

Thesis:
Volatility Modelling Using Long-Memory-GARCH Models,
Applications in S&P/TSX Composite Index

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**A Thesis Submitted to TELFER School of Management, University of
Ottawa, Ontario in Partial Fulfilment of the Requirements for the Degree of
Master of Science in Management.**

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Abstract

The statements that include sufficient detail to identify the probability distributions of future prices are asset price dynamics. In this research, using the empirical methods that could explain the historical prices and discuss about how prices change we investigate various important characteristics of the dynamics of asset pricing. The volatility changes can explain very important facts about the asset returns. Volatility could gauge the variability of prices over time. In order to do the volatility modelling we use the conditional heteroskedastic models. One of the most powerful tools to do so is using the idea of autoregressive conditional heteroskedastic process or ARCH models, which fill the gap in both academic and practical literature.

In this work we detect the asymmetric volatility effect and investigate long memory properties in volatility in Canadian stock market index, using daily data from 1979 through 2015. On one hand, we show that there is an asymmetry in the equity market index. This is an important indication of how information impacts the market. On the other hand, we investigate for the long-range dependency in volatility and discuss how the shocks are persistence. By using the long memory-GARCH models, we not only take care of both short and long memory, but also we compute the d parameter that stands for the fractional decay of the series. By considering the breaks in our dataset, we compare our findings on different conditions to find the most suitable fit. We present the best fit for GARCH, EGARCH, APARCH, GJR-GARCH, FIGARCH, FIAPARCH, and FIEGARCH models.

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ACKNOWLEDGMENTS

I would first like to thank my thesis advisor Dr. Yuri Khoroshilov of the Telfer school of management at University of Ottawa. The door to Yuri's office was always open whenever I ran into a trouble spot or had a question about my research or writing. He consistently allowed this thesis to be my own work, but steered me in the right the direction whenever he thought I needed.

I would like to also thank to the review board: my examiners, Dr. Anna Dodonova and Dr. François-Éric Racicot for their comments and suggestions. I am gratefully indebted to them for their very valuable comments on this thesis. I would like to thank Dr. François-Éric Racicot for providing insightful inputs. I would also like to thank Professor Leila Hamzaoui for taking the time to serve as chair to my thesis defense, and for her encouragement as well.

I would like express my gratitude to my first econometric teacher and advisor, professor Liudas Giraitis for his important role throughout my academic life and graduate studies. His patience and guidance shaped my academic life and motivated me to peruse research.

I cannot be more thankful to my parents, Farideh and Khalil and my sister, Soodeh for all their love and support. Without my parents, living abroad and studying at postgraduate level would be impossible for me, so I dedicate this thesis to them. I am very grateful to my kind bestmates: Giorgia for the time we have shared during the writing of this thesis, Ali and Camilla who specially helped me with proofreading and editing some parts of the thesis, my colleagues in the graduate programme at Telfer for accompanying and helping me during this whole time, and all that have been by my side as friends.

I would like to thank again to Dr. Leila Hamzaoui as the caring and supportive director of the MSc program. Also many thanks to the professors with whom I was very fortunate to take courses throughout my programme: Dr. Mark Freel, Dr. Ajax Persaud, Dr. Tyler Chamberlin, and Dr. Daniel Lane.

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Chapter1

Introduction

Volatility is a measure of variability in financial asset price over a period of time. Volatility is a very important feature in pricing financial assets in both academic and practical literature. After the introduction of ARCH model by (Engle & Bollerslev, 1986)¹, there is a considerable amount of literature on ARCH-type models. (Bollerslev, Chou, & Kroner, 1992) provides a review of the ten years of applications in different types of financial markets, such as equities, foreign exchange, and interest rates. (Bollerslev, Engle, & Nelson, 1994) assesses the most important theoretical developments in ARCH-family models of time varying conditional variances.

The main ideas behind the ARCH-type models are conditional probability density functions (PDF) that describe the density of returns conditional on all information that is known up to the present time. There has been a considerable amount of interest in models of changing conditional variance in econometrics, since the development of the ARCH family models, where finance scholars have applied such methodology in asset pricing models.

Heteroskedasticity does not comply with the assumptions of the Least Square Method. One of the essential propositions of least square models is homoscedasticity that assumes that the standard deviation of error terms is constant. This implies that the expected value of all error terms, when squared, is the same at any point. Data, in which the variance of error is changing, where the error terms may change for some ranges of

¹ He held the Sveriges Riksbank Prize in Economic Sciences in memory of Alfred Nobel 2003 (That was equally divided, also allocated to Granger for his work on cointegration) for his contribution in analyzing of time series with time-varying volatility -Auto Regressive Conditional Heteroskedastic process. A time series is a set of observations organised regarded to the time.

data, are heteroskedastic by definition. When dealing with heteroskedasticity using the Ordinary Least Square Regression (OLS), the estimated standard errors and confidence intervals tend to be too thin.

In other words, a sequence or a vector of a random variable is defined as heteroskedastic if there are different variances for the random variables. One of the most powerful tools to deal with heteroskedasticity is using the idea of autoregressive heteroskedastic process or ARCH-type models (GARCH-family models), which fill the gap in both academic and practical literature.

