IRREDUCIBLE REALIZATIONS OF TRANSFER FUNCTION MATRICES

by

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ABSTRACT

This thesis is primarily concerned with the realization of a transfer function matrix whose elements are rational functions.It is realized by a completely controllable and observable set of state equations.

There are two algorithms presented.The first is based on the Smith-MacMillan normal form of a rational matrix. The second is based on the partial fraction expansion of of the transfer matrix.It is shown that both techniques can be used in sampled data systems.

CHAPTER ONE

INTRODUCTION

The concept of state, represents a radical departure from the transfer function representation of linear systems. The state transition equations are of fundamental importance in that they provide a total description of the dynamic state of a system in contrast to the transfer function which characterizes only an input/output relationship. The primary purpose of this thesis is to find a simple algorithm to construct a minimal state space representation of systems described by transfer function matrices. This basic work initiated in a large measure by R. E. Kalman [5], [14], and followed by the work of Heyman and Thorpe [7] has led to such algorithms. However most of them are difficult to apply or are multi-step procedures.

Whenever one tackles a problem in the structure theory of linear dynamical systems there are certain concepts that must be well examined and understood before proceeding. Two of these concepts are controllability and observability.

Chapter II deals entirely with these concepts. There are few new ideas introduced but conditions for controllability and observability are well formalized. Conditions are given for general systems and also for special canonical structures such as Jordan, and companion forms. Furthermore, this chapter includes a detailed discussion of single-input single-output systems. Finally the concept of minimal order n of a system is introduced and defined.

Chapter III and IV deal with the main problem of the thesis. The former gives a technique which is based on the Smith-MacMillan form of a rational polynomial matrix. The technique which was introduced by Heyman and Thorpe [7]is the most general technique existing. It can be shown that Kalman's method [14] is a special case of this algorithm.

Its application however becomes impractical for transfer functions matrices of order larger than three or four.

Chapter IV introduces a single step technique based on the partial fraction expansion of the transfer function matrix. The resulting system Matrix A is in Jordan canonical form. In some ways this algorithm is similar to that of Panda and Chen [8] but it is more powerful since their method is a two step realization because they first find a controllable but not necessarily observable realization then reduce it to one which is controllable and observable.

Chapter V extends the algorithm of Chapter III and IV to sampled data systems. This extension is very important since most identification schemes describe a system through a pulse transfer function matrix. The resulting state space representation is a set of first order difference equations.

Finally Chapter VI summarizes the result of the thesis and also mentions work being done, with regard to this problem, in time-varying systems. This latter consideration provides insight to another method of solving this problem which would be very simple. This is suggested as an avenue of further research in the problem.

CHAPTER II

CONTROLLABILITY AND OBSERVABILITY

In this chapter the primary objective is to give conditions for the controllability and observability of a system. Systems are transformed to canonical forms and conditions are given on these special forms.

II - 1 DEFINITIONS

DEFINITION II - 1 SYSTEM DESCRIPTION. Let a time - invariant multivariable system S be represented by

$$\underline{\dot{x}} = A\,\underline{x} + B\,\underline{u}$$
$$\underline{y} = C\,\underline{x} + D\,\underline{u} \qquad 2.1$$

where:

$\underline{u} = \underline{u}\,(t)$, p - dimensional input vector

$\underline{y} = \underline{y}\,(t)$, q - dimensional output vector

$\underline{x} = \underline{x}\,(t)$, n - dimensional state vector

Note n is the order of the system

$\underline{\dot{x}} = \underline{\dot{x}}\,(t)$, time derivative of the state vector

A, constant n x n system matrix

B, constant n x p input matrix

C, constant q x n output matrix

D, constant q x p transmission matrix

NOTATION - Capital letter will designate a matrix, - underline lower case letter a vector.

DEFINITION II - 2 STATE CONTROLLABILITY A state $\underline{x}$ (o) is controllable if there exist some input $\underline{u}$ (t) defined over a closed interval [O,T] such that $\underline{x}$ (T) is the zero state.

More generally a system is completely controllable if and only if every state is controllable. From this point on state controllability will be equivalent to controllability. The distinction is made because one can talk of out put controllability which is defined as follows:

DEFINITION II - 3 OUTPUT CONTROLLABILITY An output $\underline{y}$ (o) is said to be controllable if there exists some input $\underline{u}$ (t) defined over a closed interval [O,T] such that $\underline{y}$ (T) is zero. If every output is controllable then the system is said to be completely output controllable.

DEFINITION II - 4 STATE OBSERVABILITY A State $\underline{x}$ (o) is said to be observable if any only if for some T>0 it may be determined from the knowledge of the zero - input response, $\underline{y}'(t)$, $0 \le t \le T$.

More generally a system is completely observable if an only if every state is observable. Again, as in the previous case, state observability will now be referred to simply as observability.

(NOTE In all three defenitions T is free.)

II - 2 CONDITIONS FOR CONTROLLABILITY AND OBSERVABILITY

The first test given below is the classical one first introduced by R. E. Kalman, Y.C. Ho and K. S. Narendra [3]. It has appeared in many subsequent papers and textbooks, so consequently it will just be given here and not proven. However. it will be regarded as the basic criteria and will be used as such in the proof of other tests.

THEOREM II - 1 The system S in eq. 2.1 is completely controllable if and only if the following matrix L_c is of rank n

$$L_c = [B : AB : A^2B : - \quad - \quad - \quad : A^{n-1}B] \qquad 2.2$$

Note

1) L_c is an $[n \times n \cdot p]$ matrix

2) L_c will now be referred to as the controllability matrix of the system

The above theorem refers to <u>state</u> controllability and the following one will refer to <u>output</u> controllability.

<u>THEOREM II - 2</u> The system S in eq. 2.1 is completely output controllable if and only if the following matrix L_{oc} is of rank q

$$L_{oc} = C\,[B \;\; AB \;\; A^2B \ldots A^{n-1}B] \qquad 2.3$$

To distinguish these two concepts of controllability consider the following example.

<u>EXAMPLE II - 1</u> A system S is described as follows

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \underline{u} \qquad 2.4$$

$$\underline{y} = [1 \;\; 1]\, \underline{x}$$

Its simulation diagram is

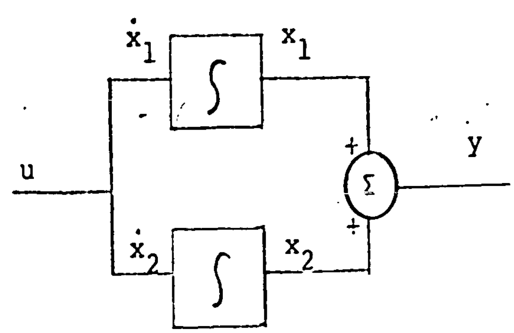

FIGURE II-1 - SIMULATION DIAGRAM OF EXAMPLE II-1

$$L_c = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \text{rank one}$$

$$L_{oc} = [1 \;\; 1] \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \text{rank one}$$

therefore the system is output controllable but is not state controllable.

Having distinguished output and state controllability it is noted that all the remaining work will only concern STATE controllability.

THEOREM II - 3 A system S defined in eq. 2. 1 is completely observable if and only if the rank of the following matrix L_o is n

$$L_o = [C^T \;\; A^T C^T \;\; \ldots . \; (A^T)^{n-1} C^T] \qquad 2.5$$

Note

1) L_o is an $[n \times n.\ q]$ matrix

2) L_o will now be referred to as the observability matrix of the system

3) If A is a complex matrix we have to take the complex conjugate transpose

II - 3 CANONICAL FORMS

Conditions for controllability and observability are now given when the matrix A has certain canonical forms

a) SYSTEM WITH DISTINCT EIGENVALUES

In this case it is known that A can be diagonalized by using a non-singular matrix T such that

$$\Lambda = T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & & & \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & \cdots & & \lambda_n \end{bmatrix}$$

where: λ_i $i = 1, 2, \ldots n$ is an eigenvalue of A and the system S described by eq. 2.1 is transformed to S_D such that

$$\dot{\underline{z}} = \Lambda \underline{z} + \beta \underline{u}$$

$$\underline{y} = \gamma \underline{z} + D \underline{u} \qquad 2.6$$

where:

$$\underline{x} = T \underline{z}$$

$$\beta = T^{-1} B$$

$$\gamma = C T$$

The controllability and observability requirement can be restated as follows:

THEOREM II - 4 A system S_D in the diagonal form and with distinct eigenvalues is controllable (observable) if and only if the matrix β (γ^T) has no rows which are zero.

NOTE For every theorem on controllability there is always a dual one for observability so to avoid repetition both theorems are incorporated in one as above.

Proof

i) necessity

First observe that

$$\begin{bmatrix} x & 0 & \cdots & & 0 \\ 0 & y & 0 & \cdots & 0 \\ 0 & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & 0 \\ 0 & 0 & \cdots & & z \end{bmatrix}^q = \begin{bmatrix} x^q & 0 & 0 & \cdots & 0 \\ 0 & y^q & 0 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & \cdots & & & z^q \end{bmatrix}$$

i.e. a diagonal matrix raised to a power is still a diagonal matrix

Also

$$\begin{bmatrix} a & 0 & \cdots & & 0 \\ 0 & b & 0 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ & & & & 0 \\ 0 & \cdots & & & c \end{bmatrix} \begin{bmatrix} \underline{m}_1 \\ \underline{m}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{m}_n \end{bmatrix} = \begin{bmatrix} a\ \underline{m}_1 \\ b\ \underline{m}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ c\ \underline{m}_n \end{bmatrix}$$

Where $\underline{m}_i$ is the i^{th} row of matrix M

It is now clear to see that the controllability matrix L_c

$$[\beta \quad \Lambda\beta \quad \Lambda^2\beta \ldots\ldots \Lambda^{n-1}\beta]$$

and observability matrix L_o

$$[\gamma^T \quad \Lambda^T\gamma^T \qquad (\Lambda^T)^{n-1}\gamma^T]$$

are of the form

$$P = \begin{bmatrix} \underline{m}_1 & \lambda_1\underline{m}_1 & \cdots & \lambda_1^{n-1} & \underline{m}_1 \\ \underline{m}_2 & \lambda_2\underline{m}_2 & \cdots & \lambda_2^{n-1} & \underline{m}_2 \\ \vdots & \vdots & & \vdots & \\ \underline{m}_n & \lambda_n\underline{m}_n & \cdots & \lambda_n^{n-1} & \underline{m}_n \end{bmatrix} \qquad 2.8$$

Where $\underline{m}_i$ is the i^{th} row of the matrix $\beta\,[\gamma^T]$.
Now if a row $\underline{m}_i$ is zero, it means a whole row of P is zero therefore rank of P is at most n-1

Q.E.D.

ii) Sufficiency

If all $\underline{m}_i \neq o$ then for $\lambda_i \neq \lambda_j$, the rows i and j of P are linearly independent. Since we assumed that all the eigenvalues of A are distinct this holds for all pairs.

Therefore n rows of P are linearly independent and so the rank of P is n. For a more explicit proof see [2].

REMARK II-1 Theorem II-4 holds only for an A matrix that has distinct eigenvalues as required by the sufficiency conditions. If we wish to extend this theorem to include all diagonal systems then we must add the following requirement.

The rows of $\beta\,[\gamma^T]$ $\underline{m}_{i,1}, \underline{m}_{i,2}, \ldots, \underline{m}_{i\ell_i}$ that correspond to the eigenvalues λ_i with multiplicity ℓ_i must be linearly independent.

This is due to the fact that the n rows of P must be linearly independent for P to be of rank n. If say $\underline{m}_{i,1} = k\underline{m}_{i,2}$, then the corresponding rows of P

$$\begin{matrix} \underline{m}_{i,1} & \lambda_i\,\underline{m}_{i,1} & \cdots & \lambda_i^{n-1}\,\underline{m}_{i,1} \\ \underline{m}_{i,2} & \lambda_i\,\underline{m}_{i,2} & \cdots & \lambda_i^{n-1}\,\underline{m}_{i,2} \end{matrix}$$

are also linearly dependent so that rank of $P < n$. Furthermore if $\underline{m}_{i,1} \neq k\,\underline{m}_{i,2}$, then the above two rows are linearly independent.

REMARK II-2 At this point let us examine a little closer the physical meaning of controllability.

i) If we consider S_D such that all the eigenvalues are distinct then it can be said that the system is uncontrollable if there exists a a term of the form

$$\dot{z}_j = \lambda_j z_j \qquad 2.10$$

that is, $\underline{m}_j = o$

which means that the input $\underline{u}(t)$ has no way to effect the element z_j of the state vector. To make this point clearer let us examine a simulation diagram of the system S_D defined in eq. 2.6.

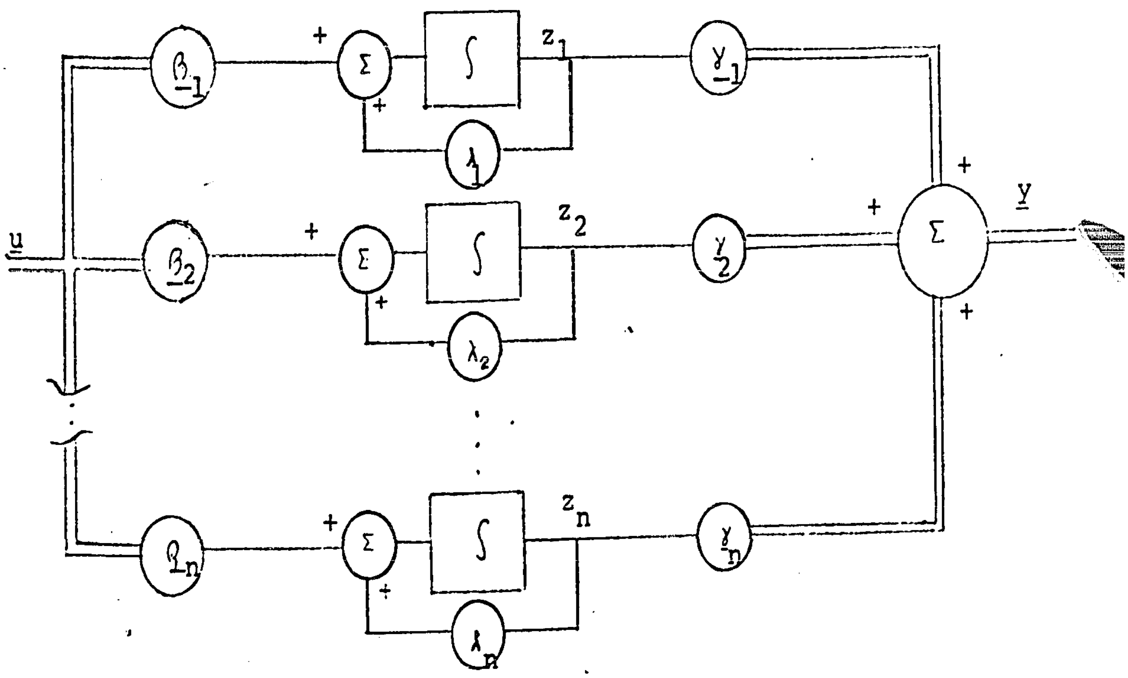

FIGURE II-2. SIMULATION DIAGRAM OF A DIAGONAL SYSTEM.

Note that in Figure II-2 the double lines represent vector quantities while single lines represent scalars.

Note that REMARK II-2 is not only a necessary condition but it is both a necessary and sufficient condition for controbillity. For proof of this statement see [19].

From Figure II-2 it is clear that if any $\underline{\beta}_i = \underline{o}$ the corresponding state z_i cannot be altered at will. Therefore system S_D is not completely controllable.

