AN APPLICATION OF FOURIER ANALYSIS AND REPRESENTATION THEORY TO RANDOM WALK ON REFLEXION GROUPS

by

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To my parents
and my husband

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ABSTRACT

This thesis presents a criterion for recurrence for the random walk on reflexion groups. First, the ultimate behavior of a random walk with a given transition probability is considered. Next, the case with a general probability measure is dealt with. The techniques of representation theory are used for the generalization.

INTRODUCTION

This thesis deals with a problem concerning a random walk on an infinite group, motivated by a result obtained by Polya [1921]. Polya considered a random walk of a particle moving on an n-dimensional lattice and showed that the ultimate behavior of this walk is recurrent if $n \le 2$ and transitory if $n \ge 3$. We obtain a similar result for a random walk on our particular infinite group.

In our problem we consider the set

$$E = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0\} \simeq \mathbb{R}^n$$

The set of vectors $\Delta = \{e_i - e_j \mid i \neq j, 1 \le i, j \le n+1\}$ (which is called the system of roots) spans E, where the vectors e_i, $1 \le i \le n+1$ form a canonical basis in $\mathbb{R}^{n+1}$.

Call $e_i - e_{i+1} = r_i$, $1 \le i \le n$ and

let $r_o = e_1 - e_{n+1} = \sum_{i=1}^{n} r_i$.

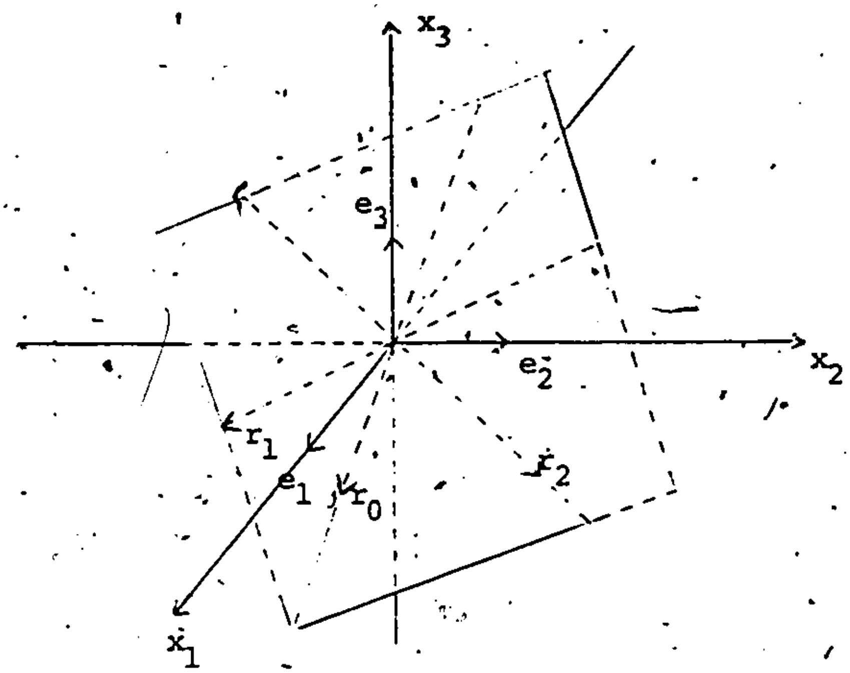

Figure 1. The space E for $n = 2$.

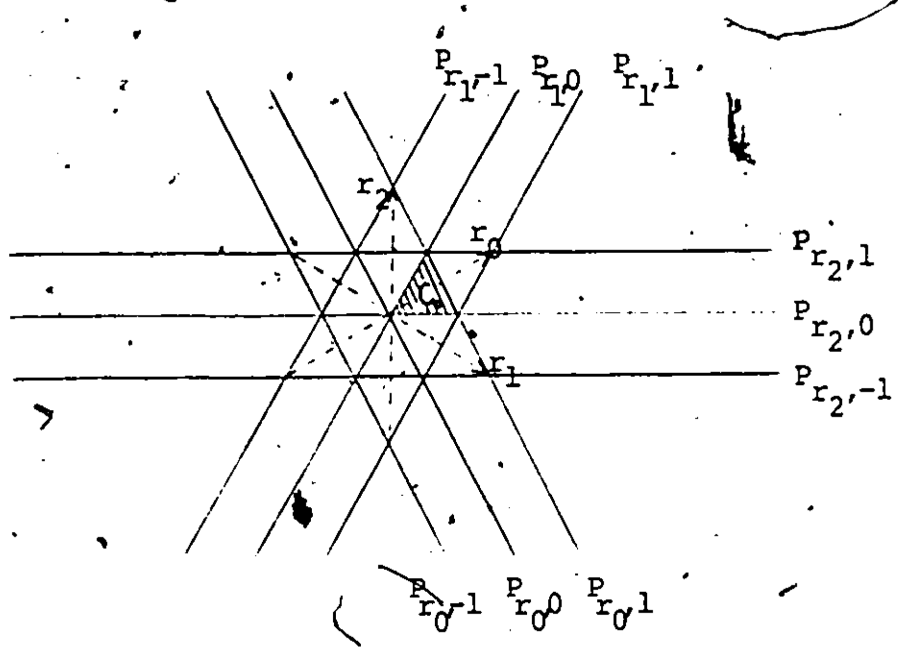

Figure 2. The chambers for $n = 2$.

Now consider the E-hyperplanes

$$P_{r,k} = \{x \in E \mid (r,x) = k \ , \ r \in \Delta \ , \ k \in \mathbb{Z}\}$$

and the "chambers" C_i in E which are regions in E cut out by the E-hyperplanes $P_{r,k}$.

The infinite group in concern is the group A generated by the reflexions with respect to these hyperplanes $P_{r,k}$.

Suppose that a particle is moving from chamber to chamber according to the following rule:

The particle starts at time zero at the chamber

$$C_0 = \{x \in E \mid (r_1,x)>0, \ (r_2,x)>0 \ \dots \ (r_n,x)>0, \ (r_0,x)<1\}$$

and moves at time $m \geq 1$ by a step across a wall into a neighbouring chamber. The steps are statistically independent.

Thus we have a random walk on the collection $\mathcal{C}$ of chambers.

In Chapter I we give a brief review of general results concerning the geometry of reflexion groups required in this thesis. The main point of this chapter is that the set of chambers can be identified with the group A.

Chapter II deals with a particular random walk on the infinite group A , where the transition probability is given by

$$\mu(C,C') = \begin{cases} \frac{1}{n+1} & \text{if } C' \text{ is one of the } n+1 \text{ chambers adjacent to } C , \\ 0 & \text{otherwise .} \end{cases}$$

Here we are led to the conclusion that the random walk is recurrent if $n = 2$ and transitory if $n > 2$.

In Chapter III we generalize our result using a general probability measure μ with the assumption that the support of μ generates the infinite group A . We obtain our final result which says that the random walk is recurrent or transitory accordingly as

$$\lim_{\theta \uparrow 1} \int_{E/L^*} tr((I - \theta\tilde{\mu}(\rho_x))^{-1}\rho_x(a^{-1}))dx \text{ is infinite or}$$

finite, respectively.[¤]

There is a vast literature on random walks on groups in general. For example, the paper by Flatto and Odlyzko (see [FlA]) is about random shuffles on group representations where the group in concern is finite. Woess ([WOE]) has written an article about random walks on certain discrete groups, but his arguments are combinatorial.

¤ The definitions of L^* , $\tilde{\mu}$ and ρ_x are given in page 6, page 53 and page 55, respectively.

It should be emphasized here that the problem is not dealt with using the Theory of Probability. Indeed, the problem itself and the techniques used come from Fourier Analysis and the Theory of Group Representation.

CHAPTER I

GEOMETRY OF REFLEXION GROUPS

It is the purpose of this chapter to review certain aspects of the geometry of the reflexion group required in this thesis. Some standard results are quoted without proof; complete proofs can be found in [BOU] and in [BEN].

Let L be the lattice in E generated by the system of roots Δ ;

$$L = \{x \in E \mid x = \sum_{i=1}^{n} m_i r_i \; , \; m_i \in \mathbb{Z}, \; r_i \in \Delta \text{ for } i=1, \dots n\}$$

where E and Δ are the sets described in the Introduction. Accordingly, we define the dual lattice of L as

$$\begin{aligned} L^* &= \{x \in E \mid (v,x) \in \mathbb{Z} \text{ for all } v \in L\} \\ &= \{x \in E \mid (r,x) \in \mathbb{Z} \text{ for all } r \in \Delta\} . \end{aligned}$$

The reflexion $s_{r,k}$ with respect to the E-hyperplane $P_{r,k}$ is given by

$$s_{r,k}(x) = x - 2\left[\frac{(r,x)-k}{(r,r)}\right] r \qquad \text{for } x \in E$$

1.1 $$= x - [(r,x) - k]r .$$

The translation t_v by the vector v in L is given by

(1.2) $$t_v(x) = x + v \qquad \text{for } x \in E .$$

Let S be the group generated by $s_{r,o}$, where $r \in \Delta$. It is known (see e.g., [BEN] Cha 5.3) that the group S is isomorphic to the symmetric group S_{n+1}. Let T be the group generated by t_r for $r \in \Delta$ and A be the group generated by $s_{r,k}$ for $r \in \Delta$ and $k \in \mathbb{Z}$.

LEMMA 1.1

(1.3) (i) $T = \{t_v \mid v \in L\} \simeq L$.

(1.4) (ii) $s_{r,k} = t_{kr} \cdot s_{r,0}$ for $r \in \Delta$ and $k \in \mathbb{Z}$.

(1.5) (iii) $t_r = s_{r,1} \cdot s_{r,0}$ for $r \in \Delta$.

(1.6) (iv) $st_v = t_{sv} \cdot s$ for all $s \in S$ and $v \in E$.

(1.7) (v) $s \cdot L = L$ for all $s \in S$.

PROOF.

(i) Let v be a vector in L.

we have

$$t_v \in T \quad \text{iff} \quad t_v = \sum_{i=1}^{n} m_i t_{r_i} \quad \text{for some } m_i \in \mathbb{Z},$$

$1 \leq i \leq n$.

This implies that

$$t_v \in T \quad \text{iff} \quad t_v = \sum_{i=1}^{n} t_{m_i r_i} = t_{\sum_{i=1}^{n} m_i r_i}$$

Hence

$$t_v \in T \quad \text{iff} \quad v \in L$$

which gives

$$T = \{t_v \mid v \in L\} .$$

Also, for each $v \in L$, there exist a unique t_v in T. Therefore

$$T \cong L.$$

■

(ii) Let x be an element in E.

By (1.1)

$$\begin{aligned} s_{r,k}(x) &= x - [(r,x) - k]r \quad \text{for } r \in \Delta \text{ and } k \in \mathbb{Z} \\ &= x - (r,x)r + kr. \end{aligned}$$

Let us write s_r for $s_{r,0}$ for all $r \in \Delta$.

Using (1.1) and (1.2) we have

$$s_{r,k}(x) = t_{kr}\cdot s_r(x)$$

This is true for all $x \in E$.

Hence

$$s_{r,k} = t_{kr}\cdot s_r \qquad \text{for } r \in \Delta \text{ and } k \in \mathbb{Z}.$$

∎

(iii). Let $x \in E$.

By (1.1) we have

$$\begin{aligned}(s_{r,1}\cdot s_r)(x) &= s_{r,1}[x-(r,x)r] \quad \text{for } r \in \Delta \\ &= x - (r,x)r - [(r,x) - 2(r,x) - 1]r \\ &= x + r \\ &= t_r(x)\end{aligned}$$

Since this is true for all $x \in E$, we have

$$s_{r,1}\cdot s_r = t_r \qquad \text{for } r \in \Delta.$$

∎

(iv) Let $x \in E$.

Using (1.2) we get

$$st_v(x) = s(v+x) = sv + s(x) = t_{sv}(s(x)) = t_{sv}\cdot s(x)$$

which is true for all $x \in E$.

Hence

$$st_v = t_{sv}\cdot s \qquad \text{for all } s \in \mathcal{S} \text{ and } v \in L.$$

∎

(v) Let $x \in L$ and $s \in S$

Then

$$x = \sum_{i=1}^{n} m_i r_i \quad \text{for some } m_i \in \mathbb{Z},\ 1 \le i \le n$$

and we have

$$s(x) = s\left(\sum_{i=1}^{n} m_i r_i\right) = \sum_{i=1}^{n} m_i s(r_i) = \sum_{i=1}^{n} \ell_i r_i$$

where $\ell_i \in \mathbb{Z}$, $1 \le i \le n$.

Hence

$$s \cdot L \subseteq L .$$

Conversely, for a given $x = \sum_{i=1}^{n} \ell_i r_i \in L$ where $\ell_i \in \mathbb{Z}$, $1 \le i \le n$, we can find $s \in S$ and $y \in L$ such that

$$sy = \sum_{i=1}^{n} \ell_i r_i .$$

This says that

$$s \cdot L \supseteq L$$

Hence we have

$$s \cdot L = L . \qquad \blacksquare$$

LEMMA 1.2.

