

A MODIFIED ITERATIVE PROCEDURE FOR
COMPUTING TIME -OPTIMAL CONTROLS FOR LINEAR SYSTEMS

by

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TO MY MOTHER

ABSTRACT

This thesis presents a modified iterative procedure for computing the time-optimal control for linear dynamical systems with amplitude constraints on the control. The procedure is based on a new iterative algorithm for computing the minimum of a quadratic form on a convex set. The iterative procedure is free from any empirical methods for step-size evaluation and has rapid exponential convergence.

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TABLE OF CONTENTS

	Page
ABSTRACT	--- i
ACKNOWLEDGEMENTS	--- ii
<u>CHAPTER I</u> . INTRODUCTION	--- 1
<u>CHAPTER II</u> BASIC THEORY OF TIME-OPTIMAL CONTROL AND ITS COMPUTATIONAL ALGORITHM.	
2.1 Statement of the Problem	--- 5
2.2 Basic Principle of Computational Algorithm	--- 12
<u>CHAPTER III</u> . ITERATIVE ALGORITHM AND COMPUTATIONAL RESULTS .	
3.1 The Algorithm	--- 15
3.2 Pecsvaradi and Narendra's Minimization Procedure.	--- 21
3.3 The Iteration Steps.	--- 28
3.4 Proof of Convergence	--- 36
3.5 Computational Results.	--- 41
CONCLUSIONS .	--- 62
REFERENCES.	--- 63
VITA.	--- 67

CHAPTER I

INTRODUCTION

Recent advances in engineering and science, specially in space technology, have stimulated much interests in time-optimal control problems. The availability of modern highspeed general purpose computers make it feasible to consider the actual computation of these controls. Several computational methods for time-optimal controls for systems whose motions can be described by a set of ordinary linear differential equations are now available. The main theme of this thesis is the improvement of a method which is referred to as convexity method, applicable to linear time-optimal control problems.

Since early fifties problems of this type began to receive considerable attention with the works of Bushaw[8], Feldbaum[12], Hopkins[19] and others. The time-optimal control problem was extensively studied by mathematicians in United States and Soviet Union. The outstanding works of La Salle and others helped the development of the basic theory of time-optimal control problems in the period from 1953 to 1957. La Salle [22] presented the general results concerning the existence and uniqueness of time-optimal controls for such problems. Classical variational theory could not readily handle the typical constraints usually encountered in optimal control problems. This difficulty was completely removed after the enunciation of Pontryagin's celebrated maximum principle [28] in 1958. This pioneering work of Pontryagin actually has established the basis of modern control theory. While the maximum principle may be viewed as an outgrowth of Hamiltonian approach to variational problems, the method of dynamic programming, which was developed by Bellman[6] around 1953-1957, may be viewed as an outgrowth of the Hamilton-Jacobi approach.

The necessity of actual computation of time-optimal controls of physical systems gave rise to a variety of computational techniques. The basic methods employed in these procedures vary widely. They include, for example non-linear and dynamic programming [5], [20], [21], gradient methods in function space [7], [21], and methods based on convexity of reachable set [24]-[26], [9], [10], [15].

Dynamic programming method has application to many control problems. In dynamic optimization the goal is to determine the control signals or the parameter settings as functions of the independent variable, generally the time, to control the system within specified constraints while simultaneously extremalizing some index of performance. Application of this technique to optimal control problems appear to be limited practically to a small class of such problems, because of the large storage capacity and long computation times that are usually required.

The steepest descent technique has many excellent applications to optimal control problems [11], [18], [16], [21]. The general mathematical theory of steepest method is discussed in papers of Goldstein [16] Kantorovich [20], and Rosenbloom [30]. But this method also exhibits slow convergence.

Most of the convexity methods [24]-[26], [9], [10], [15] are based on a general idea which was described first by Neustadt [24]-[26]. These techniques involve gradient-type minimization of scalar functions of n variables, and possess inherent difficulties due to step size determination and slow convergence rates. One such convexity method, which provided the motivation of this work is that due to Eaton [9]. He considered

the general problem of taking the output of a system to a moving target in minimum time and presented an iterative procedure for finding an optimal control policy for normal systems. The iterative procedure developed in this thesis follows the same basic idea due to Eaton [7]. In each stage of iteration this method involves the minimization of a quadratic function on a convex constraint set. This method as well as algorithms for general and time-optimal control problems due to Fadden [10], Fujisawa and Yasuda [14], Gilbert [15], Barr and Gilbert [4], may appear to be quite different in their approach but they can be viewed in a common setting [9]. These methods also involve the problem of minimization of a quadratic function on convex set. Gilbert [15], in 1966 presented an iterative procedure for computing the minimum of a quadratic function on a convex set. Gilbert's procedure for computing the minimum of a quadratic function on a convex set has been modified in stages by Barr [3], and Pecsvaradi and Narendra [27].

This thesis presents a modified iterative procedure [1] applicable to time-optimal control of linear dynamical systems with amplitude constraints on the control. The procedure is based on the new algorithm for minimization of a quadratic function on a convex set presented by Pecsvaradi and Narendra [27] in 1970.

The most striking features of this modified algorithm for time-optimal control problems are that

- i) it has rapid exponential convergence, and
- ii) it does not involve any initial approximation and step-size determination.

The outline of the thesis is as follows:

In chapter I, a brief discussion on the previous computational works in time-optimal control problems is given.

In chapter II , the time-optimal control problem is introduced with a brief discussion of the underlying theory. A theoretical basis of the computational algorithm for solving linear time-optimal control problems is also discussed in this chapter.

In chapter III , the iterative procedure is presented in detail. Convergence of the iterative procedure and numerical results for two examples of time-optimal control problems solved by this procedure are also given in this chapter.

CHAPTER II

BASIC THEORY OF TIME-OPTIMAL CONTROL AND ITS
COMPUTATIONAL ALGORITHM

In this chapter the time-optimal control problem is introduced. A brief discussion of the underlying theory is presented in section 2.1. The iterative procedure for computing optimal time and optimal control is based on the celebrated bang-bang principle of La Salle[22] and others[8],[28]. A theoretical basis of the computational algorithm is also described in section 2.2.

It is interesting to mention that this algorithm is very suitable for computation on digital computer and does not require the use of hybrid computer as many other convexity methods[10],[11],[15] do.

2.1. Statement of the Problem

The time-optimal control problem for the linear dynamical system to be considered in this thesis is described by the following vector differential equation:

$$\begin{aligned} S: \quad \dot{x}(t) &= A(t)x(t) + B(t)u(t) ; x(0) = x_0 \\ &\text{on } R_0 \triangleq \{t : t \geq 0\} \end{aligned} \quad \text{--- (2.1.1)}$$

where x is an n -dimensional vector valued function ($x(t)$ is the state of the systems at time t), A is a continuous $n \times n$ matrix valued function and similarly B is a $n \times r$ continuous matrix valued function. The ability to control the system S lies in the freedom one has, in choice of the control function u .

Let V be a subset of E^r defined by
$$V \triangleq \{u \in E^r : |u^i| \leq 1 ; i = 1, 2, \dots, r\}$$

and M , the space of measurable functions defined on R_0 and taking values in E^r . The following definition is useful in the statement of the problem.

Definition 2.1 : For the class of admissible controls we define the set

$$U \triangleq \{ u \in M : u(t) \in V \text{ almost everywhere on } R_0 \}.$$

Thus a control u defined on R_0 and taking values in E^r is said to be admissible if $u \in U$.

Given a control function $u \in U$ and initial state $x_0 \in E^n$, the equation (2.1.1) has unique and absolutely continuous solution $x(t, u)$ on every finite interval and is given by,

$$x(t, u) = \bar{\phi}(t) \left[x_0 + \int_0^t \bar{\phi}^{-1}(\tau) B(\tau) u(\tau) d\tau \right] ; t \in R_0 \quad - - - (2.1.2)$$

where $\bar{\phi}(t)$ is the $n \times n$ non-singular transition matrix which satisfies the matrix differential equation

$$\dot{\bar{\phi}}(t) = A(t) \bar{\phi}(t) \quad \text{with} \quad \bar{\phi}(0) = I.$$

The time-optimal control problem may now be stated as follows:

Given an initial state $x_0 \in E^n$, find the admissible control $u \in U$ which drives the system S to the origin (in E^n) in the shortest possible time t . The optimal control, if it exists, is denoted by u^* and the corresponding (optimal) time is denoted by t^* .

Since $\bar{\phi}^{-1}(t)$ exists, an equivalent statement of the time-optimal control problem is described as follows:

Given $x_0 \in E^n$ as an initial state, it is required to find a control $u^* \in U$ such that the following equality

$$-x_0 = \int_0^{t^*} \bar{\phi}^{-1}(\tau) B(\tau) u^*(\tau) d\tau \quad - - - (2.1.3)$$

is satisfied for the smallest value of $t = t^*$.

The following definitions will be useful in the sequel.

Definition 2.2 : For every $t \geq 0$, a set $C(t) \subseteq E^n$ corresponding to the dynamical system S and the admissible controls U defined by

$$C(t) \triangleq \left\{ x : x = - \int_0^t \Phi(t, \tau) B(\tau) u(\tau) d\tau ; u \in U \right\} \quad - - - (2.1.4.)$$

is called the recoverable set .

For each $t \geq 0$ the set $C(t)$ consists of all those initial states from which the system S can be driven back to zero in finite time t .

Definition 2.3 : A subset A of E^n is convex if for any x and y in A and r, s in R with $r \geq 0, s \geq 0; r+s=1$, the point $(rx+sy)$ is in A .

Definition 2.4 : A subset A of a metric space is called closed if it contains all its limit points.

Definition 2.5 : A set $A \subseteq E^n$ is compact if and only if it is closed and bounded.

In order to present the computational algorithm developed in this thesis we need the following theorems.

Theorem 2.1. (La Salle [22])

For each $t \geq 0$, the recoverable set $C(t)$ has the following properties :

- i) $C(t)$ is compact and convex set in E^n .
- ii) $C(t_1) \subseteq C(t_2)$ for $t_1 < t_2$.
- iii) If q is an interior point of $C(t_2)$, there exists a $t_1 < t_2$ such that q is an interior point of $C(t_1)$.

The following set of controls

$$U^0 \triangleq \{ u : u \text{ measurable and } u(t) \in \partial V \text{ for } t \geq 0 \},$$

where ∂V is the boundary of the set V as defined earlier, is known in control theory as the set of bang-bang controls.

The set

$$C^0(t) \triangleq \left\{ \int_0^t \Phi^{-1}(\tau) B(\tau) u^0(\tau) d\tau ; u^0 \in U^0 \right\} \subset E^n$$

is the set of states recoverable by use of bang-bang controls.

Theorem 2.2. (The Bang-Bang Principle, La Salle.[23]; Theorem 12.1.)

For each $t \geq 0$, $C(t) = C^0(t)$.

The significance of the theorem 2.2. is as follows, if the system S can be driven from a given initial state to the origin in time t by use of a control $u \in U$ then it can also be done by use of an appropriate control $u^0 \in U^0$ within the same time. The intuitive feeling is that, if full power is not being used, the use of the additional power available can always speed up the process. In his paper [8] in 1952 Bushaw accepted this hypothesis (and for his problem this was true), but this is not always correct. There are cases where there can be more power available than can be used effectively and optimal control is not necessarily always bang-bang. However, the bang-bang principle does say that if there is an optimal control, there is always an optimal bang-bang control for systems where control appears linearly.

Using the continuity and compactness properties of the set valued functions $C(t)$, $t \geq 0$, La Salle has established the following:

Theorem 2.3. (Existence of an optimal control)

If there is a control $u \in U$ and a $t \geq 0$ for which $x(t, u) = 0$ (null vector) then there is an optimal control.

Since existence of a control implies by theorem 2.3. the existence of an optimal control, it will be assumed throughout this thesis that there exists a control. The following definition will be useful in the sequel.

Definition 2.6. : Let $r(x)$ be a real valued function on E^n , defined by

$$r(x) \triangleq (a, x) - b = \sum_{i=1}^n a_i x_i - b ,$$

where a is a given nonzero element of E^n and b is a given real number. Then we call the subset h of E^n given by

$$H \triangleq \{ x : H(x) = \sum_{i=1}^n a_i x_i - b = 0 \}$$

a hyperplane in E^n .

Let,

$$H^+ \triangleq \{ x : H(x) = \sum_{i=1}^n a_i x_i - b > 0 \}$$

and $H^- \triangleq \{ x : H(x) = \sum_{i=1}^n a_i x_i - b < 0 \}$

be the open half spaces determined by the hyperplane H .

Let A and B be any two subsets of E^n . If A is contained in one of the two open half spaces and B is contained in the other, then it is said that H separates A and B strictly. If $H \cap \bar{A}$ (\bar{A} represents closure of A) is not empty and if H does not contain A , and A is contained either $H^+ \cup H$ or $H^- \cup H$, then H is said to be a Support hyperplane of A .

For a convex set $C(t) \subset E^n$ the support hyperplane $H(t, \lambda)$ of $C(t)$

with a fixed outward normal λ is given by

$$H(t, \lambda) = \{ x : (x, \lambda) = \max_{z \in C(t)} (z, \lambda) ; \lambda \neq 0 \} \quad \text{--- (2.1.5)}$$

Let $H_{z^*}(t, \lambda) ; \lambda \neq 0$ represent the support hyperplane of $C(t)$ containing the point $z^* \in C(t)$ with λ as the outward normal vector.

The result (iii) of theorem 2.1. implies that when an optimal control exists, x_0 belongs to the boundary of $C(t^*)$, where t^* is the optimal time. Let p be a boundary point of the set $C(t) \in E^n$ for some $t \geq 0$. Since $C(t)$ is convex, there exist at least one support hyperplane of $C(t)$ at p with outward normal $\lambda (\neq 0)$. The following two theorems have direct application in the computational algorithm developed in the next chapter.

Theorem 2.4. (La Salle [23], Lemma 13.1)

A point $p \triangleq z(t, \lambda) = - \int_0^t \bar{\phi}^{-1}(\tau) B(\tau) u_b(\tau) d\tau$, is a boundary point of $C(t)$ with λ an outward normal to a support hyperplane of $C(t)$ through p if and only if u_b is of the form;

$$u_b(\tau) = - \operatorname{sgn} \{ [\bar{\phi}^{-1}(\tau) B(\tau)]^T \lambda \} \text{ on } R_0 ; \lambda \neq 0 \quad \text{--- (2.1.6.)}$$

$$\text{where, } u_b^i(\tau) = +1 \text{ for } \{ [\bar{\phi}^{-1}(\tau) B(\tau)]^T \lambda \}^i < 0 ; 0 \leq \tau \leq t$$

$$u_b^i(\tau) = -1 \text{ for } \{ [\bar{\phi}^{-1}(\tau) B(\tau)]^T \lambda \}^i > 0 ; 0 \leq \tau \leq t$$

$$u_b(\tau) = \text{arbitrary value in } [-1, 1] \text{ for}$$

$$\{ [\bar{\phi}^{-1}(\tau) B(\tau)]^T \lambda \}^i = 0 \text{ for a positive interval of time.}$$

From the result (iii) of theorem 2.1. and theorem 2.4., follows :

Theorem 2.5. (La Salle [23], Theorem 13.2)

If u^* is an optimal control with t^* the minimum time then u^* is of the form

$$u^*(t) = - \operatorname{sgn} \left\{ \left[\bar{\varphi}^{-1}(\tau) B(\tau) \right]^T \lambda^* \right\} \quad \text{On } 0 \leq \tau \leq t^*$$

for some nonzero vector λ^* .

