GENERAL VIBRATION THEORY OF DEEP SPHERICAL SANDWICH SHELLS

by

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ABSTRACT

A complete system of differential equations of motion for free non-symmetric vibrations of deep spherical sandwich shells, stress-resultants and displacement relationships and the boundary conditions are derived using variational technique. The sandwich shell considered herein consists of three layers. The central thick core is assumed to be incompressible in the radial direction and face sheets are assumed to be made of the same isotropic material with equal thickness. The face parallel stresses in the core are neglected as compared to the inplane stresses of the face sheets. The flexural and extensional rigidities of the face sheets are taken into account. The effects of transverse shear deformation and rotary inertia have also been included in this analysis.

New deformation functions have been introduced which considerably simplify the system of differential equations. The final solution is expressed in terms of associated Legendre functions for the non-symmetric case.

The differential equations of motion and their solution for some special cases, such as the axisymmetric case of deep spherical sandwich shells in which two types of face sheets (flexural members and membranes) are considered, homogeneous spherical shells and circular sandwich plates, are also presented in this thesis.

The frequency equations of the homogeneous and sandwich spherical shells for various boundary conditions have been obtained in terms of Legendre functions and their derivatives. For the clamped edge circular sandwich plate frequency equation can be expressed in terms of Bessel's functions. The roots of highly transcendental frequency equations are generated with the help of high speed digital computer using iterative technique. Numerical computations have been performed for the axi-symmetric case and fixed boundary condition only. Non-dimensionalized frequency parameter has been plotted against thickness of the face sheets to radius ratio of the sandwich shells and plates. The effects of various parameters on natural frequencies have been studied in detail. The values of Ω for clamped edge homogeneous spherical shells are compared with their values available in the literature. Since no work on vibrations of deep spherical sandwich shells is available, in the literature, the numerical values of the natural frequencies of the sandwich shell have not been compared. The mode shapes of the sandwich shells and plates and also of homogeneous spherical shells have been plotted up to three modes. The computer programs in FORTRAN IV for the numerical computation of the natural frequency and mode shapes are listed in the Appendix.

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NOMENCLATURE

a	radius of the sandwich plate
$2c, h$	thickness of core and face sheets respectively
D, K	flexural and extensional rigidities respectively of the face sheets
E, ν	elastic modulus and Poisson's ratio of the face sheets
G, G_c	shear modulus of the face sheets and core respectively
$I_o(g_\alpha x)$	modified Bessel function
$J_o(g_\alpha x)$	Bessel function
M_ϕ, M_θ	moment resultants of sandwich shell in ϕ and θ directions respectively
M_r, M_θ	moment resultants of circular sandwich plate in r and θ directions respectively
$M_{\theta\phi}$	twisting moment resultant of sandwich shell
N_ϕ, N_θ	normal stress resultants of sandwich shell in ϕ and θ directions respectively
N_r, N_θ	normal stress resultants of circular sandwich plate in r and θ directions
$N_{\theta\phi}$	shear stress resultants of the sandwich shell
$P^m_{\nu_\alpha}(\cos\phi)$, $Q^m_{\nu_\alpha}(\cos\phi)$	associated Legendre functions of first and second kinds respectively
Q_r, Q_θ	shear stress resultants of circular sandwich plate
Q_ϕ, Q_θ	shear stress resultants of sandwich shell

R	radius of the middle surface of the sandwich shell
r, ϕ, θ	spherical coordinates
t	instant of time
u, v, w	displacement components in θ, ϕ and r directions of the sandwich shell
v_1, v_2	displacement components in ϕ direction for face sheets 1 and 2 respectively
u_1, u_2	displacement components in θ direction for face sheets 1 and 2 respectively
β_ϕ, β_θ	rotation of the normal to the middle surface of the face sheets in ϕ and θ directions respectively
β_r	rotation of the normal to the middle surface of the face sheets of circular sandwich plate in r direction
$\gamma_{r\phi}, \gamma_{r\theta}, \gamma_{\theta\phi}$	shearing strain components
$\varepsilon_r, \varepsilon_\theta, \varepsilon_{\theta\phi}$	normal strain components
ν_α	order of Legendre function
ρ, ρ_c	mass densities of face sheets and core respectively
$\sigma_r, \sigma_\phi, \sigma_\theta$	normal stresses in r, ϕ and θ directions respectively
$\tau_{r\phi}, \tau_{r\theta}, \tau_{\theta\phi}$	shear stresses
ω, ω_p	circular frequencies of spherical shells and circular sandwich plate respectively
Ω, Ω_p	non-dimensionalized frequency parameters of spherical shells and circular sandwich plates

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CHAPTER 1

INTRODUCTION

The object of investigation in this thesis is the free vibrations of deep spherical sandwich shell and circular sandwich plates. A sandwich structure consists of three or more layers of material bonded together in such a manner that they act as a single unit. In a three layered plate or shell, the outer layers are called the face sheets or skins and are made of high strength material such as aluminum, plywood, fiber glass, etc. These face sheets take most of the outer fiber loads. The thicker central layer, normally known as the core, is usually a low strength and low density material such as rigid urethane foam, polystyrene foam or honeycomb. The core separates the face sheets in a similar manner as the web does in an I beam. The major part of the shear force developed by the external load is taken by the core which also provides good thermal insulation and strength to prevent the face sheets from buckling. Sandwich structures have received a tremendous amount of recognition for their commercial use in the aircraft and aerospace industries due to their high strength-to-weight ratio, better stability, increased fatigue life and high load carrying capacity.

The vibrational characteristic of sandwich structures has presented a big challenge and a fair degree of complexity to design engineers in the aircraft and aerospace industries. The increased use

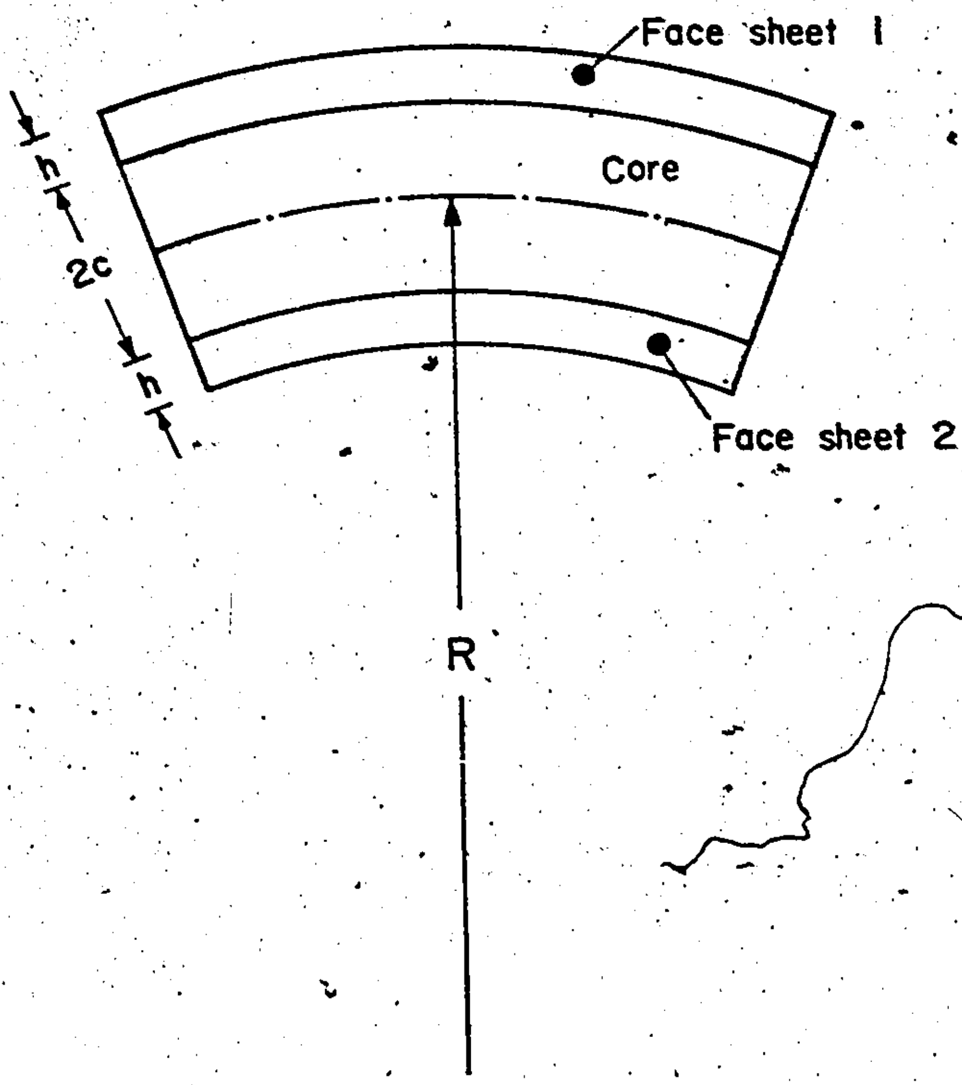

Fig. 1 Spherical Sandwich Shell Element.

of sandwich type construction in plates and shells has drawn the attention of many investigators to deal with their free and forced vibrations. Knowledge of forced and free vibrations of a system is quite essential to avoid the failures of structures caused by resonance. Resonance occurs due to time varying loads with a frequency which coincides with the natural frequency of the system. These time dependent loads at this frequency produce dangerously large deformations which cause material failure. The study of mode shapes reveals the exact deformation patterns of the vibrating system.

In the past, a considerable amount of work has been reported in the literature on spherical shells. This is partly due to the fact that it is widely used in civil and mechanical engineering related problems. A.E.H. Love, for the first time, established the general dynamic theory of elastic shells including the effects of both flexural and extensional deformations. Love also included in his work, as special cases, the previous treatment of inextensional vibrations given by Lord Rayleigh and the extensional vibrations of closed spherical shells discussed by Lamb. No further noticeable progress was made in the dynamic behaviour of shells until 1937when Federhofer reconstructed the basic differential equations of motion for spherical shells in terms of displacements. The references of the above-mentioned works are illustrated in a paper by Naghdi [18].

In 1946, E. Reissner [21] solved the differential equations of motion for axisymmetric vibrations of shallow spherical shell and advanced the theory for shallow shells. The system of equations has been reduced to two simultaneous differential equations in terms of the tangential and normal components of displacements. The solution for the displacement components was expressed in terms of Bessel functions. Neglecting the tangential inertia in comparison with the transverse inertia, the inextensional vibrations of shallow spherical shells was investigated in [22] for axisymmetric case and in [7] for the asymmetric case. The system of differential equations of motion has been reduced in terms of transverse displacement component and a stress function. The exact solution for the torsionless axisymmetric vibrations of shallow spherical shell segments has been presented in [9] with various edge conditions. The numerical values of lowest natural frequencies for the free, fixed and simply supported edge conditions have also been compared in [9] with the previous known results in which longitudinal inertia had not been taken into account. The non-linear forced vibrations of shallow spherical shells have also been studied in [30] and [5]. Van Fo Fy and Baivol [30] obtained in their analysis the general solution of differential equations of motion for forced vibrations and calculated numerical values of the natural frequencies for the axisymmetric case. In reference [5], an investigation of the axisymmetric vibrations of spherical caps with various edge conditions has been made by carrying out a consistent sequence of approximations with respect to space and time. Numerical results have also been obtained for both free and forced vibrations involving finite deflection.

During the 1960's, the vibration of non-shallow spherical homogeneous shells received substantial attention of the research workers in this area. The natural frequencies and mode shapes for homogeneous spherical deep shells have been examined thoroughly by many authors and their published work is available in the literature. In 1962, Naghdi and Kalnins [18] published the torsionless vibrations of thin elastic deep spherical shells with numerical values of the lowest natural frequency as function of the thickness of the shell. The effect of bending on free vibrations of deep spherical shells closed at one pole and open at the other has been reported in [8]. The mode shapes and natural frequencies have been presented for opening angles ranging from a shallow to a closed spherical shell. The presence of flexural and membrane modes in the frequency spectrum has also been discussed in [8]. The distinction between these two modes is made from the comparison of strain energies due to bending and stretching. The flexural modes were found to be dependent on the thickness of the shell whereas membrane modes being practically independent of thickness. It has also been shown in [8] that the set of membrane modes is a degenerate case of bending modes. The vibration of deep spherical shells under the action of a concentrated force has been reported by Manasyan [13]. An approximate analysis of axially symmetric dynamics of deep spherical elastic shells was published by Ross [24]. The approximation in his report was based on the asymptotic formulas for Legendre functions of large degree.

Non-symmetric vibrations of elastic spherical shells have also been reported by a few authors in [29,32]. Valikov and Gots proposed in their paper [29] the method to determine the natural frequencies of a spherical homogeneous shell with a single cut and the shell fixed at the boundary. The solutions of differential equations are represented in the form of associated Legendre function $P_{\nu}^{m}(\cos\phi)$. The results have been presented in a table for opening angles $\pi/3$, $\pi/2$ and $2\pi/3$ for four values of m from zero to 3. The value of m equal to zero represents the axisymmetric case whereas non-symmetric cases are represented by non-zero values of m. The authors in reference [32] deal with the non-symmetric case with a rigid insert subjected to horizontal force and moment. The effectsof transverse shear and rotary inertia are fully taken into account.

The experimental work on the spherical shell dynamics has also appeared in literature [4,6]. Hwang published his work [6] in 1966 describing the experiments on the vibration of a thin hemispherical shell having a free edge and constant thickness. Later on, Ekrem and Doige [3,4] also carried out theoretical and experimental investigations to study the natural frequencies and mode shapes of thin hemispherical shells with a clamped edge. From the above discussion, it is seen that the theoretical and experimental work on vibrations of shallow as well as deep homogeneous shells has been carried out fairly extensively for both axisymmetric and non-symmetric modes.

The dynamics of sandwich plates and shells has also been studied in the past but to a very limited extent. Y.Y. Yu [33,34] has analyzed the vibrations of rectangular sandwich plates. In his analysis, the effects of shear deformation and rotary inertia have also been included. The free axisymmetric vibrations of shallow sandwich spherical shells has been studied by Koplik and Yu [10]. A complete system of differential equations was derived from the associated variational principle and the effect of thickness shear deformation in the core was also included. They calculated numerical values of frequencies up to sixteen natural modes for a clamped edge shallow spherical sandwich shell with membrane face sheets. Mirza and Doige [15] presented the transverse vibrational characteristics of a three layered shallow spherical sandwich shell. In their analysis, the longitudinal inertia terms have been neglected and first three modes of the frequency response are shown. No work on the free vibrations of deep spherical sandwich shells was available in the literature until a paper [16] published recently by Mirza and the present author.

In this paper, the differential equations of motion for the sandwich spherical shell with membrane face sheets have been obtained by integrating the equations of motion in a continuum for the axisymmetric case. The numerical results for frequencies and mode shapes up to five natural modes are presented for clamped edge conditions.

In this thesis, the variational method, similar to the one used by Naghdi [19] and Reissner [23], has been utilized to derive the differential equations of motion, stress resultants and displacement relationships and the boundary conditions for the free non-symmetric vibrations of deep spherical sandwich shells. The shell is composed of three layers. The central thick core is assumed to be incompressible in the transverse direction. The outer two face layers are assumed to be made of the same isotropic material and are of equal thickness. It is possible to include the thickness shear deformation[1] in the constitutive equations for the sandwich shell by assuming displacements in the plane of the shell to vary linearly over the thickness, with slope in face sheets considered to be different from that in the core. The face parallel stresses in the core are assumed to be negligible as compared to the in-plane stresses of the face sheets. The flexural and extensional rigidities of the face sheets about their middle surfaces are included in the analysis, together with rotary inertia and shear deformation terms.

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1. Usually called the transverse shear deformation.

The total energy expressions in each of the three layers are generated and then added together to yield the total energy expression for the composite shell. While obtaining the strain energy in each of the individual layers, the above-mentioned assumptions are taken into consideration. The stress continuity conditions at the interfaces of core and face sheets are also utilized here. The total energy of the composite shell is then varied with respect to the displacement components and stress resultants as well as moment resultants in a similar manner as given in [19,23]. From this variational equation, constitutive equations for the free non-symmetric vibrations of deep spherical shells are derived. These equations are also presented in this thesis for special cases such as axisymmetric case of deep spherical sandwich shells, spherical homogeneous shells and circular sandwich plates. The sandwich shells are analyzed by considering face sheets as flexural members and also as membranes. The plate equations are derived as a special case from the shell equation by considering the radius of curvature of the shell to be infinity. The flexural vibrations of homogeneous circular plates have been studied quite extensively with rotary inertia and shear deformation [1,14] and also without these terms [12,31]. The vibrations of circular sandwich plates with or without rotary inertia and shear deformation terms are not available in the literature.

The system of differential equations for both sandwich plates and shells are reduced to considerably simplified forms by introducing new variables which are the functions of displacement components. The solutions of the differential equations for normal modes of vibration of the spherical sandwich shell are expressed in terms of associated Legendre functions for the non-symmetric case. In a similar manner, solution for the case of circular sandwich plates is written in terms of Bessel functions. The detailed method of solution for all the cases are presented in Chapter 3 of the thesis. The symbols Ω and (ω_p/ω_o) represent the non-dimensionalized frequency parameters of the sandwich shells and plates, respectively. The geometric and elastic parameters of the sandwich structures have also been expressed in non-dimensional forms.

The numerical values of frequency parameters Ω and (ω_p/ω_o) have been calculated on a digital computer using an iterative procedure for clamped edge sandwich plates and shells. Since there was no numerical results available for circular sandwich plates and deep spherical sandwich shells, a comparison of the results was not made directly. However, the system of differential equations of the sandwich shells was reduced to the case of homogeneous spherical shells by substituting core thickness equal to zero and the two face sheets were superimposed on each other. The numerical values of natural frequency were obtained for homogeneous shells and were found to be in excellent agreement with the values available in literature [3,8,29].

As expected, it has been found from the results that flexural and extensional rigidities of the face sheets are strongly effective, on the first and lowest mode while at the higher modes, the effect of these rigidities diminishes and the inertia quantities become predominant. The effect of various parameters on the natural frequencies has been discussed in detail in Chapter 6.

CHAPTER 2

FORMULATION OF THE PROBLEM

2.1 Notations and Fundamental Relationships

Let θ and ϕ be the coordinates of a point on the middle surface of the ith layer of a sandwich shell in the circumferencial and meridional directions and z_i be the distance measured along the outward normal to the surface. Further, u_i', v_i' and w_i' are the components of displacement at a point, which is at a distance z_i from the middle surface of the layer, in circumferential, meridional and radial directions respectively. Thus, u_i', v_i' and w_i' can be expressed as

$$u_i' = u_i + z_i\beta_{\theta_i} ; \; v_i' = v_i + z\beta_{\phi_i} ; \; w_i' = w_i \tag{2.1}$$

In Eqs. (2.1), u_i, v_i and w_i are the mid-surface displacement components in the ith layer; β_{θ_i} and β_{ϕ_i} are changes of slope of the normal to the middle surface in θ and ϕ directions. Fig. 2 shows v_1, v_2, β_{ϕ_1}, etc. in the meridional direction for the purpose of illustration. The strain components ε'_{ϕ_i}, ε'_{θ_i} ... etc. at an arbitrary point (θ,ϕ,z_i) are written as [19].

$$\begin{aligned}
(1+z_i/R_i)\varepsilon'_{\phi_i} &= \varepsilon_{\phi_i} + z_i k_{\phi_i} \\
(1+z_i/R_i)\varepsilon'_{\theta_i} &= \varepsilon_{\theta_i} + z_i k_{\theta_i} \\
(1+z_i/R_i)\gamma'_{\theta\phi_i} &= \gamma_{\theta\phi_i} + z_i k_{\theta\phi_i}
\end{aligned} \tag{2.2}$$

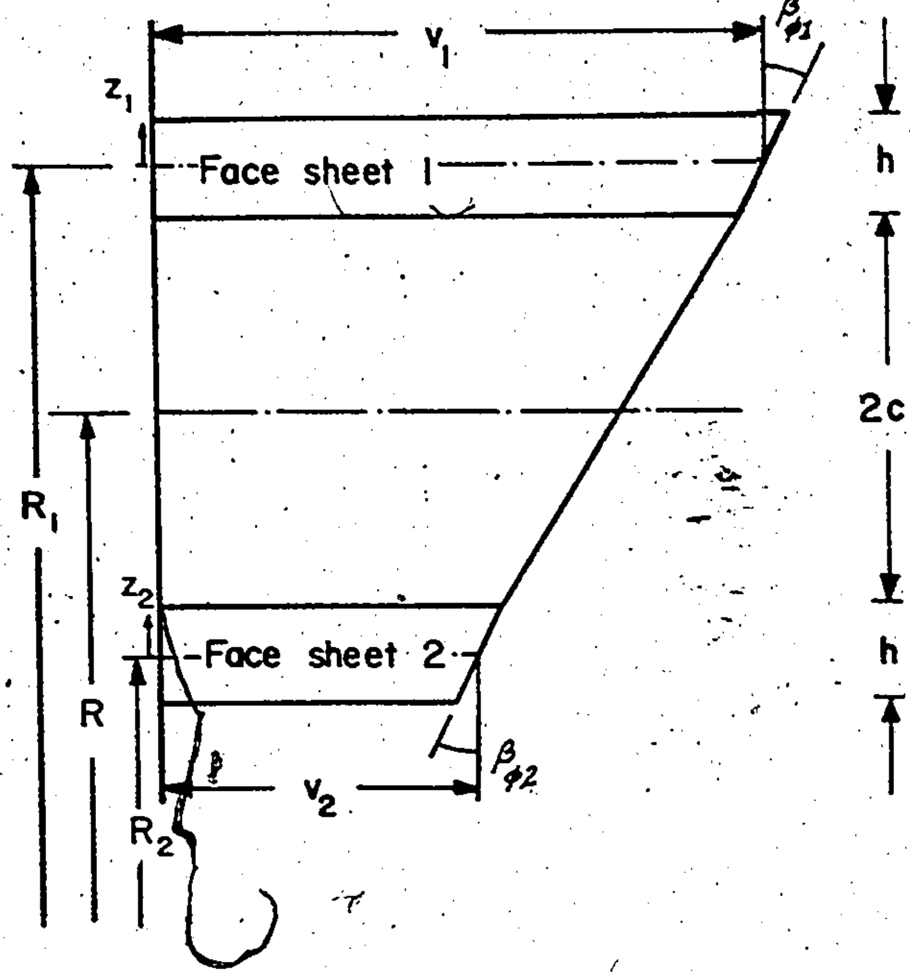

Fig. 2 Distribution of meridional displacement component along the thickness of the sandwich shell.

$$(1+z_i/R_i)\gamma'_{r\phi_i} = \gamma_{r\phi_i}$$

$$(1+z_i/R_i)\gamma'_{r\theta_i} = \gamma_{r\theta_i}$$

In Eqs. (2.2), R_i is the radius of curvature of the middle surface. The mid-surface strains ε_{ϕ_i}, ε_{θ_i}, $\gamma_{\theta\phi_i}$, k_ϕ, ... etc. are expressed in terms of u_i, v_i, w_i, β_ϕ ... etc. as follows:

$$
\begin{aligned}
R_i\,\varepsilon_{\phi_i} &= v_{i,\phi} + w_i \\
R_i\varepsilon_{\theta_i} &= \operatorname{cosec}\phi\; u_{i,\theta} + \cot\phi\; v_i + w_i \\
R_i\gamma_{\theta\phi_i} &= u_{i,\phi} - \cot\phi\; u_i + \operatorname{cosec}\phi\; v_{i,\theta} \\
R_i\gamma_{r\phi_i} &= w_{i,\phi} - v_i + R_i\beta_{\phi_i} \\
R_i\gamma_{r\theta_i} &= \operatorname{cosec}\phi\; w_{i,\theta} - u_i + R_i\beta_{\theta_i} \\
R_i k_{\phi_i} &= \beta_{\phi_{i,\phi}} \\
R_i k_{\theta_i} &= \operatorname{cosec}\phi\;\; \beta_{\theta_{i,\theta}} + \cot\phi\;\; \beta_{\phi_i} \\
R_i k_{\theta\phi_i} &= \beta_{\theta_{i,\phi}} - \cot\phi\;\; \beta_{\theta_i} + \operatorname{cosec}\phi\;\; \beta_{\phi_{i,\theta}}
\end{aligned}
\qquad (2.3)
$$

The notation $v_{i,\phi}$ etc. in the above equations stands for the partial differentiation of v_i with respect to ϕ.

The normal stresses σ'_{ϕ_i}, σ'_{θ_i} and shear stresses $\tau'_{r\theta_i}$, $\tau'_{r\phi_i}$, $\tau'_{\theta\phi_i}$ are expressed in terms of stress resultants N_{ϕ_i}, N_{θ_i}, $N_{\theta\phi_i}$; moment resultants M_{ϕ_i}, M_{θ_i}, $M_{\phi\theta_i}$; and shear stress resultants Q_{ϕ_i} and Q_{θ_i} in the following manner [19].

$$(1+z_i/R_i)\sigma'_{\phi_i} = N_{\phi_i}/h_i + M_{\phi_i} z_i/I_i$$

$$(1+z_i/R_i)\sigma'_{\theta_i} = N_{\theta_i}/h_i + M_{\theta_i} z_i/I_i$$

$$(1+z_i/R_i)\tau'_{\theta\phi_i} = N_{\theta\phi_i}/h_i + M_{\theta\phi_i} z_i/I_i$$

$$\begin{aligned}(1+z_i/R_i)\tau'_{r\phi_i} &= \frac{3}{2}\{1-(2z_i/h_i)^2\}Q_{\phi_i}/h_i - \\ &\Big\{p^+_{\phi_i}(1+h_i/2R_i)\left[1-4z_i/h_i-12(z_i/h_i)^2\right]+ \\ &p^-_{\phi_i}(1-h_i/2R_i)\left[1+4z_i/h_i-12(z_i/h_i)^2\right]\Big\}/4 \end{aligned} \quad (2.4)$$

$$\begin{aligned}(1+z_i/R_i)\tau'_{r\theta_i} &= \frac{3}{2}\{1-(2z_i/h_i)^2\}Q_{\theta_i}/h_i - \\ &\Big\{p^+_{\theta_i}(1+h_i/2R_i)\left[1-4z_i/h_i-12(z_i/h_i)^2\right]+ \\ &p^-_{\theta_i}(1-h_i/2R_i)\left[1+4z_i/h_i-12(z_i/h_i)^2\right]\Big\}/4 \end{aligned}$$

where h_i is the thickness of the layer and $I_i=h_i^3/12$. The quantities

$p^{+}_{\phi_i}$, $p^{-}_{\phi_i}$, $p^{+}_{\theta_i}$ and $p^{-}_{\theta_i}$ are given as

$$\tau'_{r\phi_i}\Big|_{z_i=h_i/2} = p^{+}_{\phi_i},\ \tau'_{r\phi_i}\Big|_{z_i=-h_i/2} = p^{-}_{\phi_i}$$

$$\tau'_{r\theta_i}\Big|_{z_i=h_i/2} = p^{+}_{\theta_i},\ \tau'_{r\theta_i}\Big|_{z_i=-h_i/2} = p^{-}_{\theta_i} \tag{2.5}$$

The variational theorem by Reissner [37] for the dynamic problems can be written as

$$\delta\ \psi = 0$$

where the functional ψ can be represented in spherical coordinates as follows.

$$\psi = \iiint_{vol.} \Big[\sigma'_{\phi_i}\varepsilon'_{\phi_i} + \sigma'_{\theta_i}\varepsilon'_{\theta_i} + \tau'_{\theta\phi_i}\gamma'_{\theta\phi_i} + \tau'_{r\phi_i} + \gamma'_{r\phi_i}\tau'_{r\theta_i} - \gamma'_{r\theta_i} - \Lambda$$
$$+ \rho_i\{(u'_{i,t})^2 +)v'_{i,t})^2 + (w'_{i,t})^2\}/2\Big] R_i^2(1+z_i/R_i)^2 \sin\phi\ d\phi\ d\theta\ dz_i$$
$$-\oint\Big[\int_{-h_i/2}^{h_i/2} (\sigma''_{\phi_i}v''^{*}_i + \tau''_{\theta\phi_i}u''_i + {}''_{r\phi_i}w''_i)(1+z_i/R_i)dz_i\Big] R_i\ \sin\phi\ d\theta \tag{2.6}$$

Here, ρ_i is the mass density of the material and the symbol Λ represents the energy term which is given as

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* The edge stresses and displacements are denoted by double primed.

$$\Lambda = \Big\{\sigma'^{\,2}_{\phi_i}+\sigma'^{\,2}_{\theta_i}-2\nu_i\sigma'_{\phi_i}\sigma'_{\theta_i}+2(1+\nu_i)$$

$$(\tau'^{\,2}_{\theta\phi_i} + \tau'^{\,2}_{r\phi_i} + \tau'^{\,2}_{r\theta_i})\Big\}/2E_i$$

E_i and ν_i are the modulus of elasticity and Poisson's ratio of the material. The line integral in eq. (2.6) represents the energy associated with the edge stresses and displacements.

Carrying out the integration of eq. (2.6) over the thickness h_i, the following expression for total energy in the ith layer is obtained. During the process of integration, use has been made of eqs. (2.1) to (2.5).

$$\iint\Big(R_i^2(N_{\phi_i}\varepsilon_{\phi_i}+N_{\theta_i}\varepsilon_{\theta_i}+N_{\theta\phi_i}\gamma_{\theta\phi_i}+Q_{\phi_i}\gamma_{r\phi_i}+Q_{\theta_i}\gamma_{r\theta_i}+M_{\phi_i}k_{\phi_i}+M_{\theta_i}k_{\theta_i}+M_{\theta\phi_i}k_{\theta\phi_i})$$

$$+\ \rho_i h_i\Big[(R_i^2+h_i^2/12)(u^2_{i,t}+v^2_{i,t}+w^2_{i,t}) + h_i^2R_i(u_{i,t}\beta_{\theta_{i,t}}+v_{i,t}\beta_{\phi_{i,t}})/3$$

$$+\ h_i^2(R_i^2+3h_i^2/20)(\beta^2_{\theta_{i,t}}+\beta^2_{\phi_{i,t}})/12\Big]/2$$

$$-\ (R_i^2/2E_ih_i)\Big[N^2_{\phi_i}+N^2_{\theta_i}+2(1+\nu_i)N^2_{\theta\phi_i}-2\nu_iN_{\theta_i}N_{\phi_i}$$

$$+\ 12(1+\nu_j)(Q^2_{\phi_i}+Q^2_{\theta_i})/5\Big] - (R_i^2/E_iI_i)\Big[M^2_{\phi_i}+M^2_{\theta_i}+2(1+\nu_i)M^2_{\theta\phi_i}-2\nu_iM_{\theta_i}M_{\phi_i}\Big]$$

$$+\ (1/2G_i)\Big[\{R_iR_i^+(Q_{\phi_i}p^+_{\phi_i}+Q_{\theta_i}p^+_{\theta_i}) + R_iR_i^-(Q_{\phi_i}p^-_{\phi_i}+Q_{\theta_i}p^-_{\theta_i})\}/5$$

$$2h_i\{(p^+_{\phi_i} R^+_i)^2+(p^+_{\theta_i} R^+_i)^2+(p^-_{\phi_i} R^-_i)^2+(p^-_{\theta_i} R^-_i)^2\}/15$$

$$+ h_i R^+_i R^-_i (p^+_{\phi_i} p^-_{\phi_i} + p^+_{\theta_i} p^-_{\theta_i})/15\Big]\Big]\Big) \sin\phi \, d\phi \, d\theta$$

$$- \oint \Big[N''_{\phi_i} v''_i + M''_{\phi_i} \beta''_{\phi_i} + N''_{\theta\phi_i} u''_i + M''_{\theta\phi_i} \beta_{\theta_i} + Q''_{\phi_i} w''_i\Big] R_i \sin\phi \, d\theta \qquad (2.7)$$

Equation (2.7) will be utilized later in Section (2.2.1) to obtain the total energy expression for the three layered sandwich shell.

2.2 Equations for Spherical Sandwich Shells

The spherical sandwich shell considered in the present investigation consists of three layers. The two face sheets are made of the same isotropic elastic material. The thickness of each face sheet is h and its modulus of elasticity and Poisson's ratio are E and ν, respectively. The change in slopes of the normals to the middle surfaces are assumed to be equal in both the face sheets and are given by the symbols β_ϕ and β_θ in ϕ and θ directions. The face sheets have been assumed to take loads due to bending moments, shear forces and membrane forces.

The thick central layer between the facings is a low density and low strength material. The compression or stretching of the core in the radial direction is assumed to be negligible. The face parallel stresses in the core are very small as compared to those in the face sheets and can be neglected. Thus,

$$\sigma_\phi = \sigma_\theta = \tau_{\phi\theta} = 0 \qquad \text{(i)}$$

The variation of displacement components in θ and ϕ directions along the thickness of the core is represented by the following expressions:

$$u = u^* + (\bar{u} - \beta_\theta h/2)\, z/c$$
$$v = v^* + (\bar{v} - \beta_\phi h/2)\, z/c \qquad (2.8)$$

where 2c is the thickness of the core and z is measured in the direction of the outer normal to the surface passing through the middle of the core. The displacement components u^*, $\bar{u}$, v^* and $\bar{v}$ are given in eq. (2.9).

$$u^* = (u_1 + u_2)/2, \quad \bar{u} = (u_1 - u_2)/2$$
$$v^* = (v_1 + v_2)/2, \quad \bar{v} = (v_1 - v_2)/2 \qquad (2.9)$$

Also, u_1 and u_2 are the mid-surface displacement components in θ direction of face sheets 1 and 2 respectively.

For convenience, the stress and moment resultants of the composite shell are defined by:

$$N_\phi^* = (N_{\phi_1} + N_{\phi_2})/2, \quad \bar{N}_\phi = (N_{\phi_1} - N_{\phi_2})/2$$

$$N_{\theta\phi}^* = (N_{\theta\phi_1} + N_{\theta\phi_2})/2, \quad \bar{N}_{\theta\phi} = (N_{\theta\phi_1} - N_{\theta\phi_2})/2$$

$$Q^*_\phi = (Q_{\phi_1} + Q_{\phi_2})/2, \quad \bar{Q}_\phi = (Q_{\phi_1} - Q_{\phi_2})/2 \tag{2.10}$$

$$M^*_\phi = (M_{\phi_1} + M_{\phi_2})/2, \quad \bar{M}_\phi = (M_{\phi_1} - M_{\phi_2})/2$$

$$p^*_\phi = (p_{\phi_1} + p_{\phi_2})/2, \quad \bar{p}_\phi = (p_{\phi_1} - p_{\phi_2})/2, \text{ etc.}$$

Since only free vibration of the sandwich spherical shell is being investigated, the external surface of the face sheet 1 and internal surface of the face sheet 2 are traction free. The quantities p_{ϕ_1}, p_{θ_1}, p_{ϕ_2} and p_{θ_2} are the shear stresses τ_{r_ϕ} and τ_{r_θ} at the two interfaces joining the core and the face sheets. Thus,

$$p_{\phi_1} = \tau_{r_\phi}\Big|_{z=c}, \quad p_{\phi_2} = \tau_{r_\phi}\Big|_{z=-c}$$
$$p_{\theta_1} = \tau_{r_\theta}\Big|_{z=c} \quad p_{\theta_2} = \tau_{r_\theta}\Big|_{z=-c} \tag{ii}$$

2.2.1 Derivation of Energy Expression

With the help of eq. (2.7), the dynamic Reissner functionals are obtained for each of the three layers. Use is also made of the appropriate assumptions for the face sheets and for the core. These are then added together to yield the total energy for the composite shell. The total energy expression can be simplified by the introduction of eqs. (2.8) to (2.10). The resulting equation takes the form:

$$
\begin{aligned}
\Pi = \iint \Big[& 2R(\overset{*}{N}_\phi \overset{*}{\varepsilon}_\phi + \bar{N}_\phi \bar{\varepsilon}_\phi + \overset{*}{N}_\theta \overset{*}{\varepsilon}_\theta + \bar{N}_\theta \bar{\varepsilon}_\theta + \overset{*}{N}_{\theta\phi} \overset{*}{\gamma}_{\theta\phi} + \bar{N}_{\theta\phi} \bar{\gamma}_{\theta\phi} + \overset{*}{Q}_\phi \overset{*}{\gamma}_{r\phi} + \bar{Q}_\phi \bar{\gamma}_{r\phi} + \\
& \overset{*}{Q}_\theta \overset{*}{\gamma}_{r\theta} + \bar{Q}_\theta \bar{\gamma}_{r\theta} + \overset{*}{M}_\phi k_\phi + \overset{*}{M}_\theta k_\theta + \overset{*}{M}_{\theta\phi} k_{\theta\phi}) + 2s(\overset{*}{N}_\phi \bar{\varepsilon}_\phi + \\
& \bar{N}_\phi \overset{*}{\varepsilon}_\phi + \overset{*}{N}_\theta \bar{\varepsilon}_\theta + \bar{N}_\theta \overset{*}{\varepsilon}_\theta + \overset{*}{N}_{\theta\phi} \bar{\gamma}_{\theta\phi} + \bar{N}_{\theta\phi} \overset{*}{\gamma}_{\theta\phi} + \overset{*}{Q}_\phi \bar{\gamma}_{r\phi} + \bar{Q}_\phi \overset{*}{\gamma}_{r\phi} + \\
& \overset{*}{Q}_\theta \bar{\gamma}_{r\theta} + \bar{Q}_\theta \overset{*}{\gamma}_{r\theta} + \bar{M}_\phi k_\phi + \bar{M}_\theta k_\theta + \bar{M}_{\theta\phi} k_{\theta\phi}) + \rho h R^2 \{ s_1 (\overset{*}{u},_t^2 + \overset{*}{v},_t^2 + w,_t^2) + \\
& s_2 (\bar{u},_t^2 + \bar{v},_t^2) + 2s_3 (\overset{*}{u},_t \bar{u},_t + \overset{*}{v},_t \bar{v},_t) + 2s_4 R(\overset{*}{u},_t \beta_{\theta,t} + \overset{*}{v},_t \beta_{\phi,t}) + \\
& 2s_5 R(\bar{u},_t \beta_{\theta,t} + \bar{v},_t \beta_{\phi,t}) + s_6 R^2 (\beta^2_{\theta,t} + \beta^2_{\phi,t}) \} - \\
& \{\Lambda_1 + 12(1+\nu)(\overset{*}{Q}{}_\phi^2 + \overset{*}{Q}{}_\theta^2 + \bar{Q}_\phi^2 + \bar{Q}_\theta^2)/5\} \, R^2(1+s^2/R^2)/Eh - \\
& \{\Lambda_2 + 12(1+\nu)(\overset{*}{Q}_\phi \bar{Q}_\phi + \overset{*}{Q}_\theta \bar{Q}_\theta)/5\} \, 4sR/Eh - \Lambda_3 R^2(1+s^2/R^2)/EI - \\
& \Lambda_4 4Rs/EI + R(Q_{\phi c}\gamma_{r\phi c} + Q_{\theta c}\gamma_{r\theta c}) - (Q^2_{\phi c} + Q^2_{\theta c}) \, 3R^2/10G_c c + \\
& \Big\{ r_G(1+sc/R^2)(\overset{*}{Q}_\phi \overset{*}{p}_\phi + \bar{Q}_\phi \bar{p}_\phi + \overset{*}{Q}_\theta \overset{*}{p}_\theta + \bar{Q}_\theta \bar{p}_\theta) + r_G(s/R + c/R) \\
& (\bar{Q}_\phi \overset{*}{p}_\phi + \bar{Q}_\theta \overset{*}{p}_\theta + \overset{*}{Q}_\phi \bar{p}_\phi + \overset{*}{Q}_\theta \bar{p}_\theta) + Q_{\phi c} \overset{*}{p}_\phi + Q_{\theta c} \overset{*}{p}_\theta + (Q_{\phi c}\bar{p}_\phi + Q_{\theta c}\bar{p}_\theta) \, c/R - \\
& \Big[2r_G h(1+c^2/R^2)/3 + c(1+5/3 \, c^2/R^2) \Big] \, (\overset{*}{p}{}_\phi^2 + \overset{*}{p}{}_\theta^2) + \Big[2r_G h(1+
\end{aligned}
\qquad (2.11)
$$

$$c^2/R^2)/3 + c(5/3 + c^2/R^2)\Big] (\bar{p}_\phi^2+\bar{p}_\theta^2) + (\overset{*}{p}_\phi\bar{p}_\phi+\overset{*}{p}_\theta\bar{p}_\theta)\ (r_G+2r_h)8ch/3R\Big\}$$

$$(R^2/5G_c)\Big]\sin\phi\ d\phi\ d\theta\ -2\oint (N'_\phi\overset{*}{v}+N''_\phi\bar{v}+N'_{\theta\phi}\overset{*}{u}+N''_{\theta\phi}\bar{u}+Q'_\phi w+M'_\phi\beta_\phi+M'_{\theta\phi}\beta_\theta)$$

$$R\sin\phi\ d\theta \tag{2.11}$$

where

$$s_1 = 1 + s^2/R^2 + h^2/12R^2 + r_\rho r_h(1+c^2/3R^2)$$

$$s_2 = 1 + s^2/R^2 + h^2/12R^2 + r_\rho r_h(1+3c^2/5R^2)/3$$

$$s_3 = 2\, s/R + 2\, r_\rho r_h c/3R \qquad \text{(iii)}$$

$$s_4 = (1-2r_\rho r_h^2)h^2/6R^2$$

$$s_5 = \{sh^2/R^3 - r_\rho r_h(1+3c^2/5R^2)h/R\}/6$$

$$s_6 = \{1+s^2/R^2+3h^2/20R^2+r_\rho r_h(1+3c^2/5R^2)\}h^2/12R^2$$

$$r_\rho = \rho_c/\rho,\ r_h=c/h,\ r_G=G_c/G,\ s=c+h/2$$

G and ν are the modulus of rigidity and Poisson's ratio of the face sheet; ρ is a mass density of face sheet; G_c and ρ_c are modulus of rigidity and mass density of the core material and t represents time and R is the radius of the spherical surface passing through the middle of the core. The quantities N'_ϕ, N''_ϕ ... $M'_{\theta\phi}$ etc. are the prescribed stress and moment resultants at the boundary $\phi=\phi_o$. Also,

$$
\begin{aligned}
&\overset{*}{\varepsilon}_\phi = \overset{*}{v}_{,\phi} + w, \quad \bar{\varepsilon}_\phi = \bar{v}_{,\phi} \\
&\overset{*}{\varepsilon}_\theta = \operatorname{cosec}\phi\, \overset{*}{u}_{,\theta} + \cot\phi\, \overset{*}{v} + w \\
&\bar{\varepsilon}_\theta = \operatorname{cosec}\phi\, \bar{u}_{,\theta} + \cot\phi\, \bar{v} \\
&\overset{*}{\gamma}_{\theta\phi} = \overset{*}{u}_{,\phi} - \cot\phi\, \overset{*}{u} + \operatorname{cosec}\phi\, \overset{*}{v}_{,\theta} \\
&\bar{\gamma}_{\theta\phi} = \bar{u}_{,\phi} - \cot\phi\, \bar{u} + \operatorname{cosec}\phi\, \bar{v}_{,\theta} \\
&\overset{*}{\gamma}_{r\phi} = w_{,\phi} - \overset{*}{v} + R\beta_\phi, \quad \bar{\gamma}_{r\phi} = -\bar{v} + s\beta_\phi \\
&\overset{*}{\gamma}_{r\theta} = \operatorname{cosec}\phi\, w_{,\theta} - \bar{u} + R\beta_\theta, \quad \bar{\gamma}_{r\theta} = -\bar{u} + s\beta_\theta \\
&\gamma_{r\phi c} = w_{,\phi} - \overset{*}{v} + \bar{v}\, R/c - \beta_\phi Rh/2c \\
&\gamma_{r\theta c} = \operatorname{cosec}\phi\, w_{,\theta} - \overset{*}{u} + \bar{u}\, R/c - \beta_\theta Rh/2c
\end{aligned}
\qquad (2.12)
$$

$$
\begin{aligned}
&\Lambda_1 = \overset{*}{N}_\phi^{\,2} + \bar{N}_\phi^{\,2} + \overset{*}{N}_\theta^{\,2} + \bar{N}_\theta^{\,2} + 2(1+\nu)(\overset{*}{N}{}^2_{\theta\phi} + \bar{N}^2_{\theta\phi}) - 2\nu(\overset{*}{N}_\phi \overset{*}{N}_\theta + \bar{N}_\phi \bar{N}_\theta) \\
&\Lambda_2 = \overset{*}{N}_\phi \bar{N}_\phi + \overset{*}{N}_\theta \bar{N}_\theta + 2(1+\nu)\overset{*}{N}_{\theta\phi} \bar{N}_{\theta\phi} - \nu(\overset{*}{N}_\phi \bar{N}_\theta + \bar{N}_\phi \overset{*}{N}_\theta) \\
&\Lambda_3 = \overset{*}{M}{}^2_\phi + \bar{M}^2_\phi + \overset{*}{M}{}^2_\theta + \bar{M}^2_\theta + 2(1+\nu)(\overset{*}{M}{}^2_{\theta\phi} + \bar{M}^2_{\theta\phi}) - 2\nu(\overset{*}{M}_\phi \overset{*}{M}_\theta + \bar{M}_\phi \bar{M}_\theta) \\
&\Lambda_4 = \overset{*}{M}_\phi \bar{M}_\phi + \overset{*}{M}_\theta \bar{M}_\theta + 2(1+\nu)\overset{*}{M}_{\theta\phi} \bar{M}_{\theta\phi} - \nu(\overset{*}{M}_\phi \bar{M}_\theta + \bar{M}_\phi \overset{*}{M}_\theta)
\end{aligned}
\qquad (2.13)
$$

2.2.2 Variation of Energy

In the theory of elasticity, the two well-known variational theorems or principles are the principle of minimum potential energy and the principle of minimum complementary energy. For the solution of elastostatic problems, variation of integral expression for either potential energy or complementary energy is carried out and then equated to zero. In the first theorem, variation of displacements are admitted to yield the differential equations of motion as Euler equations. Whereas, in the later case, the stress-displacement relationships as Euler equations are derived from the variations of stresses in the complementary energy.