Also if any $\underline{\gamma}_i = \underline{o}$ then the corresponding state z_i cannot be observed from output measurement.

ii) If system has repeated eigenvalues then the system is not completely controllable if

$$\dot{z}_{i,1} = \lambda_i z_{i,1} + \underline{\beta}_{i,1}\underline{u}$$

$$\dot{z}_{i,2} = \lambda_i z_{i,2} + k\,\underline{\beta}_{i,2}\underline{u}$$

because in this case $z_{i,1}$ and $z_{i,2}$ cannot be driven independently. Again this remark becomes clear from the simulation diagram

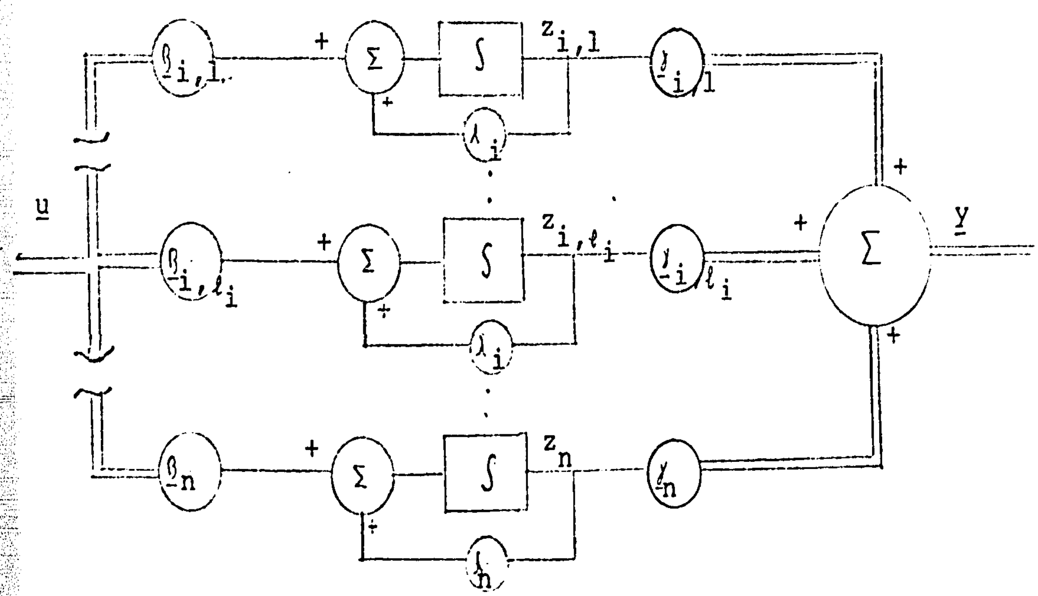

FIGURE II-3. SIMULATION DIAGRAM OF DIAGONAL SYSTEM WITH REPEATED ROOTS.

It is clear to see from the simulation diagram that if the set of vectors $\underline{\beta}_{i,1}, \underline{\beta}_{i,2}, \ldots, \underline{\beta}_{i,\ell_i}$ are not linearly independent then the states $z_{i,1}, z_{i,2}, \ldots, z_{i,\ell_i}$ cannot be excited independently. Therefore the system is not completely controllable.

Also by the same reasoning if the set of vectors $\underline{y}_{i,1}, \underline{y}_{i,2}, \ldots, \underline{y}_{i,\ell_i}$ are not linearly independent then the system is not completely observable.

b) REPEATED EIGENVALUES IN A.

In this case A can be transformed to a Jordan canonical form such that

$$J = T^{-1} A T$$

and system S described by eq. (2.1) is transformed to S_J such that

$$\dot{\underline{z}} = J\,\underline{z} + B_J\,\underline{u}$$

$$\underline{y} = C_J\,\underline{z} + D\,\underline{u} \qquad 2.11$$

where

$$\underline{x} = T\,\underline{z}$$

$$B_J = T^{-1} B$$

$$C_J = C\,T$$

NOTATION FOR JORDAN CANONICAL FORM

$$J_{nxn} = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{bmatrix} \qquad B_{nxp} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_q \end{bmatrix}$$

$$C_{qxn} = [\,C_1 \quad C_2 \cdots C_q\,]$$

Where $J_1, J_2, \ldots, J_q$ represent all the Jordan blocks corresponding to the repeated eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_q$.

Note that B and C matrices are also partioned accordingly. Now if the Jordan blocks corresponding to λ_i with multiciplicity ℓ_i are considered we denote

$$\underset{\ell_i \times \ell_i}{J_i} = \begin{bmatrix} J_{i,1} & 0 & 0 \\ 0 & J_{i,2} & \\ \vdots & & \\ 0 & & J_{i,r_i} \end{bmatrix} \qquad \underset{\ell_i \times p}{B_i} = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,r_i} \end{bmatrix}$$

$$\underset{q \times \ell_i}{C_i} = [\, C_{i,1} \;\; C_{i,2}, \;\; \cdots\cdots \;\; C_{i,r_i} \,]$$

$$n = \sum_{i=1}^{q} \ell_i$$

where $J_{i,1}, J_{i,2}, \ldots, J_{i,r_i}$ represent a single Jordan block of order $\ell_{i,1}, \ell_{i,2}, \ldots, \ell_{i,r_i}$ so that

$$\ell_i = \sum_{j=1}^{r_i} \ell_{i,j}$$

Note that the submatrices B_i and C_i are also partioned accordingly. Now if we consider more closely the submatrices $B_{i,j}$ and $C_{i,j}$ we denote them by

$$\underset{\ell_{ij} \times p}{B_{i,j}} = \begin{bmatrix} \underline{b}_{ij,1} \\ \underline{b}_{ij,2} \\ \vdots \\ \underline{b}_{ij,\ell_{ij}} \end{bmatrix} \qquad \underset{q \times \ell_{ij}}{C_{ij}} = [\, \underline{c}_{ij,1}, \cdots\cdots \;\; \underline{c}_{ij,\ell_{ij}} \,]$$

where $\underline{b}_{ij,k}$ is a row of the matrix B and $\underline{c}_{ij,k}$ is a column of the C matrix.

Now in terms of this notation we can state the controllability and observability conditions.

<u>THEOREM II-5.</u> A system S_J in the Jordan canonical form described by eq. 2.11 is controllable (observable) if and only if $\underline{b}_{ij,\ell_{ij}} \neq o$ ($c_{ij,1} \neq o$) $i = 1,2,\dots, q$ $j = 1, 2,\dots, r_i$ and if and only each set of r_i, $i = 1,2,\dots q$ vectors

$$\{\underline{b}_{i1,\ell_{i1}}, \underline{b}_{i2,\ell_{i2}}, \dots, \underline{b}_{ir_i,\ell_{ir_i}}\} \quad (\{\underline{c}_{i1,1}, \underline{c}_{i2,1}, \dots, \underline{c}_{ir_i,1}\})$$

form a set of linearly independent vectors.

<u>Proof:</u>

i) Necessity

If we have compatible partions we can write

$$\begin{bmatrix} x & 0 & & 0 \\ 0 & Y & & 0 \\ \vdots & & & \vdots \\ 0 & & & z \end{bmatrix}^q \begin{bmatrix} M_1 \\ M_2 \\ \\ M_\ell \end{bmatrix} = \begin{bmatrix} x^q \, M_1 \\ Y^q \, M_2 \\ \\ z^q \, M_\ell \end{bmatrix}$$

and furthermore if we consider a single Jordan block we get

$$\begin{bmatrix} \lambda & 1 & 0 & & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ \vdots & & & & \vdots \\ \cdot & & & & 0 \\ \cdot & & & & 1 \\ 0 & 0 & & & \lambda \end{bmatrix}^q = \begin{bmatrix} \lambda^q & k_{q-1}\lambda^{q-1} & k_{q-2}\lambda^{q-2} & \dots & 1 \\ 0 & \lambda^q & k_{q-1}\lambda^{q-1} & & \vdots \\ \vdots & \vdots & \lambda^q & & \vdots \\ \vdots & 0 & & & k_{q-1}\lambda^{q-1} \\ \vdots & \vdots & & \cdot & \\ 0 & 0 & & & \lambda^q \end{bmatrix}$$

where k_{q-i} are the coefficients of the q^{th} order polynomial

$$(\lambda+1)^q = \lambda^q + k_{q-1}\lambda^{q-1} + \dots, + k_1 \lambda + 1$$

Now if we consider the matrix $[\,B_J \quad JB_J \quad \dots\dots \quad J^{n-1} B_J]$ and if we let J_1 be the Jordan blocks corresponding to λ_1 we obtain the rows

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\underline{b}_{11,\ell_{11}} & \lambda\underline{b}_{11,\ell_{11}} & \lambda^{n-1}\underline{b}_{11,\ell_{11}} \\
\vdots & \vdots & \vdots \\
\underline{b}_{12,\ell_{12}} & \lambda\underline{b}_{12,\ell_{12}} & \lambda^{n-1}\underline{b}_{12,\ell_{12}} \\
\vdots & \vdots & \vdots \\
\underline{b}_{1r_1,\ell_{1r_1}} & \lambda\underline{b}_{1r_1,\ell_{1r_1}} & \lambda^{n-1}\underline{b}_{1r_1,\ell_{1r_1}}
\end{array}$$

Where linear dependence or linear independence depends on the linear dependence or independence of the set of vectors

$$\{\underline{b}_{11,\ell_{11}}, \underline{b}_{12,\ell_{12}}, \ldots, \underline{b}_{1r_1,\ell_{1r_1}}\}$$

This proves the necessity conditions for controllability.

Now if we consider the matrix $[C_J^T \quad J^T C_J^T \ldots, (J^T)^{n-1} C_J^T]$ and we let J_1 be the Jordan blocks corresponding to λ_1, we get the rows:

$$\begin{array}{ccc}
\underline{c}_{11,1} & \lambda\underline{c}_{11,1} & \lambda^{n-1}\underline{c}_{11,1} \\
\vdots & \vdots & \vdots \\
\underline{c}_{12,1} & \lambda\underline{c}_{12,1} & \lambda^{n-1}\underline{c}_{12,1} \\
\vdots & \vdots & \vdots \\
\underline{c}_{1r_1,1} & \lambda\underline{c}_{1r_1,1} & \lambda^{n-1}\underline{c}_{1r_1,1}
\end{array}$$

Where linear dependence or independence depends on the linear dependence or independence of the set of vectors $\{\underline{c}_{11,1}, \underline{c}_{12,1}, \ldots, \underline{c}_{1r_1,1}\}$.

ii) <u>Sufficiency</u>

Here we wish to show that a system which is not controllable (observable) must have the following set of vectors linearly dependent

$$\{\underline{b}_{i1,\ell_{i1}}, \underline{b}_{i2,\ell_{i2}}, \ldots, \underline{b}_{ir_i,\ell_{ir_i}}\}(\{\underline{c}_{i1,1}, \underline{c}_{i2,1}, \ldots, \underline{c}_{ir_i,1}\})$$

Assume that the system is not completely controllable. Then it follows from REMARK II-2 of the previous theorem that there exists $\dot{x}_\ell = \lambda x_\ell$ as one of the components of the state equation. For simplicity of notation let $\lambda = \lambda_1$, where the first r_1 Jordan blocks are the only block containing eigenvalue λ_1. Then x_ℓ must be a non-trivial linear combination of the states corresponding to the first r_1 Jordan blocks, i.e

$$x_\ell = \sum_{i=1}^{\ell_{11}} \delta_{1,i}\, z_{1,i} + \dots + \sum_{i=1}^{\ell_{1r_1}} \delta_{r_1,i}\, z_{r_1,i} \qquad 2.15$$

then the state equation for x_ℓ is

$$\dot{x}_\ell = \lambda x_\ell + \sum_{i=1}^{\ell_{11}-1} \delta_{1,i}\, z_{1,i+1} + \dots + \sum_{i=1}^{\ell_{1r_1}-1} \delta_{r_1,i}\, z_{r_1,i+1}$$

$$+ \left[\sum_{i=1}^{\ell_{11}} \delta_{1,i}\, \underline{b}_{11,i} + \dots + \sum_{i=1}^{\ell_{1r_1}} \delta_{r_1,i}\, \underline{b}_{1r_1,i} \right] \underline{u} \qquad 2.16$$

because in the Jordan form

$$\dot{z}_{1j,i} = \lambda z_{1j,i} + z_{1j,i+1} + \underline{b}_{1j,i}\, \underline{u} \qquad \begin{matrix} j = 1,2,\dots,r_1 \\ i = 1,2,\dots,\ell_{1j} \end{matrix} \qquad 2.17$$

this implies that in order to get

$$\dot{x}_\ell = \lambda x_\ell$$

$$\delta_{1,i} = 0 \qquad i = 1,2,\dots,\ell_{11}-1$$

$$\vdots \qquad\qquad \vdots$$

$$\delta_{r_1,i} = 0 \qquad i = 1,2,\dots,\ell_{1r_1}-1$$

hence

$$\sum_{i=1}^{r_1} \delta_{i,\ell_{1i}}\, \underline{b}_{1i,\ell_{1i}} = 0 \qquad 2.18$$

Q.E.D.

In a similar way we can show sufficiency for observability because if the system is not completely observable then there must exist a representation of the system where a particular state x_ℓ is not transferred to the output

i.e. $\underline{y} = \underline{y}_\ell \; x_\ell = o$

This x_ℓ must be a non-trivial linear combination of the states corresponding to the r_1 blocks with same eigenvalue λ_1 (see eq. 2.15). Thus the output due to x_ℓ is equal to the sum of the outputs of the other states, i.e.

$$\underline{y} = \sum_{i=1}^{\ell_{11}} \delta_{1,i} \, \underline{c}_{11,i} \, z_{1,i} + \ldots + \sum_{i=1}^{\ell_{1r_1}} \delta_{r_1,i} \, \underline{c}_{1r_1,i} \, z_{r_1,i} \qquad 2.19$$

but by equation 2.17

$$\underline{y} = \sum_{i=2}^{\ell_{11}} \delta_{1,i} (\dot{z}_{1,i-1} - \lambda z_{1,i-1} - \underline{b}_{11,i} \underline{u}) \underline{c}_{11,i} + \cdots$$
$$+ \sum_{i=2}^{\ell_{1r_1}} \delta_{r_1,i} \, \underline{c}_{1r_1,i} (\dot{z}_{r_1,i-1} - \lambda z_{r_1,i-1} - \underline{b}_{1r_1,i} \underline{u}) \qquad 2.20$$
$$+ \sum_{j=1}^{r_1} \delta_{j,1} \, \underline{c}_{1j,1} \, z_{1j,1}$$

now in order that $\underline{y} = o$

$$\delta_{1,i} = o \qquad i = 2,3,\ldots,\ell_{11}$$
$$\vdots$$
$$\delta_{r_1,i} = o \qquad i = 2,3,\ldots,\ell_{1r_1}$$

so we are left with the term

$$\underline{y} = \sum_{j=1}^{r_1} \delta_{j,1} \, \underline{c}_{1j,1} \, z_{1j,1} \qquad 2.21$$

now $\underline{y} = \underline{o}$ implies that the vectors

$$\underline{c}_{1j,1} \qquad j = 1,2,\ldots,r_1 \qquad 2.22$$

are linearly dependent.

Q.E.D.

COROLLARY II-1 The minimal number of inputs (outputs) for complete controllability (observability) of a system S is equal to the largest number of Jordan blocks containing the same eigenvalue.

Proof:

We know by theorem II-5 that the last rows of B (first column of C) corresponding to the last rows (first rows ·) of each block with same eigenvalue must be linearly independent. If there are r_i such blocks, for r_i rows (columns) to be linearly independent you need r_i inputs (outputs).

Again to obtain a better insight of the physical meaning of THEOREM II-5 consider the following partial simulation diagram of S_J

Assume there is only one repeated eigenvalue with r_1 blocks

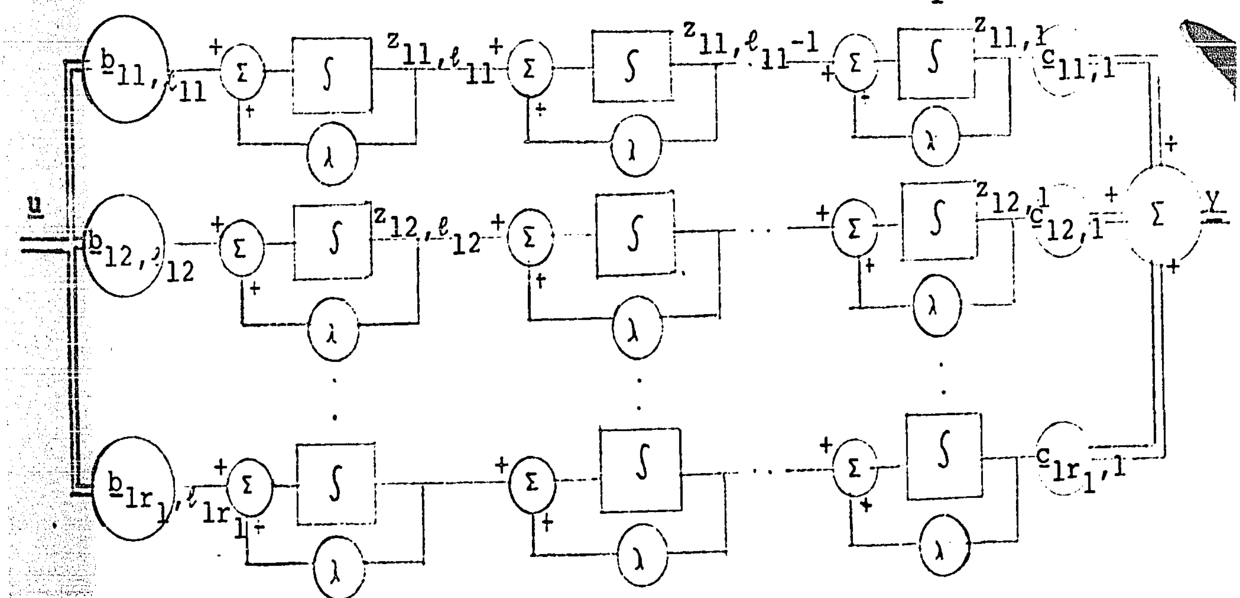

FIGURE II-4. PARTIAL SIMULATION DIAGRAM OF A JORDAN SYSTEM.

