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ALMOST PERIODIC DIFFERENTIAL EQUATIONS

By

LIGANG ZHENG

A thesis submitted to the
School of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in Mathematics

at the
University of Ottawa

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for all the love
she gave me

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Preface

In this thesis, we will study almost periodic differential equations. The motivation to study such a subject is mainly due to its wide applications. We will focus our attention on the topics of boundedness, almost periodicity, disconjugacy and the non-existence of periodic solutions for the n-body problem. Our main investigation in chapter 1 deals with Bohr almost periodic differential equations. In chapter 2, we will study Stepanov almost periodic differential equations, which is a wider class than Bohr's class and we will give a general Floquet theorem in some special cases. We devote our effort in the last chapter to the special n-body problem-if the configuration remains similar throughout the motion and show some applications of oscillation theory of differential equations to the n-body problem.

Contents

Chapter 1

Bohr Almost Periodic Second Order Equations

1.1 Introduction

The study of the solutions of equations of the form

$$x''(t) + a(t)x(t) = 0 \tag{1.1}$$

where $a(t)$ is an almost periodic function and t varies over the whole real line is of widespread importance in various branches of pure and applied mathematics. For example, when $a(t)$ is a real periodic function, i.e. when the equation (1.1) is a Hill equation, there are lots of beautiful results due to the powerful Floquet theory, see, for example, the book by W. Magnus and S. Winkler[22]. Also, there is a complete study[23] of a special Hill equation-the Mathieu equation, in terms of stability and instability regions of this equation, when $a(t) = -\alpha + \beta cos2t$, and here α, β are real parameters.

An almost periodic function is a natural generalization of a periodic function. The first such successful study is due to H. Bohr[1]. He expanded the concept of a periodic function into a wider class of functions in his definition of almost periodic functions by requiring that $f(t)$ be continuous and that the set

$$T(f,\varepsilon) = \{\tau : \quad |f(t+\tau) - f(t)| < \varepsilon, \quad for \quad all \quad t \in R\}$$

be relatively dense in R [S, $p7$] for each $\varepsilon > 0$. Such an $f(t)$ is called almost periodic (a.p. for brevity) in the sense of Bohr or Bohr almost periodic functions. Such a class of functions, of course, contains periodic functions, but it also contains many non-periodic functions. For example, $cos x + cos\sqrt{2}x$ is not periodic, since the only root of the equation $cos x + cos\sqrt{2}x = 2$ is $x = 0$, but it can be shown that it is an almost periodic function in the sense of Bohr. As we have just seen, the set of all periodic functions is not closed under summation and multiplication, but the set of all almost periodic functions is a ring under these two operations[S,p6]. If $f(t)$ is almost periodic, and $\inf |f(t)| > 0$, then $1/f(t)$ and $\sqrt{|f(t)|}$ are almost periodic as well. Furthermore, if $F(x)$ is uniformly continuous on the range of $f(t)$ to R, then $F(f(t))$ is also an almost periodic function. Also, if $f_n(t)$, $n = 1, 2, \ldots$, is a sequence of almost periodic functions and

$$f_n(t) \rightarrow f(t)$$

uniformly on the real line, then $f(t)$ is an almost periodic function as well.

It is well known that a Bohr almost periodic function $f(t)$ is bounded, uniformly continuous, and that $\int f(s)ds$ is almost periodic if and only if it is bounded while $f'(t)$ is almost periodic if and only if it is uniformly continuous. Furthermore, for any $\lambda \in R$,

$$m(f,\lambda) = \lim_{T \to \infty} \frac{1}{T} \int_r^{r+T} f(s) e^{-i\lambda s} ds$$

exists uniformly for all $r \in R$ and is independent of r, [S, $p.30$]. We call

$$M(f) = m(f,0) = \lim_{T \to \infty} \frac{1}{T} \int_r^{r+T} f(s) ds$$

the mean value of an almost periodic function $f(t)$.

It is known that

$$m(f,\lambda) = 0$$

for all $\lambda \in R$ except for a countable set $\Lambda = \{\lambda_n\}_{n \in N}$, [S, p.34]. Note that the Bessel inequality

$$\sum_{i=1}^{K} |m(f, \lambda_i)|^2 \leq M(f^2)$$
$$K \in N \quad and \quad \lambda_i \in \Lambda$$

along with Parseval's equality

$$\sum_{i=1}^{\infty} |m(f, \lambda_i)|^2 = M(f^2) \quad \lambda_i \in \Lambda$$

hold for the Fourier representations of such a function and

$$\sum_{k=1}^{\infty} m(f, \lambda_k) e^{i\lambda_k t} = f(t)$$

see [S, pp1 − 55]. The uniqueness theorem is true here. i.e., if $f(t)$ is a nonnegative almost periodic function and the mean value of f, $M(f) = 0$, then $f \equiv 0$.

The set Λ is called the set of <u>exponents</u> of $f(t)$ on R and denoted by $exp(f)$. The <u>module</u> of $f(t)$, $mod(f)$, is the smallest additive group of real numbers that contains $exp(f)$, the exponents of f.

If $f(t)$, $g(t)$ are both almost periodic functions, then the following are equivalent (see,[S, p.61]):

a. $mod(f)$ contains $mod(g)$;

b. for every $\varepsilon > 0$, there is a $\delta > 0$ such that $T(g, \varepsilon)$ contains $T(f, \delta)$;

c. $\lim_{n \to \infty} f(t + \alpha_n)$ exists implies $\lim_{n \to \infty} g(t + \alpha_n)$ exists;

d. $\lim_{n \to \infty} f(t + \alpha_n) = f(t)$ implies that $\lim_{n \to \infty} g(t + \alpha_n) = g(t)$;

e. $\lim_{n \to \infty} f(t + \alpha_n) = f(t)$ implies that $\lim_{i \to \infty} g(t + \alpha_{n_i}) = g(t)$ for some subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$.

In 1927, S. Bochner showed that a function $f(t)$ is almost periodic in the sense of Bohr if and only if for every sequence $\{\alpha'_n\}$ one can extract a subsequence $\{\alpha_n\}$ such that

$$\lim_{n \to \infty} f(t + \alpha_n)$$

exists uniformly for all $t \in R$. According to some, [S, p.1], this is the most convenient definition of an almost periodic function for the study of almost periodic differential equations. We call

$$H(f) = \{g(t) | g(t) = \lim_{n \to \infty} f(t + \alpha_n)\}$$

for some sequence α_n and uniformly for all $t \in R$, the <u>hull of $f(t)$</u>.

There is another equivalent definition of an almost periodic function. $f(t)$ is almost periodic if and only if from every pair of sequences $\{\alpha_n\}$, $\{\beta_n\}$, there are common subsequences $\{\alpha_{n_k}\}$, $\{\beta_{n_k}\}$ such that

$$\lim_{k \to \infty} f(t + \alpha_{n_k} + \beta_{n_k}) = \lim_{i \to \infty} \lim_{j \to \infty} f(t + \alpha_{n_i} + \beta_{n_j})$$

pointwise. [S, p.12].

After Bohr's work, there came many other generalizations and hence many other classes of almost periodic functions named by the discover's names: e.g., Stepanov's, Besicovitch's, Weyl's and so on. All these classes relax the condition of continuity of such a function which Bohr always supposed. In this chapter, we will study the Bohr almost periodic differential equations of the second order and in the next chapter we will study Stepanov's extension of this concept and its application to differential equations.

The remarkable success of Floquet's theory for the linear periodic equation

$$X'(t) = A(t)X(t)$$

where $A(t)$ is an $n \times n$ periodic matrix and X is an n-dimensional vector, has led to investigations of such a corresponding theory for Bohr almost periodic differential equations and other generalizations of such almost periodic differential equations. However, even in the Bohr almost periodic case there is no complete Floquet theory[S,p.191]. Still, there is an interesting theorem of Markus and Moore [2, p.111] dealing with a Floquet representation of the solutions of (1.1) when the equation is <u>disconjugate</u> (see below for this definition).

Equation (1.1) is said to be oscillatory on $(-\infty, \infty)$ if one nontrivial(not identically zero), and thereby every(by the famous Sturm's theorem), solution vanishes for arbitrarily large positive and negative values of t. It is said to be non-oscillatory on $(-\infty, \infty)$ if one, and hence every nontrivial, solution has at most a finite number of zeros on $(-\infty, \infty)$. A special instance of a non-oscillatory equation is one which is disconjugate, i.e., every nontrivial solution has at most one zero on the real line. It is well known that (1.1) is disconjugate if and only if there is a solution which is positive everywhere [28, p.5] . Furthermore, disconjugacy and non-oscillation for (1.1) are equivalent when $a(t)$ is an almost periodic function [2].

Also note that if $(\xi, \eta) \in R^2$ are parameters, then the collection of $(\xi, \eta) \in R^2$ such that

$$x''(t) + (-\xi + \eta a(t))x(t) = 0$$

is disconjugate on R is called the disconjugacy domain and denoted by D. It is known [3] that D is always a closed, convex and unbounded subset of the $\xi\eta$-plane. The differential equation is said to be in the interior or on the boundary of disconjugacy domain if (ξ, η) is in the interior or on the boundary of D. More generally, it is known [3] that if

$$y'' + q_1(x)y = 0$$

and

$$y'' + q_2(x)y = 0$$

are disconjugate, then the equation

$$y'' + ((1-r)q_1(x) + rq_2(x))y = 0$$

is also disconjugate for all $0 \leq r \leq 1$. If for each n, the equation

$$y'' + q_n(x)y = 0$$

is disconjugate and

$$q_n(x) \rightarrow q(x)$$

in L_1 sense over each compact interval of R, then the limit equation

$$y'' + q(x)y = 0$$

is disconjugate too.

In this chapter, we will study some properties of solutions of real linear second order differential equations with Bohr almost periodic(a.p.) coefficients:

$$x''(t) + a(t)x(t) = 0 \tag{1.2}$$

and its related Riccati equation:

$$r'(t) = -r^2(t) - a(t), \tag{1.3}$$

where $a(t)$ is a real almost periodic function (we suppress the name of Bohr when dealing with the classical a.p. functions).

In this chapter, we will assume that $a(t)$ is a real almost periodic function. We will show the connections between the boundedness of the solutions of (1.2) and the a.p. solutions of (1.2), and also the relationship between the a.p. solutions of (1.3) and the disconjugacy of (1.2).

Central to our study are the papers of L. Markus and R.A. Moore[2], in 1956, and a recent book by A.B. Mingarelli and S.G. Halvorsen[3].

1.2 Boundedness and Almost Periodicity

As is shown by the well-known Floquet theory, the bounded solutions of

$$x''(t) + a(t)x(t) = 0$$

are precisely the almost periodic solutions if $a(t)$ is a periodic function(see, for example, [8, p101]). Because of the apparent similarity between periodic and almost periodic functions, there are some feelings that the above should be true if $a(t)$ is replaced by an almost periodic function. There are some mathematicians who have claimed that this is indeed true. For example, J. M. Abel in [15] claimed that J. Favard [32] states that "the boundedness of the solution of the equation is equivalent to their almost periodicity". And furthermore L. Cesari in his famous book [11, p48] stated 'It should be mentioned here that if $a_i(t)$ are real almost periodic functions in $(-\infty, \infty)$, and hence bounded, then every bounded solution $x(t)$ of the differential equation

$$x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = 0$$

is necessarily almost periodic together with its first n derivatives'. He refers to a "proof" by G. Sansone [24], which however seems only to apply in the case when the $a_i(t)$ are constants, which Sansone actually supposed.

However, this result, as stated, is incorrect. We will construct a counter example by using the ideas in the book of A. M. Fink [8]. There, an example is given of a first order scalar differential equation $x'(t) = a(t)x(t)$, where $a(t)$ is almost periodic and the boundedness of a solution of the equation does not necessarily imply its almost periodicity.