The present work is concerned with the formulation of suitable stress strain relations, appropriate boundary conditions and a system of differential equations of motion for the spherical sandwich shell. In order to derive these constitutive equations, use has been made of a variational theorem due to Reissner [37]. The variational equation is derived by admitting variations to the displacements and the stress resultants simultaneously in the integral expression given by eq. (2.11). The variations of displacements u^*, $\bar{u}$, v^*, $\bar{v}$, w, β_θ and β_ϕ yield the system of differential equations of motion and the boundary conditions. The stress resultants and displacements relationships are derived by varying N_ϕ^*, $\overset{*}{\bar{N}}_\theta$, ... $\overset{*}{\bar{M}}_\phi$, $\overset{*}{\bar{M}}_{\theta\phi}$, $\bar{M}_\phi$, ... etc. in eq. (2.11) for the spherical sandwich shells. The variation of ($\sin\phi\ \overset{*}{\bar{N}}_\phi\ \overset{*}{\bar{\varepsilon}}_\phi$) with respect to $\overset{*}{\bar{N}}_\phi$, $\overset{*}{v}$ and w is illustrated below.

$$\delta(\sin\phi\, \overset{*}{\ddot{N}}_{\phi}\overset{*}{\ddot{\varepsilon}}_{\phi}) = \sin\phi\, \overset{*}{\ddot{\varepsilon}}_{\phi}\, \delta\overset{*}{\ddot{N}}_{\phi} + \overset{*}{\ddot{N}}_{\phi}\, \delta(\sin\phi\, \overset{*}{\ddot{\varepsilon}}_{\phi})$$

$$= \sin\phi\, (\overset{*}{\ddot{\varepsilon}}_{\phi}\delta\overset{*}{\ddot{N}}_{\phi}+\overset{*}{\ddot{N}}_{\phi}\delta w) + \sin\phi\, \overset{*}{\ddot{N}}_{\phi}\, \delta(\overset{*}{\ddot{v}}_{,\phi}) \qquad (2.14a)$$

$$= \sin\phi\, (\overset{*}{\ddot{\varepsilon}}_{\phi}\delta\, \overset{*}{\ddot{N}}_{\phi}+\overset{*}{\ddot{N}}_{\phi}\delta w) + (\sin\phi\, \overset{*}{\ddot{N}}_{\phi}\delta\overset{*}{\ddot{v}})_{,\phi} - (\sin\phi\, \overset{*}{\ddot{N}}_{\phi})_{,\phi}\, \delta\overset{*}{\ddot{v}}$$

Similarly, $(\rho h R^2 \sin\phi\, w^2_{,t})$ is varied with respect to w as follows:

$$\delta(\rho h R^2 \sin\phi w^2_{,t}) = \rho h R^2 \sin\phi \delta (w_{,t})^2$$

$$= 2\rho h R^2 \sin\phi\, w_{,t}\, \delta(w_{,t}) = 2\rho h R^2 \sin\phi\, w_{,t}\, \frac{\partial w_{,t}}{\partial t}\, \delta t$$

$$= 2\rho h R^2 \sin\phi\, w_{,tt}\, (w_{,t}\delta t)$$

$$= 2\rho h R^2 \sin\phi\, w_{,tt}\, \delta w \qquad (2.14b)$$

Carrying out the variations of displacement components, stress and moment resultants as illustrated in eq. (2.14) and collecting the coefficients of δu^{*}, δv^{*}, ... $\delta\overset{*}{\ddot{M}}$ etc., the following variational equation is obtained.

$$\iint\Big\{2R\Big[\{\sin\phi(\overset{*}{\ddot{N}}_{\phi}+\bar{N}_{\phi}s/R)\delta v^{*}\}_{,\phi} + \{\sin\phi(\bar{N}_{\phi}+\overset{*}{\ddot{N}}_{\phi}s/R)\delta\bar{v}\}_{,\phi} +$$

$$\{(N^{*}_{\theta}+\bar{N}_{\theta}s/R)\delta u^{*}\}_{,\theta} + \{(\bar{N}_{\theta}+\overset{*}{\ddot{N}}_{\theta}s/R)\delta\bar{u}\}_{,\theta} + \{\sin\phi(\overset{*}{\ddot{N}}_{\theta\phi}+\bar{N}_{\theta\phi}s/R)\delta\overset{*}{\ddot{u}}\}_{,\phi} +$$

$$\{\sin\phi(\bar{N}_{\theta\phi}+\overset{*}{\ddot{N}}_{\theta\phi}s/R)\delta\bar{u}\}_{,\phi} + \{(\overset{*}{\ddot{N}}_{\theta\phi}+\bar{N}_{\theta\phi}s/R)\delta\overset{*}{\ddot{v}}\}_{,\theta} + \{(\bar{N}_{\theta\phi}+\overset{*}{\ddot{N}}_{\theta\phi}s/R)\delta\bar{v}\}_{,\theta} +$$

$$\{\sin\phi(\overset{*}{Q}_\phi+\bar{Q}_\phi s/R+Q_{\phi c}/2)\delta w\}_{,\phi} + \{(\overset{*}{Q}_\theta+\bar{Q}_\theta s/R+Q_{\theta c}/2)\delta w\}_{,\theta} +$$

$$\{\sin\phi(\overset{*}{M}_\phi+\bar{M}_\phi s/R)\delta\beta_\phi\}_{,\phi} + \{(\overset{*}{M}_\theta+\bar{M}_\theta s/R)\delta\beta_\theta\}_{,\theta} +$$

$$\{\sin\phi\ (\overset{*}{M}_{\theta\phi}+\bar{M}_{\theta\phi}s/R)\delta\beta_\theta\}_{,\phi} + \{(\overset{*}{M}_{\theta\phi}+\bar{M}_{\theta\phi}s/R)\delta\beta_\phi\}_{,\theta}\Big]+$$

$$2R\Big(-\delta\overset{*}{u}\Big[\overset{*}{N}_{\theta,\theta}+(\sin\phi\ \overset{*}{N}_{\theta\phi})_{,\phi} + \cos\phi\ \overset{*}{N}_{\theta\phi}$$

$$+ \sin\phi\ (\overset{*}{Q}_\theta+Q_{\theta c}/2) + \{\bar{N}_{\theta,\theta}+(\sin\phi\ \bar{N}_{\theta\phi})_{,\phi} + \cos\phi\ \bar{N}_{\theta\phi}$$

$$+ \sin\phi\ \bar{Q}_\theta\}\ s/R - \rho hR\ \sin\phi\ (s_1\overset{*}{u}_{,tt}+s_3\bar{u}_{,tt}+s_4R\beta_{\theta,tt})\Big] -$$

$$\delta\overset{*}{v}\Big[(\sin\phi\ \overset{*}{N}_\phi)_{,\phi} - \cos\phi\ \overset{*}{N}_\theta + \overset{*}{N}_{\theta\phi,\theta} + \sin\phi\ (\overset{*}{Q}_\phi+Q_{\phi c}/2)$$

$$+ \{(\sin\phi\ \bar{N}_\phi)_{,\phi} - \cos\phi\ \bar{N}_\theta + \bar{N}_{\theta\phi,\theta} + \sin\phi\ \bar{Q}_\phi\}s/R -$$

$$\rho hR\ \sin\phi\ (s_1\overset{*}{v}_{,tt}+s_3\ \bar{v}_{,tt}+s_4R\beta_{\phi,tt})\Big] - \delta w\Big[-\sin\phi\ (\overset{*}{N}_\phi+\overset{*}{N}_\theta) +$$

$$\{\sin\phi\ (\overset{*}{Q}_\phi+Q_{\phi c}/2)\}_{,\phi} + (\overset{*}{Q}_\theta+Q_{\theta c}/2)\}_{,\theta} + \{-\sin\phi\ (\bar{N}_\phi+\bar{N}_\theta) +$$

$$(\sin\phi\ \bar{Q}_\phi)_{,\phi} + \bar{Q}_{\theta,\theta}\}\ s/R - \rho hR\ \sin\phi\ s_1 w_{,tt}\Big] -$$

$$\delta\bar{u}\Big[\bar{N}_{\theta,\theta}+(\sin\phi\ \bar{N}_{\theta\phi})_{,\phi} + \cos\phi\ \bar{N}_{\theta\phi} + \sin\phi\ (\bar{Q}_\theta-Q_{\theta c}R/2c) +$$

$$\{\overset{*}{\ddot{N}}_{\theta,\theta}+(\sin\phi\ \overset{*}{\ddot{N}}_{\theta\phi})_{,\phi} + \cos\phi\ \overset{*}{\ddot{N}}_{\theta\phi} + \sin\phi\ \overset{*}{\ddot{Q}}_{\theta}\}\ s/R - \rho hR\ \sin\phi$$

$$(s_3\overset{*}{\ddot{u}}_{,tt}+s_2\bar{u}_{,tt}+s_5R\beta_{\theta,tt})\Big]$$

$$-\delta\bar{v}\Big[(\sin\phi\ \bar{N}_{\phi})_{,\phi}-\cos\phi\ \bar{N}_{\theta} + \bar{N}_{\theta\phi,\theta} + \sin\phi\ (\bar{Q}_{\phi}-Q_{\phi c}R/2c)$$

$$+\{(\sin\phi\ \overset{*}{\ddot{N}}_{\phi})_{,\phi} - \cos\phi\ \overset{*}{\ddot{N}}_{\theta} + \overset{*}{\ddot{N}}_{\theta\phi,\theta} + \sin\phi\ \overset{*}{\ddot{Q}}_{\phi}\}s/R$$

$$-\rho hR\ \sin\phi\ (s_3\overset{*}{\ddot{v}}_{,tt}+s_2\bar{v}_{,tt}+s_5R\beta_{\phi,tt})\Big]$$

$$-\delta\beta_{\theta}\Big[\overset{*}{\ddot{M}}_{\theta,\theta} + (\sin\phi\ \overset{*}{\ddot{M}}_{\theta\phi})_{,\phi} + \cos\phi\ \overset{*}{\ddot{M}}_{\theta\phi}$$

$$-R\ \sin\phi\ (\overset{*}{\ddot{Q}}_{\theta}+\bar{Q}_{\theta}\ s/R) + (hR/4c)\ \sin\phi\ Q_{\phi c}$$

$$+\{\bar{M}_{\theta,\theta}+(\sin\phi\ \bar{M}_{\theta\phi})_{,\phi} + \cos\phi\ M_{\theta\phi} - R\ \sin\phi\ (\bar{Q}_{\theta}+ \overset{*}{\ddot{Q}}_{\theta}\ s/R)\}\ s/R$$

$$- \rho hR^2\ \sin\phi\ (s_4\overset{*}{\ddot{u}}_{,tt}+s_5\bar{u}_{,tt}+s_6R\beta_{\theta,tt})\Big]$$

$$-\delta\beta_{\phi}\Big[(\sin\phi\ \overset{*}{\ddot{M}}_{\phi})_{,\phi} - \cos\phi\ \overset{*}{\ddot{M}}_{\theta} + \overset{*}{\ddot{M}}_{\theta\phi,\theta} - R\ \sin\phi\ (\overset{*}{\ddot{Q}}_{\phi}+\bar{Q}_{\phi}\ s/R) + (hR/4c)\sin\phi\ Q_{\phi c}$$

$$+ \{(\sin\phi\ \bar{M}_{\phi})_{,\phi} - \cos\phi\ \bar{M}_{\theta} + \bar{M}_{\theta\phi,\theta} - R\ \sin\phi\ (\bar{Q}_{\phi}+\overset{*}{\ddot{Q}}_{\phi}\ s/R)\}s/R$$

$$- \rho hR^2\ \sin\phi\ (s_4\overset{*}{\ddot{v}}_{,tt}+s_5\bar{v}_{,tt}+s_6R\beta_{\phi,tt})\Big]\Big)$$

$$+ 2R\ \sin\phi\Big(\delta\overset{*}{\ddot{N}}_{\phi}\Big[\overset{*}{\ddot{\varepsilon}}_{\phi}+\bar{\varepsilon}_{\phi}\ s/R - (\overset{*}{\ddot{N}}_{\phi}-\nu\ \overset{*}{\ddot{N}}_{\theta})(1+s^2/R^2)\ R/Eh - (\bar{N}_{\phi}-\nu\ \bar{N}_{\theta})2s/Eh\Big]$$

$$+ \delta\overset{\ast}{\ddot{N}}_{\theta}\left[\overset{\ast}{\varepsilon}_{\theta}+\bar{\varepsilon}_{\theta}\; s/R - (\overset{\ast}{\ddot{N}}_{\theta}-\nu\,\overset{\ast}{\ddot{N}}_{\phi})(1+s^2/R^2)\; R/Eh - (\bar{N}_{\theta}-\nu\,\bar{N}_{\phi})2s/Eh\right]$$

$$+ \delta\bar{N}_{\phi}\left[\bar{\varepsilon}_{\phi} + \overset{\ast}{\varepsilon}_{\phi}\; s/R - (\bar{N}_{\phi}-\nu\,\bar{N}_{\theta})(1+s^2/R^2)R/Eh - (\overset{\ast}{\ddot{N}}_{\phi}-\nu\,\overset{\ast}{\ddot{N}}_{\theta})2s/Eh\right]$$

$$+ \delta\bar{N}_{\theta}\left[\bar{\varepsilon}_{\theta} + \overset{\ast}{\varepsilon}_{\theta}\; s/R - (\bar{N}_{\theta}-\nu\,\bar{N}_{\phi})(1+s^2/R^2)R/Eh - (\overset{\ast}{\ddot{N}}_{\theta}-\nu\,\overset{\ast}{\ddot{N}}_{\phi})\; 2s/Eh\right]$$

$$+ \delta\overset{\ast}{\ddot{N}}_{\theta\phi}\left[\overset{\ast}{\gamma}_{\theta\phi}+\bar{\gamma}_{\theta\phi}\; s/R - \overset{\ast}{\ddot{N}}_{\theta\phi}(1+s^2/R^2)\; R/Gh - \bar{N}_{\theta\phi}2s/Gh\right]$$

$$+ \delta\bar{N}_{\theta\phi}\left[\bar{\gamma}_{\theta\phi} + \overset{\ast}{\gamma}_{\theta\phi}\; s/R - \bar{N}_{\theta\phi}(1+s^2/R^2)\; R/Gh - \overset{\ast}{\ddot{N}}_{\theta\phi}\; 2s/Gh\right]$$

$$+ \delta\overset{\ast}{\ddot{M}}_{\phi}\left[k_{\phi} - (\overset{\ast}{\ddot{M}}_{\phi} - \nu\,\overset{\ast}{\ddot{M}}_{\theta})(1+s^2/R^2)\; R/EI - (\bar{M}_{\phi} - \nu\,\bar{M}_{\theta})\; 2s/EI\right]$$

$$+ \delta\bar{M}_{\phi}\left[k_{\phi}\; s/R - (\bar{M}_{\phi} - \nu\,\bar{M}_{\theta})(1+s^2/R^2)\; R/EI - (\overset{\ast}{\ddot{M}}_{\phi}-\nu\,\overset{\ast}{\ddot{M}}_{\theta})\; 2s/EI\right]$$

$$+ \delta\overset{\ast}{\ddot{M}}_{\theta}\left[k_{\theta} - (\overset{\ast}{\ddot{M}}_{\theta}-\nu\,\overset{\ast}{\ddot{M}}_{\theta})(1+s^2/R^2)\; R/EI - (\bar{M}_{\theta}-\nu\,\bar{M}_{\phi})\; 2s/EI\right]$$

$$+ \delta\bar{M}_{\theta}\left[k_{\theta}\; s/R - (\bar{M}_{\theta}-\nu\,\bar{M}_{\theta})(1+s^2/R^2)\; R/EI - (\overset{\ast}{\ddot{M}}_{\theta}-\nu\,\overset{\ast}{\ddot{M}}_{\phi})\; 2s/EI\right]$$

$$+ \delta\overset{\ast}{\ddot{M}}_{\theta\phi}\left[k_{\theta\phi} - \overset{\ast}{\ddot{M}}_{\theta\phi}(1+s^2/R^2)\; R/GI - \bar{M}_{\theta\phi}\; 2s/GI\right]$$

$$
\begin{aligned}
&+\delta\bar{M}_{\theta\phi}\Big[k_{\theta\phi}\ s/R - (\bar{M}_{\theta\phi}(1+s^2/R^2)\ R/GI - \overset{*}{M}_{\theta\phi}\ 2s/GI\Big] \\
&+\delta\overset{*}{Q}_{\phi}\Big[\overset{*}{\gamma}_{r\phi} + \bar{\gamma}_{r\phi}\ s/R - \overset{*}{Q}_{\phi}(1+s^2/R^2)\ 6R/5Gh - \bar{Q}_{\phi}\ 12s/5Gh \\
&+ (d_1\overset{*}{p}_{\phi}+d_2\bar{p}_{\phi})\ R/10G_c\Big] + \delta\bar{Q}_{\phi}\Big[\bar{\gamma}_{r\phi}+\overset{*}{\gamma}_{r\phi}\ s/R - \bar{Q}_{\phi}(1+s^2/R^2) \\
&\quad 6R/5Gh - \overset{*}{Q}_{\phi}\ 12s/5Gh + (d_2\overset{*}{p}_{\phi}+d_1\bar{p}_{\phi})\ R/10G_c\Big] \\
&+ \delta\overset{*}{Q}_{\theta}\Big[\overset{*}{\gamma}_{r\theta}+\bar{\gamma}_{r\theta}\ s/R - \overset{*}{Q}_{\theta}(1+s^2/R^2)\ 6R/5Gh - \bar{Q}_{\theta}\ 12s/5Gh \\
&+ (d_1\overset{*}{p}_{\theta}+d_2\bar{p}_{\theta})\ R/10G_c\Big] + \delta\bar{Q}_{\theta}\Big[\bar{\gamma}_{r\theta}+\overset{*}{\gamma}_{r\theta}\ s/R - \bar{Q}_{\theta}(1+s^2/R^2)\ 6R/5Gh \\
&- \overset{*}{Q}_{\theta}\ 12s/5Gh + (d_2\overset{*}{p}_{\theta}+d_1\bar{p}_{\theta})\ R/10G_c\Big] \\
&+ 1/2\ \delta\ Q_{\phi c}\Big[\gamma_{r\phi c}-Q_{\phi c}\ 3R/5G_c c + (\overset{*}{p}_{\phi}+\bar{p}\ c/R)\ R/5G_c\Big] \\
&+ 1/2\ \delta\ Q_{\theta c}\Big[\gamma_{r\theta c}-Q_{\theta c}\ 3R/5G_c c + (\overset{*}{p}_{\theta}+\bar{p}_{\theta}\ c/R)\ R/5G_c\Big]. \qquad (2.15) \\
&+ \Big\{\delta\overset{*}{p}_{\phi}\Big[d_1\overset{*}{Q}_{\phi}+d_2\bar{Q}_{\phi}+Q_{\phi c}-(d_3\overset{*}{p}_{\phi}+d_4\bar{p}_{\phi})\ 4h/3\Big] \\
&+ \delta\bar{p}_{\phi}\Big[d_2\overset{*}{Q}_{\phi}+d_1\bar{Q}_{\phi}+Q_{\phi c}\ c/R - (d_4\overset{*}{p}_{\phi}+d_5\bar{p}_{\phi})\ 4h/3\Big] \\
&+ \delta\overset{*}{p}_{\theta}\Big[d_1\overset{*}{Q}_{\theta}+d_2\bar{Q}_{\theta}+Q_{\theta c}-(d_3\overset{*}{p}_{\theta}+d_4\bar{p}_{\theta})\ 4h/3\Big]
\end{aligned}
$$

$$\delta\bar{p}_\theta\left[d_2\mathring{\bar{Q}}_\theta+d_1\bar{Q}_\theta+Q_{\phi c}\ c/R - (d_4\mathring{\bar{p}}_\theta+d_5\bar{p}_\theta)\ 4h/3\right]\Big\}R/10G_c\Big)\Big\}\ d\phi\ d\theta$$

$$-2R\oint(N'_\phi\ \delta\mathring{\bar{v}} + N''_\phi\ \delta\bar{v} + N'_{\theta\phi}\ \delta\mathring{\bar{u}} + N''_{\theta\phi}\ \delta\bar{u} +$$

$$Q'_\phi\ \ \delta w + M'_\phi\delta\beta_\phi + M'_{\theta\phi}\ \delta\beta_\theta\ )\ \sin\phi_o\ d\theta = 0$$

where

$$d_1 = r_G(1+sc/R^2)$$

$$d_2 = r_G(s/R+c/R)$$

$$d_3 = r_G(1+c^2/R^2) + r_h(3+5c^2/R^2)/2 \qquad \text{(iv)}$$

$$d_4 = (r_G+2r_h)\ 2c/R$$

$$d_5 = r_G(1+c^2/R^2) + r_h(5+3c^2/R^2)/2$$

In eq. (2.15), the contents in the first bracket [ ] are reduced to a line integral as follows:

$$\iint 2R\left[\{\sin\phi\ (\mathring{\bar{N}}_\phi+N_\phi\ s/R)\ \delta\mathring{\bar{v}}\}_{,\phi} + \{(\mathring{\bar{N}}_\theta+N_\theta\ s/R)\delta\mathring{\bar{u}}\}_{,\theta}\right.$$

$$\left.\cdots\cdots\right]d\phi\ \ d\theta\ = \oint 2R\left[\{\sin\phi\ (\mathring{\bar{N}}_\phi+N_\phi\ s/R)\delta\mathring{\bar{v}}\}\Big|_{\phi=0}^{\phi=\phi_o}\right. \qquad (2.16)$$

$$\left.\cdots\right]d\theta = \oint 2\ \left[(\mathring{\bar{N}}_\phi+N_\phi\ s/R)\delta\mathring{\bar{v}} + \cdots\right]\ R\ \sin\phi_o\ d\theta$$

The term $\int 2R \left[\int_0^{2\pi} \{(\overset{\star}{\overset{\star}{N}}_\theta + \bar{N}_\theta\ s/R)\delta\overset{\star}{\overset{\star}{u}}\}_{,\theta}\ d\theta \right] = 0$

due to uniqueness of the forces and the displacements.

Substituting eq. (2.16) in eq. (2.15), results in the variational equation

$$\iint_S \Big\{ 2R \Big[-\delta\overset{\star}{\overset{\star}{u}}[\overset{\star}{\overset{\star}{N}}_{\theta,\theta} + (\sin\phi\ \overset{\star}{\overset{\star}{N}}_{\theta\phi})_{,\phi} + \ldots] - \delta\overset{\star}{\overset{\star}{v}}$$

$$[(\sin\phi\ \overset{\star}{\overset{\star}{N}}_\phi)_{,\phi} - \cos\phi\ \overset{\star}{\overset{\star}{N}}_\theta + \ldots] - \delta w[-\sin\phi\ (\overset{\star}{\overset{\star}{N}}_\phi + \overset{\star}{\overset{\star}{N}}_\theta)$$

$$+ \{\sin\phi\ (\overset{\star}{\overset{\star}{Q}}_\phi + Q_{\phi c}/2)\}_{,\phi} + \ldots] - \delta\bar{u}[\bar{N}_{\theta,\theta} + (\sin\phi\ \bar{N}_{\theta\phi})_{,\phi} + \ldots]$$

$$- \delta\bar{v}\ [(\sin\phi\ \bar{N}_\phi)_{,\phi} - \cos\phi\ \bar{N}_\theta + \ldots] - \delta\beta_\theta[\overset{\star}{\overset{\star}{M}}_{\theta,\theta}$$

$$+ (\sin\phi\ \overset{\star}{\overset{\star}{M}}_{\theta\phi})_{,\phi} + \ldots] - \delta\beta_\phi[(\sin\phi\ \overset{\star}{\overset{\star}{M}}_\phi)_{,\phi} - \cos\phi\ \overset{\star}{\overset{\star}{M}}_\theta + \ldots$$

$$\ldots] \Big] + 2R\sin\phi \Big[\delta\overset{\star}{\overset{\star}{N}}_\phi[\overset{\star}{\overset{\star}{\varepsilon}}_\phi + \bar{\varepsilon}_\phi\ s/R \ldots]$$

$$+ \delta\overset{\star}{\overset{\star}{N}}_\theta[\overset{\star}{\overset{\star}{\varepsilon}}_\theta + \bar{\varepsilon}_\theta\ s/R \ldots] + \ldots \Big] \Big\}\ d\phi\ d\theta$$

$$+ 2R \oint [(\overset{\star}{\overset{\star}{N}}_\phi + \bar{N}_\phi\ s/R - N'_\phi)\delta\overset{\star}{\overset{\star}{v}} + (\bar{N}_\phi + \overset{\star}{\overset{\star}{N}}_\phi\ s/R - N''_\phi)\ \delta\bar{v}$$

$$+ (\overset{\star}{\overset{\star}{N}}_{\theta\phi} + \bar{N}_{\theta\phi}\ s/R - N'_{\theta\phi})\delta\overset{\star}{\overset{\star}{u}} + (\bar{N}_{\theta\phi} + \overset{\star}{\overset{\star}{N}}_{\theta\phi}\ s/R - N''_{\theta\phi})\delta\bar{u}$$

$$+ (\overset{\star}{\overset{\star}{Q}}_\phi + \bar{Q}_\phi\ s/R + Q_{\phi c}/2 - Q'_\phi)\delta w + (\overset{\star}{\overset{\star}{M}}_\phi + \bar{M}_\phi\ s/R - M'_\phi)\delta\beta_\phi$$

$$+ (\overset{*}{M}_{\theta\phi}+\overline{M}_{\theta\phi}\ s/R-M'_{\theta\phi})\delta\beta_\theta \Big]\sin\phi_o\ d\theta = 0 \qquad (2.17)$$

The coefficients of δu^*, δv^*, ... δM^*_ϕ, $\delta M^*_{\theta\phi}$, ... δp^*_θ, etc. of the surface and line integrals in eq. (2.17) must vanish. This condition will yield the differential equations of motion, stress resultants and displacements relationships and the boundary conditions.

2.2.3 Differential Equations of Motion

The differential equations of motion are obtained by equating to zero the coefficients of δu^*, δv^*, $\delta\bar{u}$, $\delta\bar{v}$, δw, $\delta\beta_\phi$ and $\delta\beta_\theta$ in eq. (2.17). The seven differential equations in terms of stresses for non-symmetric vibration of the sandwich shell are given below.

$$\operatorname{cosec}\phi\ \overset{*}{N}_{\theta,\theta} + \overset{*}{N}_{\theta\phi,\phi} + 2\cot\phi\ \overset{*}{N}_{\theta\phi} + \overset{*}{Q}_\theta + Q_{\theta c}/2 + (\operatorname{cosec}\phi\ \overline{N}_{\theta,\theta} +$$

$$\overline{N}_{\theta\phi,\phi} + 2\cot\phi\ \overline{N}_{\theta\phi} + \overline{Q}_\theta)\ s/R = \rho\ h\ R(s_1 u^* + s_3\bar{u} + s_4 R\beta_\theta)_{,tt}$$

$$\overset{*}{N}_{\phi,\phi} + \operatorname{cosec}\phi\ \overset{*}{N}_{\theta\phi,\theta} + \cot\phi\ (\overset{*}{N}_\phi - \overset{*}{N}_\theta) + \overset{*}{Q}_\phi + Q_{\phi c}/2 + \Big\{\overline{N}_{\phi,\phi} + \operatorname{cosec}\phi\ \overline{N}_{\theta\phi,\theta} +$$

$$\cot\phi\ (\overline{N}_\phi - \overline{N}_\theta) + \overline{Q}_\phi\Big\} s/R = \rho h\ R(s_1 \overset{*}{v} + s_3\bar{v} + s_4 R\beta_\phi)_{,tt}$$

$$\operatorname{cosec}\phi\ \overline{N}_{\theta,\theta} + \overline{N}_{\theta\phi,\phi} + 2\cot\phi\ \overline{N}_{\theta\phi} + \overline{Q}_\theta - Q_{\theta c}\ R/2c + (\operatorname{cosec}\phi\ \overset{*}{N}_{\theta,\theta} +$$

$$\overset{*}{N}_{\theta\phi,\phi} + 2\cot\phi\ \overset{*}{N}_{\theta\phi} + \overset{*}{Q}_\theta)\ s/R = \rho\ h\ R(s_3\overset{*}{u} + s_2\bar{u} + s_5 R\beta_\theta)_{,tt}$$

$$\bar{N}_{\theta,\phi} + \operatorname{cosec}\phi\ \bar{N}_{\theta\phi,\theta} + \cot\phi\ (\bar{N}_\phi - \bar{N}_\theta) + \bar{Q}_\phi - Q_{\phi c}\ R/2c + \left\{\overset{*}{N}_{\phi,\phi} + \right.$$

$$\left. \operatorname{cosec}\phi\ \overset{*}{N}_{\theta\phi,\phi} + \cot\phi\ (\overset{*}{N}_\phi - \overset{*}{N}_\theta) + \overset{*}{Q}_\phi \right\} s/R = \rho\ h\ R(s_3\overset{*}{v} + s_2\bar{v} + s_5 R\beta_\phi)_{,tt} \qquad (2.18)$$

$$(\overset{*}{Q}_\phi + Q_{\phi c}/2)_{,\phi} + \operatorname{cosec}\phi\ (\overset{*}{Q}_\theta + Q_{\theta c}/2)_{,\theta} + \cot\phi\ (\overset{*}{Q}_\phi + Q_{\phi c}/2) - (\overset{*}{N}_\phi + \overset{*}{N}_\theta) +$$

$$\left\{Q_{\phi,\phi} + \operatorname{cosec}\phi\ \bar{Q}_{\theta,\theta} + \cot\phi\ \bar{Q}_\phi - (\bar{N}_\phi + \bar{N}_\theta)\right\} s/R = \rho\ h\ R\ s_1 w_{,tt}$$

$$\operatorname{cosec}\phi\ \overset{*}{M}_{\theta,\theta} + \overset{*}{M}_{\theta\phi,\phi} + 2\cot\phi\ \overset{*}{M}_{\theta\phi} - R(\overset{*}{Q}_\theta + \bar{Q}_\theta\ s/R) + Q_{\theta c}\ h\ R/4c +$$

$$\left\{\operatorname{cosec}\phi\ \bar{M}_{\theta,\theta} + \bar{M}_{\theta\phi,\phi} + 2\cot\phi\ \bar{M}_{\theta\phi} - R(\bar{Q}_\theta + \overset{*}{Q}_\theta\ s/R)\right\} s/R =$$

$$\rho\ h\ R^2(s_4\overset{*}{u} + s_5\bar{u} + s_6 R\beta_\theta)_{,tt}$$

$$\overset{*}{M}_{\phi,\phi} + \operatorname{cosec}\phi\ \overset{*}{M}_{\theta\phi,\theta} + \cot\phi\ (\overset{*}{M}_\phi - \overset{*}{M}_\theta) - R(\overset{*}{Q}_\phi + \bar{Q}_\phi\ s/R) + Q_{\phi c}\ h\ R/4c +$$

$$\left\{M_{\phi,\phi} + \operatorname{cosec}\phi\ \bar{M}_{\theta\phi,\theta} + \cot\phi\ (\bar{M}_\phi - \bar{M}_\theta) - R(\bar{Q}_\phi + \overset{*}{Q}_\phi\ s/R)\right\} s/R =$$

$$\rho\ h\ R^2(s_4\overset{*}{v} + s_5\bar{v} + s_6 R\beta_\phi)_{,tt}$$

2.2.4 Stress Resultants and Displacement Relationships

Equating to zero the coefficients of $\delta\overset{*}{N}_\phi$, $\delta\overset{*}{N}_\theta$, $\delta\overset{*}{N}_{\theta\phi}$, ..., $\delta\overset{*}{M}_\phi$, ... $\delta\overset{*}{Q}_\phi$, $\delta\overset{*}{Q}_\theta$, $\delta Q_{\phi c}$, ..., $\delta\overset{*}{p}_\phi$, $\delta\overline{p}_\phi$, etc., a set of twenty-two equations are obtained. These equations are written here first. Subsequently, they are combined together in order to obtain relationships between $\overset{*}{N}_\phi$-strain, $\overline{N}_\phi$-strain, etc.

$$(\overset{*}{N}_\phi-\nu\overset{*}{N}_\theta)(1+s^2/R^2)+(\overline{N}_\phi-\nu\overline{N}_\theta)2s/R = (Eh/R)(\overset{*}{\varepsilon}_\phi+\overline{\varepsilon}_\phi s/R)$$

$$(\overline{N}_\phi-\nu\overline{N}_\theta)(1+s^2/R^2)+(\overset{*}{N}_\phi-\nu\overset{*}{N}_\theta)2s/R = (Eh/R)(\overline{\varepsilon}_\phi+\overset{*}{\varepsilon}_\phi s/R)$$

$$(\overset{*}{N}_\theta-\nu\overset{*}{N}_\phi)(1+s^2/R^2)+(\overline{N}_\theta-\nu\overline{N}_\phi)2s/R = (Eh/R)(\overset{*}{\varepsilon}_\theta+\overline{\varepsilon}_\theta s/R)$$

$$(\overline{N}_\theta-\nu\overline{N}_\phi)(1+s^2/R^2)+(\overset{*}{N}_\theta-\nu\overset{*}{N}_\phi)2s/R = (Eh/R)(\overline{\varepsilon}_\theta+\overset{*}{\varepsilon}_\theta s/R)$$

$$\overset{*}{N}_{\theta\phi}(1+s^2/R^2)+\overline{N}_{\theta\phi}\ 2s/R = (Gh/R)(\overset{*}{\gamma}_{\theta\phi}+\overline{\gamma}_{\theta\phi}s/R)$$

$$\overline{N}_{\theta\phi}(1+s^2/R^2)+\overset{*}{N}_{\theta\phi}\ 2s/R = (Gh/R)(\overline{\gamma}_{\theta\phi}+\overset{*}{\gamma}_{\theta\phi}\ s/R)$$

$$(\overset{*}{M}_\phi\ -\nu\overset{*}{M}_\theta)(1+s^2/R^2)+(\overline{M}_\phi-\nu\overline{M}_\theta)2s/R = (EI/R)k_\phi$$

$$(\overline{M}_\phi-\nu\overline{M}_\theta)(1+s^2/R^2)+(\overset{*}{M}_\phi-\ \overset{*}{M}_\theta)2s/R = (EI/R)k_\phi\ s/R$$

$$(\overset{*}{M}_\theta-\ \overset{*}{M}_\phi)(1+s^2/R^2)+(\overline{M}_\theta-\ \overline{M}_\theta)2s/R = (EI/R)k_\theta$$

$$(\overline{M}_\theta - \nu\overline{M}_\phi)(1+s^2/R^2) + (\overset{*}{M}_\theta - \overset{*}{M}_\phi)2s/R = (EI/R)k_\theta\ s/R$$

$$\overset{*}{M}_{\theta\phi}(1+s^2/R^2) + \overline{M}_{\theta\phi}\ 2s/R = (GI/R)k_{\theta\phi}$$

$$\overline{M}_{\theta\phi}(1+s^2/R^2) + \overset{*}{M}_{\theta\phi}\ 2s/R = (GI/R)k_{\theta\phi}\ s/R$$

$$\overset{*}{Q}_\phi(1+s^2/R^2) + \overline{Q}_\phi\ s/R - (h/12r_G)(d_1\overset{*}{p}_\phi + d_2\overline{p}_\phi) = (5Gh/6R)(\overset{*}{\gamma}_{r\phi} + \overline{\gamma}_{r\phi}\ s/R)$$

$$\overline{Q}_\phi(1+s^2/R^2) + \overset{*}{Q}_\phi\ s/R - (h/12r_G)(d_1\overline{p}_\phi + d_2\overset{*}{p}_\phi) = (5Gh/6R)(\overline{\gamma}_{r\phi} + \overset{*}{\gamma}_{r\phi}\ s/R)$$

$$\overset{*}{Q}_\theta(1+s^2/R^2) + \overline{Q}_\theta\ s/R - (h/12r_G)(d_1\overset{*}{p}_\theta + d_2\overline{p}_\theta) = (5Gh/6R)(\overset{*}{\gamma}_{r\theta} + \overline{\gamma}_{r\theta}\ s/R)$$

$$\overline{Q}_\theta(1+s^2/R^2) + \overset{*}{Q}_\theta\ s/R - (h/12r_G)(d_1\overline{p}_\theta + d_2\overset{*}{p}_\theta) = (5Gh/6R)(\overline{\gamma}_{r\theta} + \overset{*}{\gamma}_{r\theta}\ s/R)$$

$$Q_{\phi c} - (\overset{*}{p}_\phi + \overline{p}_\phi\ s/R) = (5G_c c/3R)\ \gamma_{r\phi c} \qquad (2.19)$$

$$Q_{\theta c} - (\overset{*}{p}_\theta + \overline{p}_\theta\ s/R) = (5G_c c/3R)\ \gamma_{r\theta c}$$

$$d_3\overset{*}{p}_\phi + d_4\overline{p}_\phi = (3/4h)(d_1\overset{*}{Q}_\phi + d_2\overline{Q}_\phi + Q_{\phi c})$$

$$d_3\overline{p}_\phi + d_4\overset{*}{p}_\phi = (3/4h)(d_1\overline{Q}_\phi + d_2\overset{*}{Q}_\phi + Q_{\phi c}\ c/R)$$

$$d_3\overset{*}{p}_\theta + d_4\overline{p}_\theta = (3/4h)(d_1\overset{*}{Q}_\theta + d_2\overline{Q}_\theta + Q_{\theta c})$$

$$d_3\overline{p}_\theta + d_4\overset{*}{p}_\theta = (3/4r_h)(d_1\overline{Q}_\theta + d_2\overset{*}{Q}_\theta + Q_{\phi c}\ c/R)$$

By solving eqs. (2.19)(a-d), four unknowns $\overset{*}{N}_\phi$, $\overset{*}{N}_\theta$, $\overline{N}_\phi$ and $\overline{N}_\theta$ can be obtained in terms of $\overset{*}{\varepsilon}_\phi$, $\overset{*}{\varepsilon}_\theta$, $\overline{\varepsilon}_\phi$ and $\overline{\varepsilon}_\theta$. Equations (2.19)(e-f) yield the relationships between $\overset{*}{N}_{\theta\phi}$, $\overline{N}_{\theta\phi}$ and $\overset{*}{\gamma}_{\theta\phi}$ and $\overline{\gamma}_{\theta\phi}$. Similarly, from eqs. (2.19)(g-l), $\overset{*}{M}_\phi$, $\overset{*}{M}_\theta$, $\overline{M}_\phi$, $\overline{M}_\theta$, $\overset{*}{M}_{\theta\phi}$, $\overline{M}_{\theta\phi}$ are expressed as functions of k_ϕ, k_θ and $k_{\theta\phi}$. Before, solving eqs. (2.19)(m-r) to obtain $\overset{*}{Q}_\phi$, $\overline{Q}_\phi$, $\overset{*}{Q}_\theta$, $\overline{Q}_\theta$, $Q_{\phi c}$ and $Q_{\theta c}$, the last four equations (2.19)(s-v) are solved first. These four equations yield $\overset{*}{p}_\phi$, $\overline{p}_\phi$, $\overset{*}{p}_\theta$ and $\overline{p}_\theta$ as functions of $\overset{*}{Q}_\phi$, $\overline{Q}_\phi$, $\overset{*}{Q}_\theta$, $\overline{Q}_\theta$, $Q_{\phi c}$ and $Q_{\theta c}$. Inserting $\overset{*}{p}_\phi$, $\overset{*}{p}_\theta$, $\overline{p}_\phi$ and $\overline{p}_\theta$ in eqs. (2.19)(m-r), $\overset{*}{Q}_\phi$, $\overline{Q}_\theta$, $\overset{*}{Q}_\theta$, $\overline{Q}_\phi$, $Q_{\phi c}$, $Q_{\theta c}$ are obtained as functions of $\overset{*}{\gamma}_{r\phi}$, $\overset{*}{\gamma}_{r\theta}$, $\overline{\gamma}_{r\phi}$, $\overline{\gamma}_{r\theta}$, $\gamma_{r\phi c}$. Thus, from twenty-two equations, eighteen unknowns are solved after eliminating $\overset{*}{p}_\phi$, $\overset{*}{p}_\theta$, $\overline{p}_\phi$ and $\overline{p}_\theta$. It has also been found, while solving eqs. (2.19)(g-l), $\overline{M}_\phi$, $\overline{M}_\theta$ and $\overline{M}_{\theta\phi}$ are proportional to $\overset{*}{M}_\phi$, $\overset{*}{M}_\theta$ and $\overset{*}{M}_{\theta\phi}$ respectively. In this manner, the following simplified equations are achieved finally.

$$\begin{aligned}
&\overset{*}{N}_\phi + \overline{N}_\phi\, s/R = K(\overset{*}{\varepsilon}_\phi + \nu\,\overset{*}{\varepsilon}_\theta)/R, && \overline{N}_{\theta\phi} + \overset{*}{N}_{\theta\phi}\, s/R = (Gh/R)\overline{\gamma}_{\theta\phi} \\
&\overset{*}{N}_\theta + \overline{N}_\theta\, s/R = K(\overset{*}{\varepsilon}_\theta + \nu\overset{*}{\varepsilon}_\phi)/R, && \overset{*}{M}_\phi + \overline{M}_\phi\, s/R = D(k_\phi + \nu\, k_\theta)/R \\
&\overline{N}_\phi + \overset{*}{N}_\phi\, s/R = K(\overline{\varepsilon}_\phi + \nu\overline{\varepsilon}_\theta)/R, && \overset{*}{M}_\theta + \overline{M}_\theta\, s/R = D(k_\theta + \nu\, k_\phi)/R \\
&\overline{N}_\theta + \overset{*}{N}_\theta\, s/R = K(\overline{\varepsilon}_\theta + \nu\,\overline{\varepsilon}_\phi)/R, && \overset{*}{M}_{\theta\phi} + \overline{M}_{\theta\phi}\, s/R = D(1-\nu)k_{\theta\phi}/2R \\
&\overset{*}{N}_{\theta\phi} + \overline{N}_{\theta\phi}\, s/R = (Gh/R)\,\overset{*}{\gamma}_{\theta\phi}, && \overset{*}{Q}_\phi + \overline{Q}_\phi\, s/R = 5Gh(r_1\overset{*}{\gamma}_{r\phi} + r_3\gamma_{r\phi c})/6R
\end{aligned} \tag{2.20}$$

$$\bar{Q}_\phi + \overset{*}{Q}_\phi\ s/R = 5Gh\ r_2\ \bar{\gamma}_{r\phi}/6R \qquad Q_{\phi c} = 5Gh(r_3\ \overset{*}{\gamma}_{r\phi} + r_4\ \gamma_{r\phi c})/3R$$

$$\overset{*}{Q}_\theta + \bar{Q}_\theta\ s/R = 5Gh(r_1\ \overset{*}{\gamma}_{r\theta} + r_3\gamma_{r\theta c})/6R \qquad Q_{\theta c} = 5Gh(r_3\ \overset{*}{\gamma}_{r\theta} + r_4\ \gamma_{r\theta c})/3R$$

$$\bar{Q}_\theta + \overset{*}{Q}_\theta\ s/R = 5Gh\ r_2\ \bar{\gamma}_{r\theta}/6R$$

The quantities r_1, ... r_4, K and D are given by equations (v).

$$r_1 = 1 + r_G/(15r_G+20r_h), \qquad r_3 = 2r_Gr_h/(15r_G+20r_h)$$

$$r_2 = 1 + r_G/(15r_G+40r_h), \qquad r_4 = 2r_Gr_h(15r_G+24r_h)/(15r_G+20r_h)$$

$$K = Eh/(1-\nu^2), \qquad D = Eh^3/12(1-\nu^2) \qquad \text{(v)}$$

2.2.5 Boundary Conditions

The boundary conditions for the non-symmetric vibrations of the sandwich spherical shell are also obtained from the variational eq. (2.17). The expressions inside the brackets of the line integral are equated to zero to yield the following conditions at the boundary $\phi=\phi_o$.

$$\overset{*}{N}_\phi + \bar{N}_\phi\ s/R = N'_\phi, \quad \bar{N}_{\theta\phi} + \overset{*}{N}_{\theta\phi}\ s/R = N''_{\theta\phi}$$

$$\bar{N}_\phi + \overset{*}{N}_\phi\ s/R = N''_\phi, \quad \overset{*}{M}_\phi + \bar{M}_\phi\ s/R = M'_\phi$$

$$\overset{*}{N}_{\theta\phi} + \bar{N}_{\theta\phi}\ s/R = N'_{\theta\phi}, \quad \overset{*}{M}_{\theta\phi} + \bar{M}_{\theta\phi}\ s/R = M'_{\theta\phi}$$

$$\overset{*}{Q}_\phi + \bar{Q}_\phi\ s/R + Q_{\phi c}/2 = Q'_\phi$$

Also, the prescribed values of displacement components, for which $\delta\overset{*}{u}$, $\delta\overset{*}{v}$, $\delta\overline{u}$, ..., $\delta\beta_\theta$ vanish, at $\phi=\phi_o$, make the line integral in eq. (2.17) equal to zero.