(i) S *is a subgroup of* A *and* T *is a normal subgroup of* A

(ii) $A = T \cdot S$ *in the sense that every* $a \in A$ *can be uniquely written as*

$$a = t_v \cdot s \quad \textit{with } t_v \in T \textit{ and } s \in S .$$

(iii) Given $t_{v'}$, $t_{v''} \in T$ *and* s' , $s'' \in S$,
write

$$(t_{v'} \cdot s') \cdot (t_{v''} \cdot s'') = t_v \cdot s$$

Then

$$v = v' + s'v'' \qquad s = s' \cdot s'' .$$

PROOF .

(i) The set $\{s_{r_i} \mid i = 1, \ldots, n\}$ generates the group S and each $s_{r_i} \in A$.
Therefore $S \subset A$ and hence S is a subgroup of A .
By (1.5)

$$t_r = s_{r,1} \cdot s_r \quad \text{for all} \quad r \in \Delta .$$

This implies that $t_r \in A$ for all $r \in \Delta$.
Therefore T is a subgroup of A.
By (1.6)

$$s t_v s^{-1} = t_{sv} \quad \text{for all } s \in S \text{ and } v \in L$$

Also, by (1.7), $sv \in L$
This together with (1.3) gives

$$t_{sv} \in T .$$

Therefore $s t_v s^{-1} \in T$ for all $s \in S$ and $t_v \in T$. Hence T is a normal subgroup of A .

∎

(ii) An element $a \in A$ is the composition of the elements $s_{r_i,k}$, $r_k \in \Delta$, $k \in \mathbb{Z}$:

$$a = s_{r_i,k_1} \cdot s_{r_j,k_2} \cdots s_{r_m,k_n}$$

where $1 \le i, j, \dots, m \le n$ and $i, j, \dots, m, k_1, k_2, \dots, k_n \in \mathbb{Z}$.
By (1.4) we have

$$a = (t_{k_1 r_i} \cdot s_{r_i}) \cdot (t_{k_2 r_j} \cdot s_{r_j}) \quad \cdots \quad (t_{k_n r_m} \cdot s_{r_m})$$

Using (1.6) this can be written as

$$a = t_{\ell_1 r_i} \cdot t_{\ell_2 r_j} \cdots t_{\ell_n r_m} \cdot s_{r_i} \cdot s_{r_j} \cdots s_{r_m}$$

for $\ell_i \in \mathbb{Z}$, $1 \le i \le n$.
Therefore

$$a = t_v \cdot s \quad \text{where} \quad v \in L \quad \text{and} \quad s \in S .$$

Now if

$$t_{v'} \cdot s' = t_{v''} \cdot s'' \quad \text{for} \quad v', v'' \in L \quad \text{and} \quad s', s'' \in S,$$

we have

$$s' \cdot (s'')^{-1} = t_{-v'} \cdot t_{v''}$$

But $S \cap T = \{1\}$

Therefore

$$s' \cdot (s'')^{-1} = 1$$

and

$$t_{-v'} \cdot t_{v''} = 1 .$$

which implies

$$s = s'' \quad \text{and} \quad t_{v'} = t_{v''}$$

Hence the result. ∎

(iii) Using (1.6) we have

$$t_v \cdot s = (t_{v'} \cdot s') \cdot (t_{v''} \cdot s'') = t_{v'} \cdot t_{s'v''} \cdot s' \cdot s''$$

$$= t_{v' + s'v''} \cdot s' \cdot s''$$

which implies

$$v = v' + s'v'' \quad \text{and} \quad s = s' \cdot s'' .$$

∎

LEMMA 1.3.

The group A *permutes the chambers simply transitively. For a fixed chamber* C_o *, the rule* $C = aC_o$ *, where* $a \in A$ *and* $C \in \mathcal{C}$ *is a one to one mapping from the elements of* A *to the set of chambers* $\mathcal{C}$ *.*

PROOF.

This is proved in detail in [BOU, chap. VI, §2]. Here we only show that the group A permutes the chambers.

A chamber C in $\mathcal{C}$ is bounded by $n+1$ E-hyperplanes, P_{r_i,k_i}, $0 \le i \le n$.

Since any $a \in A$ can be written as

$$a = t_v \cdot s \quad \text{for} \quad v \in L \quad \text{and} \quad s \in S ,$$

$$a \cdot P_{r_i,k_j} = t_v \cdot s \cdot (P_{r_i,k_j}) .$$

Now P_{r_i,k_j} will be mapped onto some other P_{r_ℓ,k_m} by the element s and then it will be translated to another P_{r_ℓ,k_q} by the element v. Therefore aC is the chamber bounded by the $n+1$ lines $a \cdot P_{r_i,k_i}$, $i = 0, 1, \ldots n$. In other words, the group A permutes the chambers C in $\mathcal{C}$. Since the permutation is simply transitive, the rule $C = aC_o$ is a one to one mapping from the elements of A to the set of chambers $\mathcal{C}$.

∎

CHAPTER II

A PARTICULAR RANDOM WALK

Our main objective in this chapter is to study a random walk on the infinite group A with a given transition probability μ. We intend to obtain a criterion for the ultimate behavior of the walk.

§1. INTRODUCTION TO THE PARTICULAR RANDOM WALK

Let μ be the transition probability of a random walk on the collection of chambers given by

$$(2.1) \qquad \mu(C,C') = \begin{cases} \frac{1}{n+1} & \text{if } C' \text{ is one of the } n+1 \text{ adjacent chambers to } C, \\ 0 & \text{otherwise}. \end{cases}$$

Suppose that the particle starts at time zero at the chamber

$$C_0 = \{x \in E \mid (r_i, x) > 0 \quad \text{for} \quad i = 1 \dots n \quad \text{and} \quad (r_o, x) < 1\}.$$

By the Lemma 1.3, we know that the set of chambers can be identified with the group A. Therefore we can write $\mu(a,a')$ for $\mu(C,C')$, if $C = aC_0$ and $C' = a'C_0$ for $a, a' \in A$ and $C, C' \in \mathcal{C}$.

We have the following properties of μ .

LEMMA 2.1

Let a, a_1 *and* a_2 *be elements of* A .

(2.2) (*i*) $\mu(aa_1, aa_2) = \mu(a_1, a_2)$.

(2.3) (*ii*) $\mu(a_1, a_2) = \mu(e_A, a_1^{-1}a_2)$

$$= \begin{cases} \frac{1}{n+1} & \text{if } a_1^{-1}a_2 \in \{s_{r_1}, s_{r_2}, \dots s_{r_n}, s_{r_0}, 1\}, \\ 0 & \textit{otherwise}. \end{cases}$$

where e_A *is the identity element of* A .

write

$$\mu(a) = \mu(e_A, a) .$$

(*iii*) *The probability of the particle arriving at* a *after* m *steps (starting at* e_A*) is given by*

(2.4) $$\mu^{\times m}(a) = \underbrace{\mu\times\mu\times \dots \mu\times\mu}_{m \text{ times}}(a)$$

where '$\times$' *denotes the convolution product of the functions on the group* A :

$$f \times g(a) = \sum_{b\in A} f(ab^{-1})g(b) = \sum_{b\in A} f(b)g(b^{-1}a)$$

for the functions $f, g \in \mathbb{C}^A = \{\phi \mid \phi : A \to \mathbb{C}\}$.

PROOF.

(i) Let $a_1C_0 = C_1$ and $a_2C_0 = C_2$ and for $C_1, C_2 \in \mathcal{C}_\ell$.
Then

$$\mu(a_1,a_2) = \mu(C_1,C_2)$$

$$= \begin{cases} \frac{1}{n+1} & \text{if } C_1 \text{ and } C_2 \text{ are adjacent chambers ,} \\ 0 & \text{otherwise .} \end{cases}$$

Suppose that $\mu(a_1,a_2) = \frac{1}{n+1}$.

Then the chambers C_1 and C_2 have a common boundary. Now for $a \in A$

$$\mu(aa_1,aa_2) = \mu(aC_1,aC_2) .$$

Since the common boundary of C_1 and C_2 will be mapped onto a common boundary of aC_1 and aC_2 by the element $a \in A$, we have

$$\mu(aC_1,aC_2) = \mu(aa_1,aa_2) = \frac{1}{n+1} .$$

Hence

$$\mu(a_1,a_2) = \mu(aa_1,aa_2) .$$

Now suppose that $\mu(a_1,a_2) = 0$.

Then the chambers C_1 and C_2 are not adjacent. Therefore aC_1 and aC_2 are also not adjacent; for if they are, so are

C_1 and C_2 (applying the previous argument replacing a by a^{-1}).

Therefore

$$\mu(a_1,a_2) = 0 \text{ implies } \mu(aa_1,aa_2) = 0 \text{ for } a \in A .$$

Hence we have

(2.2) $$\mu(a_1,a_2) = \mu(aa_1,aa_2) \quad \text{for all } a,\ a_1,\ a_2 \in A .$$

∎

(ii) By (2.2)

$$\mu(a_1,a_2) = \mu(a_1,a_1a_1^{-1}a_2) = \mu(e_A,a_1^{-1}a_2) .$$

Now using the definition of μ we have

$$\mu(a_1,a_2) = \mu(e_A,a_1^{-1}a_2) = \begin{cases} \frac{1}{n+1} & \text{if } a_1^{-1}a_2 \in \{s_{r_1}, s_{r_2}, \ldots, s_{r_n}, s_{r_\sigma}\}, \\ 0 & \text{otherwise} . \end{cases}$$

∎

(iii) The proof is by induction on the number of steps m.

The probability that the particle arrives at a after the first step $= \mu(a)$

Therefore the result is true for $m = 1$. Assume the result holds for $m = p$. Then the probability of the particle arriving at a after $p+1$ steps is given by

$$\sum_{c \in A} \mu(c,a)\mu^{\times p}(a) .$$

By (2.3) we have

$$\sum_{c \in A} \mu(c,a)\mu^{\times p}(a) = \sum_{c \in A} \mu(c^{-1}a)\,\mu^{\times p}(a) = \mu^{\times(p+1)}(a) ,$$

Hence the result is true for $m = p + 1$. By mathematical induction the result is true for all $m \in \mathbb{Z}$. ∎

Our aim here is to obtain a formula for the probability of the particle returning to the identity after m steps. Before proceeding further, let us derive some important results of the functions on A and their Fourier transforms.

Consider the function ε_1 defined by

$$\varepsilon_1(a) = \begin{cases} 1 & \text{if } a = e_A , \\ 0 & \text{otherwise} . \end{cases}$$

Then, for any function f on A

$$f(a) = \varepsilon_1 \times f(a) \quad \text{for all} \quad a \in A .$$

Denote an element $a = t_v \cdot s$ in A by (v,s) . Then the identity of e_A of A can be written as (o,e) where e is the identity of S and a function $f(a)$ on A becomes a function $f(v,s)$ of two variables $v \in L$ and $s \in S$.

The Fourier transform of $f(v,s)$ with respect to the variable $v \in L$ is

$$\hat{f}(x,s) = \sum_{v \in L} f(v,s)\, e^{-2\pi i (x,v)} \text{ , defined for all}$$

$x \in E$ and $s \in S$.

Observe that $\hat{f}(x,s)$ is periodic with respect to the lattice L^* ; for if $y \in L^*$,

$$\hat{f}(x+y,s) = \sum_{v \in L} f(v,s)\, e^{-2\pi i (x+y,v)}$$

$$= \sum_{v \in L} f(v,s)\, e^{-2\pi i (x,v)}\, e^{-2\pi i k}$$

where $k = (y,v) \in \mathbb{Z}$.

This implies

$$\hat{f}(x+y,s) = \sum_{v \in L} f(v,s)\, e^{-2\pi i (x,v)} = \hat{f}(x,s)$$

Therefore we can think of $\hat{f}(x,s)$ as a function on $(E/L^*) \times S$.

Thus the probability of the particle returning to the identity of 1_A after m steps is

$$\mu^{\times m}(1_A) = \mu^{\times m}(o,e) = \varepsilon_1 \times \mu^{\times m}(o,e)$$

$$(2.5) \qquad = \int_{E/L^*} (\varepsilon_1 \times \mu^{\times m})^\wedge (x,e)\, dx \ .$$

To simplify this further let us look at the Fourier transform of $f \times \mu^{\times m}$ for a function f on A.

LEMMA 2.2.

Let f *be a function on* A *and consider the function* μ *on* A:

$$\mu(v,s) = \begin{cases} \frac{1}{n+1} \text{ if } (v,s) \in \{(0,s_{r_1}),(0,s_{r_2}), \dots (0,s_{r_n}),(r_0,s_{r_0})\}, \\ 0 \quad \text{otherwise}. \end{cases}$$

Then we have

(2.6) (*i*) $\mu^{\wedge}(x,s) = \frac{1}{n+1}\{[\sum_{i=1}^{n} \varepsilon_i(s)] + e^{-2\pi i(r_0,x)}\varepsilon_0(s)\}$

where ε_i, $0 \le i \le n$, *are the indicator functions of* $s_{r_i} \in S$.