First, we give a lemma which describes the character of almost periodic solutions of systems of linear differential equations.

Lemma 1 *If $A(t)$ is an almost periodic matrix (i.e., every component is a.p.) and $X(t)$ is an almost periodic solution of $X'(t) = A(t)X(t)$, then either $\inf_{t\in R}|X(t)| > 0$ or $X(t) \equiv 0$ (here $|X(t)|$ refers to the Euclidean norm on R^N).*

Proof: See A. M. Fink[8, p85].

By using this lemma, we are able to show the following:

Theorem 2 *The boundedness and almost periodicity of solutions of equation (1.2) are not necessarily equivalent.*

Proof: For $n \geq 2$, let

$$f_n(t) = \begin{cases} -\frac{n}{2^{n-1}-1} sin(\frac{\pi}{2}t) & \text{if } 0 \leq t \leq 1; \\ -\frac{n}{2^{n-1}-1} & \text{if } 1 \leq t \leq 2^{n-1}-1; \\ -\frac{n}{2^{n-1}-1} sin(2^{n-1}-t)\frac{\pi}{2} & \text{if } 2^{n-1}-1 \leq t \leq 2^{n-1}. \end{cases}$$

Then extend $f_n(t)$ to be odd and periodic of period 2^n, i.e.,

$$f_n(-t) = -f_n(t) \quad for \quad -2^{n-1} \leq t \leq 0$$

and

$$f_n(t+2^n) = f_n(t) \quad for \quad all \quad t.$$

It is easy to see that

$$\int_0^{2^n} f_n(s)ds = 0, \tag{1.4}$$

$$\int_0^{2^{n-1}} f_n(s)ds = -\frac{n}{2^{n-1}-1}(\frac{4}{\pi} + 2^{n-1} - 2) < -n, \tag{1.5}$$

and if $||f_n||$ denotes the uniform norms of f_n over R, then

$$||f_n|| = \frac{n}{2^{n-1}-1}, \tag{1.6}$$

$$\int_0^t f_n(s)ds \leq 0, \qquad for \quad all \quad t \geq 0, \tag{1.7}$$

and

$$f_n'(t) = \begin{cases} -\frac{n\pi}{2(2^{n-1}-1)} cos(\frac{\pi}{2}t) & \text{if } 0 \leq t \leq 1; \\ 0 & \text{if } 1 \leq t \leq 2^{n-1}-1; \\ \frac{n\pi}{2(2^{n-1}-1)} cos(2^{n-1}-t)\frac{\pi}{2} & \text{if } 2^{n-1}-1 \leq t \leq 2^{n-1}. \end{cases}$$

Furthermore,

$$f_n'(-t) = f_n'(t),$$

$$f_n'(t+2^n) = f_n'(t),$$

and

$$||f_n'|| = \frac{n\pi}{2(2^{n-1}-1)}.$$

Since both $f_n(t)$ and $f_n'(t)$ are periodic, and

$$\sum_{n=2}^{\infty} \|f_n\| < \infty, \quad \sum_{n=2}^{\infty} \|f_n'\| < \infty,$$

so the functions defined by the uniformly convergent series

$$g(t) = \sum_{n=2}^{\infty} f_n(t)$$

and

$$g'(t) = \sum_{n=2}^{\infty} f_n'(t)$$

are almost periodic, [S.p.7].

Since $\int_0^t g(s)ds = \sum_{n=2}^{\infty} \int_0^t f_n(s)ds \leq 0$ by (1.7), for all $t \geq 0$, and all $f_n(t)$ are odd, $g(t)$ is odd hence its integral is even, and thus $\int_0^t g(s)ds \leq 0$, for all t. Note that by definition of g and (1.7),

$$\int_0^{2^{n-1}} g(s)ds \leq \int_0^{2^{n-1}} f_n(s)ds < -n$$

(by (1.5)). So this means

$$\lim_{n\to\infty} \int_0^{2^{n-1}} g(s)ds = -\infty$$

and, therefore,

$$\lim_{n\to\infty} e^{\int_0^{2^{n-1}} g(s)ds} = 0. \tag{1.8}$$

Furthermore, since g is uniformly bounded on R,

$$\lim_{n\to\infty} e^{\int_0^{2^{n-1}} g(s)ds} g(2^{n-1}) = 0. \tag{1.9}$$

Now,

$$x(t) = e^{\int_0^t g(s)ds}$$

is a nontrivial bounded solution(as $\int_0^t g(s)ds \leq 0$, for all $t \in R$) of the equation

$$x''(t) + (-g^2(t) - g'(t))x(t) = 0$$

and

$$\inf_{t\in R}|x^2(t)+x'^2(t)|=0$$

by (1.8) and (1.9). Note that here $X(t)=col(x(t),x'(t))$ solves $X'(t)=A(t)X(t)$ for an appropriate $A(t)$, as is well known. Thus

$$|X(t)|^2=x^2(t)+x'^2(t)$$

and by lemma 1, this solution (being nontrivial) cannot be almost periodic, yet it is bounded on R.

It is worthwhile mentioning that the other linearly independent solution of the above equation must be unbounded by a theorem of L. Markus and R. Moore[2, p104]. It is known from [2] that a disconjugate almost periodic equation has at least one unbounded solution and, furthermore, if the equation is in the interior of the disconjugacy domain then the only bounded solution is the trivial one. Since the equation is disconjugate as $x=e^{\int_0^t g(s)ds}$ is a positive solution which is bounded everywhere, the equation is on the boundary of the disconjugacy domain, see [3].

1.3 When All Solutions Are Bounded

As we pointed out in the last section, boundedness of a solution does not necessarily imply its almost periodicity for an almost periodic second order linear differential equation. In this section, we will study the case when all the solutions of the differential equation are bounded and its relationship with almost periodic solutions. Such a case was greatly studied by S. Bochner , R. H. Cameron in the 30's and some other investigators later on(see, for example, [9, 17, 18]). It is unfortunate that Prof. Cameron passed away last summer when the author found his great work. Now we point out that there are some minor errors in [18]. Some definitions and theorems of this paper regarding the terminology of stationary, twistable and almost periodicity are inconsistent. For example, the author defined twistable as follows: a vector function $y(x)$ will be called **twistable** with respect to a certain module M if there exists a constant $c \neq 1$ whose absolute value is unity such that for all x

$$\lim_{j\to\infty}\lim_{i\to\infty} y(x+h_i+k_j) = cy(x)$$

for some pair of sequences $h_1, h_2, \cdots$ and $k_1, k_2, \cdots$ such that the above iterated limit exists for all x and such that $h_1, h_2, \cdots$ converges to $-k_1, -k_2, \cdots$ with respect to M. (A sequence $h_1, h_2, \ldots$ is said to **converge with respect to a module** M if there exists an almost periodic function $f(t)$ with M as its module such that the sequence $f(t+h_1), f(t+h_2), \ldots$ converges uniformly for all $t \in R$. Moreover, **the sequence** $k_1, k_2, \ldots$ **will be said to converge to** $l_1, l_2, \ldots$ **with respect to** M if the sequence $k_1, l_1, k_2, l_2, \ldots$ converges with respect to M.) If the sequences h_i, k_j can be chosen so that all of their terms are positive then the function is called **positive twistable.** He (Cameron) defined a function $y(x)$ as **stationary** with respect to a module M if for all x

$$\lim_{j\to\infty}\lim_{i\to\infty} y(x+h_i+k_j) = y(x)$$

whenever $h_1, h_2, \cdots$ and $k_1, k_2, \cdots$ are sequences such that the above iterated limit exists for all x and such that $h_1, h_2, \cdots$ converges to $-k_1, -k_2, \cdots$ with respect to M. If the sequences are restricted to positive numbers, $y(x)$ is called **positive stationary.**

From the above definitions, it is very clear that a function **cannot be both stationary and twistable.** However, in his *Lemma* 6[18, p33], he showed that an almost periodic function is necessarily stationary. This implies, by the above, that an almost periodic function cannot be twistable and vice versa. But however, in his *Theorem IV*[18, p37], he stated that "**all solutions of $X'(t) = A(t)X(t)$ (here $A(t)$ is a $n \times n$ almost periodic matrix and X is a n-dimensional vector) are almost periodic if and only if $\int_0^t Re[a_{11}(s) + \cdots + a_{nn}(s)]ds$ is bounded, all solutions of this equation are bounded and there are no solutions of this equation (neither real or complex) which are twistable with respect to M except almost periodic ones**". This apparently implies that there are a.p. solutions which are twistable, but this is impossible as we have seen above.

In order to save *Theorem IV*$^{+}$ of [18], it is necessary to allow $c = 1$ in the definition of twistable [18, *p*33]. Then *Theorem IV*$^{+}$, of [18], is true and the proof which the author gave works. The last condition of *Theorem IV*$^{+}$ can be replaced by "every twistable solution is necessarily stationary". Below we will give a sufficient condition which ensures that all solutions of (1.2) are almost periodic when they are all bounded.

Let

$$x''(t) + a(t)x(t) = 0 \tag{1.10}$$

be an almost periodic differential equation, i.e., let $a(t)$ be an almost periodic fuction. From [2], we know that the equation (1.10) must be oscillatory when all of its solutions are bounded since the disconjugate equation at least has one nontrivial unbounded solution. In the case when $a(t)$ is a real periodic function, by the well-known Floquet theory, we actually know that all solutions are indeed almost periodic functions(for example, see [8, p101]) . Is this true when $a(t)$ is an almost periodic function? We give as an answer the following:

Lemma 3 *(R.H. Cameron[18]) Let $A(t)$ be $n \times n$ almost periodic matrix. Assume that $\int_0^t Re[tr(A(s))]ds$ is bounded, and all solutions of $X' = A(t)X$ are bounded. Then all solutions of $X' = A(t)X$ are almost periodic if every positive twistable solution is necessarily positive stationary.*

Proof: This was basically shown in [18, $p37$, $Theorem IV^{+}$], however we must allow $c = 1$ in the definition of twistable.

Lemma 4 *(R.H. Cameron[18]) Let $A(t)$ satisfy the conditions in lemma 3. If $h_1, h_2, \ldots$ converges to $-k_1, -k_2, \ldots$ with respect to the module of $A(t)$, and for each t, and each solution $X(t)$ of $X' = A(t)X$*

$$T[X(t)] = \lim_{j\to\infty} \lim_{i\to\infty} X(t + h_i + k_j)$$

exists, then T is one-to-one and linear. Moreover, there exists a linearly independent set of solutions $X^{(1)}(t), X^{(2)}(t), \ldots, X^{(n)}(t)$, such that

$$T[X^{(i)}(t)] = \lambda_i X^{(i)}(t)$$

where λ_i are constants of absolute value one.

Proof: See [18, *p*35, *lemma* 8].

By using these two lemmas, we are able to show the following:

Theorem 5 *If all solutions of (1.10) are bounded, then all solutions are almost periodic if and only if one nontrivial solution is almost periodic.*

Proof: The "necessity" part is trivial. For the "sufficiency" part, if (1.10) has one nontrivial almost periodic solution $x(t)$, let $z(t)$ be a positive twistable (allowing $c = 1$ in the definition) solution of (1.10), i.e.,

$$\lim_{j\to\infty}\lim_{i\to\infty} z(t+h_i+k_j) = \lambda z(t)$$

for some constant λ, $|\lambda| = 1$. As it was shown in [18, *p*38], subsequences $h'_1, h'_2, \ldots$, $k'_1, k'_2, \ldots$ of $h_1, h_2, \ldots$, $k_1, k_2, \ldots$ can be chosen such that

$$T[w(t)] = \lim_{j\to\infty}\lim_{i\to\infty} w(t+h'_i+k'_j)$$

exists for every solution $w(t)$ of (1.10) and also at the same time such that $T[w'(t)]$ exists too. By Lemma 4, a solution basis $x_1(t)$, $x_2(t)$ of (1.10) can be chosen such that

$$T[x_1(t)] = \lambda_1 x_1(t), \qquad T[x_2(t)] = \lambda_2 x_2(t),$$
$$T[x'_1(t)] = \lambda_1 x'_1(t), \qquad T[x'_2(t)] = \lambda_2 x'_2(t),$$

where $|\lambda_1| = 1$, $|\lambda_2| = 1$.