2.3 Special Cases

The differential equations of motion (2.18) and the stress resultants and displacement relationships (2.20) have been derived for the general case of the spherical sandwich shells. These equations can be reduced for the case of axisymmetric motion of the spherical sandwich and homogeneous shells and also the constitutive equations for the vibrations of circular sandwich plates can be derived. These cases along with some more are discussed in this section.

2.3.1 Equations for Axisymmetric Motion of Spherical Sandwich Shell with Face Sheets as Flexural Members

For this case, eqs. (2.18) and (2.20) are simplified considerably. The variables involved in the analysis become independent of the circumferential angle θ. Thus, all the derivations with respect to θ vanish. Also, the displacements $\overset{*}{u}$, $\overline{u}$ and β_θ in θ direction are zero.

The equations of motion (2.18) a, c and f are identically satisfied. The remaining differential equations of motion assume the form written below.

$$\overset{*}{N}_{\phi,\phi} + \cot\phi\,(\overset{*}{N}_{\phi}-\overset{*}{N}_{\theta}) + \overset{*}{Q}_{\phi} + Q_{\phi c}/2 + \left\{\bar{N}_{\phi,\phi}+\cot\phi\,(\bar{N}_{\phi}-\bar{N}_{\theta}) + \bar{Q}_{\phi}\right\}s/R$$

$$= \rho\, h\, R\,(s_1\overset{*}{v}+s_3\bar{v}+s_4R\beta_{\phi})_{,tt}$$

$$\bar{N}_{\phi,\phi} + \cot\phi\,(\bar{N}_{\phi}-\bar{N}_{\theta}) + \bar{Q}_{\phi} - Q_{\phi c}\,R/2c + \left\{\overset{*}{N}_{\phi,\phi}+\cot\phi\,(\overset{*}{N}_{\phi}-\overset{*}{N}_{\theta}) + \overset{*}{Q}_{\phi}\right\}s/R$$

$$= \rho\, h\, R(s_3\overset{*}{v}+s_2\bar{v}+s_5R\beta_{\phi})_{,tt}$$

$$(\overset{*}{Q}_{\phi}+Q_{\phi c}/2)_{,\phi} + \cot\phi\,(\overset{*}{Q}_{\phi}+Q_{\phi c}/2) - (\overset{*}{N}_{\phi}+\overset{*}{N}_{\theta}) + \left\{\bar{Q}_{\phi,\phi} + \cot\phi\,\bar{Q}_{\phi} -\right.$$

$$\left.(\bar{N}_{\phi}+\bar{N}_{\theta})\right\}s/R = \rho\, h\, R\, s_1 w_{,tt}$$

$$\overset{*}{M}_{\phi,\phi} + \cot\phi\,(\overset{*}{M}_{\phi}-\overset{*}{M}_{\theta}) - R(\overset{*}{Q}_{\phi}+\bar{Q}_{\phi}\,s/R-Q_{\phi c}h/4c) + \left\{\bar{M}_{\phi,\phi} + \cot\phi\,(\bar{M}_{\phi}-\bar{M}_{\theta})\right.$$

$$\left.- R(\bar{Q}_{\phi}+\overset{*}{Q}_{\phi}\,s/R)\right\}s/R = \rho\, h\, R^2(s_4\overset{*}{v}+s_5\bar{v}+s_6R\beta_{\phi})_{,tt} \qquad (2.21)$$

The stress resultants and displacement relationships which for the general case are given by eq. (2.20) reduce to:

$$\overset{*}{N}_\phi + \bar{N}_\phi\, s/R = K\{\overset{*}{v}_{,\phi}+w+\nu(\cot\phi\ \overset{*}{v}+w)\}/R$$

$$\overset{*}{N}_\theta + \bar{N}_\theta\, s/R = K\{\cot\phi\ \overset{*}{v}+w+\nu(\overset{*}{v}_{,\phi}+w)\}/R$$

$$\bar{N}_\phi + \overset{*}{N}_\phi\, s/R = K(\bar{v}_{,\phi}+\nu\ \cot\phi\ \bar{v})/R$$

$$\bar{N}_\theta + \overset{*}{N}_\theta\, s/R = K(\cot\phi\ \bar{v}+\nu\,\bar{v}_{,\phi})/R$$

$$\overset{*}{M}_\phi + \bar{M}_\phi\, s/R = D(\beta_{\phi,\phi}+\nu\ \cot\phi\ \beta_\phi)/R$$

$$\overset{*}{M}_\theta + \bar{M}_\theta\, s/R = D(\cot\phi\ \beta_\phi+\nu\,\beta_{\phi,\phi})/R$$

$$\overset{*}{Q}_\phi + \bar{Q}_\phi\, s/R = (5Gh/6R)\{r_1(w_{,\phi}-\overset{*}{v}+R\beta_\phi)+r_3(w_{,\phi}-\overset{*}{v}+\bar{v}\ R/c-\beta_\phi Rh/2c)\}$$

$$\bar{Q}_\phi + \overset{*}{Q}_\phi\, s/R = (5Gh/6R)r_2(-\bar{v}+s\beta_\phi)$$

$$Q_{\phi c} = (5Gh/3R)\{r_3(w_{,\phi}-\overset{*}{v}+R\beta_\phi)+r_4(w_{,\phi}-\overset{*}{v}+\bar{v}\ R/c-\beta_\phi Rh/2c)\} \qquad (2.22)$$

2.3.2 Equations for Axisymmetric Motion of Spherical Sandwich Shell with Membrane Face Sheets

For a sandwich shell in which the central core layer is very thick as compared to the face sheets, the face sheets behave as membranes. Their flexural rigidity D becomes negligible and they take only the in plane loads. The differential equations of motion (2.21) further simplify to a great extent.

The terms $\overset{*}{M}_\phi$, $\bar{M}_\phi$, $\overset{*}{Q}$ and $\bar{Q}$ are dropped from equations of motion and β_ϕ is set equal to zero. The simplified form of the differential equations of motion are obtained from eqs. (2.21) and are written below:

$$\overset{*}{N}_{\phi,\phi} + \cot\phi\,(\overset{*}{N}_\phi-\overset{*}{N}_\theta) + Q_{\phi c}/2 + \left\{\bar{N}_{\phi,\phi}+\cot\phi\,(\bar{N}_\phi-\bar{N}_\theta)\right\}s/R$$

$$= \rho\, h\, R(s_1\overset{*}{v}+s_3\bar{v})_{,tt}$$

$$\bar{N}_{\phi,\phi} + \cot\phi\,(\bar{N}_\phi-\bar{N}_\theta) - Q_{\phi c}\,R/2c + \left\{\overset{*}{N}_{\phi,\phi}+\cot\phi\,(\overset{*}{N}_\phi-\overset{*}{N}_\theta)\right\}s/R$$

$$= \rho\, hR(s_3\overset{*}{v}+s_2\bar{v})_{,tt} \qquad (2.23)$$

$$(Q_{\phi c,\phi}+\cot\phi\, Q_{\phi c})/2 - (\overset{*}{N}_\phi+\overset{*}{N}_\theta) - (\bar{N}_\phi+\bar{N}_\theta)s/R = \rho\, h\, R\, s_1 w_{,tt}$$

Parameters s_1, s_2, s_3, r_G and r_h are given by eq. (iii) on pg. 22 and the term h^2/R^2 can be dropped from s_1 and s_2 for very thin face sheets.

The stress resultants $\overset{*}{N}_\phi$, $\bar{N}_\phi$, $\overset{*}{N}_\theta$ and $\bar{N}_\theta$ are expressed in terms of displacements by eq. (2.22). The shear stress resultant $Q_{\phi c}$ in the core is obtained as a function of displacement components in the following manner.

Setting $\overset{*}{Q}_\phi$ and $\bar{Q}_\phi$ equal to zero in eq. (2.22)g, the following condition is achieved.

$$r_1(w_{,\phi}-\overset{*}{v}) + r_3(w_{,\phi}-\overset{*}{v}+\bar{v}\,R/c) = 0$$

The symbol β_ϕ is dropped for membrane face sheets of the sandwich shell.

Therefore, $w_{,\phi}-\overset{*}{v} = -(r_3/r_1)(w_{,\phi}-\overset{*}{v}+\bar{v}\ R/c)$ (2.24)

Inserting (2.24) in (2.22)i,

$$Q_{\phi c} = (5Gh/3R)(-r_3^2/r_1+r_4)(w_{,\phi}-\overset{*}{v}+\bar{v}\ R/c) \quad (2.25)$$

Expression $(-r_3^2/r_1+r_4)$ is further simplified with the help of eq. (v) on pg. 37.

$$-r_3^2/r_1 + r_4 = (-r_3^2+r_1r_4)/r_1$$

$$= r_3\left\{-r_3+(15r_G+24r_h)r_1/2\right\}/r_1$$

$$= r_Gr_h(8r_G+12r_h)/(8r_G+10r_h)$$

$$= r_Gr_h(6/5) \quad (2.26)$$

for a low-strength and thick core.

Substitution of (2.26) in (2.25) results in the expression for $Q_{\phi c}$ in eq. (2.27).

$$Q_{\phi c} = (2G_c\ c/R)(w_{,\phi}-\overset{*}{v}+\bar{v}\ R/c) \quad (2.27)$$

2.3.3 Equations for Homogeneous Spherical Shells (Axisymmetric Case)

The constitutive equations for homogeneous isotropic spherical shellsare derived from the equations of sandwich shells. For this case, the face sheets are superimposed on each other and the core layer in Fig. 1 is not considered. The stresses and displacements representing the core layer are dropped from the equations of motion. Due to the superposition of face sheets 1 and 2,

$$s=0,\ \overline{V}=\overline{N}_{\phi}=\overline{N}_{\phi}= \ldots =\overline{Q}_{\phi}= \ldots =\overline{M}_{\theta\phi}=0 \tag{2.28}$$

and also the terms $\overset{*}{N}_{\phi}$, $\overset{*}{N}_{\theta\phi}$, $\overset{*}{v}$, ... etc. will be expressed without *. Therefore, the differential equations of motion obtained from the simplification of eqs. (2.21) are written as:

$$\begin{aligned}
&N_{\phi,\phi} + \cot\phi\ (N_{\phi}-N_{\theta}) + Q_{\phi} = \rho\ h\ R(s_1 v + s_4 R\beta_{\phi})_{,tt} \\
&Q_{\phi,\phi} + \cot\phi\ Q_{\phi} - (N_{\phi}+N_{\theta}) = \rho\ h\ R\ s_1 w_{,tt} \\
&M_{\phi,\phi} + \cot\phi\ (M_{\phi}-M_{\theta}) - RQ_{\phi} = \rho\ h\ R^2(s_4 v + s_6 R\beta_{\phi})_{,tt}
\end{aligned} \tag{2.29}$$

where s_1, s_4 and s_6 can be written by inserting s/R=0 in eq.(iii) on pg. 22. Also, the stress resultants and displacement relations are:

$$N_{\phi} = K\{v_{,\phi}+w+\nu(\cot\phi\ v+w)\}/R$$

$$N_{\theta} = K\{\cot\phi\ v+w+\nu(v_{,\phi}+w)\}/R$$

$$M_\phi = D(\beta_{\phi,\phi} + \nu\cot\phi\ \beta_\phi)/R \qquad (2.30)$$

$$M_\theta = D(\cot\phi\ \beta_\phi + \nu\ \beta_{\phi,\phi})/R$$

$$Q_\phi = (5Gh/6R)(w_{,\phi} - v + R\beta_\phi)$$

since $r_1=1$ and $r_3=0$.

2.3.4 Equations for Axisymmetric Vibration of Circular Sandwich Plates

When the radius of curvature of the sandwich spherical shell becomes infinitely large, a shell becomes a flat plate. In order to obtain the differential equations of motion, stress resultants and displacement relationships for the circular sandwich plate, the terms multiplied by 1/R are deleted from the equations. Also, the expression R sinϕ is simplified in the following form:

$$R\sin\phi \approx R\phi = r \qquad (2.31)$$

for small angle ϕ. Here r is the distance measured in the radial direction of the plate. The diameter of the plate is assumed to be 2a.

For symmetric bending of the sandwich plate about the middle of the core, i.e. neutral plane is passing through the centre of the thickness of the composite plate, the radial displacement components in face sheets 1 and 2 are equal but opposite in direction. The distribution of the radial displacement along the thickness of the sandwich plate is shown in Fig. 3. Symbolically, it can be written as:

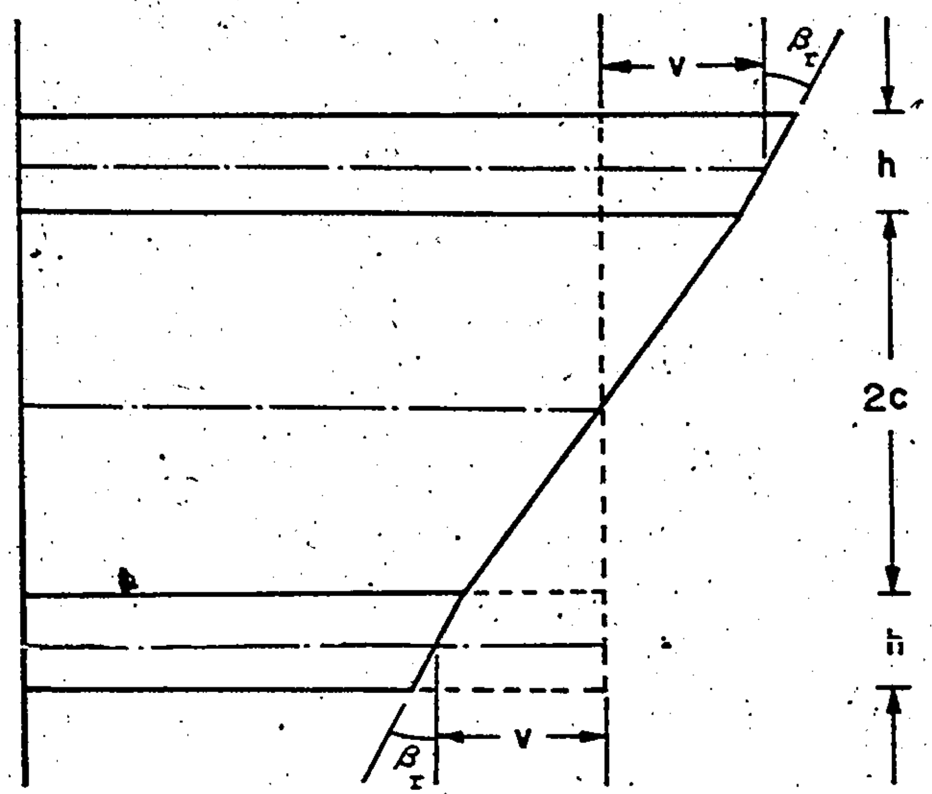

Fig.3 Distribution of radial displacement component along the thickness of the sandwich plate.

$$v_1 = -v_2 = v$$
$$\overset{*}{v}=0 \text{ and } \overline{v}=v \qquad (2.32)$$

With the help of eqs. (2.31) and (2.32), the equations of motion and stress resultants and displacement relationships for the axisymmetric case are obtained from eqs. (2.21) and (2.22) respectively. These equations are written in the sets (2.33) and (2.34).

$$N_{r,r} + (N_r - N_\theta)/r - Q_{rc}/2c = \rho h \, (\alpha_2 v + \alpha_3 a\beta_r)_{,tt}$$
$$(Q_r + Q_{rc}/2)_{,r} + (Q_r + Q_{rc}/2)/r = \rho h \alpha_1 w_{,tt} \qquad (2.33)$$
$$M_{r,r} + (M_r - M_\theta)/r - Q_r + Q_{rc} h/4c = \rho ha\{\alpha_3 v + (h^2/a^2)\alpha_1 a\beta_r/12\}_{,tt}$$

and

$$N_r = K(v_{,r} + \nu v/r), \; N_\theta = K(v/r + \nu v_{,r})$$
$$M_r = D(\beta_{r,r} + \nu \beta_r/r), \; M_\theta = D(\beta_r/r + \nu \beta_{r,r}) \qquad (2.34)$$
$$Q_r = K(1-\nu)\{r_1(w_{,r} + \beta_r) + r_3(w_{,r} + v/c - \beta_r h/2c)\} 5/12$$
$$Q_{rc} = K(1-\nu)\{r_3(w_{,r} + \beta_r) + r_4(w_{,r} + v/c - \beta_r h/2c)\} 5/6$$

The symbols r_ρ, r_G, r_h, r_1, r_3, r_4, K and D are defined in eqs. (iii) and (v)*. Also, α_1, α_2, and α_3 used in eq. (2.33) are given below.

$$\alpha_1 = 1 + r_\rho r_h$$

$$\alpha_2 = 1 + r_\rho r_h/3 \tag{2.35}$$

$$\alpha_3 = -r_\rho (c/a)/6$$

---

* Eqs. (iii) and (v) are given on pages 22 and 37 respectively.

CHAPTER 3

SOLUTION OF THE DIFFERENTIAL EQUATIONS

3.1 Non-Symmetric Case

In this chapter, the differential equations of motion (2.18) are reduced in terms of the displacement components. For the simple harmonic motion of the sandwich shell, the method of separation of variables is used to get the solution of equations. A set of new variables $\overset{*}{H}$, $\overline{H}$, $\overset{*}{X}$, $\overset{*}{\Gamma}$ and $\overline{\Gamma}$ have been introduced in this section which simplify the displacement differential equations of motion. The solution of these new functions are expressed in terms of associated Legendre functions.

For normal modes of vibration, the displacement components $\overset{*}{u}$, $\overline{u}$, $\overset{*}{v}$, $\overline{v}$, w, β_ϕ and β_θ are expressed as:

$$
\begin{aligned}
\overset{*}{u} &= \exp(j\omega t)\ \overset{*}{U} \\
\overline{u} &= \exp(j\omega t)\ \overline{U} \\
\overset{*}{v} &= \exp(j\omega t)\ \overset{*}{V} \\
\overline{v} &= \exp(j\omega t)\ \overline{V} \\
w &= \exp(j\omega t)\ W \\
R\beta_\phi &= \exp(j\omega t)\ X \\
R\beta_\theta &= \exp(j\omega t)\ Y
\end{aligned}
\qquad (3.1)
$$

where ω is the circular frequency of the sandwich shell and $j=\sqrt{-1}$.

At this stage, a set of new functions $\overset{*}{H}$, $\overline{H}$, $\overset{*}{X}$, $\overset{*}{\Gamma}$, $\overline{\Gamma}$ are introduced which are defined by eqs. (3.2).

$$\begin{aligned}
\overset{*}{H} &= \overset{*}{V}_{,\phi} + \cot\phi\ \overset{*}{V} + \operatorname{cosec}\phi\ \overset{*}{U}_{,\theta} \\
\overline{H} &= \overline{V}_{,\phi} + \cot\phi\ \overline{V} + \operatorname{cosec}\phi\ \overline{U}_{,\theta} \\
\overset{*}{X} &= \overset{*}{U}_{,\phi} + \cot\phi\ \overset{*}{U} - \operatorname{cosec}\phi\ \overset{*}{V}_{,\theta} \\
\overline{X} &= \overline{U}_{,\phi} + \cot\phi\ \overline{U} - \operatorname{cosec}\phi\ \overline{V}_{,\theta} \\
\overset{*}{\Gamma} &= X_{,\phi} + \cot\phi\ X + \operatorname{cosec}\phi\ Y_{,\theta} \\
\overline{\Gamma} &= Y_{,\phi} + \cot\phi\ Y - \operatorname{cosec}\phi\ X_{,\theta}
\end{aligned} \tag{3.2}$$

As will be seen later, the use of these special functions considerably simplifies the differential equations of motion. Replacing eqs. (2.20), (2.12), (3.1) and (3.2) together with the non-dimensional frequency parameter $\Omega=(\rho\omega^2R^2/E)^{\frac{1}{2}}$ in eqs. (2.18) results in the following system of equations:

$$A_1\overset{*}{H}_{,\phi} - \operatorname{cosec}\phi\ \overset{*}{X}_{,\theta} + A_2\overset{*}{V} + A_3\overline{V} + A_4X + A_5W_{,\phi} = 0$$

$$A_1 \operatorname{cosec}\phi\ \overset{*}{H}_{,\theta} + \overset{*}{X}_{,\phi} + A_2\overset{*}{U} + A_3\overline{U} + A_4Y + A_5 \operatorname{cosec}\phi\ W_{,\theta} = 0$$

$$A_1\overline{H}_{,\phi} - \operatorname{cosec}\phi\ \overline{X}_{,\theta} + A_3\overset{*}{V} + A_6\overline{V} + A_7X + A_8W_{,\phi} = 0$$

$$A_1 \operatorname{cosec}\phi\, \bar{H}_{,\theta} + \bar{X}_{,\phi} + A_3 \overset{*}{\ddot{U}} + A_6 \bar{U} + A_7 Y + A_8 \operatorname{cosec}\phi\, W_{,\theta} = 0$$

$$B_1 \overset{*}{\ddot{H}} + B_2 \bar{H} + B_3 \overset{*}{\ddot{\Gamma}} + B_4 \nabla^2 W + B_5 W = 0 \qquad (3.3)$$

$$A'_1 \overset{*}{\ddot{\Gamma}}_{,\phi} - \operatorname{cosec}\phi\, \bar{T}_{,\theta} + B_6 \overset{*}{\ddot{V}} + B_7 \bar{V} + B_8 X + B_9 W_{,\phi} = 0$$

$$A'_1 \operatorname{cosec}\phi\, \overset{*}{\ddot{\Gamma}}_{,\theta} + \bar{T}_{,\phi} + B_6 \overset{*}{\ddot{U}} + B_7 \bar{U} + B_8 Y + B_9 \operatorname{cosec}\phi\, W_{,\theta} = 0$$

In the above equations, ∇^2 is Laplace operator in spherical coordinates given by:

$$\nabla^2 = \partial^2/\partial\phi^2 + \cot\phi\, \partial/\partial\phi + \operatorname{cosec}^2\phi\, \partial^2/\partial\theta^2 \qquad \text{(vi)}$$

and the quantities A_1, A_2, ..., B_9 are:

$$A_1 = 2/(1-\nu),\ A_2 = 2+2\Omega^2 s_1(1+\nu) - 5\, B_4/6$$

$$A_3 = 2\Omega^2\, s_3(1+\nu) + 5\, B_2/6,\ A_4 = 2\Omega^2\, s_4(1+\nu) + 5\, B_3/6$$

$$A_5 = A_1(1+\nu) + 5\, B_4/6,\ A_6 = 2 + 2\Omega^2\, s_2(1+\nu) - 5/6\, (r_2 + r_4\, R^2/c^2)$$

$$A_7 = 2\Omega^2\, s_5(1+\nu) + 5/6\, \{r_2 s/R - (r_3 - r_4/2r_h)R/c\} \qquad (3.4)$$

$$A_8 = -\, (r_3 + r_4)\, 5R/6c,\ B_1 = -\, (17+7\nu)/5\, (1-\nu)$$

$$B_2 = (r_3+r_4)R/c, \; B_3 = r_1 + r_3 - (r_3+r_4)/2r_h$$

$$B_4 = r_1 + 2r_3 + r_4, \; B_5 = \{\Omega^2 s_1 - 2/(1-\nu)\} \frac{12}{5} (1+\nu)$$

$$B_6 = 10B_3 + 24(1+\nu)\Omega^2 s_4$$

$$B_7 = 24(1+\nu)\Omega^2 s_5 - 10\{(r_3 - r_4/2r_h)R/c - r_2 s/R\}$$

$$B_8 = 24(1+\nu)\Omega^2 s_6 - 10\{r_1 - r_3/r_h + r_4/4r_h^2 + r_2 s^2/R^2\} + 2h^2/R^2$$

$$B_9 = -10B_3 R^2/h^2, \; A_1' = A_1 h^2/R^2$$

Equations (3.3) can be further simplified to a more suitable form from where the solution can be directly obtained.

$$\begin{aligned} &(A_1\nabla^2 + A_2)\overset{*}{\bar{H}} + A_3\bar{H} + A_4\overset{*}{\bar{\Gamma}} + A_5\nabla^2 W = 0 \\ &A_3\overset{*}{\bar{H}} + (A_1\nabla^2 + A_6)\bar{H} + A_7\overset{*}{\bar{\Gamma}} + A_8\nabla^2 W = 0 \\ &B_1\overset{*}{\bar{H}} + B_2\bar{H} + B_3\overset{*}{\bar{\Gamma}} + (B_4\nabla^2 + B_5)W = 0 \\ &B_6\overset{*}{\bar{H}} + B_7\bar{H} + (A_1'\nabla^2 + B_8)\overset{*}{\bar{\Gamma}} + B_9\nabla^2 W = 0 \end{aligned} \qquad (3.5)$$

and

$$(\nabla^2+A_2)\overset{*}{\chi} + A_3\bar{\chi} + A_4\bar{\Gamma} = 0$$

$$A_3\overset{*}{\chi} + (\nabla^2+A_6)\bar{\chi} + A_7\bar{\Gamma} = 0 \qquad (3.6)$$

$$B_6\overset{*}{\chi} + B_7\bar{\chi} + (\nabla^2+B_8)\bar{\Gamma} = 0$$

The general solution of eqs. (3.5) and (3.6) is expressed in terms of associated Legendre functions of the first and second kinds.

$$W = \sum_{m=0}^{\infty} \sum_{\alpha=1}^{4} W_{\alpha}^{m} \cos m\theta$$

$$\overset{*}{H} = \sum_{m=0}^{\infty} \sum_{\alpha=1}^{4} \overset{*}{H}{}_{\alpha}^{m} \cos m\theta$$

$$\bar{H} = \sum_{m=0}^{\infty} \sum_{\alpha=1}^{4} \bar{H}_{\alpha}^{m} \cos m\theta \qquad (3.7)$$

$$\overset{*}{\Gamma} = \sum_{m=0}^{\infty} \sum_{\alpha=1}^{4} \overset{*}{\Gamma}{}_{\alpha}^{m} \cos m\theta$$

and

$$\overset{*}{\chi} = \sum_{m=0}^{\infty} \sum_{\beta=1}^{3} \overset{*}{\chi}{}_{\beta}^{m} \cos m\theta$$

$$\bar{X} = \sum_{m=0}^{\infty} \sum_{\beta=1}^{3} \bar{X}_{\beta}^{m} \cos m\theta \tag{3.8}$$

$$\bar{T} = \sum_{m=0}^{\infty} \sum_{\beta=1}^{3} \bar{T}_{\beta}^{m} \cos m\theta$$

where

$$\overset{*}{W}_{\alpha}^{m} = A_{\alpha}^{m} P_{\nu_\alpha}^{m} (\cos\phi) + B_{\alpha}^{m} Q_{\nu_\alpha}^{m} (\cos\phi)$$

$$\overset{*}{H}_{\alpha}^{m} = C_{\alpha}^{m} P_{\nu_\alpha}^{m} (\cos\phi) + D_{\alpha}^{m} Q_{\nu_\alpha}^{m} (\cos\phi) \tag{vii}$$

$$\overset{*}{X}_{\beta}^{m} = M_{\beta}^{m} P_{\eta_\beta}^{m} (\cos\phi) + N_{\beta}^{m} Q_{\eta_\beta}^{m} (\cos\phi)$$

etc.

The symbols A_{α}^{m}, B_{α}^{m}, C_{α}^{m}, D_{α}^{m}, ..., N_{β}^{m} are the arbitrary constants. The functions $P_{\nu_\alpha}^{m}(\cos\phi)$ and $Q_{\nu_\alpha}^{m}(\cos\phi)$ are the associated Legendre functions of first and second kind respectively.

The parameters λ_α and $\bar{\lambda}_\beta$ are introduced such that

$$\nu_\alpha(\nu_\alpha+1) = \lambda_\alpha \quad \text{and} \quad \eta_\beta(\eta_\beta+1) = \bar{\lambda}_\beta \tag{viii}$$

Therefore,

$$\nu_\alpha = -1/2 + (\lambda_\alpha+1/4)^{\frac{1}{2}} \quad \text{and} \quad \eta_\beta = -\frac{1}{2} + (\bar{\lambda}_\beta + \frac{1}{4})^{1/2} \tag{ix}$$

Substituting eqs. (3.7) and (3.8) in eqs. (3.5) and (3.6) respectively and also observing that

$$
\begin{aligned}
\nabla^2\overset{*}{H} &= -\lambda_\alpha\overset{*}{H}\\
\nabla^2\overline{H} &= -\lambda_\alpha\overline{H}\\
\nabla^2\overset{*}{\Gamma} &= -\lambda_\alpha\overset{*}{\Gamma}\\
\nabla^2 W &= -\lambda_\alpha W\\
\nabla^2\overset{*}{X} &= -\lambda_\beta\overset{*}{X}\\
\nabla^2\overline{X} &= -\lambda_\beta\overline{X}\\
\nabla^2\overline{\Gamma} &= -\lambda_\beta\overline{\Gamma}
\end{aligned}
\tag{x}
$$

two sets of simultaneous equations in terms of $\overset{*}{H}$, $\overline{H}$, W, $\overset{*}{\Gamma}$, $\overset{*}{X}$, $\overline{X}$ and $\overline{\Gamma}$ are obtained.

$$
\begin{aligned}
(-A_1\lambda_\alpha+A_2)\overset{*}{H} + A_3\overline{H} + A_4\overset{*}{\Gamma} - A_5\lambda_\alpha W &= 0\\
A_3\overset{*}{H} + (-A_1\lambda_\alpha+A_6)\overline{H} + A_7\overset{*}{\Gamma} - A_8\lambda_\alpha W &= 0\\
B_1\overset{*}{H} + B_2\overline{H} + B_3\overset{*}{\Gamma} + (-B_4\lambda_\alpha+B_5)W &= 0\\
B_6\overset{*}{H} + B_7\overline{H} + (-A'_1\lambda_\alpha+B_8)\overset{*}{\Gamma} - B_9\lambda_\alpha W &= 0
\end{aligned}
\tag{3.9a}
$$

and

$$(-\lambda_\beta+A_2)\overset{*}{X} + A_3\bar{X} + A_4\bar{\Gamma} = 0$$

$$A_3\overset{*}{X} + (-\lambda_\beta+A_6)\bar{X} + A_7\bar{\Gamma} = 0 \qquad (3.9b)$$

$$B_6\overset{*}{X} + B_7\tilde{X} + (-\lambda_\beta+B_8)\bar{\Gamma} = 0$$

For the non-trivial solution of $\overset{*}{H}$, $\bar{H}$, $\overset{*}{\Gamma}$, W and $\overset{*}{X}$, $\bar{X}$ and $\bar{\Gamma}$, the determinants of the coefficient matrix must vanish. From eqs. (3.9a) and (3.9b), two determinants a 4x4 and another 3x3 respectively, can be obtained. After the expansion of these two determinants, a fourth order polynomial equation in terms of λ_α and another third order polynomial equation in λ_β are obtained.

$$\lambda_\alpha^4 + a_1\lambda_\alpha^2+a_2\lambda_\alpha^2+a_3\lambda_\alpha+a_4 = 0 \qquad (3.10a)$$

and

$$\lambda_\beta^3+b_1\lambda_\beta^2+b_2\lambda_\beta+b_3 = 0 \qquad (3.10b)$$

where

$$a_1 = (A_1AA_2-A_2AA_1-A_5AD_1)/DA$$

$$a_2 = (A_1AA_3+A_3AB_1-A_2AA_2-A_4AC_1-A_5AD_2)/DA$$

$$a_3 = (A_1 AA_4 - A_2 AA_3 + A_3 AB_2 - A_4 AC_2 - A_5 AD_3)/DA$$

$$a_4 = (A_3 AB_3 - A_2 AA_4 - A_4 AC_3)/DA$$

$$b_1 = -(A_2 + A_6 + B_8)$$

$$b_2 = A_7 B_7 - A_6 B_8 + A_2 (B_8 + A_6) + A_3^2 + A_4 B_4$$

$$b_3 = A_2 (A_6 B_6 - A_7 B_7) + A_3 (A_7 B_6 - A_3 B_8) + A_4 (A_3 B_7 - A_6 B_6)$$

$$DA = A_1 AA_1 , \quad AA_1 = A_1^2 \, h^2/R^2$$

$$AA_2 = A_1 \left[B_3 B_9 - B_4 B_8 - (A_1 B_5 + A_7 B_4 + A_9 B_2) h^2/R^2 \right]$$

$$AA_3 = A_1 (B_5 B_8 + A_7 B_5 h^2/R^2) + A_7 (B_4 B_8 - B_3 B_9) + A_8 (B_2 B_9 - B_4 B_7)$$

$$+ A_9 (B_3 B_7 - B_2 B_8) \qquad (3.11)$$

$$AA_4 = B_5 (A_8 B_7 - A_7 B_8)$$

$$AB_1 = A_1 (A_9 B_1 - A_6 B_4) h^2/R^2$$

$$AB_2 = A_6 (B_4 B_8 - B_3 B_9 + A_1 B_5 h^2/R^2) + A_8 (B_1 B_9 - B_4 B_6) + A_9 (B_3 B_6 - B_1 B_8)$$

$$AB_3 = B_5(A_8B_6-A_6B_8)$$

$$AC_1 = A_1(B_4B_6-B_1B_9)$$

$$AC_2 = A_6(B_4B_7-B_2B_9) + A_7(B_1B_9-B_4B_6) + A_9(B_2B_6-B_1B_7) - A_1B_5B_6$$

$$AC_3 = B_5(A_7B_6-A_6B_7)$$

$$AD_1 = - AA_1B_1$$

$$AD_2 = A_1[B_1B_8-B_3B_6+(A_7B_1-A_6B_2)h^2/R^2]$$

$$AD_3 = A_6(B_2B_8-B_3B_7) + A_7(B_3B_6-B_1B_6) + A_8(B_1B_7-B_2B_6)$$

Equations (3.10) are two characteristic equations whose roots will generate the order of the Legendre functions ν_α appearing in the solutions (3.7) and (3.8) of the differential equations (3.5) and (3.6). These roots may be real or complex quantities. Thus, the order of the Legendre functions ν_α may also be real or complex in character. The values of associated Legendre functions $P^m_{\nu_\alpha}(\cos\phi)$, for non-zero m and real fractional and complex values of ν_α are not available in the literature.

3.2 Axisymmetric Case

In this section, solutions of differential equations of motion will be discussed for the four special cases given in Section 2.3.

3.2.1 Spherical Sandwich Shell with Face Sheets as Flexural Members

For the axisymmetric case, displacement components $\overset{*}{\bar{U}}$, $\overline{V}$ and $\overline{Y}$ given by Eq. (3.1) and the functions $\overset{*}{\bar{X}}$, $\overline{X}$ and $\overline{\Gamma}$ defined by eqs. (3.2) are equal to zero. The non-zero displacement paraemters $\overset{*}{v}$, $\overline{v}$ and β_ϕ are expressed in terms of $\overset{*}{\bar{V}}$, $\overline{V}$ and X as given in eq. (3.1) for normal mode of vibration. The function $\overset{*}{\bar{H}}$, $\overline{H}$ and $\overset{*}{\bar{\Gamma}}$ takes the following form.

$$\begin{aligned} \overset{*}{\bar{H}} &= \overset{*}{\bar{V}}_{,\phi} + \cot\phi\ \overset{*}{\bar{V}} \\ \overline{H} &= \overline{V}_{,\phi} + \cot\phi\ \overline{V} \\ \overset{*}{\bar{\Gamma}} &= X_{,\phi} + \cot\phi\ \overline{V} \end{aligned} \tag{3.12}$$

The system of differential eqs. (2.21) reduces to four equations given by:

$$\begin{aligned} & A_1\overset{*}{\bar{H}}_{,\phi} + A_2\overset{*}{\bar{V}} + A_3\overline{V} + A_4X + A_5W_{,\phi} = 0 \\ & A_1\overline{H}_{,\phi} + A_3\overset{*}{\bar{V}} + A_6\overline{V} + A_7X + A_8W_{,\phi} = 0 \\ & B_1\overset{*}{\bar{H}} + B_2\overline{H} + (B_4\nabla^2+B_5)W = 0 \\ & A_1'\overset{*}{\bar{\Gamma}}_{,\phi} + B_6\overset{*}{\bar{V}} + B_7\overline{V} + B_8X + B_9W_{,\phi} = 0 \end{aligned} \tag{3.13}$$

Eq. (3.13) can also be simplified to the form given by (3.5). Equations (3.6) are identically satisfied for this case.

