

**Higher - order eigenvalue asymptotics for Sturm -
Liouville problems with one simple turning point .**

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To my wife
and
my children Sara and Mahdee

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Abstract

Consider the Sturm–Liouville equation

$$y'' + (\lambda r(t) - q(t))y = 0, \quad t \in [a, b], \quad -\infty < a < b < +\infty,$$

with Dirichlet boundary conditions $y(a) = 0 = y(b)$. In this thesis the higher order approximation for the eigenvalues in the classical case i.e, $r(t) > 0$, and in the one simple turning point case is derived as well as the higher order approximation for the eigenvalues of the Dirichlet problem associated with

$$y'' = \frac{(l+2)^2}{4} \lambda^2 t^l y \quad c \leq t \leq d$$

with boundary conditions

$$y(c) = 0 = y(d) \quad c < 0 < d$$

where l is an odd integer. By using the above results, an infinite product representation of a particular solution of the Sturm–Liouville equation in the one simple turning point case is obtained. The last chapter of this thesis is devoted to the recovery of the potential function $q(t)$ from the spectrum and the weight function $r(t)$ in the one simple turning point case. We obtain a seemingly novel, characterization of $q(t)$, assumed non negative, in term of various *traces* and their derivatives.

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Introduction

In the mathematical literature , there are thousands of research papers devoted entirely or in part to the study of the Sturm Liouville equation i.e.,

$$y'' + (\lambda r(t) - q(t))y = 0 \quad a \leq t \leq b \quad (1)$$

where $r, q : [a, b] \rightarrow R$ are continuous functions and λ is a real or complex parameter. Indeed ,this equation is distinguished for the wealth of the methods used in studying its solutions and because of its applications to the wave equation .

In 1836 both Sturm and Liouville published articles concerning boundary value problems for the differential equation (1) with $r(t) \equiv 1$. They asked whether there exists nontrivial solutions of (1) satisfying boundary conditions of the form

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0$$

$$y(1) \cos \beta + y'(1) \sin \beta = 0$$

where α , β are real numbers between 0 and π . The parameter λ is called an *eigenvalue* if the boundary value problem can be solved . The corresponding

nontrivial solution is called an *eigenfunction* for λ . The collection of all eigenvalues is called the (point)*spectrum* of the boundary value problem .

In many physical problems which in their mathematical formulation lead to Sturm - Liouville equation, one can determine the spectrum experimentally . Often the function $q(x)$ in (1) is unknown , and the question arises as to what extent $q(x)$ can be reconstructed from given spectral data .This type of problem is known as an *inverse Sturm - Liouville problem* .

The aim of this thesis is to study the higher order approximations to the eigenvalues of one turning point problems . We then apply these results to the problem of recovering the function $q(x)$ from the spectrum and the weight function $r(t)$ in (1) with Dirichlet boundary conditions

$$y(a) = 0 = y(b)$$

where $r(t)$ has only one simple zero x_0 within (a , b) and $\frac{r(t)}{t-t_0}$ is positive and twice continuously differentiable in $[a , b]$. In the literature , the zeros of the weight function $r(t)$ are called *turning points* or *transition points* of the differential equation (1) .

Turning now more specifically to the contents of this thesis , chapter 1 considers the asymptotic solution of equation (1) in the classical case i.e.; $r(t) > 0$ on $[a , b]$, and we summarize briefly and without proof the derivation of the well known formal solutions . By making use of the formal solutions the higher order approximation for the eigenvalues is obtained (without the assumption $\int_a^b q(v)dv = 0$).

In the second chapter , Langer's transformation is briefly described . Then , by

applying this transformation to equation (1), in the one simple turning point case, it is transformed into the simple form

$$W'' + (\lambda\xi - R(\xi))W = 0, \quad c \leq \xi \leq d \quad (2)$$

where $c < 0$ and $d > 0$. By using the formal solution constructed by Olver, the higher-order approximation for the eigenvalues of equation (2) with boundary conditions $W(c) = 0 = W(d)$ and similarly with $W(c) = 0 = W(0)$ is derived. Moreover it is shown that the positive eigenvalues of equation (1) in the one turning point case with Dirichlet boundary conditions depends only on the positive part of the weight function $r(t)$. The third and fourth term of the asymptotic distribution of the eigenvalues appears to be new here.

The third chapter is devoted to the study of the asymptotic distribution of eigenvalues for the boundary value problem

$$y'' - \frac{(l+2)^2}{4} \lambda^2 t^l y = 0$$

$$y(c) = 0 = y(d)$$

as well as with boundary conditions

$$y(c) = 0 = y(0)$$

where $c < 0$, $d > 0$ and l is an odd integer.

Let $U(\xi, \lambda)$ be the solution of equation (2) which satisfies the initial condition

$$U(-1, \lambda) = 0 \quad \frac{\partial U}{\partial \xi}(-1, \lambda) = 1.$$

In chapter four, by making use of the results obtained in the previous three chapters, an infinite product representation of $U(\xi, \lambda)$ is given. Furthermore, the leading term of the asymptotic formula for $\frac{\partial U}{\partial \lambda}(x, \lambda_n(x))$, $\lambda'_n(x)$ and $\int_{-1}^x v U^2(v, \lambda_n(x)) dv$ is obtained where $\lambda_n(x)$ is a negative eigenvalue of the Dirichlet problem (2) on

$[-1, x]$ with fixed $x < 0$. Similarly , the leading term of the asymptotic expansion of $\frac{\partial U}{\partial \lambda}(x, u_n(x))$, $\frac{\partial U}{\partial \xi}(x, u_n(x))$, $u'_n(x)$, $\int_{-1}^x vU^2(v, u_n(x))dv$ and $U(\xi, u_n(x))$ for large n is derived ,where $u_n(x)$ is the sequence of positive eigenvalues of the Dirichlet problem (2) on $[-1, x]$, for fixed $x > 0$. In seeking the asymptotic form of the above- mentioned integrals we do not use Watson's lemma , but calculate these forms directly . The above results are used in the next chapter .

The fifth chapter deals with the inverse problem in the one simple turning point case . More specifically , since the Sturm Liouville equation (1) in the one simple turning point case can be transformed to the simple form (2) , it suffices to consider the Dirichlet problem for equation (2) on $[-1, 1]$. It is shown that given the initial conditions $\{u_n(1)\}$, $\{u'_n(1)\}$, $\{r_n(1)\}$ and $\{r'_n(1)\}$ with the *dual equation* of (2) one can recover the potential function $R(\xi)$ in equation (2), where $u_n(x)$ is the sequence of positive eigenvalues and $r_n(x)$ is the sequence of negative eigenvalues of the Dirichlet problem for (2) on $[-1, x]$ with fixed $x > 0$.

It seems that the above idea can be extended to more general cases. For instance, whenever the weight function $r(x)$ has one or more zeros in $[a, b]$ or whenever the boundary condition is different from the Dirichlet boundary condition .

Chapter 1

Higher - order eigenvalue asymptotics for right - definite problems

1.1 Introduction

Let $r, q : [a, b] \rightarrow \mathbb{R}$ be in $C^\infty(a, b)$ and $r(x) > 0$ on $[a, b]$ where $-\infty < a < b < \infty$.

The weighted regular Sturm-Liouville problem consists in finding the values a parameter λ for which the equation

$$y'' + (\lambda r(x) - q(x))y = 0 \quad a \leq x \leq b \quad (1.1)$$

has a solution y (non identically zero) satisfying a pair of homogeneous separated boundary conditions.

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In this chapter we consider the Dirichlet problem , in which , the boundary conditions are of the form

$$y(a) = 0 = y(b).$$

In 1837 , Liouville investigated the behaviour of solutions of ordinary linear differential equations containing a large parameter. He transforms (1 . 1)by setting

$$\xi(x) = \int \sqrt{r(x)}dx \quad y(x) = r^{-1/4}(x)W(\xi) \quad (1.2)$$

and obtains

$$\frac{d^2W}{d\xi^2} + (\lambda - R(\xi))W(\xi) = 0 \quad (1.3)$$

where

$$\begin{aligned} R(\xi) &= \frac{r''(x)}{4r^2(x)} - \frac{5r'^2}{16r^3(x)} + \frac{q(x)}{r(x)} \\ &= -\frac{d^2}{r^{3/4}dx^2}\left(\frac{1}{r^{1/4}}\right) + \frac{q(x)}{r(x)} \end{aligned} \quad (1.4)$$

The transformation (1 . 2) is usually called *the Liouville - Green (LG) transformation*. It should be noticed that the LG transformation may fail whenever $r(x)$ has some zero in [a , b].

When λ is large and positive $R(\xi)W(\xi)$, in (1.3) , can be neglected in the first approximation so that solutions of (1 .3) are approximately a linear combination of $\cos \sqrt{\lambda}\xi$ and $\sin \sqrt{\lambda}\xi$. More exactly ,Liouville uses the method of variation of parameters to show that solutions of (1.3) satisfy what is now called a Volterra integral equation ,

$$W(\xi) = c_1 \cos \sqrt{\lambda}\xi + c_2 \sin \sqrt{\lambda}\xi + \frac{1}{\sqrt{\lambda}} \int_a^\xi \sin \sqrt{\lambda}(\xi - t)R(t)W(t)dt$$

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and proposes to solve this equation by the method of successive approximations where c_1 and c_2 are some constants.

In studying the eigenvalues of the equation (1.3), one may apply the above Volterra integral equation. In this method, the asymptotic expression for the eigenvalues of the equation (1.3) on $[0, \pi]$ with the boundary condition $y(0) = 0 = y(\pi)$, is of the form

$$\lambda_n = n + 1 + \frac{1}{2n\pi} \int_0^\pi q(x)dx + O\left(\frac{1}{n^2}\right)$$

where the corresponding eigenfunction of the eigenvalue λ_n vanishes n times within the interval $(0, \pi)$. More details can be found in ([34], P.173), ([14], P.270), ([24], P.148), ([20], P.60), ([21]).

Hochstadt has assumed that the equation (1.3) has a solution of the form

$$\begin{aligned} W(\xi) &= A(\xi) \sin \omega(\xi) \\ W'(\xi) &= \sqrt{\lambda - R(\xi)} A(\xi) \cos \omega(\xi) \end{aligned}$$

By deriving a new differential equation, one can determine $A(\xi)$ and $\omega(\xi)$. By assuming that $R(\xi)$ has mean value 0, he found the fifth approximation for the eigenvalues of the equation (1.3) with the boundary condition $W(0) = 0 = W(\pi)$. It is of the form

$$\sqrt{\lambda_n} = n + \frac{\int_0^\pi q^2(x)dx - q'(\pi) + q'(0)}{8n^3\pi} + O\left(\frac{1}{n^4}\right)$$

For more details see ([13], P. 154).

By using a *Prüfer* transformation, Borg in [4], for the differential equation (1.3) with the periodic boundary conditions

$$W(a) = W(0) \quad W'(a) = W'(0),$$

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derived the following estimates for both eigenvalues λ_{2m+1} and λ_{2m+2} ,

$$\sqrt{\lambda} = \frac{2(m+1)\pi}{a} + \sum_{k=1}^{r+1} A_k \{2(m+1)\pi/a\}^{-k} + O(m^{-r-2})$$

where the A_k are independent of m and involve $R(\xi)$ and its derivatives up to order $r-1$. In particular ,

$$\begin{aligned} A_1 &= \frac{1}{2a} \int_0^a R(x) dx \\ A_2 &= 0 \\ A_3 &= \frac{1}{8a} \int_0^a R^2(x) dx - A_1^2 \end{aligned}$$

For more details see [4] or ([7], P. 58) .

Now, in section 1.2, we shall derive the formal solution of the equation (1.1). In section 1.4 of this chapter we shall study the distribution of eigenvalues of (1.1) without using Liouville's transformation by means of the formal solutions of the equation (1.1) .

1.2 The Formal Solutions

In most differential equations with variable coefficients it is impossible to obtain an exact solution . So one must resort to various approximate methods of solution . One of the most useful mathematical methods of achieving this is by representing the solution by an asymptotic form which we will study in this section .

For the existence of the solution of (1.1) depending on a parameter λ , we would like to call the reader's attention to a complete historical review by Mingarelli in [24] .

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Let the solutions of (1.1) be given by the formulae

$$y_j(x, \lambda) = e^{\pm i\xi} A_j(x, \lambda) \quad j = 1, 2, \quad (1.5)$$

in which the factors $A_j(x, \lambda)$ remain for the moment unspecified and $\xi = \xi(x, \lambda)$ is given by

$$\xi = \sqrt{\lambda} \int_{x_0}^x \sqrt{r(t)} dt \quad (1.6)$$

where x_0 is a fixed point in $[a, b]$. The upper sign in (1.5) is to be associated with $j = 1$ and the lower one with $j = 2$.

It is then found by direct substitution of $y_j(x, \lambda)$ in (1.1), that

$$y_j'' + (\lambda r(x) - q(x))y_j = e^{\pm i\xi} C_j(x, \lambda) \quad (1.7)$$

in which

$$C_j(x, \lambda) = \pm i\sqrt{\lambda}(2\sqrt{r(x)}A_j' + \frac{r'(x)}{2\sqrt{r(x)}}A_j) + A_j'' - q(x)A_j \quad (1.8)$$

A glance at the Volterra integral equation constructed by Liouville, one expects that the $A_j(x, \lambda)$ in (1.5) be of the form,

$$A_j(x, \lambda) = \sum_{k=0}^{\infty} \frac{a_{j,k}(x)}{\sqrt{\lambda}^k} \quad (1.9)$$

Effecting the substitution (1.9) in the formula (1.8) we have

$$C_j(x, \lambda) = \pm i\sqrt{\lambda}(2\sqrt{r(x)} \sum_{k=0}^{\infty} \frac{a'_{j,k}}{\sqrt{\lambda}^k} + \frac{r'(x)}{2\sqrt{r(x)}} \sum_{k=0}^{\infty} \frac{a_{j,k}}{\sqrt{\lambda}^k}) + \sum_{k=0}^{\infty} \frac{a''_{j,k}}{\sqrt{\lambda}^k} - q(x) \sum_{k=0}^{\infty} \frac{a_{j,k}}{(\sqrt{\lambda})^k}$$

or

$$\begin{aligned} C_j(x, \lambda) &= \pm\sqrt{\lambda}[2i\sqrt{r(x)}a'_{j,0} + \frac{ir'(x)}{2\sqrt{r(x)}}a_{j,0}] \\ &\pm(2i\sqrt{r(x)}a'_{j,1} + \frac{ir'(x)}{2\sqrt{r(x)}}a_{j,1}) + a''_{j,0} - q(x)a_{j,0} + \\ &\vdots \\ &+ \frac{1}{(\sqrt{\lambda})^{m-1}}[\pm(2i\sqrt{r(x)}a'_{j,m} + \frac{ir'(x)}{2\sqrt{r(x)}}a_{j,m}) + a''_{j,m-1} - q(x)a_{j,m-1}] + \\ &\frac{1}{\sqrt{\lambda}^m}(a''_{j,m} - q(x)a_{j,m}) + \dots \end{aligned} \quad (1.10)$$

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Now , in the formula (1.10),the terms in $\sqrt{\lambda}^{(1-k)}$,for $k = 0, 1, \dots$, will all vanish if the relations

$$2\sqrt{r(x)}a'_{j,0} + \frac{r'(x)}{2\sqrt{r(x)}}a_{j,0} = 0$$

$$\pm (2i\sqrt{r(x)}a'_{j,k} + i\frac{r'(x)}{2\sqrt{r(x)}}a_{j,k}) + a''_{j,k-1} - q(x)a_{j,k-1} = 0 \quad (1.11)$$

are fulfilled. These will be fulfilled if suitable values are assigned successively to $a_{j,0}, \dots, a_{j,m}, \dots$. Thus , we define $a_{j,k}$ by means of the relations

$$2\sqrt{r(x)}a'_{j,0} + \frac{r'(x)}{2\sqrt{r(x)}}a_{j,0} = 0 \quad \Rightarrow a_{j,0} = r^{-1/4}(x)$$

and , in order to compute $a_{j,k}$, $k = 1, 2, \dots$, we mutiply (1.11) by $r^{-1/4}(x)$ so that

$$\pm \frac{d}{dx}(2ia_{j,k}r(x)^{1/4}) = -\frac{a''_{j,k-1} - q(x)a_{j,k-1}}{r(x)^{1/4}} \quad k = 1, 2, \dots$$

Therefore we obtain $a_{j,k}$ recursively by

$$a_{j,k} = \frac{\pm i}{2r(x)^{1/4}} \int_{x_k}^x \frac{a''_{j,k-1} - q(t)a_{j,k-1}}{r^{1/4}(t)} dt \quad k = 1, 2, \dots \quad (1.12)$$

where the lower limit x_k , is arbitrary but fixed .In order for $a_{j,k} \in C^2$, for each $j = 1, 2$, it is sufficient that, $r, q \in C^\infty(a, b)$, which is the assumption in the introduction .

Note that the choice (1.12) serves to reduce each of the $C_j(x, \lambda)$ to zero . The resulting $y_j(x, \lambda)$ defined by (1.7) and (1.5) now formally satisfies (1.1) .The latter thus admits the pair of formal solutions

$$e^{\pm i\xi} \sum_{k=0}^{\infty} \frac{a_{j,k}(x)}{\sqrt{\lambda}^k} \quad (1.13)$$

with coefficients that are given by (1.12) .

The above series is uniformly convergent on compact x intervals as $\lambda \rightarrow \infty$. For a

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proof of the asymptotic nature of the formal solution see ([7] , P.61) ,([8] , P.80) , ([29] , 193) .

In general , the formal solution may not converge whenever the interval is not compact . For contra example see ([29], P.365).

1.3 Distribution of eigenvalues in the classical case

We know that the eigenvalues of (1.1) with boundary conditions

$$y(a) = 0 = y(b)$$

are zeros of

$$\Delta(\lambda) = \begin{vmatrix} y_1(a) & y_2(a) \\ y_1(b) & y_2(b) \end{vmatrix}$$

where y_1 and y_2 are two linearly independent solutions of equation (1.1). One can find the distribution of the eigenvalues by using the formal solutions

$$\begin{aligned} y_1 &= e^{i\xi} \sum_{k=0}^{\infty} \frac{g_k}{\sqrt{\lambda^k}} \\ y_2 &= e^{-i\xi} \sum_{k=0}^{\infty} \frac{(-1)^k g_k}{\sqrt{\lambda^k}} \\ g_k = a_{1,k} &= (-1)^k a_{2,k} = \frac{i}{2r(x)^{1/4}} \int_{x_k}^x \frac{a''_{1,k-1} - q(t)a_{1,k-1}}{r^{1/4}(t)} dt \quad k = 1, 2, \dots \end{aligned} \tag{1.14}$$

Now we find the form of the transcendental equation $\Delta(\lambda) = 0$. By the Cauchy product of two series , we may write,

$$y_1(a)y_2(b) = e^{i(\xi(a)-\xi(b))} \sum_0^{\infty} \frac{c_m}{\sqrt{\lambda^m}}$$

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$$y_1(b)y_2(a) = e^{i(\xi(b)-\xi(a))} \sum_0^{\infty} \frac{d_m}{\sqrt{\lambda^m}}$$

where

$$c_0 = d_0 = (r(a)r(b))^{-1/4} \quad (1.15)$$

and

$$c_m = \sum_0^m (-1)^{m-k} g_k(a)g_{m-k}(b) \quad (1.16)$$

$$d_m = \sum_0^m (-1)^{m-k} g_k(b)g_{m-k}(a). \quad (1.17)$$

Consequently , $\Delta(\lambda) = 0$ implies

$$e^{2i(\xi(b)-\xi(a))} = \frac{\sum_{m=0}^{\infty} \frac{c_m}{\sqrt{\lambda^m}}}{\sum_{m=0}^{\infty} \frac{d_m}{\sqrt{\lambda^m}}}. \quad (1.18)$$

Note that we may take $y_1(b)y_2(a) \neq 0$ and $y_1(a)y_2(b) \neq 0$, so that, in particular, the denominator in (1.18) never vanishes . A glance at (1.16) and (1.17) ,shows that

$$c_m = (-1)^m d_m. \quad (1.19)$$

Therefore , from (1.18) and (1.19) we get

$$e^{2i(\xi(b)-\xi(a))} = \frac{\sum_{m=0}^{\infty} (-1)^m u^m d_m}{\sum_{m=0}^{\infty} u^m d_m} \quad (1.20)$$

where $u = \frac{1}{\sqrt{\lambda}}$. Now suppose

$$e^{2i(\xi(b)-\xi(a))} = \sum_{m=0}^{\infty} u^m R_m. \quad (1.21)$$

The R_m may be determined by equating coefficients of equal powers of u in the following equality

$$\left(\sum_{m=0}^{\infty} u^m R_m \right) \left(\sum_{m=0}^{\infty} u^m d_m \right) = \sum_{m=0}^{\infty} (-1)^m u^m d_m.$$

Thus

$$R_0 = 1$$

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and

$$(-1)^m d_m = \sum_{k=0}^m R_k d_{m-k} \quad (1.22)$$

From (1.6) we write

$$\xi(b) - \xi(a) = \sqrt{\lambda} P$$

where

$$P = \int_a^b \sqrt{r(t)} dt. \quad (1.23)$$

In order to obtain the distribution of the eigenvalues we see that from (1.21) , we get

$$2i\sqrt{\lambda}P = 2n\pi i + \log\left(1 + \sum_{m=1}^{\infty} \frac{R_m}{\sqrt{\lambda^m}}\right), \quad (1.24)$$

where $n = 0, \pm 1, \pm 2, \dots$

a) First and second approximation

The expansion of $\log(1 + x)$ for $|x| \leq 1$ is of the form

$$\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \quad (1.25)$$

Therefore

$$\log\left(1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right) = O(1/\sqrt{\lambda}) \quad \text{as } \lambda \rightarrow \infty$$

where for two functions $f(x), g(x)$ we define

$$f(x) = O(g(x)) \quad \text{as } x \rightarrow \infty$$

to mean that $|f(x)/g(x)|$ is bounded.

From (1.24) and the expansion (1.25) we get the classical result ([14] , p.272)

$$\sqrt{\lambda_n} = \frac{n\pi}{P} + O(1/n) \quad (1.26)$$

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where P is defined in (1.23).

b) Third approximation

From (1.24) and (1.25) we write (as $\lambda \rightarrow \infty$),

$$\sqrt{\lambda} = \frac{n\pi}{P} + \frac{1}{2iP} \left(\frac{R_1}{\sqrt{\lambda}} + O(1/\lambda) \right) \quad (1.27)$$

From (1.17) and (1.22) we get $R_1 = \frac{-2d_1}{d_0}$ where d_0 is given by (1.15) and d_1 is given by (1.17). Indeed we have

$$\begin{aligned} d_1 &= g_1(b)g_0(a) - g_0(b)g_1(a) \\ &= \frac{i}{2(r(a)r(b))^{1/4}} \left(\int_{x_1}^b \frac{(r^{-1/4}(x))'' - q(x)r^{-1/4}(x)}{r^{1/4}(x)} dx - \int_{x_1}^a \frac{(r^{-1/4}(x))'' - q(x)r^{-1/4}(x)}{r^{1/4}(x)} dx \right) \\ &= \frac{i}{2(r(a)r(b))^{1/4}} \int_a^b \frac{(r^{-1/4}(x))'' - q(x)r^{-1/4}(x)}{r^{1/4}(x)} dx. \end{aligned}$$

Thus (1.27) is of the form

$$\sqrt{\lambda} = \frac{n\pi}{P} - \frac{1}{2\sqrt{\lambda}P} \int_a^b \frac{(r^{-1/4}(x))'' - q(x)r^{-1/4}(x)}{r^{1/4}(x)} dx + O(1/\lambda).$$

From (1.26) we write

$$\sqrt{\lambda_n} = \frac{n\pi}{P} - \frac{1}{2P \left(\frac{n\pi}{P} + O(1/n) \right)} \int_a^b \frac{(r^{-1/4}(x))'' - q(x)r^{-1/4}(x)}{r(x)^{1/4}} dx + O\left(\frac{1}{n^2}\right),$$

consequently,

$$\sqrt{\lambda_n} = \frac{n\pi}{P} - \frac{1}{2n\pi} F(a, b) + O\left(\frac{1}{n^2}\right) \quad (1.28)$$

where

$$F(z, y) = \int_z^y \frac{(r^{-1/4}(x))'' - q(x)r^{-1/4}(x)}{r^{1/4}(x)} dx. \quad (1.29)$$

In the literature

$$F(x) = \int \frac{(r^{-1/4}(x))'' - q(x)r^{-1/4}(x)}{r^{1/4}(x)} dx$$

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is called the *error - control* function (see [29],P.196).

From the properties of series collected in ([1] , §3.6.24) the formula (1.24) can be written of the form

$$2i\sqrt{\lambda}P = 2n\pi i + \frac{c_1}{\sqrt{\lambda}} + \frac{c_2}{\lambda} + \frac{c_3}{\lambda\sqrt{\lambda}} + \frac{c_4}{\lambda^2} + O\left(\frac{1}{\lambda^2\sqrt{\lambda}}\right) \quad (1.30)$$

where

$$\begin{aligned} c_1 &= R_1 \\ c_2 &= R_2 - \frac{1}{2}R_1c_1 \\ c_3 &= R_3 - \frac{1}{3}(R_2c_1 + 2R_1c_2) \\ c_4 &= R_4 - \frac{1}{4}(R_3c_1 + 2R_2c_2 + 3R_1c_3) \end{aligned}$$

Now , by (1.17) and (1.22) , we have

$$\begin{aligned} R_1 &= -\frac{2d_1}{d_0} \\ R_2 &= \frac{2d_1^2}{d_0^2} \\ R_3 &= -\frac{2d_3}{d_0} + \frac{2d_1d_2}{d_0^2} - \frac{2d_1^3}{d_0^3} \\ R_4 &= \frac{4d_1d_3}{d_0^2} - \frac{4d_1^2d_2}{d_0^3} + \frac{2d_1^4}{d_0^4} \end{aligned}$$

Therefore we get , $c_2 = 0$, and

$$\begin{aligned} c_3 &= -\frac{2d_3}{d_0} + \frac{2d_1d_2}{d_0^2} - \frac{2d_1^3}{3d_0^3} \\ c_4 &= R_4 - \frac{1}{4}(R_3R_1 + 3R_1c_3) \\ &= 0 \end{aligned}$$

c) Fourth approximation

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We know that $c_2 = 0$, therefore by (1.28)and (1.30)

$$\sqrt{\lambda_n} = \frac{n\pi}{P} - \frac{1}{2n\pi}F(a, b) + O\left(\frac{1}{n^3}\right). \quad (1.31)$$

d) Fifth approximation

In order to find the fifth approximation we first find d_1, d_2 and d_3 . Suppose in (1.14) $x_1 = x_2 = a$ and x_3 is an arbitrary point in (a, b) . Then from (1.17) we have

$$\begin{aligned} d_0 &= (r(a)r(b))^{-1/4} \\ d_1 &= \frac{i}{2(r(a)r(b))^{1/4}} \int_a^b \frac{(r^{-1/4}(x))'' - q(x)r^{-1/4}(x)}{r^{1/4}(x)} dx \\ d_2 &= \frac{i}{2\{r(a)r(b)\}^{1/4}} \int_a^b \frac{a''_{1,1} - q(t)a_{1,1}}{r^{1/4}(t)} dt \\ d_3 &= \frac{i}{2\{r(a)r(b)\}^{1/4}} \int_a^b \frac{a''_{1,2} - q(t)a_{1,2}}{r^{1/4}(t)} dt. \end{aligned}$$

From (1.29) and (1.31) , we find

$$\begin{aligned} \sqrt{\lambda_n} &= \frac{n\pi}{P} - \frac{1}{2n\pi}F(a, b) + \frac{c_3}{2iP\left\{\frac{n\pi}{P} - \frac{F(a,b)}{2n\pi} + O\left(\frac{1}{n^3}\right)\right\}^3} + O\left(\frac{1}{n^4}\right) \\ &= \frac{n\pi}{P} - \frac{1}{2n\pi}F(a, b) + \frac{c_3P^2}{2i\pi^3n^3} + O\left(\frac{1}{n^4}\right) \end{aligned} \quad (1.32)$$

Note that $\frac{c_3}{i}$ is a real number .

e) Sixth approximation

Since $c_4 = 0$ from (1.30) and (1.32) ,we write

$$\sqrt{\lambda_n} = \frac{n\pi}{P} - \frac{1}{2n\pi}F(a, b) + \frac{c_3P^2}{2i\pi^3n^3} + O\left(\frac{1}{n^5}\right) \quad (1.33)$$

where c_3 and $F(a, b)$ is determined before .In fact we have obtained the following theorem:

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Theorem 1.1 *Let $r, q \in C^\infty(a, b)$, and r be a positive function, then the eigenvalues of the Dirichlet problem (1.1) on $[a, b]$ satisfy (1.24), and (1.33) gives the sixth approximation for $\sqrt{\lambda_n}$.*

Equation (1.3) is called the normal form of equation (1.1). In many mathematical physics problems one encounters equations of type (1.3). By determining eigenvalues experimentally, some information regarding the nature of $R(\xi)$ has then been found. Let us investigate the sixth approximation for normal type. For this we put $r(x) = 1$, and

$$E(t) = \int_a^t q(x) dx. \quad (1.34)$$

Then,

$$\begin{aligned} P &= b - a \\ F(a, b) &= - \int_a^b q(x) dx = -E(b) \\ d_0 &= 1 \\ d_1 &= -\frac{1}{2}E(b) \\ d_2 &= \frac{1}{4}\{q(b) - q(a)\} - \frac{1}{8}E^2(b) \\ d_3 &= \frac{1}{8}\{q'(b) - q'(a)\} - \frac{1}{8}q(b) \int_a^b q(t) dt \\ &\quad - \frac{1}{8} \int_a^b q^2(t) dt + \frac{1}{8}q(a) \int_a^b q(t) dt + \frac{1}{48}E^3(b) \end{aligned}$$

whence

$$c_3 = -\frac{1}{4}\{q'(b) - q'(a)\} - \frac{1}{2}E(b)\{q(a) - q(b)\} + \frac{1}{4} \int_a^b q^2(t) dt - \frac{1}{3}E^3(b)$$

Consequently the sixth approximation for the eigenvalues of equation (1.3) i.e, in normal form, with boundary condition $W(a) = 0 = W(b)$ is of the form (see (1.33)),

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$$\begin{aligned}\sqrt{\lambda_n} &= \frac{n\pi}{b-a} + \frac{1}{2n\pi}E(b) \\ &+ \frac{(b-a)^2}{8\pi^3n^3}\{q'(a) - q'(b) - E(b)(q(a) - q(b)) + \int_a^b q^2(t)dt - E^3(b)\} + O\left(\frac{1}{n^5}\right)\end{aligned}$$

where $E(t)$ is determined in (1.34) .