The Wronskian:

$$x_1(t)x'_2(t) - x'_1(t)x_2(t) = c \neq 0,$$

and so

$$T[x_1(t)x'_2(t) - x_2(t)x'_1(t)] = \lambda_1\lambda_2(x_1(t)x'_2(t) - x'_1(t)x_2(t)),$$

i.e.,

$$\lambda_1\lambda_2 c = c$$

since $T[c] = c$, and hence

$$\lambda_1 \lambda_2 = 1.$$

This implies that

$$\lambda_2 = \overline{\lambda_1}.$$

Let $x(t) = c_1 x_1(t) + c_2 x_2(t)$, for some constants c_1, c_2. Since $x(t)$ is almost periodic, $T[x(t)] = x(t)$ (since an almost periodic function is necessarily stationary, see [18, p33]). So this means

$$T[x(t)] = c_1 T[x_1(t)] + c_2 T[x_2(t)] = c_1 \lambda_1 x_1(t) + c_2 \lambda_2 x_2(t),$$

hence,

$$c_1 x_1(t) + c_2 x_2(t) = c_1 \lambda_1 x_1(t) + c_2 \lambda_2 x_2(t).$$

Since $x_1(t), x_2(t)$ are linearly independent, this implies that

$$\lambda_1 c_1 = c_1 \qquad \lambda_2 c_2 = c_2.$$

Because at least one of c_1, c_2 is not zero, we find

$$\lambda_1 = 1 \quad or \quad \lambda_2 = 1.$$

As $\lambda_2 = \overline{\lambda_1}$, we have

$$\lambda_1 = 1 \quad and \quad \lambda_2 = 1.$$

therefore T is the identity. Hence

$$\lim_{j \to \infty} \lim_{i \to \infty} z(t + h_i + k_j) = z(t)$$

i.e. $z(t)$ is positive stationary. Since every positive twistable solution is necessarily positive stationary, as we just showed, so by lemma 3, all solutions of (1.10) are almost periodic. This completes the proof.

Next, we suppose that $a'(t)$ is also an almost periodic function. It is well-known (by Lindemann, and it is easy to check) that the product of any two solutions of (1.10) is a solution of

$$y'''(t) + 4a(t)y'(t) + 2a'(t)y(t) = 0. \tag{1.11}$$

Furthermore, if $x_1(t), x_2(t)$ are linearly independent solutions of (1.10) with the Wronskian $W(x_1(t), x_2(t)) = c \neq 0$, then $x_1^2(t), x_2^2(t), x_1(t)x_2(t)$ are linearly independent solutions of equation (1.11) with the Wronskian $W(x_1^2(t), x_1(t)x_2(t), x_2^2(t)) = 2c^3 \neq 0$. Thus all solutions of (1.10) are bounded if and only if all solutions of (1.11) are bounded. According to what we just proved, if all solutions of (1.10) are bounded and (1.10) has one nontrivial almost periodic solution, then all solutions of (1.10) are almost periodic, and thereby, every solution of (1.11) is necessarily almost periodic. We can describe more relationships between (1.10) and (1.11) as follows:

Theorem 6 *Let $a(t), a'(t)$ be almost periodic and let all solutions of (1.10) (or of (1.11)) be bounded. Then the following are equivalent:*

(1).All solutions of (1.11) are almost periodic;

(2).Any square of a nontrivial solution of (1.10) is almost periodic;

(3).There is a nontrivial solution of (1.10) such that its square is almost periodic.

Proof: (1) ⇒ (2) is obvious as the square of any solution of (1.10) is necessarily a solution of (1.11); (2) ⇒ (3) is trivial; for (3) ⇒ (1), let $x(t)$ be the nontrivial solution of (1.10) such that $x^2(t)$ is almost periodic and let $z(t)$ be a positive twistable solution of (1.11) i.e.,

$$\lim_{j\to\infty}\lim_{i\to\infty} z(t + h_i + k_j) = \lambda z(t)$$

for some constant $\lambda, |\lambda| = 1$ and $h_1, h_2, \ldots$ converges to $-k_1, -k_2, \ldots$ with respect to the module of $a(t), a'(t)$.

By the same argument as in theorem 5, subsequences $h_1', h_2', \ldots, k_1', k_2', \ldots$ can be chosen such that

$$T[f(t)] = \lim_{j\to\infty}\lim_{i\to\infty} f(t + h_i' + k_j')$$

exists for all solutions of (1.10) and (1.11) and their derivatives. By lemma 4, there is a solution basis $x_1(t), x_2(t)$ of (1.10) such that

$$T[x_1(t)] = \lambda_1 x_1(t), \qquad T[x_2(t)] = \lambda_2 x_2(t),$$

$$T[x_1'(t)] = \lambda_1 x_1'(t), \qquad T[x_2'(t)] = \lambda_2 x_2'(t),$$

where $|\lambda_1| = 1, |\lambda_2| = 1$.

Applying the operator T to the Wronskian

$$x_1(t)x_2'(t) - x_1'(t)x_2(t) = constant \neq 0,$$

we get $\lambda_1\lambda_2 = 1$ and hence $\lambda_2 = \overline{\lambda}_1$.

Since $x_1(t), x_2(t)$ are linearly independent solutions of (1.10), there are constants d_1, d_2 such that

$$x(t) = d_1x_1(t) + d_2x_2(t)$$

and at least one of d_1, d_2 is not zero. Since $x^2(t)$ is an almost periodic function, we have

$$T[x^2(t)] = x^2(t)$$

i.e.,

$$d_1^2x_1^2(t) + 2d_1d_2x_1(t)x_2(t) + d_2^2x_2^2(t) = \lambda_1^2d_1^2x_1^2(t) + 2d_1d_2x_1(t)x_2(t) + \lambda_2^2d_2^2x_2^2(t).$$

So, by the linear independence of $x_1^2(t), x_2^2(t)$, we have

$$d_1^2 = \lambda_1^2d_1^2 \quad and \quad d_2^2 = \lambda_2^2d_2^2$$

since at least one of d_1, d_2 is not zero, therefore,

$$\lambda_1^2 = 1 \quad or \quad \lambda_2^2 = 1.$$

Since $\lambda_1 = \overline{\lambda}_2$, the above implies that

$$\lambda_1^2 = \lambda_2^2 = 1.$$

As $x_1^2(t), x_2^2(t), x_1(t)x_2(t)$ are linearly independent solutions of (1.11), there are constants e_1, e_2, e_3, such that

$$z(t) = e_1x_1^2(t) + e_2x_2^2(t) + e_3x_1(t)x_2(t).$$

Thus

$$\begin{aligned} T[z(t)] &= c_1 T[x_1^2(t)] + c_2 T[x_2^2(t)] + c_3 T[x_1(t)x_2(t)] \\ &= c_1 x_1^2(t) + c_2 x_2^2(t) + c_3 x_1(t) x_2(t) \\ &= z(t) \end{aligned}$$

and this means that $z(t)$ is positive stationary. So, every positive twistable solution of (1.11) is necessarily positive stationary. By lemma 4, all solutions of (1.11) are almost periodic. This completes the proof.

For the non-homogeneous almost periodic equation

$$x''(t) + a(t)x(t) = h(t) \tag{1.12}$$

where $a(t), h(t)$ are almost periodic functions, we have:

Theorem 7 *Suppose all solutions of (1.12) are bounded. Then all solutions of (1.12) are almost periodic if and only if the homogeneous equation (1.10) of (1.12) has a nontrivial almost periodic solution.*

Proof: The "necessity" part is trivial since if $x_1(t)$ and $x_2(t)$ are two different almost periodic solutions of (1.12), then the difference $x_1(t) - x_2(t)$ is a nontrivial almost periodic solution of (1.10). For the "sufficiency" part, if all the solutions of (1.12) are bounded, it is easy to see that this implies that all the solutions of (1.10) are bounded as well. Furthermore, by theorem 5, we know that all the solutions of (1.10) are almost periodic since there is one nontrivial solution of (1.10) which is almost periodic. Suppose $z_1(t), z_2(t)$ are the linearly independent solutions of (1.10) with the Wronskian being 1. Since every solution of (1.12) is bounded,

$$\int_0^t (z_1(t)z_2(s) - z_2(t)z_1(s))h(s)ds$$

is almost periodic, and this is enough to imply that all solutions of (1.12) are almost periodic since the general solution of (1.12) can be expressed as

$$x(t) = c_1 z_1(t) + c_2 z_2(t) + \int_0^t (z_1(t)z_2(s) - z_2(t)z_1(s))h(s)ds.$$

This completes the proof.

R. H. Cameron in his paper [18] requires that all the solutions of the homogeneous equation be almost periodic in order to get that all the solutions of the nonhomogeneous equation be almost periodic. But according to what we just showed here, one nontrivial a.p. solution of the homogeneous equation is good enough.

We can look at the questions in the last two sections from another point of view. Certainly there is some similarity between almost periodic functions and periodic functions, so it is reasonable to expect some kind of general Floquet theory in the almost periodic case.

In 1956, L. Markus and R. Moore [2] showed that, when the equation (1.10) is disconjugate, in the interior of the disconjugacy domain, there exists a solution basis of (1.10) of the form

$$x_1(t) = e^{\alpha t} e^{\int_0^t \phi(s) ds} \tag{1.13}$$

$$x_2(t) = e^{-\alpha t} e^{\int_0^t \varphi(s) ds} \tag{1.14}$$

where $\alpha > 0$, and $\phi(t)$ and $\varphi(t)$ are almost periodic functions with zero mean value. So in this case, the general Floquet theory is true indeed.

However, the general Floquet theory is not true when (1.10) is disconjugate and on the boundary of the disconjugacy domain as the counter-example found in section 2 shows. In that example, the equation has a nontrivial bounded solution which is not almost periodic.

But how about the situation when the equation (1.10) is oscillatory? There is no answer and little is known even in the case when all the solutions are bounded. The author asked the question of whether all solutions have to be almost periodic if all solutions are bounded to Professor Barry Simon (Pasadena). His reply to this question is that this problem is 'subtle and difficult' and he 'would not be shocked either way'. As we have seen, in this case, if one solution (nontrivial) is almost periodic, then the rest are as well. But the question of what kind of conditions will be sufficient to guarantee the existence of one nontrivial almost periodic solution is still largely open. The case when equation (1.10) is oscillatory and at least one solution is unbounded is even harder.

1.4 When The Equation Is Disconjugate

If $a(t)$ is a.p., we know [3] that the non-oscillation of (1.1) is equivalent to the disconjugacy of (1.1). In this section, we study the relationship of the disconjugacy of (1.1) and the a.p. solutions of its related Riccati equation.

In 1978, Prof. S.S. Chern posed the following problem when he addressed the question of the closeness of the space curve at Academia Sinica, Beijing: Under what conditions, does a periodic coefficient Riccati equation have periodic solutions[12]? It is known that the general Riccati equation

$$r' = A(t)r^2 + B(t)r + C(t)$$

can be changed into the simple form:

$$r' = -r^2 - a(t) \tag{1.15}$$

if A(t), B(t), C(t) are smooth enough, where now $a(t)$ would be periodic/ almost periodic if $A(t), B(t), C(t)$ were periodic/almost periodic. By the simple transformation $r = x'/x$ in (1.15), we get

$$x''(t) + a(t)x(t) = 0, \tag{1.16}$$

and we know that if (1.16) is disconjugate, then (1.15) will have solutions which exist everywhere. Conversely, if (1.15) has solutions which exist everywhere, then (1.16) is disconjugate.