GARCH-family models could successfully take into account volatility clustering. The main contribution of the ARCH idea is the way it captures the clustering effect and the persistence of shocks in the series. Basically, large changes bundled together and small changes bundled together and as a result, the amplitude of the changes in returns differs over time and in other words asset prices move more rapidly over some prolonged periods than during the other ones. This phenomenon is called Volatility Clustering, and was initially addressed by (Mandelbrot, 1963). Much related literature is concerned with the apparent persistence of the estimated conditional variance of returns; see (Bollerslev, Chou, & Kroner, 1992), and also Integrated-GARCH model of (Engle & Bollerslev, 1986).

ARCH (q) of (Engle R. F., 1982), and GARCH (p, q) model of (Bollerslev T. , 1986) are the first and the most well-known models of the GARCH-family models.

One of distinctive elements of the GARCH (p, q) model is its simple structure and its small number of predictors (compared to ARCH (q) and other variations of GARCH-family models). However, this simple structure imposes some significant limitations on the GARCH (p, q) model for changing conditional variance – volatility.

Considering return $(r_t)^2$ as our dependent variable, the mean value and the variance could be defined relative to a set of previous information. Thus the present return is equal to mean value, which is the expected value, taking into account the past information plus the standard deviation of return and with considering the error term for the present time. This will essentially estimate the mean return and variance, conditional on the past information.

There are some issues while using the GARCH (p, q) model. By addressing the drawbacks of GARCH (p, q) model we aim to cover two main deficiencies of such a model, namely asymmetry in volatility and long memory.

(Nelson, 1991) argues that the GARCH (p, q) models capture the magnitude and not the response of volatility to positive or negative excess return shocks. (Black, 1976) and (Christie, 1982) introduce the leverage effect, where the decrease in a stock price led to an increase in equity ratio and ultimately makes the stock riskier and increase its volatility. Basically, the volatility responds asymmetrically to the negative and positive returns. In other words, with the higher (lower) excess returns than the expected, which could be define as “good news” (“bad news”), volatility tends to decrease (increase).

As a result, in the GARCH models, conditional variance is only dependent on the size of the residuals (shock) and not the sign. Subsequently, GARCH (p, q) models tend to ignore the information on the direction of returns. Nelson (1991) expresses the need for the models that respond asymmetrically to the positive and negative shocks that for asset pricing purposes, when modelling the volatility of asset returns.

² With changes in the asset price, the return of the asset will also change. We get the return in per cent:

$$r_t = 100[\text{Log}(P_t) - \text{Log}(P_{t-1})]$$

, where P_t is price in time t .

As the news can impact the volatility in various ways, capturing the difference in volatility modelling is a very important and relevant task, which also has an impact on the risk premium. As a result, in literature, the term ‘leverage effect’ is used synonymously with ‘asymmetric volatility’³.

In this research we try to analyse the asymmetrical impacts of positive and negative shocks on volatility of asset returns, using Canadian stock market index daily data. We investigate such evidence by applying the asymmetric-GARCH models, which take into account both magnitude and sign of the residuals (shock). Here we use three popular models, EGARCH, GJR-GARCH, and Power-ARCH, described below.

Nelson (1991) finds that the US market fall has a larger effect on the next day’s volatility than a rise with the same magnitude. Accordingly, he introduced the Exponential-GARCH model. (Glosten, Jagannathan, & Runkle, 1993) offered another variation of asymmetric GARCH models: GJR-GARCH. Also (Ding, Granger, & Engle, 1993) introduced the Power-ARCH model that was tested on 17 bilateral exchange rates.

The various specifications of the asymmetric GARCH models, such as EGARCH, GJR, and APARCH models, suggest different relationships between the conditional variance of returns and the information known up to current time. Although all three of these models take into account the asymmetric effects on volatility, the ratio of weights given to positive and negative residuals of the same absolute value are defined differently for each model.

³ There is another fact known as volatility feedback, where news increases the current volatility, and as volatility is persistence, cause an increase in the future volatility, which led to more demand for return and decline in the stock price see (French, Schwert, & Stambaugh, 1987) and (Campbell & Hentschel, 1992)

We aim to predict and compare the asymmetric parameters (or predictors) of the three asymmetric-GARCH models on the S&P/TSX Composite Index data (chapter 2 discusses the method and models in details). Thus, firstly, we aim to investigate the existence of such asymmetric effect in the volatility in our dataset, and secondly, we tend to discuss the differences in magnitude of such effect and compare the asymmetric predictors among the three models.

In addition to the asymmetric effects on volatility of asset returns in our dataset, in this paper, we are going to investigate the long-memory property in the volatility series.

Another important finding in related literature is the persistence of volatility in stock market – or long memory⁴. Although the mentioned GARCH models could take into account the autocorrelations of both absolute returns and squared return, they are unable to explain how the autocorrelations decay to zero (Bollerslev & Mikkelsen, 1996), (Baillie, Bellerslev, & Mikkelsen, 1996), and (Andersen & Bollerslev, 1997). GARCH and EGARCH models have short-memory that the autocorrelations of conditional variance and squared returns⁵ exponentially die out, and not slowly.

This concept is initially discovered in Hydrology by (Hurst, 1951), and later entered in the field of economics and finance by (Mandelbort & Van Ness, 1968). Empirical literature shows that the imported shocks (that are sharp changes in volatility) have long-lasting influences on the underlying variable, in this case, the conditional variance of the return.