Now we will illustrate theorem 1.1 with the following example .

Example 1 We wish to find the eigenvalues of the following boundary - value problem

$$\begin{aligned}y'' + \lambda(1+x)^{-2}y &= 0 & 0 \leq x \leq 1 \\ y(0) = 0 &= y(1).\end{aligned}$$

Solutions of this differential equation are of the form

$$y = (1+x)^d$$

where

$$d = \frac{1 \pm \sqrt{1-4\lambda}}{2}.$$

The exact eigenvalues of this problem are

$$\lambda_n = \frac{n^2\pi^2}{\log^2 2} + \frac{1}{4}.$$

By the Binomial theorem one can find for fixed n ,

$$\begin{aligned}\sqrt{\lambda_n} &= \frac{n\pi}{\log 2} \sqrt{1 + \frac{\log^2 2}{4n^2\pi^2}} \\ &= \frac{n\pi}{\log 2} \left\{ 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-k+1)}{k!} \left\{ \frac{\log^2 2}{4n^2\pi^2} \right\}^k \right\} \\ &= \frac{n\pi}{\log 2} + \frac{\log 2}{8n\pi} - \frac{\log^3 2}{128n^3\pi^3} + \frac{\log^5 2}{1024n^5\pi^5} + O\left(\frac{1}{n^7}\right).\end{aligned}$$

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By computing P , $F(0, 1)$ and c_3 in (1.44) one can find the same approximation for the eigenvalues. In fact we have

$$d_0 = \sqrt{2}$$

$$d_1 = -\frac{i\sqrt{2}}{8} \log 2$$

$$d_2 = -\frac{\sqrt{2}}{128} \log^2 2$$

$$d_3 = \frac{i\sqrt{2}}{3072} \log^3 2 + \frac{i\sqrt{2}}{128} \log 2$$

$$c_3 = -\frac{i}{64} \log 2$$

$$P = \log 2$$

$$F(0, 1) = -\frac{1}{4} \log 2.$$

Chapter 2

Eigenvalue asymptotics in the case of one simple turning point

2.1 Introduction

This chapter is concerned with the differential equation

$$y'' - (\lambda^2 f(x) + q(x))y = 0 \quad a \leq x \leq b \quad (2.1)$$

and the boundary condition

$$y(a) = 0 = y(b),$$

where

- (i) $f, q : [a, b] \rightarrow R$ are n times continuously differentiable functions, $2 \leq n$.
- (ii) $f(x_0) = 0$, and $\frac{f(x)}{(x-x_0)^l} = h(x)$ is positive and twice continuously differentiable within (a, b) , where x_0 is an interior point of (a, b) and l is odd.
- (ii) λ is a real parameter.

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In this chapter we will study the distribution of the eigenvalues of equation (2.1) when $f(x)$ has only one zero in the interior of $[a, b]$.

The following points should be noted . First, as a consequence of condition (i), one can find higher approximations for eigenvalues λ . Secondly , hypothesis (ii) implies that the function $\tilde{f}(x)$, defined in (2.4) , is a positive and twice continuously differentiable function (see section 2.2) . Consequently , the function $R(\xi(x))$, defined in (2.11) , is continuous in $[a, b]$. Thirdly , since l is an odd integer , the function $-f(x)$ is positive in $[a, x_0)$ and negative in $(x_0, b]$.

In the literature , the zeros of our function $f(x)$ are called *turning points* or *transition points* of the differential equation (2.1) . Geometrically , the oscillatory / non oscillatory character of a solution is altered to non -oscillatory / oscillatory as one proceeds through a turning point of $f(x)$.

In [17] Langer studied the differential equation (2.1) in which $f(x)$ was assumed to be of the form $x^\alpha g(x)$, where α is a positive number and $g(x)$ is a twice continuously differentiable function without zero . In this paper he found the asymptotic behavior of a solution , simply described in theory but rather cumbersome to apply. So, in order to find the distribution of the eigenvalues in the one *simple* turning point case , we prefer to apply Olver's method .

In section 2.2 , we give a short description of Langer' transformation . In section 2.3 the formal solution introduced by Olver in [29] is discussed and used in order

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to compute the higher order asymptotics of the eigenvalues of (2.1) in which $f(x)$ has a simple turning point, (section 2.4). We note that the leading term of the eigenvalue distribution for (2.1) has been found recently by Atkinson Mingarelli in [2].

2.2 Langer's transformation

Let us define

$$\xi(x) = -\left\{\int_x^{x_0} (-f(t))^{1/2} dt\right\}^{\frac{2}{l+2}} \quad x \leq x_0 \quad (2.2)$$

$$\xi(x) = \left\{\int_{x_0}^x f^{1/2}(t) dt\right\}^{\frac{2}{l+2}} \quad x_0 \leq x \quad (2.3)$$

From (2.2) and (2.3) it is clear that $\xi(x)$ is an increasing function of x . Let $\xi = c, d$ correspond to the endpoints $x = a, b$, respectively, under this transformation. We define

$$\tilde{f}(x) \equiv \left(\frac{d\xi}{dx}\right)^2 = \frac{4f(x)}{(l+2)^2(\xi(x))^l}. \quad (2.4)$$

Now we prove that $\tilde{f}(x)$ is a positive, twice continuously differentiable function. Since by (ii), $f(x) < 0$ for $x \leq x_0$, use of (2.2) shows that

$$\tilde{f}(x) = -\frac{4f(x)}{(l+2)^2\left(\int_x^{x_0} (-f(t))^{1/2} dt\right)^{\frac{2l}{l+2}}} \quad x < x_0, \quad (2.5)$$

and

$$\tilde{f}(x) = \frac{4f(x)}{(l+2)^2\left(\int_{x_0}^x f^{1/2}(t) dt\right)^{\frac{2l}{l+2}}}, \quad x_0 < x. \quad (2.6)$$

The Taylor series for $h^{1/2}(x) = \frac{(-f(x))^{1/2}}{(x_0-x)^{l/2}}$ for $x \leq x_0$, is of the form

$$h^{1/2}(x) = h^{1/2}(x_0) + (x - x_0) \frac{h'(x_0)}{2h^{1/2}(x_0)} + \frac{1}{2}(x - x_0)^2 (h^{1/2})''(\alpha)$$

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where α is some point in $[a, x_0]$. Therefore,

$$\int_x^{x_0} (x_0 - t)^{l/2} h^{1/2}(t) dt = \frac{2}{l+2} (x_0 - x)^{\frac{l+2}{2}} h^{1/2}(x_0) - \frac{1}{l+4} (x_0 - x)^{\frac{l+4}{2}} \frac{h'(x_0)}{h^{1/2}(x_0)} + O(x^{\frac{l+6}{2}})$$

whence

$$\int_x^{x_0} (-f)^{1/2}(t) dt = \int_x^{x_0} (x_0 - t)^{l/2} h^{1/2}(t) dt = (x_0 - x)^{\frac{l+2}{2}} P(x) \quad (2.7)$$

where

$$P(x) = b_0 + b_1(x_0 - x) + O((x_0 - x)^2)$$

$$b_0 = \frac{2}{l+2} h^{1/2}(x_0) \quad b_1 = -\frac{h'(x_0)}{(l+4)h^{1/2}(x_0)}.$$

Therefore

$$\begin{aligned} \lim_{x \rightarrow x_0^-} \tilde{f}(x) &= \lim_{x \rightarrow x_0^-} \frac{4(x_0 - x)^l h(x)}{(l+2)^2 (\int_x^{x_0} (-f)^{1/2}(t) dt)^{\frac{2l}{l+2}}} = \frac{4h(x_0)}{(l+2)^2 P^{\frac{2l}{l+2}}(x_0)} \\ &= \frac{4h^{\frac{2}{l+2}}(x_0)}{2^{\frac{2l}{l+2}} (l+2)^{\frac{4}{l+2}}}. \end{aligned}$$

Similarly, we can show that

$$\lim_{x \rightarrow x_0^+} \tilde{f}(x) = \frac{4h(x_0)}{(l+2)^2 P^{\frac{2l}{l+2}}(x_0)}.$$

Therefore defining $\tilde{f}(x_0)$ to be the common value of the last two limits we find that

$\tilde{f}(x)$ is continuous at $x = x_0$. From (2.7) we have

$$P(x) = \frac{\int_x^{x_0} (-f)^{1/2}(t) dt}{(x_0 - x)^{\frac{l+2}{2}}}.$$

Since $P(x)$ is twice continuously differentiable, $\tilde{f}(x)$ defined in (2.5) is of class $C''(a, b)$ as is not difficult to see.

Now, by using the transformation (2.2) and (2.3), we transform equation (2.1) to a simple form.

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We define a new dependent variable,

$$W(\xi) = \left(\frac{dx}{d\xi}\right)^{-1/2} y(x).$$

Equation (2.1) then reduces to the form

$$\frac{d^2 W}{d\xi^2} = \left\{ \frac{\lambda^2 f(x)}{\left(\frac{dx}{d\xi}\right)^{-2}} + R(\xi) \right\} W(\xi) \quad (2.8)$$

where

$$R(\xi) = \left(\frac{dx}{d\xi}\right)^{1/2} \frac{d^2}{d\xi^2} \left\{ \frac{1}{\left(\frac{dx}{d\xi}\right)^{1/2}} \right\} + \left(\frac{dx}{d\xi}\right)^2 q(x(\xi)). \quad (2.9)$$

By substituting (2.4) in (2.8) we obtain

$$\frac{d^2 W}{d\xi^2} = \left(\frac{l+2}{4}\right)^2 \lambda^2 \xi^l + R(\xi) W \quad (2.10)$$

where, by (2.9);

$$\begin{aligned} R(\xi) &= \frac{1}{\tilde{f}^{1/4}} \frac{d^2}{d\xi^2} (\tilde{f}^{1/4}) + \frac{q(x)}{\tilde{f}(x)} \\ &= -\frac{1}{\tilde{f}^{3/4}} \frac{d^2}{dx^2} (\tilde{f}^{-1/4}) + \frac{q(x)}{\tilde{f}(x)}. \end{aligned} \quad (2.11)$$

2.3 The formal solution in the one turning point case

In (2.10) we put $l = 1$ and $3\lambda/2 = u$ to find

$$\frac{d^2 W}{d\xi^2} = (u^2 \xi + R(\xi)) W(\xi). \quad (2.12)$$

Now we find the distribution of the eigenvalues of (2.12) by means of the asymptotic solutions of the equation with boundary condition $W(c) = W(d) = 0$, where $c < 0 < d$.

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Theorem 2.1 *The differential equation (2.12) has ,for each value of u and each nonnegative integer n , a pair of infinitely differentiable solutions $W_1(u, \xi)$ and $W_2(u, \xi)$ given by their approximations $W_{2n+1,1}(u, \xi)$, $W_{2n+1,2}(u, \xi)$ where*

$$W_{2n+1,1}(u, \xi) = Bi(u^{2/3}\xi) \sum_{s=0}^n \frac{A_s(\xi)}{u^{2s}} + \frac{Bi'(u^{2/3}\xi)}{u^{4/3}} \sum_{s=0}^{n-1} \frac{B_s(\xi)}{u^{2s}} + \epsilon_{2n+1,1}(u, \xi) \quad (2.13)$$

$$W_{2n+1,2}(u, \xi) = Ai(u^{2/3}\xi) \sum_{s=0}^n \frac{A_s(\xi)}{u^{2s}} + \frac{Ai'(u^{2/3}\xi)}{u^{4/3}} \sum_{s=0}^{n-1} \frac{B_s(\xi)}{u^{2s}} + \epsilon_{2n+1,2}(u, \xi) \quad (2.14)$$

where

$$A_0(\xi) = 1,$$

$$B_s(\xi) = \frac{1}{2\xi^{1/2}} \int_0^\xi (R(v)A_s(v) - A_s''(v)) \frac{dv}{v^{1/2}} \quad 0 < \xi, \quad (2.15)$$

$$B_s(\xi) = \frac{1}{2(-\xi)^{1/2}} \int_\xi^0 (R(v)A_s(v) - A_s''(v)) \frac{dv}{(-v)^{1/2}} \quad 0 > \xi, \quad (2.16)$$

$$A_{s+1}(\xi) = -\frac{1}{2}B_s'(\xi) + \frac{1}{2} \int^\xi R(v)B_s(v)dv, \quad (2.17)$$

$\epsilon_{2n+1,i}(u, \xi), i = 1, 2$, are error bounds, and $Ai(u^{2/3}\xi)$, $Bi(u^{2/3}\xi)$ are two independent solutions of

$$\frac{d^2W}{d\xi^2} = u^2\xi W(\xi).$$

called Airy functions .

Proof: See ([29] , chapter 11 ,§7.2) .

Note that $A_s(\xi)$ and $B_s(\xi)$ are infinitely differentiable . This is a consequence of the following result .

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Lemma 2.2 *In a given interval (α, β) containing the origin , assume that $h(\xi)$ and its first n derivatives are continuous real or complex functions , and let $I(\xi)$ be defined by*

$$\frac{1}{\xi^{1/2}} \int_0^\xi h(v) \frac{dv}{v^{1/2}} , 2h(0) , \text{ or } \frac{1}{(-\xi)^{1/2}} \int_0^\xi h(v) \frac{dv}{(-v)^{1/2}}$$

accordingly as $\xi > 0$, $\xi = 0$, or $\xi < 0$. Then $I(\xi)$ is n times continuously differentiable in (α, β) , also , $I^{n+1}(\xi)$ is continuous except possibly at $\xi = 0$.

Proof: See ([29] , chapter 11 ,§7.2) .

Remark 1 : Note that in ([29], chapter 11 ,§7.4),it is proved that the error bounds for large u are uniform with respect to ξ .

Remark 2 : The asymptotic form of $Ai(u^{2/3}\xi)$ and $Bi(u^{2/3}\xi)$ for $\xi > 0$ is not the same as for $\xi < 0$. Therefore , the solutions $W_{2n+1,1}(u, \xi)$, or $W_{2n+1,2}(u, \xi)$ have different asymptotic forms for $\xi > 0$ and $\xi < 0$. In the rest of this chapter we shall use the symbols $W^+(u, \xi), W^-(u, \xi)$ to signify the asymptotic form of $W(u, \xi)$ for $\xi > 0$, $\xi < 0$, respectively, as $u \rightarrow \infty$.

Lemma 2.3 *The asymptotic forms of $Ai(u^{2/3}\xi)$, $Ai'(u^{2/3}\xi)$, $Bi(u^{2/3}\xi)$ and $Bi'(u^{2/3}\xi)$ are given by (for $u \rightarrow \infty$),*

$$Ai(u^{2/3}\xi) \sim \frac{e^{-\frac{2}{3}u\xi^{3/2}}}{2\pi^{1/2}u^{1/6}\xi^{1/4}} \sum_{s=0}^{\infty} (-1)^s \frac{u_s}{(\frac{2}{3}u\xi^{3/2})^s} \quad \text{for } \xi > 0 \quad (2.18)$$

$$Ai'(u^{2/3}\xi) \sim -\frac{u^{1/6}\xi^{1/4}e^{-\frac{2}{3}u\xi^{3/2}}}{2\pi^{1/2}} \sum_{s=0}^{\infty} (-1)^s \frac{v_s}{(\frac{2}{3}u\xi^{3/2})^s} \quad \text{for } \xi > 0 \quad (2.19)$$

$$Ai(u^{2/3}\xi) \sim \frac{1}{\pi^{1/2}u^{1/6}(-\xi)^{1/4}} \left\{ \cos\left(\frac{2}{3}u(-\xi)^{3/2} - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s}}{(\frac{2}{3}u(-\xi)^{3/2})^{2s}} \right. \\ \left. + \sin\left(\frac{2}{3}u(-\xi)^{3/2} - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s+1}}{(\frac{2}{3}u(-\xi)^{3/2})^{2s+1}} \right\} \quad \text{for } \xi < 0 \quad (2.20)$$

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$$\begin{aligned}
 Ai'(u^{2/3}\xi) &\sim \frac{u^{1/6}(-\xi)^{1/4}}{\pi^{1/2}} \left\{ \sin\left(\frac{2}{3}u(-\xi)^{3/2} - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \frac{v_{2s}}{\left(\frac{2}{3}u(-\xi)^{3/2}\right)^{2s}} \right. \\
 &\left. - \cos\left(\frac{2}{3}u(-\xi)^{3/2} - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \frac{v_{2s+1}}{\left(\frac{2}{3}u(-\xi)^{3/2}\right)^{2s+1}} \right\} \quad \text{for } \xi < 0 \quad (2.21)
 \end{aligned}$$

$$Bi(u^{2/3}\xi) \sim \frac{e^{\frac{2}{3}u\xi^{3/2}}}{\pi^{1/2}u^{1/6}\xi^{1/4}} \sum_{s=0}^{\infty} \frac{u_s}{\left(\frac{2}{3}u\xi^{3/2}\right)^s} \quad \text{for } \xi > 0 \quad (2.22)$$

$$Bi'(u^{2/3}\xi) \sim \frac{u^{1/6}\xi^{1/4}e^{\frac{2}{3}u\xi^{3/2}}}{\pi^{1/2}} \sum_{s=0}^{\infty} \frac{v_s}{\left(\frac{2}{3}u\xi^{3/2}\right)^s} \quad \text{for } \xi > 0 \quad (2.23)$$

$$\begin{aligned}
 Bi(u^{2/3}\xi) &\sim \frac{1}{\pi^{1/2}u^{1/6}(-\xi)^{1/4}} \left\{ -\sin\left(\frac{2}{3}u(-\xi)^{3/2} - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s}}{\left(\frac{2}{3}u(-\xi)^{3/2}\right)^{2s}} \right. \\
 &\left. + \cos\left(\frac{2}{3}u(-\xi)^{3/2} - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s+1}}{\left(\frac{2}{3}u(-\xi)^{3/2}\right)^{2s+1}} \right\} \quad \text{for } \xi < 0 \quad (2.24)
 \end{aligned}$$

$$\begin{aligned}
 Bi'(u^{2/3}\xi) &\sim \frac{u^{1/6}(-\xi)^{1/4}}{\pi^{1/2}} \left\{ \cos\left(\frac{2}{3}u(-\xi)^{3/2} - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \frac{v_{2s}}{\left(\frac{2}{3}u(-\xi)^{3/2}\right)^{2s}} \right. \\
 &\left. + \sin\left(\frac{2}{3}u(-\xi)^{3/2} - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \frac{v_{2s+1}}{\left(\frac{2}{3}u(-\xi)^{3/2}\right)^{2s+1}} \right\} \quad \text{for } \xi < 0 \quad (2.25)
 \end{aligned}$$

and

$$Ai(0) = \frac{1}{3^{2/3}\Gamma(2/3)} = \frac{Bi(0)}{\sqrt{3}}, \quad Ai'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)} = -\frac{Bi'(0)}{\sqrt{3}}$$

where $u_0 = 1 = v_0$

$$u_s = \frac{(2s+1)(2s+3)(2s+5)\dots(6s-1)}{(216)^s s!} \quad (2.26)$$

$$v_s = -\frac{6s+1}{6s-1} u_s, \quad s \geq 1 \quad (2.27)$$

Proof: See ([29], chapter 11, §1.1 and §1.2)

2.4 Distribution of the eigenvalues in the one turning point case

In order to find the distribution of the eigenvalues in the one turning point case we need the following theorem which we call the Atkinson – Mingarelli theorem:

Theorem 2.4 *Consider the general weighted Sturm – Liouville equation*

$$(p(x)y')' + (\lambda r(x) - q(x))y = 0 \quad 0 \leq x \leq b \quad (2.28)$$

on $[0, b]$ where $0 < b < \infty, p, q, r : [0, b] \rightarrow \mathbb{R}, 1/p, q, r \in L(0, b)$ and $p > 0$ a.e on $(0, b)$. Whenever

$$\int_0^b \left[\frac{r(x)}{p(x)} \right]_{\pm}^{1/2} dx > 0,$$

the boundary problem (2.28),

$$y(0) \cos \alpha - (py')(0) \sin \alpha = 0,$$

$$y(b) \cos \beta + (py')(b) \sin \beta = 0,$$

has a doubly infinite sequence of real eigenvalues whose corresponding eigenfunctions have precisely n zeros in $(0, b)$, for all sufficiently large n . The eigenvalues admit the asymptotic representation

$$\lambda_n^{\pm} \sim \pm \frac{n^2 \pi^2}{\left\{ \int_0^b \left[\frac{r(x)}{p(x)} \right]_{\pm}^{1/2} dx \right\}^2}$$

as $n \rightarrow \infty$, where $0 \leq \alpha, \beta < \pi$ and f_+ (respectively f_-) denotes the positive (respectively negative part of f), i.e., $f_{\pm}(x) \equiv \max\{f_{\pm}(x), 0\}$.

Proof: See [2]

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The eigenvalues of equation (2.12) with boundary condition

$$W(c) = 0 = W(d)$$

are the zeros of $\Delta(u) = 0$ where

$$\Delta(u) \equiv \begin{vmatrix} W_1(u, d) & W_2(u, d) \\ W_1(u, c) & W_2(u, c) \end{vmatrix}$$

or those u for which

$$\Delta(u) \equiv W_1(u, d)W_2(u, c) - W_2(u, d)W_1(u, c) = 0$$

where W_1 and W_2 are two linearly independent solutions of (2.12) . Two such solutions are given in theorem 2.1 where $W_{2n+1,1}(u, \xi)$ (respectively $W_{2n+1,2}(u, \xi)$), given in (2.13), (respectively (2.14)), denote the approximations to W_1 and W_2 , valid for all sufficiently large u , with the error terms being uniform with respect to ξ for ξ in $[c, d]$.

Thus the eigenvalues are given asymptotically by the roots of

$$\Delta_n(u) \equiv W_{2n+1,1}^-(u, c)W_{2n+1,2}^+(u, d) - W_{2n+1,2}^-(u, c)W_{2n+1,1}^+(u, d) = 0$$

as $n \rightarrow \infty$, where we are using the notation of remark 2, above. In order to compute the asymptotics of these eigenvalues we use the following notations :

$$A(\xi) = \sum_{s=0}^n \frac{A_s(\xi)}{u^{2s}} \tag{2.29}$$

$$B(\xi) = \sum_{s=0}^{n-1} \frac{B_s(\xi)}{u^{2s}} \tag{2.30}$$

$$M_1(\xi) = \sum_{s=0}^n \frac{u_s}{(\frac{2}{3}u\xi^{3/2})^s}, \quad M_2(\xi) = \sum_{s=0}^n (-1)^s \frac{u_s}{(\frac{2}{3}u\xi^{3/2})^s} \tag{2.31}$$

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$$P_1(\xi) = \sum_{s=0}^n \frac{v_s}{\left(\frac{2}{3}u\xi^{3/2}\right)^s}, \quad P_2(\xi) = \sum_{s=0}^n (-1)^s \frac{v_s}{\left(\frac{2}{3}u\xi^{3/2}\right)^s} \quad (2.32)$$

$$R_1(\xi) = \sum_{s=0}^n \frac{u_{2s}}{\left(\frac{2}{3}u\xi^{3/2}\right)^{2s}}, \quad R_2(\xi) = \sum_{s=0}^n (-1)^s \frac{u_{2s}}{\left(\frac{2}{3}u\xi^{3/2}\right)^{2s}} \quad (2.33)$$

$$K_1(\xi) = \sum_{s=0}^n \frac{v_{2s}}{\left(\frac{2}{3}u\xi^{3/2}\right)^{2s}}, \quad K_2(\xi) = \sum_{s=0}^n (-1)^s \frac{v_{2s}}{\left(\frac{2}{3}u\xi^{3/2}\right)^{2s}} \quad (2.34)$$

$$T_1(\xi) = \sum_{s=0}^n \frac{u_{2s+1}}{\left(\frac{2}{3}u\xi^{3/2}\right)^{2s+1}}, \quad T_2(\xi) = \sum_{s=0}^n (-1)^s \frac{u_{2s+1}}{\left(\frac{2}{3}u\xi^{3/2}\right)^{2s+1}} \quad (2.35)$$

$$L_1(\xi) = \sum_{s=0}^n \frac{v_{2s+1}}{\left(\frac{2}{3}u\xi^{3/2}\right)^{2s+1}}, \quad L_2(\xi) = \sum_{s=0}^n (-1)^s \frac{v_{2s+1}}{\left(\frac{2}{3}u\xi^{3/2}\right)^{2s+1}} \quad (2.36)$$

$$\omega(\xi) = \frac{2}{3}u\xi^{3/2} - \frac{\pi}{4} \quad (2.37)$$

By theorem (2.1), using the above notation we write (by lemma 2.3),

$$\begin{aligned} W_{2n+1,1}^-(u, c) &= \frac{A(c)}{\pi^{1/2}u^{1/6}(-c)^{1/4}} \{(-R_2(-c) \sin \omega(-c) + T_2(-c) \cos \omega(-c))\} \\ &+ \frac{(-c)^{1/4}B(c)}{u^{7/6}\pi^{1/2}} \{(K_2(-c) \cos \omega(-c) + L_2(-c) \sin \omega(-c))\} + O\left(\frac{1}{u^{4n+19/6}}\right) \end{aligned}$$

$$\begin{aligned} W_{2n+1,2}^-(u, c) &= \frac{A(c)}{\pi^{1/2}u^{1/6}(-c)^{1/4}} \{R_2(-c) \cos \omega(-c) + T_2(-c) \sin \omega(-c)\} \\ &+ \frac{(-c)^{1/4}B(c)}{u^{7/6}\pi^{1/2}} \{K_2(-c) \sin \omega(-c) - L_2(-c) \cos \omega(-c)\} \\ &+ O\left(\frac{1}{u^{4n+19/6}}\right) \quad (2.38) \end{aligned}$$

$$W_{2n+1,1}^+(u, d) = \frac{e^{\frac{2}{3}ud^{3/2}}}{\pi^{1/2}u^{1/6}d^{1/4}} \left\{ (M_1(d)A(d) + O\left(\frac{1}{u^{n+1}}\right)) \right\} + \frac{d^{1/4}e^{\frac{2}{3}ud^{3/2}}}{\pi^{1/2}u^{7/6}} \left\{ P_1(d)B(d) + O\left(\frac{1}{u^{n+1}}\right) \right\}$$

$$W_{2n+1,2}^+(u, d) = \frac{e^{-\frac{2}{3}ud^{3/2}}}{2\pi^{1/2}u^{1/6}d^{1/4}} \left\{ M_2(d)A(d) + O\left(\frac{1}{u^{n+1}}\right) \right\} - \frac{d^{1/4}e^{-\frac{2}{3}ud^{3/2}}}{2\pi^{1/2}u^{7/6}} \left\{ P_2(d)B(d) + O\left(\frac{1}{u^{n+1}}\right) \right\}$$

whence

$$W_{2n+1,1}^-(u, c)W_{2n+1,2}^+(u, d) = e^{-\frac{2}{3}ud^{3/2}} O\left(\frac{1}{u^{1/3}}\right) \quad (2.39)$$

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and

$$\begin{aligned} W_{2n+1,2}^-(u, c)W_{2n+1,1}^+(u, d) &= e^{\frac{2}{3}ud^{3/2}} \left\{ \frac{M_1(d)A(d)}{\pi^{1/2}u^{1/6}d^{1/4}} + \frac{P_1(d)B(d)d^{1/4}}{\pi^{1/2}u^{7/6}} \right. \\ &\quad \left. + O\left(\frac{1}{u^{n+1}}\right) \right\} W_{2n+1,2}^-(u, c) \end{aligned} \quad (2.40)$$

Now by (2.39) and (2.40) , we write

$$\Delta_n(u) = e^{-\frac{2}{3}ud^{3/2}} O\left(\frac{1}{u^{1/3}}\right) - W_{2n+1,2}^-(u, c)W_{2n+1,1}^+(u, d) = 0. \quad (2.41)$$

Dividing throughout by $u^{-1/6}e^{\frac{2}{3}ud^{3/2}}$ and letting $u \rightarrow \infty$, we see that the zeros of $\Delta_n(u)$ satisfy

$$\left\{ \frac{M_1(d)A(d)}{\pi^{1/2}d^{1/4}} + \frac{P_1(d)B(d)d^{1/4}}{\pi^{1/2}u} + O\left(\frac{1}{u^{n+5/6}}\right) \right\} W_{2n+1,2}^-(u, c) \rightarrow 0$$

as $u \rightarrow \infty$. Now since $M_1(d) \rightarrow 1$ and $A(d) \rightarrow 1$ as $u \rightarrow \infty$, it follows that the zeros of $\Delta_n(u)$ satisfy

$$W_{2n+1,2}^-(u, c) \rightarrow 0. \quad (2.42)$$

as $u \rightarrow \infty$.