(2.7) (*ii*) $(f \times \mu)^{\wedge}(x,s) = \frac{1}{n+1}\{[\sum_{i=1}^{n} f^{\wedge}(x,ss_i)] + e^{2\pi i(x,sr_0)} f^{\wedge}(x,ss_0)\}$

PROOF.

(i) The Fourier transform of the function μ with respect to the variable v in L is

$$\mu^{\wedge}(x,s) = \sum_{v \in L} \mu(v,s)e^{-2\pi i(x,v)}$$

for all $x \in E$ and $s \in S$.

Now using the definition of μ we have

$$\mu^\wedge(x,s) = \mu(0,s) + \mu(r_0,s)e^{-2\pi i(x,r_0)}$$

$$= \frac{1}{n+1}\{[\sum_{i=1}^{n} \varepsilon_i(s)] + e^{-2\pi i(x,r_0)}\varepsilon_0(s)\}$$

where ε_i, $0 \le i \le n$, are the indicator functions of $s_{r_i} \in S$.

∎

(ii) The Fourier transform of the function $f \times \mu$ with respect to the variable $v \in L$ is

$$(f \times \mu)^\wedge(x,s) = \sum_{v \in L} (f \times \mu)(v,s)e^{-2\pi i(x,v)}$$

$$= \sum_{v \in L} [f(b)\mu(b^{-1}a)]e^{-2\pi i(x,v)}$$

where $a = (v,s)$.

Let $b = (v',s')$ for $v' \in L$ and $s' \in S$.

Then

$$b^{-1}a = (s')^{-1}\cdot t_{-v'}\cdot t_v \cdot s = (s')^{-1}t_{v-v'}\cdot s$$

Using (1.6) this can be simplified to

$$b^{-1}a = t_{(s')^{-1}(v-v')}\cdot(s')^{-1}s$$

Now

$$\mu(b^{-1}a) = \mu((s')^{-1}(v-v'),(s')^{-1}s)$$

$$= \begin{cases} \frac{1}{n+1} \text{ if } (s')^{-1}(v-v') = 0 \text{ and } (s')^{-1}s = s_{r_i},\ 1 \le i \le \\ \quad\text{or } (s')^{-1}(v-v') = r_0 \text{ and } (s')^{-1}s = s_{r_0}, \\ 0 \text{ otherwise}. \end{cases}$$

Therefore, $\mu(b^{-1}a)$ will be nonzero for the elements $b = (v',s')$ such that $v' = v$ and $s' = ss_{r_i}^{-1} = ss_{r_i}$ for $i = 1, \dots n$ or $v' = v - ss_{r_0}r_0$ and $s' = ss_{r_0}$.

Therefore we have

$$(f \times \mu)^\wedge(x,s) = \frac{1}{n+1} \sum_{v \in L} [(\sum_{i=1}^{n} f(v,ss_i)) + f(v - ss_0 r_0, ss_0)] e^{-2\pi i (x,v)}$$

where we have written s_i for s_{r_i}, $0 \le i \le n$.

Now using the definition of the Fourier transform and changing the variable v to $v - ss_0r_0$ in the last summation, we get

$$(f \times \mu)^\wedge(x,s) = \frac{1}{n+1} \{ [\sum_{i=1}^{n} f^\wedge(x,ss_i)] + \sum_{v \in L} f(v,ss_0) e^{-2\pi i (x, v + ss_0 r_0)} \}$$

This along with the fact that $s_0 r_0 = -r_0$ gives

$$(f \times \mu)^\wedge(x,s) = \frac{1}{n+1} \{ [\sum_{i=1}^{n} f^\wedge(x,ss_i)] + f^\wedge(x,ss_0) e^{2\pi i (x,sr_0)} \}$$

∎

Think of $\hat{f}(x,s)$ as a function of $x \in E/L^*$ with values in the space $\mathbb{C}^S$ of complex valued functions on S .

For each $x \in E/L^*$, we wish to define an operator $P(x)$ on the space $\mathbb{C}^S$ so that

$$P(x) f^\wedge(x,s) = (f \times \mu)^\wedge(x,s)$$

Using the formula (2.7), we can accomplish this by defining the operation of $P(x)$ on a function ϕ in $\mathbb{C}^S$ as

$$(P(x)\phi)(s) = \frac{1}{n+1}\{[\sum_{i=1}^{n}(S_i\phi)(s)] + e^{2\pi i(x,sr_0)}(S_0\phi)(s)\}$$

where S_i operates on functions ϕ of $\mathbb{C}^S$ by

$$(S_i\phi)(s) = \phi(ss_i) \quad \text{for all} \quad s \in S \quad \text{and} \quad 0 \le i \le n .$$

Accordingly, for any positive integer m

$$(f \times \mu^{\times m})^{\wedge}(x,s) = P(x)^m f^{\wedge}(x,s) \quad \text{for all} \quad s \in S .$$

Thus the equation (2.5) can be written as

$$\begin{aligned} \mu^{\times m}(0,e) &= \int_{E/L^*} (\varepsilon_1 \times \mu^{\times m})^{\wedge}(x,e)\,dx \\ &= \int_{E/L^*} P(x)^m \varepsilon_1{}^{\wedge}(x,e)\,dx . \end{aligned}$$

Now

$$\varepsilon_1{}^{\wedge}(x,s) = \sum_{v \in L} \varepsilon_1(v,s)e^{-2\pi i(x,v)} \quad \text{for all} \quad s \in S .$$

and since ε_1 is the indicator function of the identity $\Omega_A = (0,e)$ we have

$$(2.8) \qquad \varepsilon_1{}^{\wedge}(x,s) = \begin{cases} 1 & \text{if } s = e , \\ 0 & \text{otherwise} . \end{cases}$$

Therefore $\varepsilon_1{}^{\wedge}(x) = \varepsilon_e$ where $\varepsilon_e \in \mathbb{C}^S$ is the indicator function of the identity $e \in S$.

We have

$$(2.9) \qquad \mu^{\times m}(0,e) = \int_{E/L^*} (P(x)^m \varepsilon_e)(e)\,dx$$

Now we wish to express the integrand of (2.9) as an inner product of two functions, where the inner product is defined on the space of functions on S, as follows

Observe that the set of functions $\{\varepsilon_s\}_{s \in S}$ defined by

$$\varepsilon_s(t) = \begin{cases} 1 & \text{for } t = s , \\ 0 & \text{otherwise} . \end{cases}$$

forms a basis for the space $\mathbb{C}^S$. Any function ϕ in $\mathbb{C}^S$ can be written uniquely as $\phi = \sum_{s \in S} \phi(s)\varepsilon_s$ and therefore, can be identified with the $(n+1)!$-tuple

$$(\phi(s_1), \dots, \phi(s_{(n+1)!}))$$

with respect to the basis $\{\varepsilon_s\}_{s \in S}$

Accordingly, we can define the inner product on the space $\mathbb{C}^S$ as

$$<\phi,\psi> = \frac{1}{|S|} \sum_{s \in S} \phi(s)\overline{\psi(s)} \quad \text{for} \quad \phi,\psi \in \mathbb{C}^S .$$

Here $\overline{\psi(s)}$ denotes the complex conjugate of $\psi(s)$.

Now using this inner product we can rewrite (2.9) as

$$(2.10) \qquad \mu^{\times m}(0,e) = |S| \int_{E/L^*} <p(x)^m \varepsilon_e, \varepsilon_e> dx$$

Now we are in a position to state one of our main results:

THEOREM 2.3 .

The expected number of times that the particle visits the identity can be expressed as

$$(2.11) \qquad \sum_{m=0}^{\infty} \mu^{\times m}(0,e) = \lim_{\theta \uparrow 1} \int_{E/L^*} |S| \langle (I - \theta P(x))^{-1} \varepsilon_e, \varepsilon_e \rangle \, dx,$$

$0 \leq \theta \leq 1$.

To prove the theorem 2.3, we use the following properties of the operator $P(x)$, which will be justified in the section 3.

LEMMA 2.4 .

(i) *The operator* $P(x)$ *is self-adjoint.*

(ii) *If* λ *is an eigenvalue of* $P(x)$ *then* $|\lambda| \leq 1$ *and the operator* $(I - P(x))$ *is invertible for* $x \neq 0$ *in* E/L^*.

$P(0)$ *has 1 as a simple eigenvalue with the constant function as the eigenvector.*

PROOF OF THEOREM 2.3 .

We have obtained the equation

$$(2.10) \qquad \mu^{\times m}(0,e) = |S| \int_{E/L^*} \langle P(x)^m \varepsilon_e, \varepsilon_e \rangle \, dx$$

where m is a positive integer.

Therefore the expected number of times that the particle visits the identity is

$$\sum_{m=0}^{\infty} \mu^{\times m}(0,e) = \lim_{\theta \uparrow 1} \sum_{m=0}^{\infty} \theta^m \mu^{\times m}(0,e) .$$

i.e.

$$\sum_{m=0}^{\infty} \mu^{\times m}(0,e) = \lim_{\theta \uparrow 1} \sum_{m=0}^{\infty} \int_{E/L^*} \theta^m < P(x)^m \varepsilon_e, \varepsilon_e > dx$$

since

$$| < P(x)^m \varepsilon_e, \varepsilon_e > | \leq \| p(x) \|^m \| \varepsilon_e \|^2 \leq \frac{1}{(n+1)!} < 1 ,$$

we have

$$\sum_{m=0}^{\infty} \mu^{\times m}(0,e) = \lim_{\theta \uparrow 1} \int_{E/L^*} \sum_{m=0}^{\infty} \theta^m < p(x)^m \varepsilon_e, \varepsilon_e > dx$$

(2.12)
$$= \lim_{\theta \uparrow 1} \int_{E/L^*} < \sum_{m=0}^{\infty} \theta^m p(x)^m \varepsilon_e, \varepsilon_e > dx .$$

Now let $B_k = \sum_{m=0}^{k} \theta^m p(x)^m$

Then for $k > \ell$,

$$B_k - B_\ell = \sum_{m=\ell+1}^{k} \theta^m p(x)^m$$

Therefore

$$\| B_k - B_\ell \| \leq \sum_{m=\ell+1}^{k} \theta^m \| p^m(x) \| \leq \sum_{m=\ell+1}^{\infty} \theta^m \| p(x) \|^m$$

Since $\|P(x)\| \leq 1$, $\{B_k\}$ is a Cauchy sequence, and hence it is convergent.

Let $\lim_{k\to\infty} B_k = B$

Then

$$B(I - \theta P(x)) = \sum_{m=0}^{\infty} \theta^m P(x)^m (I - \theta P(x))$$

$$= \sum_{m=0}^{\infty} \theta^m P(x)^m - \sum_{m=0}^{\infty} \theta^{m+1} P(x)^{m+1}$$

$$= I$$

Therefore

$$\sum_{m=0}^{\infty} \theta^m P(x)^m = (I - \theta P(x))^{-1} \text{ for all } x .$$

This simplifies (2.10) to

$$\sum_{m=0}^{\infty} \mu^{\times m}(0,e) = \lim_{\theta \uparrow 1} \int_{E/L^*} \langle (I - \theta P(x))^{-1} \varepsilon_e, \varepsilon_e \rangle \, dx$$

∎

§2. CRITERION FOR RECURRENCE AND TRANSCIENCE

In this section we obtain the criterion for recurrence and transcience.

First we expect to express the operator $(I - P(x))^{-1}$ using the projection operators onto the eigenspaces.

Let $\lambda_0(x)$ be the eigenvalue of $P(x)$ which tends to one as x tends to zero.

Define the operator $E_0(x)$ by

$$E_0(x) = \frac{1}{2\pi i} \oint_C (zI - P(x))^{-1} dz$$

where C is a closed curve in $\mathbb{C}$ with only the eigenvalue $\lambda_0(x)$ in the interior. (We can show that there is only one eigenvalue which tends to one as x tends to zero. Therefore the curve C can be chosen so that it does not contain points corresponding to any other eigenvalue except $\lambda_0(x)$. See Appendix (B)).

Now if $\psi_i(x)$ is an eigenvector of $P(x)$ with an eigenvalue $\lambda_i(x)$,

$$(zI - P(x))^{-1}\psi_i(x) = (z - \lambda_i(x))^{-1} \psi_i(x)$$

Therefore

$$E_0(x)\,\psi_i(x) = \frac{1}{2\pi i} \left[\oint_C (zI - P(x))^{-1} dz\right] \psi_i(x)$$

$$= \frac{1}{2\pi i} \oint_C (zI - P(x))^{-1} \psi_i(x)\, dz$$

$$= \frac{1}{2\pi i} [\oint_C (z - \lambda_i(x))^{-1} dz] \psi_i(x)$$

$$= \begin{cases} \psi_i(x) & \text{if } i = 0 , \\ 0 & \text{otherwise} . \end{cases}$$

This shows that $E_0(x)$ is the projection operator onto the eigenspace corresponding to the eigenvalue $\lambda_0(x)$. We know that the constant function 1_S defined by

$$1_S(s) = 1 \quad \text{for all } s \in S$$

is an eigenvector for the operator $p(0)$, corresponding to the eigenvalue one.