The solutions for $\overset{*}{\ddot{H}}$, $\overline{H}$, $\overset{*}{\ddot{\Gamma}}$ and W are:

$$\begin{aligned} W &= \sum_{\alpha=1}^{4} W_\alpha \\ \overset{*}{\ddot{H}} &= \sum_{\alpha=1}^{4} n_{1\alpha} W_\alpha \\ \overline{H} &= \sum_{\alpha=1}^{4} n_{2\alpha} W_\alpha \\ \overset{*}{\ddot{\Gamma}} &= \sum_{\alpha=1}^{4} n_{3\alpha} W_\alpha \end{aligned} \qquad (3.14)$$

where

$$W_\alpha = A_{\nu_\alpha} P_{\nu_\alpha}(\cos\phi) + B_{\nu_\alpha} Q_{\nu_\alpha}(\cos\phi)$$

The expressions for $\overset{*}{\ddot{V}}$, $\overline{V}$ and X may be obtained from the solution of simultaneous eqs. (3.13) a, b and d and are written below in terms of W_α with the help of eqs. (3.14).

$$\begin{aligned} \overset{*}{\ddot{V}} &= \sum_{\alpha=1}^{4} \psi_{1\alpha} W_{\alpha,\phi} \\ \overline{V} &= \sum_{\alpha=1}^{4} \psi_{2\alpha} W_{\alpha,\phi} \\ X &= \sum_{\alpha=1}^{4} \psi_{3\alpha} W_{\alpha,\phi} \end{aligned} \qquad (3.15)$$

The symbols $\eta_{i\alpha}$ and $\psi_{i\alpha}$ (i=1,2,3) as introduced in eqs. (3.14) and (3.15) are defined as:

$$\eta_{i\alpha} = CC_{i\alpha}/CC_{\alpha}$$

$$\psi_{i\alpha} = DD_{i\alpha}/DD \tag{3.16}$$

where $CC_{i\alpha}$, CC_{α}, DD and $DD_{i\alpha}$ are given by the following determinants.

$$CC_{\alpha} = \begin{vmatrix} -A_1\lambda_\alpha+A_2 & A_3 & A_4 \\ A_3 & -A_1\lambda_\alpha+A_6 & A_7 \\ B_1 & B_2 & B_3 \end{vmatrix}, \quad CC_{1\alpha} = \begin{vmatrix} A_5\lambda_\alpha & A_3 & A_4 \\ A_8\lambda_\alpha & -A_1\lambda_\alpha+A_6 & A_7 \\ B_4\lambda_\alpha-B_5 & B_2 & B_3 \end{vmatrix}$$

$$CC_{2\alpha} = \begin{vmatrix} -A_1\lambda_\alpha+A_2 & A_5\lambda_\alpha & A_4 \\ A_3 & A_8\lambda_\alpha & A_7 \\ B_1 & B_4\lambda_\alpha-B_5 & B_3 \end{vmatrix}, \quad CC_{3\alpha} = \begin{vmatrix} -A_1\lambda_\alpha+A_2 & A_3 & A_5\lambda_\alpha \\ A_3 & -A_1\lambda_\alpha+A_2 & A_8\lambda_\alpha \\ B_1 & B_2 & B_4\lambda_\alpha-B_5 \end{vmatrix} \tag{3.17}$$

$$DD = \begin{vmatrix} A_2 & A_3 & A_4 \\ B_3 & B_6 & A_7 \\ B_6 & B_7 & B_8 \end{vmatrix}, \quad DD_{1\alpha} = \begin{vmatrix} -A_1\eta_{1\alpha}-A_5 & A_3 & A_4 \\ -A_1\eta_{2\alpha}-A_8 & A_6 & A_7 \\ -A_1\eta_{3\alpha}-B_9 & B_7 & B_8 \end{vmatrix}$$

$$DD_{2\alpha}=\begin{vmatrix} A_2 & -A_1\eta_{1\alpha}-A_5 & A_4 \\ A_3 & -A_1\eta_{2\alpha}-A_8 & A_7 \\ B_6 & -A_1\eta_{3\alpha}-B_9 & B_9 \end{vmatrix}, \quad DD_{3\alpha}=\begin{vmatrix} A_2 & A_3 & -A_1\eta_{1\alpha}-A_5 \\ A_3 & A_6 & -A_1\eta_{2\alpha}-A_8 \\ B_6 & B_7 & -A_1\eta_{3\alpha}-B_9 \end{vmatrix}$$

The order of Legendre function can be obtained from eq. (ix) on pg. 53 and $\lambda_\alpha(\alpha=1,2,3,4)$ are the roots of the characteristic eq. (3.10a).

3.2.2 Spherical Sandwich Shell with Membrane Face Sheets

The differential eqs. (2.23) are simplified in terms of displacement components W, $\overset{*}{V}$, $\bar{V}$ and functions $\overset{*}{H}$, $\bar{H}$ by using eqs. (2.22), (3.1) and (3.12).

$$\begin{aligned} &(M_1\nabla^2+M_2)W + M_3\overset{*}{H} + M_4\bar{H} = 0 \\ &M_5W_{,\phi} + \overset{*}{H}_{,\phi} + M_6\overset{*}{V} + M_7\bar{V} = 0 \qquad (3.18) \\ &M_8W_{,\phi} + M_9\overset{*}{V} + \bar{H}_{,\phi} + M_{10}\bar{V} = 0 \end{aligned}$$

where

$$\begin{aligned} &M_1 = (1-\nu^2)r_g r_h, \quad M_2 = \Omega^2 s_1(1-\nu^2) - 2(1+\nu) \\ &M_3 = -\left[1+\nu+r_g r_h(1-\nu^2)\right], \quad M_4 = r_g(1-\nu^2)R/h \\ &M_5 = -M_3, \quad M_6 = 1-\nu+(\Omega^2 s_1-r_g r_h)(1-\nu^2) \qquad (3.19) \end{aligned}$$

$$M_7 = (\Omega^2 s_3 + r_g R/h)(1-\nu^2),\ M_8 = -M_4$$

$$M_9 = -M_7,\ M_{10} = 1-\nu+(\Omega^2 s_2 - r_g R^2/hc)(1-\nu^2)$$

$$r_g = G_c/E = r_G/2(1+\nu)$$

Equations (3.18) and (3.19) are identical to eq. (3.3) and (3.4) of the previous work as reported in [12].

The solution of eqs. (3.18) is expressed in terms of Legendre functions $P_{\nu_\alpha}(\cos\phi)$ for a sandwich spherical shell closed at one pole and open at the other. The functions W, $\overset{*}{H}$ and $\bar{H}$ are assumed as:

$$W = \sum_{\alpha=1}^{3} C_{\nu_\alpha} P_{\nu_\alpha}(\cos\phi)$$

$$\overset{*}{H} = \sum_{\alpha=1}^{3} r r_{1\alpha} C_{\nu_\alpha} P_{\nu_\alpha}(\cos\phi) \tag{3.20}$$

$$\bar{H} = \sum_{\alpha=1}^{3} r r_{2\alpha} C_{\nu_\alpha} P_{\nu_\alpha}(\cos\phi)$$

The order of the Legendre function ν_α is obtained from the roots of the characteristic equation:

$$M_1\lambda_\alpha^3 + a_1'\lambda_\alpha^2 + a_2'\lambda_\alpha + a_3' = 0 \qquad (3.21)$$

and eq. (ix), where

$$a_1' = -M_2+M_3M_5+M_4M_8-M_1M_6-M_1M_{10}$$

$$a_2' = M_1(M_6M_{10}-M_7M_9) + M_2M_6+M_2M_{10} + M_3(M_7M_8-M_5M_{10}) + M_4(M_5M_9-M_6M_8) \qquad (3.22)$$

$$a_3' = M_2M_7M_9 - M_2M_6M_{10}$$

The displacement $\overset{*}{V}$ and $\overline{V}$ are:

$$\overset{*}{V} = \sum_{\alpha=1}^{3} \beta\beta_\alpha C_{\eta_\alpha} \{P_{\nu_\alpha}(\cos\phi)\}_{,\phi}$$

$$\overline{V} = \sum_{\alpha=1}^{3} \eta\eta_\alpha C_{\eta_\alpha} P_{\nu_\alpha}(\cos\phi)\}_{,\phi} \qquad (3.23)$$

The symbols $rr_{1\alpha}$, $r_{2\alpha}$, $\beta\beta_\alpha$ and $\eta\eta_\alpha$ are given as:

$$rr_{1\alpha} = [-M_5\lambda_\alpha^2+(M_5M_{10}-M_7M_8)\lambda_\alpha]/rr$$

$$rr_{2\alpha} = [-M_8\lambda_\alpha^2+(M_6M_8-M_5M_9)\lambda_\alpha]/rr$$

$$\beta\beta_\alpha = (M_5\lambda_\alpha+M_7M_8-M_5M_{10})/rr \qquad (3.24)$$

$$nn_\alpha = (M_8\lambda_\alpha + M_5M_9 - M_6M_8)/rr$$

$$rr = \lambda_\alpha^2 - (M_6+M_{10})\lambda_\alpha + M_6M_{10} - M_7M_9$$

3.2.3 Homogeneous Spherical Shells

The differential equations of motion (2.29) are simplified in this section with the help of eqs. (2.30), (3.1) and (3.2) to the following form.

$$(K_1\nabla^2+K_2)W + K_3H + K_4\Gamma = 0$$

$$K_5W_{,\phi} + H_{,\phi} + K_6V + K_7X = 0 \qquad (3.25)$$

$$K_8W_{,\phi} + K_9V + \Gamma_{,\phi} + K_{10}X = 0$$

where

$$K_1 = K_4 = 1$$

$$K_2 = \left[\Omega^2 s_1 - 2/(1-\nu)\right](1+\nu)12/5$$

$$K_3 = -(17+7\nu)/\left[5(1-\nu)\right]$$

$$K_5 = 1 + \nu + (5/12)(1-\nu)$$

$$K_6 = (7/12)(1-\nu) + \Omega^2 s_1(1-\nu^2) \qquad (3.26)$$

$$K_7 = (5/12)(1-\nu) + \Omega^2 s_4 (1-\nu^2)$$

$$K_8 = -5(1-\nu)R^2/h^2$$

$$K_9 = -K_8 + 12\,\Omega^2 s_4 (1-\nu^2) R^2/h^2$$

$$K_{10} = K_8 + 12\,\Omega^2 s_6 (1-\nu^2) R^2/h^2$$

The differential eqs. (3.25) and (3.18) are identical in nature. Thus, the complete solutions for eqs. (3.25) can be represented by eqs. (3.20) to (3.24).

3.2.4 Circular Sandwich Plate

The differential equations of motion (2.33) are expressed in terms of displacement components by inserting eq. (2.34) and are written in the following simplified form:

$$\nabla^2 W + c_1 W + c_2 \Phi + c_3 \Gamma = 0$$

$$c_4 W_{,x} + c_5 V + \Gamma_{,x} + c_6 U = 0 \qquad (3.27)$$

$$c_7 W_{,x} + \Phi_{,x} + c_8 V + c_9 U = 0$$

where

$$\Phi = V_{,x} + V/x$$

$$\Gamma = U_{,x} + U/x$$

$$w = \exp(j\omega_p t)W \qquad (3.28)$$

$$v = \exp(j\omega_p t)V$$

$$a\beta_r = \exp(j\omega_p t)U$$

Here, ω_p is the circular frequency of the sandwich plate. The non-dimensional length parameter x and the Laplacian operator in terms of variable x are defined as:

$$x = r/a$$

$$\nabla^2 = \partial^2/\partial x^2 + (1/x)\partial/\partial x \qquad (3.29)$$

The coefficients c_1, c_2, ... c_9 used in eqs. (3.27) are given as:

$$c_1 = (12/5)(1+\nu)(\omega_p/\omega_o)^2(h/a)^2\alpha_1/r_5$$

$$c_2 = (r_3+r_4)(a/c)/r_5$$

$$c_3 = [r_1+r_3-(r_3+r_4)/2r_h]/r_5 \qquad (3.30)$$

$$c_4 = -5(1-\nu)c_3r_5(a/h)^2$$

$$c_5 = -5(1-\nu)(r_3-r_4/2r_h)(a^3/h^2c) + 12\alpha_3(1-\nu^2)(\omega_p/\omega_o)^2$$

$$c_6 = -5(1-\nu)(r_1 - r_3/r_h + r_4/4r_h^2)(a/h)^2 + \alpha_1(1-\nu^2)(\omega_p/\omega_o)^2(h/a)^2$$

$$c_7 = -(5/12)(1-\nu)(r_3+r_4)a/c$$

$$c_8 = -\ (5/12)(1-\nu)(a/c)^2 r_4 + \alpha_2(1-\nu^2)(\omega_p/\omega_o)^2(h/a)^2$$

$$c_9 = -\ (5/12)(r_3 - r_4/2r_h)a/c + \alpha_3(1-\nu^2)(\omega_p/\omega_o)^2(h/a)^2$$

$\omega_o^2 = Eh^2/\rho a^4$ and $r_5 = r_1 + 2r_3 + r_4$

Since eqs. (3.27) contain too many variables, an attempt is made to further simplify these equations with the help of the special functions Φ and Γ so that a solution of these differential equations can be readily obtained.

$$\begin{aligned} &(\nabla^2 + c_1)W + c_2\Phi + c_3\Gamma = 0 \\ &c_4\,\nabla^2 W + c_5\Phi + (\nabla^2 + c_6)\Gamma = 0 \qquad (3.31) \\ &c_7\,\nabla^2 W + (\nabla^2 + c_8)\Phi + c_9\Gamma = 0 \end{aligned}$$

The solutions of eqs. (3.31) are now obtained in terms of Bessel functions for the circular sandwich plate with no hole at the centre. Thus,

$$W = \sum_{\alpha=1}^{3} L_\alpha \, J_o(g_\alpha x)$$

$$\phi = \sum_{\alpha=1}^{3} L_\alpha \, \xi_\alpha \, J_o(g_\alpha x) \qquad (3.32)$$

$$\Gamma = \sum_{\alpha=1}^{3} L_\alpha \, \eta_\alpha \, J_o(g_\alpha x)$$

where g_α^2 (α=1,2,3) are the roots of the characteristic equation

$$g_\alpha^6 + \ell_1 g_\alpha^4 + \ell_2 g_\alpha^2 + \ell_3 = 0 \qquad (3.33)$$

In eq. (3.33), ℓ_1, ℓ_2 and ℓ_3 are functions of the coefficients C_i defined earlier in eq. (3.30).

$$\ell_1 = c_3c_4 + c_2c_7 - c_1 - c_6 - c_8$$

$$\ell_2 = c_1(c_6+c_8) + c_6c_8 - c_5c_9 + c_2(c_4c_9-c_6c_7) + c_3(c_5c_7-c_4c_8)$$

$$\ell_3 = c_1(c_5c_9-c_6c_8)$$

Also, the parameters ξ_α and η_α are

$$\xi_\alpha = [(c_3c_7-c_9)g_\alpha^2 + c_1c_9]/(-c_3g_\alpha^2+c_3c_8-c_2c_9) \qquad (3.34)$$

$$\eta_\alpha = [-g_\alpha^4+(c_1+c_8-c_2c_7)g_\alpha^2-c_1c_8]/(-c_3g_\alpha^2+c_3c_8-c_2c_9)$$

The expressions for U and V are now obtained from eqs. (3.27) b and c and (3.32).

$$U = \sum_{\alpha=1}^{3} p_\alpha W_{,x}$$
$$V = \sum_{\alpha=1}^{3} z_\alpha W_{,x} \qquad (3.35)$$

Here, p_α and z_α are

$$p_\alpha = (c_5\xi_\alpha - c_8\eta_\alpha + c_5c_7 - c_4c_8)/(c_6c_8 - c_5c_9)$$
$$z_\alpha = (c_9\eta_\alpha - c_6\xi_\alpha + c_4c_9 - c_6c_7)/(c_6c_8 - c_5c_9) \qquad (3.36)$$

Utilizing eqs. (3.32) and (3.35), the solutions for stress and moment resultants (2.34) can be easily established in terms of Bessel functions.

CHAPTER 4

BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

4.1 Boundary Conditions

The prime object of studying a vibrating system is to know the fundamental frequencies and mode shapes. The natural frequencies of continuous systems such as beams, plates, shells, etc. are highly dependent on the boundary conditions. The boundary conditions play an important role in the investigation of the dynamic behaviour of beams, plates and shells. The frequency equations, as will be seen later in this Chapter, are completely different for various edge conditions of a continuous system.

Referring back to the variational eq. (2.17), it can be seen that by equating all the brackets in the line integral to zero, the stress and moment resultants of the sandwich shell at the edge $\phi=\phi_o$ must be equal to their prescribed values. Thus, the following stress boundary conditions are obtained.

$$\begin{aligned}
&\overset{*}{N}_\phi + \bar{N}_\phi\, s/R = N'_\phi, \quad \bar{N}_\phi + \overset{*}{N}_\phi\, s/R = N''_\phi \\
&\overset{*}{N}_{\theta\phi} + \bar{N}_{\theta\phi}\, s/R = N'_{\theta\phi}, \quad \bar{N}_{\theta\phi} + \overset{*}{N}_{\theta\phi}\, s/R = N''_{\theta\phi} \\
&\overset{*}{Q}_\phi + \bar{Q}_\phi\, s/R + Q_{\phi c}/2 = Q'_\phi \\
&\overset{*}{M}_\phi + \bar{M}_\phi\, s/R = M'_\phi \quad \text{and} \quad \overset{*}{M}_{\theta\phi} + \bar{M}_{\theta\phi}\, s/R = M'_{\theta\phi}
\end{aligned} \tag{4.1}$$

The other condition to satisfy the variational eq. (2.17) is that the displacements $\overset{*}{u}$, $\overset{*}{v}$, w, $\bar{u}$, $\bar{v}$, β_ϕ and β_θ should be prescribed at the edge $\phi=\phi_o$. Thus,

$$\delta\overset{*}{u} = \delta\overset{*}{v} = \delta\bar{u} = \delta\bar{v} = \delta w = \delta\beta_\phi = \delta\beta_\theta = 0 \tag{4.2}$$

Therefore, the displacement boundary conditions can be written as:

$$\overset{*}{u} = \bar{u} = \overset{*}{v} = \bar{v} = w = \beta_\phi = \beta_\theta = 0 \tag{4.3}$$

for a fixed spherical sandwich shell at $\phi=\phi_o$.

4.1.1 Axisymmetrix Case

i. Sandwich Spherical Shell

For this case, only four of the seven boundary conditions in eq. (4.1) and (4.3) are required at $\phi=\phi_o$.

$$\begin{aligned}
&\overset{*}{N}_\phi + \bar{N}_\phi\, s/R = N'_\phi; \text{ or } \overset{*}{v} \text{ prescribed}\\
&\bar{N}_\phi + \overset{*}{N}_\phi\, s/R = N''_\phi; \text{ or } \bar{v} \text{ prescribed}\\
&\overset{*}{M}_\phi + \bar{M}_\phi\, s/R = M'_\phi; \text{ or } \beta_\phi \text{ prescribed}\\
&\overset{*}{Q}_\phi + \bar{Q}_\phi\, s/R + Q_{\phi c}/2 = Q'_\phi; \text{ or } w \text{ prescribed:}
\end{aligned} \tag{4.4}$$

The fixed edge condition results in vanishing displacement components

$$\overset{*}{v} = \bar{v} = w = \beta_\phi = 0 \tag{4.5a}$$

whereas, in the case of free edge, stress and moment resultants are equated to zero.

$$\overset{\star}{Q}_\phi + \overline{Q}_\phi \, s/R + Q_{\phi c}/2 = \overset{\star}{N}_\phi + \overline{N}_\phi \, s/R = \overline{N}_\phi + \overset{\star}{N}_\phi \, s/R = \overset{\star}{M}_\phi + \overline{M}_\phi \, s/R = 0 \tag{4.5b}$$

The other classical boundary conditions can be expressed in terms of any possible combination of four stress resultants, moment resultants and displacements which vanish at the edge of the spherical sandwich shell. The set given by eq. (4.4) is required for a spherical sandwich shell in which the face sheets are capable of taking bending, membrane, and shear loads. For a sandwich shell with membrane face sheets, only three of the four conditions in eq. (4.4) are needed. Thus, the essential edge conditions for this case are:

$$\overset{\star}{N}_\phi + \overline{N}_\phi \, s/R = N'_\phi \text{ or } \overset{\star}{v} \text{ prescribed}$$

$$\overline{N}_\phi + \overset{\star}{N}_\phi \, s/R = N''_\phi \text{ or } \overline{v} \text{ prescribed} \tag{4.6}$$

$$Q_{\phi c}/2 = Q'_\phi \text{ or } w \text{ prescribed.}$$

For a fixed edge sandwich shell,

$$\overset{\star}{v} = \overline{v} = w = 0 \tag{4.7a}$$

and for a free edge,

$$\overset{**}{N}_\phi + \bar{N}_\phi\, s/R = \bar{N}_\phi + \overset{*}{N}_\phi\, s/R = Q_{\phi c} = 0 \qquad (4.7b)$$

at $\phi=\phi_o$.

ii. Homogeneous Spherical Shell

Due to identical nature of the differential equations and their solutions for the homogeneous spherical shells and sandwich spherical shells with face sheets taken as membranes, there are only three conditions needed at the boundary $\phi=\phi_o$.

$$\begin{aligned} N_\phi &= N_\phi' \text{ or } v \text{ prescribed} \\ M_\phi &= M_\phi' \text{ or } \beta_\phi \text{ prescribed} \\ Q_\phi &= Q_\phi' \text{ or } w \text{ prescribed} \end{aligned} \qquad (4.8)$$

For a fixed edge homogeneous spherical shell,

$$v = \beta_\phi = w = 0 \qquad (4.9a)$$

and for a free edge,

$$N_\phi = M_\phi = Q_\phi = 0 \qquad (4.9b)$$

at $\phi=\phi_o$.

iii. Circular Sandwich Plate

The boundary conditions for a clamped sandwich plate with face sheets having flexural stiffness at the edge r=a or in terms of non-dimensional paramter x=1 are:

$$w = v = u = 0 \tag{4.10a}$$

and for a circular sandwich plate free at the boundary r=a (x=1),

$$Q_r + Q_{rc}/2 = N_r = M_r = 0 \tag{4.10b}$$

4.2 Frequency Equations

In this section, the frequency equation will be discussed only for the axisymmetric case of sandwich and homogeneous spherical shells and also for the circular sandwich plates.

4.2.1 Spherical Sandwich Shell

From the examination of the solution of differential eqs. (3.14) and (xi), it can be observed that there are eight unknown arbitrary constants A_{ν_α} and B_{ν_α} (α=1,2,3,4). The values of A_{ν_α} and B_{ν_α} can be gotten by using the boundary conditions (4.4) at two edges, the inner edge at $\phi=\phi_1$ and the outer at $\phi=\phi_o$, and solving eight simultaneous equations. For a sandwich shell closed at one pole ϕ=0 and open at the other $\phi=\phi_o$, Legendre function of the second kind $Q_{\nu_\alpha}(\cos\phi)$ becomes singular at ϕ=0. Thus, the values of B_{ν_α} (α=1,2,3,4) in eq. (3.14) are zero due to finiteness of the displacement W. Therefore, only four constants A_{ν_α} (α=1,2,3,4) remain to be determined. Since the stress and moment resultants can be expressed in terms of displacements by means of eqs. (2.20) and (2.12), the solutions for W, $\ddot{V}$, $\overline{V}$ and X as functions of $P_{\nu_\alpha}(\cos\phi)$ and its derivatives with respect to ϕ are inserted in four homogeneous boundary conditions given by eqs. (4.4). In this manner, the following simultaneous equations are obtained.

$$\left[D_{i\alpha}\right]\left\{A_{\nu_\alpha}\right\} = 0; \quad (i,\alpha=1,2,3,4) \tag{4.11}$$

For the non-trivial solution of A_{ν_α}, the determinant of the coefficient matrix $[D_{i\alpha}]$ must vanish. Symbolically, it can be written as:

$$\left|D_{i\alpha}\right| = 0; \quad (i,\alpha=1,2,3,4) \tag{4.12}$$

The above determinantal equation (4.12) is the frequency equation for the spherical sandwich shell.

The roots of the frequency equation (4.12) are the values of the natural frequencies of the sandwich shells. The expressions for the elements of the determinant given in eq. (4.12) are written in the following sections for various edge conditions.

<u>Fixed Edge Condition</u>

For this case, the boundary conditions at $\phi=\phi_o$ are in terms of displacements and are given by eq. (4.5). The values of $D_{i\alpha}$ $(i,\alpha=1,2,3,4)$ are:

$$\begin{aligned} D_{1\alpha} &= P_{\nu_\alpha}(\cos\phi_o)/\{P_{\nu_\alpha}(\cos\phi_o)\}_{,\phi} \\ D_{i\alpha} &= \psi_{i\alpha} \quad (i=2,3,4) \end{aligned} \tag{4.13}$$

where $\psi_{i\alpha}$ are obtained from eq. (3.16b).

Free Edge Condition

The relevant boundary conditions at the free edge $\phi=\phi_o$ are given in eq. (4.6). The elements of the determinant $|D_{i\alpha}|$ are:

$$D_{1\alpha} = (r_1+r_3)(1-\psi_{1\alpha}+\psi_{3\alpha}) + (r_3+r_4)(1-\psi_{1\alpha}+\psi_{2\alpha}R/c-\psi_{3\alpha}/2r_H)$$

$$D_{2\alpha} = (1+\nu+n_{1\alpha})P_{\nu_\alpha}(\cos\phi_o)/\{P_{\nu_\alpha}(\cos\phi_o)\}_{,\phi} - (1-\nu)\cot\phi_o\,\psi_{1\alpha}$$

$$D_{3\alpha} = n_{2\alpha}\,P_{\nu_\alpha}(\cos\phi_o)/\{P_{\nu_\alpha}(\cos\phi_o)\}_{,\phi} - (1-\nu)\cot\phi_o\,\psi_{2\alpha}$$

$$D_{4\alpha} = n_{3\alpha}\,P_{\nu_\alpha}(\cos\phi_o)/\{P_{\nu_\alpha}(\cos\phi_o)\}_{,\phi} - (1-\nu)\cot\phi_o\,\psi_{3\alpha} \quad (4.15)$$

Simply Supported Edge Condition

For this case,

$$W = \overset{*}{N}_\phi + \overline{N}_\phi\, s/R = \overline{N}_\phi + \overset{*}{N}_\phi\, s/R = \overset{*}{M}_\phi + \overline{M}_\phi\, s/R = 0 \quad (4.16)$$

$D_{1\alpha}$ and $D_{i\alpha}$ (i=2,3,4) can be expressed by eqs. (4.10a) and (4.15)(b-d) respectively.

Pinned Edge Condition

$$W = \overset{*}{V} = \overline{V} = \overset{*}{M}_\phi + \overline{M}_\phi\, s/R = 0 \quad (4.17)$$

at $\phi=\phi_o$. $D_{i\alpha}$ (i=1,2,3) and $D_{4\alpha}$ can be obtained from eqs. (4.13)(a-c) and (4.11)d respectively.

4.2.2 Sandwich Spherical Shell with Membrane Face Sheets and Homogeneous Spherical Shell

For a sandwich shell with membrane face sheets or a homogeneous spherical shell, there are only three unknown arbitrary constants C_{ν_α} (α=1,2,3) as given in eq. (3.20). Therefore, the following 3x3 determinantal equation is deduced in a similar manner as eq.(4.12).

$$|D_{i\alpha}| = 0; \; (i,\alpha=1,2,3) \tag{4.18}$$

The elements $D_{i\alpha}$ of the above frequency equation are written in eq. (4.19) for a clamped sandwich spherical shell closed at one pole and open at the other.

$$\begin{aligned} D_{1\alpha} &= P_{\nu_\alpha}(\cos\phi_o)/\{P_{\nu_\alpha}(\cos\phi_o)\}_{,\phi} \\ D_{2\alpha} &= \beta\beta_\alpha \text{ and } D_{3\alpha} = \eta\eta_\alpha \end{aligned} \tag{4.19}$$

where $\beta\beta_\alpha$ and $\eta\eta_\alpha$ are given by eq. (3.24).

Equation (4.19) will also represent the elements of a frequency equation for a clamped edge homogeneous spherical shell by simply replacing K_i in place of M_i (i=1, ...10) in eqs. (3.22) and (3.24).

4.2.3 Circular Sandwich Plate

The frequency equation for clamped edge circular sandwich plates will be generated in this section. The edge conditions are given in eqs. (4.10). Inserting eqs. (3.32) and (3.35) in (4.10a) and considering the non-trivial solution of arbitrary constants L_α ($\alpha = 1,2,3$), the determinant of the coefficients must vanish which in turn yields the following determinantal equation.

$$f_1(g_\alpha, \omega_p/\omega_o) = \begin{vmatrix} J_o(g_1) & I_o(g_2) & I_3(g_3) \\ -z_1 g_1 J_1(g_1) & z_2 g_2 I_1(g_2) & z_3 g_3 I_1(g_3) \\ -p_1 g_1 J_1(g_1) & p_2 g_2 I_1(g_2) & p_3 g_3 I_1(g_3) \end{vmatrix} = 0 \qquad (4.20)$$

The presence of modified Bessel functions in place of Bessel function are because the two roots g_2^2 and g_3^2 of the characteristic equation (333) are negative quantities for the prescribed values of geometric and elastic parameters of the sandwich plate.

All the different boundary conditions reveal that the frequency equations for sandwich plates and shells are highly transcendental in nature. The closed form solutions for these equations are not possible. The roots of the frequency equations for various boundary conditions can be obtained by using the numerical iteration process. The detailed method for the calculation of natural frequencies will be discussed in the next chapter.

CHAPTER 5

NUMERICAL COMPUTATIONS

The numerical computations have been carried out for the non-dimensional frequency parameter Ω for axisymmetric vibrations of the sandwich spherical shell and (ω_p/ω_o) for the circular sandwich plates. The variations of Ω and (ω_p/ω_o) have been studied with the geometric and elastic properties of the core and the face sheets. The mode shapes in the radial direction of the sandwich shells and in the transverse direction of the circular plate have also been investigated. Due to a highly transcendental nature of the frequency equations (4.12), (4.18) and (4.20), it is not possible to obtain a closed form solution. Therefore, high speed digital computer has been used to generate the values of natural frequencies and the mode shapes of the sandwich plates and shells. The iteration process, which has been utilized to compute the roots of eqs. (4.12), (4.18) and (4.20), will be explained in this chapter. All the calculations have been done with the aid of IBM (360/65) available at the Computing Center of the University of Ottawa.

Computational Procedure

The detailed method to generate the roots of the frequency eq. (4.12) for the axisymmetric vibration of spherical sandwich shells with face sheets as flexural members will be described here.

Some details of the computer program in Fortran IV language have been presented in the Appendix.

First, the non-dimensional parameters r_G and r_ρ, which are defined in eq. (iii)*, and Poisson's ratio ν together with h/R and c/R are selected for a particular combination of materials and geometry of the shell. These values have been presented in Table 5.1. Then, quantities s_1, s_2, s_3, s_4, s_5 and s_6 also given in eq. (iii)* and r_1, r_2, r_3 and r_4 given in eq. (v) are calculated. The symbols A_1, A_2, A_3 ... B_1, B_2, ... B_9 are functions of the above-mentioned parameters and the non-dimensional frequency Ω of the sandwich shell. It can be seen that the values of A_1, A_2 ... B_9 given by eq. (3.4) can not be calculated without a value of Ω. Thus, by specifying a value of Ω, the coefficients a_1, a_2, a_3 and a_4 of the fourth order characteristic equation are obtained. With these values of a_1, a_2, a_3 and a_4 which are functions of A_1, A_2, ... B_9 as input to the subroutine QUATIC, all the four roots λ_1, λ_2, λ_3 and λ_4 are calculated. The subroutine QUATIC uses other two subroutines CUBIC and QUAD, while calculating the roots of the characteristic eq. (3.10a). Subroutines QUAD, CUBIC and QUATIC have been developed to calculate the roots of quadratic, cubic and quartic algebraic equations, respectively. Since eq. (3.10a) is a real fourth order polynomial, its roots fall in one of the following possible categories:

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* Eq. (iii) is given on page 22.

TABLE 5.1. Description of the non-dimensional parameters for the sandwich spherical shells

Description	r_G	r_ρ	Various Parameters
Sandwich Shell	1/1680	1/26.7	$C/R=0.02$, $\nu=0.3$ $h/R=0.0025-0.0100$ (interval 0.0025) $\phi_0=60^\circ-120^\circ$ (interval 30°)
	1/60	1/28	
Sandwich Plate	1/1680	1/26.7	$c/a=0.005-0.015$ (interval 0.005) $h/a=0.005-0.015$ $\nu=0.3$
	1/60	1/28	

a. all the four roots are real;

b. first two real roots and the other two are complex conjugates;

c. all the four roots are complex but in two sets of complex conjugates.

Substitutions of these four roots $\lambda_\alpha (\alpha=1,2,3,4)$ in eq. (ix)* result in the orders of $\nu_\alpha (\alpha=1,2,3,4)$ of the Legendre function $P_{\nu_\alpha}(\cos\phi)$ appearing in eqs. (3.14) and (3.15) which are the solutions of the differential equations for axisymmetric vibration of the spherical sandwich shell, with face sheets as flexural member. If λ_α has a real and positive value, ν_α is also real and positive whereas in case of a negative real value of λ_α, ν_α takes the following complex form:

$$\nu_\alpha = -1/2 + i\, p_\alpha \tag{5.1}$$

where p_α is real. The Legendre functions, whose order ν_α is given by eq. (5.1), are known as conical functions. This type of Legendre function and its derivative with respect to ϕ are real. A pair of complex conjugates for the roots of the characteristic equation also yields a pair of complex conjugates for ν_α. The orders $\nu_\alpha (\alpha=1,2,3,4)$ of the Legendre functions are written as V_1, V_2, V_3 and V_4 and $\lambda_\alpha (\alpha=1,2,3,4)$ are denoted by $Z(\alpha)$ in the main computer program.

The determinants CC_α, $CC_{1\alpha}$, $CC_{2\alpha}$, $CC_{3\alpha}$, DD, $DD_{1\alpha}$, $DD_{2\alpha}$, $DD_{3\alpha}$ given by eqs. (3.17) are calculated next. These determinants are

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* Eq. (ix) is given on page 53.

functions of A_1, A_2, ... B_9 and λ_α and appear in the main program in their expanded form as U(α), U1(α), U2(α), U3(α), DD, TD1(α), TD2(α), TD3(α) respectively.

The numerical values of $P_{\nu_\alpha}(\cos\phi)$ are calculated using subroutines CM1RLF, AVS and CONFN. Subroutine CM1RLF generates the values of $P_{\nu_\alpha}(\cos\phi)$ for complex as well as real ν_α with the aid of Mehler's integral representation given by eq. (5.2). The complex numerical integration is carried out by trapezoidal rule.

$$P^{\mu}_{\nu_\alpha}(\cos\phi) = \frac{(2/\pi)^{\frac{1}{2}}\sin^{\mu}\phi}{\sqrt{\frac{1}{2}-\mu}} \int_0^{\phi} \frac{\cos(\nu+\frac{1}{2})x}{(\cos x-\cos\phi)^{\mu+\frac{1}{2}}}\,dx \tag{5.2}$$

Eq. (5.2) can generate values of $P_{\nu_\alpha}(\cos\phi)$ for either real or complex ν_α as well as the conical function whose order is given by eq. (5.1), for small and large values of ν_α. It should be realized that the process of numerical integration involved in evaluating from eq. (5.2) takes a considerable amount of computer time. In order to save the computer time, asymptotic and series expansions of Legendre functions have also been used. When ν_α is a very large quantity, real or complex, the following asymptotic equations are of considerable interest [2,25].

$$P_{\nu_\alpha}(\cos\phi) = \frac{\sqrt{\nu_\alpha+1}}{\sqrt{\nu_\alpha+3/2}} \cdot \frac{\cos\{(\nu_\alpha+\frac{1}{2})\phi-\pi/4\}}{(\pi/2\,\sin\phi)^{\frac{1}{2}}} \tag{5.3}$$

and

$$P_{\nu_\alpha}(\cos\phi) = \frac{\cos(k\phi-\pi/4)}{(\pi/2\ \beta\ \sin\phi)^{\frac{1}{2}}}$$

where

$$k = \{(\nu_\alpha + \tfrac{1}{2})^2 + \tfrac{1}{4}\}^{\frac{1}{2}},\ \beta = \{\nu_\alpha\ (\nu_\alpha+1)\}^{\frac{1}{2}}$$

The simpler and faster method to compute the numerical values of conical functions is by taking the summation of their series expansion given below.

$$P_{-\frac{1}{2}+ip}(\cos\phi) = 1 + \varphi_1 + \varphi_1\varphi_2 + \varphi_1\varphi_2\varphi_3 + \cdots \qquad (5.4)$$

where

$$\varphi_n = \left[p^2+\{(2n-1)/2\}^2\right]\sin^2(\phi/2)/n^2$$

The summation of the series (5.4) and its derivative with respect to ϕ is carried out using the subroutine CONFN. The function of subroutine SETUP is to compute the ratio of $P_{\nu_\alpha}(\cos\phi)$ and its derivative with respect to ϕ. This subroutine examines the nature of ν_α and calls CMIRLF, AVS and CONFN accordingly.

Thus, with the knowledge of the values of Legendre functions, their derivatives with respect to ϕ and the determinants given by eqs. (3.17), the value of frequency determinant (4.12) is evaluated. Keeping non-dimensional parameters r_G, r_ρ, c/R, ν, and h/R constant, the values of frequency determinant are calculated for a series of values of Ω by calling subroutine DETERM. This procedure is repeated by giving increments in Ω until a change in sign of the values of the frequency determinant takes place. Therefore, using various boundary conditions, the values of the natural frequencies of the spherical sandwich shells are obtained.

During the process of computation, it has been observed that the roots of characteristic equation play an important role. The frequency determinants are strongly dependent on the order ν_α of the Legendre function. The roots $\lambda_\alpha (\alpha=3,4)$ have been found to be a pair as complex conjugates for some values of Ω. In the event, when a pair of complex conjugate roots of the quartic polynomial are obtained, the corresponding ν_α and $P_{\nu_\alpha}(\cos\phi)$ are also complex conjugates.

$$P_{a\pm ib}(\cos\phi) = x \pm iy \tag{5.5}$$

where a, b, x and y are real. The choice of arbitrary constants A_{ν_3} and A_{ν_4} as complex conjugates will make the frequency equation real. The roots of the characteristic polynomial have also been observed as large negative real quantities, thus ν_α takes the form given in (5.1) with large value of p_α. With this large value of p_α as input to the subroutine CONFN, serious computational difficulties as overflows on the computer print output for the values of $P_{\nu_\alpha}(\cos\phi)$ and its derivative were felt. The subroutine CONFN, in this case, failed to generate the ratio of Legendre function and its derivative. These computational difficulties were overcome by using asymptotic expansion

(5.3)b. Substituting $\nu_\alpha=-\frac{1}{2}+i\ p_\alpha$ in eq. (5.3)b and differentiating with respect to ϕ, the following equation is obtained after simplification.

$$P'_{-\frac{1}{2}+i\ p_\alpha}(\cos\phi)/\{P_{-\frac{1}{2}+i\ p_\alpha}(\cos\phi)\}_{,\phi} = 1/(p_\alpha-\tfrac{1}{2}\cot\phi) \qquad (5.6)\text{a}$$

Eq. (5.6)a does not involve the coefficient exp $(p_\alpha\phi)$, which creates problems for very large values of p_α.

The presented form of eq. (5.2) also creates computational difficulties as the denominator becomes zero at $x=\phi$. Therefore, eq. (5.2) has been changed to the following suitable form [27,36] which makes the computations possible for any value of x.

$$P_\nu(\cos\phi) = (2\ 2/\pi\sin^2\phi)\int_0^\phi \{(\nu+2)\cos(\nu+3/2)x - \nu\cos\phi\cos(\nu+1/2)x\}(\cos x - \cos\phi)^{\frac{1}{2}}dx \qquad (5.6)\text{b}$$

The natural frequencies for the sandwich spherical shells with membrane face sheets and also for the homogeneous spherical shells can be computed in the same manner as discussed above for the composite shell with face sheets as flexural members. All the above mentioned subroutines can be used except for QUATIC because the characteristic equation (3.21) in this case is only a third order polynomial.

Numerical computations for the natural frequency parameter (ω_p/ω_o) of the circular sandwich plate have also been done using exactly the same iteration technique which has been explained earlier in this chapter. The roots g_α^2 $(\alpha=1,2,3)$ of the characteristic equation (3.33) are calculated with the help of the subroutine CUBIC and the values of $J_o(g_1)$, $I_o(g_2)$, ... etc. are generated by using subroutines BESJ, IO and INUE. Subroutine BESJ calculates the values of Bessel function $J_n(x)$ for a given argument x and order n. Program IO computes modified Bessel function $I_o(x)$ and INUE obtains $I_n(x)$ for orders 1 to n.

Mode Shape Calculation

For sandwich spherical shells with face sheets as flexural members, the expression for W is taken from eq. (3.14). The fixed edge condition at $\phi=\phi_o$ yields

$$W = \overset{*}{V} = X = 0$$

and at the pole $\phi=0$, $W=W_o$.

Substitution of the above condition in eq. (3.14) and eq. (3.15) yields the following simultaneous equations.

$$\sum_{\alpha=1}^{4} A_{\nu_\alpha} = W_o$$

$$\sum_{\alpha=1}^{4} P_{\nu_\alpha}(\cos\phi_o)A_{\nu_\alpha} = 0 \tag{5.7}$$

$$\sum_{\alpha=1}^{4} \psi_{1\alpha}P'_{\nu_\alpha}(\cos\phi_o)A_{\nu_\alpha} = 0$$

$$\sum_{\alpha=1}^{4} \psi_{3\alpha}P'_{\nu_\alpha}(\cos\phi_o)A_{\nu_\alpha} = 0$$

Solving eqs. (5.7), the values of arbitrary constants A_{ν_α} (α=1,2,3,4) are obtained and are written below.

$$
\begin{aligned}
A_{\nu_1} &= W_o \, |DP1_{ij}|/|BM_{ij}| \\
A_{\nu_2} &= W_o \, |DP2_{ij}|/|BM_{ij}| \\
A_{\nu_3} &= W_o \, |DP3_{ij}|/|BM_{ij}| \\
A_{\nu_4} &= W_o \, |DP4_{ij}|/|BM_{ij}|
\end{aligned}
\tag{5.8}
$$

where the determinant $|BM_{ij}|$ is given as:

$$
|BM_{ij}| = \begin{vmatrix}
1 & 1 & 1 & 1 \\
P_{\nu_1}(\cos\phi_o) & P_{\nu_2}(\cos\phi_o) & P_{\nu_3}(\cos\phi_o) & P_{\nu_4}\cos\phi_o \\
\psi_{11}P'_{\nu_1}(\cos\phi_o) & \psi_{12}P'_{\nu_2}(\cos\phi_o) & \psi_{13}P'_{\nu_3}(\cos\phi_o) & \psi_{14}P'_{\nu_4}(\cos\phi_o) \\
\psi_{31}P'_{\nu_1}(\cos\phi_o) & \psi_{32}P'_{\nu_2}(\cos\phi_o) & \psi_{33}P'_{\nu_3}(\cos\phi_o) & \psi_{34}P'_{\nu_4}(\cos\phi_o)
\end{vmatrix}
\tag{5.9}
$$

The determinants $|DP1_{ij}|$, $|DP2_{ij}|$, $|DP3_{ij}|$ and $|DP4_{ij}|$ can be written by replacing first, second, third and fourth columns by column {1 0 0 0} in the above determinant $|BM_{ij}|$ respectively. Thus, the normalized form of W is written as:

$$W/W_o = \{|DP1_{ij}|P_{\nu_1}(\cos\phi) + |DP2_{ij}|P_{\nu_2}(\cos\phi)$$

$$+|DP3_{ij}|P_{\nu_3}(\cos\phi) + |DP3_{ij}|P_{\nu_4}(\cos\phi)\}/|BM_{ij}| \qquad (5.10)$$

The computer program for the calculation of mode shape has also been presented in the main program in a simple form and need not be explained.