From this expression and (2.38) we see that the asymptotic behavior of the eigenvalues depends essentially on the properties of the equation (2.12) on the interval $[c, 0]$, i.e., these depend on the positive part of the weight-function $-\xi$ on $[c, d]$ i.e, on $-\xi$ on $[c, 0]$. This is to be expected by the theorem of Atkinson Mingarelli since we know , a priori , that

$$u_n^2 \sim \frac{n^2 \pi^2}{\left[\int_c^0 (-\xi)^{1/2} d\xi \right]^2}$$

as $n \rightarrow \infty$.

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From (2.38) and (2.42), we find that for large u ,

$$e^{i\{2\omega(-c)+\pi\}} = \frac{M - N/i}{M + N/i} + O\left(\frac{1}{u^{4n+19/6}}\right) \quad (2.43)$$

where

$$\begin{aligned} M &= \frac{A(c)R_2(-c)}{\pi^{1/2}u^{1/6}(-c)^{1/4}} - \frac{B(c)(-c)^{1/4}L_2(-c)}{u^{7/6}\pi^{1/2}} \\ N &= \frac{A(c)T_2(-c)}{\pi^{1/2}u^{1/6}(-c)^{1/4}} + \frac{B(c)(-c)^{1/4}K_2(-c)}{u^{7/6}\pi^{1/2}} \end{aligned} \quad (2.44)$$

Consequently, using (2.33–2.37) and taking the logarithm in (2.43) we obtain

$$u_m = \frac{m\pi - \pi/4}{\frac{2}{3}(-c)^{3/2}} + \frac{1}{\frac{4}{3}(-c)^{3/2}i} \log\left[\frac{M - N/i}{M + N/i} + O\left(\frac{1}{u^{4n+19/6}}\right)\right] \quad (2.45)$$

where

$$\begin{aligned} M - \frac{N}{i} &= \frac{A(c)}{\pi^{1/2}u^{1/6}(-c)^{1/4}} \left\{1 - \sum_{s=1}^n (i)^{2+s} \frac{u_s}{\left(\frac{2}{3}u(-c)^{3/2}\right)^s}\right\} \\ &\quad - \frac{B(c)(-c)^{1/4}}{u^{7/6}\pi^{1/2}} \left\{\frac{1}{i} + \sum_{s=1}^n (i)^{s+3} \frac{v_s}{\left(\frac{2}{3}u(-c)^{3/2}\right)^s}\right\} \\ M + \frac{N}{i} &= \frac{A(c)}{\pi^{1/2}u^{1/6}(-c)^{1/4}} \left\{1 + \sum_{s=1}^n (-i)^s \frac{u_s}{\left(\frac{2}{3}u(-c)^{3/2}\right)^s}\right\} \\ &\quad + \frac{B(c)(-c)^{1/4}}{u^{7/6}\pi^{1/2}} \left\{\frac{1}{i} + \sum_{s=1}^n (-i)^{1+s} \frac{v_s}{\left(\frac{2}{3}u(-c)^{3/2}\right)^s}\right\} \end{aligned}$$

or

$$\begin{aligned} M - N/i &= \frac{1}{\pi^{1/2}(-c)^{1/4}u^{1/6}} \sum_{k=0}^n \sum_{s=0}^k i^{k-s} \frac{A_s(c)u_{k-s}}{u^{k+s} \left\{\frac{2}{3}(-c)^{3/2}\right\}^{k-s}} \\ &\quad - \frac{(-c)^{1/4}}{u^{7/6}\pi^{1/2}} \sum_{k=0}^{n-1} \sum_{s=0}^k i^{k-s+3} \frac{B_s(c)v_{k-s}}{u^{k+s} \left\{\frac{2}{3}(-c)^{3/2}\right\}^{k-s}} \\ &\quad - \frac{(-c)^{1/4}i^{3+n}v_n}{u^{7/6}\pi^{1/2} \left\{\frac{2}{3}u(-c)^{3/2}\right\}^n} \sum_{s=0}^{n-1} \frac{B_s(c)}{u^{2s}}. \end{aligned}$$

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Now

$$\begin{aligned}
 (M - N/i)\pi^{1/2}u^{1/6} &= \frac{1}{(-c)^{1/4}} + \frac{1}{u} \left\{ (-c)^{1/4} {}_i B_0 + \frac{2u_1}{3(-c)^{7/4}} \right\} \\
 &+ \frac{1}{u^2} \left\{ \frac{A_1}{(-c)^{1/4}} - \frac{3B_0 v_1}{2(-c)^{5/4}} - \frac{9u_2}{4(-c)^{13/4}} \right\} \\
 &+ O\left(\frac{1}{u^3}\right)
 \end{aligned} \tag{2.46}$$

$$\begin{aligned}
 M + N/i &= \frac{1}{\pi^{1/2}(-c)^{1/4}u^{1/6}} \sum_{k=0}^n \sum_{s=0}^k (-i)^{k-s} \frac{A_s(c)u_{k-s}}{u^{k+s} \left\{ \frac{2}{3}(-c)^{3/2} \right\}^{k-s}} \\
 &+ \frac{(-c)^{1/4}}{\pi^{1/2}u^{7/6}} \sum_{k=0}^{n-1} \sum_{s=0}^k (-i)^{s+1} \frac{B_s(c)v_{k-s}}{u^{k+s} \left\{ \frac{2}{3}(-c)^{3/2} \right\}^{k-s}} \\
 &+ \frac{(-c)^{1/4}(-i)^{1+n}v_n}{u^{7/6}\pi^{1/2} \left\{ \frac{2}{3}u(-c)^{3/2} \right\}^n} \sum_{s=0}^{n-1} \frac{B_s(c)}{u^{2s}}.
 \end{aligned}$$

$$\begin{aligned}
 (M + \frac{N}{i})\pi^{1/2}u^{1/6} &= \frac{1}{(-c)^{1/4}} + \frac{1}{u} \left\{ -(-c)^{1/4} {}_i B_0 - \frac{2u_1}{3(-c)^{7/4}} \right\} \\
 &+ \frac{1}{u^2} \left\{ \frac{A_1}{(-c)^{1/4}} - \frac{3B_0 v_1}{2(-c)^{5/4}} - \frac{9u_2}{4(-c)^{13/4}} \right\} \\
 &+ O\left(\frac{1}{u^3}\right)
 \end{aligned} \tag{2.47}$$

Consequently,

$$\frac{M - N/i}{M + N/i} = 1 + \frac{2iT}{u} - \frac{2T^2}{u^2} + O\left(\frac{1}{u^3}\right)$$

where

$$T = \frac{3u_1}{2(-c)^{3/2}} + (-c)^{1/2} B_0(c)$$

Since from (2.26), $u_1 = 5/72$, and from (2.16)

$$B_0(c) = \frac{1}{2(-c)^{1/2}} \int_c^0 \frac{R(v)}{(-v)^{1/2}} dv,$$

we see that ,

$$T = \frac{5}{72 \left\{ \frac{2}{3}(-c)^{3/2} \right\}} + \frac{1}{2} \int_c^0 \frac{R(v)}{(-v)^{1/2}} dv. \tag{2.48}$$

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By using the expansion of the logarithm (1.31) and putting

$$x = \frac{2iT}{u} - \frac{2T^2}{u^2} + O\left(\frac{1}{u^3}\right)$$

we have, from(2.45),

$$\log\left[\frac{M - N/i}{M + N/i} + O\left(\frac{1}{u^{4n+19/6}}\right)\right] = \frac{2iT}{u} + O\left(\frac{1}{u^3}\right) \quad (2.49)$$

a) First and second approximation

From (2.45) as $u \rightarrow \infty$, we have that the eigenvalues are given approximately by

$$u_m = \frac{m\pi - \pi/4}{\frac{2}{3}(-c)^{3/2}} + O(1/m) \quad (2.50)$$

Note : The eigenvalues are labeled here in such a way that the eigenfunction corresponding to $u = u_m^2$ has precisely m zeros in (c, d) . This is the Haupt-Richardson oscillation theorem (see [24]). This labeling implies, in most cases, that u_m^2 is not, in fact, the m -th positive eigenvalue of the problem.

b) Third and fourth approximation

From (2.45) and (2.49), we get a further approximation

$$u_m = \frac{m\pi - \pi/4}{\frac{2}{3}(-c)^{3/2}} + \frac{T}{\frac{2}{3}(-c)^{3/2}u} + O\left(\frac{1}{u^3}\right) \quad (2.51)$$

By substituting u from (2.50), we get

$$u_m = \frac{m\pi - \pi/4}{\frac{2}{3}(-c)^{3/2}} + \frac{T}{m\pi} + \frac{T}{4m^2\pi} + O\left(\frac{1}{m^3}\right) \quad (2.52)$$

where T is determined in (2.48).

In summary we have obtained the following theorem.

2. Eigenvalue asymptotics in the case of one simple turning point 31

Theorem 2.5 *Let the real function $R(\xi)$ be infinitely differentiable . Then the distribution of positive Dirichlet eigenvalues u of the differential equation (2.12) satisfy (2.45) and the fourth approximation is given by (2.52) .*

Note that by using the transformation (2.2) and (2.3) we can find the distribution of the eigenvalues , λ , of the differential equation (2.1) in the one turning point case .

Theorem 2.6 *Consider the differential equation*

$$y'' = (\lambda^2 f(x) + q(x))y, \quad a \leq x \leq b$$

with boundary conditions $y(a) = 0 = y(b)$, where for some point $x_0 \in [a, b]$:

- 1) $f(x_0) = 0$
- 2) $\frac{f(x_0)}{x-x_0} = h(x)$ is a positive and twice continuously differentiable function within (a, b) .
- 3) $q(x)$ is continuously differentiable ,
- 4) The integral

$$H(a) = - \int_a^{x_0} \left\{ \frac{1}{(-f)^{1/4}} \frac{d^2}{dx^2} \left(\frac{1}{(-f)^{1/4}} \right) - \frac{q}{(-f)^{1/2}} - \frac{5(-f)^{1/2}}{16(-\xi)^3} \right\} dx$$

converges where $\xi(x)$ is defined in (2.2) .

Then the asymptotic distribution of the positive eigenvalues of this problem is given by

$$\begin{aligned} \lambda_m = & \frac{m\pi - \pi/4}{\int_a^{x_0} (-f(t))^{1/2} dt} + \frac{1}{m\pi} \left\{ \frac{5}{72 \int_a^{x_0} (-f(t))^{1/2} dt} + \frac{1}{6} H(a) \right\} + \\ & + \frac{1}{4m^2\pi} \left\{ \frac{5}{72 \int_a^{x_0} (-f(t))^{1/2} dt} + \frac{1}{6} H(a) \right\} + O\left(\frac{1}{m^3}\right) \end{aligned} \quad (2.53)$$

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Proof : By means of Langer's transformation (2.2 - 2.3) with $l = 1$, we transform equation (2.1) to the simple form (2.12) , where $\frac{3\lambda}{2} = u$. By theorem 2.5 , we know that

$$u_m = \frac{m\pi - \pi/4}{\frac{2}{3}(-c)^{3/2}} + \frac{T}{m\pi} + \frac{T}{4m^2\pi} + O\left(\frac{1}{m^3}\right)$$

where T is specified in (2.48) and $c = \xi(a)$. using this in (2.2) with $l = 1$, we have

$$(-c)^{3/2} = \int_a^{x_0} (-f(t))^{1/2} dt.$$

We also have

$$\int_c^0 \frac{R(\xi)}{(-\xi)^{1/2}} d\xi = \int_a^{x_0} \frac{R(\xi(x))}{(-\xi(x))^{1/2}} \frac{d\xi}{dx} dx = -H(a).$$

Consequently , making use of the above results , (2.53) follows .

Now we find the distribution of the eigenvalues of equation (2.2) when the turning point is at the end point .

The distribution of the eigenvalues of the equation (2.12) , with boundary condition $W(0) = W(c) = 0$ can be obtained by using the transcendental equation $\Delta(u) = 0$, where

$$\Delta(u) = \begin{vmatrix} W_1(u, 0) & W_2(u, 0) \\ W_1(u, c) & W_2(u, c) \end{vmatrix}$$

and , as before , W_1, W_2 are two linearly independent solutions of (2.12) .The approximations to W_1 and W_2 are given in theorem 2.1 . Thus the eigenvalues are given asymptotically by the roots of

$$\Delta_n(u) = \begin{vmatrix} W_{2n+1,1}(u, 0) & W_{2n+1,2}(u, 0) \\ W_{2n+1,1}(u, c) & W_{2n+1,2}(u, c) \end{vmatrix}$$

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as $n \rightarrow \infty$.

From the asymptotic form of $W_{2n+1,i}(u, c)$, $i = 1, 2$, we see that

$$\begin{aligned} \Delta_n(u) &= W_{2n+1,1}(u, 0) \left\{ M \cos \omega(-c) + N \sin \omega(-c) + O\left(\frac{1}{u^{4n+19/6}}\right) \right\} \\ &\quad - W_{2n+1,2}(u, 0) \left\{ N \cos \omega(-c) - M \sin \omega(-c) + O\left(\frac{1}{u^{4n+19/6}}\right) \right\}, \end{aligned} \quad (2.54)$$

where M and N are defined in (2.44). Consequently, the eigenvalues are given asymptotically by those u for which

$$\frac{W_{2n+1,1}(u, 0)}{W_{2n+1,2}(u, 0)} = \frac{N \cos \omega(-c) - M \sin \omega(-c) + O\left(\frac{1}{u^{4n+19/6}}\right)}{M \cos \omega(-c) + N \sin \omega(-c) + O\left(\frac{1}{u^{4n+19/6}}\right)}.$$

The expression on the right simplifies to

$$\frac{W_{2n+1,1}(u, 0)}{W_{2n+1,2}(u, 0)} = \frac{e^{2i\omega(-c)} [N - M/i] + N + M/i + O\left(\frac{1}{u^{4n+19/6}}\right)}{e^{2i\omega(-c)} [M + N/i] + M - N/i + O\left(\frac{1}{u^{4n+19/6}}\right)}.$$

Using the latter in (2.54) we find after a straightforward calculation, that

$$e^{2i\omega(-c)} = \frac{N + \frac{M}{i} - \frac{W_{2n+1,1}(u, 0)}{W_{2n+1,2}(u, 0)} (M - N/i) + O\left(\frac{1}{u^{4n+19/6}}\right)}{\frac{W_{2n+1,1}(u, 0)}{W_{2n+1,2}(u, 0)} (M + N/i) - (N - M/i)}$$

Dividing the numerator and denominator by $-(M + N/i)$ and noting that $M + N/i = O(u^{-1/6})$, by (2.47), we find that

$$e^{2i\omega(-c)} = \frac{\left(\frac{M-N/i}{N+M/i}\right) \left(i + \frac{W_{2n+1,1}(u, 0)}{W_{2n+1,2}(u, 0)}\right) + O\left(\frac{1}{u^{4n+3}}\right)}{-\frac{W_{2n+1,1}(u, 0)}{W_{2n+1,2}(u, 0)} + i}. \quad (2.55)$$

Now, we need to estimate the ratio

$$\frac{W_{2n+1,1}(u, 0)}{W_{2n+1,2}(u, 0)}.$$

From (2.13 -14) we get,

$$\frac{W_{2n+1,1}(u, 0)}{W_{2n+1,2}(u, 0)} = \frac{Bi(0) \left\{ 1 + \frac{A_1(0)}{u^2} + \dots + \frac{A_n(0)}{u^{2n}} + O\left(\frac{1}{u^{2n+2}}\right) \right\} + \frac{Bi'(0)}{u^{4/3}} \left\{ B_0(0) + \dots + \frac{B_{n-1}(0)}{u^{2n-2}} + O\left(\frac{1}{u^{2n}}\right) \right\}}{Ai(0) \left\{ 1 + \frac{A_1(0)}{u^2} + \dots + \frac{A_n(0)}{u^{2n}} + O\left(\frac{1}{u^{2n+2}}\right) \right\} + \frac{Ai'(0)}{u^{4/3}} \left\{ B_0(0) + \dots + \frac{B_{n-1}(0)}{u^{2n-2}} + O\left(\frac{1}{u^{2n}}\right) \right\}}$$

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$$= \sqrt{3} \frac{\{1 + \frac{A_1(0)}{u^2} + \dots + \frac{A_n(0)}{u^{2n}} + O(\frac{1}{u^{2n+2}})\} + \frac{\alpha}{u^{4/3}} \{B_0(0) + \dots + \frac{B_{n-1}(0)}{u^{2n-2}} + O(\frac{1}{u^{2n}})\}}{\{1 + \frac{A_1(0)}{u^2} + \dots + \frac{A_n(0)}{u^{2n}} + O(\frac{1}{u^{2n+2}})\} - \frac{\alpha}{u^{4/3}} \{B_0(0) + \dots + \frac{B_{n-1}(0)}{u^{2n-2}} + O(\frac{1}{u^{2n}})\}} \quad (2.56)$$

where we have used the relations $A_i(0) = \frac{B_i(0)}{\sqrt{3}}$, $\frac{B_i'(0)}{B_i(0)} = \alpha$, $\frac{A_i'(0)}{B_i(0)} = -\frac{\alpha}{\sqrt{3}}$. Dividing out the expression (2.56) we find

$$\frac{W_{2n+1,1}(u, 0)}{W_{2n+1,2}(u, 0)} = \sqrt{3} \left\{ 1 + \frac{2\alpha B_0(0)}{u^{4/3}} + \frac{2\alpha^2 B_0^2(0)}{u^{8/3}} + O\left(\frac{1}{u^{10/3}}\right) \right\} \quad (2.57)$$

Combining (2.55) and (2.57) and using the display following (2.47), viz.,

$$\frac{M - N/i}{M + N/i} = 1 + \frac{2iT}{u} - \frac{2T^2}{u^2} + O\left(\frac{1}{u^3}\right),$$

we obtain, after a lengthy but simple calculation,

$$e^{2i\omega(-c)} = \frac{(i + \sqrt{3}) \left\{ 1 + \frac{2iT}{u} + \frac{2\sqrt{3}}{i + \sqrt{3}} \left(\frac{\alpha B_0}{u^{4/3}} \right) - \frac{2T^2}{u^2} + \frac{4\alpha\sqrt{3}iTB_0}{(i + \sqrt{3})u^{7/3}} + O\left(\frac{1}{u^{8/3}}\right) \right\}}{i - \sqrt{3} - \frac{2\alpha\sqrt{3}B_0}{u^{4/3}} - \frac{2\alpha^2\sqrt{3}B_0^2}{u^{8/3}} + O\left(\frac{1}{u^{10/3}}\right)},$$

where we replace $B_0(0)$ by B_0 , for simplicity. We factor the quantity $i - \sqrt{3}$ from the denominator and divide out the resulting expression. We obtain,

$$e^{2i\omega(-c)} = \frac{i + \sqrt{3}}{i - \sqrt{3}} \left\{ 1 + \frac{2iT}{u} - \frac{i\sqrt{3}\alpha B_0}{u^{4/3}} - \frac{2T^2}{u^2} \right\} + O\left(\frac{1}{u^{7/3}}\right).$$

By using the preceding formula we can obtain the fourth approximation, e.g.,

$$u_m = \frac{m\pi - \pi/12}{\frac{2}{3}(-c)^{3/2}} + \frac{T}{m\pi} - \frac{\alpha\sqrt{3}}{2} B_0 \frac{\left\{ \frac{2}{3}(-c)^{3/2} \right\}^{1/3}}{(m\pi)^{4/3}} + O\left(\frac{1}{m^2}\right) \quad (2.58)$$

where T is determined in (2.48) and B_0 is given in (2.16).

By means of (2.58) and Langer's transformation one can get the fourth approximation of the eigenvalues of equation (2.1) with boundary conditions $y(a) = 0 = y(x_0)$. In fact we have the following corollary:

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Corollary 2.7 *Let $f(x)$ and $H(a)$ satisfy the assumptions of theorem 2.6 . Then the fourth approximation for the eigenvalues of equation (2.1) with boundary conditions $y(a) = 0 = y(x_0)$, is of the form*

$$\lambda_m = \frac{m\pi - \pi/12}{\int_a^{x_0} (-f(t))^{1/2} dt} + \left\{ \frac{5}{72 \int_a^{x_0} (-f(t))^{1/2} dt} + \frac{1}{6} H(a) \right\} \frac{1}{m\pi} + O\left(\frac{1}{m^{4/3}}\right) \quad (2.59)$$

where

$$H(a) = - \int_a^{x_0} \left\{ \frac{1}{(-f)^{1/4}} \frac{d^2}{dx^2} \left(\frac{1}{(-f)^{1/4}} \right) - \frac{q}{(-f)^{1/2}} - \frac{5(-f)^{1/2}}{16(-\xi)^3} \right\} dx$$

and $\xi = \xi(x)$ is defined in (2.2) .

Chapter 3

Eigenvalue asymptotics in the case of one turning point of order l

3.1 Introduction

In this chapter we will seek the distribution of the real eigenvalues of the equation

$$\frac{d^2W}{d\xi^2} = \left(\frac{(l+2)^2}{4}\lambda^2\xi^l\right)W \quad c \leq \xi \leq d \quad (3.1)$$

with boundary condition

$$W(c) = 0 = W(d) \quad -\infty < c < 0 < d < +\infty$$

where l is an odd integer and λ is a real parameter .

In the preceding chapter , our research on the asymptotic behaviour of the Dirichlet eigenvalues of the equation (2.1) shows that the formal solutions constructed by Olver and the Atkinson - Mingarelli theorem are key factors. Studying

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the distribution of the eigenvalues of the equation (3.1) by the same method represents the major motivation of this chapter.

Langer in ([16], theorem 1a) showed that the eigenvalues of (3.1) are given asymptotically by the formula

$$\lambda_m^2 = (-c)^{-(l+2)} \left\{ (m - 1/4)^2 \pi^2 + \left(\frac{1}{l+2}, 1 \right) \right\} + \frac{B(m)}{m^2}$$

where $B(m)$ designates a bounded function and the symbol (r, m) , which was introduced by Hankel, represents the quantity

$$\begin{aligned} (r, m) &= \frac{\Gamma(r + m + 1/2)}{m! \Gamma(r - m + 1/2)} \\ &= \frac{(4r^2 - 1)(4r^2 - 9) \dots (4r^2 - (2m - 1)^2)}{2^{2m} m!} \\ (r, 0) &= 1 \end{aligned}$$

3.2 The formal solution

Olver in ([30], theorem 3) showed that the equation (3.1) has two twice continuously differentiable solutions $\bar{W}(\lambda^{2/l+2} \xi)$ and $W(\lambda^{2/l+2} \xi)$ which can be identified in term of Bessel functions (or modified Bessel functions) of order $\frac{1}{l+2}$. The asymptotic expansion of the solutions are of the form, [[30], §2.3], where we set $m = l + 2$, for convenience,

$$W(\lambda^{2/m} \xi) = \left\{ \frac{\pi \lambda^{2/m} (-\xi)}{2} \right\}^{1/2} \left\{ \cot\left(\frac{\pi}{2m}\right) J_{\frac{1}{m}}(\lambda(-\xi)^{m/2}) - Y_{\frac{1}{m}}(\lambda(-\xi)^{m/2}) \right\} \quad \xi < 0 \quad (3.2)$$

$$W(\lambda^{2/m} \xi) = \left\{ \frac{2\lambda^{2/m} \xi}{\pi} \right\}^{1/2} K_{\frac{1}{m}}(\lambda \xi^{m/2}) \quad \xi > 0$$

$$W(0) = \pi^{-1/2} 2^{2/m} \Gamma\left(\frac{1}{m}\right) \quad (3.3)$$

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$$\bar{W}(\lambda^{\frac{2}{m}}\xi) = -\left\{\frac{\pi\lambda^{\frac{2}{m}}(-\xi)}{2}\right\}^{1/2}\left\{\tan\left(\frac{\pi}{2m}\right)J_{\frac{1}{m}}(\lambda(-\xi)^{\frac{m}{2}}) + Y_{\frac{1}{m}}(\lambda(-\xi)^{\frac{m}{2}})\right\} \quad \xi < 0 \quad (3.4)$$

$$\bar{W}(\lambda^{\frac{2}{m}}\xi) = \left\{\frac{2\lambda^{\frac{2}{m}}\xi}{\pi}\right\}^{1/2}\left\{\frac{\pi}{\sin\frac{\pi}{2m}}I_{\frac{1}{m}}(\lambda\xi^{\frac{m}{2}}) + K_{\frac{1}{m}}(\lambda\xi^{\frac{m}{2}})\right\} \quad \xi > 0$$

$$\bar{W}(0) = \pi^{-\frac{1}{2}}2^{\frac{2-m}{2m}}\Gamma\left(\frac{1}{m}\right) \quad (3.5)$$

where J, I, K, Y are Bessel functions .

In ([1] , §9.2.5 and §9.7.1) one may find the asymptotic expansions for large argument of Bessels functions. We have as $\lambda \rightarrow \infty$,

$$J_{\frac{1}{m}}(\lambda\xi^{\frac{m}{2}}) = \left\{\frac{2}{\pi\lambda\xi^{\frac{m}{2}}}\right\}^{1/2}\{P \cos \chi - Q \sin \chi\} \quad \xi > 0 \quad (3.6)$$

$$Y_{\frac{1}{m}}(\lambda\xi^{\frac{m}{2}}) = \left\{\frac{2}{\pi\lambda\xi^{\frac{m}{2}}}\right\}^{1/2}\{P \sin \chi + Q \cos \chi\} \quad \xi > 0 \quad (3.7)$$

$$K_{\frac{1}{m}}(\lambda\xi^{\frac{m}{2}}) \sim \left\{\frac{\pi}{2\lambda\xi^{\frac{m}{2}}}\right\}^{1/2}e^{-\lambda\xi^{\frac{m}{2}}}R \quad \xi > 0$$

$$I_{\frac{1}{m}}(\lambda\xi^{\frac{m}{2}}) \sim \left\{\frac{1}{2\lambda\xi^{\frac{m}{2}}\pi}\right\}^{1/2}e^{\lambda\xi^{\frac{m}{2}}}T \quad \xi > 0$$

where

$$\chi = \lambda\xi^{\frac{m}{2}} - \left(\frac{1}{2m} + 1/4\right)\pi \quad m = l + 2$$

$$P \sim \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{1}{m}, 2k)}{(2\lambda\xi^{m/2})^{2k}} = 1 - \frac{(\alpha - 1)(\alpha - 9)}{2!(8\lambda\xi^{m/2})^2} + \dots$$

$$Q \sim \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{1}{m}, 2k + 1)}{(2\lambda\xi^{m/2})^{2k+1}} = \frac{\alpha - 1}{8\lambda\xi^{m/2}} - \dots$$

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$$\begin{aligned}
 R &= 1 + \frac{\alpha - 1}{8(\lambda\xi^{m/2})} + \frac{(\alpha - 1)(\alpha - 9)}{2!(8\lambda\xi^{m/2})^2} + \dots \\
 T &= 1 - \frac{\alpha - 1}{8(\lambda\xi^{m/2})} + \frac{(\alpha - 1)(\alpha - 9)}{2!(8\lambda\xi^{m/2})^2} - \dots \\
 \alpha &= \frac{4}{m^2} = \frac{4}{(l+2)^2}
 \end{aligned} \tag{3.8}$$

3.3 Distribution of the eigenvalues

The eigenvalues of the equation (3.1) with the boundary condition

$$W(c) = 0 = W(d)$$

are zeros of $\Delta(\lambda) = 0$ where

$$\Delta(\lambda) = \begin{vmatrix} W(\lambda \frac{2}{m} d) & \bar{W}(\lambda \frac{2}{m} d) \\ W(\lambda \frac{2}{m} c) & \bar{W}(\lambda \frac{2}{m} c) \end{vmatrix}.$$

From the asymptotic expansion of the solutions given above, we get

$$\begin{aligned}
 \Delta(\lambda) &= -\frac{1}{\cos \frac{\pi}{2m}} K_{\frac{1}{m}}(\lambda d^{\frac{m}{2}}) J_{\frac{1}{m}}(\lambda(-c)^{\frac{m}{2}}) \\
 &+ \pi I_{\frac{1}{m}}(\lambda d^{\frac{m}{2}}) \{Y_{\frac{1}{m}}(\lambda(-c)^{\frac{m}{2}}) - \cot(\frac{\pi}{2m}) J_{\frac{1}{m}}(\lambda(-c)^{\frac{m}{2}})\} = 0.
 \end{aligned}$$

By substituting the asymptotic expansion of the Bessel functions into the transcendental equation $\Delta = 0$, it is found that, for large λ , the eigenvalues satisfy the asymptotic relation :

$$O\left(\frac{1}{\lambda}\right)e^{-\lambda d^{m/2}} + (O\left(\frac{1}{\sqrt{\lambda}}\right)e^{\lambda d^{m/2}} T) \{Y_{\frac{1}{m}}(\lambda(-c)^{\frac{m}{2}}) - \cot(\frac{\pi}{2m}) J_{\frac{1}{m}}(\lambda(-c)^{\frac{m}{2}})\} = 0$$

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Multiplying throughout by $\sqrt{\lambda}e^{-\lambda d^{m/2}}$ and letting $\lambda \rightarrow \infty$, we see that the eigenvalues are given asymptotically by the roots of the equation

$$O(1)T\{Y_{\frac{1}{m}}(\lambda(-c)^{\frac{m}{2}}) - \cot(\frac{\pi}{2m})J_{\frac{1}{m}}(\lambda(-c)^{\frac{m}{2}})\} = 0$$

where T is given in (3.8). By the very nature of T it follows that as $\lambda \rightarrow \infty$, the eigenvalues correspond asymptotically to the large roots of

$$Y_{\frac{1}{m}}(\lambda(-c)^{\frac{m}{2}}) - \cot(\frac{\pi}{2m})J_{\frac{1}{m}}(\lambda(-c)^{\frac{m}{2}}) = 0$$

consequently, by (3.6) and (3.7) we find

$$\tan(\chi + \frac{\pi}{2m}) \sim \frac{P}{Q} \quad (3.9)$$

where P, Q and χ are determined in (3.8). Thus,

$$\begin{aligned} \lambda(-c)^{m/2} - \pi/4 &\sim n\pi - \pi + \arctan \frac{P}{Q} \\ &\sim n\pi - \pi + \arctan\{\beta_1\lambda b + \frac{\beta_2}{\lambda b} + O(\frac{1}{\lambda^3})\} \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} \beta_1 &= \frac{4}{\alpha - 1} \\ \beta_2 &= -\frac{(\alpha - 9)(\alpha + 11)}{3(\alpha - 1)2^2} \\ b &= 2(-c)^{m/2} \\ \alpha &= \frac{4}{m^2} = \frac{4}{(l+2)^2}. \end{aligned} \quad (3.11)$$

The series expansion of $\arctan z$ for $|z| > 1$ is of the form

$$\arctan z = \frac{\pi}{2} - \frac{1}{z} + \frac{1}{3z^3} - \frac{1}{5z^5} + \dots$$

See ([1], §4.4).

