ON INVERSE PROBLEMS FOR A BEAM WITH ATTACHMENTS

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Dedicated to my parents,

The most precious things I have in this world.

Abstract

The problem of determining the eigenvalues of a vibrational system having multiple lumped attachments has been investigated extensively. However, most of the research conducted in this field focuses on determining the natural frequencies of the combined system assuming that the characteristics of the combined vibrational system are known (forward problem). A problem of great interest from the point of view of engineering design is the ability to impose certain frequencies on the vibrational system or to avoid certain frequencies by modifying the characteristics of the vibrational system (inverse problem). In this thesis, the effects of adding lumped masses to an Euler-Bernoulli beam on its frequencies and their corresponding mode shapes are investigated for simply-supported as well as fixed-free boundary conditions. This investigation paves the way for proposing a method to impose two frequencies on a system consisting of a beam and a lumped mass by determining the magnitude of the mass as well as its position along the beam.

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1 Chapter 1- Introduction

1.1 Background

The problem of determining the natural frequencies of a continuous system has been investigated extensively. One of the major continuous structural elements whose natural frequencies are of great interest in engineering is a beam.

The derivation of the equation of motion for a beam typically results in a fourth-order Partial Differential Equation (PDE) with respect to displacement and time which can often be solved using the method of separation of variables. The application of the method of separation of variables results in a fourth-order Ordinary Differential Equation (ODE) with respect to displacement as well as a second-order ODE with respect to time.

The solution of the fourth-order displacement-dependant ODE yields the equations for natural frequencies as well as the corresponding normal modes (eigenfunctions) of vibration. For any beam, there will be an infinite number of normal modes with one natural frequency associated with each normal mode. On the other hand, the solution of the time-dependant ODE results in the transient part of the response.

As indicated before, there exist an infinite number of normal modes as well as corresponding natural frequencies for a beam. This lays the foundation for introducing a method of discretization via superimposing a finite number of mode shapes to represent the transverse vibrations of a continuous beam namely, the assumed-mode method.

Having derived the equations to calculate the natural frequencies and their corresponding mode shapes of a continuous beam, a new problem is raised to investigate the effects of adding discrete elements in the form of point masses, stiffness elements (linear and rotary springs) or damping elements (both linear and rotary) on the natural frequencies and corresponding mode shapes of the now-modified beam. This problem has been the subject of extensive research due to its widespread range of applications, from musical instruments to offshore oil platforms and aircraft wings. However, most of the research conducted in this area focuses on the determination of the natural frequencies of the combined system, assuming that the values and mounting positions of the discrete elements are known variables (forward problem). A more important problem from the point of view of engineering design is the ability to impose certain frequencies on the combined system by finding the values and mounting positions of the added lumped elements (inverse problem).

1.2 Problem Definition

The main purpose of this thesis is to consider the inverse eigenvalue (frequency) problem of beams with mass attachments. A method to impose two fundamental frequencies on the combined system of a beam and a single lumped mass attachment is proposed and investigated. In other words, the known variables in this problem are the two fundamental frequencies of the combined system while the unknown variables are the values of the lumped mass and its position on the beam. Two cases of boundary conditions are considered, namely, simply-supported and fixed-free (cantilever).

In order to solve the inverse problem, a comprehensive insight of the forward eigenvalue problem is required in order to gain an understanding of the possible range of the effect of the addition of a lumped mass on both the frequencies and mode shapes of the modified beam. Moreover, considering the forward problem and deriving the frequency spectrum allow for educated guesses for the design variables of the inverse problem. Therefore, the thesis starts with investigating the effects of adding a lumped mass to a beam on its frequencies and their corresponding mode shapes.

1.3 Thesis Contribution

As previously mentioned, most of the research in this area focuses on determining natural frequencies assuming that the values of the discrete elements and their mounting positions are known. This thesis investigates the inverse eigenvalue (frequency) problem of imposing certain natural frequencies on a beam by adding a lumped mass element to the beam; in particular, a novel method to achieve this goal is investigated for both simply supported and cantilevered beams.

Moreover, while most of the research in this area focuses on natural frequencies, this thesis conducts a comprehensive investigation on the effects of adding a single mass attachment on the mode shapes of vibration. The first five mode shape plots are derived for nine equally spaced locations along the beam. Each plot includes the mode shapes for five masses plus that of the unconstrained beam which greatly facilitates the comparisons of the mode shapes. Two cases of boundary conditions are considered, namely, simply-supported and fixed-free (cantilever).

The principal means of research and simulation in this thesis is Maple V14. The eigenvalues and eigenvectors calls in Maple are utilized in order to derive the frequencies and mode shapes, respectively. As far as the inverse eigenvalue problem is concerned, the built-in *fsolve* as well as *DirectSearch* packages are used to solve the system of two equations and two unknowns.

1.4 Thesis Outline

In Chapter 2, a comprehensive literature review is performed regarding beams with single or multiple attachments as well as stiffened plates that can be regarded as an extension of the beam with attachments problem. It is observed that most researchers focused on proposing methods to derive the frequencies of a combined system whose characteristics are already known, whilst little attention has been paid to the effect on the mode shapes nor on the inverse problem of imposing frequencies on the combined system.

In Chapter 3, the theoretical foundation of the thesis is developed by deriving the fundamental equations of motion of the combined system using the assumed-mode method and by substituting the kinetic and potential energy terms into Lagrange's equations. A comparison is made between two methods of solving the equation of motions namely, Cha's method [1] and the direct eigenvalue method. It is observed that Cha's method reduces the order of the eigenvalue equation to be solved by two which, given the fact that it does not account for the frequencies of cases where the mass is positioned on a node of a mode, does not justify the use of this method. Therefore, the direct eigenvalues and eigenvectors are utilized to consider the forward problem.

In chapter 4, the effects of adding a lumped mass to a beam on its frequencies and their corresponding mode shapes are considered. In the first problem, a single lumped mass is positioned at nine equally spaced spots on the beam and its effect on frequency is plotted for the first five fundamental frequencies. This problem is repeated for five masses. In the second problem, a single lumped mass is positioned at nine equally spaced locations along the beam and the first five mode shape plots are derived. Each plot includes the mode shapes for five different masses plus the case of an unconstrained beam. Two cases of simply-supported and fixed-free cantilever beam are considered.

In Chapter 5, the inverse problem of imposing two natural frequencies on a combined system of a beam to which a single mass is attached is considered. The design variables are the two desired natural frequencies and the unknown variables are the value of the mass and its position on the beam. The two frequencies are chosen from the results of the forward problem obtained in chapter 4 and are substituted into a system of two determinant equations. Using *fsolve* and *DirectSearch* packages, a set of results for mass and position coefficients is obtained, including the expected result obtained via the forward modelling in Chapter 4. Unexpected additional results are also obtained. The unexpected results must be substituted back into the forward problem in order to verify the solution and to verify whether the order of frequencies is conserved in the hierarchy of the frequencies of the system. Both cases of simply-supported and fixed-free (cantilever) boundary conditions are considered.

2 Chapter 2 - Literature Review

The problem of free vibrations of combined dynamical systems has been investigated extensively. This problem concerns many areas of engineering design, from musical instruments to offshore oil platforms and aircraft wings. Most of the research work performed in this area and related to the subject of this thesis can be divided into two categories:

- **Beams with attachments:** This problem involves the free transverse vibrations of Euler-Bernoulli as well as Timoshenko beams to which one or several lumped attachments are attached. The attachments can be in the form of point masses, lumped stiffness elements (linear springs), rotary inertia or rotary stiffness elements as well as damping elements. Moreover, beams may have both conventional (simply supported, cantilever, clamped) and elastic boundary conditions.
- **Stiffened plates:** This problem involves plates that are stiffened using stiffener bars or braces. This problem can be regarded as an extension of the beam analysis.

In the following sections, a comprehensive review of the research performed in both categories is presented.

2.1 Beams with attachments

The majority of the research performed in this area involves the development and evaluation of methods to determine the natural frequencies of the combined system. Cha et al in [1] proposed a method to calculate the natural frequencies (eigenvalues) of a beam with multiple miscellaneous lumped attachments more easily by reducing the order of the matrices whose determinants are to be solved from N (number of assumed modes) to S (number of attachments). The results obtained through this method were compared with corresponding results obtained through FEA and excellent agreement was observed. In another paper Cha et al [2] considered the free and forced vibration of beams carrying lumped elements in the form of point masses, translational as well as torsional springs and dampers. They introduced the Sherman-Morisson formula to explicitly find the equations of motion for a beam with a single attachment and the Sherman-Morisson-Woodbury formula to find the equations of motion for a beam with multiple attachments. The results obtained by this method were easy to code and led to frequency equations that could be solved either graphically or numerically and could be easily extended to accommodate any support type and miscellaneous attachment type. In turn, this led to explicit equations that permitted investigation of the sensitivity of the eigenvalues of the combined system to the attachment parameters. This method can also be used to determine the steady-state deformed shape of a linear structure subject to localized harmonic excitation, to obtain the frequency response at any point along the linear structure and to solve the inverse problem of imposing nodes to quench vibration.

In another paper by Cha et al [3], the determination of the frequency of a dynamical system consisting of a linear elastic structure to which a chain of masses and springs is attached was considered. By using the assumed-mode method, the authors proposed a solution for determining the frequencies of a linear elastic structure (beam) to which a system of masses and springs was attached at a particular point, which proved to be an efficient alternative to the Lagrange multipliers method. By developing the secular equation, the number of equations to be solved was reduced compared to the general eigenvalue problem. This method provided a method of solving the inverse problem of imposing nodes at a specific location. For this to happen, the frequency of the combined system must be equal to the frequency of the grounded mass-spring system that is, the natural frequency of the isolated system of the mass and the linear spring $\left(\omega = \sqrt{\frac{k}{m}}\right)$.

Cha in [4] analyzed the inverse problem of imposing nodes along a beam using a combination of elastically mounted masses. An analytical method was developed to make it possible to impose nodes in a desired location along a beam with arbitrary boundary conditions, using a system of masses and springs. Two cases were considered, the first one being the case where the mass and spring system mounting point coincided with that of the node (collocation) and the second where the mounting position differed from the node location. It is realized that in order for the node to be imposed on one of the vibrating modes, the natural frequency of the grounded spring-mass system must be the same as the natural frequency of the combined system in that mode. In another similar paper [5], Cha considered imposing nodes in desired locations along a linear elastic structure using a system of spring-masses. He realized that if the natural frequency of the combined system equals the natural frequency of the grounded spring-mass system, then a node will be created in the mounting position of the spring-mass system. Based on this fact, simultaneous nodes may be imposed along a beam by choosing appropriate springs and masses. In this method, the location of the nodes and mounting points were known variables and thus the required frequencies for each mode could be obtained. Having determined the frequencies, the characteristics of the mounted spring-mass system could then be determined using $\omega^2 = \frac{k}{m}$. It is worth noting that the solution is not unique. This method was applied to a fixed-free (cantilevered) and a simply supported Euler-Bernoulli beam.

In [6], Cha proposed an intuitive approach to solving the dynamic behavior of a combined system of linear elastic beams carrying miscellaneous lumped attachments. This novel approach was presented to simplify the derivation of the equations governing combined dynamical systems by reducing the order of matrices involved in the calculations. The results obtained were then used to solve for natural frequency problems in a variety of scenarios including: a cantilever beam with an undamped, no rigid body degree of freedom element, a beam with an undamped, single rigid body degree of freedom element, a simply supported plate with a lumped mass of 1 DOF and a linear elastic beam with miscellaneous lumped attachments. In all of the cases, excellent agreement with exact, Galerkin and FEM results was observed.

In [7], Wang considered the effect and sensitivity of positioning lumped (concentrated) masses on an Euler-Bernoulli beam on the beam's natural frequencies. Using finite element analysis, a closed-form expression for the frequency sensitivity with respect to the mass location was obtained. Numerical results were obtained for two specific cases of a cantilever beam and an unrestrained beam with two lumped masses. They concluded that the sensitivities associated with each of the inertias are independent and can be added together in calculation. The rotary inertia of the lumped mass also has a considerable impact on the sensitivity as well as frequency (especially for higher modes). The effect of sliding the lumped mass along the beam on the beam frequencies (from middle to the tip) was investigated in this paper.

Pritchard et al. [8] also considered sensitivity and optimization studies with regard to the node locations of a beam to which lumped masses were attached. Analytical and Finite Element Method results were compared with corresponding results using the finite difference method. The known variable was the nodal point and the unknown variable was the value of the lumped masses. An analytical method was derived to allow for the determination of the nodal points where they were most needed using the lumped masses on fixed locations. The span within which the nodal point must lie was considered in addition to the minimization of the required lumped masses. Sensitivity analysis refers to the derivative of the nodal point location with respect to the added mass, which gives a good idea of the changes in the nodal point when a perturbation takes place. A negative sensitivity means that the nodal point will shift to the left and vice versa.

Chang et al. in [9] considered free vibration of a simply supported beam with a concentrated mass in the middle. In their analysis, transverse shear deformation was neglected, the rotary inertia effect of the lumped mass was taken into account, the mass was in the middle of the plate and the beam cross section was uniform. The method of separation of variables was used. Due to symmetry, just half of the beam was considered and the continuity equations were applied in the middle of the beam where the lumped mass was laid. They proposed an analytical solution for the free vibration of a beam with a mass in the middle for both symmetric and anti-symmetric modes. The effects of considering the rotary inertia of the lumped mass was considered and it was understood that for higher modes the deviation was considerable.

In [10], Dowell et al. generalized the results of the Rayleigh's method for the calculation of the frequency of combined mechanical systems. Unlike Rayleigh's method, this approach states that the natural frequency of the combined systems increases in every condition. For the case of a mass on the beam, the frequency of the beam remains unchanged when the mass is positioned on a node. The left and right-hand sides of the resulting equations were depicted and the intersection points were considered as the frequencies of the combined system. This method was applied to the case of a single mass on a beam, a mass-spring system mounted on a beam and in the most general case, a beam mounted on another beam. The same authors in [11] investigated the application of Lagrange multiplier method in analyzing the free vibrations of different structures including beams. A method was presented for the analysis of free vibration of arbitrary structures using the vibration modes of component members. The Rayleigh- Ritz method was used along with the Lagrange multipliers to account for the continuity of displacement and slope at interfaces. The eigenvalue equations were obtained by assuming single harmonic motion.

The problem of determining the frequencies of beams with elastically mounted masses was also addressed by Kukla and B. Posiadala in [12]. The exact solution for the frequency of the transversal vibrations of the beam was obtained in closed-form using the green function method. This method can accommodate all possible boundary conditions. The number of sprung masses was finite but undetermined. The numerical results were shown for three different scenarios: (i) a simply supported beam with a sprung mass in between, (ii) a beam with torsional spring at both ends and a sprung mass in between, (iii) a simply supported beam with equally-spaced sprung masses. It was seen that adding each mass and spring added another frequency to the combined system and that attached masses can either increase or decrease the frequencies compared to the case of unconstrained beams. Moreover, the special cases of the attached masses and a grounded spring can be accommodated by making k and m tend to infinity, respectively.

Gürgöze et al[13] considered the effect of changing the parameters and position of a mass-spring system hung from a cantilevered Euler-Bernoulli beam with a tip mass. The partial differential equation (PDE) for the lateral vibrations of the beam was solved by incorporating displacements in the form of a separate steady-state part and a time dependant part into the PDE. The resulting ordinary differential equation (ODE) can be solved and the constants of the general solution can be determined by using the corresponding boundary and compatibility conditions. This would lead to a determinant that must equal zero in order to yield a non-trivial result. The effects of manipulating the oscillator parameters, including mass and stiffness, and the mounting position on the frequency spectrum of the system were considered qualitatively.

Maiz et al., [14], considered the exact solution to the problem of determining the frequency of a beam with attached masses. They took into account the rotatory inertia of the attached masses and the boundary conditions that were represented by translational and torsional springs which can accommodate any variation of boundary conditions. The general response to the ODE governing the eigenvalue problem was obtained as a piecewise function and its constants were determined using boundary and compatibility conditions. The solution was applied to different casual boundary conditions with masses placed either symmetrically or asymmetrically along the beam. The results were tabulated for different magnitudes of masses and radii of gyration. It was observed that in all cases where the rotatory inertia of the mass was neglected, adding a mass would decrease the natural frequency of the whole system compared to an unconstrained beam, unless the mass was on a node. If the rotary inertia of the mass was considered, in all cases the frequency decreased. The effect of rotatory inertia of the mass was greater in the upper frequencies. The effect of the linear inertia had its highest influence over a natural frequency when the mass was located at an antinode of the corresponding normal mode. In that situation, the rotatory inertia had no effect. The effect of the rotatory inertia had its highest influence when the mass was located at a node of the normal mode.

Naguleswaran et al [15] also considered the transverse vibrations of a beam with a mass at an intermediate location. The lateral vibration eigenvalue equation was non-dimensionalized and the general solution was obtained. Boundary conditions and compatibility were enforced which led to the solution of a determinant equal to zero. In that paper, the choice of two separate coordinate systems led to the solution of a 4 by 4 determinant equated to zero. Moreover, two additional constants of integration may be omitted using compatibility with regard to deflection and slope at the mounting point of the concentrated mass, although it was found this was not a great advantage. The first three frequencies were tabulated as a function of the position of the particle on the beam and for three different masses and sixteen combinations of boundary conditions. The corresponding mode shapes for two different magnitudes of masses and three positions of mass and different boundary conditions were depicted.

In another paper [16],the same authors extended their research to determine the frequencies of a beam with any number of lumped masses attached to it. The frequency equation was presented as a second-order determinant equal to zero, the general responses for each interval (part of the beam between two particles) was derived, and the constants of integration were determined using the compatibility between the adjacent parts and the boundary conditions. The final frequency equation was solved using a trial and error iterative method searching for roots by narrowing the range. The first three frequencies for four to nine particles and sixteen variations of boundary conditions were obtained.

Jacquot and Gibson, [17], developed a general method to calculate the natural frequencies as well as mode shapes of an Euler-Bernoulli beam with lumped mass and stiffness element attachments and elastic boundary conditions with no damping effect included. The equation of motion for the beam was written in which the effect of each lumped mass or stiffness element is considered as an external concentrated force. Assuming harmonic motion response of the beam in terms of the product of eigenfunctions of the unconstrained beam and the temporal sinusoidal part, the modal amplitudes were obtained by substituting the harmonic response into the equation of motion. The authors applied the method to two commonly used boundary conditions namely, simply supported and fixed-free. Taking advantage of the Jacquot's method, Ercoli and Laura, [18], extended the method to solve the problem of frequency determination of transverse vibrations of a beam constrained by elastically hung masses. They proposed an exact solution for the determination of the natural frequencies of transversal vibrations of beam with different kinds of attachments. This method, alongside two other approximate methods (Ritz & Rayleigh- Schmidt), was applied to different beams and attachment configurations and the effects of changing the positions of these attachments on fundamental frequencies were investigated.

The natural frequencies of a Timoshenko beam with a lumped mass attachments was considered in addition to an Euler-Bernoulli beam by Maurizi and Bellés in [19]. They utilized both Timoshenko as well as Euler-Bernoulli theories for beams with a mass whose value was a fraction of the value of the mass of the beam. They found the fundamental frequency coefficients for the choice of different masses and location ratios, taking into account an Euler-Bernoulli beam assumption and two cases of a Timoshenko beam with different shape factors. They concluded that for the fundamental frequency determination, the choice of the Euler-Bernoulli assumption was reasonable. However, this was not the case for higher order frequencies of the beam.

The forward problem of determining the frequencies of a beam with mass attachments was also considered by K. H. Low et al. in [20]. They performed the frequency analysis of a beam with concentrated masses attached to it and the effects of position and values of the mass on the fundamental frequencies of the combined system. The exact solution to the eigenvalue problem of the frequency of a beam and concentrated masses was established. The constants of the general solution to the eigenfunction differential equation were obtained using compatibility at the point of attached mass as well as the boundary conditions that could include ten distinct scenarios. The results of the analytical method were compared to the Rayleigh's method with two static shape functions as well as the experimental results.

The same authors in [21] took on the task of deriving a transcendental equation for frequency calculation of a beam with single mass attachments and comparing this method with Rayleigh's method. They assumed a single mass which was arbitrarily positioned along an Euler-Bernoulli beam. The effects of rotary inertia, transverse shear and second warping were ignored. The transcendental frequency equation was obtained for a single mass arbitrarily positioned along a beam for ten distinct boundary conditions. The effects of changing the lumped mass and its corresponding position on the first two fundamental frequencies were investigated using 3D plots. The frequencies of the combined system obtained through this method were compared with the Rayleigh's method as well as experimental data. The conclusion was that for quick engineering design purposes, Rayleigh's method was favorable.

K. H. Low in [22] compared two methods of deriving the frequency equation of a beam with lumped mass attachments , namely, a determinant method and using the Laplace transform. The frequencies were obtained for the case of a clamped-clamped beam with two lumped masses attached to it using both methods. The frequency equation was obtained and was solved for different combinations of masses and two cases of positions. It was found that although the equation derived using the Laplace transform was more compact compared to using the determinant equation, it took longer to solve with the Laplace transform. K. H. Low in [23] compared the eigenanalysis (exact)and Rayleigh's methods to solve for the frequencies of a beam with multiple mass attachments. The problem considered consisted of a beam with three mass attachments. Three kinds of boundary conditions were considered: clamped-clamped, clamped-free and pinned-pinned. They concluded that although the eigenfrequency method was an analytical method yielding exact results, it was computationally very time-consuming and the number of terms in the equation to be solved increased dramatically as the number of attached masses increased. On the other hand, the comparison of two methods showed that in the worst case scenario, the error of the Rayleigh's method was within 8% of the exact solution. Therefore, for engineering design purposes, Rayleigh's method was recommended.

In [24] Nicholson and Bergman derived the exact solution of the free vibration of a combined dynamical system using green functions. In this paper, the exact solution for two types of linear un-damped systems, one with one rigid body degree of freedom (a spring-mass system hung from the beam) and the other with no rigid body degree of freedom (a grounded spring attached to a lumped mass), using separation of variables and Green's function was obtained. The time-dependant part obtained via separation of variables revealed the harmonic nature of the vibrations and was used to obtain the natural frequencies while the spatial part revealed the generalized differential equation used to obtain the eigenfunctions. It could be seen that the equality between system natural frequency and the frequency of the attached part would result in the creation of a node at the mounting position.

2.2 Stiffened Plates

The free transverse vibrations of a plate with braces as attachments can be regarded as an extension of the free transverse vibrations of a beam with lumped masses attached to it. Therefore, in this section a brief review on the research conducted in this area is presented.

In [25], Cha et al. considered the free vibrations of a plate with a single lumped mass attachment. They assumed linear elastic structure with simply supported boundary conditions. They employed the assumed modes method with a degree of discretization of $N=30$. The equations governing the free vibration of a plate with attached discrete, lumped elements were obtained. The main advantage of this method was the reduction of the number of equations from N (the number of modes incorporated) to R (the number of attached elements), which required less computational time. For comparison, this method was used to solve the case of a simply supported rectangular plate with an attached lumped mass. In this case, since the mass was on the nodal line of the third mode, the corresponding frequencies of the constrained and unconstrained plates were the same.

In [26], Dozio and Ricciardi proposed a semi-analytical method for the quick prediction of the modal characteristics of rectangular ribbed plates. They assumed continuity of displacements and rotations between the plate and the beam, pure bending deformation of the plate (in-plane displacements were neglected). The effects of shear deformation and rotary inertia were neglected for the plate. The interface of the plate and the stiffener was assumed to be a line (narrow stiffener). The equations of motion for the plate and the beams were obtained independently and the compatibility and continuity were enforced in the interface of the beam and the stiffener. Using the assumed-modes method, the problem reduced to an eigenvalue problem for the natural frequency. The equations were solved for different boundary conditions. According to the authors, the main feature of this method was its capability to give a trend and consequently a way to a priori predict the changes in natural frequency by alternating geometric characteristics of the plate and beams such as the aspect ratio of the plate and stiffener height ratio. This method was valid as long as the beam was considered narrow.

In [27], Xu et al. considered the natural frequencies of a rectangular plate stiffened by any number of arbitrarily dimensioned and oriented rectangular beams. They derived an analytical method using Fourier series to describe the flexural and in-plane displacements of the plate and the beam. These displacement functions were solved using the Rayleigh- Ritz method. To account for all possible boundary conditions, this method replaced the boundaries with corresponding linear, torsional and bending springs. The results of this method were compared with the results of other research. w The effects of the aspect ratio of the plate, along with the ratio of the width of the plate to the width of the stiffener, and the ratio of the depth of the stiffener to the thickness of the plate on natural frequency of the first mode were considered. Unlike FEM, where the continuity between 2-D meshes of the plate and 1-D meshes of the stiffeners was problematic and the only conceivable condition was the full continuity between stiffener and plate elements, this analytical method could consider more realistic conditions such as a stiffener spot-welded to the plate. Since this method used springs to express boundary conditions, it would be easier to change the boundary conditions and include more complicated boundary conditions. Because of the fact that no nodes were involved, this method could also easily accommodate changes in stiffeners orientation. The beams could be placed on the edges.

The Finite Element Analysis (FEA) was utilized by Harik et al. in [28]. They performed a finite element analysis of the stiffened plate under free vibration. The effects of neglecting or considering the eccentricities – equivalent to membrane force in the plate and displacement along the stiffener - on the natural frequencies, were discussed. The interpolation functions and consistent mass and stiffness matrices were derived. Compatibility and continuity at the interface of the beam and plate were achieved by matching bending and in-plane displacement for the plate and the beam and by assuming that sections normal to the neutral plane remain normal after bending. They concluded that for low frequencies, the result of neglecting the eccentricities had little impact on the results, but in higher modes the neglect of the membrane forces (equivalent to neglecting the eccentricity) would overly underestimate the frequency.

