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**FREE VIBRATION
OF
PARTIALLY SUPPORTED
PLATES AND SHELLS**

by
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A thesis submitted to
the School of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Ph.D. in Mechanical Engineering

DEPARTMENT OF MECHANICAL ENGINEERING
UNIVERSITY OF OTTAWA
OTTAWA, ONTARIO

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FREE VIBRATION OF PARTIALLY SUPPORTED PLATES AND SHELLS

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ABSTRACT

First-order transverse shear-deformation Mindlin theory has been used to predict the free vibration frequencies and modal shapes for isotropic, laminated and composite plates or shells. A finite element model based on the small deflection linear theory has been developed to obtain numerical solutions for this class of problems. The results for some of the degenerate cases are compared with other results available in the literature. These analyses involve a wide number of variables, namely; material properties, aspect ratios, support conditions and also radius to base ratio. The cracked base plates, shells and blades are idealized as partially supported models with varying support lengths. The effects of the detached base length on natural frequencies, modal shapes and nodal lines of these types of structures are investigated.

Although the expected decrease in frequency with increase in the detached base length is observed almost for all modes it is seen that this behavior is very pronounced for higher modes in both plates and shells. Analysis also showed that the variation of the detached base length has a small effect on the natural frequencies of plates and shells with large aspect ratios ($b/a > 2, r/a > 2$).

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Notation

A list of principal symbols, superscripts and subscripts used frequently in this work are summarized below. Locally used notation and modifications are defined where used. The symbols are listed roughly in the order of occurrence in chapter sequence.

Symbols

Mathematical symbols

$\{\}_{top}$	coordinates of top point of node i
$\{\}_{bottom}$	coordinates of bottom point of node i
$[]^T, \{\}^T$	transpose of matrix, vector
$_{,x}$	differential with respect to x
$ $	absolute value
$\{\}$	unit vector
$\int \int, \int_{(s)}$	integration over area
$\int \int \int, \int_{(v)}$	integration over volume
C^0	symbol for displacement continuity
C^1	symbol for displacement and first differential of displacement continuity
E, E_{ij}	Young's modulus, Young's modulus in plane ij
G, G_{ij}	shear modulus, shear modulus in plane ij

ν, ν_{ij}	Poisson's ratio, Poisson's ratio in plane ij
c/a	ratio of detached length to the base length of plate or shell
$\{F^b\}, \{f^b\}$	body forces vector
$\{F^s\}, \{f^s\}$	surface forces vector
$\{F^i\}, \{f^i\}$	concentrated forces vector
$\{U^s\}, \{u^s\}$	surface displacement vector
$\{U\}, \{u\}$	general displacement vector
$\{\bar{F}\}, \{\bar{f}\}$	virtual displacement vector
$\{U^i\}, \{u^i\}$	displacement vector for point i
$\{\bar{U}^i\}, \{\bar{u}^i\}$	virtual displacement vector for point i
$\{R\}$	general load vector
$\{R_b\}$	body forces vector
$\{R_s\}$	surface forces vector
$\{R_{i\sigma}\}$	initial stress forces vector
$\{R_c\}$	concentrated forces vector
$\{R_{i\epsilon}\}$	initial strain forces vector
$\{R_t\}$	thermal loads vector
$\{N_p\}$	in-plane forces vector
$\{M_b\}$	bending moments vector
$\{V_s\}$	shear forces vector
$\{\epsilon^o\}$	in-plane middle surface strain
$\{\kappa\}$	element curvature vector
$\{\sigma\}^{(k)}$	stress vector in k^{th} layer
$\{\sigma\}_{x,y,z}$	stress vector in load direction
$\{\sigma\}_{1,2,3}$	stress vector in principal direction
$\{\epsilon\}, \{\epsilon'\}$	strain vector

$\{\epsilon_m\}, \{\epsilon'_m\}$	strain vector for membrane
$\{\epsilon_b\}, \{\epsilon'_b\}$	strain vector for bending
$\{\epsilon_s\}, \{\epsilon'_s\}$	strain vector for shear
$\{\sigma\}, \{\sigma'\}$	stress vector
$\{\epsilon_o\}, \{\epsilon'_o\}$	initial strain vector
$\{\sigma_o\}, \{\sigma'_o\}$	initial stress vector
$\{\epsilon_t\}, \{\epsilon'_t\}$	thermal strain vector
$\{\Delta\}$	generalized displacement vector
$\{\Delta_i\}$	generalized displacement vector at node i
$\{\ddot{\Delta}\}$	generalized acceleration vector
$\{\dot{\Delta}\}$	generalized velocity vector
$[C]$	constitutive matrix
$[L]$	differential operation matrix
$[A]$	extensional stiffness matrix
$[D]$	flexural stiffness matrix
$[B]$	strain-displacement matrix, bending stretching coupling matrix
$[A_s]$	shear stiffness matrix
$[B_{pi}]$	in-plane displacement matrix
$[B_{bi}]$	bending displacement matrix
$[B_{si}]$	shear strain displacement matrix
$[J]$	Jacobian matrix
$[K]$	global stiffness matrix
$[K]^{(m)}$	element stiffness matrix
$[M]$	general mass matrix
$[M]^{(m)}$	element mass matrix
$[C_d]$	damping matrix of structure

$[H]$	displacement interpolation matrix
$[N]$	isoparametric interpolation matrix
$[Q]$	material stiffness matrix
$[\bar{Q}]$	reduced material stiffness matrix
$[T]_\sigma$	transformation matrix for stress
$[T]_\epsilon$	transformation matrix for strain
$[\Theta]$	transformation matrix from global to local coordinates
C_{ijkl}	fourth-order elasticity tensor
H_i, H_j, H_k	weight coefficients of the Gaussian quadrature
ng	number of Gauss point
nn	number of nodes per element
D	isotropic plate and shell rigidity
(x, y, z)	Global Cartesian Coordinates
(x', y', z')	Local Cartesian Coordinates
(V_{1i}, V_{2i}, V_{3i})	Nodal Cartesian Coordinates
$N_i, N_i(\xi, \eta)$	shape function for node i
(ξ, η, ζ)	3-D isoparametric coordinates
(ξ, η)	2-D isoparametric coordinates
(ξ_i, η_i)	nodal value at node i
ζ	linear coordinate in the thickness direction of shell element
σ_{ii}, σ_i	normal stress parallel to i^{th} axis
σ, σ_{ij}	stress, second-order stress tensor
ϵ, ϵ_{kl}	strain, second-order strain tensor
μ_d	damping property parameter
ρ, ρ, ρ^m	mass density
α_{1i}, α_{2i}	rotations at node i for a shell element

τ_{ij}, τ_{xy}	shear stress in ij, xy plane
ω, Ω	frequency of vibration, non-dimensional frequency
$\lambda = \omega^2$	eigenvalue
Φ	eigenvector
γ_{xy}, γ_{ij}	shear strain in xy, ij plane
Π	potential energy
a	length of plate or arch length of shell at base
b	length of plate or shell
c	detached length of plate or shell
D_e	effective plate rigidity
t	time, thickness of isotropic plate or shell
h, t_e	thickness of plate or shell
t_c, t_f	thickness of core, face in sandwich plates or shells
r	radius of shell
$\frac{b}{a}, \frac{c}{a}, \frac{a}{t}$	nondimensional parameters in plates or shells
$\frac{a}{t_e}$	ratio of base length to total thickness in sandwich plate or shell
DOF	degree of freedom
P-F-F-F	partially supported, free, free, free, boundary of plate or shell
C-F-F-F	clamped, free, free, free, boundary of plate or shell
$ipl\phi 1$	isotropic plate with $b/a = 1$
$opl\phi 1.5$	orthotropic plate with $b/a = 1.5$
$spl\phi 2$	sandwich plate with $b/a = 2$
$ish\psi 1\phi 1$	isotropic shell with $r/a=1$ and $b/a=1$
$osh\psi 2\phi 2$	orthotropic shell with $r/a=2$ and $b/a=2$
$ssh\psi 3\phi 3$	sandwich shell with $r/a=3$ and $b/a=3$
$\phi = \frac{b}{a}$	aspect ratio for plate or shell

$\psi = \frac{r}{a}$ radius to base length ratio for shell

$\frac{\Omega}{\phi^2}$ frequency parameter

Superscripts

([']) indices of local coordinate system

(^m) indices of element m

(^o) indices of in plane strain

(^{..}) indices of acceleration

([.]) indices of velocity

Subscripts

(_o) indices of initial strain

Chapter 1

INTRODUCTION

1.1 Finite Element Concepts

A considerable amount of research on the vibration analysis of plates and shells has been carried out using various numerical methods, such as the finite difference method, finite strip method and finite element method. It has been found that the finite element method is a powerful method for the solution of a wide scope of problems in engineering.

Finite element method enables us to convert a problem with an infinite number of degrees of freedom to one with a finite number in order to simplify the solution process. It yields an approximate analysis based upon an assumed displacement field, a stress field, or a mixture of these within each element. Finite element method is a convenient method for the plate and shell type structures having irregular geometries, or complex boundary conditions. Assumed displacement field or function is widely used in finite element analysis. The prescribed displacement

functions are constructed in the following manner:

- 1- Select a set of reference or node points on the plate.
- 2- Associate with each node point a given number of degrees of freedom (displacement, slope, etc.) .
- 3- Construct a set of functions each giving a unit value for one degree of freedom and zero values for all the others.

These functions are referred to as element displacement functions; or shape functions. In order to satisfy the convergence criteria the shape functions should satisfy the following conditions:

- - linearly independent.
- - continuous and have continuous derivatives up to order $(p - 1)$ both within the element and across element boundaries (an element which satisfies these conditions is referred to as a conforming element).
- - satisfy the geometric boundary conditions.

1.2 Composite Lamina Concepts

Composite materials are finding increasing applications in various fields of advanced structures such as space craft, high-speed aircraft, naval vessels and petrochemical industries. These numerous applications are because of several of its advantages, some of these are: high strength to weight ratio, high stiffness to weight ratio, efficient structural design, good thermal and acoustical insulation, etc.

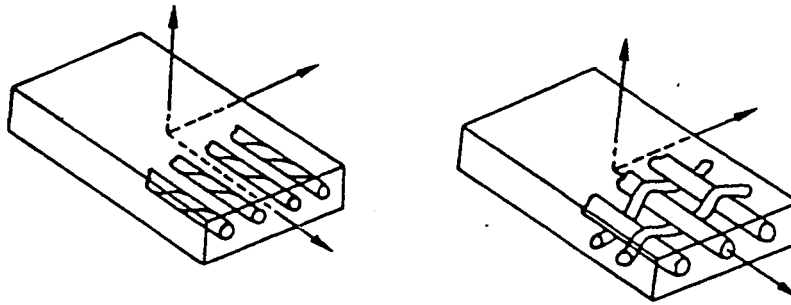


Figure 1.1: Laminar plate

A multiphase or two-material composite lamina consists of a stiff filament material embedded in a compatible matrix material. The filament material is used in composite laminae such as glass, boron, silicon carbide, carbon, graphite, beryllium, and steel. The matrix materials have included such modern materials as phenolic, polyesters, aluminum, epoxies, epoxy phenolic, and epoxy novalacs. Figure 1.1 shows two typical laminars along with their principal material axes which are parallel and perpendicular to the fiber direction. The fibers, or filaments, are the principal reinforcing or load-carrying agents. Determination of vibration characteristics of structural components made of composite materials is an important prerequisite for the design of such components. The vibration characteristics of composite plates and shells have been studied by a number of researchers. Some

of these papers [76],[77],[78],[85], [86],[87], [94],[95],[100],[101], [104],[106],[109],[110] deal with review literature on dynamic analysis of composite plates and shells.

In the structural analysis of plates and shells of composite materials, the equilibrium equation and the strain-displacement relations do not change from those for plates and shells of isotropic materials, but the stress-strain relations as well as stiffness matrix components will be different. References. [25],[26],[27],[28] give details of composite material concepts, classical and transverse shear deformation theory of plates and shells. Reference [23] gives physical and mechanical properties of composite materials and also contains some valuable information for design and analysis of advanced composite structures.

1.3 Plate Analysis by the Finite Element Method

Flat panels are used in many structures from buildings to ground and ocean vehicles, air craft, and space craft. These uses have generated considerable interest in vibration problems. The words "panel" and "plate" are used in this study to describe a structural element that is relatively thin in one dimension and flat in the mid-plane of the other two dimensions. Such structures may be analyzed by finite element method by dividing up the plate into an assemblage of two-dimensional finite elements called plate bending elements. These elements may be either triangular, rectangular or quadrilateral in shape.

Linear elastic analysis of plates using the finite element displacement method may be formulated either by the *Kirchhoff* or the *Mindlin* plate theory.

- -*Kirchhoff theory*;

In the Kirchhoff theory or classical thin plate theory, it is assumed that the normal to the mid surface before deformation remain straight and normal to the mid surface after deformation as shown in Fig. 1.2. This assumption implies that transverse shear deformation effects are negligible in plates, consequently free vibration frequencies calculated using C^1 classical thin plate theory are always higher than those obtained by more precise means, the deviation becoming larger with increasing mode number. Furthermore in a finite element context, shape functions have to ensure C^1 continuity or otherwise a non-conforming formulation must be adopted.

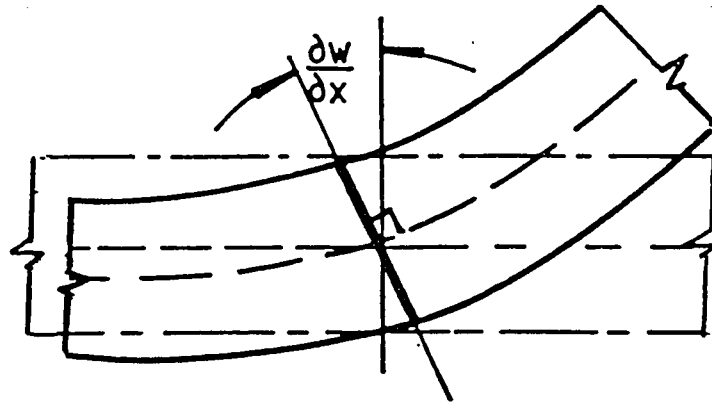


Figure 1.2: Model for *Kirchhoff theory*.

• - *Mindlin theory*;

In the *Mindlin* plate theory, which allows for shear deformation, the normal remains straight, but not necessarily normal to the mid surface after deformation. Figure 1.3 shows deflection model for *Mindlin* theory. More significantly, shape functions only require C^0 continuity. This latter requirement makes *Mindlin* plate elements particularly easy to formulate with the result that they have been successfully applied to many linear problems. Such formulations readily accommodate thick or thin plates, variable thickness, curved boundaries and laminated materials. The plate assumptions adopted for the finite element model described in this study are essentially those introduced by *Mindlin* :

- (a)- The deflection of the mid surface of the plate is small in comparison with the plate thickness.
- (b)- The transverse normal stresses are negligible.
- (c)- Normals to the mid-surface of the plates before deformation remain straight but not necessarily normal to the mid-surface after deformation.

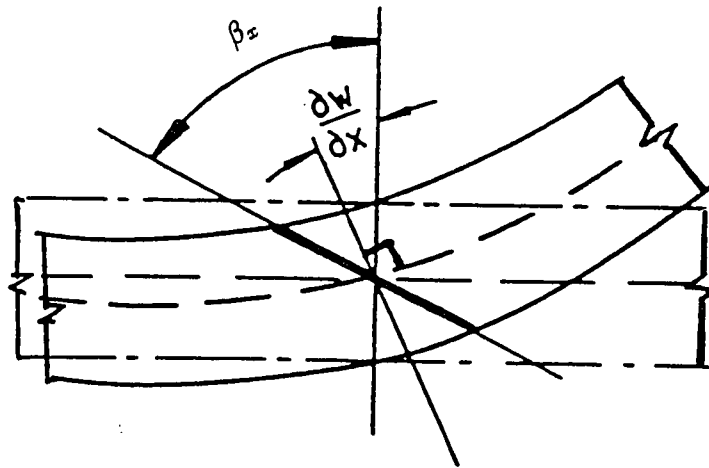


Figure 1.3: Model for *Mindlin* theory.

1.4 Shell Analysis by the Finite Element Method

Several different finite element techniques have been used by various researchers to study shells with arbitrary geometry. The selection of a particular approach depends on the shell theory used.

1- Element formulation based upon the thin shell theories of classical mechanics, [4], [31], [114], [122], and [123].

2- Flat-faced triangular or quadrilateral elements,

the earlier shell studies involved facetting the shell by plane triangular plate elements. Some aspects of flat-faced shell elements are discussed in references [4], [9], and [124].

3- The degenerate shell elements method,

the third approach employed the formulation of shell elements from the three-dimensional element. This type of shell element and some numerical techniques have been discussed in several references [4], [8], [10], [14], [15], [16], [17], [46], and [48].

1.5 Anisotropic Plates and Shells

In anisotropic plates and shells, material properties are different in all directions so that these type of plates and shells contain no planes of material property symmetry.

In the structural analysis of plates and shells of anisotropic or composite materials, the equilibrium equations and the strain-displacement relations do not change

from those for plates and shells of isotropic materials. However, it is necessary to drastically alter the stress-strain relations to account for the anisotropy of the materials system.

The difference in material property symmetry in isotropic and anisotropic plates/ shells is reflected in the mechanics and design of these types of structures. Two examples are given below.

1- The classical theory of plates and shells, based on neglecting transverse shear deformation, is not justifiable for most composite structures, including the geometrically thin members. It could be argued that for isotropic materials

$$2 \leq \frac{E}{G} = 2(1 + \nu) \leq 3$$

while, for most practical filamentary composite materials

$$20 \leq \frac{E_{11}}{G_{13}} \leq 50$$

Therefore, for a given in-plane modulus, the plate or shell is very weak in transverse shear resistance, and the effects of shear deformation in the transverse direction are significant, and hence cannot be neglected. Therefore, in order to generate a realistic model it is necessary to include transverse shear deformation in the analysis of most plate and shell structures fabricated with composite materials. Unfortunately, most of the literature on the subject of plate or shell dynamics available today is based on the classical theory.

2- For plates/ shells with isotropic material the sign convention for shear stresses and shear strains is of little practical significance, since its mechanical behavior is independent of the direction of shear stress. For plates/ shells with anisotropic material the direction of shear stress is critically important in determining their strength and modulus [74].

The rapid development of composite materials requires the discussion of better analytical models and solution techniques. A comprehensive literature survey on this topic covering a period until 1973 can be found in reference [27].

1.6 Sandwich Plates and Shells

Structural sandwich construction is one of the first forms of composite structures which acquired broad acceptance and usage. Mostly all aircrafts and space vehicles make extensive usage of sandwich construction. In addition to air and space vehicles, sandwich construction is commonly used in the manufacture of cargo containers, movable shelters and airfield surfacing, navy ship interiors, small boats, in the automobile industry, snow skis, residential construction materials, etc...

A sandwich plate consists of two or more layers, as shown in Figure 1.4:

- 1- A pair of thin, strong facings.
- 2- A thick, lightweight core to separate the facings and carry loads from one facing to the other.
- 3- An attachment which is capable of transmitting shear and axial loads to and from the core.

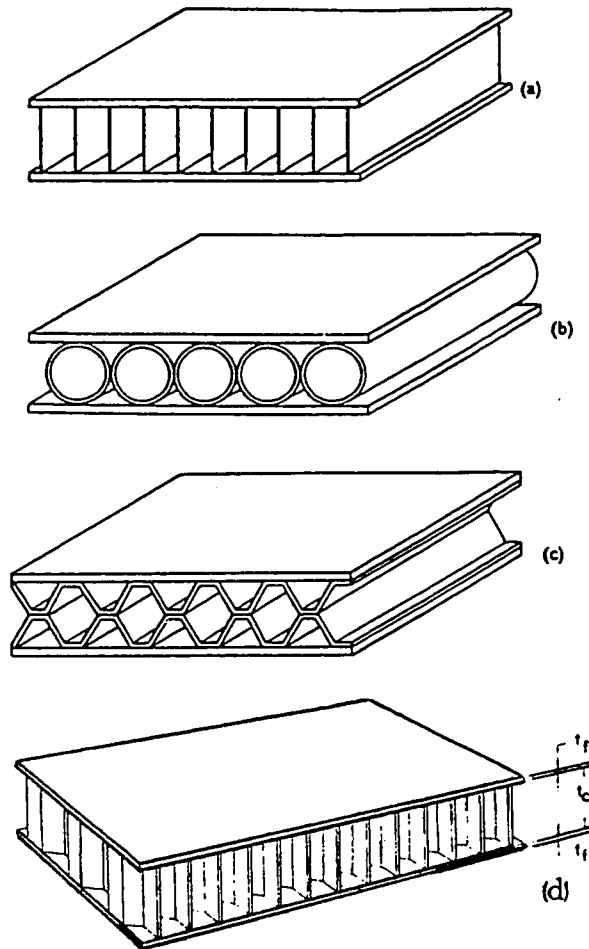


Figure 1.4: Sandwich plate

- - facing material:

The primary function of the face sheets is to provide the required bending and in-plane shear stiffness and to carry the axial, bending, and in-plane shear loading. In the aerospace field, the most commonly chosen facings are resin impregnated fiberglass cloth, graphite prepreg, 2024 or 7075 aluminum alloy, titanium, or stainless steel.

- -Core materials:

The primary function of a core in sandwich structures is that of stabilizing the facings and carrying most of the shear loads through the thickness. Several different materials have been used as sandwich cores, these include wood, foam, paper honeycomb, aluminum honeycomb, etc.

Reference [23] gives the mechanical properties, cell shape, cell size, thickness specimen geometry and test method of all kinds of cores in sandwich structures.

1.7 Laminate Plates and Shells

A laminate is a stack of laminae with various orientations of principal material direction in the laminae as shown in Figure 1.5

The laminates are usually bound together by the same matrix material that is used in the laminae. References [21],[22],[23],[24], [26],[27],[28],etc. give detailed information of macromechanical behavior of laminates, classical theory and analytical procedure for composite plates and shells structures.

1.8 Significance of the Present Study

In recent years several failures in the aircraft industry have been attributed to the lack of information on the interaction between fracture and vibration of wings due to aerodynamic forces. Another area where such information is lacking is turbine and fan blades. This necessitates further research investigating the effect of cracked

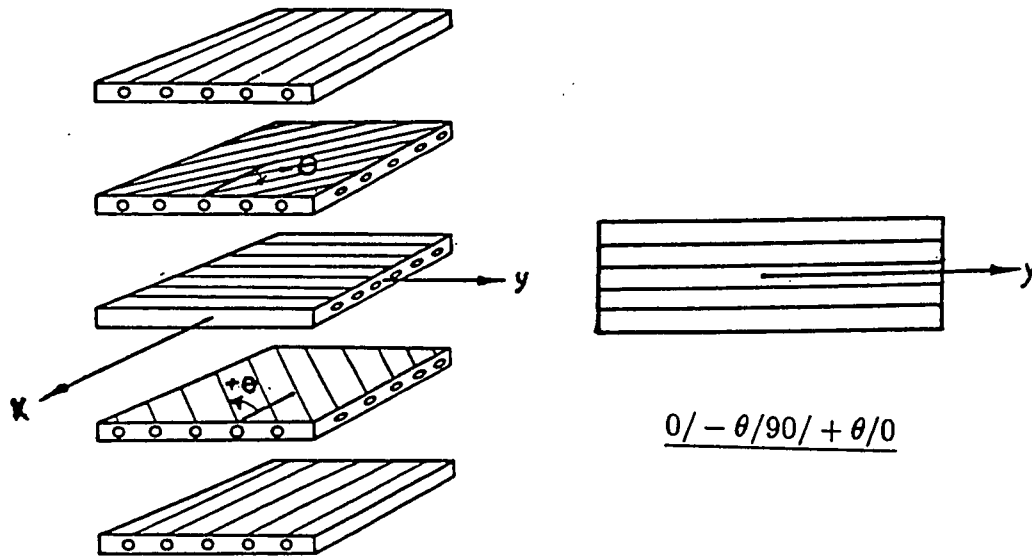


Figure 1.5: Laminate plate

length on dynamic behavior of these types of structures.

The scope includes modeling of partially supported plates and shells. The finite element method has been used to analyze the behavior of the plates/ shells and to find the effect of the supports on mode shapes and frequencies of the plates and shells. By idealizing the cracked edge as a partially supported model, one may be able to find some primary information for dynamic analysis and design usage.

1.9 Objective and Scope of this Investigation

The objective of this study is to investigate the free vibration of partially clamped cantilever plates and shells. The displacement type finite element method has been

used for analyzing natural frequencies and modal shapes of plates with different aspect ratio and different material type. The main scope of this study is to find the effect of partial support (c/a) on the variation of natural frequencies and modal shapes for plate and shell vibration.

The results in this study are divided in two parts :

- Part I

Part one deals with free vibration of partially supported triangular plates. For all geometry of plates the 8-node isoparametric element has been used.

A computer program was written and several examples of free vibration of triangular plates have been studied.

- Part II

The next part of this study deals with the free vibration analysis of partially supported curved panels, such as blades and shells. Several factors namely, frequency parameter, modal shapes and displacement patterns for shells with detached support length for isotropic, orthotropic, and sandwich materials have been investigated.

A computer code was prepared for finite element dynamic analysis of general shells and several examples of free vibration of cylindrical fan blades or shells have been studied. The degenerate cases have been compared with those available in the literature.

Another purpose of this investigation is to add important information to the data base on the topic of plate and shell dynamics of various material types.

Chapter 2

LITERATURE REVIEW

In this literature review, first the comments on the papers published on the general topic have been made and then the discussion on specific papers related to this thesis has been presented.

2.1 Finite Element Modeling

In 1977 Krishna Murty [79] reviewed several publications on F.E.M. covering a period from 1944 to 1977. these papers included lumped parameter, finite element displacement, finite element force, transfer matrix, hybrid and quadratic eigenvalue methods. In this review some finite element models have been compared and Rayleigh-Ritz finite element method was found to be the most convenient and efficient.

An extensive survey of the finite element modeling of structural vibrations has been given by Reddy [84]. This survey concentrates on the basic structural elements,

such as plates, shells and beams from 1967 to 1979.

2.2 Plate Vibrations

Leissa [82],[83],[91],[92], [102],[103] gives a comprehensive survey of plate vibration review of literature published from 1973 to 1985. Reference [82] reviews the literature published during the period 1973-1976 which mostly deals with free, undamped vibration of plates, governed by the classical theory of plates. Reference [83] is a review of literature dealing with the complicating effects of free, undamped vibrations of plates, such as: anisotropy, in-plane forces, variable thickness, surrounding media, large deflections, shear deformation, rotary inertia and non-homogeneity. In references [91] and [92] the works published during 1976-1980 are summarized. The first one deals with problems governed by classical theory of plates, and the second one considers complicating effects of anisotropy, in-plane forces, variable thickness, large transverse displacements, shear deformation, surrounding media, rotary inertia and non-homogeneity. References [102] and [103], also summarize research studies that have been carried out on the subject of free vibrations of plates during the years 1981-1985. The first article briefly summarizes 76 publications on plates which are governed by classical plate theory, dealing with thin, isotropic, homogeneous, linearly elastic, with small displacement. Publications on circular, rectangular, other quadrilateral, triangular, polygonal, sectorial plates, and other shapes are also included in these references. The second one briefly summarizes 213 papers for plates with complicating effects. These studies give valuable information about the effects of anisotropy, laminated composite material, in-plane stresses, variable

thickness, non-homogeneity, surrounding media, large deflection, large thickness, nonhomogeneous elastic foundation, initial imperfection and transverse magnetic field.

2.3 Eigenvalue Solution

Jennings [88] and [99] reviewed the methods for solving dynamic equations to determine characteristic response. In the first reference the numerical methods for eigenvalue solution, for undamped and damped systems are considered. In the second, the article updates a previous review of methods for solving dynamic equations to determine characteristic response.

2.4 Composite and Sandwich Plates

An extensive literature survey on composite and sandwich plate dynamics has been given by Bert [76],[77],[87], [94],[101]. The fundamentals of the mechanics of composite and laminate plates have been discussed in [76] and [77] which was updated in 1979 by Ref. [87]. References [94] and [101] present surveys of the literature concerning dynamics of plate-type structural elements of either composite material or sandwich construction. In the first, particular attention is given to experimental research, linear and nonlinear analysis. The reviewed material deals with rectangularly orthotropic, cylindrically orthotropic, an-isotropic, laminated and sandwich plates. Free and forced vibration, flutter and impact are also considered. In the second article emphasis is given to transverse impact and subsequent damage, en-

vironmental aspects, linear and nonlinear analysis and design.

Reddy [93],[100],[105] presented a significant survey of composite and sandwich plates. The first paper reviews finite element papers published on the static bending and free vibration of layered, an-isotropic and composite plates and shells. This publication also included large-deflection bending and large-amplitude free oscillation of layered composite plates and shells. The second review deals with modeling of natural vibrations of plates during the period from 1980 to 1985. An historical background of the development of shear deformation theories is also presented. Reddy in his third survey [105] presented an extensive review on the classical and shear deformation theories of laminated composite plates.

2.5 Layered Shell Vibration

Due to the wide industrial application of layered shells in various industries, the interest in their static and dynamic response has been increased. Most of the studies reported in this area which deal with cylindrical, spherical and conical layered shells were reviewed by Habip [107], Kapania [109] and more recent researches by Mirza [110].

2.6 Nonlinear Plate Vibration

Nonlinear problems concerning plates of various geometries have received considerable attention in the literature in recent years. An excellent monograph by

Sathyamoorthy [98] deals with nonlinear vibrations. This review is limited to papers published from 1979 to 1982. The research described in these publications included; geometric, material and a combination of these nonlinearities, complicating effects of anisotropy, attached masses, cutouts, elastic foundation, nonclassical boundary conditions, stiffeners, thermal stresses, variable thickness, transverse shear deformation, and rotatory inertia. The review article [108] is concerned with the recent developments in the nonlinear vibrational analysis of plates of various geometries. In addition of material and geometric type nonlinearities, particular attention is given to the recent developments in the analytical and numerical methods. Complicating effects such as the effects of transverse shear deformation, rotatory inertia, anisotropy, initial imperfections and variable rigidity on the vibration behavior of plates are also included.

2.7 Turbine Blade Vibration

Rao [80],[89],[97] reviewed the literature on blade excitation forces, vibration of blades with large aspect ratios, disk blade interaction, vibration of blades with small aspect ratios, blade group vibration, blade damping and response, free vibrations and experimental methods.

An extensive literature survey on vibrations of turbine engine blades by shell analysis has given by Leissa [90].

2.8 Closely Related Research Works

The literature survey shows that very little work has been done so far on the partially clamped plates and shells. But within the frame work of free vibration of partially clamped plates and shells, some investigations have been carried out to study the triangular, rectangular and other type of plates and shells with fully supported or complex supported conditions.

- -Triangular plates:

For triangular plates, most of the analyses presented in the literature so far have been reviewed by Leissa [91], [102]. In these review papers the analytical and computational complexity of the various methods of solution have been discussed. Gorman [38] used the analytical superposition technique for the free vibration analysis of simply supported right triangular plates. More classical methods have been used by Kim and Dickinson [43] and Bhat [41], and the Ritz method by Anderson et al. [116].

Some experimental results have been provided by a number of researchers [71],[60].

Finite element method has been employed by Mirza and Bijlani [63], Cowper et al.[61] and Utjes et al.[40] , and the grid-work approximation by Christensen [59].

- -Rectangular plates:

Fan and Cheung [39] used the spline finite strip method to study the flexural free vibration response of thin rectangular plates with complex supported conditions.

Kurata and Okamura [64] used the theoretical method for analysis of natural vibrations of partially clamped rectangular plates.

- -Cylindrical Fan Blades:

The finite element method is a suitable method for dynamic structural analysis of turbo-machinery blades. The earlier works in this area have been reviewed in [36] and [46]. More recent works on vibrations of fan blades can be found in References [27],[33] and [113].

Chapter 3

FINITE ELEMENT THEORY FOR PLATES AND SHELLS

3.1 Finite Element Formulation of the General Elastic Body

3.1.1 Introduction

Several types of formulations of finite element method have been used by researchers. The most important formulation, which is widely used for the solution of practical problems is the displacement-based finite element method. This is because of its simplicity, generality and special adaptability to computer techniques. Practically all major general-purpose analysis programs have been written using this type formulation. In this chapter we shall first concentrate on the general formulation of three-dimensional elastic body. In this formulation the differential equation of equilibrium and boundary conditions could be established by using the *principle of virtual displacements*. After establishing the general equations of equilibrium

governing the linear dynamic response of a system i.e,

$$[M]\{\ddot{\Delta}\} + [C_d]\{\dot{\Delta}\} + [K]\{\Delta\} = \{R\}$$

we will consider the solution of the eigenvalue problem i.e,

$$([K] - \lambda_i[M])\{\Phi\} = \{0\}$$

Then, we will derive the finite element properties for special cases such as plates and shells. Finally, bending rigidity, element shape functions and numerical integration for plates and shells are discussed.

3.1.2 The Principle of Displacement-Based Finite Element Method

The basic steps in the analysis of a structure using the displacement-based finite element method are:

- 1- Idealize an actual physical problem into an assemblage of discrete elements which are interconnected at the structural joints (nodal points).
- 2- Identify the unknown joint displacements that completely define the displacement response of the structural idealization.
- 3- Establish force balance equations corresponding to the unknown joint displacements and solve these equations.
- 4- With the element nodal-displacements known, calculate the internal element stress distributions.
- 5- Interpret the displacements and stresses predicted by the solution of the structural idealization when considering the assumptions used and correct interpretation of the results.

The establishment of an appropriate finite element model of an actual physical problem depends on a large number of factors. References [10], [14] and [29] contain a good discussion of these factors and of modeling considerations.

3.1.3 Relations of Strain-Stress and Strain-Displacement

In this general formulation we review the finite element principle and the solution method. This method is a fairly standard method discussed by several authors, for example in references [4],[5],[6],[10],[14],[18], [30],etc.

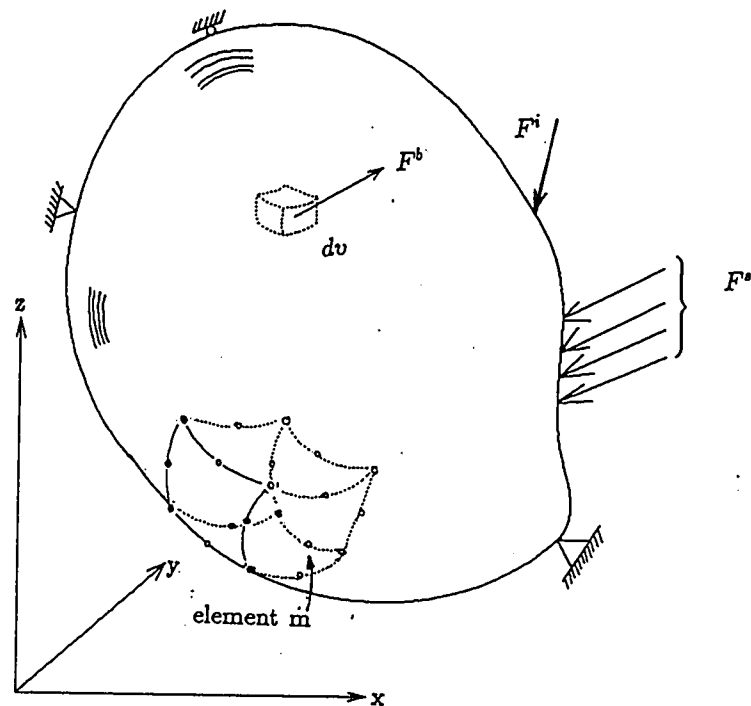


Figure 3.1: General elastic body

Let us consider the equilibrium of a general body. The external forces acting on the body corresponding to the three coordinate axis (Fig. 3.1) can be written in

the form

$$\{F^b\} = \{F_x^b, F_y^b, F_z^b\}^T$$

$$\{F^s\} = \{F_x^s, F_y^s, F_z^s\}^T$$

$$\{F^i\} = \{F_x^i, F_y^i, F_z^i\}^T$$

where,

$$\{F^b\} = \text{body forces}$$

$$\{F^s\} = \text{surface forces}$$

$$\{F^i\} = \text{concentrated forces}$$

The displacement of the system are denoted by,

$$\{U\} \equiv \{\vec{U}\}$$

where,

$$\{U\} = \{U_x(x, y, z), U_y(x, y, z), U_z(x, y, z)\}^T$$

The strains corresponding to the displacement field $\{U\}$ are

$$\{\epsilon\} = \{\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}\}^T$$

The strain-displacement relations can be represented as

$$\{\epsilon\} = [L]\{U\} \quad (3.1)$$

where $[L]$ is the differential operator matrix:

$$[L] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} \end{bmatrix} \quad (3.2)$$

The components of the stress tensor corresponding to $\{\epsilon\}$ are denoted by $\{\sigma\}$ i.e.,

$$\{\sigma\} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz}\}^T$$

and the stress-strain relation is in the form,

$$\{\sigma\} = [C](\{\epsilon\} - \{\epsilon_0\}) + \{\sigma_0\} \quad (3.3)$$

where,

$\{\sigma_0\}$ = initial stresses

$\{\epsilon_0\}$ = initial strains

and,

$[C]$ = is the constitutive matrix

In the case of three-dimensional isotropic elastic materials [29].

$$[C] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ & & & & \frac{1-2\nu}{2(1-\nu)} & 0 \\ & Sym. & & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \quad (3.4)$$

where E and ν are Young's modulus and Poisson's ratio respectively. The constitutive matrices for orthotropic materials are gives in [29].

3.1.4 The Principle of Virtual Work

According to the *Principle of Virtual Work*,

the total internal virtual work = total external virtual work;

$$\int_V \{\bar{\epsilon}\}^T \{\sigma\} dV = \int_V \{\bar{U}\} \{F^b\} dV + \int_S \{\bar{U}^s\}^T \{F^s\} dS + \sum_i \{\bar{U}^i\}^T \{F^i\} \quad (3.5)$$

where,

$\{\bar{\epsilon}\}$ = virtual strain

$\{\bar{U}\}$ = virtual displacement

The superscripts s and i denote respectively the surface displacement and point displacements. It should be emphasized that the Eq.(3.5) embodies all requirements that need be fulfilled in the analysis of problems in solid mechanics such as:

- 1- Compatibility, which requires displacements should be continuous and compatible and also satisfy the displacement boundary conditions.
- 2- Constitutive requirements, which require appropriate constitutive relations between stress and strain.
- 3- The equations of equilibrium.
- 4- Although the virtual work equation has been written in the global coordinate system x,y,z of the body, it is equally valid in any other system of coordinates.
- 5- The principle of virtual displacements can be directly related to the principle that the total potential Π of the system must be stationary.

3.1.5 Derivation of the Finite Element Equation by Virtual Work.

In finite element analysis, we idealize the elastic body in Fig.(3.1) as an assemblage of discrete finite elements which are interconnected at nodal points. At this point, we rewrite Eq.(3.5) as a sum of integrations over the volume and areas of all elements.

i.e.,

$$\begin{aligned} \sum_m \int_{v(m)} \{\bar{\epsilon}\}^T \{\sigma\} dv &= \sum_m \int_{v(m)} \{\bar{u}\}^T \{f^b\} dv \\ &+ \sum_m \int_{s(m)} \{\bar{u}^s\}^T \{f^s\} ds \\ &+ \sum_i \{\bar{u}^i\}^T \{F^i\} \end{aligned} \quad (3.6)$$

where $m = 1, 2, \dots, ne$ and $ne = \text{number of elements}$. Assuming the displacement of each element to be a function of the displacements at the n nodal points, for element m we have,

$$\{u\} = [H]\{\Delta\} \quad (3.7)$$

where

$[H]$ is the displacement interpolation matrix.

$\{u\}$ is the displacement within each element in local coordinate system.

$\{\Delta\}$ is a displacement vector at all nodal points in global coordinate system.

$$\{\Delta\} = \{\{\Delta_1\}, \{\Delta_2\}, \dots, \{\Delta_i\}, \dots, \{\Delta_n\}\}^T$$

We may note that, $\{\Delta_i\}$ is the generalized displacement at node i , representing displacement or rotation in the directions of assumed degrees of freedom. Using the definition of strain and displacement relation (Eq.3.1), we can now evaluate the corresponding element strains as follows:

$$\{\epsilon\} = [B]\{\Delta\} \quad (3.8)$$

where

$$[B] = [L][H]$$

If we substitute Eqs. (3.3), (3.7) and (3.8) into Eq. (3.6) we obtain,

$$\begin{aligned} \{\bar{\Delta}\}^T \left(\sum_m \int_{v(m)} [B]^T [C] [B] dv \right) \{\bar{\Delta}\} &= \{\bar{\Delta}\}^T \left(\sum_m \int_{v(m)} [H]^T \{f^b\} dv \right. \\ &+ \sum_m \int_{s(m)} [H]^T \{f^s\} ds \\ &- \left. \sum_m \int_{u(m)} [B]^T \{\sigma_o\} dv \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_m \int_{v^{(m)}} [B]^T [C] \{\epsilon_o\} dv \\
& + \sum_m \int_{v^{(m)}} [B]^T [C] \{\epsilon_i\} dv \\
& + \{F\}
\end{aligned} \tag{3.9}$$

The superscript (m) denotes the element m and the integrals $\int_{v^{(m)}}$ and $\int_{s^{(m)}}$ integrations over the element volumes and surfaces. It is important to note that all the parameters under the integrals such as $[H]$, $\{f^b\}$, $\{f^s\}$ and etc. are defined in local element coordinate system.

Finally, the equilibrium equations of the finite element assemblage can be written in the following form:

$$[K]\{\Delta\} = \{R\} \tag{3.10}$$

where

$$\{R\} = \{R_b\} + \{R_s\} - \{R_{i\sigma}\} + \{R_c\} + \{R_{ie}\} + \{R_t\} \tag{3.11}$$

The matrix $[K]$ is the stiffness matrix given by

$$[K] = \sum_m \int_{v^{(m)}} [B]^T [C] [B] dv \tag{3.12}$$

The vector $\{R\}$ is the load vector which includes the effects of the body forces $\{R_b\}$, the surface forces $\{R_s\}$, the initial stress forces $\{R_{i\sigma}\}$, the concentrated loads $\{R_c\}$, the initial strain forces $\{R_{ie}\}$ and the thermal loads $\{R_t\}$. These are defined by Eqs. (3.13) to (3.18).

$$\{R_b\} = \sum_m \int_{v^{(m)}} [H]^T \{f^b\} dv \tag{3.13}$$

$$\{R_s\} = \sum_m \int_{v(m)} [H^s]^T \{f^s\} dv \quad (3.14)$$

$$\{R_{i\sigma}\} = \sum_m \int_{v(m)} [B]^T \{\sigma_o\} dv \quad (3.15)$$

$$\{R_{i\epsilon}\} = \sum_m \int_{v(m)} [B]^T [C] \{\epsilon_o\} dv \quad (3.16)$$

$$\{R_i\} = \sum_m \int_{v(m)} [B]^T [C] \{\epsilon_i\} dv \quad (3.17)$$

$$\{R_c\} = \{F\} \quad (3.18)$$

Equation (3.10) is a statement of the general form of equilibrium equation, valid for static as well as dynamic problems.

We apply d'Alembert's Principle to convert the general Eq. (3.10) to the standard form of the dynamic equilibrium equation. For this purpose we can simply include the element inertia forces as part of the body forces and also, introducing the damping forces as additional contributions to the body forces, Equation (3.13) can be written as

$$\{R_b\} = \sum_m \int_{v(m)} [H]^T (\{f^b\} - \rho[H]\{\ddot{\Delta}\} - \mu_d[H]\{\dot{\Delta}\}) dv \quad (3.19)$$

where the vector $\{f^b\}$ no longer includes inertia and velocity dependent damping forces. Therefore, we will obtain the standard form of dynamic equilibrium equations as follows:

$$[M]\{\ddot{\Delta}\} + [C_d]\{\dot{\Delta}\} + [K]\{\Delta\} = \{R\} \quad (3.20)$$

where the vector $\{R\}$ and $\{\Delta\}$ are time dependent. The matrix $[M]$ is the mass matrix of the structure and the matrix $[C_d]$ is the damping matrix of the structures. i.e.,

$$[M] = \sum_m \int_{v(m)} \rho [H]^T [H] dv \quad (3.21)$$

$$[C_d] = \sum_m \int_{v(m)} \mu_d [H]^T [H] dv \quad (3.22)$$

3.2 Free Vibration

In the case of free vibration of an undamped system Eq.(3.20) becomes,

$$[M]\{\ddot{\Delta}\} + [K]\{\Delta\} = \{0\} \quad (3.23)$$

or for the eigenvalue problem the convenient form is

$$([K] - \lambda[M])\{\Phi\} = \{0\} \quad (3.24)$$

where λ is the eigenvalue ($\lambda = \omega^2$) and $\{\Phi\}$ is the eigenvector.

The condition that the Eq.(3.24) should have non-zero solution is

$$\det([K] - \lambda[M]) = 0 \quad (3.25)$$

3.2.1 Solution of Eigenvalue Problems

The eigenvalue problem given by Eq.(3.24) can be rewritten as

$$[K]\{\Phi\} = \lambda[M]\{\Phi\} \quad (3.26)$$

Two general methods, namely, transformation methods and iterative methods, are available for solving these type of problems.

Transformation Methods

The transformation methods such as Jacobi, Givens and Householder Schemes are preferable when all the eigenvalues and eigenvectors are required. A detailed solution algorithm has been discussed in references [5],[7],[11] and [14].

Iterative Methods

The iterative methods such as the power method, are preferable when only a few eigenvalues and eigenvectors are required. Another technique available is the subspace iteration method. This method is very effective for finding the first few eigenvalues and eigenvectors of large eigenvalue problems arising in structural and solid mechanics. The subspace iteration method allows the simultaneous approximate calculation of several eigenvalues and eigenvectors of Eq.(3.26), using inverse iteration.

The space is reduced from the original n dimensions of $[K]$ and $[M]$ matrices to a subspace of dimension q , where q is the number of iteration vectors. These and some of the other iteration techniques have been discussed in references [5], [7], [11], [14],[54],[57],[65],etc.

The subspace iteration method developed by Bathe [14] consists of the following

three steps:

- 1- Establish q starting iteration vectors, $q > p$, where p is the number of eigenvalues and vectors to be calculated.
- 2- Use simultaneous inverse iteration on the q vectors and Ritz analysis to extract the best eigenvalue and eigenvector approximations from the q iteration vectors.
- 3- After iteration convergence, use the Sturm sequence check to verify that the required eigenvalues and corresponding eigenvectors have been calculated.

The theoretical basis of the algorithm is well documented in the literature [13],[52],[65] and therefore only a brief overview will be presented. The flow chart of the subspace iteration scheme, for calculating p eigenvalues and eigenvectors, is set out below.

Step 1- Establish a set of starting vectors $\{X\}_1, \{X\}_2, \dots, \{X\}_q$, where $q > p$. Bathe and Wilson suggested a value of $q = \min(2p, p + 8)$ for good convergence. A computer algorithm for calculating efficient initial vectors for subspace iteration method was given in [57]. Define the initial modal matrix $[X]_0$ as

$$[X]_0 = [\{X\}_1, \{X\}_2, \dots, \{X\}_q] \quad (3.27)$$

and set the iteration number as $k = 0$

Step 2- Improve modal matrix $[X]_{k+1}$, by using the following subspace iteration procedure

(a)-Find $[\bar{X}]_{k+1}$ from the relation

$$[K][\bar{X}]_{k+1} = [M][X]_k \quad (3.28)$$

(b) compute

$$[\bar{K}]_{k+1} = [\bar{X}]_{k+1}^T [K] [\bar{X}]_{k+1} \quad (3.29)$$

and

$$[\bar{M}]_{k+1} = [\bar{X}]_{k+1}^T [M] [\bar{X}]_{k+1} \quad (3.30)$$

(c)- Solve the reduced eigenproblem

$$[\bar{K}]_{k+1} [Q]_{k+1} = [\bar{M}]_{k+1} [Q]_{k+1} [\Lambda]_{k+1} \quad (3.31)$$

where $[Q]_{k+1}$ contains the matrix of eigenvectors and $[\Lambda]_{k+1}$ is a diagonal matrix of eigenvalues. This equation can be converted to the standard eigenvalue problem and solved using the Jacobian iteration method.

(d)- Calculate an improved approximation to the eigenvectors of the original system as

$$[X]_{k+1} = [\bar{X}]_{k+1}^T [Q]_{k+1} \quad (3.32)$$

Note:

1- It is assumed that the vectors in $[X]_0$ are not orthogonal to one of the required eigenvectors.

2- It is assumed that the iteration vectors converging to the exact eigenvectors $\{X\}_1^{(exact)}$, $\{X\}_2^{(exact)}$, ..., are stored as the first, second, ..., columns of the matrix $\{X\}_{k+1}$.

Step 3- If the column vectors in $[X]_0$ are not orthogonal to one of the required eigenvectors, then the i th diagonal entry in $[\Lambda]_{k+1}$ converges to λ_i and i th vector in

matrix $[X]_{k+1}$ converges to Φ_i . The iteration is terminated whenever the following criteria are satisfied :

$$\left| \frac{\lambda_i^{(k+1)} - \lambda_i^{(k)}}{\lambda_i^{(k+1)}} \right| \leq \varepsilon, \quad i = 1, 2, \dots, p \quad (3.33)$$

where ε is a predefined tolerance. In the numerical examples presented in this work a tolerance of $\varepsilon = 10^{-6}$ is assumed.

However, once the convergence limit in Eq.(3.33) is satisfied, the final stage of the solution of Eq.(3.26) involves using the Sturm sequence to verify that the required eigenvalues and corresponding eigenvectors have been obtained.

In the present study, for solving the eigenvalue problem defined by Eq.(3.26). we have used the subspace iteration method to generate the frequencies and modal shapes of plates, where stiffness matrix $[K]$ and mass matrix $[M]$ have been arranged in symmetric banded matrices. For shell problems we used the eigenvalue solver, described by Bathe and Wilson [13], that employs a subspace iteration method with Sturm sequence checks. This code presents solution of the smaller eigenvalues and corresponding eigenvectors of the generalized eigenproblem. In this code the stiffness matrix $[K]$ and mass matrix $[M]$ are arranged in compacted form and both are positive definite.

3.3 Constitutive Behavior in Composites

The generalized Hooke's relation can be written as

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} \quad (i, j, k, l = 1, 2, 3) \quad (3.34)$$

where

σ_{ij} is the second-order stress tensor.

ϵ_{kl} is the second-order strain tensor.

C_{ijkl} is the fourth-order elasticity tensor.

The 1, 2 and 3 directions form a right-handed orthogonal co-ordinate system. It has been shown that the stress, strain, and elasticity tensors must be symmetrical [118].

i.e.,

$$\sigma_{ij} = \sigma_{ji}, \quad \epsilon_{ij} = \epsilon_{ji} \quad \text{and} \quad C_{ijkl} = C_{klij}$$

Hence, for materials having one plane of symmetry the strain-stress relations can be written as,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{21} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{31} & c_{32} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{54} & c_{55} & 0 \\ c_{61} & c_{62} & c_{63} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \quad (3.35)$$

3.3.1 Constitutive Relations for Orthotropic Plates and Shells

Materials which have three mutually orthogonal planes of elastic symmetry are called orthotropic. It can be noted that for orthotropic materials addressed in this investigation, the following simplification can be made. $c_{16} = c_{61} = 0$, $c_{26} = c_{62} = 0$,

$$c_{36} = c_{63} = 0 \text{ and } c_{45} = c_{54} = 0$$

Hence, the Eqs. (3.35) are further simplified for orthotropic plates and shells, where, the stress $\sigma_3 = 0$ leading to Eqs. (3.36):

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{Bmatrix} \quad (3.36)$$

or

$$\{\sigma\}_{(1,2,3)} = [Q]\{\epsilon\}_{(1,2,3)} \quad (3.37)$$

where

$$\begin{aligned} Q_{11} &= \frac{E_{11}}{(1 - \nu_{12}\nu_{21})} \\ Q_{22} &= \frac{E_{22}}{(1 - \nu_{12}\nu_{21})} \\ Q_{12} &= \frac{\nu_{12}E_{22}}{(1 - \nu_{12}\nu_{21})} \\ Q_{21} &= Q_{12} \\ Q_{66} &= G_{12} \\ \nu_{21}E_{11} &= \nu_{12}E_{22} \\ Q_{44} &= k_1 G_{13} \\ Q_{55} &= k_2 G_{23} \end{aligned} \quad (3.38)$$

The terms k_1 and k_2 are shear correction factors. This equation refers to a specially orthotropic lamina in which the material and geometry axes are aligned.

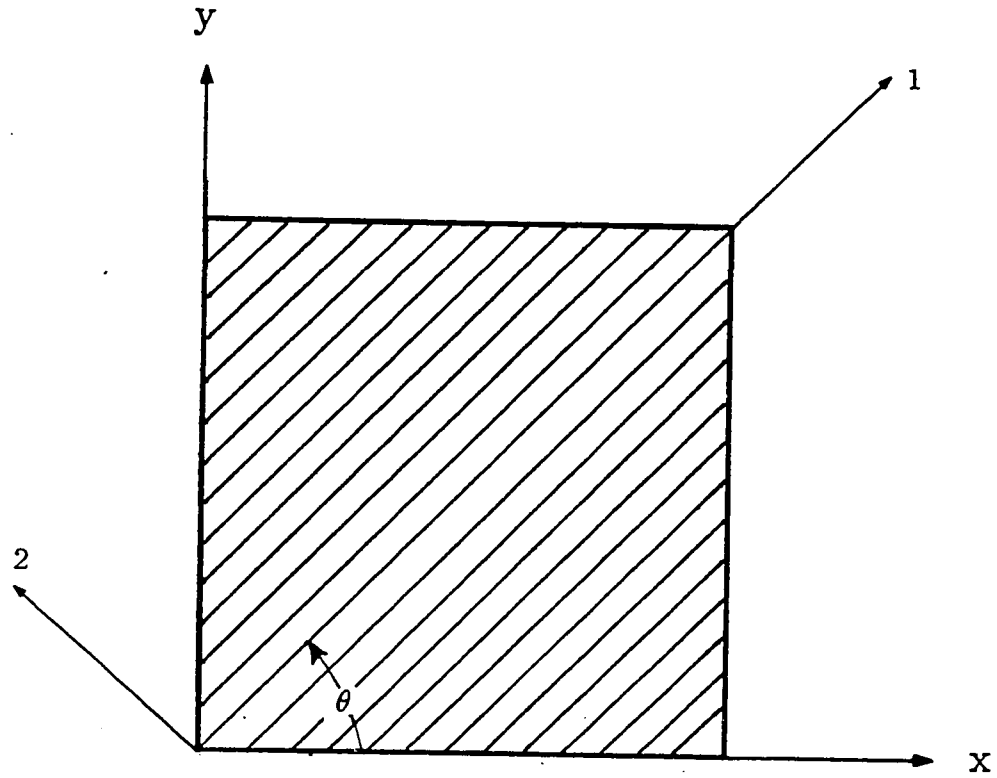


Figure 3.2: Material and geometrical axes

In general, if the principal material axes (1,2,3) are not aligned with the geometric axes (x,y,z) as shown in Fig. 3.2 certain transformations are necessary before an equation of the type (3.37) can be reached. Using the transformation relations for stress and strain,

$$\{\sigma\}_{(1,2,3)} = [T]_{\sigma} \{\sigma\}_{(x,y,z)} \quad (3.39)$$

and

$$\{\epsilon\}_{(1,2,3)} = [T]_{\epsilon} \{\epsilon\}_{(x,y,z)} \quad (3.40)$$

where

$$\{\sigma\}_{(x,y,z)} = \{\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}\}^T \quad (3.41)$$

$$\{\epsilon\}_{(x,y,z)} = \{\epsilon_x, \epsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}\}^T \quad (3.42)$$

In the above expressions the stress and strain transformation matrices $[T]_\sigma$ and $[T]_\epsilon$ are defined as follows:

$$[T]_\sigma = \begin{Bmatrix} c^2 & s^2 & 2cs & 0 & 0 \\ s^2 & c^2 & -2cs & 0 & 0 \\ -cs & cs & c^2 - s^2 & 0 & 0 \\ 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & -s & c \end{Bmatrix} \quad (3.43)$$

$$[T]_\epsilon = \begin{Bmatrix} c^2 & s^2 & cs & 0 & 0 \\ s^2 & c^2 & -cs & 0 & 0 \\ -2cs & 2cs & c^2 - s^2 & 0 & 0 \\ 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & -s & c \end{Bmatrix} \quad (3.44)$$

where

$$c = \cos\theta, \quad s = \sin\theta$$

Substituting Eq. (3.37) then Eq. (3.40) into Eq. (3.39) results in the following,

$$\{\sigma\}_{(x,y,z)} = [T]_\sigma^{-1} [Q] \{\epsilon\}_{(1,2,3)} \quad (3.45)$$

$$\{\sigma\}_{(x,y,z)} = [T]_\sigma^{-1} [Q] [T]_\epsilon \{\epsilon\}_{(x,y,z)} \quad (3.46)$$

this can be written as

$$\{\sigma\}_{(x,y,z)} = [\bar{Q}] \{\epsilon\}_{(x,y,z)} \quad (3.47)$$

The reduced stiffness matrix, $[\bar{Q}]$ for generally orthotropic laminae ([22],[26],[27]),

can be represented as:

$$[\bar{Q}] = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \quad (3.48)$$

where

$$\begin{aligned} \bar{Q}_{11} &= Q_{11}c^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}s^4 \\ \bar{Q}_{22} &= Q_{11}s^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}c^4 \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{12}(c^4 + s^4) \\ \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})c^2s^2 + Q_{66}(c^4 + s^4) \\ \bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})sc^3 + (Q_{12} - Q_{22} + 2Q_{66})s^3c \\ \bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})s^3c + (Q_{12} - Q_{22} + 2Q_{66})sc^3 \\ \bar{Q}_{44} &= Q_{44}c^2 + Q_{55}s^2 \\ \bar{Q}_{55} &= Q_{44}s^2 + Q_{55}c^2 \\ \bar{Q}_{45} &= (Q_{55} - Q_{44})cs \end{aligned} \quad (3.49)$$

3.3.2 Constitutive Representation for Laminates

Having formulated the constitutive relations for an orthotropic laminae we now address the question of the constitutive relations for a laminated plate formed by bonding several laminae together (Fig.3.3). We write Eq.(3.47) again for the k^{th}

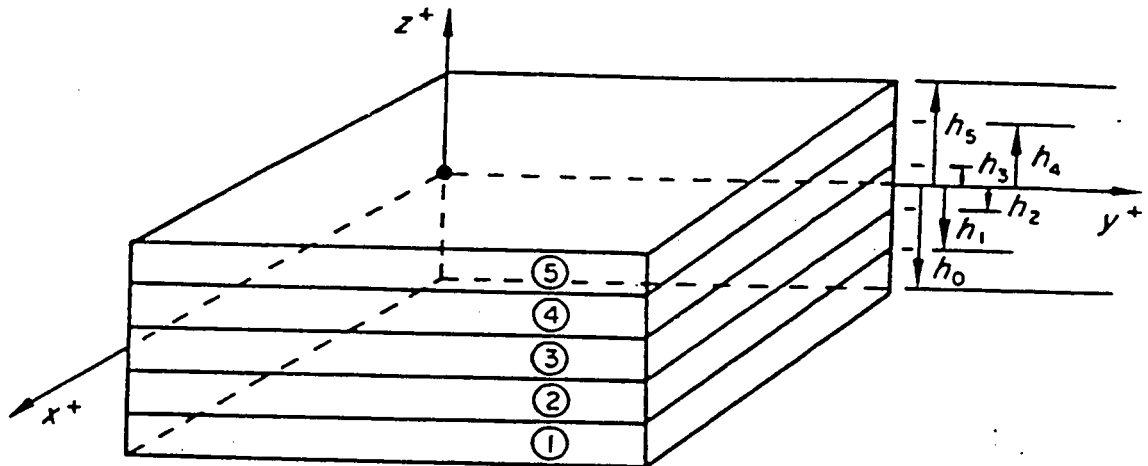


Figure 3.3: Laminated element

laminae

$$\{\sigma\}^{(k)} = [\bar{Q}]^{(k)}\{\epsilon\} \quad (3.50)$$

Strain at the point z can be written as

$$\{\epsilon\} = \{\epsilon^o\} + z\{\kappa\} \quad (3.51)$$

The in-plane mid surface strain $\{\epsilon^o\}$ and the curvature $\{\kappa\}$ are defined as follows:

$$\{\epsilon^o\} = \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} \quad (3.52)$$

$$\{\kappa\} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \quad (3.53)$$

Substituting Eq. (3.51) into Eq. (3.50) gives

$$\{\sigma\}^{(k)} = [\bar{Q}]^{(k)}\{\epsilon^o\} + z[\bar{Q}]^{(k)}\{\kappa\} \quad (3.54)$$

For a laminated plate or shell the in-plane stress resultants are obtained by integrating the stresses along the thickness. i.e.,

$$\begin{aligned} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} &= \sum_{k=1}^{nl} \int_{h_{k-1}}^{h_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^{(k)} dz \\ &= \sum_{k=1}^{nl} \left(\int_{h_{k-1}}^{h_k} [\bar{Q}]^{(k)} \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} dz + \int_{h_{k-1}}^{h_k} [\bar{Q}]^{(k)} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} z dz \right) \end{aligned} \quad (3.55)$$

where nl is the number of layers in the laminated plate, h_k and h_{k-1} are the z -coordinates of the upper and lower surfaces of the k^{th} laminae. It should be noted that h_k is negative below the mid-plane and positive above the mid-plane.

Alternatively we write,

$$\{N_p\} = [A]\{\epsilon^o\} + [B]\{\kappa\} \quad (3.56)$$

where

$$A_{ij} = \sum_{k=1}^{nl} \bar{Q}_{ij}^{(k)} (h_k - h_{k-1}) \quad (3.57)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{nl} \bar{Q}_{ij}^{(k)} (h_k^2 - h_{k-1}^2) \quad (3.58)$$

Similarly, for bending moment M_b we write.

$$\begin{aligned} \begin{Bmatrix} M_{bx} \\ M_{by} \\ M_{bxy} \end{Bmatrix} &= \sum_{k=1}^{nl} \int_{h_{k-1}}^{h_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^{(k)} z dz \\ &= \sum_{k=1}^{nl} \left(\int_{h_{k-1}}^{h_k} [\bar{Q}]^{(k)} \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} z dz + \int_{h_{k-1}}^{h_k} [\bar{Q}]^{(k)} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} z^2 dz \right) \end{aligned} \quad (3.59)$$

or

$$\{M_b\} = [B]\{\epsilon^o\} + [D]\{\kappa\} \quad (3.60)$$

where the elements in $[B]$ are defined by Eq. (3.58) and elements in $[D]$ are given by

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{nl} \bar{Q}_{ij}^{(k)} (h_k^3 - h_{k-1}^3) \quad (3.61)$$

The shear force results can be obtained in a similar manner by integrating along the thickness. i.e.,

$$\begin{Bmatrix} V_{sx} \\ V_{sy} \end{Bmatrix} = \sum_{k=1}^{nl} \left(\int_{h_{k-1}}^{h_k} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^{(k)} dz \right) \quad (3.62)$$

In finding the transverse shear forces V_{sx} , V_{sy} , it is assumed that the transverse shear stresses are distributed parabolically across the laminate thickness. However, in

spite of the discontinuities at the interfaces between laminae, a continuous function is assumed [121] as follows.

$$f(z) = \frac{5}{4} \left[1 - \left(\frac{z}{h/2} \right)^2 \right] \quad (3.63)$$

Using the stress strain relations (Eq.(3.50)), Eq.(3.62) can be rewritten as

$$\begin{Bmatrix} V_{sx} \\ V_{sy} \end{Bmatrix} = \sum_{k=1}^{nl} \left(\int_{h_{k-1}}^{h_k} [\bar{Q}]^{(k)} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(k)} dz \right) \quad (3.64)$$

After integrating we obtain,

$$\begin{Bmatrix} V_{sx} \\ V_{sy} \end{Bmatrix} = \begin{bmatrix} A_{s44} & A_{s45} \\ A_{s45} & A_{s55} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (3.65)$$

or, simply

$$\{V_s\} = [A_s] \{\epsilon_s\} \quad (3.66)$$

where

$$A_{sij} = \frac{5}{4} \sum_{k=1}^{nl} \bar{Q}_{ij}^{(k)} \left[h_k - h_{k-1} - \frac{4}{3} \frac{(h_k^3 - h_{k-1}^3)}{h^2} \right] \quad (i, j = 4, 5) \quad (3.67)$$

Alternatively, the laminate stiffness coefficients may be written as

$$\begin{aligned} (A_{ij}, B_{ij}, D_{ij}) &= \sum_{k=1}^{nl} \left(\int_{h_{k-1}}^{h_k} \bar{Q}_{ij}^{(k)} (1, z, z^2) dz \right) \\ A_{sij} &= k_s^2 A_{ij} \quad (i, j = 4, 5) \end{aligned} \quad (3.68)$$

Here k_s is the shear correction factor (taken to be 5/6 for a homogeneous cross-section).