Therefore from (3.9) and (3.10) we have,

$$\begin{aligned} \lambda(-c)^{m/2} - \pi/4 &\sim n\pi - \frac{\pi}{2} - \left\{ \frac{1}{\lambda\beta_1 b} - \frac{\beta_2}{\lambda^3\beta_1^2 b^3} + \dots \right\} \\ &+ \frac{1}{3} \left\{ \frac{1}{\lambda^3\beta_1^3 b^3} - \frac{3\beta_2}{\lambda^5\beta_1^4 b^5} + \dots \right\} - \frac{1}{5} \left\{ \frac{1}{\lambda^5\beta_1^5 b^5} + \dots \right\} + \dots \end{aligned} \quad (3.12)$$

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By applying the method of the first chapter we get the fifth approximation of the n th eigenvalue as

$$\lambda_n = \frac{n\pi - \pi/4}{(-c)^{m/2}} - \frac{1}{n\pi\beta_1 b} - \frac{1}{4\pi\beta_1 b n^2} + \frac{(-c)^{3m/2}}{n^3 b^3 \pi^3 \beta_1^2} \left\{ \beta_2 + \frac{1}{3\beta_1} \right\} + O\left(\frac{1}{n^4}\right).$$

Finally , from (3.11) we get

$$\lambda_n = \frac{n\pi - \pi/4}{(-c)^{m/2}} - \frac{4 - m^2}{8n\pi m^2 (-c)^{m/2}} - \frac{4 - m^2}{32\pi n^2 m^2 (-c)^{m/2}} - \frac{(4 - m^2)(4 - 25m^2)}{384m^4 n^3 \pi^3 (-c)^{m/2}} + O\left(\frac{1}{n^4}\right) \quad (3.13)$$

We then have the following theorem :

Theorem 3.1 *Let l be an odd number in equation (3.1) . The positive eigenvalues are given asymptotically by the formulae (3.9) and the fifth approximation for the n th positive eigenvalue is of the form (3.13).*

We shall now begin the investigation of the asymptotic distribution of the positive real eigenvalues of the equation (3.1) with boundary condition

$$W(c) = 0 = W(0).$$

Repeating the argument used in proving theorem 3.1 , we find that the transcendental equation

$$\Delta(\lambda) = 0 = \begin{vmatrix} W(0) & \bar{W}(0) \\ W(\lambda^{2/(l+2)} c) & \bar{W}(\lambda^{2/(l+2)} c) \end{vmatrix}$$

gives the eigenvalues . From (3.2 - 3.5), we get that these eigenvalues are given asymptotically by those λ 's such that

$$J_{1/m}(\lambda(-c)^{m/2}) = 0.$$

By using Stokes' method for calculating the zeros of Bessel functions , we get

$$\lambda_n(-c)^{m/2} = \beta - \frac{\alpha - 1}{8\beta} - \frac{4(\alpha - 1)(7\alpha - 31)}{3(8\beta)^3} - \dots$$

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where

$$\beta = \left(n + \frac{1}{2m} - 1/4\right)\pi \quad \alpha = \frac{4}{m^2}$$

For more details see ([1] , §9.5.12) .

We have the following lemma:

Lemma 3.2 *Let*

$$r(\xi) = -\frac{(l+2)^2}{4}\xi^l \quad c \leq \xi \leq 0$$

and l be an odd integer . Then the asymptotic formula for the n th positive eigenvalue of the boundary value problem

$$y'' + \lambda^2 r(\xi)y = 0 \quad c \leq \xi \leq 0$$

$$y(c) = 0 = y(0)$$

is of the form

$$\lambda_n = \frac{n\pi}{\int_c^0 \sqrt{r(\xi)}d\xi} + \frac{\left(\frac{1}{2(l+2)} - 1/4\right)\pi}{\int_c^0 \sqrt{r(\xi)}d\xi} - \frac{\alpha - 1}{8\beta \int_c^0 \sqrt{r(\xi)}d\xi} - \frac{4(\alpha - 1)(7\alpha - 31)}{3(8\beta)^3 \int_c^0 \sqrt{r(\xi)}d\xi} - \dots$$

where

$$\beta = \left(n + \frac{1}{2(l+2)} - 1/4\right)\pi \quad \alpha = \frac{4}{(l+2)^2}.$$

Chapter 4

The infinite product representation

4.1 Introduction

In the second chapter we found that the distribution of positive eigenvalues u_n , $1 \leq n$, of the boundary value problem

$$y'' + (\lambda t - q(t))y = 0 \quad -1 \leq t \leq 1 \quad (4.1)$$

with boundary conditions (the Dirichlet problem on $[-1, 1]$),

$$y(1) = 0 = y(-1)$$

where q is m times continuously differentiable, is of the form

$$\sqrt{u_n} = \frac{n\pi - \pi/4}{\int_0^1 \sqrt{x} dx} + \frac{1}{2n\pi} T + O\left(\frac{1}{n^2}\right)$$

where

$$T = \frac{5}{72 \int_0^1 \sqrt{x} dx} + \frac{1}{2} \int_0^1 \frac{q(x)}{\sqrt{x}} dx$$

and the distribution of the negative eigenvalues λ_n is of the form

$$\sqrt{-\lambda_n} = \frac{n\pi - \pi/4}{\int_{-1}^0 \sqrt{-x} dx} + \frac{1}{2n\pi} T + O\left(\frac{1}{n^2}\right)$$

where

$$T = \frac{5}{72 \int_{-1}^0 \sqrt{-x} dx} + \frac{1}{2} \int_{-1}^0 \frac{q(x)}{\sqrt{-x}} dx.$$

It is natural to ask whether these eigenvalues actually characterize the solution of (4.1). For instance, let, for fixed $x > 0$, $u_n(x)$ and $r_n(x)$ be respectively positive and negative eigenvalues of the Dirichlet problem on $[-1, x]$. Can one represent the solution of (4.1) in the form of an infinite product? What does the asymptotic distribution of $u'_n(x)$ look like?

Let $U(t, u_n(x))$ be eigenfunction corresponding to the eigenvalue $u_n(x)$. Can we find the asymptotic expansion of the integral $\int_{-1}^x v U^2(v, u_n) dv$ for large u_n ? These questions will occupy us in this chapter.

In section 4.2 there appears the representation of the solution of (4.1) in the form of an infinite product when $0 \leq q(t)$.

Section 4.3 and 4.4 is devoted to the study of some properties of the eigenfunctions.

4.2 Representation of the solution in the form of an infinite product .

Before representing the solution in the form of an infinite product , we make a slight digression into complex analysis.

An *entire function* is a function that is analytic at each point of the complex plane .

There is a connection between the growth of an entire function and the distribution of its zeros .In order to estimate the growth of an entire function $f(z)$ we introduce the function

$$M_f(r) = \max_{|z|=r} |f(z)|$$

An entire function $f(z)$ is said to be a function of *finite order* if there exists a positive constant k such that the inequality

$$M_f(r) < e^{r^k}$$

is valid for all sufficiently large values of r ($r > r_0(k)$) .

The greatest lower bound of such numbers k is called the *order* of the entire function $f(z)$.

It follows from this definition that if l is the order of the entire function $f(z)$, and if ϵ is an arbitrary positive number , then

$$e^{r^{l-\epsilon}} < M_f(r) < e^{r^{l+\epsilon}}$$

where the inequality on the right is satisfied for all sufficiently large values of r , and the inequality on the left holds for some sequence $\{r_n\}$ of values of r , tending to infinity. *An inequality that holds for all sufficiently large values of r will be called an asymptotic inequality* [19]. For functions of a given order a more precise characterization of the growth is given by the *type* of the function. By the type, σ , of an entire function $f(z)$ of order l we mean the greatest lower bound of positive numbers A for which asymptotically

$$M_f(r) < e^{Ar^l}$$

holds. If $\sigma = 0$, the function $f(z)$ is said to be of *minimal type* while, if $0 < \sigma < \infty$, it is said to be of *normal type*, and if $\sigma = \infty$, it is of *maximal type*.

Let $f(z)$ be an entire function, with zeros a_n , arranged in order of increasing magnitude. For the sake of simplicity we will assume that $f(0) \neq 0$. The *genus* of an entire function is the smallest integer h such that $f(z)$ can be represented in the form

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \dots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h}$$

where $g(z)$ is a polynomial of degree $\leq h$. (See the next theorem).

Theorem 4.1 {Hadamard's theorem} *The entire function $f(z)$ of finite order l can be represented in the form*

$$f(z) = z^m e^{g(z)} \prod_n \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \dots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h}$$

where $a_n \neq 0$ are the zeros of $f(z)$, $h \leq l$, $g(z)$ is a polynomial whose degree q does not exceed l and m is the multiplicity of the zero of $f(z)$ at the origin.

Proof: See [19] page 24, or [33] page 250.

The genus and the order are closely related , as seen by the following theorem :

Theorem 4.2 *The genus and the order of an entire function satisfy the double inequality $h \leq l \leq h + 1$.*

Proof : See [19] , page 24 .

By means of Hadamard's theorem , one can find an infinite expansion for e.g., $\sinh z$, $J_\nu(z)$ and $J'_\nu(z)$.

Lemma 4.3 *Let $c > 0$ be fixed . Then*

$$\sinh c\sqrt{z} = c\sqrt{z} \prod_{m=1}^{\infty} \left(1 + \frac{z}{z_m^2}\right)$$

where $z_m = \frac{m\pi}{c}$, $1 \leq m$, and the domain of the function $f(z) = z^{1/2}$, is the complement of negative real axis $z \leq 0$, while the range of $z^{1/2}$ is the right half of the z plane with the imaginary axis excluded .

Proof : It is well known that [1] , §4.5.68 , for any z ,

$$\sinh z = z \prod_{m=1}^{\infty} \left(1 + \frac{z^2}{m^2\pi^2}\right)$$

Therefore

$$\sinh c\sqrt{z} = c\sqrt{z} \prod \left(1 + \frac{zc^2}{m^2\pi^2}\right) = c\sqrt{z} \prod \left(1 + \frac{z}{z_m^2}\right).$$

Lemma 4.4 *Let $J_{1/3}$ be the Bessel function of order $1/3$, and b be a positive number . Then*

$$J_{1/3}(ib\sqrt{\lambda}) = \frac{\{i\sqrt{\lambda}b/2\}^{1/3}}{\Gamma(4/3)} \prod_{m=1}^{\infty} \left(1 + \frac{b^2\lambda}{j_m^2}\right)$$

where j_m 's, $m = 1, 2, \dots$, are the positive zeros of $J_{1/3}(z)$ and for complex λ , $\sqrt{\lambda}$ is defined as in lemma 4.3 .

Proof : From [1] , §9.5.10 , we have

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} \prod \left(1 - \frac{z^2}{j_m^2}\right)$$

where

$$j_m \sim \beta - \frac{\alpha-1}{8\beta} - \frac{4(\alpha-1)(7\alpha-31)}{3(8\beta)^3} - \dots$$

and

$$\beta = (m + \nu/2 - 1/4)\pi, \quad \alpha = 4\nu^2.$$

By inserting $z = ib\sqrt{\lambda}$, and $\nu = 1/3$, we get

$$J_{1/3}(ib\sqrt{\lambda}) = \frac{\{ib\sqrt{\lambda}/2\}^{1/3}}{\Gamma(4/3)} \prod \left(1 + \frac{b^2\lambda}{j_m^2}\right)$$

where

$$j_m^2 = m^2\pi^2 - \frac{m\pi^2}{6} + O(1) \quad (4.2)$$

Lemma 4.5 *Let $J_1'(z)$ be the derivative of the Bessel function of order 1 and c be a positive constant . Then*

$$J_1'(c\sqrt{\lambda}) = \frac{1}{2} \prod \left(1 - \frac{\lambda c^2}{\tilde{j}_m^2}\right)$$

$$J_1'(ic\sqrt{\lambda}) = \frac{1}{2} \prod \left(1 + \frac{\lambda c^2}{\tilde{j}_m^2}\right)$$

where the \tilde{j}_m 's , $m = 1, 2, \dots$ are the positive zeros of $J_1'(z)$ and for λ complex , $\sqrt{\lambda}$ is defined as in lemma 4.3 .

Proof : From [1] , §9.5.11 , we have

$$J_\nu'(z) = \frac{(z/2)^{\nu-1}}{2\Gamma(\nu)} \prod \left(1 - \frac{z^2}{\tilde{j}_m^2}\right) \quad (\nu > 0)$$

where

$$\tilde{j}_m \sim \beta' - \frac{\alpha+3}{8\beta'} - \frac{4(7\alpha^2+82\alpha-9)}{3(8\beta')^3} - \dots$$

$$\beta' = (m + \nu/2 - 3/4)\pi \quad \alpha = 4\nu^2$$

by putting $z = c\sqrt{\lambda}$, $\Gamma(1) = 1$ we have

$$J_1'(c\sqrt{\lambda}) = \frac{1}{2} \prod (1 - \frac{\lambda c^2}{\tilde{j}_m^2})$$

and similarly we can get

$$J_1'(\iota c\sqrt{\lambda}) = \frac{1}{2} \prod (1 + \frac{\lambda c^2}{\tilde{j}_m^2})$$

where

$$\tilde{j}_m^2 = m^2\pi^2 - \frac{m\pi^2}{2} + O(1) \quad (4.3)$$

The following theorems play an important role in estimating the infinite product.

Theorem 4.6 $\prod_0^\infty (1 + p_n)$ converges absolutely if and only if $\sum_0^\infty p_n$ converges absolutely, where the p_n are arbitrary complex constants.

Proof : See [11] §18.13 .

Theorem 4.7 If $p_n(z)$ is analytic in a simply connected domain D , and if $\sum_0^\infty |p_n(z)|$ converges uniformly in every closed region R of D , then

$$\prod_0^\infty (1 + p_n(z))$$

converges uniformly to $f(z)$ in every such R and $f(z)$ is analytic in D .

Proof : See [11] §18.19 .

Theorem 4.8 (a) Suppose $a_{mn}, m, n > 1$, are complex numbers satisfying

$$|a_{mn}| = O\left(\frac{1}{|m^2 - n^2|}\right) \quad m \neq n$$

then, for each $1 \leq n$,

$$\prod_{1 \leq m, m \neq n} (1 + a_{mn}) = 1 + O\left(\frac{\log n}{n}\right)$$

(b) In addition, if $b_n, 1 \leq n$, is a square summable sequence of complex numbers, then

$$\prod_{m, n > 1, m \neq n} (1 + a_{mn} b_n) < \infty$$

Proof : See [35], p.165 .

Now we consider the equation

$$y'' + (\lambda r(t) - q(t))y = 0 \tag{4.4}$$

when λ is a complex parameter and $r(t)$, $q(t)$ are real functions locally Lebesgue integrable on a real open interval (a, b) , and $r(t) \neq 0$ a.e in (a, b) . Basic existence theory, see e.g, [6] p. 37 contains the fact (based on the uniform convergence of successive approximations) that every solution of (4.4) is an entire function of λ for any fixed $t \in (a, b)$, (y, y' having fixed values, independent of λ , at some fixed $c \in (a, b)$). Halvorsen, [12], proved the following theorem.

Theorem 4.9 For $a < c < t < b$ assume $\int_c^t |r| \neq 0$, and consider a solution $y(t; \lambda)$ of (4.4) determined by fixed values of y, y' at c . Then $y(t; \lambda)$ is an entire function of λ of order $1/2$ and normal type .

In equation (4.4) if r changes sign in (a, b) , on sets of positive Lebesgue measure, by the Atkinson - Mingarelli theorem the equation (4.4) has a doubly

infinite sequence of positive and negative real eigenvalues , in the case of the Dirichlet problem on the finite interval (a , b) .

If r is a *positive function* on $[a , b]$, the equation has infinite number of positive eigenvalues for the Dirichlet problem on $[a , b]$, and in fact we have the following theorem.

Theorem 4.10 *There is an infinite number of eigenvalues $\lambda_0, \lambda_1, \dots$, forming a monotone increasing sequence with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Moreover , the eigenfunction corresponding to λ_n has exactly n zeros on (a , b) .*

Proof: See [6] , p.212 .

Now let $U(t, \lambda)$ solve the initial value problem (4.1) with initial condition

$$U(-1, \lambda) = 0 \quad \frac{\partial U}{\partial t}(-1, \lambda) = 1. \quad (4.5)$$

By theorem (4.9) , $U(x, \lambda)$ is an entire function of order $1/2$ for each x . The function $U(x, \lambda)$ has a zero set for each x , say $\{\lambda_n(x)\}$, so that $U(x, \lambda_n(x)) = 0$, which corresponds to eigenvalues of the Dirichlet problem for equation (4.1) on the closed interval $[-1, x]$. Note that $\lambda_n(x) \neq 0$ for any x by Sturm's comparison theorem since we assume that $0 \leq q(t)$.

Indeed , each *non-negative continuous function* $q(x)$ defines $U(x, \lambda, q)$ which is C^2 in x and , $\lambda_n(x, q)$, solves $U(x, \lambda_n(x, q)) = 0$. It is known that for a non-negative continuous function $q(x)$, the eigenvalues of the Dirichlet problem for (4.1) on $[-1, x]$, are real and simple (See [14] , §10.61) .

$$\frac{\partial U}{\partial \lambda}(x, \lambda_n(x)) \neq 0$$

for each $x \in (-1, 0)$. It follows from the implicit function theorem that $\lambda_n(x, q)$ is

C^2 in x and

$$\lambda'_n(x) = -\left\{ \frac{\frac{\partial U}{\partial x}(x, \lambda)}{\frac{\partial U}{\partial \lambda}(x, \lambda)} \right\}_{\lambda=\lambda_n(x)}.$$

We consider the Dirichlet problem corresponding to equation (4.1) on $[-1, x]$ where $x < 0$, is fixed. This problem has an infinite number of negative eigenvalues, (see theorem 4.10), $\{\lambda_n(x)\}$, and there holds, by Hadamard's theorem, the product formula

$$U(x, \lambda) = c \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n(x)}\right)$$

where U satisfies (4.5) and c is a constant independent of λ but may depend on x , because the genus of U is zero. By theorem 1.3, each function $\lambda_n(x)$ is of the form

$$\sqrt{-\lambda_n(x)} = \frac{n\pi}{\int_{-1}^x \sqrt{-t} dt} + O\left(\frac{1}{n}\right), \quad x < 0$$

and

$$\lim_{x \rightarrow -1} \lambda_n(x) = -\infty \quad \lambda_1(x) > \lambda_2(x) > \dots$$

In order to estimate c we rewrite the infinite product as

$$\begin{aligned} U(x, \lambda) &= c \prod \left(1 - \frac{\lambda}{\lambda_n(x)}\right) \\ &= c \prod \frac{\lambda_n(x) - \lambda}{\lambda_n(x)} \\ &= c_1 \prod \frac{\lambda - \lambda_n(x)}{z_n^2} \end{aligned} \tag{4.6}$$

with

$$c_1 = c \prod \frac{-z_n^2}{\lambda_n(x)}$$

where $z_n = \frac{n\pi}{p(x)}$,

$$p(x) = \int_{-1}^x \sqrt{-t} dt, \quad x \neq -1. \tag{4.7}$$

Note that since

$$\frac{-z_m^2}{\lambda_m(x)} = 1 + O(1/m^2),$$

the infinite product $\prod \frac{-z_m^2}{\lambda_m(x)}$ is absolutely convergent on any compact subinterval of $(-1, 0)$ by theorem 4.6 . The function $\frac{-z_m^2}{\lambda_m(x)}$ is continuous and so the O-term is uniformly bounded in x .

For $x = 0$, by corollary 2.7 the distribution of the eigenvalues of equation (4.1) is of the form

$$\sqrt{-\lambda_n(0)} = \frac{n\pi - \pi/12}{\int_{-1}^0 \sqrt{-t} dt} + O\left(\frac{1}{n}\right).$$

By Hadamard's theorem we also have

$$U(0, \lambda) = c \prod \left(1 - \frac{\lambda}{\lambda_n(0)}\right).$$

Now let $j_n, n = 1, 2, \dots$ be the positive zeros of the Bessel function of order $1/3$.

Then (see 4.2)

$$\frac{-9j_n^2}{4\lambda_n(0)} = 1 + O(1/n^2)$$

and so the infinite product $\prod \frac{-9j_n^2}{4\lambda_n(0)}$ is absolutely convergent . Consequently we may write , as before ,

$$U(0, \lambda) = c_2 \prod \frac{4(\lambda - \lambda_n(0))}{9j_n^2}, \tag{4.8}$$

where $c_2 = c \prod \left(\frac{-9j_n^2}{4\lambda_n(0)}\right)$.

For $x \in (0, 1]$, fixed , the Dirichlet problem for (4.1) on $[-1, x]$ has an infinite number of positive and negative eigenvalues , by theorem 2.4 , which we denote by $\{u_n(x)\}$, $\{r_n(x)\}$ respectively .

By Theorem 2.5 , $u_n(x)$ is of the form

$$\sqrt{u_n(x)} = \frac{n\pi - \pi/4}{\int_0^x \sqrt{t} dt} + O\left(\frac{1}{n}\right) \quad 0 < x$$

and $r_n(x)$ is of the form

$$\sqrt{-r_n(x)} = \frac{n\pi - \pi/4}{\int_{-1}^0 \sqrt{-t} dt} + O\left(\frac{1}{n}\right) \quad , 0 < x.$$

By Hadamard's theorem , the solution on $[-1, x]$ for $x > 0$ is of the form:

$$U(x, \lambda) = c \prod \left(1 - \frac{\lambda}{r_n(x)}\right) \prod \left(1 - \frac{\lambda}{u_n(x)}\right)$$

Now let $\tilde{j}_n, n = 1, 2, \dots$, be the positive zeros of $J_1'(z)$. Then by (4.3)

$$\frac{\tilde{j}_n^2}{f^2(x)u_n(x)} = 1 + O(1/n^2)$$

$$\frac{-\tilde{j}_n^2}{p^2(0)r_n(x)} = 1 + O(1/n^2)$$

where

$$f(x) = \int_0^x \sqrt{t} dt \quad x > 0 \quad (4.9)$$

and $p(x)$ is defined in (4.7) .

Consequently the infinite products $\prod \frac{\tilde{j}_n^2}{f^2(x)u_n(x)}$ and $\prod \frac{-\tilde{j}_n^2}{p^2(0)r_n(x)}$ are absolutely convergent for each $x > 0$. Therefore we may write

$$U(x, \lambda) = c_3 \prod \frac{(\lambda - r_n(x))p^2(0)}{\tilde{j}_n^2} \prod \frac{f^2(x)(u_n(x) - \lambda)}{\tilde{j}_n^2} , \quad (4.10)$$

where

$$c_3 = c \prod \frac{\tilde{j}_n^2}{f^2(x)u_n(x)} \prod \frac{-\tilde{j}_n^2}{p^2(0)r_n(x)} .$$

Now we will first approximate the infinite products , then by using the asymptotic form of $U(x, \lambda)$, we will determine $c_i, i = 1, 2, 3$.

Lemma 4.11 *Let $z_m = \frac{m\pi}{p(x)}$ and $\lambda_m(x), 1 \leq m$, be a sequence of continuous functions such that, for each x ,*

$$\lambda_m(x) = -\frac{m^2\pi^2}{p^2(x)} + O(1) \quad -1 < x \leq a < 0$$

where $p(x) = \int_{-1}^x \sqrt{-t} dt$. Then the infinite product

$$\prod_{1 \leq m} \frac{\lambda - \lambda_m(x)}{z_m^2}$$

is an entire function of λ for fixed x in $(-1, 0)$ whose roots are precisely $\lambda_m(x), 1 \leq m$. Moreover

$$\prod_{1 \leq m} \left(\frac{\lambda - \lambda_m(x)}{z_m^2} \right) = \frac{\sinh p(x)\sqrt{\lambda}}{p(x)\sqrt{\lambda}} \left(1 + O\left(\frac{\log n}{n}\right) \right)$$

uniformly on the circles $|\lambda| = \frac{(n+1/2)^2\pi^2}{p^2(x)}$.

Proof : Let x be fixed . By the uniform boundedness of $\lambda_m(x) + \frac{m^2\pi^2}{p^2(x)}$ for $1 \leq m$,

$$\sum_{1 \leq m} \left| \frac{\lambda - \lambda_m(x)}{z_m^2} - 1 \right| = \sum_{1 \leq m} \left| \frac{\lambda + O(1)}{z_m^2(x)} \right|$$

converges uniformly on bounded subsets of the complex plane . Therefore by theorem 4.7 the infinite product converges to an entire function of λ , whose roots are precisely $\lambda_m(x), 1 \leq m$.

Now, by theorem 4.3,

$$\frac{\sinh \sqrt{\lambda} p(x)}{p(x)\sqrt{\lambda}} = \prod \left(1 + \frac{\lambda}{z_m^2} \right)$$

thus the quotient of the infinite products is

$$\frac{\prod \frac{\lambda - \lambda_m(x)}{z_m^2}}{\prod \left(1 + \frac{\lambda}{z_m^2} \right)} = \prod_{1 \leq m} \frac{\lambda - \lambda_m}{\lambda + z_m^2}.$$

Furthermore,

$$\left| \frac{\lambda - \lambda_m}{\lambda + z_m^2} - 1 \right| = \left| \frac{-\lambda_m - z_m^2}{\lambda + z_m^2} \right| \leq \frac{|O(1)|}{\left| \lambda - \frac{m^2\pi^2}{p^2} \right|}.$$

Therefore on the circles $|\lambda| = \frac{(n+1/2)^2\pi^2}{p^2(x)}$, the uniform estimates

$$\frac{-\lambda_m + \lambda}{z_m^2 + \lambda} = \begin{cases} 1 + O(1/n) & \text{if } m = n \\ 1 + O(\frac{1}{|m^2 - n^2|}) & m \neq n \end{cases}$$

hold . By theorem 4.8 ,

$$\prod_{1 \leq m} \frac{-\lambda_m + \lambda}{\lambda + z_m^2} = (1 + O(\frac{1}{n}))(1 + O(\frac{\log n}{n})) = \{1 + O(\frac{\log n}{n})\}$$

uniformly on these circles . Therefore

$$\prod_{1 \leq m} \frac{\lambda - \lambda_m(x)}{z_m^2} = \frac{\sinh p(x)\sqrt{\lambda}}{p(x)\sqrt{\lambda}}(1 + O(\frac{\log n}{n}))$$

Lemma 4.12 *Let \tilde{j}_m be the positive zeros of $J_1'(z)$ and $u_m(x), 1 \leq m$, be a sequence of continuous functions defined on any compact subinterval of $(0, 1)$ such that*

$$u_m(x) = \frac{m^2\pi^2}{f^2(x)} - \frac{m\pi^2}{2f^2(x)} + O(1) \quad 1 \leq m$$

where $f(x) = \int_0^x \sqrt{t} dt$. Then the infinite product

$$\prod_{1 \leq m} \frac{(u_m(x) - \lambda)f^2(x)}{\tilde{j}_m^2}$$

is an entire function of λ for fixed x , whose roots are precisely $u_m(x), 1 \leq m$.

Moreover

$$\prod_{1 \leq m} \frac{(u_m(x) - \lambda)f^2(x)}{\tilde{j}_m^2} = 2J_1'(\sqrt{\lambda}f(x))(1 + O(\frac{\log n}{n}))$$

uniformly on the circles $|\lambda| = \frac{n^2\pi^2}{f^2(1)} = \frac{9n^2\pi^2}{4}$.

Proof : Let x be fixed . Since from (4.3)

$$\tilde{j}_m^2 = m^2\pi^2 - \frac{m\pi^2}{2} + O(1),$$

therefore

$$\sum_{1 \leq m} \left| \frac{(u_m(x) - \lambda)f^2(x)}{\tilde{j}_m^2} - 1 \right| = \sum_{1 \leq m} \left| \frac{\lambda + O(1)}{\tilde{j}_m^2} \right|$$

converges uniformly on bounded subsets of complex plane . Therefore by theorem 4.7 , the infinite product converges to an entire function of λ , whose roots are precisely $u_m(x)$, $1 \leq m$.