Long memory process is discussed by (Granger, 1980), (Granger & Joyeux, 1980), (Hosking, 1981). The existence of long memory in our underlying variable (e.g.

⁴ Also known as long-range dependency

⁵ One can study returns and absolute returns series, as well.

market return, or volatility series) we observe a gradual response to shocks (or information coming) and over a “long” period of time. The importance of such process, which has slower decay than exponential, is well addressed in the research in stock markets. Ding, Granger, and Engle (1993) investigated the long memory properties of stock market returns. See also (Baillie & Bollerslev, 1993), (Baillie & Chung, 1993), (Baillie, Chung, & Tieslau, 1996), and (Lo, 1991).

There are the applications of long memory using the Fractionally Integrated GARCH model and Fractionally Integrated Exponential GARCH model of Baillie, Bollerslev, and Mikkelsen (1996) and Bollerslev, and Mikkelsen (1996), respectively, as well as and Fractionally Integrated APARCH Model of (Tse, 1998) that specifically capture such a property, namely persistence in long range dependency in volatility.

In this research we focus on the Canadian stock market, on which there is limited research about volatility modelling. By using different varieties of GARCH-family models, we capture long memory and leverage effects, and compare the predicted parameters of the models with each other. We aim to use the models in order to address the volatility-return relationship and dynamics of asset prices in the market. To the best of our knowledge, there is no similar work on such dataset on market volatility modelling using GARCH (-family) models since their development, specifically when it comes to study the volatility using FIGARCH, FIEGARCH and, FIAPARCH models. There is some research on the similar dataset, however it does not measure the long-memory properties of volatility, nor does it compare the discussed parameters.

(Christensen, Nielsen, & Zhu, 2015) investigate the risk-return tradeoff and leverage effect using a specific variation of FIEGARCH-M of (Christensen, Nielsen, & Zhu, 2010) on several markets. (Sivakumar & Mohandas, 2009) shows applications of FIGARCH models in Indian stock market and (Han, 2003) investigates Korean won-

US dollar Exchange Rates, respectively. (Rambaccussing, 2014) discusses GARCH, GJR-GARCH and FIGARCH models' parameters in the Romanian and UK stock market data.

This paper examines the long-memory property in the conditional variance and leverage effect in the S&P/TSX Composite Index. The daily data on the market from the beginning of the index in 1979 until present are employed. The GARCH, EGARCH, APARCH, and GJR-GARCH frameworks also will be estimated for comparative studies on the parameters and best fits of models.

In order to estimate the models, we use the log likelihood, Schwartz, and Akaike Information Criteria to distinguish among models to find the most suitable model to capture the persistence of shocks in the volatility series.

We will also discuss the potential conditional mean and volatility relationships by using the In-Mean-GARCH idea. We expect our results to confirm the long memory in the conditional variance.

This paper fills the gap in the Canadian stock market data⁶, on which, to the best of our knowledge, there is a not similar paper. It specifically examines the market's volatility modelling on the similar dataset using asymmetric and long memory GARCH (-family) models since the development of these models. By investigating for the return and volatility, we address the important features in dynamics of asset prices.

We contribute to the current literature mainly by volatility modelling using FIGARCH, FIEGARCH, and FIAPARCH on the S&P/TSX composite index. We provide a discussion on the long-memory in the volatility series on the Canadian stock market with consideration of breaks. Additionally, we make suggestions on how volatility may or may not influence the mean of the GARCH process and its

⁶ S&P/TSX composite index

significance, mostly when taking into account the long-memory property in the volatility series (such influence is known as volatility spill-over in mean equation). Finally, we provide a comparative discussion on the parameters of the applied models using the various residual distributions.

Chapter2

Background and Methods

In this chapter, first we provide a brief background on the topic of asset price changes. Then we discuss volatility modelling. After that we introduce the related econometric models in detail, with equations and their concepts. After that, we go over the GARCH models and finally, in the last section we provide the equations for the GARCH models and discuss their parameters.

The asset pricing area is one of the most famous areas in the whole economics and finance.

The statements that include sufficient detail to identify the probability distributions of future prices are asset price dynamics (Taylor, 2005). Here we will investigate various important characteristics of the dynamics of asset pricing using the empirical methods that could explain the historical prices and discuss about how prices change.

There has been so many works that try to show the mechanisms of financial markets such as CAPM see (Sharpe, 1964) and (Lintner, 1965), and (Black & Scholes, 1973), which make grounds for classical finance theory and asset pricing.

One of the main ideas in finance is that an intrinsic value of an asset is the sum of discounted future cash flows, see (Fisher, 1930), and (Williams, 1938). Among all

differences in the field, the whole literature agrees that the prices could fluctuate higher or lower than the intrinsic value, though.

The traditional framework states that the market will adjust itself by the flow of information and reaching to the equilibrium of demand and supply that is the fundamental value (Fama, 1965) and (Fama, 1970). However, there are evidences and findings that challenge such a view, in theory and in practice.

The evidence of failure of the traditional framework is well-established area, e.g. difficulties such as jumps in the stock market index after a shock are added by (Shleifer & Vishny, 1986). In addition, behavioural framework also makes suggestions that challenge those of the traditional one.