Combining Eqs. (3.56), (3.60) and (3.66) and assembling in matrix form gives

$$\begin{Bmatrix} \{N_p\} \\ \{M_b\} \\ \{V_s\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [0] \\ [B] & [D] & [0] \\ [0] & [0] & [A_s] \end{bmatrix} \begin{Bmatrix} \{\epsilon_o\} \\ \{\kappa\} \\ \{\epsilon_s\} \end{Bmatrix} \quad (3.69)$$

where:

$[A]$ is the extensional stiffness matrix,

$[D]$ is the flexural stiffness matrix,

$[B]$ is the bending stretching coupling matrix,

$[A_s]$ is the shear stiffness matrix.

Equations (3.69) represents the full system of constitutive equations necessary to study orthotropic laminated plates.

As the above equations show, a laminated plate or shell exhibits a variety of couplings, namely between in-plane loading and out-of-plane deflection, and also between applied bending load and twisting response. Such couplings complicate the analysis of laminated plates and shells and also reduce the overall stiffness of the laminate. Elimination of coupling effects not only overcomes the above difficulties in laminated structures but also eliminates the temperature effect on warping of the laminate. It is shown that a selection of a suitable ply stacking, by balancing the laminate about its mid-plane, by a suitable choice of thickness ratio and laminate stacking sequence, and finally, by controlling the angle of fibre-directions, the coupling in laminated structures can be completely eliminated. Therefore, several simplifications of Eqs.(3.69) are possible:

- 1- Balanced laminate, consider the balanced laminate which contains an equal number of laminae of $+\theta^\circ$ and $-\theta^\circ$ fiber orientation. for such a laminate the shear coupling stiffness terms, A_{16} , and A_{26} , vanish. i.e, $A_{16} = A_{26} = 0$.
- 2- Symmetric laminate, if the laminae within the laminate are positioned symmetrically with respect to the laminate mid-plane, then the coupling terms, B_{ij} , vanish. i.e, $B_{ij} = 0$ ($i, j = 1, 2, 6$).

It is important to note that while the inplane shear and normal terms are uncoupled for the symmetric and balanced laminate the twisting-bending terms (D_{16}, D_{26}) do not vanish.

References [27],[34],[73] and [72] discuss some aspects of the elimination of the coupling in the laminated plates and shells.

3.4 Laminated Anisotropic Plate Theory

3.4.1 Introduction

The classical plate theory or Kirchhoff theory ignores the transverse shear deformation effect. As we discussed earlier in chapter 1 the effect of transverse shear deformation on laminated plates is very important. Thick plate theory or Mindlin plate theory [115] which takes into account shear deformation effect, provides a more realistic alternative to Kirchhoff theory.

Recent developments in the analysis of multilayered composite plate indicate the pronounced effect of transverse shears. For this type of construction Mindlin theory is best suited.

A number of shear deformation theories for composite plates have been proposed to date. The first generalized theory for laminated anisotropic plate is called "YNS" theory which has been given in reference [66]. The plate assumptions used in the formulations of plate theory in this study are essentially those introduced by Mindlin [115].

3.4.2 Formulation of Mindlin Plate Theory

Some of the important assumptions used in the development of the Mindlin plate theory are:

- (1)- The deflection of the midsurface of the plate is small.
- (2)- Transverse normal stresses are negligible.

(3)- Normals to the midsurface of the plate remains straight after deformation but not necessarily normal.

Based on the above assumption, the displacement field for a first-order shear deformation theory ([45],[75],etc.) can be expressed as

$$\begin{aligned} u(x, y, z, t) &= u^o(x, y, t) + z\beta_x(x, y, t) \\ v(x, y, z, t) &= v^o(x, y, t) + z\beta_y(x, y, t) \\ w(x, y, z, t) &= w^o(x, y, t) \end{aligned} \quad (3.70)$$

where u , v and w are displacement components in the x , y and z directions, respectively, u^o , v^o and w^o are the displacements of the midsurface, β_x and β_y are the normal rotations in xz and zy planes respectively as shown in Fig. 3.4.

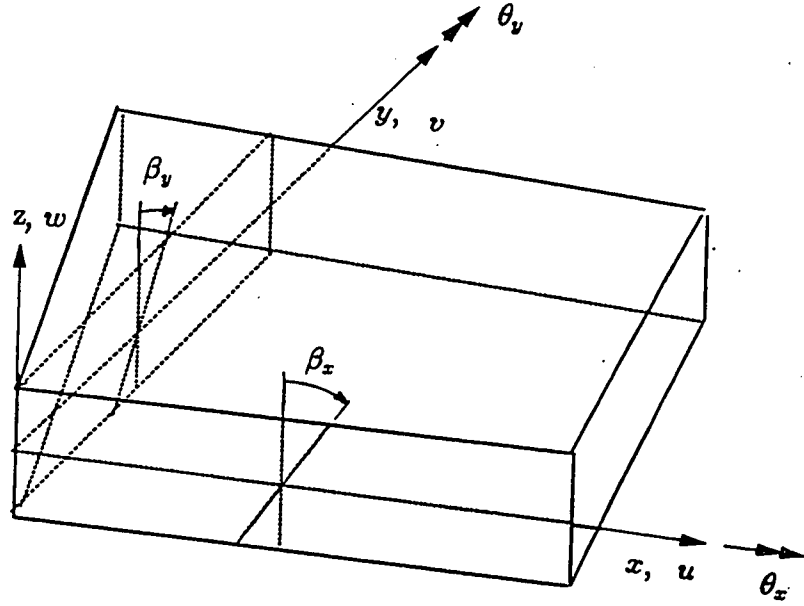


Figure 3.4: Mindlin plate model

3.4.3 Strain-Displacement Relations

Using the infinitesimal strain definition

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{Bmatrix} \quad (3.71)$$

then strain at point z can be expressed in the form

$$\{\epsilon\} = \begin{Bmatrix} \frac{\partial u^o}{\partial x} + z \frac{\partial \beta_x}{\partial x} \\ \frac{\partial v^o}{\partial y} + z \frac{\partial \beta_y}{\partial y} \\ (\frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x}) + z(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}) \\ \beta_x + \frac{\partial w^o}{\partial x} \\ \beta_y + \frac{\partial w^o}{\partial y} \end{Bmatrix} \quad (3.72)$$

The in-plane strain components may be written

$$\{\epsilon\} = \{\epsilon_p\} + \{\epsilon_f\} \quad (3.73)$$

where $\{\epsilon_p\}$ and $\{\epsilon_f\}$ are the extensional and flexural parts of the in-plane strains which can be represented as

$$\{\epsilon_p\} = \left\{ \frac{\partial u^o}{\partial x}, \frac{\partial v^o}{\partial y}, \left(\frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x} \right) \right\}^T$$

$$\{\epsilon_f\} = z\{\kappa\} = z \left\{ \frac{\partial \beta_x}{\partial x}, \frac{\partial \beta_y}{\partial y}, \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \right) \right\}^T$$

where $\{\kappa\}$ is the vector of curvatures. Finally, the transverse shear strains are given as

$$\begin{aligned} \{\epsilon_s\} &= \{\gamma_{xz}, \gamma_{yz}\}^T \\ &= \left\{ \left(\frac{\partial w^o}{\partial x} + \beta_x \right), \left(\frac{\partial w^o}{\partial y} + \beta_y \right) \right\}^T \end{aligned} \quad (3.74)$$

3.4.4 Constitutive Relations for Plates

A detailed derivation of the constitutive relations has been given in section 3.3. In this section we discuss, briefly, the constitutive relations for layered plates. Constitutive relations for any layer in the (x,y) system are given by,

$$\{\sigma\}_{(x,y,z)}^{(k)} = [\bar{Q}]^{(k)} \{\epsilon\}_{(x,y,z)} \quad (3.75)$$

where $[\bar{Q}]^{(k)}$ is the reduced stiffness matrix of the k^{th} layer of a layered material. Introducing the stress and moment resultants per unit length, we can write,

$$(N_x, N_y, N_{xy}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x, \sigma_y, \tau_{xy})^{(k)} dz \quad (3.76)$$

$$(V_{sx}, V_{sy}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\tau_{xz}, \tau_{yz})^{(k)} dz \quad (3.77)$$

$$(M_x, M_y, M_{xy}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x, \sigma_y, \tau_{xy})^{(k)} z dz \quad (3.78)$$

Finally, the constitutive relations in terms of the resultants and displacements can be written as,

$$\begin{Bmatrix} \{N_p\} \\ \{M_b\} \\ \{V_s\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [0] \\ [B] & [D] & [0] \\ [0] & [0] & [A_s] \end{bmatrix} \begin{Bmatrix} \{\epsilon_p\} \\ \{\epsilon_f\} \\ \{\epsilon_s\} \end{Bmatrix} \quad (3.79)$$

The laminate stiffnesses A_{ij} , B_{ij} and D_{ij} are given by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij}^{(k)} (1, z, z^2) dz \quad (i, j = 1, 2, 6) \quad (3.80)$$

$$(A_{sij}) = -k_s \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij}^{(k)} dz \quad (i, j = 4, 5) \quad (3.81)$$

The stiffness coefficient $\bar{Q}_{ij}^{(k)}$ depends on the material properties and orientation of the k^{th} layer and was given in section 3.3. The parameter k_s is the shear correction coefficient.

It should be noted that variable thicknesses can be accounted for by considering variable limits of integration. Thus,

$$A_{ij} = A_{ij}(x, y), \quad B_{ij} = B_{ij}(x, y), \quad D_{ij} = D_{ij}(x, y), \quad A_{sij} = A_{sij}(x, y)$$

3.4.5 Equation of Motion for Plates

The equations of motion according to the "YNS" Theory [66] are,

$$\begin{aligned} N_{x,x} + N_{xy,y} + R_x^b &= P\ddot{u} \\ N_{xy,x} + N_{y,y} + R_y^b &= P\ddot{v} \\ V_{sx,x} + V_{sy,y} + R_z^b &= P\ddot{w} \\ M_{x,x} + M_{xy,y} - V_{sx} &= I\ddot{\theta}_x - R_x^b \\ M_{xy,x} + M_{y,y} - V_{sy} &= I\ddot{\theta}_y - R_y^b \end{aligned} \quad (3.82)$$

where P and I are the normal and rotary inertia coefficients,

$$(P, I) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z^2) \rho^{(m)} dz \quad (3.83)$$

$\rho^{(m)}$ being the material density of layer m and the body forces are given by

$$(R_x^b, R_y^b, R_z^b) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (F_x^b, F_y^b, F_z^b) dz \quad (3.84)$$

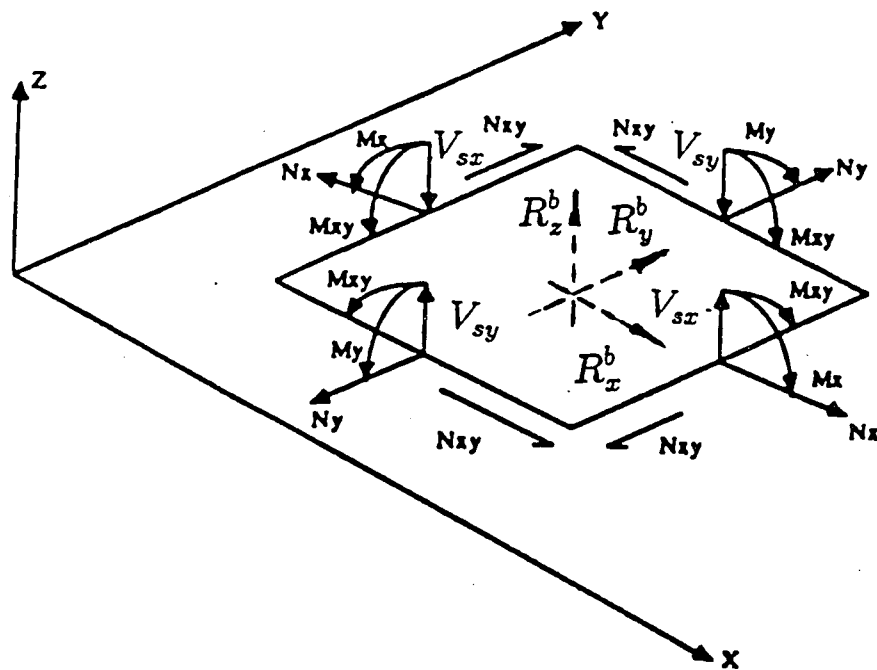


Figure 3.5: Equilibrium of plate

3.5 Finite Element Model for Plates

For the plate element in the general case, five nodal displacements ($u^o, v^o, w^o, \beta_x, \beta_y$) may be considered. The displacement and normal rotations at any point in a typical element can be expressed as

$$\begin{Bmatrix} u^o \\ v^o \\ w^o \\ \beta_y \\ \beta_x \end{Bmatrix} = \sum_{i=1}^{nn} \begin{bmatrix} N_i & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & N_i & 0 & 0 \\ 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} u_i^o \\ v_i^o \\ w_i^o \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} \quad (3.85)$$

or

$$\{u\} = \sum_{i=1}^{nn} [N_i] \{\Delta_i\} \quad (3.86)$$

where, Δ_i are the nodal values u^o, v^o, w^o, β_y , and β_x respectively, at the i^{th} node of the element, and nn is the number of nodes in each element. The N_i are shape functions associated with node i and expressed in terms of the natural local coordinate system (ξ, η) i.e, $N_i \equiv N_i(\xi, \eta)$.

In an isoparametric formulation, the coordinates of any point x and y in the element can be described by the same interpolation function $N_i(\xi, \eta)$, thus

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \sum_{i=1}^{nn} \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \end{Bmatrix} \quad (3.87)$$

where x_i and y_i are the coordinates of the i^{th} node of the element.

The strain displacement relation can be expressed as

$$\{\epsilon\} = [L]\{u\} \quad (3.88)$$

substituting Eq. (3.86) in Eq. (3.88) we obtain,

$$\{\epsilon\} = [L][N]\{\Delta_i\}$$

this can be written as

$$\{\epsilon\} = \sum_{i=1}^{nn} [B_i] \{\Delta_i\} \quad (3.89)$$

where,

$$[B_i] = \begin{bmatrix} N_{i,x} & 0 & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 & 0 \\ N_{i,y} & N_{i,x} & 0 & 0 & 0 \\ 0 & 0 & 0 & N_{i,x} & 0 \\ 0 & 0 & 0 & 0 & N_{i,y} \\ 0 & 0 & 0 & N_{i,y} & N_{i,x} \\ 0 & 0 & N_{i,x} & N_i & 0 \\ 0 & 0 & N_{i,y} & 0 & N_i \end{bmatrix} \quad (3.90)$$

The strain tensor can be split in three parts as follows:

- -the in-plane strain

$$\{\epsilon_p\} = \sum_{i=1}^{nn} [B_{pi}] \{\Delta_i\} \quad (3.91)$$

where $[B_{pi}]$ is the in-plane displacement matrix, given by

$$[B_{pi}] = \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix} \quad (3.92)$$

- -the flexural strain

$$\{\kappa\} = \sum_{i=1}^{nn} [B_{bi}] \{\Delta_i\} \quad (3.93)$$

where $[B_{bi}]$ is the bending-displacement matrix, written in the form

$$[B_{bi}] = \begin{bmatrix} 0 & N_{i,x} & 0 \\ 0 & 0 & N_{i,y} \\ 0 & N_{i,y} & N_{i,x} \end{bmatrix} \quad (3.94)$$

- -the shear strain

$$\{\epsilon_s\} = \sum_{i=1}^{nn} [B_{si}] \{\Delta_i\} \quad (3.95)$$

where $[B_{s,i}]$ is the shear-displacement matrix, expressed as

$$[B_{s,i}] = \begin{bmatrix} N_{i,x} & N_i & 0 \\ N_{i,y} & 0 & N_i \end{bmatrix} \quad (3.96)$$

The shape functions N_i are given in natural coordinate system. Their derivatives $N_{i,x}$ and $N_{i,y}$ in cartesian coordinates, can be obtained by the chain rule as follows:

$$\begin{Bmatrix} N_{i,x} \\ N_{i,y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{Bmatrix} \quad (3.97)$$

where, $[J]^{-1}$ is the inverse of the Jacobian matrix $[J]$, the Jacobian matrix of the element is given by

$$[J] = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix} \quad (3.98)$$

or

$$[J] = \sum_{i=1}^{nn} \begin{bmatrix} N_{i,\xi} & N_{i,\eta} \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \end{Bmatrix} \quad (3.99)$$

Hence, the inverse of Jacobian matrix $[J]$ may be obtained as,

$$[J]^{-1} = \begin{bmatrix} \xi_{,x} & \eta_{,x} \\ \xi_{,y} & \eta_{,y} \end{bmatrix} \quad (3.100)$$

and the element stiffness matrix and mass matrix of the element are given as,

$$[K]^{(m)} = \int_{-1}^1 \int_{-1}^1 [B]^T [C] [B] \det[J] d\xi d\eta \quad (3.101)$$

$$[M]^{(m)} = \int_{-1}^1 \int_{-1}^1 [N]^T \rho [N] \det[J] d\xi d\eta \quad (3.102)$$

where $[B]$, $[C]$ and $[N]$ are the strain-displacement, stress-strain and element shape matrices for the plate element. Finally, the total stiffness matrix and the total mass

matrix are given by

$$\begin{aligned} [K] &= \sum_{m=1}^{ne} [K]^{(m)} \\ [M] &= \sum_{m=1}^{ne} [M]^{(m)} \end{aligned} \quad (3.103)$$

where ne is the number of elements in the plate.

3.6 Laminated Anisotropic Shell Theory

3.6.1 Introduction

As mentioned in the chapter 1 there are three different approaches to formulating shell elements. As discussed elsewhere in this thesis the effect of shear deformation on thick and composite shells is significant in comparison to thin homogenous shells. Therefore, for the above reasons a degenerate, Mindlin-type, curved element has been adopted in this study for finite element formulation of shell structures.

3.6.2 Formulation of Shell Element

Coordinate System

In the analysis of shell structures, by the degenerate curved shell elements, we need to define four coordinate systems as shown in Fig.3.6.

1- Global Coordinate System (x, y, z)

A global cartesian coordinate system is used to define the geometry of the structure (Fig.3.6). Nodal coordinates, displacements, global stiffness matrix, and applied forces are all defined in this coordinate system.

2- Nodal Coordinate System (V_{1i}, V_{2i}, V_{3i})

The nodal cartesian coordinate system is attached at each nodal point of the shell element and its origin is located at the shell midsurface. This local coordinate system is used to define the direction of the normal at node i . The vector $\vec{V}_{3i}$ is

constructed from the nodal coordinates of the top and bottom surfaces at node i , thus

$$\vec{V}_{3i} = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} - \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom} = \begin{Bmatrix} \Delta x_i \\ \Delta y_i \\ \Delta z_i \end{Bmatrix} \quad (3.104)$$

The vector $\vec{V}_{1i}$ is perpendicular to $\vec{V}_{3i}$ and parallel to the xz -plane,

$$\vec{V}_{1i} = \hat{i} \times \vec{V}_{3i} \quad (3.105)$$

$\vec{V}_{2i}$ is a vector orthogonal to both $\vec{V}_{3i}$ and $\vec{V}_{1i}$, therefore,

$$\vec{V}_{2i} = \vec{V}_{3i} \times \vec{V}_{1i} \quad (3.106)$$

The unit vectors in the nodal coordinate system are,

$$\hat{v}_{1i} = \frac{\vec{V}_{1i}}{|\vec{V}_{1i}|}, \quad \hat{v}_{2i} = \frac{\vec{V}_{2i}}{|\vec{V}_{2i}|}, \quad \hat{v}_{3i} = \frac{\vec{V}_{3i}}{|\vec{V}_{3i}|} \quad (3.107)$$

The nodal coordinate system is used not only to define the direction of the normal at each node but also the two rotations α_1, α_2 are defined about directions $\vec{V}_{2i}, \vec{V}_{1i}$ respectively in this coordinate system.

3-Natural Coordinate System (ξ, η, ζ)

The natural coordinate system is used to define shape function N_i . In the natural coordinate system ξ, η are curvilinear coordinates in the midsurface of the shell element and ζ a linear coordinate in the thickness direction. It is to be noted that ζ is only approximately perpendicular to the shell midsurface. In this system the coordinates ξ, η and ζ vary between -1 and +1.

4-Local Coordinate System (x', y', z')

This cartesian coordinate system is defined at the sampling points to define local

stresses and strains within the shell element. The direction z' is taken to be perpendicular to the surfaces $\zeta = \text{constant}$. However, it is to be noted that orthogonal axes x', y', z' do not coincide with natural coordinates ξ, η, ζ although x', y' are in the ξ, η plane. The vector $\vec{V}'_3$ defines the z' direction and it can be obtained from the cross product of the vectors which are tangential to the ξ and η direction. Thus,

$$\vec{V}'_3 = \begin{Bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{Bmatrix} \times \begin{Bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{Bmatrix} \quad (3.108)$$

The vector $\vec{V}'_1$ which defines the x' direction can be taken tangent to the ξ -direction at the sampling point,

$$\vec{V}'_1 = \begin{Bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{Bmatrix} \quad (3.109)$$

Finally the vector $\vec{V}'_2$ which defines the y' direction can be simply obtained by the cross product of $\vec{V}'_3$ and $\vec{V}'_1$

$$\vec{V}'_2 = \vec{V}'_3 \times \vec{V}'_1 \quad (3.110)$$

To evaluate stresses and strains in the local cartesian coordinates (x', y', z') we need a further transformation matrix between global and local coordinates. This is obtained by constructing a matrix of unit vectors in the local coordinate system, i.e.

$$[\Theta] = [\hat{V}_{x'}, \hat{V}_{y'}, \hat{V}_{z'}] \quad (3.111)$$

where $\hat{V}_{x'}$, $\hat{V}_{y'}$, and $\hat{V}_{z'}$ are unit vectors in the direction of the x' , y' and z' axes respectively.

The alternative formula for the direction cosine matrix $[\Theta]$ which relates the transformation between the local and the global coordinate systems can be written as,

$$[\Theta] = \begin{bmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & \frac{\partial x}{\partial z'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & \frac{\partial y}{\partial z'} \\ \frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial z'} \end{bmatrix} \quad (3.112)$$

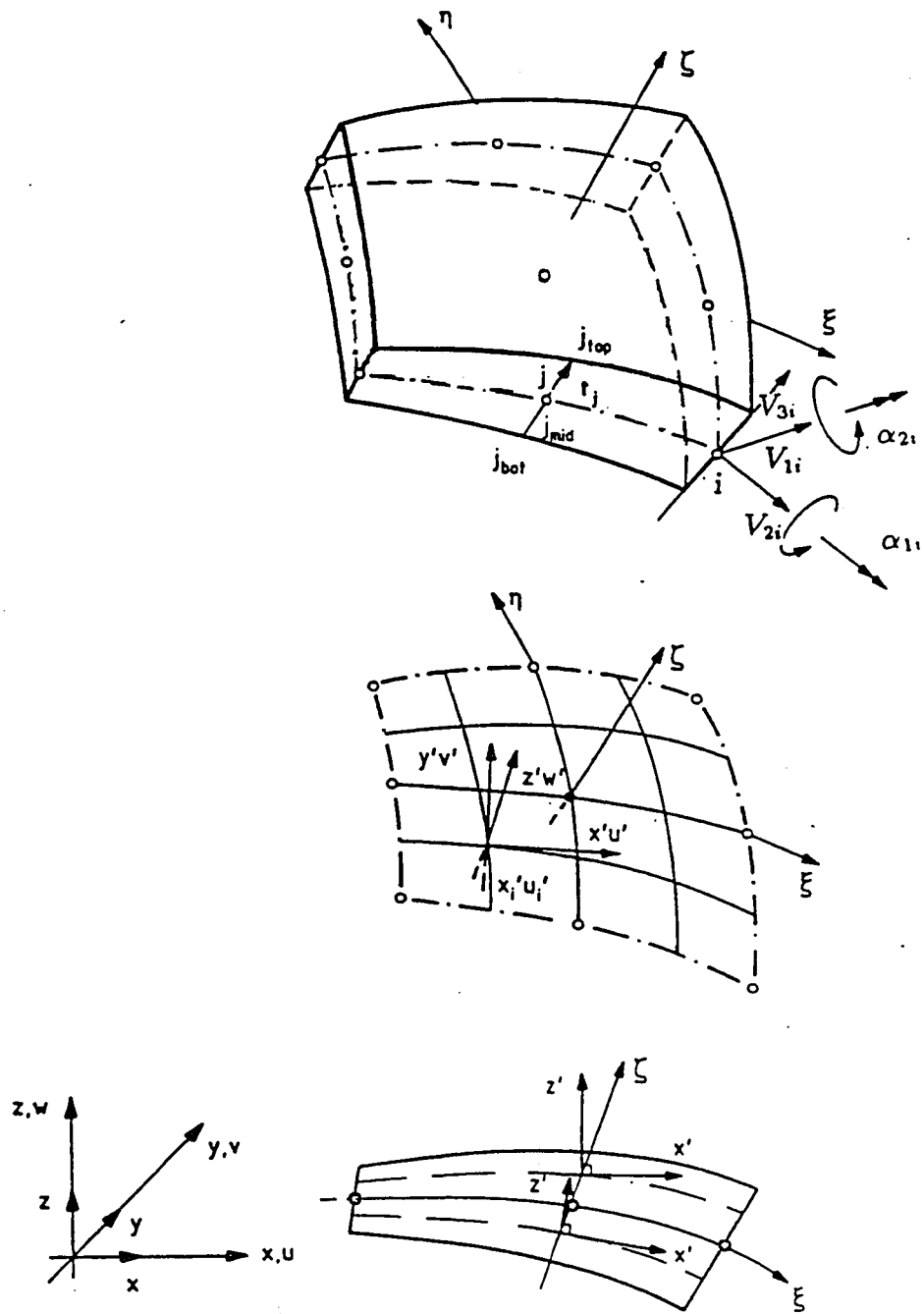


Figure 3.6: Coordinate systems for shell element

3.6.3 Geometric Definition of the Shell Element

The geometry of the element is described by a set of pairs of points i_{top} , i_{bottom} , each with given global coordinates. The global coordinates of any point of the shell can be written as,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{nn} N_i(\xi, \eta) \frac{1+\zeta}{2} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} + \sum_{i=1}^{nn} N_i(\xi, \eta) \frac{1-\zeta}{2} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom} \quad (3.113)$$

Here, the shape function $N_i(\xi, \eta)$ takes a value of unity at the node i and zero at all other nodes and nn = the number of nodes in midsurface of each element. Equation (3.113) can be further simplified as follows:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{nn} N_i(\xi, \eta) \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{mid} + \sum_{i=1}^{nn} N_i(\xi, \eta) \frac{\zeta}{2} \vec{V}_{3i} \quad (3.114)$$

where

$$\begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{mid} = \frac{1}{2} \left(\begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} + \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom} \right) \quad (3.115)$$

and

$$\vec{V}_{3i} = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} - \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom} = \begin{Bmatrix} \Delta x_i \\ \Delta y_i \\ \Delta z_i \end{Bmatrix} \quad (3.116)$$

In the case of relatively thin shells, the vector $\vec{V}_{3i}$ may be replaced by a unit vector $\hat{v}_{3i}$, in the direction perpendicular to the midsurface. Now the last term can be written simply as

$$\sum_{i=1}^{nn} N_i(\xi, \eta) \zeta \frac{t_i}{2} \hat{v}_{3i}$$

where t_i is the shell thickness at the node i .

3.6.4 Displacement Field for Shell Element

Using the shape functions the displacement field can be obtained as follows ([1], [4], [14], [55] etc.):

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{nn} N_i(\xi, \eta) \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix}_{mid} + \sum_{i=1}^{nn} N_i(\xi, \eta) \frac{\zeta t_i}{2} \{ \hat{v}_{1i} \quad -\hat{v}_{2i} \} \begin{Bmatrix} \alpha_{1i} \\ \alpha_{2i} \end{Bmatrix} \quad (3.117)$$

where $\hat{v}_{1i}$ and $\hat{v}_{2i}$ are the unit vectors in the directions of $\vec{V}_{1i}$, $\vec{V}_{2i}$ respectively, and u_i , v_i and w_i are the displacements at the i th node. Furthermore, α_{1i} , α_{2i} are the two rotations about direction $\vec{V}_{2i}$, $\vec{V}_{1i}$ respectively. Equation (3.117) may be expressed in matrix form. i.e.,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{nn} \begin{bmatrix} N_i & 0 & 0 & N_i \zeta \frac{t_i}{2} v_{1i}^x & -N_i \zeta \frac{t_i}{2} v_{2i}^x \\ 0 & N_i & 0 & N_i \zeta \frac{t_i}{2} v_{1i}^y & -N_i \zeta \frac{t_i}{2} v_{2i}^y \\ 0 & 0 & N_i & N_i \zeta \frac{t_i}{2} v_{1i}^z & -N_i \zeta \frac{t_i}{2} v_{2i}^z \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \alpha_{1i} \\ \alpha_{2i} \end{Bmatrix} \quad (3.118)$$

where v_{1i}^x , v_{2i}^x are x components of unit vectors $\hat{v}_{1i}$ and $\hat{v}_{2i}$ respectively.

In a more general form the above equation can be written as

$$\{u\} = [N]\{\Delta\} \quad (3.119)$$

where

$$[N] = [[N_1], \dots, [N_k], \dots, [N_n]]$$

and

$$\{\Delta\} = \{\{\Delta_1\}^T, \dots, \{\{\Delta_i\}^T, \dots, \{\{\Delta_n\}^T, \dots, \}$$

is the vector of the element nodal variables, where

$$\{\Delta_i\}^T = \{u_i, v_i, w_i, \alpha_{1i}, \alpha_{2i}\}$$

3.6.5 Definition of Strains

Once the displacement field for the element is known, the local coordinate system is the most convenient system for expressing the stress and strain components in shell structures. Thus, the strain components are given by

$$\{\epsilon'\} = \begin{Bmatrix} \epsilon_{x'} \\ \epsilon_{y'} \\ \gamma_{x'y'} \\ \gamma_{x'z'} \\ \gamma_{y'z'} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u'}{\partial x'} \\ \frac{\partial v'}{\partial y'} \\ \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} \\ \frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial x'} \\ \frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial y'} \end{Bmatrix} \quad (3.120)$$

or

$$\{\epsilon'\} = [L']\{u'\} \quad (3.121)$$

where $[L']$ is linear operator matrix given by

$$[L'] = \begin{bmatrix} \frac{\partial}{\partial x'} & 0 & 0 \\ 0 & \frac{\partial}{\partial y'} & 0 \\ \frac{\partial}{\partial y'} & \frac{\partial}{\partial x'} & 0 \\ \frac{\partial}{\partial z'} & 0 & \frac{\partial}{\partial x'} \\ 0 & \frac{\partial}{\partial z'} & \frac{\partial}{\partial y'} \end{bmatrix} \quad (3.122)$$

and

$$\{u'\} = \begin{Bmatrix} u' \\ v' \\ w' \end{Bmatrix} \quad (3.123)$$

Using the same procedure as followed earlier in obtaining Eq. (3.119) we write,

$$\{u'\} = [N]\{\Delta'\} \quad (3.124)$$

Substitution of Eq.(3.124) in Eq.(3.121) yields,

$$\{\epsilon'\} = [B]\{\Delta'\} \quad (3.125)$$

where

$$[B] = [L'] [N] \quad (3.126)$$

The local derivatives can be obtained from the global derivatives by the following operation,

$$\begin{bmatrix} \frac{\partial u'}{\partial x'} & \frac{\partial v'}{\partial x'} & \frac{\partial w'}{\partial x'} \\ \frac{\partial u'}{\partial y'} & \frac{\partial v'}{\partial y'} & \frac{\partial w'}{\partial y'} \\ \frac{\partial u'}{\partial z'} & \frac{\partial v'}{\partial z'} & \frac{\partial w'}{\partial z'} \end{bmatrix} = [\Theta]^T \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} [\Theta] \quad (3.127)$$

where $[\Theta]$ is given by Eq.(3.111) or Eq.(3.112).

The global derivatives of the displacements u , v , and w are given by,

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial v}{\partial \xi} & \frac{\partial w}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} & \frac{\partial w}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} & \frac{\partial v}{\partial \zeta} & \frac{\partial w}{\partial \zeta} \end{bmatrix} \quad (3.128)$$

where $[J]$ is the Jacobian matrix, i.e.,

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \quad (3.129)$$

3.6.6 Definition of Stresses

Using the assumption of zero stress in the direction normal to the shell midsurface,

($\sigma_{z'} = 0$) the stress components in the local system are:

$$\{\sigma'\} \begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \\ \tau_{x'z'} \\ \tau_{y'z'} \end{Bmatrix} = [C](\{\epsilon'\} - \{\epsilon'_o\}) + \{\sigma'_o\} \quad (3.130)$$

where $\{\epsilon'\}$ is defined in Eq.(3.125), $\{\epsilon'_o\}$ and $\{\sigma'_o\}$ are the initial strain vector and initial stress vector, respectively. The elasticity matrix $[C]$ in case of shell element is a 5×5 matrix, which is given in section 3.3.

3.6.7 Equations of Motion For Shells

The principle of virtual displacements may be used to formulate general equations of motion for shell structures. According to the principle of virtual work, the total internal virtual work is equal to the total external virtual work; i.e.,

$$\begin{aligned} \sum_m \int_{v(m)} \{\bar{\epsilon}\}^T \{\sigma\} dv &= \sum_m \int_{v(m)} \{\bar{u}\}^T \{f^b\} dv \\ &- \sum_m \int_{v(m)} \mu_d \{\bar{u}\}^T \{\dot{u}\} dv \\ &- \sum_m \int_{v(m)} \{\bar{u}\}^T \rho \{\ddot{u}\} dv \end{aligned} \quad (3.131)$$

Substituting for strains and displacements relations

$$\{\sigma\} = [C]\{\epsilon\}$$

$$\{\epsilon\} = [B]\{\Delta\}$$

$$\{u\} = [N]\{\Delta\}$$

in Eq. (3.131), we obtain

$$\begin{aligned} \{\bar{\Delta}\}^T \left(\sum_m \int_{v(m)} [B]^T [C] [B] dv \right) \{\bar{\Delta}\} &= \{\bar{\Delta}\}^T \left(\sum_m \int_{v(m)} [N]^T \{f^b\} dv \right. \\ &- \sum_m \int_{v(m)} [N]^T \mu_d [N] \{\dot{\Delta}\} dv \\ &- \left. \sum_m \int_{v(m)} [N]^T \rho [N] \{\ddot{\Delta}\} dv \right) \end{aligned} \quad (3.132)$$

This could be written in a convenient form as follows:

$$[M]\{\ddot{\Delta}\} + [C_d]\{\dot{\Delta}\} + [K]\{\Delta\} = \{R_B\} \quad (3.133)$$

where the mass matrix $[M]$, the damping matrix $[C_d]$, the stiffness matrix $[K]$ and the force vector $\{R_B\}$ are defined as

$$[M] = \sum_m \int_{v(m)} \{N\}^T \rho \{N\} dv \quad (3.134)$$

$$[C_d] = \sum_m \int_{v(m)} [N]^T \mu_d [N] dv \quad (3.135)$$

$$[K] = \sum_m \int_{v(m)} [B]^T [C] [B] dv \quad (3.136)$$

$$\{R_B\} = \sum_m \int_{v(m)} [N]^T \{f^b\} dv \quad (3.137)$$

Using an isoparametric mapping for element stiffness and mass matrices we can write,

$$[K]^m = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [B]^T [C] [B] \det[J] d\xi d\eta d\zeta \quad (3.138)$$

$$[M]^m = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [N]^T \rho [N] \det[J] d\xi d\eta d\zeta \quad (3.139)$$

The individual stiffness and mass matrices for all the finite elements are evaluated numerically using Gaussian quadrature weighted coefficients. i.e,

$$[K]^m = \sum_{i=1}^{ng} \sum_{j=1}^{ng} \sum_{k=1}^{ng} H_i H_j H_k [B]^T [C] [B] \det[J] \quad (3.140)$$

$$[M]^m = \sum_{i=1}^{ng} \sum_{j=1}^{ng} \sum_{k=1}^{ng} H_i H_j H_k [N]^T \rho [N] \det[J] \quad (3.141)$$

in which H is the weight coefficient of the Gaussian quadrature, ng is the order of integration; i.e, the number of the Gauss point in the three directions of integration.

It may be noted that the order of integration in the thickness direction is less than the order of integration in the plane ξ, η . If we use the same order of integration in all the three directions it results in an over stiff element [47]. Therefore, a selectively reduced integration scheme has been recommended in the thickness direction [44], [47], [18] and etc. However, the reduced or selective integration techniques are not always successful in overcoming the over stiff or so-called *locking* phenomenon specially in thin membrane or shells with highly constrained boundaries. In recent years much attention has been focussed on the membrane and shear locking phenomenon [18] and [55].

The total stiffness and mass matrices are

$$[K] = \sum_m [K]^{(m)} \quad (3.142)$$

$$[M] = \sum_m [M]^{(m)} \quad (3.143)$$

Similarly, other matrices for the element properties can be evaluated using the procedure outlined above.

3.7 Element Interpolation Functions

3.7.1 Introduction

Depending upon the geometry of a given physical body we have to use engineering judgment in selecting appropriate interpolation functions in finite element analysis. The choice of the type of element is dictated by the geometry of the body and the number of independent special coordinates necessary to describe the system.

The essential procedure in the displacement-based isoparametric finite element formulation is to express the element coordinates and element displacements in the form of interpolations or shape functions using the natural coordinate system of the element. For example in a general three-dimensional element,

$$\begin{aligned}x &= \sum_{i=1}^{nn} N_i x_i & u &= \sum_{i=1}^{nn} N_i u_i \\y &= \sum_{i=1}^{nn} N_i y_i & v &= \sum_{i=1}^{nn} N_i v_i \\z &= \sum_{i=1}^{nn} N_i z_i & w &= \sum_{i=1}^{nn} N_i w_i\end{aligned}$$

where, x , y , and z are the coordinates at any point of the element, and x_i , y_i and z_i ($i = 1, \dots, nn$) are the coordinates of the nn element nodes, the u , v , w are the displacement at any point of the element and u_i , v_i and w_i are the displacements at nodal points. The shape functions N_i are defined in the natural coordinate system of the element (ξ, η, ζ) , which variables ξ , η and ζ vary from -1 to +1. The element interpolation functions N_i is unity at node i and is zero at all other nodes.

The literature survey on shells and plates analysis shows that much emphasis

has been placed in the development of a versatile, reliable and cost effective bending element for shells and plates. Some of these are given in References [2], [4], [12], [14], [15], [55], [58], etc.

In this research work rectangular type elements have been used for plates and shells analysis. In addition to their efficiency, generality and simplicity, the rectangular element families for plates and shells analysis could also be used as degenerate triangular elements.

3.7.2 Isoparametric Rectangular Family

8-Node Serendipity Elements

To derive the shape functions, for 8-node serendipity elements (i.e, boundary nodes only) as shown in Fig. 3.7, the second degree displacement interpolation functions are assumed

$$N_i(\xi, \eta) = c_1 + c_2\xi + c_3\eta + c_4\xi\eta + c_5\xi^2 + c_6\eta^2 + c_7\xi^2\eta + c_8\xi\eta^2 \quad (3.144)$$

Based on the assumed displacement interpolation functions, the quadratic shape functions are ([1], [4],[5], [12], etc.):

For nodes 1 to 4

$$N_i(\xi, \eta) = \frac{1}{4}(1 + \xi_i\xi)(1 + \eta_i\eta)(\xi_i\xi + \eta_i\eta - 1) \quad (3.145)$$

For nodes 5 and 7

$$N_i(\xi, \eta) = \frac{1}{2}(1 - \xi^2)(1 + \eta_i\eta) \quad (3.146)$$

For nodes 6 and 8

$$N_i(\xi, \eta) = \frac{1}{2}(1 + \xi_i\xi)(1 - \eta^2) \quad (3.147)$$

where (ξ_i, η_i) are the coordinates of node i , and the values of ξ_i and η_i are listed in Table 3.1.

Table 3.1: Nodal Coordinate for ξ_i, η_i

i	1	2	3	4	5	6	7	8
ξ_i	-1	1	1	-1	0	1	0	-1
η_i	-1	-1	1	1	-1	0	1	0

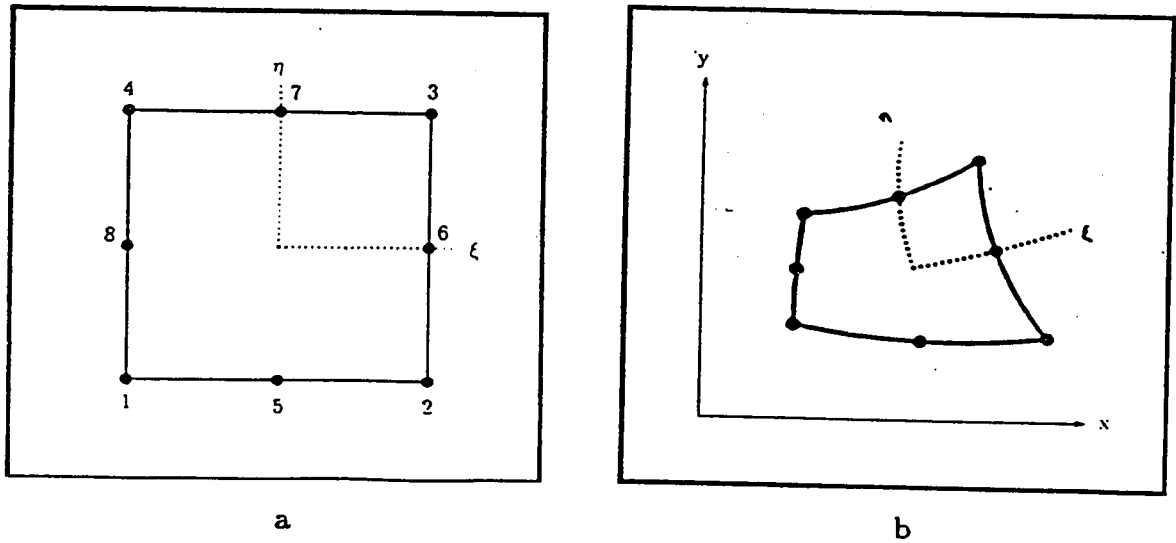


Figure 3.7: 8-node isoparametric element: (a) parent element (b) actual element

9-Node Quadratic Elements

The foregoing element, and any other isoparametric elements having boundary nodes only, are called "Serendipity" elements. Addition of an internal node, as shown in Fig. 3.8 makes the element a "Lagrange" quadratic element. The formulations of 9-node Lagrange quadratic element shape functions are given in references [1],[2], [3], [4], [5], [14]. The quadratic 9-node Lagrange element has been used for shell analysis in this study. The shape functions are written as

$$N_i = \{\xi\xi_i(1 + \xi\xi_i)/2 + (1 - \xi^2)(1 - \xi_i^2)\}\{\eta\eta_i(1 + \eta\eta_i)/2 + (1 - \eta^2)(1 - \eta_i^2)\} \quad (3.148)$$

or in the expanded form,

$$\begin{aligned} N_1 &= \frac{1}{4}\xi\eta(1 - \xi)(1 - \eta) \\ N_2 &= -\frac{1}{2}\eta(1 - \xi^2)(1 - \eta) \\ N_3 &= -\frac{1}{4}\xi\eta(1 + \xi)(1 - \eta) \\ N_4 &= \frac{1}{2}\xi(1 + \xi)(1 - \eta^2) \\ N_5 &= \frac{1}{4}\xi\eta(1 + \xi)(1 + \eta) \\ N_6 &= \frac{1}{2}\eta(1 - \xi^2)(1 + \eta) \\ N_7 &= -\frac{1}{4}\xi\eta(1 - \xi)(1 + \eta) \\ N_8 &= -\frac{1}{2}\xi(1 - \xi)(1 - \eta^2) \\ N_9 &= (1 - \xi^2)(1 - \eta^2) \end{aligned} \quad (3.149)$$

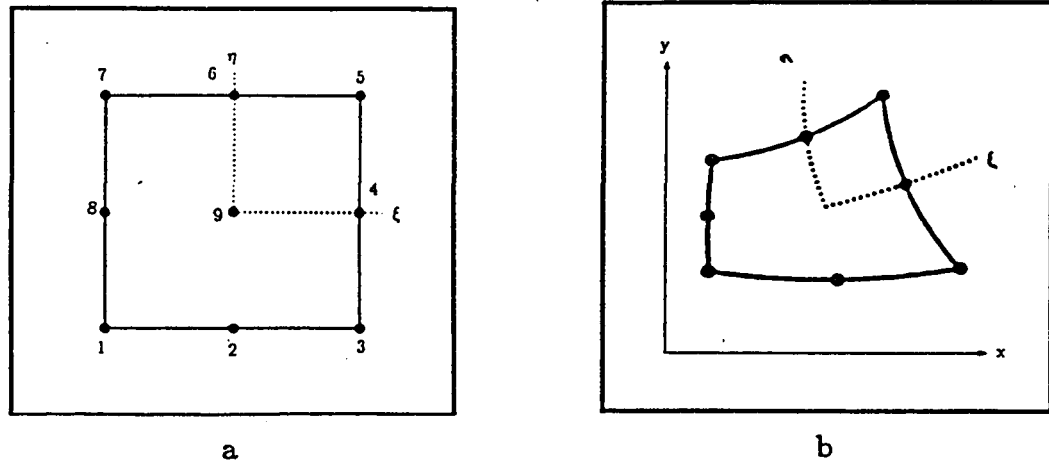


Figure 3.8: 9-node isoparametric element: (a) parent element (b) actual element

Variable-number-nodes from 4 to 9

The establishment of an appropriate finite element model for an actual practical problem may require some mesh grading. i.e., a finer or different finite element mesh in certain areas than in others. The variable-number-nodes element enables us the mesh grading in such transition zones as shown in Fig. 3.9. More details may be found in several references [1],[2],[10],[12],[14], etc.

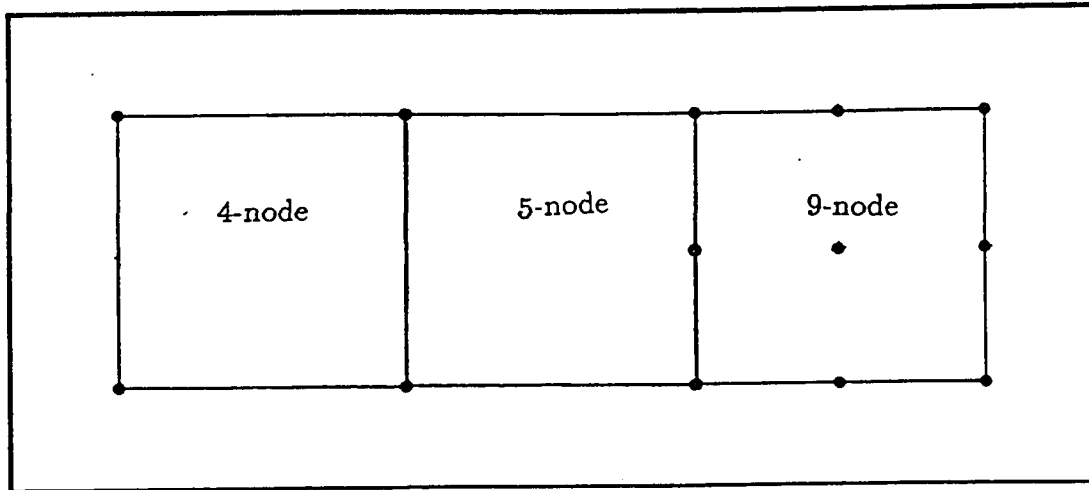


Figure 3.9: 4-node to 9-node element transition region

Table 3.2: Interpolation Functions of 4 to 9 Variable-Number-Nodes in Two-Dimensional Element

		include only if node i is defined				
		$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$
N_1	$\frac{1}{4}(1 - \xi)(1 - \eta)$	$-\frac{1}{2}N_5$			$-\frac{1}{2}N_8$	$-\frac{1}{4}N_9$
N_2	$\frac{1}{4}(1 + \xi)(1 - \eta)$	$-\frac{1}{2}N_5$	$-\frac{1}{2}N_6$			$-\frac{1}{4}N_9$
N_3	$\frac{1}{4}(1 + \xi)(1 + \eta)$		$-\frac{1}{2}N_6$	$-\frac{1}{2}N_7$		$-\frac{1}{4}N_9$
N_4	$\frac{1}{4}(1 - \xi)(1 + \eta)$			$-\frac{1}{2}N_7$	$-\frac{1}{2}N_8$	$-\frac{1}{4}N_9$
N_5	$\frac{1}{2}(1 - \xi^2)(1 - \eta)$					$-\frac{1}{4}N_9$
N_6	$\frac{1}{2}(1 + \xi)(1 - \eta^2)$					$-\frac{1}{4}N_9$
N_7	$\frac{1}{2}(1 - \xi^2)(1 + \eta)$					$-\frac{1}{4}N_9$
N_8	$\frac{1}{2}(1 - \xi^2)(1 - \eta^2)$					$-\frac{1}{4}N_9$
N_9	$(1 - \xi^2)(1 - \eta^2)$					

3.8 Locking Phenomenon in Plates and Shells

For thin shells, when exact numerical integration is used with Mindlin based finite elements, very disappointing results are obtained, especially if the boundaries are highly constrained. This problem is referred to in the literature as the *locking* phenomenon and has been investigated extensively by a number of researchers [35], [42], [55] and [62]. There are usually two types of locking behavior, namely; (i) shear locking and (ii) membrane locking. Both shear and membrane locking occur when extremely thin shells are considered. The membrane stiffness and the shear stiffness both are usually of the same order of magnitude. However, membrane locking is a somewhat different phenomenon than shear locking. To avoid locking behavior several alternative techniques have been suggested such as: reduced integration method, selective integration method, and the assumed displacement method.

- Reduced integration method

This method, although successful in some cases, [47] [50], produces other problems of rank deficiency of the stiffness matrix [51],[126].

- Selective Integration Method

Selective integration procedure [127], in which a reduced integration scheme is followed, is used to obtain stiffness matrix associated with the shear strain energy. Although, a lot of research has been done to correct the over stiffness created by the selective integration techniques [47], [51] and [44], these techniques are not always successful in overcoming locking behavior for the cases where highly constrained boundaries and coarse meshes exist. On the other hand, in case of lightly

constrained boundaries, element mechanisms may form. These element mechanisms may spread from element to element and cause either rank deficiency in the stiffness matrix or the solution obtained may be polluted by a nearmechanism. In addition, the type of selective-reduced integration suggested in Reference [44] can only be used, when the material properties of the shell do not vary through the plate/ shell thickness or vary symmetrically about the plate/ shell mid-surface. Otherwise, the coupling terms between the membrane and the bending stiffness will not disappear and results in a stiffness matrix which cannot be split into two parts in order to use selective integration.

- Assumed Strains Method.

The assumed strains method has been preferred by many authors [53],[117],[125].

In this study an enhanced interpolation of the membrane strains is employed in the local coordinate system (x', y', z') to avoid membrane locking, and an enhanced interpolation of the transverse shear strains in the natural coordinate (ξ, η, ζ) is used to overcome the shear locking behavior in shell elements. These two techniques [55],[125] have been reviewed in the following:

1-Elimination of shear locking

To avoid shear locking in the 9-node degenerated shell element, Huang and Hinton [117] for the shear strains fields suggested the polynomial terms of the following form in the natural coordinate system:

$$\bar{\gamma}_{\xi\zeta} = b_1 + b_2\xi + b_3\eta + b_4\xi\eta + b_5\eta^2 + b_6\xi\eta^2. \quad (3.150)$$

$$\bar{\gamma}_{\eta\zeta} = c_1 + c_2\xi + c_3\eta + c_4\xi\eta + c_5\xi^2 + c_6\eta\xi^2 \quad (3.151)$$

The above strain fields are chosen as

$$\bar{\gamma}_{\xi\zeta} = \sum_{i=1}^3 \sum_{j=1}^2 L_j(\xi) H_i(\eta) \bar{\gamma}_{\xi\zeta}^{ij} \quad (3.152)$$

$$\bar{\gamma}_{\eta\zeta} = \sum_{i=1}^3 \sum_{j=1}^2 H_i(\xi) L_j(\eta) \bar{\gamma}_{\eta\zeta}^{ij} \quad (3.153)$$

where

$$H_1(z) = z(z/b + 1)/2b \quad H_2(z) = 1 - (z/b)^2 \quad H_3(z) = z(z/b - 1)/2b$$

$$L_1(z) = (z/a + 1)/2 \quad L_2(z) = (1 - z/a)/2$$

In the above equations, $a = 3^{-0.5}$, $b = 1$ and the locations of i and j are shown in Fig.3.10. and the terms $\bar{\gamma}_{\eta\zeta}^{ij}$ and $\bar{\gamma}_{\xi\zeta}^{ij}$ are the transverse shear strain evaluated from the displacement field. The six unknown parameters for $\bar{\gamma}_{\xi\zeta}$ are the values $\bar{\gamma}_{\xi\zeta}^{ij}$ ($i = 1 - 3, j = 1, 2$) at the two Gauss point locations ($\xi = a$) on the three lines $\eta = b$, $\eta = 0$ and $\eta = -b$. Thus $\bar{\gamma}_{\xi\zeta}$ is linear in ξ and quadratic in η . Similarly the six unknown parameters defining $\bar{\gamma}_{\eta\zeta}$ are the values $\bar{\gamma}_{\eta\zeta}^{ij}$ ($i = 1 - 3, j = 1, 2$) on the two Gauss point locations ($\eta = a, \eta = -a$) on the three lines $\xi = b$, $\xi = 0$ and $\xi = -b$. Thus $\bar{\gamma}_{\eta\zeta}$ is linear in η and quadratic in ξ .

2-Enhanced interpolation of the membrane strains

In shells, although the stiffness associated with the membrane strain energy is important, especially when the shell thickness becomes very small, it should not dominate the total stiffness. The membrane strains can only be separated from the bending strains in the local cartesian coordinate system. Therefore, using the membrane strain in this coordinate system can help to eliminate locking behavior.

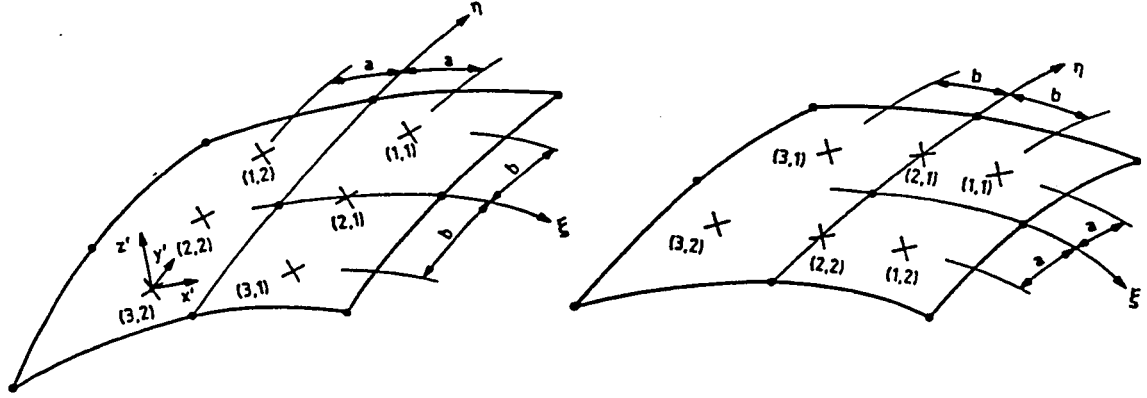


Figure 3.10: Interpolation points(i,j)

In the local coordinate system the following interpolation is used

$$\bar{\epsilon}_{mx'x'} = \sum_{i=1}^3 \sum_{j=1}^2 L_j(\xi) H_i(\eta) \bar{\epsilon}_{mx'x'}^{ij} \quad (3.154)$$

$$\frac{1}{2} \bar{\epsilon}_{mx'y'} = \sum_{i=1}^3 \sum_{j=1}^2 L_j(\xi) H_i(\eta) \bar{\epsilon}_{mx'y'}^{ij} / 2 \quad (3.155)$$

$$\bar{\epsilon}_{my'y'} = \sum_{i=1}^3 \sum_{j=1}^2 H_i(\xi) L_j(\eta) \bar{\epsilon}_{my'y'}^{ij} \quad (3.156)$$

$$\frac{1}{2} \bar{\epsilon}_{mx'y'} = \sum_{i=1}^3 \sum_{j=1}^2 H_i(\xi) L_j(\eta) \bar{\epsilon}_{mx'y'}^{ij} / 2 \quad (3.157)$$

Finally, the in-plane shear strain is quadratic in both directions, that is

$$\bar{\epsilon}_{mx'y'} = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^2 L_j(\xi) H_i(\eta) \bar{\epsilon}_{mx'y'}^{ij} + \frac{1}{2} \sum_{l=1}^3 \sum_{k=1}^2 H_l(\xi) L_k(\eta) \bar{\epsilon}_{mx'y'}^{lk} \quad (3.158)$$

where $H_i(z)$ and $L_j(z)$ have the same meaning as in shear locking formula and $a = 3^{-0.5}$ and $b = 1$.

Part I

PLATES

Chapter 4

NUMERICAL RESULTS FOR PLATES

4.1 Computation Procedure

In this chapter, vibration of plates with partial supports along a boundary are discussed. The present formulation as discussed earlier in detail, is based on the Q-8(eight nodes quadrilateral) plate bending element with three DOF(degree of freedom)at each node.

Unless otherwise indicated, the symbols F, S, C and P generally denote free, simply, clamped and partially supported wedges, respectively. These symbols for a triangular plate has been shown in Figure 4.1. The figure also shows a typical finite element grid for a sample problem.

The computations in this investigation are carried out by varying two essential parameters. These are the aspect ratio b/a and the unsupported length c/a as

shown in Figure 4.2. In order to demonstrate the accuracy of the analysis, the results for several degenerate case of triangular plates with full support ($c/a = 0$), are presented and compared with previously published results. Results are then given for partially supported right triangular cantilever plates.

For generating a data base, the non-dimensional frequency parameter used in tables is defined as:

$$\Omega = \omega a^2 \phi^2 \left(\frac{\rho_e t_e}{D_e} \right)^{0.5}$$

where

ϕ = aspect ratio (b/a)

Ω = non-dimensional frequency parameter

ω = frequency(rad./sec.)

t_e = plate thickness

ρ_e = effective mass density

D_e = effective plate rigidity

Trial computations for several alternative grid forms and number of elements are conducted and it is concluded that good convergence is achieved with 55 Q-8 elements. The data presented in the tables is given for the first four frequencies. Computations for each plate geometry of plate are carried out in three parts. The first part deals with isotropic plates, the second with orthotropic plates, and the third with composite plates such as, sandwich, laminate or other anisotropic plates.

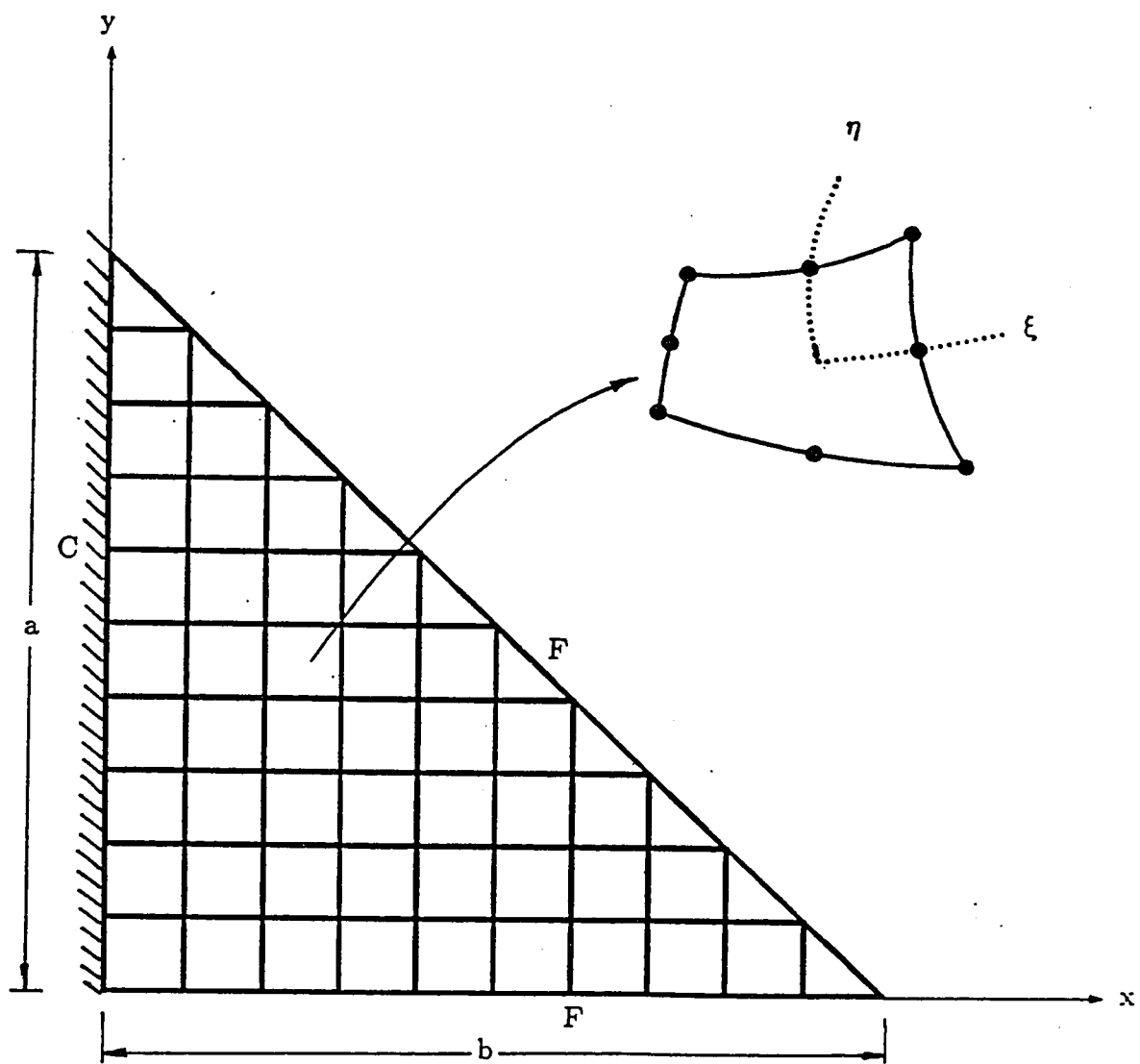


Figure 4.1: Right triangular C-F-F plate.

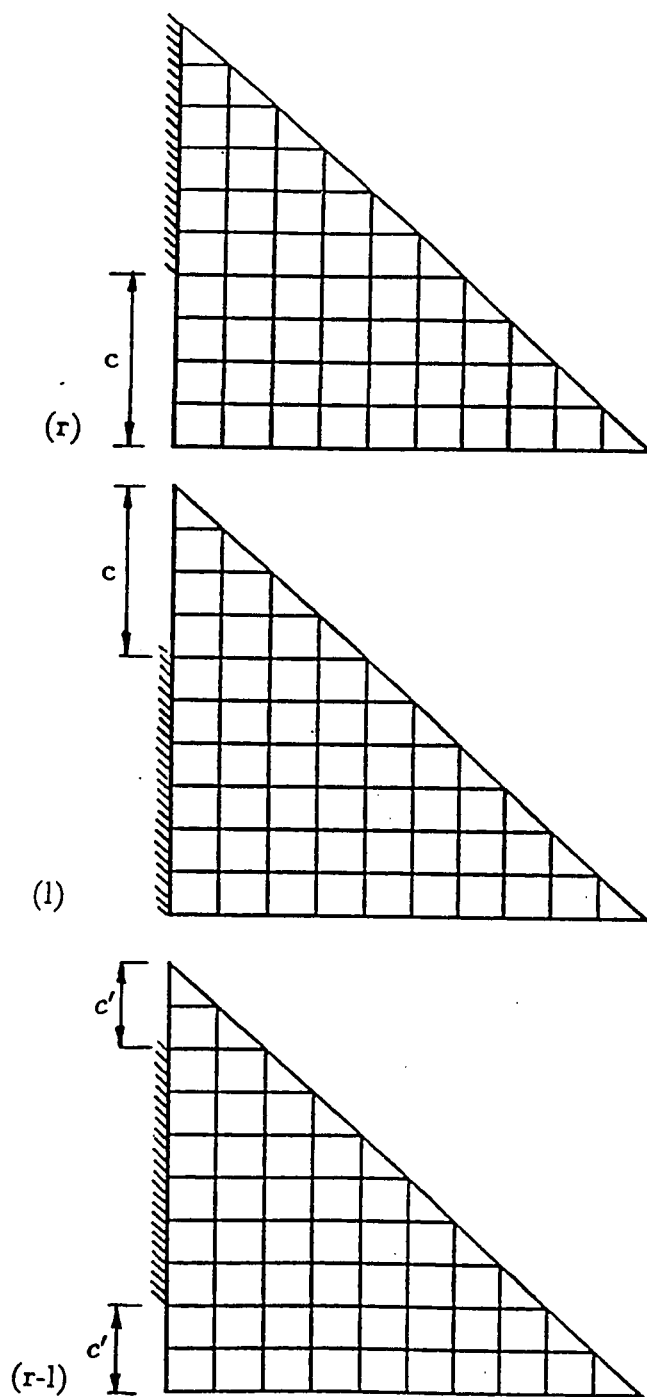


Figure 4.2: Right triangular plate with detached support:
 (r) detached length from right angle end
 (l) detached length from acute angle end
 (r-l) detached length from both ends.