Now , by theorem 4.5 ,

$$J_1'(\sqrt{\lambda}f(x)) = \frac{1}{2} \prod \left(1 - \frac{\lambda f^2(x)}{\tilde{j}_m^2} \right)$$

thus the quotient of the products is

$$\frac{\prod_{1 \leq m} \frac{(u_m(x) - \lambda)f^2(x)}{\tilde{j}_m^2}}{\frac{1}{2} \prod \left(1 - \frac{\lambda f^2(x)}{\tilde{j}_m^2} \right)} = 2 \prod_{1 \leq m} \frac{u_m - \lambda}{\frac{\tilde{j}_m^2}{f^2(x)} - \lambda} .$$

Furthermore ,

$$\begin{aligned} \left| \frac{u_m(x) - \lambda}{\frac{\tilde{j}_m^2}{f^2(x)} - \lambda} - 1 \right| &= \\ \frac{|u_m(x) - \frac{\tilde{j}_m^2}{f^2(x)}|}{\left| \frac{\tilde{j}_m^2}{f^2(x)} - \lambda \right|} &\leq \frac{|O(1)|}{\left| \frac{\tilde{j}_m^2}{f^2(x)} - |\lambda| \right|} . \end{aligned}$$

Therefore on the circles $|\lambda| = \frac{n^2 \pi^2}{f^2(1)}$, the uniform estimates

$$\frac{u_m - \lambda}{\frac{\tilde{j}_m^2}{f^2(x)} - \lambda} = \begin{cases} 1 + O(1/n) & \text{if } m = n \\ 1 + O\left(\frac{1}{|m^2 - n^2|}\right) & m \neq n \end{cases}$$

hold . By theorem 4.8

$$\prod_{1 \leq m} \frac{u_m - \lambda}{\frac{\tilde{j}_m^2}{f^2(x)} - \lambda} = (1 + O\left(\frac{\log n}{n}\right))(1 + O(1/n)) = 1 + O\left(\frac{\log n}{n}\right)$$

whence

$$\prod_{1 \leq m} \frac{(u_m(x) - \lambda)f^2(x)}{\tilde{j}_m^2} = 2J_1'(\sqrt{\lambda}f(x))(1 + O\left(\frac{\log n}{n}\right))$$

uniformly on these circles .

Similarly we can prove the following lemma .

Lemma 4.13 *Let $\tilde{j}_m, m = 1, 2, \dots$ be the positive zeros of $J_1'(z)$, and for fixed x in $(0, 1)$,*

$$r_m(x) = -\frac{m^2\pi^2}{p^2(0)} + \frac{m\pi^2}{2p^2(0)} + O(1) \quad 1 \leq m$$

be a negative sequence of continuous functions where $p(x) = \int_{-1}^x \sqrt{-t} dt$. Then the infinite product

$$\prod_{1 \leq m} \frac{(\lambda - r_m(x))p^2(0)}{\tilde{j}_m^2}$$

is an entire function of λ for fixed x , whose roots are precisely $r_m(x), 1 \leq m$.

Moreover

$$\prod_{1 \leq m} \frac{(\lambda - r_m(x))p^2(0)}{\tilde{j}_m^2} = 2J_1'(\iota\sqrt{\lambda}p(0))(1 + O(\frac{\log n}{n}))$$

uniformly on the circles $|\lambda| = \frac{n^2\pi^2}{p^2(0)} = \frac{9n^2\pi^2}{4}$.

Proof : This follows from theorem 4.8 and use of the method of the proof of the preceding lemma . Similarly we have

Lemma 4.14 *Let $j_m, m = 1, 2, \dots$ be the positive zeros of $J_{1/3}(z)$ and*

$$\lambda_m(0) = -\frac{m^2\pi^2}{p^2(0)} + \frac{m\pi^2}{6p^2(0)} + O(1) \quad 1 \leq m$$

be a negative sequence where $p(0) = \int_{-1}^0 \sqrt{-t} dt$. Then the infinite product

$$\prod_{1 \leq m} \frac{(\lambda - \lambda_m(0))p^2(0)}{j_m^2}$$

is an entire function of λ , whose roots are precisely $\lambda_m(0), 1 \leq m$. Moreover

$$\prod_{1 \leq m} \frac{(\lambda - \lambda_m(0))p^2(0)}{j_m^2}$$

$$= \frac{\Gamma(4/3)}{\{i\sqrt{\lambda}b/2\}^{1/3}} J_{1/3}(ib\sqrt{\lambda})(1 + O(\frac{\log n}{n}))$$

uniformly on the circles $|\lambda| = \frac{n^2\pi^2}{p^2(0)} = \frac{9n^2\pi^2}{4}$, where $b = p(0) = 2/3$.

Proof : This follows from lemma 4.4 and use of the method of lemma 4.12 .

Now in order to determine $c_i, i = 1, 2, 3, \dots$ we find the asymptotic form of $U(t, \lambda)$. The asymptotic nature of the approximate solutions of the differential equation (4.2) in which $r(t)$ has a simple zero , $q(t)$ is a continuous real or complex function in the interval $a \leq t \leq b$ and $\lambda \rightarrow \infty$, has been investigated by many writers , particularly Langer {[17] , [18]} , Cherry [5] , Jeffreys [15] , Erdelyi {[9], [10]} and Olver {[26] , [27]} . In [28] Olver established , for the first time , explicit strict bounds for the errors of the Airy-function approximation . Now we apply Olver's result in order to determine the asymptotic form of $U(t, \lambda)$. First we state a number of relevant properties of the functions Ai and Bi (the Airy functions) . Then we also introduce certain auxiliary functions and constants associated with Ai and Bi which are used in Olver's paper [28].

Relevant properties of Airy functions.

The Airy functions are the solutions of the differential equation

$$y'' = ty.$$

For real values of t the standard solutions are denoted by $Ai(t)$ and $Bi(t)$. They have the initial values

$$Ai(0) = \frac{1}{\sqrt{3}}Bi(0) = \frac{1}{3^{2/3}\Gamma(2/3)},$$

$$Ai'(0) = -\frac{1}{\sqrt{3}}Bi'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)},$$

and satisfy the Wronskian relation

$$Ai(t)Bi'(t) - Ai'(t)Bi(t) = 1/\pi \quad (4.11)$$

When t is positive $Ai(t)$, $Bi(t)$, $-Ai'(t)$ and $Bi'(t)$ are all positive and monotonic; when t is negative these functions are oscillatory, with diminishing period as $t \rightarrow -\infty$. Their precise asymptotic behavior is given in [36] by

$$Ai(t) = \frac{1}{2}\pi^{-1/2}t^{-1/4}e^{-(2/3)t^{3/2}}\{1 + O(t^{-3/2})\},$$

$$Ai'(t) = -\frac{1}{2}\pi^{-1/2}t^{1/4}e^{-(2/3)t^{3/2}}\{1 + O(t^{-3/2})\},$$

$$Bi(t) = \pi^{-1/2}t^{-1/4}e^{(2/3)t^{3/2}}\{1 + O(t^{-3/2})\},$$

$$Bi'(t) = \pi^{-1/2}t^{1/4}e^{(2/3)t^{3/2}}\{1 + O(t^{-3/2})\},$$

as $t \rightarrow +\infty$, and

$$Ai(t) = \pi^{-1/2}(-t)^{-1/4}\{\cos(\frac{2}{3}(-t)^{3/2} - \pi/4) + O(|t|^{-3/2})\}$$

$$Ai'(t) = \pi^{-1/2}(-t)^{1/4}\{\sin(\frac{2}{3}(-t)^{3/2} - \pi/4) + O(|t|^{-3/2})\}$$

$$Bi(t) = -\pi^{-1/2}(-t)^{-1/4}\{\sin(\frac{2}{3}(-t)^{3/2} - \pi/4) + O(|t|^{-3/2})\}$$

$$Bi'(t) = \pi^{-1/2}(-t)^{1/4}\{\cos(\frac{2}{3}(-t)^{3/2} - \pi/4) + O(|t|^{-3/2})\}$$

as $t \rightarrow -\infty$.

In order to have a simple way of estimating the magnitudes of the Airy functions, we define

$$\begin{aligned} E(t) &= e^{\frac{2}{3}t^{3/2}}, & (t > 0) \\ E(t) &= 1, (t \leq 0); & E^{-1}(t) \equiv 1/E(t), \end{aligned} \quad (4.12)$$

and introduce four auxiliary functions $M(t), \chi(t), N(t)$ and $\psi(t)$, defined for all real values of t by the equations

$$\begin{aligned} Ai(t) &= E^{-1}(t)M(t) \sin \chi(t), \\ Bi(t) &= E(t)M(t) \cos \chi(t), \\ Ai'(t) &= E^{-1}(t)N(t) \sin \psi(t), \\ Bi'(t) &= E(t)N(t) \cos \psi(t). \end{aligned} \quad (4.13)$$

Thus

$$\begin{aligned} M^2(t) &= E^2(t)Ai^2(t) + E^{-2}(t)Bi^2(t), \\ N^2(t) &= E^2(t)Ai'^2(t) + E^{-2}(t)Bi'^2(t), \end{aligned}$$

From the asymptotic forms of Ai and Bi we deduce that(see [28])

$$\begin{aligned} M(t) &\sim \left(\frac{5}{4\pi}\right)^{1/2} \frac{1}{t^{1/4}}, & N(t) &\sim \left(\frac{5}{4\pi}\right)^{1/2} t^{1/4}, & t &\rightarrow +\infty, \\ M(t) &\sim \frac{1}{\pi^{1/2}|t|^{1/4}}, & N(t) &\sim \frac{|t|^{1/4}}{\pi^{1/2}}, & t &\rightarrow -\infty. \end{aligned} \quad (4.14)$$

The following constants occur in the subsequent analysis, (see [28])

$$\begin{aligned} \lambda_1 &= \max_{(-\infty, \infty)} \{\pi|t|^{1/2}M^2(t)\} = 1.430\dots, \\ \lambda_2 &= \max_{(-\infty, \infty)} \{\pi|t|^{1/2}Bi(t)|E^{-1}(t)M(t)\} = 1.315\dots, \end{aligned} \quad (4.15)$$

Approximation theorem for the equation 4.1

By theorem 2.1 ,the differential equation 4.1 has solutions $W_1(t, \lambda), W_2(t, \lambda)$ given by

$$\begin{aligned} W_1(t, \lambda) &= Ai(-\lambda^{1/3}t) + \epsilon_1 \\ W_2(t, \lambda) &= Bi(-\lambda^{1/3}t) + \epsilon_2 \\ W_1'(t, \lambda) &\equiv \frac{\partial W_1(t, \lambda)}{\partial t} = -\lambda^{1/3}\{Ai'(-\lambda^{1/3}t) + \eta_1\} \end{aligned}$$

$$W_2'(t, \lambda) \equiv \frac{\partial W_2(t, \lambda)}{\partial t} = -\lambda^{1/3} \{B_2'(-\lambda^{1/3}t) + \eta_2\} \quad (4.16)$$

where from [28] error bounds $\epsilon_i, \eta_i, i = 1, 2$ are of the form

$$\begin{aligned} \frac{E(-t\lambda^{1/3})}{M(-t\lambda^{1/3})} |\epsilon_1|, \frac{E(-t\lambda^{1/3})}{N(-t\lambda^{1/3})} |\eta_1| &\leq \frac{1}{\lambda_1} \{e^{\frac{\lambda_1 F_1(-t)}{\sqrt{\lambda}}} - 1\}, \\ \frac{E^{-1}(-t\lambda^{1/3})}{M(-t\lambda^{1/3})} |\epsilon_2|, \frac{E^{-1}(-t\lambda^{1/3})}{N(-t\lambda^{1/3})} |\eta_2| &\leq \frac{\lambda_2}{\lambda_1} \{e^{\frac{\lambda_1 F_2(-t)}{\sqrt{\lambda}}} - 1\}. \end{aligned} \quad (4.17)$$

In the above result, the functions F_1, F_2 are given by, for $|t| < 1$,

$$\begin{aligned} 0 \leq F_1(t) &= \int_t^1 \frac{q(v)}{v^{1/2}} dv \\ 0 \leq F_2(t) &= \int_{-1}^t \frac{q(v)}{v^{1/2}} dv \end{aligned}$$

(note that these integrals are convergent) and λ_1, λ_2 are constants defined in (4.15) while $E(t), M(t)$ and $N(t)$ are auxiliary functions defined earlier.

If λ is large then the right side of (4.17) is clearly $O(\frac{1}{\sqrt{\lambda}})$ uniformly with respect to t . Hence we have

$$\begin{aligned} |\epsilon_1| &= E^{-1}(-t\lambda^{1/3})M(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right), \\ |\eta_1| &= E^{-1}(-t\lambda^{1/3})N(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right), \\ |\epsilon_2| &= E(-t\lambda^{1/3})M(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right), \\ |\eta_2| &= E(-t\lambda^{1/3})N(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right). \end{aligned} \quad (4.18)$$

Now, the solution of the equation (4.1) with initial condition (4.5) is of the form

$$U(t, \lambda) = \frac{W_1(-1, \lambda)W_2(t, \lambda) - W_1(t, \lambda)W_2(-1, \lambda)}{W_1(-1, \lambda)W_2'(-1, \lambda) - W_1'(-1, \lambda)W_2(-1, \lambda)}.$$

It is well known that the Wronskian of two linearly independent solutions of (4.1) is a nonzero constant, (for each λ), whence we can say that

$$U(t, \lambda) = \frac{W_1(-1, \lambda)W_2(t, \lambda) - W_1(t, \lambda)W_2(-1, \lambda)}{W_1(0, \lambda)W_2'(0, \lambda) - W_1'(0, \lambda)W_2(0, \lambda)} \quad (4.19)$$

Now from (4.16) and (4.18) we have

$$\begin{aligned}
& W_1(0, \lambda)W_2'(0, \lambda) - W_1'(0, \lambda)W_2(0, \lambda) \\
&= -\lambda^{1/3}\{Ai(0) + \epsilon_1(0)\}\{Bi'(0) + \eta_2(0)\} + \lambda^{1/3}\{Ai'(0) + \eta_1(0)\}\{Bi(0) + \epsilon_2(0)\} \\
&= -\lambda^{1/3}\{Ai(0)Bi'(0) - Ai'(0)Bi(0)\} + \lambda^{1/3}O\left(\frac{1}{\sqrt{\lambda}}\right) \\
&= -\lambda^{1/3}/\pi + O\left(\frac{1}{\lambda^{1/6}}\right) \quad \text{by(4.11)}. \tag{4.20}
\end{aligned}$$

Now let

$$p(t) = \int_{-1}^t \sqrt{-v}dv = (-2/3)(-t)^{3/2} + 2/3 \quad \text{for } t \leq 0.$$

Then by (4.12) and (4.14) for $-1 \leq t < 0$ we have

$$\begin{aligned}
E^{-1}(\lambda^{1/3})E(-t\lambda^{1/3}) &= e^{-\sqrt{\lambda}p(t)}, \\
E(\lambda^{1/3})E^{-1}(-t\lambda^{1/3}) &= e^{\sqrt{\lambda}p(t)}, \\
M(\lambda^{1/3})M(-t\lambda^{1/3}) &\sim \frac{5}{4\pi(-t)^{1/4}\lambda^{1/6}} \quad \lambda \rightarrow \infty. \tag{4.21}
\end{aligned}$$

We need the following estimates for determining the asymptotic form of the solutions. From (4.13), (4.18) and (4.21) we have, noting that $\cos \chi(t) = O(1)$,

$$\begin{aligned}
\epsilon_1(-1)Bi(-t\lambda^{1/3}) &= E^{-1}(\lambda^{1/3})M(\lambda^{1/3})E(-t\lambda^{1/3})M(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right) = \\
&\begin{cases} e^{-p(t)\sqrt{\lambda}}\frac{1}{(-t)^{1/4}}O(\lambda^{-2/3}) & \text{if } -1 \leq t < 0 \\ e^{-\frac{2}{3}\sqrt{\lambda}}O(\lambda^{-7/12}) & t = 0 \\ e^{-\frac{2}{3}\sqrt{\lambda}}\frac{1}{t^{1/4}}O(\lambda^{-2/3}) & \text{if } 0 < t \leq 1 \end{cases} \\
\epsilon_2(t)Ai(\lambda^{1/3}) &= E^{-1}(\lambda^{1/3})M(\lambda^{1/3})E(-t\lambda^{1/3})M(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right) = \\
&\begin{cases} e^{-p(t)\sqrt{\lambda}}\frac{1}{(-t)^{1/4}}O(\lambda^{-2/3}) & \text{if } -1 \leq t < 0 \\ e^{-\frac{2}{3}\sqrt{\lambda}}O(\lambda^{-7/12}) & t = 0 \\ e^{-\frac{2}{3}\sqrt{\lambda}}\frac{1}{t^{1/4}}O(\lambda^{-2/3}) & \text{if } 0 < t \leq 1 \end{cases}
\end{aligned}$$

$$\begin{aligned} \epsilon_1(t)Bi(\lambda^{1/3}) &= E(\lambda^{1/3})M(\lambda^{1/3})E^{-1}(-t\lambda^{1/3})M(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right) = \\ &\begin{cases} e^{p(t)\sqrt{\lambda}}\frac{1}{(-t)^{1/4}}O(\lambda^{-2/3}) & \text{if } -1 \leq t < 0 \\ e^{\frac{2}{3}\sqrt{\lambda}}O(\lambda^{-7/12}) & t = 0 \\ e^{\frac{2}{3}\sqrt{\lambda}}\frac{1}{t^{1/4}}O(\lambda^{-2/3}) & \text{if } 0 < t \leq 1 \end{cases} \\ \epsilon_2(-1)Ai(-t\lambda^{1/3}) &= E(\lambda^{1/3})M(\lambda^{1/3})E^{-1}(-t\lambda^{1/3})M(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right) = \\ &\begin{cases} e^{p(t)\sqrt{\lambda}}\frac{1}{(-t)^{1/4}}O(\lambda^{-2/3}) & \text{if } -1 \leq t < 0 \\ e^{\frac{2}{3}\sqrt{\lambda}}O(\lambda^{-7/12}) & t = 0 \\ e^{\frac{2}{3}\sqrt{\lambda}}\frac{1}{t^{1/4}}O(\lambda^{-2/3}) & \text{if } 0 < t \leq 1 \end{cases} \end{aligned}$$

From the asymptotic behavior of Ai and Bi we also have :

$$\begin{aligned} Ai(\lambda^{1/3})Bi(-t\lambda^{1/3}) &= \\ &\begin{cases} \frac{1}{2\pi}(-t)^{-1/4}\lambda^{-1/6}e^{-p(t)\sqrt{\lambda}}\{1 + O(\lambda^{-1/2})\} & \text{if } -1 \leq t < 0 \\ \frac{1}{2\sqrt{\pi}}\lambda^{-1/12}Bi(0)e^{-\frac{2}{3}\sqrt{\lambda}}\{1 + O(\frac{1}{\sqrt{\lambda}})\} & t = 0 \\ -\frac{1}{2\pi}t^{-1/4}\lambda^{-1/6}e^{-\frac{2}{3}\sqrt{\lambda}}\{\sin(\frac{2}{3}t^{3/2}\sqrt{\lambda} - \pi/4) + O(\frac{1}{\sqrt{\lambda}})\} & \text{if } 0 < t \leq 1 \end{cases} \end{aligned} \quad (4.22)$$

and

$$\begin{aligned} Bi(\lambda^{1/3})Ai(-t\lambda^{1/3}) &= \\ &\begin{cases} \frac{1}{2\pi}(-t)^{-1/4}\lambda^{-1/6}e^{p(t)\sqrt{\lambda}}\{1 + O(\lambda^{-1/2})\} & \text{if } -1 \leq t < 0 \\ \frac{1}{\sqrt{\pi}}\lambda^{-1/12}Ai(0)e^{\frac{2}{3}\sqrt{\lambda}}\{1 + O(\frac{1}{\sqrt{\lambda}})\} & t = 0 \\ \frac{1}{\pi}t^{-1/4}\lambda^{-1/6}e^{\frac{2}{3}\sqrt{\lambda}}\{\cos(\frac{2}{3}t^{3/2}\sqrt{\lambda} - \pi/4) + O(\frac{1}{\sqrt{\lambda}})\} & \text{if } 0 < t \leq 1 \end{cases} \end{aligned} \quad (4.23)$$

From above calculation we get

$$\begin{aligned} &W_1(-1, \lambda)W_2(t, \lambda) - W_1(t, \lambda)W_2(-1, \lambda) = \\ &Ai(\lambda^{1/3})Bi(-t\lambda^{1/3}) - Ai(-t\lambda^{1/3})Bi(\lambda^{1/3}) + \\ &\epsilon_1(-1)Bi(-t\lambda^{1/3})\epsilon_2(t)Ai(\lambda^{1/3}) - \epsilon_1(t)Bi(\lambda^{1/3})\epsilon_2(-1)Ai(-t\lambda^{1/3}) + \\ &\epsilon_2(-1)\epsilon_1(t)\epsilon_2(t)\epsilon_1(-1) = \end{aligned}$$

$$\left\{ \begin{array}{l} \frac{1}{2\pi}(-t)^{-1/4}\lambda^{-1/6}e^{-p(t)\sqrt{\lambda}}\{1+O(\lambda^{-1/2})\}-\frac{1}{2\pi}(-t)^{-1/4}\lambda^{-1/6}e^{p(t)\sqrt{\lambda}}\{1+O(\lambda^{-1/2})\}+ \\ e^{p(t)\sqrt{\lambda}}(-t)^{-1/4}O(\lambda^{-2/3})+e^{-p(t)\sqrt{\lambda}}(-t)^{-1/4}O(\lambda^{-2/3}) \quad t \in (-1, 0), \\ \frac{1}{2\sqrt{\pi}}\lambda^{-1/12}Bi(0)e^{-\frac{2}{3}\sqrt{\lambda}}\{1+O(\frac{1}{\sqrt{\lambda}})\}-\frac{1}{\sqrt{\pi}}\lambda^{-1/12}Ai(0)e^{\frac{2}{3}\sqrt{\lambda}}\{1+O(\frac{1}{\sqrt{\lambda}})\} \quad t = 0, \\ -\frac{1}{2\pi}(t)^{-1/4}\lambda^{-1/6}e^{-\frac{2}{3}\sqrt{\lambda}}\{\sin(\frac{2}{3}t^{3/2}\sqrt{\lambda}-\pi/4)+O(\frac{1}{\sqrt{\lambda}})\} \\ -\frac{1}{\pi}t^{-1/4}\lambda^{-1/6}e^{\frac{2}{3}\sqrt{\lambda}}\{\cos(\frac{2}{3}t^{3/2}\sqrt{\lambda}-\pi/4)+O(\frac{1}{\sqrt{\lambda}})\} \\ +t^{-1/4}e^{\frac{2}{3}\sqrt{\lambda}}O(\lambda^{-2/3})+t^{-1/4}e^{-\frac{2}{3}\sqrt{\lambda}}O(\lambda^{-2/3}) \quad t \in (0, 1] \end{array} \right. \quad (4.24)$$

By using the asymptotic form for the Wronskian, we can find the leading term of $U(t, \lambda)$. For $-1 \leq t < 0$, the above results and (4.19) show that

$$\begin{aligned} U(t, \lambda) &= \frac{1}{2(-t)^{1/4}\sqrt{\lambda}} \left\{ e^{\sqrt{\lambda} \int_{-1}^t \sqrt{-tdt}} (1 + O(\frac{1}{\sqrt{\lambda}})) - e^{-\sqrt{\lambda} \int_{-1}^t \sqrt{-tdt}} (1 + O(\frac{1}{\sqrt{\lambda}})) \right\} \\ &= \frac{1}{(-t)^{1/4}\sqrt{\lambda}} (1 + O(\frac{1}{\sqrt{\lambda}})) \sinh(p(t)\sqrt{\lambda}) \quad \lambda \rightarrow \infty \end{aligned} \quad (4.25)$$

where, as usual,

$$p(t) = \int_{-1}^t \sqrt{-tdt}$$

Similarly by using (4.19), (4.22) and (4.23) with $t = 0$ we have,

$$\begin{aligned} U(0, \lambda) &= \left\{ \frac{\pi^{1/2}Ai(0)}{\lambda^{5/12}}e^{\frac{2}{3}\sqrt{\lambda}} - \frac{\pi^{1/2}Bi(0)}{2\lambda^{5/12}}e^{-\frac{2}{3}\sqrt{\lambda}} \right\} \left\{ 1 + O(\frac{1}{\sqrt{\lambda}}) \right\} \\ &= \frac{\pi^{1/2}Ai(0)}{\lambda^{5/12}} \left\{ e^{\frac{2}{3}\sqrt{\lambda}} - \frac{\sqrt{3}}{2}e^{-\frac{2}{3}\sqrt{\lambda}} \right\} \left\{ 1 + O(\frac{1}{\sqrt{\lambda}}) \right\} \quad \lambda \rightarrow \infty \end{aligned} \quad (4.26)$$

because $Ai(0) = \frac{Bi(0)}{\sqrt{3}}$.

For $0 < t$, we have by (4.22)–(4.23),

$$\begin{aligned} U(t, \lambda) &= \frac{1}{t^{1/4}\sqrt{\lambda}} \left\{ \frac{e^{-\frac{2}{3}\sqrt{\lambda}} \sin(\frac{2}{3}t^{3/2}\sqrt{\lambda} - \pi/4)}{2} + e^{\frac{2}{3}\sqrt{\lambda}} \cos(\frac{2}{3}t^{3/2}\sqrt{\lambda} - \pi/4) \right\} \left\{ 1 + O(\frac{1}{\sqrt{\lambda}}) \right\} \\ &= \frac{1}{t^{1/4}\sqrt{\lambda}} \left\{ e^{\frac{2}{3}\sqrt{\lambda}} \cos(\frac{2}{3}t^{3/2}\sqrt{\lambda} - \pi/4) \right\} (1 + O(\frac{1}{\sqrt{\lambda}})) \quad \lambda \rightarrow \infty \end{aligned} \quad (4.27)$$

Now we find $\frac{\partial U}{\partial t}(t, \lambda)$. Indeed we have

$$\frac{\partial U}{\partial t}(t, \lambda) = \frac{W_1(-1, \lambda)W_2'(t, \lambda) - W_1'(t, \lambda)W_2(-1, \lambda)}{W_1(0, \lambda)W_2'(0, \lambda) - W_1'(0, \lambda)W_2(0, \lambda)}. \quad (4.28)$$

From (4.16) we may write

$$W_1(-1, \lambda)W_2'(t, \lambda) = -\lambda^{1/3}\{Ai(\lambda^{1/3}) + \epsilon_1(-1)\}\{Bi'(-\lambda^{1/3}t) + \eta_2(t)\} .$$

Now we calculate the error bounds in the last display. From (4.18), (4.14) and (4.21) we may write, for large λ ,

$$\lambda^{1/3}Ai(\lambda^{1/3})\eta_2(t) = \lambda^{1/3}M(\lambda^{1/3})E^{-1}(\lambda^{1/3})N(-t\lambda^{1/3})E(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right) =$$

$$\begin{cases} e^{-p(t)\sqrt{\lambda}}(-t)^{1/4}O(\lambda^{-1/6}) & \text{if } -1 \leq t < 0 \\ e^{-\frac{2}{3}\sqrt{\lambda}}O(\lambda^{-1/4}) & t = 0 \\ e^{-\frac{2}{3}\sqrt{\lambda}}t^{1/4}O(\lambda^{-1/6}) & \text{if } 0 < t \leq 1 \end{cases}$$

$$\lambda^{1/3}\epsilon_1(-1)Bi'(-t\lambda^{1/3}) = \lambda^{1/3}E^{-1}(\lambda^{1/3})M(\lambda^{1/3})E(-t\lambda^{1/3})N(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right) =$$

$$\begin{cases} e^{-p(t)\sqrt{\lambda}}(-t)^{1/4}O(\lambda^{-1/6}) & \text{if } -1 \leq t < 0 \\ e^{-\frac{2}{3}\sqrt{\lambda}}O(\lambda^{-1/4}) & t = 0 \\ e^{-\frac{2}{3}\sqrt{\lambda}}t^{1/4}O(\lambda^{-1/6}) & \text{if } 0 < t \leq 1 \end{cases}$$

Similarly we have by (4.13), (4.12) and (4.14),

$$W_1'(t, \lambda)W_2(-1, \lambda) = -\lambda^{1/3}\{Ai'(-\lambda^{1/3}t) + \eta_1(t)\}\{Bi(\lambda^{1/3}) + \epsilon_2(-1)\} ,$$

$$\lambda^{1/3}\eta_1(t)Bi(\lambda^{1/3}) = \lambda^{1/3}E^{-1}(-t\lambda^{1/3})M(\lambda^{1/3})E(\lambda^{1/3})N(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right) =$$

$$\begin{cases} e^{p(t)\sqrt{\lambda}}(-t)^{1/4}O(\lambda^{-1/6}) & \text{if } -1 \leq t < 0 \\ e^{\frac{2}{3}\sqrt{\lambda}}O(\lambda^{-1/4}) & t = 0 \\ e^{\frac{2}{3}\sqrt{\lambda}}t^{1/4}O(\lambda^{-1/6}) & \text{if } 0 < t \leq 1 , \end{cases}$$

$$\lambda^{1/3}\epsilon_2(-1)Ai'(-t\lambda^{1/3}) = \lambda^{1/3}E(\lambda^{1/3})M(\lambda^{1/3})E^{-1}(-t\lambda^{1/3})N(-t\lambda^{1/3})O\left(\frac{1}{\sqrt{\lambda}}\right) =$$

$$\begin{cases} e^{p(t)\sqrt{\lambda}}(-t)^{1/4}O(\lambda^{-1/6}) & \text{if } -1 \leq t < 0 \\ e^{\frac{2}{3}\sqrt{\lambda}}O(\lambda^{-1/4}) & t = 0 \\ e^{\frac{2}{3}\sqrt{\lambda}}t^{1/4}O(\lambda^{-1/6}) & \text{if } 0 < t \leq 1 , \end{cases}$$

The leading terms in the display following (4.28) are found next .

$$\begin{aligned}
 & -\lambda^{1/3} Ai(\lambda^{1/3}) Bi'(-t\lambda^{1/3}) = \\
 & \left\{ \begin{array}{ll} -\frac{1}{2\pi}(-t)^{1/4} \lambda^{1/3} e^{-p(t)\sqrt{\lambda}} \{1 + O(\lambda^{-1/2})\} & \text{if } -1 \leq t < 0 \\ -\frac{1}{2\sqrt{\pi}} \lambda^{1/4} Bi'(0) e^{-\frac{2}{3}\sqrt{\lambda}} \{1 + O(\frac{1}{\sqrt{\lambda}})\} & t = 0 \\ -\frac{1}{2\pi} t^{1/4} \lambda^{1/3} e^{-\frac{2}{3}\sqrt{\lambda}} \{ \cos(\frac{2}{3}t^{3/2}\sqrt{\lambda} - \pi/4) + O(\frac{1}{\sqrt{\lambda}}) \} & \text{if } 0 < t \leq 1 \end{array} \right. \quad (4.29)
 \end{aligned}$$

$$\begin{aligned}
 & -\lambda^{1/3} Bi(\lambda^{1/3}) Ai'(-t\lambda^{1/3}) = \\
 & \left\{ \begin{array}{ll} \frac{1}{2\pi}(-t)^{1/4} \lambda^{1/3} e^{p(t)\sqrt{\lambda}} \{1 + O(\lambda^{-1/2})\} & \text{if } -1 \leq t < 0 \\ -\frac{1}{\sqrt{\pi}} \lambda^{1/4} Ai'(0) e^{\frac{2}{3}\sqrt{\lambda}} \{1 + O(\frac{1}{\sqrt{\lambda}})\} & t = 0 \\ -\frac{1}{\pi} t^{1/4} \lambda^{1/3} e^{\frac{2}{3}\sqrt{\lambda}} \{ \sin(\frac{2}{3}t^{3/2}\sqrt{\lambda} - \pi/4) + O(\frac{1}{\sqrt{\lambda}}) \} & \text{if } 0 < t \leq 1 . \end{array} \right. \quad (4.30)
 \end{aligned}$$

Combining the last six displays we get

$$\begin{aligned}
 & \frac{\partial U}{\partial t}(t, \lambda) = \\
 & \left\{ \begin{array}{ll} (-t)^{1/4} \{1 + O(\lambda^{-1/2})\} \cosh(p(t)\sqrt{\lambda}) & \text{if } -1 \leq t < 0 \\ \pi^{1/2} \lambda^{-1/12} Bi'(0) \{ \frac{1}{2} e^{-\frac{2}{3}\sqrt{\lambda}} + \frac{1}{\sqrt{3}} e^{\frac{2}{3}\sqrt{\lambda}} \} \{1 + O(\frac{1}{\sqrt{\lambda}})\} & t = 0 \\ t^{1/4} \{ \frac{1}{2} e^{-\frac{2}{3}\sqrt{\lambda}} \cos(\frac{2}{3}t^{3/2}\sqrt{\lambda} - \pi/4) - e^{\frac{2}{3}\sqrt{\lambda}} \sin(\frac{2}{3}t^{3/2}\sqrt{\lambda} - \pi/4) \} \{1 + O(\frac{1}{\sqrt{\lambda}})\} & \text{if } 0 < t \leq 1 \end{array} \right. \quad (4.31)
 \end{aligned}$$

where the error terms are uniform in t if t is in some compact set which excludes $t = 0$.