In [29], Zeng and Bert considered free vibration of eccentrically stiffened plates using a Differential Quadrature (DQ) method. The equilibrium equations were derived for both plate and the stiffener separately, leading to partial differential equations. The boundary conditions were determined by taking into account the type of restraints at the edges (simply supported or clamped) and by the compatibility at the interface between the plate and stiffener. The natural frequencies were calculated. Their method was applied to the stiffened plate and it was validated against other theoretical and numerical methods (FEM and FDM). The proposed method reduced the computational load and was as exact as are the other methods.

In [30], Varadan considered large amplitude flexural vibrations of a symmetrically stiffened plate, taking into account the effects of in-plane displacements (non-linearity). The governing differential equations, as well as boundary conditions, were obtained using the principal of minimum potential energy. In-plane boundary conditions were either movable or immovable. Galerkin's method was used to solve the governing differential equation. Two mode shape functions were suggested. The phenomenon of an increase in frequency with increasing amplitude of vibration (a hardening type of non-linearity) was observed. It was observed that the relationship between non-dimensional frequency and amplitude for any stiffened plate was always of a less hardening nature than that of the corresponding unstiffened plate, for most practical cases of interest. The hardening effect was substantially larger for the immovable case than for the movable case. The hardening was found to increase with aspect ratio, as a general rule.

Based on this literature review and to the best of the author's knowledge, the inverse problem of imposing certain frequencies on a combined dynamical system has not been considered so far.

3 Chapter 3: Modelling the Forward Problem

3.1 Modelling Assumptions

In this chapter, the forward problem of the free vibrations of a simply supported and a cantilever Euler-Bernoulli beam carrying a number of lumped masses will be considered. In particular, the method of assumed modes and also the method proposed by Cha in [1] will be evaluated for finding the frequencies of a simply supported and cantilever beam carrying two or more lumped masses.

3.2 The Method of Assumed Modes

The approaches utilized to solve continuous problems in engineering involve the discretization of the continuous system into elements for which analytical or numerical solutions can be found. One of the most widely used methods is Finite Element Analysis (FEA) which involves the discretization of the continuous system into a number of small, discrete elements and the application of compatibility conditions at the interface of the adjacent elements as well as the application of boundary conditions. The greater the number of elements utilized, the more accurate the results obtained.

For the special case of vibrational analysis, there exists another commonly used discretization approach, called the assumed modes method. The logic behind this method is the principle of superposition of different vibrational modes that the system may undergo. As with the case of FEA, the greater the number of modes utilized, the more accurate the results obtained. However, in contrast to FEA, assumed modes is a superposition of global elements, with each element often defined over the entire domain of the problem. Usually, the vibrational modes of a related but simpler problem are superimposed to find approximate solutions to a more complicated problem. A good introduction to the assumed modes can be found in [31].

Both of these methods, when applied to a continuous, conservative vibrational system, will result in two matrices, namely mass and stiffness matrices. The dimensions of these matrices are determined by the degree of discretization selected for the problem. Here lies the main advantage of the assumed modes method over finite element analysis. It has been shown in [1] that the same level of accuracy can be reached by the assumed mode method using smaller degrees of discretization than with FEA. This implies mass and stiffness matrices that are smaller and can be handled more easily as far as computational issues are concerned.

For this thesis, the assumed modes method was chosen to derive the equations of motion for the case of a Euler-Bernoulli beam to which a number of discrete elements are attached. The discretization process starts with modelling the transverse vibrations of an Euler-Bernoulli beam as a finite series whose elements are the product of an eigenfunction and a generalized coordinate so that the transverse vibrations can be written as

$$w(x,t) = \sum_{j=1}^{N} \phi_j(x)\,\eta_j(t) \tag{3.1}$$

Here, $w(x,t)$ is the transverse displacement of the beam, ϕ is the space-dependent eigenfunction, η is generalized coordinate and *N* is the number of assumed modes chosen for the problem. It is important to note that ϕ varies with the choice of the beam and any ϕ should be chosen to satisfy the required boundary conditions of the selected beam.

As can be seen in (3.1), the eigenfunctions are functions of position, *x*, and the generalized coordinates are just a function of time (*t*), which demonstrates the application of separation of variables in this method.

3.3 Derivation of Equations of Motion

In order to derive the equations of motion for the one dimensional Euler-Bernoulli beam with multiple lumped point-mass attachments, expressions for kinetic and potential energies must first be found. The kinetic energy of the beam is given by

$$T = \frac{1}{2}\sum_{j=1}^{N} M_j \dot{\eta}_j^{\,2}(t) + \frac{1}{2} m_1 \dot{w}^2(x_1,t) + ... + \frac{1}{2} m_s \dot{w}^2(x_s,t) \tag{3.2}$$

where M_j are generalized masses of the bare beam (no attachments), an over dot indicates derivatives with respect to time and $m_1 \ldots m_s$ are *s* lumped point masses positioned at $x_1 \ldots x_s$, respectively.

Using the same procedure, the equation for the potential energy can be written as

$$V = \frac{1}{2}\sum_{j=1}^{N} K_j\, \eta_j^{\,2}(t) \tag{3.3}$$

where K_j ‘s are the generalized stiffnesses of the bare beam.

Substituting (3.1) into equation (3.2) , the following equation for kinetic energy is obtained

$$T=\frac{1}{2}\sum_{j=1}^{N}M_j\dot{\eta}_j^{\,2}(t)+\frac{1}{2}m_1\left[\sum_{j=1}^{N}\phi_j(x_1)\dot{\eta}_j(t)\right]^2+....+\frac{1}{2}m_s\left[\sum_{j=1}^{N}\phi_j(x_s)\dot{\eta}_j(t)\right]^2 \qquad (3.4)$$

Equation (3.3) for the potential energy remains the same as no elastic element is added to the beam.

Having found the expressions for kinetic and potential energies in terms of ϕ and η, these are then substituted into the Lagrange's equations to yield the equations of motion. Lagrange's equations are given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\eta}_i}\right)-\frac{\partial T}{\partial \eta_i}+\frac{\partial V}{\partial \eta_i}=0 \qquad i=1,2,..,N \qquad (3.5)$$

where *N* corresponds to the number of generalized coordinates and hence the number of differential equations.

Substituting equations (3.4) and (3.3) into (3.5) and converting the system of equations into a matrix representation, the matrix equation of motion will be given by

$$\mathbf{M}\underline{\ddot{\eta}}+\mathbf{K}\underline{\eta}=\underline{0}, \qquad (3.6)$$

where $\mathbf{M}$ and $\mathbf{K}$ are the system mass and stiffness matrices respectively and are given by

$$\mathbf{M}=\mathbf{M}^d+m_1\cdot\underline{\phi_1}\cdot\underline{\phi_1}^T+...+m_s\cdot\underline{\phi_s}\cdot\underline{\phi_s}^T \ . \qquad (3.7)$$

In equation (3.7), $\underline{\phi_1} \ldots \underline{\phi_s}$ are N-dimensional column vectors of the *N* eigenfunctions evaluated at point $x_1 \ldots x_s$, so that for example

$$\underline{\phi_1}=\begin{bmatrix}\phi_1(x_1)\\ .\\ .\\ .\\ \phi_N(x_1)\end{bmatrix} \qquad (3.8)$$

$\mathbf{M}^d$ is a diagonal matrix whose diagonal components are the generalized masses M_i and $m_1 \ldots m_s$ are the masses of the lumped attachments.

As far as the stiffness matrix is concerned, since elastic elements are not being added to the beam, it remains a diagonal matrix whose elements are the generalized stiffnesses of the beam. Hence, the stiffness matrix is given by

$$\mathbf{K} = \mathbf{K}^d . \tag{3.9}$$

3.4 Frequencies and Mode shapes

In order to solve equation (3.6), a system of N second-order differential equations, the vector of generalized coordinates $\underline{\eta}$ is written as

$$\underline{\eta} = \vec{\eta} e^{i\omega t} \tag{3.10}$$

Here, ω is the frequency of vibration of the system. Moreover; the inclusion of the complex number "i" is justified given the fact that the system is conservative and it is expected that the vibrations are purely oscillatory and thus undamped.

Substituting equation (3.10) into (3.6) and taking derivatives yields

$$\left(-\omega^2 \mathbf{M} + \mathbf{K}\right)\vec{\eta} = \underline{0} \tag{3.11}$$

In order for equation (3.11) to have a non-trivial solution, the following equation must hold

$$\det(-\omega^2 \mathbf{M} + \mathbf{K}) = 0 \tag{3.12}$$

The solution of equation (3.12) has been the subject of ongoing research, in particular for continuous systems with lumped attachments such as the one being considered here. In [1], a method to decrease the dimensions of the matrices involved was proposed by Cha and is explained in the next section. According to this method, the dimension of the final determinant is a function of the number of attachments rather than the number of assumed modes chosen. This method promised to be very useful for solving forward and inverse problems for beams with attachments since according to this method, the order of the determinant depends on the number of attachments only and thus should be of lower order than if obtained via a traditional characteristic determinant-based method. Furthermore, it is known that to increase accuracy of the result, the number of modes must be increased, thus a method that would permit an increase in the number of modes without sacrifice in complexity would be very appealing. Thus, for this reason, this method is investigated below and compared to the approach using a traditional determinant. Current state-of-the-art mathematical and simulation programs such as Maple (Maplesoft) and Matlab (Mathworks) are powerful tools to solve equation (3.12) and typically have built-in solvers for finding the generalized eigenvalues and eigenvectors of matrices. In this thesis, the "Eigenvalues" and "Eigenvectors" function calls in Maple are used to solve this equation and are referred to as the "direct determinant method". This direct method of finding the system determinant and eigenvalues shall be compared to using Cha's method with a reduced-order determinant.

3.4.1 Cha's Method for Frequencies of a Beam with Miscellaneous Attachments

In this section, the method proposed by Cha in [1] is explained. This method was proposed and used to solve eigenvalue problems of vibrations of a beam with various discrete attachments. Here, it is outlined by considering a simply supported beam to which several lumped point masses are attached.

Substituting equations (3.7) and (3.9) into (3.11), we obtain

$$\left(-\omega^2\mathbf{M}^d - \omega^2 m_1 \underline{\phi_1}\cdot\underline{\phi_1}^T - \ldots - \omega^2 m_s \underline{\phi_s}\cdot\underline{\phi_s}^T + \mathbf{K}^d\right)\vec{\eta} = \underline{0} \tag{3.13}$$

Or, after rearranging

$$\left(-\omega^2\mathbf{M}^d + \mathbf{K}^d + \sum_{i=1}^{s}\sigma_i\underline{\phi_i}\cdot\underline{\phi_i}^T\right)\vec{\eta} = \underline{0} \tag{3.14}$$

where, in this case

$$\sigma_i = -m_i\omega^2 . \tag{3.15}$$

In order for equation (3.14) to have a non-trivial solution, the following equation must hold

$$\det\left(-\omega^2\mathbf{M}^d + \mathbf{K}^d + \sum_{i=1}^{s}\sigma_i\underline{\phi_i}\cdot\underline{\phi_i}^T\right) = 0 \tag{3.16}$$

The procedure for solving equation (3.16) is where the hallmark of the Cha's method lies.

From [32], it is shown that the following relation concerning the determinant of a square matrix holds

$$\det\begin{bmatrix}\mathbf{A}_{n\times n} & \mathbf{B}_{n\times m}\\ \mathbf{C}_{m\times n} & \mathbf{D}_{m\times m}\end{bmatrix} = \det\mathbf{A}\det\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right) \tag{3.17}$$

where $\det\mathbf{A}\neq 0$.

If $\det\mathbf{D}\neq 0$ also holds, then the following relation holds as well

$$\det\begin{bmatrix} \mathbf{A}_{n\times n} & \mathbf{B}_{n\times m} \\ \mathbf{C}_{m\times n} & \mathbf{D}_{m\times m} \end{bmatrix} = \det\mathbf{D}\det\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right) \tag{3.18}$$

Now, in order for equation (3.17) to be compatible with the form of equation (3.16), the following substitutions are performed;

$$\mathbf{B} = -\mathbf{X}, \mathbf{C} = \mathbf{Y}^T, \mathbf{D} = \mathbf{I}_m \tag{3.19}$$

By substituting (3.19) into equations (3.17) and (3.18) and equating the results, it follows that

$$\det\left(\mathbf{A}+\mathbf{X}\mathbf{Y}^T\right) = \det\mathbf{A}\det\left(\mathbf{I}_m+\mathbf{Y}^T\mathbf{A}^{-1}\mathbf{X}\right) \tag{3.20}$$

where $\mathbf{X}$ and $\mathbf{Y}$ matrices are defined as

$$\mathbf{X} = \begin{bmatrix} \underline{x}_1 & . & . & \underline{x}_m \end{bmatrix} \tag{3.21}$$

$$\mathbf{Y} = \begin{bmatrix} \underline{y}_1 & . & . & \underline{y}_m \end{bmatrix} \tag{3.22}$$

Here, each $\underline{x}_i$ and $\underline{y}_i$ are $n\times 1$ column vectors. It can be shown that

$$\mathbf{X}\mathbf{Y}^T = \sum_{i=1}^{m}\underline{x}_i\underline{y}_i^T \tag{3.23}$$

Substituting equation (3.23) into (3.20), then it follows that

$$\det\left(\mathbf{A}+\sum_{i=1}^{m}\underline{x}_i\underline{y}_i^T\right) = \det\mathbf{A}\det\left(\mathbf{I}_m+\bar{\mathbf{G}}\right) \tag{3.24}$$

Based on the definitions of $\mathbf{X}$ and $\mathbf{Y}$, the matrix $\bar{\mathbf{G}}$ can be calculated as follows

$$\bar{\mathbf{G}} = \mathbf{Y}^T\mathbf{A}^{-1}\mathbf{X} = \begin{bmatrix} \underline{y}_1^T\mathbf{A}^{-1}\underline{x}_1 & . & \underline{y}_1^T\mathbf{A}^{-1}\underline{x}_j & . & \underline{y}_1^T\mathbf{A}^{-1}\underline{x}_m \\ . & . & . & . & . \\ \underline{y}_i^T\mathbf{A}^{-1}\underline{x}_1 & . & \underline{y}_i^T\mathbf{A}^{-1}\underline{x}_j & . & \underline{y}_i^T\mathbf{A}^{-1}\underline{x}_j \\ . & . & . & . & . \\ \underline{y}_m^T\mathbf{A}^{-1}\underline{x}_1 & . & . & . & \underline{y}_m^T\mathbf{A}^{-1}\underline{x}_m \end{bmatrix} \tag{3.25}$$

By comparing equations (3.24) and (3.25) with equation (3.16), the following analogies can be made;

$$\mathbf{A} = -\omega^2 \mathbf{M}^d + \mathbf{K}^d \tag{3.26}$$

$$\underline{x}_i = \sigma_i \underline{\phi}_i \tag{3.27}$$

$$\underline{y}_i = \underline{\phi}_i \tag{3.28}$$

$$m = s \tag{3.29}$$

Substituting equations (3.26), (3.27) and (3.28) into equation (3.24), then

$$\det\left[\left(\mathbf{K}^d - \omega^2 \mathbf{M}^d\right) + \sum_{i=1}^{s} \sigma_i \underline{\phi}_i \underline{\phi}_i^T\right] = \det\left(\mathbf{K}^d - \omega^2 \mathbf{M}^d\right) \det \mathbf{G} \tag{3.30}$$

where $\mathbf{G} = \mathbf{I}_s + \bar{\mathbf{G}}$, $\mathbf{I}_s$ is the *s*-dimensional identity matrix and $\mathbf{G}$ is defined as follows

$$\mathbf{G} = \left[g_{ij}\right] = \left[\delta_i^{\,j} + \underline{\phi}_i^T \left(\mathbf{K}^d - \omega^2 \mathbf{M}^d\right)^{-1} \sigma_j \underline{\phi}_j\right] \tag{3.31}$$

In the previous equation, $\delta_i^{\,j}$ is the standard Kronecker delta. By expanding equation (3.31), each coefficient of $\mathbf{G}$ can be determined as

$$g_{ij} = \delta_i^{\,j} + \sigma_j \sum_{r=1}^{N} \frac{\phi_r\left(x_i\right) \cdot \phi_r\left(x_j\right)}{\left(K_r - \omega^2 M_r\right)} \qquad i, j = 1 \ldots s \tag{3.32}$$

In order for equation (3.30) to have a non-trivial solution, the following condition must be satisfied

$$\det \mathbf{G} = 0 \tag{3.33}$$

This implies that

$$\det\left(g_{ij}\right) = 0 \tag{3.34}$$

where the notation $\det\left(g_{ij}\right) = 0$ implies the determinant of the matrix whose entries are given by g_{ij}. This equation is still valid if the first column of the determinant is divided by σ_1, the second by σ_2 and so forth. Considering equation (3.32), this leads to

$$\det\left(g_{ij}^{'}\right) = \det\left(\frac{1}{\sigma_j}\delta_i^{\,j} + \sum_{r=1}^{N}\frac{\phi_r\left(x_i\right)\cdot\phi_r\left(x_j\right)}{\left(K_r - \omega^2 M_r\right)}\right) = 0 \qquad i,\, j = 1\ldots\text{s} \tag{3.35}$$

where $g_{ij}^{'} = \dfrac{g_{ij}}{\sigma_j}$. Subsequently, for the simple case of only 2 masses attached to the beam, the resulting determinant is given by

$$\begin{vmatrix} \dfrac{1}{-m_1\omega^2} + \sum_{r=1}^{N}\dfrac{\phi_r^{\,2}\left(x_1\right)}{\left(K_r - \omega^2 M_r\right)} & \sum_{r=1}^{N}\dfrac{\phi_r\left(x_1\right)\cdot\phi_r\left(x_2\right)}{\left(K_r - \omega^2 M_r\right)} \\ \sum_{r=1}^{N}\dfrac{\phi_r\left(x_2\right)\cdot\phi_r\left(x_1\right)}{\left(K_r - \omega^2 M_r\right)} & \dfrac{1}{-m_2\omega^2} + \sum_{r=1}^{N}\dfrac{\phi_r^{\,2}\left(x_2\right)}{\left(K_r - \omega^2 M_r\right)} \end{vmatrix} = 0 \tag{3.36}$$

One major reservation regarding (3.30) and thus(3.33), is the fact that it assumes that none of the masses is positioned on a node of any mode, meaning it assumed $\det\left(\mathbf{K}^d - \omega^2\mathbf{M}^d\right) \neq 0$. Therefore, the frequency spectrum derived using this method does not include all the frequencies of the system. In order to obtain the full span of frequencies in the case where one of the masses is positioned on the node of any mode, the following additional equation must also be solved to account for the missing frequencies

$$\det\left(\mathbf{K}^d - \omega^2\mathbf{M}^d\right) = 0 \tag{3.37}$$

The method outlined above will be referred to as Cha's method and the traditional way of obtaining the spectrum of the system via obtaining generalized eigenvalues of the $\left(\mathbf{K},\mathbf{M}\right)$ system will be referred to as the "direct determinant method". The characteristics of the method outlined in this section are further explored in the next section by considering a specific eigenvalue problem and solving it using both the direct determinant method as well as Cha's method.

3.5 Comparison of Direct Determinant and Cha's Methods

To investigate the capabilities and limitations of the method proposed by Cha in [1], a specific problem involving a beam to which several masses have been attached is considered. The system under consideration is shown in Figure 1.

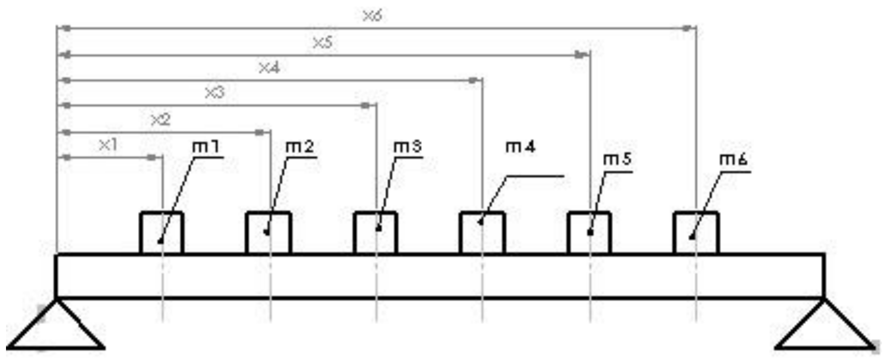


Figure 1 Beam with mass attachments

3.5.1 Assumptions and definition of the problem

The following assumptions are made for the implementation of both methods:

- The beam is an Euler-Bernoulli beam with length L.
- The boundary conditions are of the simply-supported type.
- The number of vibrational (assumed) modes utilized is 10 (N=10).
- The number of lumped masses attached to the beam is 6 (s=6).
- The six masses are of masses $\rho L, 2\rho L, 3\rho L, 4\rho L, 5\rho L$ and $6\rho L$ positioned at $0.2L, 0.3L, 0.4L, 0.6L, 0.8L$ and $0.9L$, respectively (ρ represents the mass per unit length).
- The *jth* eigenfunction (assumed mode) utilized for a Simply-supported (SS) beam is

$$\phi_j(x) = \sqrt{\frac{2}{\rho L}} \sin\left(\frac{j\pi x}{L}\right) \tag{3.38}$$

It should be noted that though the assumption of masses heavier than the mass of the beam itself may be unrealistic for some problems, these results are used for demonstrative and comparison purposes of the methods and as such it is desirable to use a range of values in the simulations.

3.5.2 Coding and the results

Using Maple v14, the code to solve the problem defined in the previous subsection was written. implementing both the direct determinant method and the method proposed by Cha in [1],the results obtained for different combinations of mode numbers and attachments are tabulated in Table 1 and are discussed below. Although both methods seek to solve the same problem, the implementation of both methods was different and shall be outlined in the following subsections.

3.5.2.1 Coding of Cha's method

The process of coding Cha's method involves the following major steps:

- Cha's method assumes a solution of the form $\underline{\eta} = \vec{\eta} e^{\lambda t}$ instead of $\underline{\eta} = \vec{\eta} e^{i\omega t}$, so that the characteristic equation will be in terms of λ. Thus, Cha's method solves for λ, which will have different forms depending on the kind of attachments to the beam.
- The numbers of modes and lumped masses involved must first be determined which in this case are *N=10* and *s=6.*
- Two vectors are defined to account for the values of the masses and their respective positions along the beam, namely, $\underline{m}$ and $\underline{x}$.
- Equation (3.38) is the eigenfunction used, and is defined as a bi-variable function in Maple $\phi(j, y) = \sqrt{\dfrac{2}{\rho L}} \sin\left(\dfrac{j\pi y}{L}\right)$
- Two $n \times 1$ vectors representing the generalized masses and stiffnesses are defined. In this case, these are defined so that their *i*th entries are $M_i = 1$ and $K_i = \dfrac{(i\pi)^4 EI}{\rho L^4}$ respectively.
- A new matrix called $\mathbf{B}_0$ is constructed whose coefficients are determined by the following indexing function

$$\mathrm{B}_0(i, j) \rightarrow \sum_{r=1}^{N} \frac{\phi_r(x_i)\phi(x_j)}{\lambda^2 M_r + K_r} \qquad i, j = 1..s \tag{3.39}$$

where *i* and *j* represent the rows and columns of the matrix respectively and λ is the exponent in the assumed form of the solution $\vec{\eta} e^{\lambda t}$.

- A new vector called $\underline{B}_1$ is formed by mapping the inverse function $y \to \dfrac{1}{y}$ over the $\underline{m}$ vector. This vector is then multiplied by the scalar $\dfrac{1}{\lambda^2}$ to form vector $\underline{B}_2$. $\underline{B}_2$ is then transformed into a diagonal matrix called $\mathbf{B}_3$ whose diagonal elements are the coefficients of $\underline{B}_2$.
- By adding $\mathbf{B}_0$ and $\mathbf{B}_3$, a new matrix called $\mathbf{B}$ is formed whose determinant is equivalent to the determinant of equation (3.34).

By taking the determinant of $\mathbf{B}$, a polynomial fraction is achieved.