4.2 Isotropic Plate

4.2.1 Triangular Isotropic Plates

In computations for triangular isotropic plates the length a , which is along the support, is taken as 10.0 in. (25.4 cm) and the thickness $t=0.063$ in.(0.16 cm).

The material is assumed to be aluminium and for all these calculations Poisson's ratio is taken $\nu = 0.3$. The material properties are given in a tabular form below (Table 4.1). Computations have been carried out for the non-dimensionalized frequency with applicability for any kind of isotropic plate .

For the isotropic case the plate rigidity D is defined as:

$$D = Et^3/12(1 - \nu^2)$$

Table 4.1: Material Properties for Isotropic Plate

Material property	Assumed value	System of units
ν	0.3	
ρ	1.0 (arbitrary)	$lb.sec^2/in^4$
E_{11}/E_{22}	1.0 (arbitrary)	psi/psi
E_{22}/G_{12}	2.6	psi/psi

4.2.2 Fully Supported Triangular Isotropic Plates

Table 4.2 shows the comparison of the lowest four frequency parameters for cantilever plates with two aspect ratios, $b/a=1$ and 2. For both aspect ratios, it can be seen that the present results are in close agreement with other published results. This particular case has been essentially, generated to provide a check on the computer program, convergence etc.

Table 4.2: Comparison of Frequency Parameters $\Omega = \omega a^2 \phi^2 (\frac{\rho t}{D})^{0.5}$ for C-F-F Triangular Plate ($\nu = 0.3$)

$a = 10.in, a/t = 158.73.$				
$b/a = 1$				
Source of results	Ω_1	Ω_2	Ω_3	Ω_4
Present(55 elements)	6.162	23.445	32.625	56.203
Kim et al. [43]	6.164	23.457	32.664	56.149
F.E.M.(25 elements)[43]	6.455	22.937	31.473	55.074
F.E.M.(100 elements)[43]	6.122	23.020	31.853	54.881
Mirza (25 elements)[63]	6.159	23.061	33.289	55.915
Christensen [59]	6.160	23.700	32.540	55.010
Bhat [41]	6.173	23.477	32.716	56.405
Experiment [71]	5.930	23.400	32.700	55.900
$b/a = 2$				
Source of results	Ω_1	Ω_2	Ω_3	Ω_4
Present (55 elements)	6.641	28.552	49.667	70.036
Kim et al. [43]	6.622	28.435	49.398	69.651
F.E.M.(25 elements)[43]	7.054	29.033	50.534	70.417
F.E.M.(100 elements)[43]	6.608	28.185	49.500	69.070
Mirza (25 elements)[63]	6.696	29.499	51.011	77.759

4.2.3 Partially Supported Cantilever Triangular Plates

The frequency parameters for P-F-F, right triangular plates with aspect ratios (b/a) 1.0, 1.5 and 2.0 are presented in Tables 4.3 to 4.5 for several values of the ratio c/a which indicates the extent of detached support. In each tables, table (r) contains frequency parameters for detached length from the right angle end, the table (l) for detached length from acute angle end and (r-l) for detached length from both ends. Furthermore, the symbols $c-r$, $c-l$ and $c-r-l$ denote the unsupported boundary from right angle end, the acute angle end and both ends.

Figures 4.3 to 4.5 show variation of the frequency parameter with c/a for the lowest four modes for a triangular P-F-F plate with aspect ratios 1., 1.5 and 2.0. Modal shapes corresponding to these frequencies have been given in Fig. A.1 to A.18.

Table 4.3: Non-dimensional Frequency $\Omega = \omega a^2 \phi^2 (\frac{g}{D})^{0.5}$ for P-F-F
Triangular Isotropic Plate (*ipl* ϕ 1)

$a = 10.0in, \phi = b/a = 1.0, a/t = 158.73.$				
r	Detached length from right angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	6.1628	23.4524	32.6357	56.2332
0.20	5.7118	23.2154	27.2022	55.3499
0.40	4.5470	16.3487	22.5726	43.9650
0.60	3.2212	10.0867	20.7558	36.9024
0.80	2.0280	6.8295	17.3768	31.0417
1.00	0.7964	3.3326	11.6859	25.3488
l	Detached length from acute angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	6.1628	23.4524	32.6357	56.2332
0.20	6.1570	23.2970	32.5590	55.4390
0.40	6.0590	21.2530	29.2990	33.2260
0.60	5.6280	12.0340	21.1320	32.3380
0.80	4.5680	06.6020	19.5690	32.1770
1.00	2.0380	03.2980	18.7700	26.9160
r-l	Detached length from both ends			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	6.1628	23.4524	32.6357	56.2332
0.20	6.0570	23.3837	31.2409	55.9632
0.40	5.7062	23.0503	27.1612	54.6209
0.60	5.1539	20.7998	22.8810	48.1147
0.80	4.4312	15.4641	20.9957	30.1222
1.00	3.0609	8.6740	17.5473	21.7838

Table 4.4: Non-dimensional Frequency $\Omega = \omega a^2 \phi^2 (\frac{\rho t}{D})^{0.5}$ for P-F-F
Triangular Isotropic Plate ($\phi = 1.5$)

$a = 10.0in, \phi = b/a = 1.5, a/t = 158.73.$				
r	Detached length from right angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	6.4697	27.2658	40.1730	66.8654
0.20	6.2252	26.8185	36.8189	65.1707
0.40	5.4601	23.6994	29.5514	57.6153
0.60	4.4028	16.2749	26.5725	48.5898
0.80	3.1652	10.7873	24.3870	41.4429
1.00	1.4074	4.8806	19.5185	32.3882
l	Detached length from acute angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	6.4697	27.2638	40.1730	66.8654
0.20	6.4550	27.0760	39.7680	66.4080
0.40	6.3120	25.6090	37.0310	57.9180
0.60	5.8700	19.6710	29.3930	42.6130
0.80	4.9530	11.8220	26.1640	41.0870
1.00	2.7790	05.5210	24.0580	38.1790
r-l	Detached length from both ends			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	6.4697	27.2658	40.1730	66.8654
0.20	6.4137	27.1302	39.2575	66.3642
0.40	6.2096	26.6688	36.4254	64.7461
0.60	5.8307	25.6014	31.5567	61.0506
0.80	5.2494	23.5366	25.5675	53.7161
1.00	3.9819	14.7917	21.4750	42.4543

Table 4.5: Non-dimensional Frequency $\Omega = \omega a^2 \phi^2 (\frac{g}{D})^{0.5}$ for P-F-F
Triangular Isotropic Plate (*ipl* ϕ 2)

$a = 10.0in, \phi = b/a = 2.0, a/t = 158.73.$				
r	Detached length from right angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	6.6418	28.5620	49.7287	70.0671
0.20	6.4879	28.0971	47.3072	68.5476
0.40	5.9446	26.1350	39.8763	64.3432
0.60	5.1111	21.3438	32.2500	58.6410
0.80	3.9643	15.0798	28.4440	52.0667
1.00	1.9309	6.6809	23.3974	41.5462
l	Detached length from acute angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	6.6418	28.5620	49.7287	70.0671
0.20	6.6130	28.3680	48.9730	69.7180
0.40	6.4540	27.3860	45.1130	68.2370
0.60	6.0420	24.2530	35.9530	59.7100
0.80	5.2180	17.0680	29.5460	54.4920
1.00	3.2280	08.3980	25.9940	50.2350
r-l	Detached length from both ends			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	6.6418	28.5620	49.7287	70.0671
0.20	6.5957	28.3763	48.8642	69.5502
0.40	6.4577	27.9303	46.6369	68.1738
0.60	6.1689	26.9803	42.0038	65.6086
0.80	5.6852	25.5975	34.7646	62.0520
1.00	4.5578	21.6774	23.7242	56.2654

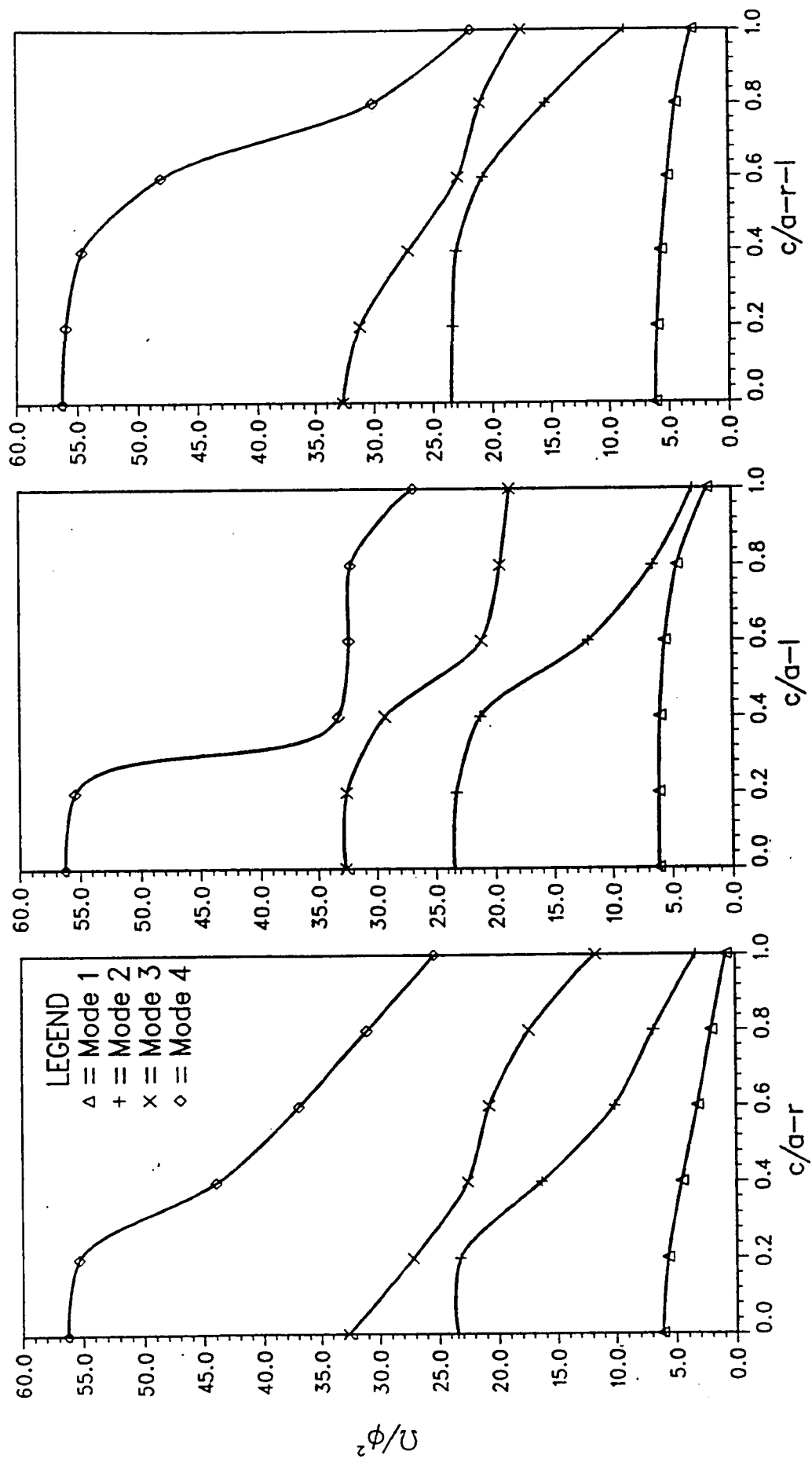


Figure 4.3: Variation of the non-dimensional frequency with c/a , for P-F-F isotropic plate. $a = 10.in$, $b/a = 1.0$, $a/t = 158.73$.

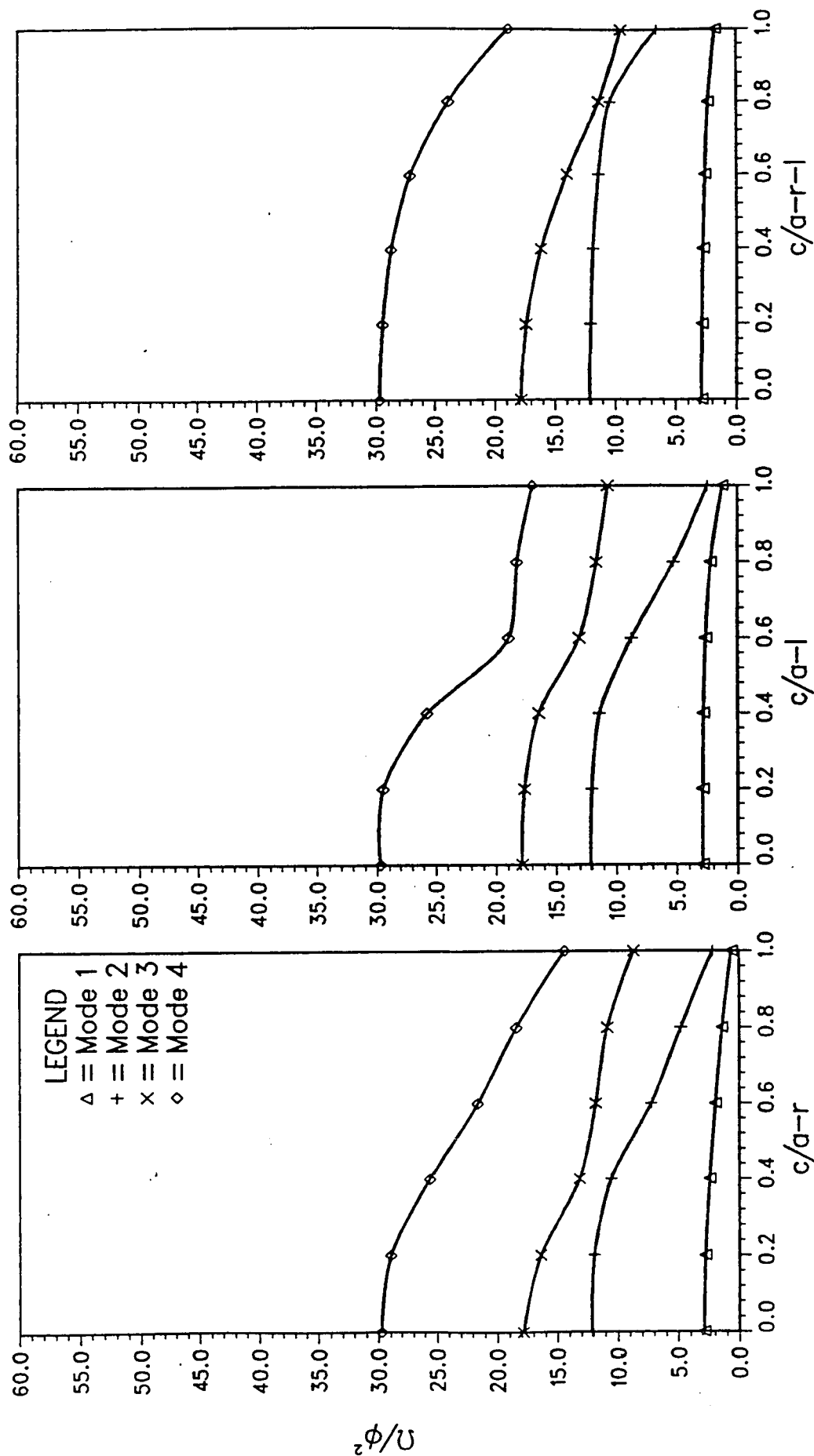


Figure 4.4: Variation of the non-dimensional frequency with c/a , for P-F-F isotropic plate. $a = 10.in$, $b/a = 1.5$, $a/t = 158.73$.

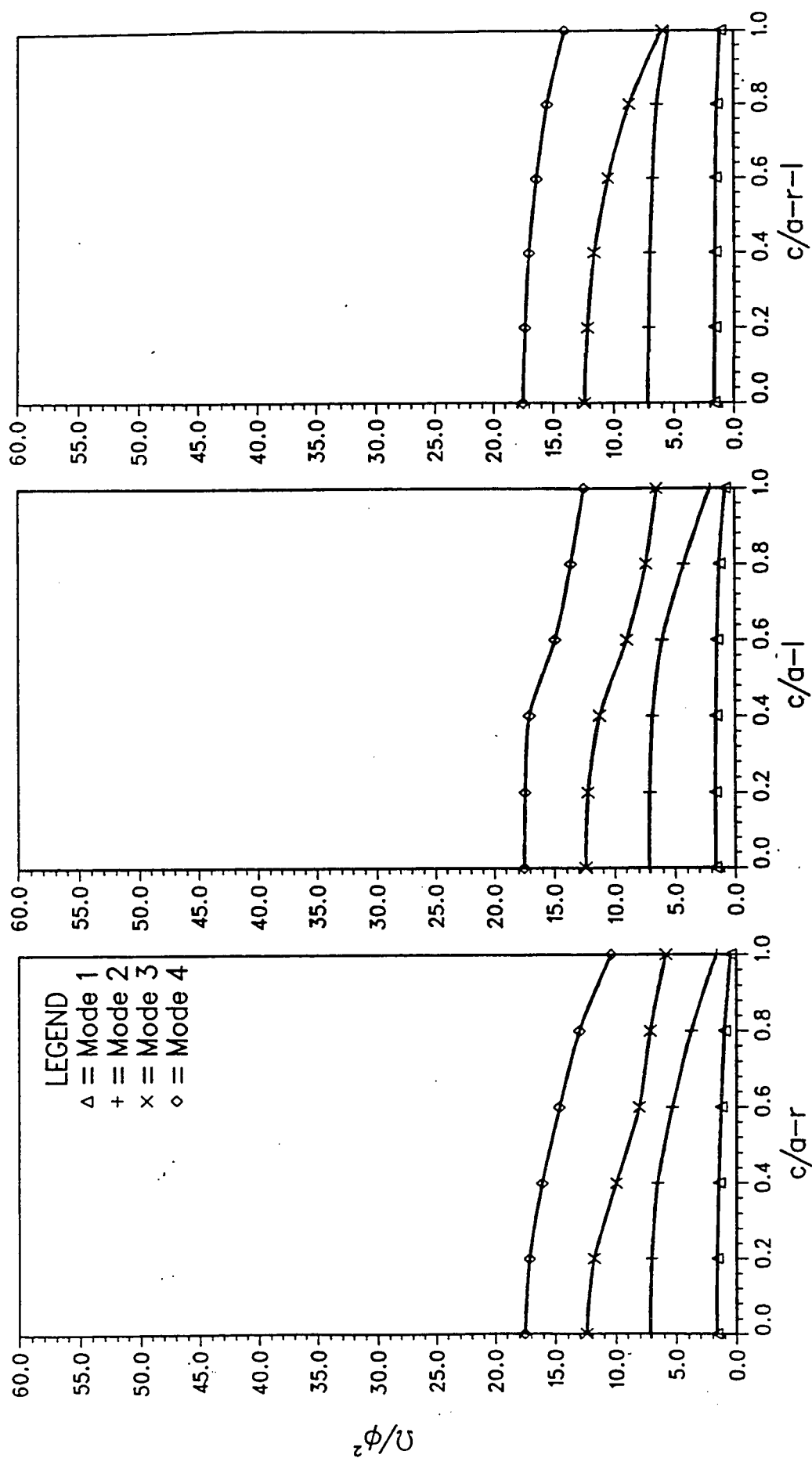


Figure 4.5: Variation of the non-dimensional frequency with c/a , for P-F-F isotropic plate. $a = 10.0$ in, $b/a = 2.0$, $a/t = 158.73$.

4.3 Orthotropic Plates

4.3.1 Triangular Orthotropic Plates

It should be noted that the geometry of the plate has been kept the same as in the case of isotropic triangular plate. This will provide us with an additional parameter for comparison of results. Carbon-Epoxy has been selected as a sample material for the plate whose material properties are summarized in Table 4.6. The selection of this material was also guided by the fact that some of the results available for fully clamped plates use these material properties.

The non-dimensional frequency is given by

$$\Omega = \omega a^2 \phi^2 \left(\frac{\rho_e t_e}{D_e} \right)^{0.5} \quad (4.1)$$

in which

$$D_e = \nu_{12} D_{22} + 2D_{66} \quad (4.2)$$

where

$$D_{22} = \frac{E_{22} t_e^3}{12(1 - \nu_{12}\nu_{21})}$$

$$D_{66} = G_{12} \frac{t_e^3}{12}$$

and other parameters are defined as before.

Table 4.6: Material Properties for Orthotropic Plates (Carbon-Epoxy).

Material property	Assumed value	System of units
ν_{12}	0.315	
ρ	259.2E-6	<i>lb.sec²/in⁴</i>
E_{11}/E_{22}	16.5	<i>psi/psi</i>
E_{22}/G_{12}	2.628	<i>psi/psi</i>

4.3.2 Fully Supported Triangular Orthotropic Plates

Table 4.7 gives the first four frequency parameters Ω for cantilever orthotropic plates with aspect ratios $b/a = 1, 1.5, 2$. It may be seen that the present results are in close agreement with results from the other sources. The available results in the literature are only for fully supported plates.

Table 4.7: Comparison of Frequency Parameters $\Omega = \omega a^2 \phi^2 \left(\frac{\rho t^3}{D_c} \right)^{0.5}$ for C-F-F Orthotropic Triangular Plate (Carbon-Epoxy).

Source of results	Ω_1	Ω_2	Ω_3	Ω_4
$\phi = b/a = 1.0$				
Present (55 elements)	19.42	46.75	96.58	112.06
Kim et al. [43]	19.57	47.00	97.12	113.10
$\phi = b/a = 1.5$				
Present (55 elements)	20.81	60.35	109.37	155.61
Kim et al. [43]	20.97	60.67	110.30	156.20
$\phi = b/a = 2.0$				
Present (55 elements)	21.91	71.97	116.30	190.66
Kim et al. [43]	22.06	72.34	117.10	191.70

4.3.3 Partially Supported Triangular Orthotropic Plates

Frequency parameters for several cases of partially supported orthotropic plate with the three aspect ratios $b/a = 1, 1.5, 2.0$ have been studied and the results are given in Tables 4.8 to 4.10.

Figures 4.6 to 4.8 show variation of frequency parameter with c/a . As it has been mentioned earlier the symbols (r) , (l) and $(r-l)$ in all figures and tables denote the detached length measured from the right angle end, the acute angle end and both ends respectively.

Table 4.8: Non-dimensional Frequency $\Omega = \omega a^2 \phi^2 \left(\frac{\rho_c t_c}{D_c} \right)^{0.5}$ for P-F-F Triangular Orthotropic Plate (Carbon-Epoxy)(*opl* ϕ 1).

$a = 10.0, \phi = b/a = 1.0., a/t = 158.7301.$				
r	Detached length from right angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	19.4168	46.7511	96.5825	112.0576
0.20	13.2409	45.7162	60.0173	100.9603
0.40	7.3137	23.7515	41.7071	100.8224
0.60	4.0629	13.8690	31.0516	64.6688
0.80	2.2285	9.2042	20.8530	44.7382
1.00	0.8087	4.4669	12.0446	32.4219
l	Detached length from acute angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	19.4168	46.7511	96.5825	112.0576
0.20	19.4166	46.7445	96.5579	112.0292
0.40	19.3938	32.5556	45.4597	76.3048
0.60	14.0900	19.3068	30.8039	61.0719
0.80	7.2529	15.2805	23.7914	46.0258
1.00	2.3492	6.9383	21.6415	29.4561
r-l	Detached length from both ends			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	19.4168	46.7511	96.5825	112.0576
0.20	17.2119	46.2652	88.9280	104.4374
0.40	13.2408	45.7092	60.0173	100.9269
0.60	9.8443	35.0100	45.3037	58.7731
0.80	7.1713	23.6373	32.5595	40.7918
1.00	4.2575	13.1889	21.6036	27.5770

Table 4.9: Non-dimensional Frequency $\Omega = \omega a^2 \phi^2 (\frac{\rho_c t_c}{D_c})^{0.5}$ for P-F-F
Triangular Orthotropic Plate (Carbon-Epoxy)($opl\phi 1.5$).

$a = 10.0, \phi = b/a = 1.5., a/t = 158.7301.$				
r	Detached length from right angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	20.8147	60.3502	109.3689	155.6143
0.20	16.5139	60.0879	84.1952	151.3186
0.40	10.6971	41.8955	57.6633	119.2280
0.60	6.6962	23.8176	49.4013	104.1998
0.80	3.9335	15.5281	37.6875	75.6553
1.00	1.4847	7.4059	22.9287	52.8831
l	Detached length from acute angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	20.8147	60.3502	109.3689	155.6143
0.20	20.8114	60.2844	109.2487	154.7216
0.40	20.7928	53.1234	67.9579	104.8013
0.60	20.3348	26.4066	43.6142	107.2887
0.80	12.7722	17.8044	35.7407	89.9323
1.00	4.1082	9.5290	34.9774	53.6966
r-l	Detached length from both ends			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	20.8147	60.3502	109.3689	155.6143
0.20	19.5075	60.1797	101.1728	154.4506
0.40	16.5068	60.0190	84.1620	150.4070
0.60	13.3200	58.1219	60.3948	115.6511
0.80	10.4820	40.7413	51.4836	67.6608
1.00	6.8818	20.7422	37.5563	46.8818

Table 4.10: Non-dimensional Frequency $\Omega = \omega a^2 \phi^2 \left(\frac{\rho_c t_c}{D_c} \right)^{0.5}$ for P-F-F Triangular Orthotropic Plate (Carbon-Epoxy)($opl\phi 2.$).

$a = 10.0, \phi = b/a = 2.0., a/t = 158.7301.$				
r	Detached length from right angle end			
0.00	21.9065	71.9703	116.3016	190.6557
0.20	18.7176	71.8814	97.8486	187.5495
0.40	13.3036	60.3282	71.3930	136.4095
0.60	8.9837	35.2803	64.4647	120.8816
0.80	5.6203	22.6344	53.4503	103.1486
1.00	2.2265	10.5616	34.3657	74.2442
l	Detached length from acute angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	21.9065	71.9703	116.3016	190.6557
0.20	21.9031	71.4891	115.8162	189.3602
0.40	21.8660	61.5035	103.5855	120.0224
0.60	21.0694	37.5656	61.5263	110.6902
0.80	15.9148	22.8410	49.1980	109.5945
1.00	5.7332	12.5510	47.0872	78.8593
r-l	Detached length from both ends			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	21.9065	71.9703	116.3016	190.6557
0.20	21.0464	71.9201	110.2763	190.0183
0.40	18.7056	71.4067	97.4629	186.2224
0.60	15.8665	68.2701	79.3683	157.9042
0.80	13.1066	56.8915	61.5892	110.0057
1.00	9.1143	29.1851	48.6905	76.9514

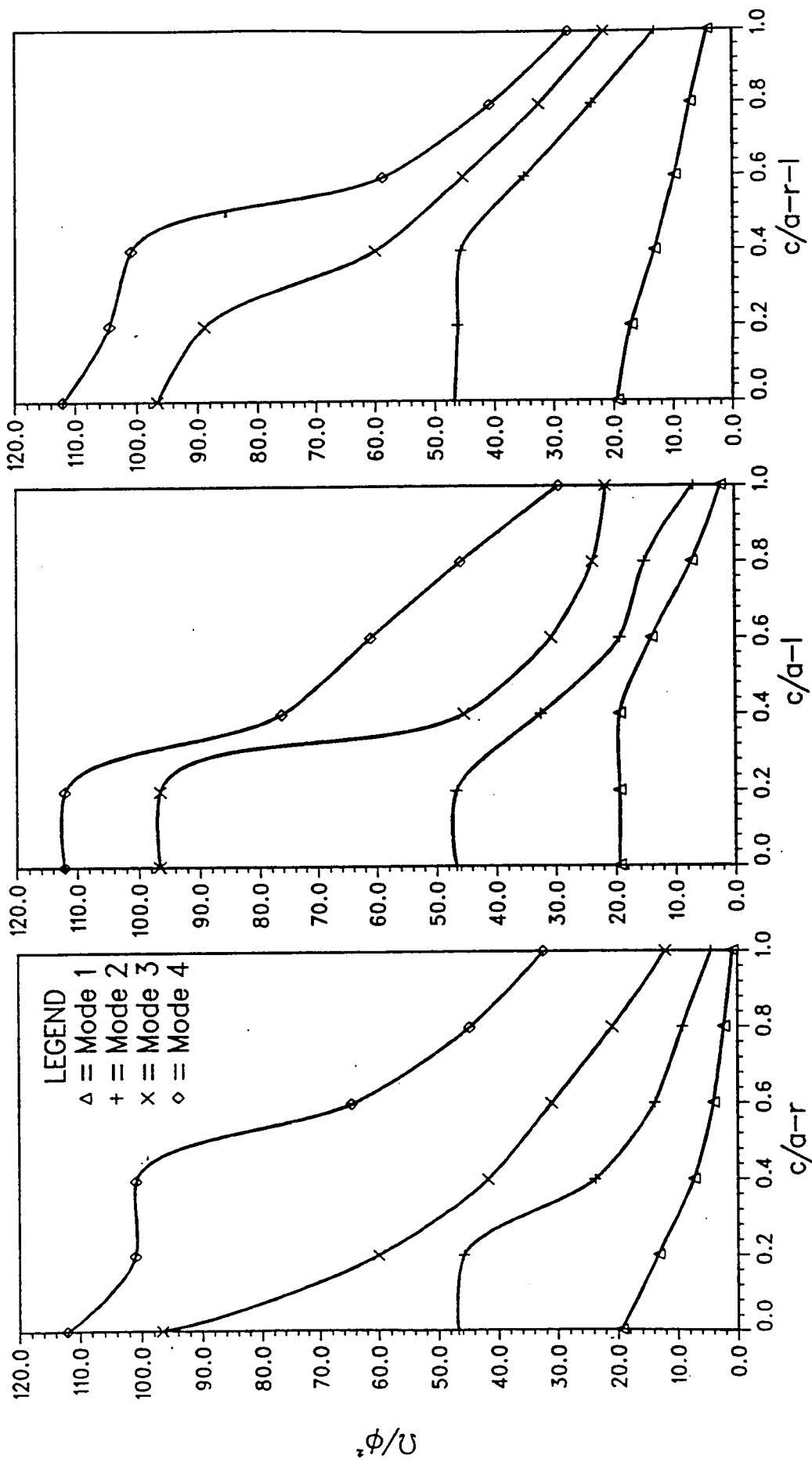


Figure 4.6: Variation of the non-dimensional frequency with c/a , for P-F-F triangular orthotropic plate (carbon-epoxy).
 $a = 10.in$, $b/a = 1.0$, $a/t = 158.73$.

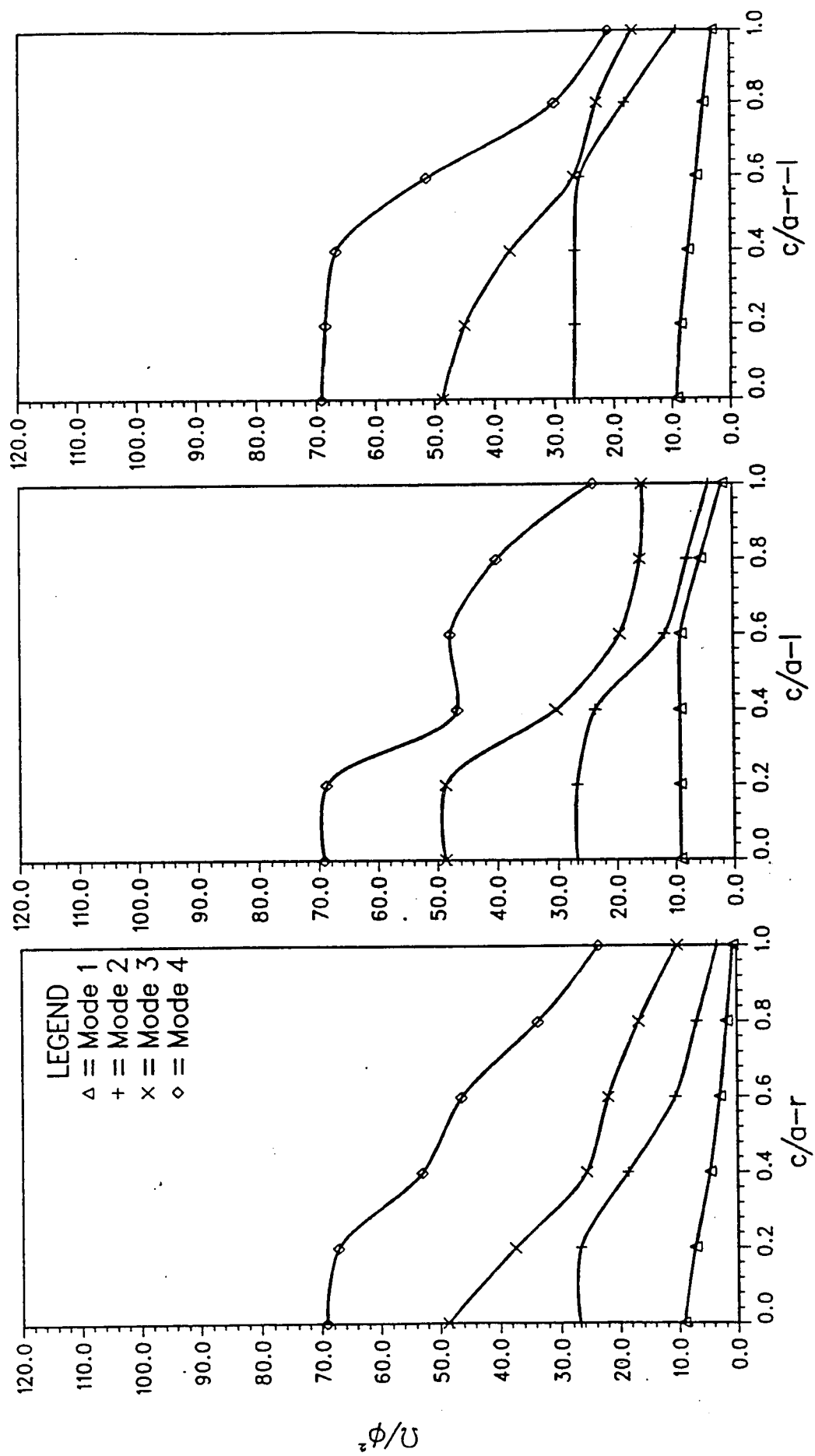


Figure 4.7: Variation of the non-dimensional frequency with c/a , for P-F-F triangular orthotropic plate (carbon-epoxy).
 $a = 10 \text{ in}$, $b/a = 1.5$, $a/t = 158.73$.

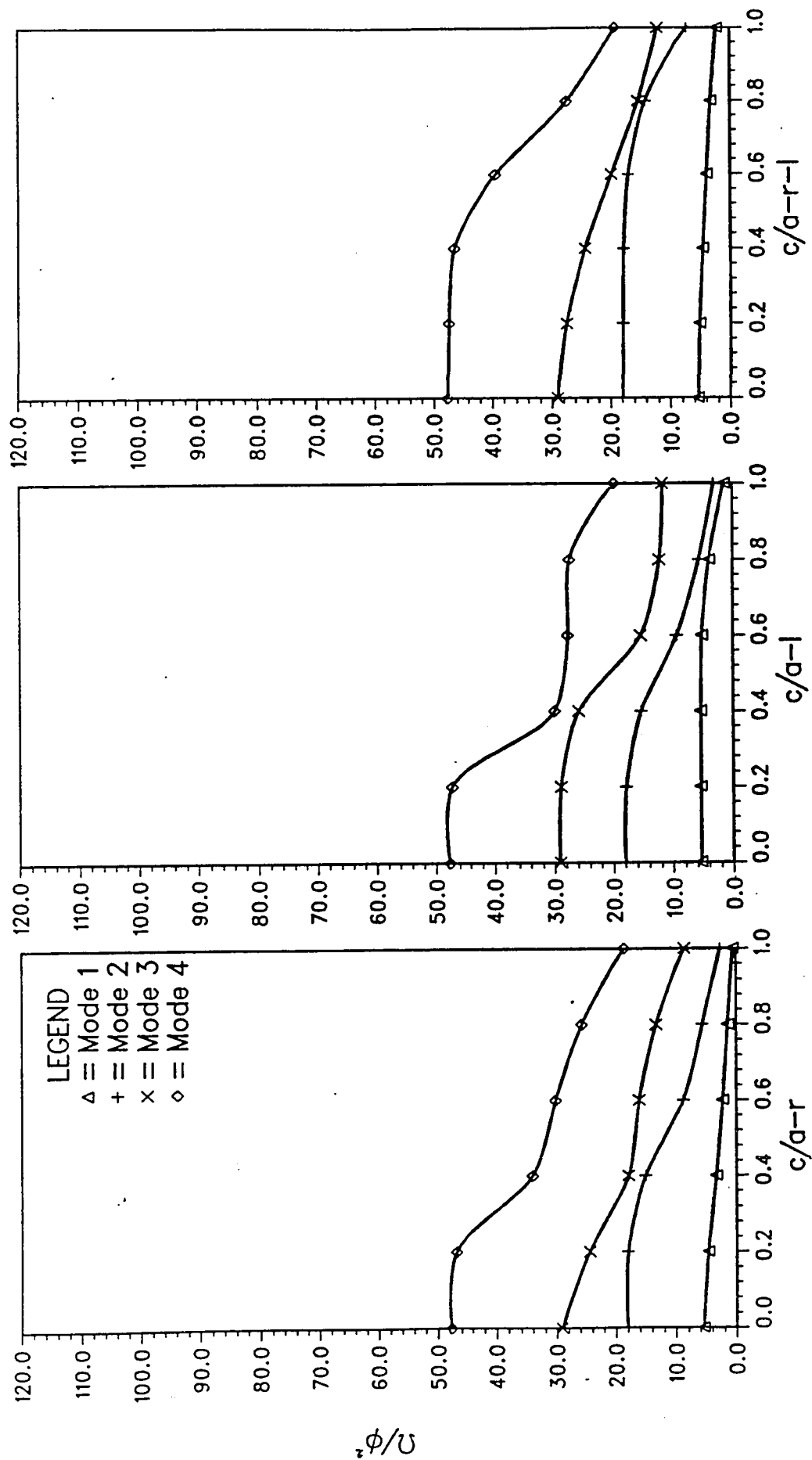


Figure 4.8: Variation of the non-dimensional frequency with c/a , for P-F-F triangular orthotropic plate (carbon-epoxy).
 $a = 10.in$, $b/a = 2.0$, $a/t = 158.73$.

4.4 Composite Plate

4.4.1 Triangular Sandwich Plates

The numerical data in this section has been generated for three layered triangular sandwich plates. In order to compare the results for sandwich plate with other published results in literature, an orthotropic sandwich triangular plate has been taken as a special case where the thickness of the core has been assumed to be zero. This produces the results for a degenerate case of two layers of the same material for which the results are available. Computations have been carried out for the fully supported and then for the partially supported plate. The length a , is taken the same as in case one (10.0 in). The core and face materials properties are given in Table 4.11.

Table 4.11: Material Properties for Sandwich Plates.

face	Carbon-Epoxy	
Material property	Assumed value	System of units
ν_{12}	0.315	
ρ	259.2E-6	$lb.sec^2/in^4$
E_{11}	3.3E7	psi
E_{11}/E_{22}	16.5	psi/psi
E_{22}/G_{12}	2.628	psi/psi
core	HRH-10-1/8 aramid fiber reinforced honeycomb	
Material property	Assumed value	System of units
ρ	8.9861E-6	$lb.sec^2/in^4$
E_{11}	60E3	psi
E_{11}/E_{22}	1	psi/psi
E_{22}/G_{23}	10.0000	psi/psi
E_{22}/G_{13}	4.6154	psi/psi

4.4.2 Fully Supported Triangular Sandwich Plates

For this case also as for the two other cases i.e, isotropic and orthotropic plates, the values of various parameters have been condensed. The obvious change is the addition of the parameter defining the geometry of the core. Figure 4.9 shows a typical sandwich triangular plate with a honeycomb core .

For comparison of results a degenerate case of two layered plate ($t_c = 0$) has been condensed. Table 4.12 gives the first four frequency parameters for this case (core thickness=0) of a triangular sandwich plate with aspect ratios $b/a = 1, 1.5, 2$. It was observed that the results were exactly the same as given in Table 4.7 which are for one layer orthotropic plate. This enables the checking of the computer outputs. This adds confidence not only to the computer program developed here but also in the computation procedure.

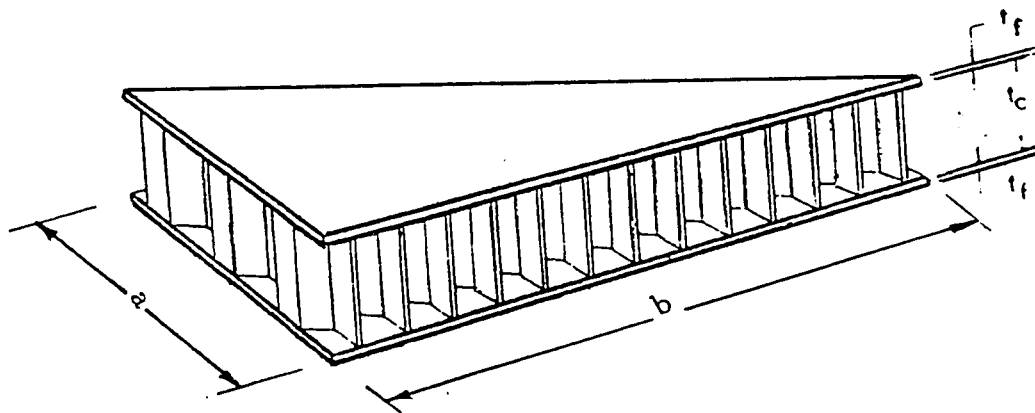


Figure 4.9: Typical triangular sandwich plate.

Table 4.12: Comparison of Frequency Parameters $\Omega = \omega a^2 \phi^2 (\frac{\rho_c t_c}{D_e})^{0.5}$ for C-F-F Triangular Sandwich Plate (Carbon-Epoxy Faces).

Source of results	Ω_1	Ω_2	Ω_3	Ω_4
$\phi = b/a = 1.0$	core thickness=0.			
Present (55 elements)	19.42	46.75	96.58	112.06
Kim et al. [43]	19.57	47.00	97.12	113.10
$\phi = b/a = 1.5$	core thickness=0.			
Present (55 elements)	20.81	60.35	109.37	155.61
Kim et al. [43]	20.97	60.67	110.30	156.20
$\phi = b/a = 2.0$	core thickness=0.			
Present (55 elements)	21.91	71.97	116.30	190.66
Kim et al. [43]	22.06	72.34	117.10	191.70

The non-dimensional frequency is defined as

$$\Omega = \omega a^2 \phi^2 \left(\frac{\rho_c t_c}{D_e} \right)^{0.5}$$

where

$$t_e = t_c + 2t_f$$

$$\rho = \frac{\rho_c t_c + 2\rho_f t_f}{t_e}$$

$$D_e = (\nu_{12}D_{22} + 2D_{66})_{face} + (\nu_{12}D_{22} + 2D_{66})_{core}$$

For the face material:

$$D_{22} = \frac{2E_{22}}{3(1 - \nu_{12}\nu_{21})}((t_e/2)^3 - (t_c/2)^3)$$

$$D_{66} = \frac{2G_{12}}{3}((t_e/2)^3 - (t_c/2)^3)$$

For the core material:

$$D_{22} = \frac{2E_{22}}{3(1 - \nu_{12}\nu_{21})}(t_c/2)^3$$

$$D_{66} = \frac{2G_{12}}{3}(t_c/2)^3$$

4.4.3 Partially Supported Triangular Sandwich Plates

Several cases dealing with the partially detached support conditions for triangular sandwich plates have been generated. Frequency parameters for some sample problems of partially supported Sandwich plates with aspect ratios $b/a = 1., 1.5, 2.0$ are given in Tables 4.13 to 4.15 for various values of the c/a . Figures 4.10 to 4.12 show variation of the frequency parameter with c/a ranging from 0 to 1.0 at intervals of 0.2. In each set of tables the top table (r) contains data of the detached support length from the right angle, the next set marked (l) for the detached length from acute angle, and the last set marked (r-l) is for the support with detached lengths from both ends.

Table 4.13: Non-dimensional Frequency $\Omega = \omega a^2 \phi^2 (\frac{\rho_c t_c}{D_c})^{0.5}$ for P-F-F Triangular Sandwich Plate (Carbon-Epoxy Faces)(*spl* ϕ 1).

$a = 10.0, \phi = b/a = 1.0, a/t = 31.9489$				
r	Detached length from right angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	18.6311	43.5318	85.3947	97.6569
0.20	12.1461	42.4162	51.2355	90.4456
0.40	6.7605	21.5831	38.3813	88.3950
0.60	3.7741	12.8141	28.5545	58.9212
0.80	2.0709	8.3512	19.3121	41.7067
1.00	0.7730	2.4372	11.3711	30.6212
l	Detached length from acute angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	18.6311	43.5318	85.3947	97.6569
0.20	18.6306	43.5217	85.3348	97.5484
0.40	18.5988	28.8499	41.6788	66.5501
0.60	12.9276	18.4448	27.8755	57.0010
0.80	6.5864	14.0539	22.7744	43.0114
1.00	2.0682	5.1204	20.5127	27.2751
r-l	Detached length from both ends			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	18.6311	43.5318	85.3947	97.6569
0.20	15.8690	43.0985	76.6497	92.7904
0.40	12.1457	42.4057	51.2354	90.4705
0.60	9.0511	31.0467	41.7434	49.8692
0.80	6.5496	21.2938	28.8966	37.0146
1.00	3.6075	10.2820	20.4833	23.6136

Table 4.14: Non-dimensional Frequency $\Omega = \omega a^2 \phi^2 (\frac{\rho_c t_c}{D_c})^{0.5}$ for P-F-F
Triangular Sandwich Plate (Carbon-Epoxy Faces)(*spl*1.5).

$a = 10.0, \phi = b/a = 1.5, a/t = 31.9489$				
r	Detached length from right angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	20.2330	57.0989	100.8404	140.2686
0.20	15.3475	56.8194	75.4758	136.1456
0.40	9.9122	38.0946	54.0294	111.9740
0.60	6.1845	22.1146	45.7980	96.4460
0.80	3.6419	14.2808	34.5730	70.7115
1.00	1.4326	4.2845	21.1149	50.5501
l	Detached length from acute angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	20.2330	57.0989	100.8404	140.2686
0.20	20.2290	57.0066	100.6632	138.9815
0.40	20.2041	48.1334	61.0501	96.1127
0.60	19.5614	23.8613	40.8361	93.9666
0.80	11.3718	16.7336	34.6075	83.7905
1.00	3.7688	6.6207	33.8597	49.7180
r-l	Detached length from both ends			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	20.2330	57.0989	100.8404	140.2686
0.20	18.3866	56.9390	91.4976	139.2014
0.40	15.3378	56.7227	75.4297	134.9067
0.60	12.3239	53.2410	54.9611	97.0675
0.80	9.6202	36.0488	46.9248	60.4521
1.00	5.6904	15.9400	32.8184	44.6180

Table 4.15: Non-dimensional Frequency $\Omega = \omega a^2 \phi^2 (\frac{\rho_s t_c}{D_c})^{0.5}$ for P-F-F
Triangular Sandwich Plate (Carbon-Epoxy Faces)(*spl* ϕ 2.).

$a = 10.0, \phi = b/a = 2.0, a/t = 31.9489$				
r	Detached length from right angle end			
0.00	21.3986	68.4983	109.7143	173.9334
0.20	17.5460	68.3424	90.0297	170.7245
0.40	12.3517	55.1821	67.2653	128.6021
0.60	8.2881	32.7437	60.1553	114.7827
0.80	5.1782	20.9736	48.9209	97.1534
1.00	2.1222	6.5061	30.9719	71.5136
l	Detached length from acute angle end			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	21.3986	68.4983	109.7143	173.9334
0.20	21.3949	67.9218	109.1079	172.0917
0.40	21.3369	56.4825	93.5590	111.2295
0.60	20.2722	33.9010	58.0461	104.9873
0.80	14.2804	21.2474	47.6899	103.0624
1.00	5.2750	8.7355	45.8265	72.2195
r-l	Detached length from both ends			
c/a	Ω_1	Ω_2	Ω_3	Ω_4
0.00	21.3986	68.4983	109.7143	173.9334
0.20	20.0483	68.4690	101.7443	173.2244
0.40	17.5292	67.7836	89.5540	168.8866
0.60	14.7519	63.9984	71.9633	141.3415
0.80	12.0593	49.6174	57.1995	98.4669
1.00	7.3473	22.7762	42.9591	73.1543

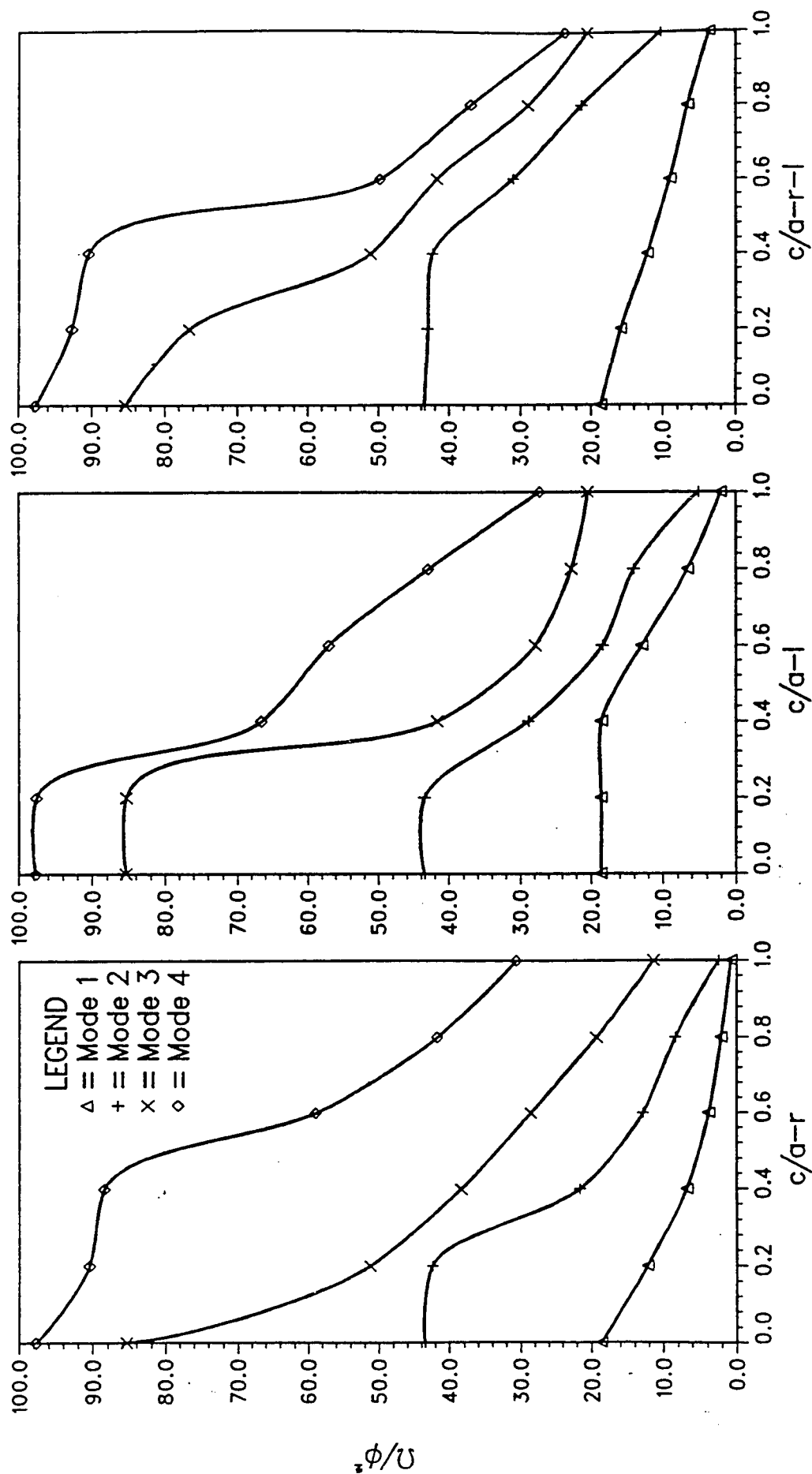


Figure 4.10: Variation of the non-dimensional frequency with c/a , for P-F-F triangular sandwich plate (carbon-epoxy faces).
 $a = 10.0 \text{ in}$, $\phi = b/a = 1.0$, $a/t = 31.9489$.

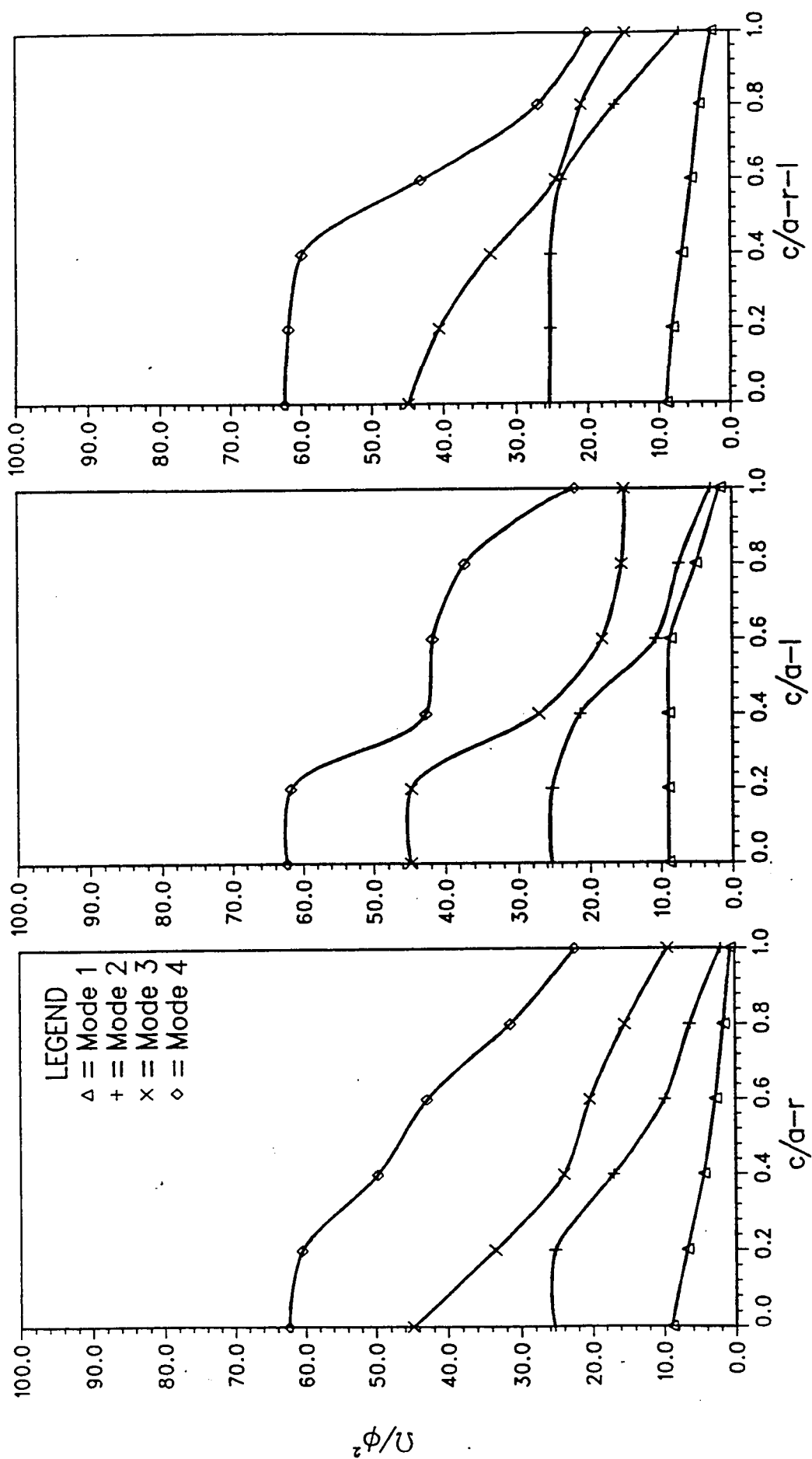


Figure 4.11: Variation of the non-dimensional frequency with c/a , for P-F-F triangular sandwich plate (carbon-epoxy faces).
 $a = 10.0\text{in}$, $\phi = b/a = 1.5$, $a/t = 31.9489$.

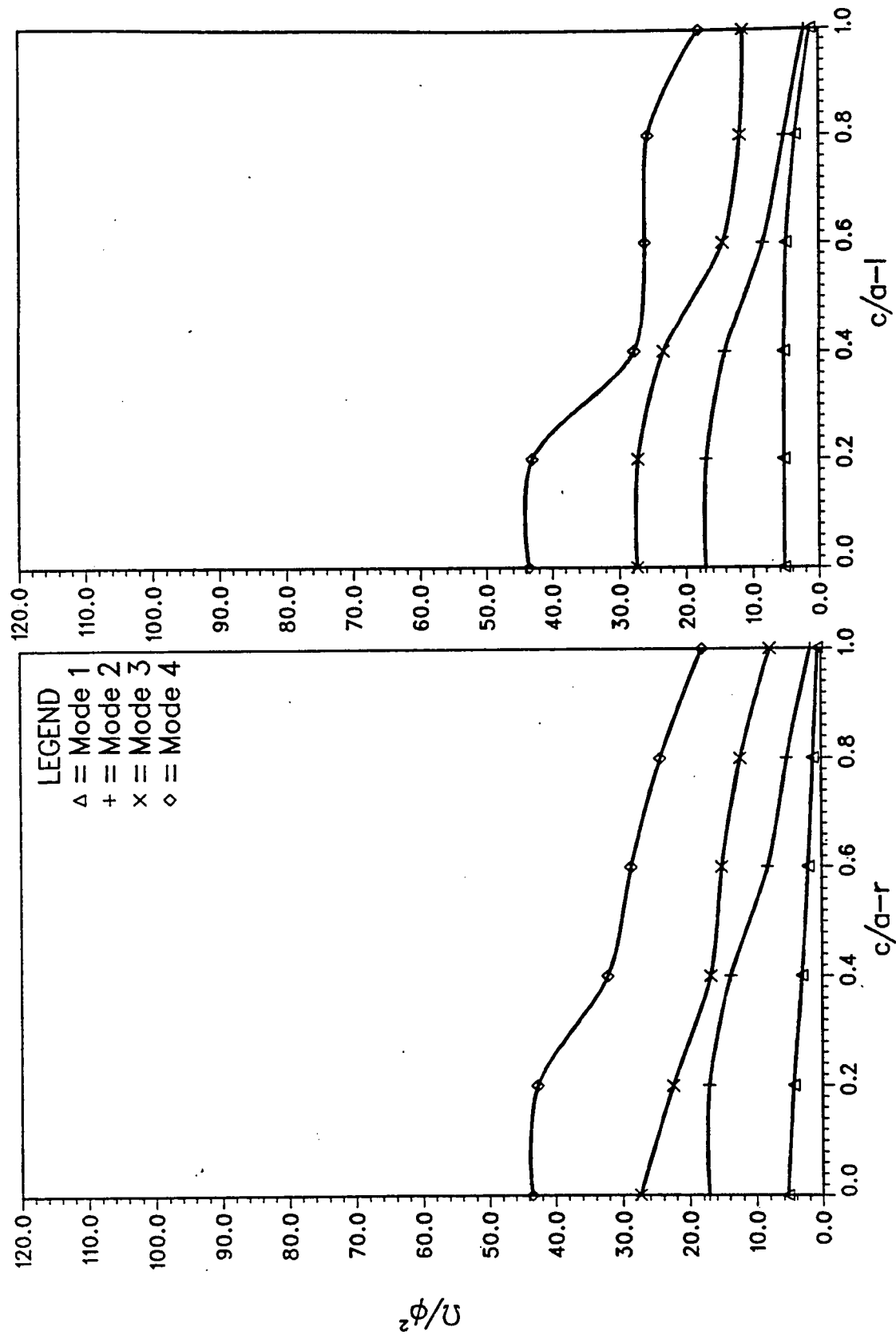


Figure 4.12: Variation of the non-dimensional frequency with c/a , for P-F-F triangular sandwich plate (carbon-epoxy faces).
 $a = 10.0in$, $\phi = b/a = 2.0$, $a/t = 31.9489$.

4.5 Discussion of Results and Conclusions for Plates

From the results presented in the graphical and tabular form the following conclusions are made:

1- On the basis of this study the isoparametric quadrilateral 8-node element with reduced and selective integration emerges as the best suited for use with Mindlin plate theory. For this technique, integration in the transverse direction could be done numerically and individual sampling points could be allowed to behave elastically as governed by the constitutive law. This approach would also facilitate the inclusion of in-plane stresses to which plates are generally subjected. Furthermore, the transverse shear effects in thin plates can be effectively eliminated by employing a reduced (2x2) Gaussian integration rule. However, for thick plates the optimum numerical integration order is also a two point rule. Therefore, for all thick and thin plates a two point quadrature is recommended.

Although there are several different schemes for deriving element mass matrices, in this study the consistent mass matrix has been evaluated using a 3x3 Gaussian quadrature rule.

The numerical problems presented in degenerate cases give results which compare well with solutions from other sources.

2- In order to establish a general data base, for the case of triangular plates the dimensionless frequencies are presented in all the tables and graphs. Therefore, these graphs and tables can be used to estimate the natural frequencies of plates

with similar geometries but with different materials. for a numerical example, let us generate the first four modes of vibration of a steel plate assuming the elastic and geometric properties as :

$$a = 10in, \phi = \frac{b}{a} = 1, t = 0.063in$$

$$E = 29 \times 10^6 psi, \nu = 0.3, \text{ density} = 0.282lb/in^3$$

Thus;

$$\rho = \frac{0.282}{386} = 7.31 \times 10^{-4} lb.sec^2.in^{-4}$$

$$D = \frac{Et^3}{12(1-\nu^2)} = \frac{29 \times 10^6 \times 0.063^3}{12(1-0.3^2)} = 686.94$$

and

$$\Omega = \omega a^2 \phi^2 \left(\frac{\rho t^3}{D_e} \right)^{0.5} = 0.0259\omega$$

The dimensionless frequencies Ω for the first to fourth modes of this plate can be obtained from Table 4.3. These values are,

6.162, 23.445, 32.625 and 56.203.

Therefore,

$$\omega_1 = 37.896 \text{ rad/sec} \quad \omega_2 = 144.187 \text{ rad/sec}$$

$$\omega_3 = 200.644 \text{ rad/sec} \quad \omega_4 = 345.634 \text{ rad/sec}$$

3- (a) Isotropic Triangular Plates

For the case of isotropic triangular plates the natural frequencies and corresponding modal shapes have been considered. Computations have been carried out for the following three cases by varying the aspect ratio

- detached length from right angle end or case (r),
- detached length from acute angle end or case (l),
- detached length from both ends or case (r - l).

For the case (r), the variation of frequency with $\frac{r}{a}$ is higher than those for the other

two cases. This is true because most of the mass distribution is along the right angle side of the plate. Therefore, the detached length from right angle end has a significant effect on the natural frequencies and modal shapes. Another general observation for all cases ($r, l, r - l$) is that as the aspect ratio increases the variation of frequency with $\frac{c}{a}$ severely decreases. As can be seen from Table 4.5 and Figure 4.5, for an aspect ratio 2 even for the 80 percent detached length there is little change in modal frequencies of plates. It is obvious that when the aspect ratio increases the plate behavior becomes similar to the beam response. It is to be noted that the beam frequency is independent of the beam width [32].

The third general observation that is common among all three types of detached lengths and various aspect ratios is that, the greatest effect of detached length on natural frequencies takes place in the higher modes. Therefore, regardless of plate aspect ratio, all the examples illustrate the sensitivity of natural frequency in higher modes to the detached length. In other words we may get a clearer understanding from comparing fundamental modes with higher modes. For example, the fundamental frequency of plates with any $\frac{b}{a}$ ratio is very slightly affected by detached length. An interesting relationship occurs in the fundamental frequency of plates with aspect ratio 2, where the frequencies of the fully supported and the point supported plate are almost the same.

3- (b) Orthotropic Triangular Plates.

Comparing results from orthotropic plates with isotropic triangular plates which have the same geometry we find almost the same general trend as the isotropic plate. The only difference that can be noted is that the natural frequency is more

sensitive to detached length for orthotropic plates than for isotropic plates. The observation may be explained by the fact that in the isotropic case the rigidity is equal in all directions in the x-y plane. But in orthotropic plates the stiffnesses are greatly different in the longitudinal and lateral directions. Hence, when the detached length increases the contribution of stiffness should be different in these two cases. Therefore, when the plate goes into higher modes the vibration behavior for orthotropic material shows different patterns from those for isotropic material.