Theorem 4.15 *Let $U(t, \lambda)$ be the solution of the initial value problem (4.1-5) .*

Then for $0 < x$,

$$U(x, \lambda) = \frac{\pi\sqrt{x}}{6} \prod \frac{(\lambda - r_k(x))p^2(0)}{\tilde{j}_k^2} \prod \frac{f^2(x)(u_k(x) - \lambda)}{\tilde{j}_k^2} ,$$

where $f(x) = \int_0^x \sqrt{t} dt$, $p(v) = \int_{-1}^v \sqrt{-t} dt$, the sequence $\{u_k(x)\}$ represents the positive eigenvalues and $\{r_k(x)\}$ the negative eigenvalues of the Dirichlet problem associated with (4.1) on $[-1, x]$.

Proof : By theorem 2.5 we have,

$$r_m(x) = -\frac{m^2\pi^2}{b^2} + \frac{m\pi^2}{2b^2} + O(1) \quad 1 \leq m$$

$$u_m(x) = \frac{m^2\pi^2}{f^2(x)} - \frac{m\pi^2}{2f^2(x)} + O(1) \quad 1 \leq m$$

where $f(x) = \int_0^x \sqrt{t} dt$, $b = p(0) = 2/3$. Therefore by (4.10) and (4.27) we have

$$\begin{aligned} U(x, \lambda) &= c_3 \prod \frac{(\lambda - r_k(x))p^2(0)}{\tilde{j}_k^2} \prod \frac{f^2(x)(u_k(x) - \lambda)}{\tilde{j}_k^2}, \\ &= \frac{1}{x^{1/4}\sqrt{\lambda}} \{e^{\frac{2}{3}\sqrt{\lambda}} \cos(\frac{2}{3}x^{3/2}\sqrt{\lambda} - \pi/4)\} (1 + O(\frac{1}{\sqrt{\lambda}})) \quad \lambda \rightarrow \infty. \end{aligned}$$

By lemma 4.12 and 4.13, on the circles $|\lambda| = \frac{n^2\pi^2}{b^2}$ we have

$$\begin{aligned} \prod \frac{(\lambda - r_k(x))p^2(0)}{\tilde{j}_k^2} \prod \frac{f^2(x)(u_k(x) - \lambda)}{\tilde{j}_k^2} &= \\ 4J_1'(f(x)\sqrt{\lambda})J_1'(ib\sqrt{\lambda}) \{1 + O(\frac{\log n}{n})\} &. \end{aligned}$$

It is known that([1], §9.2.11)

$$J_\nu'(z) = \sqrt{\frac{2}{z\pi}} \{-R(\nu, z) \sin X - S(\nu, z) \cos X\} \quad (|\arg z| < \pi)$$

where ν is fixed and

$$\begin{aligned} X &= z - \left(\frac{\nu}{2} + 1/4\right)\pi, \\ R(\nu, z) &\sim \sum_{k=0}^{\infty} (-1)^k \frac{4\nu^2 + 16k^2 - 1}{4\nu^2 - (4k-1)^2} \left\{ \frac{(\nu, 2k)}{(2z)^{2k}} \right\} \\ &= 1 - \frac{(\alpha-1)(\alpha+15)}{2(8z)^2} + \dots, \\ S(\nu, z) &\sim \sum_{k=0}^{\infty} (-1)^k \frac{4\nu^2 + 4(2k+1)^2 - 1}{4\nu^2 - (4k+1)^2} \left\{ \frac{(\nu, 2k+1)}{(2z)^{2k+1}} \right\} \\ &= \frac{\alpha+3}{8z} - \frac{(\alpha-1)(\alpha-9)(\alpha+35)}{3!(8z)^3} + \dots, \end{aligned}$$

as $|z| \rightarrow \infty$, where $\alpha = 4\nu^2$. Now, after some lengthy but straightforward calculations we find,

$$J_1'(f(x)\sqrt{\lambda})J_1'(ib\sqrt{\lambda}) = \frac{2 \sin(ib\sqrt{\lambda} - \frac{3\pi}{4})}{i^{1/2}\pi f^{1/2}(x)b^{1/2}\sqrt{\lambda}} \left\{ \sin(f(x)\sqrt{\lambda} - \frac{3\pi}{4}) + O\left(\frac{1}{\sqrt{\lambda}}\right) \right\}.$$

Since

$$\sin(f(x)\sqrt{\lambda} - \frac{3\pi}{4}) \sin(ib\sqrt{\lambda} - \frac{3\pi}{4}) = \frac{1}{2i} \cos(f(x)\sqrt{\lambda} - \pi/4) \{e^{b\sqrt{\lambda} + \frac{3i\pi}{4}} - e^{-b\sqrt{\lambda} - \frac{3i\pi}{4}}\},$$

it follows that

$$J_1'(f(x)\sqrt{\lambda})J_1'(ib\sqrt{\lambda}) = \frac{e^{b\sqrt{\lambda} + \frac{3i\pi}{4}}}{\pi i^{3/2} b^{1/2} f^{1/2}(x) \sqrt{\lambda}} \left\{ \cos(f(x)\sqrt{\lambda} - \pi/4) + O\left(\frac{1}{\sqrt{\lambda}}\right) \right\}$$

whence for $|\lambda| = \frac{n^2 \pi^2}{b^2}$,

$$\begin{aligned} U(x, \lambda) &= 4c_3 \frac{e^{b\sqrt{\lambda} + \frac{3i\pi}{4}}}{\pi i^{3/2} \sqrt{bf(x)\lambda}} \left\{ \cos(f(x)\sqrt{\lambda} - \pi/4) + O\left(\frac{1}{\sqrt{\lambda}}\right) \right\} \left\{ 1 + O\left(\frac{\log n}{n}\right) \right\} \\ &= \frac{1}{x^{1/4} \sqrt{\lambda}} \left\{ e^{\frac{2}{3}\sqrt{\lambda}} \cos\left(\frac{2}{3}x^{3/2}\sqrt{\lambda} - \pi/4\right) \right\} \left(1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right) \quad \lambda \rightarrow \infty \end{aligned}$$

by (4.27). Consequently

$$\begin{aligned} c_3 &= \frac{i^{3/2} \pi f^{1/2}(x) b^{1/2}}{4x^{1/4} e^{3i\pi/4}} \left(1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right) \\ &= \frac{\pi \sqrt{x}}{6} \left(1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right) \end{aligned}$$

where the O-term may be made uniform in x for $x \in (0, 1)$. Since c_3 depends only on x , by Hadamard's theorem, we may let $\lambda \rightarrow \infty$, and find

$$c_3 = \frac{\pi \sqrt{x}}{6}$$

Theorem 4.16 *Let $U(t, \lambda)$ be the solution of the initial value problem (4.1-5).*

Then for $x = 0$,

$$U(0, \lambda) = \prod \frac{4(\lambda - \lambda_k(0))}{9j_k^2},$$

where $\{\lambda_k(0)\}$ is the sequence of negative eigenvalues of the Dirichlet problem associated with (4.1) on $[-1, 0]$. As before j_k represents the sequence of positive zeros of the Bessel function of order $1/3$.

Proof : From (4.8) and (4.26), we have

$$\begin{aligned} U(0, \lambda) &= c_2 \prod \frac{4(\lambda - \lambda_k(0))}{9j_k^2} = \\ &= \frac{\pi^{1/2} Ai(0)}{\lambda^{5/12}} \left\{ e^{\frac{2}{3}\sqrt{\lambda}} - \frac{\sqrt{3}}{2} e^{-\frac{2}{3}\sqrt{\lambda}} \right\} \left\{ 1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right\} \quad \lambda \rightarrow \infty \\ &= \frac{\pi^{1/2} Ai(0)}{\lambda^{5/12}} e^{\frac{2}{3}\sqrt{\lambda}} \left\{ 1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right\} . \quad \lambda \rightarrow \infty \end{aligned}$$

By lemma 4.14 on the circles $|\lambda| = \frac{9n^2\pi^2}{4}$ we have

$$\begin{aligned} &\prod \frac{4(\lambda - \lambda_k(0))}{9j_k^2} \\ &= \frac{\Gamma(4/3)}{(i\sqrt{\lambda}b/2)^{1/3}} J_{1/3}(ib\sqrt{\lambda}) \left(1 + O\left(\frac{\log n}{n}\right) \right) \end{aligned}$$

From [1], §9.2.1, for fixed ν and $|z| \rightarrow \infty$, the asymptotic form of $J_\nu(z)$ is of the form

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z - \pi\nu/2 - \pi/4\right) + e^{i|Imz|} O\left(\frac{1}{|z|}\right) \right\} \quad (|\arg z| < \pi) .$$

Therefore

$$J_{1/3}(ib\sqrt{\lambda}) = \sqrt{\frac{2}{i\pi b\sqrt{\lambda}}} \left\{ \cos\left(ib\sqrt{\lambda} - \frac{5\pi}{12}\right) + e^{b\sqrt{\lambda}} O\left(\frac{1}{\sqrt{\lambda}}\right) \right\} .$$

Since

$$\begin{aligned} \cos\left(ib\sqrt{\lambda} - \frac{5\pi}{12}\right) &= \frac{1}{2} \left\{ e^{b\sqrt{\lambda} + \frac{5i\pi}{12}} + e^{-b\sqrt{\lambda} - \frac{5i\pi}{12}} \right\} , \\ J_{1/3}(ib\sqrt{\lambda}) &= e^{b\sqrt{\lambda} + \frac{5i\pi}{12}} \sqrt{\frac{1}{2i\pi b\sqrt{\lambda}}} \left(1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right) . \end{aligned}$$

Since $Ai(0) = \frac{1}{3^{2/3}\Gamma(2/3)}$,

$$c_2 = \frac{U(0, \lambda)}{\prod \frac{4(\lambda - \lambda_k(0))}{9j_k^2}}$$

$$= \frac{\pi^{1/2} Ai(0) e^{\frac{2}{3}\sqrt{\lambda}} (b/2)^{1/3} \lambda^{1/6} (2i\pi b)^{1/2} \lambda^{1/4}}{\lambda^{5/12} \Gamma(4/3) e^{b\sqrt{\lambda} + \frac{5i\pi}{12}}} (1 + O(\frac{\log n}{n}))$$

whence

$$c_2 = \frac{2\pi}{\sqrt{3}\Gamma(1/3)\Gamma(2/3)} (1 + O(\frac{\log n}{n}))$$

or

$$c_2 = 1 + O(\frac{\log n}{n})$$

on the circles $|\lambda| = \frac{9n^2\pi^2}{4}$, because $\Gamma(1/3)\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}$ (See [1], §6.1.17). Using an argument similar to the one for c_3 we find that $c_2 = 1$.

Theorem 4.17 For $-1 \leq x < 0$,

$$U(x, \lambda) = \frac{p(x)}{(-x)^{1/4}} \prod \frac{\lambda - \lambda_k(x)}{z_k^2}$$

where $p(x) = \int_{-1}^x \sqrt{-t} dt$ and $U(t, \lambda)$ is a solution of the initial value problem (4.5) for the equation (4.1) and $\{\lambda_k(x)\}$ is the sequence of eigenvalues, for the Dirichlet problem associated with (4.1) on $[-1, x]$, i.e.,

$$y(-1, \lambda) = 0 = y(x, \lambda)$$

here $z_m = \frac{m\pi}{p(x)}$, as usual.

Proof: From (4.6) and (4.25) we have

$$U(x, \lambda) = c_1 \prod \frac{\lambda - \lambda_k(x)}{z_k^2}$$

$$= \frac{1}{(-x)^{1/4}\sqrt{\lambda}} (1 + O(\frac{1}{\sqrt{\lambda}})) \sinh(p(x)\sqrt{\lambda}) \quad \lambda \rightarrow \infty$$

where $p(x) = \int_{-1}^x \sqrt{-t} dt$.

From lemma 4.11, uniformly on the circles $|\lambda| = \frac{(n+1/2)^2 \pi^2}{p^2(x)}$, we have

$$\prod \frac{\lambda - \lambda_k(x)}{z_k^2} = \frac{\sinh(p(x)\sqrt{\lambda})}{p(x)\sqrt{\lambda}} (1 + O(\frac{\log n}{n}))$$

whence on the circles $|\lambda| = \frac{(n+1/2)^2 \pi^2}{p^2(x)}$

$$c_1 = \frac{U(x, \lambda)}{\prod \frac{\lambda - \lambda_k(x)}{z_k^2}} = \frac{p(x)}{(-x)^{1/4}} (1 + O(\frac{\log n}{n}))$$

As before we get

$$c_1 = \frac{p(x)}{(-x)^{1/4}}.$$

We will often use the abbreviated notation $\dot{U} \equiv \frac{\partial U}{\partial \lambda}$.

Theorem 4.18 *Let $U(t, \lambda)$ be the solution of boundary value problem*

$$y'' + (\lambda t - q)y = 0 \quad -1 \leq t \leq d$$

where d is arbitrary but fixed and

$$y(-1) = 0 = y(d) \quad \frac{\partial y}{\partial t}(-1) = 1.$$

Then

$$\dot{U}(d, \lambda)U'(d, \lambda) = \int_{-1}^d tU^2 dt.$$

Proof : Differentiating the equation with respect to λ yields

$$\dot{U}'' + tU + (\lambda t - q)\dot{U} = 0$$

Multiplying this equation by U , the original equation by \dot{U} and taking the difference, we obtain

$$\dot{U}''U - \dot{U}U'' + tU^2 = 0,$$

hence

$$\int_{-1}^d tU^2 dt = \dot{U}(d, \lambda)U'(d, \lambda)$$

since $\dot{U}(-1, \lambda) = 0$ by (4.6) .

4.3 Some properties of the eigenfunctions in the classical case

In this section we study some results in connection with the eigenfunctions corresponding to the eigenvalues in the classical case. Similar results with different methods have been obtained by Walter Pranger in [31] , for the equation

$$U'' + \lambda r(x)U = 0 \quad \text{on} \quad 0 \leq x \leq 1 \quad (4.32)$$

with the Dirichlet conditions

$$U(0) = U(1) = 0$$

where we assume , without loss of generality that

$$\frac{\partial U}{\partial x}(0, \lambda) = 1,$$

where $r(x)$ is a positive twice continuously differentiable function on $(0, 1)$. The first lemma is classical .

Lemma 4.19 *Let $U(t, \lambda)$ solve the initial value problem*

$$y'' + (\lambda t - q(t))y = 0$$

with initial condition

$$U(-1, \lambda) = 0 \quad \frac{\partial U}{\partial t}(-1, \lambda) = 1$$

for $-1 \leq t < 0$. Then

a)

$$U(t, \lambda_n(x)) = \frac{1}{(-t)^{1/4} \sqrt{-\lambda_n(x)}} \sin(p(t) \sqrt{-\lambda_n(x)}) + O\left(\frac{1}{-\lambda_n}\right) \quad \lambda_n \rightarrow -\infty$$

b)

$$\frac{\partial U}{\partial t}(x, \lambda_n(x)) = (-x)^{1/4} \cos p(x) \sqrt{-\lambda_n(x)} \left\{ 1 + O\left(\frac{1}{\sqrt{-\lambda_n}}\right) \right\} \quad \lambda_n \rightarrow -\infty$$

where $p(x)$ is defined in (4.7) and $\lambda_n \equiv \lambda_n(x)$ is the sequence of eigenvalues of the Dirichlet problem (4.1) on $[-1, x]$ for $x < 0$.

Proof : Let U be the solution of equation (4.1) which satisfies the given initial condition. If the following transformation is made, i.e.,

$$\xi = p(t) = \int_{-1}^t \sqrt{-v} dv,$$

$$W = (-t)^{1/4} U$$

$$-\lambda = \omega^2$$

then the equation assumes the normal form

$$\frac{d^2 W}{d\xi^2} + (\omega^2 - R(\xi))W(\xi) = 0$$

where

$$R(\xi) = \frac{5}{16t^3(\xi)} - \frac{q(t(\xi))}{t(\xi)}.$$

It is clear that W satisfies the initial condition

$$W(0, \omega) = 0 \quad \frac{dW(0, \omega)}{d\xi} = 1$$

By the basic estimates in {[35] , p.13} we have

$$W(\xi, \omega) = \frac{\sin(\xi\omega)}{\omega} + O\left(\frac{1}{\omega^2}\right),$$

$$\frac{dW(\xi, \omega)}{d\xi} = \cos(\xi\omega) + O\left(\frac{1}{\omega}\right).$$

Therefore , by applying the above transformation we get

$$U(t, \lambda_n(x)) = \frac{1}{(-t)^{1/4} \sqrt{-\lambda_n(x)}} \sin(p(t)\sqrt{-\lambda_n(x)}) + O\left(\frac{1}{-\lambda_n}\right) \quad \lambda_n \rightarrow -\infty$$

and

$$\frac{\partial U}{\partial t}(x, \lambda_n(x)) = (-x)^{1/4} \cos p(x)\sqrt{-\lambda_n(x)} + O\left(\frac{1}{\sqrt{-\lambda_n(x)}}\right) \quad \lambda_n \rightarrow -\infty$$

Lemma 4.20 *Let $U(t, \lambda)$ solve the initial value problem (4.1) with initial condition (4.5) for $-1 \leq t < 0$. Then*

$$\frac{\partial U}{\partial \lambda}(x, \lambda_n(x)) = \frac{p^3(x)(-1)^{n+1}}{2n^2\pi^2(-x)^{1/4}} \left\{1 + O\left(\frac{\log n}{n}\right)\right\}$$

where $p(x)$ is defined in (4.7) and $\lambda_n \equiv \lambda_n(x)$ is the sequence of eigenvalues of the Dirichlet problem (4.1) on $[-1, x]$ for $x < 0$.

Proof : From lemma 4.3 we have

$$\sinh p(x)\sqrt{\lambda} = p(x)\sqrt{\lambda} \prod \left(1 + \frac{\lambda}{z_k^2}\right)$$

where $z_k = \frac{k\pi}{p(x)}$, $k \in N$, and $p(x)$ is defined in (4.7) . we have

$$\frac{d}{d\lambda} \left\{ \frac{\sinh p(x)\sqrt{\lambda}}{p(x)\sqrt{\lambda}} \right\} \Big|_{\lambda=-z_n^2} = \frac{p^2(x)}{n^2\pi^2} \prod_{1 \leq k, n \neq k} \left(1 - \frac{z_n^2}{z_k^2}\right)$$

whence

$$\prod_{k \neq n} \left(1 - \frac{z_n^2}{z_k^2}\right) = \frac{(-1)^{n+1}}{2}$$

For fixed x , from theorem 4.17 we have

$$U(x, \lambda) = \frac{p(x)}{(-x)^{1/4}} \prod \frac{\lambda - \lambda_n(x)}{z_n^2}$$

therefore

$$\frac{\partial U}{\partial \lambda}(x, \lambda_n) = \frac{p^3(x)}{(-x)^{1/4} n^2 \pi^2} \prod_{1 \leq k, k \neq n} \frac{\lambda_n - \lambda_k(x)}{z_k^2}$$

Whence

$$\frac{\frac{\partial U}{\partial \lambda}(x, \lambda_n)}{\frac{(-1)^{n+1} p^2(x)}{2n^2 \pi^2}} = \frac{p(x)}{(-x)^{1/4}} \prod_{1 \leq k, k \neq n} \frac{\lambda_n - \lambda_k(x)}{z_k^2 - z_n^2}$$

Since

$$\frac{\lambda_n - \lambda_k}{z_k^2 - z_n^2} = 1 + O\left(\frac{1}{|k^2 - n^2|}\right) \quad k \neq n$$

therefore by theorem 4.8 we get

$$\frac{\partial U}{\partial \lambda}(x, \lambda_n) = \frac{p^3(x)(-1)^{n+1}}{2n^2 \pi^2 (-x)^{1/4}} \left\{1 + O\left(\frac{\log n}{n}\right)\right\}$$

Lemma 4.21 For fixed $x < 0$ let $\lambda_n(x)$ be the sequence of negative eigenvalues of the equation (4.1) for the Dirichlet problem on $[-1, x]$ where $0 \leq q(t)$, so that $U(x, \lambda_n(x)) = 0$. Then we have

a) $\lambda_n(x)$ is twice continuously differentiable and

$$\lambda'_n(x) = \frac{2(-x)^{1/2} n^2 \pi^2}{p^3(x)} \left\{1 + O\left(\frac{\log n}{n}\right)\right\} \quad n \rightarrow \infty$$

b)

$$\lim_{k \rightarrow \infty} \frac{d}{dx} \log \lambda_k(x) = \frac{-2z'_k}{z_k},$$

c) The series

$$\sum_{n \neq k, 1 \leq k} \frac{-\lambda_n \lambda'_k(x)}{(\lambda_k(x) - \lambda_n(x)) \lambda_k(x)},$$

is uniformly convergent on any compact subset of $(-1, 0)$,

d)

$$\int_{-1}^x v U^2(v, \lambda_n) dv = -\frac{p^3(x)}{2n^2 \pi^2} \left\{1 + O\left(\frac{\log n}{n}\right)\right\} \quad n \rightarrow \infty$$

where $z_k(x) = \frac{n\pi}{p(x)}$ and $p(x) = \int_{-1}^x \sqrt{-t} dt$.

Proof : a) It is known that for a non-negative continuous function $q(x)$, the eigenvalues of the Dirichlet problem for (4.1) on $[-1, x]$, are real and simple (see [14] , §10.61) , i.e.,

$$\frac{\partial U}{\partial \lambda}(x, \lambda_n(x)) \neq 0$$

for each $x \in (-1, 0)$. It follows from the implicit function theorem that $\lambda_n(x, q)$ is C^2 in x and

$$\lambda'_n(x) = -\left\{ \frac{\frac{\partial U}{\partial x}(x, \lambda)}{\frac{\partial U}{\partial \lambda}(x, \lambda)} \right\}_{\lambda=\lambda_n(x)}.$$

From the lemmas 4.19 and 4.20 we have

$$\lambda'_n(x) = \frac{2(-x)^{1/2}n^2\pi^2 \cos\{p(x)\sqrt{-\lambda_n}\}}{p^3(x)(-1)^n} \left\{ 1 + O\left(\frac{\log n}{n}\right) \right\} \quad n \rightarrow \infty$$

By inserting

$$\sqrt{-\lambda_n} = \frac{n\pi}{p(x)} + O\left(\frac{1}{n}\right)$$

in the above formula we get

$$\lambda'_n(x) = \frac{2(-x)^{1/2}n^2\pi^2 \cos\{n\pi + O(\frac{1}{n})\}}{p^3(x)(-1)^n} \left\{ 1 + O\left(\frac{\log n}{n}\right) \right\} \quad n \rightarrow \infty.$$

By the mean value theorem we have

$$\cos O\left(\frac{1}{n}\right) = 1 + O\left(\frac{1}{n^2}\right)$$

therefore

$$\lambda'_n(x) = \frac{2(-x)^{1/2}n^2\pi^2}{p^3(x)} \left\{ 1 + O\left(\frac{\log n}{n}\right) \right\} \quad n \rightarrow \infty$$

(b) From (a) and the distribution of $\sqrt{-\lambda_n(x)}$ we immediately obtain (b) .

(c) By (a) and (b) the sequence

$$\frac{\lambda'_k(x)}{\lambda_k(x)}$$

is uniformly bounded on any compact subset of $(-1, 0)$. Thus the above series is uniformly convergent by the M-test .

d) From Theorem 4.18 ,

$$\dot{U}(x, \lambda_n)U'(x, \lambda_n) = \int_{-1}^x vU^2 dv$$

Substituting the asymptotic form of $\dot{U}(x, \lambda_n)$ and $U'(x, \lambda_n)$ from theorem 4.20 and theorem 4.1 , respectively , and using the mean value theorem for $\cos(n\pi + O(1/n))$ we finally obtain the result .

4.4 Some properties of the eigenfunctions in the one turning point case

The method of the previous section is applicable to the one turning point case .

Lemma 4.22 *Let $U(t, \lambda)$ solve the initial value problem (4.1) with initial condition (4.5) for $-1 \leq t \leq 1$. Then for fixed $x > 0$,*

$$\frac{\partial U}{\partial \lambda}(x, u_n) = \frac{(-1)^{n-1} \sqrt{x} f^{5/2}(x) e^{\sqrt{u_n} p(0)}}{3p^{1/2}(0)n^2\pi^2} \{1 + O(1/n)\}$$

where $p(0) = 2/3$ and $f(x)$ is defined in (4.9) and $u_n \equiv u_n(x)$ is the sequence of eigenvalues of the Dirichlet problem (4.1) on $[-1, x]$ for $x > 0$.

Proof : For fixed $0 < x$, from theorem 4.15 we have

$$\frac{\partial U}{\partial \lambda}(x, u_n) = \frac{-f^2(x)\pi\sqrt{x}}{6\tilde{j}_n^2} \prod_{k=1}^n \frac{(u_n - r_k(x))p^2(0)}{\tilde{j}_k^2} \prod_{k \neq n, 1 \leq k \leq n} \frac{f^2(x)(u_k(x) - u_n)}{\tilde{j}_k^2} ,$$

Now we estimate the two infinite products . By lemma 4.13 we can write

$$\prod_{1 \leq k} \frac{(u_n - r_k(x))p^2(0)}{\tilde{j}_k^2} = 2J_1'(i\sqrt{u_n}p(0))(1 + O(\frac{\log n}{n}))$$

as $n \rightarrow \infty$.

