution P'_∞ of the quadratic equation

$$0 = PA + A^T P - P b b^T P + Q' \tag{12}$$

such that

$$P'_\infty b = P b \tag{13}$$

then P'_∞ is also a solution for $t \to \infty$ of

$$\dot{P} = PA + A^T P - Pbb^T P + Q' : \qquad P(0) = 0 \tag{14}$$

Proof:- We need to show that P'_∞ is a completely stable equilibrium point of (14). Consider a translation of (5) to the equilibrium point P_∞. Let

$$\bar{P}(t) = P(t) - P_\infty, \tag{15}$$

which satisfies

$$\dot{\bar{P}} = \bar{P}A + A^T\bar{P} - \bar{P}bb^T\bar{P} - P_\infty bb^T\bar{P} - \bar{P}bb^T P_\infty + P_\infty A + A^T P_\infty - P_\infty bb^T P_\infty + Q$$

that is

$$P = \bar{P}A_k + A_k^T\bar{P} - \bar{P}bb^T\bar{P} : \qquad \bar{P}(0) = - P_\infty \tag{16}$$

where

$$A_k = A - bk^T : \qquad k = P_\infty b \tag{17}$$

By assumption of complete controllability of [A, b] and the admissibility of Q, the matrix A_k of the closed loop system is asymptotically stable and equation (5) for P is completely stable with respect to $P = P_\infty$. This stability is not affected by translation (15) of the origin of (5) to the point P_∞,[18] and to $\bar{P} = 0$. Consider now a translation of the origin of (14) to P'_∞. This equation for

$$(\bar{\bar{P}}_t) = P(t) - P'_\infty$$

is identical to equation (16) for $\bar{P}(t)$:

$$\dot{\bar{\bar{P}}} = \bar{\bar{P}}A_k + A_k^T\bar{\bar{P}} - \bar{\bar{P}}bb^T\bar{\bar{P}}, \qquad \bar{\bar{P}}(0) = - P'_\infty,$$

and is therefore also completely stable with respect to $\bar{\bar{P}} = 0$. Thus P'_∞ is a completely stable equilibrium point of (14).

We will now state a few theorems[21], which are relevant and may prove useful for design purpose of single-input systems.

Consider Q_c in (10) to be a diagonal matrix denoted by Q^*. The $1/2\ n(n-1)$ off-diagonal equations of the symmetric matrix equn (10) are

$$-q_{ij} = \sum_{k=1}^{n} (m_{ik} a_{kj} + a_{ki} m_{kj}), \quad i=1,2,\ldots,n-1. \quad j=i+1,\ldots,n. \tag{18}$$

and together with the n equations in (11), they form $1/2\ n(n+1)$ equations for $1/2\ n(n+1)$ unknown m's, the elements of the symmetric matrix M. Having M, one can solve for Q^* in terms of Q from the diagonal equns in (10).

$$q^*_{ii} = q_{ii} + 2 \sum_{k=1}^{n} m_{ik}\, a_{ki} : \ i = 1, 2, \ldots, n \tag{19}$$

We can now state the following result:-

Theorem. 1 (Kriendler and Hedric) For a single-input plant given by (1) and a performance index (2) with R=1, there exists a unique equivalent diagonal matrix Q^* for all admissible matrices Q, if and only if the set of $1/2\ n(n+1)$ equations of (11) and (18) has the rank of $1/2\ n(n+1)$.

The proof is presented in the reference 21.

The condition Theorem 1, imposes on A and b has no relation to such other properties of the plant as complete controllability, and stability. The condition of the theorem is met by the important special case when (1) is in the phase-variable cannonical form

$$\dot{x}_i = x_{i+1}, \qquad i = 1, 2, \ldots\ldots\ldots, n-1$$

$$\dot{x}_n = -\sum_{i=1}^{n} a_i x_i + u \tag{20}$$

This is shown in the reference 20.

Theorem 2. (Kriendler and Hedric) In the case of a plant given by (20) and a performance index (2) with R=1, for each admissible matrix Q, there exists a unique equivalent diagonal matrix Q^*, and Q^* is related to Q by

$$q^*_{ii} = q_{ii} - 2\, q_{i-1,i+1} + 2\, q_{i-2,i+2} - \ldots, \quad i=1, 2, \ldots n \tag{21}$$

where the alternating sum is continued, until all the available q's are exhausted.