It is obvious from this simulation that the only way that all the states in a particular block be controllable (observable) we must have

$$\underline{b}_{1j,\ell_{1j}} \neq 0 \quad (\underline{c}_{1j,1} \neq 0) \qquad j = 1,2,\ldots, r_1$$

Also to insure that the states in different blocks, but with same eigenvalue, be indepedently controlled (observed) then the set of vectors

$$\{\underline{b}_{11,\ell_{11}},\ \underline{b}_{12,\ell_{12}},\ \ldots,\ \underline{b}_{1r_1,\ell_{1r_1}}\}$$

$$(\{\underline{c}_{11,1},\ \underline{c}_{12,1},\ \ldots\ldots,\ \underline{c}_{1r_1,1}\}$$

must be linearly independent.

c) COMPANION FORM

So far in our investigation we only considered the Jordan canonical form. We will now investigate the generalized companion matrix form.

System described by eq. 2.1 is now transformed to a companion form such that

$$\dot{\underline{z}} = A_c\,\underline{z} + B_c\,\underline{u}$$
$$\underline{y} = C_c\,\underline{z} + D\,\underline{u} \qquad 2.23$$

where:

$$A_c = \begin{bmatrix} A_1 & 0 & & 0 \\ 0 & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{bmatrix} \qquad B_c = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{bmatrix}$$

$$C = [\,C_1 \quad C_2 \quad \cdots \quad C_k\,]$$

where each A_i is in companion form

$$\underset{\ell_i \times \ell_i}{A_i} = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ -a_{i,o} & -a_{i,1} & & & -a_{i,\ell_i-1} \end{bmatrix} \qquad \underset{\ell_i \times p}{B_i} = \begin{bmatrix} \underline{b}_{i,1} \\ \underline{b}_{i,2} \\ \vdots \\ \underline{b}_{i,\ell_i} \end{bmatrix}$$

$$\underset{q \times \ell_i}{C_i} = [\,\underline{c}_{i,1} \quad \underline{c}_{i,2} \ \cdots\cdots\ \underline{c}_{i,\ell_i}\,]$$

$$n = \sum_{i=1}^{k} \ell_i$$

Note that each A_i represents an invariant (Cyclic) subspace of A.

THEOREM II-6 A system S_c in companion matrix form as described by eq. 2.23 is controllable (observable) if and only if each $\underline{b}_{i,\ell_i} \neq o$ ($\underline{c}_{i,1} \neq o$) $i = 1, 2, \ldots, k$ and if and only if each set of k vectors

$$\{\underline{b}_{1,\ell_1}, \underline{b}_{2,\ell_2}, \ldots, \underline{b}_{k,\ell_k}\} \; (\{\underline{c}_{1,1}, \underline{c}_{2,1}, \ldots, \underline{c}_{k,1}\})$$

form a set of linearly independent vectors.

For a formal proof see [15]

However a simulation diagram of the system S_c will be given and it should indicate quite clearly the proof of the theorem.

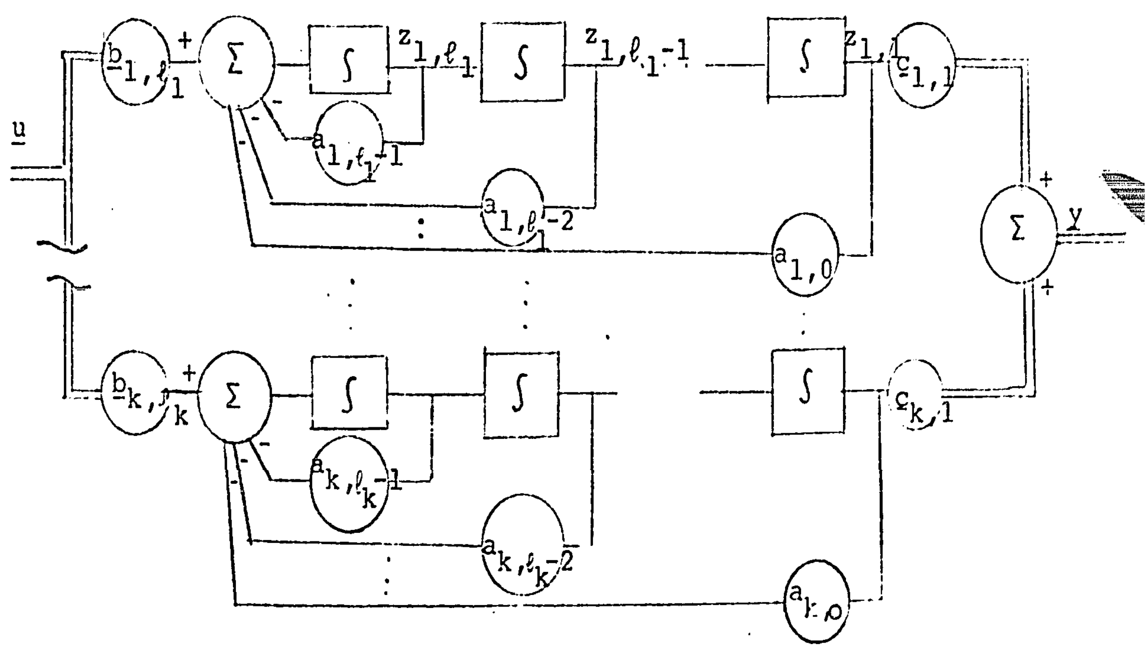

FIGURE II-5. SIMULATION DIAGRAM OF A SYSTEM IN THE COMPANION STRUCTURE.

From this diagram it is obvious that for each state in a particular block to be controllable (observable) it is required that

$$\underline{b}_{i,\ell_i} \neq o \quad (\underline{c}_{i,1} \neq o) \qquad i = 1, 2, \ldots, k$$

Also for each state of different blocks to be controlled independently the set of vectors $\{\underline{b}_{1,\ell_1}, \underline{b}_{2,\ell_2}, \ldots, \underline{b}_{k,\ell_k}\}$ must be linearly independent.

This is so because in the compansion form the submatrices A_i $i = 1, 2, .., k$ are, as mentioned before, the invariant subspaces of A. Now if A_1 corresponds to the minimal polynomial then it contains all the leading elementary divisors so that all the other subspaces are made up from these elementary divisors. This is why the above vectors must be linearly independent. The same remark can be made for observability.

d) SINGLE - INPUT SINGLE - OUTPUT SYSTEM.

Finally one might want to consider the special case of a single-input single-output system.

THEOREM II-7 A necessary condition for controllability (observability) is that no two Jordan blocks contain the same eigenvalue.

Proof:

This follows directly from Corollary II-1 and Theorem II-5 because we only have one input (output). Therefore you can only have one independent row of B (C^T). Then there can be only one Jordan block with same eigenvalue. To include the sufficiency condition in the theorem one must add the following requirement.

For complete controllability (observability) the rows of B, (C^T) corresponding to the last row of each Jordan block(corresponding to the first row of each Jordan block) must be different from zero.

We can get a similar theorem if A is in companion form.

THEOREM II-8 A necessary condition for controllability is that A have only one companion block or in a mathematical sense that the minimum polynomial m(A) should be equal to the characteristic polynomial p(A)..

Proof:

Proving either statement is the same because if m(A) = P(A) then A has only one cyclic subspace. Therefore A has one companion block.

Consider the requirement that

$$[\underline{b} \quad A\underline{b} \quad A^2\underline{b} \quad \cdots \quad A^{n-1}\underline{b}] \qquad 2.27$$

be of rank n, that is, the set of vectors in 2.27 should be linearly independent

$$\text{i.e.} \qquad \sum_{i=o}^{n-1} a_i A^i \underline{b} = 0 \qquad \forall\ a_i \neq o \quad \text{where } A^0 = I$$

but this can be rewritten as

$$[a_o I + a_1 A + a_2 A^2, \ldots + a_{n-1} A^{n-1}]\ \underline{b} = o$$

and if $m(A) \neq p(A)$ there is a polynomial of order less than n such that $m(A) = o$

or

$$k_o I + k_1 A + \ldots. + k_m A^m = o$$

but this means that for these coefficients $\forall\ \underline{b}$ we have

$$[k_o I + k_1 A + \ldots + k_m A^m]\ \underline{b} = o$$

Thus the maximal rank of our matrix is m. Q.E.D.

The proof of observability goes exactly in the same form.

Again in order to include the sufficiency condition this theorem it is required that the last row of B (first column of C) be different from zero.

II-4 THE MINIMAL ORDER OF A SYSTEM.

Up to this point we obtained some mathematical conditions for controllability and observability. Physically this seems to indicate the following partitioning of the states of a System S.

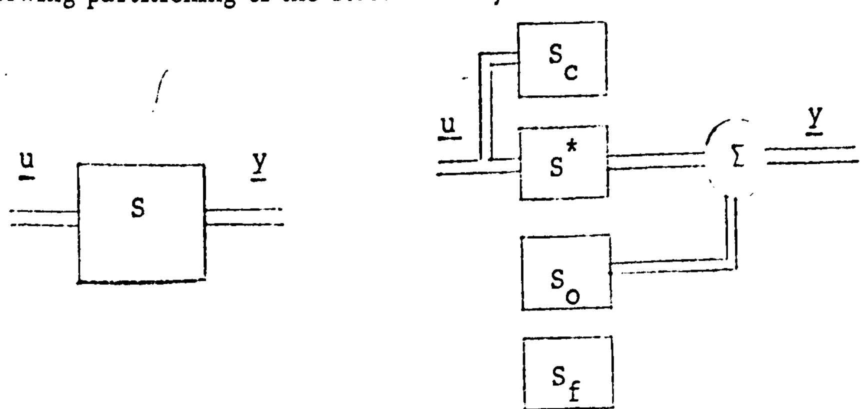

FIGURE II-6. SYSTEM S AND ITS PARTITIONED REPRESENTATION.

where

S_c = controllable but unobservable states

S^* = controllable and observable states

S_o = uncontrollable but observable states

S_f = uncontrollable and unobservable states.

From the above figure it can be seen that if we only consider the black box (input / output) behaviour of the system only S^* (i.e. controllable and observable part) need be considered. This would infer that the transfer function matrix representation of a system (2.1) only includes S^*.

This leads us to define the minimal order of a system.

<u>DEFINITION II-5</u> The state space representation

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$$\underline{y} = C\underline{x} + D\underline{u}$$

of order n is the minimal representation of a transfer function matrix

If

a) it is controllable and observable

b) $H(s) = C(sI - A]^{-1} B$.

<u>THEOREM II-9</u> The minimal order q of an n^{th} order system, that is, the number of states which are both controllable and observable, is given by the rank of the following matrix,

$$\Gamma = L_o^T L_c \qquad 2.30$$

where L_c = controllability matrix

L_o = observability matrix

Proof

Any system of order n, as defined in equation 2.1, can be transformed by a non-singular matrix T, to an equivalent system of the form.

$$\dot{\underline{z}} = A_T\underline{z} + B_T\underline{u}$$

$$\underline{y} = C_T\underline{z} \qquad 2.31$$

where $\underline{x} = T\underline{z}$, and the vector $\underline{z}$ is made up of three sets of components, that is:

$$\underline{z} = \begin{bmatrix} \underline{z}_1 \\ \underline{z}_2 \\ \underline{z}_3 \end{bmatrix} \qquad \begin{aligned} \underline{z}_1 &= \text{q-dimensional vector} \\ \underline{z}_2 &= \text{l-dimensional vector} \\ \underline{z}_3 &= \text{m-dimensional vector} \end{aligned}$$

$$A_T = T^{-1}\,AT = \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}$$

$$B_T = T^{-1}\,B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} \qquad C_T = CT = \begin{bmatrix} C_1 & 0 & C_3 \end{bmatrix}$$

For a detailed proof of this transformation see [9]. However, this transformation is evident if one considers the following remarks. System 2.31 can be subdivided into three subsystems:

(a) $\dot{\underline{z}}_1 = A_{11}\underline{z}_1 + A_{13}\underline{z}_3 + B_1\underline{u}$

$\underline{y}_1 = C_1\underline{z}_1$ 2.32

This system is completely controllable and observable.

(b) $\dot{\underline{z}}_2 = A_{22}\underline{z}_2 + A_{21}\underline{z}_1 + B_2\underline{u} + A_{23}\underline{z}_3$

$\underline{y}_2 = 0$ 2.33

This system is completely controllable but unobservable.

(c) $$\dot{\underline{z}}_3 = A_{33}\underline{z}_3$$

$$\underline{y}_3 = C_3\underline{z}_3 \qquad 2.34$$

This system is completely observable but uncontrollable.

If system defined in equation 2.1 is first reduced to a system which is completely controllable, but not observable, and then further reduced to a completely controllable and observable system, one gets the above three subsystems. The non-singular transformation required to do these reductions can be found in [9].

We now wish to find the rank of the matrix Γ_T defined by:

$$\Gamma_T = \begin{bmatrix} C_T \\ C_T A_T \\ \cdot \\ \cdot \\ \cdot \\ C_T A_T^{n-1} \end{bmatrix} \begin{bmatrix} B_T & A_T B_T \ldots\ldots A_T^{n-1} B_T \end{bmatrix}$$

and show that it is q

First observe that:

$$\begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}^P = \begin{bmatrix} D_{11} & 0 & D_{13} \\ D_{21} & D_{22} & D_{23} \\ 0 & 0 & D_{33} \end{bmatrix}$$

where the matrices Dij $i=1,3$ $j=1,3$ are functions of the matrices Aij $i=1,3$ $j=1,3$. It follows that L_c and L_o will be of the form:

$$L_c = \begin{bmatrix} B_1 & G_1 & \cdots & H_1 \\ B_2 & G_2 & \cdots & H_2 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$L_o = \begin{bmatrix} C_1 & P_1 & Q_1 \\ 0 & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ C_3 & P_3 & Q_3 \end{bmatrix}$$

therefore:

$$\Gamma_T = \begin{bmatrix} C_1 & 0 \cdot \cdot & C_3 \\ P_1 & 0 \cdot \cdot & P_3 \\ Q_1 & 0 \cdot \cdot & Q_3 \end{bmatrix} \begin{bmatrix} B_1 & G_1 \cdot \cdot \cdot & H_1 \\ B_2 & G_2 \cdot \cdot \cdot & H_2 \\ 0 & 0 \cdot \cdot \cdot & 0 \end{bmatrix}$$

has only q independent rows, hence the rank of Γ_T is q.

This could be seen also by considering Γ_T in terms of the controllability matrices of the sybsystems $[A_{11}, B_1, C_1]$, $[A_{22}, B_2, 0]$, $[A_{33}, 0, C_3]$ where the upper left qxq partition of Γ_T is the matrix $L_{10}^T L_{1C}$ of rank q. L_{1C} and L_{10} are the controllability and observability matrices of subsystem defined in equation 2.32.

By using the result of the next theorem (whose proof is not dependent on this theorem), we see that:

$$\Gamma = \Gamma_T$$

since the rank of Γ_T is q, then the rank of Γ is also q.