Now let

$$(2.13) \qquad \phi_0(x) = E_0(x) 1_S .$$

Then $\phi_0(x)$ is an eigenvector for $p(x)$ with the simple eigenvalue $\lambda_o(x)$, for small x.

Having this information at hand, we prove the following theorem, which leads us to the criterion for recurrence.

THEOREM 2.5.

The expected number of times that the particle visits the identity is finite if $n > 2$ *and infinite if* $n = 2$.

Here we use the following property of the eigenvalue $\lambda_0(x)$, the proof of which is given in the section 3.

THEOREM 2.6.

Let $\lambda_0(x)$ *be the eigenvalue of* $p(x)$ *which tends to one as* x *tends to zero. Then*

(2.22) $$\lambda_0(x) = 1 - \gamma(x,x) + O(\|x\|^4)$$

where

$$\gamma = \frac{(2\pi)^2}{n(n+1)^2} .$$

PROOF OF THEOREM 2.5

Equation (2.11) says

$$\sum_{n=0}^{\infty} \mu^{\times m}(0,e) = \lim_{\theta \uparrow 1} \int_{E/L^*} \langle (I - \theta P(x))^{-1} \varepsilon_e, \varepsilon_e \rangle dx .$$

Now for some positive δ ,

$$\int_{E/L^*} \langle (I - \theta P(x))^{-1}\varepsilon_e, \varepsilon_e \rangle dx$$

$$= \int_{\substack{|x|<\delta \\ x\in E/L^*}} \langle (I - \theta P(x))^{-1}\varepsilon_e, \varepsilon_e \rangle dx + \int_{\substack{|x|\geq\delta \\ x\in E/L^*}} \langle (I - \theta P(x))^{-1}\varepsilon_e, \varepsilon_e \rangle dx$$

$\int_{\substack{|x|\geq\delta \\ x\in E/L^*}} \langle (I - \theta P(x))^{-1}\varepsilon_e, \varepsilon_e \rangle dx$ is bounded for all θ, $0 \leq \theta \leq 1$.

Therefore let us consider $\int_{\substack{|x|<\delta \\ x\in E/L^*}} \langle (I - \theta P(x))^{-1}\varepsilon_e, \varepsilon_e \rangle dx$.

Since $(I - P(x)) = \sum_i (1 - \lambda_i(x))E_i(x)$ where $\lambda_i(x)$ are the eigenvalues of $P(x)$ and $E_i(x)$ are the corresponding orthogonal projections to the eigenspaces, we have

$$\int_{\substack{|x|<\delta \\ x\in E/L^*}} \langle (I - \theta P(x))^{-1}\varepsilon_e, \varepsilon_e \rangle dx$$

$$= \sum_i \int_{\substack{|x|<\delta \\ x\in E/L^*}} \langle (1 - \theta\lambda_i(x))^{-1}E_i(x)\varepsilon_e, \varepsilon_e \rangle dx$$

$$= \int_{\substack{|x|<\delta \\ x\in E/L^*}} \langle (1 - \theta\lambda_0(x))^{-1} E_0(x)\varepsilon_e, \varepsilon_e \rangle dx \;+$$

$$+ \sum_{i\neq 0} \int_{\substack{|x|<\delta \\ x\in E/L^*}} \langle (1 - \theta\lambda_i(x))^{-1} E_i(x)\varepsilon_e, \varepsilon_e \rangle dx$$

Again, $\sum_{i\neq 0} \int_{\substack{|x|<\delta \\ x\in E/L^*}} \langle (1- \theta\lambda_i(x))^{-1} E_i(x)\varepsilon_e, \varepsilon_e \rangle dx$ is bounded

for all θ, $0 \le \theta \le 1$, since $\lambda_0(x)$ is the only eigenvalue which tends to one as x tends to zero.

Now

$$\lim_{x\to 0} E_0(x)\varepsilon_e = E_0(0)\varepsilon_e = \langle \varepsilon_e, 1_S \rangle 1_S = \frac{1}{(n+1)!} 1_S .$$

Therefore for small x,

$$\langle (1 - \theta\lambda_0(x))^{-1} E_0(x)\varepsilon_e, \varepsilon_e \rangle \sim \frac{1}{((n+1)!)^2} \frac{1}{(1 - \theta\lambda_0(x))}$$

and therefore

$$\int_{\substack{|x|<\delta \\ x\in E/L^*}} \langle (1 - \theta\lambda_0(x))^{-1} E_0(x)\varepsilon_e, \varepsilon_e \rangle dx \sim \frac{1}{((n+1)!)^2} \int_{\substack{|x|<\delta \\ x\in E/L^*}} \frac{1}{(1-\theta\lambda_0(x))}$$

Now using the result (2.14),

$$\frac{1}{((n+1)!)^2} \int_{\substack{|x|<\delta \\ x\in E/L^*}} \left(\frac{1}{1-\theta\lambda_0(x)}\right) dx = \frac{1}{((n+1)!)^2} \int_{\substack{|x|<\delta \\ x\in E/L^*}} \frac{1}{(1-\theta(1-\gamma(x,x)+0||x||^4)} dx$$

where $\gamma = \frac{(2\pi)^2}{n(n+1)^2}$.

Now for small $|x|$, $(1 - \theta(1 - \gamma(x,x) + 0||x||^4))$ is a decreasing function of θ , and hence its reciprocal is an increasing function of θ .

By an application of monotone convergence, for small $|x|$ we get

$$\lim_{\theta\uparrow 1} \int_{\substack{|x|<\delta \\ x\in E/L^*}} \frac{dx}{(1-\theta(1-\gamma(x,x)+0||x||^4))} = \int_{\substack{|x|<\delta \\ x\in E/L^*}} \frac{1}{(\gamma(x,x)+0||x||^4)} dx .$$

Therefore $\sum_{m=0}^{\infty} \mu^{\times m}(0,e)$ is finite or infinite according to the value of the integral

$$\int_{\substack{|x|<\delta \\ x\in E/L^*}} \frac{1}{(\gamma(x,x) + 0||x||^4)} dx ,$$

which is equal to

$$\int_{\substack{|x|<\delta \\ x\in E/L^*}} \frac{dx}{(x,x)(\gamma + 0||x||^2)} .$$

But $\gamma + O\|x\|^2$ is bounded as x tends to zero. Hence we conclude that $\sum_{m=0}^{\infty} \mu^{\times m}(0,e)$ is finite or infinite according to the value of the integral

$$\int_{\substack{|x|<\delta \\ x\in E/L^*}} \frac{1}{(x,x)}\,dx$$

Thus, we have gained the result we claimed, since the above integral is finite for $n = 2$ and infinite for $n > 2$. ∎

Finally we derive the criterion for recurrence for the particular random walk, as in the classical case of Polya ([POL], [DYM])

THEOREM 2.7 .

The random walk on the infinite group A *with the given transition probability* μ *is recurrent for* $n = 2$ *and transitory for* $n > 2$.

PROOF FOR $n > 2$.

In this case we have that the expected number of times that the particle visits the identity is finite. This is the same as to say that the actual number of visits to the identity is finite with probability one. Since this result is true for any element $a \in A$, we have that for any $R < \infty$

the particle ultimately stops visiting the chambers within a distance R of e_A , which says that the random walk is transitory.

PROOF FOR $n = 2$.

Suppose that the probability of the particle ultimately returning to the identity is p . Then the probability of visiting the identity for at least m times is p^{m-1} , including the visit at time $m = 0$.

Then the probability of visiting the identity for

$$\text{exactly } m \text{ times} = p^{m-1} - p^m = p^{m-1}(1 - p)$$

Therefore if $p < 1$, the expected number of visits

$$= \sum_{m=1}^{\infty} m\, p^{m-1} (1-p)$$

$$= (1 - p)^{-1} < \infty \quad .$$

But this is a contradiction, since we have shown in Theorem 2.5 that the expected number of visits is infinite.

Therefore the particle visits the identity infinitely often with probability one.

We conclude that the random walk is recurrent for the case $n = 2$. ∎

§3 THE SPECTRAL ANALYSIS OF P(x)

PROOF OF THEOREM 2.4

(i) Let ϕ and ψ be two functions in $\mathbb{C}^S$.
Using the definition of the inner product on $\mathbb{C}^S$ we get

$$<P(x)\phi,\psi> = \frac{1}{|S|} \sum_{s\in S} (P(x)\phi)(s)\overline{\psi(s)}$$

Now using the definition of $P(x)$,

$$<P(x)\phi,\psi> = \frac{1}{|S|} \sum_{s\in S} [\sum_{i=1}^{n} \phi(ss_i)\overline{\psi(s)} + e^{2\pi i(x,sr_0)}\phi(ss_0)\overline{\psi(s)}]$$

Changing the variable s to ts_i in each summation and using the fact that $s_0 r_0 = -r_0$, we have

$$<P(x)\phi,\psi> = \frac{1}{|S|} \sum_{t\in S}[\sum_{i=1}^{n}\phi(t)\overline{\psi(ts_i)} + e^{-2\pi i(x,tr_0)}\phi(t)\overline{\psi(ts_0)}]$$

$$= \frac{1}{|S|} \sum_{t\in S}\phi(t)\overline{(P(x)\psi)(t)}$$

$$= <\phi,P(x)\psi>$$

Hence the operator $P(x)$ is self-adjoint ∎

Consequently, all the eigenvalues of $P(x)$ are real.

(ii) Suppose that there exists a non zero function ϕ in $\mathbb{C}^S$ such that

$$P(x)\phi = \lambda\phi \quad \text{for } |\lambda| \geq 1$$

We have

$$< P(x)\phi, \phi > = \frac{1}{|S|} \sum_{s \in S} (P(x)\phi)(s)\overline{\phi(s)}$$

which is the same as

$$\lambda \|\phi\|^2 = < \lambda\phi, \phi > = \frac{1}{|S|} \frac{1}{n+1} \sum_{s \in S} [\sum_{i=1}^{n} (S_i \phi)(s)\overline{\phi(s)} + e^{2\pi i (x, sr_0)}(S_0 \phi)(s)\overline{\phi(s)}]$$

Defining an operator K on $\mathbb{C}^S$ by

$$(K\phi)(s) = e^{2\pi i (x, sr_0)}(S_0 \phi)(s) \quad \text{for all } s \in S .$$

We can rewrite the above formula for $< P(x)\phi, \phi>$ as

$$\lambda \|\phi\|^2 = \frac{1}{|S|} \frac{1}{n+1} \sum_{s \in S} [\sum_{i=1}^{n} (S_i \phi)(s)\overline{\phi(s)} + (K\phi)(s)\overline{\phi(s)}]$$

$$= \frac{1}{n+1} [\sum_{i=1}^{n} <S_i \phi, \phi> + <K\phi, \phi>] . \tag{2.15}$$

Therefore

$$|\lambda| \|\phi\|^2 = |< P(x)\phi, \phi>| \leq \frac{1}{n+1} [\sum_{i=1}^{n} |< S_i \phi, \phi>| + |<K\phi, \phi>|] . \tag{2.16}$$

Now using the Schwarz inequality we have

$$|< S_i \phi, \phi>| \leq \|S_i \phi\| \|\phi\| \qquad \text{for } i = 0, 1, \ldots, n \tag{2.17}$$

and

$$|<K\phi, \phi>| \leq \|K\phi\| \|\phi\| . \tag{2.18}$$

But

$$\|S_i\phi\| = \|\phi\| \qquad \text{for } i = 0,1,\dots, n$$

and

$$\|K\phi\| = \frac{1}{|S|}\left|\sum_{s\in S} e^{2\pi i(x,sr_0)}(S_0\phi)(s)\, e^{-2\pi i(x,sr_0)}\overline{(S_0\phi)(s)}\right|$$

$$= \frac{1}{|S|}\left|\sum_{s\in S}(S_0\phi)(s)\overline{(S_0\phi)(s)}\right|$$

$$= \|S_0\phi\| = \|\phi\| .$$

Therefore we have

$$|\langle S_i\phi,\phi\rangle| \le \|\phi\|^2 \text{ and } |\langle K\phi,\phi\rangle| \le \|\phi\|^2 .$$

By (2.16) we get

$$|\lambda|\,\|\phi\|^2 = |\langle P(x)\phi,\phi\rangle| \le \frac{1}{n+1}\left[\sum_{i=1}^{n}|\langle S_i\phi,\phi\rangle| + |\langle K\phi,\phi\rangle|\right]$$

$$\le \|\phi\|^2 .$$

Hence we conclude that λ cannot be greater than one and also that we have equality in (2.16), (2.17) and (2.18), if $\lambda = 1$

Now the equalities of (2.15) and (2.17) implies

(2.19) $\quad S_i(\phi) = \alpha_i\phi \quad$ for $i = 1, \dots, n$ and

(2.20) $\quad K(\phi) = \alpha_0\phi$

where $\alpha_i \in \mathbb{C}$, with $|\alpha_i| = 1$ for $i = 0, 1, \dots, n$.