The expression for W/W_o can also be obtained in a similar manner for the sandwich spherical shells with membrane face sheets, homogeneous spherical shells and circular sandwich plates.

CHAPTER 6

NUMERICAL RESULTS AND DISCUSSION

Numerical values of the natural frequencies of spherical sandwich shells and circular sandwich plates have been presented as functions of the ratio of thickness of the face sheet to radius. The values of frequencies have been calculated from the simultaneous solution of characteristic eqs. (3.10a), (3.21) and (3.33) and the determinantal frequency eqs. (4.12), (4.1) and (4.20). These have been worked out for several special cases; spherical sandwich shells (i) with face sheets having flexural stiffness, (ii) with face sheets as membranes, (iii) limiting case of homogeneous shell and (iv) circular sandwich plates. The calculations were done for the two sets of ratios defined by r_G and r_ρ of the sandwich structure with aluminum face sheets and aluminum honeycomb core. These values are given in Table 5.1. The results for spherical sandwich shells, homogeneous spherical shells and circular sandwich plates will be discussed separately in the following sections.

6.1 Results for Spherical Sandwich Shells

On close examination of the frequency equations for the sandwich and homogeneous spherical shells, it is observed that their roots

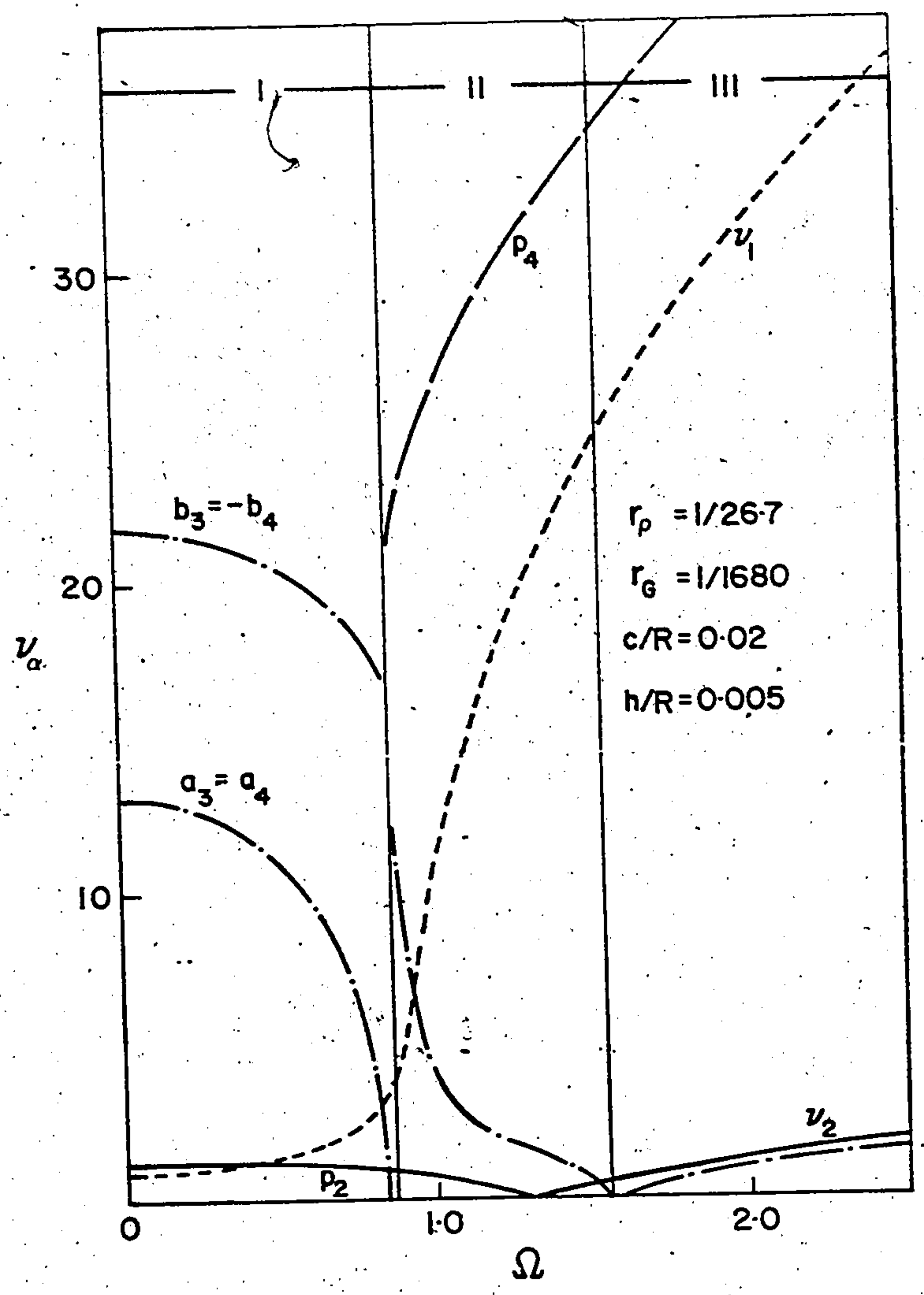

Fig. 4 Order of Legendre functions vs Ω for spherical sandwich shell.

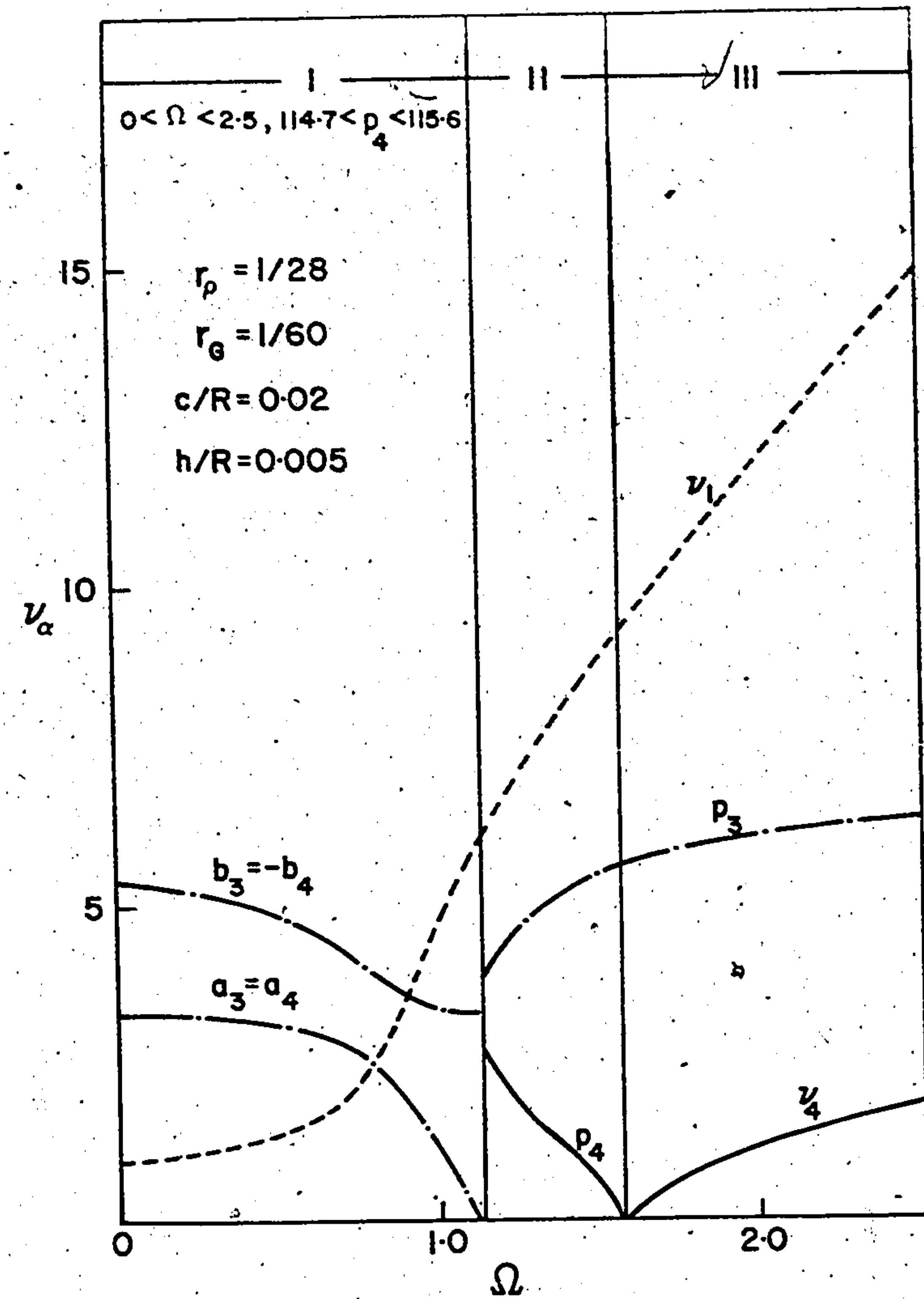

Fig. 5 Orders of Legendre functions vs Ω for spherical sandwich shell.

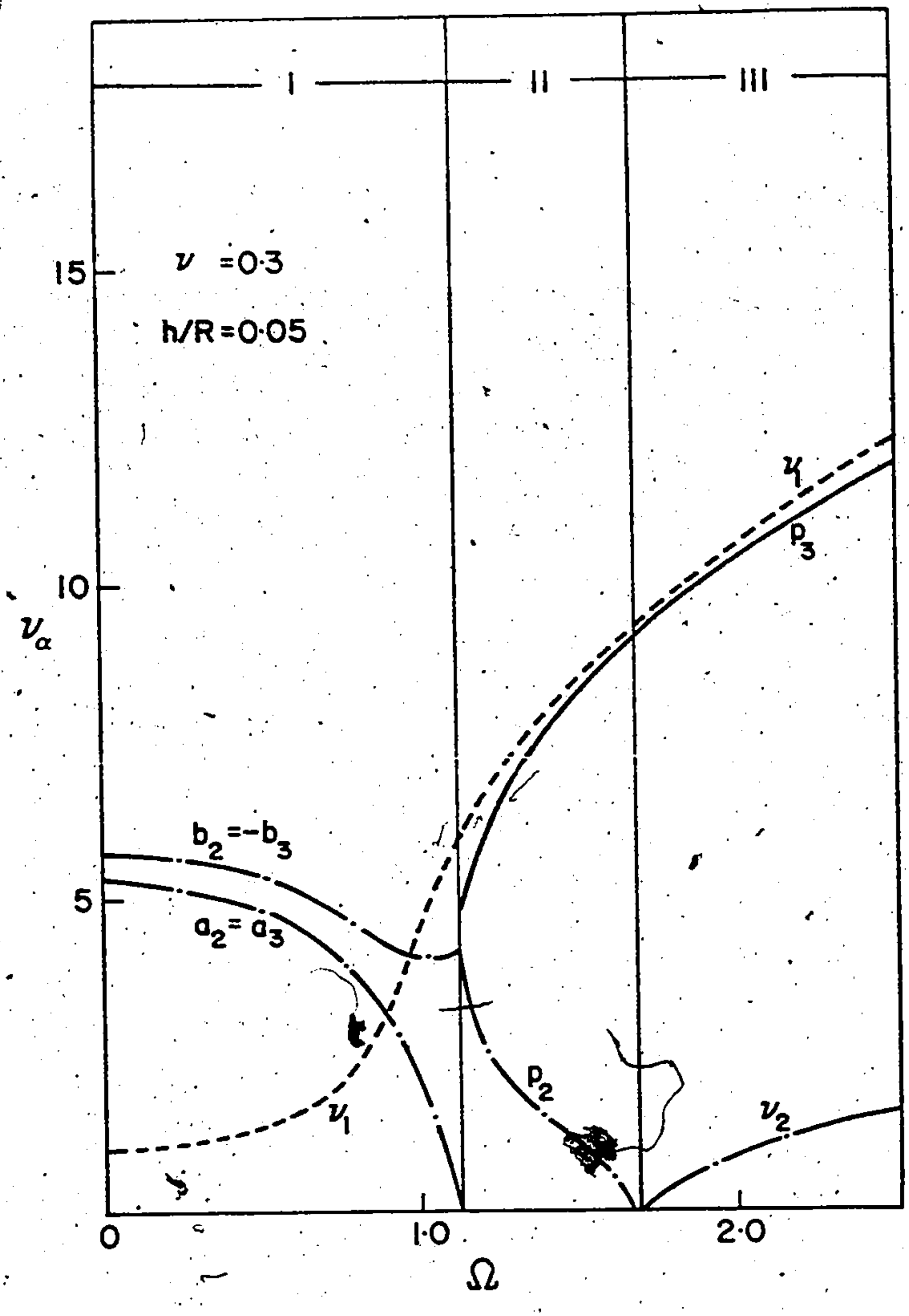

Fig. 6 Orders of Legendre functions vs Ω for spherical homogeneous shell.

TABLE 6.1. Summary of the variation of ν_α for Legendre function P_{ν_α} $(\cos\phi)$.

Zone	Sandwich Shell $r_\rho = 1/26.7$, $r_G = 1/1680$	Sandwich Shell $r_\rho = 1/28$, $r_G = 1/60$	Homogeneous Shell
I	$0<\Omega<0.84$ $\nu_3=a_3+i\ b_3$ $\nu_4=a_3-i\ b_3$	$0<\Omega<1.12$ $\nu_3=a_3+i\ b_3$ $\nu_4=a_3-i\ b_3$	$0<\Omega<1.12$ $\nu_2=b_2+i\ b_3$ $\nu_3=b_2-i\ b_3$
II	$0.85<\Omega<1.53$ $\nu_3=-\frac{1}{2}+i\ p_3$ $\nu_4=-\frac{1}{2}+i\ p_4$	$1.13<\Omega<1.54$ $\nu_3=-\frac{1}{2}+i\ p_3$ $\nu_4=-\frac{1}{2}+i\ p_4$	$1.13<\Omega<1.65$ $\nu_2=-\frac{1}{2}+i\ p_2$ $\nu_3=-\frac{1}{2}+i\ p_3$
III	$1.54<\Omega<2.50$ ν_3 is real $\nu_4=-\frac{1}{2}+i\ p_4$	$1.55<\Omega<2.5$ ν_3 is real $\nu_4=-\frac{1}{2}+i\ p_4$	$1.66<\Omega<2.50$ ν_3 is real $\nu_3=-\frac{1}{2}+i\ p_3$

are strongly dependent on the character of orders $\nu_\alpha(\alpha=1,2,3,4)$ of the Legendre function. These indices $\nu_\alpha(\alpha= 1,2,3,4)$ have been plotted against non-dimensional frequency Ω in Figs. 4 and 5 for the case of composite shell with stiff face sheets. In these two cases, the index ν_1 is always real and positive for the real and positive λ_1. The variations of ν_2, ν_3 and ν_4 are not continuous and are seen to be different in various distinct zones. The second root λ_2 of the characteristic equation (3.10a) is also real. The root λ_2 is negative for $0<\Omega<1.22$ and positive for $\Omega>1.22$ which is observed from the variation of p_2 and ν_2 in Fig. 4. Whereas in Fig. 5, the examination of the value of p_2 ($114.7<p_2<115.6$ for $0<\Omega<2.5$) reveals that λ_2 is a real large negative quantity. The indices ν_1, ν_2 and ν_3 for the homogeneous shell are plotted in Fig. 6 for $\nu=0.3$ and $h/R=0.05$. The order ν_1 is always positive in this case also. The variations of ν_3 and ν_4 for the spherical sandwich shells and ν_2 and ν_3 for the spherical homogeneous shell are summarized in Table 6.1. The indices ν_1, ν_2 and ν_3 for the spherical homogeneous shell have been plotted by Kalnins in [8] and Fig. 6 is in excellent agreement with his results. In Table 6.1, a_3, a_4, b_2, b_3, p_3 and p_4 are real quantities.

Figures 7-9 show the plot of Ω versus h/R up to five modes for $r_\rho=1/26.7$, $r_G=1/1680$ and $\nu=0.3$ for three different values of $\phi_0=60^\circ$, 90° and 120°. Similarly, Figs. 10-12 present the variation of Ω with h/R

for r_ρ=1/28, r_G=1/60 and ν=0.3 for the same three values of ϕ_0. The curves drawn by full lines represent the variation of natural frequency of the sandwich spherical shell, when both flexural and extensional stiffnesses are considered in the face sheets. The curves shown by broken lines are the variation of Ω of the sandwich spherical shell in which faces are assumed as membranes. In order to simplify the discussion, sandwich shells are classified in four types A, B, C and D as illustrated in Table 6.2.

TABLE 6.2. Properties of the sandwich shells.

Type	Face sheets	r_ρ	r_G	Remarks
A	Flexural Members	1/26.7	1/1680	weak core
B	Membranes	1/26.7	1/1680	
C	Flexural Members	1/28	1/60	strong core
D	Membranes	1/28	1/60	

For the opening angle ϕ_0=60°, non-dimensionalized frequency parameter Ω increases with h/R in all the five modes in Fig. 7 for type A whereas Ω increases only up to three modes for type B and then starts decreasing with the increase in h/R in fourth and fifth modes. The nature of variation of the frequency curve for shell types A and B

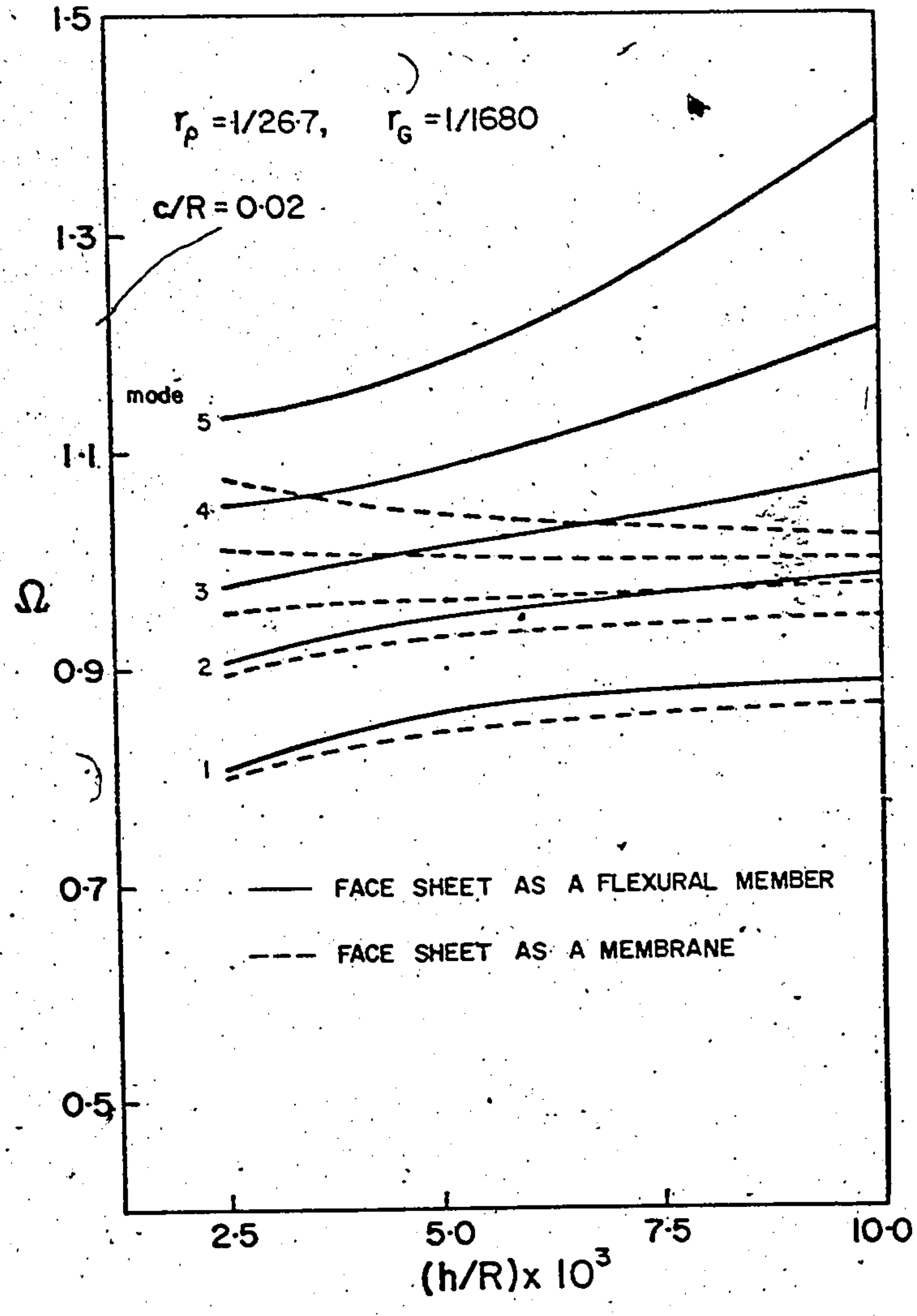

Fig. 7 Frequency variation with h/R for sandwich spherical shell clamped at $\phi_o = 60°$

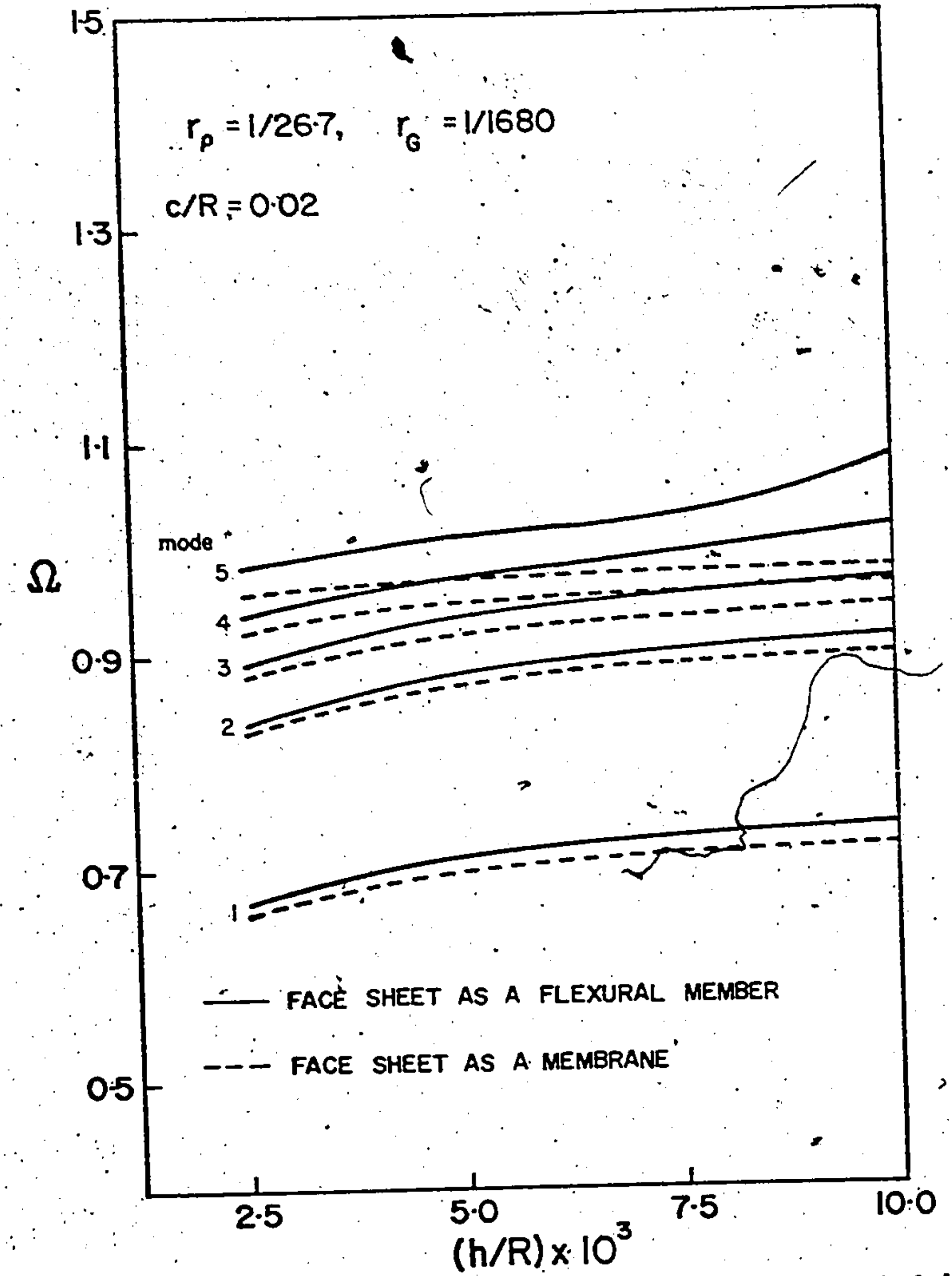

Fig.8 Frequency variation with h/R for sandwich spherical shell clamped at ϕ_0=90°

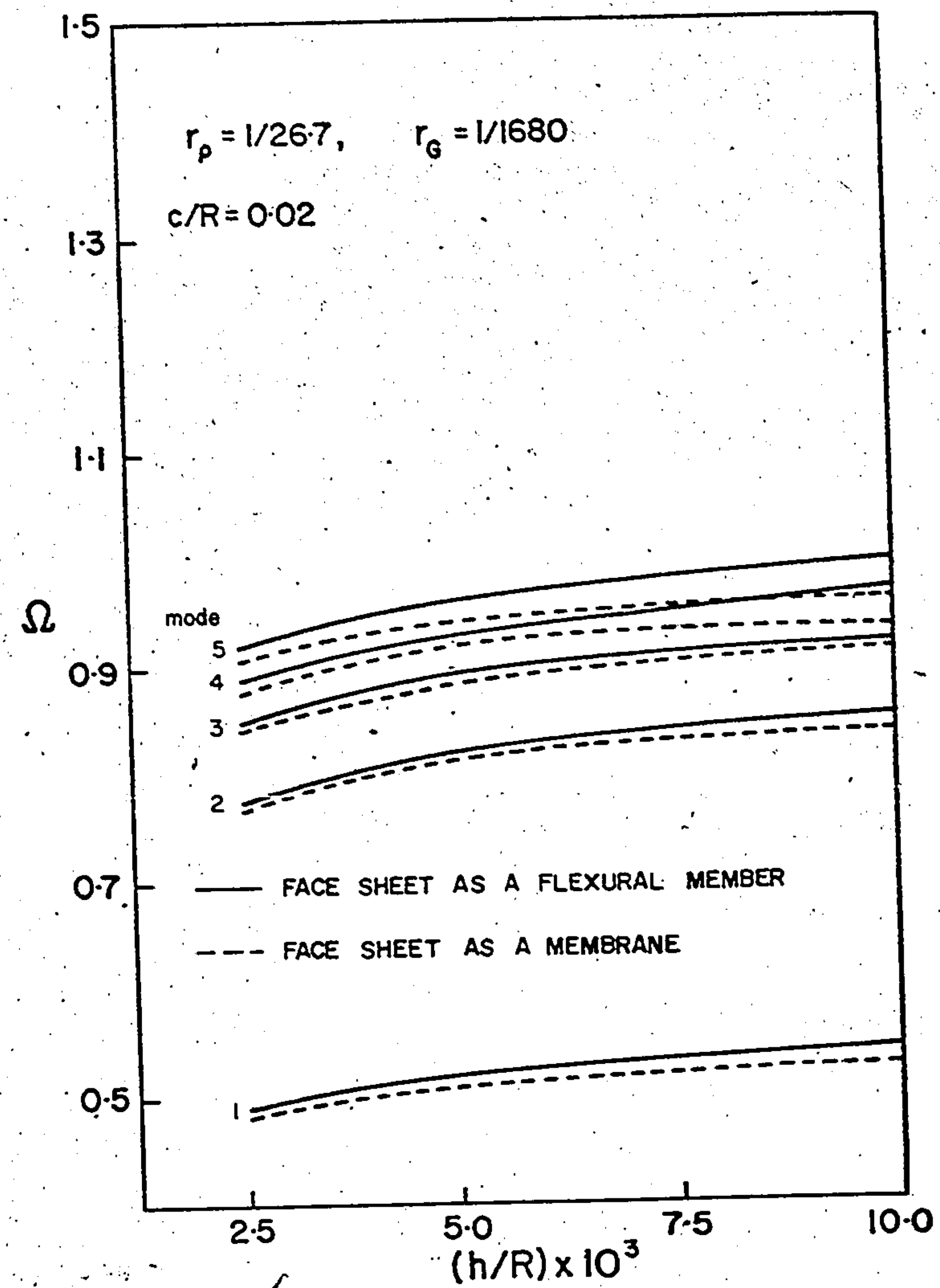

Fig. 9 Frequency variation with h/R for sandwich spherical shell clamped at ϕ_o=120°

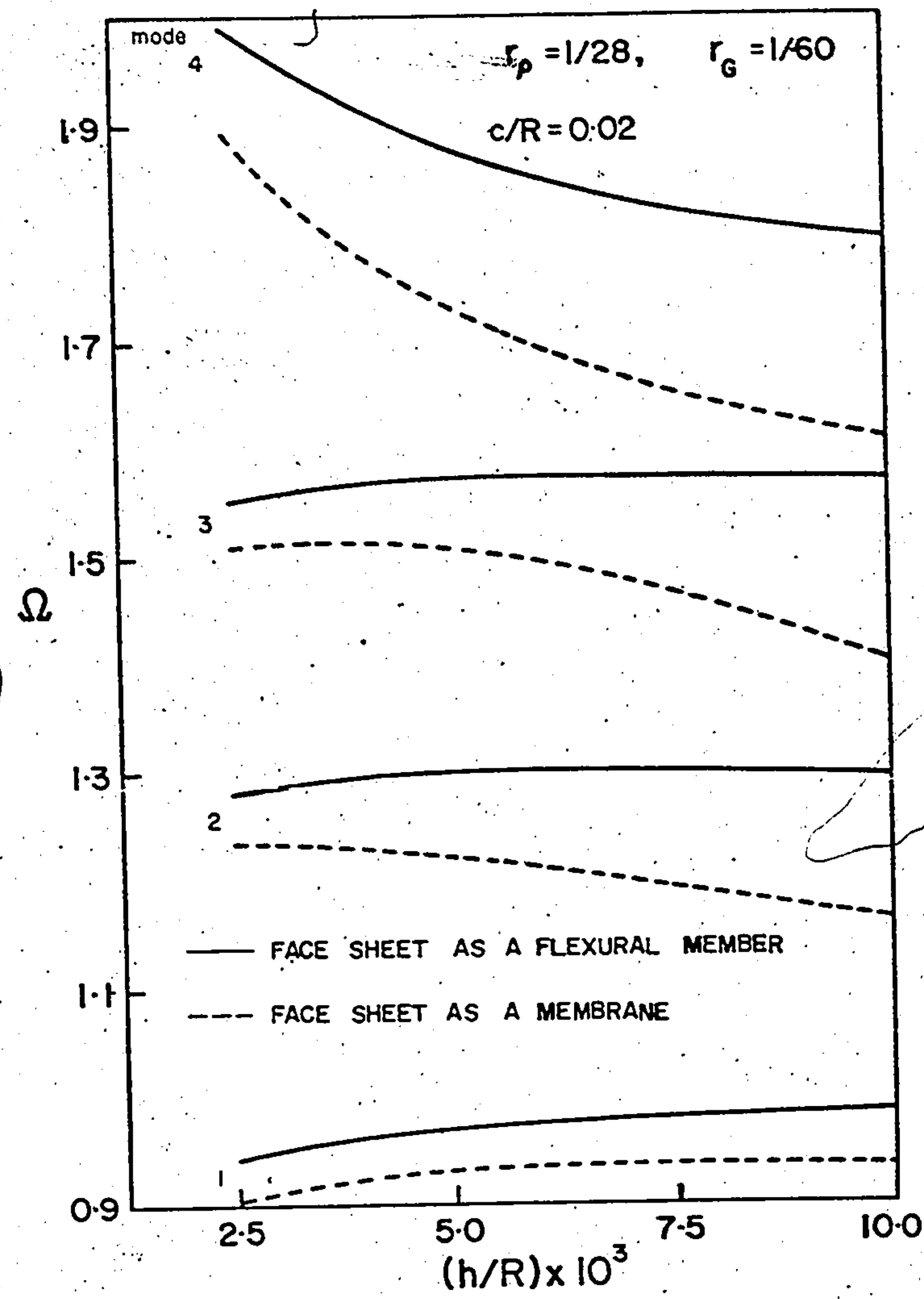

Fig. 10 Frequency variation with h/R for sandwich spherical shell clamped at $\phi_o = 60°$

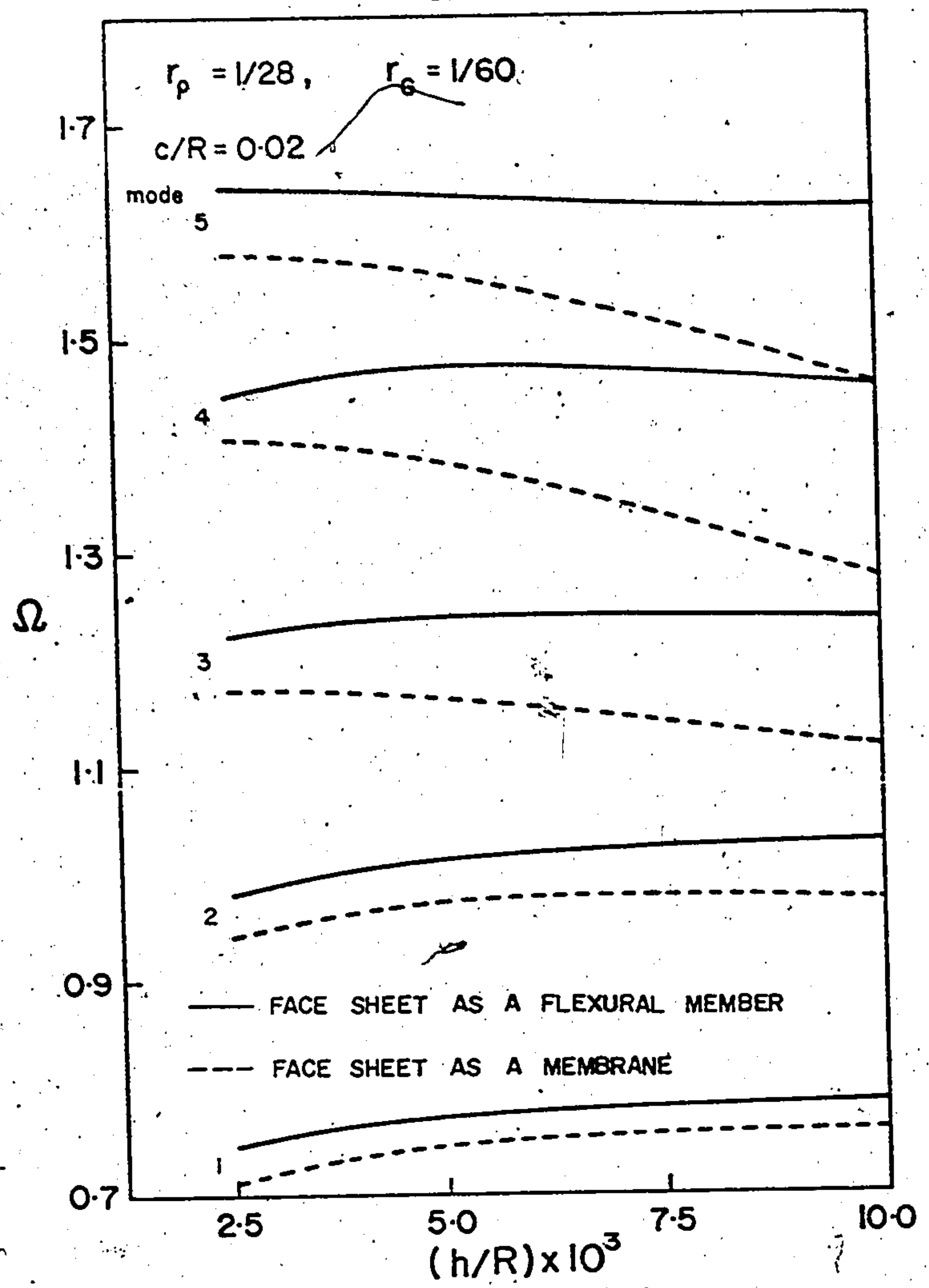

Fig. 11 Frequency variation with h/R for sandwich spherical shell clamped at ϕ_o=90°

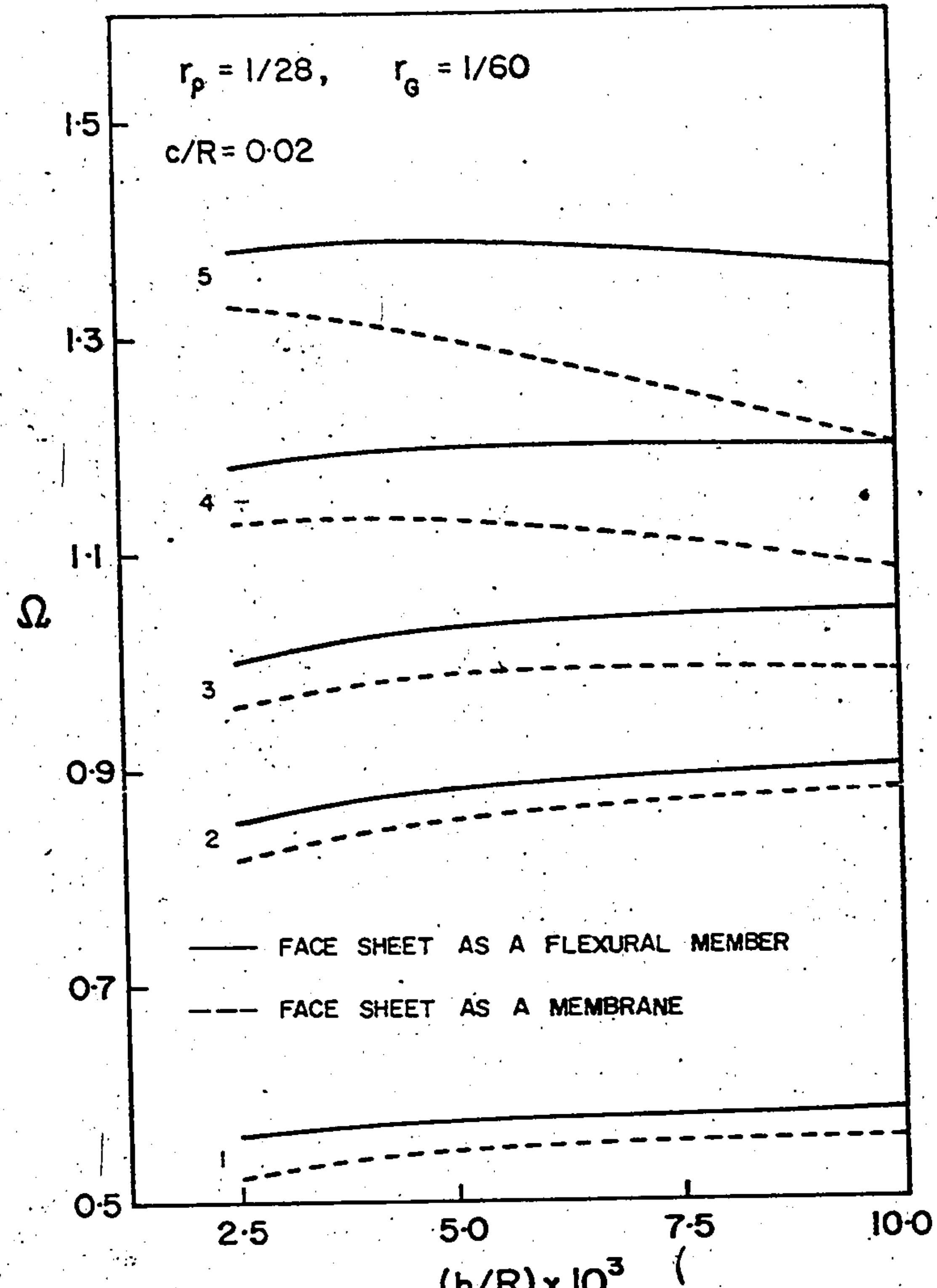

Fig. 12 Frequency variation with h/R for sandwich spherical shell clamped at $\phi_o = 120°$

remains similar only up to the second mode. In Fig. 8, Ω increases with the increase in h/R for both types A and B. The frequency behaviour shown by broken and solid lines in Fig. 9 is the same for all the five modes.

Numerical results for the shell types C and D are presented in Figs. 10-12. Fig. 10 shows the plot of Ω vs h/R up to four natural modes for the spherical sandwich shell with opening angle 60°. The opening angles of the composite shell, whose results are presented in Figs. 11 and 12 are 90° and 120°. The values of Ω increase in the first mode for 60°, the first two modes for 90° and up to third mode for the shell with opening angle 120°. It is seen that at higher modes Ω starts decreasing as h/R increases in all three cases. The significance of these results is discussed in the following.

The present formulation of the problem includes transverse shear effect in the core; rotary and translatory inertias of the core; rotary inertia of the faces about the middle plane of the sandwich shell (which is actually contributed by the translatory inertia of the faces) as well as about their own middle planes; and flexural and extensional rigidities of the faces.

It has been discussed earlier by Yu in his work [33] that the first and lowest frequency of the ordinary sandwich structures is strongly affected by the flexural and extensional rigidities, whereas in case of higher modes, the frequencies are practically independent of these rigidities. At higher frequency levels, it depends much on the various inertia quantities such as rotary and translatory inertia of the core and the face sheets. The shear effect, in fact, is far more important for sandwich plates and shells than for homogeneous plates and shells vibration studies.