Theorem 3. (kalman) Two matrices Q and Q_e are equivalent only if

$$b^T \Phi^T(-s)\, Q_e \Phi(s) b = b^T \Phi^T(-s)\, Q\, \Phi(s)\, b, \tag{22}$$

$$\Phi(s) = [sI - A]^{-1}$$

if the control laws k and k_e corresponding to Q and Q_e, respectively are stable (that is the eigen values of $A - bk^T$ and $A - bk_c^T$ have negative real parts), then (22) is also a sufficient condition for equivalence of Q and Q_e.

The proof is given in reference 18.

VI.3. The multi-input case.

Given an arbitrary admissible pair [Q, R] in (2) leading to the control law (3), we wish to find the number of parameters needed for a parameterization of the equivqlent pair $[Q_e, R_e]$ that must satisfy:

$$P_eA+A^TP_e-P_eBR_e^{-1}B^TP_e+Q_e=0 \tag{23}$$

$$R_e^{-1}B^TP_e=K \tag{24}$$

combining (23) and (24) we have

$$P_eA+A^TP_e=K^TR_eK-Q_e \tag{25}$$

Assuming the real parts of the sum of any two eigen values of A is not zero, (25) can be solved for P_e in the form

$$P_e=N(K^TR_eK)-N(Q_e) \tag{26}$$

where $N(K^TR_eK)$ and $N(Q_e)$ are non-singular matrices N, whose elements are linear combinations of elements K^TR_eK and Q_e respectively. Substituting (26) into (24) gives

$$R_eK-B^TN(K^TR_eK)+B^TN(Q_e)=0 \tag{27}$$

The matrix equation (27) represents a set of nr linear homogeneous equations in the elements of R_e and Q_e, assuming that none of the elements of the matrix on the left side of (27) are identically zero for all K's in the set $\{K\}$ corresponding to the set $\{Q_e, R_e\}$ of all admissible couples $[Q_e, R_e]$.

Consider an element of $R_eK-B^TN(K^TR_eK)$. It is a linear combination of elements of R_e and the coefficients are of the form $k_\alpha + \Sigma d k_\beta k_\gamma$, where k_α, k_β, and k_γ are some elements of K, and d is some number. For those coefficients to be identically zero, the corresponding k's of all K in K must be confined to a quadratic surface. This cannot occurr if some K has a neighbourhood in alinear sub-space of nr space. Consider the couple $[Q, R=I]$ where Q is positivedefinite, and consider the corresponding steady state solution of P of (4) and the matrix K given by (3). For a sufficiently small ϵ the matrix $Q+\epsilon\delta Q$ remains positive definite for all δQ bounded by some number. In the limit $\epsilon \to 0$, δP corresponding to δQ is given by

$$\delta P \, A_K + A_K^T \, \delta P = - \delta Q : \; A_K = A - B \, B^T P = A - B \, K \tag{28}$$

which, since the eigen values of A_K have all a negative real part, has a unique solution δP for all δQ. Thus, corresponding to a neighbourhood of O, there is a neighbourhood of P, and hence, provided B of (1) has maximum rank, also a neighbourhood of K in nr space. If B has less than maximum rank, the neighbourhood of K is confined to a linear sub-space.

Thus there are nr equations in (27) and therefore $[Q_e, R_e]$ must depend on at least nr parameters. For the linear set (27) to have a non-trivial solution, it must have a nullity ofat least one and thus more than nr parameters may be needed. The solution $[Q_e, R_e]$ of (27) is non-unique to a degree equal to the nullity of (27). (since (2) can always be multiplied by some positive number without changing K, $[Q_e, R_e]$ has always a one- degree non-uniqueness.). The type of parameterization or structure of Q_e, R_e that requires the smallest number of parameters or the least degree of non-uniqueness of $[Q_e, R_e]$

is not known.

The theoretical findings as outlined in this chapter are gradually being accepted as important design aids. Although no numerical work incorporating these theoretical observations has yet appeared in the literature, several authors have expressed their optimism with regards to the optimal control system design.

Other approaches on selection of weighting matrices are through the Eigen value approach[22], Butterworth function, and Root Locus[23] approach. The most common difficulty with these approaches however, is the increased complexity for higher order systems. Applications of these are quite common in the literature in the optimal control system design, and also in sensitivity reduction techniques. It is felt that a better understanding of the role of the weighting matrices in the performance index is almost mandatory, before any effective sensitivity reduction design approach can be proposed.

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