3- (c) Sandwich Triangular Plates.

The vibrational behavior of sandwich plates is similar to that of orthotropic plates with similar geometry. The reason for this similarity is that the face sheets considered in the sandwich plates were exactly the same as those for orthotropic plates. Therefore, it could be concluded that for sandwich plates the vibration behaviors are most similar to the vibration behavior of face sheets.

Part II

SHELLS

Chapter 5

NUMERICAL RESULTS FOR SHELLS

5.1 Computation Procedure

In this chapter, we consider the vibration of cylindrical shell structures, which are partially supported along a boundary and free along other three boundaries. The present study is based on the assumed strain 9-node finite element formulation [55], [125]. As mentioned in chapter 3, this element is one of the degenerate shell elements, which has five degrees of freedom per node(5-DOF), i.e. three translational displacements in the direction of the global axes and two rotations with respect to the axes in the planes of the middle surfaces.

All through the discussion the symbols F, S, C and P denote free, simple, clamped and partially supported edges, respectively, as shown in Figure 5.1. The figure also shows a typical finite element grid. The symbols c/a , denote unsupported length ratio as shown in Figure 5.2. This may be from one or both sides.

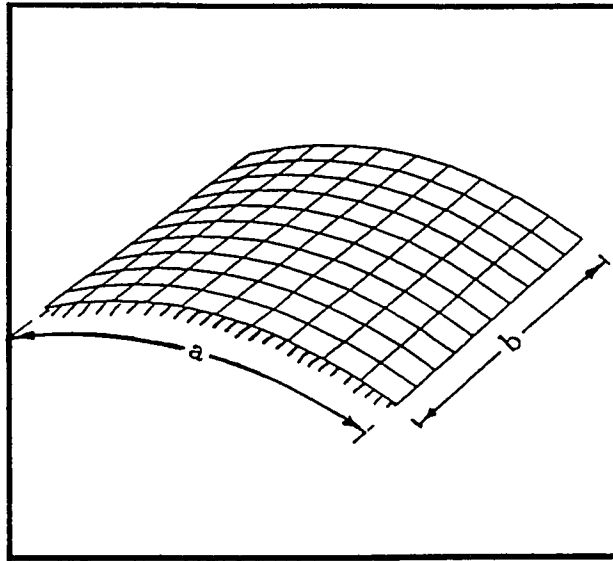


Figure 5.1: Typical C-F-F-F cylindrical shell.

All the computations in this investigation have been carried out by varying three essential parameters under study. These are the r/a , the aspect ratio b/a and unsupported length c/a as shown in Figure 5.2. In order to demonstrate the accuracy of the analysis, the results for several degenerate case of cylindrical shell with full support ($c/a = 0$), are presented and compared with previously published results. Results are then given for partially supported cylindrical cantilever shells.

For generating a data base, the non-dimensional frequency parameter used in the tables is defined as:

$$\Omega = 2\pi\omega a^2 \phi^2 \left(\frac{\rho_e t_e}{D_e} \right)^{0.5} \quad (5.1)$$

where

ϕ = aspect ratio (b/a)

Ω = nondimensional frequency

ω = frequency(Hz.)

t_e = shell thickness

ρ_e = effective mass density

D_e = effective shell rigidity

Trial computations were conducted for several alternative grid forms, CPU time, and the number of elements, it was concluded that good convergence is achieved with 25 nine-node elements. This computation has not been shown here. The tables in this section contain frequencies for the first six modes. Computations for each geometry of shell are carried out for the following three cases:

- 1) Isotropic shells
- 2) Orthotropic shells
- 3) Layered shells such as, sandwich laminated or other anisotropic shells.

In each case, the data has been generated by varying the three parameters r/a , b/a and c/a .

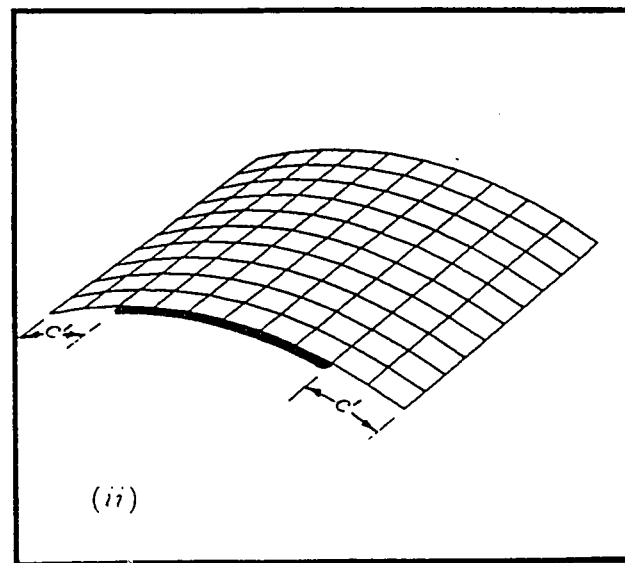
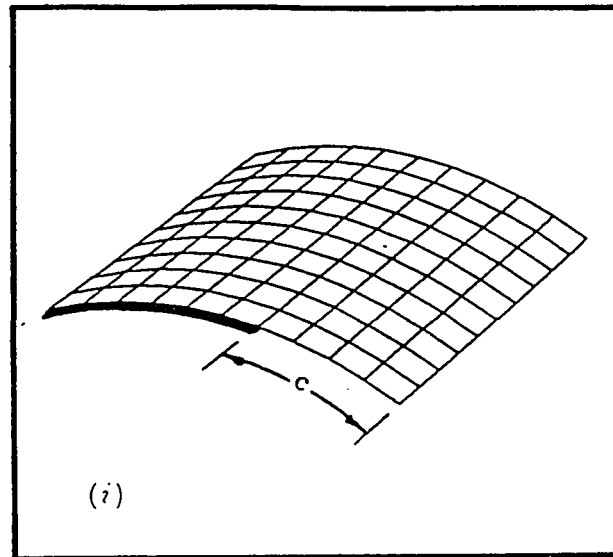


Figure 5.2: Partially supported cylindrical shell,
 (i) support detachment from one side (c/a),
 (ii) support detachment from both sides ($2c'/a$),

5.2 Cylindrical Isotropic Shells

For computation of data for cylindrical isotropic shells, the curved length ' a ' along the support is taken as 12.0 in.(304.8 cm) and the thickness $t=0.12$ in.(0.3048 cm) is the same for all cases, Figure 5.1. For this type of shell the dimensionless frequency Ω is defined as:

$$\Omega = 2\pi\omega a^2 \phi^2 \left(\frac{12\rho}{Et^2} \right)^{0.5} \quad (5.2)$$

The material properties for isotropic shells material have been summarized in Table 5.1.

Table 5.1: Material Properties for Isotropic Shell

Material property	Assumed value	System of units
ν	0.3	
ρ	1.0 (arbitrary)	$lb.sec^2/in^4$
E_{11}/E_{22}	1.0 (arbitrary)	psi/psi
E_{22}/G_{12}	2.6	psi/psi

5.2.1 Fully Supported Cylindrical Isotropic Shells

Almost all the results available in literature are in this category. These cases have been generated to test the accuracy of the element and computations. This is why only one case is presented here for comparative study.

Table 5.2 shows the comparison of the lowest six frequency parameters for uniform cylindrical blades with those obtained by several other researchers. It may be seen that the present results are in close agreement with other published results with the largest discrepancy of 1.4 % occurring at the higher modes.

Table 5.2: Comparison of Frequency Parameters $\Omega = 2\pi\omega a^2\phi^2(\frac{12\rho}{Et^3})^{0.5}$ for C-F-F-F Cylindrical Shell.

	$a = 12.in, r/a = 2.0, \phi = b/a = 1.0, a/t = 100.$					
Source of results	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
Present(5 × 5)	11.153	17.958	31.973	44.671	50.400	69.090
Ref.[33]	11.080	17.909	31.824	44.064	49.604	67.642
Exp.[33]	11.080	17.264	33.370	45.223	50.892	68.414
Ref.[36]	11.158	17.934	32.377	44.914	50.686	68.723
Exp.[36]	11.029	17.329	33.370	45.223	50.891	68.414
Ref.[37]	11.158	17.986	32.082	45.223	51.343	68.865
Ref.[113]	11.003	17.702	31.670	45.340	52.078	67.976
Ref.[120]	12.043	19.017	32.867	50.647	54.563	68.840
Exp.[120]	11.158	17.458	33.356	45.172	50.917	68.427

5.2.2 Partially Detached Cantilever Cylindrical Shells.

The frequency parameters for P-F-F-F , cylindrical shells with r/a (1.0, 2.0 and 3.0) and for different aspect ratios b/a (1.0, 2.0 and 3.0) are presented in Tables 5.3 to 5.11 for several values of the partial support ratios c/a and $2c'/a$. The latter ratio is used for the case when the shell has detached boundaries from the two sides as shown in Figure 5.2. Each of these group of tables contains two tables i.e, (i) and (ii), where the first one contains frequency parameters for support detachment from one side, and the second for detachment from both sides. Figures 5.3 to 5.11 show variation of frequency parameter with c/a and $2c'/a$ for the lowest six modes for a P-F-F-F shell with aspect ratios 1, 2 and 3.

Modal shapes and displacement patterns with nodal lines are shown in Figures B.1 to B.24.

Table 5.3: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{12\rho}{Et^2})^{0.5}$ for P-F-F-F
Cylindrical Isotropic Shell(ish $\psi 1\phi 1$).

$a = 12.in, r/a = 1.0, \phi = b/a = 1.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.671	23.615	47.523	59.655	60.891	70.911
0.2	11.101	21.355	38.259	48.434	60.167	70.065
0.4	6.633	18.129	22.844	29.969	59.882	67.993
0.6	4.019	10.518	14.894	26.785	54.111	60.611
0.8	2.254	5.448	11.640	24.918	34.829	40.332
0.9	1.777	4.404	10.833	22.399	23.631	32.764
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.671	23.615	47.523	59.655	60.891	70.911
0.2	13.676	21.771	44.381	56.038	57.559	69.764
0.4	9.461	15.572	33.363	48.173	48.458	68.106
0.6	7.727	10.317	27.833	39.478	40.324	66.922
0.8	4.786	6.063	20.978	23.861	30.648	53.940

Table 5.4: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{12\rho}{Et^2})^{0.5}$ for P-F-F-F Cylindrical Isotropic Shell(ish $\psi 1\phi 2$).

$a = 12.in, r/a = 1.0, \phi = b/a = 2.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	21.878	41.795	91.433	98.040	116.295	187.387
0.2	18.652	33.430	84.024	97.152	111.407	171.906
0.4	12.543	24.031	65.650	93.743	103.105	126.164
0.6	7.490	17.548	38.397	88.219	93.205	119.253
0.8	3.996	11.937	26.128	55.503	91.891	94.717
0.9	3.128	10.442	23.860	36.105	79.445	93.651
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	21.878	41.795	91.433	98.040	116.295	187.387
0.2	20.686	38.969	86.385	97.939	112.347	183.023
0.4	17.111	25.628	76.169	96.344	105.579	170.632
0.6	15.217	15.967	70.777	93.754	101.882	155.318
0.8	7.388	13.310	55.144	75.006	87.658	96.346

Table 5.5: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{12\rho}{Et^2})^{0.5}$ for P-F-F-F Cylindrical Isotropic Shell($ish\psi 1\phi 3$).

$a = 12.in, r/a = 1.0, \phi = b/a = 3.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	28.512	43.930	109.341	186.882	213.310	245.843
0.2	25.697	37.219	102.641	174.608	210.438	232.576
0.4	17.234	29.339	95.305	148.914	190.135	204.451
0.6	9.948	24.374	68.336	107.942	138.158	197.386
0.8	5.308	18.874	44.086	66.839	114.422	188.652
0.9	4.154	17.082	39.026	46.761	105.166	178.578
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	28.512	43.930	109.341	186.882	213.310	245.843
0.2	27.597	42.068	105.154	181.687	212.518	236.479
0.4	24.489	30.407	94.296	167.843	210.187	210.353
0.6	19.923	22.482	87.907	161.498	185.715	209.283
0.8	9.725	20.401	75.632	114.117	142.046	171.364

Table 5.6: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{12\rho}{Et^2})^{0.5}$ for P-F-F-F Cylindrical Isotropic Shell($ish\psi 2\phi 1$).

$a = 12.in, r/a = 2.0, \phi = b/a = 1.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	11.153	17.959	31.974	44.671	50.401	69.090
0.2	9.092	14.650	29.813	39.297	48.178	68.678
0.4	5.874	11.334	21.563	29.360	46.877	56.551
0.6	3.531	7.837	14.408	27.381	45.619	52.391
0.8	2.094	5.071	11.620	25.401	36.225	37.340
0.9	1.696	4.401	10.964	21.152	24.324	33.355
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	11.153	17.959	31.974	44.671	50.401	69.090
0.2	10.406	16.161	31.387	42.789	48.415	68.866
0.4	8.510	10.415	28.474	37.335	42.523	67.988
0.6	6.911	7.479	26.162	32.463	37.374	62.138
0.8	3.868	5.901	20.567	22.202	30.635	52.423

Table 5.7: Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2(\frac{12\rho}{Et^2})^{0.5}$ for P-F-F-F Cylindrical Isotropic Shell(*ish* $\psi 2\phi 2$).

$a = 12.in, r/a = 2.0, \phi = b/a = 2.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	17.835	22.064	64.276	91.023	102.507	137.380
0.2	15.598	19.201	60.858	86.874	99.869	130.553
0.4	9.775	17.074	53.210	77.964	97.688	103.434
0.6	5.727	14.611	36.369	66.911	92.631	96.311
0.8	3.432	11.316	25.608	53.379	71.636	95.805
0.9	2.827	10.329	23.602	35.852	62.124	93.640
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	17.835	22.064	64.276	91.023	102.507	137.380
0.2	17.322	20.792	62.409	88.801	101.146	133.955
0.4	14.559	15.707	57.520	82.686	98.858	124.392
0.6	9.962	14.610	54.273	79.005	98.060	116.494
0.8	5.744	12.804	46.398	66.766	87.477	93.118

Table 5.8: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{12\rho}{Et^2})^{0.5}$ for P-F-F-F Cylindrical Isotropic Shell($ish\psi 2\phi 3$).

$a = 12.in, r/a = 2.0, \phi = b/a = 3.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	22.614	24.495	82.844	122.972	167.976	214.721
0.2	19.079	24.093	79.620	112.793	162.855	214.007
0.4	12.172	23.312	74.520	100.029	148.196	198.995
0.6	7.262	21.401	60.681	85.678	117.021	161.175
0.8	4.417	18.124	42.193	66.566	92.181	127.818
0.9	3.659	17.069	37.981	47.249	83.029	122.370
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	22.614	24.495	82.844	122.972	167.976	214.721
0.2	21.805	24.062	81.235	118.785	164.595	214.548
0.4	16.757	22.644	76.613	105.763	155.399	213.566
0.6	12.155	21.543	73.179	99.455	147.819	211.432
0.8	7.397	19.664	66.204	93.480	109.588	135.799

Table 5.9: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{12\rho}{Et^2})^{0.5}$ for P-F-F-F Cylindrical Isotropic Shell(*ish* $\psi 3\phi 1$).

$a = 12.in, r/a = 3.0, \phi = b/a = 1.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	10.013	13.300	29.373	38.728	45.585	68.705
0.2	8.391	11.092	28.217	35.119	43.314	66.974
0.4	5.324	9.256	20.819	29.063	41.248	50.812
0.6	3.253	7.020	14.240	27.492	39.630	48.930
0.8	2.037	4.878	11.582	25.442	33.966	38.119
0.9	1.670	4.310	10.970	20.993	24.344	32.368
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	10.013	13.300	29.373	38.728	45.585	68.705
0.2	9.592	12.017	29.127	37.576	44.017	68.571
0.4	7.956	8.269	27.353	33.238	39.409	66.614
0.6	5.634	7.398	25.483	29.432	35.544	57.660
0.8	3.534	5.824	20.072	21.325	30.404	47.283

Table 5.10: Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2(\frac{12\rho}{Et^2})^{0.5}$ for P-F-F-F
Isotropic Cylindrical Isotropic Shell(ish $\psi 3\phi 2$).

$a = 12.in, r/a = 3.0, \phi = b/a = 2.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.081	16.705	57.274	73.775	99.222	117.896
0.2	12.364	16.324	54.725	69.012	98.491	112.953
0.4	7.932	15.492	48.180	64.144	89.970	97.783
0.6	4.945	13.762	34.941	58.733	76.861	96.581
0.8	3.218	10.977	25.202	51.367	66.819	80.897
0.9	2.712	10.092	23.322	36.305	55.824	77.478
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.081	16.705	57.274	73.775	99.222	117.896
0.2	14.307	16.401	56.241	71.161	98.940	115.954
0.4	10.553	15.288	52.770	64.957	98.075	108.879
0.6	7.814	14.399	50.052	61.967	97.583	102.709
0.8	5.080	12.626	43.517	55.925	84.982	87.866

Table 5.11: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{12\rho}{Et^2})^{0.5}$ for P-F-F-F
Isotropic Cylindrical Isotropic Shell(*ish* $\psi 3\phi 3$).

$a = 12.in, r/a = 3.0, \phi = b/a = 3.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.370	23.342	76.204	87.073	147.046	196.001
0.2	13.527	22.835	73.318	81.493	143.255	187.758
0.4	9.456	22.028	66.255	76.506	131.414	171.862
0.6	6.132	20.517	54.965	72.166	105.309	147.313
0.8	4.069	17.675	40.316	64.408	87.439	99.608
0.9	3.459	16.722	36.667	48.283	74.300	95.435
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.370	23.342	76.204	87.073	147.046	196.001
0.2	14.880	23.081	75.293	84.401	145.226	191.984
0.4	11.927	22.118	72.167	75.755	139.120	182.718
0.6	9.321	21.259	69.460	71.307	133.659	177.072
0.8	6.392	19.462	63.372	67.513	108.974	123.826

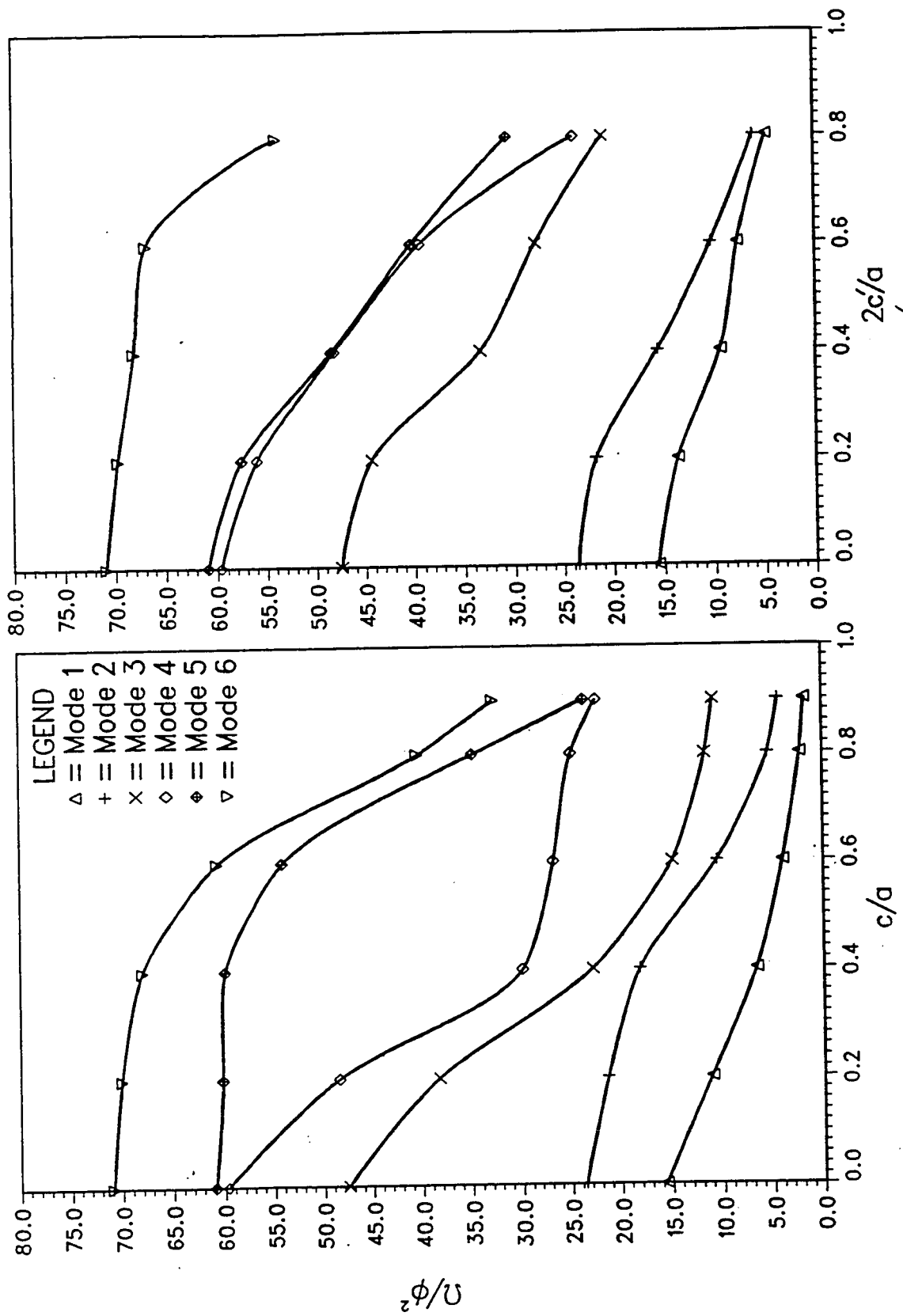


Figure 5.3: Variation of the Ω/ϕ^2 vs c/a , for P-F-F isotropic shell.
 $r/a = 1.0$, $\phi = b/a = 1.0$, $a/t = 100.00$

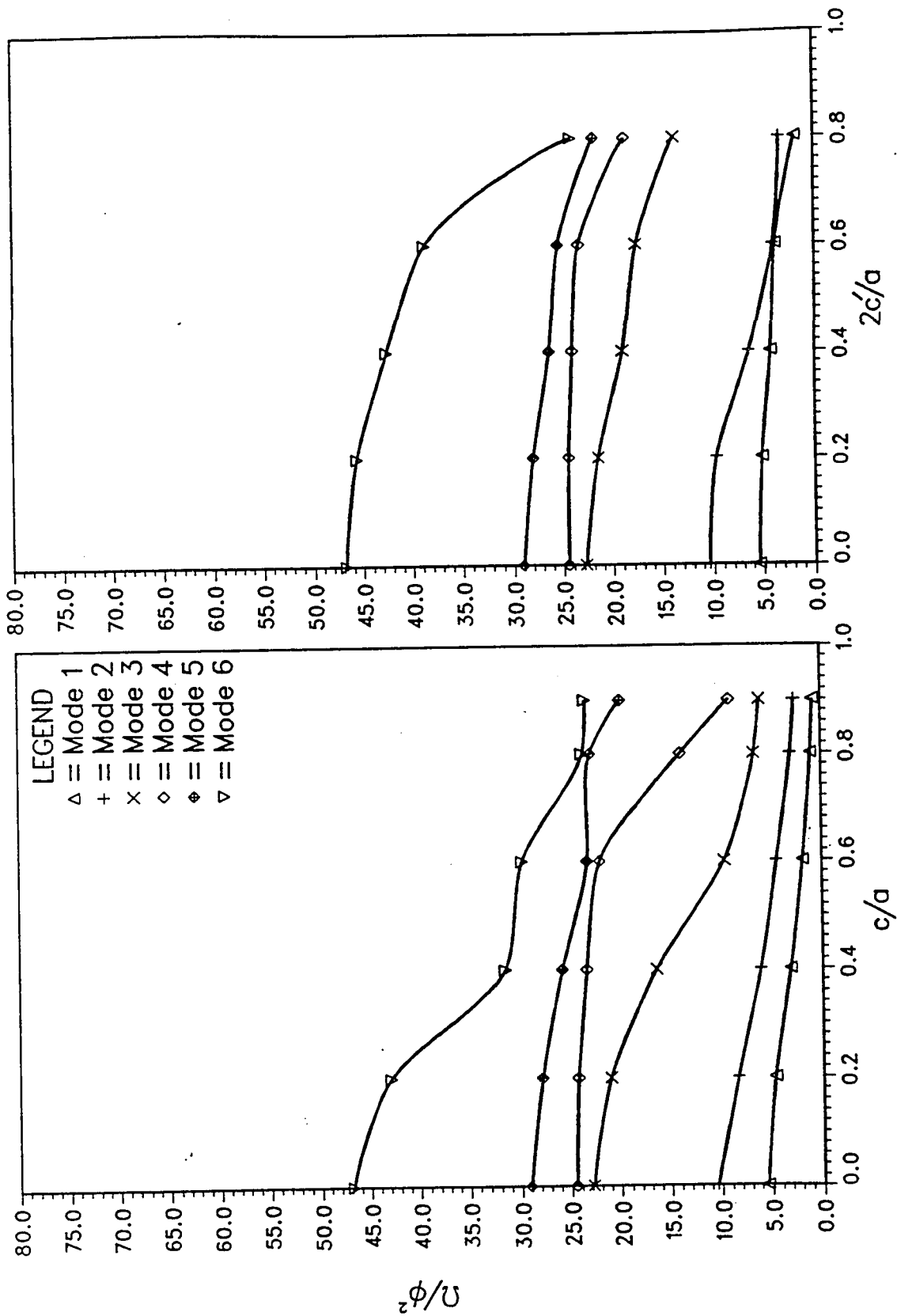


Figure 5.4: Variation of the Ω/ϕ^2 vs c/a , for P-F-F isotropic shell.
 $r/a = 1.0$, $\phi = b/a = 2.0$, $a/t = 100.00$

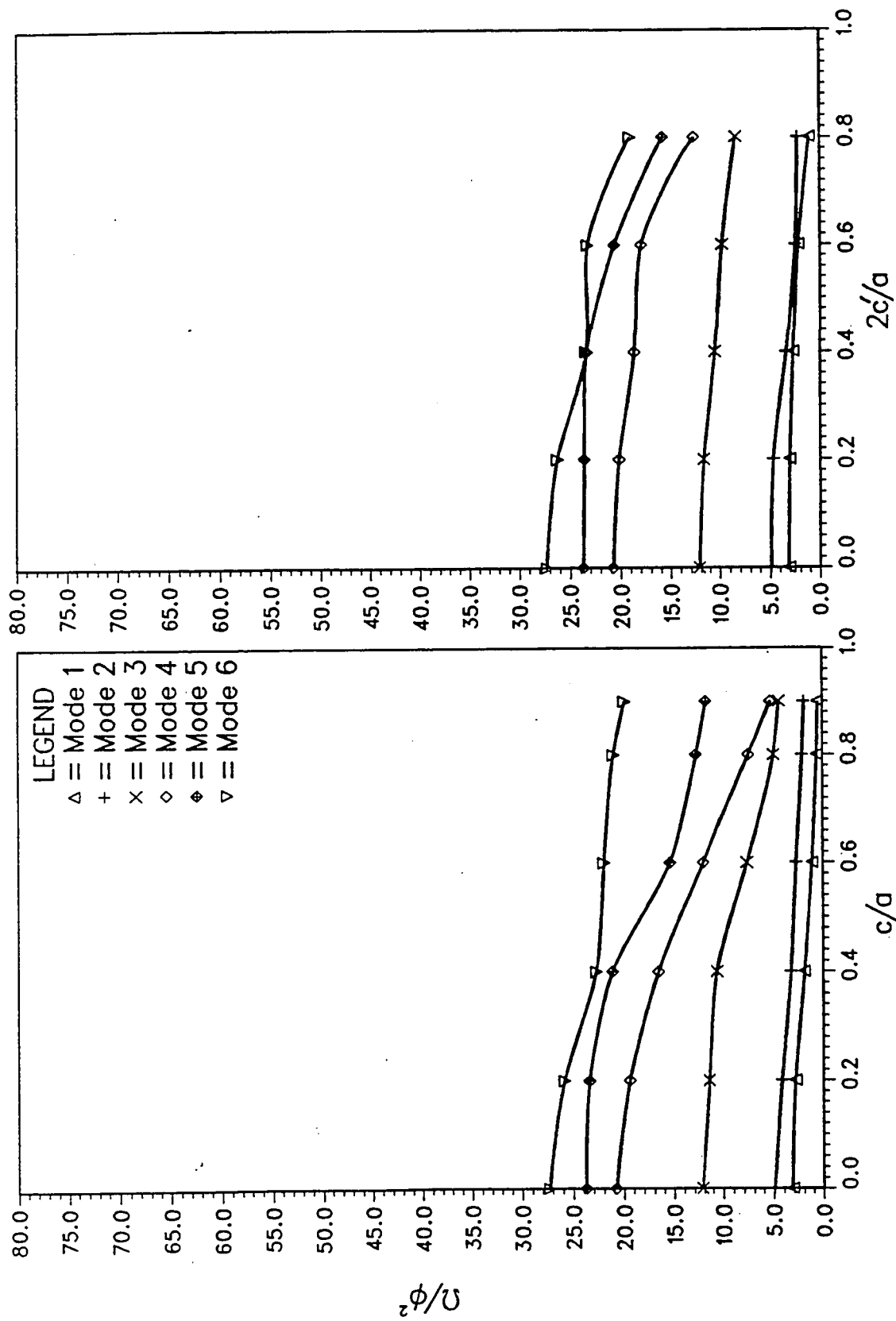


Figure 5.5: Variation of the Ω/ϕ^2 vs c/a , for P-F-F isotropic shell.

$r/a = 1.0$, $\phi = b/a = 3.0$, $a/t = 100.00$

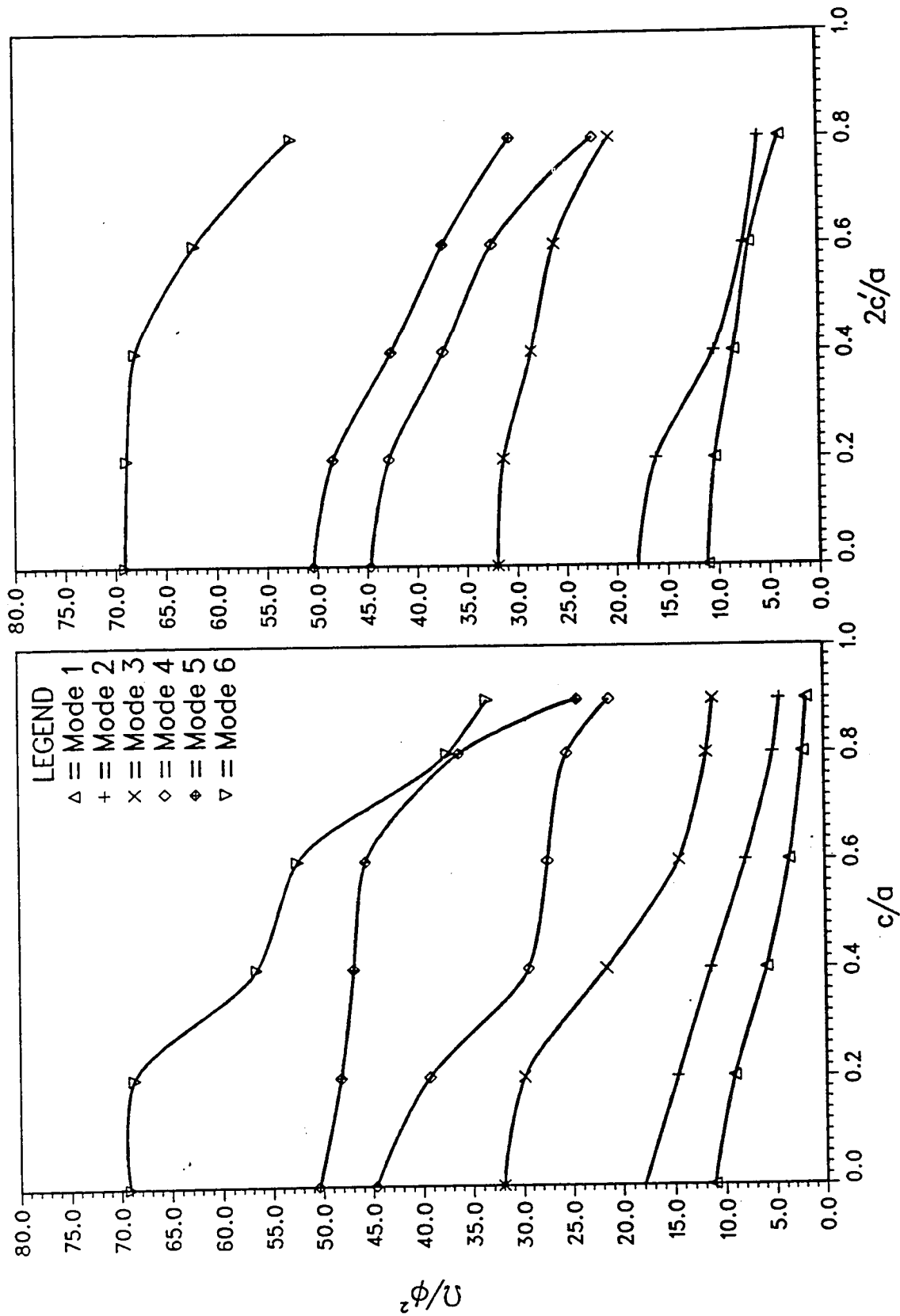


Figure 5.6: Variation of the Ω/ϕ^2 vs c/a , for P-F-F isotropic shell.
 $r/a = 2.0$, $\phi = b/a = 1.0$, $a/t = 100.00$

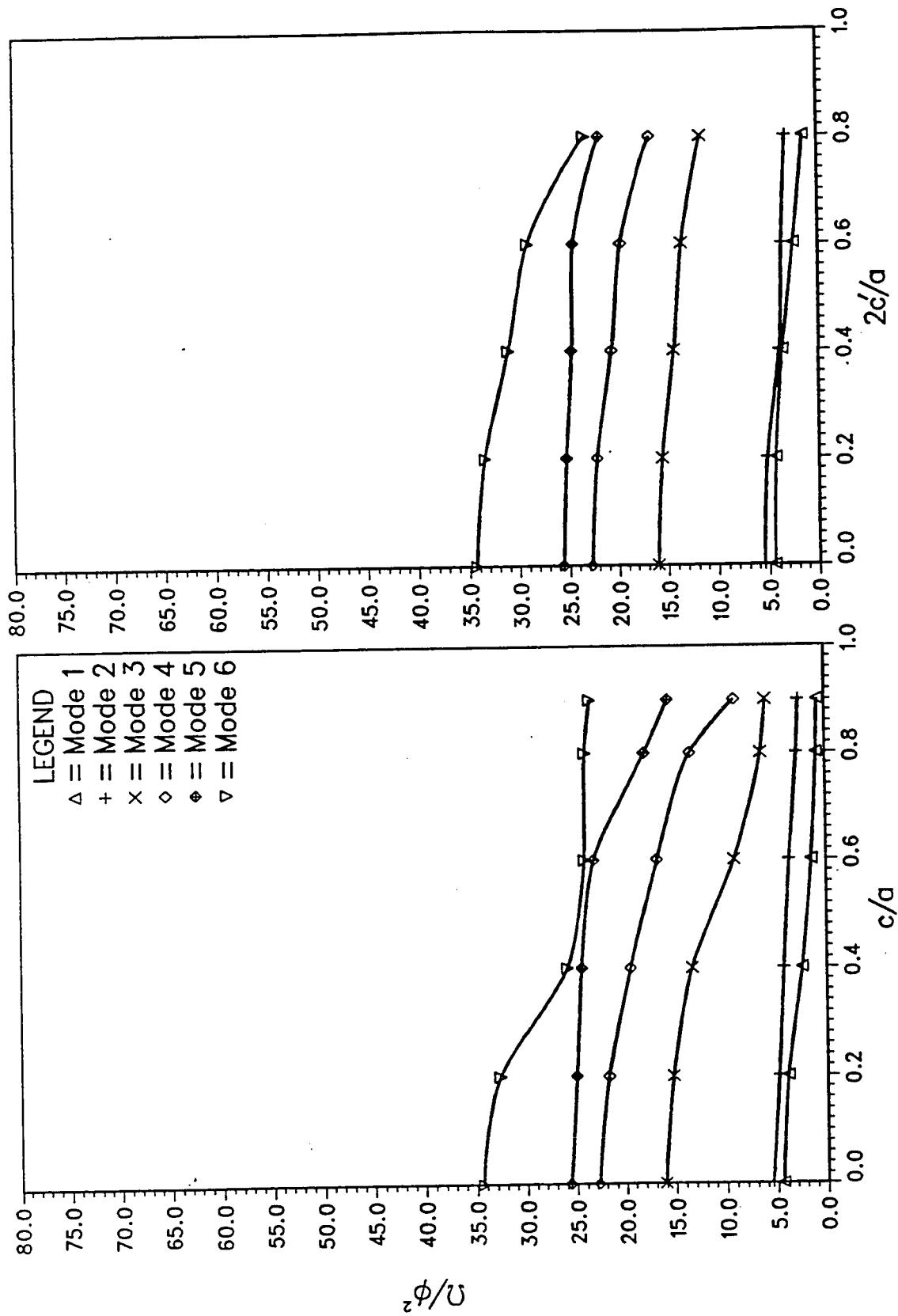


Figure 5.7: Variation of the Ω/ϕ^2 vs c/a , for P-F-F isotropic shell.
 $r/a = 2.0$, $\phi = b/a = 2.0$, $a/t = 100.00$

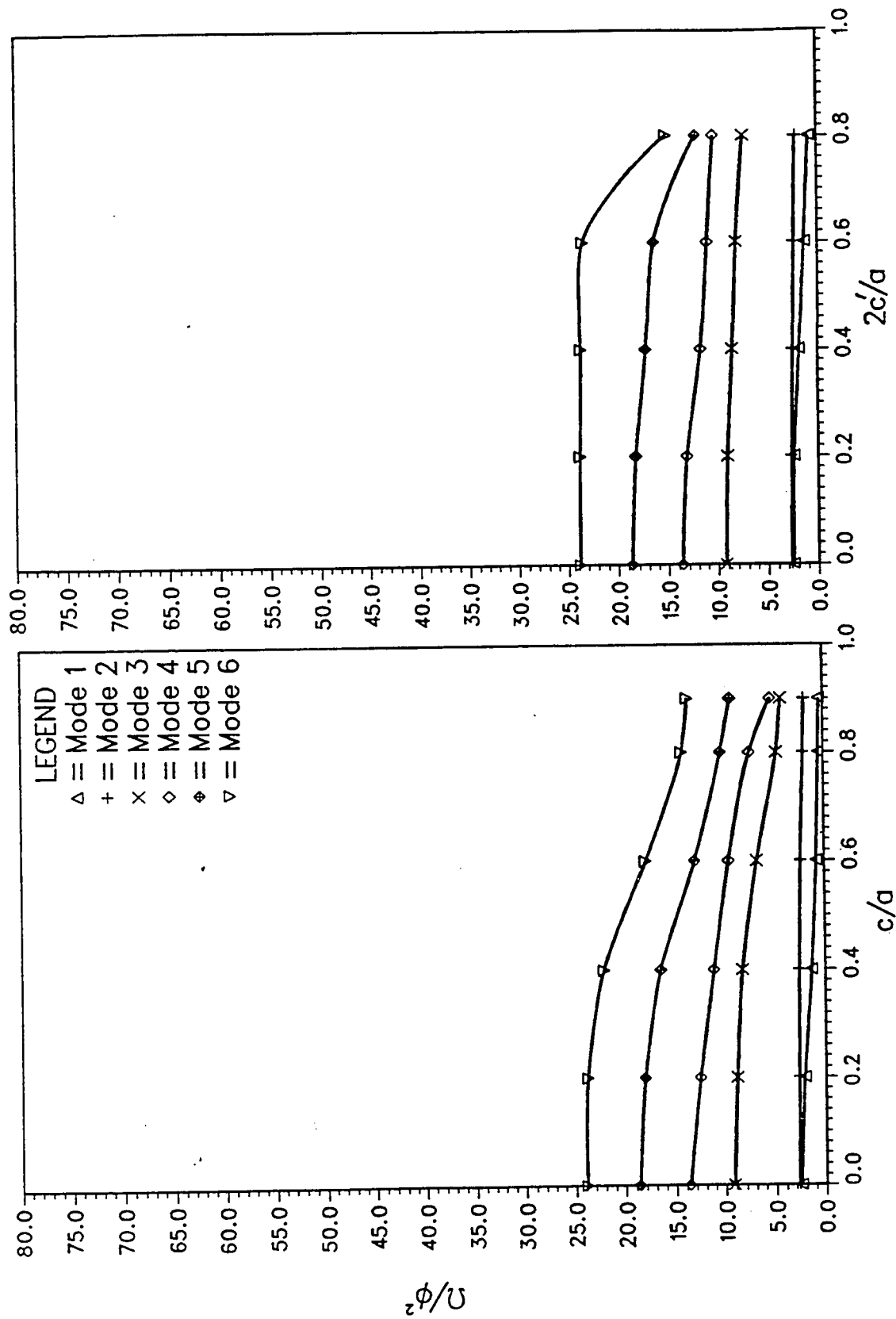


Figure 5.8: Variation of the Ω/ϕ^2 vs c/a , for P-F-F-F isotropic shell.
 $r/a = 2.0$, $\phi = b/a = 3.0$, $a/t = 100.00$

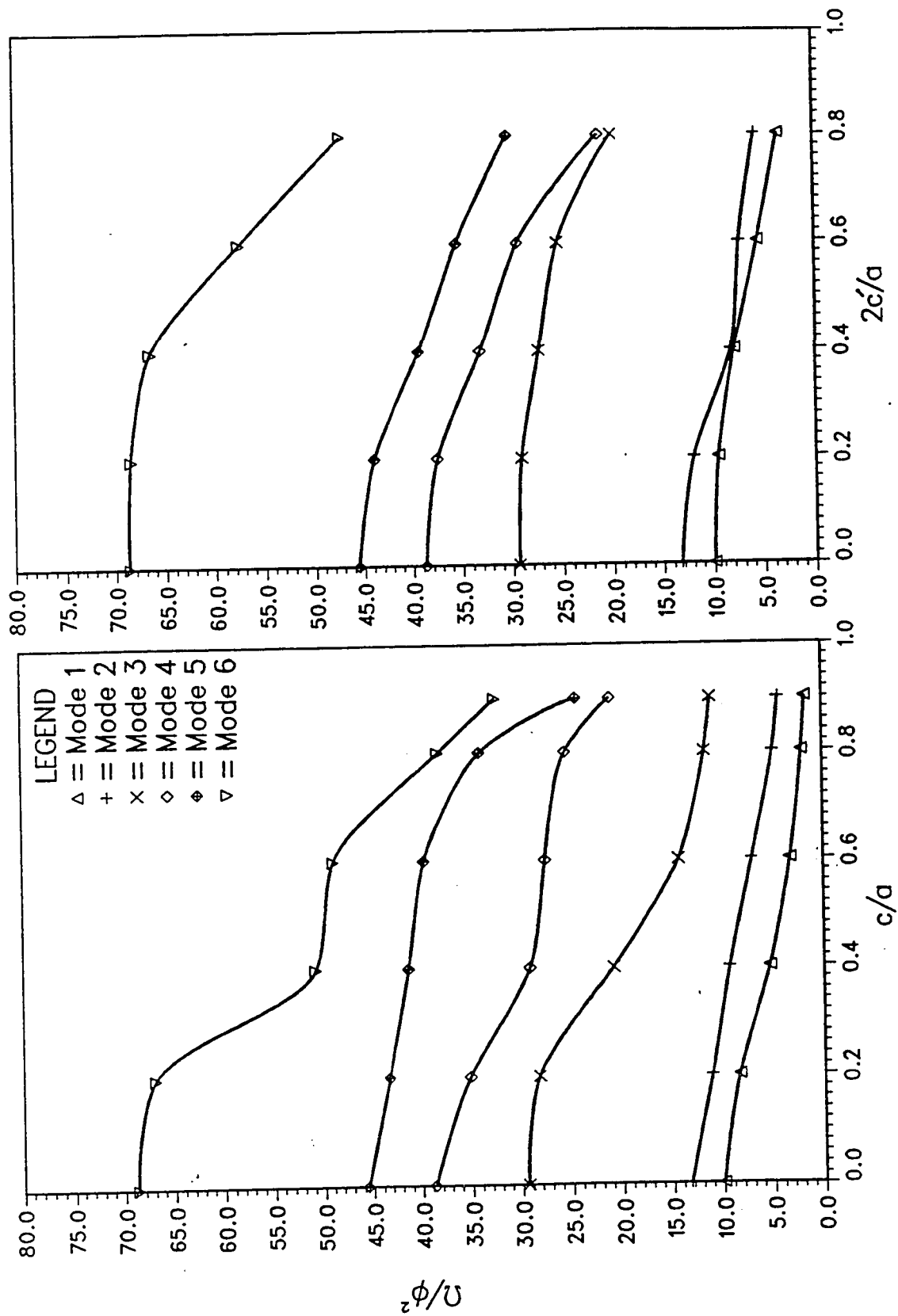


Figure 5.9: Variation of the Ω/ϕ^2 vs c/a , for P-F-F isotropic shell.
 $r/a = 3.0$, $\phi = b/a = 1.0$, $a/t = 100.00$

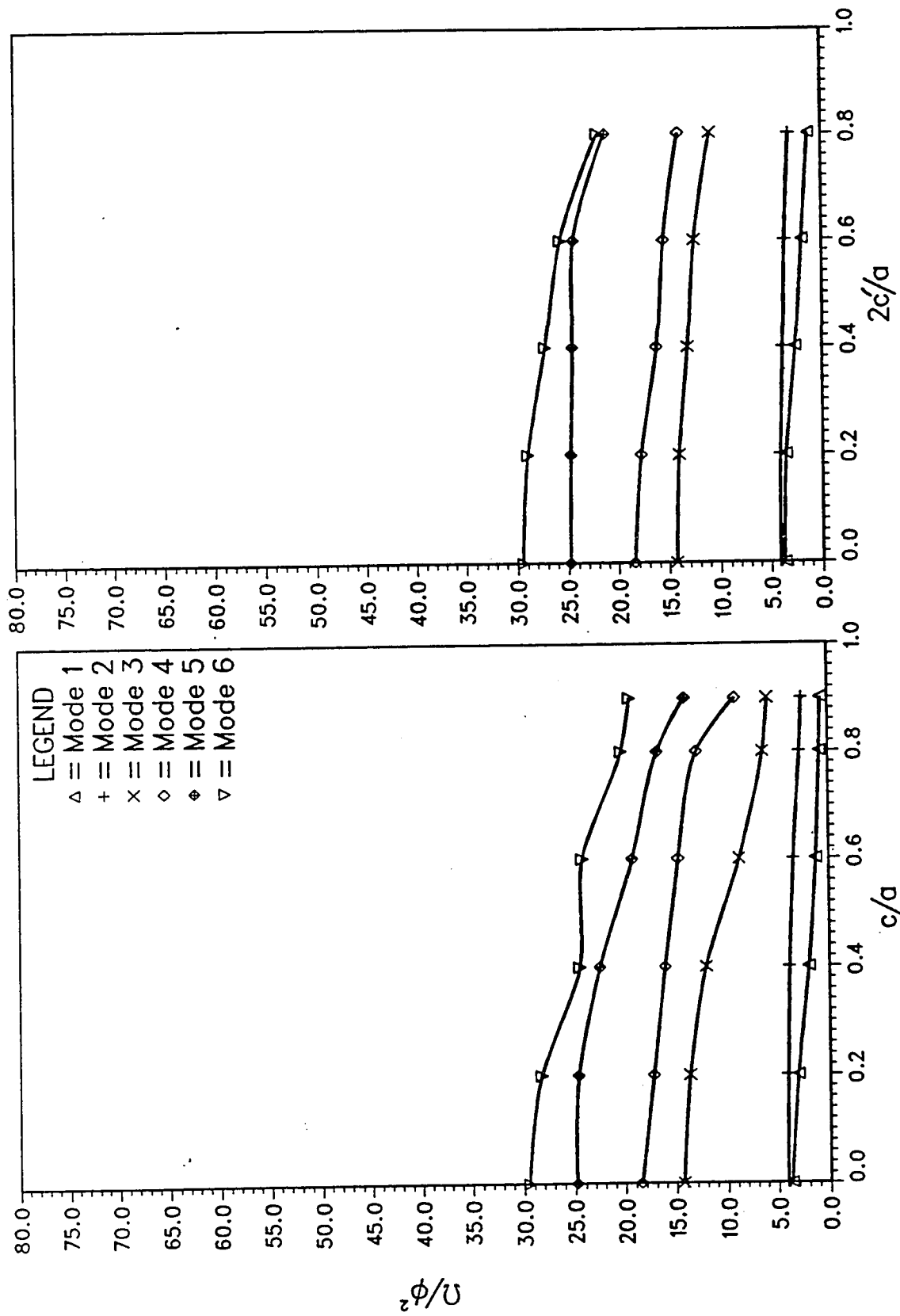


Figure 5.10: Variation of the Ω/ϕ^2 vs c/a , for P-F-F-F isotropic shell.
 $r/a = 3.0, \phi = b/a = 2.0, a/t = 100.00$

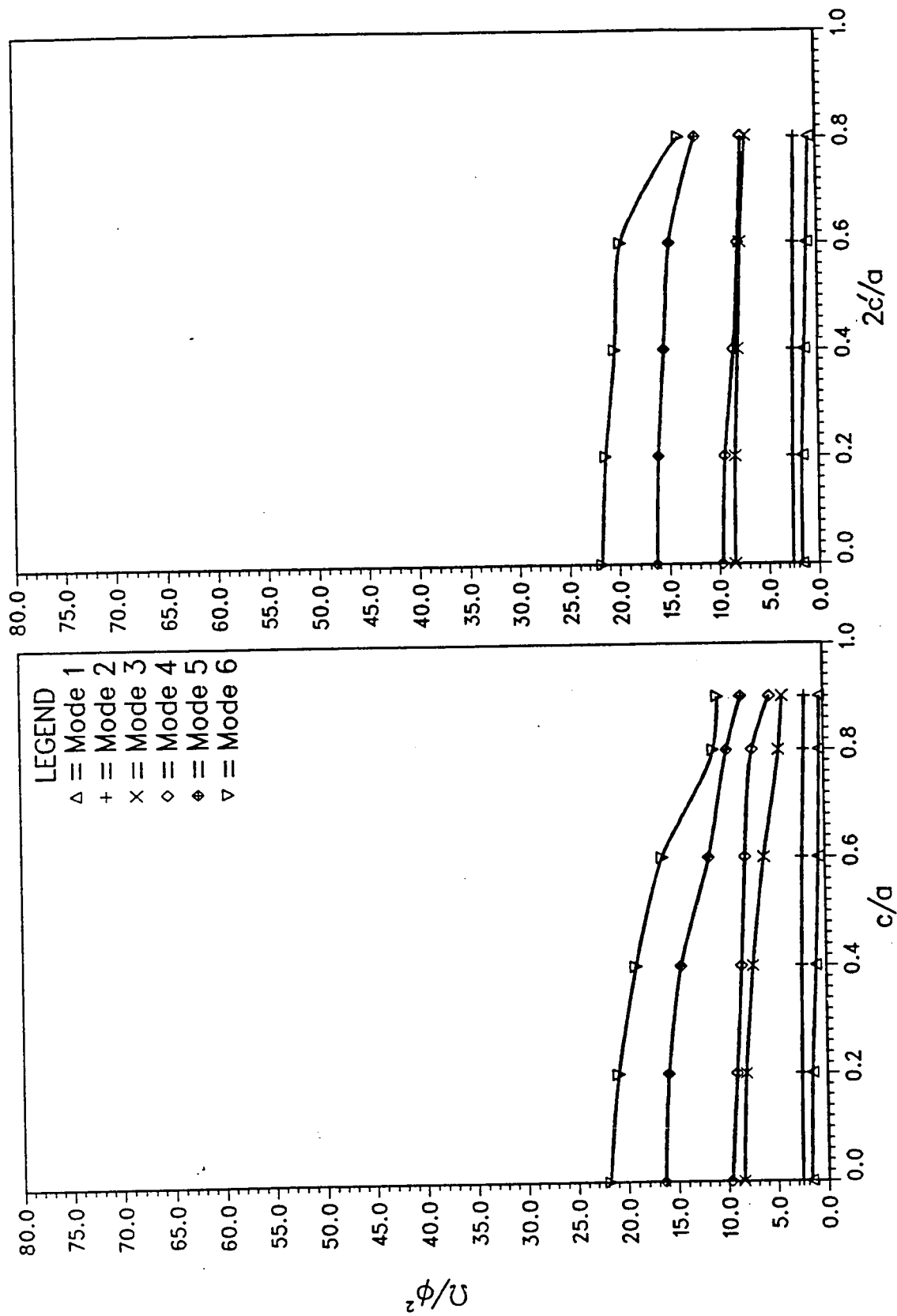


Figure 5.11: Variation of the Ω/ϕ^2 vs c/a , for P-F-F-F isotropic shell.

$r/a = 3.0$, $\phi = b/a = 3.0$, $a/t = 100.00$

5.3 Cylindrical Orthotropic Shells

For computation of data for cylindrical orthotropic shells, similar to the isotropic shells, the curved length along the support $a = 12.0in.(304.8 \text{ cm})$, and the thickness $t_e = 0.12in.(0.3048 \text{ cm})$ are assumed constant for all cases, as shown in Figure 5.1.

For these type of shells the dimensionless frequency Ω is defined as:

$$\Omega = 2\pi\omega a^2 \phi^2 \left(\frac{\rho_e t_e}{D_e} \right)^{0.5} \quad (5.3)$$

In this equation

$$D_e = \nu_{12}D_{22} + 2D_{66}$$

where

$$D_{22} = \frac{E_{22}t_e^3}{12(1 - \nu_{12}\nu_{21})}$$

$$D_{66} = G_{12} \frac{t_e^3}{12}$$

and other parameters are defined as before. The material properties for orthotropic shells have been summarized in Table 5.12.

5.3.1 Partially Supported Orthotropic Shells.

The frequency parameters for P-F-F-F , cylindrical orthotropic shells with r/a (1.0, 2.0 and 3.0) and for different aspect ratios b/a (1.0, 2.0 and 3.0) are presented in Tables 5.13 to 5.21 for several values of the partial support ratios c/a and

Table 5.12: Material Properties for Orthotropic Shell

Material property	Assumed value	System of units
ν_{12}	0.315	
ρ_c	0.260×10^{-3}	$lb.sec^2/in^4$
E_{11}/E_{22}	16.5	psi/psi
E_{22}/G_{12}	2.632	psi/psi

$2c'/a$. The latter ratio is used for the case when the shell has detached boundaries from the two sides as shown in Figure 5.2. Similar to the isotropic shells, each of these group tables contains two tables (i) and (ii), of which the first one contains frequency parameters for detached base length from one side and the second one denotes to equal detached base lengths from both sides. Figures 5.12 to 5.20 show variation of frequency parameter with c/a and $2c'/a$ for the lowest six modes for a P-F-F-F orthotropic shell with aspect ratios 1, 2 and 3.

Table 5.13: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{\rho_1 t_1}{D_1})^{0.5}$ for P-F-F-F Cylindrical Orthotropic Shell (*osh* $\psi 1\phi 1$).

$a = 12.in, r/a = 1.0, \phi = b/a = 1.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	32.227	33.442	73.252	85.886	114.403	116.045
0.2	17.267	32.382	59.483	66.773	76.020	115.069
0.4	8.912	23.604	31.664	40.912	70.981	109.503
0.6	4.964	13.550	23.907	30.340	54.670	66.787
0.8	2.856	8.740	14.020	26.016	34.958	62.679
0.9	2.207	7.456	12.033	23.643	31.902	51.821
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	32.227	33.442	73.252	85.886	114.403	116.045
0.2	25.949	27.041	71.317	75.459	105.575	107.235
0.4	14.934	19.552	52.180	61.268	64.975	70.224
0.6	10.228	15.772	36.632	42.934	45.074	64.981
0.8	6.594	9.755	21.633	24.071	31.033	61.749
1.0	0.000	4.487	6.540	16.753	18.166	28.525

Table 5.14: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{\rho_e t_e}{D_e})^{0.5}$ for P-F-F-F
Cylindrical Orthotropic Shell (*osh*1 ϕ 2).

$a = 12.in, r/a = 1.0, \phi = b/a = 2.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	47.643	77.041	145.343	169.155	178.299	244.986
0.2	30.802	69.384	118.963	145.079	174.035	243.835
0.4	17.088	57.235	71.085	93.797	170.969	199.112
0.6	10.384	34.585	38.669	88.473	162.714	191.983
0.8	6.151	18.821	26.253	87.037	109.066	141.822
0.9	4.819	15.826	23.953	78.863	86.238	96.538
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	47.643	77.041	145.343	169.155	178.299	244.986
0.2	38.904	69.862	133.605	160.658	170.345	244.005
0.4	23.701	49.006	105.423	141.719	149.781	240.601
0.6	17.867	32.456	94.165	122.178	125.497	218.001
0.8	13.959	15.577	67.768	73.041	91.338	166.163
1.0	0.001	9.639	11.337	45.117	59.167	89.741

Table 5.15: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{\rho_s t_s}{D_s})^{0.5}$ for P-F-F-F
Cylindrical Orthotropic Shell (*osh* $\psi 1\phi 3$).

$a = 12.in, r/a = 1.0, \phi = b/a = 3.0, a/t = 100.$						
i	Support detachment from one side.					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	54.938	126.149	220.281	230.896	245.937	436.174
0.2	39.259	98.284	205.567	214.322	240.702	393.949
0.4	23.673	67.860	137.464	194.670	239.252	273.836
0.6	15.082	41.849	72.584	189.993	235.203	263.877
0.8	8.854	26.172	44.855	162.999	196.199	217.717
0.9	6.935	22.992	40.449	97.668	180.541	194.086
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	54.938	126.149	220.281	230.896	245.937	436.174
0.2	47.241	111.702	216.378	218.404	236.596	424.536
0.4	31.552	67.792	201.660	202.529	224.251	384.509
0.6	25.311	41.937	188.191	196.095	212.890	343.640
0.8	19.710	21.203	121.745	152.495	194.182	212.188
1.0	-	-	-	-	-	-

Table 5.16: Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2(\frac{E_c t_c}{D_c})^{0.5}$ for P-F-F-F
Cylindrical Orthotropic Shell ($osh\psi 2\phi 1$).

$a = 12.in, r/a = 2.0, \phi = b/a = 1.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	22.887	27.639	49.279	67.951	100.737	101.039
0.2	14.622	25.876	40.906	59.603	68.608	100.845
0.4	8.147	22.658	24.822	32.492	66.396	99.463
0.6	4.762	13.063	19.179	28.301	55.838	67.135
0.8	2.791	8.605	14.023	25.927	37.308	63.732
0.9	2.158	7.611	12.613	24.485	33.947	46.133
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	22.887	27.639	49.279	67.951	100.737	101.039
0.2	19.489	24.422	45.133	67.007	94.013	94.774
0.4	12.714	17.481	35.774	57.767	60.922	68.852
0.6	9.708	13.510	29.930	41.235	43.219	65.890
0.8	6.532	8.636	20.997	23.483	31.043	63.526
1.0	0.000	4.383	6.208	16.585	17.903	29.034

Table 5.17: Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2(\frac{\rho_s t_s}{D_c})^{0.5}$ for P-F-F-F
Cylindrical Orthotropic Shell ($osh\psi 2\phi 2$).

$a = 12.in, r/a = 2.0, \phi = b/a = 2.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	30.541	56.927	103.258	129.207	143.619	244.865
0.2	23.207	45.581	97.852	115.210	138.323	233.323
0.4	14.983	34.371	66.660	93.309	135.389	157.766
0.6	9.502	24.580	37.361	91.004	134.622	150.461
0.8	5.755	17.342	25.974	89.029	109.476	129.739
0.9	4.565	15.582	23.973	75.062	87.562	98.296
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	30.541	56.927	103.258	129.207	143.619	244.865
0.2	27.244	50.024	101.228	125.024	138.731	244.543
0.4	20.195	31.982	95.077	110.784	123.462	231.919
0.6	17.044	22.031	91.564	98.093	110.618	192.439
0.8	12.826	13.735	61.922	71.128	92.932	153.045
1.0	0.001	8.781	10.944	43.123	57.719	91.800

Table 5.18: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 \left(\frac{\rho_c t_c}{D_c}\right)^{0.5}$ for P-F-F-F Cylindrical Orthotropic Shell ($osh\psi 2\phi 3$).

$a = 12.in, r/a = 2.0, \phi = b/a = 3.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	36.757	71.792	153.641	202.627	212.515	349.177
0.2	30.187	55.606	144.753	199.574	205.373	318.730
0.4	20.716	40.477	116.805	180.464	198.967	229.822
0.6	13.166	30.753	68.819	164.404	196.696	221.699
0.8	7.973	24.456	43.709	143.637	182.892	199.295
0.9	6.391	22.836	39.533	94.164	147.148	196.431
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	36.757	71.792	153.641	202.627	212.515	349.177
0.2	33.951	63.982	148.448	201.717	207.081	341.070
0.4	27.378	40.068	136.464	193.154	201.722	311.419
0.6	24.195	27.284	128.309	184.279	200.136	284.642
0.8	15.921	20.770	100.512	144.000	198.547	205.978
1.0	0.001	10.939	17.814	73.089	117.590	188.053

Table 5.19: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 \left(\frac{\rho_c t_c}{D_c}\right)^{0.5}$ for P-F-F-F Cylindrical Orthotropic Shell (*osh* $\psi 3\phi 1$).

$a = 12.in, r/a = 3.0, \phi = b/a = 1.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	19.602	25.125	38.210	66.972	94.870	97.090
0.2	13.583	22.654	34.008	58.016	67.906	95.826
0.4	7.890	20.287	22.470	31.441	65.717	94.568
0.6	4.695	12.654	17.426	28.271	55.713	67.145
0.8	2.777	8.506	13.973	25.870	37.831	63.949
0.9	2.148	7.599	12.746	24.581	34.540	44.750
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	19.602	25.125	38.210	66.972	94.870	97.090
0.2	17.490	22.259	36.299	66.096	90.091	90.752
0.4	12.135	15.699	31.408	56.340	59.678	68.500
0.6	9.583	12.264	28.110	40.498	42.721	66.063
0.8	6.517	8.239	20.671	23.267	31.032	63.886
1.0	0.000	4.354	6.078	16.467	17.822	29.135

Table 5.20: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{\rho_s t_s}{D_s})^{0.5}$ for P-F-F-F Cylindrical Orthotropic Shell ($osh\psi 3\phi 2$).

$a = 12.in, r/a = 3.0, \phi = b/a = 2.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	26.014	42.169	96.914	113.522	133.150	198.031
0.2	21.116	33.981	94.255	102.854	127.266	196.212
0.4	14.150	27.005	63.945	93.166	121.009	146.069
0.6	9.036	21.611	36.838	91.575	118.277	140.792
0.8	5.624	16.659	25.890	89.165	103.362	125.556
0.9	4.491	15.216	23.976	74.588	86.431	96.186
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	26.014	42.169	96.914	113.522	133.150	198.031
0.2	24.173	37.442	96.143	110.390	129.247	196.626
0.4	19.331	24.819	92.944	97.498	115.618	194.796
0.6	16.836	18.265	87.173	90.225	105.007	181.412
0.8	11.781	13.652	58.888	69.236	93.182	136.333
1.0	0.001	8.415	10.797	42.174	56.578	92.183

Table 5.21: Frequency Parameter $\Omega = 2\pi\omega a^2 \phi^2 (\frac{\rho_c t_c}{D_c})^{0.5}$ for P-F-F-F Cylindrical Orthotropic Shell (*osh* $\psi 3\phi 3$).