For the case under consideration, the required determinant returns a polynomial fraction whose numerator is given by following polynomial.

$$
\begin{aligned}
&\Big(-365357052 L^{36}\rho^{9}\cos\left(\tfrac{2}{5}\pi\right)+365357052\cos\left(\tfrac{1}{5}\pi\right)L^{36}\rho^{9} \\
&+730714185 L^{36}\rho^{9} \Big)\lambda^{18} + \Big(-1952757611298 L^{32}\rho^{8} I E\pi^{4}\cos\left(\tfrac{2}{5}\pi\right) \\
&+1964216826738\cos\left(\tfrac{1}{5}\pi\right)L^{32}\rho^{8} IE\pi^{4} + 3916975525305 L^{32}\rho^{8} I E\pi^{4} \Big)\lambda^{16} \\
&+ \Big(-49761551744000\cos\left(\tfrac{3}{10}\pi\right)L^{28}\rho^{7} I^{2}E^{2}\pi^{8} + 6390541319887042 L^{28}\rho^{7} I^{2}E^{2}\pi^{8} \\
&+3397426005770996\cos\left(\tfrac{1}{5}\pi\right)L^{28}\rho^{7} I^{2}E^{2}\pi^{8} + 58304887872000 L^{28}\rho^{7}\cos\left(\tfrac{1}{10}\pi\right)I^{2}E^{2}\pi^{8} \\
&-2993110120386116 L^{28}\rho^{7} I^{2}E^{2}\pi^{8}\cos\left(\tfrac{2}{5}\pi\right) \Big)\lambda^{14} \\
&+ \Big(2129452916497104480\cos\left(\tfrac{1}{5}\pi\right)L^{24}\rho^{6} I^{3}E^{3}\pi^{12} \\
&+87107794559424000 L^{24}\rho^{6}\cos\left(\tfrac{1}{10}\pi\right)I^{3}E^{3}\pi^{12} + 3600125636973144602 L^{24}\rho^{6} I^{3}E^{3}\pi^{12} \\
&-1470661813666581680 L^{24}\rho^{6} I^{3}E^{3}\pi^{12}\cos\left(\tfrac{2}{5}\pi\right) \\
&-105376459320704000\cos\left(\tfrac{3}{10}\pi\right)L^{24}\rho^{6} I^{3}E^{3}\pi^{12} \Big)\lambda^{12} \\
&+ \Big(-209296905130396686000 L^{20}\rho^{5} I^{4}E^{4}\pi^{16}\cos\left(\tfrac{2}{5}\pi\right) \\
&+264777766905260403200\cos\left(\tfrac{1}{5}\pi\right)L^{20}\rho^{5} I^{4}E^{4}\pi^{16} \\
&+2448582323539552000 0 L^{20}\rho^{5}\cos\left(\tfrac{1}{10}\pi\right)I^{4}E^{4}\pi^{16} + 474085131146614363205 L^{20}\rho^{5} I^{4}E^{4}\pi^{16} \\
&-60883088624729728000\cos\left(\tfrac{3}{10}\pi\right)L^{20}\rho^{5} I^{4}E^{4}\pi^{16} \Big)\lambda^{10} \\
&+ \Big(98027713505892726427 02\cos\left(\tfrac{1}{5}\pi\right)L^{16}\rho^{4} I^{5}E^{5}\pi^{20} \\
&-9236151106946874050702 L^{16}\rho^{4} I^{5}E^{5}\pi^{20}\cos\left(\tfrac{2}{5}\pi\right) \\
&+285502002007240256000 L^{16}\rho^{4}\cos\left(\tfrac{1}{10}\pi\right)I^{5}E^{5}\pi^{20} \\
&-5234587023924680320000\cos\left(\tfrac{3}{10}\pi\right)L^{16}\rho^{4} I^{5}E^{5}\pi^{20} \\
&+19043198911927774233725 L^{16}\rho^{4} I^{5}E^{5}\pi^{20} \Big)\lambda^{8} \\
&+ \Big(1395088313743630028800 0 L^{12}\rho^{3}\cos\left(\tfrac{1}{10}\pi\right)I^{6}E^{6}\pi^{24} \\
&+2124315341958447079705 60 L^{12}\rho^{3} I^{6}E^{6}\pi^{24} \\
&+980227953114937577898 08\cos\left(\tfrac{1}{5}\pi\right)L^{12}\rho^{3} I^{6}E^{6}\pi^{24} \\
&-4129080292632949555200 0\cos\left(\tfrac{3}{10}\pi\right)L^{12}\rho^{3} I^{6}E^{6}\pi^{24} \\
&-1137308536508885635779 68 L^{12}\rho^{3} IM^{6}E^{6}\pi^{24}\cos\left(\tfrac{2}{5}\pi\right) \Big)\lambda^{6} \\
&+\Big(-8936257308033024000000 0\cos\left(\tfrac{3}{10}\pi\right)L^{8}\rho^{2} I^{7}E^{7}\pi^{28} \\
&-3620558397654644536834 56 L^{8}\rho^{2} I^{7}E^{7}\pi^{28}\cos\left(\tfrac{2}{5}\pi\right) \\
&+2495688988810882558671 36\cos\left(\tfrac{1}{5}\pi\right)L^{8}\rho^{2} I^{7}E^{7}\pi^{28} + \\
&6401148893703825600683 52 L^{8}\rho^{2} I^{7}E^{7}\pi^{28} \\
&+6515274866245632000000 0 L^{8}\rho^{2}\cos\left(\tfrac{1}{10}\pi\right)I^{7}E^{7}\pi^{28} \Big)\lambda^{4} \\
&+\Big(-8438515420563689472000 0 L^{4}\rho I^{8}E^{8}\pi^{32}\cos\left(\tfrac{2}{5}\pi\right) \\
&+2798637219186573312000 0\cos\left(\tfrac{1}{5}\pi\right)L^{4}\rho I^{8}E^{8}\pi^{32} \\
&+4126013344606875301969 92 L^{4}\rho IM^{8}E^{8}\pi^{32} \Big)\lambda^{2} + 1734012131277275136000 0\pi^{36}E^{9}I^{9}
\end{aligned}
\tag{3.40}
$$

As can be seen from the preceding equation, the polynomial is of order 18 in the variable λ.

3.5.2.2 Coding of the Direct Eigenvalue Method

As in the previous section, here the procedure of coding the direct eigenvalues method is explained which includes the following major steps:

- The first step is determining of the number of modes as well as the number of attachments, that is, $N=10$ and $s=6$.
- Defining the eigenfunction of the unconstrained beam for the case of simply supported boundary conditions is done using the bi-variable function $\phi(i,x)=\sqrt{\frac{2}{\rho L}}\sin\left(\frac{i\pi x}{L}\right)$.
- In addition to $\underline{m}$ and $\underline{x}$ vectors representing masses and their corresponding positions respectively, a new vector $\underline{N}$ is introduced containing the sequence of integers from 1 to N that is, the number of modes.
- A Maple procedure f is introduced to calculate matrices $m_1 \cdot \underline{\phi_1}.\underline{\phi_1}^T \ldots \mathrm{m}_s \cdot \underline{\phi_s}.\underline{\phi_s}^T$.
- In order to construct matrices $m_1 \cdot \underline{\phi_1}.\underline{\phi_1}^T$ to $m_s \cdot \underline{\phi_s}.\underline{\phi_s}^T$, Maple procedure f must work within a loop which is repeated s times and in each execution, it takes a coefficient of $\underline{m}$ and its corresponding coefficient in $\underline{x}$ as input to the procedure *f*. The outputs are $m_1 \cdot \underline{\phi_1}.\underline{\phi_1}^T$ to $m_s \cdot \underline{\phi_s}.\underline{\phi_s}^T$ which are represented by $\mathbf{X}_q,\ q=1..s$.
- A new matrix $\mathbf{M}_2$ is introduced by adding the outputs of the loop

$$\mathbf{M}_2=\sum_{q=1}^{s}\mathbf{X}_q \tag{3.41}$$

- To make the complete mass matrix representing the system, The *n*th-order identity matrix I_n must be added to $\mathbf{M}_2$ that is,

$$\mathbf{M}_t=\mathbf{I}_n+\mathbf{M}_2 \tag{3.42}$$

- To form the stiffness matrix $\mathbf{K}_t$, initially a vector $\underline{K}_{n\times 1}$ is generated whose coefficients are derived using the following sequence function

$$seq\left(\frac{p^4\pi^4 EI}{\rho L^4}, p=1..n\right) \tag{3.43}$$

- Using the $\underline{K}$ vector, $\mathbf{K}_t$ is constructed as a diagonal matrix whose diagonal coefficients are elements of $\underline{K}$.

- Call Eigenvalues$\left(\mathbf{K}_t,\mathbf{M}_t\right)$. Using the built-in eigenvalues call in Maple for $\mathbf{M}_t$ and $\mathbf{K}_t$ is equivalent to solving for the squared frequencies of a conservative system represented by $\mathbf{M}_t$ and $\mathbf{K}_t$:

$$\det\left(\mathbf{K}_t - \omega^2\mathbf{M}_t\right) = 0 \tag{3.44}$$

- The natural frequencies of the system are found by taking the square root of the generalized eigenvalues returned by Maple.

3.5.3 Comparing and Analyzing the Results

The results for both Cha's method and the direct determinant method for the case with *N*=10, *s*=6 discussed above are shown in Table 1. The cases *N*=10, *s*=4 and *s*=3 are also considerd. For the *s*=4 case, the masses are ρL , $2\rho L$, $3\rho L$, $5\rho L$ and are positioned at 0.2*L*,*0.3L*,*0.4L*,*0.8L*, respectively. The results for this case are shown in Table 2. For the *s*=3 case, the masses are ρL , $3\rho L$, $5\rho L$ and are positioned at 0.2*L*,*0.3L*,*0.7L* respectively. The results are shown in Table 3.

Due to the fact that the general response utilized by Cha in [1] is of the form $\underline{\eta} = \vec{\eta}e^{\lambda t}$ in which $i = \sqrt{-1}$ is not included in the exponential term, the results obtained are complex, purely imaginary and appear in complex conjugate pairs. This is consistent with the initial prediction that the system is conservative and undergoes harmonic oscillation. On the other hand, since the general response used by the built-in eigenvalues function in Maple is a priori assumed to be of the form $\underline{\eta} = \vec{\eta}e^{i\omega t}$ in which $i = \sqrt{-1}$ is already included in the exponential term, the results obtained in the latter case are positive real numbers.

By comparing the two vectors of frequencies, it is seen that Cha's method does not yield the highest natural frequency of the system. These are highlighted in bold in Table 1,

Table 2 and Table 3. This can be attributed to the fact that Cha's method does not account for the case where the mass is located on a node of a mode. In order to derive the missing frequencies using Cha's method equation (3.37) must be solved separately.

As far as the simplicity of the Cha's method is concerned, it is observed that the (reduced) order of the polynomial resulting from this method is 18, that is, only a two degree reduction compared to the direct eigenvalues method which would yield a characteristic polynomial of order 20. Thus, although Cha's method did yield a reduced-order characteristic equation as it claimed, the reduction in the order of the polynomial was not large.

Table 1 Comparison of Cha's and Direct Methods for the case *N*=10, *s*=6

	Frequencies obtained using Cha's Method	*Frequencies obtained using Eigenvalues call in Maple*
N=10 and s=6	2.120794116*i* –2.120794116*i* 7.95380551462965*i* –7.95380551462965*i* 18.4693672793564*i* –18.4693672793564*i* 38.1407199603821*i* –38.1407199603821*i* 52.3291169834460*i* –52.3291169834460*i* 101.580716086930*i* –101.580716086930*i* 329.043761391471*i* –329.043761391471*i* 357.089316434398*i* –357.089316434398*i* 521.669478786281*i* –521.669478786281*i*	2.12079411705346 7.95380551378547 18.4693672704206 38.1407199785536 52.3291169086411 101.580715998471 329.043761767604 357.089316372872 521.669479577096 **986.9604403**

Table 2 Comparison of Cha's and Direct Methods for *N*=10 and *s*=4

	Frequencies obtained using Cha's Method	*Frequencies obtained using Eigenvalues call in Maple*
N=10 and s=4	$\begin{bmatrix} 2.710095636i \\ -2.710095636i \\ 9.215747234i \\ -9.215747234i \\ 29.99015426i \\ -29.99015426i \\ 86.47091852i \\ -86.47091852i \\ 122.6284860i \\ -122.6284860i \\ 273.2641574i \\ -273.2641574i \\ 348.2297693i \\ -348.2297693i \\ 382.1531076i \\ -382.1531076i \\ 670.3638316i \\ -670.3638316i \end{bmatrix}$	$\begin{bmatrix} 2.71009563678465 \\ 9.21574723499000 \\ 29.9901545172188 \\ 86.4709107577214 \\ 122.628498364147 \\ 273.264203297751 \\ 348.229457277760 \\ 382.153385635185 \\ 670.363831036907 \\ \mathbf{986.9604403} \end{bmatrix}$

Table 3 Comparison of Cha's and Direct Methods for N=10 and s=3

	Frequencies obtained using Cha's Method	*Frequencies obtained using Eigenvalues call in Maple*
N=10 and s=3	$\begin{bmatrix} 2.821315270i \\ -2.821315270i \\ 9.498380445i \\ -9.498380445i \\ 54.61361634i \\ -54.61361634i \\ 102.6498824i \\ -102.6498824i \\ 155.9326016i \\ -155.9326016i \\ 308.2554133i \\ -308.2554133i \\ 359.5185341i \\ -359.5185341i \\ 498.1507730i \\ -498.1507730i \\ 686.0743704i \\ -686.0743704i \end{bmatrix}$	$\begin{bmatrix} 2.82131526729543 \\ 9.49838045292375 \\ 54.6136162161018 \\ 102.649883625710 \\ 155.932600316251 \\ 308.255424784912 \\ 359.518516766900 \\ 498.150782334268 \\ 686.074370148055 \\ \mathbf{986.9604403} \end{bmatrix}$

3.6 Conclusion

In this chapter, the problem of determining the natural frequencies of a conservative one-dimensional system (beam) to which several masses has been attached was considered. Two methods were developed in order to solve for the natural frequencies of the combined system, namely Cha's method and the direct method. A specific problem was considered in order to further explore the advantages and shortcomings of the two methods and the following observations were made:

- Although Cha's Method is successful in reducing the dimensions of matrices whose determinant should be solved by relating the dimensions to the number of attachments rather than to the number of assumed modes, the reduction in the degree of polynomial to be solved is not sufficient to compensate for the additional complexity of the implementation of the method. The resulting polynomial using Cha's method is of order 18, which means a reduction in order of the original polynomial by only two. The resulting characteristic polynomial of the system must be of order 20 since 10 assumed modes are chosen.

- Using Cha's method, some of the natural frequencies may be missing because this method does not account for cases where mass is located on a node or in other words this method does not yield trivial answers. In order to get the full span of frequencies, equation (3.37) must be solved separately as well.

- The results obtained using this method are in complex form, purely imaginary and appear in complex conjugate pairs, while the results obtained by the direct method are positive real numbers. This difference is because of the initial assumption in the selection of the general solution to the system of second-order differential equations.

Based on these observations, the author believes that using the built-in Eigenvalues and Eigenvectors function calls in Maple is a more efficient way to solve the forward natural frequency problems of the physical systems and this is the method that was chosen for implementation in the rest of the thesis.

4 Chapter 4 - Effects of Adding a Mass to a Beam

4.1 Defining the problem

In this chapter, the effects of adding a mass to a beam are considered. Adding a mass can influence two major vibrational parameters of a beam that interact with each other; that is, a change in one parameter can bring about corresponding changes in the other. These two parameters are **frequencies** and **mode shapes.**

To explore these effects, the problem is attacked from two different angles:

- In the first case, the effects on the beam frequencies of placing a lumped mass at various locations along the beam are explored. This analysis is repeated for various masses. The process of changing the location of the mass from one end of the beam to the other shall be referred to as sliding the mass along the beam.
- In the second case, the effects on the mode shapes of adding various masses on a particular spot on the beam are investigated.

It is worth noting that these cases are applied to beams with two different sets of boundary conditions that are of great interest in the engineering world, namely, **simply-supported** and **fixed-free (cantilever)** beams.

4.2 Effect of an Added Mass on Beam Frequencies

In this section, the effect of adding a mass to the beam on the frequencies of the beam is investigated. This is achieved by adding a mass at various locations along the beam and investigating the resulting effect on the first five natural frequencies, as compared to the frequencies when no mass is added.

4.2.1 Assumptions and Modelling

Here, the major assumptions regarding the system under consideration are listed:

- The beam is an Euler-Bernoulli beam with length L.
- Only a single (point) mass is attached to the beam.
- The number of vibrational (assumed) modes utilized in the assumed modes method is 10 (N=10) for the simply-supported beam and N=17 for the cantilever beam.
- Five cases are considered with masses $m = 0.1\rho L, 0.5\rho L, \rho L, 5\rho L, 10\rho L$.

- Each of these masses is placed at various locations along the beam from *x=0* to *L* in 0.1*L* increments.
- Both simply-supported and cantilever boundary conditions are considered.
- The vectors of resulting natural frequencies are sorted in ascending order.
- The frequencies of the combined system are normalized with respect to the frequencies of the unconstrained beam.
- The investigation is confined to the first five natural frequencies of the combined system.

4.2.2 A Fixed Mass at Various Locations Along a Simply-supported Beam

Using Maple v14 (Maplesoft), the code to generate the vectors of frequencies is written.

4.2.2.1 Coding the simply-supported beam case

The steps to follow in order to code the simply-supported beam case are outlined here:

- The number of modes are *N=10.*
- The eigenfunction used for the case of a simply-supported beam is given by

$$\phi_i(x) = \sqrt{\frac{2}{\rho L}} \sin \frac{i\pi x}{L} \quad . \tag{4.1}$$

- The generalized masses and stiffnesses of the unconstrained beam which are used to build mass and stiffness matrices $\mathbf{M}^d$ and $\mathbf{K}^d$ respectively, of the unconstrained beam are defined as follows:

$$M_i = 1 \tag{4.2}$$

  and

$$K_i = \frac{(i\pi)^4 EI}{\rho L^4} \tag{4.3}$$

  where E is Young's modulus and I is the second-order moment of inertia of the cross-section of the beam.

- Based on the number of vibrational (assumed) modes taken, a vector $\underline{N}$ is built containing the sequence of natural numbers from 1 to *N*. This vector is further used to build the eigenfunction vectors.

- The $\mathbf{M}^d$ and $\mathbf{K}^d$ matrices are built using (4.2) and (4.3) along their main diagonal. These two matrices are used to derive the frequencies of the unconstrained beam.
- The built-in eigenvalues call in Maple is used to solve for the generalized eigenvalue problem with $\mathbf{M}^d$ and $\mathbf{K}^d$

$$\det\left(\mathbf{K}^d - \lambda\mathbf{M}^d\right) = 0\,. \tag{4.4}$$

- A list is defined whose elements are the different masses that are considered, namely $m = 0.1\rho L, 0.5\rho L, \rho L, 5\rho L, 10\rho L$.

- To simulate the effect of changing the location of each mass along the beam, a nested loop is written which takes a single element of the mass list and then using the nested loop, builds the mass matrix for the combined beam-and-mass system and calculates the absolute and relative natural frequencies with respect to the frequencies of the unconstrained beam for positions *x=0.1L* to *x=0.9L*.

- The vectors of relative frequencies obtained are organized into a matrix whose rows and columns correspond to the various masses and positions, respectively. So for example, based on this organization, calling coefficient (1, 5) of the matrix will give the vector of relative frequencies for the first entry of the mass list (in this case $m = 0.1\rho L$) positioned at the fifth entry of the position list (in this case $x = 0.5L$) and so on.
- Using data obtained in previous steps, the effects on the first five natural frequencies, of changing the position of each mass along the length of the beam are depicted using relative frequency (ratio of the beam with added mass frequency to bare beam natural frequency) versus position plots. These plots are shown below in Figure 2, Figure 3, Figure 4, Figure 5 and Figure 6.

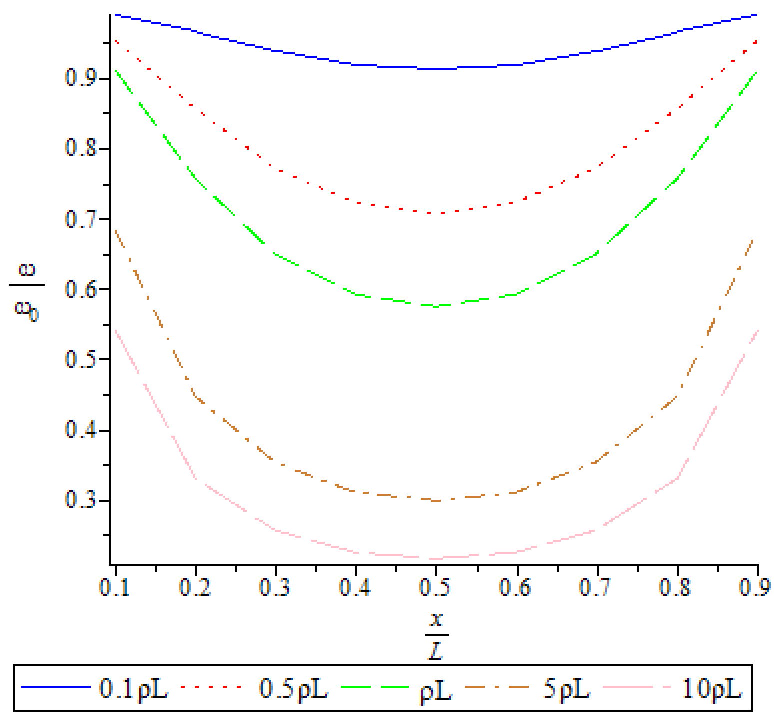


Figure 2 Changes of 1st Natural Frequency of a Simply-supported Beam with a Sliding Mass

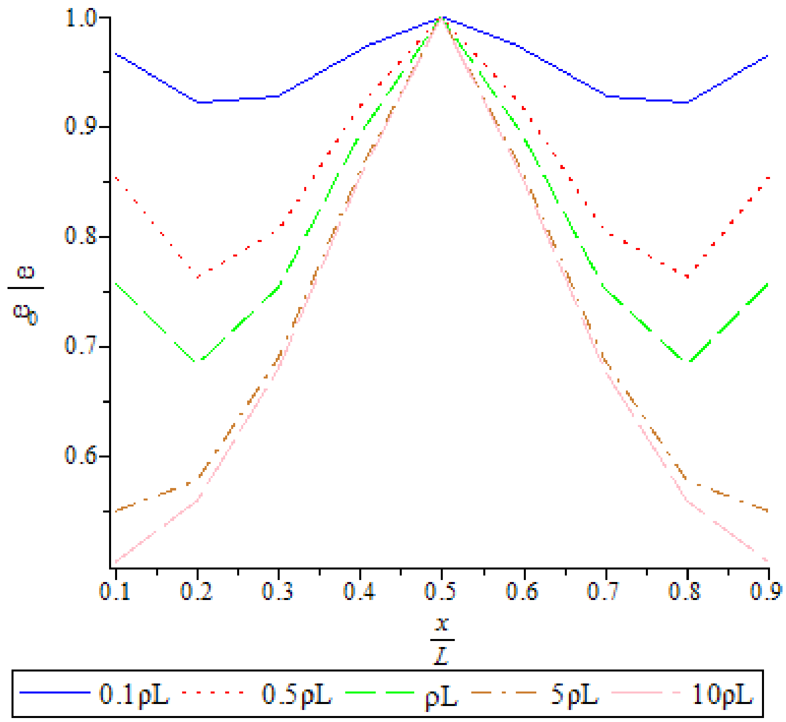


Figure 3 Changes of the 2nd Natural Frequency of a Simply-supported Beam with a Sliding Mass

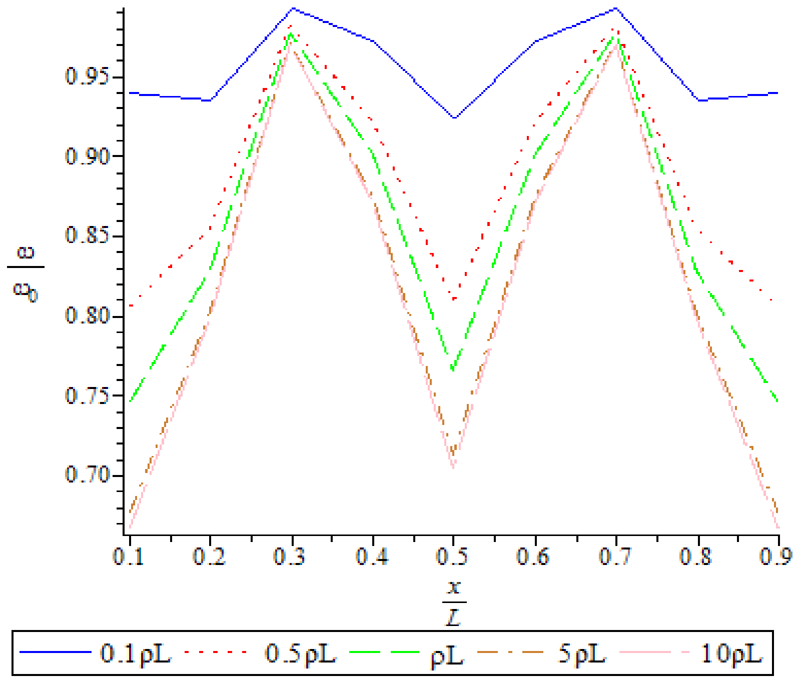


Figure 4 Changes of the 3rd Natural Frequency of a Simply-supported Beam with a Sliding Mass

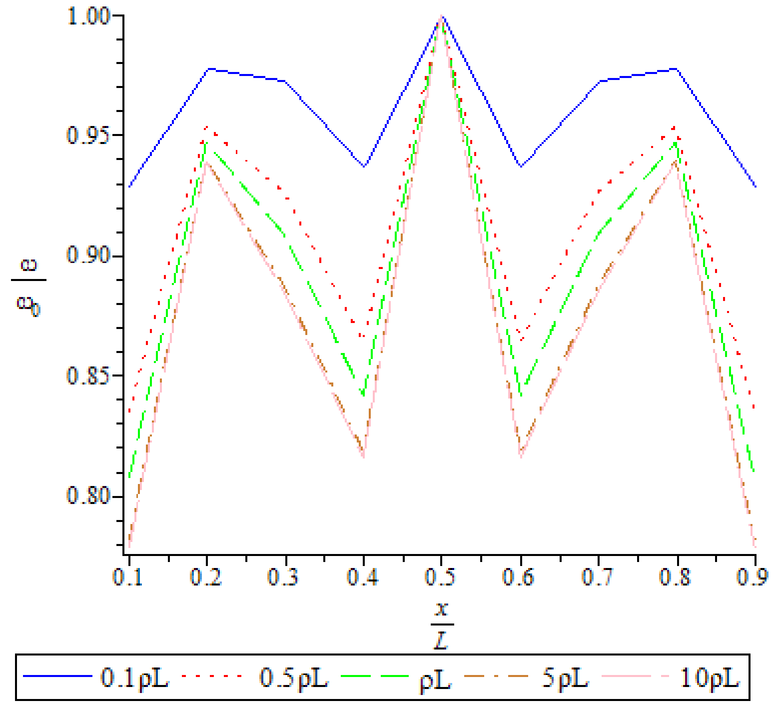


Figure 5 Changes of the 4th Natural Frequency of a Simply-supported Beam with a Sliding Mass

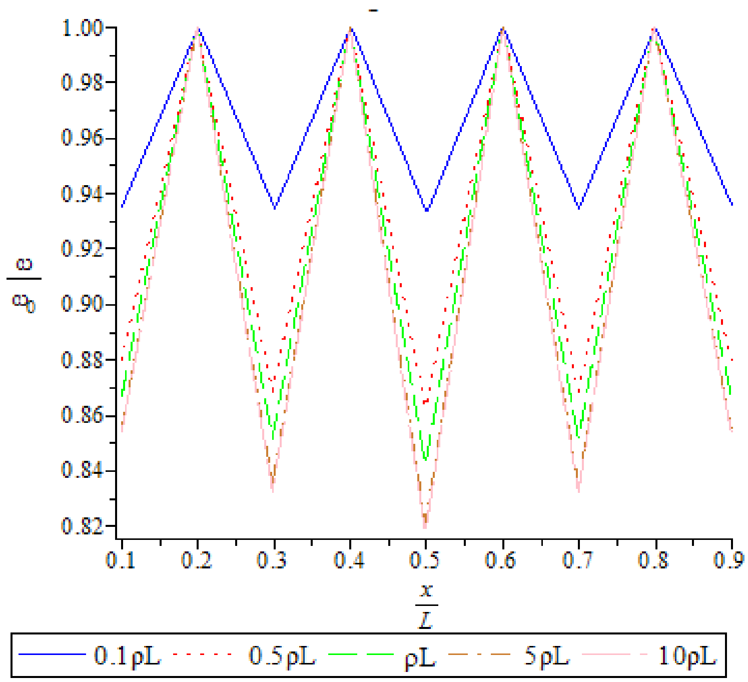


Figure 6 Changes of the 5th Natural Frequency of a Simply-supported Beam with a Sliding Mass

4.2.2.2 Observations and Analysis

Figure 2 to Figure 6 reveals several important aspects of how the position of a mass along a beam can affect its resulting natural frequencies;

- The first observation is the symmetry which exists around the mid-point of the beam. This implies that positioning the mass at, for example, $\frac{x}{L} = 0.2$, the same frequency is obtained as when the mass is positioned at $\frac{x}{L} = 0.8$. It should be noted that this is true only for cases where the boundary conditions at the two ends of the beam are identical. Therefore, this symmetry is not expected for the cantilever beam. Thus, in situations where such symmetry exists, it is enough to consider half of the beam and extend the results to the other half.
- In all of these plots, there exist situations where all five curves approach unity. The physical interpretation of this phenomenon is that when the position of a mass coincides with the node of any vibrational mode of the beam, the corresponding frequency remains unaltered regardless of the value of the added mass. As can be seen by considering the frequency plots for the second and

fifth modes, it is evident that as the mode number increases so does the probabilities of putting the mass on a node since higher modes have greater numbers of nodes.