The next theorem shows the strong connection between the existence of periodic/almost periodic solutions of (1.15) and the disconjugacy of (1.16).

Theorem 8 *Let $a(t)$ be a real almost periodic function. Then the equation (1.15) has almost periodic solutions if and only if the equation (1.16) is disconjugate. Furthermore, the module of the a.p. solution of (1.15) is contained in the module of $a(t)$.*

Proof: If (1.15) has an a.p. solution $r(t)$, then $x(t) = e^{\int_0^t r(s)ds}$ is a positive solution of (1.16), thus (1.16) is disconjugate. Conversely, if (1.16) is disconjugate, suppose that $x(t)$ is the positive solution of (1.16). Then $r(t) = x'(t)/x(t)$ is a solution of (1.15) which exists for all $t \in R$. Thus, it is bounded (this is a well-known fact about the Riccati equation, for example, see [2, p101]). So, by a result of Opial [5], (1.15) has an almost periodic solution with the module contained in the module of $a(t)$. Thus, if $a(t)$ is periodic, then the a.p. solution of (1.15) is, indeed, periodic. This completes the proof.

The above theorem also answers an open question in [2, p.119]. It is known that if (1.16) is in the interior of the disconjugacy domain, then the related Riccati equation (1.15) will have a.p. solutions. Whether this is true or not when (1.16) is on the boundary of the disconjugacy domain is not known. In [2] there are examples which indicated that this could be true. But, by theorem 8, this is clear and the answer is yes.

Chapter 2

Stepanov Almost Periodic Second Order Equations

2.1 Introduction

In this chapter, the general Floquet theory for the equation

$$x''(t) + (-\xi + \eta p(t))x(t) = 0, \qquad t \in R \tag{2.1}$$

is investigated. Here (ξ, η) are real parameters and $p(t)$ is a real bounded almost periodic function in the sense of Stepanov [7].

Stepanov almost periodic functions are a generalization of Bohr almost periodic functions: The restrictive requirement that the function be continuous is waived in the class of Stepanov almost periodic functions. We say that $p(t) \in S_L$, the class of Stepanov almost periodic functions if $p(t) \in L^1_{loc}(R)$ and there exists a real number $L > 0$ such that for each $\varepsilon > 0$, there is a relatively dense set of Stepanov translation numbers $\tau_p(\varepsilon)$, i.e., numbers such that

$$\sup_{x \in R}\{\frac{1}{L}\int_x^{x+L} |p(s + \tau_p(\varepsilon)) - p(s)|ds\} < \varepsilon$$

is relatively dense in R. There is another way of defining the Stepanov almost periodic functions, that is, by taking the closure relative to the metric D_S defined by

$$D_S[f(x), g(x)] = \sup_{x\in R} \frac{1}{L} \int_x^{x+L} |f(s) - g(s)|ds$$

of the class of all finite trigonometric polynomials.

It is known that functions in the Stepanov almost periodic class S_L are bounded in the Stepanov metric, i.e.

$$||p||_S = \sup_{x\in R} \frac{1}{L} \int_x^{x+L} |p(s)|ds < \infty$$

and, as in the case of Bohr almost periodic functions, the functions of S_L also admit a mean-value

$$M(p) = \lim_{T\to\infty} \frac{1}{T} \int_r^{r+T} p(s)ds$$

and this limit exists uniformly with respect to r, and is independent of r.

All the solutions we are considering are in the Carathéodory sense [6]. Since $p(t)$ is a Stepanov almost periodic function, it is known that equation (2.1) is disconjugate if it is nonoscillatory (for example, see [3, p38]). The collection of all $(\xi, \eta) \in R^2$ for which (2.1) is disconjugate on $(-\infty, +\infty)$ will be dubbed the disconjugacy domain of (2.1) and denoted by D. It turns out that D is a closed, convex and unbounded subset of the $ab-plane$ which we call the parameter space and label it R^2 for simplicity. We will show that if (ξ, η) is in the interior of the disconjugacy domain D, then (2.1) has a basis of solutions of the form

$$x_1(t) = e^{\alpha t} e^{\int_0^t \phi(s)ds} \tag{2.2}$$

$$x_2(t) = e^{-\alpha t} e^{\int_0^t \varphi(s)ds} \tag{2.3}$$

where $\alpha > 0$ and $\phi(t)$ and $\varphi(t)$ are almost periodic functions in the sense of Bohr! It is somewhat surprising since one would expect that $\phi(t)$ and $\varphi(t)$ be almost periodic functions in the sense of Stepanov, if compared with [2]. It is well known that (2.1) is disconjugate as long as the related Riccati equation

$$r'(t) + r^2(t) + (-\xi + \eta p(t)) = 0 \tag{2.4}$$

has solutions which are well defined for all $t \in R$. Since the investigation of (2.1) depends heavily on the Riccati equation, we give several lemmas in the next section to expose the properties of solutions of the Riccati equation.

2.2 Investigation of the Riccati Equation

Lemma 9 *The solution $r(t)$ of the Riccati equation*

$$\frac{dr}{dt} = M^2 - r^2(t) \tag{2.5}$$

with initial value $r(t_0)$, can be expressed as

$$r(t) = \frac{2M(r(t_0)+M)}{2M + (r(t_0)+M)(e^{2M(t-t_0)}-1)} - M \tag{2.6}$$

or

$$r(t) = \frac{2M(r(t_0)-M)}{2M - (r(t_0)-M)(e^{-2M(t-t_0)}-1)} + M \tag{2.7}$$

where $M > 0$ is a constant.

Proof: It is easy to see that $r = M$ and $r = -M$ are solutions of (2.5), and a substitution $r = z \pm M$ will lead (2.5) into a simple linear differential equation of the first order which is easily solvable and, as a result, we get the solutions of (2.5) in the form of (2.6) or (2.7).

Lemma 10 *The solutions of (2.5) will exist for all $t \in R$ if and only if $|r(t_0)| \leq M$. If $|r(t_0)| > M$, then the solution will have a finite escape time.*

Proof: If $r(t_0) < -M$, then by (2.6), it is easy to see that there is a $t_1 > t_0$, such that

$$e^{2M(t_1-t_0)} - 1 = \frac{2M}{-(r(t_0)+M)}.$$

But this means that

$$\lim_{t \to t_1^-} r(t) = -\infty$$

i.e. $r(t)$ has finite escape time. A similar argument also shows that if $r(t_0) > M$, $r(t)$ will have finite escape time too. Also, from the two expressions for the solutions we can see clearly that $r(t)$ is well defined for all $t \in R$ if and only if $|r(t_0)| \leq M$.

Lemma 11 *Let $f(t)$ be a bounded function, $|f(t)| \le M^2$, for all $t \in R$, and assume that the solutions of*

$$r'(t) = -r^2(t) + f(t) \tag{2.8}$$

exist in the sense of Carathéodory (i.e., $r \in AC_{loc}(R)$ and satisfies the differential equation almost everywhere). If $r(t)$ is a solution of (2.8), then $r(t)$ is well defined for all $t \in R$ if and only if $|r(t)| \le M$ for all $t \in R$.

Proof: Since $|f(t)| \le M^2$, we have

$$-r^2 + f(t) \le -r^2 + M^2. \tag{2.9}$$

If $|r(t_0)| > M$, for some t_0, let $u(t)$ be the solution of $u' = -u^2 + M^2$, $u(t_0) = r(t_0)$. Then, by a well known differential inequality(for example, see [10, p.43]), we have

$$r(t) \le u(t), \qquad if \quad t \ge t_0, \tag{2.10}$$

$$r(t) \ge u(t), \qquad if \quad t \le t_0. \tag{2.11}$$

So if $r(t_0) > M$, i.e., $u(t_0) > M$, then, by lemma 10, we have $u(t) \to \infty$, as $t \to t_2^+, t_2 < t_0$. Thus by (2.11), we have $r(t) \to \infty$, as $t \to t_2^+$, so that $r(t)$ has finite escape time; also if $r(t_0) < -M$, i.e., $u(t_0) < -M$, then, as above, $u(t) \to -\infty$, as $t \to t_1^-$, $t_1 > t_0$, so, by (2.10), we also have $r(t) \to -\infty$, as $t \to t_1^-$.

Secondly, if $|r(t)| \le M$ whenever it exists, then $r(t)$ must exist for all $t \in R$ by a well-known property of ODE, (for example, see[6, p.18]). This completes the proof.

Lemma 12 *Let $u_i(t)$, $i = 1, 2, 3, 4$, be 4 different solutions of*

$$r'(t) = -r^2(t) + f(t)$$

where $f \in L^1_{loc}(R)$. Then the following equality holds as long as all the solutions are well defined at the time t:

$$\frac{u_1(t) - u_2(t)}{u_1(t) - u_3(t)} = \lambda \frac{u_4(t) - u_2(t)}{u_4(t) - u_3(t)}$$

for some real constant λ.

Proof: See [16, p6].

Now we begin a study leading to a Floquet-type theorem. The proofs of theorems 13, 14, 15 is very similar to the proofs of theorems 8, 9, 10 of [3, pp108 – 111].

Theorem 13 *Let $K(t)$ be a bounded function such that*

$$u' + u^2 + K(t) + \varepsilon = 0 \qquad (R_\varepsilon)$$

has well defined solutions on R for all $\varepsilon \in [0, c]$, where c is a positive constant. (Hence, by lemma 11, those well defined solutions are bounded). Then for each such ε, there are infinitely many bounded solutions of (R_ε) and these solutions fill a closed bounded domain B_ε in the (t, u) plane (bounded in the sense of u). Also B_{ε_2} lies interior to B_{ε_1} if $\varepsilon_1 < \varepsilon_2$. The upper and lower bounded solutions $u_u(t)$ and $u_l(t)$, respectively, of (R_0) are separated, that is,

$$\inf_{-\infty<t<+\infty} |u_u(t) - u_l(t)| \geq \eta > 0 \tag{2.12}$$

where η is a constant. Furthermore, for any two bounded solutions $u_1(t)$ and $u_2(t)$ of (R_0), which are not both the extreme solutions, we have

$$\lim |u_1(t) - u_2(t)| = 0 \tag{2.13}$$

as $t \to +\infty$ or $t \to -\infty$.