Geometrically speaking, the vector λ^* is the outward normal vector of a supporting hyperplane of $C(t^*)$ passing through x_0 , which is a boundary point of $C(t^*)$. Thus it is clear from the theorem 2.4 that for any time $t \geq 0$, a boundary point $z(t, \lambda) \in C(t)$ contained in the supporting hyperplane of $C(t)$ with outward normal $\lambda (\neq 0)$ satisfy the following relation,

$$z(t, \lambda) = \int_0^t \left\{ \left[\bar{\varphi}^{-1}(\tau) B(\tau) \right] \operatorname{sgn} \left\{ \left[\bar{\varphi}^{-1}(\tau) B(\tau) \right]^T \lambda \right\} \right\} d\tau \quad \text{--- (2.1.7)}$$

It has been shown that for each such boundary point $z(t, \lambda)$ of $C(t)$, the corresponding control is unique (even though the direction of the vector λ at $z(t, \lambda) \in C(t)$ may not be unique. that is $C(t)$ has a corner point at $z(t, \lambda)$). Furthermore if $z(t, \lambda) \in \partial C(t)$ for some $\lambda (\neq 0)$, then $z(t, \lambda) \notin C(\tau)$ for any $\tau \neq t$, ($\tau, t \geq 0$).

From the above discussion the following conclusion can be drawn. If an optimal control exists, it is unique for almost all t , provided the system is normal [22], and is of the form (2.1.5) for some $\lambda = \lambda^*$, and that the optimal time t^* is also unique. It should be noted that the optimal control u^* depends only on the direction of λ^* and not on its magnitude.

2.2 Basic Principle of Computational Algorithm

In this section the theoretical background of the algorithm is established by using the properties of the recoverable set $C(t)$ discussed in section 2.1.

With reference to the relation (2.1.7), any boundary point $z(t, \lambda)$ in $C(t)$ is represented by

$$z(t, \lambda) = \int_0^t \bar{\phi}^{-1}(\tau) B(\tau) \operatorname{sgn} \{ [\bar{\phi}^{-1}(\tau) B(\tau)]^T \lambda \} d\tau, \quad \lambda \neq 0 \quad \text{---(2.2.1)}$$

where λ is the normal to the support hyperplane of $C(t)$ at $z(t, \lambda) \in C(t)$ and is directed in the half space not containing the set $C(t)$. For brevity of description this half space may be denoted for each $t \geq 0$ by N_t^+ . Since $z(t, \lambda)$ is contained in the support hyperplane of $C(t)$, the following relation holds (definition 2.6) for any point $q \in C(t)$.

$$(\lambda, z(t, \lambda)) \geq (\lambda, q) \quad \text{for all } q \in C(t) \quad \text{--- (2.2.2)}$$

Definition 2.7: Let $\Omega_0 \in E^n$ be the set of all initial states from which the system S can be driven to the origin in finite time by utilization of an admissible control. The set Ω_0 is called the set of controllability.

In the development of the algorithm it is assumed that the initial state $x_0 \in \Omega_0$. For a fixed but arbitrary $x_0 \in \Omega_0$ let us define the real valued function f on $R_0 \otimes E^n$ by

$$f(t, \lambda) \triangleq (\lambda, z(t, \lambda) - x_0) \quad \text{--- (2.2.3)}$$

where $z(t, \lambda)$ is given by the expression (2.2.1.). It is clear that for any $\lambda \in E^n$, $z(0, \lambda) = 0$. For $t=0$, one can choose a λ_0 in E^n so that the

value of the function f at $(t, \lambda) = (0, \lambda_0)$ given by

$$f(0, \lambda_0) = (\lambda_0, -x_0) < 0 \quad \text{--- (2.2.3)}$$

For some time $t > 0$, relation (2.2.2) implies that for some $\lambda \neq 0$ which is directed in the half space N_t^+ ,

$$(\lambda, q) - (\lambda, z(t, \lambda)) \leq 0 \quad \text{for all } q \in C(t) \quad \text{--- (2.2.4)}$$

for all $q \in C(t)$. Since $x_0 \in \Omega_0$ (definition 2.7) there exists an admissible control that drives the system S from the initial state x_0 to the origin. So the point x_0 must lie on the boundary of a set $C(t^*)$ for some $t^* \in R_0$.

For some time $t' > t^*$, the property (ii) of theorem 2.1. and the relation (2.2.4.) imply that,

$$(\lambda, x_0) - (\lambda, z(t', \lambda)) \leq 0$$

Thus,

$$(\lambda, z(t', \lambda) - x_0) \geq 0$$

for any λ directed into the half space $N_{t'}^+$. Thus by definition of the function f we have, $f(t', \lambda) \geq 0$ for $t' > t^*$.

Again from relations (2.2.3.) and (2.2.1.), for a fixed $\lambda \neq 0$,

$$\begin{aligned} \frac{\partial f(t, \lambda)}{\partial t} &= \frac{\partial}{\partial t} \left\{ (\lambda, \int_0^t [\bar{e}^{-1}(\tau) B(\tau)] \operatorname{sgn} \{ [\bar{e}^{-1}(\tau) B(\tau)]^T \lambda \} d\tau - x_0) \right\} \\ &= \left| [\bar{e}^{-1}(t) B(t)]^T \lambda \right| \geq 0 \quad \text{for all } t \geq 0 \end{aligned}$$

--- (2.2.5)

So from relations (2.2.3.), (2.2.4.) and (2.2.5) one can conclude that there exists a smallest $t^* \in R_0$ so that

$$f(t^*, \lambda) = 0 \quad ; \quad 0 < t^* \leq t' \quad \text{--- (2.2.6)}$$

Geometrically $f(t, \lambda)$ is proportional to the distance from x_0 to the support hyperplane of $C(t)$ at $z(t, \lambda)$. Therefore at $t=t^*$, x_0 is contained in the support hyperplane of $C(t^*)$. Unless $z(t, \lambda) = x_0$, x_0 is outside $C(t)$ for $t < t^*$. It follows from theorem 2.5. that the optimal control $u^*(t)$ is given by

$$u^*(t) = - \operatorname{sgn} \left\{ \left[\frac{d}{dt} B(t) \right]^T \lambda^* \right\} \quad ; \quad 0 \leq t \leq t^*, \quad \lambda^* \neq 0 \quad \text{--- (2.2.7)}$$

where the vector λ^* , directed in the half space N_t^+ is normal to the support hyperplane of $C(t^*)$ that passes through the point $x_0 \in \partial C(t^*)$.

CHAPTER III

ITERATIVE ALGORITHM AND COMPUTATIONAL RESULTS

In this chapter the iterative algorithm for the time-optimal control problem stated in chapter I is discussed.

In section 3.1, the algorithm consisting of two main phases is described. This algorithm is based on an iterative procedure for minimization of a quadratic form on a convex set. In section 3.2 this minimization technique is discussed. In section 3.3 the detailed iterative steps are presented with a flow chart. The rate of convergence of the algorithm is investigated in section 3.4. For illustration, two examples of linear time-optimal control problems are solved using this algorithm. In section 3.5, the computational results are presented with critical discussions.

3.1. The Algorithm

The algorithm may be described as a successive approximation method to achieve the optimal time t^* and the corresponding optimal control $u^*(t)$, $0 \leq t \leq t^*$. It is based on the properties of the recoverable set $C(t)$ discussed in chapter II.

Let $H(t, \lambda)$ denote the support hyperplane of $C(t)$ with outward normal λ as in definition 2.6. The following definitions will be used in the sequel.

Definition 3.1. : The set $F(t, \lambda) \triangleq H(t, \lambda) \cap C(t)$; for a fixed $\lambda \neq 0$ is defined as the contact set of $C(t)$ and its elements are called contact points of $C(t)$.

It follows that $F(t, \lambda)$ is not empty and $F(t, \lambda) \subset \mathcal{J}C(t)$. If $C(t)$ is

strictly convex then $F(t, \lambda)$ contains only a single element.

For an arbitrary but fixed $t \geq 0$, a function $K_t(\lambda)$, defined on E^n is a contact function of $C(t)$ if $K_t(\lambda) \in F(t, \lambda)$, $\lambda \neq 0$ and $K_t(0) \subset C(t)$.

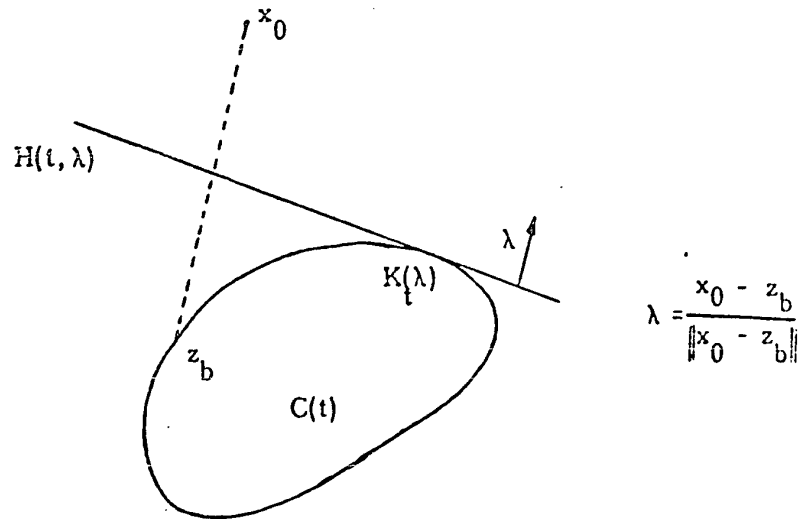


Fig. 3.1 The contact function $K_t(\lambda)$ of the convex set $C(t)$ determined by a boundary point z_b of $C(t)$.

Clearly it follows from the relation (2.1.5) that $K_t(\lambda)$ satisfies the relation

$$(K_t(\lambda), \lambda) = \max_{z \in C(t)} (z, \lambda)$$

Apparently, determination of $K_t(\lambda)$ involves the use of linear programming

technique. But it follows from theorem 2.4 and the relation

$$(\lambda, K_t(\lambda)) = \max_{u \in U} (\lambda, - \int_0^t \bar{g}^{-1}(\tau) B(\tau) u(\tau) d\tau)$$

that $K_t(\lambda)$ is given by

$$K_t(\lambda) = \int_0^t \bar{g}^{-1}(\tau) B(\tau) \operatorname{sgn} \{ [\bar{g}^{-1}(\tau) B(\tau)]^T \lambda \} d\tau \quad \text{--- (3.1.1.)}$$

which is precisely equal to $z(t, \lambda)$ given in the expression (2.1.7).

With this preparation the iterative procedure may be described as follows :

The iteration process starts with $t(1)=0$, for $k=1$. Any k th. stage ($k=1, 2, \dots$) of iteration consists of the following two phases A and B.

Phase A: The value of $t(k)$ is known from the previous stage (i.e. from $(k-1)$ th stage). Corresponding to $t(k)$, the recoverable set $C(t(k))$ and its boundary $\partial C(t(k))$ is determined.

For $t=t(k)$, the function $f(t(k), \lambda)$ on E^n to the reals as defined by (2.2.2) is

$$f(t(k), \lambda) \triangleq (\lambda, z(t(k), \lambda) - x_0)$$

where $z(t(k), \lambda) \in \partial C(t(k))$ for every $\lambda \in E^n$. From the definition of the boundary points $z(t, \lambda) \in \partial C(t)$ as given in (2.1.7) and the expression for the contact function $K_t(\lambda)$ given by (3.1.1) it is clear that $K_t(\lambda) = z(t, \lambda)$.

Note: This is true since it is assumed that $C(t)$, $t \geq 0$ is strictly convex. Define a vector $\lambda \in E^n$ by

$$\lambda \triangleq \frac{x_0 - z}{\|x_0 - z\|} \quad \text{for } z \in \partial C(t) \quad \text{--- (3.1.2)}$$

clearly $\lambda \in S_1 \triangleq \{x : \|x\| = 1\}$

For the fixed $t(k) \geq 0$ it is now required to find a $z_k \in \partial C(t(k))$ so that

$$\|x_0 - z_k\| = \min_{z \in C(t(k))} \|x_0 - z\| \quad \text{--- (3.1.3)}$$

For this $z_k \in \partial C(t(k))$, the vector $\lambda_k \in E^n$ is determined from the relation (3.1.2) and $K(\lambda_k)$ from the relation (3.1.1). Further it is to be observed that the function $f(t(k), \lambda)$ attains its infimum at $\lambda = \lambda_k$ and is given by

$$\begin{aligned} f(t(k), \lambda_k) &= \inf_{\lambda \in S_1} (\lambda, K(\lambda)_{t(k)} - x_0) \\ &= (\lambda_k, K(\lambda_k)_{t(k)} - x_0) \\ &= - \|x_0 - K(\lambda_k)_{t(k)}\| \end{aligned} \quad \text{--- (3.1.4)}$$

Geometrically the optimal choice of $\{\lambda_k\}$ at each stage ($k=1, 2, \dots$) leads $\{K(\lambda_k)_{t(k)}\}$ to converge (fig. 3.2) to the point x_0 along the steepest path. It is interesting to mention that the control $u_k(t)$, $0 \leq t \leq t(k)$, obtained by using λ_k for λ^* in theorem 2.5 gives the optimal control that drives the system S from the point $K(\lambda_k)_{t(k)}$ to the origin in minimum time $t(k)$.

Phase B : Corresponding to this fixed $t(k)$, construct a hyperplane

$$H(t(k), \lambda_k) \triangleq \{x : (x, \lambda_k) = (x_0, \lambda_k)\} \text{ through the point } x_0 \text{ with } \lambda_k$$

as the outward normal. If $K(\lambda_k) \neq x_0$, the intersection $H(t(k), \lambda_k) \cap C(t(k), x_0)$ is empty. Find out the shortest $t(k+1) > t(k)$ such that $H(t(k+1), \lambda_k) \cap C(t(k+1), x_0)$ is not empty. This $t(k+1)$ can be found out by solving the equation,

$$(\lambda_k, K(\lambda_k) - x_0) = 0$$

for the shortest value of $\tau > t(k)$, where $K(\lambda_k)$ belongs to the hyperplane $H(t(k), \lambda_k)$ and is the contact function of $C(\tau)$. At this phase, the time $t(k)$ is modified to $t(k+1)$, which is closer to t^* and the iterative procedure starts again with phase A.