As expected, it has been observed from the present results shown in Figs. 7-12 that the lowest frequency of the shell increases with the increase in h/R. This behaviour of frequency variation is substantial due to the influence of flexural and extensional stiffnesses of the face sheets for lower range of Ω. In all cases, the effect of flexural rigidity of the face sheets on the natural frequency is significant. The frequencies are higher for sandwich shells with face sheets as flexural members than the shells in which faces are taken as membranes. The difference between the values of Ω for shells with two types of face sheets (flexural and membrane) widens with the mode numbers. The percentage differences between the frequencies of the sandwich shells, one with flexural face sheets and the other with membrane face sheets, are shown in Table 6.3. The percentage is calculated on the basis of the numerical values of Ω for the shell with membrane face sheets. The maximum percentage error, which is as much as 27%, occurs in the fifth mode of the sandwich shell with opening angle 60°;

TABLE 6.3. Percentage difference between the frequencies for sandwich shells with two types of face sheets (flexural and membrane).

h/R \ ϕ_0	60°	90°	120°	Other Parameters
0.0025	4.5 (m=1) 5.2 (m=4)	5.0 (m=1) 2.0 (m=5)	6.65 (m=1) 3.75 (m=5)	$r_\rho=1/28$ $r_G=1/60$ C/R=0.02
0.0100	5.35 (m=1) 11.4 (m=4)	4.0 (m=1) 12.0 (m=5)	3.6 (m=1) 16.2 (m=5)	
0.0025	1.2 (m=1) 3.0 (m=5)	1.5 (m=1) 2.8 (m=5)	1.5 (m=1) 1.3 (m=5)	$r_\rho=1/26.7$ $r_G=1/1680$ C/R=0.02
0.0100	3.0 (m=1) 27.0 (m=5)	2.6 (m=1) 10.2 (m=5)	3.8 (m=1) 3.65 (m=5)	

h/R=0.010, r =1/26.7 and r_G=1/1680. This analysis clearly shows that it is important to include the flexural rigidity of the face sheets about their own middle planes and their extensional rigidity simultaneously in the differential equations of motion. In case of very thin face sheets and lower frequency level, it appears that the effects of flexural rigidity of the face sheets about their own middle planes can be dropped. This will help in the reduction of the order of differential equations of motion and further simplification of their solution will also take place simultaneously. But, the inaccuracy increases very rapidly for thicker face sheets and higher frequency level.

The effect of rotary inertia of the face sheets about their own middle plane has been found to be negligible on the natural frequencies of the sandwich shells. This inertia term has been omitted by setting s_4, s_5 and s_6 in eqs. (2.21) equal to zero. However, the rotary inertia of the core and the face sheets about the middle plane of the sandwich shell is obviously very important.

The effects of shear modulus G_c of the core on the natural frequency of the sandwich shell has been found to be of considerable significance. When the shear modulus of the core is increased, its transverse shear resistance also increases. Thus, for a sandwich shell made of identical face sheets, the percentage of transverse shear taken by a core with higher value of G_c is more than the other core with lower value of G_c, when the distribution of total shear load is considered between the core and the face sheets. Also, by increasing the values of shear modulus G_c, overall stiffness of the sandwich shell is increased and the transverse shear deformation is reduced. In the limiting case, when total transverse shear load is taken by the core, for the simple bending of sandwich structures, there will be only membrane forces in the face sheets. On the other hand, if the shear modulus of the core is zero and all the shear force is taken by the face sheets, the two face sheets of the sandwich shell will behave like two different homogeneous shells with the same phase in the radial direction. In this case, their flexural rigidity will dominate over the extensional rigidity. The core is weaker in transverse shear resistance in the sandwich shell types A and B as compared with types C and D due to its lower value of G_c. The value of shear modulus of the core in types C and D is twenty-eight times the value of G_c of types A and B when face sheets are assumed to be made of the same material.

It is found from Figs. 7-12 that due to increased overall stiffness and lower thickness shear deformation in the core of shell types C and D, the values of Ω are higher in this case than shell types A and B for identical geometry of the sandwich shells.

The values of natural frequencies of the sandwich shell with membrane face sheets increased with h/R for the lower frequency level $\Omega<1.0$ as can be seen from Figs. 7-12. But, at higher frequency range $\Omega>1.0$, frequency decreases with the increase in h/R. The extensional rigidity of the face sheets is predominant in the lower range of the frequency spectrum. The mass of the sandwich shell is increased by 230% when h/R is varied from 0.0025 to 0.0100. Therefore, the increase in mass and also increase in the value of Ω increases the inertia quantity due to kinetic energy of the shell. This inertia term becomes more effective than the extensional stiffness and frequency decreases.

The flexural rigidity of the face sheets is found to be predominant even for the higher modes in the shell type A due to its lower value of G_c as can be seen from Figs. 7-9 that frequency curves shown by solid lines increase with the increase in h/R. In case of shell type C, the frequency curves in Figs. 10-12 decrease with h/R in higher modes. This can be explained because of the higher value of G_c and core takes the major portion of transverse shear and consequently, membrane forces are predominant in the face sheets over the flexural forces. Even though for stronger core, the percentage of transverse shear taken by the face sheets is lower and membrane forces are far more important, it will not be advisable to neglect the flexural rigidity of the face sheets.

Since no work on vibrations of deep spherical sandwich shells is available in the literature, the numerical results for the natural frequencies of the sandwich shell can not be compared. However, as discussed earlier in this thesis, published works are available on vibrations of deep spherical homogeneous shells. The author has simplified the present general case and presented some values of natural frequency of clamped edge spherical homogeneous shells and compared these with the previous results in the following section.

6.2 Results for Homogeneous Spherical Shell

The numerical values of the natural frequency Ω up to five modes have been calculated for homogeneous spherical shells clamped at the edge ϕ_0. The effects of rotary inertia and shear deformation have

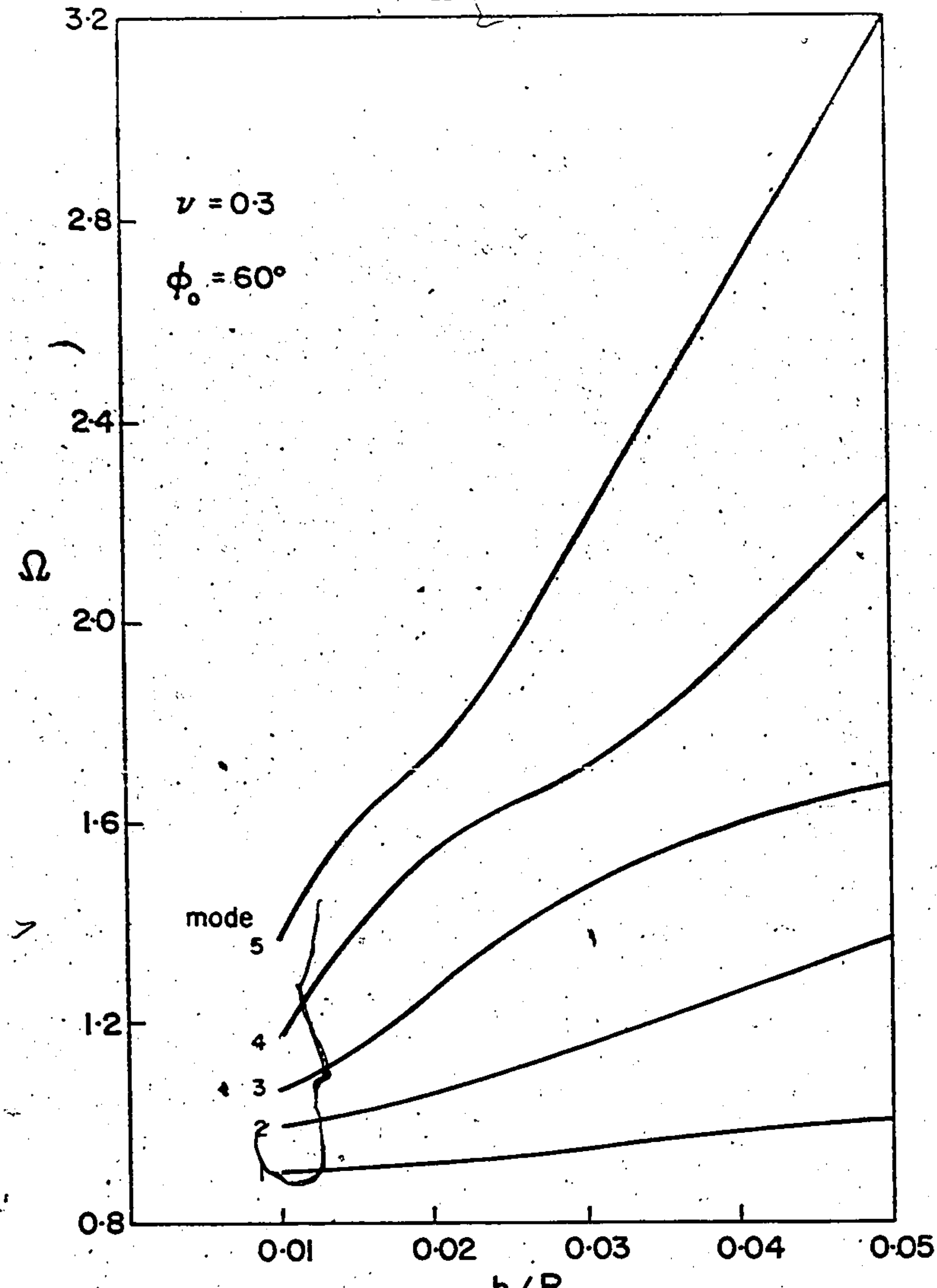

Fig. 13 Frequency variation with h/R for homogeneous spherical shell.

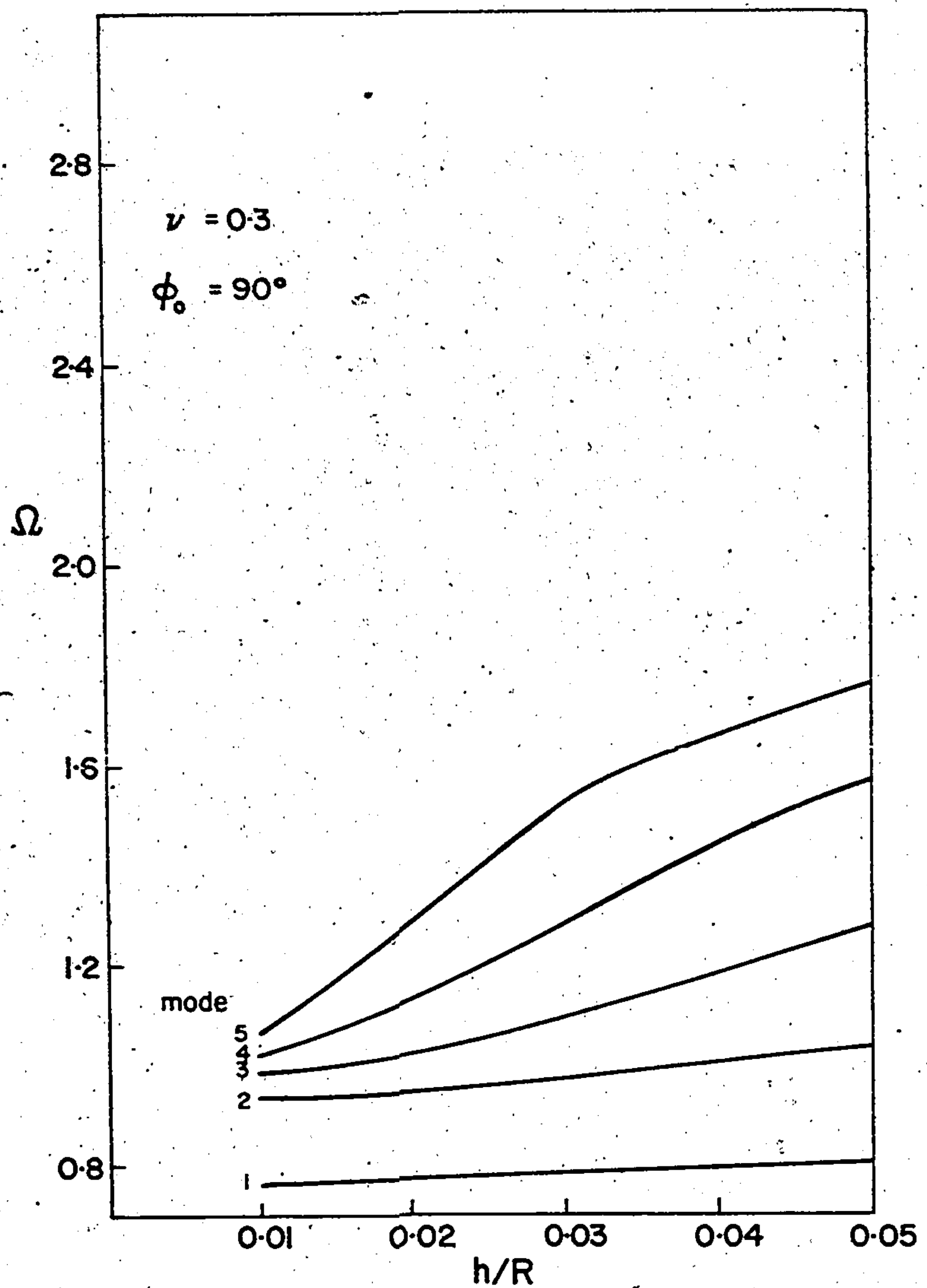

Fig. 14 Frequency variation with h/R for homogeneous spherical shell.

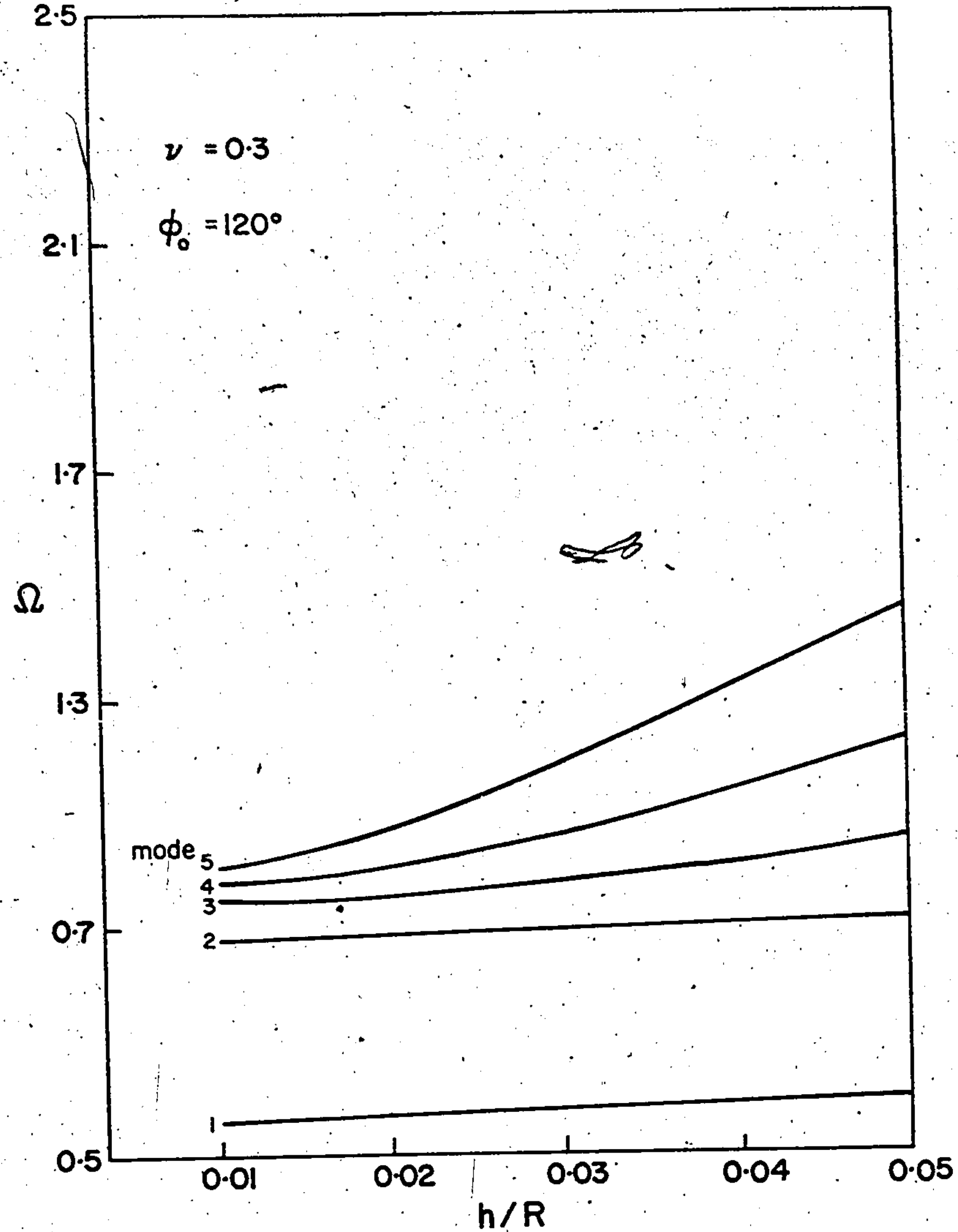

Fig.15 Frequency variation with h/R for homogeneous spherical shell.

been included in the differential equations of motion. These results are presented in Figs. 13-15 for three values of ϕ_0 which are 60°, 90° and 120°. Frequency Ω increases with thickness h/R and also with the mode number for all three values of ϕ_0. For smaller values of ϕ_0, it has been found that the gradient of the frequency curve is more pronounced than for larger values of opening angles.

The values of Ω have also been presented in this thesis in tabular form. Table 6.4 shows the values of Ω for h/R from 0.01 to 0.05 and angle ϕ_0 equal to 60° and 90°. The values in column I represent the natural frequencies of the homogeneous spherical shell in which the effects of rotary inertia and shear deformation are included. The values of Ω in column II have been calculated by equating s_1=1.0 and s_4=s_6=0 in eqs. (2.29) to eliminate the effects of rotary inertia. As expected, the values of Ω are lower when rotary inertia and shear deformation have been taken into account. But, it can be pointed out here that the effects of rotary inertia and shear deformation are negligible on the natural frequencies of the deep homogeneous spherical shells for all practical purposes, which is apparent from Table 6.4.

The values of the natural frequencies of homogeneous spherical shells have also been calculated in the past. In Table 6.5, the values of Ω, calculated on the basis of the present formulation of the problem which also includes effects of rotary inertia and shear deformation,

Table 6.4. Natural Frequencies Ω_m for Clamped Edge Spherical Homogeneous Shells. ν=0.3

ϕ_0	m	h/R=0.01 I	h/R=0.01 II	h/R=0.02 I	h/R=0.02 II	h/R=0.03 I	h/R=0.03 II	h/R=0.04 I	h/R=0.04 II	h/R=0.05 I	h/R=0.05 II
60^o	1	0.909	0.909	0.926	0.927	0.947	0.947	0.971	0.972	1.000	1.001
	2	0.995	0.995	1.062	1.063	1.159	1.162	1.269	1.273	1.368	1.373
	3	1.065	1.065	1.265	1.267	1.477	1.481	1.589	1.593	1.672	1.679
	4	1.183	1.184	1.542	1.545	1.704	1.709	1.959	1.963	2.243	2.253
	5	1.365	1.367	1.727	1.730	2.191	2.205	2.701	2.711	3.183	3.190
90^o	1	0.761	0.761	0.773	0.773	0.783	0.783	0.792	0.792	0.801	0.801
	2	0.937	0.938	0.952	0.953	0.973	0.974	1.000	1.001	1.033	1.035
	3	0.983	0.983	1.027	1.027	1.095	1.097	1.180	1.184	1.273	1.270
	4	1.020	1.020	1.128	1.130	1.282	1.285	1.445	1.451	1.565	1.570
	5	1.069	1.070	1.281	1.280	1.529	1.534	1.652	1.656	1.767	1.770

I = Rotary Inertia Included

II = Without Rotary Inertia

Table 6.5. Natural Frequencies Ω_m for Clamped Edge Hemispherical Homogeneous Shell. $\nu=0.3$

Mode	R/h=20 I	R/h=20 III	R/h=50 I	R/h=50 III	R/h=100 I	R/h=100 III
1	0.801	0.803	0.773	0.773	0.761	0.762
2	1.033	1.039	0.952	0.953	0.937	0.938
3	1.273	1.294	1.027	1.028	0.983	0.983
4	1.565	1.586	1.128	1.132	1.020	1.021
5	1.767	1.813	1.281	1.289	1.069	1.071
6	2.222	2.334	1.490	1.500	1.142	1.144

I = Present Theory

III = Elkrem's Results

Table 6.6. Natural Frequencies Ω_m of the Homogeneous Spherical Shell for $R/h=80$ and $\nu=0.3$

m	$\phi_0=60^o$		$\phi_0=90^o$		$\phi_0=120^o$	
	I	IV	I	IV	I	IV
1	0.91	0.928	0.76	0.771	0.56	0.560
2	1.00	1.027	0.94	0.970	0.87	0.905
3	1.10	1.094	0.99	1.023	0.94	0.984
4	1.27	1.216	1.04	1.062	0.98	1.021

I = Present Calculation

IV = Results from Valikov and Gots paper have been compared with the results obtained by Eikrem and Doige in their work [3] for the axisymmetric vibrations of clamped hemispherical shells. The present values of Ω are in excellent agreement. The maximum discrepancy in the value of Ω occurs in the fifth mode of the shell with R/h=20 and is less than 3%. The values of Ω in column I are smaller than those in column III for all R/h values. The values of obtained for R/h=80 and the opening angles 60^{o}, 90^{o} and 120^{o} are presented in Table 6.6 which also shows the results from the published work by Valikov and Gots [29]. The differences in the values of Ω when compared with the results in reference [29] lie within 5%. Kalnins in his paper [8] has presented some values of natural frequencies Ω of the homogeneous spherical shell with opening angle 60^{o} and h/R=0.05 for various edge conditions. He has also tabulated the values of Ω for clamped edge conditions in a column as 1.006, 1.391, 2.371 and 3.486 for mode numbers one to four respectively. The values of Ω calculated by the author for the same conditions of the shell are 1.000, 1.368, 1.672 and 2.242 for the first four modes. It seems here that Kalnins has failed to locate one root of the frequency equation between 1.391 and 2.371. This missing root should be in the vicinity of 1.672 which according to the present calculation is the value of Ω for the third natural mode of a clamped homogeneous spherical shell with opening angle 60^{o} and h/R=0.05.

6.3 Results for Circular Sandwich Plate

The dimensionless frequency parameter ω_p/ω_o has been plotted against h/a in Figs. 16 and 17 for the case of circular sandwich plates. These values are shown for the different values of c/a. The values of r_ρ, r_G are indicated in these Figures. It is interesting to note the effect of c/a on the natural frequency. For h/a=0.005, the values of ω_p/ω_o are higher for larger value of c/a but as h/a increases, these curves start getting closer. Fig. 16 and 17 both show the same trend of variation. This reveals that the effect of core thickness reduces for thicker face sheets. As seen from Fig. 16, the values of frequencies for different values of c/a and for all the three modes are almost equal near h/a=0.015. In this case, the core is softer in transverse shear resistance and the bending of face sheets is predominant at higher values of h/a and ω_p/ω_o nearly becomes independent of the core thickness. Due to absence of curvature in the sandwich plate, flexural rigidity of the face sheets is more important than their extensional rigidities for transverse vibration of the circular sandwich plate. When the shear modulus of the core increases, as in the case of results shown in Fig. 17, (ω_p/ω_o) vs h/a curve is steeper and the effect of core thickness on the natural frequency is present even at higher values of h/a. For r_G=1/60, the values of natural frequencies of the sandwich plate is higher for thicker core layers.

In Fig. 18, the non-dimensionalized form of natural frequency $\Omega_p = (\rho\omega^2 a^2/E)^{\frac{1}{2}}$ of the circular sandwich plate has been plotted vs h/a. The frequency parameter Ω_p does not contain thickness of the face sheet h. It is seen that in case of $r_G = 1/1680$, Ω_p increases with h/a for all the three modes plotted in Fig. 18 and the slope of the curves also increases with the mode number. The values of Ω_p increase only up to second mode with h/a for $r_G = 1/60$. The frequency curve for third mode decreases with increase in h/a first and then starts increasing slowly. From the examination of the results shown in this figure, it is obvious that the flexural stiffness of the face sheets is more effective on the natural frequencies because of the lower shear modulus G_c of the core. Higher values of G_c lower the effect of flexural stiffness of the face sheets on the natural frequency for thicker core, but increases the capacity of shear resistance in the central layer of the sandwich plate and also the overall rigidity of the sandwich structures. It can be seen from Fig. 18 that values of Ω_p are greater for $r_G = 1/60$ than those for $r_G = 1/1680$. The transverse shear deformation, which is more for weaker cores, plays an important role in the dynamics of sandwich plates. The same conclusion was reached while discussing the effect of core properties on the natural frequencies of the sandwich spherical shell.

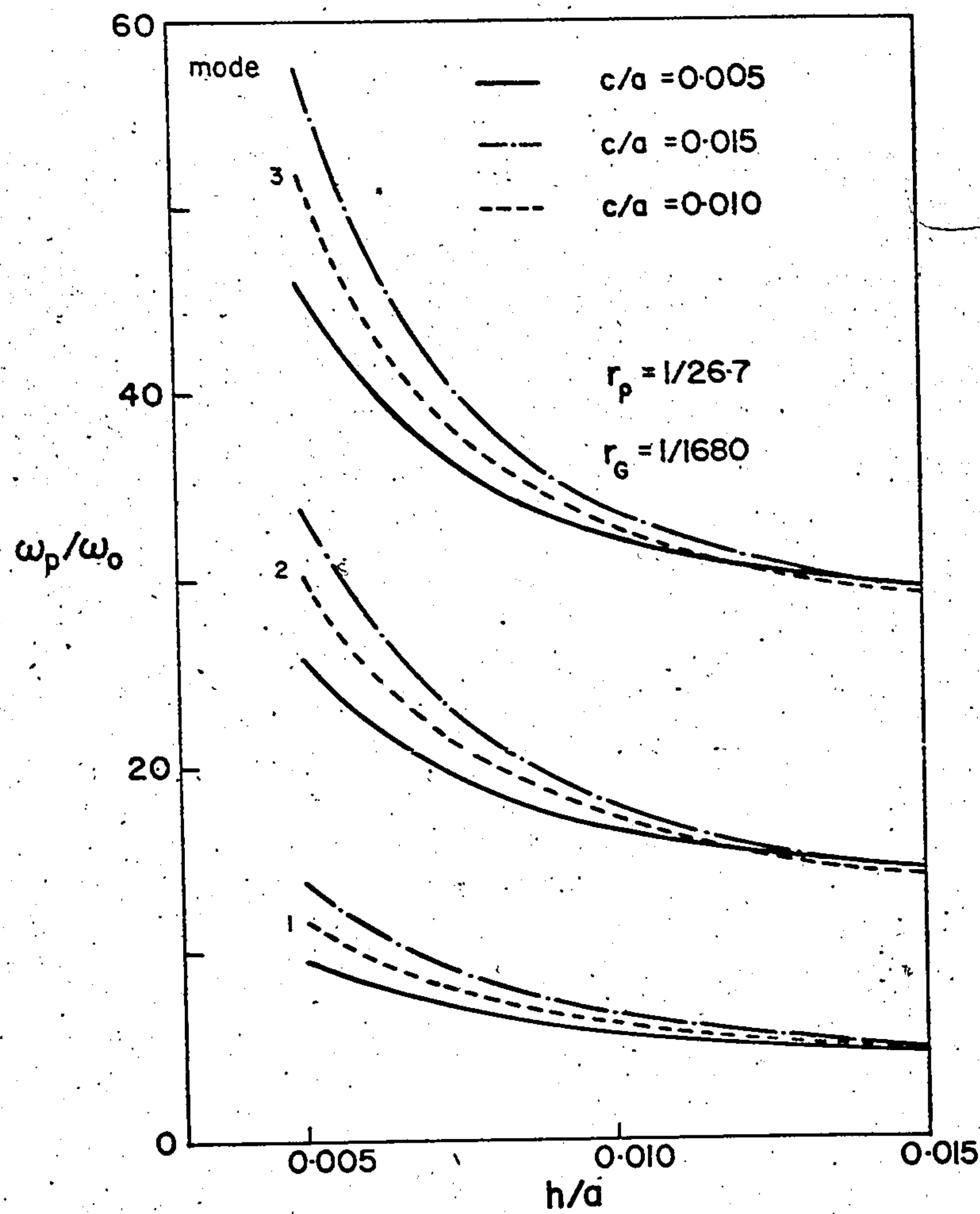

Fig.16 Frequency ω_p/ω_o vs h/a of circular sandwich plate clamped at $r = a$.

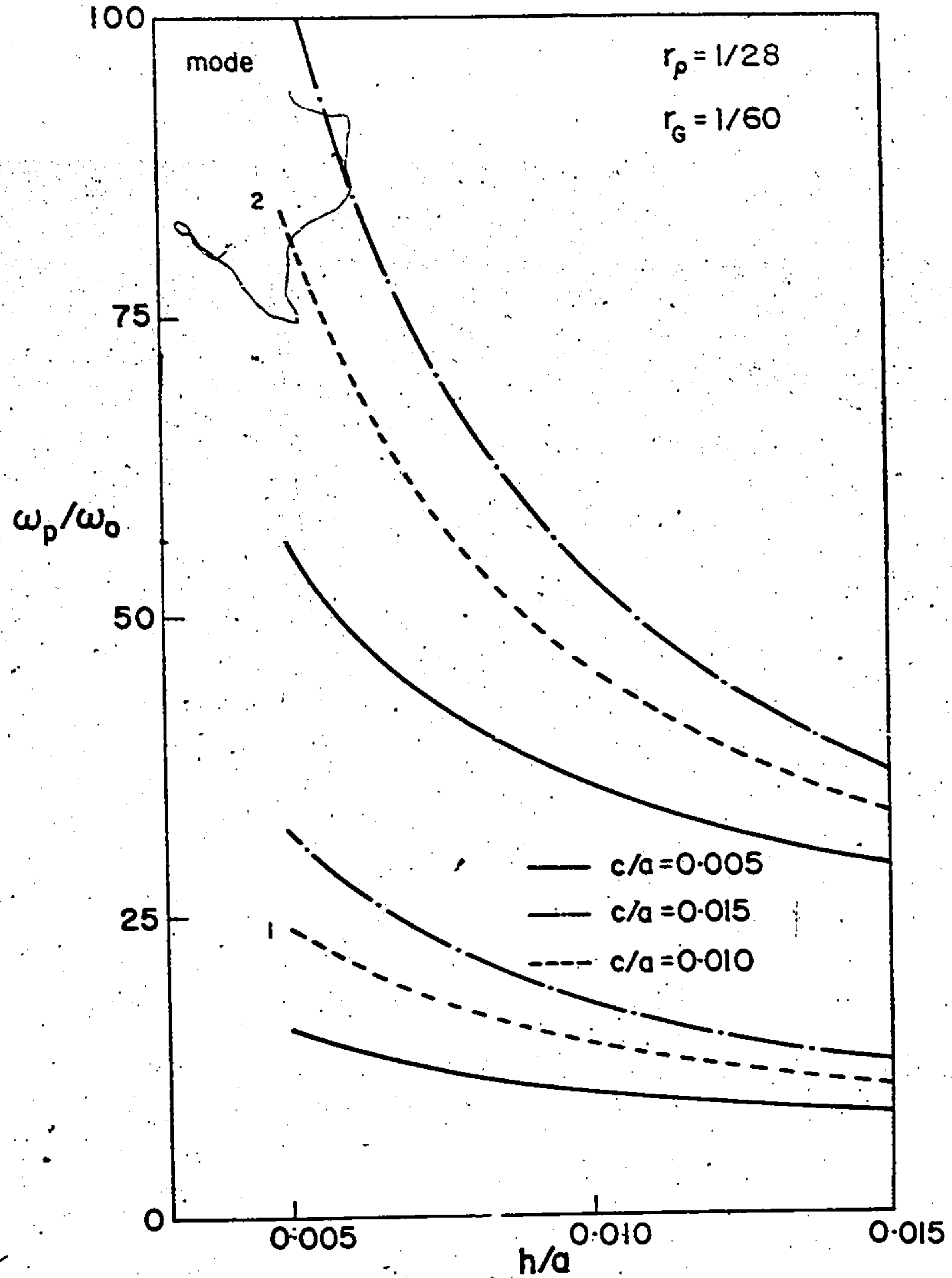

Fig.17 Frequency ω_p/ω_o vs h/a of circular sandwich plate clamped at $r = a$.

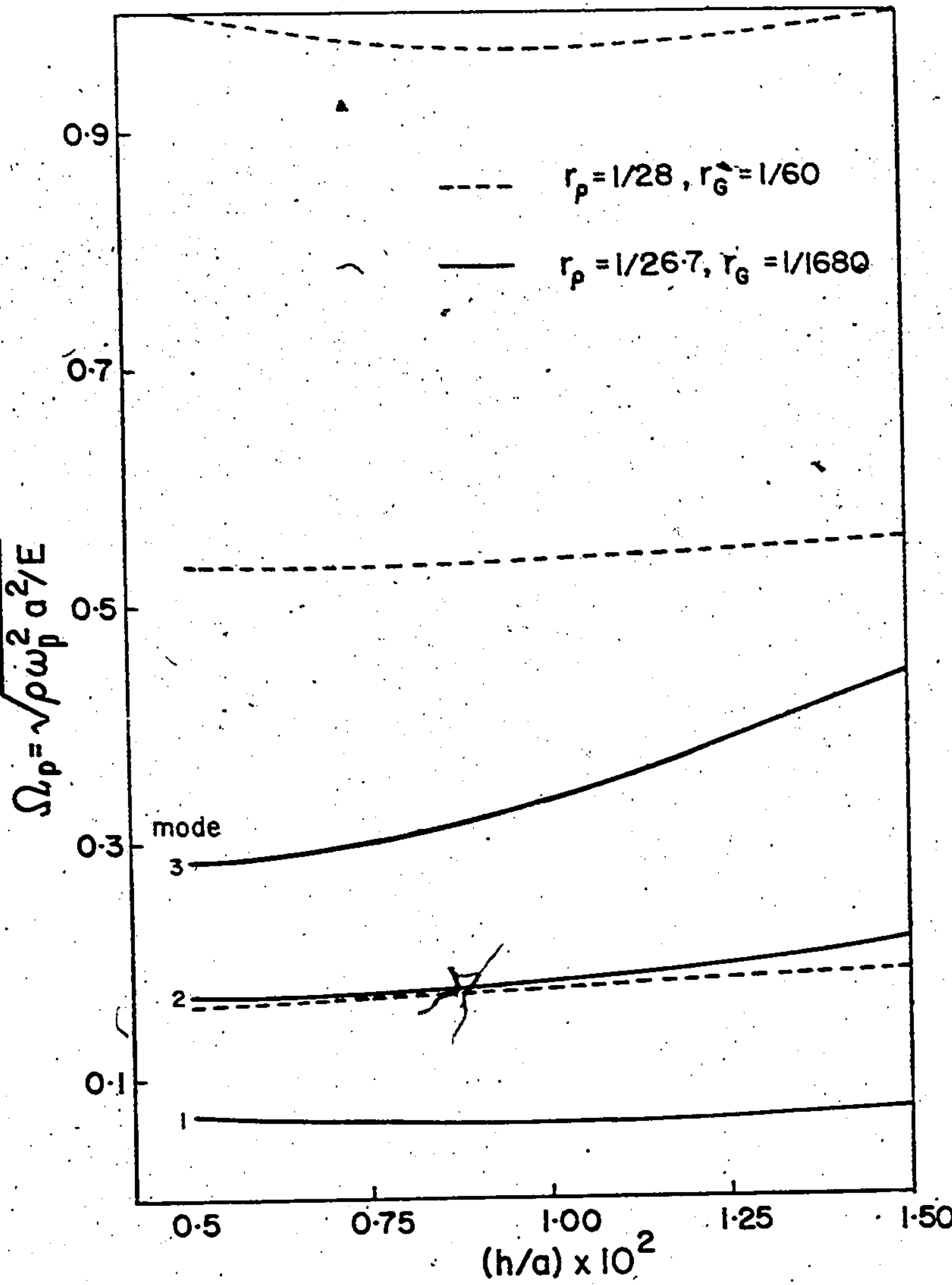

Fig 18 Frequency Ω_p vs h/a of circular sandwich plate clamped at r = a.

TABLE 6.8. Natural frequencies ω_p/ω_o for clamped edge circular sandwich plates.

$r_\rho=1/28$, $r_G=1/60$, $\nu=0.3$

m	c/a	h/a=0.010 I	h/a=0.010 II	h/a=0.015 I	h/a=0.015 II
1	0.010	14.2	14.2	10.7	10.7
2		45.7	45.7	33.3	33.3
3		85.3	85.5	61.4	61.5
4		129.0	129.2	92.9	93.1
5		175.3	175.7	127.7	128.0
1	0.015	17.6	18.1	12.6	12.8
2		53.7	55.2	37.1	37.8
3		97.0	99.7	66.3	67.6
4		143.5	147.4	98.3	100.3
5		192.0	197.3	133.6	136.0

TABLE 6.7. Natural frequencies ω_p/ω_o for clamped edge circular sandwich plates.

$r_\rho=1/26.7$, $r_G=1/1680$, $\nu=0.3$

m	c/a	h/a=0.010 I	h/a=0.010 II	h/a=0.015 I	h/a=0.015 II
1	0.10	6.1	6.2	4.6	4.7
2		17.1	17.5	14.3	14.5
3		32.9	33.5	29.3	29.7
4		53.9	54.8	49.9	50.5
5		80.4	81.8	76.2	77.0
1	0.015	6.6	6.8	4.8	4.9
2		17.9	18.4	14.5	14.7
3		33.7	34.6	29.4	29.9
4		54.6	56.1	49.9	50.8
5		81.0	83.2		

I = Various inertia terms included

II = Only transverse inertia included

Effect of Rotary Inertia in Sandwich Plates

The values of ω_p/ω_o have also been presented in Tables 6.7 and 6.8 to demonstrate the effect of rotary inertia. The values of natural frequencies for the transverse vibrations of the circular sandwich plate have been calculated by dropping out the inertia terms from eqs. (2.33)a and c. The values of ω_p/ω_o are presented in the columns under II in these tables. The values of natural frequencies for the transverse vibration of the circular sandwich plate are higher than the values of ω_p/ω_o when rotary inertia terms are included. But, the difference does not exceed even 2% up to five higher modes.

6.4 Discussion of the Mode Shapes

The mode shapes for the axisymmetric vibrations of the spherical sandwich shell and circular sandwich plate have been presented in Figs. 19-25. With the help of eqs. (5.8) to (5.10), the variations of W/W_ϕ vs ϕ have been plotted in Figs. 19 and 20 for the first three modes of the spherical sandwich shells clamped at ϕ_0 equal to 60° and 90°, respectively. The number of nodal circles in a particular mode is the same as the mode number.

Fig. 21 shows the first three mode shapes of the circular sandwich plates clamped at r=a. In this case, the first mode does not have a nodal circle, whereas second and third mode shapes have respectively one and two nodal circles.

Since the mode shapes of the sandwich constructions have been plotted by assuming face sheets as flexural members, the slopes at the clamped edges of the plate and shells are tending to zero. The mode shapes are also plotted in Figs. 22 and 23 for the clamped edge spherical sandwich shells with membrane face sheets and core capable of taking shear deformations. In these figures, the slopes are not zero at the edge. As the boundary conditions for pinned and fixed boundary conditions of the sandwich shells with membrane face sheets are the same as given by eq. (4.7)a, it is improper to distinguish between these conditions. The edge conditions for these two cases are different as given in eqs. (4.5)a and (4.13) when face sheets are considered as flexural members.

The modal shapes for clamped spherical homogeneous shells with opening angles of 60° and 90° have also been plotted in Figs. 24 and 25. In these cases also, the number of nodal circles is the same as the mode number. The slope $w_{,\phi}$ is zero for homogeneous shells at the clamped edge.