Proof: Since

$$-u^2 - K(t) - \varepsilon_1 > -u^2 - K(t) - \varepsilon_2 \tag{2.14}$$

if $\varepsilon_2 > \varepsilon_1 \geq 0$, by a differential inequality [10, p43], we have

$$u_{\varepsilon_1}(t) \geq u_{\varepsilon_2}(t) \tag{2.15}$$

for $t \geq \alpha$, and

$$u_{\varepsilon_1}(t) < u_{\varepsilon_2}(t) \tag{2.16}$$

for $t < \alpha$. Here $u_{\varepsilon_1}(t)$ and $u_{\varepsilon_2}(t)$ are the solutions of (R_{ε_1}) and (R_{ε_2}) with initial value $u_{\varepsilon_1}(\alpha) = u_{\varepsilon_2}(\alpha)$. So, if (R_{ε_1}) had just one bounded solution $u_{\varepsilon_1}(t)$, then (R_{ε_2})

would not have any bounded solutions, since if $u_{\varepsilon_2}(t)$ is a bounded solution of (R_{ε_2}), then we can find a solution $u(t)$ of (R_{ε_1}) such that $u(t) \not\equiv u_{\varepsilon_1}(t)$, and $u(\beta) = u_{\varepsilon_2}(\beta)$ for some real β. Since $u_{\varepsilon_1}(t)$ is the only bounded solution of (R_{ε_1}), so by Lemma 11, $u(t)$ must have finite escape time, i.e., $u(t) \to -\infty$, as $t \to t_1^-$ for some $t_1 > \beta$ or $u(t) \to +\infty$, as $t \to t_2^+$, for some $t_2 < \beta$, but this means, by (2.15) or (2.16), that $u_{\varepsilon_2}(t)$ would have finite escape time too, which contradicts the assumption. This proves that each (R_ε) has a band B_ε of bounded solutions. Moreover, each band B_ε is closed by the "continuous dependence" property of solutions . So B_ε has an upper and lower bounded solution for the corresponding equation (R_ε). If $\varepsilon_2 > \varepsilon_1 \geq 0$, using the same method as above, it is easy to see that each solution curve of (R_{ε_2}) which intersects either extremal solution of (R_{ε_1}) must become unbounded. Thus B_{ε_2} lies in the interior of B_{ε_1}. Let $u_\varepsilon(t)$ be the solution of (R_ε), $\varepsilon > 0$, through $(t_0, u_u(t_0))$ for some t_0. Then by (2.15), we have $u_u(t) > u_\varepsilon(t)$ for $t > t_0$, since $u_\varepsilon(t)$ lies above the lower edge of B_ε, so $u_\varepsilon(t) > u_l(t)$, for all $t > t_0$ (here $u_u(t)$ and $u_l(t)$ are the extremal bounded solutions of (R_0)). That is,

$$u_u(t) \geq u_\varepsilon(t) \geq u_l(t) \tag{2.17}$$

for all $t \geq t_0$. Let

$$M = \max\{||u_u(t)||, ||u_l(t)||\} \tag{2.18}$$

where

$$||f|| = \max_{-\infty<t<+\infty} |f(t)|,$$

and the existence of $||u_u||$ and $||u_l||$ are guaranteed by lemma 11. From (R_0) and (R_ε), we have

$$u_u(t) - u_\varepsilon(t) = \int_{t_0}^{t} (-u_u^2(s) + u_\varepsilon^2(s) + \varepsilon)ds \tag{2.19}$$

for $t \geq t_0$. By (2.18) and (2.19) we have

$$u_u(t) - u_\varepsilon(t) \leq (2M^2 + \varepsilon)(t - t_0) \tag{2.20}$$

for $t \geq t_0$. On the other hand, we have, by (2.19),

$$u_u(t) - u_\varepsilon(t) = \varepsilon(t - t_0) + Q(t) \tag{2.21}$$

for $t \geq t_0$. Here

$$Q(t) = \int_{t_0}^{t} (u_\varepsilon(s) - u_u(s))(u_\varepsilon(s) + u_u(s))ds$$

and

$$|Q(t)| \leq 2M(2M^2 + \varepsilon) \int_{t_0}^{t} (s - t_0)ds, \quad by \quad (2.20),$$

or

$$|Q(t)| \leq M(2M^2 + \varepsilon)(t - t_0)^2. \tag{2.22}$$

Next, let $t = t_0 + h$, where

$$h = \frac{\varepsilon}{2M(2M^2 + \varepsilon)}.$$

Then by (2.21) and (2.22), we have

$$u_u(t_0 + h) - u_\varepsilon(t_0 + h) \geq \varepsilon h - M(2M^2 + \varepsilon)h^2 = \frac{\varepsilon^2}{4M(2M^2 + \varepsilon)}. \tag{2.23}$$

Now, we set

$$\varepsilon = \frac{c}{2}$$

to find

$$u_u(t_0 + h) - u_{\frac{c}{2}}(t_0 + h) \geq \frac{c^2}{16M(2M^2 + \frac{c}{2})} = \frac{c^2}{M(32M^2 + 8c)}. \tag{2.24}$$

Let

$$\eta = \frac{c^2}{M(32M^2 + 8c)}.$$

Then we have

$$u_u(t_0 + h) - u_l(t_0 + h) \geq \eta. \tag{2.25}$$

Since t_0 is arbitrary, this means

$$u_u(t) - u_l(t) \geq \eta \tag{2.26}$$

for all t. Now, let $u_1(t)$ and $u_2(t)$ be two bounded solutions of (R_0) and say that

$$u_1(t) < u_2(t)$$

and that $u_1(t)$ is not the lowest bounded solution of (R_0) (the other cases are similar). We will show that

$$\lim_{t\to\infty} |u_u(t) - u_1(t)| = 0.$$

Hence we will have

$$\lim_{t\to\infty} |u_2(t) - u_1(t)| = 0.$$

Let $u_\infty(t)$ be a solution of (R_0) lying below the band of bounded solutions. By the well-known property of the Riccati equation (lemma 12), we have

$$\frac{u_1(t) - u_u(t)}{u_1(t) - u_l(t)} = \lambda \frac{u_\infty(t) - u_u(t)}{u_\infty(t) - u_l(t)} \tag{2.27}$$

for some real constant λ.

Now choose $u_\infty(0)$ so near to $u_l(0)$, that $|\lambda|$ is determined smaller than a prescribed positive number ξ. For each number $-n^2$ ($n \in N$), there is a $\bar{t}$ such that

$$u_\infty(t) < -n^2$$

whenever $u_\infty(t)$ is defined for $t \geq \bar{t}$. Thus one can choose n^2 so large that for $t \geq \bar{t}$,

$$\frac{u_\infty(t) - u_u(t)}{u_\infty(t) - u_l(t)}$$

is arbitrarily near 1.

But this means that the ratio

$$\left|\frac{u_1(t) - u_u(t)}{u_1(t) - u_l(t)}\right|$$

becomes smaller than ξ. Thus

$$\liminf |u_u(t) - u_1(t)| = 0.$$

Furthermore, since

$$\frac{d}{dt}\left(\frac{u_\infty(t) - u_u(t)}{u_\infty(t) - u_l(t)}\right) = \frac{(u_u(t) - u_\infty(t))(u_u(t) - u_l(t))}{u_\infty(t) - u_l(t)} < 0, \tag{2.28}$$

the ratio

$$\frac{u_1(t) - u_u(t)}{u_1(t) - u_l(t)}$$

decreases monotonely to zero as $t \to \infty$. Thus

$$\lim_{t \to \infty} |u_u(t) - u_1(t)| = 0.$$

This completes the proof.

Theorem 14 *Let $u_1(t)$ and $u_2(t)$ be bounded solutions of*

$$u'(t) + u^2(t) + K(t) = 0, \qquad (R_0)$$

where $K(t)$ is a real bounded almost periodic function in the sense of Stepanov. If

$$u_1(t) - u_2(t) \geq \delta > 0$$

on $-\infty < t < +\infty$, then $u_1(t)$ and $u_2(t)$ are almost periodic functions in the sense of Bohr and the module of their frequencies is contained in the module of $K(t)$.

Proof: Since

$$u_1(t) - u_2(t) \geq \delta > 0,$$

we find, by theorem 13, that $u_1(t)$ is the upper bounded solution and $u_2(t)$ is the lower bounded solution of (R_0). Now, we let

$$\delta = \inf_{t \in R} |u_1(t) - u_2(t)|,$$

and let $\{h_n\}$ be any given sequence of real numbers. We will prove below that we can find a subsequence $\{h_{n_k}\}$ such that $u_1(t + h_{n_k})$ converges uniformly on $-\infty < t < +\infty$ (in the norm of $C^0(R)$). This shows that $u_1(t)$ is an almost periodic function in the sense of Bohr and the same argument holds for $u_2(t)$. Since $\{u_1(0 + h_n)\}$ and $\{u_2(0 + h_n)\}$ are both bounded sequences, one can extract a subsequence (again called h_n) such that

$$u_1(0 + h_n) \to \alpha,$$

$$u_2(0+h_n) \to \beta.$$

Also, one can require that

$$K(t+h_n) \to K^*(t)$$

in the Stepanov metric.

Let $u_1^*(t)$ and $u_2^*(t)$ be the solutions of

$$u'(t) + u^2(t) + K^*(t) = 0, \qquad (R^*)$$

with initial values $u_1^*(0) = \alpha$ and $u_2^*(0) = \beta$. Then by the continuous dependence of the solutions of a differential equation upon the coefficients, we have

$$\lim_{n\to\infty} u_1(t+h_n) = u_1^*(t), \tag{2.29}$$

and

$$\lim_{n\to\infty} u_2(t+h_n) = u_2^*(t), \tag{2.30}$$

where the convergence is uniform on each compact interval. Since

$$\inf_{-\infty<t<+\infty} |u_1(t+h_n) - u_2(t+h_n)| = \delta$$

for all h_n, if it were possible that

$$|u_1^*(t_0) - u_2^*(t_0)| < \delta$$

at some point t_0, then for large n, we would have

$$|u_1(t_0+h_n) - u_2(t_0+h_n)| < \delta$$

which is impossible. Thus

$$\inf_{-\infty<t<+\infty} |u_1^*(t) - u_2^*(t)| \geq \delta$$

and by theorem 13, this means that $u_1^*(t)$ and $u_2^*(t)$ are the upper bounded solution and the lower bounded solution of (R^*) respectively. Next, we show that $\{u_1(t+h_n)\}$ is a Cauchy sequence in $C^0(-\infty,+\infty)$. Suppose the contrary. Then there exists

$\varepsilon > 0$ such that for each N_1 there are integers $n_1 > N_1$, $m_1 > N_1$, and some t_1 such that

$$|u_1(t_1 + h_{n_1}) - u_1(t_1 + h_{m_1})| > \varepsilon.$$

Choose a sequence $N_k \to +\infty$ and corresponding integers $n_k, m_k > N_k$ and numbers t_k at which

$$|u_1(t_k + h_{n_k}) - u_1(t_k + h_{m_k})| > \varepsilon. \tag{2.31}$$

By the boundedness of $u_1(t)$, we can extract subsequences n_{k_i} and m_{k_i} (again called n_k, m_k) for which

$$u_1(t_k + h_{n_k}) \to \overline{\alpha},$$

and also

$$u_1(t_k + h_{m_k}) \to \hat{\alpha},$$

and by (2.31),

$$\overline{\alpha} \neq \hat{\alpha}.$$

Also, we can require that

$$K(t + t_k + h_{n_k}) \to \overline{K^*}(t)$$

and

$$K(t + t_k + h_{m_k}) \to \hat{K}^*(t)$$

both in the Stepanov metric. Then, by the same technique as above, the upper bounded solutions of $(\overline{R^*})$ and $(\hat{R}^*)$ are $\overline{u_1^*}(t)$ and $\hat{u}_1^*(t)$, respectively, with $\overline{u_1^*}(0) = \overline{\alpha}$, $\hat{u}_1^*(0) = \hat{\alpha}$. But, in the Stepanov metric, we have

$$||K(t + t_k + h_{n_k}) - K(t + t_k + h_{m_k})|| \leq$$
$$||K(t + t_k + h_{n_k}) - K^*(t + t_k)|| + ||K^*(t + t_k) - K(t + t_k + h_{m_k})||$$

Since $K(t + h_n) \to K^*(t)$ in the distance of Stepanov, we have

$$\overline{K^*}(t) = \hat{K}^*(t)$$

in the metric of Stepanov. i.e.,

$$\overline{K^*}(t) = \hat{K}^*(t)$$

almost everywhere. But this means

$$\overline{u}^*(t) = \hat{u}^*(t)$$

everywhere, because $\overline{u}^*(t)$ and $\hat{u}^*(t)$ are both upper bounded solutions of the same equation and so

$$\overline{\alpha} = \hat{\alpha},$$

contradiction! Therefore,

$$u_1(t + h_{n_k}) \to u_1^*(t) \tag{2.32}$$

uniformly on $-\infty < t < +\infty$ and thus $u_1(t)$ is an almost periodic function in the sense of Bohr. The same argument will show that $u_2(t)$ is almost periodic also. Furthermore, for each sequence $\{h_n\}$ with $K(t + h_n) \to K(t)$, by the same argument as above, there is a subsequence h_{n_i} such that $u_1(t + h_{n_i}) \to u_1^*(t)$ with uniform convergence on $-\infty < t < +\infty$. As we showed on page 30, $u_1^*(t)$ is the upper bounded solution of (R^*):

$$u'(t) + u^2(t) + K^*(t) = 0,$$

with

$$K^*(t) \equiv K(t).$$

So, $u_1^*(t)$ is the upper bounded solution of

$$u'(t) + u^2(t) + K(t) = 0.$$

However, since we assume that $u_1(t)$ is the upper bounded solution of the above equation, so

$$u_1^*(t) \equiv u_1(t).$$

Hence

$$u_1(t + h_{n_i}) \to u_1(t)$$

uniformly on $-\infty < t < +\infty$. By (e) of page 3, the module of frequencies of $u_1(t)$ is contained in that of $K(t)$. This completes the proof.