In course of this iteration, at a certain stage k , the modified value of λ_k obtained at the end of phase A will satisfy the equation,

$$f(t(k), \lambda_k) = - \|x_0 - K(\lambda_k)\| = 0, \quad \text{---} \quad (3.1.5)$$

where $K(\lambda_k)$ is the contact point of $C(t(k))$ with the hyperplane $H(t(k), \lambda_k)$

When the relation (3.1.5) is satisfied, then the iterative process (consisting of Phase A and Phase B) terminates with,

optimal-time $t^* = t(k)$, and

optimal control $u^*(t) = - \text{sign} \{ [\bar{\varphi}^{-1}(t) B(t)]^T \lambda_{k-1} \}; 0 \leq t \leq t^*$.

This iterative process is based on another algorithm for minimization of a quadratic function on a convex set due to Pecsvaradi and Narendra[27].

3.2. Pecsvaradi And Narendra's Minimization Procedure

In this section the minimization technique due to Pecsvaradi and Narendra [27], as mentioned before, is discussed.

The basic problem PI may be stated as follows :

PI: Given a compact convex set $C(t)$ and a fixed point $x_0 \in E^n$ find a $z^* \in C(t)$ such that

$$\|x_0 - z^*\| = \min_{z \in C(t)} \|x_0 - z\|$$

The properties of the solution of the above problem is given in the following theorem.

Theorem 3.1

- i) A solution z^* exists and is unique .
- ii) $\|x_0 - z^*\| = 0$ if and only if $x_0 \in C(t)$
- iii) If $\|x_0 - z^*\| > 0$, then $z^* \in \partial C(t)$.
- iv) For $\|x_0 - z^*\| > 0$, $z = z^*$ if and only if $z \in H(x_0 - z) \cap C(t) = F(x_0 - z)$.

where $H(x_0 - z)$ represent the support hyperplane to $C(t)$ with $(x_0 - z)$ as the outward normal.

Proof : i) Since $C(t)$ is compact and the function $Q(z) \triangleq \|x_0 - z\|$ is continuous, Q has a minimum on $C(t)$. The uniqueness is proved by contradiction. Suppose Q attains its minimum at z_1^* and z_2^* in $C(t)$. Define $y_1^* = x_0 - z_1^*$ and $y_2^* = x_0 - z_2^*$. It is clear that

$$\|y_1^* - y_2^*\|^2 + \|y_1^* + y_2^*\|^2 = 2\|y_1^*\|^2 + 2\|y_2^*\|^2$$

and since both z_1^* and z_2^* minimizes the function Q , $\|y_1^*\| = \|y_2^*\| = d$ (for some $d > 0$). Thus

$$\|z_1^* - z_2^*\|^2 + 4 \|x_0 - \frac{1}{2}(z_1^* + z_2^*)\|^2 = 4d^2.$$

Since $z_1^*, z_2^* \in C(t)$ and $C(t)$ is convex $\frac{1}{2}(z_1^* + z_2^*) \in C(t)$ and $\|x_0 - \frac{1}{2}(z_1^* + z_2^*)\| \geq d$, and consequently $\|z_1^* - z_2^*\| \leq 0$ which implies $z_1^* = z_2^*$.

ii) This is obvious from the statement of the theorem.

iii) Let $0 < d = \|x_0 - z^*\|$. For the given $(x_0 - z^*)$ there exists $a^* \in E^n$ with $\|a^*\| = 1$, so that $\|x_0 - z^*\| = (a^*, x_0 - z^*)$. Define the function $f(y) = (a^*, x_0 - y)$ for $y \in C(t)$. Since f is linear on $C(t) - \{x_0\}$ and $C(t)$ is compact and convex, f attains its minimum on the boundary $\partial C(t)$ at $y = y^*$. Thus, by the property (i) $z^* = y^* \in \partial C(t)$

iv) The necessary condition i.e. if $z \in H(x_0 - z) \cap C(t)$ then $z = z^*$, can be proved as follows :

Since $z \in H(x_0 - z)$, from the definition 2.6. of the support hyperplane,

$$H(x_0 - z) = \{x : (x, x_0 - z) = \text{constant}\}$$

Again for any point $q \in C(t)$ it is known that

$$(z, x_0 - z) - (q, x_0 - z) \geq 0$$

This relation clearly indicates that the point $z \in C(t)$ is the most distant point in $C(t)$ from the origin along the outward normal vector $(x_0 - z)$ of the support hyperplane $H(x_0 - z)$ to $C(t)$. In otherwords z is at minimum distance among all points in $C(t)$ from x_0 , implying $z = z^*$.

The sufficiency condition follows from the property (iii) and the definition of $H(x_0 - z)$.

It is convenient to introduce here some basic definitions which will be required in the sequel .

Definition 3.3. : If X is an arbitrary set of points in E^n , then the convex hull of X written $Co(X)$, is the set of points which is the intersection of all the convex sets that contain X .

$Co(X)$ is convex , and a necessary and sufficient condition that X be convex is that $X = Co(X)$. Furthermore if X is compact , then $Co(X)$ is compact.

Definition 3.4. : A m-Simplex L^m , ($m > 0$) is a set of points $X = (x_i)$ defined in terms of $(m+1)$ linearly independent points P_0, P_1, \dots, P_m

by

$$x_i = \sum_{a=0}^m t_a P_{ai} \quad ; i = 1, 2, \dots, n,$$

where $\sum_{a=0}^m t_a = 1$ and $0 \leq t_a \leq 1$, $a = 0, 1, \dots, m$.

A 0-simplex is simply a point . The number m is called the dimension of the simplex , and the set of $(m+1)$ points P_0, P_1, \dots, P_m is the skeleton of the m -simplex .

The iterative procedure due to Pecsvaradi and Narendra [27] minimizes the quadratic function $Q(z) \triangleq \|x_0 - z\|^2$ over successively higher dimension m -simplexes , where the dimension of the simplexes m ranges from 0 to $(n-1)$. These simplexes are constructed by choosing linearly independent points belonging to the convex set $C(t)$.

Let j denote the index number of the iterative procedure . For $j < n$, the minimization is carried out over a sequence of successively

higher dimensional simplexes $L^{j-1} \subset C(t)$; and for $j \geq n$, the minimization at each stage is performed over the $(n-1)$ -simplex L^{n-1} . These simplexes (i.e. L^{j-1} and L^{n-1}) are properly chosen to obtain rapid convergence. The iterative procedure is initiated with an arbitrary choice of $p_0 \in C(t)$.

For $1 \leq j < n$: Let $p_0, p_1, p_2, \dots, p_{j-1}$ be j known linearly independent points in $C(t)$, and denote the $(j-1)$ simplex as

$$L_j^{j-1} \triangleq \text{Co} (p_0, p_1, p_2, \dots, p_{j-1}).$$

Find $z_{j-1} \in L_j^{j-1}$ such that

$$\|x_0 - z_{j-1}\| = \min_{z \in L_j^{j-1}} \|x_0 - z\|$$

Find the contact point $K_j(x_0 - z_{j-1})$ of $C(t)$ corresponding to its support hyperplane with $(x_0 - z_{j-1})$ as the normal and put $p_j = K_j(x_0 - z_{j-1})$. Then L_j^{j-1} is modified to L_{j+1}^j as

$$L_{j+1}^j \triangleq \text{Co} (p_0, p_1, p_2, \dots, p_j)$$

For $j \geq n$: Let p_0, p_1, \dots, p_{n-1} be n linearly independent points in $C(t)$ and denote the corresponding $(n-1)$ simplex by L_j^{n-1} as,

$$L_j^{n-1} \triangleq \text{Co} (p_0, p_1, \dots, p_{n-1}), \quad ; j \geq n.$$

Define z_{j-1} and $p_n \triangleq K_j(x_0 - z_{j-1})$ as above and define the n -simplex L_j^n as $L_j^n \triangleq \text{Co} (L_j^{n-1}, p_n)$. Find a point $z_n \in L_j^n$ such that

$$\|x_0 - z_n\| = \min_{z \in L_j^n} \|x_0 - z\|$$

Define the hyperplane H_{n+1} as

$$H_{n+1} \triangleq \{ x : (x_0 - z_n), (z_n - x) = 0 \}$$

and let d_i , $i = 0, 1, 2, \dots, n-1$ be the Euclidean distance from p_i to H_{n+1} , that is

$$d_i \triangleq \min_{x \in H_{n+1}} \|x - p_i\| \quad ; \quad i = 0, 1, 2, \dots, n-1 .$$

Assume that $\max_{0 \leq i \leq n-1} d_i$ occurs for $i = m$. Then the corresponding point p_m is replaced by the point p_n . As a consequence the $(n-1)$ -simplex L_j^{n-1} is modified to another $(n-1)$ -simplex

$$L_{j+1}^{n-1} \triangleq \text{Co} (p_0, p_1, p_2, \dots, p_n, \dots, p_{n-1}).$$

If $\max_{0 \leq i \leq n-1} d_i$ occurs for several i , then one of the corresponding p_i is chosen arbitrarily and is replaced by p_n . By this procedure we have obtained a sequence $\{z_j\} \in E^n$ satisfying the properties as stated in the following theorem.

Theorem 3.2.

- i) $z_j \in C(t)$ for $j = 1, 2, \dots$
- ii) The sequence $\{ \|x_0 - z_j\| \}$ is decreasing and $\|x_0 - z_j\| = \|x_0 - z_{j-1}\|$ holds for some j if and only if $z_j = z_{j-1} = z^*$.
- iii) $z_j \rightarrow z^*$.

Proof : The proof is given in the reference [3]

As observed in the iterative procedure discussed above, the basic problem P I of minimization of the function $Q(z) \triangleq \|x_0 - z\|$ on a convex set $C(t)$, involve at each iteration stage j the following problem :

PII : Given a $(m-1)$ -simplex L^{m-1} of m known linearly independent points $y_1, y_2, y_3, \dots, y_m$ in E^n and a fixed point $x_0 \in E^n$, find a point $y^* \in L^{m-1}$ such that,

$$\|x_0 - y^*\| = \min_{y \in L^{m-1}} \|x_0 - y\|$$

Let $H(x_0 - y)$, $(x_0 - y) \neq 0$ be the support hyperplane of L^{m-1} with outward normal $(x_0 - y)$. In view of the problem PII the theorem 3.1 reduces to

Theorem 3.1'

- i) The solution y^* (of problem PII) is unique.
- ii) $\|x_0 - y^*\| = 0$ if and only if $x_0 \in L^{m-1}$
- iii) For $\|x_0 - y^*\| > 0$; $y^* \in \partial L^{m-1}$.
- iv) For $\|x_0 - y^*\| > 0$; $y^* = y$ if and only if $y \in H(x_0 - y) \cap L^{m-1}$.

It is important to note the distinction between PI and PII. The set $C(t)$ in basic problem PI is described only by a contact function $K(\cdot)$ of $C(t)$ whereas the convex set L^{m-1} of problem PII is the $(m-1)$ -simplex of m known points. Thus the problem PII is much simpler than the problem PI.

Solution of problem PII :

This type of problem is usually described in literature (e.g. [2], [5], [17]) under the heading " Quadratic Programming ". However, the computational algorithms which are suggested always begin by assuming that the constraint set is described by a set of linear equations and/or inequalities rather than by points which form a corresponding simplex as constraint set. Thus to apply the standard quadratic programming techniques directly to PII, it is necessary to first determine from these points a description of the constraint set in terms of linear equations and/or inequalities. Such a determination involves extreme computational difficulties.

Fortunately there is an alternative method of solving the problem PII which makes possible the use of the standard algorithm. It is shown here that the solution of the problem PII is given by the solution of another reduced quadratic programming problem called PIII, which has a constraint set described by linear equations. The problem PIII is described below:

Let $p_1, p_2, p_3, \dots, p_m \in E^n$ be the known linearly independent points and L^{m-1} be the corresponding $(m-1)$ -simplex given by $L^{m-1} \triangleq \text{Co} (p_1, p_2, \dots, p_m)$ as specified in PII. Note that since $\|\cdot\|$ is the Euclidean norm, an equivalent statement of PII is as follows:

Find $p^* \in L^{m-1}$ such that

$$\|x_0 - p^*\|^2 = \min_{p \in L^{m-1}} \|x_0 - p\|^2$$

Each $p \in L^{m-1}$ has the representation $p = \sum_{i=1}^m \delta^i p_i$; where $\sum_{i=1}^m \delta^i = 1$; $\delta^i \geq 0$, $i = 1, 2, \dots, m$. Therefore

$$\begin{aligned} \|x_0 - p\|^2 &= \left\| \sum_{i=1}^m \delta^i (x_0 - p)_i \right\|^2 \\ &= \sum_{i=1}^m \sum_{j=1}^m \delta^i \delta^j ((x_0 - p)_i, (x_0 - p)_j) \end{aligned}$$

If δ is the m -vector $(\delta^1, \delta^2, \delta^3, \dots, \delta^m)$ and D is the $m \times m$ symmetric matrix with elements $d_{ij} = ((x_0 - p)_i, (x_0 - p)_j)$; $i, j = 1, 2, \dots, m$. then

$$\|x_0 - p\|^2 = (\delta, D\delta) \quad \dots \quad (3.2.1)$$

Since $\|x_0 - p\|^2 \geq 0$, the quadratic form $(\delta, D\delta)$ is nonnegative definite.

Consider now the following quadratic problem.

P III: Given D an $m \times m$ symmetric non-negative definite matrix and the constraint set, $W \triangleq \{ \delta \in E^n : \sum_{i=1}^m \delta^i = 1 ; \delta^i \geq 0 , i = 1, 2, \dots, m . \}$

Find a point $\delta^* \in W$ such that

$$(\delta^* , D\delta^*) = \min_{\delta \in W} (\delta , D\delta) .$$

It is clear from the equation (3.2.1) that minimization of $(\delta , D\delta)$ on W is equivalent to minimization of $\|x_0 - p\|^2$ on L^{m-1} . Thus if δ^* solves the problem PIII then the solution p^* of the problem PII is given by

$$p^* = \sum_{i=1}^m \delta^{*i} p_i .$$

The problem PIII is solvable by any of the well known quadratic programming techniques such as due to Beale[5], Hildreth[17], and Frank and Wolfe [13]. The standard technique due to Frank and Wolfe has been used to solve the problem PIII which in turn solves the problem PI, at each nested iteration.

3.3 The Iteration Steps.

A double subscript notation will be used, of which the first subscript k denotes the particular stages of successive approximation for the terminal time $t(k)$, and the second subscript j denotes the index for the nested iterative procedure for minimization of the quadratic form on the convex set $C(t(k))$ corresponding to the k th stage of the main iteration.

Step 1. Put $k = 0$, $t(0) = 0$.

Step 2 . Find out

$$P_0 \triangleq z_{k,0} = - \int_0^{t(k)} \phi^{-1}(\tau) B(\tau) u_k(\tau) d\tau \quad \text{--- (3.3.1)}$$

where $u_k \in U$ be arbitrary admissible control.

Step 3 . Put $j = 1$.