- The effects of adding a mass on a beam are most pronounced when the mass is positioned at a peak point (anti-node) for the original mode shape. For instance, considering the first mode of vibration, for all the masses the minimum point (largest effect of adding a mass) is $\frac{x}{L} = 0.5$ which is the antinode of the original mode shape.

4.2.3 A Fixed Mass at Various Locations along a Cantilever Beam

The procedure to derive the frequencies and plots for the cantilever beam follows that of the simply-supported beam. The only difference between the simply-supported beam and cantilever (Fixed-Free) beam case is the choice of eigenfunction which is more complicated in the case of a cantilever beam and affects the initial stages of the coding process

4.2.3.1 Coding the Fixed-Free (Cantilever) Beam Case

Due to the complexity of the eigenfuncion of the fixed-free beam, the initial steps of coding differ from those of the simply-supported beam.

The eigenfuncion used for the fixed-free beam case is defined as in [33]

$$\phi_i(x) = \frac{1}{\sqrt{\rho L}}\left(\cos\beta_i x - \cosh\beta_i x + \frac{\sin\beta_i L - \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L}\left(\sin\beta_i x - \sinh\beta_i x\right)\right), \tag{4.5}$$

where $\beta_i L$ must satisfy the following transcendental equation

$$\cos\beta_i L \cosh\beta_i L = -1 \tag{4.6}$$

Moreover, the generalized masses and stiffnesses are given by[33]

$$M_i = 1 \tag{4.7}$$

$$K_i = \frac{(\beta_i L)^4 EI}{\rho L^4} \tag{4.8}$$

- The *fsolve* package in Maple (Maplesoft) is used to derive the roots of equation (4.6). The results are organized in vector format (equation (4.9)) and will be substituted into equation (4.5) to form the $\underline{\phi}$ vectors.

$$\beta_i L = \begin{bmatrix} 1.875104069 \\ 4.694091133 \\ 7.854757438 \\ 10.99554073 \\ 14.13716839 \\ 17.27875953 \\ 20.42035225 \\ 23.56194490 \\ 26.70353756 \\ 29.84513021 \\ 32.98672286 \\ 36.12831552 \\ 39.26990817 \\ 42.41150082 \\ 45.55309348 \\ 48.69468613 \\ 51.83627878 \end{bmatrix} \tag{4.9}$$

It is important to note the following issues regarding equation (4.6)

- The number of assumed modes chosen is equivalent to the number of solutions obtained for $\beta_i L$ in equation (4.6) (in this case 17).
- Equation (4.6) is an even function which implies symmetry in the roots obtained with respect to the origin.
- If the first root is set aside, it will be seen that each root is obtained by adding approximately 3.14 (close to π) to the previous root (almost periodic function behaviour)

Having obtained the roots of equation (4.6), they should be substituted into equation (4.5). The rest of the procedure is the same as for the simply-supported beam.

The plots depicting the changes in natural frequency of the combined system by changing the location of the lumped point masses along the beam are shown for the first five natural frequencies in Figure 7, Figure 8, Figure 9, Figure 10 and Figure 11. The frequencies are calculated as a ratio to corresponding natural frequencies of the bare beam.

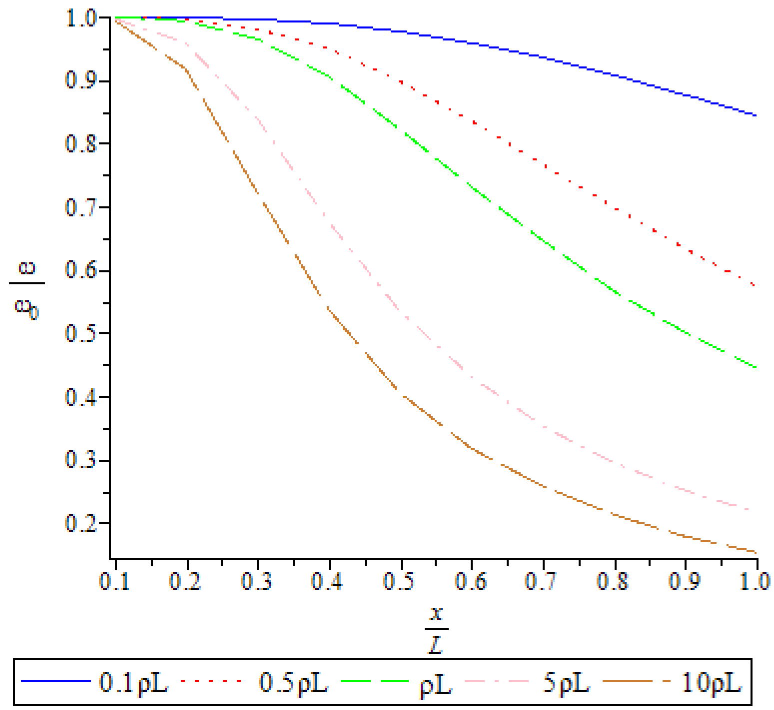


Figure 7 Changes of the 1st Natural Frequency of a Fixed-free Beam with a Sliding Mass

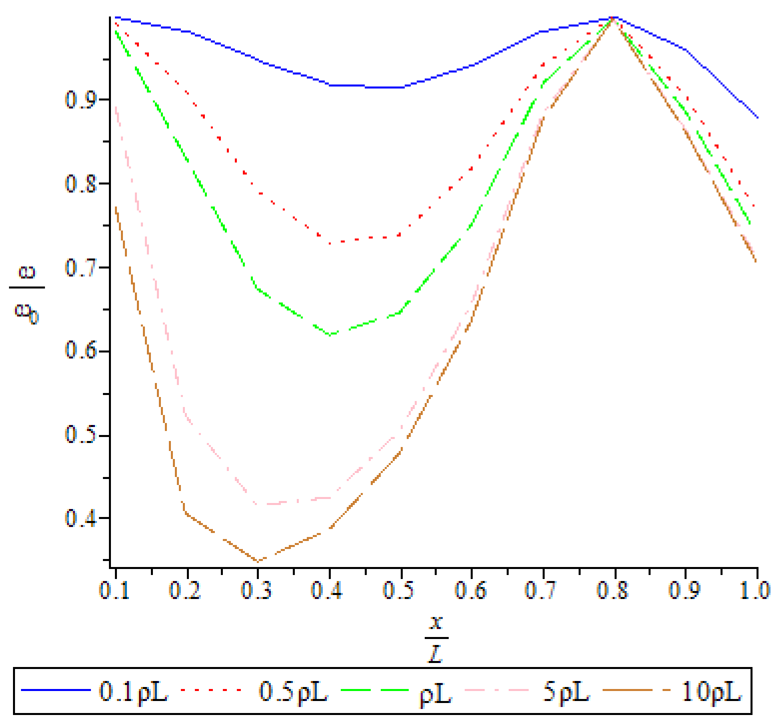


Figure 8 Changes of 2nd Natural Frequency of a Fixed-free Beam with a Sliding Mass

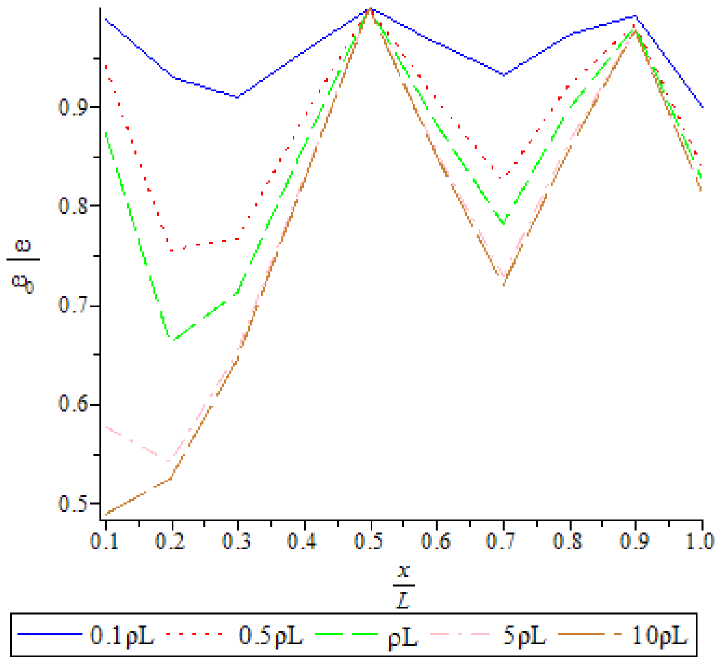


Figure 9 Changes of the 3rd Natural Frequency of a Fixed-free Beam with a Sliding Mass

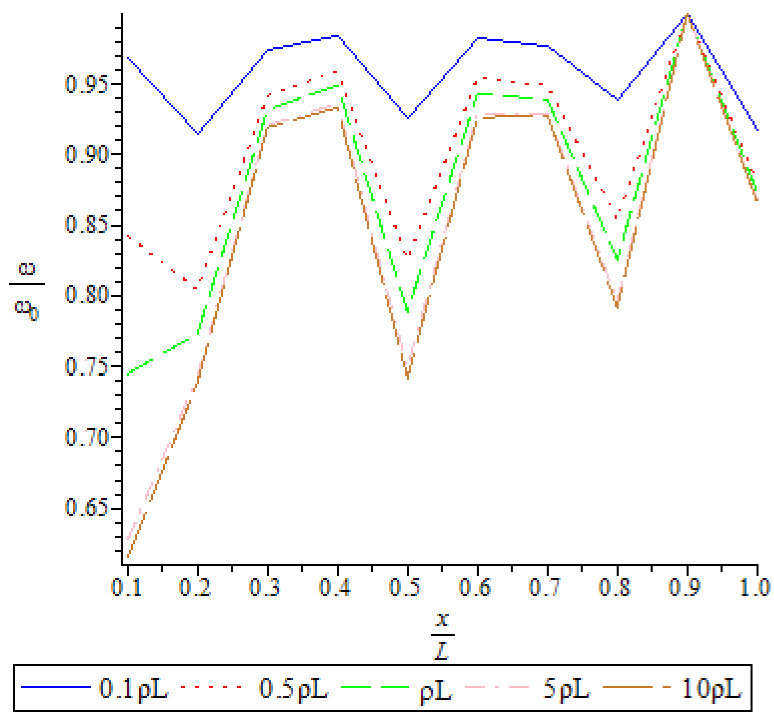


Figure 10 Changes of 4th Natural Frequency of a Fixed-free Beam with a Sliding Mass

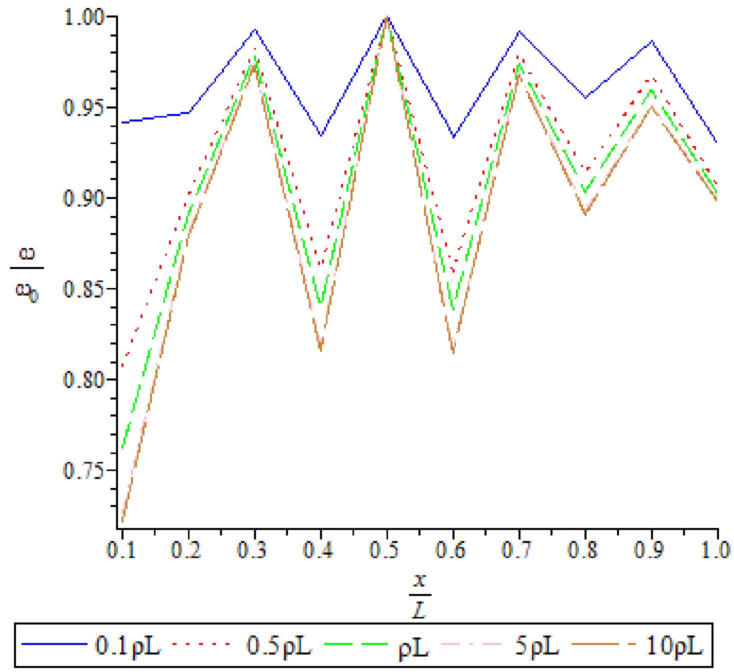


Figure 11 Changes of 5th Natural Frequency of a Fixed-free Beam with a Sliding Mass

4.2.3.2 Observations and Analysis

By inspecting the plots, the following observations and inferences can be made

- Unlike the case of a simply-supported beam, the frequency plots are not symmetrical. This is compatible with the prior prediction that symmetry arises from identical boundary conditions. Since the cantilever beam does not have identical boundary conditions, symmetry is not expected.
- There are points on the plots where all five plots converge to the same value. These points are the nodes of the original mode (mode of the beam without any added point mass).
- As the lumped mass is moved further away from the fixed end of the beam, its effects on the frequency become more pronounced.

4.3 Effect of an Added Mass on Beam Mode shapes

In this section, the effect of adding a mass to the beam on the frequencies of the beam is investigated. This is achieved by adding different masses which are positioned at various specified locations along the beam and plotting the modeshapes so that the effect on the mode shapes can be considered.

4.3.1 Assumptions and Modelling

For the sake of compatibility with the real world situations, only the cases where the mass of the attached lumped mass is a fraction of the mass of the beam shall be considered or in other words $m_{attached} = c\rho L$ where $m_{beam} = \rho L$ and $0 < c < 1$. The major assumptions used in the solution of the problem are as follows:

- The number of assumed modes utilized for discretization is 20 (*N*=20) for the case of a simply supported beam and *N=17* for fixed-free (cantilever) beam.
- The first five natural frequencies and their corresponding mode shapes are investigated.
- The attached masses considered are: $0.1\rho L, 0.2\rho L, 0.5\rho L, 0.8\rho L, \rho L$
- The mounting locations start from $x = 0.1L$ and end at $x = 0.9L$ with an $0.1L$ increment.
- The beam under investigation is a Euler-Bernoulli beam.
- The vectors of frequencies and their corresponding mode shapes are sorted in ascending order.
- To compare the modeshapes, the eigenvectors are normalized. This is achieved by dividing the *i*th eigenvector by its maximum value. This was found to produce normalized modeshapes that could be compared to each other. Note that although the eigenvectors are normalized the resulting modeshapes (an eigenvector-based linear combination of eigenfunctions) are not normalized.
- The frequencies are normalized with respect to their corresponding unconstrained beam frequencies.

4.3.2 Simply-supported Beam

Numerical simulation is accomplished using Maple (Maplesoft).

4.3.2.1 Coding procedure and description

The steps to follow in order to code the solution to the simply-supported beam are as follows:

- The simply-supported eigenfunction is defined as a bi-variable function using equation (4.1).
- The number of vibrational (assumed) modes are defined (*N=20*).
- Generalized mass and stiffness matrices $\mathbf{M}^d$ and $\mathbf{K}^d$ of the unconstrained beam are built using equations (4.2) and (4.3).
- Substituting $\mathbf{M}^d$ and $\mathbf{K}^d$ into the eigenvalues call in Maple, the frequencies of the unconstrained beam are obtained.

- A list is defined whose elements are $m = 0, 0.1\rho\mathrm{L}, 0.2\rho\mathrm{L}, 0.5\rho\mathrm{L}, 0.8\rho\mathrm{L}, \rho\mathrm{L}$ where zero stands for the case of unconstrained beam.
- A nested loop is introduced which takes a coefficient from the mass list and calculates the mass and stiffness matrices as well as sorted eigenvalues and eigenvectors for the nine positions along the beam, starting from *x=0.1L* and ending at *x=0.9L* in *0.1L* increments.
- The preceding loop enables the plotting of the first five mode shapes. Each plot represents six mode shapes corresponding to 5 masses plus the unconstrained beam. The plots are produced for the first five mode shapes and nine locations, as described in the previous step.

4.3.2.2 Results

The mode shapes plots obtained in the previous section are tabulated for the first five mode shapes and nine different positions of the added mass. The results are depicted in Table 4 through Table 9. The legends of the tables are explained in Figure 12:

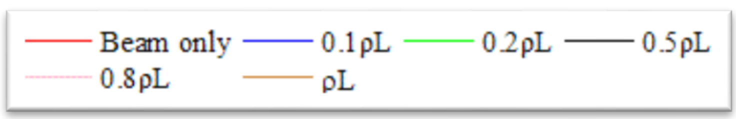


Figure 12 Legend for the Mode shape Plots

Table 4 First three simply supported mode shapes at positions *0.1L, 0.2L, 0.3L*

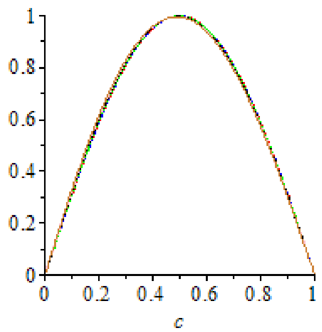
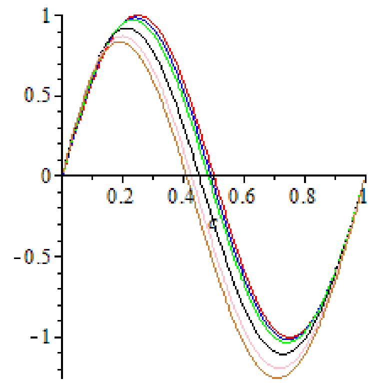
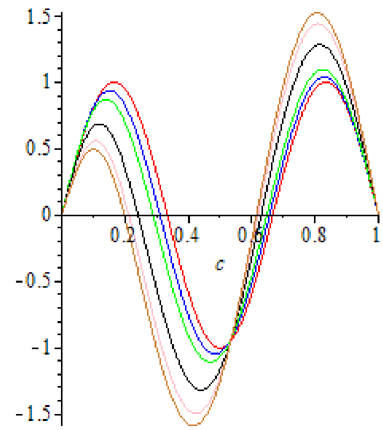
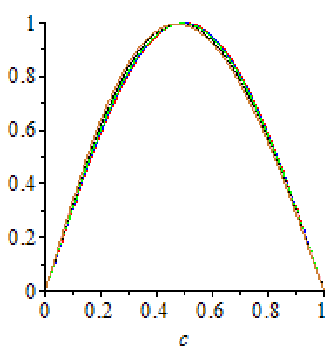
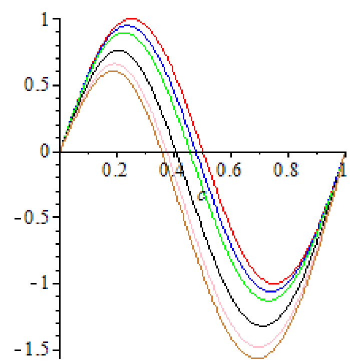
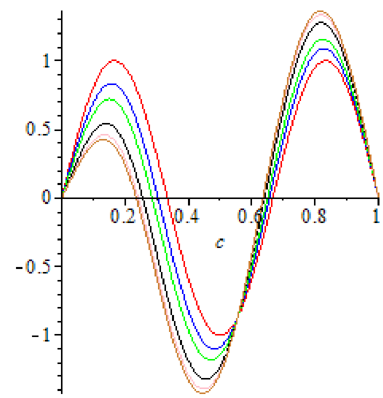
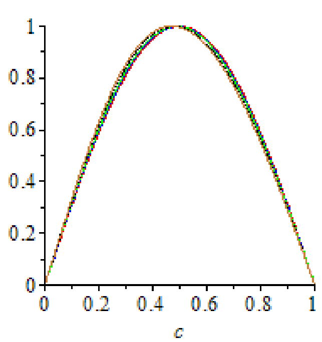
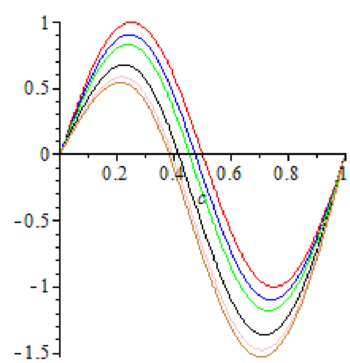
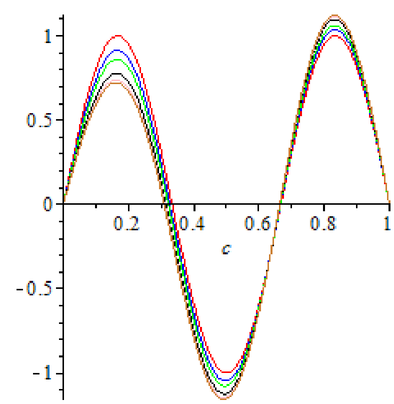
Mass Position	Mode shape 1	Mode shape 2	Mode shape 3
0.1L			
0.2L			
0.3L			

Table 5 Simply supported mode shapes 4 and 5 for Positions *0.1L, 0.2L, 0.3L*

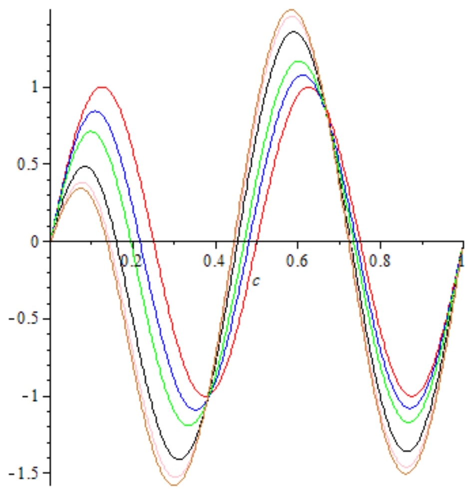
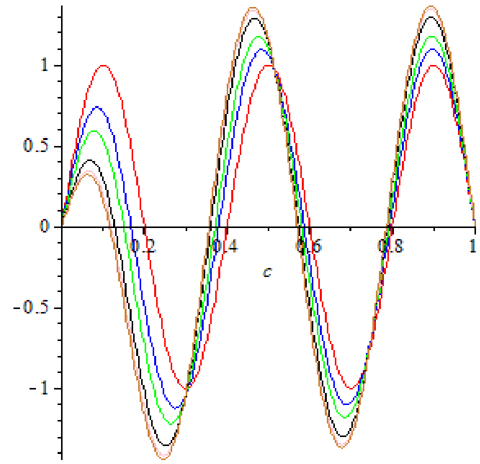
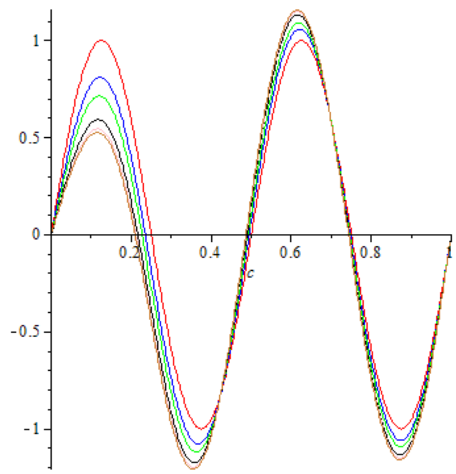
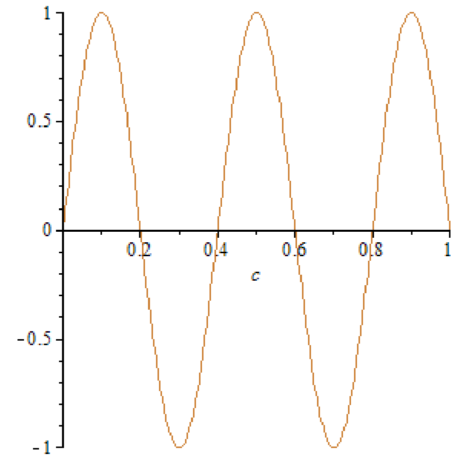
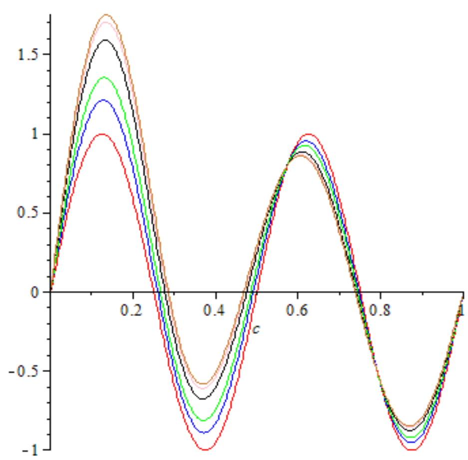
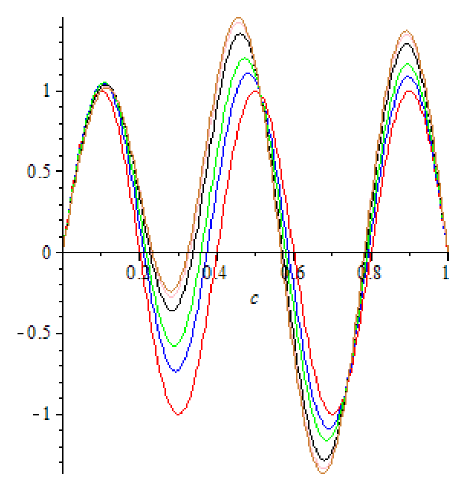
Mass Position	**Mode shape 4**	**Mode shape 5**
0.1L		
0.2L		
0.3L		