From (2.15) we have

$$\|\phi\|^2 = \frac{1}{n+1}(\alpha_1 + \alpha_2 + \ldots + \alpha_n + \alpha_0)\|\phi\|^2$$

which implies

(2.21) $$\frac{1}{n+1}(\alpha_1 + \alpha_2 + \ldots + \alpha_n + \alpha_0) = 1 .$$

Using the equation (2.21) and the fact that $|\alpha_i| = 1$ for $i = 0, 1, \ldots, n$, we can conclude

$\alpha_i = 1$ for $i = 0, 1, \ldots, n$. (For the proof of this conclusion, see Appendix (A)).

Now by (2.19) we get

(2.22) $$(S_i\phi)(s) = \phi(ss_i) = \phi(s) \quad \text{for } i = 1, 2, \ldots, n .$$

In particular, for $s = e$, we have

$$\phi(s_i) = \phi(e) \quad \text{for } i = 1, 2, \ldots, n .$$

For $s = s_j$,

$$\phi(s_j s_i) = \phi(s_j) = \phi(e) \quad \text{for } i,j = 1, 2, \ldots, n .$$

But $\{s_i\}_{i=1}^{n}$ generates the group S .

Thus $\phi(s) = \phi(e)$ for all $s \in S$, which says that the function ϕ should be a constant function.

Now from (2.20) we have

$$(K\phi)(s) = e^{2\pi i(x, sr_0)}\phi(ss_0) = \phi(s) \quad \text{for all } s \in S .$$

Now since ϕ is a constant function,

$$e^{2\pi i(x,sr_0)} = 1 \quad \text{for all } s \in S .$$

In other words $(x,sr_0) \in \mathbb{Z}$ for all $s \in S$
which implies $(x,r) \in \mathbb{Z}$ for all $r \in \Delta$

Therefore

$$x \in L^* \text{ and hence } x = 0 \pmod{L^*} .$$

Thus we conclude that the eigenvalues of $P(x)$ for $x \neq 0$ are less than one in absolute value and hence the operator $(I - P(x))$ is invertible for $x \neq 0$. ∎

PROOF OF THEOREM 2.6

By (2.13), $\phi_0(x,s) = E_0(x)1_s(s)$ for all $s \in S$.
Taking the complex conjugate ,

$$(2.23) \qquad \overline{\phi_0(x,s)} = \overline{E_0(x)1_s(s)} = \overline{E_0(-x)1_s(s)} = \overline{\phi_0(-x,s)}$$

for all $x \in E$ and $s \in S$.
If we write

$$\phi_0(x,s) = \alpha(x,s) + i\beta(x,s) ,$$

by (2.23) we have

$$\alpha(x,s) - i\beta(x,s) = \alpha(-x,s) - i\beta(-x,s)$$

Comparing real and imaginary parts

$$\alpha(x,s) = \alpha(-x,s)$$

and $\quad \beta(x,s) = -\beta(-x,s) \quad$ for all $x \in E$.

Therefore, using Taylor's formula,

$$\alpha(x,s) = 1 + \alpha_2(x,s) + 0||x||^4$$

where $\alpha_2(x,s)$ is the 2^{nd} degree term with respect to x and

$$\beta(x,s) = \beta_1(x,s) + 0||x||^3$$

where $\beta_1(x,s)$ is the 1^{st} degree term with respect to x. Now we have

$$\text{(2.24)} \qquad P(x)\phi_0(x,s) = \lambda_0(x)\phi_0(x,s).$$

Using the definition of $P(x)$,

$$P(x)\phi_0(x,s) = \frac{1}{n+1}\left[\sum_{j=1}^{n} \phi_0(x,ss_j) + e^{2\pi i(x,sr_0)}\phi(x,ss_0)\right].$$

Therefore

$$\lambda_0(x)(\alpha(x,s) + i\beta(x,s)) = \frac{1}{n+1}\{[\sum_{j=1}^{n}(\alpha(x,ss_j) + i\beta(bx,ss\ ))] +$$

$$+ [\cos 2\pi(x,sr_0) + i\sin 2\pi(x,sr_0)][\alpha(x,ss_0) + i\beta(x,ss_0)]\}.$$

Comparing the real and imaginary parts,

(2.25)

$$\lambda_0(x)\alpha(x,s) = \frac{1}{n+1}\{[\sum_{j=1}^{n}\alpha(x,ss_j)] + \cos 2\pi(x,sr_0)\alpha(x,ss_0) -$$

$$- \sin 2\pi(x,sr_0)\beta(x,ss_0)\}$$

and

$$(2.26) \quad \lambda_0(x)\beta(x,s) = \frac{1}{n+1}\{[\sum_{j=1}^{n}\beta(x,ss_j)] + \cos 2\pi\,(x,sr_0)\beta(x,ss_0)$$

$$\sin 2\pi\,(x,sr_0)\alpha(x,ss_0)\} .$$

Now from (2.24) ,

$$\overline{P(x)\phi_0(x,s)} = \overline{\lambda_0(x)\phi_0(x,s)}$$

This implies

$$(2.27) \quad P(-x)\phi_0(-x,s) = \lambda_0(x)\phi_0(-x,s) \quad \text{for all } x \in E .$$

But again from (2.24)

$$(2.28) \quad P(-x)\phi_0(-x,s) = \lambda_0(-x)\phi_0(-x,s) \quad \text{for all } x \in E .$$

By (2.27) and (2.28)

$$\lambda_0(x) = \lambda_0(-x) \quad \text{for small } x \in E .$$

By Taylor's formula

$$(2.29) \quad \lambda_0(x) = 1 + \lambda_2(x) + O\|x\|^4 \quad \text{for small } x \in E$$

where $\lambda_2(x)$ is the 2^{nd} degree term with respect to x .
Now comparing the 2^{nd} degree terms in x in (2.25) ,

$$(2.30) \quad \alpha_2(x,s) + \lambda_2(x) = \frac{1}{n+1}\{[\sum_{i=1}^{n}\alpha_2(x,ss_i)] + \alpha_2(x,ss_0) -$$

$$- \left(\frac{2\pi}{2}(x,sr_0)\right)^2 - 2\pi(x,sr_0)\beta_1(x,ss_0)\} .$$

Comparing the 1^{st} degree terms in (2.26) ,

$$(2.31) \qquad \beta_1(x,s) = \frac{1}{n+1}\{[\sum_{i=1}^{n} \beta_1(x,ss_i)] + \beta_1(x,ss_0)\} + 2\pi(x,sr_0).$$

Equation (2.30) gives

$$(2.32) \qquad \sum_{s\in S} \alpha_2(x,s) + (n+1)!\lambda_2(x) = \frac{1}{n+1}\{\sum_{s\in S} \sum_{i=1}^{n}\alpha_2(x,ss_i)$$

$$+ \sum_{s\in S}\alpha_0(x,ss_0)\} - 2\pi^2\sum_{s\in S}(x,sr_0)^2 - 2\pi\sum_{s\in S}(x,sr_0)\beta_1(x,ss_0)$$

It is clear that there exists a unique vector b_s in E such that $\beta_1(x,s) = (b_s,x)$. Let us make the assumption that $\beta_1(x,s) = (sb,x)$ for a fixed vector b in E , which will be justified later.

Then using Schurs Orthogonality Relations (see [SER]) we have

$$\frac{1}{|S|} \sum_{s\in S} (x,sr_0)^2 = \frac{1}{n}(r_0,r_0)(x,x)$$

and

$$\frac{1}{|S|} \sum_{s\in S}(x,sr_0)(ss_0b,x) = \frac{1}{|S|} \sum_{s\in S}(s^{-1}x,r_0)(s^{-1}x,s_0b)$$

$$= \frac{1}{n}(r_0,s_0b)(x,x) .$$

Now by (2.32) we have

$$D + (n+1)!\lambda_2(x) = \frac{1}{n+1}\{(n+1)D - \frac{2\pi^2(n+1)!}{n}(r_0,r_0)(x,x)$$

$$+ \frac{2\pi(n+1)!}{n(n+1)}(r_o,b)(x,x)\}$$

where $D = \sum_{s\in S} \alpha_2(x,s)$.

This simplifies to

$$\lambda_2(x) = \frac{-2\pi^2(x,x)}{(n+1)n} + \frac{2\pi}{n(n+1)}(r_0,b)(x,x)$$

(2.33) $$= -(x,x)\left[\frac{2\pi^2}{(n+1)n} - \frac{2\pi}{(n+1)n}(r_0,b)\right] .$$

Now in (2.31) putting $\beta_1(x,s) = (sb,x)$ we have

$$(sb,x) = \frac{1}{n+1}\left\{\sum_{i=1}^{n}(ss_ib,x) + (ss_0b,x) + 2\pi(sr_0,x)\right\} .$$

Since this is true for all $x \in E$, we have

$$sb = \frac{1}{n+1}\left\{\sum_{i=1}^{n} ss_ib + ss_0b + 2\pi sr_0\right\}$$

and therefore

$$b = \frac{1}{n+1}\left\{\sum_{i=1}^{n} s_ib + s_0b + 2\pi r_0\right\} .$$

Now using the definition for s_i , $i = 0, 1, \ldots, n$

$$b = \frac{1}{n+1}\left\{\sum_{i=1}^{n}(b - (b,r_i)r_i) + (b - (b,r_0)r_0) + 2\pi r_0\right\}$$

$$= b - \frac{1}{n+1}\left\{\sum_{i=1}^{n}(b,r_i)r_i + (b,r_0)r_0 - 2\pi r_0\right\}$$

This gives

$$\sum_{i=1} (b,r_i)r_i = (2\pi - (b,r_0))r_0$$

(2.34) $$= (2\pi - (b,r_0))\sum_{i=1}^{n} r_i \, .$$

Comparing the coefficients of r_i , we get

(2.35) $$(b,r_i) = 2\pi - (b,r_0) \quad \text{for } i = 1, 2, \ldots, n \, .$$

Therefore

$$\sum_{i=1}^{n} (b,r_i) = n(2\pi - (b,r_0))$$

which gives

$$(n+1)(b,r_0) = 2\pi n$$

and hence

$$(b,r_0) = \frac{2\pi n}{n+1} \, .$$

Now let $\{r^i\}_{i=1}^{n}$ be the set of vectors in $\mathbb{R}^{n+1}$ satisfying $(r^j,r_i) = \delta_{ij}$.

By (2.35) we have

(2.36) $$b = (2\pi - \frac{2\pi n}{n+1}) \sum_{i=1}^{n} r^i = \frac{2\pi}{n+1} \sum_{i=1}^{n} r^i \, .$$

By (2.33)

$$\lambda_2(x) = -(x,x) \left[\frac{(2\pi)^2}{(n+1)n} - \frac{2\pi}{n} \frac{2\pi n}{(n+1)^2}\right] .$$

Thus

$$(2.37) \qquad \lambda_2(x) = -(x,x)\,\frac{(2\pi)^2}{n(n+1)^2} .$$

Finally (2.29) together with (2.36) gives the desired result.

Now let us verify our assumption for $\beta_1(x,s)$: that it can be written as (sb,x) for a vector $b \in E$.

For fixed x , write

$$\beta_1(x,s) = h(s) , \quad \text{so that } h \in \mathbb{C}^S .$$

Then by (2.31) we have

$$h(s) = \frac{1}{n+1}\{\sum_{i=1} h(ss_i) + h(ss_0) + k(s)\}$$

where $k(s) = 2\pi(x,sr_0)$ for fixed x .

This can be written as

$$(2.38) \qquad \{(I - P(0))h\}(s) = k(s) .$$

Since $(I - P(0))$ is invertible in the space $(1_s)^{\perp}$ which is the set of functions in $\mathbb{C}^S$ that are orthogonal to 1_s , the equation (2.38)

$(I - P(0))h = k$ has a unique solution in the space $(1_S)^{\perp}$.

Let us show that the function f defined by

$$f(s) = (sb,x) \quad \text{for fixed } x,$$

belongs to the space $(1_S)^{\perp}$.

$$f(s) = (sb,x) \quad \text{for all } s \in S$$

Thus

$$\sum_{s \in S} f(s) = \sum_{s \in S} (sb,x) .$$

Using (2.36) we get

$$\sum_{s \in S} f(s) = \frac{2\pi}{(n+1)} \sum_{s \in S} (s \sum_{i=1}^{n} r^i, x)$$

It can be shown that

$$\sum_{i=1}^{n} r^i = \tfrac{1}{2} \sum_{k=1}^{n} k(n - (k-1)) r_k \qquad \text{(see Appendix (C))}.$$

Let $\tfrac{1}{2}[k(n-(k-1))] = M(k)$.