2.3 The Interior of the Disconjugacy Domain

In this section we give a general Floquet-type theorem of (2.33) when $(\xi, \eta) \in D^{\circ}$ is in the interior of the disconjugacy domain.

Theorem 15 *Let*

$$x''(t) + (-\xi + \eta p(t))x(t) = 0 \tag{2.33}$$

where $p(t)$ is a real bounded Stepanov almost periodic function and (ξ, η) belongs to the interior of the disconjugacy domain D. Then there is a solution basis of the form

$$x_u(t) = e^{\alpha t} exp(\int_0^t \varphi(s)ds) \tag{2.34}$$

$$x_l(t) = e^{-\alpha t} exp(\int_0^t \phi(s)ds) \tag{2.35}$$

where $\alpha > 0$, and $\varphi(t)$ and $\phi(t)$ are almost periodic functions in the sense of Bohr with mean value zero. Moreover, 2α is the mean of the width of the band of bounded solutions of the associated Riccati equation and $\int_0^t(\varphi(s) + \phi(s))ds$ is also a Bohr almost periodic function. Thus, $x_u(t)x_l(t)$ and its reciprocal are Bohr almost periodic functions too.

Proof: Let $u_u(t)$ and $u_l(t)$ be the upper and lower Bohr almost periodic solutions of the related Riccati equation

$$u'(t) + u^2(t) + (-\xi + \eta p(t)) = 0,$$

(they exist by theorem 14). Define

$$x_u(t) = exp(\int_0^t u_u(s)ds)$$

and

$$x_l(t) = exp(\int_0^t u_l(s)ds).$$

These are clearly linearly independent since if

$$e^{\int_0^t u_u(s)ds} = ke^{\int_0^t u_l(s)ds}$$

for all t and for some constant k, then differentiating the above will lead to

$$u_u(t) = u_l(t)$$

which is impossible. Thus $x_u(t)$, $x_l(t)$ form a basis of (2.33). Let $u_u(t) = \alpha_u + \varphi(t)$, $u_l(t) = \alpha_l + \phi(t)$, where both $\varphi(t)$ and $\phi(t)$ are almost periodic functions in the sense of Bohr with mean zero. The width of the band B of the bounded solutions of the associated Riccati equation is

$$\Delta(t) = u_u(t) - u_l(t) = \frac{W}{x_u(t)x_l(t)} \tag{2.36}$$

where the Wronskian

$$W = x_u'(t)x_l(t) - x_u(t)x_l'(t) \tag{2.37}$$

is a non-zero constant. Since

$$\Delta(t) \geq \delta > 0,$$

so we must have

$$0 < c_1 \leq x_u(t)x_l(t) \leq c_2, \qquad for \quad all \quad t \in R \tag{2.38}$$

for some constants c_1, c_2 (notice that $\Delta(t)$ is bounded above too). Since

$$x_u(t)x_l(t) = e^{(\alpha_u+\alpha_l)t+\int_0^t(\varphi(s)+\phi(s))ds}$$

and

$$(\alpha_u + \alpha_l) + \frac{1}{t}\int_0^t (\varphi(s) + \phi(s))ds \to (\alpha_u + \alpha_l)$$

as $t \to \infty$, this means that

$$\alpha_u = -\alpha_l = \alpha$$

and the almost periodic function $\Delta(t)$ has a mean of $2\alpha \neq 0$ (keep in mind that $\Delta(t) \geq \delta > 0$, so $M(\Delta(t)) \geq \delta$). Thus $x_u(t)x_l(t)$ and its reciprocal are almost periodic functions in the sense of Bohr too. But this means that $\int_0^t(\varphi(s) + \phi(s))ds$ is bounded and hence is a Bohr almost periodic function too. This completes the proof.

2.4 A Sufficient Condition For Nonoscillation

From the previous discussion, we can see that the existence of almost periodic solutions of a Riccati equation depends essentially on the disconjugacy of the related second order differential equation. In this section, we will give some conditions to ensure the disconjugacy of the second order equation.

Let $q : R \mapsto R$ be a locally Lebesgue integrable function and let c be a real constant, $c \neq 0$. It is interesting to know whether

$$x''(t) + cq(t)x(t) = 0, \qquad t \geq 0, \tag{2.39}$$

is oscillatory or nonoscillatory. There are many results regarding the oscillation of (2.39), for example, see Wong[14]. As for nonoscillation, in combining the results of L. Markus and R. Moore[2] and A.B. Mingarelli and S. Halvorsen [3], we know that (2.39) is disconjugate (nonoscillatory) for some $c \neq 0$ (or oscillatory for every $c \neq 0$) if and only if $M(q(t)) \neq 0$ (or $M(q(t)) = 0$) when $q(t)$ is a Bohr almost periodic function. Here $M(q(t))$ stands for the mean-value of $q(t)$. Furthermore, in [4], the authors gave an estimate for how big c could be so that (2.39) is nonoscillatory. This is good to know, since it is well known by the property of the convexity of the nonoscillation domain, (for example, see [3]), that if (2.39) is nonoscillatory for some $c = c_1 > 0$, and (2.39) is disconjugate when $c = 0$, then (2.39) remains nonoscillatory for all $c \in [0, c_1]$. Below, we improve the estimate of [4] under weaker conditions than those found in [4].

Theorem 16 *Let there exist positive constants M, T, ε, such that*

$$\left| \int_{nT}^{t} q(s)ds \right| \leq M \tag{2.40}$$

for all $t \in [nT, (n+1)T]$ and $n = 0, 1, 2, \cdots$, and

$$\int_{nT}^{(n+1)T} q(s)ds \leq -\varepsilon \tag{2.41}$$

for all $n = 0, 1, 2, \cdots$. Then (2.39) is nonoscillatory for all $c \in [0, \frac{4\varepsilon}{T(2M+\varepsilon)^2}]$.

Proof: By the remark above, we only need to show that (2.39) is nonoscillatory for

$$c = c_1 = \frac{4\varepsilon}{T(2M+\varepsilon)^2}.$$

It is also well-known that the nonoscillation of (2.39) is equivalent to the existence of a solution of the related Riccati equation

$$r'(t) = -cq(t) - r^2(t) \tag{2.42}$$

on some interval $[a, \infty)$, $a \geq 0$, (simply by changing the variable $r = x'/x$ in (2.39)). Below, we will show that the solution of (2.42) will exist on $[a, \infty)$, $a \geq 0$, if we chose the initial value $r(0)$ carefully.

Let $c_b = \frac{4\varepsilon}{bT(2M+\varepsilon)^2}$, $2 \geq b > 1$, set $c = c_b$ in (2.42) and choose $r(0)$ so that

$$0 < \frac{1}{T}((1 - c_bMT) - \sqrt{1 - 2c_bMT}) < r(0) \leq \frac{1}{T}(\sqrt{c_b\varepsilon T} - c_bMT) \tag{2.43}$$

Let $r(t)$ be a solution of (2.42) with $c = c_b$ and assume that $r(0)$ satisfies (2.43). We claim that

$$r(t) > -r(0) - c_bM$$

for all $t \in [0, T]$. If not, let $t_0 \leq T$ be such that $r(t) > -r(0) - c_bM$, for all $t \in [0, t_0)$ and $r(t_0) = -r(0) - c_bM$. Since

$$r(t) = r(0) - c_b \int_0^t q(s)ds - \int_0^t r^2(s)ds \tag{2.44}$$

by (2.42), it is easy to see, by condition (2.40), that we have

$$r(t) < r(0) + c_bM \tag{2.45}$$

for all $t \in [0, T]$. Thus we have

$$\begin{aligned} r(t_0) &= r(0) - c_b \int_0^{t_0} q(s)ds - \int_0^{t_0} r^2(s)ds \\ &\geq r(0) - c_bM - T(r^2(0) + 2c_bMr(0) + c_b^2M^2) \\ &> -r(0) - c_bM \end{aligned}$$

by (2.43). This contradicts the assumption that $r(t_0) = -r(0) - c_bM$. (Note that $\frac{1}{T}((1 - c_bMT) + \sqrt{1 - 2c_bMT}) > \frac{1}{T}(\sqrt{c_b\varepsilon T} - c_bMT)$). So

$$|r(t)| < r(0) + c_bM$$

for all $t \in [0, T]$. Furthermore, we have

$$\begin{aligned} r(T) &= r(0) - c_b\int_0^T q(s)ds - \int_0^T r(s)^2 ds \\ &\geq r(0) + c_b\varepsilon - T(r^2(0) + 2c_bMr(0) + c_b^2M^2) \\ &\geq r(0) \end{aligned}$$

since $0 \leq r(0) \leq \frac{1}{T}(\sqrt{c_b\varepsilon T} - c_bMT)$. Thus $r(t)$ exists on $[0, T]$,

$$|r(t)| < r(0) + c_bM$$

for all $t \in [0, T]$, and

$$r(T) \geq r(0) > 0.$$

Repeat this argument on every interval $[nT, (n+1)T]$. Induction shows that $r(t)$ exists on $[0, \infty)$ and remains positive. For example, on the interval $[T, 2T]$, let $s(t)$ be the solution of (2.42) with $c = c_b$ and $s(T) = r(0)$. Then by the same argument as above, we have $|s(t)| < r(0) + c_bM$, on $[T, 2T]$ and $s(2T) \geq s(T)$, but obviously, we have $r(t) > s(t)$ on $[T, 2T]$. Thus, by the equivalence of nonoscillation of (2.39) and the existence of a solution of (2.42) on $[a, \infty)$ for some real a, we know that (2.39) is nonoscillatory for every $c_b = \frac{4\varepsilon}{bT(2M+\varepsilon)^2}$, $2 \geq b > 1$. Actually, we can say that (2.39) is disconjugate for $c = c_b$, since $x(t) = c^{\int_0^t r(s)ds}$ is a positive solution of (2.39). And it is well-known that the disconjugacy domain is closed (for example, see [3]). Since

$$c_1 = \lim_{b \to 1} c_b$$

so (2.39) is also disconjugate when

$$c = c_1 = \frac{4\varepsilon}{T(2M + \varepsilon)^2}$$

This proves the theorem.

Remark: A comparison with [4] shows that the condition

$$\int_{nT}^{(n+1)T} \left(\int_{nT}^{t} q(s)ds\right)^2 dt \leq M^2, \qquad for \quad all \quad n = 0, 1, 2, \cdots,$$

is not necessary in order to obtain the same conclusion. Also note that the estimate here is better than the corresponding one in [4], since in [4], when $T = 1$, it gives the maximum of c for nonoscillation as

$$min\{0.01M^{-1}, 0.01\varepsilon M^{-2}\}$$

and it is easy to see that

$$\frac{4\varepsilon}{(2M+\varepsilon)^2} > 40 min\{0.01M^{-1}, 0.01\varepsilon M^{-2}\},$$

so this means that the estimate here is at least 40 times better than the one found in [4].