Behavioural finance that is basically an elaboration of the classical finance, with “more realistic” assumptions, makes some significant improvements in our understating of the market phenomena. As (Statman, 2014) says:

“ [Behavioural finance] incorporates parts of standard finance, replaces others, and includes bridges between theory, evidence, and practice.”

As (Barberis & Thaler, 2003) state, there are two blocks for the behavioural finance, the limit to the arbitrage, and psychology. Here we would like to mention that no matter which paradigm we follow or agree with, they both agree on changeability of the asset prices -which is also a common sense.

With the changes of the asset prices there would be a need for the investors to learn about the possible changes, and how the prices change. Finance scholars may need to consider the effects of price dynamics on hypothesis tests corporate events against prices volatility; portfolio managers frequently need to know the concurrent status of their investments for their asset allocations and other fund decisions; the

individual investors need to see their current trading positions and changes that has occurred for further decisions, the exchanges need to monitor the securities, and markets need to price the option contracts available regards to their maturity, and so on. In order to study the changes in the market in either individual level, corporate level, and or policy making level we need to know more about the dynamics of asset prices.

In analysis of the price dynamics, we use returns (log returns) rather than price series. The consequent prices could be correlated, but the changes in price could be a better tool to investigate the asset prices dynamics. Prices are non-stationary and have trends; it is why we define returns as changes of the logarithmic prices. The base of analysis of a time series is stationarity.

In a stationary process, means and variances do not change by time, and covariance only depends on the difference of the time subscripts; otherwise a process is a non-stationary one.

There are two types of stationary processes: strictly stationary and weak stationary. Strictly stationary is hard to reach empirically, and here with stationary processes, we are mainly concerned with weak stationary process that is defined as follows:

A time series $\{r_t\}$ is said to be weak stationary when both mean of r_t and covariance between r_t and r_{t-l} , where l is an integer, are invariant through the time. Implicitly, the first two moments of r_t are finite.

With changes in the asset price, the return of the asset will also change. We get the return in per cent:

$$r_t = 100[\text{Log}(P_t) - \text{Log}(P_{t-1})], \quad \text{Eq. (1)}$$

, Where P_t is price in time t .

The changes in the asset price have some specific features that are very interesting to study. For instance, asset prices move more rapidly over some prolonged periods than during the other ones.

There is a very important feature of volatility series that can be shown in the return series. This phenomenon is called volatility clustering that was initially addressed by Mandelbrot (1963). Basically, large changes are bundled together and small changes bundled together. As Mandelbrot (1963) presents “... large changes tend to be followed by large changes – of either sign – and small changes by small changes...”.

In order to complete our direct observations from the market, statistical and financial econometrics tools can also help us to explain how prices change (Why the prices change is also another very interesting question, though).

Changes in returns led to differences in investors trading decisions, portfolio managers' positions, banks capital requirements status, etc. These all make it really interesting to study how the return series changes.

The volatility changes could explain very important facts about the asset returns. Volatility could gauge the variability of prices over time. We cannot be sure about the sources of the volatility changes, and why there are changes. Nonetheless, from the asset returns, one can estimate time series models for volatility.

In order to understand the dynamics of financial asset prices, considering the return series does not provide a good solution. For instance, there is a gap in the classical finance theory for asset pricing that relates the return of an asset to the notion of standard deviation and variance, which does not seem as a suitable approach. Examples include Capital Asset Pricing Model, here after CAPM (Sharpe 1964), and

Black Scholes (1973) that relate the changes of the asset price directly to its individual variance.

(Baillie & DeGennaro, 1990) show that there is no evidence on a relationship between mean of the returns and the variance or standard deviation of the returns on a portfolio of stocks, leading to the conclusion that it is not appropriate to use simple mean-variance models, and suggest use of other alternative models.

In order to investigate for such properties in asset prices, we need to know about the distribution of data, the behaviour of changes in price, and characteristics of the asset returns. This information is not necessarily enough to come up with an idea to “beat the market”. However, we could use such information for our investigations on the rate of asset price changes. Financial time-series can help us investigate many features in asset prices. One of the main notions in the research on the rate of price changes is how there has been a shift from research on the mean of the stock market return to more sophisticated features of asset prices, such as volatility –that is the conditional standard deviation of the underlying assets.

For instance, on the practical importance of volatility we can mention the Volatility index (VIX) that a useful instrument in the process of financial investment decision. VIX introduced in the United States by Chicago Board of Option Exchange (CBOE) in 1993 (and reformulated in 2003) and in Canada since 2010. There is an important feature about the volatility of the asset prices, which makes it different from the assets price and return.

Asset price can be observed (or measured) instantly. In contrast (Anderson, Bollerslev, & Das, 1998) and (Andersena, Bollerslev, & Lange, 1999) show that volatility has a latent property (that is, inherently unobserved and evolves by the time) that is needed to be measured over a period of time. As a result, there is a need for

volatility modelling, which is a very well established method in finance, see Engle (1982), Bollerslev (1986), and (Christensen & Nielsen, 2007).

In order to do the volatility modelling, and “observe” volatility we use the conditional heteroskedastic models. A sequence or a vector of random variable is defined as heteroskedastic if there are different variances for the random variables. One of the most powerful tools when dealing with heteroskedasticity is using the idea of autoregressive conditional heteroskedasticity process or ARCH models. There is a well-established body of literature on the GARCH models. In order to measure volatility, the heteroskedastic models are proposed, which were filling the gap in both academic and practical literature.