$a = 12.in, r/a = 3.0, \phi = b/a = 3.0, a/t = 100.$						
i	Support detachment from one side					
c/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	32.182	50.025	131.763	181.505	202.528	293.783
0.2	27.708	40.175	124.433	175.118	201.139	283.212
0.4	19.068	32.354	105.071	158.815	199.510	206.603
0.6	12.095	27.678	66.801	140.790	193.891	198.218
0.8	7.642	23.593	43.230	127.939	174.983	194.495
0.9	6.201	22.327	39.145	94.095	127.877	186.643
ii	Support detachment from both sides					
$2c'/a$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	32.182	50.025	131.763	181.505	202.528	293.783
0.2	30.641	45.245	128.509	177.198	201.915	291.597
0.4	26.327	30.070	117.958	168.140	200.423	276.399
0.6	22.097	23.867	110.682	160.882	199.769	254.547
0.8	14.381	20.587	90.776	132.989	186.355	199.231
1.0	0.001	10.344	17.519	69.460	111.651	164.103

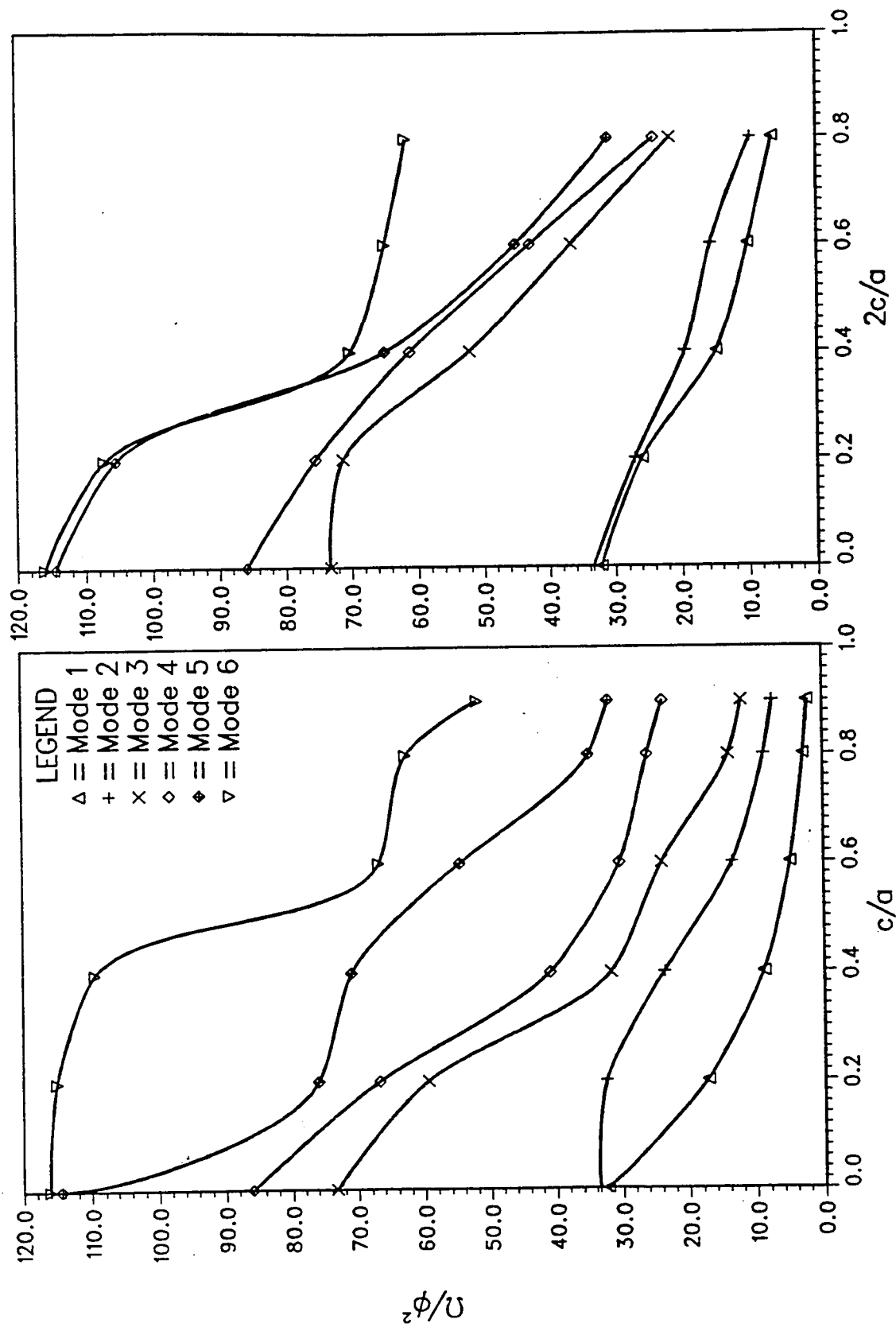


Figure 5.12: Variation of the Ω/ϕ^2 vs c/a , for P-F-F-F orthotropic shell.
 $r/a = 1.0$, $\phi = b/a = 1.0$, $a/t = 100.00$

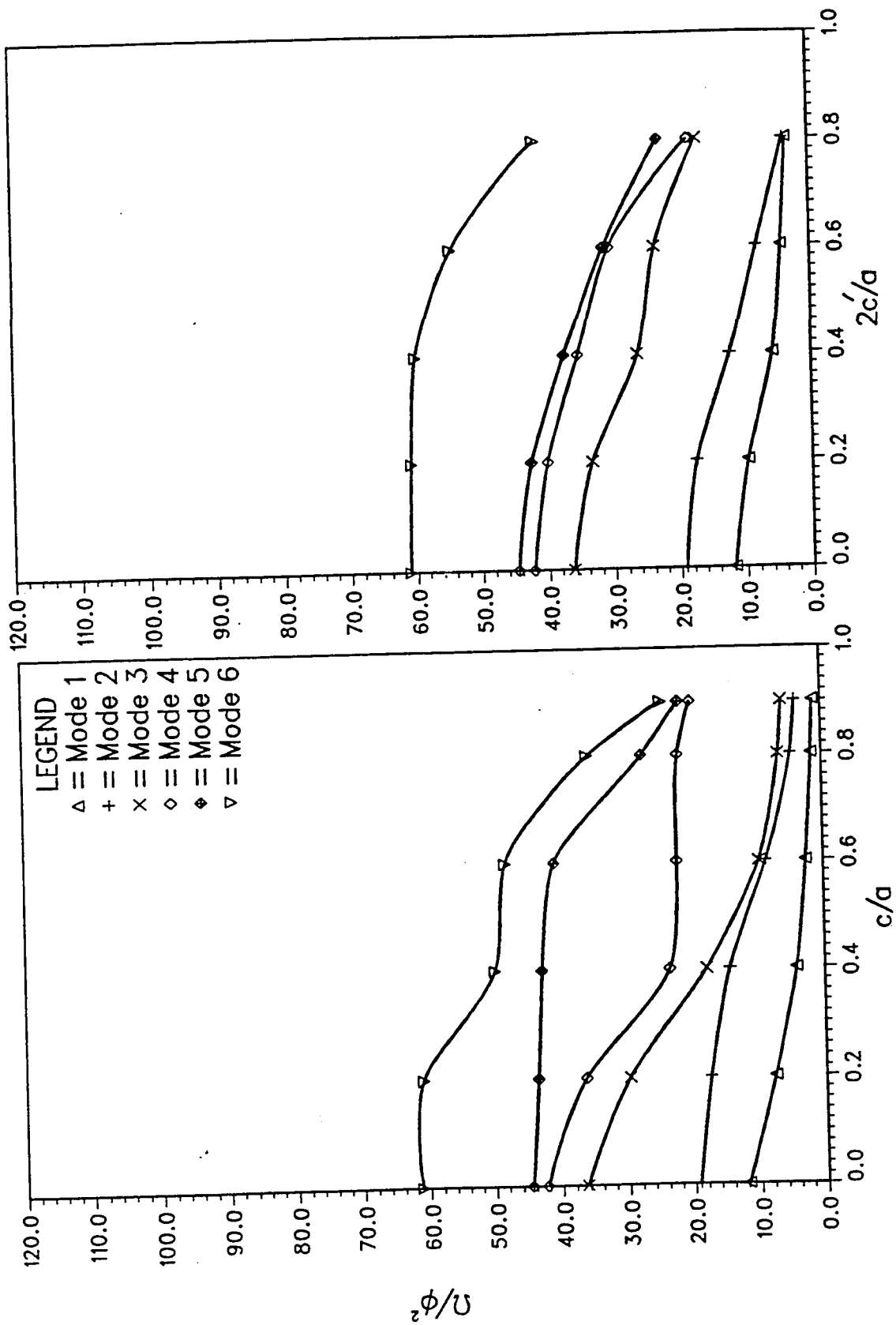


Figure 5.13: Variation of the Ω/ϕ^2 vs c/a , for P-F-F orthotropic shell.
 $r/a = 1.0$, $\phi = b/a = 2.0$, $a/t = 100.00$

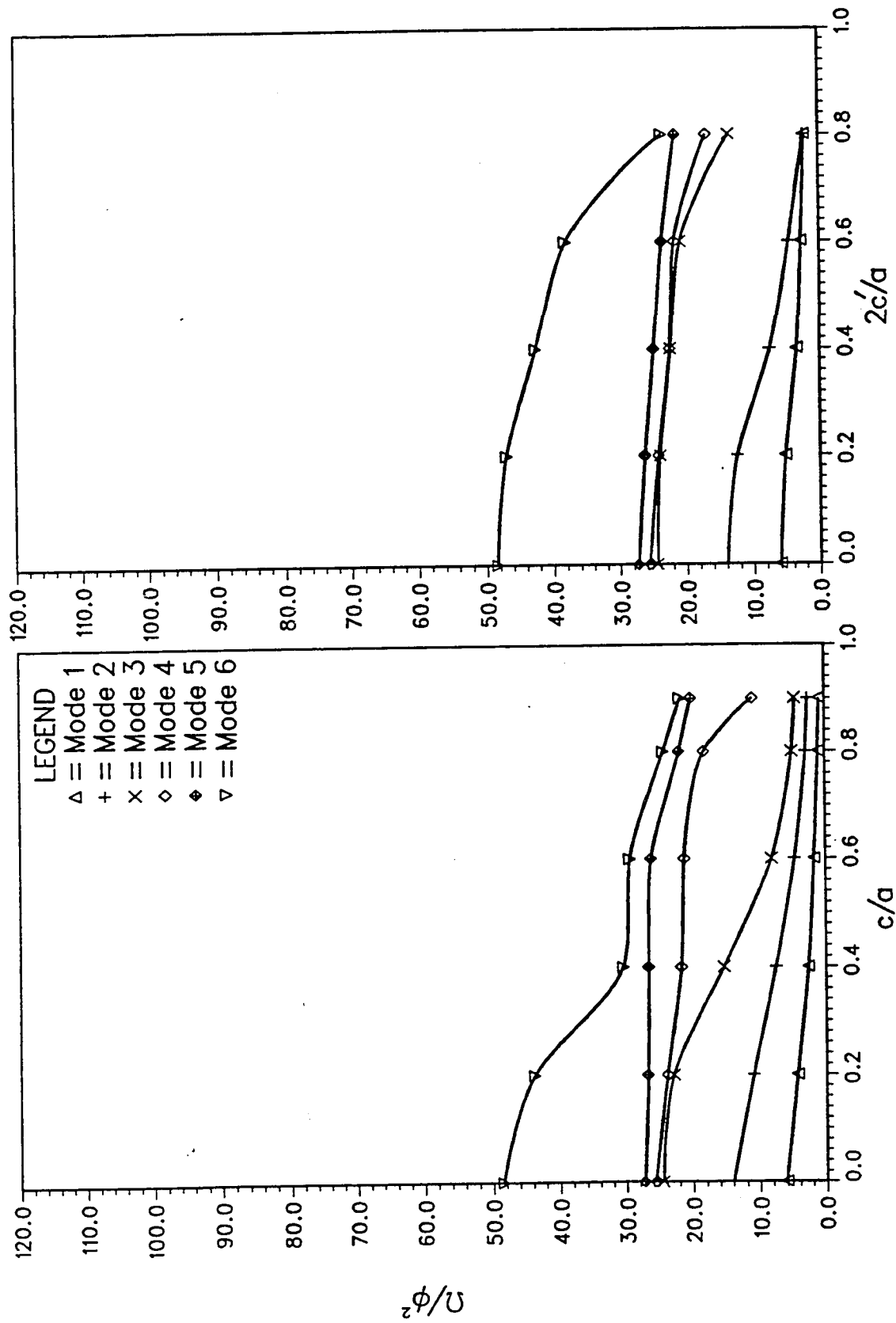


Figure 5.14: Variation of the Ω/ϕ^2 vs c/a , for P-F-F-F orthotropic shell. $r/a = 1.0$, $\phi = b/a = 3.0$, $a/t = 100.00$

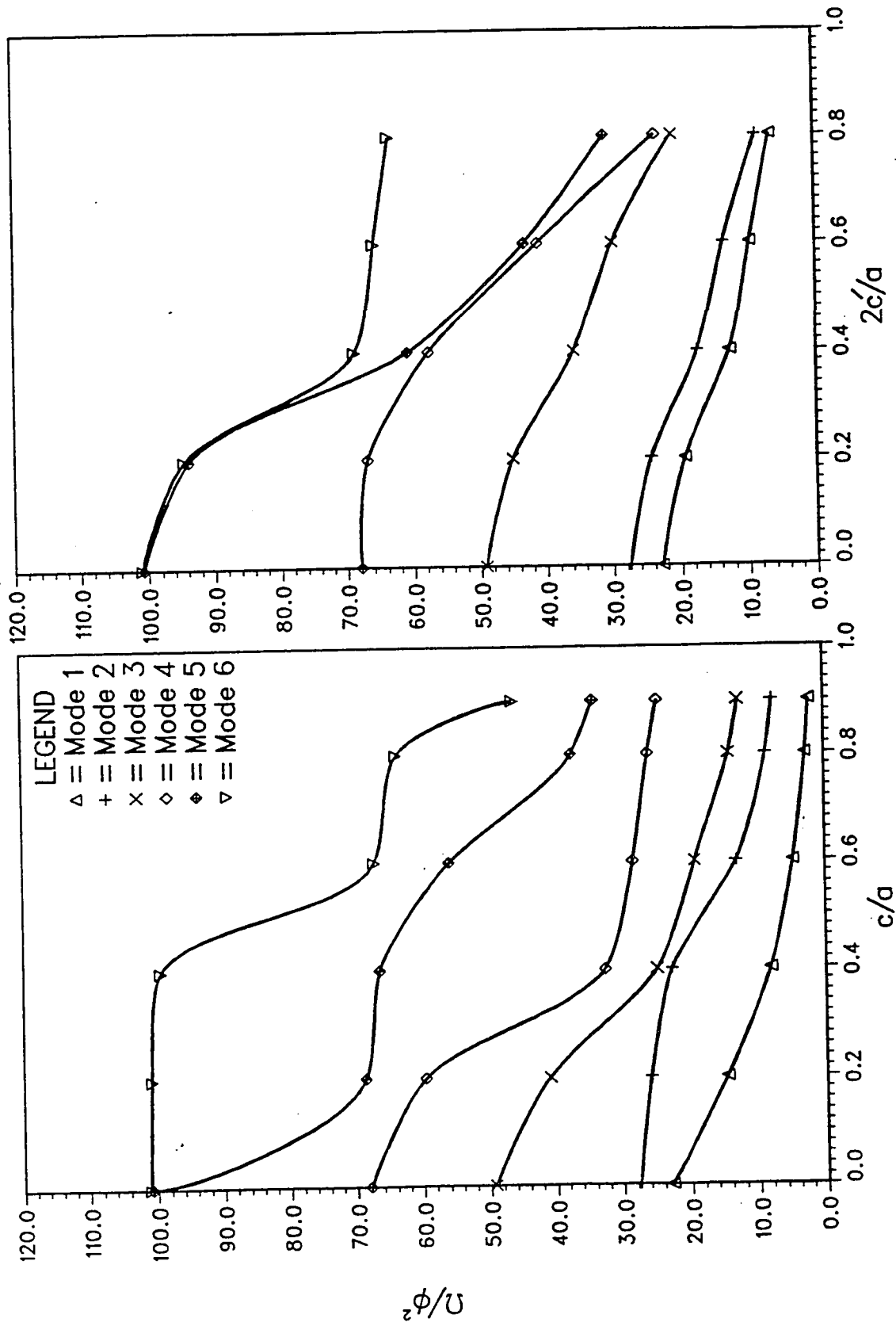


Figure 5.15: Variation of the Ω/ϕ^2 vs c/a , for P-F-F orthotropic shell.
 $r/a = 2.0$, $\phi = b/a = 1.0$, $a/t = 100.00$

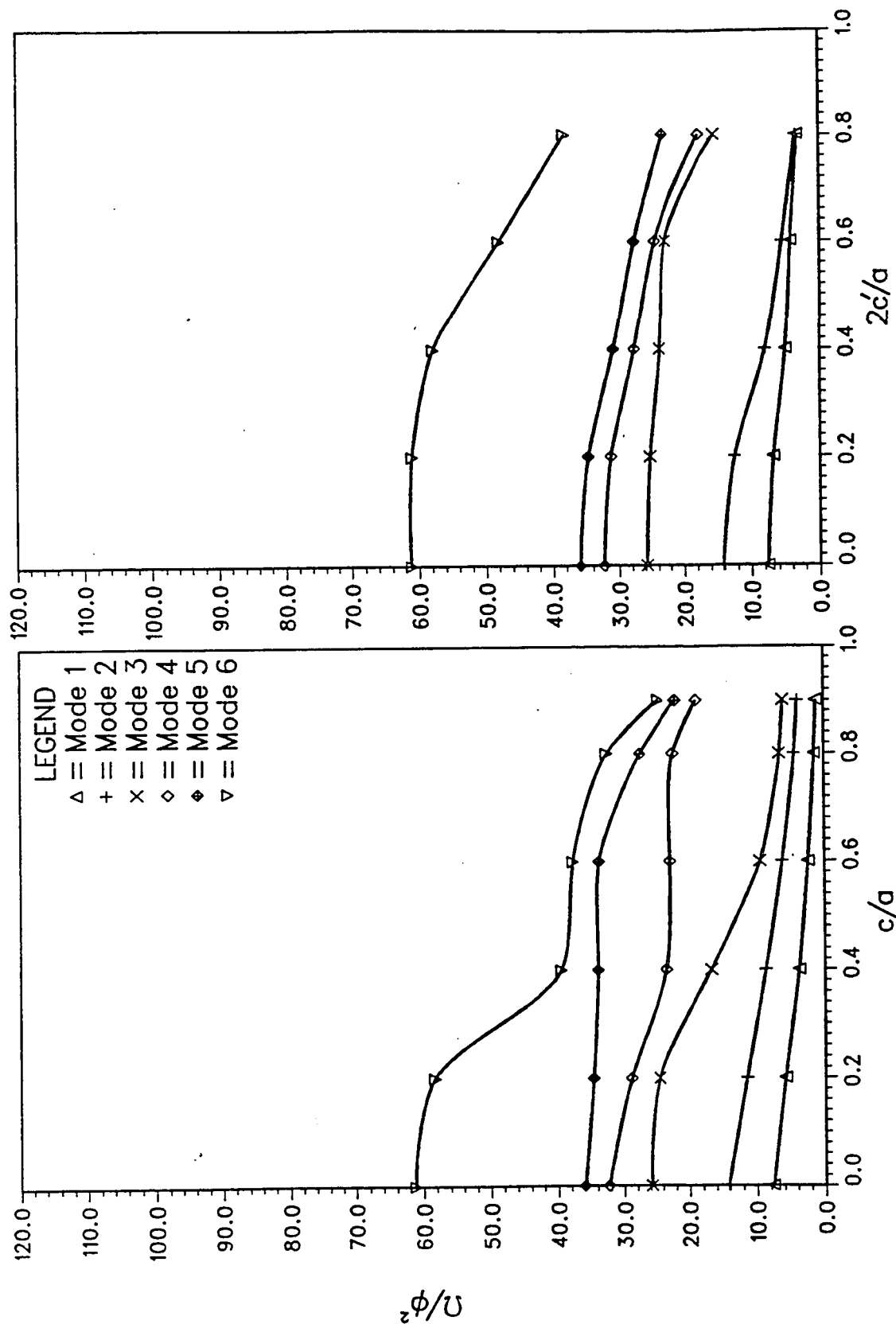


Figure 5.16: Variation of the Ω/ϕ^2 vs c/a , for P-F-F-F orthotropic shell.

$r/a = 2.0$, $\phi = b/a = 2.0$, $a/t = 100.00$

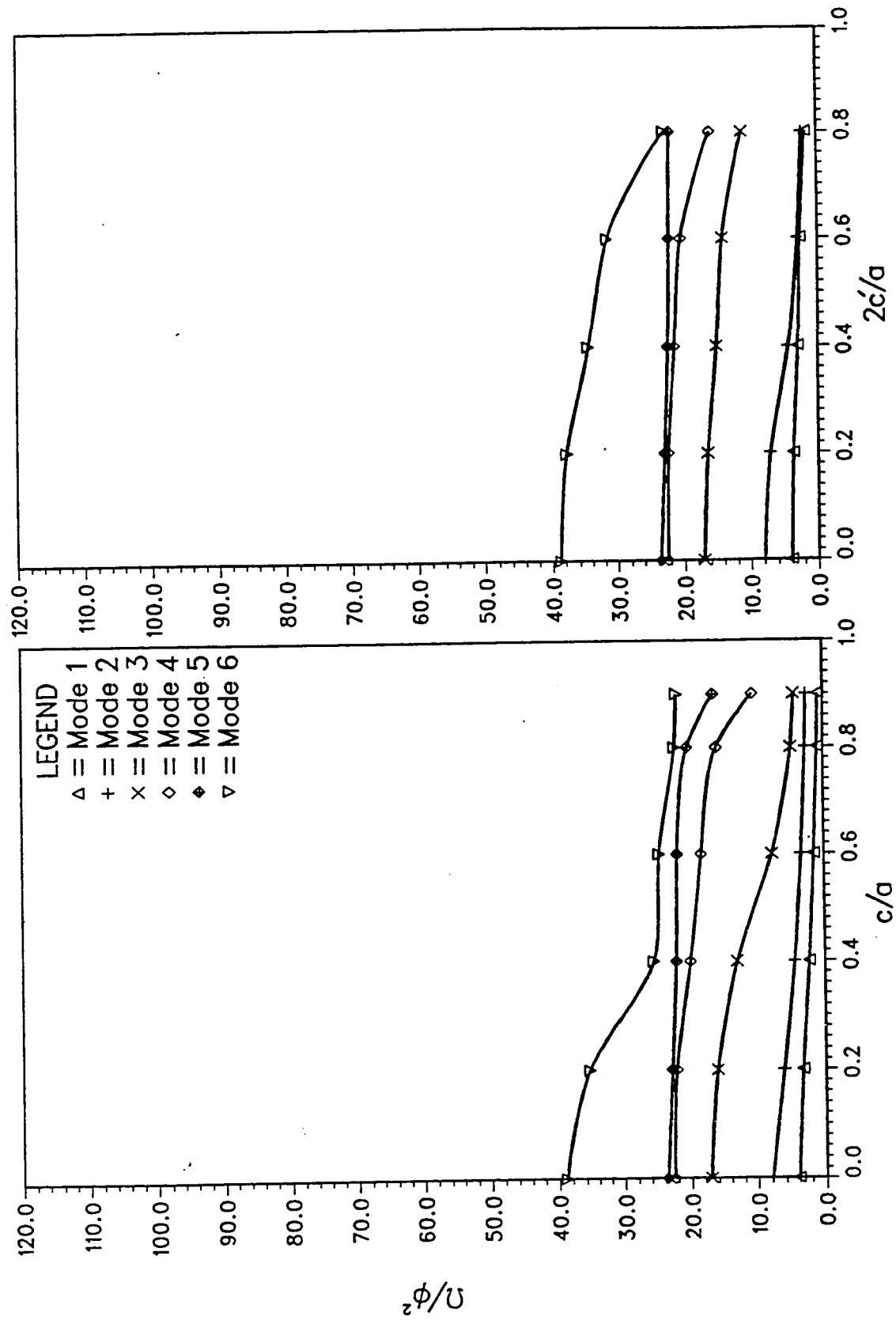


Figure 5.17: Variation of the Ω/ϕ^2 vs c/a , for P-F-F-F orthotropic shell.
 $r/a = 2.0$, $\phi = b/a = 3.0$, $a/t = 100.00$

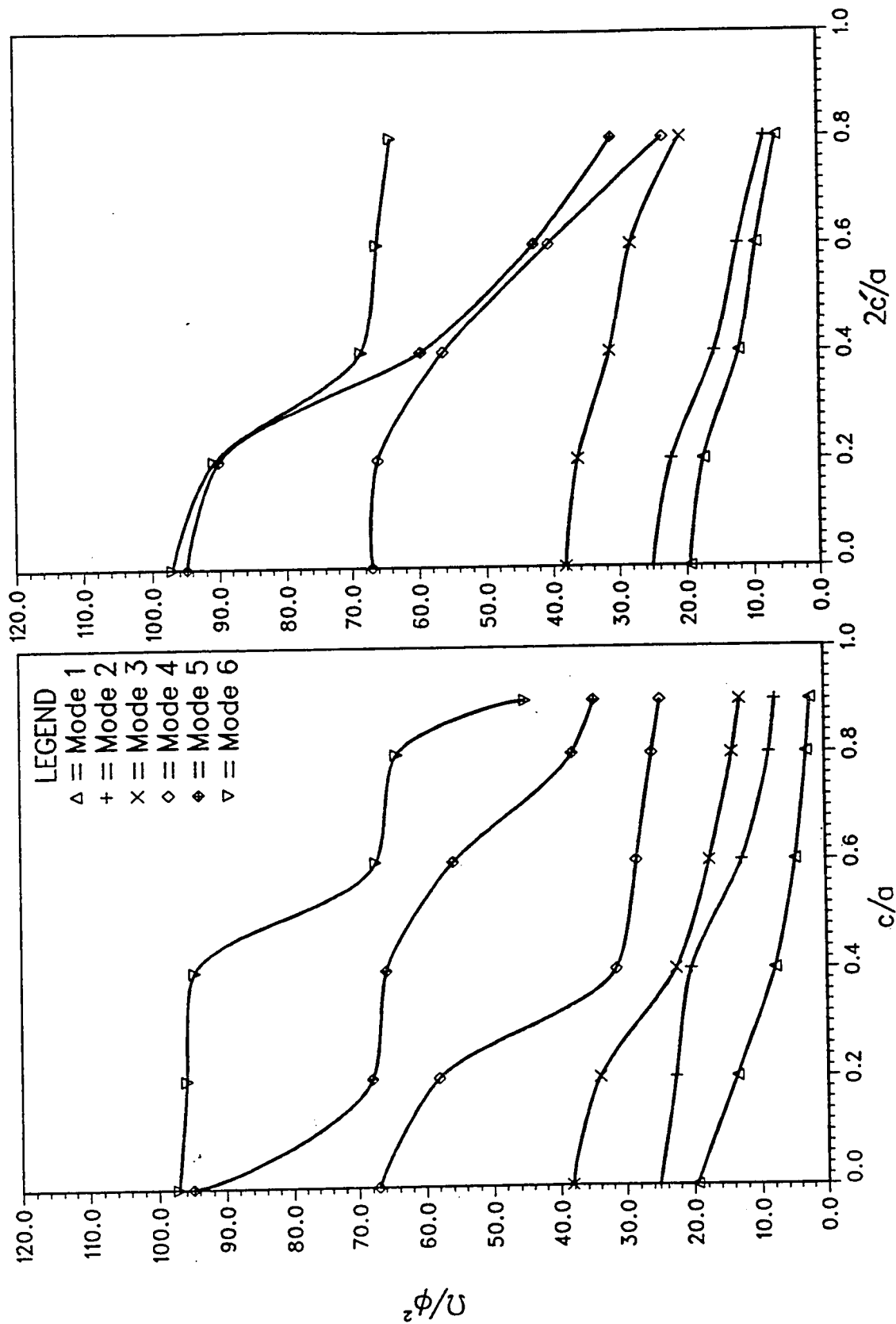


Figure 5.18: Variation of the Ω/ϕ^2 vs c/a , for P-F-F orthotropic shell.
 $\tau/a = 3.0$, $\phi = b/a = 1.0$, $a/t = 100.00$

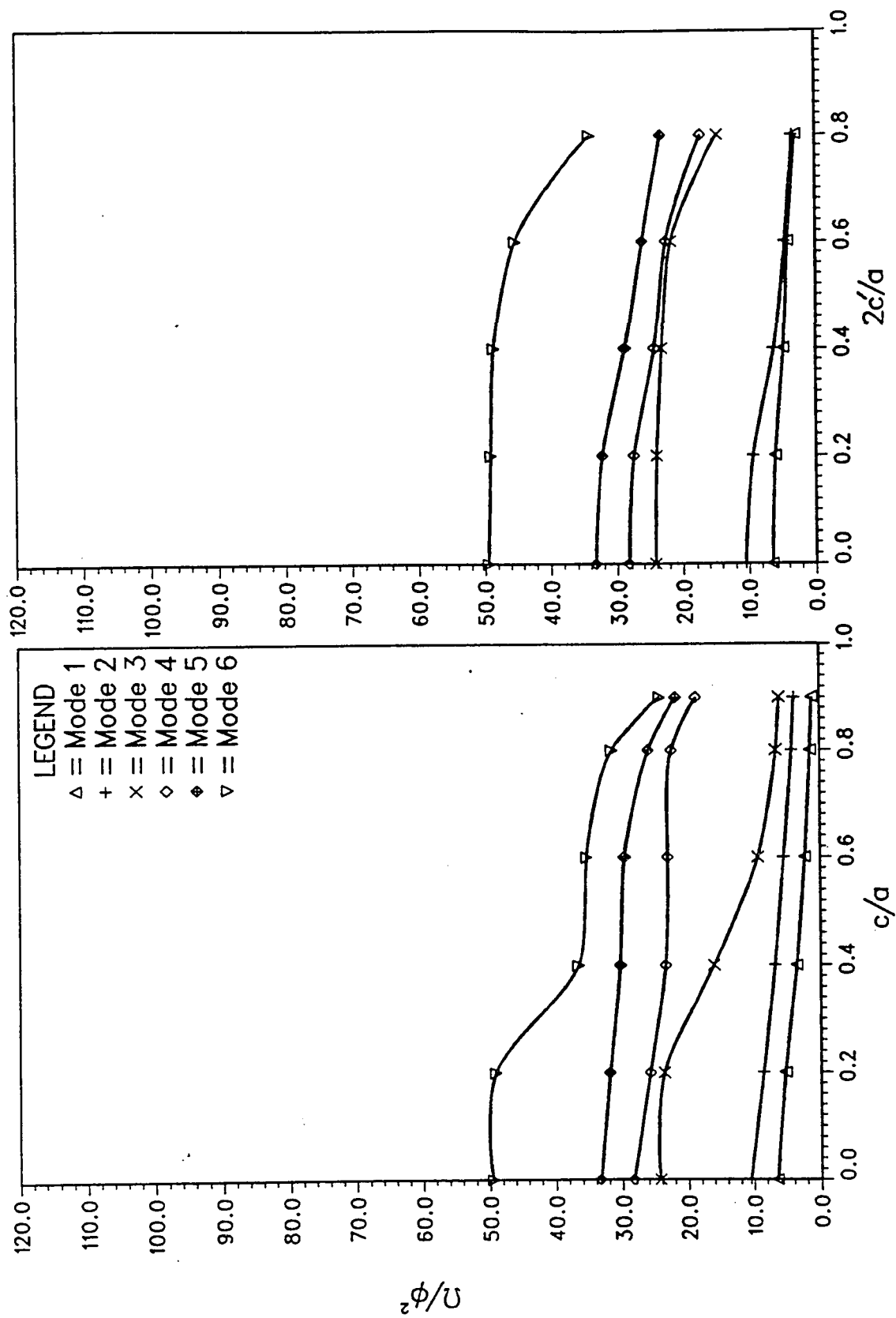


Figure 5.19: Variation of the Ω/ϕ^2 vs c/a , for P-F-F-F orthotropic shell.

$\tau/a = 3.0, \phi = b/a = 2.0, a/t = 100.00$

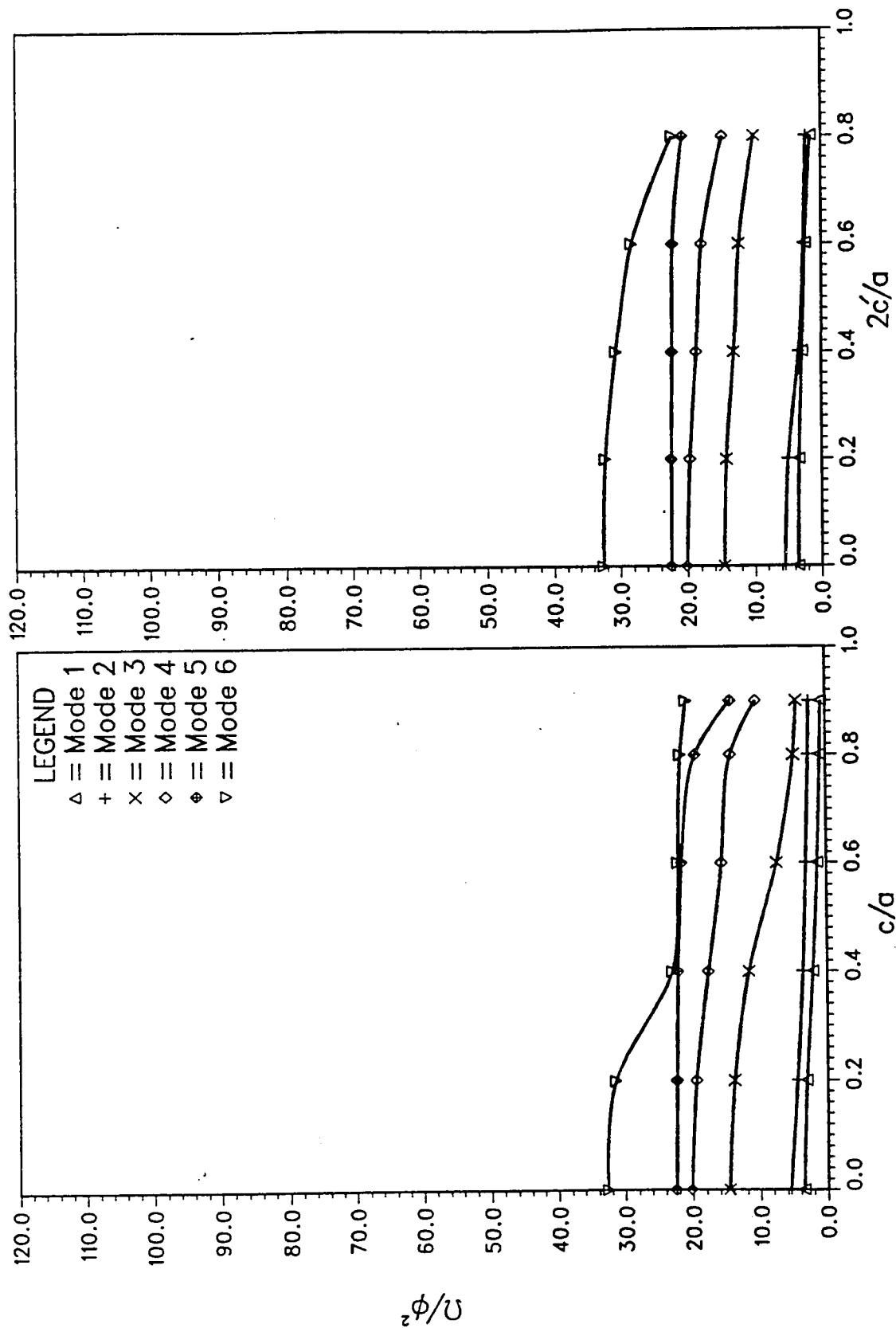


Figure 5.20: Variation of the Ω/ϕ^2 vs c/a , for P-F-F-F orthotropic shell.
 $r/a = 3.0$, $\phi = b/a = 3.0$, $a/t = 100.00$

5.4 Composite Shells

5.4.1 Comparison of Results for Sandwich Shell/Plate

Direct comparison of these results as well as those for the cracked sandwich shells was not possible since no such results are available. Most of the results available in literature are for sandwich plates simply supported on all four sides or for plates with other classical boundary conditions. In order to test the accuracy of the element and computations degenerate cases for sandwich plates, which were obtained from cylindrical shells by letting $r/a \rightarrow \infty$, have been considered. Table 5.22 shows the comparison of the lowest six frequencies for sandwich plate with those obtained by several other researchers. It may be seen that the present results are in close agreement with other published results and the largest discrepancy is about 4.5%. It should also be noted that the present results are lower than the others and because of the method used here, they should be more accurate.

5.4.2 Convergence of Results for Sandwich Shell/Plate

To find the best and effective mesh a sandwich shell with very large radius ($r/a \rightarrow \infty$) has been analyzed with different number of elements. Table 5.23 and figure 5.21 show these results for a clamped three layered composite shell/plate with n by n elements ($n \times n$) mesh. According to the *CPU* time and accuracy requirement a 5×5 mesh has been found to be the most suitable configuration.

Table 5.22: Comparison of Natural Frequency for S-S-S-S Degenerate Sandwich Shell with Sandwich Plate ($r/a \rightarrow \infty$).

$$\nu_f = 0.34, E_f = 10 \times 10^6 \text{ psi}, \rho_f = 248.0 \times 10^{-6} \text{ lb.sec}^2.\text{in}^{-4}$$

$$G_{23c} = 7500 \text{ psi}, G_{13c} = 19500 \text{ psi}, \rho_c = 4.02 \times 10^{-6} \text{ lb.sec}^2.\text{in}^{-4}$$

$$a = 48.0 \text{ in.}, r/a = 90.0, b/a = 1.5, a/t_c = 170.21, t_f = 0.016 \text{ in.}, t_c = 0.25 \text{ in.}$$

Source of results	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
Present(5×5)	23	43	68	78	89	124
Ref.[49]	23	45	71	82	92	128
Ref.[128]	23	45	71	80	92	126
Ref.[129]	23	44	70	80	90	125
Ref.[130](Exp.)	-	45	69	78	92	129
Ref.[130](Theo.)	23	45	71	80	91	126

Table 5.23: Natural Frequency for Clamped (C-C-C-C)
Sandwich Shell/ Rectangular Plate ($r/a \rightarrow \infty$).

$$\nu_f = 0.34, \quad E_f = 10 \times 10^6 \text{ psi} \quad \rho_f = 500.2 \times 10^{-6} \text{ lb.sec}^2.\text{in}^{-4}$$

$$G_c = G_{23c} = G_{13c} = 1000 \text{ psi},$$

$$a = 40.0 \text{ in.}, \quad r/a = 999., \quad a/t_e = 141.84, \quad t_f = 0.016 \text{ in.}, \quad t_c = 0.25 \text{ in.}$$

$n \times n$	$b/a = 1.0$						CPU(s)
	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	
2×2	62.332	157.385	157.386	225.413	2127.023	2127.023	5.5237
3×3	59.314	129.473	129.474	192.308	266.457	267.674	12.7493
4×4	58.905	122.399	122.400	180.725	237.830	238.972	24.3596
5×5	58.800	120.773	120.774	178.051	222.845	223.904	41.5742
6×6	58.763	120.204	120.204	177.135	218.385	219.434	64.1589
7×7	58.747	119.960	119.961	176.751	216.511	217.553	90.2666
8×8	58.740	119.842	119.843	176.567	215.599	216.638	134.5251
$n \times n$	$b/a = 1.5$						CPU(s)
	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	
2×2	46.920	82.789	148.141	170.004	1630.849	1969.499	5.8267
3×3	44.558	72.047	118.057	129.918	142.252	191.341	12.7161
4×4	44.230	69.154	110.719	118.527	133.505	178.872	24.3623
5×5	44.144	68.493	109.014	112.106	131.467	172.021	43.4661
6×6	44.114	68.265	108.411	110.217	130.760	169.919	62.1809
7×7	44.101	68.169	108.152	109.429	130.461	168.634	90.0059
8×8	44.095	68.122	108.025	109.048	130.317	166.862	143.1709
$n \times n$	$b/a = 2.0$						CPU(s)
	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	
2×2	42.842	59.466	145.533	155.494	1416.829	1621.153	5.4275
3×3	40.590	54.044	83.733	114.694	127.074	151.458	12.8914
4×4	40.270	52.531	78.291	107.282	119.069	119.322	23.8818
5×5	40.185	52.180	74.925	105.553	113.550	117.187	40.5056
6×6	40.155	52.059	73.940	104.938	107.638	116.524	62.1387
7×7	40.142	52.008	73.532	104.673	105.617	116.241	107.8787
8×8	40.136	51.984	73.336	104.544	104.665	116.104	134.1040

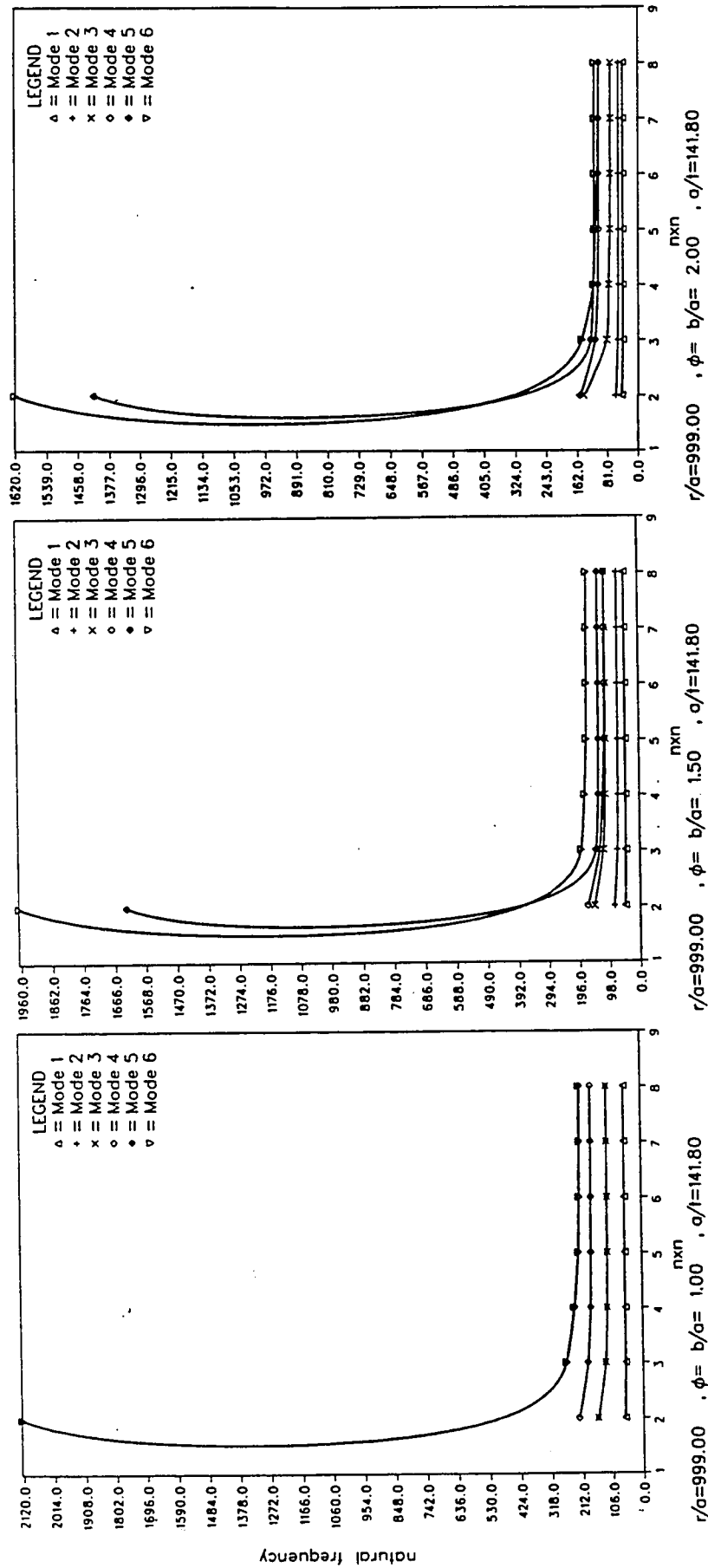


Figure 5.21: Convergence of finite element results for C-C-C-C sandwich shell/plate ($a = 40.0in$, $r/a \rightarrow \infty$).

5.4.3 Cylindrical Sandwich Shells

For computation of data, the cylindrical sandwich shells have been considered as cantilever. In all computations the curved length a is taken as 12 in. (30.48 cm) along the support. The thickness of each face $t_f = 0.016$ in. (0.041 cm) and the core thickness $t_c = 0.25$ in. (0.6350 cm) for all cases. The total thickness t_e is defined as,

$$t_e = t_c + 2t_f$$

The material properties for sandwich shells have been summarized below in Table 5.24.

Table 5.24: Material Properties for Sandwich Shells.

face		
Material property	Assumed value	System of units
ν_{12}	0.34	
ρ	248.0E-6	$lb.sec^2/in^4$
E_{11}	1.0E7	psi
E_{11}/E_{22}	1	psi/psi
E_{22}/G_{12}	2.68	psi/psi
core		
Material property	Assumed value	System of units
ρ	4.02E-6	$lb.sec^2/in^4$
G_{23}	7500	psi
G_{13}	19500	psi

5.4.4 Partially Detached Sandwich Shells.

The frequency parameters for P-F-F-F , cylindrical shells with r/a equal 1.0, 2.0 and 3.0 and for different aspect ratios b/a 1.0, 2.0 and 3.0 are presented in Tables

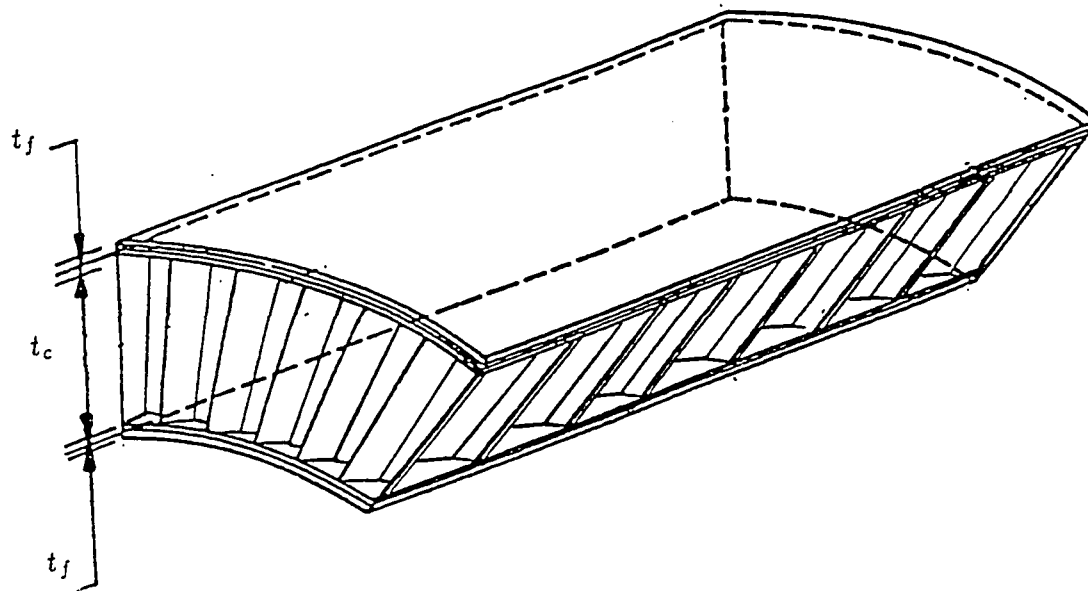


Figure 5.22: Typical Cylindrical Sandwich Shell.

5.25 to 5.33 for several values of the partial support ratio c/a and $2c'/a$. The ratio $2c'/a$ has been used for the case when the shell has detached boundaries from the two sides as shown in Figure 5.2. The tables contains data for two cases: (i) natural frequency for detached base lengths from one side and (ii) frequency for shells with equal detached base length from both sides. Figures 5.23 to 5.31 show variation of frequency parameter with c/a and $2c'/a$ for the lowest six modes for a P-F-F-F shell with aspect ratios 1, 2 and 3.

Table 5.25: Natural Frequency ω for P-F-F-F Cylindrical Sandwich Shell
(ssh ψ 1 ϕ 1).

$a = 12.in, r/a = 1.0, \phi = b/a = 1.0, a/t_e = 32.432$						
i	Detached length from one side					
c/a	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	232.256	298.867	664.790	892.264	1050.126	1581.694
0.2	194.095	250.472	639.879	806.365	994.740	1411.577
0.4	122.682	209.265	480.748	657.851	936.590	1053.175
0.6	75.059	156.678	332.460	587.910	697.764	927.670
0.8	47.080	105.891	259.777	329.250	574.983	754.165
0.9	38.499	84.839	174.445	257.753	526.643	701.337
ii	Detached length from both sides					
$2c'/a$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	232.256	298.867	664.790	892.264	1050.126	1581.694
0.2	222.887	270.086	659.698	865.577	1013.621	1555.949
0.4	179.382	192.896	621.285	759.211	905.865	1182.645
0.6	127.502	172.891	571.399	656.586	802.406	897.611
0.8	80.308	136.563	421.760	461.848	499.715	695.919
1.0	0.002	57.094	97.990	397.268	405.469	659.226

Table 5.26: Natural Frequency ω for P-F-F-F Cylindrical Sandwich Shell ($ssh\psi1\phi2$).

$a = 12.in, r/a = 1.0, \phi = b/a = 2.0, a/t_e = 32.432$						
i	Detached length from one side					
c/a	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	83.907	97.234	332.028	417.893	566.803	567.748
0.2	69.348	94.782	314.771	384.117	506.049	566.528
0.4	45.013	89.670	271.246	327.791	419.541	526.474
0.6	28.329	78.855	196.233	234.206	362.253	462.641
0.8	18.569	61.498	117.313	156.193	329.096	444.770
0.9	15.636	50.842	74.593	137.121	310.701	439.187
ii	Detached length from both sides					
$2c'/a$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	83.907	97.234	332.028	417.893	566.803	567.748
0.2	79.666	95.581	325.746	402.715	545.478	565.436
0.4	59.108	89.442	300.261	366.545	431.790	560.916
0.6	43.969	84.555	272.952	349.079	350.920	557.415
0.8	28.826	74.413	161.591	259.313	316.815	493.843
1.0	0.001	20.941	61.566	216.931	287.123	428.314

Table 5.27: Natural Frequency ω for P-F-F-F Cylindrical Sandwich Shell (*ssh ψ 1 ϕ 3*).

$a = 12.in, r/a = 1.0, \phi = b/a = 3.0, a/t_e = 32.432$						
i	Detached length from one side					
c/a	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	37.989	60.437	196.814	216.213	273.791	378.531
0.2	33.539	59.218	184.193	202.429	253.600	369.075
0.4	23.708	57.069	146.975	188.309	227.148	337.888
0.6	15.554	52.607	107.749	154.832	207.053	274.039
0.8	10.415	43.820	65.481	110.789	192.139	243.655
0.9	8.818	35.592	46.487	96.317	184.517	237.436
ii	Detached length from both sides					
$2c'/a$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	37.989	60.437	196.814	216.213	273.791	378.531
0.2	36.795	59.827	194.126	209.627	267.109	373.896
0.4	29.615	57.554	180.531	188.122	225.191	361.690
0.6	23.239	55.548	158.609	177.103	197.927	349.273
0.8	16.055	51.045	90.393	167.357	168.669	313.776
1.0	0.001	12.060	45.130	147.835	161.878	280.680

Table 5.28: Natural Frequency ω for P-F-F-F Cylindrical Sandwich Shell
(ssh ψ 2 ϕ 1).

$a = 12.in, r/a = 2.0, \phi = b/a = 1.0, a/t_e = 32.432$						
i	Detached length from one side					
c/a	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	173.652	216.226	647.496	784.939	798.544	1386.646
0.2	141.394	205.546	615.766	727.265	783.144	1273.426
0.4	97.764	182.611	453.231	667.990	764.024	953.816
0.6	67.319	145.997	325.410	608.247	662.711	774.939
0.8	45.778	105.583	263.846	307.051	590.532	701.806
0.9	37.873	88.018	162.954	254.950	549.408	679.108
ii	Detached length from both sides					
$2c'/a$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	173.652	216.226	647.496	784.939	798.544	1386.646
0.2	161.553	211.830	641.668	769.540	783.917	1367.230
0.4	121.707	189.377	596.032	697.119	736.793	1182.538
0.6	98.709	171.112	538.303	613.823	717.735	896.165
0.8	72.069	134.236	422.238	428.521	479.040	685.311
1.0	0.002	54.287	95.772	371.753	395.612	665.835

Table 5.29: Natural Frequency ω for P-F-F-F Cylindrical Sandwich Shell
(ssh ψ 2 ϕ 2).

$a = 12.in, r/a = 2.0, \phi = b/a = 2.0, a/t_e = 32.432$						
i	Detached length from one side					
c/a	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	46.522	93.181	252.971	307.424	565.716	567.095
0.2	41.181	90.697	236.884	299.562	496.360	562.263
0.4	31.404	85.113	210.035	280.824	384.551	456.311
0.6	23.262	75.896	170.626	232.738	288.561	370.686
0.8	17.115	60.825	116.095	147.385	258.401	340.895
0.9	14.792	52.167	71.178	130.009	251.144	332.808
ii	Detached length from both sides					
$2c'/a$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	46.522	93.181	252.971	307.424	565.716	567.095
0.2	44.816	92.390	246.424	304.628	544.694	562.765
0.4	36.944	87.825	228.611	288.126	429.289	549.924
0.6	31.254	83.482	219.346	270.722	338.539	525.726
0.8	24.005	73.275	163.737	205.118	243.684	450.648
1.0	0.002	18.799	60.075	195.043	209.446	412.289

Table 5.30: Natural Frequency ω for P-F-F-F Cylindrical Sandwich Shell ($ssh\psi 2\phi 3$).

$a = 12.in, r/a = 2.0, \phi = b/a = 3.0, a/t_e = 32.432$						
i	Detached length from one side					
c/a	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	20.928	58.792	122.656	186.663	273.187	311.410
0.2	19.270	57.829	115.725	183.253	246.789	304.236
0.4	15.690	55.414	103.781	174.925	196.660	286.319
0.6	12.178	51.385	89.732	142.486	160.689	239.994
0.8	9.257	44.024	63.666	95.009	142.265	210.573
0.9	8.103	38.700	42.211	83.365	135.968	203.611
ii	Detached length from both sides					
$2c'/a$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	20.928	58.792	122.656	186.663	273.187	311.410
0.2	20.446	58.500	120.080	185.546	266.310	306.371
0.4	17.829	56.778	111.309	178.469	222.194	293.345
0.6	15.671	54.987	106.242	165.803	186.211	285.786
0.8	12.642	50.480	91.991	100.319	160.882	271.812
1.0	0.001	10.301	44.193	96.961	143.344	260.138

Table 5.31: Natural Frequency ω for P-F-F-F Cylindrical Sandwich Shell
(ssh ψ 3 ϕ 1).

$a = 12.in, r/a = 3.0, \phi = b/a = 1.0, a/t_c = 32.432$						
i	Detached length from one side					
c/a	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	133.406	212.969	617.949	701.974	779.154	1353.683
0.2	115.204	201.658	578.632	691.995	738.804	1236.898
0.4	87.646	176.962	435.719	669.655	685.996	921.756
0.6	64.682	141.606	320.894	612.880	643.345	700.018
0.8	45.348	103.893	265.200	301.994	592.583	639.067
0.9	37.677	87.660	160.740	253.863	552.992	629.111
ii	Detached length from both sides					
$2c'/a$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	133.406	212.969	617.949	701.974	779.154	1353.683
0.2	126.643	209.603	608.962	696.888	766.991	1333.302
0.4	103.180	188.635	555.000	684.071	687.775	1166.328
0.6	88.915	170.704	500.849	604.461	682.519	895.908
0.8	68.720	133.691	402.615	422.779	473.903	668.579
1.0	0.002	52.959	95.279	352.701	389.650	655.193

Table 5.32: Natural Frequency ω for P-F-F-F Cylindrical Sandwich Shell
(ssh ψ 3 ϕ 2).

$a = 12.in, r/a = 3.0, \phi = b/a = 2.0, a/t_c = 32.432$						
i	Detached length from one side					
c/a	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	34.962	92.317	197.521	302.456	489.401	567.025
0.2	32.014	89.985	188.528	294.455	468.957	510.330
0.4	26.488	84.040	173.377	271.560	379.332	426.805
0.6	21.233	74.419	151.695	229.622	262.544	356.609
0.8	16.409	59.507	115.851	140.815	218.042	328.287
0.9	14.358	51.375	71.138	124.140	211.323	319.288
ii	Detached length from both sides					
$2c'/a$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	34.962	92.317	197.521	302.456	489.401	567.025
0.2	34.020	91.717	193.933	300.364	484.446	544.640
0.4	29.743	87.480	182.753	285.530	429.360	466.298
0.6	26.599	83.256	176.105	270.083	336.458	448.596
0.8	21.786	73.042	164.359	164.763	240.281	400.277
1.0	21.786	73.042	164.359	164.763	240.281	400.277

Table 5.33: Natural Frequency ω for P-F-F-F Cylindrical Sandwich Shell
(ssh ψ 3 ϕ 3).