If $q(t)$ is bounded on R and satisfies (2.41), then it is easy to see that (2.39) is nonoscillatory for some $c > 0$ by the theorem we just proved. Also, if $q(t)$ is the restriction on $[0, +\infty)$ of a real Stepanov almost periodic function on $(-\infty, +\infty)$ with non-zero mean-value,

$$M(q) = \lim_{T\to\infty} \frac{1}{T}\int_{a}^{a+T} q(s)ds = \mu \neq 0,$$

and, without loss of generality, we can assume that $\mu < 0$, then (2.39) is nonoscillatory for some $c > 0$, since in this case we have

$$||q||_S = \sup_{x\in R} \frac{1}{L}\int_{x}^{x+L} |q(s)|ds < \infty$$

for some $L > 0$. Thus (2.40) as well as (2.41) hold. Since every Bohr almost periodic function is in the class of Stepanov almost periodic functons, it follows that if q(t) is a Bohr a.p. function with non-zero mean-value, then (2.39) is nonoscillatory for some $c \neq 0$.

We have seen that under the conditions of theorem 16, there is a $c_1 > 0$ such that (2.39) is nonoscillatory for $c = c_1$. Is there some $c < 0$, such that (2.39) is nonoscillatory too? The answer is no in general. For such an example we appeal to theorem 2 of L. Markus and R. Moore [2, p103]. We can rewrite (2.39) in the form

$$x''(t) + (cM(q) + c(q(t) - M(q)))x(t) = 0 \tag{2.46}$$

where $M(q)$ stands for the mean value of $q(t)$, and [2] shows that the disconjugacy domain lies in $a = -cM(q) > 0$, i.e., $cM(q) < 0$ except the origin. So, if $M(q)$ is negative, then the only possible value c could have so that (2.39) is nonoscillatory, is positive. The proof of [2, $Theorem2, p103$] is quite long and complicated. Below, we give a different and somewhat shorter proof of [2,theorem 2, p103].

Theorem 17 *[L. Markus and R. Moore]: let $p(t) \not\equiv 0$ be a real almost periodic function with mean value zero. Then for*

$$x''(t) + (-a + bp(t))x(t) = 0 \tag{2.47}$$

the disconjugacy domain D, minus the origin, lies in $a > 0$.

Proof: Since the equation is disconjugate the related Riccati equation

$$r'(t) + r^2(t) + (-a + bp(t)) = 0 \tag{2.48}$$

has solutions which are well defined for all $t \in R$. Furthermore, those solutions are uniformly bounded for all $t \in R$ (by lemma 11). Thus, by the result of Opial [5], we know that there is an almost periodic solution $r(t)$ with its module in the module of $p(t)$. Integration of the Riccati equation leads to

$$r(t) - r(0) + \int_0^t r^2(s)ds + (-at + b\int_0^t p(s)ds) = 0 \tag{2.49}$$

so, for $t \neq 0$, we have

$$\frac{r(t)}{t} - \frac{r(0)}{t} + \frac{1}{t}\int_0^t r^2(s)ds + (-a + \frac{b}{t}\int_0^t p(s)ds) = 0. \tag{2.50}$$

Letting $t \to \infty$ and noting that $r(t)$ is almost periodic, and hence bounded, we get:

$$M(r^2) + (-a + bM(p)) = 0 \tag{2.51}$$

since $M(p) = 0$, we have $a = M(r^2)$. Since $M(r^2) > 0$ (because if $M(r^2) = 0$, then by Bohr's uniqueness theorem $r^2(t) \equiv 0$, and this contradicts the assumption), we have $a > 0$, and this completes the proof.

Theorem 18 *Let $q(t)$ be an almost periodic function in the sense of Bohr. Then the Riccati equation (2.42) has an almost periodic solution for some $c \neq 0$ if and only if $M(q) \neq 0$.*

Proof: If the Riccati equation has an almost periodic solution $r(t)$, then (2.39) will have a positive solution $x(t) = e^{\int_0^t r(s)ds}$, so (2.39) is disconjugate, i.e., $M(q) \neq 0$. (otherwise (2.39) is oscillatory by [2]). Conversely, if $M(q) \neq 0$, then as we just showed in theorem 16, (2.39) is disconjugate for some $c \neq 0$. Thus the Riccati equation (2.42) has solutions which exist for all $t \in R$, hence by the same argument as before, those solutions are bounded, so by Opial's results [5] again, (2.42) has an almost periodic solution.

Chapter 3

Some Thoughts on The Problem of N Bodies

3.1 Introduction

We will consider the motion of n particles P_k $(k = 1, 2, \ldots, n)$ in a three-dimensional Euclidean space, where $n > 1$. Let $\xi_k = (x_k, y_k, z_k)$ denote the coodinates of P_k in a fixed Cartesian coordinate system, $m_k > 0$ its mass. The motion of these n bodies is completely described by Newton's law of attraction:

$$m_k \xi_k'' = U_{\xi_k}$$

where

$$\xi_k = (x_k, y_k, z_k)$$

and

$$U = \sum_{1 \leq k < l \leq n} \frac{m_k m_l}{r_{kl}}$$

is the potential function for Newton's law, U_{ξ_k} is the partial derivative of U with respect to ξ_k and

$$r_{kl} = |\xi_k - \xi_l| = \sqrt{(x_k - x_l)^2 + (y_k - y_l)^2 + (z_k - z_l)^2}$$

is the distance between the particles P_k and P_l. Note that we assume the gravitational constant to be equal to 1, for simplicity.

Hence, by working out U_{ξ_k} in terms of ξ_k, we can write Newton's law of attraction for n bodies as the following set of nonlinear differential equations (see, for example, [19, $pp20-21$]):

$$m_k(x''_k(t), y''_k(t), z''_k(t)) = \sum_{\substack{l\neq k\\ 1\leq l\leq n}} -\frac{m_k m_l}{r_{kl}^3}(x_k(t)-x_l(t), y_k(t)-y_l(t), z_k(t)-z_l(t)) \quad (3.1)$$

$$k = 1, 2, \ldots, n$$

i.e.

$$(x''_k(t), y''_k(t), z''_k(t)) = \sum_{\substack{l\neq k\\ 1\leq l\leq n}} -\frac{m_l}{r_{kl}^3}(x_k(t)-x_l(t), y_k(t)-y_l(t), z_k(t)-z_l(t)) \quad (3.2)$$

and the center of mass is (see, for example, [20, $p.234$])

$$\xi^* = (x_0, y_0, z_0) = \frac{1}{\sum_{i=1}^n m_i}\sum_{i=1}^n m_i(x_i, y_i, z_i) = \frac{1}{\sum_{i=1}^n m_i}\sum_{i=1}^n m_i\xi_i. \quad (3.3)$$

Let

$$\Xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{pmatrix} \quad (3.4)$$

and

$$A = (a_{ij})_{n\times n}$$

where

$$a_{ij} = \begin{cases} \frac{m_j}{r_{ij}^3}, & \text{if} \quad i\neq j; \\ -\sum_{\substack{k\neq i\\ 1\leq k\leq n}} \frac{m_k}{r_{ik}^3}, & \text{if} \quad i = j. \end{cases}$$

Then the above equations can be written more conveniently as

$$\Xi''(t) = A(\Xi)\Xi(t). \quad (3.5)$$

A coordinate system ξ_k, $k = 1, 2, \ldots, n$, for the n-body problem is called **inertial** if

$$m_k\xi''_k = U_{\xi_k}$$

is valid in it. A **barycentric coordinate system** is defined as one which has the **centre of mass** at the **origin** for all time t. Any inertial system ξ_i, $i = 1, 2, \ldots, n$, can be changed into a barycentric coordinate system by setting

$$\bar{\xi}_i = \xi_i - \xi^*; \quad i = 1, 2, \ldots, n$$

(see, [20, p.243]). In what follows, we always assume that the inertial coordinate system is the barycentric coordinate system.

In general, the motion of such n bodies is very difficult to determine due to the nonlinearity of the equations. As is well known today, for $n = 2$, this problem is completely solved. But for $n \geq 3$, it is still wide open.

In the following two sections, we will look at some special cases of such systems in terms of their geometrical properties. We will analyze the special n-body problem where the configuration formed by n bodies remains similar throughout the motion, and give some applications of oscillation theory of ordinary differential equations to the problem.

3.2 Homographic Solutions

In 1772, J. Lagrange [19, p.92] showed that three masses fixed at the vertices of an equilateral triangle, rotating about their common center of mass with an appropriate angular velocity, describe a periodic solution of the planar three-body problem. Lagrange thought that the solutions of such a configuration have no significance in astronomy. However, it was discovered lately that the Sun, Jupiter, and the small planets of the Trojan group form an approximately equilateral triangle [19, p.92].

Lagrange generalized the above solutions further by asking for, and constructing, all additional solutions for which the triangles formed by the particles remain similar throughout the motion.

With the motivation of Lagrange's solution, people defined the particular case of the n body problem-the one where the configuration formed by the n bodies remains similar to itself throughout the motion, as a homographic or permanent configuration. The resulting solutions are called homographic solutions. More precisely, a given solution $\xi_i = \xi_i(t), i = 1, 2, \ldots, n$, of the problem of n bodies (under Newton's law of attraction) is called homographic if the configuration formed by the n bodies at a given time t moves in the inertial barycentric coordinate system ξ in such a way as to remain similar to itself when t varies. By this is meant that there exists a scalar $r = r(t) > 0$, an orthogonal 3×3 matrix $\Omega = \Omega(t)$ such that for every i and t one has

$$\xi_i(t) = r(t)\Omega(t)\xi_i(0) \tag{3.6}$$

$i = 1, 2, \ldots, n$, [20, p.284].

It is known that a homographic solution necessarily forms a **central configuration** [20, p.297], i.e., n position vectors ξ_i of the n bodies m_i will be said to form a central configuration if the force of gravitation acting on m_i is proportional to the mass m_i and to the barycentric position vector ξ_i, i.e., if

$$U_{\xi_i} = \sigma m_i \xi_i, \quad i = 1, 2, \ldots, n,$$

where σ is uniquely determined as

$$\sigma = -\frac{U}{J}$$

and

$$U = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{r_{ij}}, \quad J = \sum_{i=1}^{n} m_i \xi_i^2.$$

It is known that n bodies with equal masses at the vertices of a regular polygon forms a planar central configuration since the configuration in this case is homographic [25]. The requirement that all the bodies have equal mass can be waived in the case of an equilateral triangle. Four bodies at the vertices of a regular tetrahedron also form a central configuration. This is also true for the regular octahedron and the cube [30].

It is easy to see that there are two special cases of homographic solutions. One is the case where the configuration is **dilating without rotation.** i.e.,

$$\Omega(t) \equiv I \tag{3.7}$$

in (3.6), where I is the identity 3×3 matrix and the resulting solution is called homothetic. In this case, the homothetic solution is characterized by

$$\xi_i(t) = r(t)\xi_i(0) \tag{3.8}$$

$i = 1, 2, \ldots, n$.

Another case is where the configuration is **rotating without any dilation.** i.e.,

$$r(t) \equiv 1 \tag{3.9}$$

in (3.6), and it is then called a relative equilibrium configuration. The solution is characterized by

$$\xi_i(t) = \Omega(t)\xi_i(0) \tag{3.10}$$

$i = 1, 2, \ldots, n$, and it is called a solution of relative equilibrium .

However, there is no stationary solution, i.e., it is impossible that

$$r(t) \equiv 1 \quad \textit{and} \quad \Omega(t) \equiv 1 \tag{3.11}$$

in (3.6). (see, [19,p.91]).