Table 6 First three simply supported mode shapes for *0.4L, 0.5L and 0.6L*

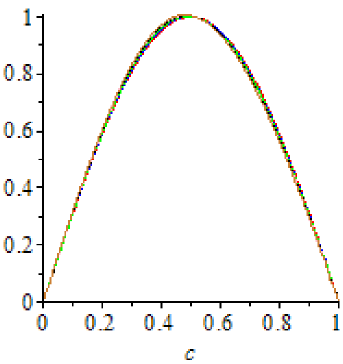
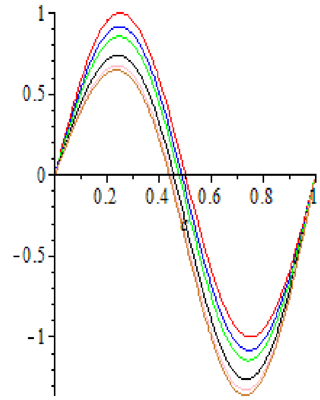
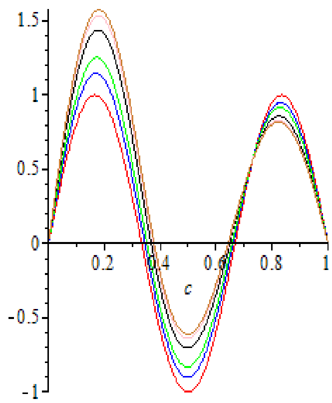
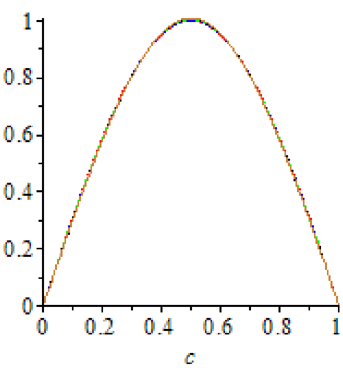
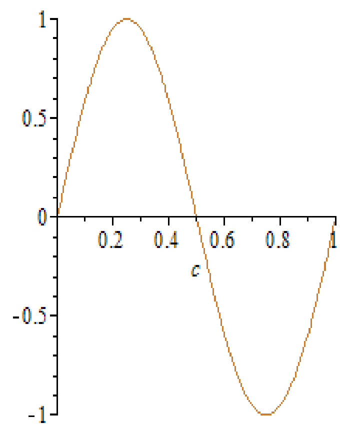
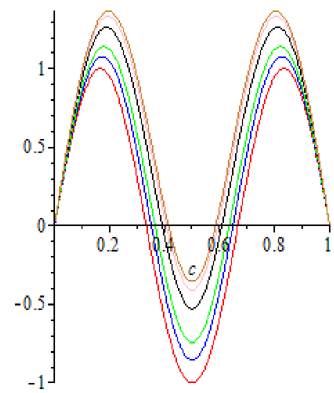
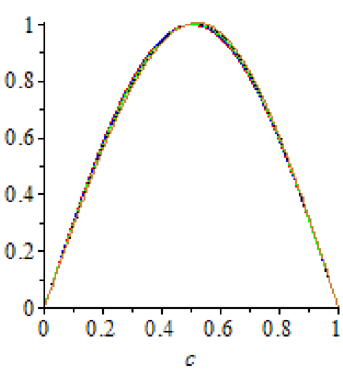
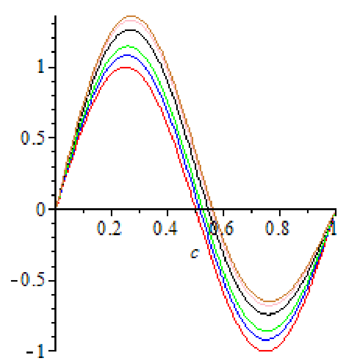
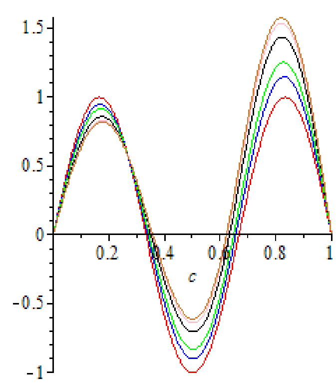
Mass Position	Mode shape 1	Mode shape 2	Mode shape 3
0.4L			
0.5L			
0.6L			

Table 7 Simply supported mode shape 4 and 5 for positions *0.4L, 0.5L and 0.6L*

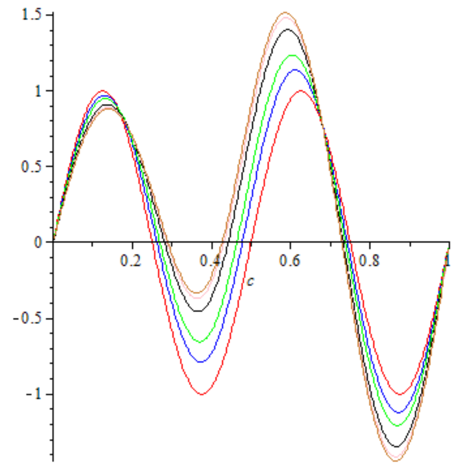
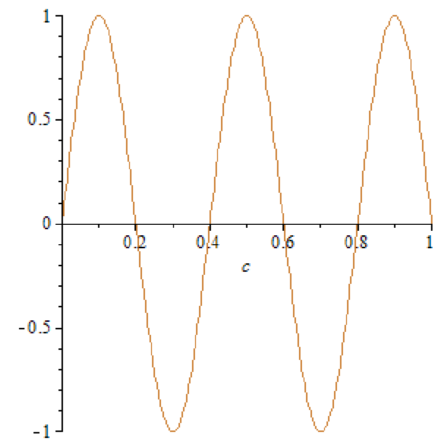
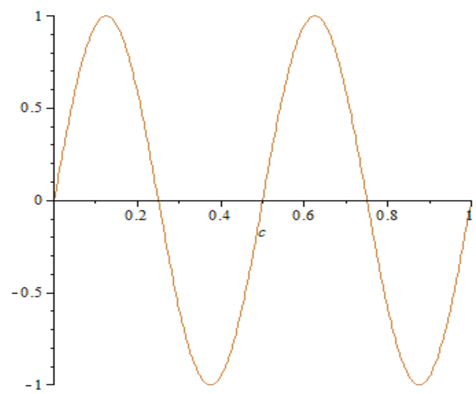
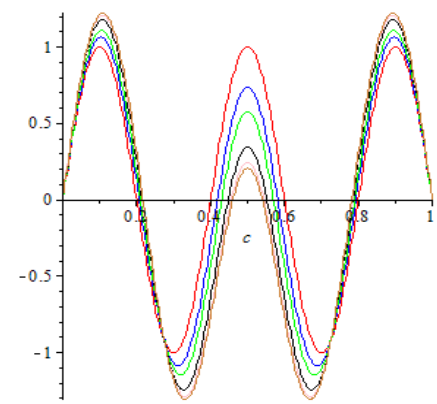
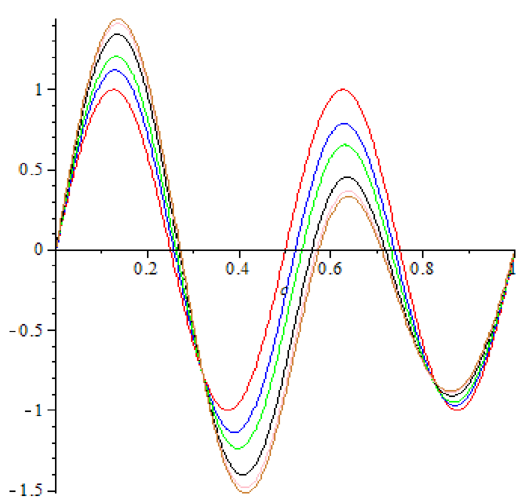
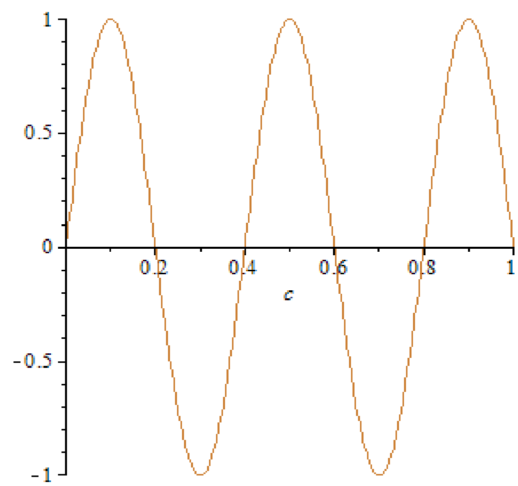
Mass Position	Mode shape4	Mode shape5
0.4L		
0.5L		
0.6L		

Table 8 First three simply supported mode shapes for positions *0.7L, 0.8L and 0.9L*

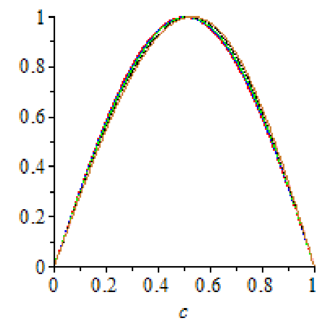
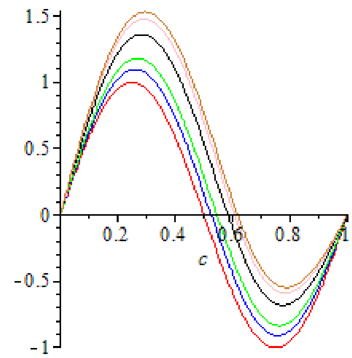
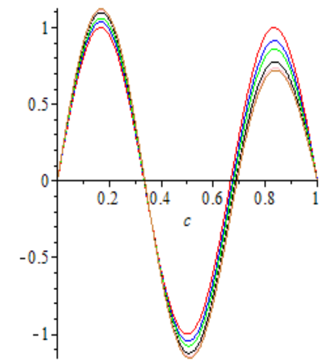
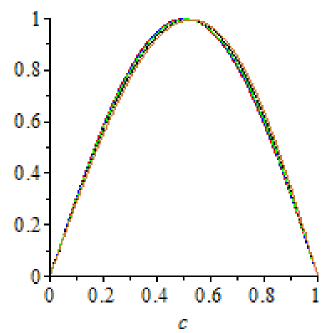
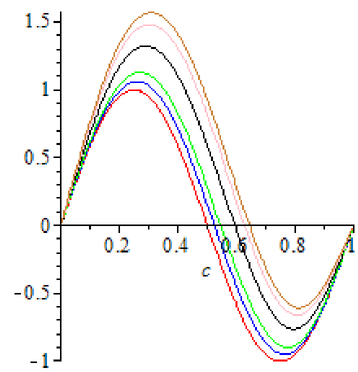
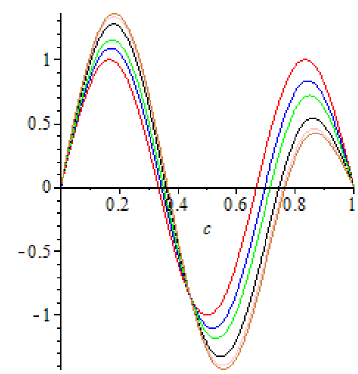
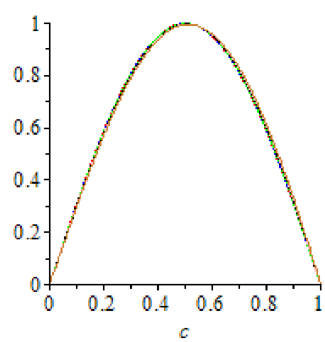
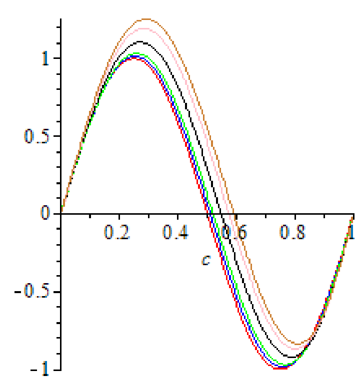
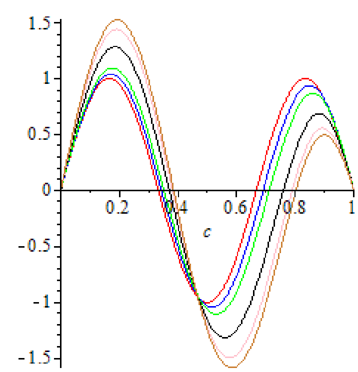
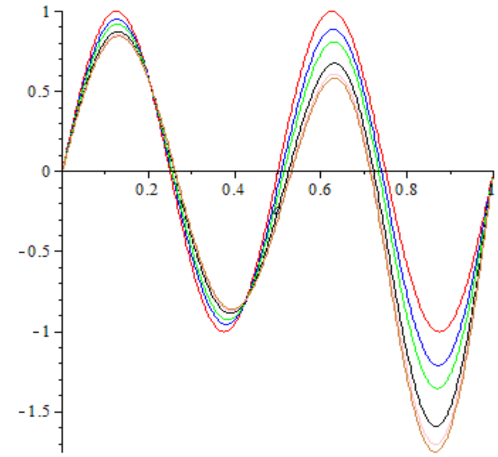
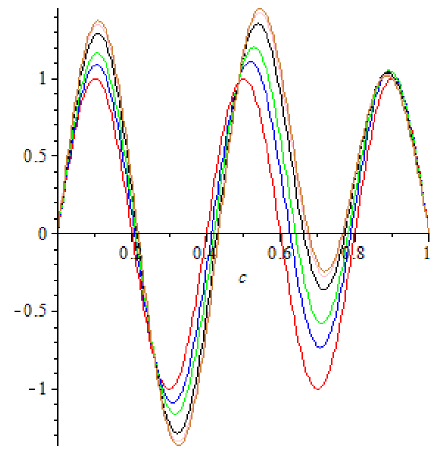
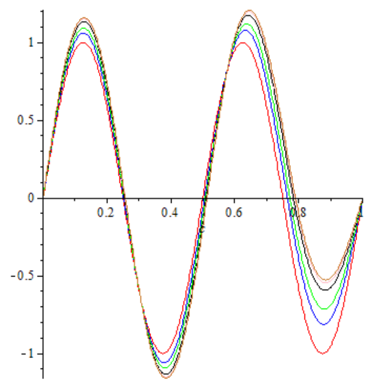
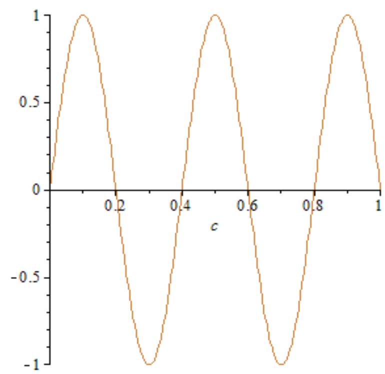
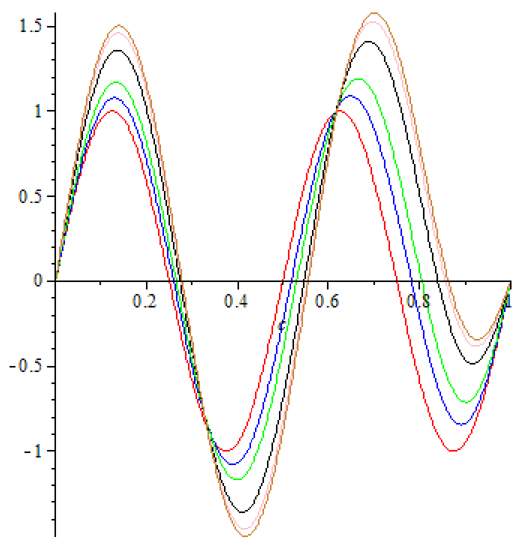
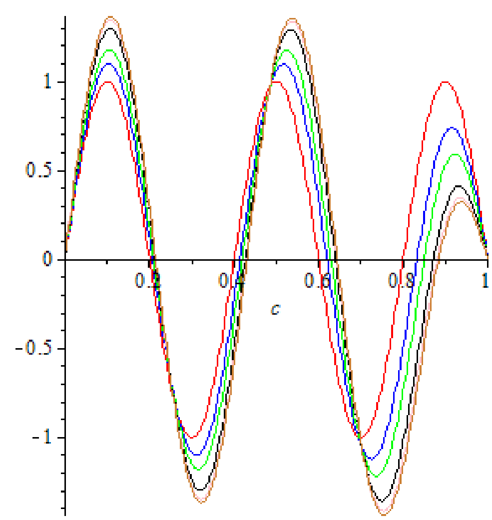
Mass Position	Mode shape1	Mode shape2	Mode shape3
0.7L			
0.8L			
0.9L			

Table 9 Simply supported mode shapes 4 and 5 for positions *0.7L, 0.8L and 0.9L*

Mass Positions	Mode shape4	Mode shape5
0.7L		
0.8L		
0.9L		

4.3.2.3 Observations and Analysis

From the numerical simulations, it can be observed that the first mode shape is insensitive to the mass and its position along the beam. In other words, the first mode shape remains approximately unaltered regardless of the value of the mass and its mounting position.

In cases where the position of the added mass coincides with the node of a certain vibrational mode of the unconstrained beam, then the original vibrational mode remains unaltered by the additional mass. In other words, the mode shape of the unconstrained beam and the mode shape of the combined system are the same regardless of the value of the added mass. This phenomenon is observable in the following mode shapes:

1. In Table 5, mode shape5 at *x=0.2L*
2. In Table 6, mode shape2 at *x=0.5L*
3. In Table 7, mode shape5 at *x=0.4L*
4. In Table 7 , mode shape4 at *x=0.5L*
5. In Table 7, mode shape5 at *x=0.6L*
6. In Table 9, mode shape5 at *x=0.8L*

Positioning a mass on or near a peak point (antinode) decreases the amplitude of the mode at that location. This decrease in amplitude is directly related to value of the mass. In other words, the heavier the mass, the greater the decrease in amplitude.

By visual examination of these plots, an interesting observation can be made regarding the positioning of a mass on a peak point (antinode). It is evident that a decrease in the amplitude of an antinode is compensated by an increase in amplitudes of the opposite antinode. For instance, considering the case of putting $m = 0.2\rho L$ at *x=0.5L*, which is an antinode for the third mode of vibration, as can be seen in Table 6, the following data are obtained:

- Amplitude of the mode shape at $x = \frac{L}{2}$ (the mounting position of the mass and an antinode) is -0.744.
- Amplitudes at $x = \frac{L}{6}$ and $x = \frac{5}{6}L$ (two opposite antinodes with respect to *x=0.5L*) are 1.133.

- In comparison, the corresponding amplitudes for the bare beam at $\frac{L}{2}$, $\frac{L}{6}$ and $\frac{5}{6}L$ are -1, +1 and +1 respectively.

Moreover, it is observed that the decrease in the amplitude of the antinode over which the mass is mounted is approximately canceled out by the sum of the increase in the amplitude at other antinodes. This implies a kind of conservation law and is depicted numerically in Table 10 to Table 13 for the case of $m = 0.2\rho L$ and the fifth mode shapes.

Table 10 Amplitude change for $m = 0.2\rho L$ positioned at *x=0.5L* for the 3rd mode shape

Position of the antinode	Decrease in amplitude due to added mass	Increase in amplitude due to added mass
$L/6$		0.133
$L/2$	0.256	
$5L/6$		0.133
Result	**0.256**	**0.266**

Table 11 Amplitude changes for $m = 0.2\rho L$ positioned at *x=0.1L* for the 5th mode shape

Position of the antinode	Decrease in amplitude due to added mass	Increase in amplitude due to added mass
0.1L	0.4611541252	
0.3L		0.042670436
0.5L		0.099013724
0.7L		0.152128483
0.9L		0.178399426
Result	**0.4611541252**	**0.472212069**

Table 12 Amplitude change for $m = 0.2\rho L$ positioned at *x=0.3L* for the 5th mode shape

Position of the antinode	Decrease in amplitude due to added mass	Increase in amplitude due to added mass
0.1L		0.041824515
0.3L	0.4308311341	
0.5L		0.100985491
0.7L		0.134872574
0.9L		0.164712200
Result	**0.4308311341**	**0.44239478**

Table 13 Amplitude change for $m = 0.2\rho L$ positioned at *x=0.5L* for the 5th mode shape

Position of the antinode	Decrease in amplitude due to added mass	Increase in amplitude due to added mass
0.1L		0.109178420
0.3L		0.103832243
0.5L	0.4141843233	
0.7L		0.103832243
0.9L		0.109178419
Results	**0.4141843233**	**0.426021326**

4.3.3 Fixed-free (Cantilever) Beam

Numerical simulation was performed using Maple v14 (Maplesoft).

4.3.3.1 Coding Procedure and Description

Since the goal of both codes is to obtain the eigenvalues, the procedures to follow are the same for both codes; however, due to the complex nature of the eigenfunction of a fixed-free (cantilever) beam, a few additional steps must be added at the outset of the program. These additional steps are as follows:

- A function must be defined representing the transcendental equation (4.6).
- Using *fsolve* command inside a set in order to avoid duplicate results, equation (4.6) is solved for $\beta_i L$ and then the results are converted to a vector format representation as depicted in (4.9). The number of modes utilized is equivalent to the number of roots obtained.
- A bi-variable function representing the eigenfunction of a fixed-free (cantilever) beam (equation (4.5)) is defined.
- The mass and stiffness matrices of the unconstrained beam are built using the generalized mass and stiffnesses as described by equations (4.7) and (4.8) respectively.

The rest of the coding process is identical to that of the simply-supported beam.

4.3.3.2 Results

The mode shapes for the case of a fixed-free (cantilever) beam are derived and tabulated for the first five frequencies and ten locations along the beam in *0.1L* increments. These results are tabulated in Table 14 to Table 19 and the legend used is as depicted in Figure 12.