$a = 12.in, r/a = 3.0, \phi = b/a = 3.0, a/t_e = 32.432$						
i	Detached length from one side					
c/a	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	15.659	58.444	93.312	184.569	244.893	273.087
0.2	14.753	57.552	89.573	181.425	233.549	255.304
0.4	12.830	54.986	83.218	171.493	193.105	237.898
0.6	10.743	50.581	76.251	137.463	153.426	217.871
0.8	8.641	42.969	62.806	83.317	129.012	200.926
0.9	7.695	38.340	41.997	72.545	120.868	195.569
ii	Detached length from both sides					
$2c'/a$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.0	15.659	58.444	93.312	184.569	244.893	273.087
0.2	15.392	58.225	91.931	183.778	242.128	266.179
0.4	14.013	56.619	86.968	177.886	221.888	233.857
0.6	12.897	54.874	83.803	168.355	182.884	228.473
0.8	11.052	50.369	79.160	92.401	159.238	218.320
1.0	0.001	9.367	44.009	76.055	142.373	210.935

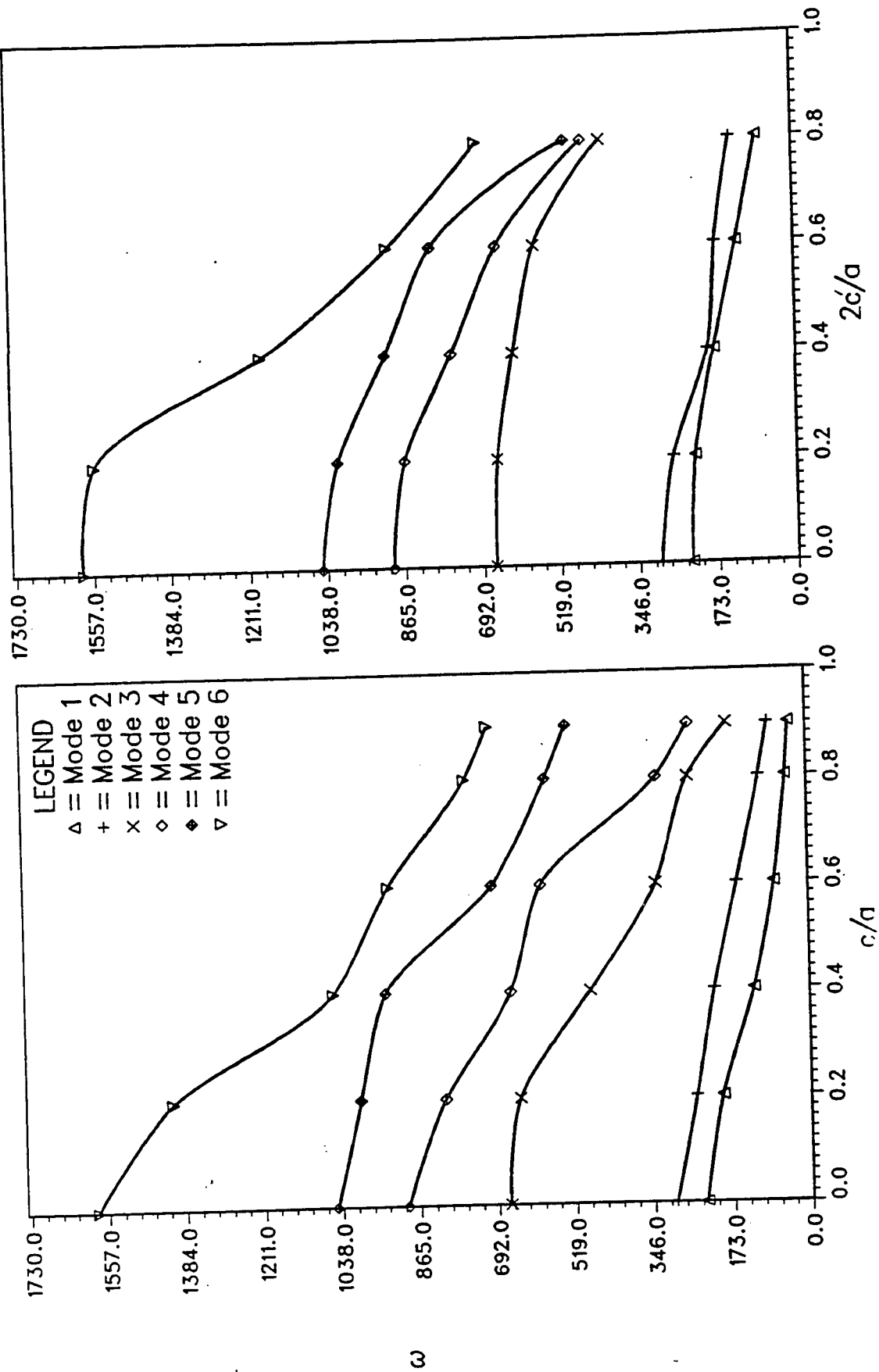


Figure 5.23: Variation of ω vs c/a , for P-F-F sandwich shell.
 $r/a = 1.0$, $\phi = b/a = 1.0$, $a/t_c = 32.432$

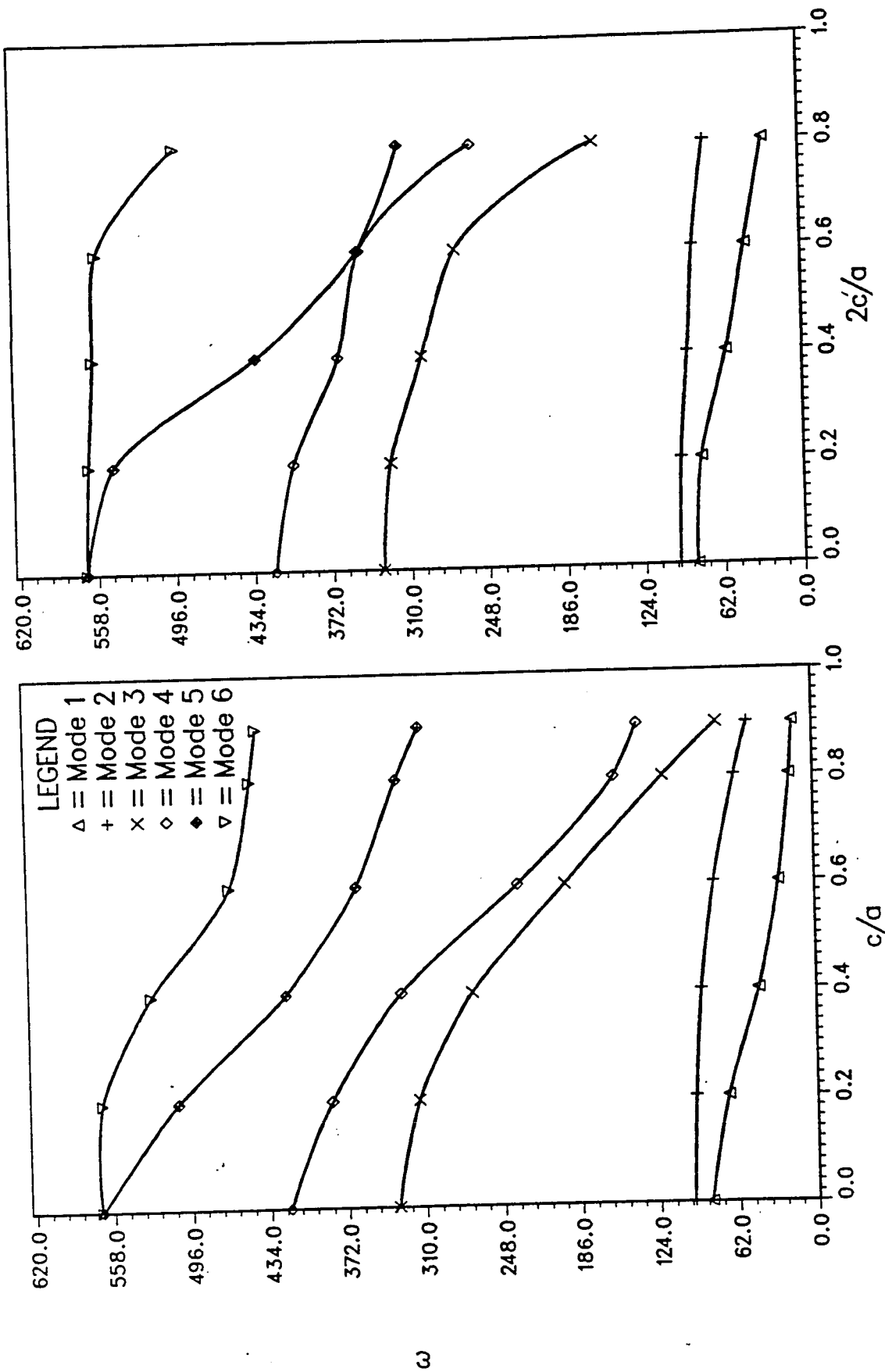


Figure 5.24: Variation of ω vs c/a , for P-F-F sandwich shell.
 $r/a = 1.0$, $\phi = b/a = 2.0$, $a/t_e = 32.432$

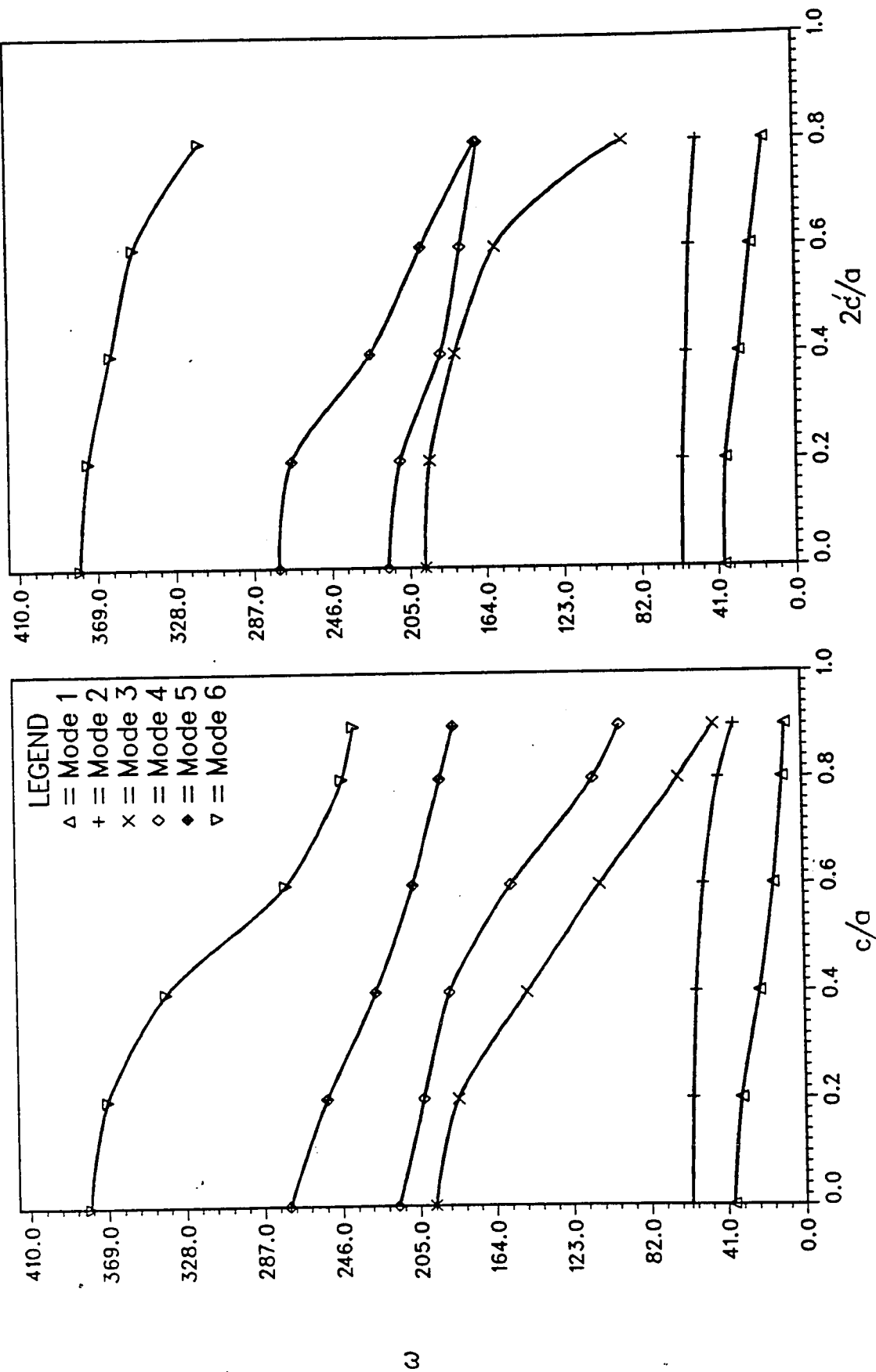


Figure 5.25: Variation of ω vs c/a , for P-F-F sandwich shell.
 $r/a = 1.0$, $\phi = b/a = 3.0$, $a/t_c = 32.432$

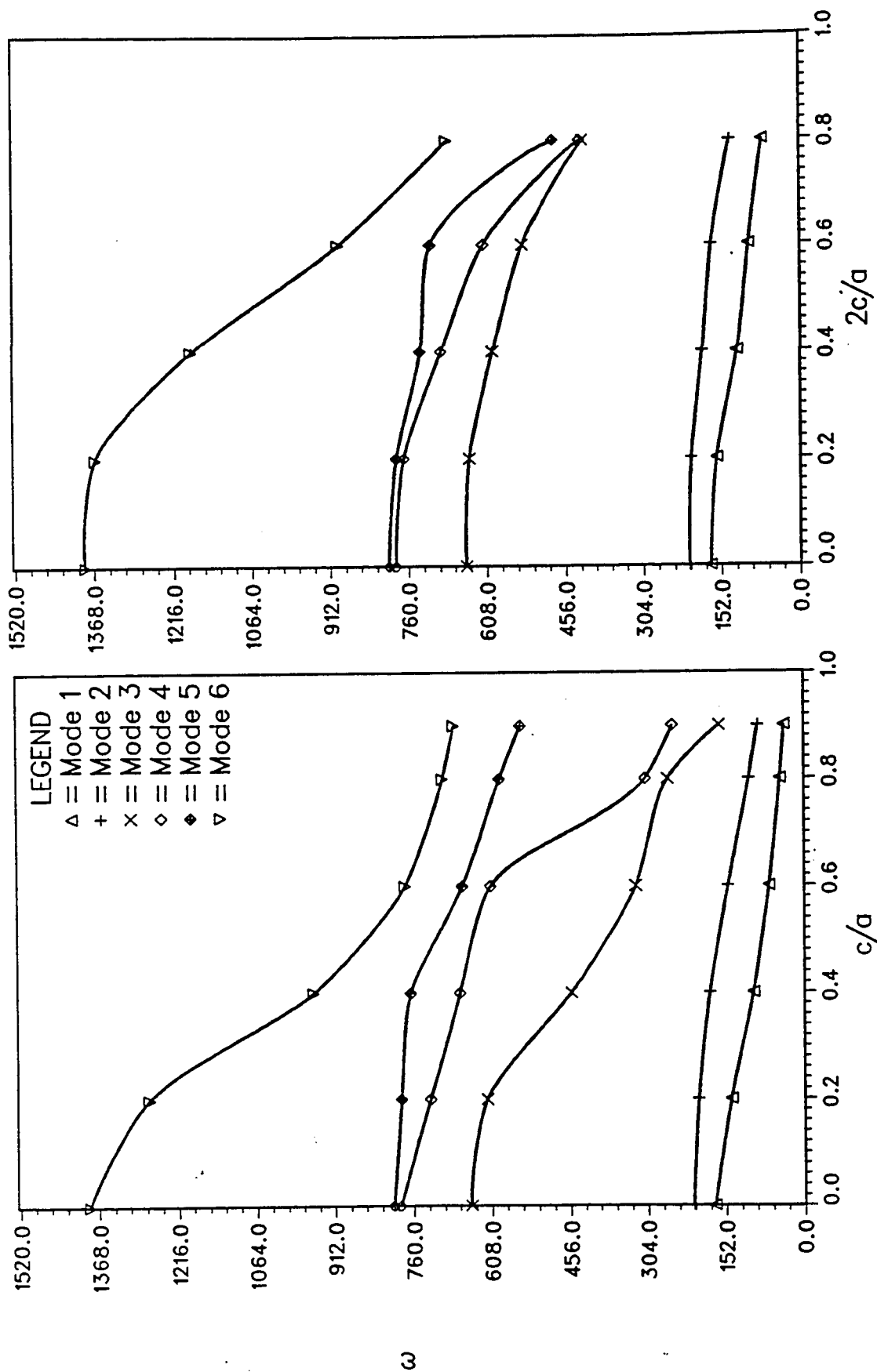


Figure 5.26: Variation of ω vs c/a , for P-F-F sandwich shell.
 $r/a = 2.0$, $\phi = b/a = 1.0$, $a/t_e = 32.432$

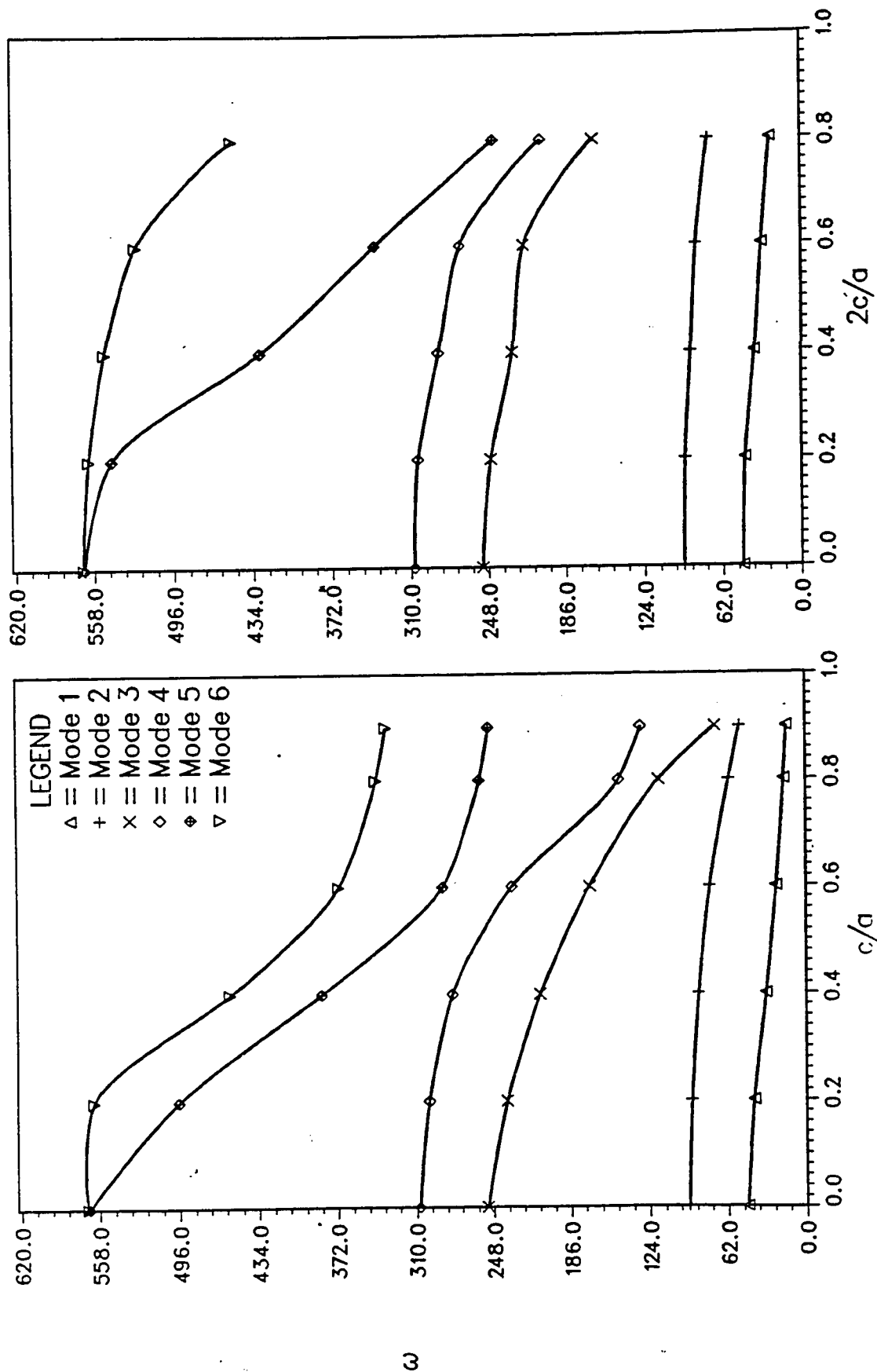


Figure 5.27: Variation of ω vs c/a , for P-F-F sandwich shell.

$r/a = 2.0$, $\phi = b/a = 2.0$, $a/t_e = 32.432$

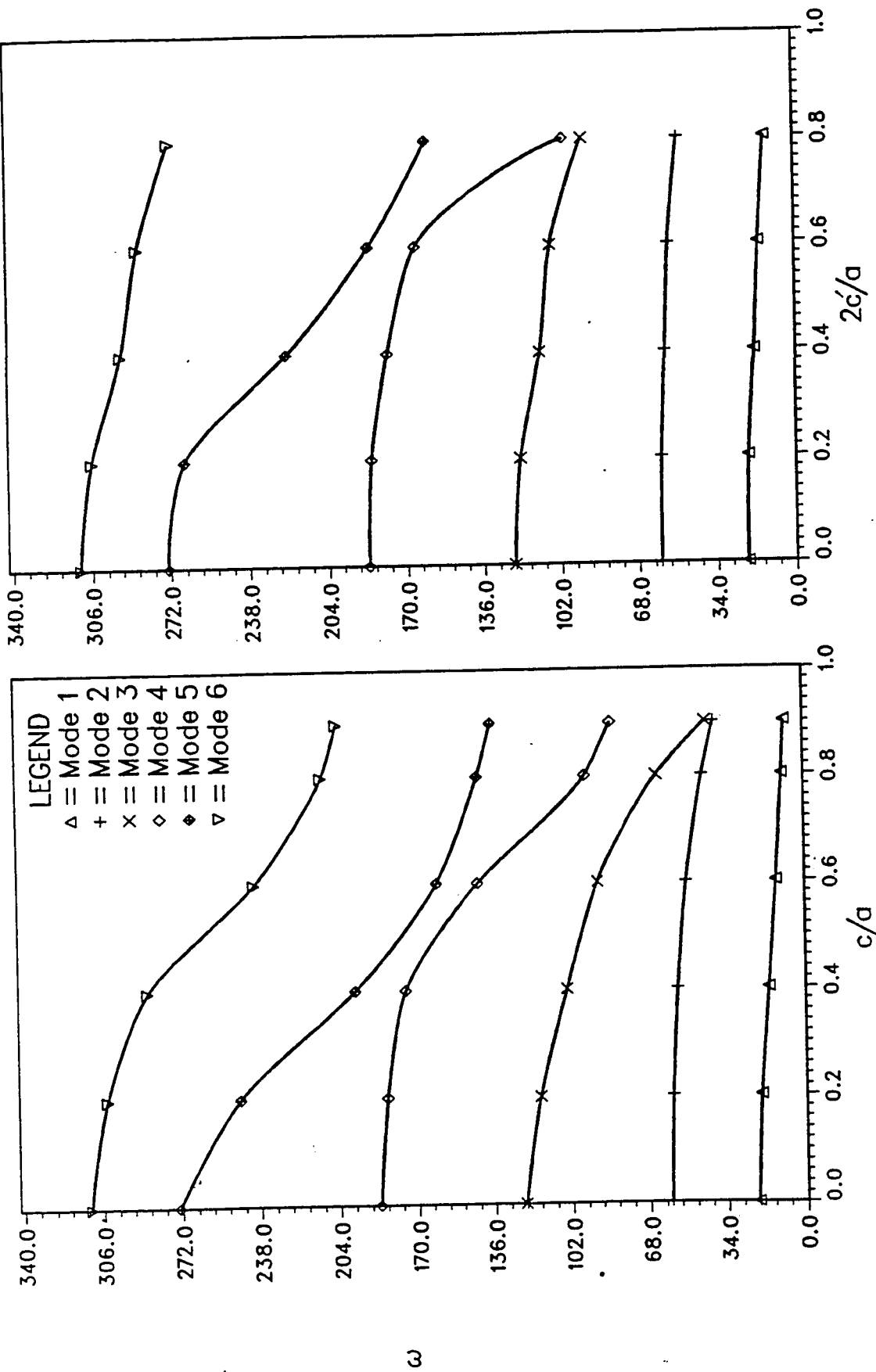


Figure 5.28: Variation of ω vs c/a , for P-F-F-F sandwich shell.
 $\tau/a = 2.0$, $\phi = b/a = 3.0$, $a/t_e = 32.432$

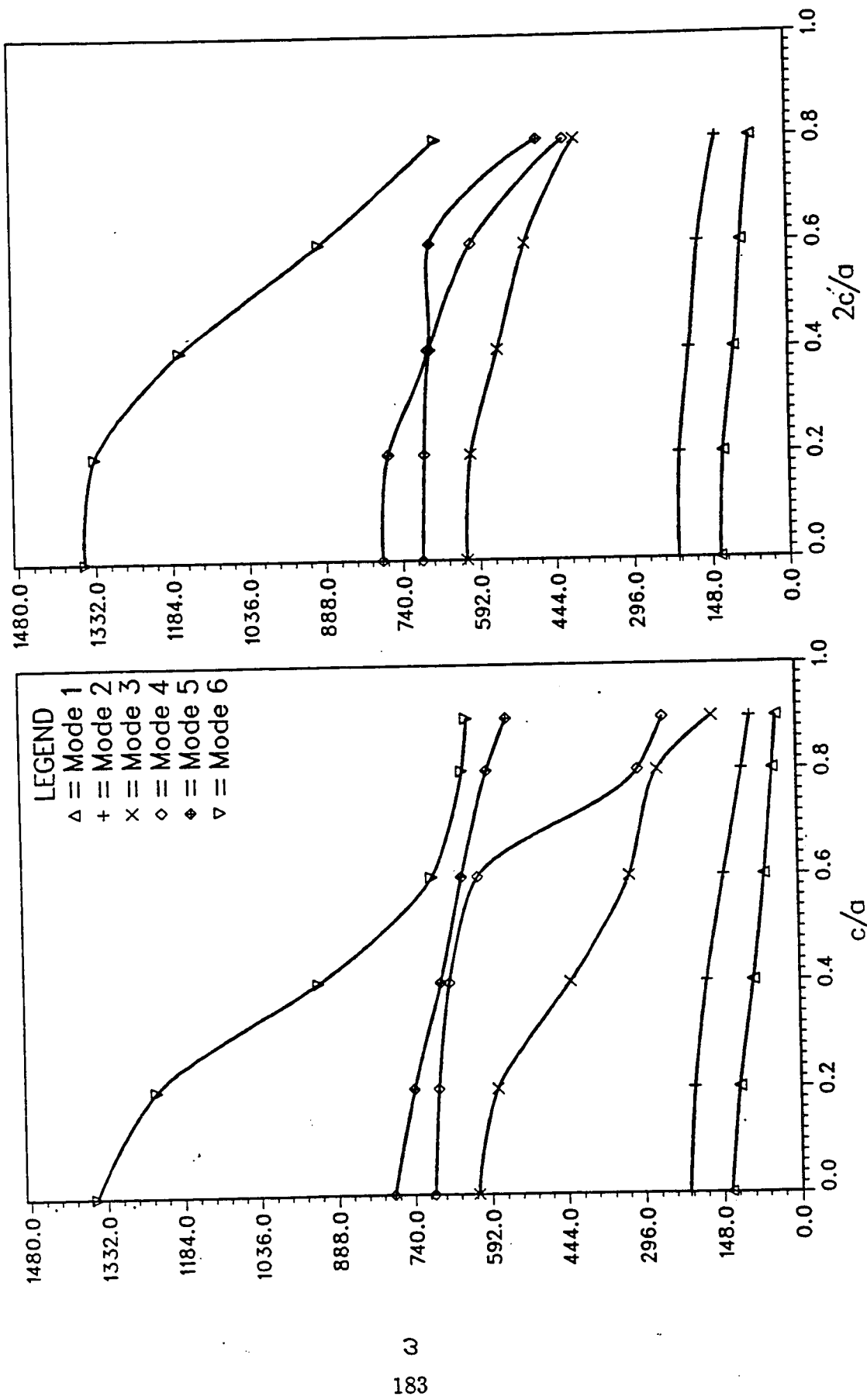


Figure 5.29: Variation of ω vs c/a , for P-F-F sandwich shell.
 $r/a = 3.0$, $\phi = b/a = 1.0$, $a/t_e = 32.432$

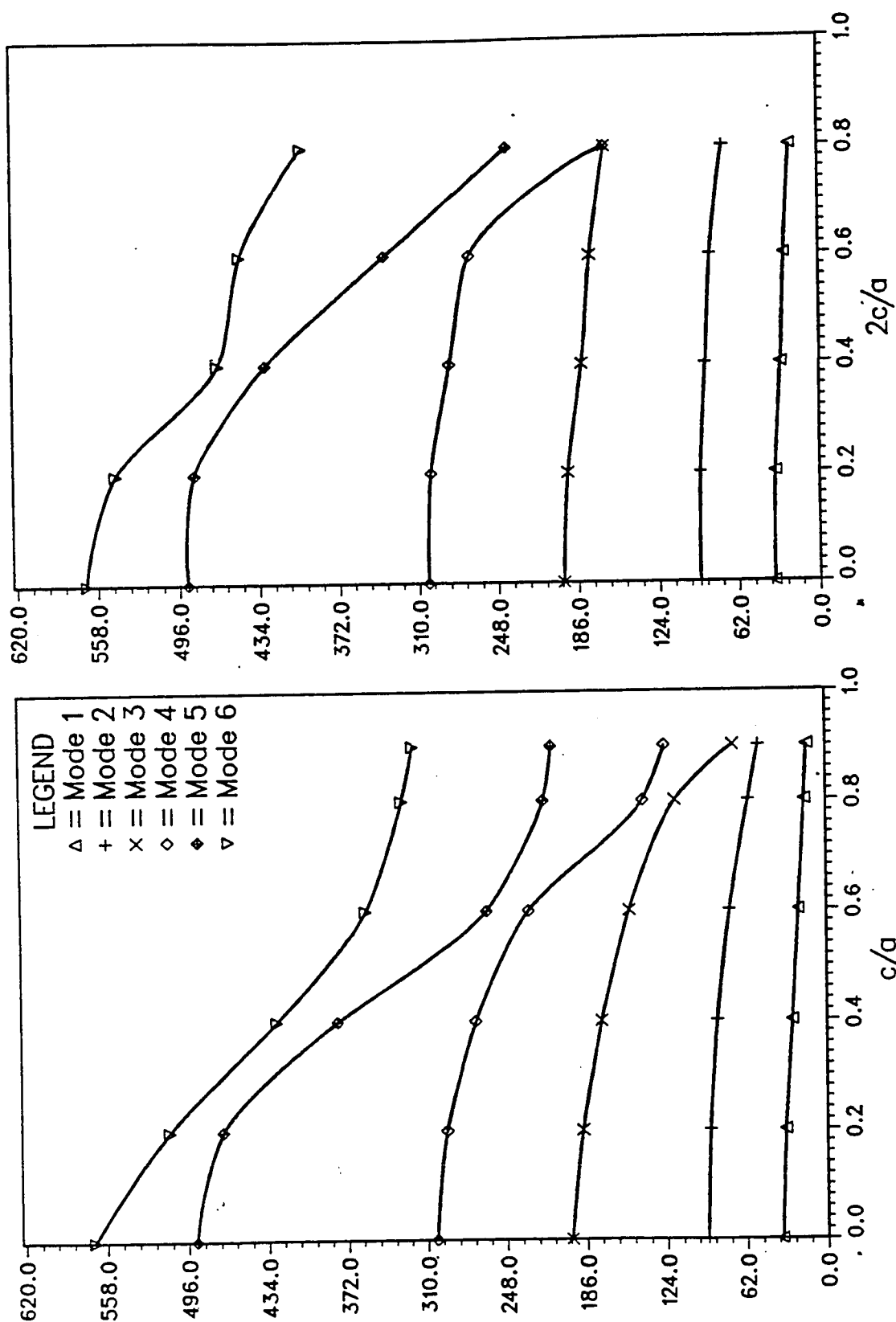


Figure 5.30: Variation of ω vs c/a , for P-F-F sandwich shell.
 $r/a = 3.0$, $\phi = b/a = 2.0$, $a/t_e = 32.432$

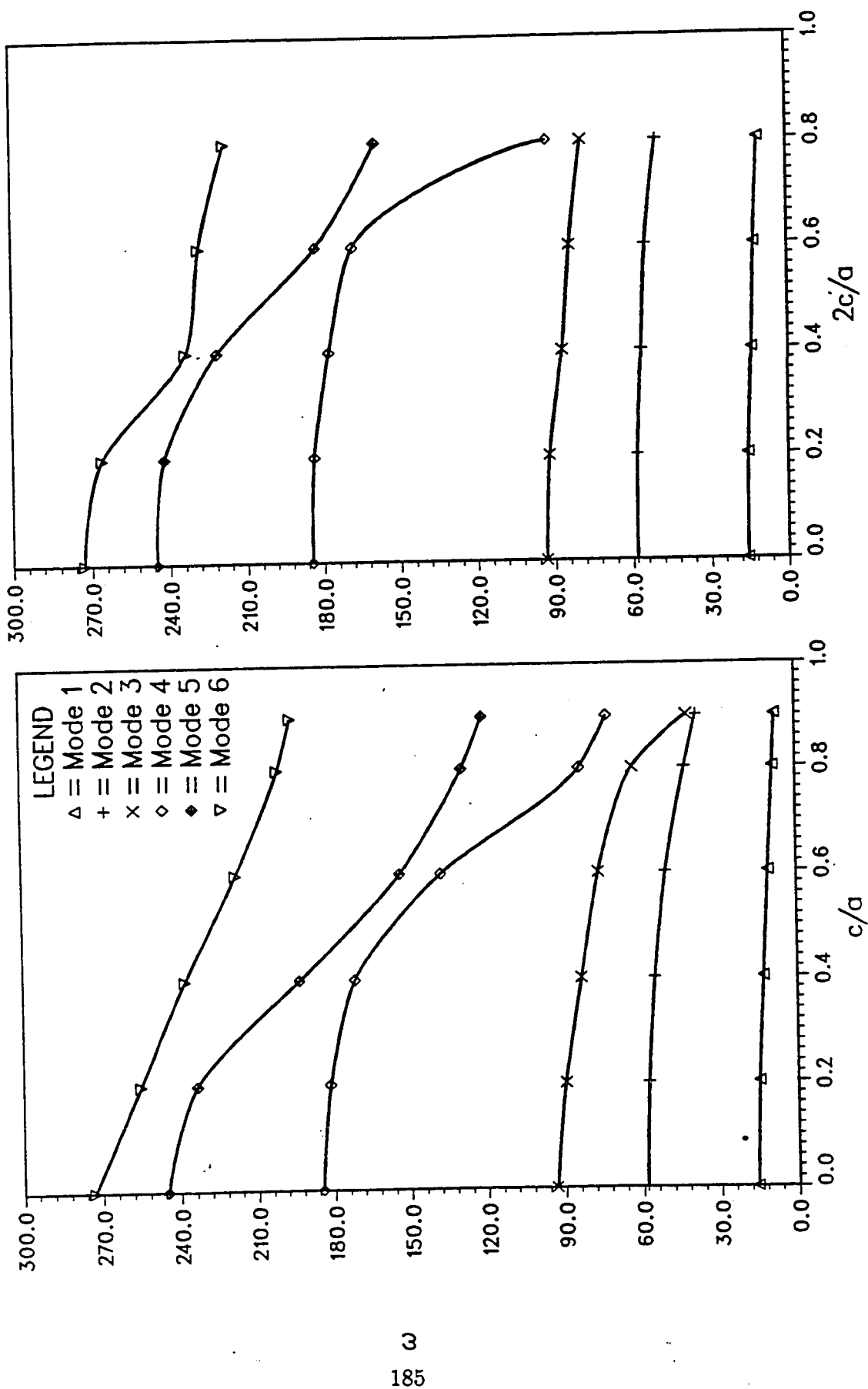


Figure 5.31: Variation of ω vs c/a , for P-F-F sandwich shell.
 $\tau/a = 3.0$, $\phi = b/a = 3.0$, $a/t_e = 32.432$

5.5 Discussion and Conclusions on Shells Results

The displacement based finite element method for analysis of shell structures is presented here. Based on the numerical results obtained in this study, the following conclusions can be drawn.

1 - A nine-node degenerate shell element has been used in finite element analysis of vibration of isotropic , orthotropic and composite shells. To overcome the shear locking problem, an enhanced interpolation of the transverse shear strains in the natural coordinate system is used. Also, to avoid membrane locking behavior, an enhanced interpolation of the membrane strains in the local cartesian coordinate system is applied. A number of cases of isotropic, orthotropic and sandwich degenerate shells ($\frac{\epsilon}{a} = 0$) have been analyzed and compared with other published results.

2 - In order to make the results general and to be able to draw general conclusions on the class of problems rather than specific shells, a non-dimensional form of frequency was used. For example, to calculate the first six natural frequencies of a curved fan blade with the following data,

$$a = 12in, \phi = \frac{b}{a} = 1, t = 0.12in, \frac{r}{a} = 2, \frac{a}{t} = 100, \frac{\epsilon}{a} = 0,$$

$$\nu = 0.3, E = 30 \times 10^6 psi, density = 0.284 \frac{lb}{in^3}$$

the following procedure may be followed.

$$\rho = \frac{0.284}{386} = 7.35 \times 10^{-4} lb.sec^2.in^{-4}$$

$$\Omega = 2\pi\omega a^2 \phi^2 \left(\frac{12\rho}{Et^2}\right)^{\frac{1}{2}}$$

$$\Omega = 2\pi\omega(12)^2(1)^2 \left(\frac{12 \times 7.35 \times 10^{-4}}{30 \times 10^6 \times 0.12^2}\right)^{\frac{1}{2}} = 0.129\omega$$

The non-dimensional frequencies Ω for this shell are obtained from Table (5.6) and are given below

11.153, 17.959, 31.974, 44.671, 50.401, 69.090

Therefore, the first six frequencies of the curved fan blade with the above geometry are,

$$\omega_1 = 86.268 \text{ Hz}, \quad \omega_2 = 138.912 \text{ Hz}$$

$$\omega_3 = 247.319 \text{ Hz}, \quad \omega_4 = 345.530 \text{ Hz}$$

$$\omega_5 = 389.852 \text{ Hz}, \quad \omega_6 = 534.411 \text{ Hz}$$

These values are very close to the results given in reference [33].

3 - (a) Isotropic Cylindrical Shells

A fan blade with a crack along its base can be idealized as a cylindrical partially supported shell. The first six natural frequencies and corresponding modal shapes, and nodal lines for variable detached base length have been generated in this work. Investigations have been carried out for two cases with three different aspect ratios as well as three different radii to base length ratio. These are identified as:

(i)- detached base length from one side ($\frac{c}{a}$).

(ii)- detached base length from both sides ($\frac{2c'}{a}$).

The results show that as the ratio $\frac{r}{a}$ increases or the shell becomes shallower the frequency ω decreases. The change in the frequency parameters ($\frac{\Omega}{\phi^2}$) for case (i) is more pronounced than for case (ii). The two cases for the constant radius of curvature ($\frac{r}{a} = \text{constant}$) and for the higher aspect ratio ($\frac{b}{a} = 3$), the frequency parameter $\frac{\Omega}{\phi^2}$ stay relatively unaffected for longer detached base length (80%). Thus, we conclude that as the aspect ratio increases, shells and plates behave more like

beams where the frequency is not sensitive to the length of base support.

In general, the response of shells with partially detached base length is similar to that of plates. The main difference in the response of short shells is that the fundamental mode is usually the twisting mode except for the large aspect ratios ($\frac{b}{a} \geq 2$) and large radii of curvature ($\frac{r}{a} \geq 3$). For these types of shells the first fundamental twisting modes are replaced by beam modes.

-(b) Orthotropic shells

On comparison of results obtained for orthotropic shells with those for isotropic shells, we find the trends to be similar in general. This observation is the same as the one noted earlier for isotropic and orthotropic plates. The important point to note is that the natural frequency is more sensitive to the detached length in case of orthotropic materials than for isotropic materials. This sensitivity is very obvious even for long and shallow shells as can be seen from graphs 5.23 to 5.31 and tables 5.25 to 5.33.

-(c) Sandwich shells.

In computation of natural frequency for sandwich shells the shell is assumed to consist of an orthotropic core and two face sheets of equal thickness which are made up of isotropic material. Here also the variation of natural frequency with detached length is similar to the variation in the case of isotropic materials. It is observed that, for sandwich shells with the same face material and for the same thickness of face sheets as for isotropic shells, the frequencies are much higher. However the variation of the frequencies with the detached length for the two shell types is the same. The increase in frequency for sandwich shells can be explained by the fact

that the overall thickness t_e of the shell is larger giving a larger value of stiffness.

5.6 General Conclusions for Plates and Shells

Several examples have been presented and the results have been compared with the available results in the literature. These have been found to be in good agreement. The crack along the base of plates, shells and curved fan blades is idealized as a partial detachment of base. The effects of variation of the detached base length on natural frequencies, modal shapes, displacement patterns for these models have been presented in this study.

A comparative study of the dynamic behavior of plates and shells, in general, has been presented in this thesis. Furthermore, this thesis also presents an in depth study of the dynamic behavior of various types of materials; for example isotropic, orthotropic and sandwich materials.

Based on the investigation carried out and the results presented here, the following conclusions can be drawn:

1 - Variation of natural frequency with detached base length is very large for both plates and shells having small aspect ratios ($\frac{b}{a} \leq 1.5$).

2 - For larger aspect ratios of plates and shallow shells ($\frac{b}{a} \geq 2$), the variation of natural frequency is very small even for 80 percent detached base length.

3 - In composite or orthotropic materials the variation of the natural frequency with detached base length is more than that for the case of isotropic plates and shells with similar geometry. This would mean that for the design and analysis

of orthotropic plates and shells we should accurately model the presence of any detachment of support.

4 - For the degenerate case ($\frac{c}{a} = 0$) the fundamental mode for plates starts with the beam mode. But in shells with small aspect ratio and small radius to base length ratio the beam mode is the second mode. This trend is reversed only for larger ($\frac{b}{a} \geq 2$, ($\frac{r}{a} \geq 3$) ratios where the beam mode shifts to the fundamental mode.

5- The computations for triangular plates have been carried out for three different cases, namely:

- detached length from right angle end, case (r),
- detached length from acute angle end, case (l),
- detached length from both ends, case ($r - l$).

In the case (r), the variation of frequency with $\frac{c}{a}$ is higher than for the other two cases.

6 - For cylindrical shells two cases have been studied:

- (i) detached base length from one side (c/a)
- (ii) detached base length from both sides ($2c'/a$).

Its shown in all the corresponding figures, the variation of frequency in case (i) is much more than the case (ii).

7 - For shallow shells as r/a becomes larger, ($\frac{r}{a} \geq 3$) the dynamic behavior is similar to that of a plate. This is quite justified and expected.

8- The examples illustrate that the effect of detached length is most significant

for higher mode frequencies.

5.7 Recommendation for Further Study

There are several practical problems in this area which need further work. It is seen that this investigation on partially supported plates and shells can be extended for the following problems:

1- Trapezoidal plates generated by cutting the tip of the triangular plates are of considerable practical importance in the aircraft industry. These problems are a logical extension of the problem studied in this thesis.

2- Further generalization of the problem noted above is the skew plate with variable skew angle and different aspect ratios. These plates are also commonly employed in the structures of aerospace vehicles such as missiles, rockets and aircrafts. The study of the effects of support detachment and crack at root, on the vibrational behavior of these types of plates is important for the design purposes in the aerospace industry.

3- The pre-twisted fan blades have many applications in the turbo-machinery industry. The effect of cracks along the support on the vibration characteristics of twisted cantilever plates is very important in the design of blades. The present technique could be logically extended to this problem.

4- Further work could also be required to the study of the effects of support detachment on fibre orientation in composite plates and shells. The results in this thesis have been generated only for one angle of fibre orientation.

5- Study may be applied to the vibrational behavior of spherical shells and circular plates with support along a part of the boundary.

6- The results obtained for modal shapes in this study can be used as a data for analysing the response of forced vibration of similar geometries by the mode summation method.

7- Work may also be extended to nonlinear analysis for the problems discussed in this thesis.

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Appendix A

AA

A.1 ISOTROPIC PLATES

1

MODAL SHAPE FOR TRIANGULAR ISOTROPIC PLATES

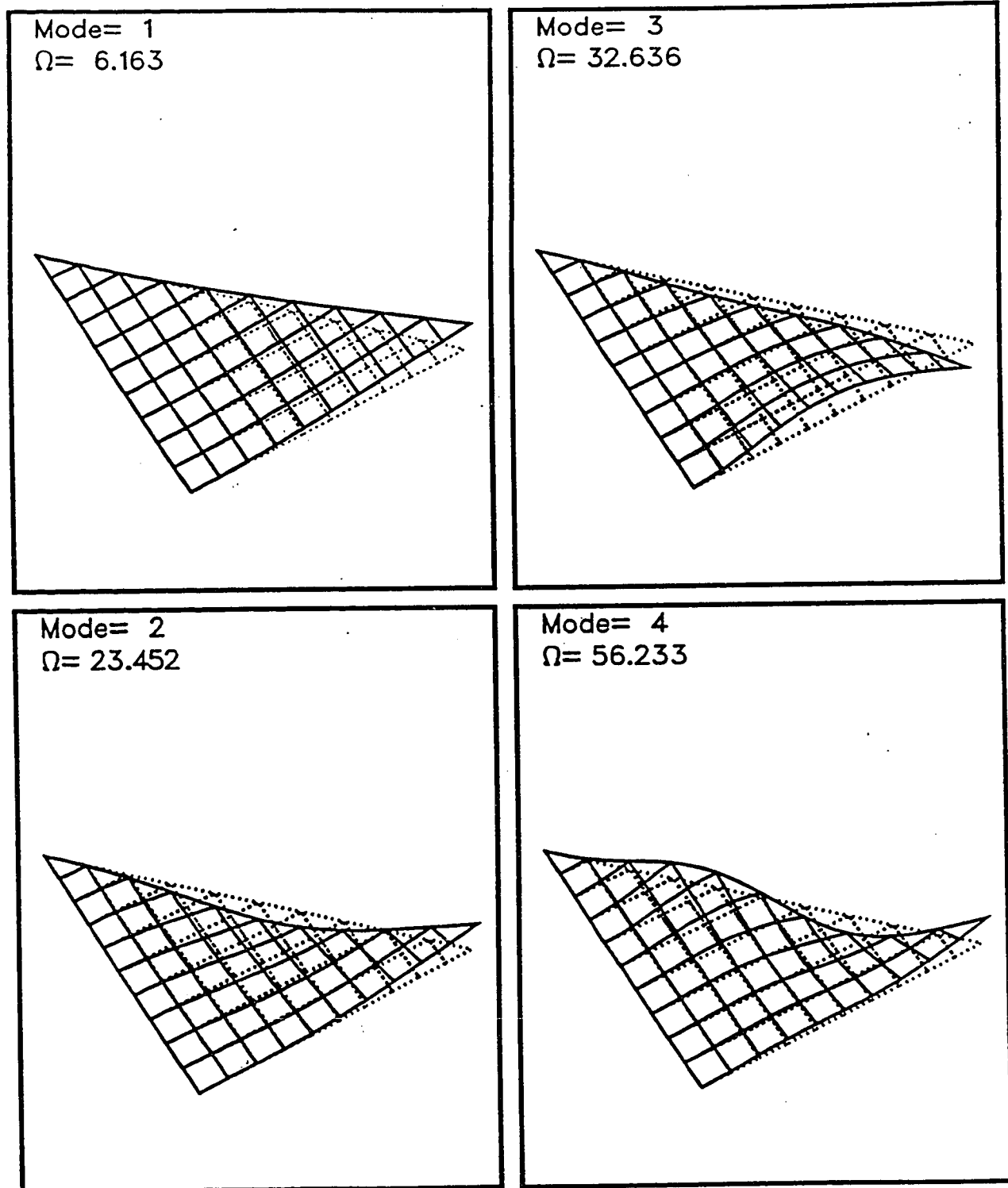


Figure A.1: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.0$, $c - r$.

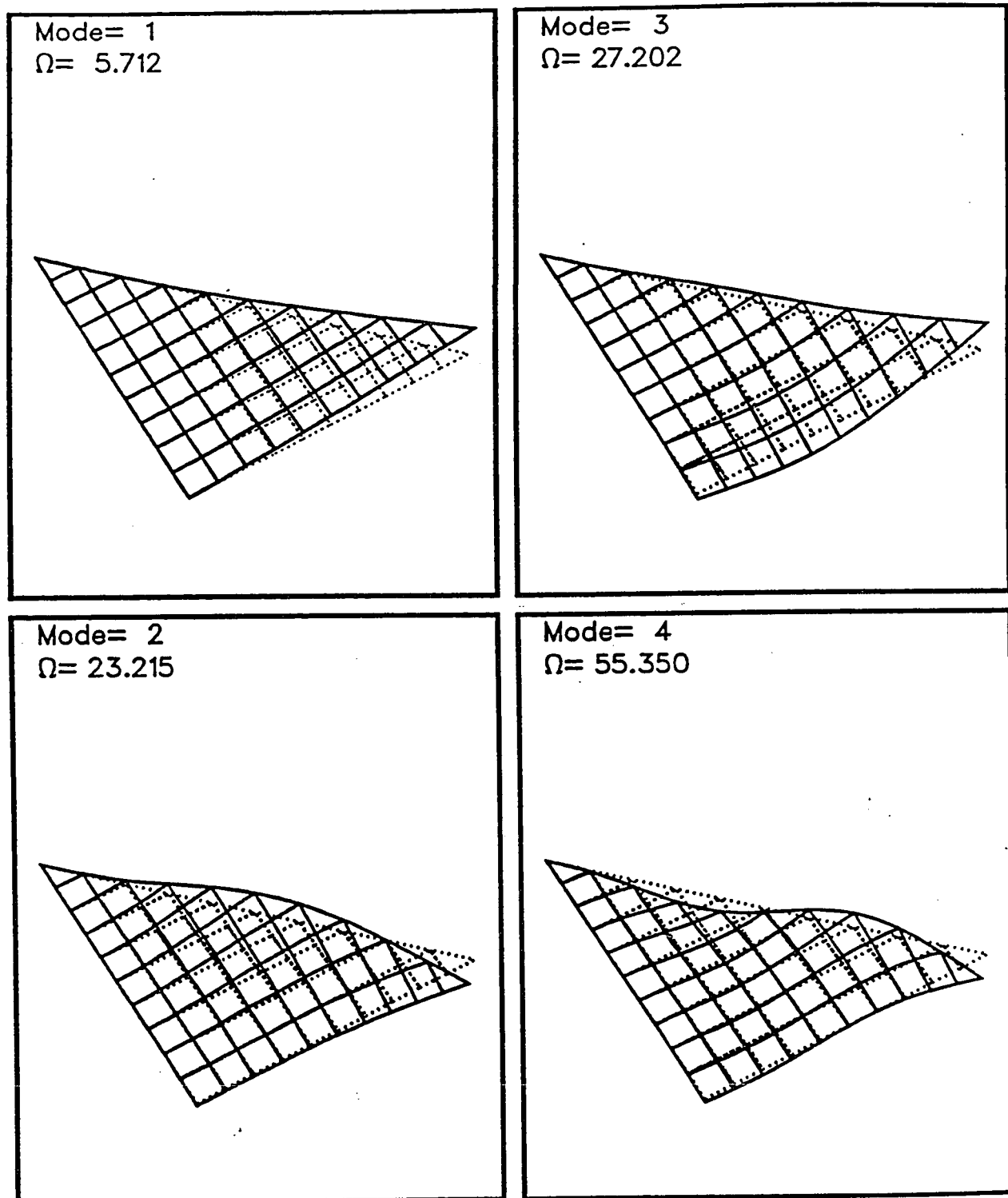


Figure A.2: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.2$, $c - r$.

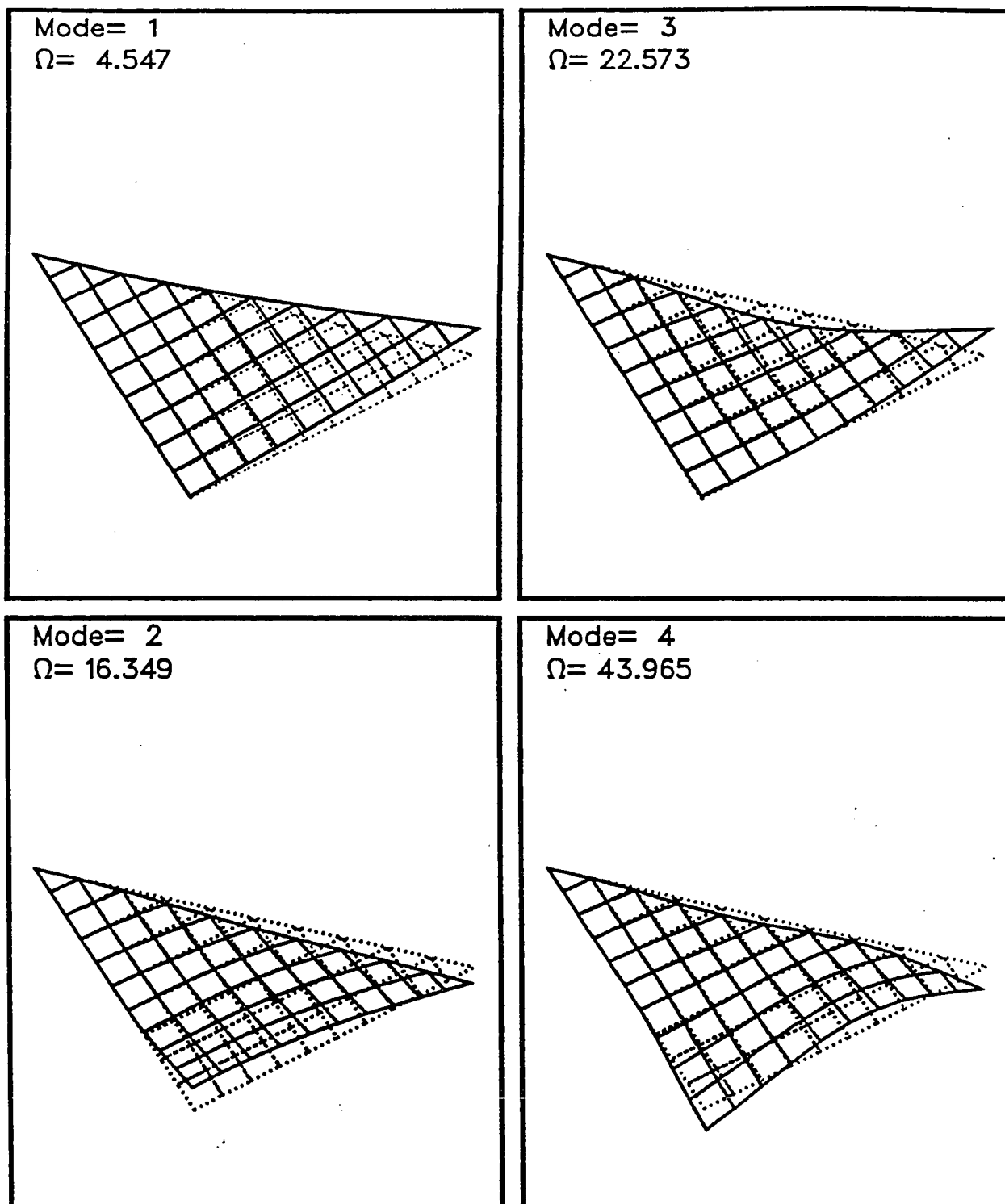


Figure A.3: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.4$, $c = r$.

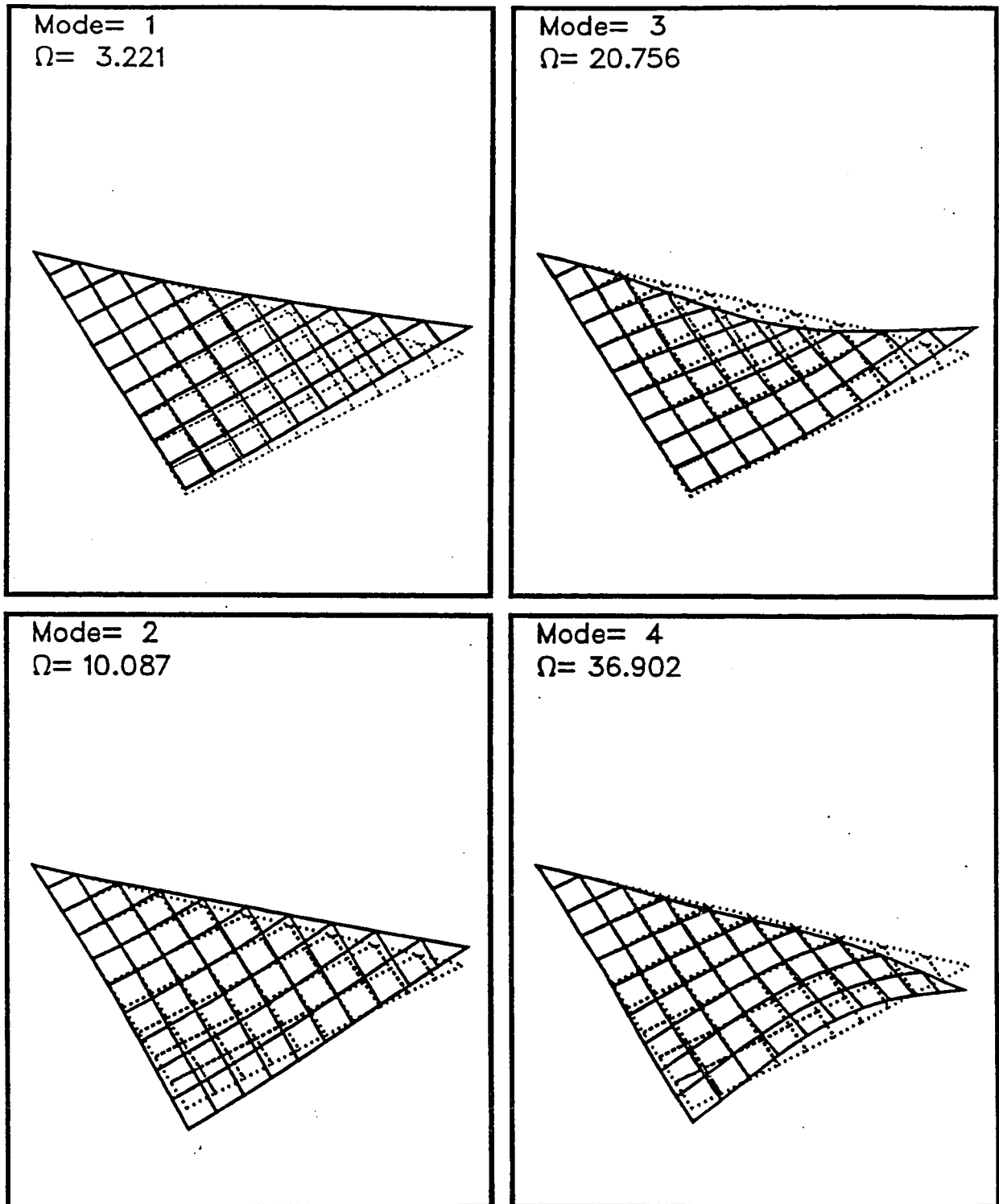


Figure A.4: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.6$, $c - r$.

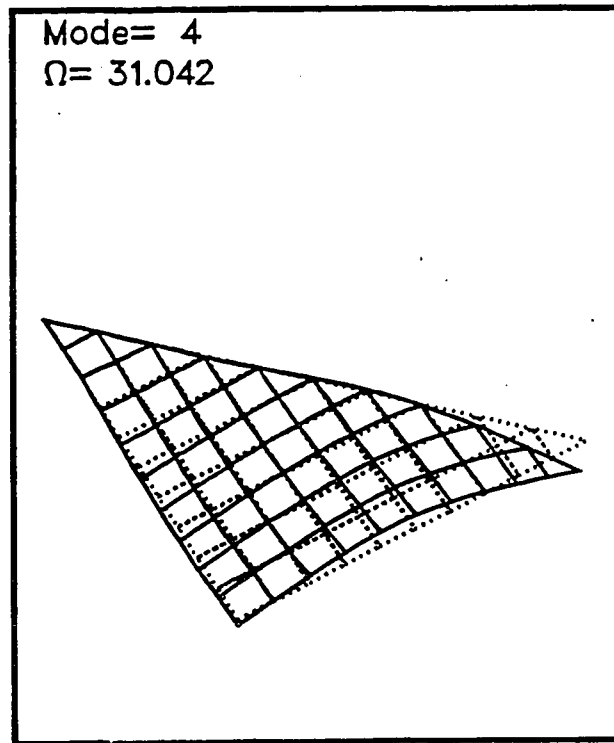
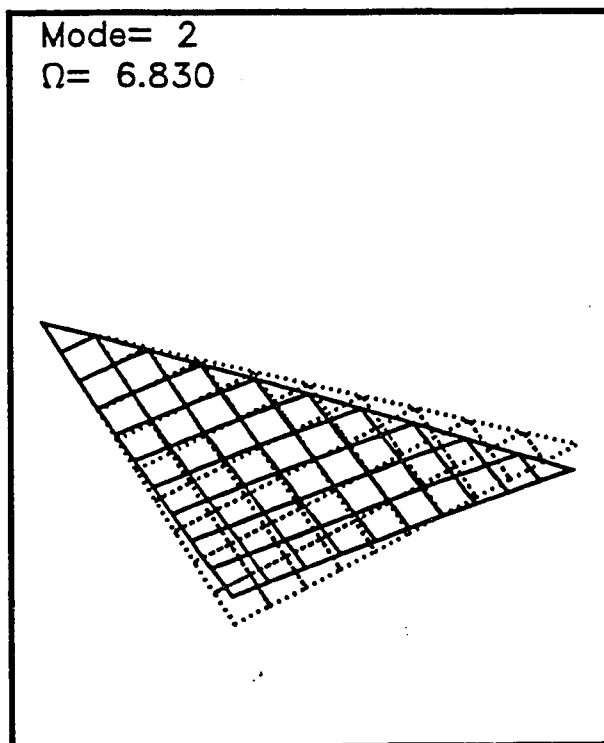
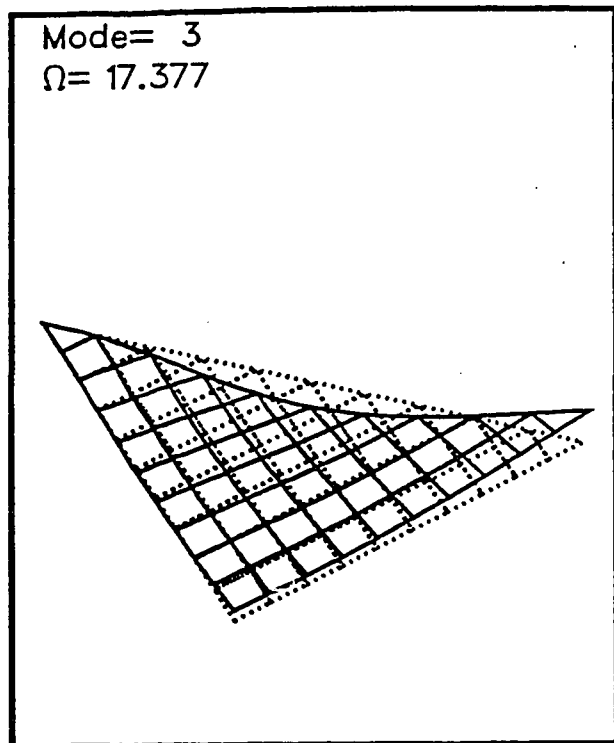
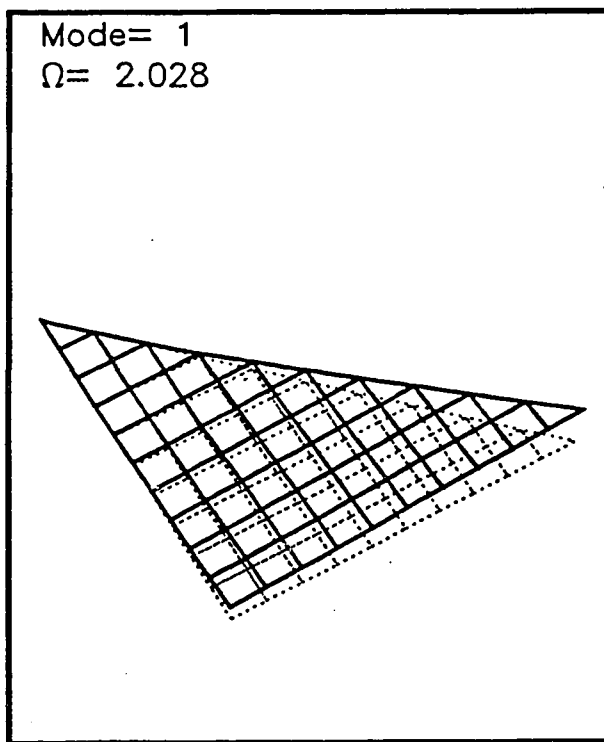


Figure A.5: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.8$, $c = r$.

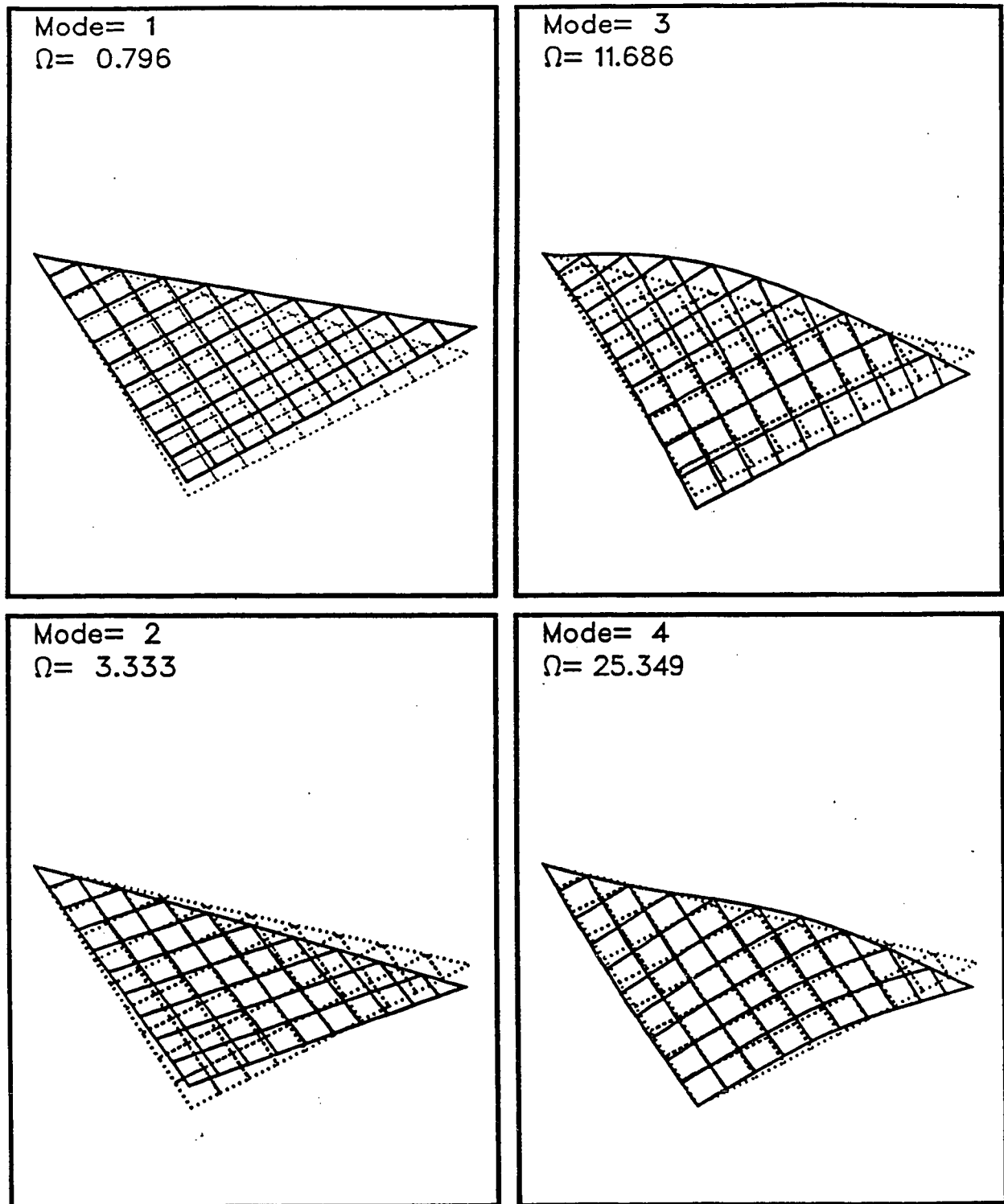


Figure A.6: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 1.0$, $c = r$.

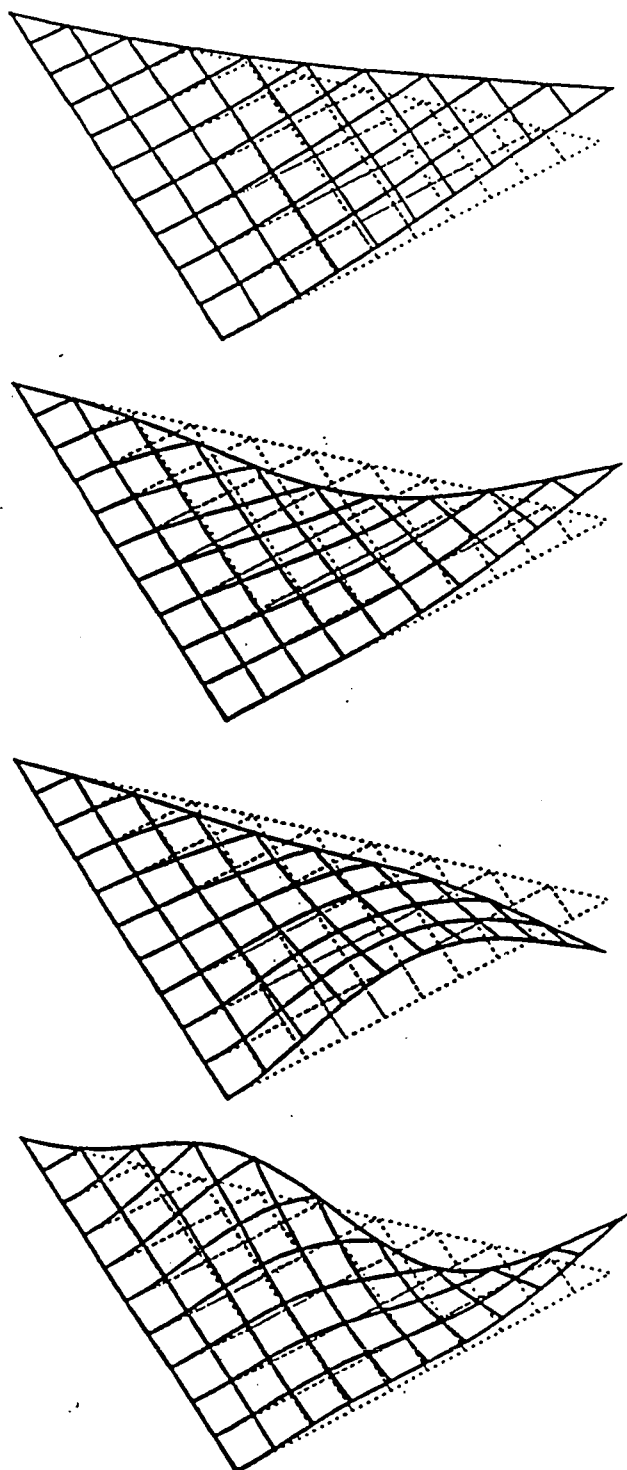


Figure A.7: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.0$, $c = l$.