From lemma 4.5 we get

$$\frac{d}{d\lambda} J_1'(c\sqrt{\lambda})|_{\lambda = \frac{i_n^2}{c^2}} = \frac{-c^2}{2\tilde{j}_n^2} \prod_{k \neq n, 1 \leq k} (1 - \frac{\tilde{j}_n^2}{\tilde{j}_k^2})$$

Since

$$\frac{\prod_{k \neq n, 1 \leq k \leq n} \frac{f^2(x)(u_k(x) - u_n)}{j_k^2}}{\prod_{k \neq n, 1 \leq k \leq n} (1 - \frac{j_n^2}{j_k^2})} = \prod_{k \neq n, 1 \leq k \leq n} \frac{f^2(x)(u_k(x) - u_n)}{\tilde{j}_k^2 - \tilde{j}_n^2},$$

from (4.3) and the form of the distribution of $u_k(x)$, $1 \leq k$, we can write

$$\left| \frac{f^2(x)(u_k(x) - u_n)}{\tilde{j}_k^2 - \tilde{j}_n^2} \right| = 1 + O\left(\frac{1}{|k^2 - n^2|}\right) \quad k \neq n.$$

Therefore by theorem 4.8 we get

$$\prod_{k \neq n, 1 \leq k \leq n} \frac{f^2(x)(u_k(x) - u_n)}{\tilde{j}_k^2} = \frac{-2\tilde{j}_n^2}{c^2} \frac{d}{d\lambda} J_1'(c\sqrt{\lambda}) \Big|_{\lambda = \frac{i\tilde{j}_n^2}{c^2}} \left\{ 1 + O\left(\frac{\log n}{n}\right) \right\}.$$

Consequently,

$$\frac{\partial U}{\partial \lambda}(x, u_n) = \frac{2f^2(x)\pi\sqrt{x}}{3c^2} J_1'(i\sqrt{u_n}p(0)) \frac{d}{d\lambda} J_1'(c\sqrt{\lambda}) \Big|_{\lambda = \frac{i\tilde{j}_n^2}{c^2}} \left(1 + O\left(\frac{\log n}{n}\right) \right)$$

The recurrence formula for $J_n(x)$ is of the form (see [1], §9.27)

$$xJ_n'(x) = -nJ_n + xJ_{n-1}$$

$$J_0'(x) = -J_1(x).$$

Therefore,

$$J_1'(c\sqrt{\lambda}) = -\frac{1}{c\sqrt{\lambda}} J_1(c\sqrt{\lambda}) + J_0(c\sqrt{\lambda}).$$

Thus, by using the recurrence formulae we find

$$\frac{d}{d\lambda} J_1'(c\sqrt{\lambda}) \Big|_{\lambda = \frac{i\tilde{j}_n^2}{c^2}} = \frac{-c^2}{2\tilde{j}_n} J_1(\tilde{j}_n) \left\{ 1 - \frac{1}{\tilde{j}_n^2} \right\}$$

The asymptotic form of $J_1(z)$ is

$$J_1(z) = \sqrt{\frac{2}{\pi z}} \{ \cos(z - 3\pi/4) + e^{|Imz|} O(|z|^{-1}) \} \quad (|\arg z| < \pi)$$

as $|z| \rightarrow \infty$. (see [1], §9.2.1)

By using the asymptotic form of $J_1'(z)$, (see the proof of theorem 4.15), and $J_1(z)$

we obtain

$$J_1'(i\sqrt{u_n}p(0)) = \frac{\sqrt{2}e^{\sqrt{u_n}p(0)}}{2\sqrt{\pi}u_n^{1/4}p^{1/2}(0)} \{ 1 + O(1/n) \}$$

$$J_1(\tilde{j}_n) = \sqrt{\frac{2}{\pi \tilde{j}_n}} \{ \cos(\tilde{j}_n - 3\pi/4) + O(1/n) \}$$

whence , use of (4.3) gives ,

$$\begin{aligned} \frac{\partial U}{\partial \lambda}(x, u_n) &= -\frac{f^2(x)\sqrt{x}}{3\tilde{j}_n^{3/2}u_n^{1/4}p^{1/2}(0)} e^{\sqrt{u_n}p(0)} \cos(\tilde{j}_n - 3\pi/4) \{1 + O(\frac{\log n}{n})\} \\ &= \frac{(-1)^n \sqrt{x} f^{5/2}(x) e^{\sqrt{u_n}p(0)}}{3p^{1/2}(0)n^2\pi^2} \{1 + O(\frac{\log n}{n})\} \end{aligned}$$

Theorem 4.23 Let $U(t, \lambda)$ be the solution of the Dirichlet problem (4.1) with variable t on $(-1, x)$, for fixed $0 < x$. Assume further that $U(t, \lambda)$ satisfies the condition (4.5) . Then

a)

$$\begin{aligned} U(t, u_n(x)) &= \frac{f(x)}{t^{1/4}n\pi} \left\{ \frac{1}{2} e^{-\pi(n-1/4)z(1)} \sin(\pi(n-1/4)z(t) - \pi/4) + \right. \\ &\quad \left. e^{\pi(n-1/4)z(1)} \cos(\pi(n-1/4)z(t) - \pi/4) \right\} \{1 + O(1/n)\} \end{aligned}$$

where $z(t)$ is defined below , in (f) ,

b)

$$\frac{\partial U}{\partial t}(x, u_n(x)) = (-1)^n x^{1/4} e^{\frac{2}{3}\sqrt{u_n(x)}} \{1 + O(1/n)\},$$

c)

$$u'_n(x) = \frac{-3n^2\pi^2}{x f^2(x)} \{1 + O(\frac{\log n}{n})\},$$

d)

$$\lim_{n \rightarrow \infty} \frac{u'_n(x)}{u_n(x)} = -\frac{3}{x} = \frac{-2f'(x)}{f(x)},$$

e) the series

$$\sum_{k \neq n} \frac{-u_n u'_k}{(u_k - u_n) u_k}$$

is uniformly convergent ,

f)

$$\int_{-1}^x v U^2(v, u_n) dv = \frac{x^{3/4} f^{5/2}(x) e^{\frac{4}{3}\sqrt{u_n}}}{3p^{1/2}(0)n^2\pi^2} \{1 + O(\frac{\log n}{n})\}$$

where $u_n(x)$ is a positive eigenvalue of the problem and

$$z(t) = \frac{1}{\int_0^x \sqrt{v} dv} \int_0^t \sqrt{v} dv = \frac{1}{f(x)} \int_0^t \sqrt{v} dv$$

Proof : a) By inserting $\lambda = u_n(x)$ in (4.27) and simplifying we obtain

$$U(t, u_n(x)) = \frac{2x^{3/2}}{3t^{1/4}n\pi} \left\{ \frac{1}{2} e^{-x^{-3/2}\pi(n-1/4)} \sin(\pi t^{3/2} x^{-3/2}(n-1/4) - \pi/4) + \right. \\ \left. e^{x^{-3/2}\pi(n-1/4)} \cos(\pi t^{3/2} x^{-3/2}(n-1/4) - \pi/4) \{1 + O(1/n)\} \right\}$$

Note that in the above long but straightforward calculation we use the following facts

$$\cos O(1/n) = 1 + O(1/n^2)$$

$$\sin O(1/n) = O(1/n)$$

for large n .

By inserting $z(t)$ in the preceding expression (a) follows.

(b) This follows by putting $\lambda = u_n(x)$ in (4.31).

(c) Since

$$u_n'(x) = -\frac{\frac{\partial U}{\partial x}(x, u_n(x))}{\frac{\partial U}{\partial \lambda}(x, u_n(x))}$$

therefore from lemma 4.20 and (b) the proof of (c) follows.

(d) Follows from (c) and the distribution of $u_n(x)$.

(e) Since $\frac{u_n'(x)}{u_n(x)}$ is uniformly bounded on any compact subset of $(-1, x)$ therefore the series is uniformly convergent.

f) This follows from theorem 4.18, 4.23 and 4.24(b).

Chapter 5

Some remarks on the inverse spectral problem

5.1 Introduction

In physics , engineering and quantum mechanics it is necessary to build a mathematical model to represent certain problems , e.g.,the determination of the inter-atomic forces for given energy-levels . This leads to the following general problem :

Given the spectrum of a second - order differential operator , to determine this operator .

This is known as the inverse Sturm-Liouville problem , in one dimension . It may be pointed out that the inverse problem may not be properly formulated . Marchenko in [22] has shown that the problem can only have a solution if the spectral distribution function $\sigma(\lambda)$ of the operator is given . The spectral distribution function is

defined in term of the normalization constants of the eigenfunctions.

Levitan and Gasymov in [20] gave a new account of the solution of the inverse problem in terms of the spectral function in the case of a classical Sturm-Liouville operator. In fact, the work of Levitan and Gasymov provides the following theorem

Theorem

If a non-decreasing function $\sigma(\lambda)$, defined for real λ , satisfies the following conditions :

For arbitrary real x the integral

$$\int_{-\infty}^0 e^{x\sqrt{|\lambda|}} d\sigma(\lambda)$$

exists .

Let

$$\tau(\lambda) = \begin{cases} \sigma(\lambda) - \frac{2}{\pi}\sqrt{\lambda} & \text{for } \lambda > 0 \\ \sigma(\lambda) & \text{for } \lambda < 0. \end{cases}$$

Then, for all x in the interval $0 \leq x < \infty$, the integral

$$\int_1^{\infty} \frac{\cos x\sqrt{\lambda}}{\lambda} d\tau(\lambda)$$

exists, and the function

$$a(x) = \int_1^{\infty} \frac{\cos x\sqrt{\lambda}}{\lambda} d\tau(\lambda)$$

has continuous derivatives up to and including the fourth order for $0 \leq x < \infty$.

Furthermore, if the set of points at which $\sigma(\lambda)$ increases has at least one finite limit point, then there exists just one differential operator of the second - order, defined by a differential expression

$$l(y) = -y'' + q(x)y \quad 0 \leq x < \infty,$$

with a continuous coefficient $q(x)$, and by a boundary condition of the form

$$y'(0) - \delta y(0) = 0$$

which has $\sigma(\lambda)$ as its spectral distribution function. The function $q(x)$ and the number δ are related by the formula

$$q(x) = \frac{1}{2} \frac{\partial k(x, x)}{\partial x}, \quad \delta = k(0, 0),$$

where $k(x, y)$ is a solution of the integral equation

$$f(x, y) + \int_0^x f(s, y)k(x, s)ds + k(x, y) = 0$$

In the above integral equation, $f(x, y)$ is of the form

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y}$$

where

$$F(x, y) = \int_{-\infty}^{\infty} \frac{\sin x \sqrt{\lambda} \sin y \sqrt{\lambda}}{\lambda} d\tau(\lambda)$$

For more details see [25].

The inverse problem for the canonical case of a two-term Sturm-Liouville operator on a finite interval is completely solved in [35]. In [35] the authors introduced the Dirichlet spectrum associated with the function q in L^2 and derived some of its properties. They found that the Dirichlet eigenvalues $u_n, 1 \leq n$ form a strictly increasing sequence of real numbers satisfying

$$u_n = n^2 \pi^2 + c + l^2(n)$$

where $c = \int_0^1 q(t)dt$, and $l^2(n)$ denotes a sequence a_n for which $\sum |a_n|^2 < \infty$. They presented these questions: ([35], p.49)

1) Do these conditions actually characterize all possible Dirichlet spectra? For instance, the sequence $n^2 \pi^2, 1 \leq n$, is the Dirichlet spectrum for $q = 0$. Suppose the

first eigenvalue π^2 is replaced by any number ν_1 below the second eigenvalue $4\pi^2$. Is the modified sequence still the Dirichlet spectrum of some q in L^2 ? Perhaps ν_1 has to be chosen in a special way?

2) To what extent is a point q in L^2 determined by its Dirichlet spectrum? For instance, are there any functions q in L^2 besides $q = 0$ with Dirichlet eigenvalues $u_n = n^2\pi^2, 1 \leq n$? More ambitiously, what does the *isospectral set*

$$M(p) = \{q \in L^2 : u_n(q) = u_n(p), 1 \leq n\}$$

of all functions q with the same Dirichlet spectrum as p look like?

These questions are completely answered in [35].

Belishev in [3] considered the homogenous boundary value problem

$$y'' + \lambda p(x)y = 0, \quad x \in [0, l], \quad (l < \infty) \quad (I)$$

with the conditions $y'(0) = y(l) = 0$. The density $p(x)$ is real-valued $p(x) \in L_1(0, l)$, $p(x) \neq 0$ almost everywhere on $(0, l)$. Let $\sigma = \{\lambda_j\}_{-\infty}^{\infty}$ be the negative and positive eigenvalues of the problem I and $\{Q_j(x)\}_{-\infty}^{\infty}$ be the corresponding eigenfunctions which are normalized by the condition $Q_j(0) = 1$ for all j . In [3] it is shown that the function $p(x)$ can be reconstructed from the spectrum σ and a knowledge of $\epsilon = \{\nu_j\}_{-\infty}^{\infty}$ where $\nu_j = \int_0^l [Q_j'(x)]^2 dx$.

In [31] Pranger considered differential equation (I) in which $p(x)$ is a positive twice continuously differentiable function on $[0, 1]$. In this article he studied the recovery of the function $p(x)$ from a knowledge of the eigenvalues.

This chapter is devoted to adapting Pranger's method to the one turning point

case. In section 5.2, we derive the dual equation and in section 5.3 the recovery of the potential function from the eigenvalues is studied. *In fact, the function $0 \leq q(x)$ is explicitly recovered by means of certain traces and their derivatives (see theorem 5.5 below).*

5.2 The dual equation

We recall that in chapter 4, for fixed $x \in (-1, 0)$ and $0 \leq q(t)$, the eigenvalues of the Dirichlet problem for the equation

$$y'' + (\lambda t - q(t))y = 0 \quad (5.1)$$

on $[-1, x]$ are denoted by $\lambda_n(x)$, $1 \leq n$. For fixed $x \in (0, 1)$, the positive eigenvalues of the same problem are denoted by $u_n(x)$, $1 \leq n$, while the negative eigenvalues by $r_n(x)$, $1 \leq n$. By the implicit function theorem $\lambda_n(x)$, $u_n(x)$ and $r_n(x)$ are twice continuously differentiable functions [chapter 4, §.4.2]. Let $U(t, \lambda)$ solve the equation (5.1) with initial conditions

$$U(-1, \lambda) = 0 \quad \frac{\partial U}{\partial t}(-1, \lambda) = 1.$$

For $x < 0$, the condition

$$U(x, \lambda_n(x)) = 0$$

gives, as usual,

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial \lambda} \cdot \lambda'_n = 0 \quad (5.2)$$

and differentiating again

$$\frac{\partial^2 U}{\partial x^2} + 2 \frac{\partial^2 U}{\partial x \partial \lambda} \cdot \lambda'_n + \frac{\partial^2 U}{\partial \lambda^2} \cdot (\lambda'_n)^2 + \frac{\partial U}{\partial \lambda} \cdot \lambda''_n = 0 \quad (5.3)$$

The first term in (5.3) is zero at $(x, \lambda_n(x))$ by virtue of (5.1) . Thus

$$2 \frac{\partial^2 U}{\partial x \partial \lambda} \cdot \lambda'_n + \frac{\partial^2 U}{\partial \lambda^2} \cdot (\lambda'_n)^2 + \frac{\partial U}{\partial \lambda} \cdot \lambda''_n = 0 \quad (5.4)$$

Similarly for , $0 < x$, the conditions

$$U(x, u_n(x)) = 0$$

$$U(x, r_n(x)) = 0$$

give the equations

$$\begin{aligned} 2 \frac{\partial^2 U}{\partial x \partial \lambda} \cdot u'_n + \frac{\partial^2 U}{\partial \lambda^2} \cdot (u'_n)^2 + \frac{\partial U}{\partial \lambda} \cdot u''_n &= 0 \\ 2 \frac{\partial^2 U}{\partial x \partial \lambda} \cdot r'_n + \frac{\partial^2 U}{\partial \lambda^2} \cdot (r'_n)^2 + \frac{\partial U}{\partial \lambda} \cdot r''_n &= 0 \end{aligned} \quad (5.5)$$

If we make use of the infinite product form of $U(x, \lambda)$, substitute this in (5.4), in the case $x < 0$ (and in (5.5) for $x > 0$) we will obtain the *dual* of the equation (5.1) . Indeed we need the various derivatives of $U(x, \lambda)$ at the points $(x, \lambda_n(x))$ for $x < 0$ and at the points $(x, u_n(x))$ and $(x, r_n(x))$ for $x > 0$.

Now , we first calculate the various derivatives of $U(x, \lambda)$ for $x < 0$. In this case, from (4.6), we can write

$$U(x, \lambda) = c \prod (1 - \frac{\lambda}{\lambda_k(x)}) \quad (5.6)$$

where c is only a function of x . In fact, by using theorem 4.17 one can determine $c(x)$. From 4.6 and theorem 4.17 ,

$$c_1 = \frac{p(x)}{(-x)^{1/4}} = c \prod \frac{-z_k^2}{\lambda_k}$$

where $z_k = \frac{k\pi}{p(x)}$ and $p(x) = \int_{-1}^x \sqrt{-t} dt$. Therefore

$$c(x) = \frac{p(x)}{(-x)^{1/4}} \prod \frac{-\lambda_k}{z_k^2} . \quad (5.7)$$

We calculate $\frac{\partial U}{\partial \lambda}$, $\frac{\partial^2 U}{\partial \lambda^2}$ and $\frac{\partial^2 U}{\partial x \partial \lambda}$ at the points $(x, \lambda_n(x))$ by using (5.6). In forming $\frac{\partial^2 U}{\partial \lambda \partial x}$ from (4.6), the interchange of summation and differentiation in

$$\frac{d}{dx} \sum \log\left(1 - \frac{\lambda}{\lambda_k(x)}\right)$$

will be valid if the differentiated series

$$\sum_{k \neq n} \frac{-\lambda_n \lambda'_k(x)}{(\lambda_k(x) - \lambda_n) \lambda_k(x)}$$

is uniformly convergent. According to lemma 4.21 (c) the above series is uniformly convergent. We define F_n by

$$F_n = F_n(x, \lambda_n(x)) = \prod_{k \neq n, 1 \leq k} \left(1 - \frac{\lambda_n(x)}{\lambda_k(x)}\right). \quad (5.8)$$

Since

$$\frac{\partial U}{\partial \lambda} = c \sum_{i=1}^{\infty} \frac{-1}{\lambda_i(x)} \prod_{k \neq i, 1 \leq k} \left(1 - \frac{\lambda}{\lambda_k(x)}\right),$$

we have

$$\begin{aligned} \frac{\partial U}{\partial \lambda}(x, \lambda_n) &= \frac{-c F_n}{\lambda_n(x)}, \\ \frac{\partial^2 U}{\partial \lambda^2}(x, \lambda_n(x)) &= \frac{2c F_n}{\lambda_n(x)} \sum_{i \neq n, 1 \leq i} \frac{1}{\lambda_i} \left(1 - \frac{\lambda_n(x)}{\lambda_i(x)}\right)^{-1}, \\ \frac{\partial^2 U}{\partial \lambda \partial x} &= \frac{-c'(x) F_n}{\lambda_n(x)} + \frac{c(x) \lambda'_n F_n}{\lambda_n^2} - c(x) F_n \sum_{i \neq n, 1 \leq i} \frac{\lambda'_i}{\lambda_i^2} \left(1 - \frac{\lambda_n(x)}{\lambda_i(x)}\right)^{-1} \\ &\quad - \frac{c(x) F_n \lambda'_n}{\lambda_n} \sum_{i \neq n, 1 \leq i} \frac{1}{\lambda_i} \left(1 - \frac{\lambda_n(x)}{\lambda_i(x)}\right)^{-1}. \end{aligned}$$

Placing these terms into (5.4) we obtain

$$\lambda''_n + \frac{2c' \lambda'_n}{c} + 2\lambda_n \lambda'_n \sum_{i \neq n, 1 \leq i} \frac{\lambda'_i}{\lambda_i^2} \left(1 - \frac{\lambda_n(x)}{\lambda_i(x)}\right)^{-1} - 2 \frac{(\lambda'_n)^2}{\lambda_n} = 0. \quad (5.9)$$

Dividing the above equation by λ'_n and integrating from a fixed number $\alpha \neq -1$ up to x , we obtain

$$\lambda'_n(x) = \frac{\lambda_n^2(x) \lambda'_n(\alpha) c^2(\alpha)}{\lambda_n^2(\alpha) c^2(x)} e^{-2S_n(x, \lambda_n)} \quad (5.10)$$

where

$$S_n(x, \lambda_n) = \sum_{i \neq n} \int_{\alpha}^x \frac{\lambda_i' \lambda_n}{\lambda_i} (\lambda_i - \lambda_n)^{-1} \quad (5.11)$$

and $c(x)$ is determined in (5.7) .

Similarly , for the case $x > 0$ from (4.10) and theorem 4.15 , we have

$$U(x, \lambda) = a(x) \prod (1 - \frac{\lambda}{r_k(x)}) \prod (1 - \frac{\lambda}{u_k(x)}) \quad (5.12)$$

with

$$a(x) = \frac{\pi \sqrt{x}}{6} \prod \frac{f^2(x) u_k(x)}{\tilde{j}_k^2} \prod \frac{p^2(0) r_k(x)}{\tilde{j}_k^2} \quad (5.13)$$

where $f(x) = \int_0^x \sqrt{t} dt, p^2(0) = 2/3$ and $\tilde{j}_k, k = 1, 2, \dots$ are the positive zeros of $J_1'(z)$.

As before , we calculate the various derivatives of $U(x, \lambda)$ and evaluate these at the fixed points $(x, u_n(x))$. Since , by theorem 4.23(e), the series

$$\sum_{k \neq n} \frac{-u_n u_k'(x)}{(u_k(x) - u_n) u_k(x)}$$

is uniformly convergent we obtain $\frac{\partial^2 U}{\partial \lambda \partial x}$ from (5.12) in terms of u_n and r_n .

Suppose

$$G_n(x, \lambda) = \prod_{k \neq n, 1 \leq k} (1 - \frac{\lambda}{u_k(x)})$$

$$H_n(x, \lambda) = \prod_{1 \leq k} (1 - \frac{\lambda}{r_k(x)}) .$$

Then ,

$$G_n = G_n(x, u_n(x)) = \prod_{k \neq n, 1 \leq k} (1 - \frac{u_n(x)}{u_k(x)}) \quad (5.14)$$

$$H_n = H_n(x, u_n(x)) = \prod_{1 \leq k} (1 - \frac{u_n(x)}{r_k(x)}) \quad (5.15)$$

so that

$$\prod_{k \neq i, 1 \leq k} (1 - \frac{u_n(x)}{r_k(x)}) = H_n (1 - \frac{u_n}{r_i})^{-1} . \quad (5.16)$$

We have

$$\begin{aligned}\frac{\partial U}{\partial \lambda}(x, u_n) &= \frac{-aH_n G_n}{u_n(x)} \\ \frac{\partial^2 U}{\partial \lambda^2}(x, u_n(x)) &= \frac{2aH_n G_n}{u_n(x)} \sum_{1 \leq i} \frac{1}{r_i(x) - u_n(x)} + \frac{2aH_n G_n}{u_n(x)} \sum_{1 \leq i, i \neq n} \frac{1}{u_i(x) - u_n(x)} \\ \frac{\partial^2 U}{\partial \lambda \partial x}(x, u_n(x)) &= \frac{-a'(x)H_n G_n}{u_n(x)} + \frac{a(x)H_n G_n u'_n}{u_n^2} \\ &\quad - \frac{a(x)H_n G_n u'_n}{u_n} \sum_{1 \leq i} \frac{1}{r_i(x) - u_n(x)} - a(x)H_n G_n \sum_{1 \leq i} \frac{r'_i}{r_i} (r_i(x) - u_n(x))^{-1} \\ &\quad - a(x)H_n G_n \sum_{1 \leq i, i \neq n} \frac{u'_i}{u_i} (u_i(x) - u_n(x))^{-1} - \frac{a(x)H_n G_n u'_n}{u_n} \sum_{1 \leq i, i \neq n} \frac{1}{u_i(x) - u_n(x)}\end{aligned}$$

Placing these terms into (5.5) we obtain

$$u_n'' + \frac{2a'u'_n}{a} + 2u_n u'_n \left\{ \sum_{i \neq n, 1 \leq i} \frac{u'_i}{u_i} (u_i(x) - u_n(x))^{-1} + \sum_{1 \leq i} \frac{r'_i}{r_i} (r_i(x) - u_n(x))^{-1} \right\} - 2 \frac{(u'_n)^2}{u_n} = 0. \quad (5.17)$$

Similarly for negative eigenvalue $r_n(x)$ we get

$$r_n'' + \frac{2a'r'_n}{a} + 2r_n r'_n \left\{ \sum_{i \neq n, 1 \leq i} \frac{r'_i}{r_i} (r_i(x) - r_n(x))^{-1} + \sum_{1 \leq i} \frac{u'_i}{u_i} (u_i(x) - r_n(x))^{-1} \right\} - 2 \frac{(r'_n)^2}{r_n} = 0. \quad (5.18)$$

Dividing the equation (5.17) by u'_n , the equation (5.18) by r'_n and integrating from x up to 1, we obtain

$$\begin{aligned}u'_n(x) &= \frac{u_n^2(x)u'_n(1)a^2(1)}{u_n^2(1)a^2(x)} e^{2T_n(x, u_n, r_n)} \\ r'_n(x) &= \frac{r_n^2(x)r'_n(1)a^2(1)}{r_n^2(1)a^2(x)} e^{2T_n(x, r_n, u_n)}\end{aligned} \quad (5.19)$$

where

$$T_n(x, u_n, r_n) = \sum_{i \neq n} \int_x^1 \frac{u'_i u_n}{u_i} (u_i - u_n)^{-1} dv + \sum_i \int_x^1 \frac{r'_i u_n}{r_i} (r_i - u_n)^{-1} dv, \quad (5.20)$$

$a(x)$ is determined in (5.13), and the variables in the integrand in (5.20) are dropped for brevity.

The system of equations (5.10) and (5.19) is dual to the original equation (5.1) and involves only the functions $\lambda_n(x)$, $u_n(x)$ and $r_n(x)$.

5.3 The Inverse problem

We now proceed with the solution of the inverse problem. We first study the relation between the eigenvalues and the potential function.

Theorem 5.1 *Let $\lambda_n(x)$ be the eigenvalues of the Dirichlet problem (5.1) on $[-1, x]$, for fixed $x < 0$. Then*

a)

$$\sigma_1(x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k(x)} = \int_{-1}^x \frac{1}{c^2(t)} \int_{-1}^t v c^2(v) dv dt \quad (5.21)$$

b)

$$q(x) = \left(\frac{x - \sigma_1''}{2\sigma_1'} \right)^2 + \left(\frac{x - \sigma_1''}{2\sigma_1'} \right)' \quad x < 0 \quad (5.22)$$

where $c(x)$ is determined in (5.7).

Proof: For fixed $x < 0$, if $U(x, \lambda)$ in (5.6) is differentiated logarithmically, then

$$-\frac{\partial}{\partial \lambda} \log U(x, \lambda) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k - \lambda} = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{\lambda^m}{\lambda_k^{m+1}}$$

provided that $|\lambda| < |\lambda_1|$. The last series is absolutely convergent because

$$\sum \frac{1}{|\lambda_k - \lambda|} \leq \sum \frac{1}{|\lambda_k| - |\lambda|} = \sum O\left(\frac{1}{k^2}\right),$$

thus we can interchange the order of summation. Put

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{m+1}} = \sigma_{m+1}$$

and inter-change the order of summation . Then

$$-\frac{\partial}{\partial \lambda} U(x, \lambda) = U(x, \lambda) \sum_{m=0}^{\infty} \sigma_{m+1} \lambda^m.$$

Replace $U(x, \lambda)$ on each side by

$$U(x, \lambda) = a_0(x) + a_1(x)\lambda + a_2(x)\lambda^2 + \dots \quad (5.23)$$

Multiply out the product on the right , and equate coefficients of the various powers of λ in the identity ; we thus obtain the system of equations

$$-a_1(x) = a_0(x)\sigma_1 \quad (5.24)$$

$$-2a_2(x) = a_0(x)\sigma_2 + a_1(x)\sigma_1$$

...

$$-na_n(x) = a_0(x)\sigma_n + a_1(x)\sigma_{n-1} + \dots + a_{n-1}(x)\sigma_1.$$

From (5.6) and (5.23) we have

$$U(x, 0) = c(x) = a_0(x).$$

Using the expansion of $U(x, \lambda)$ in (5.1) we get

$$a_0'' + \lambda a_1'' + \dots + (\lambda x - q(x))(a_0 + a_1 \lambda + \dots) = 0.$$

Consequently , by equating coefficients of the various powers of λ we obtain

$$a_0'' - qa_0 = 0$$

$$a_1'' + xa_0 - a_1q(x) = 0 \quad (5.25)$$

$$a_2'' + xa_1 - a_2q(x) = 0$$

...

Placing (5.24) in (5.25) and using $a_0'' = qa_0$ we get

$$a_0^2 \sigma_1'' + 2a_0 a_0' \sigma_1' - x a_0^2 = 0 \quad (5.26)$$

or

$$\frac{d}{dx}(a_0^2 \sigma_1') = x a_0^2.$$

Integrating this over $[-1, x]$, noting that $a_i(-1) = 0$, we see that, after another integration and use of the fact that $\sigma_1(-1) = 0$ and $a_0 = c$, that there holds (5.21).

b) From (5.26) we write

$$\sigma_1'' + 2 \frac{a_0'}{a_0} \sigma_1' - x = 0.$$

Consequently

$$\left(\frac{a_0'}{a_0}\right)' = \left(\frac{x - \sigma_1''}{2\sigma_1'}\right)'$$

whence

$$\begin{aligned} q(x) &= \frac{a_0''}{a_0} = \left(\frac{a_0'}{a_0}\right)' + \left(\frac{x - \sigma_1''}{2\sigma_1'}\right)' \\ &= \left(\frac{x - \sigma_1''}{2\sigma_1'}\right)' + \left(\frac{x - \sigma_1''}{2\sigma_1'}\right)' \quad x < 0 \end{aligned}$$

Applying arguments similar to those above, we deduce the following theorem :

Theorem 5.2 *Let the sequence $u_n(x)$ represent the positive eigenvalues and $r_n(x)$ the negative eigenvalues of the Dirichlet problem (5.1) on $[-1, x]$ for fixed $0 < x$.*

Then

a)

$$\Lambda_1(x) \equiv \sum_{k=1}^{\infty} \left[\frac{1}{u_k(x)} + \frac{1}{r_k(x)} \right] = \Lambda_1(1) - \int_x^1 \frac{1}{a^2(t)} \int_0^t v a^2(v) dv dt \quad (5.27)$$

b)

$$q(x) = \left(\frac{x - \Lambda_1''}{2\Lambda_1'}\right)' + \left(\frac{x - \Lambda_1''}{2\Lambda_1'}\right)' \quad x > 0 \quad (5.28)$$

where $a(x)$ is determined in (5.19).