The introduction of ARCH models by Engle (1982) spreads conditional variance modelling in the literature. This is an idea that allows the conditional variance to depend on the conditional mean of a random variable in regression analysis, and the ARCH model provides a parsimonious method to explain the conditional heteroskedasticity.

In order to explain heteroskedastic processes first we need to go over some basics of the econometrics models. For instance, before introducing the useful time series models, we need to define white noise.

A time series r_t is defined as a white noise if $\{r_t\}$ is a sequence of independent and identically distributed random variables with finite mean and variance. Specifically, if r_t is normally distributed with mean zero and variance σ^2 , the series could be named a Gaussian white noise.

Considering the sample autocorrelations of the value-weighted index returns denotes that for asset returns it may be necessary to model the serial dependence beforehand and then start any further analysis.

In what follows, we review the models that are useful in modeling the dynamics of our time series. Furthermore, some concepts are shown are useful in volatility modeling.

AR Models

A simple Autoregressive (AR) model is:

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t, \quad \text{Eq. (2)}$$

, Where $\{a_t\}$ is defined as a white noise series with mean zero and variance σ_a^2 . This model follows the same rules of a simple linear regression model in which the dependent variable is r_t and the explanatory variable is r_{t-1} .

In the literature of time series, equation (2) is denoted as an autoregressive (AR) model of order 1 or AR (1) model.

Similarly, an AR(p) model is:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \dots + \phi_p r_{t-p} + a_t, \quad \text{Eq. (3)}$$

, Where p is assumed to be a nonnegative integer and $\{a_t\}$ is white noise. Such model articulates that the past p variables r_{t-i} ($i = 1, \dots, p$) jointly determine the conditional expectation of r_t given all previous data. Similar to Eq. (3) in the AR (p) model the lagged values serve as the explanatory variables following a multiple linear regression model.

Properties of AR Models

Assuming that the series is weakly stationary, and considering the Eq. (2) and assuming:

$$E(r_t) = \phi_0 + \phi_1 E(r_{t-1}), \quad \text{Eq. (3)}$$

Under the stationarity condition,

$$E(r_t) = E(r_{t-1}) = \mu$$

Also

$$\text{Var}(r_t) = \phi_1^2 \text{Var}(r_{t-1}) + \sigma_a^2, \quad \text{Eq. (4)}$$

, Where σ_a^2 is the variance of a_t , and by knowing that the covariance between r_{t-1} and a_t is zero, then under the stationarity assumption, $\text{Var}(r_t) = \text{Var}(r_{t-1})$.

Using $\phi_0 = (1 - \phi_1)\mu$, we rewrite a stationary AR(1) model as:

$$r_t = (1 - \phi_1)\mu + \phi_1 r_{t-1} + a_t. \quad \text{Eq. (5)}$$

Such a model is frequently used in the finance literature, setting ϕ_1 as a measure for the persistence of the dynamic dependence of an AR(1) time series.

MA Models

Now we present another type of models that are also helpful in the process of modeling return series. There are several numbers of approaches to present the MA models. One method is to consider such model as an extension of white noise series. Another approach would be to define the model as an infinite-order AR model that has some parameter constraints. Here we employ the second approach.

We, theoretically, can write an AR model with infinite order as:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \dots + a_t. \quad \text{Eq. (6)}$$

However, because this model has infinite numerous parameters, such an AR model would not be a realistic one.

In order to be able to use this model in practice, it needs to be assumed that the coefficients ϕ_i 's fulfill some of the constraints that are determined by a finite amount of parameters. We can write this idea as:

$$r_t = c_0 + a_t - \theta_1 a_{t-1} - \dots - \theta_q a_{t-q}, \quad \text{Eq. (7)}$$

, Where c_0 is a constant and $\{a_t\}$ is a white noise series.

Properties of MA Models

Moving-average models are always in the weakly stationary form. As they are finite linear arrangements of a white noise sequence for which the first two moments are time invariant.

ARMA Models

When using the AR and MA models separately, these models may become cumbersome because one may need a very high-order model that contains many parameters to adequately describe the data. To overcome such issue, the autoregressive moving-average (ARMA) models are introduced; see Box, Jenkins, and Reinsel (1994).

An ARMA model combines the ideas of AR and MA models together in a way that the number of parameters contained is stayed small. As a result, using ARMA models we can achieve parsimony in parameterization. When considering the return series in finance, ARMA models are not very useful directly. However, the idea of ARMA models is greatly relevant in volatility modeling. As a matter of fact, the generalized autoregressive conditional heteroskedastic (GARCH) model can be regarded as an ARMA model, even though, for the a_t^2 series.

A time series r_t follows an ARMA (1,1) model if it satisfies

$$r_t - \phi_1 r_{t-1} = \phi_0 + a_t - \theta_1 a_{t-1}, \quad \text{Eq. (8)}$$

, Where $\{a_t\}$ is a white noise series. The right-hand side of Equation (8) is the MA component and the left-hand side is the AR part of the model. ϕ_0 is the constant and, we should consider $\phi_1 \neq \theta_1$; otherwise, the whole model will be simplified to a white noise series.