Table 14 First three mode shapes of a fixed-free (cantilever) beam at positions *0.1L, 0.2L and 0.3L*

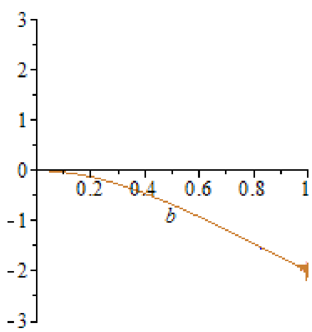
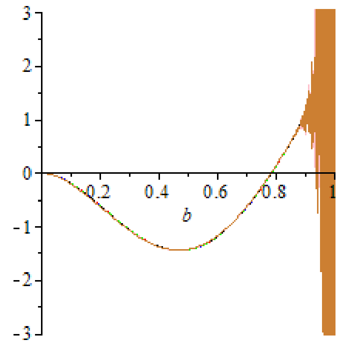
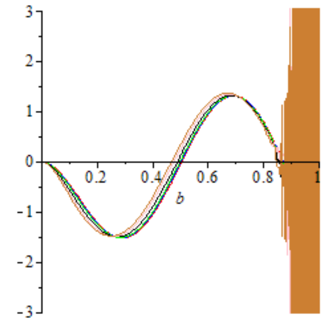
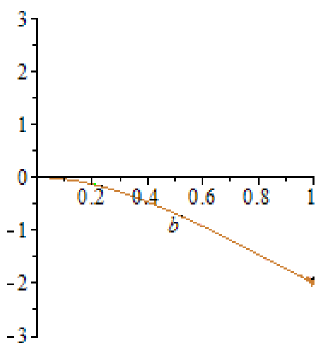
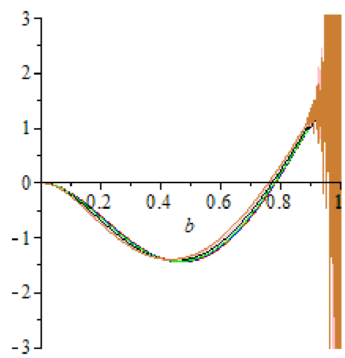
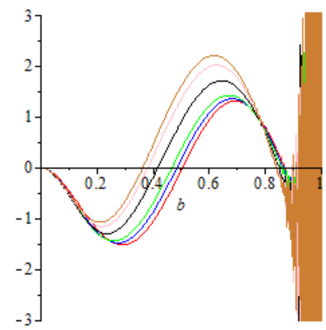
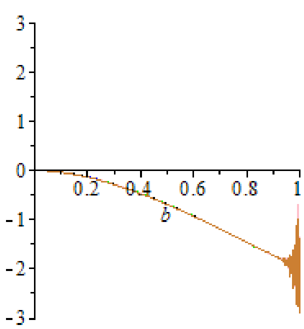
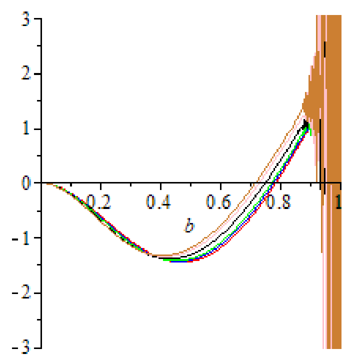
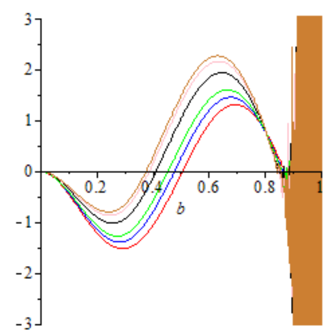
Mass Position	Mode shape 1	Mode shape 2	Mode shape 3
0.1L			
0.2L			
0.3L			

Table 15 Mode shapes 4 and 5 for a cantilever beam at positions *0.1L, 0.2L and 0.3L*

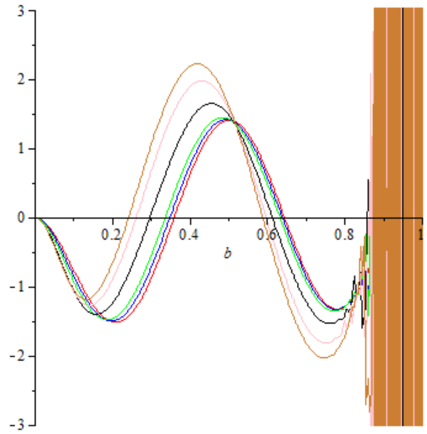
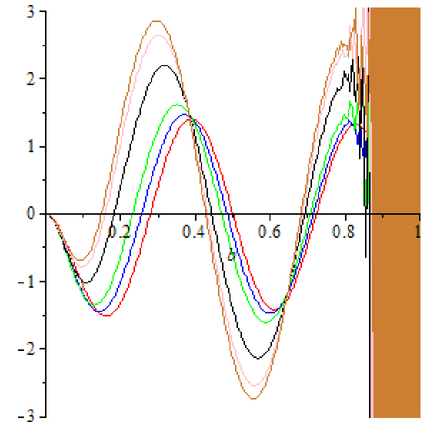
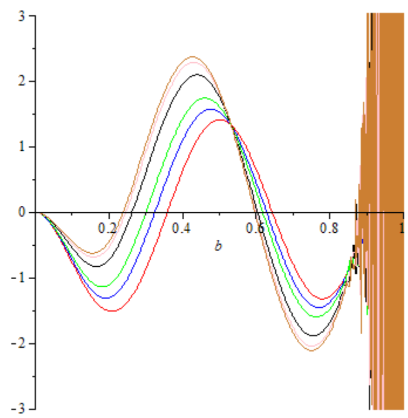
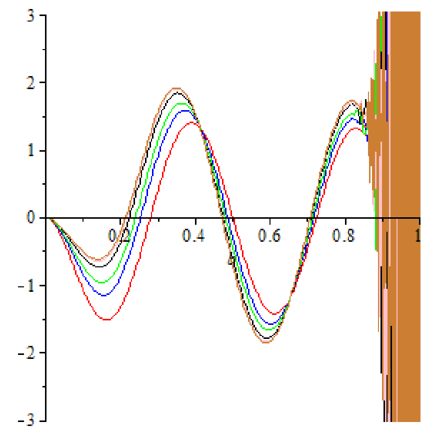
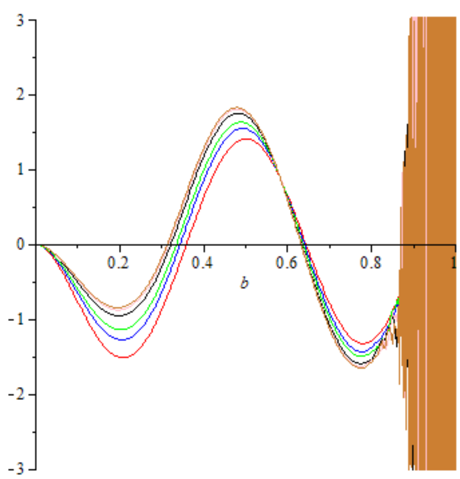
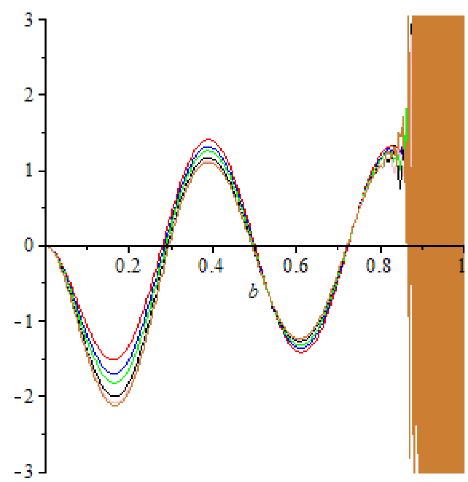
Mass Position	Mode shap 4	Mode shape 5
0.1L		
0.2L		
0.3L		

Table 16 First three mode shapes of a fixed-free beam with a single mass at positions *0.4L*, *0.5L* and *0.6L*

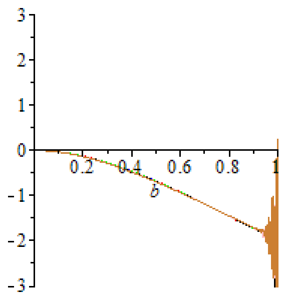
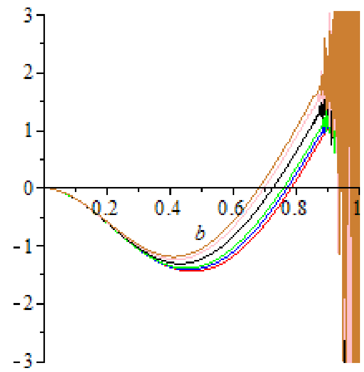
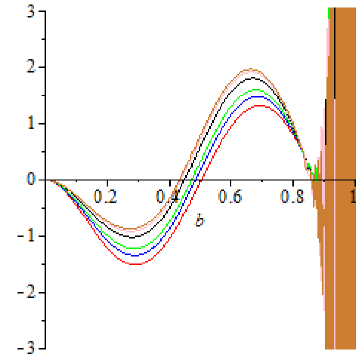
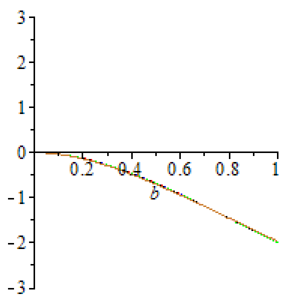
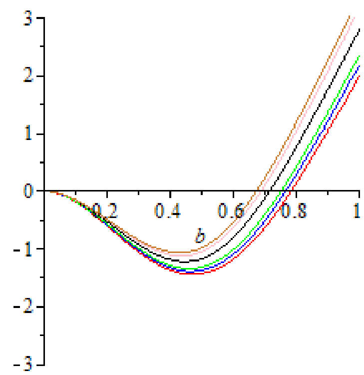
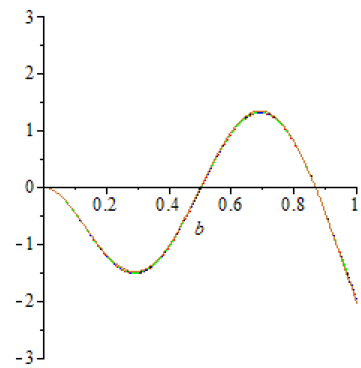
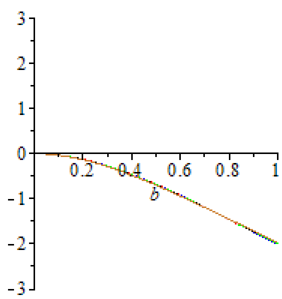
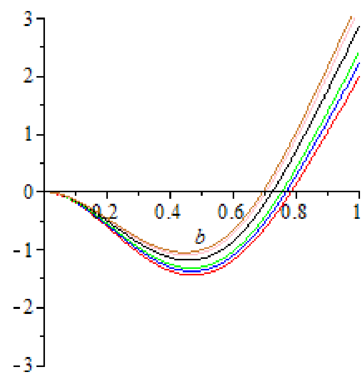
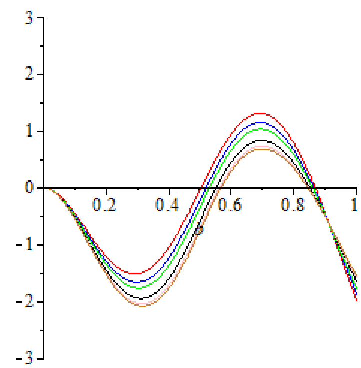
Mass Position	Mode shape1	Mode shape 2	Mode shape 3
0.4L			
0.5L			
0.6L			

Table 17 Mode shapes 4 and 5 of a fixed-free beam with single mass at positions *0.4L, 0.5L* and *0.6L*

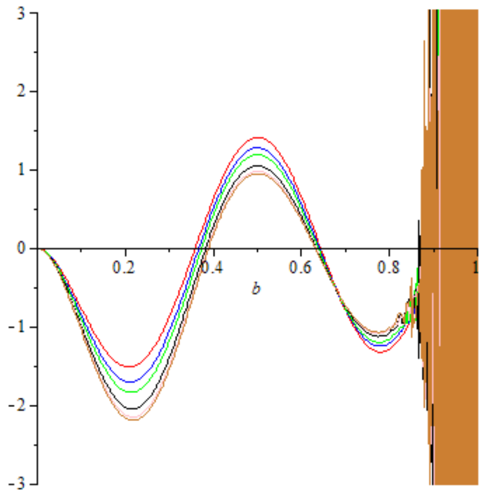
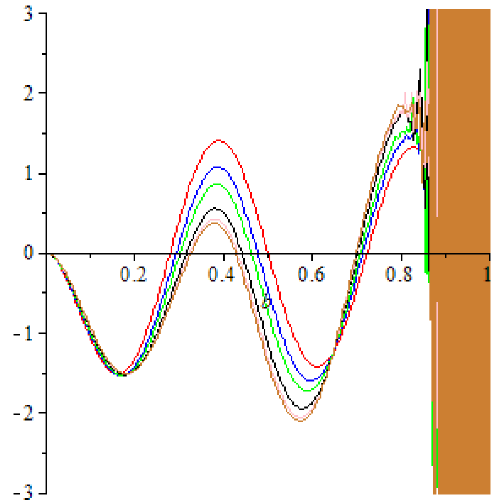
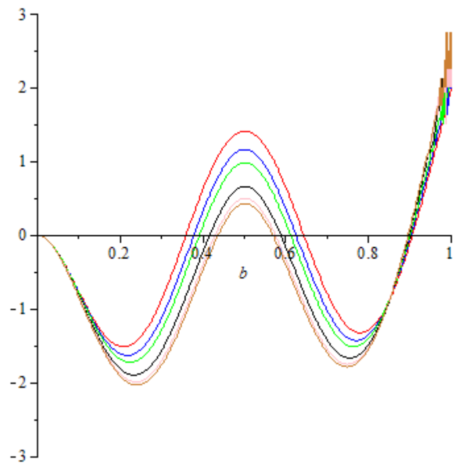
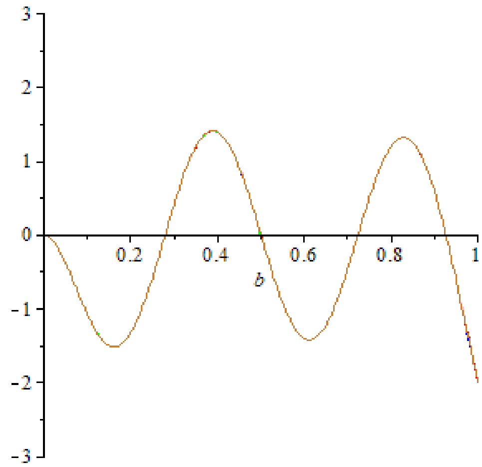
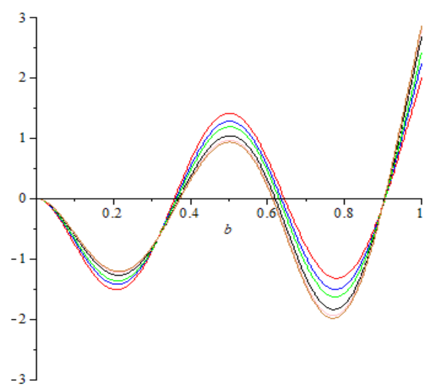
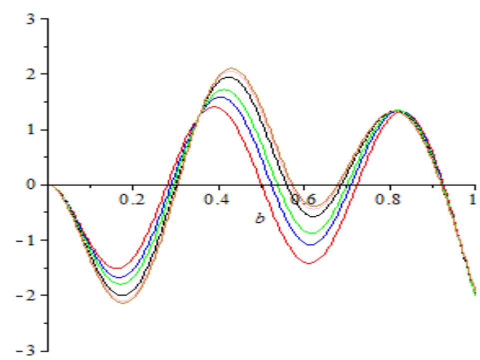
Mass Position	Mode shape 4	Mode shape 5
0.4L		
0.5L		
0.6L		

Table 18 First three mode shapes of a fixed-free beam with single attachment at positions *0.7L, 0.8L* and *0.9L*

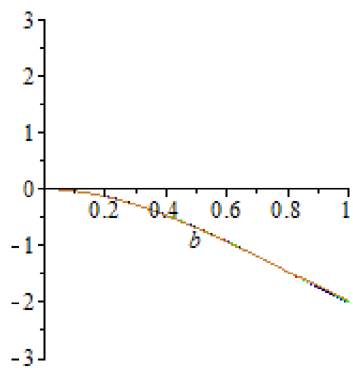
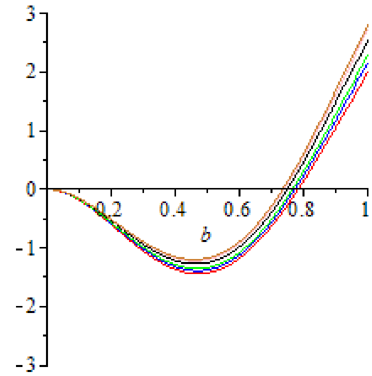
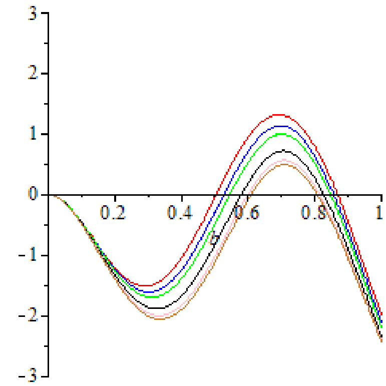
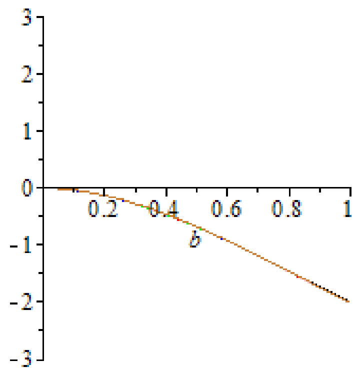
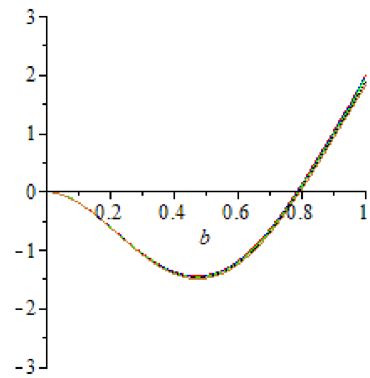
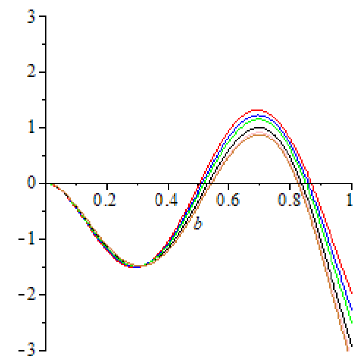
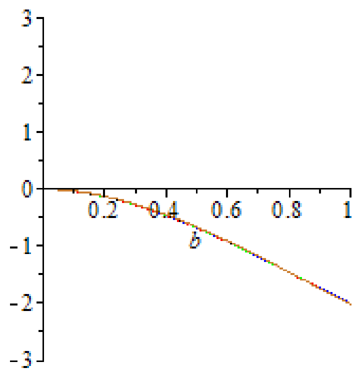
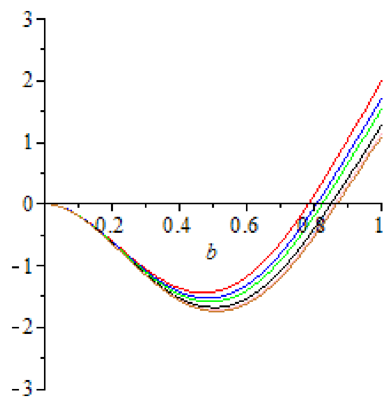
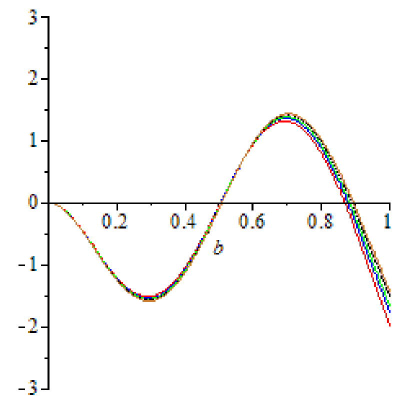
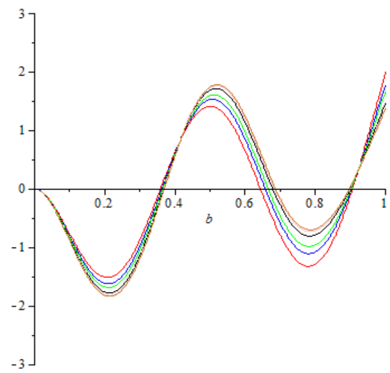
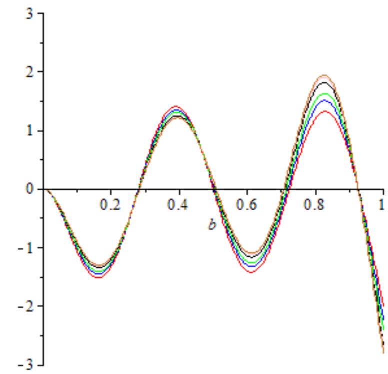
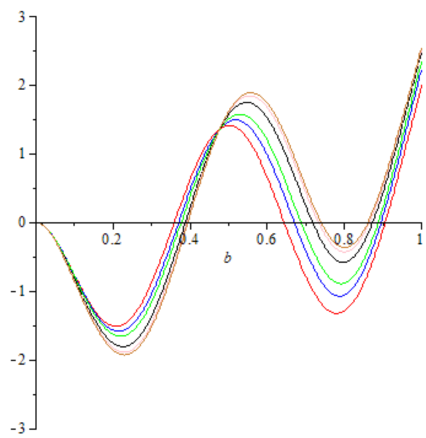
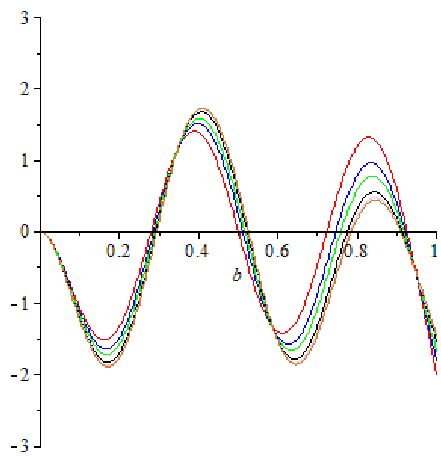
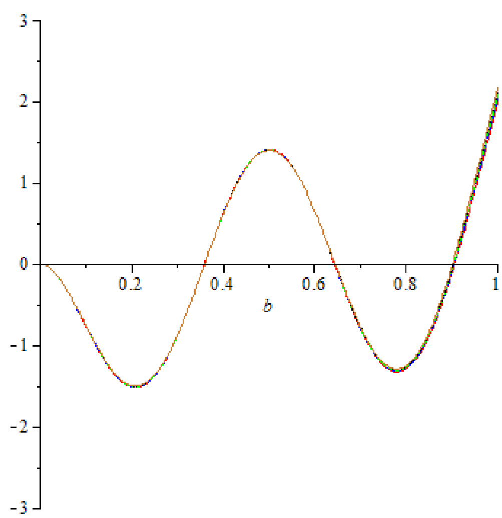
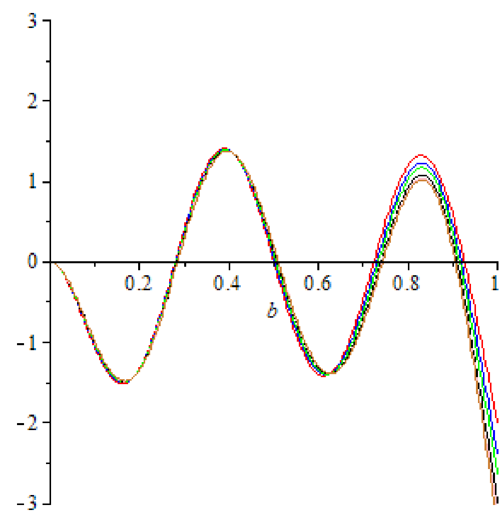
Mass Position	Mode shape 1	Mode shape 2	Mode shape 3
0.7L			
0.8L			
0.9L			

Table 19 Mode shapes 4 and 5 for a fixed-free beam with single mass at 0.7L, 0.8L and 0.9L

Mass Positions	Mode shape 4	Mode shape 5
0.7L		
0.8L		
0.9L		

4.3.3.3 Observations and Analysis

By inspecting Table 14 to Table 19, the following observations can be made regarding the free vibrations of a fixed-free (cantilever) beam with single lumped mass attachment:

- As in the case of a simply-supported beam, the first mode shape of vibration remains almost unaltered. In other words, the first mode shape is dependant neither on the mass nor on its location along the beam.
- Gibb's phenomenon is clearly present near the free end of the beam for some of the mode shapes. These are caused by the inability of Fourier series to handle and simulate discontinuity points. This phenomenon is mainly present in situations where the lumped mass is positioned between the fixed end and middle of the beam.
- As in the case of the simply-supported beam, if the lumped mass is positioned on a node of a mode shape, it has no effect on that mode shape, neither on its corresponding frequency. This issue is observable in the following mode shapes in Table 14 to Table 19.
    1. Mode shape 3 at position $x=0.5L$.
    2. Mode shape 5 at position $x=0.5L$.
    3. Mode shape 4 at position $x=0.9L$.

4.4 Conclusion

The problem of adding a single lumped mass to a simply-supported as well as a fixed-free (cantilever) beam was extensively considered in this chapter. Two different problems were defined, each of which considers the problem from different aspects. The first problem considers the effects of changing the location of a lumped mass along the beam on the frequency of the combined system while the second problem investigates the effects of adding masses at various locations on the mode shapes of the beam. The results were plotted in separate graphs and the mode shapes plots were tabulated for forty-five cases. It was observed that once a mass is positioned on an antinode, the reduction in amplitude at that point is compensated by increases in the amplitude of other antinodes of that mode. This implies a conservation law regarding the amplitude of the vibration which requires further investigation and can be the subject of an independent research.

5 Chapter 5 - Inverse Frequency Problems of a Beam with an Attachment

5.1 Defining the Problem

The work, done in previous chapters lays the foundation for considering the inverse eigenvalue problem of a beam with a single mass attachment. Unlike the problem considered in previous chapters where the unknown variables were frequencies and mode shapes, here the goal is to impose certain frequencies on the system by manipulating the value of the attached mass and its position. In other words, the unknown variables in the inverse problem are mass and its position along the beam.

In the forward problem, the built-in eigenvalues and eigenvectors calls were used to calculate the frequency spectrum. Alternatively, the eigenvalues could be found from the determinant (characteristic) equation instead. This means solving the equation

$$p(t) = \det(K - tM) = 0 \tag{5.1}$$

for *t*, where *t* represents the roots (eigenvalues, squared frequencies) of the polynomial equation in the previous equation. In the inverse problem under consideration, λ and $\lambda^{'}$ are squared natural frequencies that are to be imposed on the system. Replacing *t* with the specified λ and $\lambda^{'}$ in turn, leads to two scalar equations and two unknowns so that the following system of two equations and two unknowns must be solved

$$\begin{cases} \det\left(\mathbf{K}_t - \lambda \mathbf{M}_t\right) = 0 \\ \det\left(\mathbf{K}_t - \lambda^{'} \mathbf{M}_t\right) = 0 \end{cases} \tag{5.2}$$

Here, $\mathbf{K}_t$ and $\mathbf{M}_t$ are stiffness and mass matrices containing any unknown parameters of the combined system, and are derived as explained in previous chapters and λ and $\lambda^{'}$ are the squared natural frequencies to be imposed on the system.