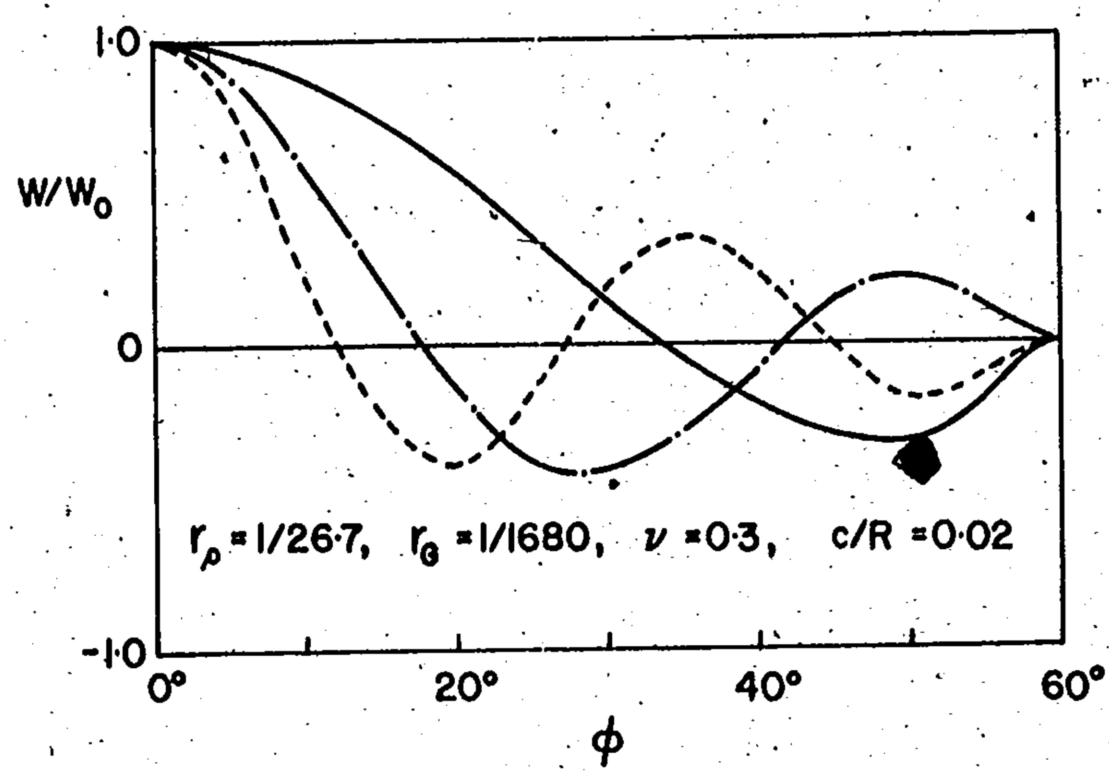

Fig. 19 Mode shapes for spherical sandwich shell clamped at $\phi_o = 60°$

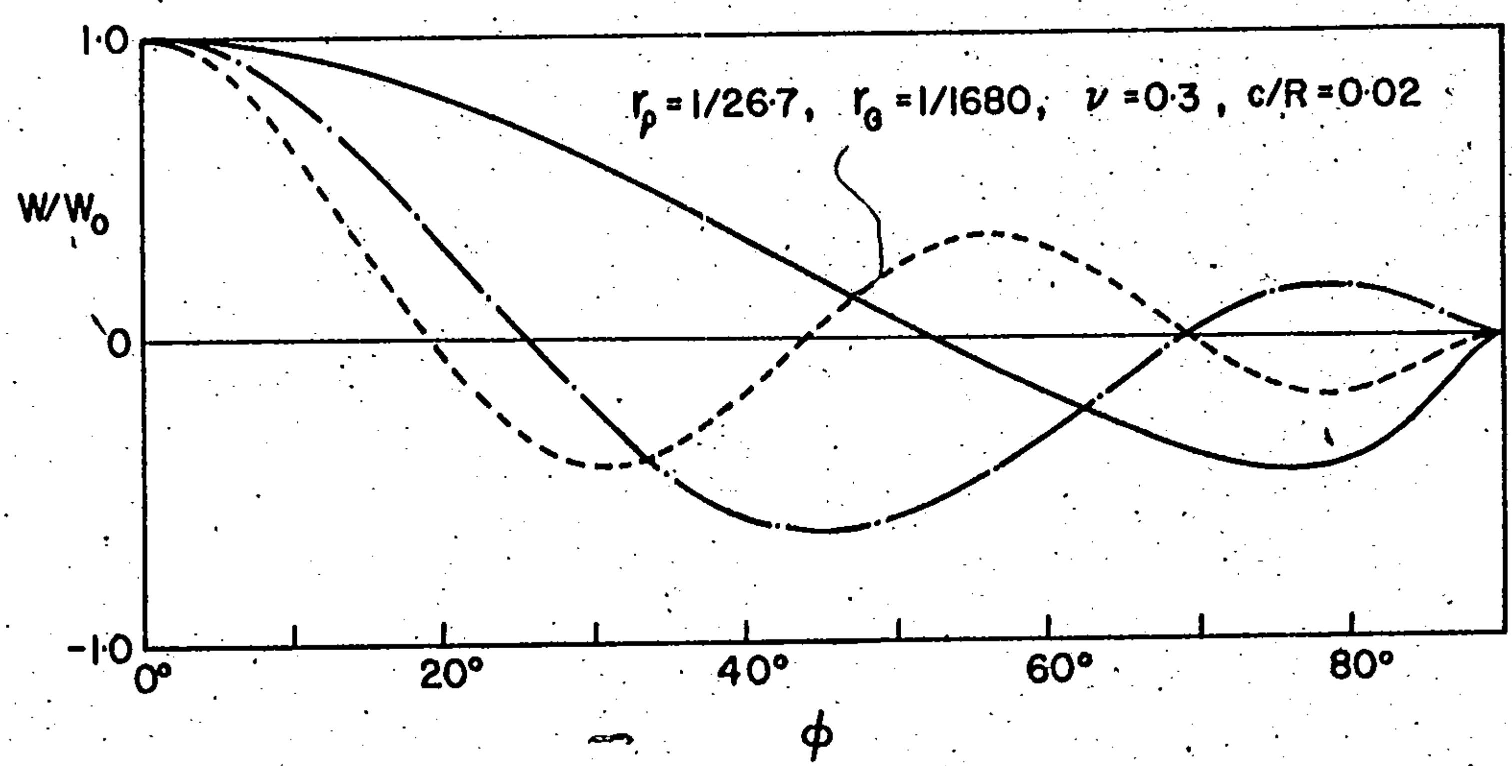

Fig. 20 Mode shapes for spherical sandwich shell clamped at $\phi_o = 90°$

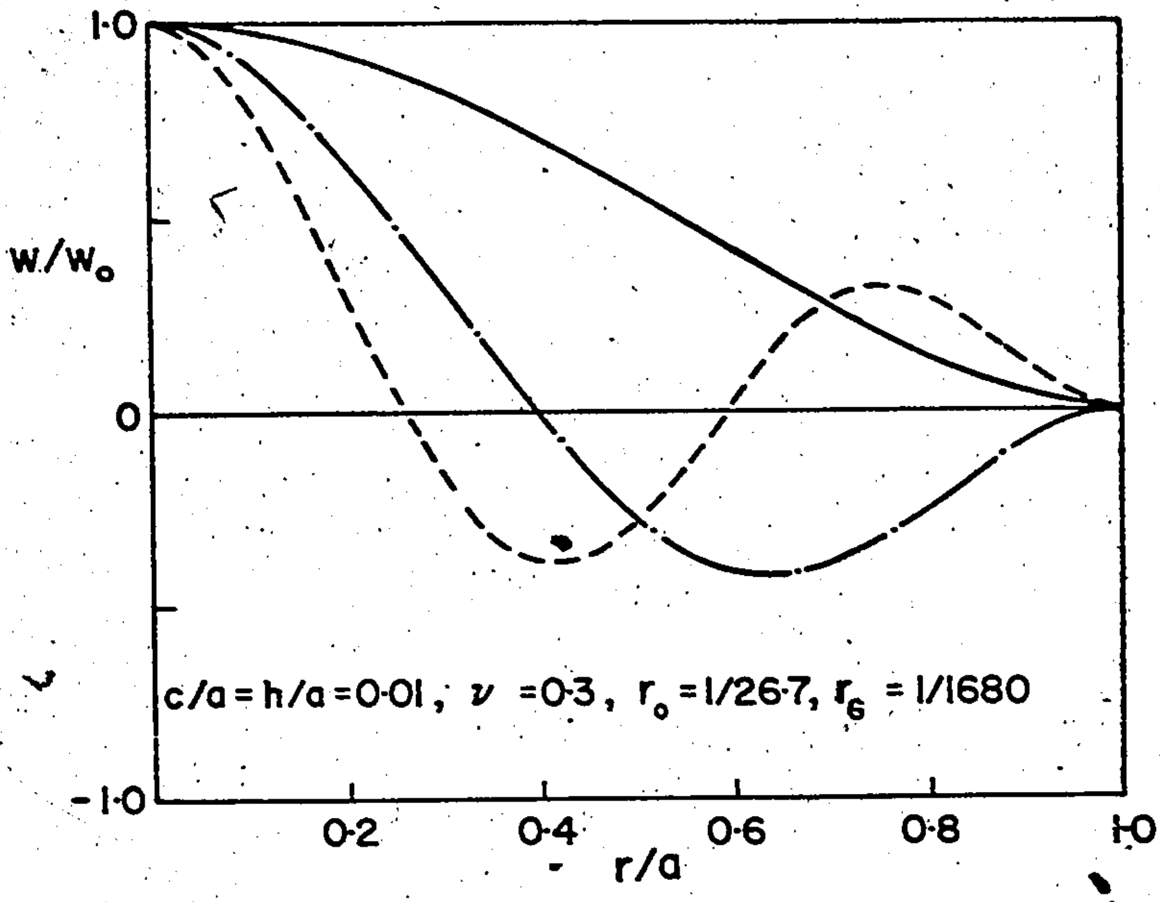

Fig.21 Mode shapes for clamped circular sandwich plate.

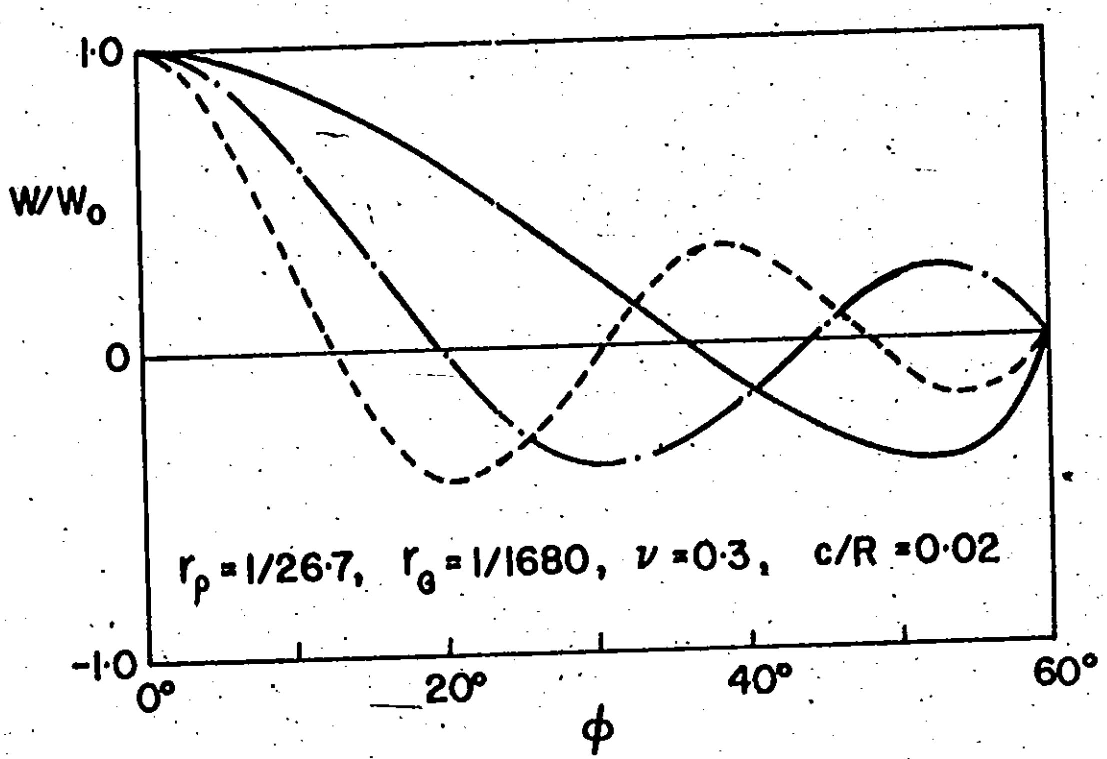

Fig.22 Mode shapes for clamped spherical sandwich shell with membrane face sheets.

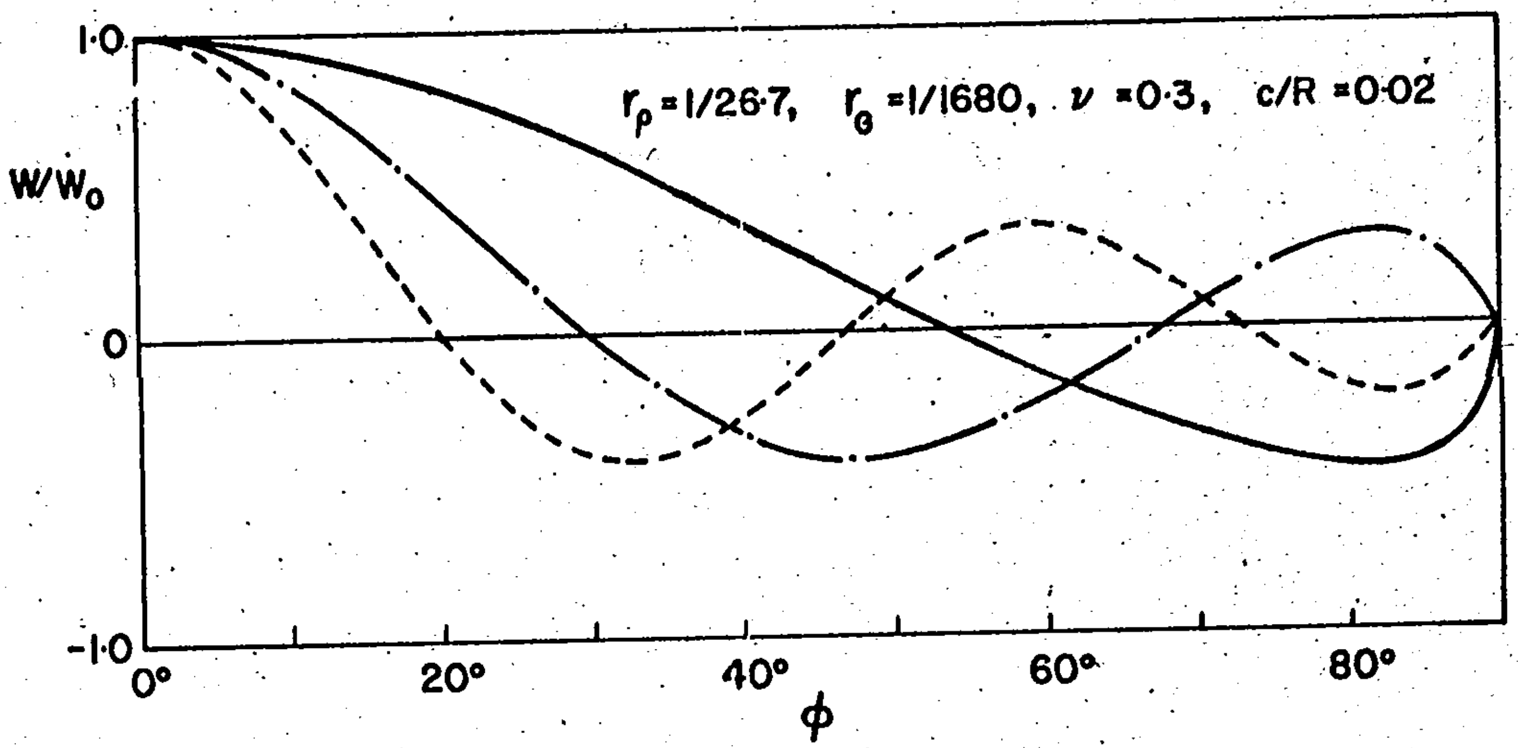

Fig. 23 Mode shapes for clamped spherical sandwich shell with membrane face sheets.

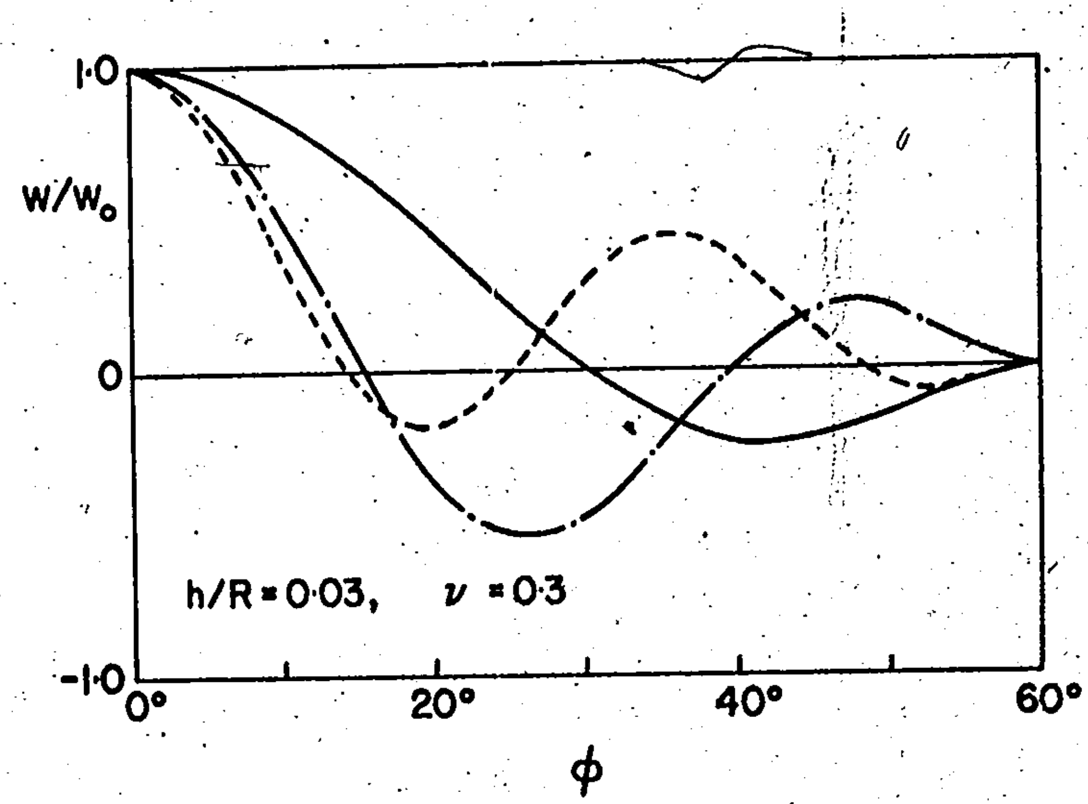

Fig.24 Mode shapes for clamped homogeneous spherical shell.

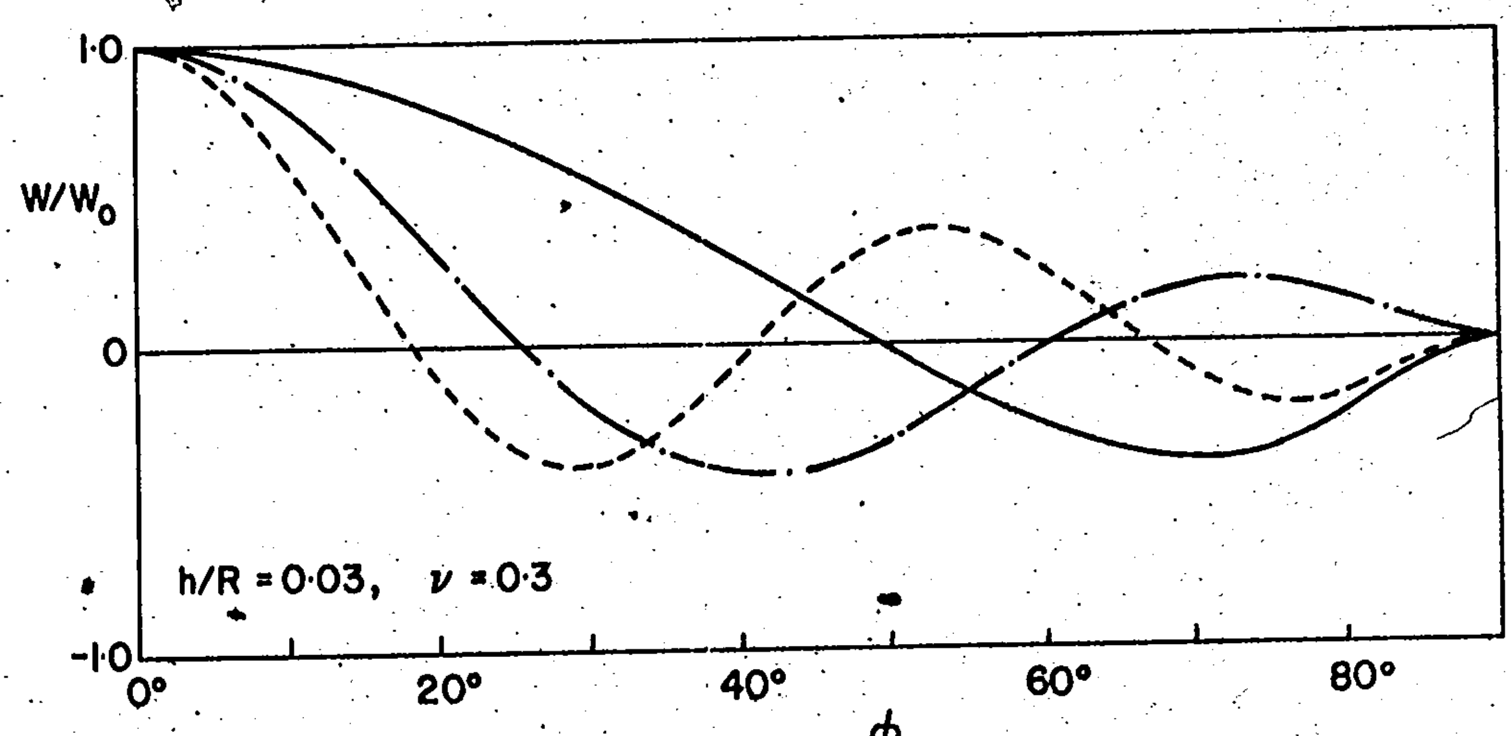

Fig.25 Mode shapes for clamped homogeneous spherical shell.

6.5 EXPERIMENTATION

Experimental investigation of the natural frequencies and mode shapes of spherical sandwich shells would prove to be a very useful contribution to the development of shell vibration theory. Unfortunately, no experimental data is reported in this thesis. However, attempts were made by the author to perform experiments for the purpose of comparison with theoretical results. The author and his supervisor communicated with Hexel Products Inc., located in the U.S.A., and Leigh Instruments of Canada, regarding the supply of materials.

The construction of a sandwich shell requires very skilled and high precision technique. If an aluminum honeycomb sheet is bent to form a spherical shell, it will invariably develop anticlastic curvatures and if it is then forced to the spherical shape, the honeycomb cells will definitely deform. This will lead to the non-uniformity of the thickness and elastic modulus of aluminum honeycomb. Therefore, it is essential that small pieces of aluminum honeycomb are attached together using a special technique to construct a smooth curved surface. In this method, the basic properties of the honeycomb material will be preserved. Another method of construction of these shells will be to form the honeycomb cells directly on the spherical surface so the cell walls are in the radial direction. Hexel Products Inc. does this kind of specialized job. But the estimated cost to manufacture sandwich shells was found to be in the vicinity of $45,000. This cost is excessive, and it was the primary reason for dropping the experimental investigation.

As an alternative, it was decided to construct and test circular sandwich plates for which the theoretical results have been obtained and presented in this work. The author designed and fabricated a structure with the help of machine shop of the Department of Mechanical Engineering of the University of Ottawa. This structure was quite strong, heavy and rigid as compared to the vibrating plate to produce clamped boundary condition of the circular plate. A specimen circular sandwich plate was also constructed for the test. Unfortunately, this experiment was also not carried out due to the failure of the shaker table which remained out of operation until after completion of this work.

CHAPTER 7

CONCLUSIONS

In this thesis, the constitutive equations for free non-symmetric vibrations of deep spherical sandwich shells have been derived using a variational method. The differential equations of motion and the stress-resultants and displacements relationships are quite general in nature. They include thickness-shear deformation, rotary inertia about the middle planes of face sheets as well as about the middle plane of the sandwich structure, extensional and flexural rigidities of the face sheets. Special cases for axisymmetric motion of (i) spherical sandwich shells with face sheets as flexural members, (ii) spherical sandwich shells with membrane face sheeets, (iii) homogeneous spherical shells and (iv) circular sandwich plates have been derived in Section 2.3 of this thesis. With the help of eqs. (2.18) and (2.20), governing equations for free axisymmetric and non-symmetric vibrations of shallow spherical sandwich and homogeneous shells can also be obtained.

The introduction of new deformation functions, $\overset{*}{H}$, $\overline{H}$, $\overset{*}{\chi}$, $\overline{\chi}$, $\overset{*}{\Gamma}$ and $\overline{\Gamma}$ in eq. (3.2) considerably simplifies the solution of the differential equations of motion. These deformation functions also reduce the complexity of the form of the solutions of equations for shallow and deep homogeneous spherical shells which have been studied using different approaches by many investigators in the past.

In the past, the computation of the numerical values of Legendre functions $P_\nu(\cos\phi)$ of non-integral or complex orders ν and their derivatives $d\, P_\nu(\cos\phi)/d\phi$ posed serious problems to the engineers working in the shell dynamics area. In this work, the subroutines CMIRLF, CONFN and AVS have been developed by the author to compute the values of Legendre functions $P_\nu(\cos\phi)$ for real and complex ν.

The flexural rigidity of the face sheets about their own middle planes has been found to have substantial influence on the natural frequencies of the sandwich spherical shells. However, if the face sheets are assumed as membranes, which eventually reduces the order of the differential equation, the error introduced is only 6%, for the lowest natural frequency calculations and also for the thinner face sheets with h/R ratio less than 0.0025. A designer who can allow an error of this order can simplify these equations without serious discrepancies. The percentage error, however, increases for the higher modes and then such flexural rigidity effects should be included.

The effects of rotary inertia about the middle planes of the face sheets are negligible on the values of the natural frequencies of the spherical sandwich shells. For homogeneous spherical shells also, the difference between the values of natural frequencies with and without rotary inertia does not exceed 0.02%.

The numerical results for the non-symmetric vibration of deep spherical sandwich shells have not been presented in this thesis. Further, numerical computations and extension of this will provide interesting results. Spherical sandwich shells with central holes and completely closed shells are the logical extension of this problem and based on the present study work can be extended in these directions. Experimental study of the natural frequencies and mode shapes of spherical sandwich shells and circular sandwich plates is, in the opinion of the author, an important part and an attempt should be made to correlate the results presented in this thesis with the values obtained from experimentation.

Free axisymmetric and non-symmetric vibration of shallow, deep spherical sandwich and homogeneous shells on elastic supports could be considered for possible future study.

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APPENDIX

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C     -------------------------------------------------------------------
C
C     CALCULATION OF NATURAL FREQUENCIES AND MODE SHAPES OF THE
C     FREE VIBRATION OF DEEP SPHERICAL SANDWICH SHELLS WITH
C     FACE SHEETS AS FLEXURAL MEMBERS.
C
C     -------------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-S)
      IMPLICIT COMPLEX*16(T-Z)
      DIMENSION A(4),Z(4),U(4),U1(4),U2(4),U3(4),T1(4),T2(4),T3(4),
     1TD1(4),TD2(4),TD3(4),B(4,4),VV(200,4),PP(4,4),AXX(6),
     2 TS2(4),TS3(4),TS4(4),PS(4,4),
     3BM(4,4),DP1(4,4),DP2(4,4),DP3(4,4),DP4(4,4)
      FC=1.00000
      PI=4.0*DATAN(FC)
      FI=60.0
      FI0=FI*PI/180.0
      S4=(1.-2.*RP*RH*RH)*HH/6.0
      CT=DCOTAN(FI0)
      AV=0.30
      AP=1.0+AV
      AM=1.0-AV
      CR=0.010
      RP=1.0/28.0
      RG=1.0/60.0
      HR=0.0150
      DO 40 IK=1,2
      WRITE(3,105)CR,HR,FI
      RH=CR/HR
      SR=CR+0.50*HR
      SS=1.0+SR*SR
      HH=HR*HR
      CC=CR*CR
      CA=RP*RH*(1.0+0.60*CC)
      S1=SS+HH/12.+RP*RH*(1.+CC/3.)
      S2=SS+HH/12.0+CA/3.
      S3=2.*(SR+RP*RH*CR/3.)
      S5=(HH*SR-CA*HR)/6.
      S6=(SS+0.150*HH+CA)*HH/12.
      R1=1.0+RG/(15.0*RG+20.0*RH)
      R2=1.0+RG/(15.0*RG+40.0*RH)
      R3=2.0*RH*RG/(15.0*RG+20.0*RH)
      R4=R3*(15.*RG+24.*RH)/2.
      R5=10.0*(R1-R3/RH+R4*0.25/(RH*RH)+R2*SR*SR)
C     READ(1,120)(AXX(II),II=1,2)
C     DO 35 IJ=1,2
      OMEGA=0.010
      DO 30 J=1,100
      P=OMEGA*OMEGA
      B1=-(17.0+7.0*AV)/(5.0*AM)
      B2=(R3+R4)/CR
      B3=R1+R3-(R3+R4)*0.50/RH
      B4=R1+2.0*R3+R4
```

```
      B5=(P*S1-2.0/AM)*12.0*AP/5.0
      B6=10.0*B3+24.0*AP*P*S4
      B7=24.0*AP*P*S5-10.0*((R3-R4*0.50/RH)/CR-R2*SR)
      B8=2.*HH+24.*AP*P*S6-R5
      B9=-10.0*B3
      A1=2.0/AM
      A2=2.+2.*P*S1*AP-5.*B4/6.
      A3=2.0*P*S3*AP+B2*5./6.0
      A4=2.*P*S4*AP+B3*5./6.0
      A5=2.*AP/AM+B4*5./6.
      A6=A3
      A7=2.+2.*P*S2*AP-(R2+R4/CC)*5./6.
      A8=2.0*P*S5*AP+(R2*SR-(R3-R4*0.50/RH)/CR)*5.0/6.0
      A9=-B2*5./6.0
      AA1=HH*A1*A1*B4
      AA2=A1*(B3*B9-B4*B8-HH*(A1*B5+A7*B4-A9*B2))
      AA3=A1*(B5*B8+HH*A7*B5)+A7*(B4*B8-B3*B9)+A8*(B2*B9-B4*B7)+
     1A9*(B3*B7-B2*B8)
      AA4=A8*B7*B5-A7*B8*B5
      AB1=A1*HH*(A9*B1-A6*B4)
      AB2=A6*(A1*B5*HH+B4*B8-B3*B9)+A8*(B1*B9-B4*B6)+A9*(B3*B6-B1*B8)
      AB3=A8*B5*B6-A6*B5*B8
      AC1=A1*(B4*B6-B1*B9)
      AC2=A6*(B4*B7-B2*B9)+A7*(B1*B9-B4*B6)+A9*(B2*B6-B1*B7)-A1*B5*B6
      AC3=A7*B5*B6-A6*B5*B7
      AD1=-HH*A1*A1*B1
      AD2=A1*(HH*(A7*B1-A6*B2)+B1*B8-B3*B6)
      AD3=A6*(B2*B6-B3*B7)+A7*(B3*B6-B1*B8)+A8*(B1*B7-B2*B6)
      A(1)=(A1*AA2-A2*AA1-A5*AD1)/(A1*AA1)
      A(2)=(A1*AA3-A2*AA2+A3*AB1-A4*AC1-A5*AD2)/(A1*AA1)
      A(3)=(A1*AA4-A2*AA3+A3*AB2-A4*AC2-A5*AD3)/(A1*AA1)
      A(4)=(A3*AB3-A2*AA4-A4*AC3)/(A1*AA1)
C
C     SUBROUTINE QUATIC CALCULATES THE ROOTS OF A 4TH ORDER
C     POLYNOMIAL.
C
      CALL QUATIC(A,Z)
C
      V1=CDSQRT(Z(1)+0.250)-0.500
      V2=CDSQRT(Z(2)+0.250)-0.500
      V3=CDSQRT(Z(3)+0.250)-0.500
      V4=CDSQRT(Z(4)+0.250)-0.500
C
C
   10 AH=AIMAG(Z(3))
      U(1)=A1*A1*B3*Z(1)*Z(1)+A1*(A8*B2+A4*B1-A7*B3-A2*B3)*Z(1)+
     1A2*(A7*B3-A8*B2)+A3*(A8*B1-A6*B3)+A4*(A6*B2-A7*B1)
      U(2)=A1*A1*B3*Z(2)*Z(2)+A1*(A8*B2+A4*B1-A7*B3-A2*B3)*Z(2)+
     1A2*(A7*B3-A8*B2)+A3*(A8*B1-A6*B3)+A4*(A6*B2-A7*B1)
      U(3)=A1*A1*B3*Z(3)*Z(3)+A1*(A8*B2+A4*B1-A7*B3-A2*B3)*Z(3)+
     1A2*(A7*B3-A8*B2)+A3*(A8*B1-A6*B3)+A4*(A6*B2-A7*B1)
      U(4)=A1*A1*B3*Z(4)*Z(4)+A1*(A8*B2+A4*B1-A7*B3-A2*B3)*Z(4)+
     1A2*(A7*B3-A8*B2)+A3*(A8*B1-A6*B3)+A4*(A6*B2-A7*B1)
```

```
 U1(1)=A1*(A4*B4-A5*B3)*Z(1)*Z(1)+(A5*(A7*B3-A8*B2)+A3*(A8*B4-
1A9*B3)+A4*(A9*B2-A7*B4-A1*B5))*Z(1)+B5*(A4*A7-A3*A8)
 U1(2)=A1*(A4*B4-A5*B3)*Z(2)*Z(2)+(A5*(A7*B3-A8*B2)+A3*(A8*B4-
1A9*B3)+A4*(A9*B2-A7*B4-A1*B5))*Z(2)+B5*(A4*A7-A3*A8)
 U1(3)=A1*(A4*B4-A5*B3)*Z(3)*Z(3)+(A5*(A7*B3-A8*B2)+A3*(A8*B4-
1A9*B3)+A4*(A9*B2-A7*B4-A1*B5))*Z(3)+B5*(A4*A7-A3*A8)
 U1(4)=A1*(A4*B4-A5*B3)*Z(4)*Z(4)+(A5*(A7*B3-A8*B2)+A3*(A8*B4-
1A9*B3)+A4*(A9*B2-A7*B4-A1*B5))*Z(4)+B5*(A4*A7-A3*A8)
 U2(1)=A1*(A8*B4-A9*B3)*Z(1)*Z(1)+(A2*(A9*B3-A8*B4)-A1*A8*B5+
1A5*(A8*B1-A6*B3)+A4*(A6*B4-A9*B1))*Z(1)+(A2*A8-A4*A6)*B5
 U2(2)=A1*(A8*B4-A9*B3)*Z(2)*Z(2)+(A2*(A9*B3-A8*B4)-A1*A8*B5+
1A5*(A8*B1-A6*B3)+A4*(A6*B4-A9*B1))*Z(2)+(A2*A8-A4*A6)*B5
 U2(3)=A1*(A8*B4-A9*B3)*Z(3)*Z(3)+(A2*(A9*B3-A8*B4)-A1*A8*B5+
1A5*(A8*B1-A6*B3)+A4*(A6*B4-A9*B1))*Z(3)+(A2*A8-A4*A6)*B5
 U2(4)=A1*(A8*B4-A9*B3)*Z(4)*Z(4)+(A2*(A9*B3-A8*B4)-A1*A8*B5+
1A5*(A8*B1-A6*B3)+A4*(A6*B4-A9*B1))*Z(4)+(A2*A8-A4*A6)*B5
 U3(1)=A1*A1*B4*(Z(1)**3)+A1*(A9*B2-A1*B5-A7*B4-A2*B4+A5*B1)*Z(1)
1*Z(1)+(A2*(A1*B5-A9*B2+A7*B4)+A1*A7*B5+A3*(A9*B1-A6*B4)+A5*(B2*A6
2-B1*A7))*Z(1)+B5*(A3*A6-A2*A7)
 U3(2)=A1*A1*B4*(Z(2)**3)+A1*(A9*B2-A1*B5-A7*B4-A2*B4+A5*B1)*Z(2)
1*Z(2)+(A2*(A1*B5-A9*B2+A7*B4)+A1*A7*B5+A3*(A9*B1-A6*B4)+A5*(B2*A6
2-B1*A7))*Z(2)+B5*(A3*A6-A2*A7)
 U3(3)=A1*A1*B4*(Z(3)**3)+A1*(A9*B2-A1*B5-A7*B4-A2*B4+A5*B1)*Z(3)
1*Z(3)+(A2*(A1*B5-A9*B2+A7*B4)+A1*A7*B5+A3*(A9*B1-A6*B4)+A5*(B2*A6
2-B1*A7))*Z(3)+B5*(A3*A6-A2*A7)
 U3(4)=A1*A1*B4*(Z(4)**3)+A1*(A9*B2-A1*B5-A7*B4-A2*B4+A5*B1)*Z(4)
1*Z(4)+(A2*(A1*B5-A9*B2+A7*B4)+A1*A7*B5+A3*(A9*B1-A6*B4)+A5*(B2*A6
2-B1*A7))*Z(4)+B5*(A3*A6-A2*A7)
 T1(1)=-(A5+A1*U1(1)/U(1))
 T1(2)=-(A5+A1*U1(2)/U(2))
 T1(3)=-(A5+A1*U1(3)/U(3))
 T1(4)=-(A5+A1*U1(4)/U(4))
 T2(1)=-(A9+A1*U2(1)/U(1))
 T2(2)=-(A9+A1*U2(2)/U(2))
 T2(3)=-(A9+A1*U2(3)/U(3))
 T2(4)=-(A9+A1*U2(4)/U(4))
 T3(1)=-(B9+HH*A1*U3(1)/U(1))
 T3(2)=-(B9+HH*A1*U3(2)/U(2))
 T3(3)=-(B9+HH*A1*U3(3)/U(3))
 T3(4)=-(B9+HH*A1*U3(4)/U(4))
 D1=A7*B8-A8*B7
 D2=A8*B6-A6*B8
 D3=A6*B7-A7*B6
 D4=A4*B7-A3*B8
 D5=A3*A8-A4*A7
 D6=A2*B8-A4*B6
 D7=A4*A6-A2*A8
 D8=A3*B6-A2*B7
 D9=A2*A7-A3*A6
 DD=A2*D1+A3*D2+A4*D3
 TD1(1)=T1(1)*D1+T2(1)*D4+T3(1)*D5
 TD1(2)=T1(2)*D1+T2(2)*D4+T3(2)*D5
 TD1(3)=T1(3)*D1+T2(3)*D4+T3(3)*D5
```

```
      TD1(4)=T1(4)*D1+T2(4)*D4+T3(4)*D5
      TD2(1)=T1(1)*D2+T2(1)*D6+T3(1)*D7
      TD2(2)=T1(2)*D2+T2(2)*D6+T3(2)*D7
      TD2(3)=T1(3)*D2+T2(3)*D6+T3(3)*D7
      TD2(4)=T1(4)*D2+T2(4)*D6+T3(4)*D7
      TD3(1)=T1(1)*D3+T2(1)*D8+T3(1)*D9
      TD3(2)=T1(2)*D3+T2(2)*D8+T3(2)*D9
      TD3(3)=T1(3)*D3+T2(3)*D8+T3(3)*D9
      TD3(4)=T1(4)*D3+T2(4)*D8+T3(4)*D9
C
C         ROOT Z(1) HAS BEEN FOUND TO BE A VERY LARGE NEGATIVE
C         QUANTITY FOR THE GIVEN DATA. THE RATIO ZZ1=PV1/PPV1
C         IS CALCULATED FROM ASYMTOTIC EXPANSION OF THE LEGE-
C         NDRE FUNCTION.
C         REFERENCE: SCHELKUNOFF S. A., APPLIED MATHEMATICS
C         FOR ENGINEERS AND SCIENTISTS, D. VAN NOSTRAND CO.
C         INC., NEW YORK, 1965, EQ. 12 PAGE. 419.
C
C     P1=AIMAG(V1)
C     A0=0.00
C     PVZ=1.0/(P1-CT*0.500)
C     ZZ1=DCMPLX(PVZ,A0)
C
      CALL SETUP(FI0,V1,ZPV1,ZPPV1,ZZ1)
      CALL SETUP(FI0,V2,ZPV2,ZPPV2,ZZ2)
      BM11=REAL(ZPPV1)
      BM12=REAL(ZPPV2)
      BM(1,1)=1.0/BM11
      BM(1,2)=1.0/BM12
      B(1,1)=REAL(ZZ1)
      B(1,2)=REAL(ZZ2)
      B(2,1)=REAL(TD1(1))/DD
      B(2,2)=REAL(TD1(2))/DD
      B(3,1)=REAL(TD2(1))/DD
      B(3,2)=REAL(TD2(2))/DD
      B(4,1)=REAL(TD3(1))/DD
      B(4,2)=REAL(TD3(2))/DD
      TS2(1)=(AP+U1(1)/U(1))*ZZ1-AM*CT*TD1(1)/DD
      TS2(2)=(AP+U1(2)/U(2))*ZZ2-AM*CT*TD1(2)/DD
      TS3(1)=(U2(1)/U(1))*ZZ1-AM*CT*TD2(1)/DD
      TS3(2)=(U2(2)/U(2))*ZZ2-AM*CT*TD2(2)/DD
      TS4(1)=(U3(1)/U(1))*ZZ1-AM*CT*TD3(1)/DD
      TS4(2)=(U3(2)/U(2))*ZZ2-AM*CT*TD3(2)/DD
      PS(1,1)=B(1,1)
      PS(1,2)=B(1,2)
      PS(2,1)=REAL(TS2(1))
      PS(2,2)=REAL(TS2(2))
      PS(3,1)=REAL(TS3(1))
      PS(3,2)=REAL(TS3(2))
      PS(4,1)=REAL(TS4(1))
      PS(4,2)=REAL(TS4(2))
      PP(4,1)=PS(4,1)
      PP(4,2)=PS(4,2)
```

```
      IF(AH.NE.0.00)GO TO 15
      CALL SETUP(FIO,V3,ZPV3,ZPPV3,ZZ3)
      CALL SETUP(FIO,V4,ZPV4,ZPPV4,ZZ4)
      BM13=REAL(ZPPV3)
      BM14=REAL(ZPPV4)
      BM(1,3)=1.0
      BM(1,4)=1.0
      B23=REAL(TD1(3))/DD
      B24=REAL(TD1(4))/DD
      B33=REAL(TD2(3))/DD
      B34=REAL(TD2(4))/DD
      B43=REAL(TD3(3))/DD
      B44=REAL(TD3(4))/DD
      B(1,3)=REAL(ZPV3)
      B(1,4)=REAL(ZPV4)
      B(2,3)=B23*BM13
      B(2,4)=B24*BM14
      B(3,3)=B33*BM13
      B(3,4)=B34*BM14
      B(4,3)=B43*BM13
      B(4,4)=B44*BM14
      TS2(3)=(AP+U1(3)/U(3))*ZPV3-AM*CT*TD3(3)*ZPPV3/DD
      TS2(4)=(AP+U1(4)/U(4))*ZPV4-AM*CT*TD3(4)*ZPPV4/DD
      TS3(3)=(U2(3)/U(3))*ZPV3-AM*CT*TD2(3)*ZPPV3/DD
      TS3(4)=(U2(4)/U(4))*ZPV4-AM*CT*TD2(4)*ZPPV4/DD
      TS4(3)=(U3(3)/U(3))*ZPV3-AM*CT*TD3(3)*ZPPV3/DD
      TS4(4)=(U3(4)/U(4))*ZPV4-AM*CT*TD3(4)*ZPPV4/DD
      PS(1,3)=B(1,3)
      PS(1,4)=B(1,4)
      PS(2,3)=REAL(TS2(3))
      PS(2,4)=REAL(TS2(4))
      PS(3,3)=REAL(TS3(3))
      PS(3,4)=REAL(TS3(4))
      PS(4,3)=REAL(TS4(3))
      PS(4,4)=REAL(TS4(4))
      PP(4,3)=PS(4,3)
      PP(4,4)=PS(4,4)
      GO TO 20
C     GO TO 70
   15 CALL SETUP(FIO,V3,ZPV3,ZPPV3,ZZ3)
      BM(1,3)=2.0
      BM(1,4)=0.0
      TP1=TD1(3)*ZPPV3/DD
      TP2=TD2(3)*ZPPV3/DD
      TP3=TD3(3)*ZPPV3/DD
      B(1,3)=2.0*REAL(ZPV3)
      B(1,4)=-2.0*AIMAG(ZPV3)
      B(2,3)=-2.0*REAL(TP1)
      B(2,4)=-2.0*AIMAG(TP1)
      B(3,3)=2.0*REAL(TP2)
      B(3,4)=-2.0*AIMAG(TP2)
      B(4,3)=2.0*REAL(TP3)
      B(4,4)=-2.0*AIMAG(TP3)
```