<u>THEOREM II-10</u> Γ defined in equation 2.30 is an invariant under any non-singular transformation T. That is, if system defined in equation 2.1 is transformed to the following system:

$$\dot{\underline{z}} = A_T \underline{z} + B_T \underline{u}$$

$$\underline{y} = C_T \underline{z}$$

where $\underline{x} = T\underline{z}$

$$A_T = T^{-1} AT$$

$$B_T = T^{-1} B$$

$$C_T = CT$$

then $\Gamma_T = \Gamma$

Proof

$$\Gamma_T = \begin{bmatrix} CT \\ CTT^{-1}AT \\ \vdots \\ CTT^{-1}A^{n-1}T \end{bmatrix} \begin{bmatrix} T^{-1}B & T^{-1}ATT^{-1}B & \cdots & T^{-1}A^{n-1}TT^{-1}B \end{bmatrix}$$

because $(T^{-1}AT)^n = T^{-1}A^nT$

therefore $\Gamma_T = \Gamma$

11-5 - COMMENTS

In this chapter , tests for controllability and observability have been given.In particular, we have given conditions for controllability and observality for system described by certain canonical forms such as Jordan or Companion.Also, single-input single-output systems have been studied.Most of the conditions are known but they are presented and proven because they are required in the following chapters.

The concept of minimal order of a system was introduced and defined. In particular, a test was given to find the minimal order of a system. This test was stated in [9] but not proven.

CHAPTER III

IRREDUCIBE REALIZATION USING SMITH-MACMILLAN FORM

In this chapter we endeavour to find a simple algorithm to construct a minimal realization of transfer function matrices. The algorithm will be based on the Smith-MacMillan form of rational matrices.

III-1 TRANSFER FUNCTION MATRIX

As in the previous chapter a linear dynamic time-invariant system S is described by the following equation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} + D\underline{u} \end{aligned} \qquad 3.1$$

where all the vectors and matrices are as defined in equation 2.1.

In all the following work it is assumed that $D \equiv o$, that is there is no instanteneous response. Assuming that the system S defined in eq. 3.1 starts at rest at time $t = o$ the Laplace transform $\underline{Y} = \underline{Y}(s)$ and $\underline{U} = \underline{U}(s)$ of $\underline{y}(t)$ and $\underline{u}(t)$ are related by

$$\underline{Y}(s) = Z(s)\,\underline{U}(s) \qquad 3.2$$

where Z(s) is the transfer function matrix of the system S(A, B, C) and is given by

$$Z(s) = C\,[\,sI - A\,]^{-1}\,B \qquad 3.3$$

Since we have made $D \equiv o$, the matrix Z(s) is a proper rational matrix, that is each entry of Z(s) is a quotient of polynomials in s with the degree of the numerator lower than that of the denominator.

The matrix Z(s) exhibits the transfer (input/output) behaviour of the system but suppresses the internal (state) behaviour. Thus to each system S(A, B, C) is associated a unique rational matrix Z(s). However, to each proper rational matrix Z(s) is associated a whole class of systems S called realizations of Z(s) each having Z(s) as transfer function matrix. These systems share the same input/output behaviour but can differ internally. In particular they can differ in the dimension of the state space (dimension of the matrix A). The realization of Z(s) having the smallest possible state space dimensions are of particular interest. These are the minimal realizations of Z(s) and have been defined in the previous chapter.

As mentioned before the approach to find the minimal state space representation of Z(s) will be based on the Smith-MacMillan form. It requires the reduction of the transfer function matrix Z(s) by means of elementary row and column operations to this special diagonal form in which certain divisibility conditions hold.

III-2 SMITH-MACMILLAN FORM [16]

A matrix P(s) consisting of polynomials in s will be written P or P(s). The polynomial $\Delta_\rho\,[P(s)]$ will denote the greatest common divisor (g.c.d.) of all $\rho \times \rho$ minors of P(s). By convention $\Delta_o = 1$. Clearly Δ_ρ divides $\Delta_{\rho+1}$ which is written, $\Delta_\rho \mid \Delta_{\rho+1}$. All Δ_ρ are made monic, that, is leading coefficient is normalized to one. If $\Delta_\rho\,[P(s)] \neq o$ but $\Delta_R[\,P(s)] = o$ for all $\rho > R$ then R is the rank of the polynomical matrix P(s).

DEFINITION III-1 Let Z(s) be a proper rational matrix and let $\psi(s)$ denote its least common denominator (l.c.d.). Then ψ Z(s) is a polynomial matrix same as P(s) above.

Any q x p polynomial matrix ψ Z(s) of rank R can be represented, after the application of elementary row and column operation by

$$\psi\, Z(s) = G\,\Gamma\,H \tag{3.4}$$

where

i) $$\Gamma(s) = \text{diag}\,[\gamma_1, \gamma_2, \cdots \gamma_R] \tag{3.5}$$

and

$$\gamma_\rho \mid \gamma_{\rho+1} \qquad \rho = 1, \ldots, R-1$$

ii) G(s) and H(s) are q x q and p x p polynomial matrices with constant non-zero determinants.

iii) $\gamma_1 \cdots \gamma_R$ are polynomials uniquely determined by $\psi Z(s)$; in fact

$$\gamma_\rho = \Delta_\rho [\psi Z(s)] \,/\, \Delta_{\rho-1} [\psi Z(s)] \qquad 3.6$$

Dividing both sides of eq. 3.4 by ψ and reducing each polynomial function

$$\gamma_i / \psi \qquad i = 1, \ldots, R-1$$

by cancellation of common factors we obtain

$$Z = G \Lambda H \qquad 3.7$$

where

$$\Lambda = \text{diag} [\, \epsilon_1/\psi_1, \epsilon_2/\psi_2, \ldots, \epsilon_R/\psi_R, \ldots \,] \qquad 3.8$$

the $\epsilon_i = \epsilon_i(s)$ and $\psi_i = \psi_i(s)$ $i = 1, 2, \ldots, R$ are monic polynomials where $\epsilon_i | \epsilon_{i+1}$ and $\psi_{i+1} | \psi_i$ such that each pair (ϵ_i, ψ_i) are relatively prime. The matrix Λ is the Smith-MacMillan form of Z(s), [16].

III-3 EQUIVALENCE RELATIONS

DEFINITION III-2 [16] Two rational matrices Z(s) and $\hat{Z}(s)$ are called strongly equivalent if there exist polynomial matrices G(s) and H(s) with constant nonzero determinants such that

$$\hat{Z} = G Z H \qquad 3.9$$

Two constant linear dynamical systems are said to be strongly equivalent if their transfer function matrices are strongly equivalent.

Clearly each rational matrix is strongly equivalent to a unique Smith-MacMillan canonical matrix. It follows therefore that the rank R together with the 2R polynomials $\psi_1, \psi_2, \ldots, \psi_R$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_R$ form a complete set of invariants for strong equivalence of rational matrices.

The set of polynomials $\psi_1, \ldots, \psi_R$ in fact are the invariant (cyclic) subspaces of the system S. So knowing these polynomials gives us the structure of the system and completely defines the A matrix of the minimal realization. With only this information we could then solve a set of equation to find the matrices B (input) and C (output) because $Z(s) = C[sI - A]^{-1}B$, and knowing A and Z(s) we could then solve this equation to get the entries of B and C.

When one finds the Smith-MacMillan form of a system transfer function matrix Z(s) there may be cases when it will not be proper, that is,for some particular j, ϵ_j may be of higher degree than ψ_j. To be able to get around this we now define weak equivalence.

DEFINITION III-2 [7] Let Z(s) and $\hat{Z}(s)$ be rational matrices and let ψ and $\hat{\psi}$ be respectively the least common demoninator of the entries of Z and $\hat{Z}$.

Z and $\hat{Z}$ are weakly equivalent if

i) $\psi = \hat{\psi}$

ii) There exist polynomial matrices G and H with constant non-zero determinants such that

$$\psi \hat{Z} \equiv G \psi Z H \pmod{\psi} \qquad 3.10$$

Two constant linear dynamical systems are said to be weakly equivalent if their transfer function matrices are weakly equivalent.

With this definition we can now get a proper Smith-MacMillan form Λ' from an improper one.

DEFINITION III-3 Λ' is defined as the reduced Smith-MacMillan form of $Z(s)$ and is of the form

$$\Lambda' = \text{diag}\ [\epsilon_1'/\psi_1,\ \epsilon_2'/\psi_2, \dots, \epsilon_r'/\psi_r, 0, \dots 0] \qquad 3.11$$

where

i) $r = \max\{ i \mid 1 \le i \le R,\ \psi_i \ne 1\}$

ii) ϵ_i' is the remainder after dividing ϵ_i by ψ_i for $i = 1, 2, \dots, R$

iii) $$\psi(z) = G[\psi \Lambda'] H \pmod{\psi} \qquad 3.12$$

REMARK III-1 Λ' is a proper diagonal matrix and also it is weakly equivalent to $Z(s)$. To realize the latter point examine equation 3.12

We now wish to investigate the relationship between weakly equivalent linear dynamical systems. The reason for this is very simple, as Λ and Λ' are weakly equivalent matrices. If we can find a state space representation $\hat{S}(\hat{A}, \hat{B}, \hat{C})$ for Λ' then knowing the relationship between weakly equivalent systems it would enable us to get a state space representation $S(A,B,C)$ for Λ. The resulting representation is strongly equivalent to $Z(s)$.

THEOREM III-1 [7] Let $\hat{S}(\hat{A}, \hat{B}, \hat{C})$ be a constant linear dynamical system. Let G and H be polynomial matrices of appropriate sizes such that the matrix product $G\hat{C}$ and $\hat{B}H$ are defined. Let $G = \Sigma G_i s^i$ and $H = \Sigma H_i s^i$ express G and H as matrix polynomials. Now define $S(A, B, C)$ such that

$$\begin{aligned} A &= \hat{A} \\ B &= \sum_j \hat{A}^j \hat{B} H_j \\ C &= \sum_i G_i \hat{C} \hat{A}^i \end{aligned} \qquad 3.13$$

Then the transfer function matrix $Z(s)$ of the system $S(A,B,C)$ is related to the transfer function matrix $\hat{Z}(s)$ of $\hat{S}(\hat{A}, \hat{B}, \hat{C})$, by

$$\psi Z = G(\psi \hat{Z}) H \pmod{\psi} \qquad 3.14$$

where ψ is the minimal polynomial of A. This relationship (3.13) between two systems is known as external equivalence.

Proof

For this proof we need the following relationship for each non-negative k.

$$s^k \psi(s)\ (sI - A)^{-1} \equiv A^k \psi(s)(sI-A)^{-1} \pmod{\psi} \qquad 3.15$$

$$\psi(s)(sI - A)^{-1} s^k \equiv \psi(s)(sI-A)^{-1} A^k \pmod{\psi} \qquad 3.16$$

3.15 follows from

$$s\,\psi(s)(sI-A)^{-1} - A\,\psi(s)(sI-A)^{-1} = (sI-A)\,\psi(s)(sI-A)^{-1} = \psi(s)\,I$$

$$\equiv 0 \pmod{\psi}$$

Then 3.15 is true for $k = 1$, and an elementary induction argument establishes 3.15 for all k. Equation 3.16 is a consequence of 3.15 and the fact that A commutes with $(sI-A)^{-1}$ [10]

$$\begin{aligned} Z(s) &= C\,(sI-A)^{-1} B \\ &= (\Sigma G_i \hat{C} \hat{A}^i)(sI-\hat{A})^{-1}(\Sigma \hat{A}^j \hat{B} H_j) \end{aligned}$$

now using 3.15 and 3.16 we get

$$\begin{aligned} Z(s) &= (\Sigma G_i \hat{C} s^i)(sI-\hat{A})^{-1}(\Sigma s^j \hat{B} H_j) \pmod{\psi} \\ &= (\Sigma G_i s^i)\,\hat{C}\,(sI-\hat{A})^{-1}\hat{B}\,(\Sigma H_j s^j) \\ &\doteq G \hat{Z} H \end{aligned}$$

Q.E.D.

The result of this theorem is that externally equivalent systems are also weakly equivalent. That is, equation 3.13 now gives us a relationship between weakly equivalent systems.

The other important property of externally equivalent system is that they have the same controllability and observability properties. This leads to the following theorem.

THEOREM III-2 [7] Let $S(A,B,C)$ and $\hat{S}(\hat{A}, \hat{B}, \hat{C})$ be externally equivalent constant linear dynamical systems. Then their controllability matrices (L_c) and observability matrices (L_o) have the same rank. That is, $\hat{S}(\hat{A},\hat{B},\hat{C})$ is controllable and observable if and only if $S(A,B,C)$ is controllable and observable.

Proof

Suppose $\underline{c}$ is a row vector such that

$$\underline{c}\ [B, AB, A^2B, \ldots, A^{n-1}B] = o$$

Then since each power of A can be expressed as a polynomial in A of degree less than n [10] it follows $C A^k B = o$ for all non-negative integers.

Hence

$$C A^k \hat{B} = \sum_j C A^{k+j} B H_j = o$$

for all $k \geq o$ and so

$$\underline{c}[\hat{B}, \hat{A}\hat{B}, \ldots, \hat{A}^{N-1}\hat{B}] = o$$

Thus, viewing L_c and $\hat{L}_c$ as linear operators acting on row vectors, we see that the null space of L_c is contained in the null space of $\hat{L}_c$. But, since external equivalence is a symmetric relation it follows that the null space of $\hat{L}_c$ is also contained in the null space L_c. That is, the null spaces are equal so therefore L_c and $\hat{L}_c$ must have the same rank since they have the same nullity space. Proof for observability is similar.

Q.E.D.

With these theorems in mind we can now form the algorithm which will give a minimal realization for $Z(s)$. As mentioned in the previous chapter the realization in order to be minimal will have to be controllable and observable and also have $Z(s)$ as its transfer function matrix.

III-4 TRANSFER FUNCTION REALIZATION

Before going on to the realization of the whole transfer function matrix $Z(s)$ let us first consider the minimal realization of a simple transfer function of the form

$$H(s) = \frac{a_o + a_1 s + a_2 s^2 + \ldots + a_{n-1} s^{n-1}}{b_o + b_1 s + \ldots + s^n} \qquad 3.20$$

THEOREM III-3 A minimal state representation for a transfer function H(s) of the form (3.20) is as follows:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 \ldots & 0 \\ \cdot \\ \cdot \\ \cdot \\ -b_o & -b_1 & \ldots & & -b_{n-1} \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 1 \end{bmatrix} \underline{u} \qquad 3.21$$

$$y = [a_o \ a_1 \ldots a_{n-1}] \ \underline{x}$$

Proof

The realization 3.21 will be minimal if

i) it is controllable and observable

ii) its transfer function is $H(s) = C(sI-A)^{-1} B$

i) This representation is controllable and observable by Theorem II-8 of the previous chapter and because the last row of B and first column of C are ≠0

ii) Now we must show that

$$H(s) = C(sI-A)^{-1} B$$

because of the form of B the only term in the product $C(sI-A)^{-1}$ that will be of any importance is the product of C times the last column of $(sI-A)^{-1}$. The last column of $(sI-A)^{-1}$ can be shown to be

$$\frac{1}{\Delta(sI-A)} \begin{bmatrix} 1 \\ s \\ s^2 \\ \cdot \\ s^{n-1} \end{bmatrix} \quad \text{where} \quad \Delta(sI-A) \equiv \det\ (sI-A)$$

Then

$$C(sI-A)^{-1}B = [a_o\ a_1 \dots a_{n-1}] \frac{1}{\Delta(sI-A)} \begin{bmatrix} x & x & \dots & x & 1 \\ \cdot & \cdot & & & s \\ \vdots & & & & s^2 \\ \vdots & & & & \vdots \\ x & x & \dots & x & s^{n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Then

$$H(s) = \frac{a_o + a_1 s + \dots + a_{n-1} s^{n-1}}{b_o + b_1 s + \dots + s^n}$$

Q.E.D.

because

$$\Delta(sI - A) = b_o + b_1 s + b_2 s^2 + \dots + s^n$$

III-5 MINIMAL REALIZATION OF THE TRANSFER MATRIX.