Step 4 . Define , $z_{k,i} = p_i$, $i = 1, 2, \dots, j$, and the simplex

$L_{k,j}^{j-1} \triangleq \text{Co} (p_0, p_1, \dots, p_{j-1})$. Find $z_{k,j} \in L_{k,j}^{j-1}$ such that

$$\|x_0 - z_{k,j}\| = \min_{s \in L_{k,j}^{j-1}} \|x_0 - s\| \quad \text{--- (3.3.2)}$$

This is precisely the problem in PII . Calculate

$$\lambda_{k,j} = \frac{\|x_0 - z_{k,j}\|}{\|x_0 - z_{k,j}\|} ; \text{ and check } \|x_0 - z_{k,j}\| < \epsilon_1 , \text{ for some preassigned } \epsilon_1 > 0 .$$

If true , put $\lambda^* = \lambda_{k,j-1}$ and go to step 10 .

Otherwise find,

$$K(\lambda_{k,j}) = \int_0^{t(k)} \phi^{-1}(\tau) B(\tau) \text{sgn} \{ [\phi^{-1}(\tau) B(\tau)]^T \lambda_{k,j} \} d\tau$$

Step 5 . Check $\|z_{k,j} - z_{k,j-1}\| < \epsilon_2$ for some prespecified $\epsilon_2 > 0$.

If true , put $z_k^* = z_{k,j}$ and

$$\lambda_k = \frac{\|x_0 - z_k^*\|}{\|x_0 - z_k^*\|} ; \text{ and go to the step 9 .}$$

Otherwise, if $j < n$ go to step 6 ,

if $j \geq n$ go to step 7 .

Step 6. Put $p_j \triangleq z_{k,j} = K(\lambda_{k,j})$.

Step 7. Replace j by $j+1$ and
 if $j < n$ go back to step 4
 if $j \geq n$ go to step 8.

Step 8. Define,

$$L_{k,j}^{n-1} = \text{Co} (p_0, p_1, p_2, \dots, p_{n-1}).$$

Find a point $z_{k,j} \in L_{k,j}^{n-1}$ such that

$$\|x_0 - z_{k,j}\| = \min_{s \in L_{k,j}^{n-1}} \|x_0 - s\| \quad \text{--- (3.3.3)}$$

This is precisely the problem PII. Check $\|x_0 - z_{k,j}\| < \epsilon_1$.

If true, put $\lambda^* = \lambda_{k,j-1}$ and go to step 10.

Otherwise, calculate

$$\lambda_{k,j} = \frac{x_0 - z_{k,j}}{\|x_0 - z_{k,j}\|}, \text{ and}$$

$$K(\lambda_{k,j}) = \int_0^{t(k)} \delta(\tau) B(\tau) \text{sgn} \{ [\delta(\tau) B(\tau)]^T \lambda_{k,j} \} d\tau \quad \text{--- (3.3.4)}$$

Put $p_n = K(\lambda_{k,j})$ and define $L_{k,j}^n \triangleq \text{Co}(p_0, p_1, \dots, p_{n-1}, p_n)$.

Find $z_n \in L_{k,j}^n$ such that

$$\|z_n - x_0\| = \min_{s \in L_{k,j}^n} \|s - x_0\|. \quad \text{--- (3.3.5)}$$

Calculate, $\mu_i = \frac{((x_0 - z_n), (z_n - p_i))}{\|x_0 - z_n\|^2}$; $i = 0, 1, 2, \dots, n-1$, and

$$D_i = \mu_i \|x_0 - z_n\|, \quad i = 1, 2, 3, \dots, n-1.$$

Find D_m for $m \leq n-1$ such that $D_m = \max_{0 \leq i \leq n-1} D_i$.

Put $p_m = p_n$, and substitute j for $j+1$ and go back to step 5.

Step 9 . Define ,

$$K_{\tau k}(\lambda_k) = \int_0^{\tau} \bar{\varphi}^{-1}(t) B(t) \operatorname{sgn} \{ [\bar{\varphi}^{-1}(t) B(t)]^T \lambda_k \} dt .$$

Solve the following equation

$$(\lambda_k , (K_{\tau k}(\lambda_k) - x_0)) = 0 \quad \text{--- (3.3.6)}$$

for the smallest value of $\tau = \tau^* > t(k)$. Put $t(k+1) = \tau^*$ and substitute k for $k+1$ and go back to step 2 .

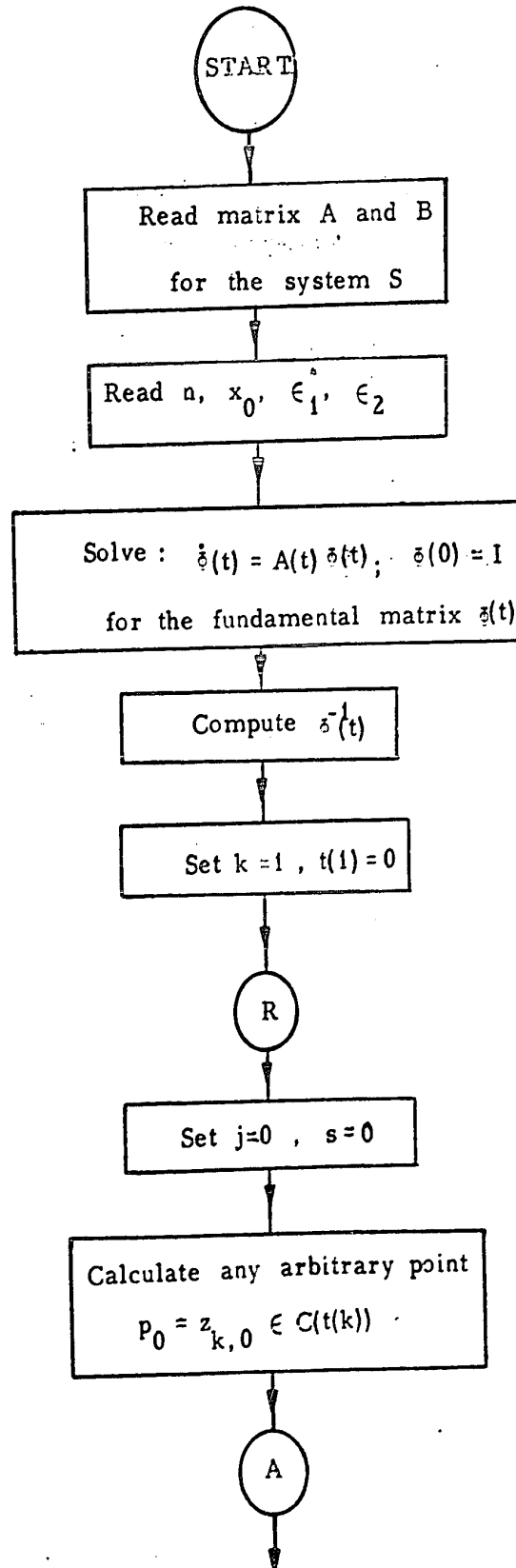
Step 10 . Iterative process is complete with

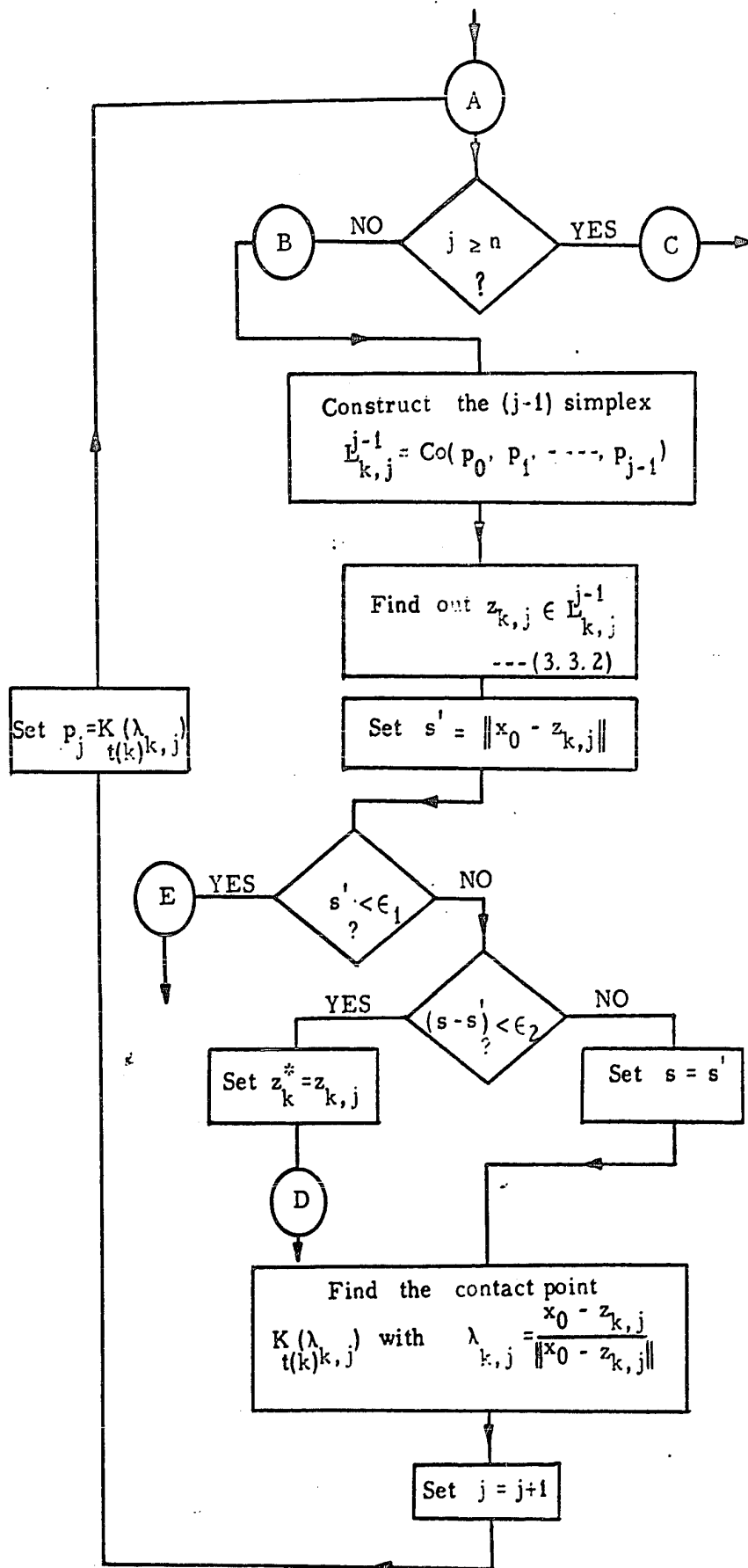
optimal time $t^* = t(k)$.

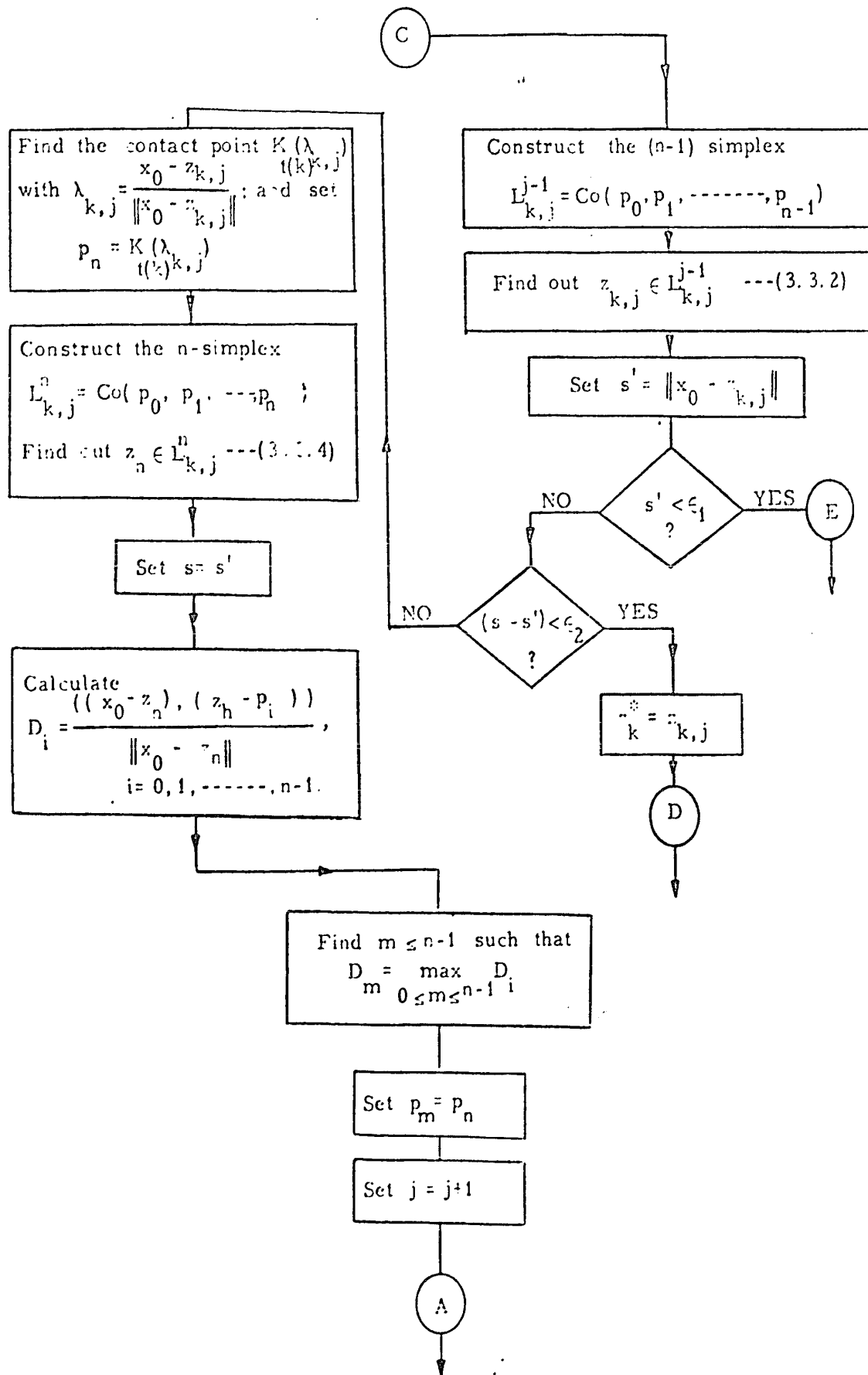
optimal control $u^*(t) = - \operatorname{sgn} \{ [\bar{\varphi}^{-1}(t) B(t)]^T \lambda_{k,j-1} \} ; 0 \leq t \leq t^*$.