```
      TS2(3)=(AP+U1(3)/U(3))*ZPV3-AM*CT*TD3(3)*ZPPV3/DD
      TS3(3)=(U2(3)/U(3))*ZPV3-AM*CT*TD2(3)*ZPPV3/DD
      TS4(3)=(U3(3)/U(3))*ZPV3-AM*CT*TD3(3)*ZPPV3/DD
      PS(1,3)=B(1,3)
      PS(1,4)=B(1,4)
      PS(2,3)=2.0*REAL(TS2(3))
      PS(2,4)=-2.0*AIMAG(TS2(3))
      PS(3,3)=2.0*REAL(TS3(3))
      PS(3,4)=-2.0*AIMAG(TS3(3))
      PS(4,3)=2.0*REAL(TS4(3))
      PS(4,4)=-2.0*AIMAG(TS4(3))
      PP(4,3)=PS(4,3)
      PP(4,4)=PS(4,4)
C     GO TO 70
   20 CALL DETERM(B,4,DET)
      DO 25 IM=1,3
      DO 25 JM=1,4
   25 PP(IM,JM)=B(IM,JM)
      CALL DETERM(PP,4,DET1)
      CALL DETERM(PS,4,DET2)
C
C        DET  = FREQUENCY EQUATION FOR FIXED EDGE CONDITION.
C        DET1 = FREQUENCY EQUATION FOR PINNED EDGE CONDITION.
C        DET2 = FREQUENCY EQUATION FOR SIMPLY SUPPORTED EDGE
C        CONDITION.
C
      WRITE(3,110)J,OMEGA,DET,DET1,DET2
   30 OMEGA=OMEGA+0.010
   40 HR=HR+0.0050
   35 CONTINUE
   45 FI=FI+30.0
   50 CR=CR+0.010
      GO TO 125
C
C        MODE SHAPE COMPUTATION FOLLOWS.
C
   70 DO 75 IA=2,3
      DO 75 IB=1,4
      IC=IA-1
   75 BM(IA,IB)=B(IC,IB)
      DO 76 KA=1,4
   76 BM(4,KA)=B(4,KA)
      DO 80 JA=1,4
      DO 80 JB=1,4
      DP1(JA,JB)=BM(JA,JB)
      DP2(JA,JB)=BM(JA,JB)
      DP3(JA,JB)=BM(JA,JB)
   80 DP4(JA,JB)=BM(JA,JB)
      DO 85 JC=2,4
      DP1(JC,1)=0.0
      DP2(JC,2)=0.0
      DP3(JC,3)=0.0
   85 DP4(JC,4)=0.0
```

```
      DP1(1,1)=1.0
      DP2(1,2)=1.0
      DP3(1,3)=1.0
      DP4(1,4)=1.0
      CALL DETERM(BM,4,BMD)
      CALL DETERM(DP1,4,DPD1)
      CALL DETERM(DP2,4,DPD2)
      CALL DETERM(DP3,4,DPD3)
      CALL DETERM(DP4,4,DPD4)
      AK1=DPD1/(BMD*BM11)
      AK2=DPD2/(BMD*BM12)
      AK3=DPD3/BMD
      AK4=DPD4/BMD
      ALFA=5.0
      DO 97 KK=1,12
      FIA=ALFA*PI/180.0
      CALL SETUP(FIA,V1,TPV1,TPPV1,TZ1)
      CALL SETUP(FIA,V2,TPV2,TPPV2,TZ2)
      APV1=REAL(TPV1)
      APV2=REAL(TPV2)
      IF(AH.NE.0.0)GO TO 90
      CALL SETUP(FIA,V3,TPV3,TPPV3,TZ3)
      CALL SETUP(FIA,V4,TPV4,TPPV4,TZ4)
      APV3=REAL(TPV3)
      APV4=REAL(TPV4)
      GO TO 95
   90 CALL SETUP(FIA,V3,TPV3,TPPV3,TZ3)
      APV3=2.0*REAL(TPV3)
      APV4=-2.0*AIMAG(TPV3)
   95 AW=AK1*APV1+AK2*APV2+AK3*APV3+AK4*APV4
      WRITE(3,110)KK,ALFA,AW
   97 ALFA=ALFA+5.0
  104 CONTINUE
  105 FORMAT(1H1,//,5X,'ROOTS OF THE FREQUENCY EQUATION',//,
     15X,'C/R=',F5.3,2X,'H/R=',F6.4,2X,'FI=',F5.1,//)
  110 FORMAT(2X,I3,2X,F8.4,2X,8D14.6)
  115 FORMAT(1H1,//,5X,'ORDERS OF THE LEGENDRE FUNCTION',//,
     15X,'C/R=',F5.3,2X,'H/R=',F6.4,//)
  120 FORMAT(6F10.3)
  125 RETURN
      END
```

```
C      -----------------------------------------------------------------------
C
C      CALCULATION OF NATURAL FREQUENCIES AND MODE SHAPES OF THE
C      FREE VIBRATION OF DEEP SPHERICAL SANDWICH SHELLS WITH
C      FACE SHEETS AS MEMBRANES.
C
C      -----------------------------------------------------------------------
       IMPLICIT REAL*8(A-D,Q-S,V-Z)
       IMPLICIT COMPLEX*16(E-H,O-P,T-U)
       DIMENSION A(4),AQ(3),XR(3),AXY(6)
       CD=0.100000D 01
       API=4.0*DATAN(CD)
       B=90.0*API/180.0
       CR=0.020
       AX1=1.0
       AX2=26.70
       AX3=4368.0
       AX4=5.0/6.0
       V=0.30
       AM=1.0-V
       AP=1.0+V
       AV=1.0-V*V
       RP=AX1/AX2
       RG=AX1/AX3
       X=0.00250
       DO 90 I=1,4
       XX=X*X
       RH=CR/X
       WRITE(3,7)KI,CR,X
     7 FORMAT(1H1,////,5X,I2,2X,'C/R=',F4.2,2X,'H/R=',F6.4,//)
       Y=0.010
       DO 100 J=1,100
       YY=Y*Y
       DR=CR+0.50*X
       DS=DR*DR
       S1=1.0+DS+RP*RH*(1.0+CR*CR/3.0)
       S2=1.0+DS+RP*RH*(1.0+0.60*CR*CR)/3.0
       S3=2.0*(DR+RP*RH*CR/3.0),
       A1=AV*RG*RH
       A2=YY*S1*AV-2.0*AP
       A3=-(AP+RG*RH*AV)
       A4=RG*AV/X
       B1=-A3
       B2=AM+(YY*S1-RG*RH)*AV
       B3=(YY*S3+RG/X)*AV
       C1=-A4
       C2=B3
       C3=AM+(YY*S2-RG/(X*CR))*AV
       A(1)=A1
       A(2)=-A2+A3*B1+A4*C1-A1*B2-A1*C3
       A(3)=A1*B2*C3-A1*B3*C2+A2*B2+A2*C3+A3*B3*C1-
      1A3*B1*C3+A4*B1*C2-A4*B2*C1
       A(4)=A2*B3*C2-A2*B2*C3
```

```
      IF(A(1).EQ.0.0)GO TO 10
C
C
      CALL CUBIC(A,XR,XI)
C
C
      GO TO 20
   10 AQ(1)=A(2)
      AQ(2)=A(3)
      AQ(3)=A(4)
      XR(1)=0.0
      CALL QUAD(AQ,XR(2),XR(3),XI)
   20 Z1=0.0
      Z=-XI
      T1=DCMPLX(XR(1),Z1)
      T2=DCMPLX(XR(2),XI)
      T3=DCMPLX(XR(3),Z)
      U1=CDSQRT(T1+0.25)-0.50
      U2=CDSQRT(T2+0.25)-0.50
      U3=CDSQRT(T3+0.25)-0.50
C
C     WRITE(3,65)Y,U1,U2,U3
C     GO TO 100
C
   24 E1=(B1*T1+B3*C1-B1*C3)/(T1*T1-(B2+C3)*T1+B2*C3-B3*C2)
      E2=(B1*T2+B3*C1-B1*C3)/(T2*T2-(B2+C3)*T2+B2*C3-B3*C2)
      E3=(B1*T3+B3*C1-B1*C3)/(T3*T3-(B2+C3)*T3+B2*C3-B3*C2)
      F1=(C1*T1+B1*C2-B2*C1)/(T1*T1-(B2+C3)*T1+B2*C3-B3*C2)
      F2=(C1*T2+B1*C2-B2*C1)/(T2*T2-(B2+C3)*T2+B2*C3-B3*C2)
      F3=(C1*T3+B1*C2-B2*C1)/(T3*T3-(B2+C3)*T3+B2*C3-B3*C2)
      Y1=REAL(U1)
      Y2=REAL(U2)
      Y3=REAL(U3)
      IF(Y1.GT.10.0)GO TO 25
      CALL PVPPV(B,U1,PV1,PPV1)
      GO TO 30
   25 CALL AVS(B,Y1,SV1,SPV1)
      PV1=DCMPLX(SV1,Z1)
      PPV1=DCMPLX(SPV1,Z1)
   30 IF(Y2.EQ.-0.500)GO TO 35
      CALL PVPPV(B,U2,PV2,PPV2)
      GO TO 40
   35 DD2=AIMAG(U2)
      CALL CONFN(B,DD2,DPV2,DPPV2)
      PV2=DCMPLX(DPV2,Z1)
      PPV2=DCMPLX(DPPV2,Z1)
   40 IF(Y3.EQ.-0.500)GO TO 45
      CALL PVPPV(B,U3,PV3,PPV3)
      GO TO 50
   45 DD3=AIMAG(U3)
      CALL CONFN(B,DD3,DPV3,DPPV3)
      PV3=DCMPLX(DPV3,Z1)
      PPV3=DCMPLX(DPPV3,Z1)
```

```
 50 T11=PV1
    T12=PV2
    T13=PV3
    T21=E1*PPV1
    T22=E2*PPV2
    T23=E3*PPV3
    T31=F1*PPV1
    T32=F2*PPV2
    T33=F3*PPV3
    IF(XI.NE.0.0)GO TO 55
    TC2=(T13*T21-T11*T23)/(T12*T23-T22*T13)
    TC3=(T11*T22-T12*T21)/(T12*T23-T22*T13)
    TDET T11 *(T22*T33-T32*T23)+T12*(T31*T23-T21*T33)+T13*(T21*T32-
   1T31*T22)
    GO TO 60
C   GO TO 120
 55 D12=2.0*REAL(T12)
    D13=-2.0*AIMAG(T12)
    D22=2.0*REAL(T22)
    D23=-2.0*AIMAG(T22)
    D32=2.0*REAL(T32)
    D33=-2.0*AIMAG(T32)
C   GO TO 115
 56 CONTINUE
    TDET=T11*(D22*D33-D32*D23)+D12*(T31*D23-T21*D33)+D13*(T21*D32-
   2T31*D22)
 60 WRITE(3,65)Y,TDET
 65 FORMAT(10X,F7.4,2X,8(D12.5,2X))
100 Y=Y+0.010
 90 X=X+0.00250
    GO TO 400
C
C        MODE SHAPE COMPUTATION FOLLOWS.
C
115 CONTINUE
    AA2=(D13*T21-T11*D23)/(D12*D23-D22*D13)
    AA3=(T11*D22-D12*T21)/(D12*D23-D22*D13)
    AA4=-AA3
    TC2=DCMPLX(AA2,AA3)
    TC3=DCMPLX(AA2,AA4)
120 TT=1.0+TC2+TC3
    ALFA=5.0
    DO 170 KK=1,18
    BB=ALFA*API/180.0
    IF(Y1.GT.10.0)GO TO 125
    CALL PVPPV(BB,U1,PVF1,PPVF1)
    GO TO 130
125 CALL AVS(BB,Y1,SVF1,SPVF1)
    PVF1=DCMPLX(SVF1,Z1)
    PPVF1=DCMPLX(SPVF1,Z1)
130 IF(Y2.EQ.-0.500)GO TO 135
    CALL PVPPV(BB,U2,PVF2,PPVF2)
    GO TO 140
135 DDF2=AIMAG(U2)
```

```
      CALL CONFN(BB,DDF2,DPVF2,DPPVF2)
      PVF2=DCMPLX(DPVF2,Z1)
      PPVF2=DCMPLX(DPPVF2,Z1)
  140 IF(Y3.EQ.-0.500)GO TO 145
      CALL PVPPV(BB,U3,PVF3,PPVF3)
      GO TO 150
  145 DDF3=AIMAG(U3)
      CALL CONFN(BB,DDF3,DPVF3,DPPVF3)
      PVF3=DCMPLX(DPVF3,Z1)
      PPVF3=DCMPLX(DPPVF3,Z1)
  150 TW1=PVF1+TC2*PVF2+TC3*PVF3
      TVS=E1*PPVF1+E2*TC2*PPVF2+E3*TC3*PPVF3
      TVB=F1*PPVF1+F2*TC2*PPVF2+F3*TC3*PPVF3
      TW11=TW1/TT
      TVSS=TVS/TT
      TVBB=TVB/TT
      WRITE(3,160)ALFA,TW11,TVSS,TVBB
  160 FORMAT(15X,F5.1,2X,6(D15.6,2X))
  170 ALFA=ALFA+5.0
  180 CONTINUE
  400 RETURN
      END
```

```
C     ------------------------------------------------------------------
C
C     CALCULATION OF NATURAL FREQUENCIES OF THE FREE VIBRATION
C     OF A CLAMPED CIRCULAR SANDWICH PLATE.
C
C     ------------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(4),B(3),XR(3),AXZ(7)
      V=0.30
      AM1=1.0+V
      AM2=1.0-V
      AM3=1.0-V*V
      RP=1.0/28.0
      RG=1.0/60.0
      CA=0.0050
      DO 30 K=1,3
      HA=0.005
      DO 20 I=1,5
      HS=HA*HA
      RH=CA/HA
      S1=1.0+RP*RH
      S2=1.0+RP*RH/3.0
      S3=-RP*CA/6.0
      RD=15.0*RG+20.0*RH
      R1=1.0+RG/RD
      R3=2.0*RG*RH/RD
      R4=R3*(15.*RG+24.*RH)/2.
      DN=R1+2.0*R3+R4
      WRITE(3,60)CA,HA
      W=0.10
      DO 10 J=1,300
      WS=W*W
      A1=12.0*AM1*S1*WS*HS/(5.0*DN)
      A2=(R3+R4)/(CA*DN)
      A3=(R1+R3-(R3+R4)*0.50/RH)/DN
      A4=-5.0*AM2*(R1+R3-(R3+R4)*0.50/RH)/HS
      A5=-5.0*AM2*(R3-R4*0.50/RH)/(HS*CA)+12.0*S3*AM3*WS
      A6=-5.0*AM2*((R1-R3/RH+R4*0.250/(RH*RH))/HS)+S1*AM3*WS*HS
      A7=-5.0*AM2*(R3+R4)/(CA*12.0)
      A8=-5.0*AM2*R4/(CA*CA*12.0)+S2*AM3*WS*HS
      A9=-5.0*AM2*(R3-R4*0.50/RH)/(CA*12.0)+S3*AM3*WS*HS
      A(1)=1.0
      A(2)=A2*A7+A3*A4-A1-A6-A8
      A(3)=A1*(A6+A8)+A6*A8-A5*A9+A2*(A4*A9-A6*A7)+A3*(A5*A7-A4*A8)
      A(4)=A1*(A5*A9-A6*A8)
C
      CALL CUBIC(A,XR,XI)
C
C     WRITE(3,65)W,XR(1),XR(2),XR(3),XI
C     GO TO 10
C
    4 X1=DSQRT(XR(1))
      X2=DSQRT(-XR(2))
      X3=DSQRT(-XR(3))
```

```
      D1=-A3*XR(1)+A3*A8-A2*A9
      D2=-A3*XR(2)+A3*A8-A2*A9
      D3=-A3*XR(3)+A3*A8-A2*A9
      E1=(-XR(1)*XR(1)+(A1+A8-A2*A7)*XR(1)-A1*A8)/D1
      E2=(-XR(2)*XR(2)+(A1+A8-A2*A7)*XR(2)-A1*A8)/D2
      E3=(-XR(3)*XR(3)+(A1+A8-A2*A7)*XR(3)-A1*A8)/D3
      F1=((A3*A7-A9)*XR(1)+A1*A9)/D1
      F2=((A3*A7-A9)*XR(2)+A1*A9)/D2
      F3=((A3*A7-A9)*XR(3)+A1*A9)/D3
      DR=A6*A8-A5*A9
      P1=(A5*F1-A8*E1+A5*A7-A4*A8)/DP
      P2=(A5*F2-A8*E2+A5*A7-A4*A8)/DP
      P3=(A5*F3-A8*E3+A5*A7-A4*A8)/DP
      Z1=(A9*E1-A6*F1+A4*A9-A6*A7)/DP
      Z2=(A9*E2-A6*F2+A4*A9-A6*A7)/DP
      Z3=(A9*E3-A6*F3+A4*A9-A6*A7)/DP
      CALL BESJ(X1,0,BJ0,D,I1)
      CALL BESJ(X1,1,BJ1,D,I2)
      D=0.001E-04
      CALL I0(X2,BI02)
      CALL I0(X3,BI03)
      CALL INUE(X2,1,BI02,BI12)
      CALL INUE(X3,1,BI03,BI13)
      C1=-BJ0/BJ1
      C2=BI02/BI12
      C3=BI03/BI13
      XX1=C1*X2*X3*(Z2*P3-Z3*P2)
      XX2=C2*X1*X3*(Z1*P3-Z3*P1)
      XX3=C3*X1*X2*(Z1*P2-Z2*P1)
      DET=XX1-XX2+XX3
      WRITE(3,55)W,DET
   10 W=W+0.10
   15 CONTINUE
   20 HA=HA+0.00250
   30 CA=CA+0.0050
   55 FORMAT(2X,F6.2,2X,3D14.6,2X,2(D14.6,2X))
   60 FORMAT(10X,'C/A=',F5.3,5X,'H/A=',F6.4,/)
   65 FORMAT(5X,F6.2,7(D12.5,2X))
   75 FORMAT(8F10.2)
   85 FORMAT(//,2X)
      RETURN
      END
```

```
      SUBROUTINE QUATIC(P,Y)
C     -------------------------------------------------------------
C     PROGRAM........(1)
C     SUBROUTINE QUATIC(P,Y)
C
C     PURPOSE:
C         THIS SUBROQURINE CALCULATES THE ROOTS OF A QUARTIC
C         EQUATION GIVEN AS:
C         Y**4 + P(1)*Y**3 + P(2)*Y**2 + P(3)*Y + P(4) = 0
C
C     -------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-S)
      IMPLICIT COMPLEX*16(T-Z)
      DIMENSION R(3),D(4),Y(4),P(4)
      A=-3.0*P(1)*P(1)/8.0+P(2)
      B=(P(1)**3)/8.0-P(1)*P(2)/2.0+P(3)
      A1=-3.*(P(1)**4)/256.+P(1)*P(1)*P(2)/16.0
      C=A1-P(1)*P(3)/4.0+P(4)
      D(1)=1.0
      D(2)=A/2.0
      D(3)=(A*A-4.0*C)/16.0
      D(4)=-B*B/64.0
      CALL CUBIC(D,R,RI)
      RZ=0.0
      R1=-RI
      T1=DCMPLX(R(1),RZ)
      T2=DCMPLX(R(2),RI)
      T3=DCMPLX(R(3),R1)
      U1=CDSQRT(T1)
      U2=CDSQRT(T2)
      U3=CDSQRT(T3)
      X1=-U1-U2-U3
      X2=-U1+U2+U3
      X3=U1-U2+U3
      X4=U1+U2-U3
      IF(B)10,15,15
   10 X1=-X1
      X2=-X2
      X3=-X3
      X4=-X4
   15 Y(1)=X1-P(1)/4.0
      Y(2)=X2-P(1)/4.0
      Y(3)=X3-P(1)/4.0
      Y(4)=X4-P(1)/4.0
      RETURN
      END
```

```
      SUBROUTINE CUBIC(A,XR,XI)
C     ------------------------------------------------------------
C     PROGRAM.......(2)
C     SUBROUTINE CUBIC(A,XR,XI)
C     PURPOSE:-
C          THIS SUBROUTINE CALCULATES THE ROOTS OF A CUBIC
C          EQUATION GIVEN AS:
C
C          A(1)*X**3 + A(2)*X**2 + A(3)*X + A(4) =0
C
C          XR(1)=REAL ROOT OF THE CUBIC REAL POLYNOMIAL
C          XR(2)=REAL PART OF THE FIRST COMPLEX ROOT
C          XR(3)=REAL PART OF THE  2ND COMPLEX ROOT
C          XI=IMAGINARY PART OF THE COMPLEX ROOTS
C
C        NOTE-
C          COMPLEX ROOTS ARE COMPLEX CONJUGATE OF EACH OTHER
C
C
C     ------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(4),XR(3),AQ(3)
  200 NPATH=2
      AX=1.0
      BX=3.0
      EX=AX/BX
      IF(A(4))1006,1004,1006
 1004 XR(1)=0
      GO TO 1034
 1006 A2=A(1)*A(1)
      A3=A(1)*A(2)*A(3)
      Q=(27.*A2*A(4)-9.*A3+2.*A(2)**3)/(54.*A2*A(1))
      IF(Q)1010,1008,1014
 1008 Z=0.0
      GO TO 1032
 1010 Q=-Q
      NPATH=1
 1014 P=(3.*A(1)*A(3)-A(2)*A(2))/(9.*A2)
      ARG=P*P*P+Q*Q
      IF(ARG)1016,1018,1020
 1016 Z=-2.*DSQRT(-P)*DCOS(DATAN(DSQRT(-ARG)/Q)/3.0)
      GO TO 1028
 1018 Z=-2.*Q**EX
      GO TO 1028
 1020 SARG=DSQRT(ARG)
      IF(P)1022,1024,1026
 1022 Z=-(Q+SARG)**EX-(Q-SARG)**EX
      GO TO 1028
 1024 Z=-(2.*Q)**EX
      GO TO 1028
```

```
1026 Z=(SARG-Q)**EX-(SARG+Q)**EX
1028 GO TO(1030,1032),NPATH
1030 Z=-Z
1032 XR(1)=(3.*A(1)*Z-A(2))/(3.*A(1))
1034 AQ(1)=A(1)
     AQ(2)=A(2)+XR(1)*AQ(1)
     AQ(3)=A(3)+XR(1)*AQ(2)
     CALL QUAD(AQ,XR(2),XR(3),XI)
     IF(XI)1040,1036,1040
1036 DO 1038 I=1,2
     M=I+1
     DO 1038 J=M,3
     IF(XR(I).GE.XR(J))GO TO 1038
     DUMMY=XR(I)
     XR(I)=XR(J)
     XR(J)=DUMMY
1038 CONTINUE
1040 RETURN
     END


      SUBROUTINE QUAD(A,XR1,XR2,XI)
C     -------------------------------------------------------
C     PROGRAM.......(3)
C     SUBROUTINE QUAD(A,XR1,XR2,XI)
C     PURPOSE:
C         THIS SUBROUTINE CALCULATES THE ROOTS OF A
C         QUADRATIC EQUATION GIVEN AS:
C
C         A(1)*X*X + A(2)*X + A(3) = 0
C
C     -------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(3)
  300 X1=-A(2)/(2.*A(1))
      DISC=X1*X1-A(3)/A(1)
      IF(DISC)10,20,20
   10 X2=DSQRT(-DISC)
      XR1=X1
      XR2=X1
      XI=X2
      GO TO 30
   20 X2=DSQRT(DISC)
      XR1=X1+X2
      XR2=X1-X2
      XI=0.0
   30 RETURN
      END
```

```
      SUBROUTINE SETUP(FI,V,ZPV,ZPPV,ZZ)
C     -----------------------------------------------------------------
C     PROGRAM.......(4)
C     SUBROUTINE SETUP(FI,V,ZPV,ZPPV,ZZ)
C     PURPOSE:
C         THIS SUBROUTINE CALCULATES THE RATIO OF LEGENDRE
C         FUNCTION AND ITS DERIVATIVE. IT FIRST EXAMINES THE
C         ORDER V AND THEN CALLS THE REQUIRED SUBROUTINE.
C
C     -----------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-S)
      IMPLICIT COMPLEX*16(T-Z)
      A=0.00
      B=1.0000
      PI=4.0*DATAN(B)
      R=REAL(V)
      C=AIMAG(V)
      IF(C)20,5,20
    5 IF(R.GT.10.0)GO TO 10
      CALL PVPPV(FI,V,ZPV,ZPPV)
      GO TO 30
   10 CALL AVS(FI,R,PV,PPV)
   15 ZPV=DCMPLX(PV,A)
      ZPPV=DCMPLX(PPV,A)
      GO TO 30
   20 IF(R.EQ.-0.5000)GO TO 25
      CALL PVPPV(FI,V,ZPV,ZPPV)
      GO TO 30
   25 CALL CONFN(FI,C,PV,PPV)
      GO TO 15
   30 ZZ=ZPV/ZPPV
      RETURN
      END
```

```
      SUBROUTINE PVPPV(B,V,PV,PPV)
C     -----------------------------------------------------------------
C     PROGRAM.......(5)
C     SUBROUTINE PVPPV(B,V,PV,PPV)
C     PURPOSE:
C         THIS SUBROUTINE CALLS CMIRLF TWICE TO GENERATE THE
C         VALUES OF LEGENDRE FUNCTION OF ORDER V AND ITS
C         DERIVATIVE WITH RESPECT TO B USING THE CONTIGUOUS
C         RELATION:
C
```

```
C
C         P (COSB) = (V+1)(P     (COSB) -   COSB P (COSB))/SINB
C          V                 V+1                  V
C
C         V    = ORDER OF THE LEGENDRE FUNCTION.
C         PV   = LEGENDRE FUNCTION  P (COSB).
C                                    V
C         PPV  = DERIVATIVE OF P (COSB).
C                                V
C
C     ------------------------------------------------------------
      IMPLICIT REAL*8(A-H,T,X-Z)
      IMPLICIT COMPLEX*16(O-S,U-W)
      V1=V+1.0
      CALL CMIRLF(B,V,PV)
      CALL CMIRLF(B,V1,PV1)
      PPV=V1*(PV1-DCOS(B)*PV)/DSIN(B)
      RETURN
      END



      SUBROUTINE CMIRLF(B,V,PV)
C     ------------------------------------------------------------
C     PROGRAM.......(6)
C     SUBROUTINE CMIRLF(V,PV)
C
C     PURPOSE:
C         NUMERICAL COMPUTATION OF LEGENDRE FUNCTION OF
C          COMPLEX ORDERS.
C
C     USAGE:
C         CALL CMIRLF(B,V,PV)
C
C     METHOD:
C         THE VALUES OF THE LEGENDRE FUNCTIONS ARE CALCULATED
C          USING MEHLER'S INTEGRAL REPRESENTATION.
C
C     DESCRIPTION OF THE PARAMETERS:
C         API=PI.
C         N   =NUMBER OF INTERVALS
C         H   =THICKNESS OF THE STRIP
C         A   =LOWER LIMIT OF INTEGRATION
C         B   =UPPER LIMIT OF INTEGRATION
C         V   =ORDER OF THE FUNCTION
C         PV  =LEGENDRE FUNCTION OF THE COMPLEX ORDER 'V'
C
C     ------------------------------------------------------------
```

```
      IMPLICIT REAL*8(A-H,T,X-Z)
      IMPLICIT COMPLEX*16(O-S,U-W)
      C=1.0
      API=4.0*DATAN(C)
      A=0.0
      N=512
C
C     COMPLEX NUMERICAL INTEGRATION FOLLOWS
C
      FN=N
      H=(B-A)/FN
      X=A
      D=2.0*(2.0**0.5)/(API*DSIN(B)*DSIN(B))
      R1=(V+1.5)*X
      R2=(V+0.5)*X
      S=(V+2.0)*CDCOS(R1)-V*DCOS(B)*CDCOS(R2)
      AB=DCOS(X)-DCOS(B)
      OLDZ=D*S*(AB**0.5)
      SUM=0.0
      DO 20 I=1,N
      X=X+H
      R1=(V+1.5)*X
      R2=(V+0.5)*X
      S=(V+2.0)*CDCOS(R1)-V*DCOS(B)*CDCOS(R2)
      AB=DCOS(X)-DCOS(B)
      W=D*S*(AB**0.5)
      SUM=SUM+OLDZ+W
   20 OLDZ=W
      PV=SUM*H/2.0
      RETURN
      END



      SUBROUTINE AVS(FI,V,PV,PPV)
C     ----------------------------------------------------------
C     PROGRAM.......(7)
C
C     PURPOSE:
C        NUMERICAL COMPUTATION OF LEGENDRE FUNCTION AND ITS
C        DERIVATIVE FOR LARGE VALUES OF V USING THE
C        ASYMTOTIC EXPANSION.
C
C     REFERENCE:
C        ERDELY, A., HIGHER TRANSCENDENTAL FUNCTIONS, VOL.1,
C        MC GRAW HILL, NY., 1953, EQ. 2 PAGE. 162.
C
C     ----------------------------------------------------------
```

```
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION U(2)
      C=0.10000D 01
      PI=4.0*DATAN(C)
      Z=V+1.0
      DO 10 I=1,2
      A1=1.0/(12.0*Z)
      A2=-1.0/(360.0*(Z**3))
      A3=1.0/(1260.0*(Z**5))
      A4=-1.0/(1680.0*(Z**7))
      A5=1.0/(1188.0*(Z**9))
      A6=-691.0/(360360.0*(Z**11))
      A7=1.0/(156.0*(Z**13))
      A8=-3617.0/(122400.0*(Z**15))
      S=A1+A2+A3+A4+A5+A6+A7+A8
      B=(Z-0.50)*DLOG(Z)-Z+0.50*DLOG(2.0*PI)+S
      U(I)=B
   10 Z=Z+0.50
      X=U(1)-U(2)
      AB=DEXP(X)
      V1=V+0.50
      ALFA=V1*FI-PI/4.0
      AC=DCOS(ALFA)
      AD=PI*DSIN(FI)/2.0
      AA=DSQRT(AD)
      PV=AB*AC/AA
      PA=(2.*V+1.)*DSIN(FI)*DSIN(ALFA)
      PB=DCOS(FI)*DCOS(ALFA)
      PC=2.*PI*(DSIN(FI)**3)
      PPV=-AB*(PA+PB)/(DSQRT(PC))
      RETURN
      END


      SUBROUTINE CONFN(BB,PP,PV,PPV)
C     -----------------------------------------------------------------
C
C     PROGRAM.......(8)
C     SUBROUTINE CONFN(BB,PP,PV,PPV)
C
C     PURPOSE:
C          NUMERICAL COMPUTATION OF CONICAL FUNCTION AND ITS
C          DERIVATIVE USING SERIES EXPANSION.
C
C     REFERENCE:
C          SINGH,A.V.; MIRZA,S.; WIDERA,O.E.; "LEGENDRE FUNCTIONS
```

```
C            OF COMPLEX ORDERS",EQ.(6),REPORT NO.73-1, UNIVERSITY
C            OF ILLINOIS AT CHICAGO CIRCLE, JUNE 1973.
C
C     ----------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(300),B(300)
      ERROR=0.12500D-08
      FI=BB*0.500
      S=DSIN(FI)*DSIN(FI)
      A(1)=(PP*PP+0.250)*S
      B(1)=A(1)
      SUM=1.0+A(1)
      SUMP=B(1)
      DO 7 K=2,150
      C=K
      D=PP*PP+((C+C-1.0)*0.50)**2
      A(K)=A(K-1)*D*S/(C*C)
      B(K)=C*A(K)
      IF(A(K).LE.ERROR)GO TO 8
      SUM=SUM+A(K)
    7 SUMP=SUMP+B(K)
    8 PV=SUM
      PPV=SUMP*DCOTAN(FI)
      RETURN
      END


      SUBROUTINE DETERM(AA,N,D)
C     ----------------------------------------------------------------
C     PROGRAM.......(9)
C     SUBROUTINE DETERM(AA,N,D)
C     PURPOSE:
C         THIS SUBROUTINE GENERATES THE NUMERICAL VALUE OF
C         DETERMINANT  AA(N,N) .
C
C     REFERENCE:
C         PENNINGTON R. H. : INTRODUCTORY COMPUTER METHODS AND
C         NUMERICAL ANALYSIS. PAGE. 289, THE MACMILLON COMPANY
CC        NEW YORK (1968).
C
C     ----------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AA(4,4),A(4,4)
      DO 20 I=1,N
      DO 20 J=1,N
```

```
 20 A(I,J)=AA(I,J)
    D=1.0
    K=1
  1 CONTINUE
    KK=K+1
    IS=K
    IT=K
    B=DABS(A(K,K))
    DO 2 I=K,N
    DO 2 J=K,N
    IF(DABS(A(I,J))-B)2,2,21
 21 IS=I
    IT=J
    B=DABS(A(I,J))
  2 CONTINUE
    IF(IS-K)3,3,31
 31 DO 32 J=K,N
    C=A(IS,J)
    A(IS,J)=A(K,J)
 32 A(K,J)=-C
  3 CONTINUE
    IF(IT-K)4,4,41
 41 DO 42 I=K,N
    C=A(I,IT)
    A(I,IT)=A(I,K)
 42 A(I,K)=-C
  4 CONTINUE
    D=A(K,K)*D
    IF(A(K,K))5,71,5
  5 CONTINUE
    DO 6 J=KK,N
    A(K,J)=A(K,J)/A(K,K)
    DO 6 I=KK,N
    W=A(I,K)*A(K,J)
    A(I,J)=A(I,J)-W
    IF(DABS(A(I,J)-0.00001*DABS(W)))61,6,6
 61 A(I,J)=0.
  6 CONTINUE
    K=KK
    IF(K-N)1,70,1
 70 D=A(N,N)*D
 71 RETURN
    END
```

```
      SUBROUTINE BESJ(X,N,BJ,D,IER)
C
C     ..................................................................
C
C        SUBROUTINE BESJ
C
C        PURPOSE
C        COMPUTE THE J BESSEL FUNCTION FOR A GIVEN ARGUMENT
C       AND ORDER.
C
C        USAGE
C           CALL BESJ(X,N,BJ,D,IER)
C
C        DESCRIPTION OF PARAMETERS
C           X  -THE ARGUMENT OF THE J BESSEL FUNCTION DESIRED
C           N  -THE ORDER OF THE J BESSEL FUNCTION DESIRED
C           BJ -THE RESULTANT J BESSEL FUNCTION
C           D  -REQUIRED ACCURACY
C           IER-RESULTANT ERROR CODE WHERE
C              IER=0  NO ERROR
C              IER=1  N IS NEGATIVE
C              IER=2  X IS NEGATIVE OR ZERO
C              IER=3  REQUIRED ACCURACY NOT OBTAINED
C              IER=4  RANGE OF N COMPARED TO X NOT CORRECT
C
C        REMARKS
C           N MUST BE GREATER THAN OR EQUAL TO ZERO, BUT IT MUST
C           BE LESS THAN
C              20+10*X-X** 2/3   FOR X LESS THAN OR EQUAL TO 15
C              90+X/2            FOR X GREATER THAN 15
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           RECURRENCE RELATION TECHNIQUE DESIRED BY H. GOLDSTEIN
C           AND R.M.THALER, 'RECURRENCE TECHNIQUES FOR THE
C           CALCULATION OF BESSEL FUNCTIONS', M.T.A.C., VOL.13,
C           PP.102-108 AND STEGUN AND ABRAMOWITZ,'GENERATION OF
C           BESSEL FUNCTIONS ON HIGH SPEED COMPUTERS',M.T.A.C.,
C           VOL.11,1957,PP.255-257.
C
C     ..................................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
      BJ=.0
      IF(N)10,20,20
   10 IER=1
      RETURN
```

```
   20 IF(X)30,30,31
   30 IER=2
      RETURN
   31 IF(X-15.)32,32,34
   32 NTEST=20.+10.*X-X** 2/3
      GO TO 36
   34 NTEST=90.+X/2.
   36 IF(N-NTEST)40,38,38
   38 IER=4
      RETURN
   40 IER=0
      N1=N+1
      BPREV=.0
C
C     COMPUTE STARTING VALUE OF M
C
      IF(X-5.)50,60,60
   50 MA=X+6.
      GO TO 70
   60 MA=1.4*X+60./X
   70 IIX=X
      MB=N+IIX/4+2
      MZERO=MAX0(MA,MB)
C
C     SET UPPER LIMIT OF M
C
      MMAX=NTEST
  100 DO 190 M=MZERO,MMAX,3
C
C     SET F(M),F(M-1)
C
      FM1=1.0D-28
      FM=.0
      ALPHA=.0
      IF(M-(M/2)*2)120,110,120
  110 JT=-1
      GO TO 130
  120 JT=1
  130 M2=M-2
      DO 160 K=1,M2
      MK=M-K
      BMK=2.*FLOAT(MK)*FM1/X-FM
      FM=FM1
      FM1=BMK
      IF(MK-N-1)150,140,150
  140 BJ=BMK
  150 JT=-JT
      S=1+JT
  160 ALPHA=ALPHA+BMK*S
```

```
      BMK=2.*FM1/X-FM
      IF(N)180,170,180
  170 BJ=BMK
  180 ALPHA=ALPHA+BMK
      BJ=BJ/ALPHA
      IF(DABS(BJ-BPREV)-DABS(D*BJ))200,200,190
  190 BPREV=BJ
      IER=3
  200 RETURN
      END

C     ..................................................................
C
C        SUBROUTINE IO
C
C        PURPOSE
C           COMPUTE THE MODIFIED BESSEL FUNCTION I OF ORDER ZERO
C
C        USAGE
C           CALL IO(X,RIO)
C
C        DESCRIPTION OF PARAMETERS
C           X     -GIVEN ARGUMENT OF THE BESSEL FUNCTION I OF ORDER 0
C           RIO   -RESULTANT VALUE OF THE BESSEL FUNCTION I OF ORDER 0
C
C        REMARKS
C           LARGE VALUES OF THE ARGUMENT MAY CAUSE OVERFLOW IN THE
C           BUILTIN EXP-FUNCTION
C
C        METHOD
C          POLYNOMIAL APPROXIMATIONS GIVEN BY E.E. ALLEN ARE
C          USED FOR CALCULATION.
C          FOR REFERENCE SEE
C          M. ABRAMOWITZ AND I.A. STEGUN,'HANDBOOK OF MATHEMATICAL
C          FUNCTIONS', U.S. DEPARTMENT OF COMMERCE, NATIONAL BUREAU OF
C          STANDARDS APPLIED MATHEMATICS SERIES, 1966, P.378.
C     ..................................................................
      SUBROUTINE IO(X,RIO)
      IMPLICIT REAL*8(A-H,O-Z)
      RIO=DABS(X)
      IF(RIO-3.75)1,1,2
    1 Z=X*X*7.111111D-2
      RIO=(((((4.5813D-3*Z+3.60768D-2)*Z+2.659732D-1)*Z+1.206749D0)*Z
     1+3.089942D0)*Z+3.515623D0)*Z+1.
      RETURN
    2 Z=3.75/RIO
      RIO=DEXP(RIO)/DSQRT(RIO)*((((((((3.92377D-3*Z-1.647633D-2)*Z
     1+2.635537D-2)*Z-2.057706D-2)*Z+9.162810D-3)*Z-1.57565D-3)*Z
     2+2.253190D-3)*Z+1.328592D-2)*Z+3.989423D-1)
      RETURN
      END
```

```
C
C     ..................................................................
C
C        SUBROUTINE INUE
C
C        PURPOSE
C           COMPUTE THE MODIFIED BESSEL FUNCTIONS I FOR ORDERS 1 TO N
C
C        USAGE
C           CALL INUE(X,N,ZI,RI)
C
C        DESCRIPTION OF PARAMETERS
C           X      -GIVEN ARGUMENT OF THE BESSEL FUNCTIONS I
C           N      -GIVEN MAXIMUM ORDER OF BESSEL FUNCTIONS I
C           ZI     -GIVEN VALUE OF BESSEL FUNCTION I OF ORDER ZERO
C                   FOR ARGUMENT X
C           RI     -RESULTANT VECTOR OF DIMENSION N, CONTAINING THE
C                   VALUES OF THE FUNCTIONS I FOR ORDERS 1 TO N
C
C        REMARKS
C           THE VALUE OF ZI MAY BE CALCULATED USING SUBROUTINE I0.
C           USING A DIFFERENT VALUE HAS THE EFFECT THAT ALL VALUES OF
C           BESSEL FUNCTIONS I ARE MULTIPLIED BY THE  FACTOR ZI/I(0,X)
C           WHERE I(0,X) IS THE VALUE OF I FOR ORDER 0 AND ARGUMENT X.
C           THIS MAY BE USED DISADVANTAGEOUSLY IF ONLY THE RATIOS OF I
C           FOR DIFFERENT ORDERS ARE REQUIRED.
C
C        METHOD
C           THE VALUES ARE OBTAINED USING BACKWARD RECURRENCE RELATION
C           TECHNIQUE. THE RATIO I(N+1,X)/I(N,X) IS OBTAINED FROM A
C           CONTINUED FRACTION.
C           FOR REFERENCE SEE
C           G. BLANCH,'NUMERICAL EVALUATION OF CONTINUED FRACTIONS',
C           SIAM REVIEW, VOL.6,NO.4,1964,PP.383-421.
C
C     ..................................................................
C
      SUBROUTINE INUE(X,N,ZI,RI)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION RI(1)
      IF(N)10,10,1
    1 FN=N+N
      Q1=X/FN
      IF(DABS(X)-5.D-4)6,6,2
    2 A0=1.
      A1=0.
      B0=0.
      B1=1.
      FI=FN
    3 FI=FI+2.
```

```
      AN=FI/DABS(X)
      A=AN*A1+A0
      B=AN*B1+B0
      A0=A1
      B0=B1
      A1=A
      B1=B
      Q0=Q1
      Q1=A/B
      IF(DABS((Q1-Q0)/Q1)-1.D-6)4,4,3
    4 IF(X)5,6,6
    5 Q1=-Q1
    6 K=N
    7 Q1=X/(FN+X*Q1)
      RI(K)=Q1
      FN=FN-2.
      K=K-1
      IF(K)8,8,7
    8 FI=ZI
      DO 9 I=1,N
      FI=FI*RI(I)
    9 RI(I)=FI
   10 RETURN
      END
```