We have

$$\sum_{s \in S} (s \sum_{i=1}^{n} r^i) = \sum_{s \in S} \sum_{k=1}^{n} M(k) s r_k$$

$$= \sum_{k=1}^{n} M(k) \sum_{s \in S} s r_k .$$

Now replacing s by ss_k in each summation, we have

$$\sum_{s\in S}\left(s\sum_{i=1}^{n} r^i\right) = \sum_{k=1}^{n} M(k) \sum_{s\in S} ss_k r_k$$

$$= -\sum_{k=1}^{n} M(k) \sum_{s\in S} sr_k$$

since $s_k r_k = -r_k$ for $k = 1, \ldots, n$.

Thus
$$\sum_{s\in S}\left(s\sum_{i=1}^{n} r^i\right) = -\sum_{s\in S}\left(s\sum_{i=1}^{n} r^i\right) ,$$

which implies

$$\sum_{s\in S}\left(s\sum_{i=1}^{n} r^i\right) = 0 .$$

Therefore we have

$$\sum_{s\in S} f(s) = \frac{2\pi}{(n+1)} \sum_{s\in S} \left(s\sum_{i=1}^{n} r^i , x\right) = 0 ,$$

and hence

$$f \in (1_S)^{\perp} .$$

Also we can show that the function h defined by $h(s) = \beta_1(s,x)$ for all $s \in S$ and for fixed x , is in the space $1_S^{\perp}$:

Let $\{E_i(x)\}$ be the set of projection operators on to the eigenspaces corresponding to the operator $p(x)$.

Then we have

$$I = E_0(x) + \sum_{i\neq 0} E_i(x)$$ where $E_0(x)$ is the projection operator corresponding to the eigenvalue $\lambda_0(x)$.

Therefore

$$I(1_S) = E_0(x)(1_s) + \sum_{i\neq 0} E_i(x)(1_s)$$

Now taking the inner product with $\phi_0(x) = E_0(x)1_S$, we have

$$<1_S, \phi_0(x)> = <\phi_0(x), \phi_0(x)>$$

since $<\phi_0(x), \phi_i(x)> = 0$ for $i \neq 0$.

Therefore

$<1_S, \phi_0(x)>$ is a real number.

But

$$<1_S, \phi_0(x)> = \frac{1}{|S|} \sum_{s\in S} \phi_0(x,s)$$

$$= \frac{1}{|S|} \sum_{s\in S} (\alpha(x,s) + i\beta(x,s)),$$

so we conclude that $\sum_{s\in S} \beta(x,s) = 0$.

Now comparing the 1^{st} degree terms in x, we get

$$\sum_{s\in S} \beta_1(x,s) = 0 .$$

Hence

$$h \in (1_S)^{\perp} .$$

Now $\beta_1(x,s)$ and (sb,x) both satisfy the equation (2.3)

$$(I - P(0))h(s) = k(s) .$$

But the solution of (2.3) should be unique in the space $(1_S)^{\perp}$, since the operator $(I - P(0))$ is invertible in that space. Therefore we conclude that $\beta_1(x,s)$ can be written as (sb,x) for a vector b in E . ∎

CHAPTER III

GENERALIZATION

This chapter is devoted to a generalization of our result for the random walk on the infinite group A. We shall be dealing with a general probability measure μ under the assumption that the support of μ generates the group A.

Since the techniques used in this generalization come from the theory of group representations, we start with a brief discussion of some of the general results of the Representation theory of groups.

§1. REPRESENTATIONS OF GROUPS

A representation ρ of a finite group G is a group homomorphism from G into the group of isomorphisms of a finite dimensional complex vector space V. A representation is irreducible or simple if V is not 0 and if no proper subspaces of V is stable under G. The dimension of V is called the degree of ρ. Two representations ρ, ρ' of the same group G in vector spaces V, V' are said to be similar or isomorphic if there exists a linear isomorphism

$\tau : V \to V'$ satisfying

$\tau \cdot \rho_s = \rho'_s \cdot \tau$ for all $s \in G$.

The representations L_s and R_s of G on the space of function $\mathbb{C}^G = \{f \mid f : G \to \mathbb{C}, \text{supp } f \text{ finite}\}$ defined by

(3.1) $(L_s f)(t) = f(s^{-1}t)$

and

(3.2) $(R_s f)(t) = f(ts)$ for all $s,t \in G$

are called the left regular representation and the right regular representation, respectively.

The complex valued function $\chi_\rho(s) = \mathrm{Tr}(\rho_s) =$ trace of ρ_s is called the character of the representation ρ . A character χ_ρ is called irreducible whenever ρ is.

$\mathbb{C}[G]$ is the set of all formal linear combinations $f = \sum_{s \in G} f(s)s$ (where f is a function on G) under pointwise addition and convolution multiplication.

Any representation V, ρ of G extends uniquely to a representation of $\mathbb{C}[G]$ by the rule

(3.3) $\rho(f) = \sum_{s \in G} f(s) \rho_s$

Let n_ρ be the degree of the irreducible representation ρ of G. Then the map

$$\mathbb{C}[G] \to \prod_{\rho \text{ irreducible}} M_{n_\rho}(\mathbb{C})$$

(where $f \to \tilde{f}$)

given by

(3.4) $\tilde{f}(\rho) = \rho(f)$ (matrix)

is the Fourier transform of the function f .

Let H be a subgroup of G, and W,σ be a representation of H. The induced representation

$$V,\rho = \mathrm{Ind}_H^G\, W,\sigma$$

is defined as

$$V = \{f \mid f: G \to W ,\ f(hg) = \sigma(h)f(g) \quad \text{for all } h \in H,\ g \in G\}$$

and

$$(\rho(g_0)f)(g) = f(gg_0) \quad \text{for all } g, g_0 \in G.$$

If G is an abelian group, all the irreducible representations of G have degree one.

If the group G is a semidirect product of an abelian group H and a group K, we can construct all the irreducible representations of G as induced representations of irreducibles of certain subgroups of K (by the method of 'little groups' of Wagner and Mackey) and these represent tions of A are unique (up to isomorphisms), (see [SER]),

§2. CONSTRUCTION OF THE IRREDUCIBLE REPRESENTATIONS OF THE GROUP A .

As we have shown in Chapter I, the group A is the semidirect product of the group L (which is a normal abelian subgroup of A) and the symmetric group S_{n+1}.

We use the method of 'little groups' to construct the irreducible representations of A.

Since the subgroup L is abelian, the irreducible characters of L are of degree one, and form a group

$$\bar{X} = \{\chi_x,\ x \in E \mid \chi_x(v) = e^{2\pi i (x,v)} \text{ for } v \in L\}$$

The group A acts on $\bar{X}$ by

(3.5) $(a\chi_x)(v) = \chi_x(ava^{-1})$ for $a \in A$, $\chi_x \in \bar{X}$ and $v \in L$.

Let S^x be the stabilizer of χ_x in S_{n+1}:

$$S^x = \{s \in S_{n+1} \; s\chi_x = \chi_x\} .$$

The little group method says that all the irreducible representations ρ_x of A is of the form $\mathrm{Ind}^{A}_{L \rtimes S^x}(\bar{\chi}_x \otimes \sigma)$ where $\chi_x \in \bar{X}$ and σ is an irreducible representation of S^x.

By (3.5)

$$s \cdot \chi_x = \chi_x \quad \text{implies} \quad \chi_x(sv) = \chi_x(v) \quad \text{for all } v \in L .$$

Therefore

$$S^x = \{s \in S_{n+1} | \chi_x(sv) = \chi_x(v) \quad \text{for all} \quad v \in L\}$$

$$= \{s \in S_{n+1} | \; e^{2\pi i(x,sv)} = e^{2\pi i(x,v)} \quad \text{for all} \quad v \in L\}$$

$$= \{s \in S_{n+1} | \; (x, sv - v) \in \mathbb{Z} \quad \text{for all} \quad v \in L\}$$

Now, if x is inside a chamber,

$$(x, sv - v) = (x, sv) - (x, v) \notin \mathbb{Z}$$

Hence, for an x inside a chamber,

$$S^x = \{e\} \quad \text{where } e \text{ is the identity of } S_{n+1} .$$

Therefore the only representation of S^x is the identity representation 1.

Define

(3.6) $\rho_x = \mathrm{Ind}^{A}_{L}(\chi_x \otimes 1)$ where $\otimes$ denotes the tensor product of representation.

Then ρ_x is a representation of the group A, and it acts on the space V of functions where

$$V = \{f \mid f \in \mathbb{C}^A,\ f(v,s) = \chi_x(v) f(0,s) \text{ for } v \in L, \text{ and } s \in S_{n+1}\}$$

Now $\rho_x(a)$ $(a \in A)$ acts on $\mathbb{C}^{S_{n+1}}$ as follows:

Let $\phi \in \mathbb{C}^{S_{n+1}}$, $v \in L$ and $e,s,t \in S_{n+1}$

Then

$$(3.7)\qquad \begin{aligned}(\rho_x(v,e)\phi)(s) = (\rho_x(v,e)\phi)(0,s) &= \phi((0,s)\ (v,e)) \\ &= \phi(sv,s) \\ &= e^{2\pi i(x,sv)}\phi(0,s) \\ &= e^{2\pi i(x,sv)}\phi(s)\end{aligned}$$

and

$$(3.8)\qquad (\rho_x(0,t)\phi)(s) = \phi((0,s)\cdot(0,t)) = \phi(0,st) = \phi(st)$$

Hence for $a = (v,t) \in A$ we have

$$\begin{aligned}(\rho_x(a)\phi)(s) &= (\rho_x((v,e)\cdot(0,t))\phi)(0,s) \quad \text{for all } s \in S_{n+1} \\ &= \rho_x(v,e)\phi(st) \\ &= e^{2\pi i(x,sv)}\phi(st) \quad \text{for all } s \in S_{n+1}\,.\end{aligned}$$

We summarize the discussion in this section as follows:

For each x inside a chamber, we have an irreducible representation ρ_x of A which acts on the space $\mathbb{C}^{S_{n+1}}$ according to the rule

$$(3.9) \qquad (\rho_x(v,t)\phi)(s) = e^{2\pi i(x,sv)}\phi(st)$$

where $\phi \in \mathbb{C}^{S_{n+1}}$, $v \in L$ and $s,t \in S_{n+1}$.

§3. GENERALIZATION

Let μ be a probability measure on the group A and assume that the support of μ generates the infinite group A.

The probability of the particle visiting the identity after

$$\begin{aligned} m \text{ steps} &= \mu^{\times m}(0,e) \\ &= (\mu^{\times m} \times \varepsilon_1)(o,e) \\ &= \int_{E/L^*} (\mu^{\times m} \times \varepsilon_1)^\wedge(x,e)\, dx , \end{aligned}$$

where $(\mu^{\times m} \times \varepsilon_1)^\wedge(x,s)$ is the Fourier transform of the function $(\mu^{\times m} \times \varepsilon_1)(v,s)$, with respect to the variable $v \in L$.

Thus

$$\mu^{\times m}(0,e) = \int_{E/L^*} (\mu^{\times m})^\wedge(x,e)\, \varepsilon_1^\wedge(x,e)\, dx .$$

By (2.8) this can be simplified to

$$\mu^{\times m}(0,e) = \int_{E/L^*} (\mu^{\times m})^\wedge(x,e)\,dx \;.$$

Now since $(\mu^{\times m})^\wedge(x,s) = \sum_{v\in L} \mu^{\times m}(v,s)e^{-2\pi i(x,v)}$,

we have

$$\mu^{\times m}(0,e) = \int_{E/L^*} \sum_{v\in L} (\mu^{\times m})(v,e)e^{-2\pi i(x,v)}\,dx$$

Changing the variable of integration from x to $-x'$,

$$\begin{aligned}\mu^{\times m}(0,e) &= \int_{E/L^*} \sum_{v\in L}(\mu^{\times m})(v,e)e^{2\pi i(x,v)}\,dx \\ &= \int_{E/L^*} \sum_{(v,s)\in A}\mu^{\times m}(v,s)e^{2\pi i(x,v)}\;\varepsilon_e(s)\,dx \;.\end{aligned}$$

By (3.9) ,

$$(\rho_x(v,s)\varepsilon_e)(e) = e^{2\pi i(x,v)}\,\varepsilon_e(s) \;.$$

Therefore

$$\mu^{\times m}(0,e) = \int_{E/L^*} \sum_{a\in A} (\mu^{\times m}(a)\rho_x(a)\varepsilon_e)(e)\,dx \;.$$

By (3.3) and (3.4)

$$\mu^{\times m}(0,e) = |S_{n+1}|\int_{E/L^*} \langle(\tilde{\mu}(\rho_x))^m\varepsilon_e,\varepsilon_e\rangle dx \;.$$

Now changing the variable x to sx , we have

$$\mu^{\times m}(0,e) = |S_{n+1}| \int_{E/L^*} < (\tilde{\mu}(\rho_{xs}))^m \varepsilon_e, \varepsilon_e > dx$$

and therefore

$$(3.10) \qquad \mu^{\times m}(0,e) = \sum_{s \in S_{n+1}} \int_{E/L^*} < (\tilde{\mu}(\rho_{sx}))^m \varepsilon_e, \varepsilon_e > dx \ .$$

It can be shown that

$$(3.11) \qquad \rho_{sx}(a) = L_s \rho_x(a) L_{s^{-1}} \ , \text{ for all } s \in S_{n+1} \text{ and } a \in A \ ,$$

where L_s is the regular representation defined by (3.1) :

For, if $\phi \in \mathbb{C}^{S_{n+1}}$ and $a = (v',s') \in A$,

$$[(L_s \rho_x(a) L_{s^{-1}}) \phi](t) = [L_s \rho_x(a)(L_{s^{-1}}\phi)](t)$$

for $t,s \in S_{n+1}$.