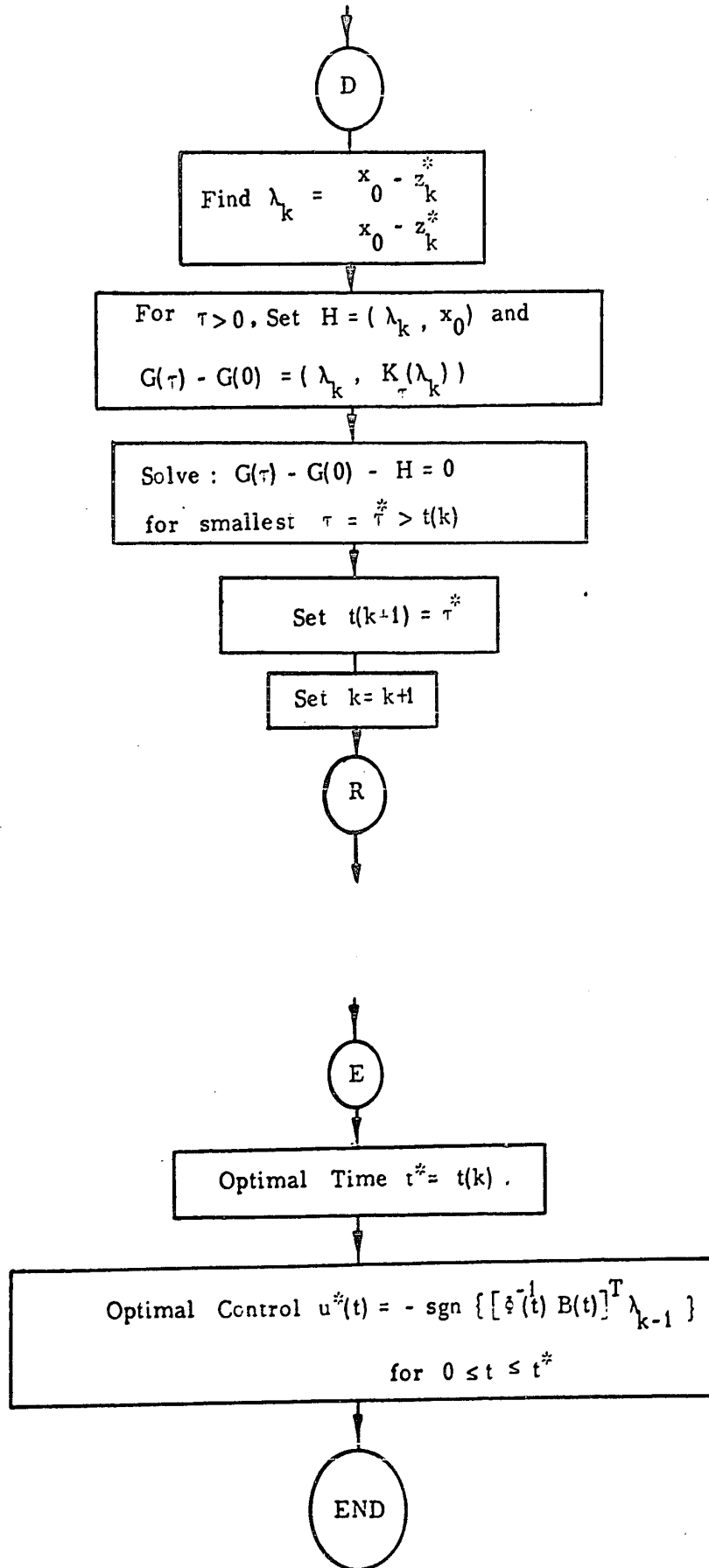
Stop.

The corresponding flow chart for solving time-optimal control problem (system S) is presented in the following figure.









3.4. Proof of Convergence.

In this section the rate of convergence of $\{t(k)\}$ to the optimal time t^* is considered. It is shown here that the rate of convergence of $\{t(k)\}$ to t^* is exponential for the iterative procedure described in this thesis.

Suppose t^* is the optimal time in which the system S (equation 2.1.1) is driven from the initial state x_0 to the origin with the optimal control $u^*(t)$ for $0 \leq t \leq t^*$. Let λ^* be the corresponding normal vector of the support hyperplane of $C(t^*)$ passing through x_0 .

For any intermediate stage k of the iterative procedure let the corresponding value $t(k) < t^*$. Now the nested iteration process will minimize the function $\|x_0 - z_{k,j}\|$ with $z_{k,j} \in C(t(k))$ for some $j = m$, that is

$$\|x_0 - z_{k,m}\| = \min_{s \in C(t(k))} \|x_0 - s\|.$$

It follows from Schwartz's inequality that

$$(\lambda^*, (x_0 - z_{k,m})) \leq \|x_0 - z_{k,m}\|.$$

Therefore

$$\|x_0 - z_{k,m}\| \geq (\lambda^*, x_0) - (\lambda^*, z_{k,m}) \quad \text{--- (3.4.1)}$$

Since $x_0 \in \partial C(t^*)$, and contained in the support hyperplane of $C(t^*)$ with λ^* as the outward normal (fig. 3.4.1), one can write from relation (2.1.7) that

$$\begin{aligned} (\lambda^*, x_0) &= (\lambda^*, \int_0^{t^*} [\bar{z}^{-1}(\tau) \exists(\tau)] \operatorname{sgn} \{ [\bar{z}^{-1}(\tau) B(\tau)]^T \lambda^* \} d\tau) \\ &= \int_0^{t^*} \left| [\bar{z}^{-1}(\tau) B(\tau)]^T \lambda^* \right| d\tau \quad \text{--- (3.4.2)} \end{aligned}$$

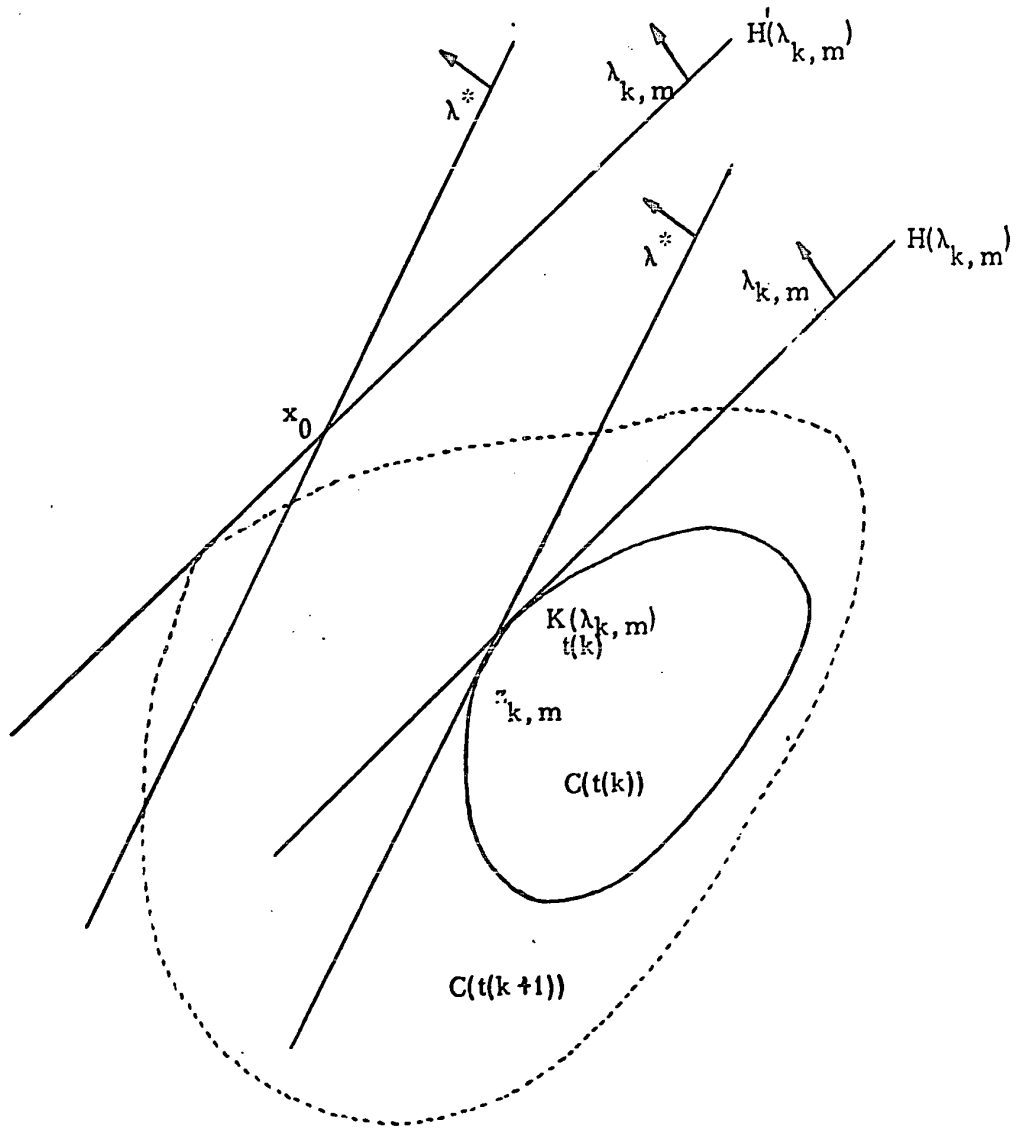


Fig. 3.4.1 Illustration of convergence rate of the iterative procedure.

Similarly, since $z_{k,m} \in \partial C(t(k))$ and contained in the support hyperplane of $C(t(k))$ with outward normal λ^* (fig. 3.4.1), it follows that

$$(\lambda^*, z_{k,m}) = \int_0^{t(k)} \left[\bar{\varepsilon}^{-1}(\tau) B(\tau) \right]^T \lambda^* d\tau \quad \text{--- (3.4.3)}$$

from equations (3.4.1), (3.4.2) and (3.4.3) it follows that

$$\|x_0 - z_{k,m}\| \geq \int_{t(k)}^{t^*} \left[\bar{\varepsilon}^{-1}(\tau) B(\tau) \right]^T \lambda^* d\tau \quad \text{--- (3.4.4)}$$

From the relation (3.4.4), and the assumption that B is continuous, there exists a number $p \in (0, 1)$ such that the following inequality holds.

$$\|x_0 - z_{k,m}\| \geq p (t^* - t(k)) \left| \left[\bar{\sigma}^{-1}(t^*) B(t^*) \right]^T \lambda^* \right| \quad \dots (3.4.5)$$

Define a unit vector ,

$$\lambda_{k,m} = \frac{x_0 - z_{k,m}}{\|x_0 - z_{k,m}\|} \quad \dots (3.4.6)$$

Now let $K(\lambda_{k,m}, t(k))$ denotes the contact function of $C(t(k))$ corresponding to the hyperplane $H(\lambda_{k,m})$, (equation 3.1.1). By Schwartz's inequality ,

$$\|x_0 - K(\lambda_{k,m}, t(k))\| \geq (\lambda_{k,m}, x_0 - K(\lambda_{k,m}, t(k))) \quad \dots (3.4.7)$$

$$\begin{aligned} &= (\lambda_{k,m}, x_0) - (\lambda_{k,m}, z_{k,m}) \\ &\quad + (\lambda_{k,m}, z_{k,m}) - (\lambda_{k,m}, K(\lambda_{k,m}, t(k))) \\ &= (\lambda_{k,m}, x_0 - z_{k,m}) - (\lambda_{k,m}, K(\lambda_{k,m}, t(k)) - z_{k,m}) \end{aligned} \quad \dots (3.4.8)$$

Now the point $K(\lambda_{k,m}, t(k)) \in H(\lambda_{k,m})$. From definition 2.6 it follows that

$$(\lambda_{k,m}, K(\lambda_{k,m}, t(k))) \geq (\lambda_{k,m}, z_{k,m}) \text{ for all } z_{k,m} \in C(t(k)).$$

Thus

$$(\lambda_{k,m}, K(\lambda_{k,m}, t(k)) - z_{k,m}) \geq 0$$

Since $\|x_0 - z_{k,m}\| \neq 0$, one can choose an $\epsilon \geq 0$ such that

$$(\lambda_{k,m}, K(\lambda_{k,m}, t(k)) - z_{k,m}) \geq \epsilon \|x_0 - z_{k,m}\| \quad \dots (3.4.9)$$

Since $(\lambda_{k,m}, x_0 - z_{k,m}) = \|x_0 - z_{k,m}\|$, --- (3.4.10)
 it follows from relations (3.4.7), (3.4.8), (3.4.9) and (3.4.10) that

$$(1 - \epsilon) \|x_0 - z_{k,m}\| \leq (\lambda_{k,m}, x_0) - (\lambda_{k,m}, K(\lambda_{k,m}, t^{(k)})_{k,m})$$

--- (3.4.11)

Since $K(\lambda_{k,m}) \in \partial C(t^{(k)})$ (fig. 3.4.1) and is contained in the support hyperplane $H(\lambda_{k,m})$ of $C(t^{(k)})$ with $\lambda_{k,m}$ as the outward normal , it can be easily shown , as in (3.4.2), that

$$(\lambda_{k,m}, K(\lambda_{k,m}, t^{(k)})_{k,m}) = \int_0^{t^{(k)}} \left| \left[\bar{\vartheta}^{-1}(\tau) B(\tau) \right]^T \lambda_{k,m} \right| d\tau$$

--- (3.4.12)

But it is clear (fig. 3.4.1) that the hyperplane $H'(\lambda_{k,m})$ which passes through the point x_0 with outward normal $\lambda_{k,m}$, does not touch $C(t^{(k)})$.
 Let $t^{(k+1)}$, $t^{(k+1)} > t^{(k)}$, be the shortest time for which $C(t^{(k+1)}) \cap H'(\lambda_{k,m})$ is not empty , where $C(t^{(k+1)}) \supset C(t^{(k)})$. Let $q \in \partial C(t^{(k+1)})$ be the point where the support hyperplane $H'(\lambda_{k,m})$ touches $C(t^{(k+1)})$. Since q and x_0 both lie on the support hyperplane $H'(\lambda_{k,m})$, it follows that

$$(\lambda_{k,m}, x_0) = (\lambda_{k,m}, q) = \text{constant} .$$

Following the same reasoning as mentioned before,

$$(\lambda_{k,m}, x_0) = (\lambda_{k,m}, q) = \int_0^{t^{(k+1)}} \left| \left[\bar{\vartheta}^{-1}(\tau) B(\tau) \right]^T \lambda_{k,m} \right| d\tau$$

--- (3.4.13)

From relations (3.4.11) , (3.4.12) and (3.4.13) it follows that,

$$(1 - \epsilon) \|x_0 - z_{k,m}\| \leq \int_0^{t^{(k+1)}} \left| \left[\bar{\vartheta}^{-1}(\tau) B(\tau) \right]^T \lambda_{k,m} \right| d\tau - \int_0^{t^{(k)}} \left| \left[\bar{\vartheta}^{-1}(\tau) B(\tau) \right]^T \lambda_{k,m} \right| d\tau$$

Therefore

$$\|x_0 - z_{k,m}\| \leq \frac{1}{(1-\epsilon)} \int_{t(k)}^{t(k+1)} |[\bar{\sigma}^{-1}(\tau)B(\tau)]^T \lambda_{k,m}| d\tau \quad \text{--- (3.4.14)}$$

$$\text{Defining } \max_{\|\lambda\|=1} \max_{\tau \in [0, t^*]} |[\bar{\sigma}^{-1}(\tau)B(\tau)]^T \lambda| = M \quad \text{--- (3.4.15)}$$

the above inequality (3.4.14) implies that

$$\|x_0 - z_{k,m}\| \leq \frac{M}{(1-\epsilon)} (t(k+1) - t(k)) \quad \text{--- (3.4.16)}$$

Since the system S in equation (2.1.1) is assumed to be controllable, optimal time $t^* < \infty$ and the sequence $\{t(k)\}$ is bounded. Further, by construction (section 3.1), $\{t(k)\}$ is an increasing sequence. Therefore the difference $\{t(k+1) - t(k)\} \rightarrow 0$ as $k \rightarrow \infty$. Consequently it follows from (3.4.16) that $\lim_{k \rightarrow \infty} z_{k,m} = x_0$. This proves the convergence of the iterative procedure developed in this thesis. The rate of convergence is discussed as follows. From relation (3.4.16) and (3.4.5), it follows

$$\frac{M(t(k+1) - t(k))}{(1-\epsilon)} \geq p(t^* - t(k))N \quad \text{--- (3.4.17)}$$

$$\text{where } N = |[\bar{\sigma}^{-1}(t^*)B(t^*)]^T \lambda^*|$$

Since $M \geq N$, there exists a constant μ , $0 < \mu < 1$ such that

$$(t^* - t(k+1)) \leq \mu(t^* - t(k)) \quad \text{--- (3.4.18)}$$

This result shows the exponential convergence of the sequence $t(k)$ to the optimal time t^* .