General ARMA Models

One can write the general ARMA (p, q) model as below:

$$r_t = \phi_0 + \sum_{i=1}^p \phi_i r_{t-i} + a_t - \sum_{i=1}^q \theta_i a_{t-i}, \quad \text{Eq. (9)}$$

, Where $\{a_t\}$ is a white noise series and p and q should be non-negative integers.

As it can be implied from above, one can consider the AR and MA processes as special kinds of the ARMA (p, q) model. The autoregressive moving average (ARMA) process that introduced by Box, Jenkins, and Reinsel (1994) that combines the autoregressive and moving average concepts, is a way to keep the number of parameters small.

ARIMA and ARFIMA

A particular class of models is autoregressive integrated moving average (ARIMA) models that includes stationary ARMA processes as a subclass.

When someone considers the conditional mean, the ARFIMA specification has been offered to fulfil the gap between short and permanent persistence, so that the short-run behaviour (or short-memory) of the time-series is taken by the ARMA process, while using a fractional differencing parameter, ARFIMA allows for modelling the long-run dependency in the series. Basically, ARFIMA is a generalized ARIMA model that allows for non-integer d parameter.

We will explain the fractional differentiating parameter and long-run dependence later, when we talk about long-memory property.

Information Criteria

In order to determine the order p of an AR process, there are several information criteria available that someone can employ. All the information criteria are based on likelihood.

The well-known Akaike information criterion (AIC) as Akaike (1973) defines is:

$$\text{AIC} = \frac{-2}{T} \ln(\text{likelihood}) + \frac{2}{T} \times (\text{number of parameters}), \quad \text{Eq. (10)}$$

, Where the likelihood function is assessed at the maximum-likelihood estimates and T is our sample size.

For a Gaussian AR (l) model, AIC reduces to

$$\text{AIC} = \ln(\hat{\sigma}_l^2) + \frac{2l}{T}, \quad \text{Eq. (11)}$$

, Where $\hat{\sigma}_l^2$ is the maximum-likelihood estimate of σ_a^2 , which is the variance of a_t , and T is the sample size. The first term of the AIC considering Eq. (10) measures the goodness of fit of the AR (l) model to the data, and also the second term is considered as the penalty function of the criterion because it uses the number of parameters in a candidate model to penalizes it. Different penalty terms led to different information criteria.

Another commonly used criterion function, and the most important one for us, is the Schwarz–Bayesian information criterion (BIC). For a Gaussian AR(1) model, the criterion is

$$\text{BIC} = \ln(\hat{\sigma}_l^2) + \frac{l \ln(T)}{T}, \quad \text{Eq. (12)}$$

The penalty function for each parameter used is 2 for AIC and $\ln(T)$ for BIC. Thus, compared with AIC, BIC tends to select a lower AR model when the sample size is moderate or large. This makes BIC the main criterion to select the best fit.

Selection Process

In order to employ AIC to select an AR model, we should calculate $\text{AIC}(l)$ for $l = 0, \dots, P$, where p is a pre-specified positive integer and selects the order k that has the minimum AIC value. The same steps are involved when working with BIC.

Autocorrelation Function (ACF)

We can consider the return series r_t , which is a weakly stationary process. There is a concept to check for the linear dependency of r_t and its previous values r_{t-l} , which

can be considered as a generalized concept of correlation, namely: autocorrelation. The correlation coefficient between r_t and r_{t-l} is referred as the lag- l autocorrelation of r_t that we show it ρ_l . Considering the weak stationarity assumptions we can write the below equation:

$$\rho_l = \frac{\text{Cov}(r_t, r_{t-l})}{\sqrt{\text{Var}(r_t)\text{Var}(r_{t-l})}} = \frac{\text{Cov}(r_t, r_{t-l})}{\text{Var}(r_t)} = \frac{\gamma_l}{\gamma_0}, \quad \text{Eq. (13)}$$

, Where we know that $\text{Var}(r_t) = \text{Var}(r_{t-l})$ in a weakly stationary series. From the assumptions that we have, we consider $\rho_0 = 1$, $\rho_l = \rho_{-l}$, and also $-1 \leq \rho \leq 1$. Furthermore, a weakly stationary series r_t cannot be serially correlated if and only if we have $\rho_l = 0$ for all $l > 0$.

It is worth mentioning that for a white noise series, all the ACFs are zero. As a result, In practice, where all sample ACFs are close to zero, the series is a white noise series.

Portmanteau Test

Usually in finance, we need to jointly test for multiple autocorrelations of r_t and see whether they are all zero or not. Box and Pierce (1970) suggest the Portmanteau statistic that is:

$$Q^*(m) = T \sum_{l=1}^m \hat{\rho}_l^2. \quad \text{Eq. (14)}$$

This is a test statistic under the null hypothesis $H_0: \rho_1 = \dots = \rho_m = 0$ and against the alternative hypothesis of $H_a: \rho_l \neq 0$ for some $l \in \{1, \dots, m\}$. As a results, is our series $\{ r_t \}$ is an *i.i.d* sequence with certain moment conditions, $Q^*(m)$ is asymptotically a chi-squared random variable that is considered to have m degrees of freedom.