It is possible that a homographic configuration is rotating and also dilating. But this is possible only in the case the solutions are planar. (i.e. if there exists a plane P which contains all n initial position vectors $\xi_i(0)$ and all n initial velocity vectors $\xi_i'(0)$, and this will imply, by the uniqueness theorem of differential equations, that all position vectors $\xi_i(t)$ along with their velocity vectors $\xi_i'(t)$ must lie on the plane P for all time t, i.e., the n bodies all lie on the same plane P for all time t).

Furthermore, a homographic solution is a solution of relative equilibrium if and only if it is planar and rotates around the center of mass with an appropriate constant angular velocity (hence the relative equilibrium solutions are periodic solutions), [20, p.287].

About 100 years after Lagrange proposed the question, in 1879, R. Hoppe [25] found that N equal masses at the vertices of a regular polygon, rotating about their center of mass (which is the center of the polygon in this case) with an appropriate angular velocity describe a periodic solution (relative equilibrium solution) of the n-body problem. After another 50 years, MacMillan and Bartky [26] found the converse for $N = 4$ in 1932; that is, a system of four masses at the vertices of a square rotating about the center of mass, has a periodic solution only if all the masses are equal (note that this is not the case for an equilateral triangle). For $N = 5$, this program was carried out by Williams [27] six years later in 1938. Finally, after about another 50 years, L. Perko and E. Walter [21] showed that for $N \geq 4$, N masses at the vertices of a regular polygon, rotating about their common center of mass with an appropriate angular velocity, will have a periodic solution if and only if all the masses are equal. This was also discovered independently by B. Elmabsout [29] in 1987.

A solution of the n-body problem will be called **flat** if there exists for every t, a plane $P(t)$ which contains all n bodies at this t. Therefore, the planar case is a special flat case in that $P(t) \equiv P$. Actually, it has been proved that any flat homographic solution is necessary planar. If a homographic solution is not flat, then it **must be homothetic**, which means the configuration can only dilate but can't rotate [20, p.287]. Furthermore, the dilating factor $r(t)$ has to satisfy the following nonlinear differential equation

$$r^2(t)r''(t) = \lambda \tag{3.12}$$

where λ is a constant. This surprising and important result was discovered by the Italian mathematician Pizzetti [30] in 1904.

Integrating (3.12) will lead to

$$r'^2(t) + \frac{2\lambda}{r(t)} = r'^2(0) + \frac{2\lambda}{r(0)}. \tag{3.13}$$

If $\lambda > 0$ then it is easy to see that

$$\inf_{t>0} r(t) > 0$$

otherwise the left-hand side of equation (3.13) will be infinite and the right-hand side is a constant which is impossible. This means that, in this case, it is impossible to have a simultaneous collision of all bodies. If $r(t)$ is bounded for all $t > 0$, say, $r(t) < B$, then, as $\lambda > 0$, we have

$$r''(t) = \frac{\lambda}{r^2(t)} \geq \frac{\lambda}{B^2}$$

and integrating this twice leads to

$$r(t) \geq \frac{\lambda}{B^2}t^2 + r'(0)t + r(0),$$

hence $r(t)$ can't be bounded. Therefore, if $\lambda > 0$, then these n bodies will not have a simultaneous collision and the distances $r_{ij}(t)$ between any two particles ξ_i and ξ_j:

$$r_{ij}(t) = |\xi_i(t) - \xi_j(t)| = r(t)|\xi_i(0) - \xi_j(0)|$$

will be as big as you wish since $r(t)$ is unbounded.

If $\lambda < 0$ and $r'(0) < 0$,

$$r''(t) = \frac{\lambda}{r^2(t)} < 0,$$

for all $t > 0$, so $r'(t)$ is decreasing, and therefore

$$r'(t) < r'(0)$$

for all $t > 0$. Hence

$$r(t) \leq r'(0)t + r(0),$$

and this means that

$$r(t_o) = 0$$

for some finite t_o. i.e., these n bodies will have a simultaneous collision at a finite time.

If $\lambda < 0$ and $r'(0) > 0$. there are two cases: First, if

$$r'^2(0) + \frac{2\lambda}{r(0)} < 0$$

then

$$\sup r(t) \leq B,$$

for some constant $B > 0$. If

$$\sup r(t) = \infty,$$

then

$$r'^2(t) + \frac{2\lambda}{r(t)} = r'^2(0) + \frac{2\lambda}{r(0)}$$

implies

$$r'^2(t) < 0,$$

for some t, which is impossible. Hence $r(t)$ is bounded. So we have from the differential equation that

$$r''(t) = \frac{\lambda}{r^2(t)} \leq \frac{\lambda}{B^2}$$

and this will imply that

$$r(t) \leq \frac{\lambda}{B^2}t^2 + r'(0)t + r(0).$$

Since $\lambda < 0$, this means that there is a finite time t_0 such that

$$r(t_0) = 0.$$

So, in this case, $r(t)$ will increase to some maximum value and then decrease to zero which means that these n bodies will have a simultaneous collision in finite time.

Secondly, if

$$r'^2(0) + \frac{2\lambda}{r(0)} \geq 0.$$

$r(t)$ will increase indefinitely because

$$\inf r'(t) > 0.$$

This is easy to see since, if $\inf r'(t) = 0$, then for some t, we will have

$$\frac{2\lambda}{r(t)} \geq 0.$$

because

$$r'^2(t) + \frac{2\lambda}{r(t)} = r'^2(0) + \frac{2\lambda}{r(0)}$$

and $r'(t)$ is very small for some t.But certainly this is impossible as $\lambda < 0$. So $r(t)$ will increase indefinitely. Hence the distances between any two particles will increase indefinitely as well.

If $\lambda = 0$, then

$$r''(t) = 0,$$

hence

$$r(t) = r'(0)t + r(0).$$

Therefore, if $r'(0) > 0$, then

$$r(t) \to +\infty \quad as \quad t \to \infty,$$

which means that the distances between any two bodies will increase indefinitely. If $r'(0) < 0$, then it is easy to see that there is a finite time t_0 such that

$$r(t_0) = 0$$

so these n bodies will have a simultaneous collision in finite time.

By the above analysis, we can see that any **non flat** homographic solutions of the problem of n bodies either will have a simultaneous collision at a finite time or the distances between any two particles will increase indefinitely. Therefore, **there is no hope of finding any almost periodic non-flat homographic solutions.** By contrast, in the planar case, any relative equilibrium solution is necessarily periodic (and so almost periodic).

3.3 An Application of Oscillation Theory to the Problem of N Bodies

We define the configuration formed by the n bodies under the influence of Newton's law of attraction as a linked n-body problem if the distances between any two particles

$$r_{ij}(t) = |\xi_i(t) - \xi_j(t)|$$

is directly a function of time t but is independent of the positions of ξ_i, ξ_j, i.e.,

$$r_{ij}(t) = b_{ij}(t)$$

and

$$\frac{\partial b_{ij}}{\partial \xi_i} = 0 \quad \frac{\partial b_{ij}}{\partial \xi_j} = 0.$$

For example, the homographic configuration is a special case of a linked n-body problem since

$$r_{ij}(t) = |\xi_i(t) - \xi_j(t)| = |r(t)\Omega(t)(\xi_i(0) - \xi_j(0))| = b_{ij}(t).$$

In the linked n-body problem, since the distances between any two particles are explicit functions of t, we have a relatively easier set of differential equations arising from (3.5), namely,

$$\Xi''(t) = A(t)\Xi(t) \tag{3.14}$$

where

$$\Xi(t) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{pmatrix} \tag{3.15}$$

and

$$A(t) = (a_{ij}(t))_{n \times n}$$

where

$$a_{ij} = \begin{cases} \frac{m_j}{b_{ij}^3(t)}, & \text{if} \quad i \neq j; \\ -\sum_{\substack{k \neq i \\ 1 \leq k \leq n}} \frac{m_k}{b_{ik}^3(t)}, & \text{if} \quad i = j. \end{cases}$$

Thus the assumption on the distances between any two particles in the linked n-body problem reduces the nonlinear equations to linear equations.

We will consider the special case here in which all the particles have the same mass, i.e.,

$$m_i = m, \quad i = 1, 2, \ldots, n. \tag{3.16}$$

So we have

$$A(t) = mC(t)$$

where $C(t) = (c_{ij})_{n \times n}$ and

$$c_{ij} = \begin{cases} \frac{1}{b_{ij}^3(t)}, & \text{if} \quad i \neq j; \\ -\sum_{\substack{k \neq i \\ 1 \leq k \leq n}} \frac{1}{b_{ik}^3(t)}, & \text{if} \quad i = j. \end{cases}$$

Also note that $A(t)$ and $C(t)$ are symmetric.

We will consider the following question: Suppose that n bodies at one time, say $t = t_1$, are on a plane P, can these n bodies come back to the same plane P again later on? We give an answer to this question by means of the next theorem:

Theorem 19 *Suppose that we have a linked n-body problem and $A(t) = mC(t)$ as we understand it above. $A(t) \leq 0$ (i.e. $-A(t)$ is positive semi-definite) and there is a time $t = t_1$ such that these n bodies are on a plane P. Then if the mass m is bigger than some number $m_0 > 0$, then these n bodies will come back to this same plane P later on for some appropriate set of initial velocities.*

Proof: Without loss of generality, we can suppose the plane P is the xy-plane. We suppose that for these n bodies located at $\xi_i = (x_i, y_i, z_i)$, $i = 1, 2, \ldots, n$, there is a time $t = t_1$ such that $z_i(t_1) = 0$, for all $1 \leq i \leq n$. Since

$$\Xi''(t) = mC(t)\Xi(t)$$

and

$$\Xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix},$$

and $\xi_i = (x_i, y_i, z_i)$, it is easy to see that

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

satisfies

$$Z''(t) = mC(t)Z(t)$$

as well. Applying a theorem of A. Mingarelli and G. Halvorsen [3, p.58], we know that there is an $m_0 > 0$ such that the above equation is not disconjugate, hence there is a time $t = t_2$ such that $Z(t_2) = 0$, for some non-trivial solution $Z(t)$, i.e., these n bodies will come back to the same plane. Since the disconjugacy domain is convex [3], so if m_0 is a conjugate point, then so are all $m \geq m_0$. Hence if $m \geq m_0$, the equation

$$Z''(t) = mC(t)Z(t)$$

is also not disconjugate. That implies there are t such that $Z(t) = 0$. This completes the proof.

Remark: In the planar case there is a time $t = t_1$ such that at this time all the position vectors $\xi_i(t_1)$ and the velocity vectors $\xi_i'(t_1)$ are on the same plane P. How about if you only know that all the position vectors are on the same plane at one time? Well, it is not that bad in the cases we studied above. If the mass m is bigger than a certain constant then the configuration will at least come back.

We can look at this question in another way. Suppose that you would like to have these n bodies come back before a certain time t_1, what kind of conditions should you put on the mass m? Well, we look at the Sturm-Liouville boundary problem

$$Z''(t) = mC(t)Z(t)$$

with Dirichlet boundary condition

$$Z(0) = 0 = Z(t_1).$$

Then by a theorem of U. Mampitiya [31, p.68], if the set

$$\{t : \quad 0 \leq t \leq t_1, \quad C_-(t) > 0\} \tag{3.17}$$

is of positive measure, then there is a $m_1 > 0$ (the first positive eigenvalue) such that the Sturm-Liouville problem has a nontrivial solution. (It is known that for any matrix A, there are unique A_+ and A_- such that $A = A_+ - A_-$, where $A_+ \geq 0$ and $A_- \geq 0$). Since $m_1(t_1)$ decreases while t_1 increases, so if we want these n bodies to come back to the same plane before the time t_1, m has to be greater than m_1. Note that we can replace the definiteness condition on $C(t)$ by the condition (3.17) and still ensure the existence of such a mass m_1 for which there is a solution of the linked problem of n bodies which returns to P. In particular, this applies to the case where the $b_{ij}(t)$ are almost periodic and $C(t)$ has a nontrivial negative part $C_-(t)$.

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