In the proof of theorem 5.2 , we follow a procedure similar to that of the proof of theorem 5.1 to find

$$\frac{d(a^2 \Lambda_1')}{dx} = x a^2 \tag{5.29}$$

where $a(x)$ is defined in (5.13) and $\Lambda_1 = \sum[\frac{1}{u_k} + \frac{1}{r_k}]$. Since $\lim_{x \rightarrow 0} a(x) = 0$, we obtain

$$a^2 \Lambda_1'(x) = \int_0^x v a^2(v) dv.$$

Consequently ,

$$\Lambda_1(x) = \Lambda_1(1) - \int_x^1 \frac{1}{a^2(t)} \int_0^t v a^2(v) dv$$

where

$$\Lambda_1(1) = \sum[\frac{1}{u_k(1)} + \frac{1}{r_k(1)}] \tag{5.30}$$

We see that if σ_1 in (5.21) , and Λ_1 in (5.27) is known , the formula (5.22) and (5.28) defines the function $q(x)$. So the main problem is to determine the functions $u_n(x)$, $r_n(x)$ and $\lambda_n(x)$.

Lemma 5.3 *Let $(u_n(x)), (r_n(x))$ be a sequence of twice continuously differentiable functions on $(0 , 1]$ such that for each $x > 0$,*

a)

$$0 < u_1(x) < u_2(x) < \dots,$$

b)

$$\sqrt{u_n(x)} = \frac{12n\pi - 3\pi}{8x^{3/2}} + O(1/n)$$

c)

$$u_n'(x) < 0 \text{ for each } n \text{ and each } x \text{ in } (0 , 1] \text{ and}$$

$$u'_n(x) = -\frac{27n^2\pi^2}{4x^4} \left\{ 1 + O\left(\frac{\log n}{n}\right) \right\}$$

d)

$$u''_n(x) = O(n^2)$$

e)

$$\dots < r_3(x) < r_2(x) < r_1(x) < 0$$

f)

$$\sqrt{-r_n(x)} = \frac{12n\pi - 3\pi}{8} + O(1/n)$$

g)

$r'_n(x) < 0$ for each n and each x in $(0, 1]$, furthermore $r'_n(x) = O(n)$

h)

$$r''_n(x) = O(n) .$$

i)

The function $a(x)$ defined in (5. 19) is twice continuously differentiable .

Let

$$\begin{aligned} U(x, \lambda) &= a(x) \prod \left(1 - \frac{\lambda}{r_k(x)}\right) \prod \left(1 - \frac{\lambda}{u_k(x)}\right) \\ &= \frac{\pi\sqrt{x}}{6} \prod \frac{(\lambda - r_k(x))p^2(0)}{\tilde{j}_k^2} \prod \frac{f^2(x)(u_k(x) - \lambda)}{\tilde{j}_k^2}, \quad x > 0 \end{aligned}$$

where \tilde{j}_k is the sequence of positive zeros of $J'_1(z)$. Then

1) $U(x, \lambda) \in C^2[0, 1]$ for each λ in the complex plane .

2) $U, \frac{\partial U}{\partial x}$ and $\frac{\partial^2 U}{\partial x^2}$ are entire functions of λ , for each x .

Proof : We establish first that U , $G_n(x, \lambda) = \prod_{k \neq n} (1 - \frac{\lambda}{u_k(x)})$, $T_n(x, \lambda) = \prod_{k \neq n} (1 - \frac{\lambda}{r_k(x)})$, $\frac{\partial G_n(x, \lambda)}{\partial x}$ and $\frac{\partial T_n(x, \lambda)}{\partial x}$ are uniformly bounded on $A \times B$ where A is a compact subset of $(0, 1]$ and B is a compact subset of the complex plane . From the inequality , $1 + v < e^v, v > 0$, we can write

$$\begin{aligned} |U(x, \lambda)| &\leq |a(x)| e^{|\lambda| \sum (\frac{1}{u_k} + \frac{1}{|r_k|})} \\ &\leq \frac{\pi \sqrt{x}}{6} e^{|\lambda| \sum (\frac{1}{u_k} + \frac{1}{|r_k|}) + \sum O(\frac{1}{k^2})} \end{aligned}$$

hence U is uniformly bounded on $A \times B$.

The convergence of $(1 - \frac{\lambda}{u_n})$ to 1 uniformly on $A \times B$ means that given $\delta > 0$ sufficiently small we have on $A \times B$, $0 < \delta < |1 - \frac{\lambda}{u_n}|$ for all but a finite number of n . Since

$$\begin{aligned} \left| \frac{1}{\prod (1 - \frac{\lambda}{r_k(x)})} \right| &= \left| \prod \left(\frac{r_k(x)}{r_k(x) - \lambda} \right) \right| \leq e^{\sum \left| \frac{r_k}{r_k - \lambda} - 1 \right|} \\ &\leq e^{\sum \frac{|\lambda|}{\| \lambda \| - |r_k|} = \sum O(\frac{1}{k^2})} \end{aligned}$$

it follows that ,

$$|G_n(x, \lambda)| \leq \left| \frac{1}{a(x)} \right| \|U\| \prod \left(\frac{r_k}{r_k - \lambda} \right) \left| \left(1 - \frac{\lambda}{u_n} \right)^{-1} \right| < \left| \frac{1}{a(x)} \right| \|U\| \delta^{-1} e^{\sum O(\frac{1}{k^2})}$$

for all but finitely many n . Taking into account finitely many terms and using the fact that U is uniformly bounded we see that $G_n(x, \lambda)$ is also uniformly bounded on $A \times B$. Similarly one can prove that $T_n(x, \lambda)$ is uniformly bounded on $A \times B$.

Consider the series

$$\sum_{i \neq n} \frac{\lambda u'_i}{u_i^2} \prod_{k \neq i, k \neq n} \left(1 - \frac{\lambda}{u_k} \right) = \sum_{i \neq n} \frac{\lambda u'_i}{u_i^2} \left(1 - \frac{\lambda}{u_i} \right)^{-1} G_n(x, \lambda).$$

We show that this series converges uniformly on $A \times B$. It then follows that , since this is the series obtained by formally differentiating the infinite product representation of $G_n(x, \lambda)$, $\frac{\partial G_n(x, \lambda)}{\partial x}$ exists and is given by the last display , i.e.,

$$\frac{\partial G_n(x, \lambda)}{\partial x} = \sum_{i \neq n} \frac{\lambda u'_i}{u_i^2} \prod_{k \neq i, k \neq n} \left(1 - \frac{\lambda}{u_k} \right)$$

A similar argument shows that $\frac{\partial T_n(x, \lambda)}{\partial x}$ exists and is given by

$$\frac{\partial T_n(x, \lambda)}{\partial x} = \sum_{i \neq n} \frac{\lambda r'_i}{r_i^2} \prod_{k \neq i, k \neq n} \left(1 - \frac{\lambda}{r_k}\right)$$

on $A \times B$. Since $\frac{u'_i}{r_i} = O(\frac{1}{i^2})$ and $\frac{r'_i}{r_i} = O(\frac{1}{i^2})$ by hypothesis, we see that by means of a previous inequality for $|G_n(x, \lambda)|$, the series is uniformly convergent and we can also bound $\frac{\partial G_n(x, \lambda)}{\partial x}$ and $\frac{\partial T_n(x, \lambda)}{\partial x}$ uniformly on $A \times B$. Consider the expression

$$\begin{aligned} a'(x) & \prod \left(1 - \frac{\lambda}{r_k(x)}\right) \prod \left(1 - \frac{\lambda}{u_k(x)}\right) + a(x) \sum \frac{\lambda r'_i}{r_i^2} T_i(x, \lambda) \prod \left(1 - \frac{\lambda}{u_k(x)}\right) + \\ a(x) & \prod \left(1 - \frac{\lambda}{r_k(x)}\right) \sum \frac{\lambda u'_i}{u_i^2} G_i(x, \lambda) \end{aligned}$$

It follows from the above analysis that each series herein converges uniformly on $A \times B$. It then follows that, since this expression is the expression obtained by formally differentiating the infinite product representation of $U(x, \lambda)$, $\frac{\partial U}{\partial x}$ exists and is given by the last display, i.e.,

$$\begin{aligned} \frac{\partial U}{\partial x} & = a'(x) \prod \left(1 - \frac{\lambda}{r_k(x)}\right) \prod \left(1 - \frac{\lambda}{u_k(x)}\right) + a(x) \sum \frac{\lambda r'_i}{r_i^2} T_i(x, \lambda) \prod \left(1 - \frac{\lambda}{u_k(x)}\right) \\ & + a(x) \prod \left(1 - \frac{\lambda}{r_k(x)}\right) \sum \frac{\lambda u'_i}{u_i^2} G_i(x, \lambda) \end{aligned}$$

A similar argument shows that $\frac{\partial^2 U}{\partial x^2}$ exists on $A \times B$ and is given by

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} & = a''(x) \prod \left(1 - \frac{\lambda}{r_k(x)}\right) \prod \left(1 - \frac{\lambda}{u_k(x)}\right) + 2a'(x) \sum \frac{\lambda r'_i}{r_i^2} T_i(x, \lambda) \prod \left(1 - \frac{\lambda}{u_n(x)}\right) \\ & + a(x) \lambda \sum \left[\left(\frac{r''_i}{r_i^2} - \frac{2(r'_i)^2}{r_i^3} \right) T_i(x, \lambda) + \frac{r'_i}{r_i^2} \frac{\partial T_i(x, \lambda)}{\partial x} \right] \prod \left(1 - \frac{\lambda}{u_k(x)}\right) \\ & + a(x) \lambda \prod \left(1 - \frac{\lambda}{r_k(x)}\right) \sum \left[\left(\frac{u''_i}{u_i^2} - \frac{2(u'_i)^2}{r_i^3} \right) G_i(x, \lambda) + \frac{u'_i}{u_i^2} \frac{\partial G_i(x, \lambda)}{\partial x} \right] + \\ & 2a'(x) \prod \left(1 - \frac{\lambda}{r_k(x)}\right) \sum \frac{\lambda u'_i}{u_i^2} G_i(x, \lambda) + 2a(x) \sum \frac{\lambda r'_i}{r_i^2} T_i(x, \lambda) \sum \frac{\lambda u'_i}{u_i^2} G_i(x, \lambda) \quad (5.31) \end{aligned}$$

The only terms which essentially warrant an additional explanation in (5.31) are the third and fourth terms (series). By assumptions (f), (g), (h) and the uniform

boundedness of T_i and $\frac{\partial T_i}{\partial x}$, it follows that the third series converges uniformly on $A \times B$. A similar argument applies to the fourth term except that we now use (b), (c), (d) etc.

Note that each one of the terms U , $\frac{\partial U}{\partial x}$, $\frac{\partial^2 U}{\partial x^2}$ are entire functions of λ , for each x , as each one of the terms in their representation is analytic for λ in a compact subset of complex plane, being the uniformly convergent limit of entire functions. Similarly we can prove

Lemma 5.4 *Let $\lambda_n(x)$ be a sequence of twice continuously differentiable functions such that for each $x < 0$,*

a)

$$0 > \lambda_1(x) > \lambda_2(x) > \lambda_3(x) > \dots,$$

b)

$$\sqrt{-\lambda_n(x)} = \frac{n\pi}{p(x)} + O(1/n)$$

c)

$$\lambda'_n(x) > 0 \text{ for each } n \text{ and each } x \text{ in } [-1, 0) \text{ and}$$

$$\lambda'_n(x) = \frac{2n^2\pi^2 p'(x)}{p^3(x)} \left\{ 1 + O\left(\frac{\log n}{n}\right) \right\} \quad n \rightarrow \infty$$

d)

$$\lambda''_n(x) = O(n^2)$$

e) *the function $c(x)$ defined in (5.7) is twice continuously differentiable function and $c'(-1) = 1$.*

Let

$$\begin{aligned} U(x, \lambda) &= c(x) \prod \left(1 - \frac{\lambda}{\lambda_k(x)}\right) & x < 0 \\ &= \frac{p(x)}{(-x)^{1/4}} \prod \frac{\lambda - \lambda_k(x)}{z_k^2} \end{aligned}$$

where $p(x) = \int_{-1}^x \sqrt{-t} dt$ and $z_k = \frac{k\pi}{p(x)}$. Then

- 1) $U(x, \lambda) \in C^2[-1, 0)$ for each λ in the complex plane .
- 2) $U, \frac{\partial U}{\partial x}$ and $\frac{\partial^2 U}{\partial x^2}$ are entire functions of λ , for each x .
- 3) We have ,

$$\begin{aligned} \frac{\partial U}{\partial x} &= c'(x) \prod \left(1 - \frac{\lambda}{\lambda_k}\right) + c(x) \sum \frac{\lambda \lambda'_k}{\lambda_k^2} F_k(x, \lambda) \\ \frac{\partial^2 U}{\partial x^2} &= c''(x) \prod \left(1 - \frac{\lambda}{\lambda_k}\right) + \\ &2c'(x) \sum \frac{\lambda \lambda'_k}{\lambda_k^2} F_k(x, \lambda) + c(x) \lambda \sum \left[\left(\frac{\lambda''_k}{\lambda_k^2} - 2 \frac{(\lambda'_k)^2}{\lambda_k^3} \right) F_k(x, \lambda) + \frac{\lambda'_k}{\lambda_k^2} \frac{\partial F_k(x, \lambda)}{\partial x} \right] \end{aligned}$$

where

$$F_k(x, \lambda) = \prod_{k \neq i} \left(1 - \frac{\lambda}{\lambda_i(x)}\right).$$

These equalities hold for x in a compact subset of $[-1, 0)$ and λ in a compact subset of the complex plane .

- 4) $U(-1, \lambda) = 0$, $\frac{\partial U}{\partial x}(-1, \lambda) = 1$ and $U(x, 0) = c(x)$.

Let the functions $u_n(x)$, $r_n(x)$ and $\lambda_n(x)$ with special properties be given . We want to find out whether a second order differential equation exists in the interval $[-1, 1]$ whose eigenvalues for the Dirichlet problem coincide with $u_n(x)$, $r_n(x)$ and $\lambda_n(x)$, and if it does exist we want to determine that equation .

Theorem 5.5 Suppose the solution of the differential equation (5.17) and (5.18) on $(0, 1]$ with initial conditions $(u_n(1), (r_n(1)), (u'_n(1))$ and $(r'_n(1))$ exists and is

given by $(u_n(x)), (r_n(x))$. Furthermore, assume that the solution of the differential equation (5.9) on $[-1, 0)$ exists, is given by $\lambda_n(x)$, and is such that $\lim_{x \rightarrow 0^-} \lambda_n(x) = \lim_{x \rightarrow 0^+} r_n(x)$ and $\lim_{x \rightarrow 0^-} \lambda_n'(x) = \lim_{x \rightarrow 0^+} r_n'(x)$. Then

1) If $(u_n(x)), (r_n(x))$ satisfies (a) through (i) of lemma 5.3, then there are functions $U(x, \lambda)$, $q(x)$ satisfying (5.1) on $(0, 1]$, and $q(x)$ is continuous everywhere except possibly at $x = 0$.

2) If $(\lambda_n(x))$ satisfies (a) through (e) of lemma 5.4, then there are functions $U(x, \lambda)$, $q(x)$ satisfying (5.1) on $[-1, 0)$, (see(1)).

3) If

$$\lim_{x \rightarrow 0^+} \left[\left(\frac{x - \Lambda_1''}{2\Lambda_1'} \right)^2 + \left(\frac{x - \Lambda_1''}{2\Lambda_1'} \right)' \right] = \lim_{x \rightarrow 0^-} \left[\left(\frac{x - \sigma_1''}{2\sigma_1'} \right)^2 + \left(\frac{x - \sigma_1''}{2\sigma_1'} \right)' \right]$$

then $q(x)$ is continuous at $x = 0$.

4) If $\frac{x - \Lambda_1''}{2\Lambda_1'}$ is an increasing function on $(0, 1]$, then $0 \leq q(x)$ on $(0, 1]$. Similarly, if $\frac{x - \sigma_1''}{2\sigma_1'}$ is an increasing function on $[-1, 0)$, then $0 \leq q(x)$ on $[-1, 0)$ where

$$\sigma_1 = \sum_{k=1}^{\infty} \frac{1}{\lambda_n(x)}$$

$$\Lambda_1 = \sum_{k=1}^{\infty} \left[\frac{1}{u_n(x)} + \frac{1}{r_n(x)} \right].$$

Proof : For $0 < x \leq 1$ let $u_n(x)$ and $r_n(x)$ be the solution of the differential equation (5.17) and (5.18) with given initial conditions $(u_n(1), r_n(1)), (u_n'(1), r_n'(1))$. For $0 < x \leq 1$ let $\lambda_n(x)$ be the solution of the differential equation (5.9) with initial condition $\lambda_n'(0) = \lim_{x \rightarrow 0} r_n'(x)$ and $\lambda_n(0) = \lim_{x \rightarrow 0} r_n(x)$. We define

$$U(x, \lambda) = \begin{cases} a(x) \prod(1 - \frac{\lambda}{r_n(x)}) \prod(1 - \frac{\lambda}{u_n(x)}) & \text{for } 0 < x \leq 1 \\ \prod(1 - \frac{\lambda}{\lambda_n(0)}) & \text{for } x = 0 \\ U(x, \lambda) = c(x) \prod(1 - \frac{\lambda}{\lambda_n(x)}) & \text{for } -1 \leq x < 0 \end{cases}$$

$$q(x) = \begin{cases} \left(\frac{x - \Lambda_1''}{2\Lambda_1'} \right)^2 + \left(\frac{x - \Lambda_1''}{2\Lambda_1'} \right)' & \text{for } 0 < x \leq 1 \\ \left(\frac{x - \sigma_1''}{2\sigma_1'} \right)^2 + \left(\frac{x - \sigma_1''}{2\sigma_1'} \right)' & \text{for } -1 \leq x < 0 \end{cases} \quad (5.32)$$

and at $x = 0$,

$$q(0) = \lim_{x \rightarrow 0} \left(\frac{x - \sigma_1''}{2\sigma_1'} \right)^2 + \left(\frac{x - \sigma_1''}{2\sigma_1'} \right)'$$

where

$$\sigma_1 = \sum_{k=1}^{\infty} \frac{1}{\lambda_n(x)},$$

$$\Lambda_1 = \sum_{k=1}^{\infty} \left[\frac{1}{u_n(x)} + \frac{1}{r_n(x)} \right],$$

$$\begin{aligned} \frac{c'(x)}{c(x)} &= \frac{x - \sigma_1''}{2\sigma_1'}, & -1 < x < 0 \\ \frac{a'(x)}{a(x)} &= \frac{x - \Lambda_1''}{2\Lambda_1'}, & x > 0. \end{aligned} \tag{5.33}$$

We prove that if $(u_n(x), r_n(x))$ satisfies (a) through (i) of lemma 5.3, and $\lambda_n(x)$ (a) through (e) of lemma 5.4, then U satisfies (5.1). By lemma 5.4, we see that $U(-1, \lambda) = 0$, $\frac{\partial U}{\partial x}(-1, \lambda) = 1$. Let us consider first the case $0 < x \leq 1$. Since $U(x, u_n(x)) = 0, U(x, r_n(x)) = 0$, equations similar to (5.3) hold :

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} + 2 \frac{\partial^2 U}{\partial x \partial \lambda} \cdot u_n' + \frac{\partial^2 U}{\partial \lambda^2} \cdot (u_n')^2 + \frac{\partial U}{\partial \lambda} \cdot u_n'' &= 0 \\ \frac{\partial^2 U}{\partial x^2} + 2 \frac{\partial^2 U}{\partial x \partial \lambda} \cdot r_n' + \frac{\partial^2 U}{\partial \lambda^2} \cdot (r_n')^2 + \frac{\partial U}{\partial \lambda} \cdot r_n'' &= 0 \end{aligned}$$

But equations (5.17) and (5.18) yield the equations (5.5) where $U(x, \lambda)$ has the above representation. Therefore $\frac{\partial^2 U}{\partial x^2}$ has $u_n(x)$ and $r_n(x)$ as zeros. Since $\frac{\partial^2 U}{\partial x^2}$ is entire and vanishes at each $u_n(x), r_n(x)$ it follows that $\prod(1 - \frac{\lambda}{r_n(x)}) \prod(1 - \frac{\lambda}{u_n(x)})$ must divide $\frac{\partial^2 U}{\partial x^2}$. Whence there is an entire function $Q(x, \lambda)$ such that

$$\frac{\partial^2 U}{\partial x^2} = Q(x, \lambda)U.$$

We show that

$$Q(x, \lambda) = q(x) - \lambda x.$$

Multiplying (5.17) with $n = i$, by $u_i^{-2}(1 - \frac{\lambda}{u_i})^{-1}$ and taking summation we get

$$\begin{aligned} & \sum [(\frac{u_i''}{u_i^2} - 2\frac{(u_i')^2}{u_i^3})(1 - \frac{\lambda}{u_i})^{-1}] = -2\frac{a'(x)}{a(x)} \sum \frac{u_i'}{u_i^2}(1 - \frac{\lambda}{u_i})^{-1} \\ & -2 \sum \frac{u_i'}{u_i}(1 - \frac{\lambda}{u_i})^{-1} \sum_{i \neq j} \frac{u_j'}{u_j}(u_j - u_i)^{-1} - 2 \sum \frac{u_i'}{u_i}(1 - \frac{\lambda}{u_i})^{-1} \sum_j \frac{r_j'}{r_j}(r_j - u_i)^{-1}. \end{aligned}$$

Similarly from (5.18) we have

$$\begin{aligned} & \sum (\frac{r_i''}{r_i^2} - 2\frac{(r_i')^2}{r_i^3})(1 - \frac{\lambda}{r_i})^{-1} = -2\frac{a'(x)}{a(x)} \sum \frac{r_i'}{r_i^2}(1 - \frac{\lambda}{r_i})^{-1} \\ & -2 \sum \frac{r_i'}{r_i}(1 - \frac{\lambda}{r_i})^{-1} \sum_{i \neq j} \frac{r_j'}{r_j}(r_j - r_i)^{-1} - 2 \sum \frac{r_i'}{r_i}(1 - \frac{\lambda}{r_i})^{-1} \sum_j \frac{u_j'}{u_j}(u_j - r_i)^{-1}. \end{aligned}$$

From the above display and the second derivative of U as given in (5.31) we get, after some simplifications, for $0 < x \leq 1$,

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} &= \frac{a''(x)}{a(x)}U - 2\lambda U \sum \frac{u_i'}{u_i}(1 - \frac{\lambda}{u_i})^{-1} \sum_{i \neq j} \frac{u_j'}{u_j}(u_j - u_i)^{-1} \\ & -2\lambda U \sum \frac{u_i'}{u_i}(1 - \frac{\lambda}{u_i})^{-1} \sum_j \frac{r_j'}{r_j}(r_j - u_i)^{-1} + 2\lambda^2 U \sum \frac{r_i'}{r_i^2}(1 - \frac{\lambda}{r_i})^{-1} \sum \frac{u_i'}{u_i^2}(1 - \frac{\lambda}{u_i})^{-1} \\ & -2\lambda U \sum \frac{r_i'}{r_i}(1 - \frac{\lambda}{r_i})^{-1} \sum_{i \neq j} \frac{r_j'}{r_j}(r_j - r_i)^{-1} - 2\lambda U \sum \frac{r_i'}{r_i}(1 - \frac{\lambda}{r_i})^{-1} \sum_j \frac{u_j'}{u_j}(u_j - r_i)^{-1} \end{aligned}$$

whence

$$\begin{aligned} Q(x, \lambda) &= \frac{a''(x)}{a(x)} - 2\lambda \sum \frac{u_i'}{u_i}(1 - \frac{\lambda}{u_i})^{-1} \sum_{i \neq j} \frac{u_j'}{u_j}(u_j - u_i)^{-1} \\ & -2\lambda \sum \frac{u_i'}{u_i}(1 - \frac{\lambda}{u_i})^{-1} \sum_j \frac{r_j'}{r_j}(r_j - u_i)^{-1} + 2\lambda^2 U \sum \frac{r_i'}{r_i^2}(1 - \frac{\lambda}{r_i})^{-1} \sum \frac{u_i'}{u_i^2}(1 - \frac{\lambda}{u_i})^{-1} \\ & -2\lambda \sum \frac{r_i'}{r_i}(1 - \frac{\lambda}{r_i})^{-1} \sum_{i \neq j} \frac{r_j'}{r_j}(r_j - r_i)^{-1} - 2\lambda \sum \frac{r_i'}{r_i}(1 - \frac{\lambda}{r_i})^{-1} \sum_j \frac{u_j'}{u_j}(u_j - r_i)^{-1}. \end{aligned}$$

Let us expand $Q(x, \lambda)$ into a power series about $\lambda = 0$. Since

$$Q(x, 0) = \frac{a''(x)}{a(x)}$$

and

$$\begin{aligned} \frac{\partial Q}{\partial \lambda}(x, 0) &= -2 \sum \frac{u'_i}{u_i} \sum_{i \neq j} \frac{u'_j}{u_j} (u_j - u_i)^{-1} - 2 \sum \frac{u'_i}{u_i} \sum_j \frac{r'_j}{r_j} (r_j - u_i)^{-1} \\ &\quad - 2 \sum \frac{r'_i}{r_i} \sum_{i \neq j} \frac{r'_j}{r_j} (r_j - r_i)^{-1} - 2 \sum \frac{r'_i}{r_i} \sum_j \frac{u'_j}{u_j} (u_j - r_i)^{-1}. \end{aligned}$$

Therefore

$$Q(x, \lambda) = \frac{a''(x)}{a(x)} + \lambda \frac{\partial Q}{\partial \lambda}(x, 0) + g(x, \lambda)$$

where $g(x, \lambda)$ is some entire function . By definition of $q(x)$ in (5.32), and $a(x)$ in (5.33) we have

$$q(x) = \frac{a''(x)}{a(x)}$$

and

$$x = \frac{2a'}{a} \Lambda_1' + \Lambda_1'' . \quad (5.34)$$

Now , dividing (5.17) by u_i^2 , and (5.18) by r_i^2 , taking the sum from the result and using the definition of Λ_1 we obtain ,

$$\begin{aligned} -\Lambda_1'' &= \frac{2a'}{a} \Lambda_1' - 2 \sum \frac{u'_i}{u_i} \sum_{i \neq j} \frac{u'_j}{u_j} (u_j - u_i)^{-1} - 2 \sum \frac{u'_i}{u_i} \sum_j \frac{r'_j}{r_j} (r_j - u_i)^{-1} \\ &\quad - 2 \sum \frac{r'_i}{r_i} \sum_{i \neq j} \frac{r'_j}{r_j} (r_j - r_i)^{-1} - 2 \sum \frac{r'_i}{r_i} \sum_j \frac{u'_j}{u_j} (u_j - r_i)^{-1}. \end{aligned}$$

Therefore , from (5.34) and the expression for $\frac{\partial Q}{\partial \lambda}(x, 0)$, we get

$$\frac{\partial Q}{\partial \lambda}(x, 0) = -x.$$

Consequently ,

$$U'' + (\lambda x - q(x) - g(x, \lambda))U = 0$$

Since the solution of $U'' + (\lambda x - q(x))U = 0$ and the previous equation can be represented by the same form of infinite products , (see theorem 4.15) , therefore $g(x, \lambda) \equiv 0$.

One can prove the corresponding results when $x < 0$ in a similar way . The proof of (3) and (4) follows directly from (5.32) .

We see that the differential equations (5.10) and (5.19) for the eigenvalues provide a solution to the inverse problem in the one turning point case . In the classical case there are some examples which provide easy formulas for the $\lambda_n(x)$.

For instance , let us consider the Dirichlet problem for

$$y'' + (\lambda t - q(t))y = 0 \tag{5.35}$$

on $[-1, x]$,for fixed $x < 0$, where

$$q(t) = \frac{5}{16t^2} - \frac{t}{4}.$$

The two linearly independent solutions of (5.35) are of the form

$$y(t) = (-t)^{-1/4} e^{\frac{1}{3}(-t)^{3/2}(1-2d)} \quad -1 \leq t < 0$$

where $d = \frac{1}{2} \pm \sqrt{\lambda + 1/4}$. The solution of (5.35) which satisfies the initial conditions

$$U(-1, \lambda) = 0 \quad \frac{\partial U}{\partial t}(-1, \lambda) = 1$$

is

$$U(t, \lambda) = \frac{(-t)^{-1/4}}{\sqrt{\lambda + 1/4}} \sinh p(t) \sqrt{\lambda + 1/4} \quad t < 0, \tag{5.36}$$

with $p(x) = \int_{-1}^x \sqrt{-v} dv$. The eigenvalues of this problem are

$$\lambda_n(x) = -\frac{n^2 \pi^2}{p^2(x)} - 1/4 \quad x < 0. \tag{5.37}$$

In order to write the solution of (5.35) in infinite product form , we obtain the function $c(x)$ in (5.7) . We have

$$c(x) = \frac{p(x)}{(-x)^{1/4}} \prod_{k=1}^{\infty} \left(1 + \frac{p^2(x)}{4k^2\pi^2}\right).$$

From the infinite product form of $\sinh z$, (see [1] , §4.5.68) , we have

$$\sinh \frac{p(x)}{2} = \frac{p(x)}{2} \prod_{k=1}^{\infty} \left(1 + \frac{p^2(x)}{4k^2\pi^2}\right).$$

Therefore

$$c(x) = 2(-x)^{-1/4} \sinh \frac{p(x)}{2}.$$

Consequently,

$$U(x, \lambda) = 2(-x)^{-1/4} \sinh \frac{p(x)}{2} \prod \left(1 - \frac{\lambda}{\lambda_n(x)}\right) \quad (5.38)$$

where $\lambda_n(x)$ is defined in (5.37) .The infinite product form of (5.35) can also be obtained directly from (5.36) by using Hadamard's factorization theorem . We also see that

$$\begin{aligned} q(x) &= \frac{5}{16x^2} - \frac{x}{4} \\ &= \frac{c''(x)}{c(x)} \quad x < 0 \end{aligned}$$

in agreement with our results .

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