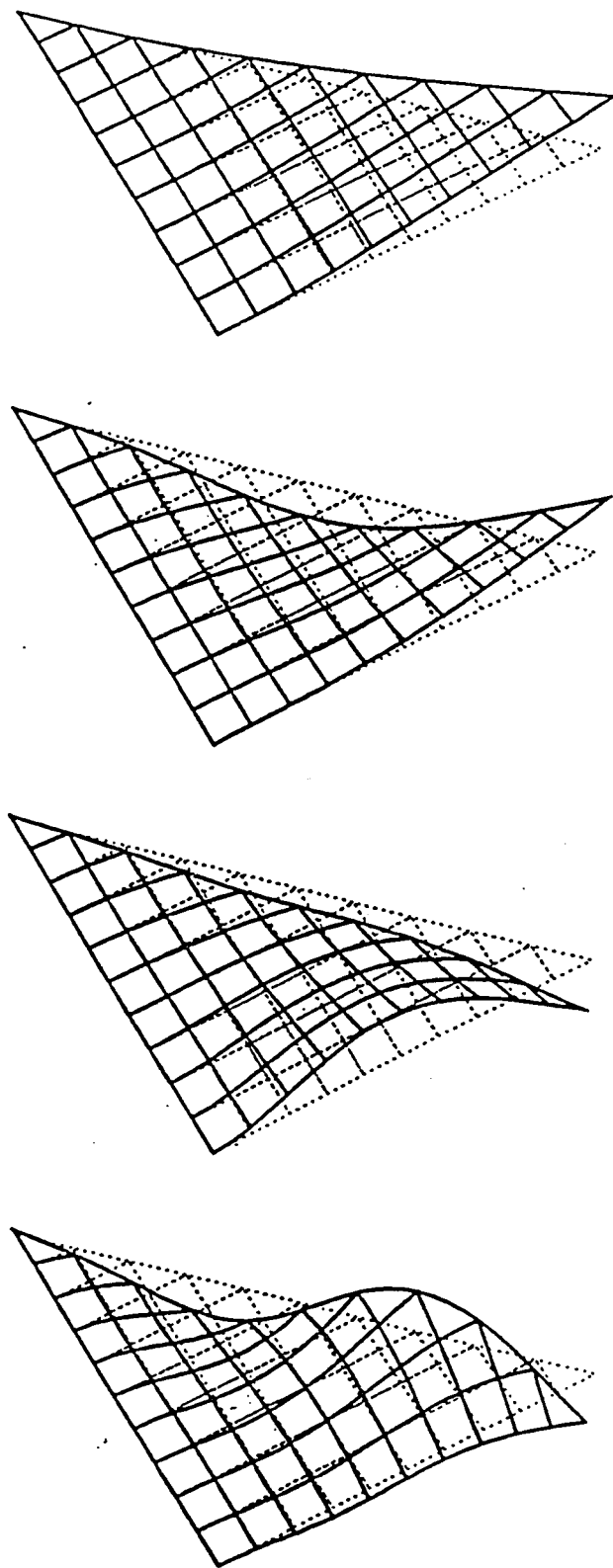


Figure A.8: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.2$, $c - l$.

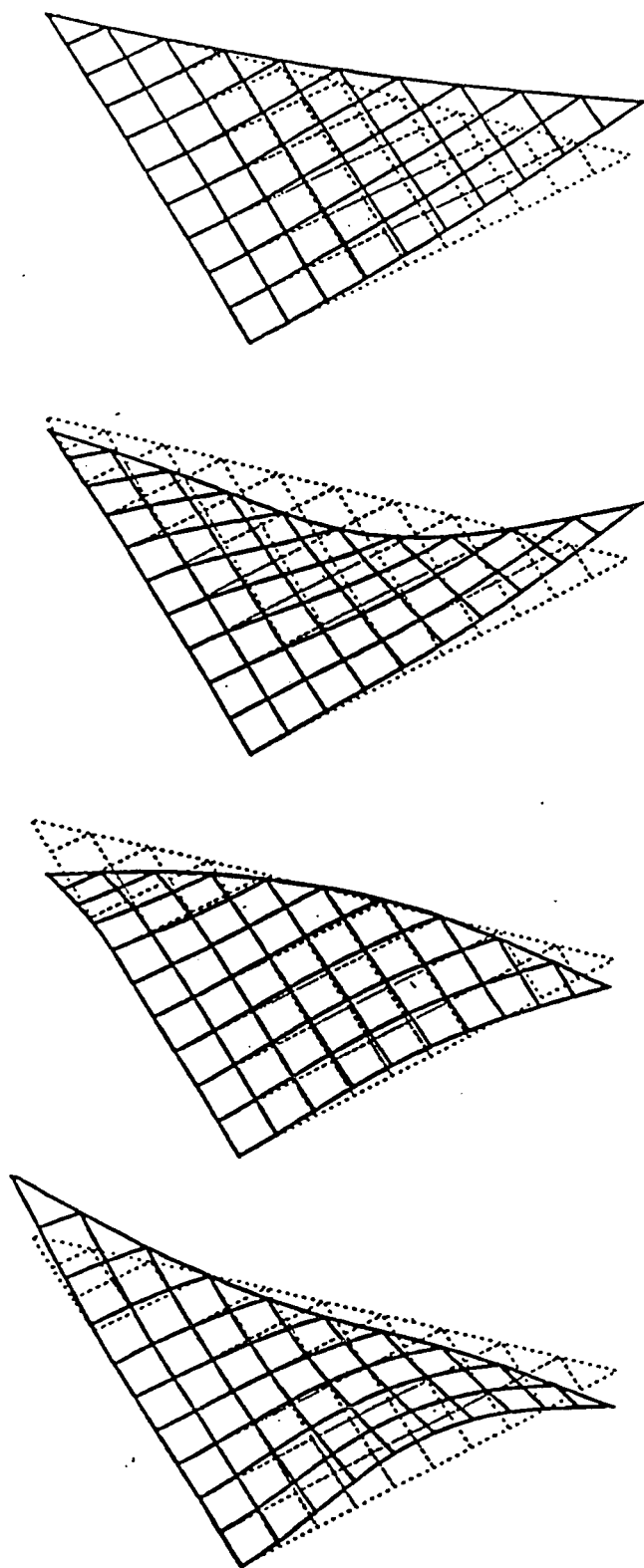


Figure A.9: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.4$, $c - l$.

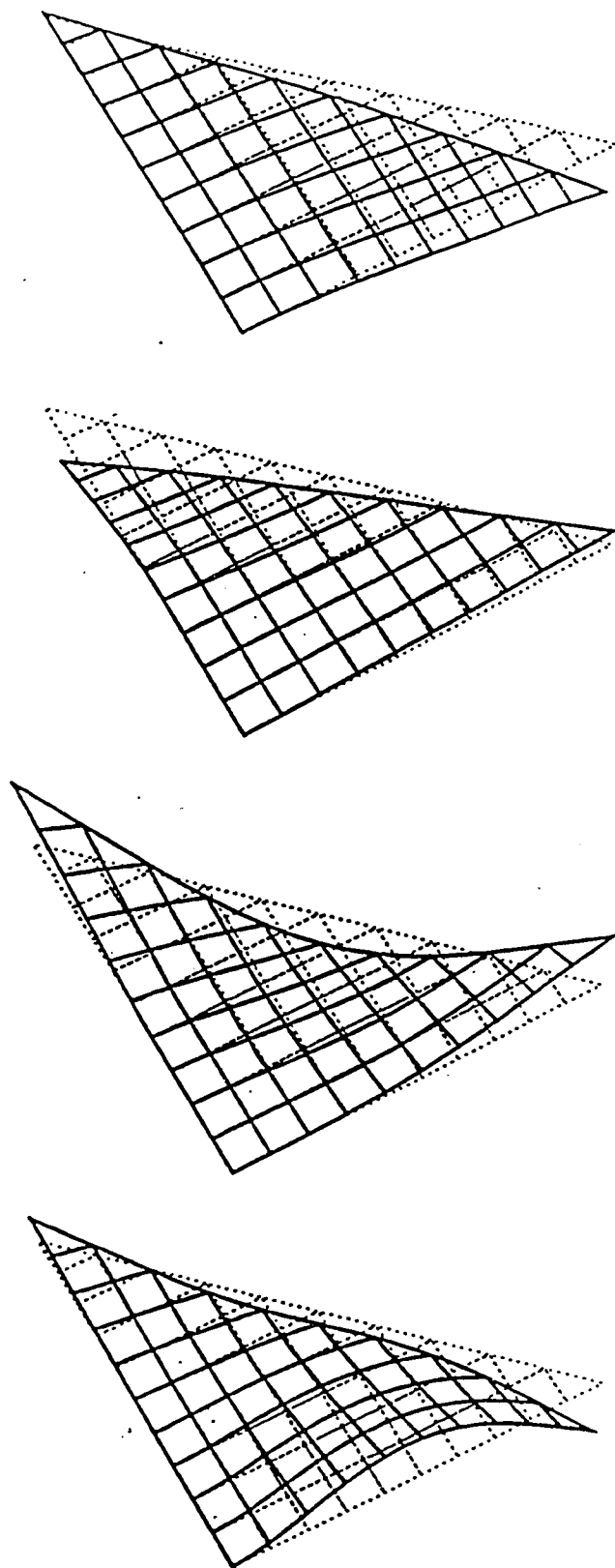


Figure A.10: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.6$, $c - l$.

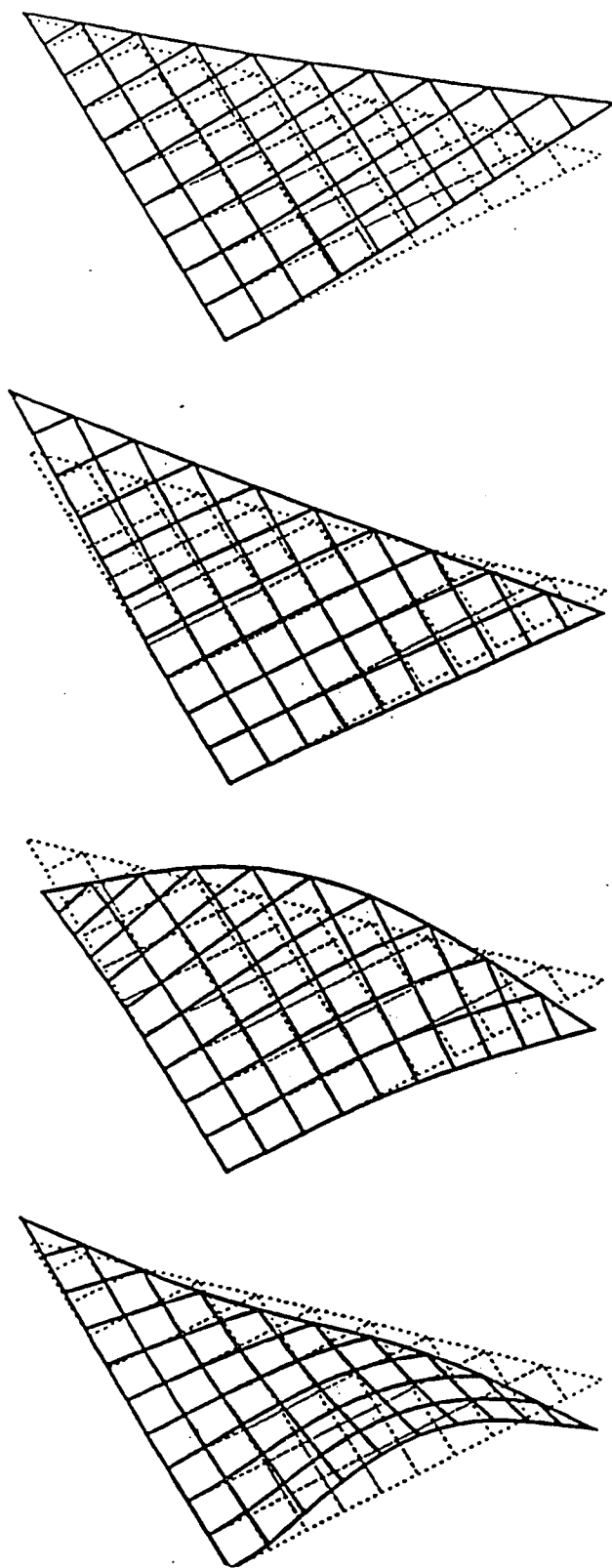


Figure A.11: Modal shapes for triangular isotropic plate with
 $b/a = 1.0, c/a = 0.8, c - l$.

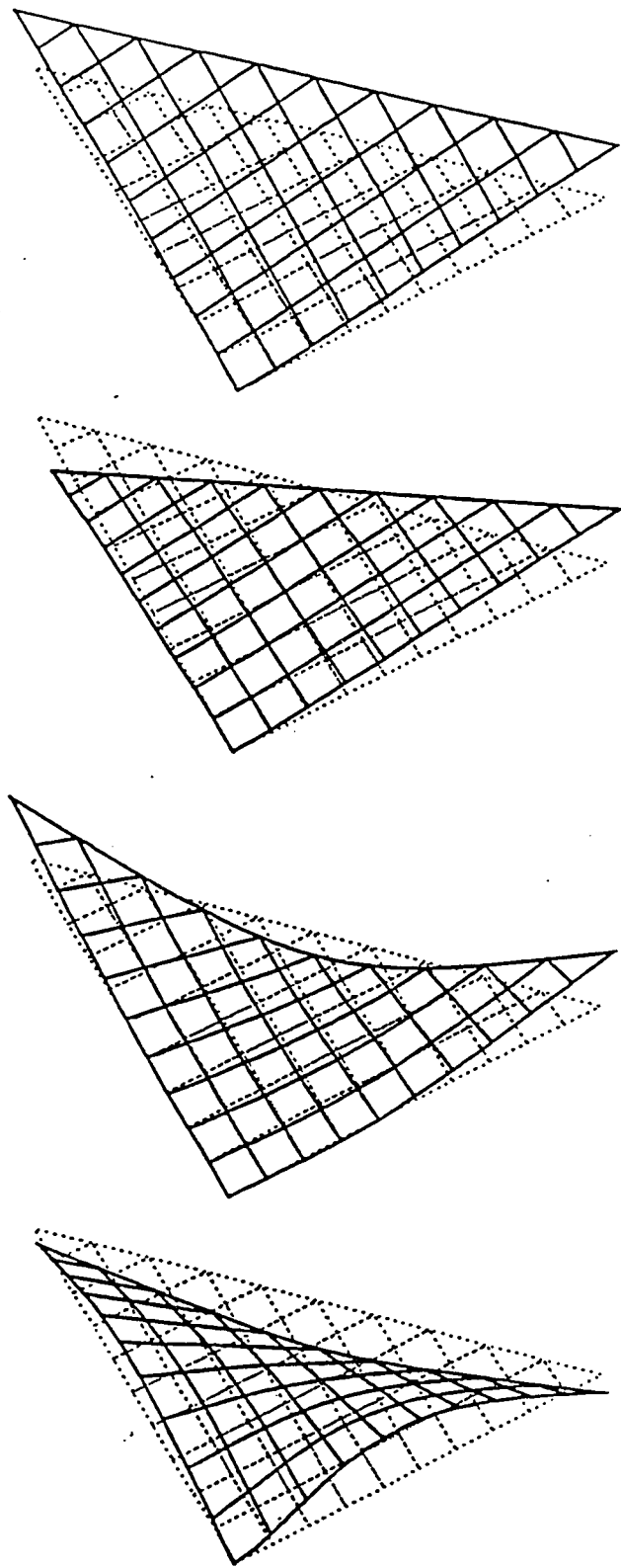


Figure A.12: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 1.0$, $c - l$.

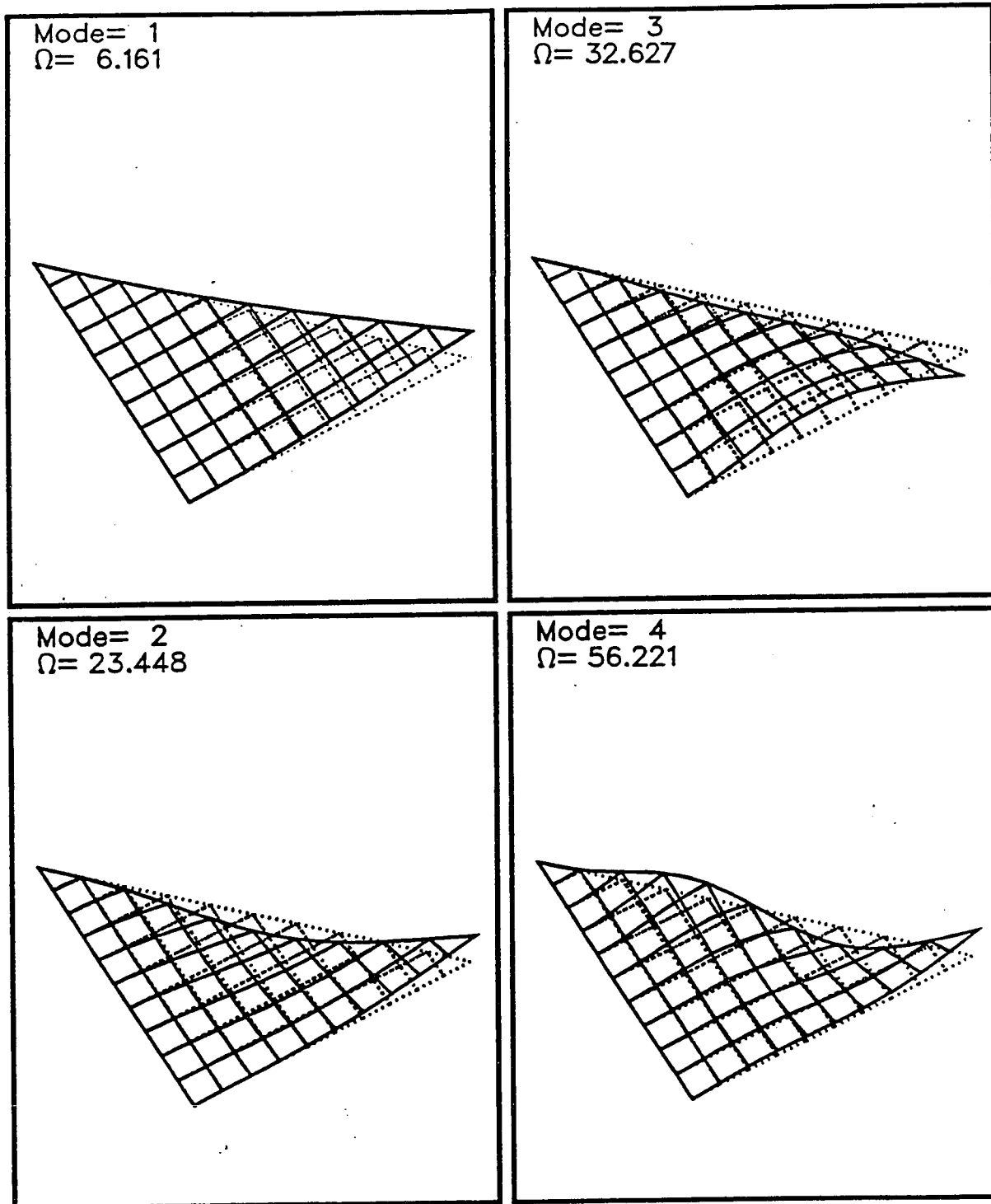


Figure A.13: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.0$, $c - r - l$.

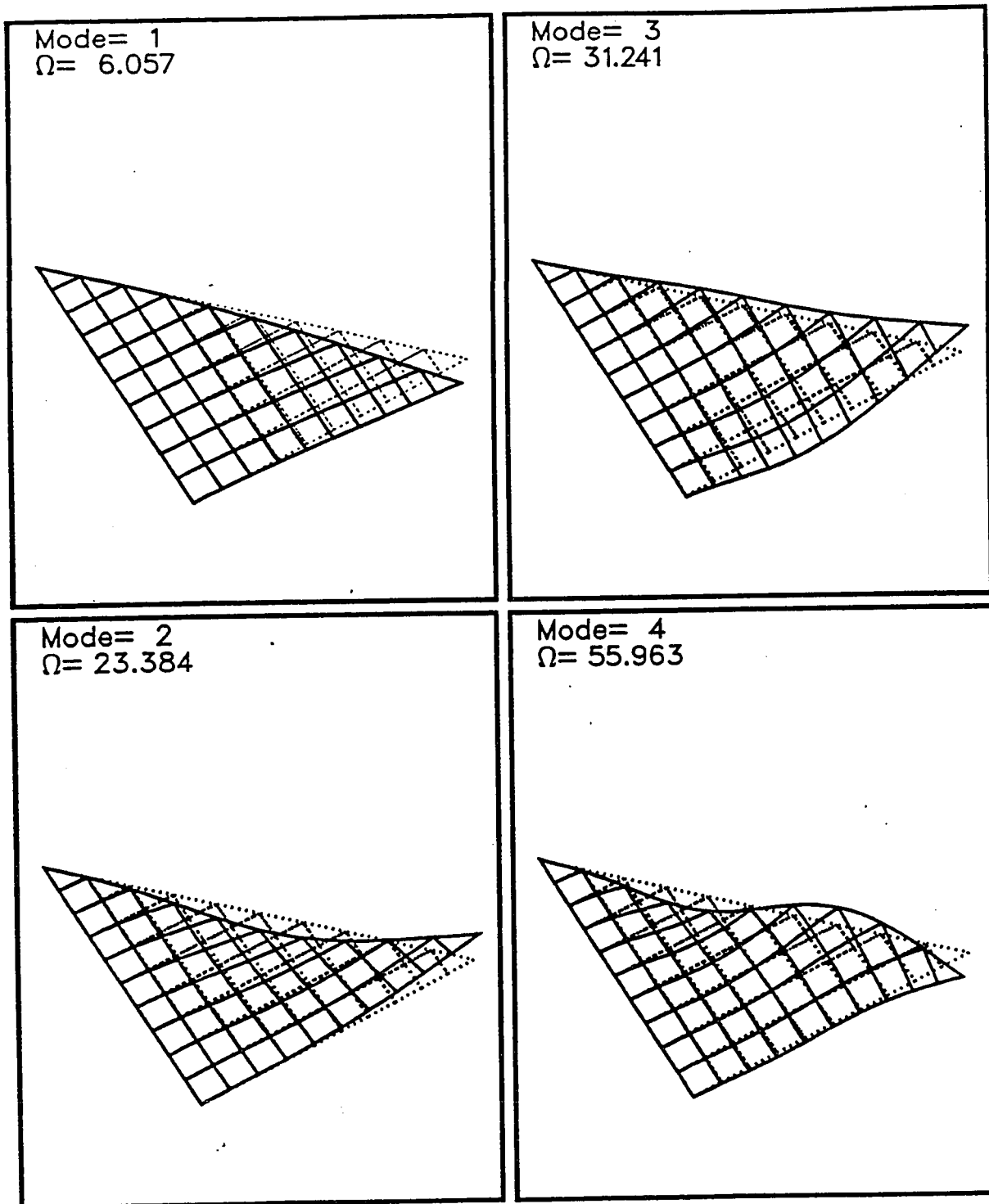


Figure A.14: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.2$, $c - r - l$.

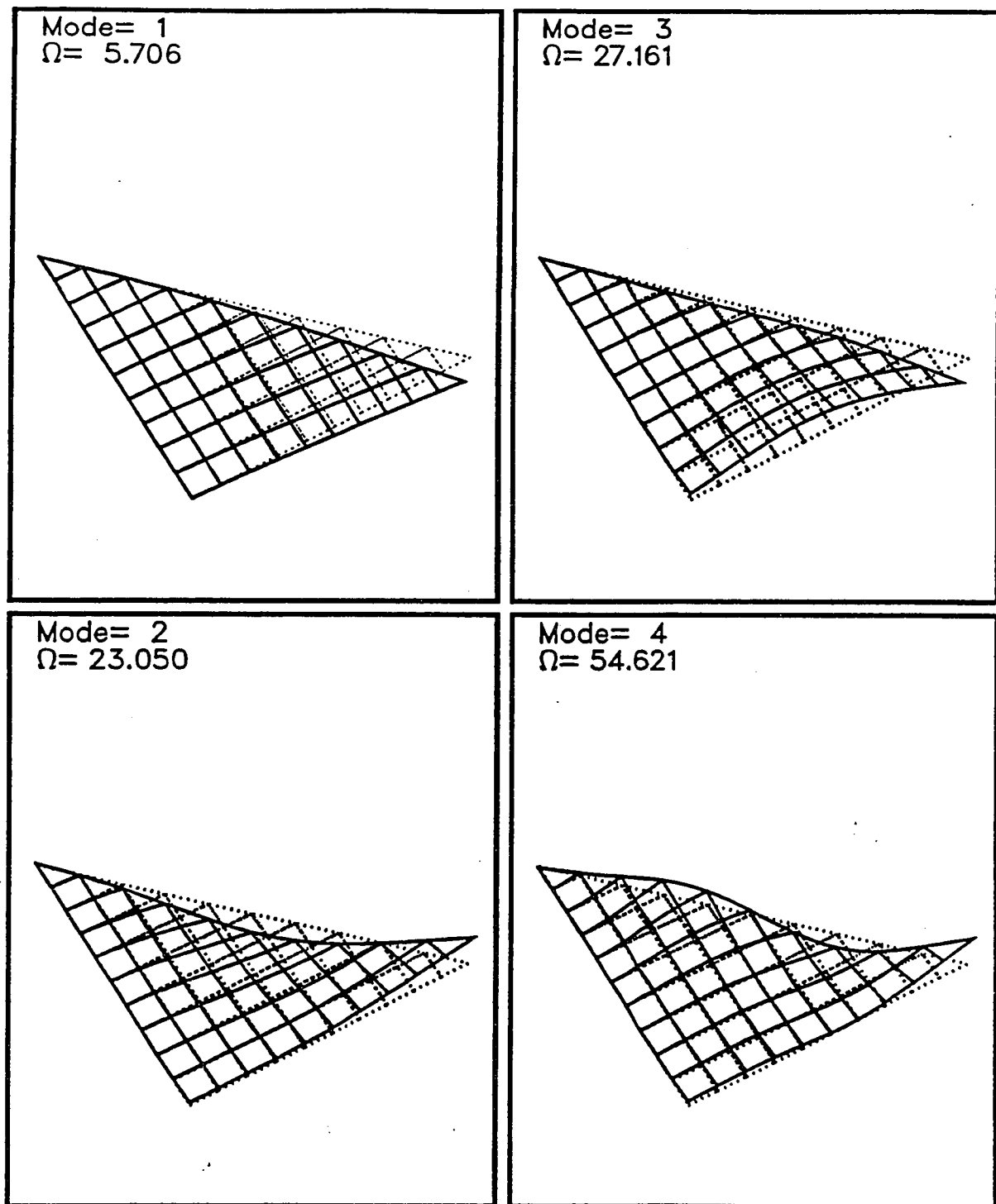


Figure A.15: Modal shapes for triangular isotropic plate with
 $b/a = 1.0$, $c/a = 0.4$, $c - r - l$.

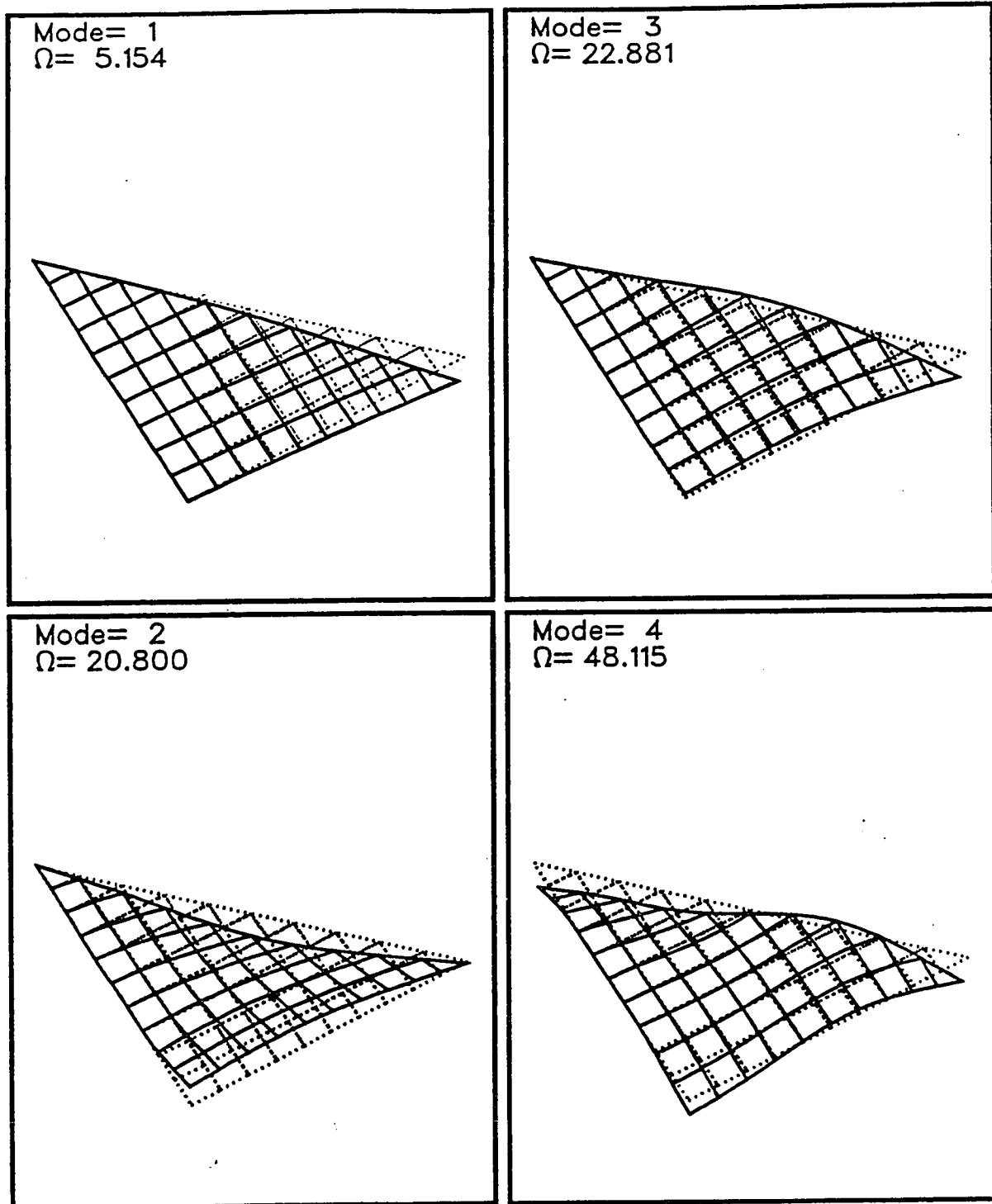


Figure A.16: Modal shapes for triangular isotropic plate with
 $b/a = 1.0, c/a = 0.6, c = r = l.$

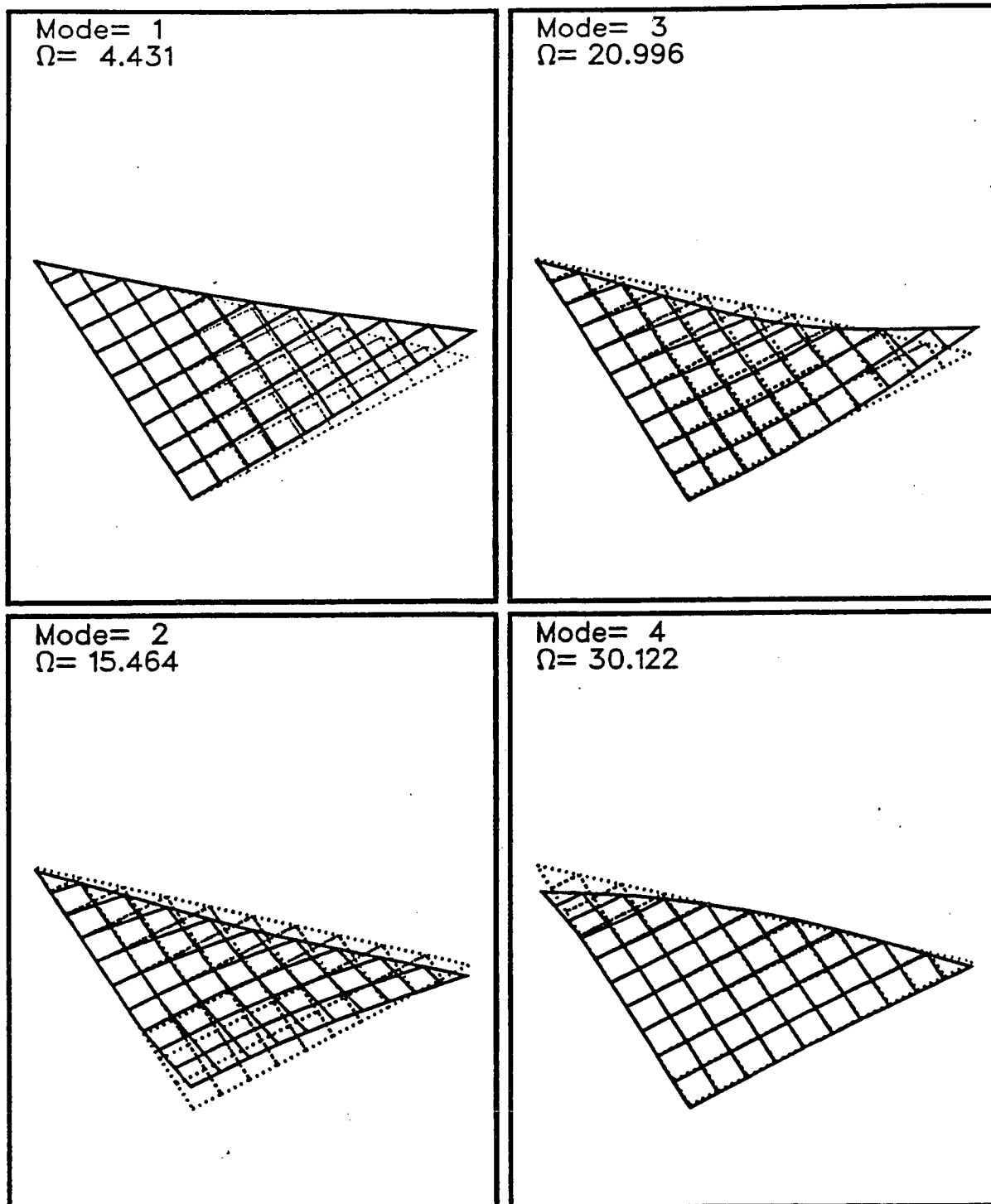


Figure A.17: Modal shapes for triangular isotropic plate with
 $b/a = 1.0, c/a = 0.8, c - r - l.$

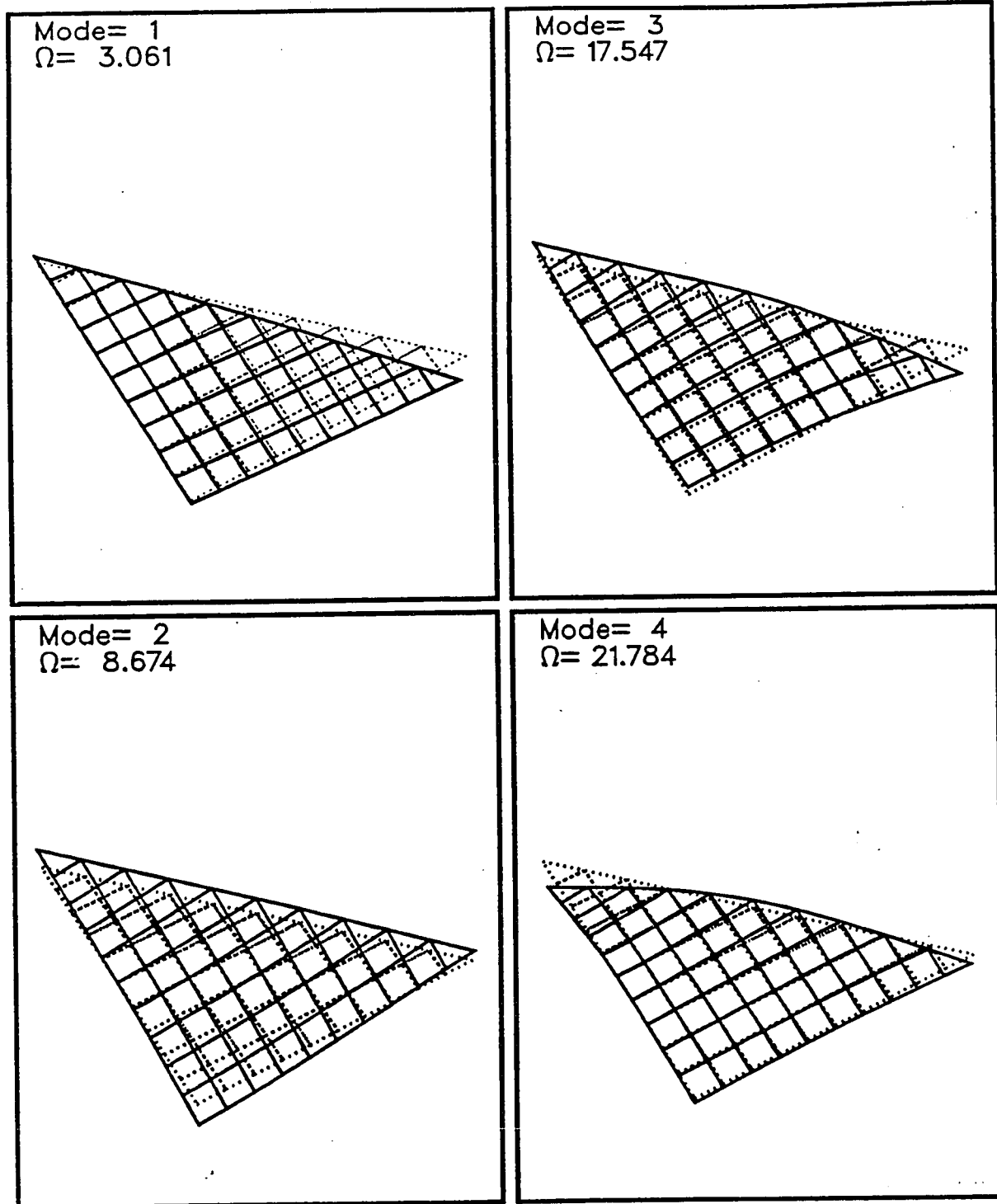


Figure A.18: Modal shapes for triangular isotropic plate with
 $b/a = 1.0, c/a = 1.0, c = r = l.$

Appendix B

BB

B.1 ISOTROPIC SHELLS

1

MODAL SHAPES FOR CYLINDRICAL ISOTROPIC SHELLS

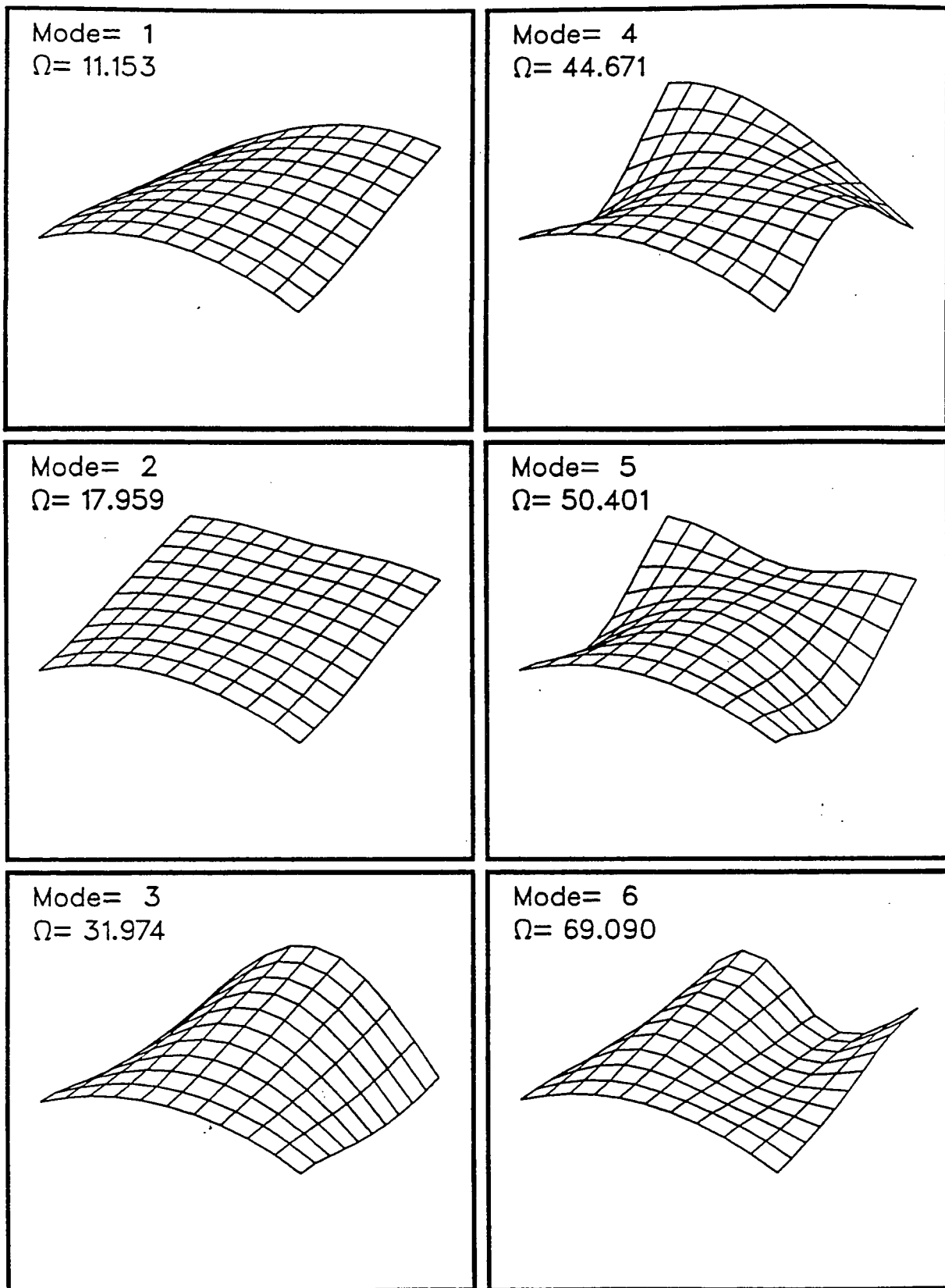


Figure B.1: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, \quad b/a = 1.0, \quad a/t = 100.00, \quad c/a = 0.0.$$

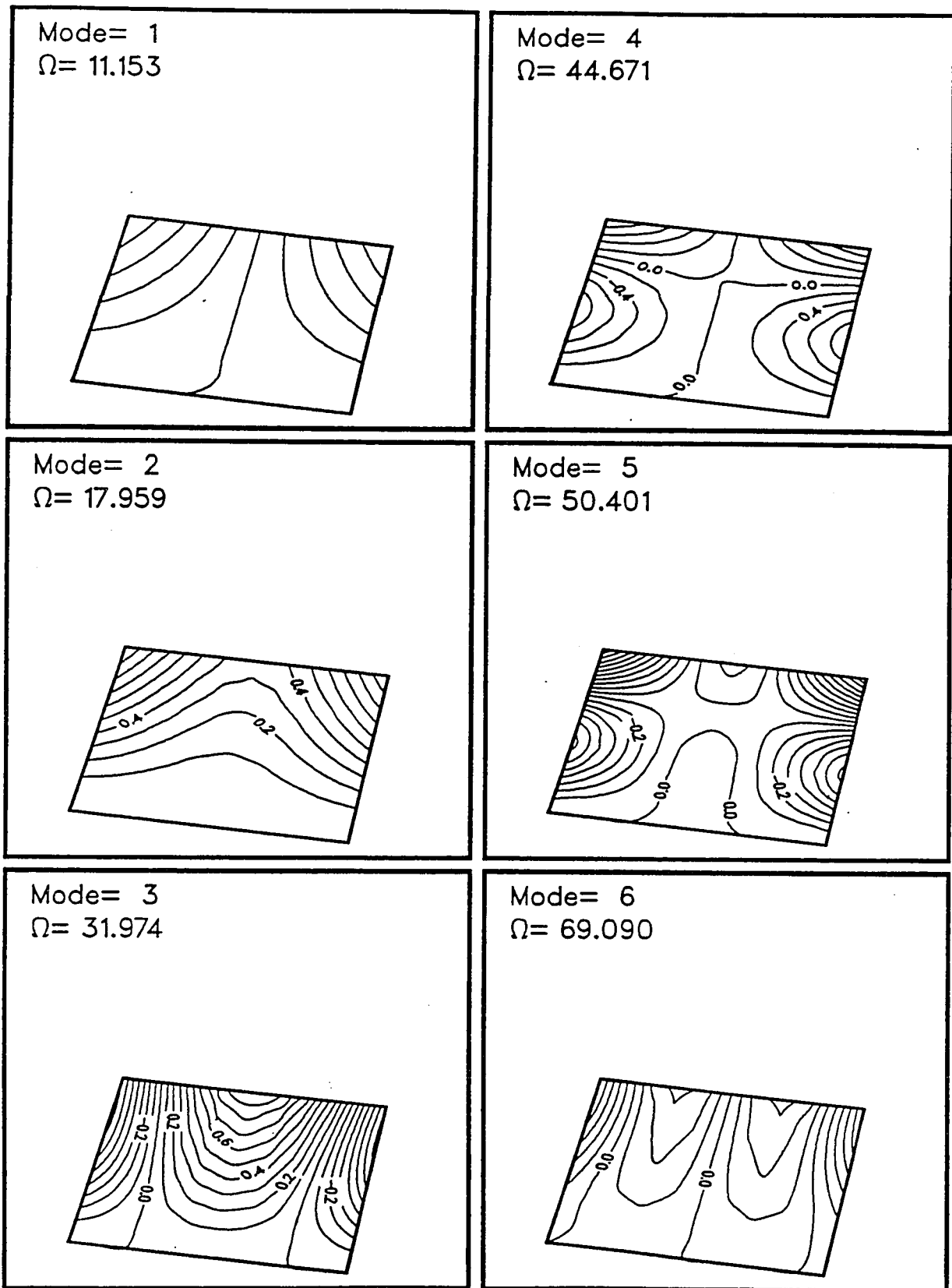


Figure B.2: Displacement pattern with nodal lines for isotropic shell.

$$r/a = 2.0, \quad b/a = 1.0, \quad a/t = 100.00, \quad c/a = 0.0.$$

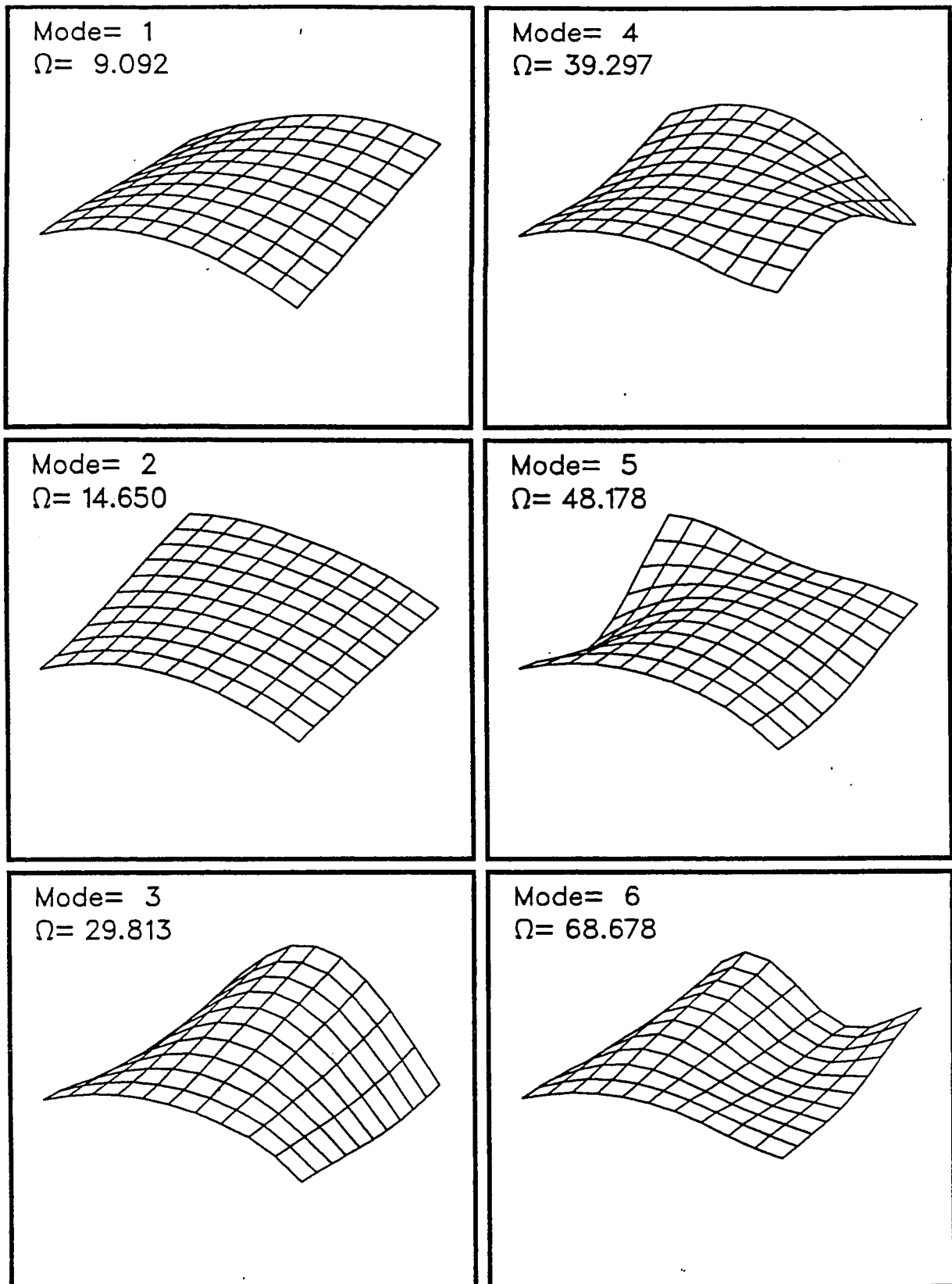


Figure B.3: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, c/a = 0.2.$$

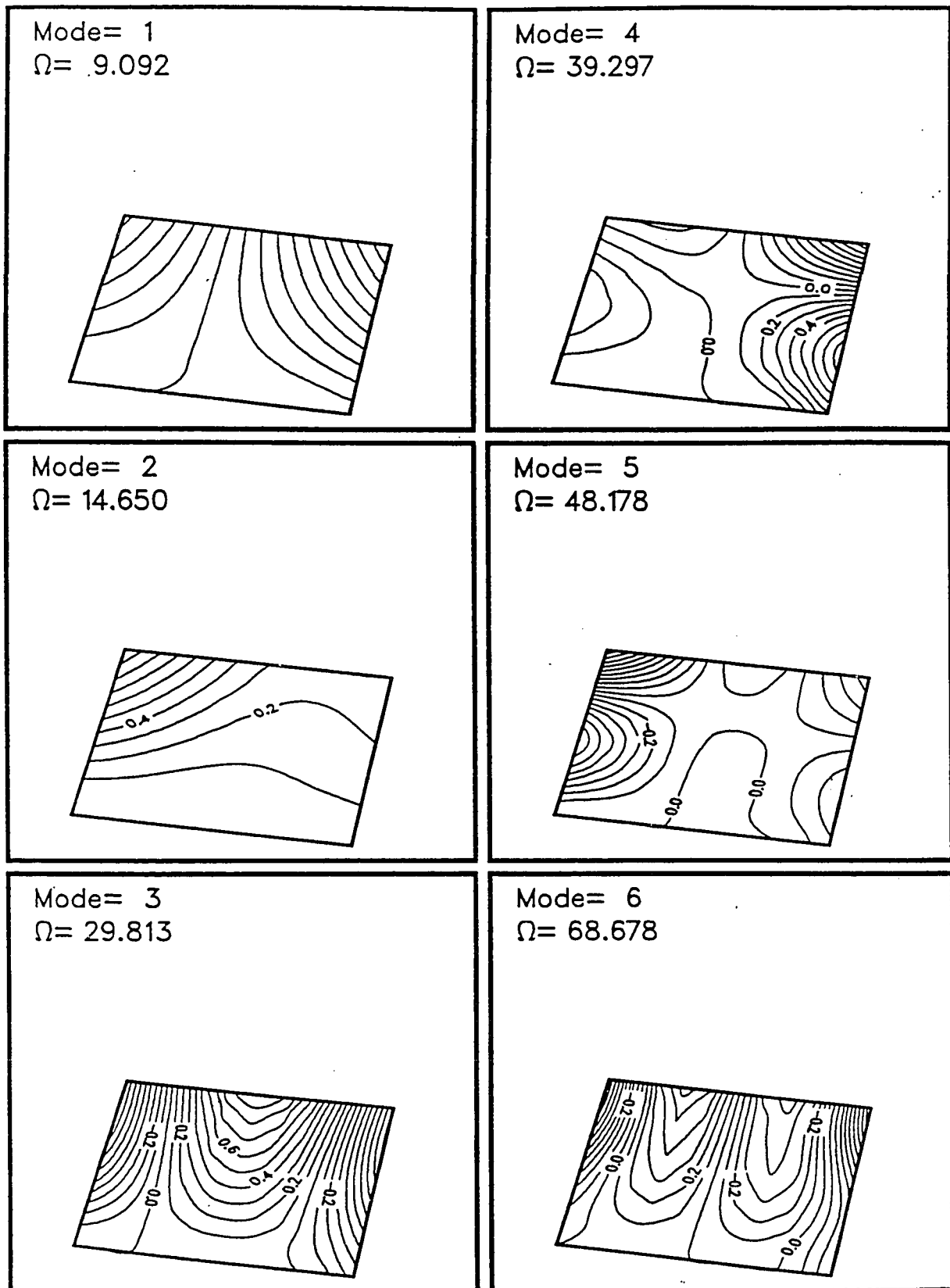


Figure B.4: Displacement pattern with nodal lines for isotropic shell.

$$r/a = 2.0, \quad b/a = 1.0, \quad a/t = 100.00, \quad c/a = 0.2.$$

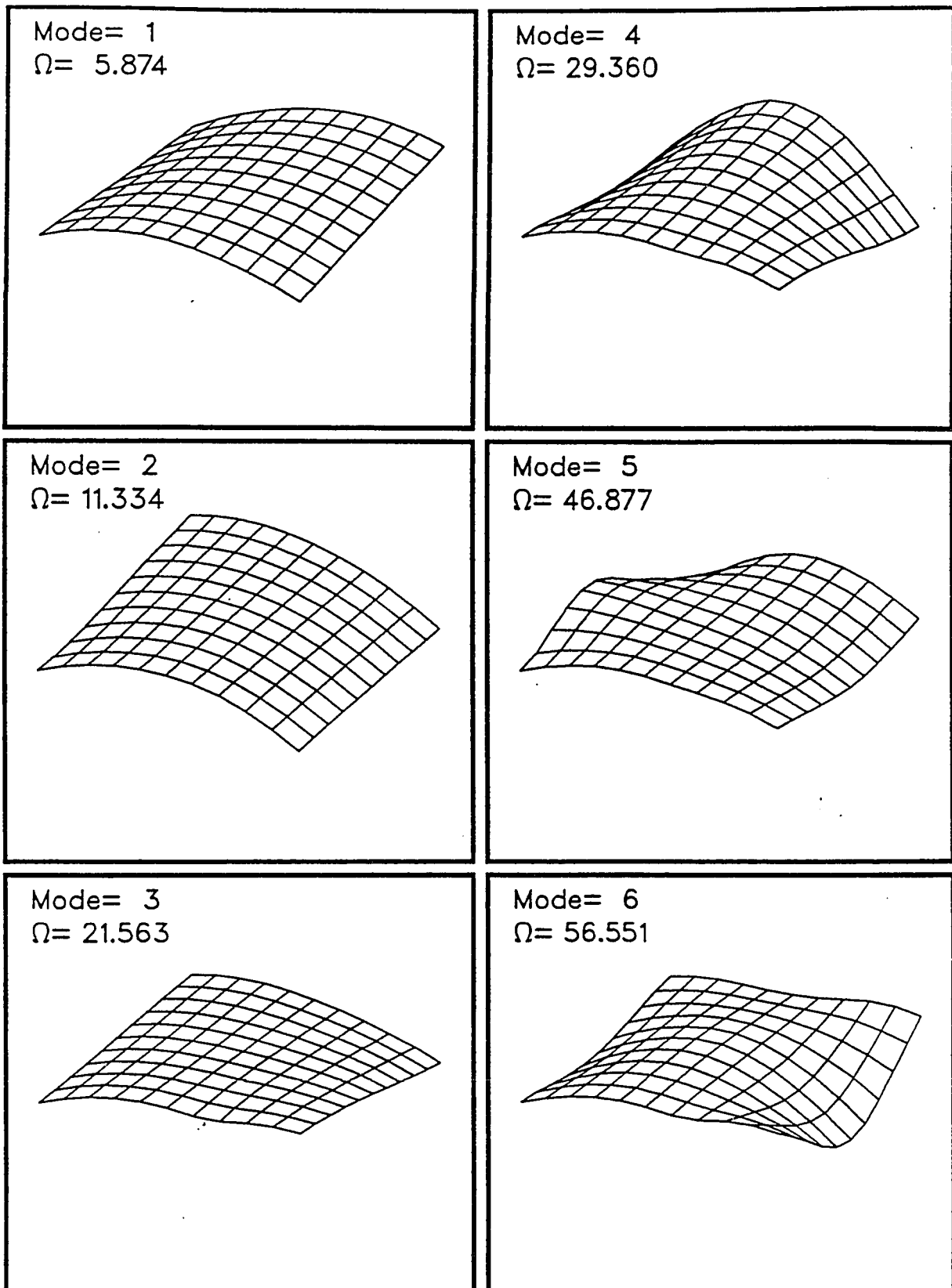


Figure B.5: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, \quad b/a = 1.0, \quad a/t = 100.00, \quad c/a = 0.4.$$

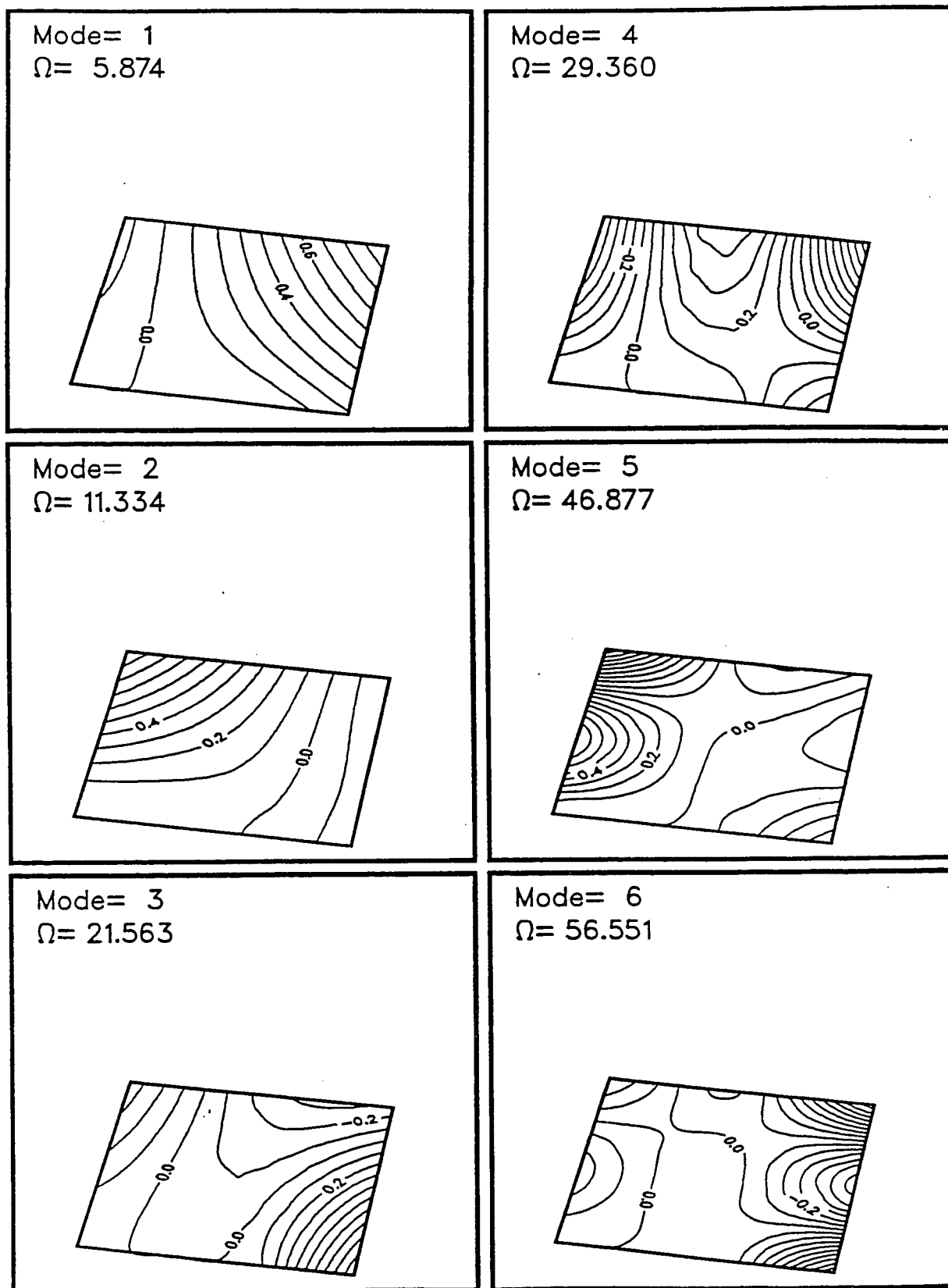


Figure B.6: Displacement pattern with nodal lines for isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, c/a = 0.4.$$

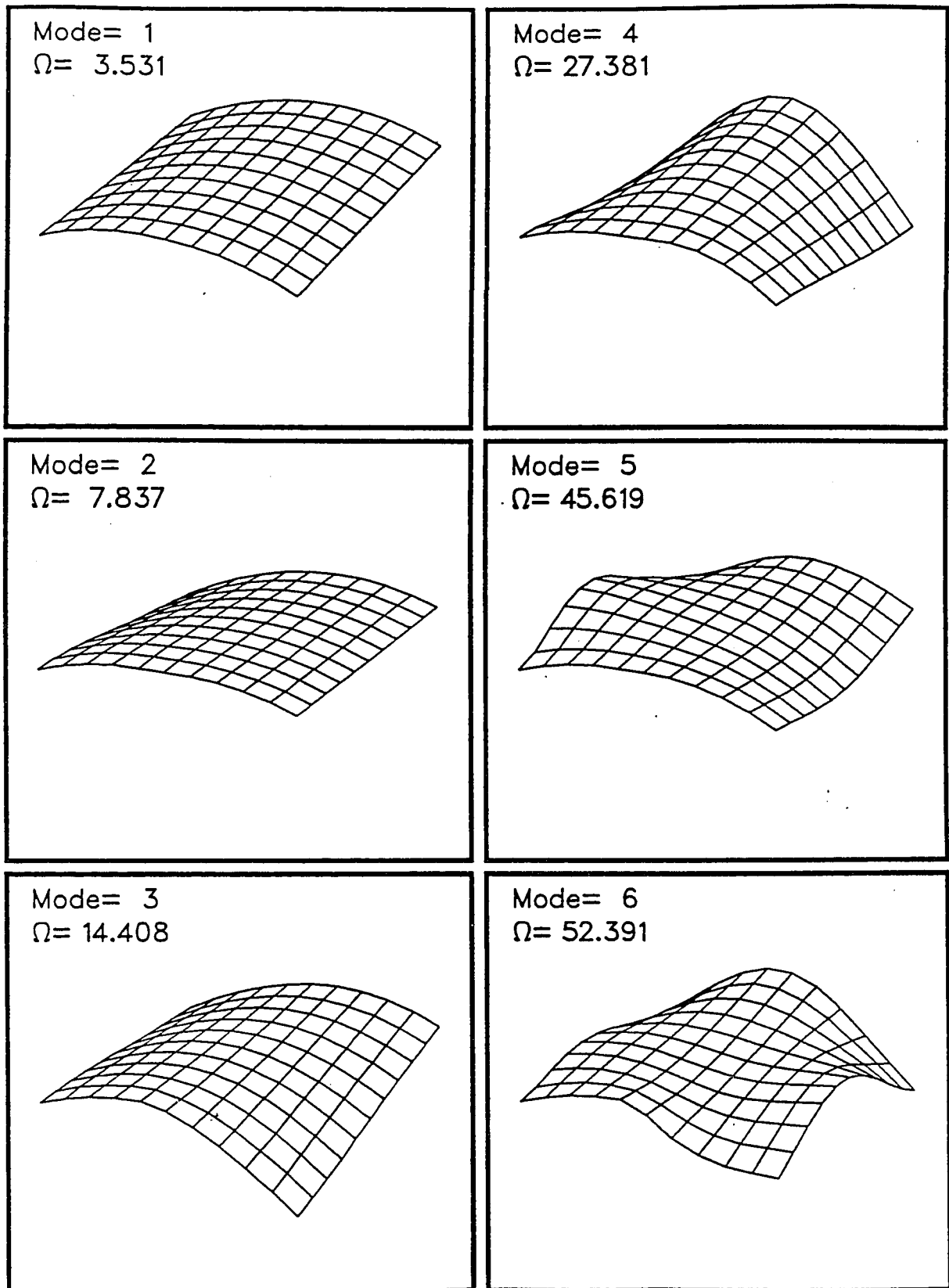


Figure B.7: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, \quad b/a = 1.0, \quad a/t = 100.00, \quad c/a = 0.6.$$