5.2 The Determinant Method

Due to the fact that the two equations are derived using equation (5.2) which includes the determinant, this method will be called the **Determinant Method**.

5.3 Assumptions and Modelling

The following assumptions are considered in defining the inverse problem:

- Only the case of adding a single mass is considered.
- The known (input) variables are two desired natural frequencies that must be imposed on the system.
- The unknown variables to be found are the mass and its position along the beam.
- The acceptable mass range is a fraction of the mass of the beam, that is, $m = c\rho L \quad 0 < c < 1$.
- The inverse problem is solved for both the cases of simply-supported and fixed-free (cantilever) boundary conditions.
- The degree of discretization using the assumed modes method as outlined in prior chapters is *N=10* for a simply-supported beam and *N=4* for a fixed-free (cantilever) beam.
- The acceptable position range is a fraction of the length of the beam *L*, that is, $l_{lumpedmass} = pL$, where $0 \le p \le 1$.

5.4 Simply-supported Beam

In this section, we consider the simulation and results of using Maple V14 to solve the inverse problem for a simply-supported beam with a single mass attachment.

5.4.1 Coding and Problem Solving Procedure

The major steps in solving, as well as coding the inverse eigenvalue problems are outlined here:

- Returning to natural frequencies obtained for the simply-supported beam obtained in Chapter 4, two frequencies were chosen as input (desired frequencies) to the inverse problem code. These are the frequencies we seek to impose on the beam with its mass attachment. To ensure a solvable problem, we choose known values from our prior forward problem.
- The generalized mass and stiffness matrices must be formed $\left(\mathbf{M}^d, \mathbf{K}^d\right)$.
- The eigenfunction vector, $\underline{\phi}$, must be built using the simply-supported beam eigenfunction

  $$\phi_i(x) = \sqrt{\frac{2}{\rho L}} \sin\left(\frac{i\pi x}{L}\right)$$

- Since no stiffness element (springs) is added to the system, the stiffness matrix of the combined system is the same as the unconstrained beam. However, the mass matrix of the combined system is affected by the presence of lumped masses. Hence, the matrices are given by

$$\mathbf{K}_t = \mathbf{K}^d \tag{5.3}$$

$$\mathbf{M}_t = \mathbf{M}^d + c\rho L \underline{\phi}_{pL} \underline{\phi}_{pL}^T \tag{5.4}$$

where c is the mass coefficient and vector $\underline{\phi}_{pL}$ is defined as

$$\underline{\phi}_{pL} = \begin{bmatrix} \phi_1(pL) \\ . \\ . \\ \phi_N(pL) \end{bmatrix} \tag{5.5}$$

- Equations (5.3) and (5.4), along with the two desired values of λ are substituted into equation (5.2). This gives two equations in two unknowns which shall be solved for the two unknowns, *c* and *p*, using *fsolve* as well as the *DirectSearch* packages in Maple. *Fsolve* is Maple's built-in equation-solving package. The details of the *DirectSearch* pacakge can be found in [34].
- The results obtained for *c* and *p* include the anticipated results (already known from the forward problem since we chose the desired λ from a known forward problem) plus additional results for *c* and *p*. To check whether the order of the frequencies in the frequency spectrum will be conserved or if the results returned by the inverse problem achieve the desired system frequencies, the unexpected results must be substituted into the forward code in order to obtain the entire spectrum of system frequencies.

5.4.2 Results

Using the results already obtained in the previous chapter through the forward problem, two squared frequencies (λ , λ') are selected and are substituted into equation (5.2), along with the mass and stiffness matrices formed as outlined in the prior subsection. The resulting masses and their corresponding positions, obtained by solving for *c* and *p*, are then substituted into the forward problem to yield the order of the frequencies and the full frequency spectrum itself. Due to the symmetry of the simply-supported boundary condition about the beam midpoint, only half of the beam is considered and the results can be extended to the other half. These results are tabulated in the left hand side column of Table 20 to Table 25.

The squared frequencies in the left hand side columns of these tables are chosen from the previously solved forward problem considered in Chapter 4 and their subscripts indicate their order in the hierarchy of the frequency spectrum. The middle columns of these tables contain the values of the masses as well as their positions on the beam obtained after substituting the squared frequencies of the first columns into the inverse problem code. Finally, the right hand side columns are the full span of frequency spectrum obtained after substituting the masses as well as their positions of the middle column into the forward problem. The desired squared frequencies are in bold face in the right-hand-column vectors to make it easier for the reader to compare them with the desired frequencies of the left hand side column.

Table 20 Inverse problem solution for imposed 2^{nd} and 4^{th} frequencies

Input given to inverse determinant method	Solution via inverse determinant method	Full span of frequency squared spectrum obtained via the solution of forward problem
$\lambda_2 = 1011.440587$, $\lambda_4 = 21396.91740$	$m = 0.3687158455453 70\rho L$ $l = 0.176445917607521L$ (This result was obtained using both ***Direct Search*** and ***fsolve*** packages)	$\begin{bmatrix} 80.2917098111304 \\ \mathbf{1011.44058820436} \\ 5653.88218498388 \\ \mathbf{21396.9174579792} \\ 59058.0067611865 \\ 1.2497747904383010^{5} \\ 2.0288682007826410^{5} \\ 3.3270657105954510^{5} \\ 5.6908898201563510^{5} \\ 9.2941561272525910^{5} \end{bmatrix}$
	m=0.5ρL $l = 0.3L$ (This result was obtained using both ***Direct Search*** and ***fsolve*** packages)	$\begin{bmatrix} 57.9743733986489 \\ \mathbf{1011.44058666463} \\ 7594.89690390518 \\ \mathbf{21396.9175111041} \\ 45936.5256083285 \\ 1.1736936534938710^{5} \\ 2.2787533609942610^{5} \\ 3.3142948441550010^{5} \\ 5.8809217314114110^{5} \\ 9.74090910810^{5} \end{bmatrix}$

Table 21 Inverse problem solution for imposed 2nd and 3rd frequencies

Input given to inverse determinant method	Solution via inverse determinant method	Full span of frequency squared spectrum obtained via the solution of forward problem
$\lambda_2 = 1210.743012,\ \lambda_3 = 7703.709313$	$m = 0.2\rho L$ $l = 0.3L$ (This result was obtained via both ***Direct Search*** and ***fsolve*** methods)	76.8682251290379 **1210.74301213888** **7703.70930996904** 22705.2480741553 49733.6142925164 1.1938488206827310^{5} 2.2957055844737010^{5} 3.4530959279126410^{5} 5.9563079377918610^{5} 9.74090910810^{5}
	$m = 0.389510562240428\rho L$ $l = 0.358709426954520L$ (This result was obtained via both ***Direct Search*** and ***fsolve*** packages)	59.0673463440848 **1210.74301320973** **7703.70929993294** 18122.3022182383 55694.1292449090 1.1884569081395810^{5} 1.9116355472115310^{5} 3.8830389363207610^{5} 5.8969858556972810^{5} 8.6916308553698610^{5}

Table 22 Inverse problem solution for imposed 3rd and 4th frequencies

Input given to inverse determinant method	Solution via inverse determinant method	Full span of frequency spectrum obtained via the solution of forward problem
$\lambda_3 = 7774.837479,\ \lambda_4 = 23571.61098$	$m = 0.0324093942548158\rho L$ $l = 0.387893812541676L$ (This result was obtained via both ***Direct Search*** and ***fsolve*** packages)	92.1360362813541 1519.29408377341 **7774.83747044827** **23571.6111048298** 60757.6494633054 1.2111830999984310^{5} 2.2664487181658410^{5} 3.9685684291734510^{5} 6.0870168625136210^{5} 9.6823052080303510^{5}
	$m = 0.1\rho L$ $l = 0.3L$ (This result was obtained via both ***Direct Search*** and ***fsolve*** packages)	86.0190764111106 1342.98217388258 **7774.83748224542** **23571.6109340115** 53147.7470615752 1.2129169651487210^{5} 2.3096110466067010^{5} 3.5935497319867610^{5} 6.0435693489765510^{5} 9.74090910810^{5}

Table 23 Inverse problem solution for imposed 2nd and 4th frequencies

Input given to inverse determinant method	Solution via inverse determinant method	Full span of frequency spectrum obtained via the solution of forward problem
$\lambda_2 = 903.8306222,\ \lambda_4 = 22676.01972$	$m = 0.5\rho L$ $l = 0.2L$ (This result was obtained using both ***Direct Search*** and ***fsolve*** packages)	71.3644833281625 **903.830621900072** 5751.74114742945 **22676.0197542973** 60880.68192 1.1064892845346410^{5} 1.8796352312610910^{5} 3.4962959557249210^{5} 6.1270648684098110^{5} 9.74090910810^{5}
	$m = 0.730956541093387\rho L$ $l = 0.285210504737460L$ (This result was obtained using both ***Direct Search*** and ***fsolve*** packages)	50.2446541680404 **903.830621670896** 7269.39641631875 **22676.0196400138** 43931.6319025254 1.1050446218692310^{5} 2.3387170301267210^{5} 3.4171952844757810^{5} 5.5555519257418310^{5} 9.5109411025788910^{5}
	$m = 0.880408480262521\rho L$ $l = 0.0181730815412715L$ (This result was obtained using ***Direct Search*** package only)	96.8520413155306 1522.91948019413 7482.74399557873 **22676.0209300952** 52852.4648410244 1.0596806432751410^{5} 1.9446403024193910^{5} 3.3544803650416210^{5} 5.5018962431811410^{5} 8.6721103211074810^{5}

Table 24 Inverse problem solution for imposed 3rd and 5th frequencies

Input given to inverse determinant method	Solution via inverse determinant method	Full span of frequency spectrum obtained via the solution of forward problem
$\lambda_3 = 7164.651108,\ \lambda_5 = 60880.68192$	$m = 0.0654834947966672\rho L$ $l = 0.8L$ (This result was obtained using ***Direct Search*** package only)	93.1547571365356 1396.19609697085 **7164.65111038463** 24106.9097395153 **60880.68192** 1.2149624720675110^5 2.1419279613860510^5 3.7237469903322910^5 6.2427712676267010^5 9.74090910810^5
	$m = 0.2\rho L$ $l = 0.4L$ (This result was obtained using ***Direct Search*** package only)	71.3526533804908 1411.90027472048 **7164.65110665533** 20420.5236289702 **60880.68192** 1.0594575391440310^5 2.2175184985882710^5 3.7738817129679910^5 5.7362969493291310^5 9.74090910810^5

Table 25 Inverse problem solution for imposed 2nd and 4th frequencies

Input given to inverse determinant method	Solution via inverse determinant method	Full span of frequency spectrum obtained via the solution of forward problem
$\lambda_2 = 1411.900275,\ \lambda_4 = 20420.52362$	$m = 0.151574270580968\rho L$ $l = 0.0971832276075668L$ (This result was obtained via ***Direct Search*** and ***fsolve*** packages)	$\begin{bmatrix} 94.7850516628762 \\ \mathbf{1411.90027721258} \\ 6626.22311098193 \\ \mathbf{20420.5235978916} \\ 51205.1149887429 \\ 1.1120990807654710^{5} \\ 2.1569758556846710^{5} \\ 3.8231447552916410^{5} \\ 6.2963832077535210^{5} \\ 9.7328764617927010^{5} \end{bmatrix}$
	$m = 0.469450520656909\rho L$ $l = 0.426816711258635L$ (This result was obtained via ***Direct Search*** package only)	$\begin{bmatrix} 51.2467286850404 \\ \mathbf{1411.90027375455} \\ 6082.12039298319 \\ \mathbf{20420.5235299165} \\ 57662.4807523578 \\ 1.0020355514575210^{5} \\ 2.3379704621212210^{5} \\ 3.3340798339811810^{5} \\ 6.1750718554363910^{5} \\ 9.0420627773358410^{5} \end{bmatrix}$
	$m = 0.2\rho L$ $l = 0.4L$ (This result was obtained via ***Direct Search*** package only)	$\begin{bmatrix} 71.3526533804908 \\ \mathbf{1411.90027472048} \\ 7164.65110665533 \\ \mathbf{20420.5236289702} \\ 60880.68192 \\ 1.0594575391440310^{5} \\ 2.2175184985882710^{5} \\ 3.7738817129679910^{5} \\ 5.7362969493291310^{5} \\ 9.74090910810^{5} \end{bmatrix}$

5.4.3 Observations and Analysis

By considering the left hand column of **Table 20** to Table 25, the following observations can be made regarding the solution to the inverse frequency (eigenvalue) problem:

- The order of the two desired system frequencies was conserved. For instance, in Table 25, the two desired input frequencies remained as the 2nd and 4th system frequencies when the full span of the frequency spectrum was found for all three possible solutions.
- The comprehensive investigation of the forward problem in the previous chapter allowed for the selection of input frequencies for which the results are known to exist and against which the results of the inverse method can be verified. This hindsight is helpful in verifying the accuracy of the **determinant method**.
- Due to the symmetry of the boundary conditions of the simply supported beam, it is evident that for each mass obtained, there must be two corresponding positions that are symmetrical with respect to the middle of the beam. This is a good criterion for filtering out the results brought up by the ***Direct Search*** method that do not meet this requirement. This is evident in Table 23 where the third result yields only one of the desired frequencies.
- A good equation solver is a requirement for this method to work properly and in this case, the use of an alternative equation solver (*DirectSearch)* yielded additional unexpected results that were not returned by Maple's built-in equation solver. For example, in Table 24, both results are obtained using the *DirectSearch* package.

5.5 Fixed-free (Cantilever) Beam

In this section, we consider the simulation and results of using Maple V14 to solve the inverse problem for a fixed-free (cantilever) beam with a single mass attachment.

5.5.1 Coding and Problem Solving Procedure

As with the case of the forward problem, the steps to follow in order to code the inverse problem for the cantilever beam follow the same steps as for the simply-supported beam. However, due to additional complexity of the eigenfunction of a fixed-free (cantilever) beam, additional steps are required at the outset of the code. In particular, the transcendental equation must be solved. Additionally, due to the complexities of the eigenfunction and thus resulting determinant, the highest degree of discretization that Maple V14 could handle for the inverse problem was found to be $N=4$. For discretization degrees greater than $N=4$, the length of the two determinants of equation (5.2) exceeds the limit of one million terms. Moreover, since the investigation of the inverse problem for the simply supported beam suggests that *DirectSearch* package yields more results compared to the built-in *fsolve* package, it is the only solver that used to solve (5.2). The rest of the procedure follows the same steps as for the case of a simply-supported beam

5.5.2 Results

A pair of frequencies was chosen from the already solved forward problem and then substituted into the inverse code as the desired system frequencies. In the same manner as for the simply supported beam, the resulting equations of motion were then solved to yield the masses and their corresponding positions from equation (5.2). The results of the inverse code were then substituted into the forward problem to determine the full span of frequency spectrum as well as the order of the desired pair of frequencies in the hierarchy of the frequency spectrum. The results of these simulations are shown in Table 26 to Table 34. Moreover, in Table 26 to Table 34, the left hand side columns are the pair of squared frequencies chosen from the solution of the forward problem with a degree of discretization $N=17$, the indices indicate their orders in the hierarchy of the frequency spectrum. The middle columns contain the values of the masses and their corresponding locations along the beam after substituting the pair of squared frequencies of the left hand side column into the inverse code whose degree of discretization is $N=4$. This degree of discretization (N=4) was chosen to ensure solvability of the inverse problem as Maple had difficulties in solving the inverse problem for the cantilever beam with higher orders of discretization. Finally, the right hand side columns contain the full span of the frequency spectrum after substituting the values of the masses and their corresponding locations along the beam of the middle columns back into the forward problem code whose degree of discretization is $N=17$.

Table 26 Inverse problem solution for imposed 2nd and 3rd frequencies

Input given to the inverse problem	Solution via inverse determinant method	Full span of squared frequency spectrum obtained via the solution of forward problem
$\lambda_2 = 435.9605825,\ \lambda_3 = 3153.618086$	$m = 0.100987559017826\rho L$ $l = 0.298901670102059L$	$\begin{bmatrix} 12.27092689 \\ \mathbf{435.8828642} \\ \mathbf{3147.473055} \\ 13829.11815 \\ 39445.98343 \\ 78488.20227 \\ 1.61139981810^{5} \\ 3.08184030910^{5} \\ 4.68642444510^{5} \\ 7.26151396310^{5} \\ 1.17102313410^{6} \\ 1.64540647110^{6} \\ 2.18054652710^{6} \\ 3.14689939410^{6} \\ 4.28020079310^{6} \\ 5.25519408610^{6} \\ 6.94723992110^{6} \end{bmatrix}$
	$m = 0.147834769363708\rho L$ $l = 0.650832393569747L$	$\begin{bmatrix} 10.6170630154127 \\ \mathbf{435.807485614025} \\ \mathbf{3234.86226747339} \\ 14601.0784380446 \\ 34510.6315904192 \\ 79566.7957037566 \\ 1.7377521236194210^{5} \\ 2.7138127997581910^{5} \\ 4.7580095962335810^{5} \\ 7.8916156752326710^{5} \\ 1.1157368212306410^{6} \\ 1.6263049067957310^{6} \\ 2.37815163610^{6} \\ 3.23544904810^{6} \\ 4.30597495710^{6} \\ 5.62245881810^{6} \\ 7.21996790910^{6} \end{bmatrix}$

Table 27 Inverse problem solution for imposed 3rd and 4th frequencies

Input given to the inverse problem	Solution via inverse determinant method	Full span of squared frequency spectrum obtained via the solution of forward problem
$\lambda_3 = 2237.822193,\ \lambda_4 = 12956.87459$	$m = 0.508691237900552\rho L$ $l = 0.297005785744171L$	11.91897844 304.0735594 **2215.017240** **12822.48711** 38901.19781 67844.13029 1.51250303910^{5} 3.07850355810^{5} 4.39888791110^{5} 6.88500101010^{5} 1.15713937410^{6} 1.62390308410^{6} 2.09105774010^{6} 3.09328395410^{6} 4.28845003310^{6} 5.13680874510^{6} 6.83403210510^{6}
	$m = 0.374200722215735\rho L$ $l = 0.821218609095489L$	6.67208135348432 475.261049637701 **3543.71809261081** **11322.7820172604** 32250.9704273657 81034.2625002964 1.7176498639483610^{5} 3.0286627364266310^{5} 4.6469946859560610^{5} 7.93403134610^{5} 1.18401358810^{6} 1.70369109110^{6} 2.37815163610^{6} 3.23544904810^{6} 4.30597495710^{6} 5.62245881810^{6} 7.21996790910^{6}

Table 28 Inverse problem solution for imposed 3rd and 4th frequencies (Continued)

Input given to the inverse problem	Solution via inverse determinant method	Full span of frequency squared spectrum obtained via the solution of forward problem
$\lambda_3 = 2237.822193,\ \lambda_4 = 12956.87459$	$m = 0.0948715522820537\rho L$ $l = 0.763022021626335L$	10.5377071255105 484.407419086569 **3459.95589448716** **13005.9225907739** 38605.4747463968 88738.4781808609 1.5919352145505410^{5} 2.7796242912520210^{5} 4.8760604737386310^{5} 7.93403134610^{5} 1.18401358810^{6} 1.70369109110^{6} 2.37815163610^{6} 3.23544904810^{6} 4.30597495710^{6} 5.62245881810^{6} 7.21996790910^{6}
	$m = 0.0767781996475414\rho L$ $l = 0.498927853155918L$	11.94098766 421.4626873 **3805.878468** **12910.38503** 39942.89521 79714.97287 1.73872176710^{5} 2.78764081610^{5} 5.08440484910^{5} 7.25361608810^{5} 1.18388385210^{6} 1.57497424510^{6} 2.37815163610^{6} 3.03108979810^{6} 4.30597495710^{6} 5.62245881810^{6} 7.21996790910^{6}

Table 29 Inverse problem solution for imposed 2nd and 4th frequencies

Input given to the inverse problem	Solution via inverse determinant method	Full span of frequency squared spectrum obtained via the solution of the forward problem
$\lambda_2 = 394.1468677,\ \lambda_4 = 13464.73868$	$m = 0.203420385672269\rho L$ $l = 0.298048359877936L$	12.18077682 **393.9109094** 2757.006451 **13393.68012** 39202.92650 73520.32289 1.56259358510^{5} 3.08095495110^{5} 4.53923842510^{5} 7.06437256510^{5} 1.16474917110^{6} 1.63186423710^{6} 2.13220773810^{6} 3.12029648410^{6} 4.28215112610^{6} 5.18607432510^{6} 6.88886501510^{6}
	$m = .308622128977385\rho L$ $l = 0.0791194816492039L$	12.36064627 **483.3178009** 3689.246572 **13121.28377** 31681.29752 66027.22597 1.31045323910^{5} 2.43566834710^{5} 4.22212285410^{5} 6.88579623210^{5} 1.06724027010^{6} 1.58540575110^{6} 2.27225274410^{6} 3.15744575110^{6} 4.26767414110^{6} 5.61864890210^{6} 7.20297525510^{6}

Table 30 Inverse problem solution for imposed 2nd and 4th frequencies (continued)

Input given to the inverse problem	Solution via inverse determinant method	Full span of frequency squared spectrum obtained via the solution of forward problem
$\lambda_2 = 394.1468677,\ \lambda_4 = 13464.73868$	$m = 0.506714278022789 \rho L$ $l = 0.0679666470446149 L$	12.36081169 **483.4656953** 3691.798459 **13021.28293** 30357.02976 61921.05112 1.23454803110^{5} 2.30882765010^{5} 4.01815338110^{5} 6.57011252010^{5} 1.02031483010^{6} 1.51850654110^{6} 2.18111798910^{6} 3.04012317710^{6} 4.12935207610^{6} 5.48329979510^{6} 7.13461352510^{6}
	$m = 0.605246977556542 \rho L$ $l = 0.0644625017337932 L$	12.36085862 **483.5078701** 3692.384023 **12980.50932** 29845.59588 60514.38892 1.21057119110^{5} 2.26989794610^{5} 3.95606316110^{5} 6.47400408710^{5} 1.00595128410^{6} 1.49779568010^{6} 2.15230210910^{6} 3.00148457910^{6} 4.07968054510^{6} 5.42297898110^{6} 7.06844560910^{6}

Table 31 Inverse problem solution for imposed 2nd and 4th frequencies (continued)

Input given to the inverse problem	Solution via inverse determinant method	Full span of frequency squared spectrum obtained via the solution of the forward problem
$\lambda_2 = 394.1468677,\ \lambda_4 = 13464.73868$	$m = 0.702676632952036 \rho L$ $l = 0.0616903438119407 L$	12.36089420 **483.5398954** 3692.769575 **12944.21880** 29403.55903 59377.00372 1.19189463410^{5} 2.23991836810^{5} 3.90841885510^{5} 6.40033961910^{5} 9.94942158010^{5} 1.48190688810^{6} 2.13014629810^{6} 2.97164406310^{6} 4.04098636010^{6} 5.37518483310^{6} 7.01431302410^{6}
	$m = 0.792417854998709 \rho L$ $l = 0.0595631726612599 L$	12.36092061 **483.5636982** 3693.017614 **12913.65815** 29040.99127 58496.33555 1.17784638710^{5} 2.21755098010^{5} 3.87296723210^{5} 6.34559419910^{5} 9.86766526910^{5} 1.47011279210^{6} 2.11370194610^{6} 2.94948588910^{6} 4.01221384910^{6} 5.33955134110^{6} 6.97390565410^{6}

Table 32 Inverse problem solution for imposed 1st and 2nd frequencies

Input given to the inverse problem	Solution via inverse determinant method	Full span of frequency squared spectrum obtained via solution of forward problem
$\lambda_1 = 12.35540051,\ \lambda_2 = 476.9707703$	$m = 0.5\rho L$ $l = 0.1L$	**12.35527388** **476.8083361** 3358.652070 10287.98578 25821.56123 61911.29633 1.32414541710^{5} 2.53984707710^{5} 4.46545989010^{5} 7.32836739110^{5} 1.13725903710^{6} 1.68249830710^{6} 2.37804498510^{6} 3.18762584810^{6} 4.06482433610^{6} 5.19077685510^{6} 6.74238770010^{6}
	$m = 0.6047218485768888\rho L$ $l = 0.09536408994150788L$	**12.35536677** **476.7892762** 3344.426485 10049.31175 25096.49564 60477.91168 1.29730862810^{5} 2.49237358310^{5} 4.38698754910^{5} 7.20790150410^{5} 1.12041578710^{6} 1.66276827210^{6} 2.36747981910^{6} 3.23050001410^{6} 4.19919009910^{6} 5.30240656910^{6} 6.77499226210^{6}

Table 33 Inverse problem solution for imposed 1st and 2nd frequencies (continued)

Input given to the inverse problem	Solution via inverse determinant method	Full span of frequency squared spectrum obtained via solution of forward problem
$\lambda_1 = 12.35540051,\ \lambda_2 = 476.9707703$	$m = 0.751096036126443\rho L$ $l = 0.0898555279401088L$	**12.35547681** **476.7626910** 3325.289145 9733.978301 24205.65002 58762.80497 1.26524190310^{5} 2.43536600710^{5} 4.29192579910^{5} 7.05968044310^{5} 1.09897272710^{6} 1.63492193310^{6} 2.33943284110^{6} 3.22931054410^{6} 4.29297421810^{6} 5.48764067210^{6} 6.89991403110^{6}
	$m = 0.445746452115740\rho L$ $l = 0.103767364483139L$	**12.35519276** **476.8229388** 3369.927981 10478.83835 26433.64236 63149.96497 1.34739782410^{5} 2.58082503510^{5} 4.53263011910^{5} 7.42969117910^{5} 1.15078531410^{6} 1.69559327210^{6} 2.37109226910^{6} 3.11121509610^{6} 3.96126047910^{6} 5.15207309110^{6} 6.75221190810^{6}

Table 34 Inverse problem solution for imposed 1st and 2nd frequencies (continued)

Input given to the inverse problem	Solution via inverse determinant method	Full span of frequency squared spectrum obtained via solution of forward problem
$\lambda_1 = 12.35540051,\ \lambda_2 = 476.9707703$	$m = 0.811076994813296\rho L$ $l = 0.0879887064151891L$	**12.35551340** **476.7526882** 3318.252126 9620.193878 23902.33218 58189.31875 1.25451779410^{5} 2.41622382010^{5} 4.25981291110^{5} 7.00913858010^{5} 1.09153062810^{6} 1.62483371010^{6} 2.32761347710^{6} 3.22067061410^{6} 4.30360809810^{6} 5.53953518710^{6} 6.96042931210^{6}
	$m = 0.9051428103353378\rho L$ $l = 0.0853963748750558L$	**12.35556362** **476.7378494** 3307.937933 9455.934835 23480.79722 57400.29742 1.23975293610^{5} 2.38980041110^{5} 4.21533077410^{5} 6.93878129110^{5} 1.08108305910^{6} 1.61041599510^{6} 2.30985342910^{6} 3.20419360810^{6} 4.30476733610^{6} 5.59093317710^{6} 7.04820604610^{6}

5.5.3 Observations and Analysis

By considering Table 26 through Table 34, the following observations can be made regarding the inverse eigenvalue (frequency) problem of a fixed-free beam:

- Despite the fact that the degrees of discretization for forward and inverse problems are very different, $N=17$ and $N=4$ respectively, The results of the full frequency spectrum obtained after substituting the mass and its corresponding position into the forward problem still show good approximation with respect to the original frequencies. This is evident by comparing the input frequencies on the left hand side columns with the bold numbers in the vector of squared frequencies in the right hand side columns of Table 26 to Table 34.
- As with the case of the simply-supported beam, the order of the frequencies in the full span of frequency spectrum remains the same for all mass and position solutions.
- The effect of the degree of discretization is most noticeable in the higher order frequencies. In other words, the higher the order of frequency, the higher the divergence from the exact solution. This implies that for situations where the lower fundamental frequencies are of concern, lower degrees of discretization suffice for the engineering design purposes which imply lower order matrices, in addition to lower order polynomials and thus less computation (See Table 32 to Table 34).