3.5. Computational Results

In this section computational results are presented for two examples solved by the iterative procedure developed in sections 3.1 and 3.2. For comparison between this method and the iterative procedure due to Fujisawa and Yasuda [14], computational results obtained for the same examples using both the methods are presented.

Example 1. The time-optimal control of a pure inertia plant S_1 characterised by the system equation

$$S_1 : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

is considered. This system is extensively studied by Pontryagin[28] and others. This system is controllable (definition 2.7).

The computational results for this system with a given initial condition $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.10 \\ -0.12 \end{bmatrix}$, obtained by using the iterative procedure developed in this thesis, is presented in table A. Each cycle of iteration (for a fixed value of $t(k)$) involves the quadratic minimization problem PI (section 3.2) on the convex set $C(t(k))$. The nested iteration that performs this minimization in each cycle stops when the value of the error (step 5 in section 3.3) is less than a preassigned value ($=0.0001$). The main iteration terminates when the value of the error (step 4 or 8 in section 3.3) is less than a preassigned value ($=0.0006$).

Table A

Assumed value of $t(1) = 0.5$

Main Iteration Number (k)	Results of Nested Iteration					Vector $\begin{bmatrix} 1 \\ \lambda_k \\ 2 \\ \lambda_k \end{bmatrix}$	Modified Time $t(k+1)$
	Vector $\begin{bmatrix} z_{k,0}^1 \\ 2 \\ z_{k,0} \end{bmatrix}$	Index of Nested Iteration (j)	Vector $\begin{bmatrix} z_{k,j}^1 \\ 2 \\ z_{k,j} \end{bmatrix}$	Value of $\ x_0 - z_{k,j}\ $	Error		
1	0.1250	1	0.1250	2.0112238	0	0.981989	2.2395592
	-0.5000		-0.5000				
2	2.503	1	1.228	1.309	1.309	0.872539	2.6525259
	-2.240	2	1.252 -.09989	0.8487	0.4603		
		3	1.507 -.4520	0.6796	0.1691		
		4	1.5070 -.4520	0.67963684	0.1013×10^{-5}		
3	3.518	1	1.397	1.168	1.168	0.838037	2.7798901
	-2.653	2	1.758 -.002339	0.3615	0.8065		
		3	1.918 -.2404	0.2179	0.1436		
		4	1.9184 -.2404	0.21790494	0.1192×10^{-6}		
4	3.854	1	1.441	1.129	1.129	0.821821	2.8201475
	-2.780	2	1.916 -.03293	0.2033	0.9257		
		3	2.043 -.1597	0.06953	0.1338		
		4	2.0429 -.1597	0.0695285	0.5960×10^{-7}		

(Table A. contd.)

Main Iteration Number (k)	Results of Nested Iteration					Vector $\begin{bmatrix} \lambda_k^1 \\ \lambda_k^2 \end{bmatrix}$	Modified Time t (k+1)
	Vector $\begin{bmatrix} 1 \\ z_{k,0} \\ 2 \\ z_{k,0} \end{bmatrix}$	Index of Nested Iteration (j)	Vector $\begin{bmatrix} 1 \\ z_{k,j} \\ 2 \\ z_{k,j} \end{bmatrix}$	Value of $\ x_0 - z_{k,j}\ $	Error		
5	3.977 -2.820	1	1.454 -1.031	1.117	1.117	0.818164 0.577948	2.8330536
		2	1.963 -.04895	0.1544	0.9626		
		3	2.082 -.1328	0.02215	0.1323		
		4	2.0819 -.1328	0.02214712	0.9425×10^{-3}		
6	4.013 -2.833	1	1.458 -1.029	1.113	1.113	0.816158 0.588958	2.8676102
		2	1.977 -.05467	0.1389	0.9741		
		3	2.094 -.1240	0.006911	0.1320		
		4	2.0944 -.1240	0.0069105	0.5585×10^{-6}		
7	4.026 -2.838	1	1.460 -1.029	1.112	1.112	0.817955 0.608541	2.9397821
		2	1.982 -.05675	0.1335	0.9785		
		3	2.099 -.1209	0.001528	0.1320		
		4	2.0988 -.1209	0.00152824	0.5923×10^{-7}		

(Table A contd.)

Main Iteration Number (k)	Results of nested Iteration				Vector $\begin{bmatrix} 1 \\ \lambda_k \\ 2 \\ \lambda_k \end{bmatrix}$	Modified Time t (k+1)	
	Vector $\begin{bmatrix} 1 \\ z_{k,0} \\ 2 \\ z_{k,0} \end{bmatrix}$	Index of Nested Iteration (j)	Vector $\begin{bmatrix} 1 \\ z_{k,m} \\ 2 \\ z_{k,m} \end{bmatrix}$	Value of $\ x_0 - z_{k,m}\ $			Error
8	4.028 -2.838	1	1.460 -1.029	1.111	1.111	0.7618015 0.5825617	2.9399157
		2	1.983 -.05713	0.1325	0.9785		
		3	2.100 -.1203	0.0006704	0.1320		
		4	2.0995 -.1203	0.00056947	0.9667x10 ⁻⁶		
9	4.045 -2.844	1	1.462 -1.028	1.110	1.110		
		2	1.984 -.05730	0.1321	0.9779		
		3	2.100 -.1201	0.0001249	0.1320		
		4	2.0999 -.1201	0.12489x10 ⁻³	0.9852x10 ⁻⁷		

* With the exception of a few important data , the rest are tabulated with accuracy upto 4th. decimal places.

Total computation time = 41.69 seconds. (IEM System 360/65.)

From the table A , it is found that the iteration process terminates at the ninth cycle when the value of $\|x_0 - z_{k,j}\| = 0.00012488855$ which is much less than its preassigned tolerance 0.0006. As discussed in section 3.3 the value of the optimal time $t^* = 2.9399157$, and value of the vector $\lambda^* = \begin{bmatrix} 0.761801 \\ 0.582561 \end{bmatrix}$. The optimal control $u^*(t)$, $0 \leq t \leq t^*$ is calculated and is given by

$$u^*(t) = -1 \text{ for } 0 \leq t \leq t_s$$

$$= +1 \text{ for } t_s < t \leq t^* ,$$

where the switching time $t_s = 0.764705$ secs. So the time-optimal control problem for the system S_1 is solved completely by the iterative procedure developed.

In figure 3.5.1 the rate of convergence of $t(k)$ to t^* with several values of $t(1)$ is shown and it is exponential as desired (section 3.4). The value of t^* and the corresponding u^* obtained with several $t(1)$'s are identical as expected. It is interesting to note from the figure 3.5.1 that the choice of the values of $t(1)$ closer to t^* reduces the number of iterations required to solve the problem thereby reduces the computational time also. Further it has been found(table A) that with the exception of the main iteration cycle 1 , the rest involve a number of nested iterations. It is interesting to note that for each of these main cycles, the number of nested iterations are exactly four though it need not be true in general.

Computational results obtained , for the same system S_1 with the initial state $\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.10 \\ - .12 \end{bmatrix}$. by using the method due to Fujisawa and Yasuda[14], is presented in table B. The results of the nested iterations involved in each stage are omitted in this table. For the sake of comparison of these results with those given in table A , this iteration process also initiated with $t(1)=0.5$, with identical error criterion.

$$S_1: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u ; \quad \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} = \begin{bmatrix} 2.10 \\ -1.12 \end{bmatrix}$$

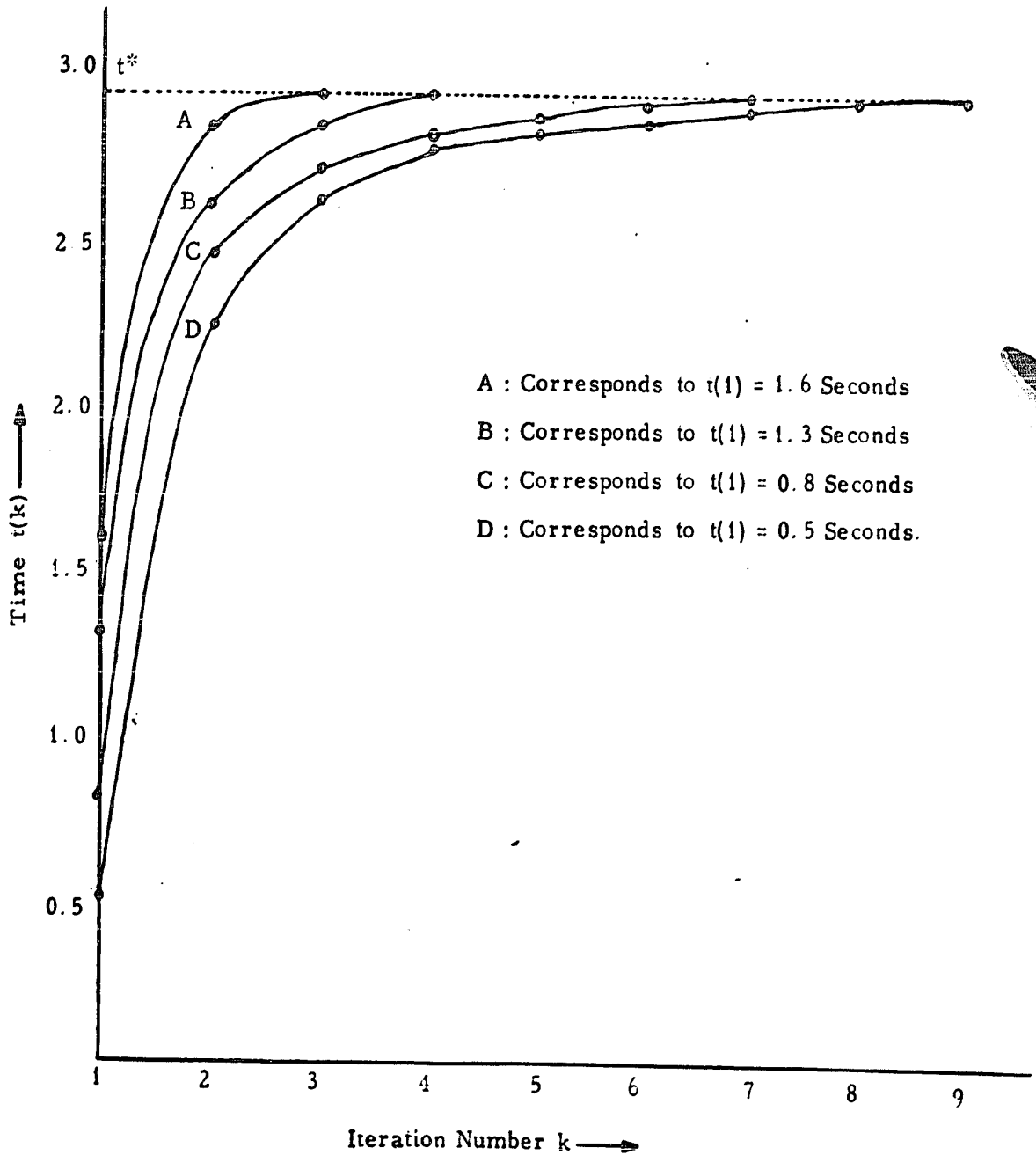


Fig. 2.5.1 Rate of convergence of $t(k)$ to t^* with several $t(1)$.

Table B

Assumed value of $t(i) = 0.5$

Number of Cycles (k)	Time t(k)	State Vector		Value of $\ x_0 - z_k\ $
		z_k^1	z_k^2	
1	0.5	0.1250	-0.5000	2.0112238
2	2.2395592	1.4778	-0.4853	0.72149909
3	2.6256699	1.9463	-0.2236	0.28532580
4	2.7138833	2.0280	-0.1715	0.088478267
5	2.7955524	2.0580	-0.1510	0.052156322
6	2.8160967	2.0728	-0.1405	0.034081470
7	2.8561163	2.0812	-0.1344	0.023649830
8	2.9100313	2.0365	-0.1304	0.017072327
9	2.9200667	2.0900	-0.1278	0.012668718
10	2.9275246	2.0925	-0.1259	0.0095865130
11	2.9331598	2.0942	-0.1246	0.0073687658
12	2.9374628	2.0955	-0.1236	0.0057359636
13	2.9408159	2.0965	-0.1223	0.0044991113
14	2.9434156	2.0972	-0.1222	0.0035584904
15	2.9454784	2.0978	-0.1218	0.0028314642
16	2.9471006	2.0982	-0.1214	0.0022677574
17	2.9484406	2.0986	-0.1211	0.0018005426
18	2.9492855	2.0988	-0.1209	0.0015170977
19	2.9518967	2.0995	-0.1204	0.00064270967
20	2.9520864	2.0995	-0.1204	0.00058052759

The table B shows that this iteration process takes 20 iterations to find out the optimal time t^* and the optimal control u^* for the system S_1 . The value of the optimal time calculated by this procedure is $t^* = 2.9520864$ seconds. The optimal control $u^*(t)$, $0 \leq t \leq t^*$ is calculated and is given by

$$u^*(t) = -1 \quad \text{for } 0 \leq t \leq t_s$$

$$= +1 \quad \text{for } t_s < t \leq t^*$$

where switching time $t_s = 0.8$ second.

The rate of convergence of $t(k)$ to t^* as calculated by the proposed method in this thesis, and the method due to Fujisawa and Yasuda are shown in figure 3.5.2 a and 3.5.2.b respectively. It is clear from the figure that, though the initial rates are almost same, the rate of convergence becomes very slow in the method due to Fujisawa and Yasuda in the later stage compared to the proposed method. Comparison between the two methods is presented in the table below.