Applying (3.9) for the function $L_{s^{-1}}\phi$,

$$[L_s \rho_x(a)(L_{s^{-1}}\phi)](t) = L_s[e^{2\pi i(x,tv')}(L_{s^{-1}}\phi)(ts')]$$

Now using the definition of L_s ,

$$L_s[^{2\pi i(x,tv')}(L_{s^{-1}}\phi)(ts')] = e^{2\pi i(x,s^{-1}tv')}(L_{s^{-1}}\phi)(s^{-1}ts')$$

$$= e^{2\pi i(sx,tv')}\phi(ts')$$

$$= (\rho_{sx}(v',s')\phi)(t) \ , \text{ by } (3.9) \ .$$

Thus

$$[(L_s \rho_x(a) L_{s^{-1}})\phi](t) = (\rho_{sx}(v', s')\phi)(t) = (\rho_{sx}(a)\phi)(t) .$$

Since this is true for all $t \in S_{n+1}$ and $\phi \in \mathbb{C}^{S_{n+1}}$,

we have

$$L_s \rho_x(a) L_{s^{-1}} = \rho_{sx}(a) \quad \text{for all} \quad a \in A \quad \text{and} \quad s \in S_{n+1} .$$

Now by (3.3)

$$\rho_{sx}(\mu^{\times m}) = \sum_{a \in A} \mu^{\times m}(a) \rho_{sx}(a)$$

$$= \sum_{a \in A} \mu^{\times m}(a) L_s \rho_x(a) L_{s^{-1}} , \text{ by } (3.11) .$$

Therefore

$$(3.12) \qquad (\tilde{\mu}(\rho_{sx}))^m = \rho_{sx}(\mu^{\times m}) = L_s(\tilde{\mu}(\rho_x))^m L_{s^{-1}} .$$

Now from (3.10) and (3.12) we have

$$\mu^{\times m}(0,e) = \sum_{s \in S_{n+1}} \int_{E/L^*} \langle L_s(\tilde{\mu}(\rho_x))^m L_{s^{-1}} \varepsilon_e, \varepsilon_e \rangle \, dx$$

$$= \sum_{s \in S_{n+1}} \int_{E/L^*} \langle (\tilde{\mu}(\rho_x))^m L_{s^{-1}} \varepsilon_e, L_{s^{-1}} \varepsilon_e \rangle \, dx$$

Now

$$(L_{s^{-1}} \varepsilon_e)(t) = \varepsilon_e(st) = \begin{cases} 1 & \text{if } st = 1 , \\ 0 & \text{otherwise} . \end{cases}$$

Hence

$$L_{s^{-1}} \varepsilon_e = \varepsilon_{s^{-1}} \quad \text{for all } s \in S_{n+1} .$$

Therefore

$$\mu^{\times m}(0,e) = \sum_{s \in S_{n+1}} \int_{E/L^*} \langle (\tilde{\mu}(\rho_x))^m \varepsilon_{s^{-1}}, \varepsilon_{s^{-1}} \rangle \, dx .$$

For the vector space $\mathbb{C}^{S_{n+1}}$ the set $\{\varepsilon_s\}_{s \in S_{n+1}}$ form a basis.

Therefore the trace of $(\tilde{\mu}(\rho_x))^m$,

$$\operatorname{tr}(\tilde{\mu}(\rho_x))^m = |S_{n+1}| \sum_{s \in S_{n+1}} \langle (\tilde{\mu}(\rho_x))^m \varepsilon_{s^{-1}}, \varepsilon_{s^{-1}} \rangle \, dx .$$

Hence we have

$$(3.13) \quad \mu^{\times m}(0,e) = \frac{1}{|S_{n+1}|} \int_{E/L^*} \operatorname{tr}(\tilde{\mu}(\rho_x))^m \, dx .$$

More generally, the formula (3.13) is valid for any function $f: A \to \mathbb{C}$ in place of μ.

Furthermore, we can obtain a formula for $\mu^{\times m}(a)$ for any element $a \in A$, using the above formula (3.13):

Observe that $\mu^{\times m}(a) = (R_a(\mu^{\times m}))(0,e)$ where R_a is the right regular representation.

Using (3.13) we get

$$\mu^{\times m}(a) = (R_a(\mu^{\times m}))(0,e) = \frac{1}{|S_{n+1}|} \int_{E/L^*} \operatorname{tr}(\rho_x(R_a \mu^{\times m})) \, dx .$$

By (3.2) and (3.3)

$$\rho_x(R_a\mu^{\times m}) = \sum_{b\in A}(R_a(\mu^{\times m}))(b)\rho_x(b)$$

$$= \sum_{b\in A}\mu^{\times m}(ba)\rho_x(b) \ .$$

Letting $ba = c$,

$$\rho_x(R_a\mu^{\times m}) = \sum_{c\in A}\mu^{\times m}(c)\rho_x(ca^{-1})$$

$$= \sum_{c\in A}\mu^{\times m}(c)\rho_x(c)\rho_x(a^{-1})$$

$$= \rho_x(\mu^{\times m})\ \rho_x(a^{-1}) \ .$$

Therefore we have

(3.14) $$\mu^{\times m}(a) = \frac{1}{|S_{n+1}|}\int_{E/L^*} tr((\tilde{\mu}(\rho_x))^m\rho_x(a^{-1}))\,dx \ .$$

Now our intention is to derive a formula for the expected number of times the particle visits the point a .

But this expected number is equal to $\sum_{m=0}^{\infty}\mu^{\times m}(a)$.

Hence by (3.14)

(3.15) $$\sum_{m=0}^{\infty}\mu^{\times m}(a) = \frac{1}{|S_{n+1}|}\lim_{\theta\uparrow 1}\sum_{m=0}^{\infty}\int_{E/L^*}\theta^m tr((\tilde{\mu}(\rho_x))^m\rho_x(a^{-1}))\,dx ,$$

$0 \le \theta < 1$.

Now we can interchange the summation and integration if $\|\tilde{\mu}(\rho_x)\| \le 1$.

Therefore, our next attempt is to show that the operator $\tilde{\mu}(\rho_x)$ does not have eigenvalues greater than or equal to one, in absolute value.

By the definition,

$$\tilde{\mu}(\rho_x) = \rho_x(\mu) = \sum_{a\in A} \mu(a)\rho_x(a)$$

and we have that $\mu(a) \ge 0$ for all $a \in A$, $\sum_{a\in A}\mu(a) = 1$ and the support of $\mu = \{a\in A \mid \mu(a) \ne 0\}$ generates A . Let us assume that $\tilde{\mu}(\rho_x)$ has a nonzero eigenvector f with eigenvalue $|\lambda| \ge 1$:

(3.16) $\quad \tilde{\mu}(\rho_x)f = \lambda f \ , \ |\lambda| \ge 1$.

Using the definition of $\tilde{\mu}(\rho_x)$, (3.16) becomes

$$\Big(\sum_{(v,s)\in A} \mu(v,s)\rho_x(v,s)f\Big)(t) = \lambda f(t) \quad \text{for all } t \in S_{n+1} .$$

which simplifies to

(3.17) $\quad \displaystyle\sum_{(v,s)\in A} \mu(v,s)e^{2\pi i(x,tv)}f(ts) = \lambda f(t)$.

Let $\max_{t\in S_{n+1}} |f(t)| = |f(t_0)|$ for $t_0 \in S_{n+1}$.

Letting $t = t_0$ in (3.17) ,

(3.18) $$\sum_{(v,s)\in A} \mu(v,s)e^{2\pi i(x,t_0v)} f(t_0s) = \lambda f(t_0) .$$

Since $f \neq 0$, $|f(t_0)| \neq 0$.

Dividing (3.18) by $f(t_0)$ we have

$$\sum_{(v,s)\in A} \mu(v,s)e^{2\pi i(x,t_0v)} \frac{f(t_0s)}{f(t_0)} = \lambda .$$

Consequently

$$|\lambda| = \left| \sum_{(v,s)\in A} \mu(v,s)e^{2\pi i(x,t_0v)} \frac{f(t_0s)}{f(t_0)} \right|$$

(3.19) $$\leq \sum_{(v,s)\in A} \mu(v,s) \left| e^{2\pi i(x,t_0v)} \frac{f(t_0s)}{f(t_0)} \right| .$$

But $$\left| e^{2\pi i(x,t_0v)} \frac{f(t_0s)}{f(t_0)} \right| < 1$$

Therefore if we have

$$\left| e^{2\pi i(x,t_0v)} \frac{f(t_0s)}{f(t_0)} \right| < 1$$

for even only one element $(v,s) \in A$ with $\mu(v,s) \neq 0$, the inequality (3.19) becomes

$$|\lambda| = \left| \sum_{(v,s)\in A} \mu(v,s)e^{2\pi i(x,t_0v)} \frac{f(t_0s)}{f(t_0)} \right| \leq \sum_{(v,s)\in A} \mu(v,s) \left| e^{2\pi i(x,t_0v)} \frac{f(t_0s)}{f(t_0)} \right| < 1$$

which is impossible, since we have assumed $|\lambda| \geq 1$. Therefore for all $(v,s) \in A$ for which $\mu(v,s) \neq 0$ we have

$$(3.20) \qquad \left| e^{2\pi i(x,t_0 v)} \frac{f(t_0 s)}{f(t_0)} \right| = 1$$

and also the eigenvalues λ of $\tilde{\mu}(\rho_x)$ cannot be greater than one in absolute value. Hence $\|\tilde{\mu}(\rho_x)\| \leq 1$.

Also, the eigenvalues of the operator $\tilde{\mu}(\rho_x)$ are strictly less than one. For, suppose the operator $\tilde{\mu}(\rho_x)$ has the eigenvector f_0 corresponding to the eigenvalue one. From (3.19) we have

$$\sum_{(v,s)\in A} \mu(v,s) e^{2\pi i(x,t_0 v)} \frac{f_0(t_0 s)}{f_0(t_0)} = 1$$

and from (3.20)

$$\left| e^{2\pi i(x,t_0 v)} \frac{f_0(t_0 s)}{f_0(t_0)} \right| = 1 \quad \text{for } (v,s) \in A \text{ for which}$$

$\mu(v,s) \neq 0$.

Therefore we conclude that

$$e^{2\pi i(x,t_0 v)} \frac{f_0(t_0 s)}{f_0(t_0)} = 1 \quad \text{for all } (v,s) \in A \text{ with}$$

$\mu(v,s) \neq 0$. (For this conclusion see Appendix (A)).

Hence for all (v,s) in A for which $\mu(v,s) \neq 0$, we have

$$(3.21) \qquad e^{2\pi i(x,t_0v)} f_0(t_0s) = f(t_0) .$$

Now the equation

$$\tilde{\mu}(\rho_x) f_0 = f_0$$

implies

$$L_u \mu(\rho_x) L_{u^{-1}} L_u f_o = L_u f_0 \quad \text{for all} \quad u \in S_{n+1} .$$

From (3.11) we get

$$\tilde{\mu}(\rho_{ux}) L_u f_0 = L_u f_o .$$

This shows that $L_u f_0$ is an eigenvector of $\tilde{\mu}(\rho_{ux})$ with the eigenvalue one.

Replacing f_0 by $L_u f_o$ in (3.21), we have

$$f_0(u^{-1}t_0s) = e^{-2\pi i(ux,t_0v)} f_0(u^{-1}t_0) \quad \text{for all } u \in S_{n+1} .$$

Letting $u^{-1}t_0 = w$

$$f_0(ws) = e^{-2\pi i(x,wv)} f_0(w) \quad \text{for all} \quad w \in S_{n+1} .$$

This gives

$$(3.22) \qquad |f_0(ws)| = |f_0(w)| \qquad \text{for all} \quad w \in S_{n+1} \quad \text{and for}$$

all $s \in S_{n+1}$ with $\mu(v,s) = 0$.

Using (3.2) we can rewrite (3.22) as

$$|R_s f_o(w)| = |f_0(w)|$$

which is the same as to say

$$R_s|f_0|(w) = |f_0|(w) \quad \text{for all} \quad w \in S_{n+1} .$$

Therefore for all s in S_{n+1} with $\mu(v,s) \neq 0$ for some $v \in L$, we have

$$R_s|f_0| = |f_0| .$$

Now according to our assumption on the measure μ, the set $\{a \in A | \mu(a) \neq 0\}$ generates $A = L \times S_{n+1}$

Also

$$S_{n+1} \approx A/L$$

Therefore the set $\{ s \in S_{n+1} | \mu(v,s) \neq 0$ for some $v \in L\}$ generates S_{n+1}.