5.6 Conclusion

In this chapter, the inverse eigenvalue (frequency) problem was considered for the case of a beam with a single lumped mass attachment for both simply-supported, as well as fixed-free (cantilever) boundary conditions. The known variables were a pair of desired frequencies and the unknowns were c and p, the mass and length coefficients of the added mass, respectively.

Unlike the forward problem for which the built-in eigenvalues and eigenvectors function calls were used, the inverse problem must be solved by using equation (5.2), which involves a system of two determinants. This equation was solved for different combinations of desired frequencies and the results were tabulated in the right hand side columns of **Table 20** to Table 34 for both the simply-supported and fixed-free (cantilever) beam.

6 Chapter 6- Summary and Conclusion

6.1 Overview

In this thesis, the inverse eigenvalue (frequency) problem of combined dynamical systems was considered. The dynamical system under consideration consists of an Euler-Bernoulli beam to which multiple lumped attachments can be attached. These attachments can be in the form of lumped masses, lumped stiffness elements (linear and rotary springs) or damping elements (both linear and rotary). Moreover, the beam may have different boundary conditions (simply supported, fixed-free, fixed-fixed, among others).

For this thesis, the case of an Euler-Bernoulli beam to which a single lumped mass is attached was considered. Two commonly used boundary conditions considered here were the simply-supported and fixed-free (cantilever) boundary conditions.

Unlike forward frequency problem which means finding the frequency spectrum of a dynamical system assuming that the characteristics of the system are known variables, the inverse problem aims to impose certain desired frequencies on the system by manipulating the characteristics of the system. In other words, in the inverse frequency problem, the known variables are the frequencies while the unknown variables are the characteristics of the system. Although the main purpose of this thesis was to consider the inverse eigenvalue problem, a comprehensive insight of the forward problem was required in order to delineate the possible solutions to the inverse problem and to understand the scope of possible solutions to the inverse problem.

In Chapter 2, a comprehensive review of the research work already done in this area was presented. It was observed that the majority of prior research work was focused on proposing more efficient methods to solve the forward problem. A few researchers took on the task of considering the inverse problem. However, they were more concerned with imposing nodes at certain locations along the beam by adding a spring-mass system rather than imposing frequencies.

In Chapter 3, the theoretical foundation of the forward method was established and the equations of motion were derived using Lagrange's equations and the assumed-mode method for the system consisting of a beam with lumped mass attachments. Two methods of solving the equations of motion were compared, namely Cha's method proposed in [1] and the direct eigenvalues method. It was realized that the computational savings claimed by Cha's method were not significant and did not justify the use of this method. Therefore, it was decided to utilize the direct eigenvalues and eigenvectors functions call in Maple V14 in order to code and solve the forward problem.

In Chapter 4, a comprehensive investigation of the forward problem was presented. The effects of adding a lumped mass to an Euler-Bernoulli beam at nine equally spaced spots along the beam on the first five fundamental frequencies and mode shapes of the beam were considered. It was realized that the effects of adding a mass to a beam on the frequencies of the beam will be annulled if the mass is positioned on a node of a vibrational mode. On the other hand, the effects of adding a mass to a beam on the frequencies of the beam are best pronounced when the mass is positioned on an antinode (peak point) of a vibrational mode. The same observations were made regarding the corresponding mode shapes of the beam, that is, if the mass is positioned on a node of a vibrational mode, the mode shape will remain unaltered. Moreover, the presence of a mass on an antinode (peak point) of a vibrational mode suppresses the antinode while making the other antinodes soar in amplitude which indicates the fact that adding only one mass may not be enough if the purpose of adding the mass is to quench excessive vibrations . It was observed that the decrease in amplitude of the antinode over which the lumped mass is positioned is canceled out by the sum of the increases in amplitudes of the other antinodes. This implies a kind of conservation phenomenon which requires further investigation.

In Chapter 5, the inverse problem of imposing two fundamental frequencies on an Euler-Bernoulli beam by adding a lumped mass to the beam was considered. In this problem, the known (design) variables are the two fundamental frequencies while the unknowns are the value of the lumped mass, as well as its position along the beam. The two frequencies were chosen from the results of the forward problem in Chapter 4. It was realized that by solving the two determinant equations a set of results can be obtained for the value of the mass and its corresponding location along the beam, including the expected result already present in the forward problem and an additional number of unexpected results. These results were obtained using two different solvers: the built-in *fsolver* in Maple as well as *DirectSearch* package. It was observed that the *DirectSearch* package produced r results compared to the built-in *fsolver*. However, some of the results obtained via *DirectSearch* may not be accurate and must be verified by substitution back into the forward problem. Having substituted the unexpected results back into the forward code, it was observed that for all cases the order of the two frequencies in the hierarchy of the frequencies of the system remains the same, for example if the two frequencies chosen from the forward problem were the 2nd and 4th frequencies, for all the results of the mass and its corresponding position obtained via the inverse method, these two frequencies remained the 2nd and 4th frequencies, respectively.

Although the ideal situation was to use the same degree of discretization for both forward and inverse problem, it was also observed that due to the more complicated nature of the inverse problem compared to the forward problem, fewer degrees of discretization can be used for the inverse problem. This issue was especially evident in the case of a fixed-free (cantilever) beam whose degree of discretization was chosen to be $N=4$ in the inverse problem, compared to $N=17$ in the forward problem. However, after substituting the mass and its corresponding position, obtained via inverse code using $N=4$, back into the forward problem and obtaining the frequency spectrum using a degree of discretization $N=17$, it was observed that the divergence between these two frequencies and the original frequencies is small. This implies that for situations where the lower order frequencies of the system are concerned, even the choice of small number of assumed modes can produce acceptable results from the point of view of engineering design. However, for higher order frequencies the divergence is significant and cannot be neglected.

6.2 Future Work

The results of this thesis pave the way for considering the transverse vibrations of combined dynamical systems more extensively. The issue of conservation of changes in the amplitudes of vibrations once the mass is positioned on an antinode of a vibrational mode can be further investigated. Moreover, the inverse problem of adding multiple lumped masses or other combination of attachments to a beam can also be investigated. Finally, the issue of stiffened plates which can be regarded as an expansion of a beam with mass attachments is another avenue of research that can be explored.

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8 Appendix A – Maple Code

8.1 Frequency code

8.1.1 Simply-supported beam

$assume(\rho, positive)$

$assume(E, positive)$
$assume(L, positive)$
$assume(IM, positive)$

$$\phi := (i, x) \rightarrow \sqrt{\frac{2}{\rho \cdot L}} \cdot \sin\left(\frac{i \cdot \pi \cdot x}{L}\right) :$$

$n := 10 :$
$interface(rtablesize = infinity) :$
$Md := Matrix(n, shape = identity) :$
$N := Vector([seq(r, r = 1 ..n)]) :$

$$K := Vector\left(\left[seq\left(\frac{j^4 \cdot \pi^4 \cdot E \cdot IM}{\rho \cdot L^4}, j = 1 ..n\right)\right]\right) :$$

$Kd := Matrix(K, shape = diagonal) :$
$with(LinearAlgebra) :$
$Z := Eigenvalues(Kd, Md) :$

$$Z1 := \frac{\rho \cdot L^4}{E \cdot IM} \cdot Z :$$

$Z2 := map(x \rightarrow \sqrt{x}, Z1) :$
$Z3 := convert(Z2, float) :$
$Z4 := sort(Z3, `<`) :$
$listm := [0.1 \cdot \rho \cdot L, 0.5 \cdot \rho \cdot L, \rho \cdot L, 5 \cdot \rho \cdot L, 10 \cdot \rho \cdot L] :$
$listm1 := convert(listm, rational) :$

```
for p from 1 to nops(listm) do
 m := listm1[p];
 for q from 1 to 9 do
 l := q·L/10;
 Φ := zip(ϕ, N, l);
 ΦT := Transpose(Φ);
 Mt := Matrix(Md + m·Φ.ΦT);
 λ := Eigenvalues(Kd, Mt);
 λ1 := (ρ·L^4)/(E·IM)·λ;
 ω[p, q] := map(x→sqrt(x), λ1);
 ω1[p, q] := convert(ω[p, q], float);
 ω2[p, q] := sort(ω1[p, q], `<`);
 ω3[p, q] := zip(`/`, ω2[p, q], Z4);
 end do:
 end do:
```

```
f := (i, j)→ω3[i, j] :
M := Matrix(p − 1, q − 1, f) :

L1 := [seq(h·L/10, h = 1 ..q − 1)] :
L2 := convert(L1, Vector) :
L3 := L2/L :
M1 := [seq(b·ρ·L/10, b = 1 ..p − 1)] :
with(plots) :
interface(rtablesize = infinity)
```

$$\infty$$

```
for r from 1 to p − 1 do
 for d from 1 to 10 do
 Ω1[r, d] := Vector([seq(ω1[r, i][d], i = 1 ..q − 1)]);
 Ω2[r, d] := Vector([seq(ω3[r, i][d], i = 1 ..q − 1)]);
 end do;
 end do;
```

for j **from** 1 **to** 5 **do**

$A[j] := plot(L3, \Omega2[1, j], legend = "0.1\rho L", linestyle = solid, color = blue);$

$B[j] := plot(L3, \Omega2[2, j], legend = "0.5\rho L", linestyle = dot, color = red);$

$C[j] := plot(L3, \Omega2[3, j], legend = "\rho L", linestyle = dash, color = green);$

$F[j] := plot(L3, \Omega2[4, j], legend = "5\rho L", linestyle = dashdot, color = gold);$

$G[j] := plot(L3, \Omega2[5, j], legend = "10\rho L", linestyle = longdash, color = pink);$

$H[j] := [A[j], B[j], C[j], F[j], G[j]];$

$display\left(H[j], title = typeset(\text{"Changes of "}, j, \text{"th Natural Frequency of a Simply-supported Beam with a Sliding Mass"}), labels = \left[\frac{x}{L}, \frac{\omega}{\omega_0}\right]\right);$

end do:

8.1.2 Cantilever beam

$assume(\rho, positive):$

$assume(E, positive):$

$assume(L, positive):$

$assume(IM, positive):$

$interface(rtablesize = infinity):$

$g := \beta \rightarrow \cos(\beta) \cdot \cosh(\beta) + 1:$

$R := \{seq(fsolve(g, i), i = 1..100)\}:$

$R1 := Vector([op(2..nops(R), R)]):$

$$\phi := (c, x) \rightarrow \frac{1}{\sqrt{\rho \cdot L}} \cdot \left(\cos\left(\frac{c \cdot x}{L}\right) - \cosh\left(\frac{c \cdot x}{L}\right) + \frac{(\sin(c) - \sinh(c))}{\cos(c) + \cosh(c)} \cdot \left(\sin\left(\frac{c \cdot x}{L}\right) - \sinh\left(\frac{c \cdot x}{L}\right)\right)\right):$$

$n := nops(R) - 1 :$

$Md := Matrix(n, shape = identity) :$

$K := map\left(x \to \frac{x^4 \cdot E \cdot IM}{\rho \cdot L^4}, R1\right) :$

$Kd := Matrix(K, shape = diagonal) :$

$with(LinearAlgebra) :$

$Kt := convert(Kd, rational) :$

$Z := Eigenvalues(Kt, Md) :$

$Z1 := \frac{\rho \cdot L^4}{E \cdot IM} \cdot Z :$

$Z2 := map\left(x \to \sqrt{x}, Z1\right) :$

$Z3 := convert(Z2, float) :$

$Z4 := sort(Z3, \text{`<`}) :$

$listm := [0.1 \cdot \rho \cdot L, 0.5 \cdot \rho \cdot L, \rho \cdot L, 5 \cdot \rho \cdot L, 10 \cdot \rho \cdot L] :$

$listm1 := convert(listm, rational) :$

for *p* **from** 1 **to** nops(listm) **do**
$m := listm1[p];$
for *q* **from** 1 **to** 10 **do**
$l := \frac{q \cdot L}{10};$
$\Phi := zip(\phi, R1, l);$
$\Phi1 := convert(\Phi, rational);$
$\Phi T := Transpose(\Phi1);$
$Mt := Matrix(Md + m \cdot \Phi1 . \Phi T);$
$\lambda := Eigenvalues(Kt, Mt);$
$\lambda1 := convert(\lambda, float);$
$\lambda2 := \frac{\rho \cdot L^4}{E \cdot IM} \cdot \lambda1;$
$\omega[p, q] := map\left(x \to \sqrt{x}, \lambda2\right);$
$\omega1[p, q] := sort(\omega[p, q], \text{`<`});$
$\omega2[p, q] := zip(\text{`/`}, \omega1[p, q], Z4);$
end do:
end do:

$f := (i, j) \rightarrow \omega 2[i, j]$:

$$(i, j) \rightarrow \omega 2_{i, j}$$

$FR := Matrix(p-1, q-1, f)$:

for r **from** 1 **to** $p-1$ **do**
for d **from** 1 **to** n **do**
$\Omega[r, d] := Vector([seq(\omega 2[r, i][d], i = 1..q-1)])$;
end do:
end do:

$L1 := Vector\left(\left[seq\left(\frac{b}{10}, b = 1..10\right)\right]\right)$:

$with(plots)$:

for j **from** 1 **to** 5 **do**
$A[j] := plot(L1, \Omega[1, j], legend =$ "0.1ρL", $linestyle = solid, color = blue)$;
$B[j] := plot(L1, \Omega[2, j], legend =$ "0.5ρL", $linestyle = dot, color = red)$;
$C[j] := plot(L1, \Omega[3, j], legend =$ "ρL", $linestyle = dash, color = green)$;
$F[j] := plot(L1, \Omega[4, j], legend =$ "5ρL", $linestyle = dashdot, color = pink)$;
$G[j] := plot(L1, \Omega[5, j], legend =$ "10ρL", $linestyle = longdash, color = gold)$;
$H[j] := [A[j], B[j], C[j], F[j], G[j]]$;
$display\Big(H[j], title = typeset($"changes of", j,
"th Natural Frequency of a Fixed-Free Beam with a Sliding Mass")
$, labels = \left[\frac{x}{L}, \frac{\omega}{\omega_0}\right]\Big)$;
end do:

8.2 Mode shape code

8.2.1 Simply supported beam

$assume(\rho, positive)$

$assume(L, positive)$

$assume(E, positive)$

$assume(IM, positive)$

$\phi := (i, x) \rightarrow \sqrt{\frac{2}{\rho \cdot L}} \cdot \sin\left(\frac{i \cdot \pi \cdot x}{L}\right):$

$interface(rtablesize = infinity):$

$with(LinearAlgebra):$

$n := 20:$

$N := Vector([seq(i, i = 1 ..n)]):$

$Md := Matrix(n, shape = identity):$

$K := Vector\left(\left[seq\left(\frac{i^4 \cdot \pi^4 \cdot E \cdot IM}{\rho \cdot L^4}, i = 1 ..n\right)\right]\right):$

$Kd := Matrix(K, shape = diagonal):$

$Z := Eigenvalues(Kd, Md):$

$Z1 := \frac{\rho \cdot L^4}{E \cdot IM} \cdot Z:$

$Z2 := map\left(x \rightarrow \sqrt{x}, Z1\right):$

$Z3 := convert(Z2, float):$

$Z4 := sort(Z3, `<`):$

$Kd1 := convert(Kd, float):$

$Kd2 := \frac{\rho \cdot L^4}{E \cdot IM} \cdot Kd1:$

$listm := [0, 0.1 \cdot \rho \cdot L, 0.2 \cdot \rho \cdot L, 0.5 \cdot \rho \cdot L, 0.8 \cdot \rho \cdot L, \rho \cdot L]:$

$V := zip(\phi, N, c \cdot L):$

$V1 := simplify\left(\sqrt{\frac{\rho \cdot L}{2}} \cdot V\right):$

$VT := Transpose(V1):$

```
for p from 1 to nops(listm) do
 m := listm[p];
 for q from 1 to 9 do
 l := 0.1·q·L;
 Φ := zip(ϕ, N, l);
 Φ1 := convert(Φ, float);
 ΦT := Transpose(Φ1);
 Mt := Matrix(Md + m·Φ1.ΦT);
 λ[p, q] := Eigenvectors(Kd2, Mt);
 P[p, q] := map(attributes, sort([seq(setattribute(evalf(abs(λ[p,
     q][1][j])), j), j = 1 ..n)], `<`));
 Sortedeigen[p, q] := simplify(Vector([seq(λ[p, q][1][j], j = P[p,
     q])]));
 Sortedeigen1[p, q] := map(x→√x, Sortedeigen[p, q]);
 Sortedeigen2[p, q] := zip(`/`, Sortedeigen1[p, q], Z4);
 Sortedvec[p, q] := simplify([seq(Column(λ[p, q][2], r), r = P[p,
     q])]);
 Sortedvec1[p, q] := [seq(map(x→ x / Sortedvec[p, q][i][i],
     Sortedvec[p, q][i]), i = 1 ..n)];
 Modef[p, q] := [seq(VT.Sortedvec1[p, q][i], i = 1 ..n)];

 end do:
 end do:
```

```
interface(rtablesize = infinity) :
```

```
for t from 1 to 5 do
 for u from 1 to q − 1 do
 A[t, u] := plot([seq(Modef[j, u][t], j = 1 ..nops(listm))], c = 0 ..1,
     color = [red, blue, green, black, pink, gold], labels = [x/L,
     φ/sqrt(2/(ρ·L))], legend = ["Beam only", "0.1ρL", "0.2ρL", "0.5ρL",
     "0.8ρL", "ρL"], title = typeset(t,
     "th Mode Shape of a Simply-Supported Beam with a Lumped Mass
     at", u·L/10));
 end do
end do
```

```
Mass := convert(listm, Vector) :

Mass1 := Mass/(ρ·L) :
```

```
for r from 1 to 5 do
 for j from 1 to q − 1 do
 Ω[r, j] := Vector([seq(Sortedeigen2[i, j][r], i = 1 ..nops(listm))]);
 B[r, j] := plot(Mass1, Ω[r, j], style = point, symbol = solidcircle,
     symbolsize = 12);
 end do
end do
```

```
for t from 1 to 5 do
 for u from 1 to q − 1 do
 Abare[t, u] := plot([seq(Modef[j, u][t], j = 1 ..nops(listm))], c = 0
      ..1, color = [red, blue, green, black, pink, gold]);
 end do
end do
```

8.2.2 Cantilever beam

$assume(\rho, positive)$
$assume(L, positive)$
$assume(E, positive)$
$assume(IM, positive)$
$interface(rtablesize = infinity) :$
$g := \beta \to \cos(\beta) \cdot \cosh(\beta) + 1 :$
$R := \{seq(fsolve(g, i), i = 1..100)\} :$
$R1 := Vector([op(2..nops(R), R)]) :$

$$\phi := (b, x) \to \frac{1}{\sqrt{\rho \cdot L}} \cdot \left(\cos\left(\frac{b \cdot x}{L}\right) - \cosh\left(\frac{b \cdot x}{L}\right) + \frac{(\sin(b) - \sinh(b))}{\cos(b) + \cosh(b)} \cdot \left(\sin\left(\frac{b \cdot x}{L}\right) - \sinh\left(\frac{b \cdot x}{L}\right)\right)\right) :$$

$with(LinearAlgebra) :$
$n := \mathrm{nops}(R) - 1 :$

$Md := Matrix(n, shape = identity) :$

$$K := map\left(x \to \frac{x^4 \cdot E \cdot IM}{\rho \cdot L^4}, R1\right) :$$

$Kd := Matrix(K, shape = diagonal) :$
$Kt := convert(Kd, rational) :$
$Z := Eigenvalues(Kt, Md) :$

$$Z1 := \frac{\rho \cdot L^4}{E \cdot IM} \cdot Z :$$

$Z2 := map\left(x \to \sqrt{x}, Z1\right) :$
$Z3 := convert(Z2, float) :$
$Z4 := sort(Z3, `<`) :$

$$Kd2 := \frac{\rho \cdot L^4}{E \cdot IM} \cdot Kd :$$

$listm := [0, 0.1 \cdot \rho \cdot L, 0.2 \cdot \rho \cdot L, 0.5 \cdot \rho \cdot L, 0.8 \cdot \rho \cdot L, \rho \cdot L] :$
$V := zip(\phi, R1, b \cdot L) :$
$V1 := simplify\left(\sqrt{\rho \cdot L} \cdot V\right) :$

$VT := Transpose(V1) :$

```
for p from 1 to nops(listm) do
 m := listm[p];
 for q from 1 to 9 do
 l := 0.1·q·L;
 Φ := zip(ϕ, R1, l);
 Φ1 := convert(Φ, float);
 ΦT := Transpose(Φ1);
 Mt := Matrix(Md + m·Φ1.ΦT);
 λ[p, q] := Eigenvectors(Kd2, Mt);
 P[p, q] := map(attributes, sort([seq(setattribute(evalf(abs(λ[p,
     q][1][j])), j), j = 1 ..n)], `<`));
 Sortedeigen[p, q] := simplify(Vector([seq(λ[p, q][1][j], j = P[p,
     q])]));
 Sortedeigen1[p, q] := map(x→√x, Sortedeigen[p, q]);
 Sortedeigen2[p, q] := zip(`/`, Sortedeigen1[p, q], Z4);
 Sortedvec[p, q] := simplify([seq(Column(λ[p, q][2], r), r = P[p,
     q])]);
 Sortedvec1[p, q] := [seq(map(x→ x/Sortedvec[p, q][i][i],
     Sortedvec[p, q][i]), i = 1 ..n)];
 Modef[p, q] := [seq(VT.Sortedvec1[p, q][i], i = 1 ..n)];

 end do:
 end do:
```

```
for t from 1 to 5 do
 for u from 1 to q − 1 do
 A[t, u] := plot([seq(Modef[j, u][t], j = 1 ..nops(listm))], b = 0 ..1,
     color = [red, blue, green, black, pink, gold], labels = [x/L, ϕ
     ·√(ρ·L)], legend = ["Beam only", "0.1ρL", "0.5ρL", "ρL", "5ρL",
     "10ρL"], title = typeset(t,
     "th Mode Shape of a Fixed-Free Beam with a Lumped Mass at",
     u·L/10));
 end do:
 end do:
```

$$Mass := convert(listm, Vector) :$$

$$Mass1 := \frac{Mass}{\rho \cdot L} :$$

```
for r from 1 to 5 do
 for j from 1 to q − 1 do
 Ω[r, j] := Vector([seq(Sortedeigen2[i, j][r], i = 1 ..nops(listm))]);
 B[r, j] := plot(Mass1, Ω[r, j], style = point, symbol = solidcircle,
     symbolsize = 12);
 end do:
 end do:

for t from 1 to 5 do
 for u from 1 to q − 1 do
 Abare[t, u] := plot([seq(Modef[j, u][t], j = 1 ..nops(listm))], b = 0
      ..1, color = [red, blue, green, black, pink, gold], view = [0 ..1, -3
      ..3]);
 end do:
 end do:
```