In order to increase the power of the test in also on finite samples Ljung and Box

(1978) adjust the $Q^*(m)$ statistic as:

$$Q^*(m) = T(T+1) \sum_{l=1}^m \frac{\hat{\rho}_l^2}{T-l}, \quad \text{Eq. (15)}$$

One should apply the decision rule to reject H_0 if $Q(m) > \chi_a^2$, where χ_a^2 denotes the 100(1 - α)th percentile of a chi-squared distribution with m degrees of freedom.

ARCH Test

Another test for conditional heteroscedasticity is the Lagrange multiplier test or LM ARCH test (Engle, 1982) that is to test the presence of ARCH effects in series. For each specified order, the squared series is regressed on q of its own lags.

This test is equivalent to the usual F statistic for testing $a_i = 0$ ($i = 1, \dots, m$) in the linear regression

$$x_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2 + \dots + \alpha_m x_{t-m}^2 + e_t, \quad \text{Eq. (16)}$$

, With $t = m + 1, \dots, T$.

Where e_t denotes the error term, m is a pre-specified positive integer, and T shows our sample size. The null hypothesis or H_0 is no ARCH effect; otherwise someone can reject the null and conclude that there is an ARCH effect in the time series.

Model Checking

We need to make sure that we examine the fitted models very carefully and check for probable model inadequacy during the modeling process.

If we already have adequate model fitted on data, then we expect that the residual series should be recognized as a white noise. We can also use other tests such as BDS test of independence on the residual series for that purpose. We explain the methods of the BDS test later in this chapter. In order to make sure that the residual series is a white noise, we use the Ljung–Box statistics in Eq. (15) to test for the closeness of \hat{a}_t to a white noise, using the ACF. On basis of the number of constraints exist on residuals

\hat{a}_t from fitting the AR (p) to an AR (0) model the degrees of freedom are adjusted.

If we realize that the fitted model is inadequate, then we should refine the model. For example, we may need to simplify the model and aim to remove the insignificant parameters from the model, and use the information criteria to make sure that we reach the suitable fit that is an adequate one. Additionally, when the residuals ACF check exhibit a serial correlation in the series, then we should extend the model to get rid of of the serial correlations.

Autocorrelation Function for MA Models

When working with a moving average process, we are confident that a MA(q) series is only linearly associated to its initial q -lagged values and henceforth is a “finite-memory” model, meaning that any correlation dies out in after a few lags.

Identifying MA Order.

The ACF is useful in identifying the order of an MA model. For a time series r_t with ACF ρ_l , if $\rho_q \neq 0$, but $\rho_l = 0$ for $l > q$, then r_t follows an MA(q) model.

Partial Autocorrelation Function (PACF) for AR Models

The partial-ACF (PACF) of a stationary series is defined as a function of its ACF and is a suitable method to choose the proper order p of our AR process.

Identifying ARMA Models

We should mention that the ACFs and PACFs are not directly giving instructions for the orders of the ARMA process. As a results, in determining the order of our ARMA models, we should apply the log-likelihood methods, and therefore, check for AIC and BIC criteria to reach to the “best” combinations of p and q order.

Unit Root Test

Unit root tests on y_t to evaluate for the stationarity of series are also necessary to run the empirical tests.

The Augmented Dickey-Fuller (ADF) test $H_0: y_t$ is $I(1)$ against $H_1: y_t$ is $I(0)$ (see Dickey and Fuller, 1981) will be provided by the t-statistic on $\hat{\theta}$ in:

$$\Delta y_t = \beta_0 + \beta_1 t + \theta y_{t-1} + \sum_{i=1}^q \gamma_i \Delta y_{i-1} + a_t$$

Eq. (17)

, Where q is the number of lagged first differences included in the ADF regression and a_t is a white noise series.

QLR Test for Stability

Investigating for structural changes is an important matter in econometrics because a myriad of financial or non-financial factors can cause the variables that are under study to change over time. There are numerous works around the issue of structural changes that are started since the works of (Chow, 1960) and (Quandt, 1960).

Under the structural break tests, when testing for break, typically under the null hypothesis there is no change, or in the other words, no break is realised. We use the Quandt Likelihood-Ratio (QLR) test.

In a similar method as (Racicot & Théoret, 2016), which used the QLR for the breaks and according to (Stock & Watson, 2011), the QLR statistic is specified by:

$$QLR = \text{MAX}[F(\tau_0), F(\tau_0 + 1), \dots, F(\tau_1)] \quad \tau_0 < \tau < \tau_1$$

, Where $F(\cdot)$ refers to the standard F statistic evaluated at time τ .

Basically, considering the F statistic that is calculated over the potential period of the breakpoints, and by maximising such F statistic one can define the QLR statistics. It is worth mentioning that this approach is a generalised on basis of Chow test, where

we pre-assume the potential breakpoints in series under investigation. In this research, we provide the test results on basis of QLR that are according to (Andrews, 1993).

Nonlinearity Test

BDS Test

We should check for the existence of the independence and identical distribution (i.i.d) assumption of a time series using Brock, Dechert, and Scheinkman (1987) introduced test statistic, that is commonly known as BDS test. As BDS test has a good power against a broad range of data, we can use this test to investigate the processes that are departed from the property of i.i.d.