Table C

Method	Number of Iterations Required	Value of Optimal Time t^* Obtained	Total Computation Time (IBM 360/65)
The Proposed Method	9	2.9399157 Seconds.	41.69 Seconds.
The Fujisawa And Yasuda's Method.	20	2.9520864	1 Min. 3.22 Seconds.

$$S : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u ; \quad \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} = \begin{bmatrix} 2.10 \\ -12 \end{bmatrix}$$

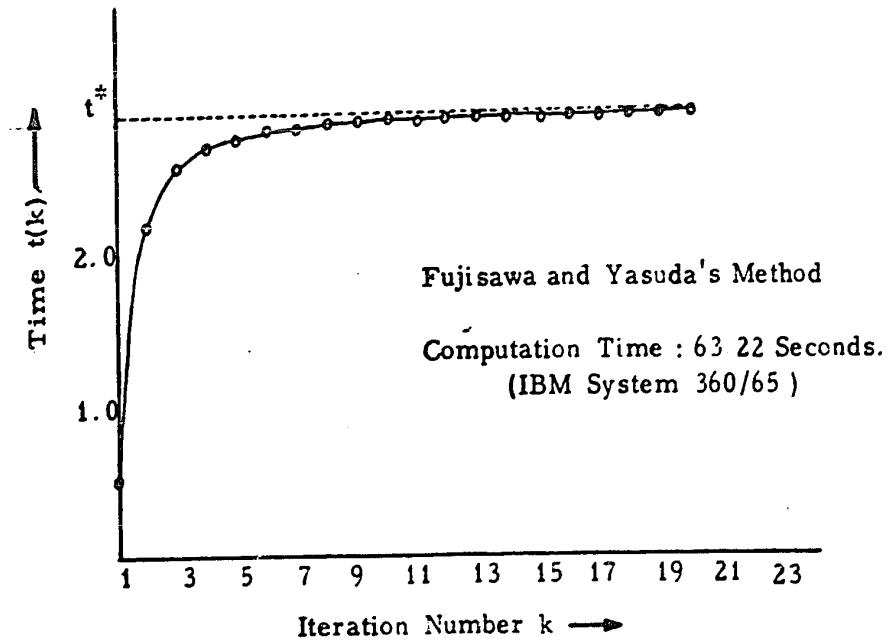
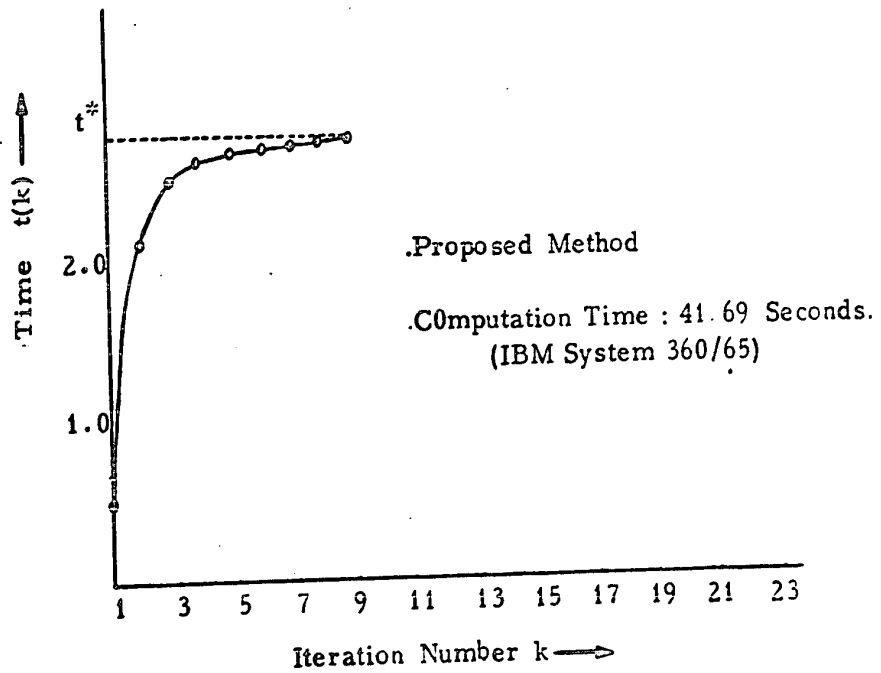


Fig. 3.5.2. Convergence of $\{t(k)\}$ to optimal time t^* .

It is found that there is 0.4% discrepancy in the value of t^* as calculated by the two methods. In figure 3.5.3 the error $\|x_0 - z_k\|$ as a function of the iteration number k , for the two methods, is shown. The curve A corresponds to the proposed method and shows clearly a much faster rate of reduction compared to the curve B corresponding to the method due to Fujisawa and Yasuda. Moreover it is to be noted that both the methods terminate, when the error becomes less than the preassigned value, 0.0006. At the ninth iteration the value of the error given by the proposed method is 1.2489×10^{-4} whereas that given by Fujisawa and Yasuda's method at the 20th iteration is 5.805×10^{-4} . It is clear that the later method would have taken quite a number of iterations more to reach the above degree of accuracy attained by the former.

Example 2: In order to present their iterative procedure Fujisawa and Yasuda solved a time-optimal control problem for the system S_2 characterised by the following differential equation

$$S_2 : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

The corresponding iteration results for one initial condition $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.20 \\ -0.55 \end{bmatrix}$ are given in their paper

The time-optimal control problem corresponding to the system S_2 is also solved by the proposed iterative procedure. The computational results with the same initial state as mentioned above are presented in table D. The iteration process is terminated when the error (step 4/8 section 3.3) becomes less than 0.00005. Each cycle of main iteration involves a nested iteration process. This nested iteration ends when it reaches the preassigned value 0.00001.

$$S : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u ; \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} = \begin{bmatrix} 2 \ 10 \\ -.12 \end{bmatrix}$$

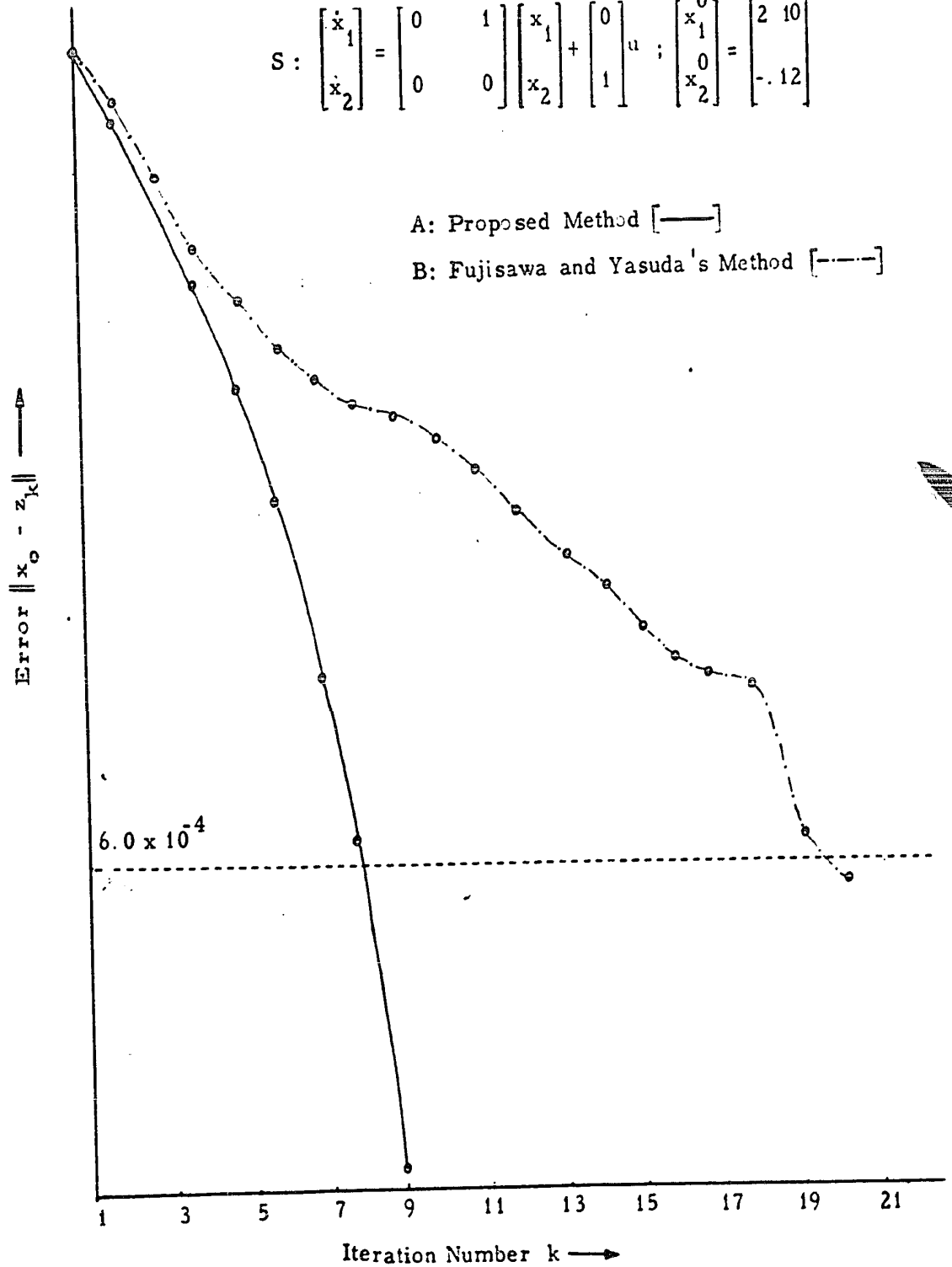


Fig. 3.5.3 Rate of Convergence

Table D

Main Iteration Number (k)	Results of Nested Iteration					Vector $\begin{bmatrix} 1 \\ \lambda_k \\ 2 \\ \lambda_k \end{bmatrix}$	Modified Time t(k+1)
	Vector $\begin{bmatrix} z_{k,0}^1 \\ z_{k,0}^2 \end{bmatrix}$	Index of Nested Iteration (j)	Vector $\begin{bmatrix} z_{k,j}^1 \\ z_{k,j}^2 \end{bmatrix}$	Value of $\ x_0 - z_{k,j}\ $	Error		
1	0.0 0.0	1	0.0 0.0	1.3200369	0.0	0.909065 -.416655	1.8374376
2	1.8374 0.8408	1	0.7841 0.3588	0.99943417	0.9994		
		2	0.5097 0.03500	0.90482551	0.09461		
		3	0.7557 0.09045	0.77945101	0.1254		
		4	0.8432 0.0790	0.72312969	0.05632		
		5	0.8230 0.05037	0.70893180	0.01420		
		6	0.8385 0.05584	0.70551270	0.003419		
		7	0.8322 0.0510	0.70459604	0.9167×10^{-3}		
		8	0.8358 0.05294	0.70439249	0.2035×10^{-3}		
		9	0.8342 0.05192	0.70433271	0.5978×10^{-4}		
		10	0.8352 0.05247	0.70432073	0.1198×10^{-4}		
		11	0.8348 0.05226	0.70431727	0.3457×10^{-5}		

(Table D contd.)

Main Iteration Number (k)	Results of Nested Iteration					Vector $\begin{bmatrix} \lambda_k^1 \\ \lambda_k^2 \end{bmatrix}$	Modified Time t(k+1)
	Vector $\begin{bmatrix} z_{k,0}^1 \\ z_{k,0}^2 \end{bmatrix}$	Index of Nested Iteration(j)	Vector $\begin{bmatrix} z_{k,j}^1 \\ z_{k,j}^2 \end{bmatrix}$	Value of $\ x_0 - z_{k,j}\ $	Error		
3	3.4037 0.9667	1	0.9659 0.2743	0.85694063	0.8569		
		2	0.8853 -.4655	0.32581884	0.5311		
		3	1.0870 -.3509	0.22903210	0.09679		
		4	1.1250 -.4672	0.19753748	0.03149		
		5	1.1410 -.3778	0.18202657	0.01551		
		6	1.1480 -.3834	0.17450869	0.007517		
		7	1.1463 -.3857	0.17287916	0.001630		
		8	1.1486 -.3858	0.17236161	0.5775×10^{-3}		
		9	1.1478 -.3861	0.17228699	0.7453×10^{-4}		
		10	1.1479 -.3862	0.17221884	0.3815×10^{-4}		
		11	1.1479 -.3861	0.17214623	0.2623×10^{-5}	0.207699 -.951544	4.0090513
4	4.0091 0.9818	1	1.0054 0.2461	0.81966591	0.8197		
		2	1.1942 -.5286	0.02216476	0.7975		
		3	1.1945 -.5285	0.02216357	0.1188×10^{-5}	0.248154 -.970060	4.1041532

(Table D contd.)

Main Iteration Number (k)	Results of Nested Iteration					Vector $\begin{bmatrix} \lambda_k^1 \\ \lambda_k^2 \end{bmatrix}$	Modified Time t(k+1)
	Vector $\begin{bmatrix} z_{k,0}^1 \\ z_{k,0}^2 \end{bmatrix}$	Index of Nested Iteration(j)	Vector $\begin{bmatrix} z_{k,j}^1 \\ z_{k,j}^2 \end{bmatrix}$	Value of $\ x_0 - z_{k,j}\ $	Error		
5	4.0091 0.9835	1	1.0103 0.2461	0.81455469	0.8146		
		2	1.199 -.5412	0.8802×10^{-2}	0.8058		
		3	1.199 -.5442	0.5841×10^{-2}	0.002961		
		4	1.199 -.5461	0.3929×10^{-2}	0.001912		
		5	1.199 -.5472	0.2857×10^{-2}	0.001072		
		6	1.1994 -.5478	0.2292×10^{-2}	0.5645×10^{-3}		
		7	1.1995 -.5478	0.2006×10^{-2}	0.2852×10^{-3}		
		8	1.1995 -.5482	0.1854×10^{-2}	0.1522×10^{-3}		
		9	1.1996 -.5482	0.1842×10^{-2}	0.1522×10^{-3}		
		10	1.1996 -.5482	0.1834×10^{-2}	0.7095×10^{-5}	0.217985 -.980932	4.1117830
6	4.1118 0.9836	1	1.0713 0.2418	0.81409442	0.8141		
		2	1.199 -.5424	0.7635×10^{-2}	0.8065		
		3	1.199 -.5457	0.2290×10^{-2}	0.2098×10^{-2}		

(Table D contd.)