By theorem III- 3 we know how to find the minimal realization of any term of the form $H(s) = \frac{\epsilon(s)}{\psi(s)}$. If we now look at the diagonal terms of Λ' (the reduced Smith-MacMillan form) we see r such terms. The matrix Λ' can then be minimally realized by taking the sum of the realization of each diagonal term. This sum is then controllable and observable since each diagonal term was realized by an independently controllable and observable system. Having a minimal realization of Λ' we can then find a minimal realization of $Z(s)$ through equation 3.13. The algorithm will now be given in detail.

The steps to find a minimal state space realization of $Z(s)$ are:

1) Find Λ' the reduced Smith-MacMillan form of which is defined in 3.11.

2) For each $i = 1, 2, \dots, r$ let $\{A_i, B_i, C_i\}$ be given by

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & 0 \\ \cdot & & & & 1 \\ -b_{io} & -b_{i1} & \cdots & & -b_{in_i-1} \end{bmatrix} \qquad B_i = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \qquad 3.22$$

$$C_i = [\, a_{io} \quad a_{i1} \cdots a_{in_i-1} \,]$$

where the a_{ij} and b_{ij}, are the coefficients of the polynomials $\epsilon_i(s)$ and $\psi_i(s)$.

Note:

This is a minimal realization (proven in Theorem III-3) of each of the r blocks.

3) Now the system $(\hat{A}, \hat{B}, \hat{C})$ which is the direct sum of the system $(\hat{A}_i, \hat{B}_i, \hat{C}_i)$ $i = 1, 2, \ldots, r$ is the minimal realization of Λ'.

4) To obtain the state space realization of $Z(s)$, use the following relations

$$\begin{aligned} A &= \hat{A} \\ B &= \sum_j \hat{A}^j \hat{B} H_j \\ C &= \sum_i G_i \hat{C} A^i \end{aligned} \qquad 3.23$$

This realization is minimal.

Proof:

To prove this let us first prove that $(\hat{A}, \hat{B}, \hat{C})$ is a minimal realization of Λ'.

From Theorem III-3 each (A_i, B_i, C_i) is a minimal realization of $\epsilon_i(s) / \psi_i(s)$ for $i = 1, 2, \ldots, r$.

But the direct sum of controllable and observable systems is controllable and observable. Also from Theorem II-6 it is obvious that the realizations $(\hat{A}, \hat{B}, \hat{C})$ is controllable and observable. Also its transfer function matrix is definitely Λ'.

Thus $\quad S(\hat{A}, \hat{B}, \hat{C})$ is the minimal realization of Λ'.

Finally by Theorem III-1 the System $S(A, B, C)$ constructed in in step 4 of the algorithm is a realization of $Z(s)$. That is S, (A,B,C) has $Z(s)$ as its transfer function matrix. Also by Theorem III-2 it is controllable and observable. Therefore (A,B,C) is a minimal realization of $Z(s)$.

Q.E.D.

To illustrate the realization procedure we consider the following examples.

Example III-1. Let

$$Z(s)=\frac{1}{s^4}\begin{bmatrix} s^3 - s^2 + 1 & 1 & -s^3 + s^2 - 2 \\ 1.5s + 1 & s + 1 & -1.5s - 2 \\ s^3 - 9s^2 - s + 1 & -s^2 + 1 & s^3 - s - 2 \end{bmatrix} \qquad 3.24$$

Step One

The normal Smith-MacMillan form is

$$\phi Z = G \Lambda H \qquad 3.25$$

Where

$$G = \begin{bmatrix} 1 & 0 & 0 \\ s + 1 & 1 & 0 \\ -s^2 + 1 & 2s^2 - 10s - 2 & 1 \end{bmatrix} \qquad 3.26$$

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -2s^6 + 9s^5 + 5s^4 - 9s^3 \end{bmatrix} \qquad 3.27$$

$$H = \begin{bmatrix} s^3 - s^2 + 1 & 1 & -s^3 + s^2 - 2 \\ -s^3 + s + .5 & 0 & s^3 - s + .5 \\ -1 & 0 & 1 \end{bmatrix} \qquad 3.28$$

Step Two

The reduced Smith-MacMillan form Λ' is weakly equivalent to Z and in fact $Z = G(\Lambda')\,H \bmod \psi$, where: G and H are as above. The matrix Λ' is given by

$$\Lambda' = \frac{1}{s^4} \begin{vmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -9s^3 \end{vmatrix} \qquad 3.29$$

Step Three

A minimal representation of Λ' is given by

$$\hat{A} = \left[\begin{array}{cccc|ccc|c} 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & & 0 & & 0 \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ \hline & & & & 0 & 1 & 0 & \\ & 0 & & & 0 & 0 & 1 & \\ & & & & 0 & 0 & 0 & \\ \hline & 0 & & & & 0 & & 0 \end{array}\right] \qquad 3.30$$

$$\hat{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \hat{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 \end{bmatrix}$$

Step Four

A minimal realization of Z(s) is given by

$$A = \hat{A}$$

$$B = \sum_j \hat{A}^j \hat{B} H_j \qquad 3.31$$

$$C = \sum_i G_i \hat{C} \hat{A}^i$$

where

$$G = \Sigma G_i S^i$$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -10 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix} s^2 \qquad 3.32$$

and $H = \Sigma H_i S^i$

$$H = \begin{bmatrix} 1 & 1 & -2 \\ .5 & 0 & .5 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} s + \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} s^3$$

3.33

$$B = \hat{B}[H_o] + \hat{A}\hat{B}[H_1] + \hat{A}^2\hat{B}[H_2] + \hat{A}^3\hat{B}[H_3] \qquad 3.34$$

$$C = [G_o]\hat{C} + [G_1]\hat{C}\hat{A} + [G_2]\hat{C}\hat{A}^2 \qquad 3.35$$

$$\therefore B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \\ .5 & 0 & .5 \\ -1 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -2 & -10 & 2 & -9 \end{bmatrix}$$

Example III-2 Let

$$Z(s) = \frac{1}{s(s+1)(s+2)} \begin{bmatrix} s+1 & 2s^2+s-1 & s^2-1 \\ -s^2-s & -s^2+s & s \end{bmatrix}$$

Step One

The normal Smith-MacMillan form is

$\psi Z = G \Lambda H$

where

$$G = \begin{bmatrix} -s^2+1 & s \\ -s & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s^4+s^3 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} s^3+s^2+s+1 & s^3+s^2+s-1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Step Two

The reduced Smith-MacMillan form Λ' is weakly equivalent to $Z(s)$ and in fact

$Z = G \Lambda' H \pmod{\psi}$

where

i) G and H are as above

ii)

$$\Lambda' = \begin{bmatrix} \frac{1}{s(s+1)(s+2)} & 0 & 0 \\ 0 & \frac{4}{s+2} & 0 \end{bmatrix}$$

Step Three

A minimal representation of Λ' is given by

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad \hat{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \hat{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Step Four

A minimal realization of $Z(s)$ is given by

$$A = \hat{A}$$

$$B = \sum_i \hat{A}^i \hat{B} H_i$$

$$C = \sum_j G_j \hat{C} \hat{A}^j$$

where $G = \sum G_i s^i$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} s + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} s^2$$

and

$$H = \Sigma H_i S^i$$

$$H = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^3$$

$$B = \hat{B}[H_o] + \hat{A}\hat{B}[H_1] + \hat{A}^2 \hat{B}[H_2] + \hat{A}^3 \hat{B}[H_3]$$

$$C = [G_o]\hat{C} + [G_1]\hat{C}\hat{A} + [G_2]\hat{C}\hat{A}^2$$

which gives

$$B = \begin{bmatrix} -2 & -2 & 0 \\ 5 & 5 & 0 \\ -10 & -12 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & -1 & -8 \\ 0 & -1 & 0 & 4 \end{bmatrix}$$

III-6 COMMENTS

A few comments may now be made on this algorithm. It should be noted that the matrix Λ' defined in equation 3.11 is a proper diagonal matrix with certain divisibility properties. In fact, the algorithm does not really require that the diagonal matrix be the reduced Smith-MacMillan form. The diagonal matrix can be the proper part of any diagonal matrix D which is strongly equivalent to Z(s). However, only the Smith-MacMillan form, as defined in equation 3.8 contains the invariant subspace of the system. As was noted before, the invariant factors $\psi_1, \psi_2, \cdots \psi_R$ found in equation 3.8 completely define the structure of the minimal state space representation. The matrix B (input) and the matrix C (output) could then be found by solving $Z(s)=C(sI-A)^{-1}B$. However, it would be very wise to use the information stored in the polynomials $\epsilon_1, \epsilon_2, \cdots \epsilon_R$ of equation 3.8, rather than attempt to solve a set of algebraic non-linear equations. The concept of external equivalence gives us a chance to make use of these polynomials. This is where the power of this algorithm is really shown. The steps are clear to follow and all the elements of matrices B and C are found by analytical means.

The big disadvantage, however, is that getting the Smith-MacMillan normal form or any other diagonal form is by no means an easy task. Also, deriving the equivalence matrices G(s) and H(s) is even more cumbersome. Unless a computer program is written or already exists (which could not be found), this method becomes impractical for cases where Z(s) is of any dimension larger than three or four.

Also it should be observed that Kalman has derived an algorithm [5] which uses the Smith-MacMillan form of Z(s). However, he derives a Jordan form rather than a companion form realization. In another paper [14] Kalman has described a realization which is in the companion form and produces the same result as the algorithm of this chapter.

The minimal realization of Z(s), where Z(s) is defined in eq. 3.7 is obtained by rewriting this equation as

$$Z=\sum_{i=1}^{r} G^i \epsilon_i/\psi_i H^i + \sum_{i=r+1}^{R} G^i \epsilon_i H^i \qquad 3.36$$

and finding the minimal realization for each term in the summation.

The algorithm is based on the following equation

$$\psi(s)(sI-A)^{-1} = \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-1} \end{bmatrix} \begin{bmatrix} P_o(s) & \cdots & P_{n-1}(s) \end{bmatrix} \bmod \psi$$

where: $P_k(s) = x^{n-k-1} + a_{n-1}x^{n-k-2} + \ldots + a_{k+1}$

$k=0,1,\ldots,n-1$

and the matrices B and C, of the minimal realization are found from

$$C \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-1} \end{bmatrix} = G(s)\epsilon(s) \bmod \psi \qquad \begin{bmatrix} P_o(s) & \cdots & P_{n-1}(s) \end{bmatrix} B = H(s) \bmod \psi$$

In fact it can easily be shown that the algorithm of this chapter satisfies the above equations. Hence it produces the same result although it requires less computations.

CHAPTER IV

IRREDUCIBLE REALIZATION USING PARTIAL FRACTION EXPANSION.

In this chapter we attempt to derive the minimal state space representation of $Z(s)$, the transfer function matrix, by expanding it in partial fraction where the denominator of each fraction represents a root of ψ, the least common denominator of each element of $Z(s)$. As in the previous chapter before introducing the algorithm itself we must give certain mathematical definitions.

IV-1 MATHEMATICAL BACKGROUND

Definition IV-1 Given a $q \times p$ proper rational transfer function matrix defined as follows:

$$Z(s) = \frac{1}{d(s)} G(s) \tag{4.1}$$

where: $d(s) = \psi(s)$ = least common denominator of each element of $Z(s)$

$G(s)$ = polynomial matrix

then its partial fraction expansion is given as follows:

$$Z(s) = \frac{G_1(s)}{(s+\lambda_1)^{n_1}} + \frac{G_2(s)}{(s+\lambda_2)^{n_2}} + \dots + \frac{G_\ell(s)}{(s+\lambda_\ell)^{n_\ell}} \tag{4.2}$$

where λ_i $i = 1,2,\dots \ell$ represent the repeated roots of $d(s)$.

Each $G_i(s)$ for $i = 1,2,\dots,\ell$ can be further expanded in fraction as follows:

$$\frac{G_i(s)}{(s+\lambda_i)^{n_i}} = \frac{M_{n_i}}{(s+\lambda_i)^{n_i}} + \frac{M_{n_i-1}}{(s+\lambda_i)^{n_i-1}} + \dots + \frac{M_{i1}}{(s+\lambda_i)} \tag{4.3}$$

Theorem IV-1 Given a set of linearly independent vectors $\{\underline{Y}_1, \underline{Y}_2, \dots, \underline{Y}_\ell\} \in R^p$ and a similar set of linearly independent vectors $\{\underline{\beta}_1, \underline{\beta}_2, \ldots, \underline{\beta}_\ell\} \in R^m$ then

the matrix $$H^j = \sum_{i=1}^{j} \underline{\gamma}_i \underline{\beta}_i^T \qquad j = 1, 2, \ldots, \ell \qquad \ell \leq \min(p, m)$$

is of rank j.

Proof:

We start by considering $H^j \underline{x} = o$ **4.5**

as a set of homogeneous equations. By computing the number of independent solutions, (that is the nullity) we have, rank = number of unknowns - number of independent solutions .

Now

$$H^j = \begin{bmatrix} \sum_{k=1}^{j} \gamma_{1k} \beta_{k1} & \sum_{k=1}^{j} \gamma_{1k} \beta_{k2} \cdots\cdots & \sum_{k=1}^{j} \gamma_{1k} \beta_{km} \\ \sum_{k=1}^{j} \gamma_{2k} \beta_{k1} & \sum_{k=1}^{j} \gamma_{2k} \beta_{k2} \cdots\cdots & \sum_{k=1}^{j} \gamma_{2k} \beta_{km} \\ \vdots & \vdots & \vdots \\ \sum_{k=1}^{j} \gamma_{pk} \beta_{k1} & \sum_{k=1}^{j} \gamma_{pk} \beta_{k2} \cdots\cdots & \sum_{k=1}^{j} \gamma_{pk} \beta_{km} \end{bmatrix} \qquad 4.6$$

Now if we consider $H^j \underline{x} = 0$ we have the following set of p equations with m unknowns

$$\begin{matrix} \sum_{s=1}^{m} \left(\sum_{k=1}^{j} \gamma_{1k} \beta_{ks} \right) x_s = o & \qquad \sum_{k=1}^{j} \gamma_{1k} \left(\sum_{s=1}^{m} \beta_{ks} x_s \right) = o \\ \vdots & \vdots \\ \sum_{s=1}^{m} \left(\sum_{k=1}^{j} \gamma_{pk} \beta_{ks} \right) x_s = o & \qquad \sum_{s=1}^{j} \gamma_{pk} \left(\sum_{s=1}^{m} \beta_{ks} x_s \right) = o \end{matrix} \qquad 4.7$$

Now since $\sum_{s=1}^{m} \beta_{ks} x_s$ is only a function of "k" we can replace it by Y_k . Thus we have now p equations in j unknowns ($j \leq p$) of the form

$$\sum_{k=1}^{j} \gamma_{1k} \, y_k = o \quad \vdots \quad \sum_{k=1}^{j} \gamma_{pk} \, y_k = o \qquad 4.8$$

And since the $\underline{y}_i$ vectors are linearly independent the matrix is of rank j there exists only the trivial solution

$$y_k = o \qquad \forall_k$$

Introducing this in our previous equations we now obtain j equations in m unknowns of the form

$$\sum_{s=1}^{m} \beta_{1s} \, x_s = o \quad \vdots \quad \sum_{s=1}^{m} \beta_{js} \, x_s = o \qquad 4.9$$

And since the $\underline{\beta}_i$ vectors are linearly independent, the matrix is of rank j, thus there are m-j independent solutions. Thus $H^j \underline{x} = o$ which is a set of p equations in m unknowns, has m-j independent solutions.

Hence rank $[H^j] = m - (m-j) = j$

Q.E.D.

IV-2 ALGORITHM TO FIND THE MINIMAL ORDER OF REALIZATION OF Z(s) [13].

An efficient method is now given to compute the number (n) which is the minimal order of the realization of Z(s).

Let Z(s) be a proper rational matrix then we write

$$Z(s) = \frac{G(s)}{d(s)} = \frac{N_o + N_1 s + \ldots\ldots + N_{r-1} s^{r-1}}{a_o + a_1 s + \ldots\ldots + s^r} \qquad 4.10$$

where:

N_o, N_1, ..., N_{r-1} are real q x p matrices.