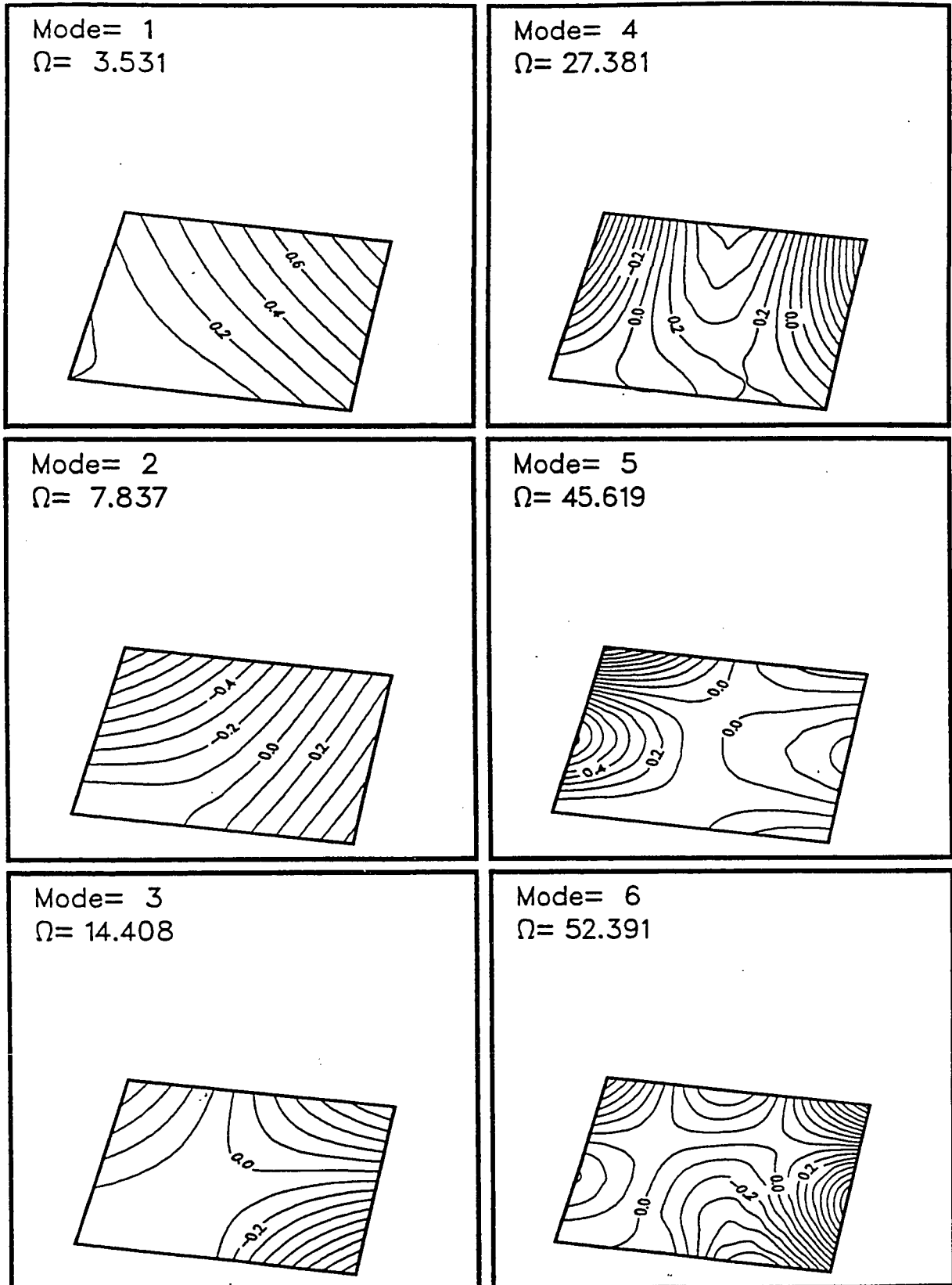


Figure B.8: Displacement pattern with nodal lines for isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, c/a = 0.6.$$

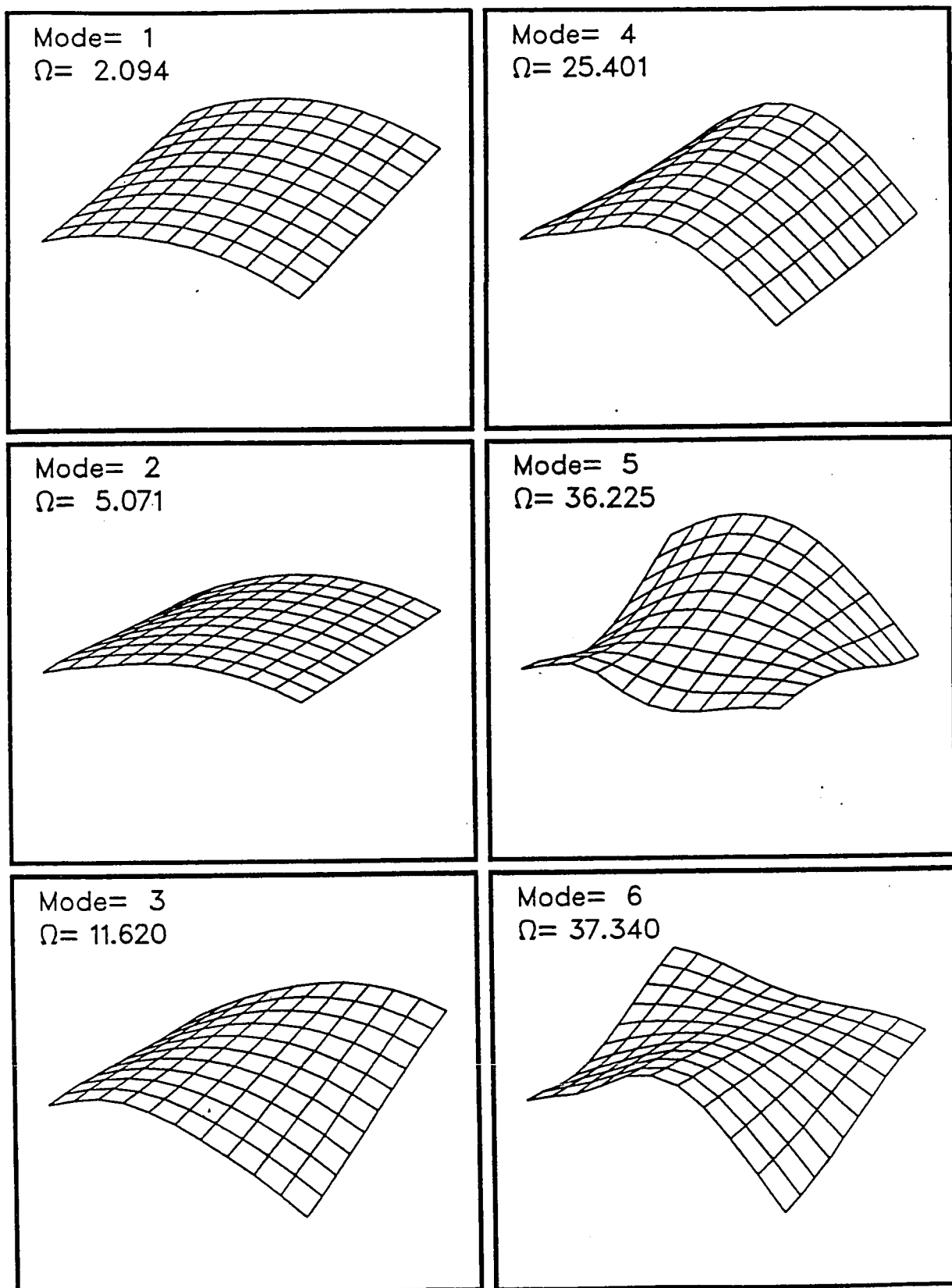


Figure B.9: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, \quad b/a = 1.0, \quad a/t = 100.00, \quad c/a = 0.8.$$

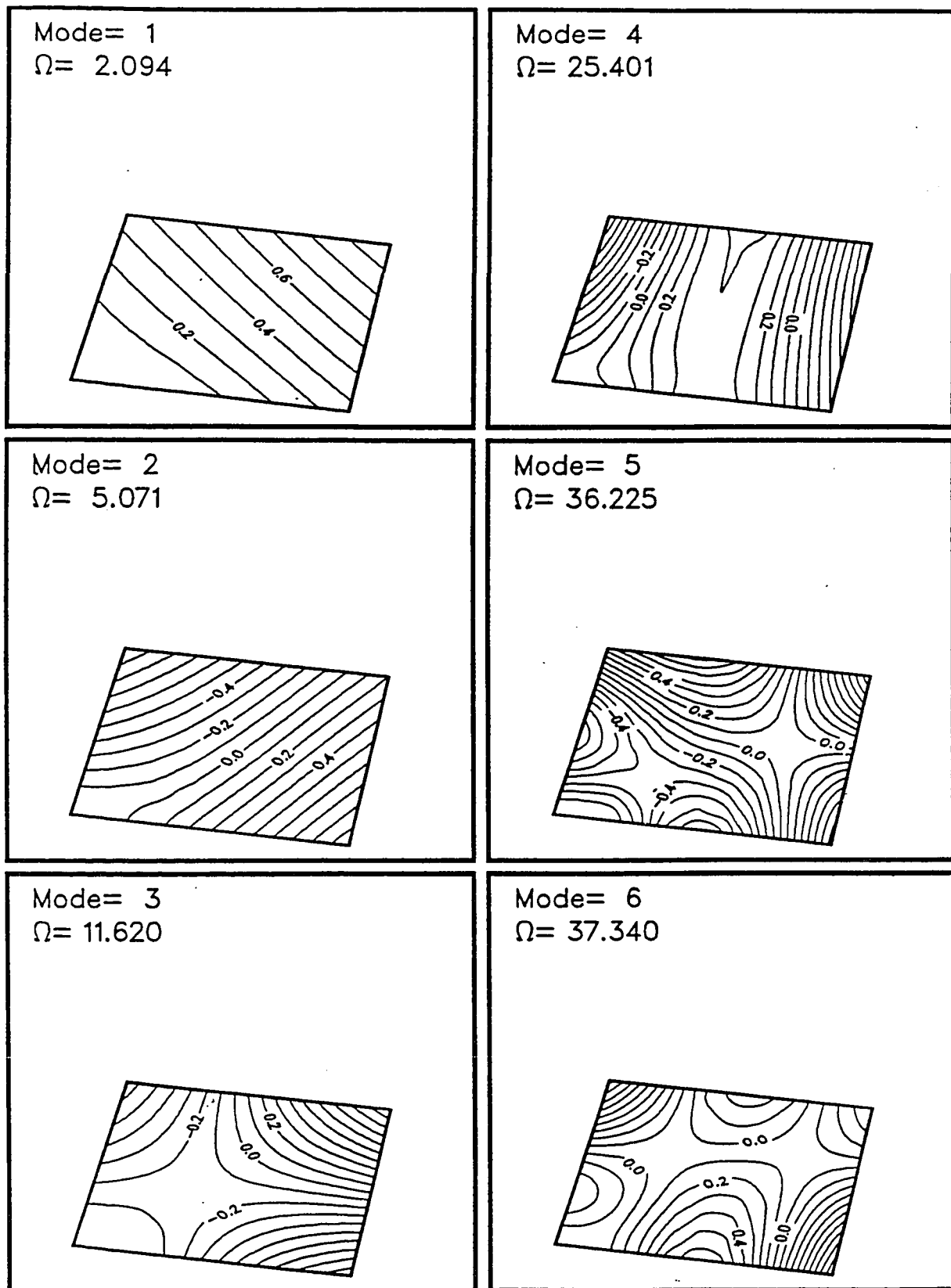


Figure B.10: Displacement pattern with nodal lines for isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, c/a = 0.8.$$

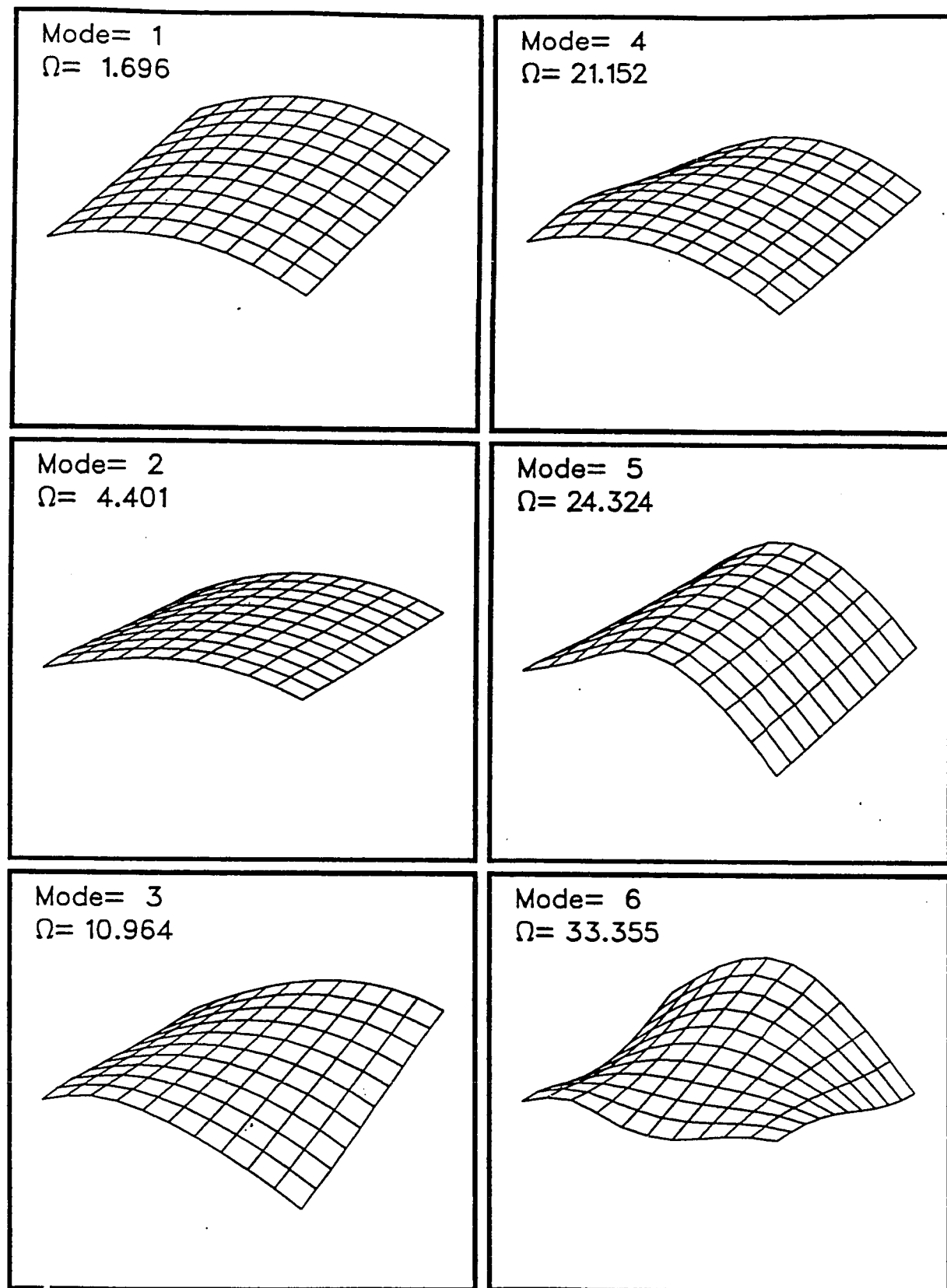


Figure B.11: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, c/a = 0.9.$$

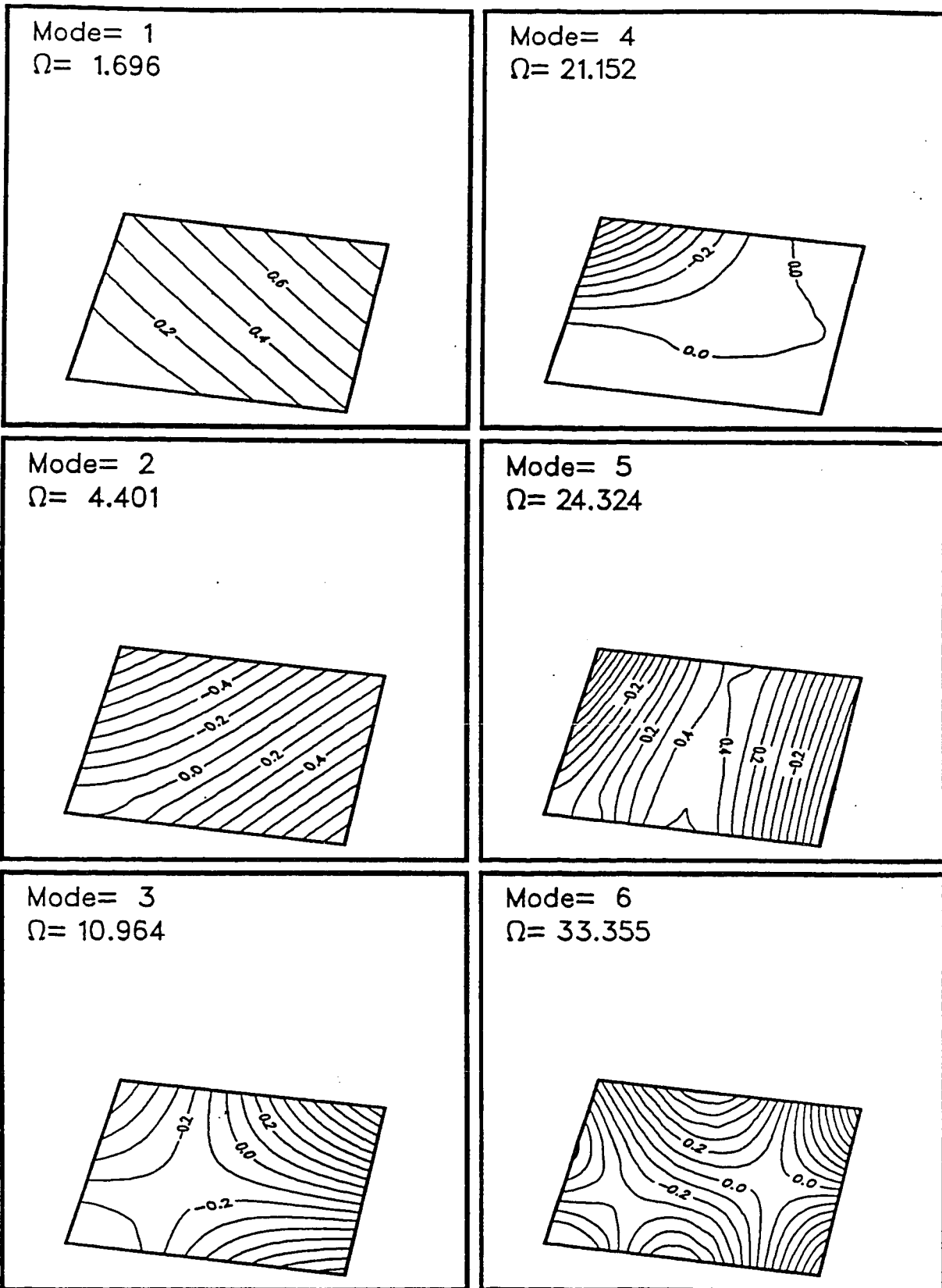


Figure B.12: Displacement pattern with nodal lines for isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, c/a = 0.9.$$

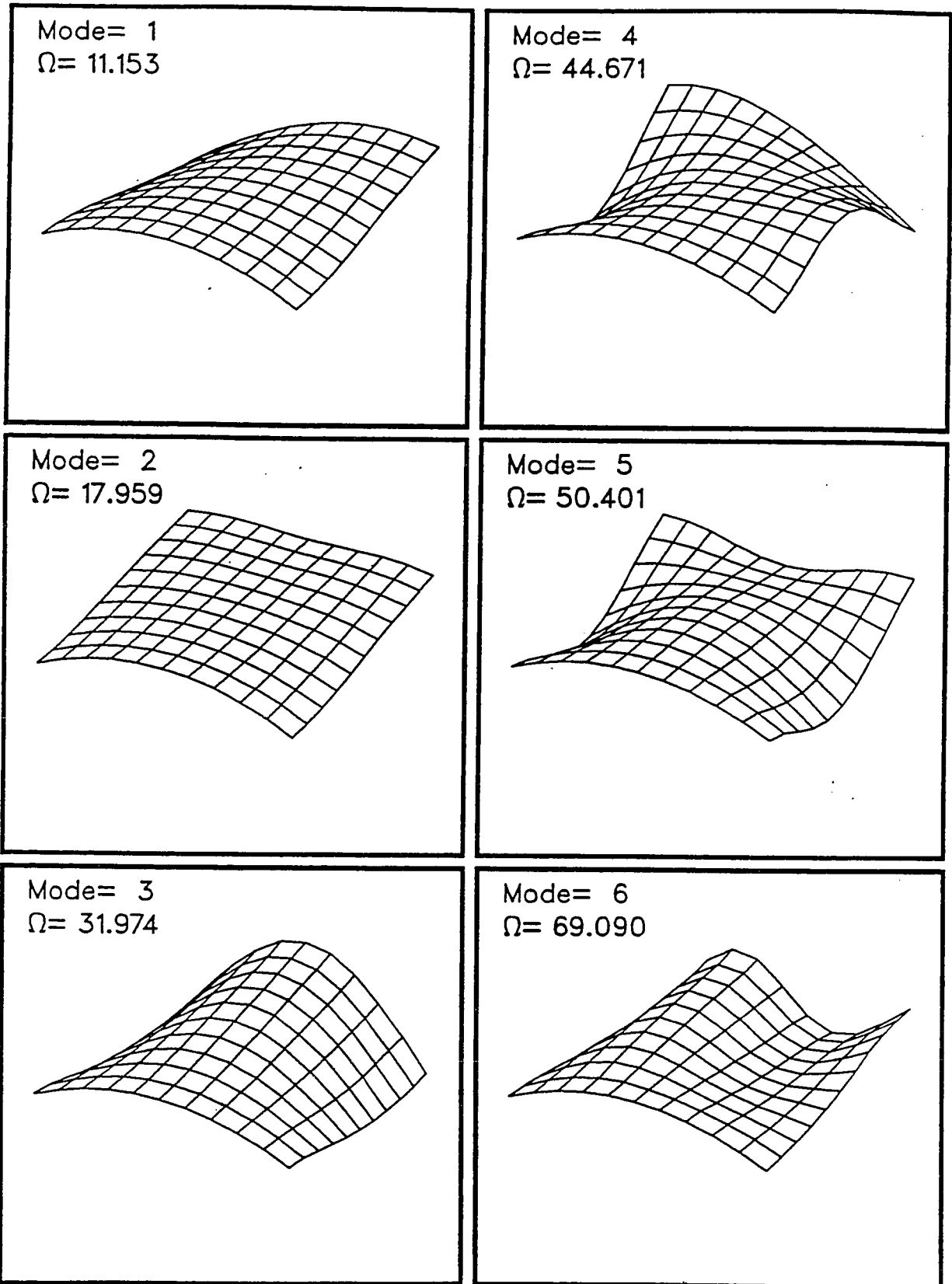


Figure B.13: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, \quad b/a = 1.0, \quad a/t = 100.00, \quad 2c'/a = 0.0.$$

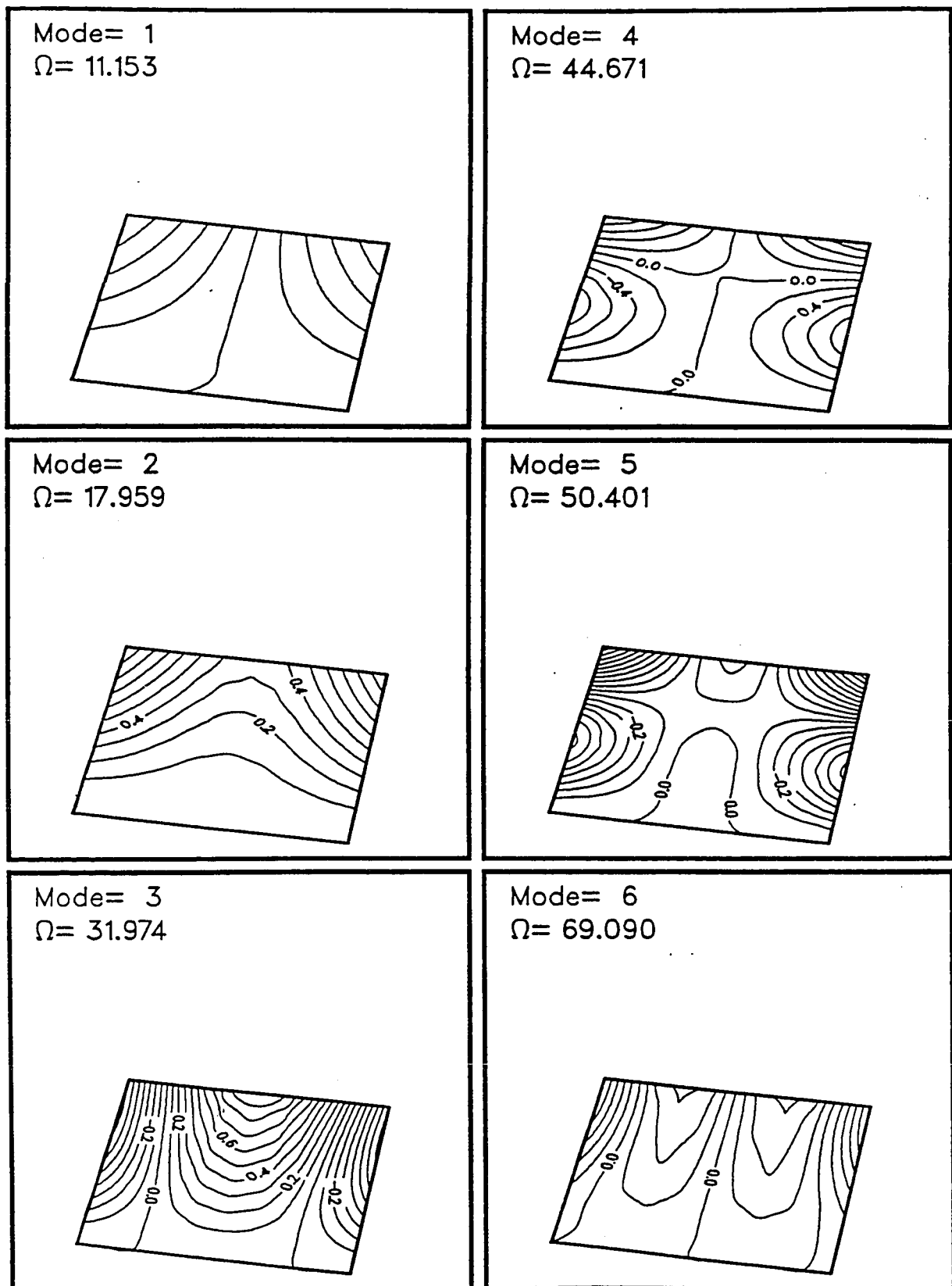


Figure B.14: Displacement pattern with nodal lines for isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, 2c'/a = 0.0.$$

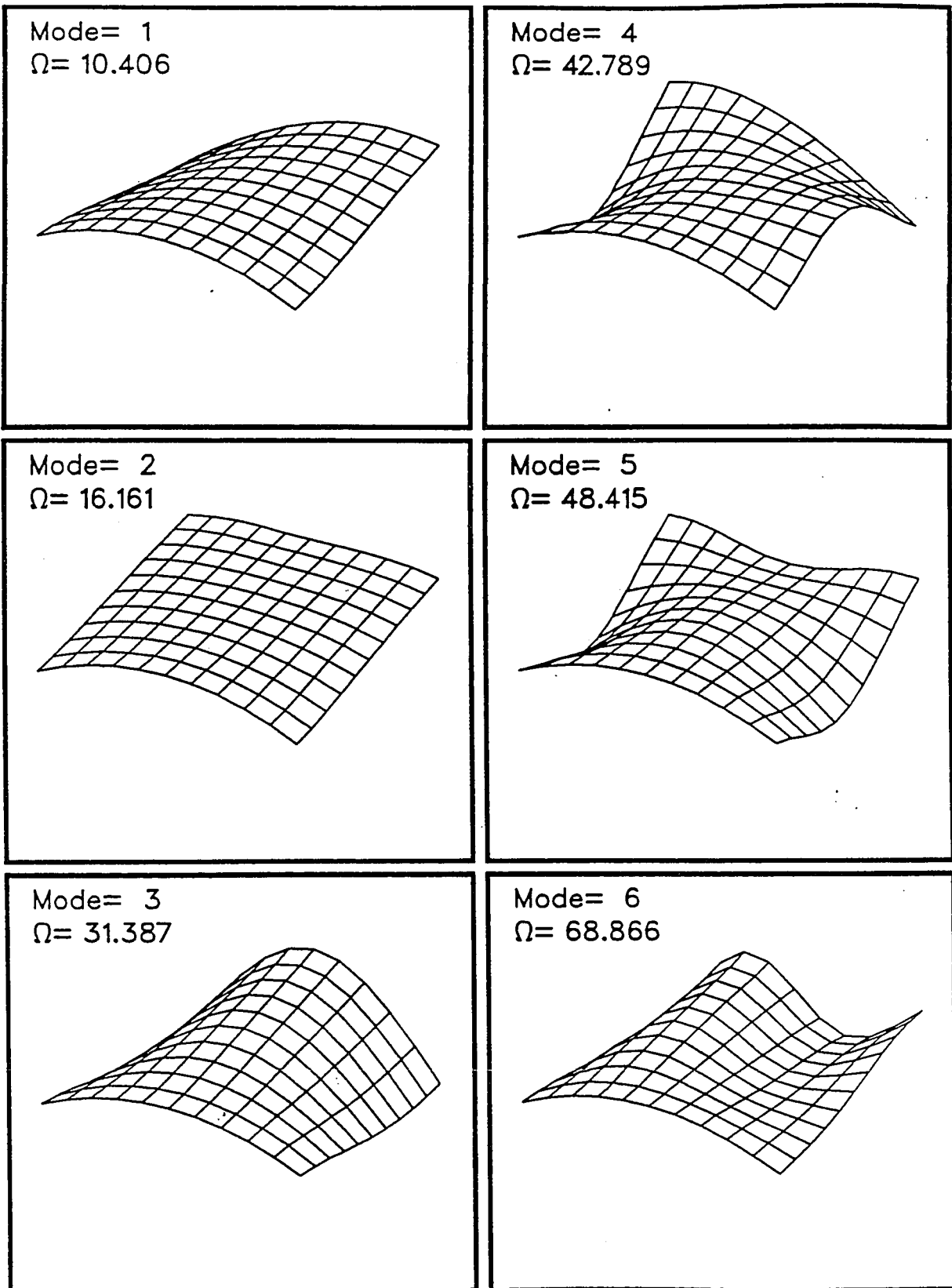


Figure B.15: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, 2c'/a = 0.2.$$

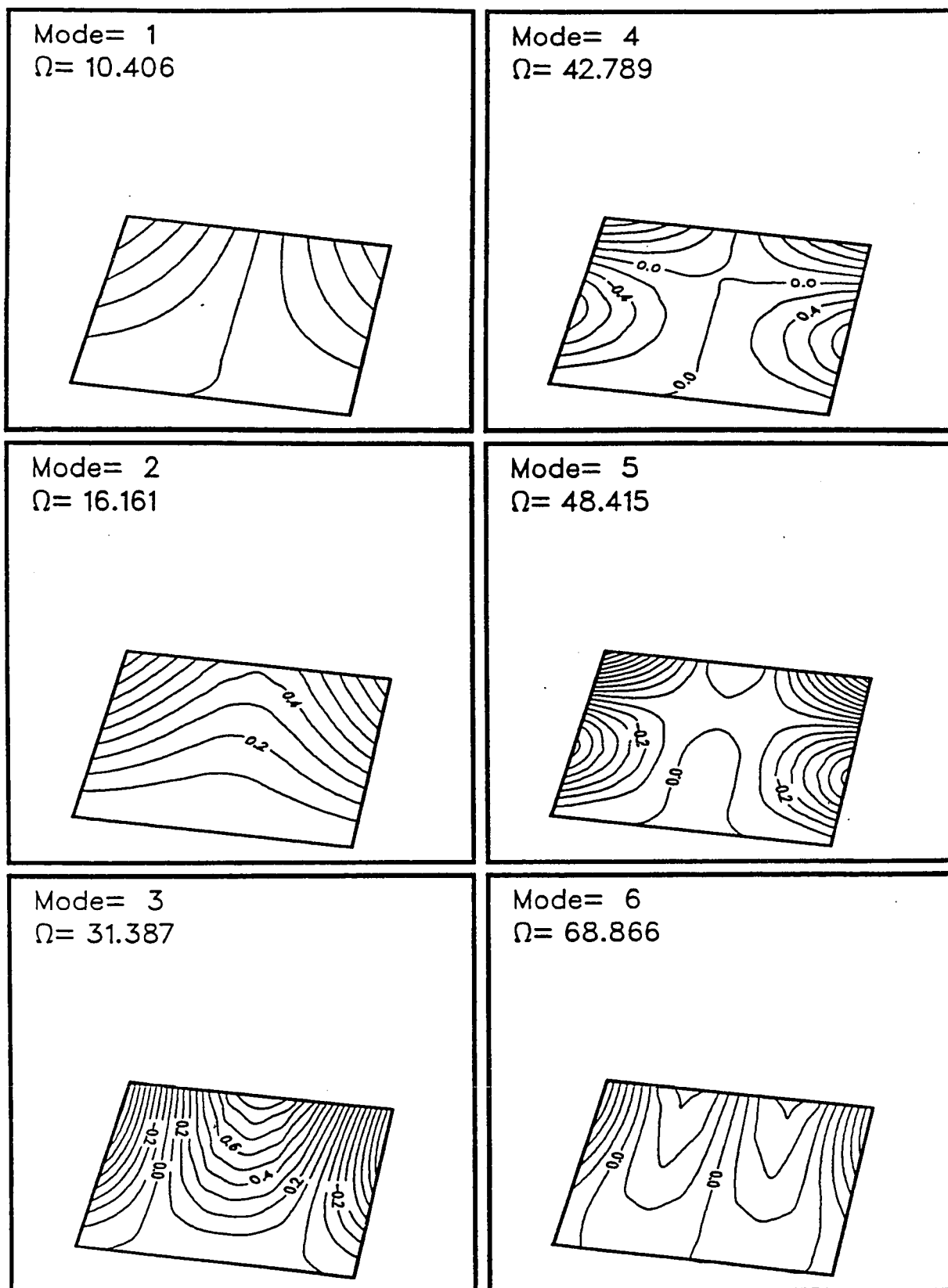


Figure B.16: Displacement pattern with nodal lines for isotropic shell. ell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, 2c'/a = 0.2.$$

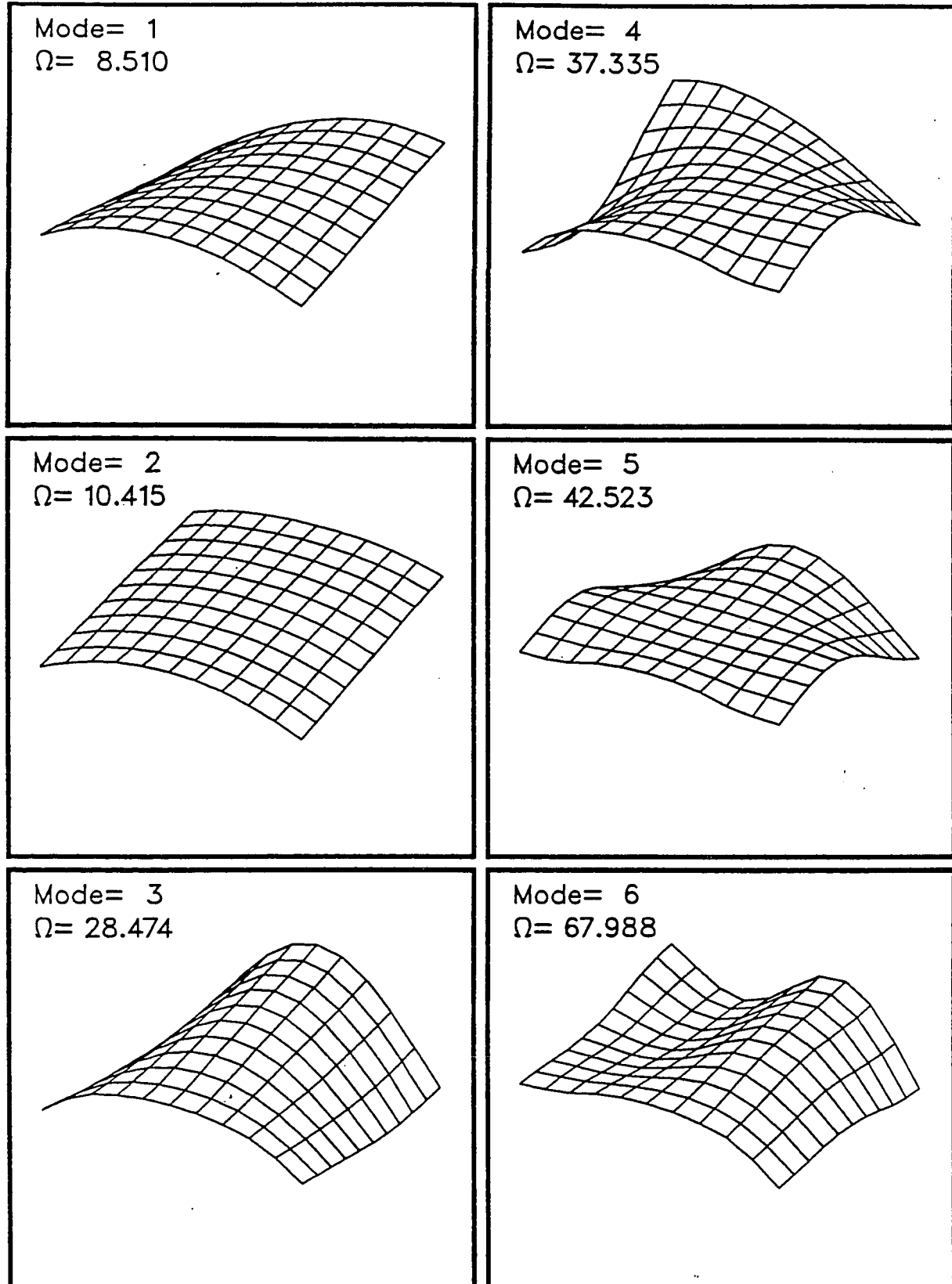


Figure B.17: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, 2c'/a = 0.4.$$

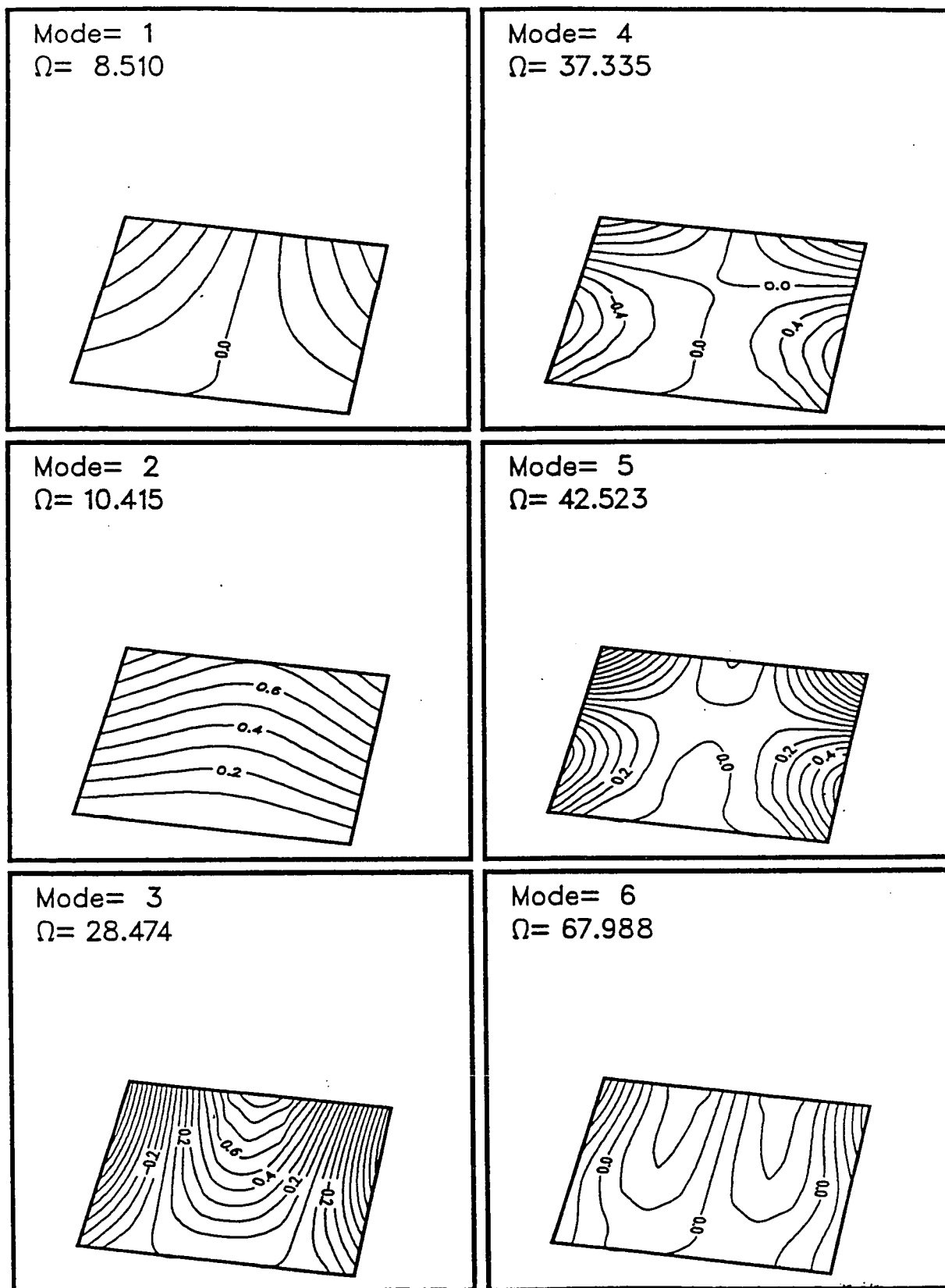


Figure B.18: Displacement pattern with nodal lines for isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, 2c'/a = 0.4.$$

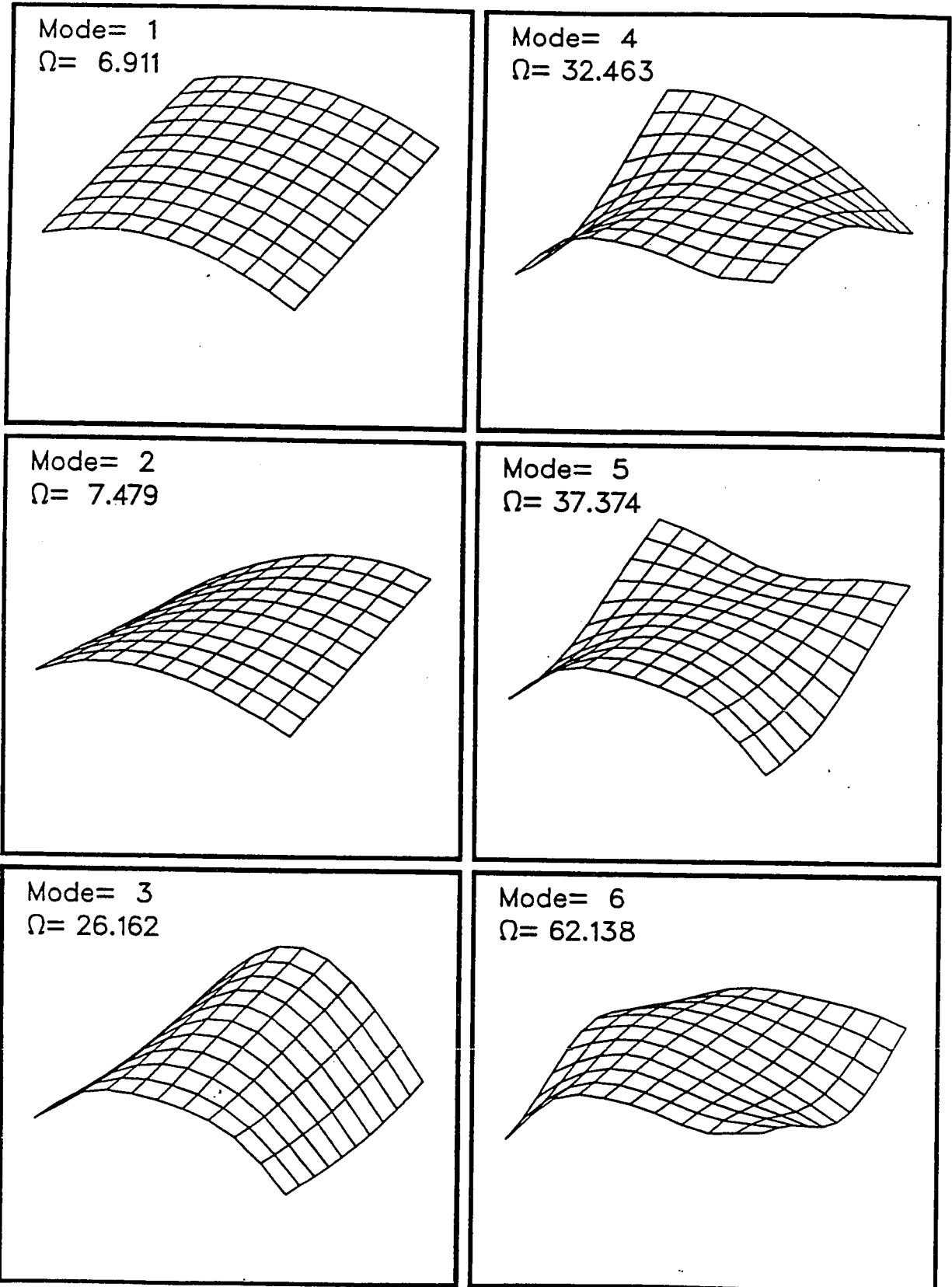


Figure B.19: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, \quad b/a = 1.0, \quad a/t = 100.00, \quad 2c'/a = 0.6.$$

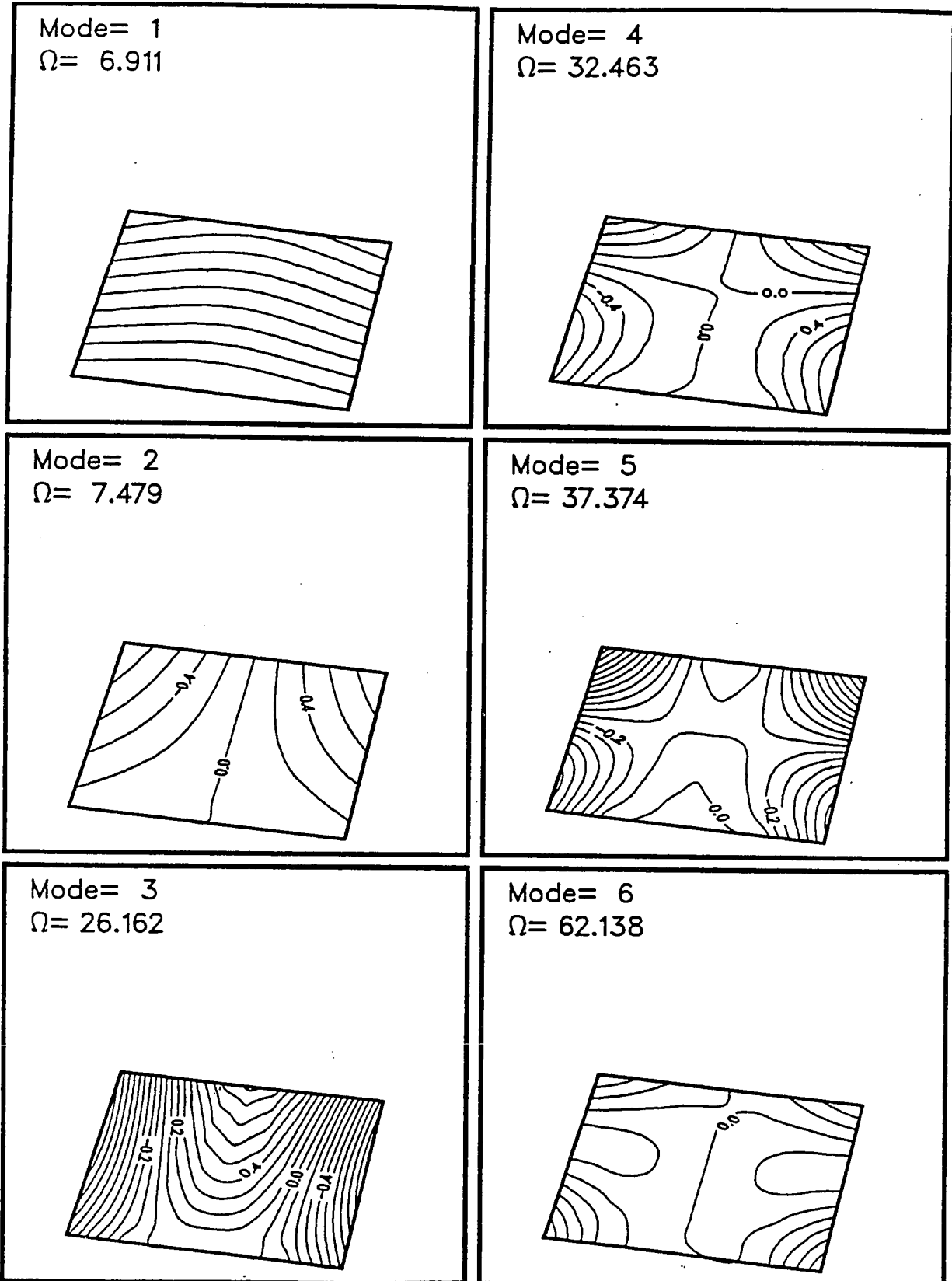


Figure B.20: Displacement pattern with nodal lines for isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, 2c'/a = 0.6.$$

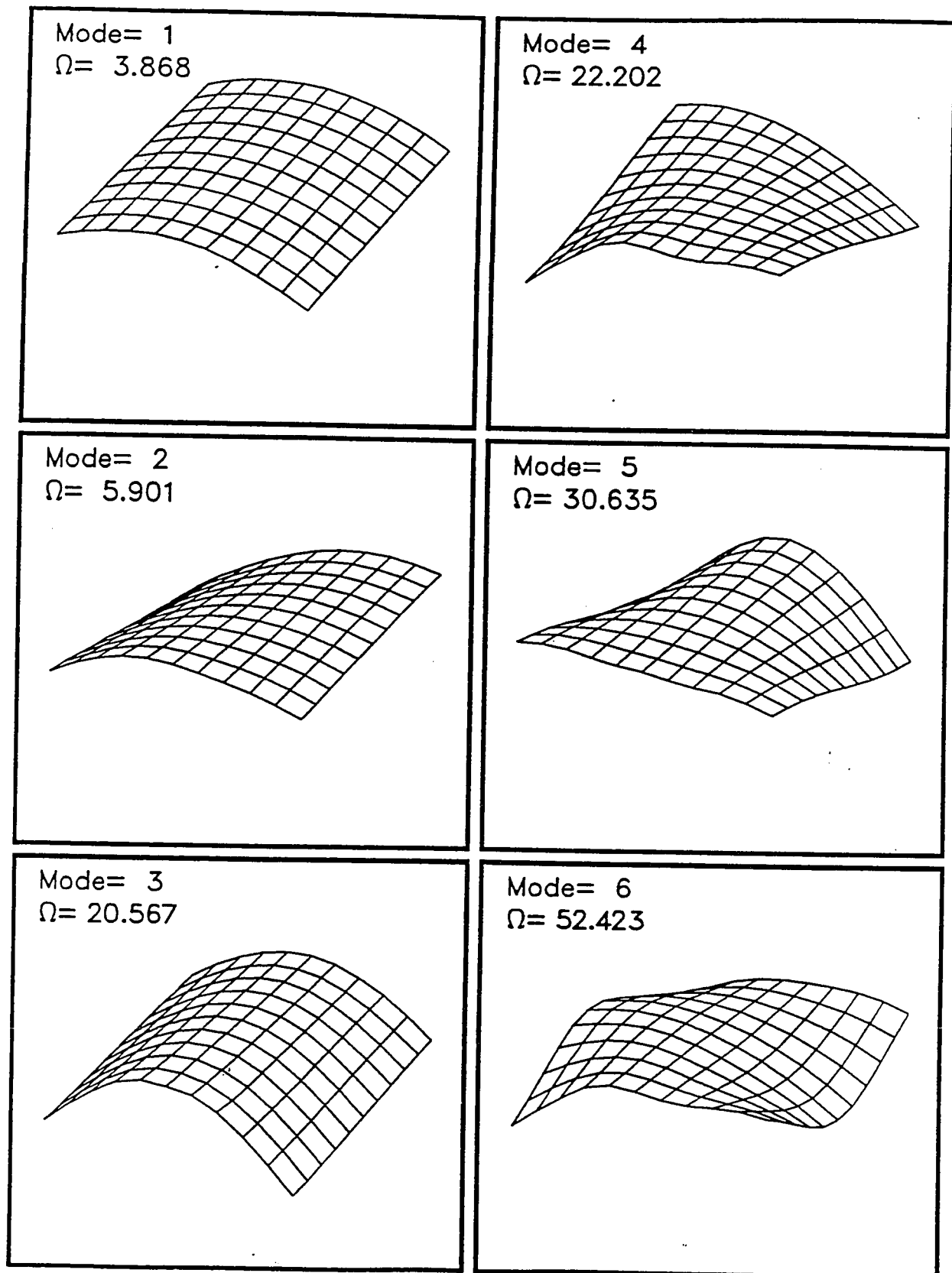


Figure B.21: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, 2c'/a = 0.8.$$

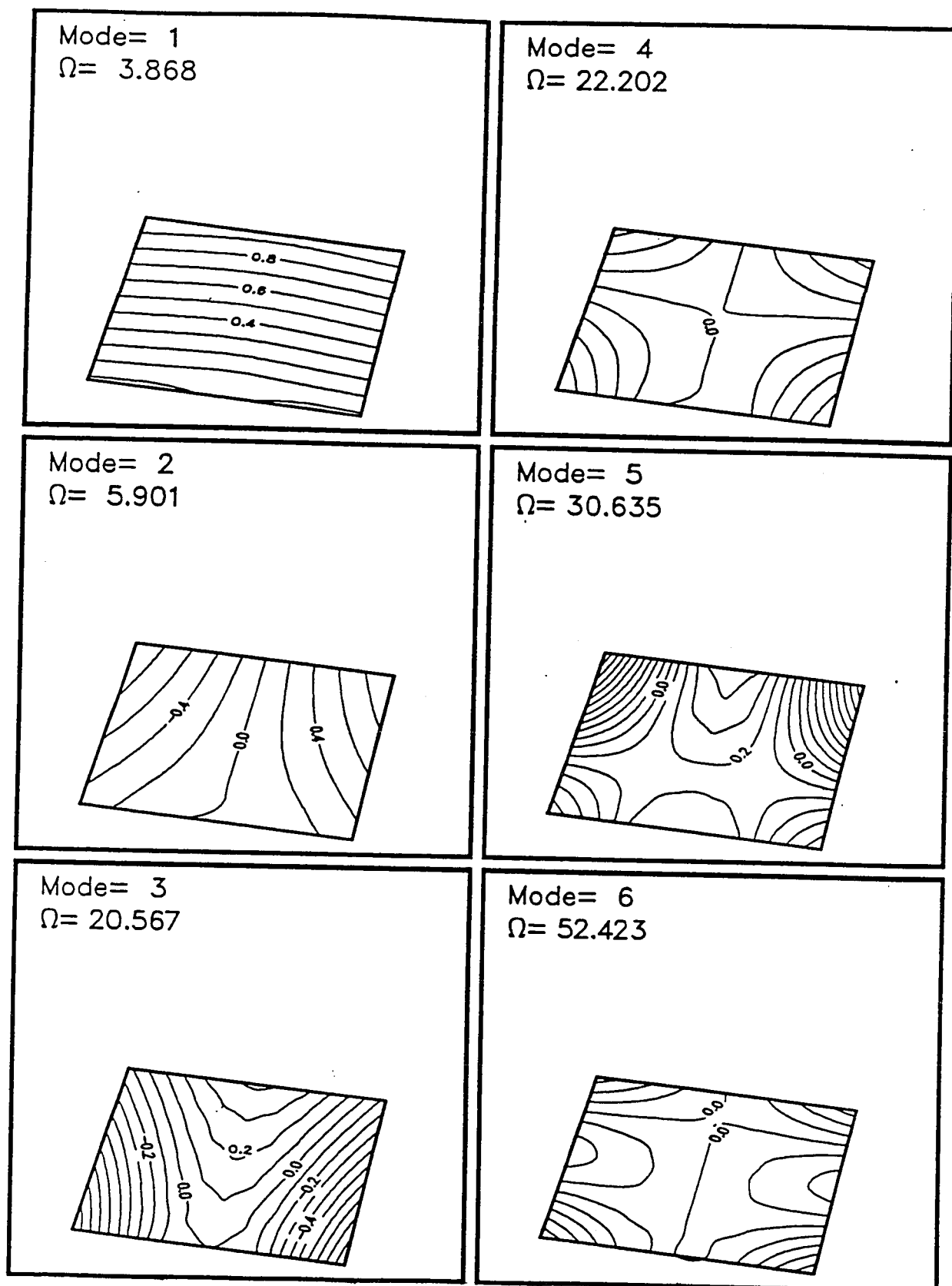


Figure B.22: Displacement pattern with nodal lines for isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, 2c'/a = 0.8.$$

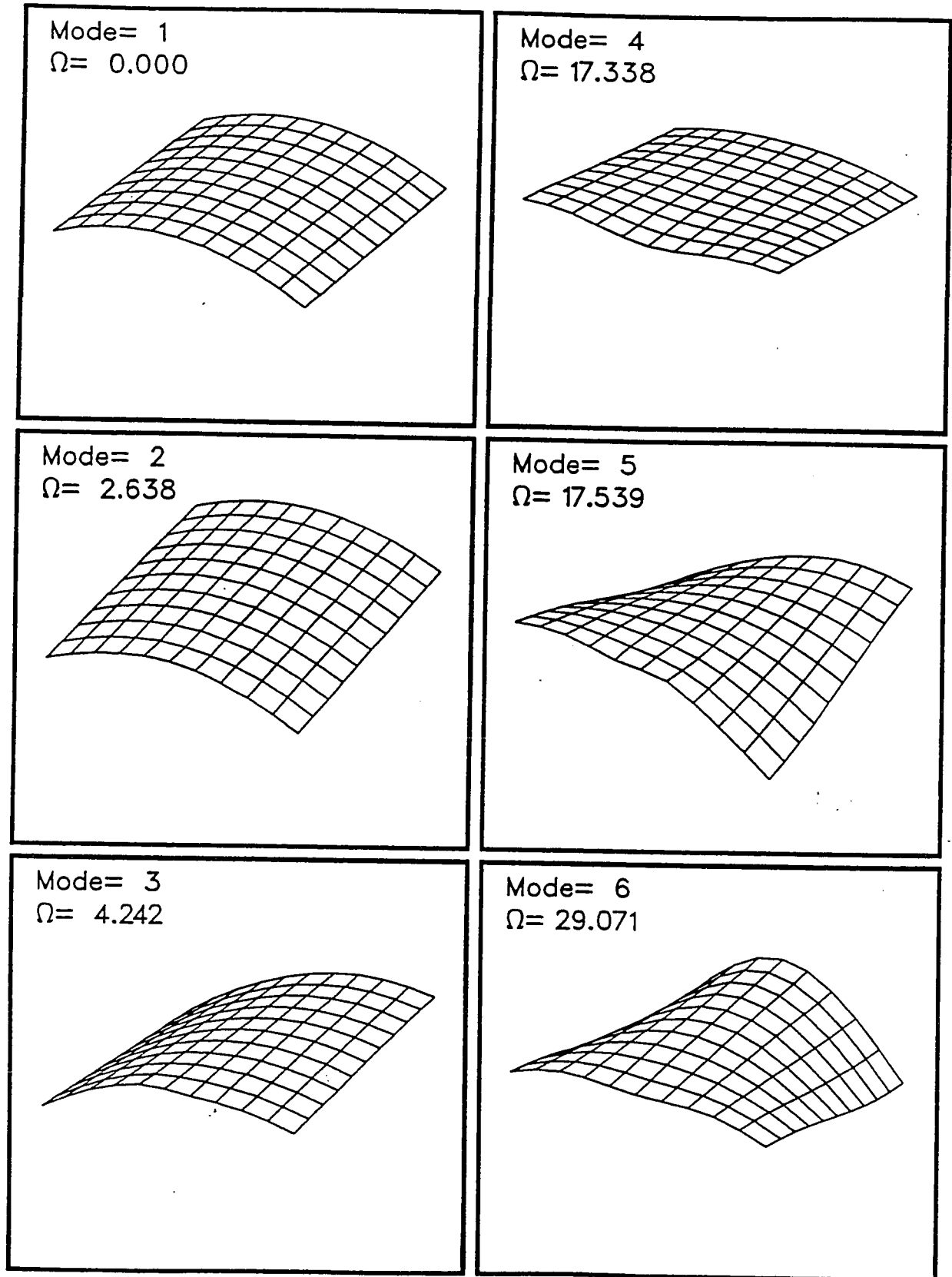


Figure B:23: Modal shapes for cylindrical isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, 2c'/a = 1.0.$$

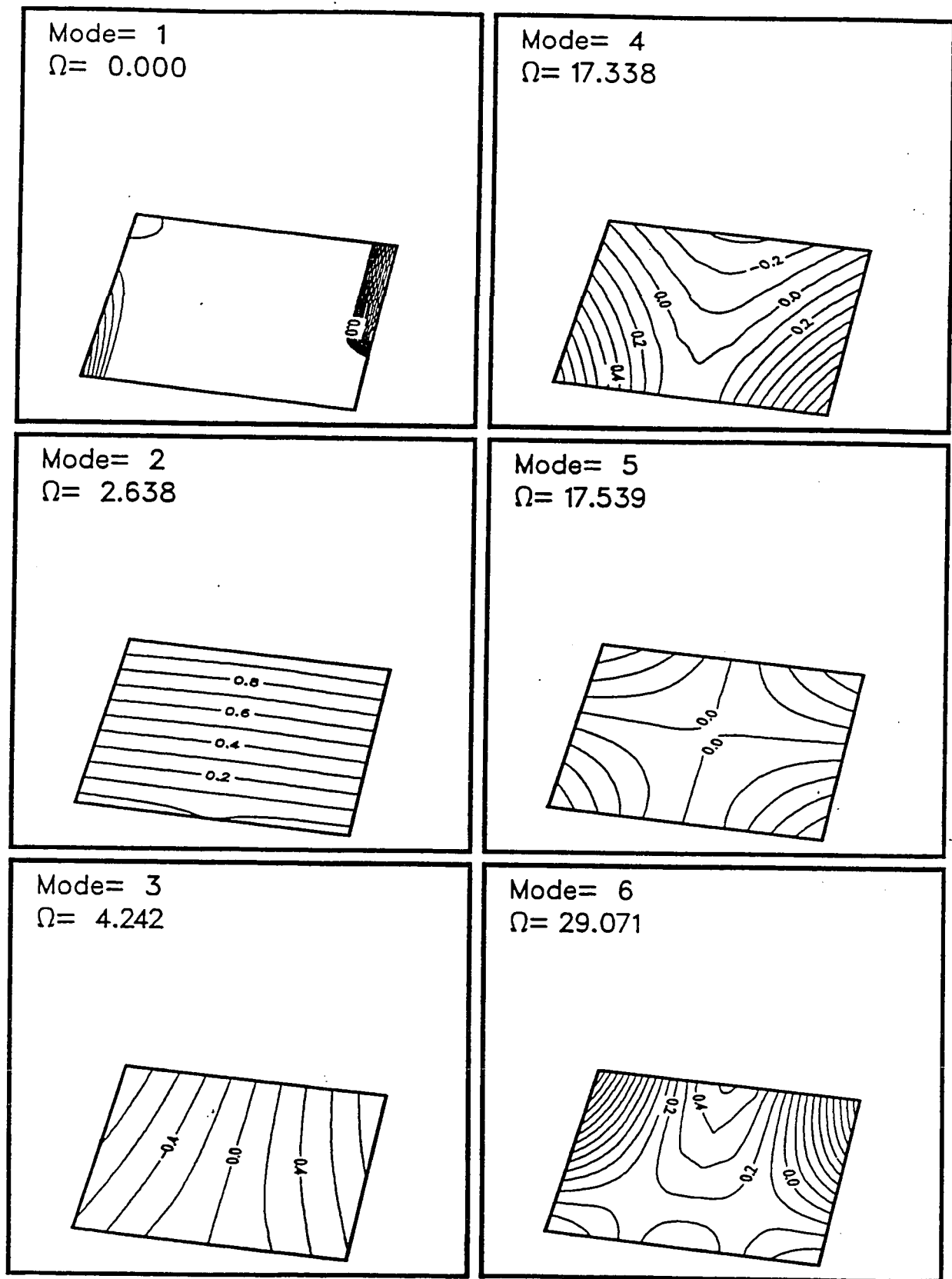


Figure B.24: Displacement pattern with nodal lines for isotropic shell.

$$r/a = 2.0, b/a = 1.0, a/t = 100.00, 2c'/a = 1.0.$$