Main Iteration Number (k)	Results of Nested Iteration				Vector $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$	Modified Time t(k+1)
	Vector $\begin{bmatrix} z_{k,0}^1 \\ z_{k,0}^2 \end{bmatrix}$	Index of Nested Iteration (j)	Vector $\begin{bmatrix} z_{k,j}^1 \\ z_{k,j}^2 \end{bmatrix}$	Value of $\ x_0 - z_{k,j}\ $		
6		4	1.2 -.5457	0.2290×10^{-2}	0.2098×10^{-2}	
		5	1.2 -.5489	0.1113×10^{-2}	0.1177×10^{-2}	
		6	1.2 -.5494	0.4952×10^{-3}	0.6182×10^{-3}	
		7	1.2 -.5498	0.1590×10^{-3}	0.3361×10^{-3}	
		8	1.2 -.5499	0.1283×10^{-3}	0.1242×10^{-3}	
		9	1.2 -.5499	0.6062×10^{-4}	0.6772×10^{-4}	
		10	1.2 -.5501	0.3579×10^{-4}	0.3482×10^{-4}	
		11	1.2 -.5500	0.2602×10^{-4}	0.2282×10^{-4}	

The table D shows that the iterative process initiates with $t(k)=0.0$ for $k=1$. The optimal time is given by $t^* = 4.1117830$ seconds. The optimal control $u^*(t)$, $0 \leq t \leq t^*$, is calculated and is given by,

$$u^*(t) = +1 \quad \text{for } 0 \leq t \leq t_s$$

$$= -1 \quad \text{for } t_s < t \leq t^*$$

where switching time $t_s = 1.48953$ seconds.

The total number of iterations required are 6 and the total computation time is 39.09 seconds. Each cycle of main iteration involves nested iterations. It is interesting to note that the number of nested iterations required in the fourth cycle of main iteration is only 3, whereas in 5th. and 6th. cycles it requires 10 and 11 iterations respectively. Referring back to example 1, it is found in table A that the number of nested iterations required for each main cycle are 4. It appears that there is no general rule governing the number of nested iterations required for each main cycle.

The system S_2 is also solved by using the method due to Fujisawa and Yasuda for the same initial condition employing identical error criteria. Iteration results are given in table E. This table does not show the results of the nested iterations in detail but show the total number of nested iterations required at each cycle of the main iteration.

Table E

Number of Main Cycle (k)	Time t(k)	Number of Nested Iterations Required	State Vector		Value of $\ x_0 - z_k\ $
			z_k^1	z_k^2	
1	0.0	1	0.0	0.0	1.320
2	1.8374376	7	0.8355	0.0527	0.7043
3	3.4067240	31	1.1472	-.3869	0.1715
4	4.0098972	3	1.1947	-.5287	0.02197
5	4.1087351	45	1.1993	-.5469	0.3173×10^{-2}
6	4.1100807	146	1.1995	-.5478	0.2266×10^{-2}
7	4.118031	11	1.200	-.5500	0.3874×10^{-5}

From the table E , it is found that the iteration process starts with $t(k)=0.0$ for $k=1$ and ends after the 7th. cycle when the error reaches the value of 0.000003874. The optimal time is found to be $t^* = 4.1118031$ seconds, and the optimal control $u^*(t)$, $0 \leq t \leq t^*$, is calculated and is given by ,

$$u^*(t) = +1 \quad \text{for } 0 \leq t \leq t_s$$

$$= -1 \quad \text{for } t_s < t \leq t^*$$

where the switching time $t_s = 1.48186$ seconds.

The rate of convergence of $t(k)$ to t^* as calculated by the proposed iterative method and the method due to Fujisawa and Yasuda are plotted in the figure 3.5.4.a and 3.5.4.b respectively. For the proposed method , the rate of convergence is a bit faster at later stages, and computation time is 39.09 seconds compared to 50.15 seconds taken by the later method.

In the table F the detail nested iteration results , which are obtained in stage 3 of the main iteration (Table E) are presented. In the proposed method , a method due to Pecsvaradi and Narendra has been used to minimize the quadratic function $Q_k(z)$ involved at each stage k . This iteration method is superior to the Fujisawa and Yasuda's method. In the figure 3.5.5 the rate of convergence of both the methods is shown. It is to be noted that the quadratic minimization technique used in the proposed iterative procedure has monotone convergence , whereas for the other, the rate of convergence is oscillatory and slow.

$$S_2: \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \quad ; \quad \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} = \begin{bmatrix} 1.20 \\ -0.55 \end{bmatrix}$$

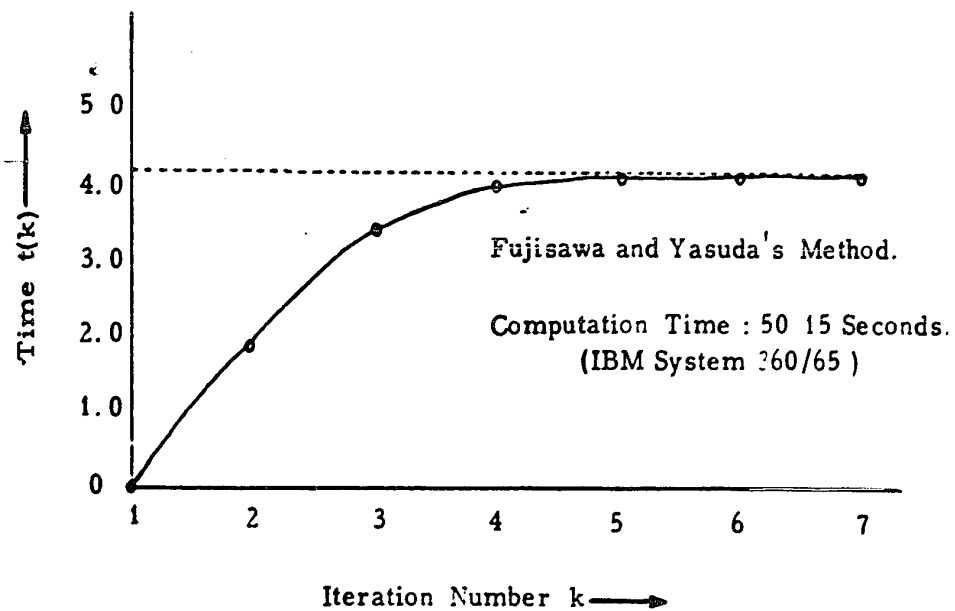
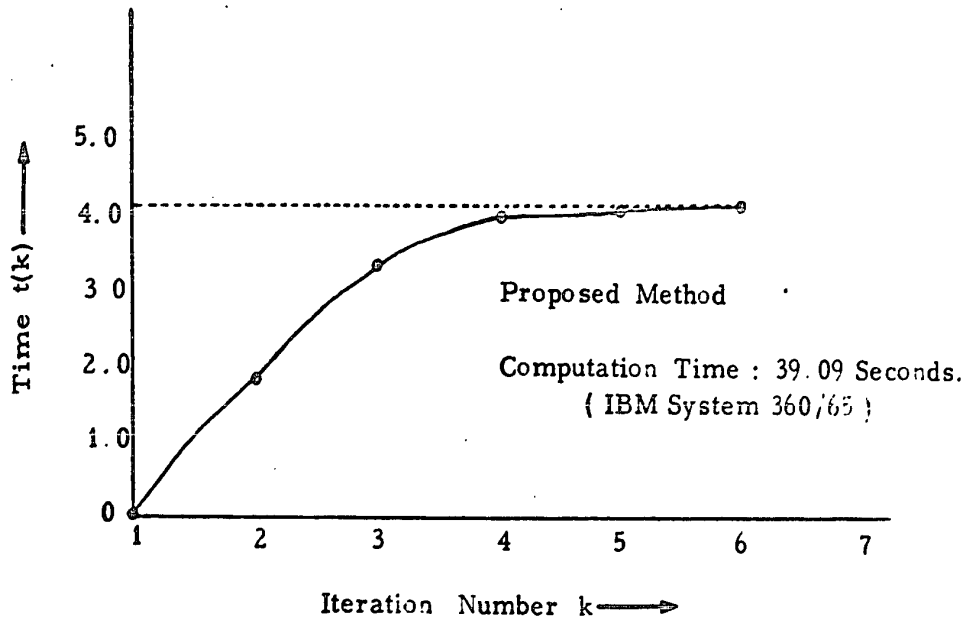


Fig. 3 5.4 Convergence of $\{t(k)\}$ to optimal time t^*

* Table F

Nested Iteration Results Corresponding to Cycle 3 of Table E

Index of Nested Iteration (j)	State Vector		Value of $\ x_0 - z_{k,j}\ $	Error
	$z_{k,j}^1$	$z_{k,j}^2$		
1	0.9661	0.2742	2.678	1.797
2	0.8868	-.4658	0.8567	0.5914
3	1.0874	-.3518	0.3243	0.6687
4	1.1195	-.3395	0.2280	0.04377
5	1.1557	-.3321	0.2254	0.01441
6	1.1354	-.3388	0.2220	0.02427
7	1.0705	-.3931	0.2209	0.01113
8	1.1805	-.3732	0.2038	0.09551
10	1.1439	-.3601	0.1979	0.007218
11	1.1315	-.3665	0.1969	0.006585
12	1.1497	-.3616	0.1961	0.006432
14	1.1553	-.3621	0.1944	0.005290
15	1.1426	-.3662	0.1932	0.008364
17	1.1249	-.3904	0.1788	0.046958
18	1.1357	-.3856	0.1767	0.011143
20	1.1527	-.3809	0.1761	0.009726
22	1.1428	-.3887	0.1757	0.0007620
24	1.1463	-.3872	0.1715	0.0001767
26	1.1487	-.3867	0.1715	0.1457×10^{-4}
28	1.1497	-.3864	0.1715	0.1138×10^{-4}
29	1.1483	-.3866	0.1715	0.5134×10^{-4}
30	1.1474	-.3869	0.1715	0.1574×10^{-4}
31	1.1472	-.3869	0.1715	0.8968×10^{-5}

* With the exception of a few data the rest are tabulated with accuracy upto 4 th. decimal places.

$$S_2: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u ; \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.20 \\ -0.55 \end{bmatrix}$$

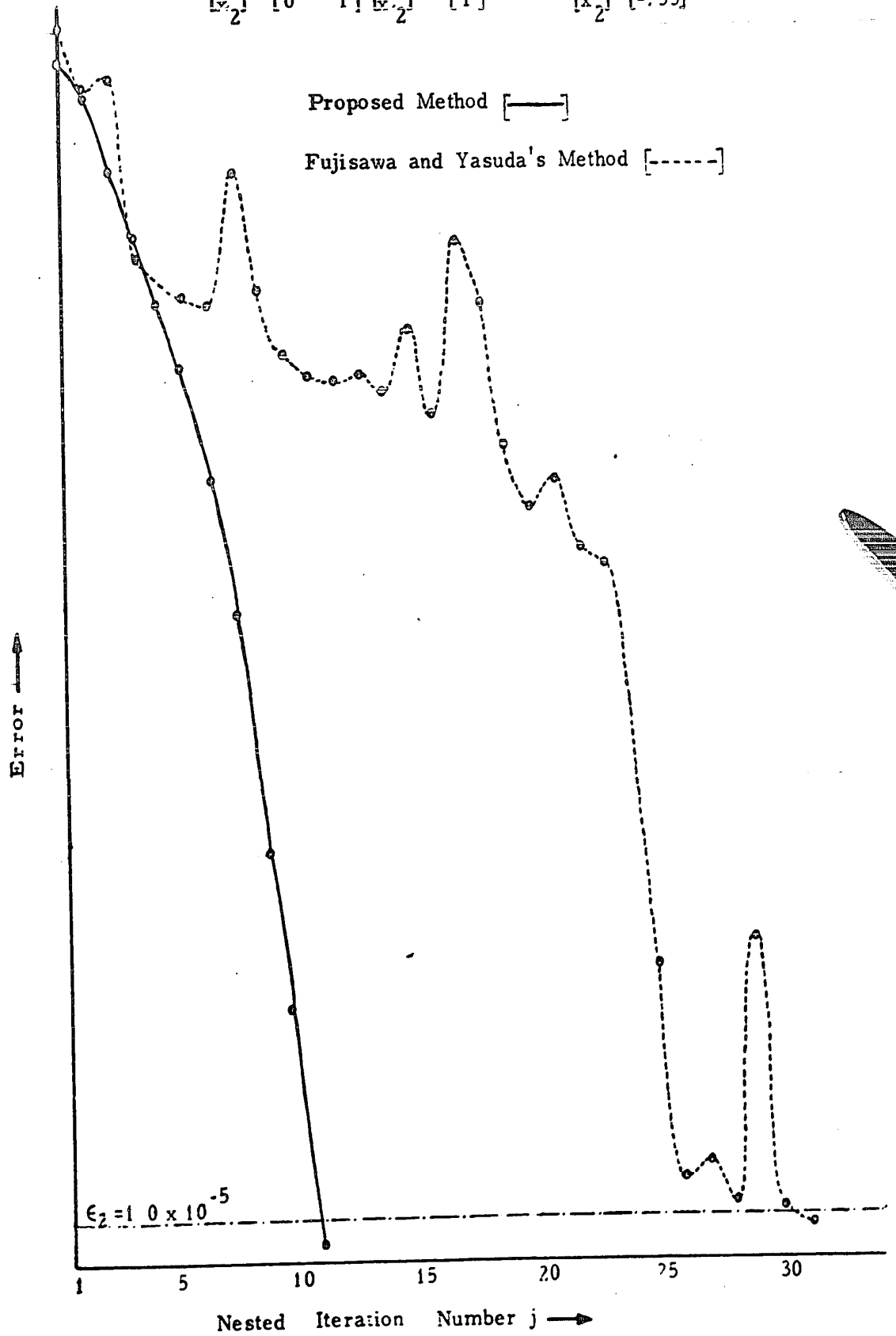


Fig. 3.5.5. Error vs. nested iteration number plot in cycle 3

The systems S_1 and S_2 have been solved for a number of initial states by the proposed method and the Fujisawa and Yasuda's method. In each case it has been observed that the rate of convergence as well as the computation time is better in the proposed method than the method due to Fujisawa and Yasuda.

CONCLUSIONS

A modified iterative procedure for computing time-optimal controls for finite dimensional linear systems (time-varying or time-invariant) has been presented in this thesis. This procedure is suitable for computation on digital computers. This algorithm does not involve any empirical procedures for step-size determination, and has rapid exponential convergence. The system under consideration is required to be controllable (definition 2.7). This iterative procedure itself has no ability to identify whether the system is controllable with respect to the given initial state. If the system under consideration is uncontrollable then the iterative procedure may fail at the ninth step (section 3.3) where no finite τ can solve equation (3.3.5). However, the iterative procedure may continue indefinitely without failure even for uncontrollable case when the sequence $\{t(k)\}$ obtained in step nine (section 3.3) increases monotonically without bound.

In this method the choice of the error value ϵ_2 appearing on page 29, section 3.3, in connection with the nested iteration process has considerable effect on computation time. If the value of ϵ_2 is allowed to be significantly high, then at each cycle the nested iteration may terminate fast but the total number of cycles required to solve the problem will increase and the computational time may also increase as observed in practice. Again if ϵ_2 is made very small, it is found that the nested iterations at each cycle take longer time. It would be interesting to find out a method that may determine the optimal value of ϵ_2 for each cycle of iteration. This is proposed for further study.

Examples of higher dimensional systems worked out using the iterative procedure proposed in this thesis and subsequent comparison of the corresponding results with those obtained by other methods [9], [14], [26], will indicate the expected superiority of the present method over the others. This is left for further study.

Since our computational algorithm is independent of any requirement of the normality of the system it is expected that the proposed method will be equally applicable for singular systems. The control however is not unique in this case.

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