Generate matrices

$$Q_{ij} \qquad 1 \le i, j, \le r$$

by the following algorithm

$$Q_{1o} = o \qquad Q_{1j} = N_{j-1} \qquad j = 1,2,\ldots,r \tag{4.11}$$

$$Q_{io} = o \quad Q_{ij} = Q_{i-1,j-1} - a_{j-1} Q_{i-1,r} \qquad i = 2,\ldots,r \quad j = 1,2,\ldots,r \tag{4.12}$$

Let R be the $q \cdot r \times p \cdot r$ block matrix formed by the Qij such that

$$R = \begin{bmatrix} Q_{11} & Q_{12} \cdots\cdots & Q_{1r} \\ Q_{21} & Q_{22} \cdots\cdots & Q_{2r} \\ Q_{r1} & Q_{r2} \cdots\cdots & Q_{rr} \end{bmatrix} \tag{4.13}$$

Then the minimal order n is the rank of R.

For proof see Rosenbrock [13].

An alternative procedure given by B. Ho and R. E. Kalman [20], is to generate matrices $Y_o, Y_1, \ldots, Y_{2r-2}$,

where:

$$G(s) = \frac{Y_o}{s} + \frac{Y_1}{s} + \ldots . \tag{4.14}$$

The minimal order n is the rank of S

where :

$$S = \begin{bmatrix} Y_o & Y_1 \cdots\cdots & Y_{r-1} \\ Y_1 & Y_2 \cdots\cdots & Y_r \\ \vdots & \vdots & \vdots \\ Y_{r-1} & Y_r \cdots\cdots & Y_{2r-2} \end{bmatrix} \tag{4.15}$$

For proof see [20].

Note: It should be observed that equation 4.3 is a partial expansion of a proper rational matrix with only one repeated eigenvalue (λ). The value of the matrices Y_o, $Y_1, \ldots, Y_{2r-2}$ will vary depending on the multiplicity of the eigenvalue. In fact these matrices $Y_o, Y_1, \ldots Y_{2r-2}$ can be generated from the matrices $M_{n_i}, M_{n_i-1}, \ldots, M_{i1}$ of equation 4.3 using the following expansion

$$\frac{M_1}{s+\lambda} = \frac{M_1}{s} - \frac{M_1\lambda}{s^2} + \frac{M_1\lambda^2}{s^3} - \frac{M_1\lambda^3}{s^4} + \frac{M_1\lambda^4}{s^5} - \frac{M_1\lambda^5}{s^6} + \frac{M_1\lambda^6}{s^7} \cdots$$

$$\frac{M_2}{s^2+2\lambda s+\lambda^2} = \frac{M_2}{s^2} - \frac{2M_2\lambda}{s^3} + \frac{3M_2\lambda^2}{s^4} - \frac{4M_2\lambda^3}{s^5} + \frac{5M_2\lambda^4}{s^6} - \frac{6M_2\lambda^5}{s^7} \cdots$$

$$\frac{M_3}{s^3+3s^2\lambda+3s\lambda^2+\lambda^3} = \frac{M_3}{s^3} - \frac{3M_3\lambda}{s^4} + \frac{6M_3\lambda^2}{s^5} - \frac{10M_3\lambda^3}{s^6} + \frac{15M_3\lambda^4}{s^7} \cdots$$

$$\frac{M_4}{s^4+4s^3\lambda+6s^2\lambda^2+4s\lambda^3+\lambda^4} = \frac{M_4}{s^4} - \frac{4M_4\lambda}{s^5} + \frac{10M_4\lambda^2}{s^6} - \frac{20M_4\lambda^3}{s^7} \cdots$$

$$\frac{M_5}{s^5+5s^4\lambda+10s^3\lambda^2+10s^2\lambda^3+5s\lambda^4+\lambda^5} = \frac{M_5}{s^5} - \frac{5\lambda M_5}{s^6} + \frac{15M_5\lambda^2}{s^7} \cdots$$

$$\frac{M_6}{s^6+6s^5\lambda+15s^4\lambda^2+20s^3\lambda^3+15s^2\lambda^4+6s\lambda^5+\lambda^6} = \frac{M_6}{s^6} - \frac{6M_6\lambda}{s^7} \cdots$$

If we note the binomial expansion

$$(1-\lambda)^n = 1 - n\lambda + \frac{(n)(n-1)}{2!}\lambda^2 - \frac{(n)(n-1)(n-2)}{3!}\lambda^3 + \frac{(n)(n-1)(n-2)(n-3)\lambda^4}{4!}$$

We see how any column of the above expansion can be generated. Also it is clear to see how this expansion can be used to find the matrices $Y_o, Y_1, \ldots, Y_{2r-2}$. To show this consider the case where the eigenvalue (λ) is repeated 4 times. Equation 4.3 is then of the form

$$\frac{G(s)}{(s+\lambda)^4} = \frac{M_4}{(s+\lambda)^4} + \frac{M_3}{(s+\lambda)^3} + \frac{M_2}{(s+\lambda)^2} + \frac{M_1}{(s+\lambda)}$$

then $r = 4$ and

$$Y_0 = M_1$$

$$Y_1 = M_2 - \lambda M_1$$

$$Y_2 = M_3 - 2 M_2 \lambda + \lambda^2 M_1$$

$$Y_3 = M_4 - 3 \lambda M_3 + 3 \lambda^2 M_2 - \lambda^3 M_1$$

$$Y_4 = -4\lambda M_4 + 6 \lambda^2 M_3 - 4 \lambda^3 M_2 + \lambda^4 M_1$$

$$Y_5 = 10 \lambda^2 M_4 - 10 \lambda^3 M_3 + 5 \lambda^4 M_2 - \lambda^5 M_1$$

$$Y_6 = -20 \lambda^3 M_4 + 15 \lambda^4 M_3 - 6 \lambda^5 M_2 + \lambda^6 M_1$$

We now have two different methods of finding the minimal order (n) of a realization of z(s).

IV-III ON THE GENERAL STRUCTURE OF THE MINIMAL SYSTEM

Before proceeding to the algorithm itself let us examine its general lines. First Z(s) is expanded in a partial fraction as given in equation 4.2. Then the system is further expanded in partial fraction to the form of equation 4.3. Now for each subsystem in the form of equation 4.3 we find a minimal realization. The realization of Z(s) is then the sum of all the realizations.

So our main concern is to find the realization of a transfer function matrix of the form

$$\frac{G(s)}{(s+\lambda)^k} = \frac{M_k}{(s+\lambda)^k} + \frac{M_{k-1}}{(s+\lambda)^{k-1}} + \dots + \frac{M_1}{(s+\lambda)} \qquad 4.20$$

The system matrix (A) which is found will be in the Jordan Canonical Form.

To find its structure we need

i) the number of Jordan blocks

ii) its dimension n (the minimal order)

Both these pieces of information can be found easily

1. The minimal order (n) can be found using either Rosenbrock or the altered B. Ho and R. E. Kalman algorithm.

2. The number of Jordan blocks is found using the following theorem.

Theorem IV-2: The rank R of the transfer function matrix G(s) is equal to the number of Jordan blocks.

Proof :

In the previous chapter where the Smith-MacMillan form was formed it was found that the rank R of a polynomial matrix was also exactly equal to the number of invariant (cyclic) subspaces of the matrix. Now since the invariant subspaces of C(s) are the same as those of the system and that A is in the Jordan Canonical form then A must have R Jordan blocks (or invariant subspaces).

Q. E. D.

Theorem IV- 3: The rank r of the matrix M_k of equation 4.20 is exactly equal to the number of k^{th} order Jordan blocks.

Proof:

Let us examine $[sI - A]^{-1}$ for a k^{th} order Jordan blocks

$$\underset{k\times k}{A} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 \cdots & 0 \\ \vdots & & & & \\ \vdots & & & & 0 \\ \vdots & & & & 1 \\ 0 & 0 & \cdots & & \lambda \end{bmatrix} \qquad 4.21$$

It can be shown that

$$[sI-A]^{-1} = \begin{bmatrix} \frac{1}{s+\lambda} & \frac{1}{(s+\lambda)^2} & \frac{1}{(s+\lambda)^3} & \cdots & \frac{1}{(s+\lambda)^k} \\ 0 & \frac{1}{s+\lambda} & \frac{1}{(s+\lambda)^2} & \cdots & \frac{1}{(s+\lambda)^{k-1}} \\ \vdots & 0 & \frac{1}{s+\lambda} & & \vdots \\ \vdots & \vdots & 0 & & \vdots \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \frac{1}{s+\lambda} \end{bmatrix} \qquad 4.22$$

but $\frac{G(s)}{(s+\lambda)^k} = C\,[sI-A]^{-1}\,B$

if matrix C is partitioned by columns

and matrix B is partitioned by rows we get

$$[\,\underline{c}_1 \;\; \underline{c}_2 \;\; \cdots \;\; \underline{c}_k\,]\;[sI-A]^{-1} \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_k \end{bmatrix}$$

$$\frac{G(s)}{(s+\lambda)^k} = \frac{M_1}{(s+\lambda)} + \frac{M_2}{(s+\lambda)^2} + \cdots + \frac{M_{k-1}}{(s+\lambda)^{k-1}} + \frac{\underline{c}_1\underline{b}_k}{(s+\lambda)^k} \qquad 4.23$$

now say you have r such k^{th} order blocks, then G(s) is given by

$$\frac{G(s)}{(s+\lambda)^k} = \frac{M_1}{s+\lambda} + \frac{M_2}{(s+\lambda)^2} + \dots + \frac{(\underline{c}_1 \underline{b}_k + \underline{c}_{k+1} \underline{b}_{2k} + \dots + \underline{c}_{(r-1)k+1} \underline{b}_{rk})}{(s+\lambda)^k} \quad 4.24$$

but from Theorem II-5 we know that for the system to be completely controllable and observable the following set of vectors

$$\{ \underline{c}_1, \underline{c}_{k+1}, \dots, \underline{c}_{(r-1)k+1} \}$$

and

$$\{ \underline{b}_k, \underline{b}_{2k}, \dots, \underline{b}_{rk} \}$$

must be linearly independent. Then from Theorem III-1 the rank of M_k is r.

Q.E.D.

The rank of the matrix M_k gives you exactly the number of highest order Jordan blocks.

With all this information we may now proceed to the algorithm itself.

IV-IV MINIMAL REALIZATIONS OF Z(s)

a) SYSTEM WHERE d(s) HAS DISTINCT ROOTS.

In this case the partial fraction expansion for Z(s) is of the form

$$Z(s) = \frac{G(s)}{d(s)} = \frac{M_1}{(s+\lambda_1)} + \frac{M_2}{(s+\lambda_2)} + \dots + \frac{M_\ell}{(s+\lambda_\ell)} \quad 4.25$$

where: M_i $i = 1, 2, \ell$ are constant matrices

denote $\rho(M_i) = n_i$ as the rank of the matrix M_i and let $\underline{b}_{1i}, \underline{b}_{2i}, \dots, \underline{b}_{n_i,i}$ be linearly independent vectors generating all the rows of M_i.

ALGORITHM

1) If d(s) has distinct roots find the partial fraction of $Z(s)$ as in equation 4.25.

1) Find the rank n_i of each matrix

$$\rho(M_i)=n_i \qquad i = 1,2,\ldots,\ell$$

then minimal order of the realization is

$$n = \sum_{i=1}^{\ell} \rho\ (M_i) = \sum_{i=1}^{\ell} n_i \qquad 4.26$$

and the representation is given by

$$\dot{\underline{x}} = \begin{bmatrix} A_1 & 0 & \cdots\cdots & 0 \\ 0 & A_2 & \cdots\cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots\cdots & A_\ell \end{bmatrix} \underline{x} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_\ell \end{bmatrix} \underline{u} \qquad 4.27$$

$$\underline{y} = [\, C_1 \ \ \cdots\cdots \ \ C_\ell \,] \ \underline{x}$$

where:

$$\underset{n_i x n_i}{A_i} = \begin{bmatrix} \lambda_i & 0 \cdots\cdots & 0 \\ 0 & \lambda_i \cdots\cdots & 0 \\ \vdots & & \vdots \\ \vdots & & 0 \\ 0 & & \lambda_i \end{bmatrix} \qquad \underset{n_i \ x \ p}{B_i} = \begin{bmatrix} \underline{b}_{1i} \\ \underline{b}_{2i} \\ \vdots \\ \underline{b}_{n_i,i} \end{bmatrix} \qquad 4.28$$

$$\underset{qxn_i}{C_i} = [\, \underline{c}_{1i}, \ \cdots \ \underline{c}_{n_i,i} \,]$$

such that;

$$M_i = \sum_{j=1}^{n_i} \underline{c}_{ji} \, \underline{b}_{ji} \qquad 4.29$$

This realization is minimal since it is both controllable and observable and does indeed have $Z(s)$ as its transfer function matrix.

It is controllable and observable for the following reason. Take any term $\frac{M_i}{(s+\lambda_i)}$ if $\rho(M_i) = n_i$ then by Theorem IV-1 there must be n_i linearly independent rows $\{\underline{b}_{1i}, \underline{b}_{2i}, \ldots, \underline{b}_{n_i,i}\}$ and n_i linearly independent columns $\{\underline{c}_{1i}, \ldots, \underline{c}_{n_i,i}\}$ such that

$$M_i = \sum_{j=1}^{n_i} \underline{c}_{ji} \, \underline{b}_{ji}$$

Now by Theorem II-5 of Chapter II we know that for n_i Jordan blocks with the same eigenvalue, the conditions for controllability and observability is that the set of vectors $\{\underline{b}_{1i}, \ldots \underline{b}_{n_i,i}\}$ and $\{\underline{c}_{1i}, \ldots \underline{c}_{n_i,i}\}$ be linearly independent. Our realization is therefore controllable and observable and also minimal.

b) SYSTEMS WHERE $d(s)$ HAS REPEATED ROOTS.

Such a system has a partial fraction expansion of the form of equation 4.2. However, as explained before the realization of $Z(s)$ is the sum of the minimal realization of transfer function matrices of the form

$$\frac{G(s)}{(s+\lambda)^k} = \frac{M_k}{(s+\lambda)^k} + \ldots\ldots\ldots + \frac{M_1}{(s+\lambda)} \qquad 4.30$$

ALGORITHM TO REALIZE TRANSFER MATRIX OF EQUATION 4.30

1) Apply Rosenbrock or B. Ho and R.E Kalman algorithm to find the minimal order of the system, denote it by n.

2) Find the rank of $G(s)$ to obtain the number of Jordan blocks

the system matrix A will have.

3) From the rank of M_k find the number of highest order Jordan blocks that will be needed.

From this information one is able to write the Jordan canonical form of the system A matrix. The rest of the procedure is constructive in nature.

4) From M_k one can derive the linearly independent rows $\{\underline{b}_{1i}, \ldots, \underline{b}_{ri}\}$ of B and columns $\{\underline{c}_{1i}, \ldots, \underline{c}_{ri}\}$ of C, corresponding to the last row and first row respectively of each Jordan block of k^{th} order.