Now let $s_1, s_2 \in S_{n+1}$ and suppose that $\mu(v_1,s_1)$ and $\mu(v_2,s_2)$ are non zero for some $v_1, v_2 \in L$. We have

$$R_{s_1}|f_0| = |f_0|$$

$$R_{s_2}|f_0| = |f_0|$$

and therefore

$$R_{s_1 s_2}|f_0| = R_{s_1}R_{s_2}|f_0| = |f_0| \ .$$

Hence we have

$$R_s|f_0| = |f_0| \quad \text{for all} \quad s \in S_{n+1} \ .$$

In other words

$$|f_0(s)| = |R_s f_o(e)| = |f_0(e)| \ .$$

Thus

$$|f_0(s)| = |f_0(e)| \qquad \text{for all} \quad s \in S_{n+1} \ .$$

Now again using the equation (3.17) with $\lambda = 1$ and $f = f_0$, we have

$$\sum_{(v,s)\in A} \mu(v,s) e^{2\pi i(x,tv)} \frac{f_0(ts)}{f_0(t)} = 1 \qquad \text{for all} \quad t \in S_{n+1}$$

and also

$$\left| e^{2\pi i(x,tv)} \frac{f_0(ts)}{f_0(t)} \right| = \left| \frac{f_0(ts)}{f_0(t)} \right| = 1 \quad \text{for all} \quad t \in S_{n+1} \ .$$

This, together with the fact that $\sum_{(v,s)\in A} \mu(v,s) = 1$ implies

$$e^{2\pi i(x,tv)} \frac{f_0(ts)}{f_0(t)} = 1 \quad \text{for all} \quad t,s \in S_{n+1} \quad \text{with} \quad \mu(v,s) \neq 0$$

for some $v \in L$. (see Appendix A).

Now this equation can be written as

(3.23) $\mu_x(a) f_0 = f_0$ where $a = (v,s) \in A$ with $\mu(v,s) \neq 0$

Let b be any element in A. Since the support of $\tilde{\mu}$ generates A,

$$\rho_x(b) f_0 = \rho_x(a_1 \dots a_k) f_0 \text{ , where } \mu(a_i) \neq 0 \text{ for } i = 1, \dots, k .$$

Therefore by (3.23)

$$\rho_x(b) f_0 = \rho_x(a_1) \dots \rho_x(a_k) f_0 = f_0 .$$

Hence we have

$$\rho_x(b) f_0 = f_0 \quad \text{for all } b \in A$$

This implies that the subspace of $\mathbb{C}^{S_{n+1}}$ generated by the function f_0 is a non trivial invariant subspace, which contradicts the fact that ρ_x is irreducible.

Therefore we conclude that the operator $\tilde{\mu}(\rho_x)$ does not have an eigenvector with the eigenvalue one.

Hence

$$\|\tilde{\mu}(\rho_x)\| < 1 .$$

Now by (3.15) we have

$$\sum_{m=0}^{\infty} \mu^{\times m}(a) = \lim_{\theta \uparrow 1} \sum_{m=0}^{\infty} \frac{1}{|S_{n+1}|} \int_{E/L^*} \theta^m \operatorname{tr}((\tilde{\mu}(\rho_x))^m \rho_x(a^{-1}))\,dx$$

$$= \lim_{\theta \uparrow 1} \frac{1}{|S_{n+1}|} \int_{E/L^*} \operatorname{tr}\left(\sum_{m=0}^{\infty} \theta^m (\tilde{\mu}(\rho_x))^m \rho_x(a^{-1})\right) dx$$

which leads us to the equation

$$\sum_{m=0}^{\infty} \mu^{\times m}(a) = \frac{1}{|S_{n+1}|} \lim_{\theta \uparrow 1} \int_{E/L^*} \operatorname{tr}[(I - \theta\tilde{\mu}(\rho_x))^{-1} \rho_x(a^{-1})]\,dx$$

We state our conclusion as follows.

THEOREM 3.1

Let μ *be a probability measure on the infinite group* A *whose support is finite. Then the random walk induced by* μ *is recurrent or transitory accordingly as*

$$\lim_{\theta \uparrow 1} \frac{1}{|S_{n+1}|} \int_{E/L^*} \operatorname{tr}[(I - \theta\tilde{\mu}(\rho_x))^{-1}\,dx$$

is finite or infinite, respectively.

If μ is the probability measure of Chapter II, this is equivalent to Theorem 2.3 (See Appendix D).

APPENDIX

(A).

Let $\{\mu_n\}_{n=1}^{N}$ be a set of positive real numbers with $\sum_{n=1}^{N} \mu_n = \mu_0$, where μ_0 is any positive real number.

Suppose that $\sum_{n=1}^{N} \mu_n e^{i\theta_n} = \mu_0$, where θ_n is a real number depending on n. Then it can be shown that each $e^{i\theta_n} = 1$ for $n = 1, \ldots, N$: For, suppose we have the two conditions

(A.1) $\sum_{n=1}^{N} \mu_n e^{i\theta_n} = \mu_0$ and $\sum_{n=1}^{N} \mu_n = \mu_0$. We use induction on N to prove this result. The case is trivial for $N = 1$. Assume it is true for $N - 1$.

By (A.1) we have

(A.2) $$\mu_0 = \left| \sum_{n=1}^{N} \mu_n e^{i\theta_n} \right| \leq \left| \sum_{n=1}^{N-1} \mu_n e^{i\theta_n} \right| + \mu_N \leq \mu_0$$

Hence we have the equality in (A.2):

(A.3) $$\left| \sum_{n=1}^{N-1} \mu_n e^{i\theta_n} \right| + \mu_N = \mu_0 ,$$

From (A.3) we have

$$\left| \sum_{n=1}^{N-1} \mu_n e^{i\theta_n} \right| = |\mu_0 - \mu_N|$$

and from (A.1)

$$\left|\sum_{n=1}^{N-1} \mu_n e^{i\theta_n}\right| = |\mu_0 - \mu_N e^{i\theta_N}|$$

Hence

$$|\mu_0 - \mu_N e^{i\theta_N}| = |\mu_0 - \mu_N|$$

This implies $e^{i\theta_N} = 1$

Therefore

$$\text{(A.4)} \qquad \sum_{n=1}^{N-1} \mu_n e^{i\theta_n} = \mu_0 - \mu_N$$

Now (A.4) along with the fact that $\sum_{n=1}^{N-1} \mu_n = \mu_0 - \mu_N$ gives us the result that

$$e^{i\theta_n} = 1 \quad \text{for each } n = 1, \ldots, N-1$$

using the induction assumption.

Therefore we have $e^{i\theta_n} = 1$ for all $i = 1, \ldots N$.

Hence by mathematical induction, the result is true for all $N \in \mathbb{Z}$.

∎

(B).

The operator $E_0(x)$ is defined by

$$E_0(x) = \frac{1}{2\pi i} \oint_C (zI - P(x))^{-1} \, dz$$

where C is a closed curve in $\mathbb{C}$ with the eigenvalue $\lambda_0(x)$ in the interior. We claim that there is only one eigenvalue of $P(x)$ which tends to one as x tends to zero; so that the curve C can be chosen in such a way that it does not contain points corresponding to any other eigenvalue except $\lambda_0(x)$.

We prove our claim as follows:

Consider the polynomial

(B.1) $\det(tI - P(x)) = \prod_i (t - \lambda_i(x))$ of t, where $\lambda_i(x)$ are the eigenvalues of $P(x)$.

$$\lim_{x \to 0} \det(tI - P(x)) = \lim_{x \to 0} \prod_i (t - \lambda_i(x))$$

(B.2) $$\det(tI - P(0)) = \prod_i (t - \lambda_i(0)) .$$

If we have more than one eigenvalue of $P(x)$ which tends to one as x tends to zero, we have

(B.3) $\det(tI - P(0)) = (t-1)^r (t - \lambda_i(0))$ where $r > 1$.

But we have shown that $P(0)$ has only one linearly-independent eigenvector (namely 1_s) corresponding to the eigenvalue one.

Therefore in (B.3), $r = 1$ and hence there is only one eigenvalue $\lambda_0(x)$ that tends to one as x tends to zero.

∎

(C).

Δ is the system of roots in $\mathbb{R}^{n+1}$:

$\Delta = \{e_i - e_j \mid 1 \le i,j \le n\}$ where $\{e_i\}$ form a canonical basis in $\mathbb{R}^{n+1}$.

and $\{r^i\}_{i=1}^n$ is the set of vectors in E satisfying

(C.1) $(r^i, r_j) = \delta_{ij}$ for $1 \le i,j \le n$.

Then

$$\sum_{i=1}^{n} r^i = \tfrac{1}{2}\sum_{k=1}^{n} k(n-(k-1))r_k :$$

For,

$$\text{let } r^i = \sum_{j=1}^{n+1} \lambda_j^i e_j .$$

Since $r^i \in E$,

$$\text{(C.2)} \qquad \sum_{j=1}^{n+1} \lambda_j^i = 0 .$$

Using (C.1) and (C.2), we get

$$r^i = \frac{(n-(i-1))}{(n+1)} \sum_{k=1}^{i} e_k - \frac{i}{n+1} \sum_{k=i+1}^{n+1} e_k$$

Therefore

$$(n+1)\sum_{i=1}^{n} r^i = \sum_{i=1}^{n} [(n-(i-1))\sum_{k=1} e_k - i \sum_{k=i+1}^{n+1} e_k]$$

$$= \sum_{k=1}^{n} e_k \sum_{i=1}^{n}(n-(i-1)) - \sum_{k=2}^{n+1} e_k \sum_{i=1}^{k-1} i$$

$$= \tfrac{1}{2}\sum_{k=1}^{n} e_k (n-k+2)(n-k+1) - \tfrac{1}{2}\sum_{k=2}^{n+1} e_k (k-1)k$$

(C.3)
$$= \tfrac{1}{2}\sum_{k=1}^{n+1} (n-2(k-1))e_k$$

Now subtracting and adding e_k's with appropriate coefficients from the left hand side of (C.3), we have

$$\sum_{i=1}^{n} r^i = \tfrac{1}{2}\sum_{k=1}^{n} k(n-(k-1))r_k .$$ ∎

(D).

Let the probability measure μ of the random walk be defined as in chapter II. Then the function μ can be written as

$$\mu = \frac{1}{n+1} \{\sum_{i=1}^{n} \varepsilon_{(0,s_i)} + \varepsilon_{(r_o,s_o)}\}$$

where $\varepsilon_{(0,s_i)}$, $i = 1, \dots n$ and $\varepsilon_{(r_0,s_o)}$ are the indicator functions of the elements $(0,s_i)$; $i = 1, \dots, n$, and (r_0,s_0) of A.

Then $\tilde{\mu}(\rho_x) = \rho_x(\mu) = \sum_{a\in A} \mu(a)\rho_x(a)$

$$= \frac{1}{n+1} \sum_{a\in A} (\varepsilon_{(0,s_1)}(a) + \ldots + \varepsilon_{(0,s_n)}(a) + \varepsilon_{(r_0,s_0)}(a))\rho_x(a)$$

$$= \frac{1}{n+1} [\sum_{i=1}^{n} \rho_x(0,s_i) + \rho_x(r_0,s_0)]$$

Therefore, for $\phi \in \mathbb{C}^{S_{n+1}}$

$$(\tilde{\mu}(\rho_x)\phi)(s) = \frac{1}{n+1}\{[\sum_{i=1}^{n} \rho_x(0,s_i) + \rho_x(r_0,s_0)]\phi\}(s)$$

Using (3.9) we have

$$(\tilde{\mu}(\rho_x)\phi)(s) = \frac{1}{n+1} \{\sum_{i=1}^{n} \phi(ss_i) + e^{2\pi i(x,sr_0)} \phi(ss_0)\}$$

$$= (P(x)\phi)(s)$$

Since this is true for all $\phi \in \mathbb{C}^{S_{n+1}}$, we have

$$\tilde{\mu}(\rho_x) = \rho(x) \quad \text{for all } x \text{ inside a chamber.}$$

Hence the integrand given in the Theorem 3.1 becomes

$$\lim_{\theta\uparrow 1} \frac{1}{|S_{n+1}|} \int_{E/L^*} \operatorname{tr}(I - \theta P(x))^{-1} dx$$

which is equal to

$$\lim_{\theta\uparrow 1} \int_{E/L^*} \sum_{s\in S_{n+1}} < (I - \theta P(x))^{-1}\varepsilon_s, \varepsilon_s > dx$$

$$= \lim_{\theta\uparrow 1} \int_{E/L^*} |S_{n+1}| < (I - \theta P(x))^{-1} \varepsilon_e, \varepsilon_e > dx \quad \text{(as on pp.59–60)}$$

which is the same as the equation (2.11) in Chapter II.

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