5) The rest of the terms in B and C can be found in two ways.

i) Solving the set of equation generated by $G(s) = C[sI - A]^{-1}B$ where we now know G(s) and A and some rows of B and columns of C.

ii) By construction, using the following steps:

a) As the structure of A is known, block diagram representation of the system is of the form

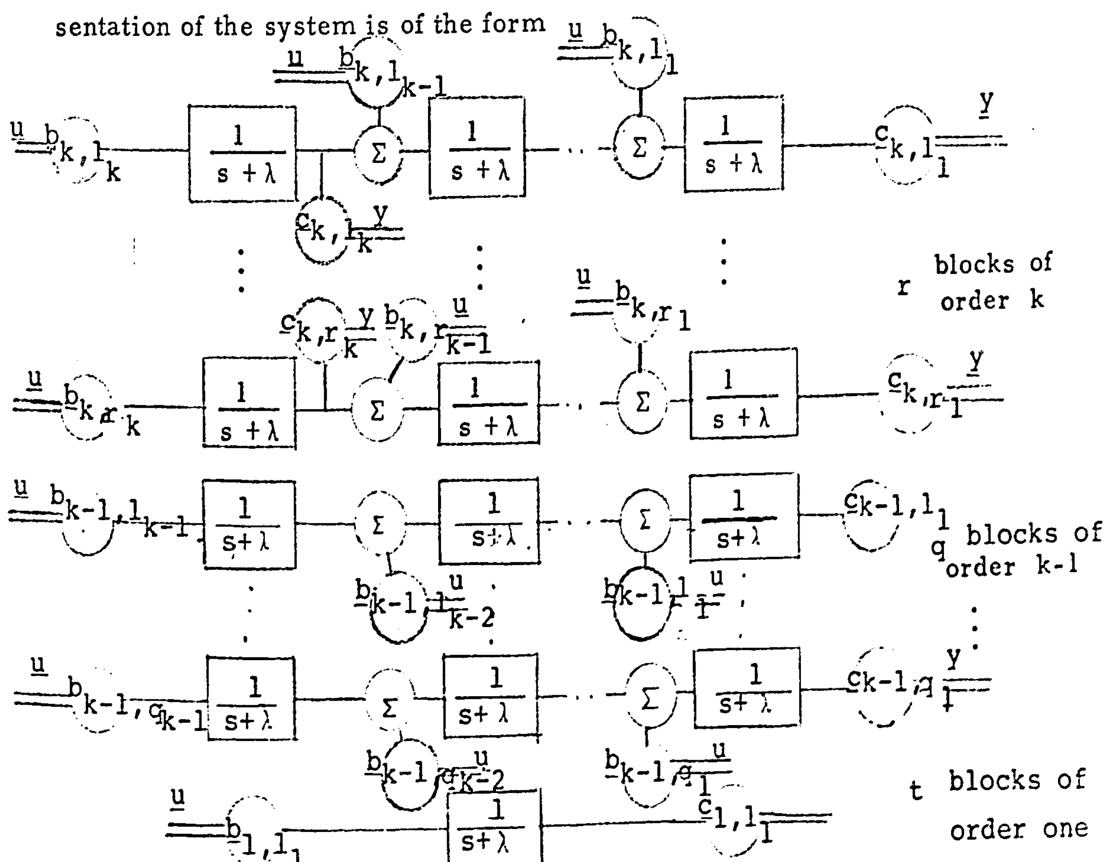

FIGURE IV-1 SIMULATION DIAGRAM OF MINIMAL REALIZATION.

NOTATION

$\underline{b}_{i,j_k}$

i = order of block

j = block number of order i

k = which state is referred to in block j of order i

b) From M_k one can generate the linearly independent set of vectors $\{\underline{b}_{k,1_k}\ \underline{b}_{k,2_k} \dots\ \underline{b}_{k,r_k}\}$, $\{\underline{c}_{k,1_1}, \underline{c}_{k,2_1}, \dots \underline{c}_{k,r_1}\}$.

c) Now from M_{k-1} and the knowledge of the above set of vectors you can generate the following set of vectors $\{\underline{b}_{k-1,1_{k-1}}, \dots \underline{b}_{k-1,p_{k-1}}\}$

$\{c_{k-1,1_1}, \dots c_{k-1,p_1}\}$ $\{\underline{b}_{k,1_{k-1}}, \dots, \underline{b}_{k,r_{k-1}}\}$ $\{\underline{c}_{k,1_2}, \dots, \underline{c}_{k,r_2}\}$

etc... .

This algorithm will now be demonstrated by using the two examples presented in the previous chapter.

Example IV-1 Let

$$Z(s) = \frac{1}{s(s+1)(s+2)} \begin{vmatrix} s+1 & 2s^2+s-1 & s^2-1 \\ -s^2-s & -s^2+s & s \end{vmatrix}$$

Step One

$$Z(s) = \frac{1}{s}\begin{bmatrix} .5 & -.5 & -.5 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{s+1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} -.5 & 2.5 & 1.5 \\ -1 & -3 & -1 \end{bmatrix}$$

since d(s) has distinct roots .

Step Two

$$\rho(M_1) = 1$$

$$\rho(M_2) = 1$$

$$\rho(M_3) = 2$$

thus minimal order is 4.

A minimal representation is thus given by

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} .5 & -.5 & -.5 \\ 0 & 2 & 1 \\ -.5 & 2.5 & 1.5 \\ -1 & -3 & -1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \underline{x}$$

Note: The rows of B and columns of C are derived from the matrices M_1 M_2 and M_3 using equation 4.29

Example IV-2 Let

$$Z(s) = \frac{1}{s^4} \begin{bmatrix} s^3 - s^2 + 1 & 1 & -s^3 + s^2 - 2 \\ 1.5s + 1 & s + 1 & -1.5s - 2 \\ s^3 - 9s^2 - s + 1 & -s^2 + 1 & s^3 - s - 2 \end{bmatrix}$$

Step One

The minimal order n is found using Rosenbrock Algorithm:

$$Z(s) = \frac{1}{s^4}\left[\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1.5 & 1 & -1.5 \\ -1 & 0 & -1 \end{bmatrix} s + \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -9 & -1 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} s^3 \right]$$

The matrix R formed by equations 4.11 and 4.12 is

$$R = \begin{bmatrix} 1 & 1 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 1 & -2 & 1.5 & 1 & -1.5 & -1.5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & -1 & 0 & -1 & -1 & -9 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 & -2 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 & -2 & 1.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & -2 & -1 & 0 & -9 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1.5 & 1 & -1.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix}.$$

Now it can be shown that $\rho(R) = 8$, therefore the minimal order of this system is 8.

Step Two

The rank of $G(s)$ is three, since the determinant of $G(s) \neq 0$, therefore by Theorem III-3 there are three Jordan blocks.

Step Three

The partial fraction expansion of $Z(s)$ is

$$Z(s) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \frac{1}{s} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -9 & -1 & 0 \end{bmatrix} \frac{1}{s^2} + \begin{bmatrix} 0 & 0 & 0 \\ 1.5 & 1 & -1.5 \\ -1 & 0 & -1 \end{bmatrix} \frac{1}{s^3} + \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix} \frac{1}{s^4}$$

$$\rho(M_4) = 1$$

therefore there is only one Jordan block of order 4.

Now $\rho(M_3 + M_4) = 2$. This means that there must be one block of order 3. The reason for this is two-fold.

1) If $\rho(M_3 + M_4) = 2$ it means there must be two columns $\{\underline{c}_1, \underline{c}_2\}$ and two rows $\{\underline{b}_1, \underline{b}_2\}$ which are linearly independent such that

$$\sum_{i=1}^{2} \underline{c}_i \underline{b}_i = M_3 + M_4 \qquad 4.31$$

But by Theorem II-5 we know that in order that a system in Jordan Canonical form be controllable (observable) certain rows (columns) of B (C) must be linearly independent. If the row of equation 4.31 are linearly independent it means that there must be two blocks one of which is of order 4 while the other is of order 3.

We now have completely determined the A matrix that is the structure of the system

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the block diagram representation of the system is of the form

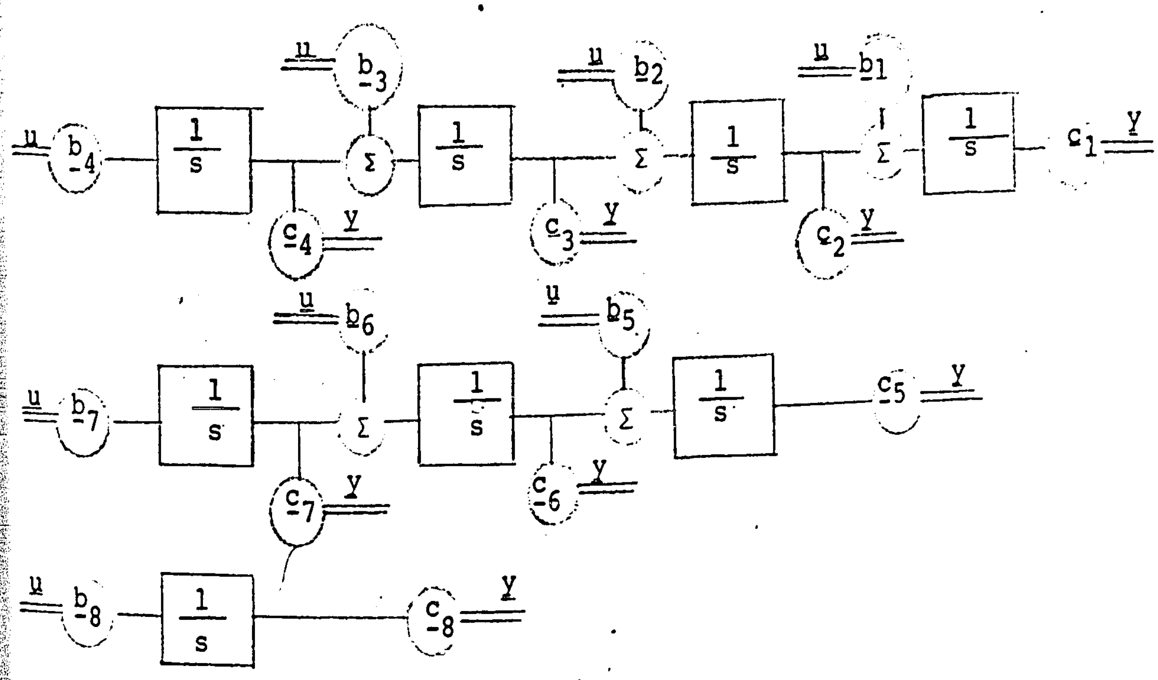

FIGURE IV-2 BLOCK DIAGRAM FOR EXAMPLE IV-2.

Since $$M_4 = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\underline{c}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{b}_4 = [1 \quad 1 \quad -2]$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1.5 & 1 & -1.5 \\ -1 & 0 & -1 \end{bmatrix} = \underline{c}_5\,\underline{b}_7 + \underline{c}_2\,\underline{b}_4 + \underline{c}_1\,\underline{b}_3$$

If we set $\underline{b}_3 = [0 \; 0 \; 0]$ then

$$\underline{c}_5 = \begin{bmatrix} 0 \\ -.5 \\ 1 \end{bmatrix} \quad \underline{c}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{b}_7 = [1 \quad 0 \quad 1]$$

$$M_2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -9 & -1 & 0 \end{bmatrix} = \underline{c}_3\,\underline{b}_4 + \underline{c}_1\,\underline{b}_2 + \underline{c}_6\,\underline{b}_7 + \underline{c}_5\,\underline{b}_6$$

$$= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [1 \quad 1 \quad -2] + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [-1 \; 0 \; 1] + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} [-1 \; 0 \; -1] + \begin{bmatrix} 0 \\ -.5 \\ 1 \end{bmatrix} [-2 \; 0 \; 2]$$

Finally from

$$M_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \underline{c}_1\underline{b}_1 + \underline{c}_2\,\underline{b}_2 + \underline{c}_4\,\underline{b}_4 + \underline{c}_5\underline{b}_5 + \underline{c}_6\underline{b}_6\,\underline{c}_7\underline{b}_7 + \underline{c}_8\,\underline{b}_8 \quad \ldots 4.15$$

We get all the remaining row of the matrix B and columns of the matrix C.

To check that the realization is indeed controllable and observable we know, from Theorem II-5 that $\{\underline{c}_1, \underline{c}_5, \underline{c}_3\}$ and $\{\underline{b}_4, \underline{b}_7, \underline{b}_8\}$ must be linearly independent. The above derived vector satisfy the condition.

Thus a minimal realization of this system is as follows:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \\ -1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \underline{v}$$

$$\underline{c} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -.5 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 5 & -1 & -10 \end{bmatrix} \underline{x}$$

IV-5 COMMENTS

This algorithm has just the opposite qualities of the previous one. Mathematically, it does not look as elegant, but for most cases it will yield the proper result and require much less computation. It is indeed interesting to note that the amount of computation is not as dependent on the dimension of the transfer matrix as the previous algorithm. The reason is that one works with a single eigenvalue at a time, and a parallel realization is obtained. At no point do we deal with the whole transfer matrix. This point is well illustrated by the first example. Using the approach of the previous chapter requires quite a few computations to find the Smith-MacMillan form of the transfer matrix and the equivalence matrices G(s) and H(s). Even after having obtained this information, there are more computations required before the minimal realization is obtained. In this algorithm, once the partial fraction expansion of the transfer matrix has been obtained, one can immediately derive the minimal realization.

This algorithm does, however, have one shortcoming. It is a result of the fact that only the following information is available:

(1) minimal order n given by Rosenbrock.
(2) number of Jordan blocks.
(3) number of highest order Jordan blocks

which may not be enough to clearly define the structure of the system matrix A, if the multiplicity of an eigenvalue is too high. When these cases occur, a variation of the technique of Panda and Chen [8] will yield proper result. This technique, however, is a two-step procedure in that a controllable, but unobservable, realization is first generated and this realization is then reduced to one which is both controllable and observable.

Consequently, the partial fraction algorithm is practical to use, providing the multiplicity of a given eigenvalue is not too high. For any physical system, this condition will usually be met.

In conclusion, we can say that this algorithm will require less computation than the previous algorithm for most practical cases. It is also more powerful than the technique of Panda and Chen [8] , since it is a one-step procedure. However, in cases where it does not generate a structure for the system, one can use either of the above methods.

CHAPTER V

IRREDUCIBLE REALIZATIONS IN SAMPLED DATA SYSTEMS

In this chapter we wish to demonstrate that the two techniques used to derive an irreducible realization of a transfer matrix, presented earlier, can be extended to sample data systems. Although this is quite easily done it is nevertheless a very important result. Most identification schemes such as ordinary least square fit [17], stochastic approximation [18] maximum likelihood [17], etc. result in a pulse transfer function representation of a system, that is in the z transform $Y(z) = H(z)\ U(z)$ 5.1

where: $H(z)$ is a $q \times p$ matrix for a p - input q - output system.

If modern optimizing techniques, in state space, are to be used one must derive a minimal realization for $H(z)$.

V-1 DEFINITIONS

a) SYSTEM DESCRIPTION

A continuous system is represented in state space by a set of first order differential equations. In sampleddata systems the differential equations become difference equations, of the form

$$\underline{x}(k+1) = A\,\underline{x}(k) + B\,\underline{u}(k)$$
$$\underline{y}(k) = C\,\underline{x}(k) + D\,\underline{u}(k) \qquad 5.2$$

where: $k = k^{th}$ instant of time

$\underline{u} = \underline{u}(k)$, p - dimensional input vector

$\underline{y} = \underline{y}(k)$, q - dimensional output vector

$\underline{x} = \underline{x}(k)$, n - dimensional state vector

n is the order of the system

A,	constant	n x n	system matrix
B,	constant	n x p	input matrix
C,	constant	q x n	output matrix
D,	constant	q x p	transmission matrix

As in the previous chapters we will assume $D = 0$. If z^{-1} is considered as a delay operator and assuming that the system starts at rest, equation 5.2 can be rewritten as

$$z \underline{X}(z) - A \underline{X}(z) = B \underline{U}(z)$$
$$\underline{Y}(z) = C \underline{X}(z) \tag{5.3}$$

therefore $\underline{X}(z) = (zI - A)^{-1} B \underline{U}(z)$ 5.4

using equations 5.3 and 5.4 we obtain

$$\underline{Y}(z) = C(zI - A)^{-1} B \underline{U}(z)$$

thus $H(z) = C(zI - A)^{-1} B$ 5.5

Equation 5-5 is the definition of a pulse transfer function matrix

We are now back at the same problem as in the previous chapter, that is, given a pulse transfer function matrix H(z) find a minimal state space representation which must be

1) controllable and observable
2) whose transfer function matrix is $H(z) = C(zI - A)^{-1} B$

b) CONTROLLABILITY AND OBSERVABILITY

The conditions for controllability and observability must be redefined in terms of sampled data systems. These definitions are exactly the same as given in DEFINITION II-2 and DEFINITION II-4 except that the phrase " Finite time interval" is replaced by "Finite numbers of time intervals" The main criteria for controllability and observability was first introduced by R. E Kalman [6] and reads as follows:

DEFINITION 5 - 1 - A discreete time system defined in equation 5.2 is completely controllable if and only if the following matrix L_c is of rank n

$$L_c = (B, A^{-1} B, \cdot \cdot \cdot, A^{-n+1} B) \qquad 5.6$$

for proof see [6]. Since this definition is similar to theorem II - 1 and that all the remaining theorems in Chapter II were proven using theorem II - 1 we will assume without proof that all the theorems on controllability proven in Chapter II can be extended to sampled data systems. Similar comments may be made for observability.

V - 2 MINIMAL REALIZATION OF A TRANSFER FUNCTION

We will derive a minimal realization for a pulse transfer function of the form

$$H(z) = \frac{a_o + a_1 z + a_2 z^2 + - - + a_m z^m}{b_o + b_1 z + - + z^n} \qquad 5.7$$

From a theorem on sampled data systems [11] regarding physical realizability it is required that $\lim_{z \to \infty} z^{-1} H(z) = 0$ for a system to be realizable For the proof of this theorem see [11]

If this theorem is applied to equation 5.7 it is seen that the order of the denomenator must be larger than or equal to the order of the numerator that is, $n \geq m$. Since we assumed that $D = o$ the largest value of m is $m = n-1$.

THEOREM 5.1 A minimal state space representation of $H(z)$. defined by equation 5.7 where $m = n-1$, is given by

$$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ -b_o & -b_1 & \cdots & \cdots & -b_{n-1} \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \underline{u}(k) \tag{5.8}$$

$$\underline{Y}(k) = (a_o, a_1, \cdots a_{n-1}) \; \underline{x}(k)$$

<u>Proof</u>

$$H(z) = C(zI - A)^{-1}B$$

$$= (a_o \; a_1 \cdots a_{n-1}) \frac{1}{\Delta(zI-A)} \begin{bmatrix} x & x & \cdots & 1 \\ \cdot & & & z \\ \cdot & & & \cdot \\ \vdots & & & \cdot \\ \cdot & & & \cdot \\ x & x & \cdots & z^{n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{a_o + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}}{b_o + b_1 z + \cdots + z^n}$$

This representation is also controllable and observable because of theorem II - 8 which could be revised for sampled data systems and the fact that the last row of B and first column of C are different from zero

V - 3 REALIZATION OF H (z)

Here it will be shown by some examples that the technique of Chapter IV can be used to derive an irreducible realization for H (z)

EXAMPLE V - 1

Let

$$H(z) = \frac{1}{(z+1)(z+2)(z+3)} \begin{bmatrix} z+1 & 1 \\ 4z^2+19z+21 & z^2+2z+1 \end{bmatrix}$$

H (z) can be expanded in a partial fraction

$$H(z) = \frac{1}{(z+1)} \begin{bmatrix} 0 & 1/2 \\ 3 & 0 \end{bmatrix} + \frac{1}{(z+2)} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + \frac{1}{(z+3)} \begin{bmatrix} -1 & 1/2 \\ 0 & 2 \end{bmatrix}$$

the minimal order of the realization will be

$$n = \rho(M_1) + \rho(M_2) + \rho(M_3)$$

$$= 2 + 1 + 2$$

$$= 5$$

a minimal realization of H (z) is

$$\underline{X}(k+1) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \underline{X}(k) + \begin{bmatrix} 0 & 1/2 \\ 3 & 0 \\ 1 & -1 \\ -1 & 1/2 \\ 0 & 2 \end{bmatrix} \underline{u}(k)$$

$$\underline{Y}(k) = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \underline{x}(k)$$

Example V·2 Let

$$H(z) = \frac{1}{(z-1)^3} \begin{bmatrix} 5z^2-7z+3 & 3z^2-5z+2 & -4z^2+6z-3 \\ 3z^2-5z+1 & z^2-3z+2 & -2z+3 \\ 2z^2-2z & 2z^2-4z+2 & -2z+2 \end{bmatrix}$$

H(z) can be expanded in partial fraction as follows

$$H(z) = \frac{1}{(z-1)} \begin{bmatrix} 5 & 3 & -4 \\ 3 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} + \frac{1}{(z-1)^2} \begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & -2 \\ 2 & 0 & -2 \end{bmatrix} + \frac{1}{(z-1)^3} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad 5.10$$

STEP ONE

The minimal order is found using ROSENBROCK algorithm

From

$$H(z) = \frac{\begin{bmatrix} 3 & 2 & -3 \\ 1 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix} + \begin{bmatrix} -7 & -5 & 6 \\ -5 & -3 & -2 \\ -2 & -4 & -2 \end{bmatrix} z + \begin{bmatrix} 5 & 3 & -4 \\ 3 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} z^2}{z^3-3z^2+3z-1}$$

we can generate the matrix R of equation 4.13 using equations 4.11 and 4.12

$$R=\begin{bmatrix} 3 & 2 & -3 & -7 & -5 & 6 & 5 & 3 & -4 \\ 1 & 2 & 3 & -5 & -3 & -2 & 3 & 1 & 0 \\ 0 & 2 & 2 & -2 & -4 & -2 & 2 & 2 & 0 \\ 5 & 3 & -4 & -12 & -7 & 9 & 8 & 4 & -6 \\ 3 & 1 & 0 & -8 & -1 & 3 & 4 & 0 & -2 \\ 2 & 2 & 0 & -6 & -4 & 2 & 4 & 2 & -2 \\ 8 & 4 & -6 & -19 & -9 & 14 & 12 & 5 & -9 \\ 4 & 0 & -2 & -9 & 1 & 6 & 4 & -1 & -3 \\ 4 & 2 & -2 & -10 & -4 & 6 & 6 & 2 & -4 \end{bmatrix}$$

It can be shown that the rank of R is 3.Therefore the minimal order of the system is 3.Since the largest block is of order 3 the structure of the system is now completely defined and all the remaining steps of the may be skipped.A block diagram of the system is as follows

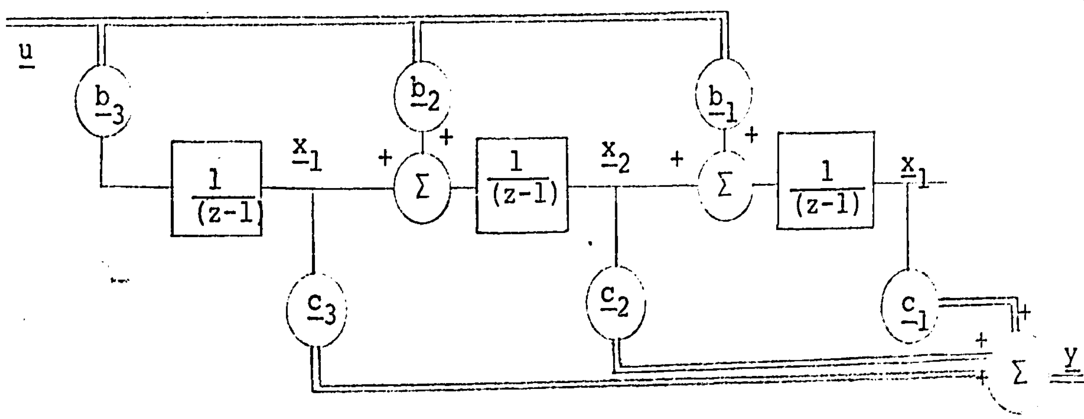

FIGURE V 1 BLOCK DIAGRAM FOR EXAMPLE V 2

The rows of the matrix B and matrix C are found as follows. From equation 5.10and the above figure we get

$$M_3=\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \underline{c}_1\underline{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

$$M_2=\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & -2 \\ 2 & 0 & -2 \end{bmatrix} = \underline{c}_1\underline{b}_2 + \underline{c}_2\underline{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 5 & 3 & -4 \\ 3 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} = \underline{c}_1\underline{b}_1 + \underline{c}_2\underline{b}_2 + \underline{c}_3\underline{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

Thus a minimal realization is

$$\underline{x}(k+1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} u(k)$$

$$\underline{y}(k) = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 1 & 3 \\ 0 & 2 & 2 \end{bmatrix} \underline{x}(k)$$

V - COMMENTS

The important point of this chapter is that given any pulse transfer function matrix we can find an irreducible realization with exactly the same techniques as used in continuous systens.Although this is easily shown it is a point which is not mentioned in current litterature.

Example V-2 also brings out a few interesting points. It is obvious that if the Smith- MacMillan algorithm had been used the number of computations required to derived a minimal realization would be greatly increased. Also if one had used the technique of Panda and Chen [8] a controllable but unobservable realization of 6 states would had first been generated.This realization would then have been reduced to one of 3 states which is controllable and observable.This example points the advantage of the partial fraction algorithm over the other two.

CHAPTER VI

CONCLUSION

So far we have examined linear time-invariant systems which can be continuous or sampled. Two algorithms were given for such systems and their relative worth was described. We may conclude that for time-invariant cases the problem is well solved. However work carried out is time-varying systems may lead to an even simpler algorithm.

VI-1 TIME-VARYING SYSTEMS.

In a recent book by H. D'Angelo [12], an algorithm is given to derive the minimal state space realization for a time-varying system. Obviously the input/output relationship of such a system will be described through an impulse response matrix and not a transfer matrix in polynomials of s. It is not our intention to give the full details of the algorithm but only point out its general ideas and how they may be used in the time-invariant case.

Recall from chapter II that the product of the observability matrix L_o and controllability matrix L_c defined as

$$\Gamma = L_o^T L_c \qquad 6.1$$

was invariant under any non-singular transformation $\underline{x} = T \underline{z}$ and the rank of Γ gives the minimal order of the state space realization. This seems to indicate that Γ possesses all the information regarding the structure of a minimal representation of the system.

In time-varying systems L_o and L_c are defined differently [12] in fact they are,

$$L_c(t) = [\, B(t),\ \Delta_c B(t), \dots,\ \Delta_c^{n-1} B(t) \,] \qquad 6.2$$

where Δ_c is an operator of the form

$$\Delta_c \equiv - A(t) + \frac{d}{dt}$$

$$L_o(t) = [\, C^T(t),\ \Delta_o C^T(t),\ \ldots,\ \Delta_o^{n-1} C^T(t) \,] \qquad 6.3$$

where Δ_o is an operator of the form

$$D_o \equiv A^T(t) + \frac{d}{dt}$$

To simplify the product $\Gamma(t)$ let us define $L_o(t)$ and $L_c(t)$ using an alternative notation

$$L_c(t) = [\, R_o(t),\ R_1(t), \ldots, R_{n-1}(t) \,] \qquad 6.3$$

$$L_o(t) = [\, S_o(t),\ S_1(t), \ldots, S_{n-1}(t) \,] \qquad 6.4$$

where : $R_{k+1}(t) = - A(t)\ R_k(t) + \dot{R}_k(t)$

$$S_{k+1}(t) \equiv A^T(t)\, S_k(t) + \dot{S}_k(t)$$

and

$$R_o(t) \equiv B(t)$$

$$S_o(t) \equiv C^T(t)$$

then

$$\Gamma_{ij}(t) = \begin{bmatrix} S_o^T(t) \\ S_1^T(t) \\ \vdots \\ S_{i-1}^T(t) \end{bmatrix} [\, R_o(t),\ R_1(t), \ldots, R_{j-1}(t) \,] \qquad 6.5$$

The impulse response of a time-varying system is defined as

$$\Omega(t,T) = C(t)\ \phi(t,T)\ B(T) \qquad 6.6$$

where $\phi(t,T)$ is the transition matrix of the system.

Now let us form the matrix $Q_{ij}(t,T)$ as follows :

$$Q_{ij}(t, T)= \begin{bmatrix} \Omega_{o,o}(t,T) & \Omega_{o,1}(t,T) & \Omega_{o,j-1}(t,T) \\ \Omega_{1,0}(t,T) & \Omega_{1,1}(t, T) & \Omega_{1,j-1}(t,T) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \Omega_{i-1,o} & \Omega_{i-1,1}(t,T) & \Omega_{i-1,j-1}(t,T) \end{bmatrix} \quad 6.7$$

where $\Omega_{ij}(t,T) = \frac{\partial^i}{\partial t^i} \frac{\partial^j}{\partial T^j} \Omega(t,T)$

it may be verified [12] that

$$\Omega_{ij}(t,T) = S_i^T(t) \phi(t,T) R_j(T) \quad 6.8$$

therefore

$$\Omega_{ij}(t,t) = S_i^T(t) R_j(t) \quad 6.9$$

this using equations 6.8 and 6.5 we obtain

$$Q_{ij}(t, t) = \Gamma_{ij}(t) \quad 6.10$$

Equation 6.10 relating $Q_{ij}(t,T)$, which is formed from the system's impulsive response matrix $\Omega(t,T)$, and $\Gamma_{ij}(t)$, which is formed from the system's differential equation characterization, is the basis of the method for synthesizing $\Omega(t,T)$. In fact the minimal system is then simply constructed using certain partitions of the matrix $Q_{ij}(t,T)$. So as suspected the matrix $\Gamma_{ij}(t)$ does contain all the information required to get a minimal realization of a transfer function matrix.

D'Angelo [12] has found a relationship in the time-domain that enabled him to get the matrix $\Gamma_{ij}(t)$ from the impulse response matrix.

Much effort and time was spent in trying to do the same for time invariant system, that is to find a relationship between Z(s) and Γ. However we were not able to come up with one.

Also if one carefully examines the matrix R in eq. 4.13 that Rosenbrock uses to find the minimal order of a system there again seems to be a relationship between R and Γ. However neither could this relationship be pinpointed. It is however our belief that some new approach to finding these relationships may lead to a new and powerful result. Once you have Γ it contains all the information regarding the structure of a system.

In conclusion we may make the following statements.

Two different algorithm have been proposed to solve the problem. The first based on the Smith - MacMillan form is the most general of the two but becomes impractical for transfer matrices of order higher than 3 or 4. The partial fraction algorithm, although not as general, is however more practical to use. In cases where it does not lead to a definite structure of A use, as mentioned before, the alternate method proposed by Chen and Panda [8].

REFERENCES

1. L.A. Zadeh and E. Polak: Book, "System Theory", McGraw-Hill/69.
2. E.G. Gilbert, "Controllability and Observability in Multivariable Control Systems", S.I.A.M., Section A, Volume 1, No. 2 (1963).
3. R.E. Kalman, Y.C. Ho and K.S. Narendra, "Controllability of Linear Dynamical Systems", appeared in book, Contribution to Differential Equations, Volume 1, John Wiley, New York (1961).
4. Y.C. Ho, "What Constitutes a Controllable System?", I.R.E. Prof. Group on Automatic Control, AC-7, No. 3 (1962).
5. R.E. Kalman, "Irreducible Realization and the Degree of a Rational Matrix", J. S.I.A.M. Control, 13 (1969), pp. 152-192.
6. R.E. Kalman, "On the General Theory of Control Systems", Proc. 1st International Congress on Automatic Control (Moscow, 1960), Volume 1, Buttersworths, London, 1961, pp. 481-492.
7. M. Heymann and J. Thorpe, "Transfer Equivalence of Linear Dynamical Systems", J. S.I.A.M. Control, Volume 8, No. 1, February 1970.
8. S. Panda and C. Chen, "Irreducible Jordan Form of a Rational Matrix", I.E.E.E. Trans. Automatic Control, February 1969.
9. C. Chen and C. Desoer, "Controllability and Observability of Composite Systems", I.E.E.E. Trans. Automatic Control, Volume AC-12, pp. 402-409.
10. P. DeRusso, R. Roy and C. Close: Book, "State Variables for Engineers", John Wiley, 1967.
11. D. Lindorff: Book, "Theory of Sampled-Data Control Systems", John Wiley, 1965.
12. H. D'Angelo: Book, "Linear Time-Varying System: Analysis and Synthesis", Allyn and Bacon, 1970.
13. H. Rosenbrock, "Efficient Computation of First Order for a Given Transfer Function", Electronics Letters, Volume 3, No. 9, September 1967, pp. 413-415.
14. R. E. Kalman, " On structural properties of linear , constant multivariable systems", Proc. 3rd IFAC Congress,London,1966, pp. 6A.1-6A.9

15. J. Farley, "Synthesis of Linear Sequential Machines Using Cyclic Subspaces", M.Sc. thesis, University of Ottawa, Dept. of Electrical Engineering, Chapter III.

16. F.R. Gantmakher: Book, "The Theory of Matrices", Chelsea, Volume 1, New York 1959.

17. J. Johnston: Book, "Econometric Methods", McGraw-Hill 1960.

18. R. C. K. Lee, "Optimal Estimation, Identification, and Control" M.I.T. Press 1964.

19. J. D. Simon and S. K. Mitter, "A Theory of Modal Control", Information and Control 13, pp. 316-353 (1968).

20. B. Ho and R. E. Kalman, "Effective Construction of Linear State-Variable Models from Input/Output Functions", Proc. Third Alberton Conf. 1965 pp. 449-459.

21. Wolovich and Falb, "On the Stucture of Multivariable Systems", Proc. JACC Aug. 1969, pp. 860.