One can write the test statistic as a correlation integral for a k-dimensional time series X_t and observations $\{X_t\}_{t=1}^{Tk}$ that is defined as:

$$C_k(\delta) = \lim_{Tk \rightarrow \infty} \frac{2}{Tk(Tk-1)} \sum I_\delta(X_i, X_j), \quad \text{Eq. (18)}$$

Where $I_\delta(u, v)$ can indicate 1 if $\|u, v\| < \delta$, and zero otherwise. The correlation integral measure the fraction of data pairs of $\{X_t\}$ that are located in a distance of δ from one another.

(Brock, Scheinkman, Dechert, & LeBaron, 1996) offers BDS test that is a common method to apply to the standardized residuals⁷ of GARCH models. The standardized residual is the residual divided by its standard deviation. Standardized residuals are used to standardize normal distributions in order to compare values.

There is some consideration on applications of the BDS test. For instance, in order to avoid committing type I error, the data should be a stationary process. Consequently, we may need to test for the unit root (Racicot, 2012). In running the empirical tests, it

⁷ A standardized residual is a ratio: The difference between the observed count and the expected count and the standard deviation of the expected count in chi-square testing.

is recommended to do some bootstraps experiments. In the use of BDS in this research we also used the bootstrap option on the test.

The null hypothesis of the test is defined as the series under investigation is an i.i.d. process.

In order to employ the BDS test on the residuals of our GARCH models, there are also some considerations that must be expressed. Someone can use the residuals of the GARCH models, however, for the GARCH models the results may not be satisfactory that the literature suggests considering the standardised residuals of the GARCH models for the BDS test (de Lima, 1996). Furthermore, besides using the 'Standardized residuals' on our GARCH models as an input of the BDS test, we can improve the test statistics using the transformed of the standardised residuals. Basically, when we work on the large data, such as our residual series with over 9000 counts, the standardized residuals won't improve the distribution assumptions of the test by increases of the sample size. Therefore, our null hypothesis will be under-rejected, and thus in order to improve such results, one can use the transformed version of the standardised residual series, namely: the natural logarithm of the squared residuals (Maddala, et. Al. 1993).

GARCH Models.

Since the seminal work of Engle (1982), there is a broad range of univariate ARCH-type models available to deal with heteroskedasticity condition. The main contributions of the ARCH idea were taking into account of the clustering effect, and persistence of shocks in the series. The differences among such different models, basically, come from the features that each single model could address regards to the type of data available. Each of the ARCH-type (also known as: GARCH) models has

some specific characteristics, and may or may not be useful when they are applied to specific type of financial data.

The ARCH model leads to the development of many other conditional heteroskedastic models. There are several works that applied the idea of ‘conditional heteroskedasticity’ for volatility modelling on data, i.e. the most popular models such as GARCH (q, p) that is a generalised version of ARCH (q) and EGARCH (q, p) (exponential-GARCH). Although in some studies we could find the evidences on forecasting performances of mentioned models as powerful and simple (parsimonious), to investigate for features of our return and volatility time-series, the use of other models are significantly recommended in various studies (e.g. Bollerslev and Mikkelsen (1996), and Baillie et al. (1996)). In the following part, we discuss the details of different variations of some of the most powerful GARCH models, their parameters, and limitations.

Each of the developed models has its own strengths, as well as weaknesses while dealing with financial data. As there are many parameters and numbers of assumptions are involved, accurately distinguishing among the models to make predict models is a challenge that we should cope with in order to find “the best fit” of the estimated models. The first model to start with is the ARCH (q) model.

The ARCH (q) model takes into account the clusters errors and nonlinearity or the residuals. ARCH has some weaknesses, i.e. it responds “too slow” to the large isolated shocks (or over-predicting). That was a motivation for further development of the process. Getting from the idea of the ARCH model to the GARCH (Generalised ARCH) model is similar to thinking of an ARMA to an AR process. The simplest GARCH model is GARCH (1,1) with only four predictors. A GARCH (q, p) process that is developed by Bollerslev (1986) offers a parsimonious model than ARCH model.

Yet, there are some problems around the use of GARCH models for volatility modelling of different assets. Firstly, there are few works on the ARCH-family models that account for the long-range dependency (see Giraitis et al. (2000) Giraitis et al. (2015), Bollerslev and Mikkelsen (1996), and Baillie et al. (1996)) and negative/positive innovations (see Nelson (1991)) in the return and volatility series. Secondly, to our knowledge no studies available on the Canadian stock market Index that contains several variations of GARCH models, comparing the parameters and significances of the models, side by side. In this paper we specifically focus on the stock market index.

Each newer model that was developed was designed to cover lacks of the previous ones, and for that reason it could be natural that the more recent ones are more advanced and will make better predictions (e.g. moving from the idea of ARCH to GARCH). Yet, as there are so many conditions that could depend on the frequency, length, and type of data one should not choose a model on the theoretical assumptions and without empirically applying the models on data.

Volatility may change over a long period of time, very slowly. This is found in Ding, Granger, and Engle (1993), as the long memory in volatility, where the impacts of a shock could decay slowly to zero. In their study using the daily data of S&P_500 index, they discover that the squared returns series has positive autocorrelations over more than 10-year period. This is the motivation behind using the fractionally integrated models to capture the long memory property in volatility.

However here we are not tending to develop any new model or theory. Rather we address the gap in the empirical literature by applying the sophisticated Fractional Integrated GARCH, Fractional Integrated Exponential GARCH, and Fractionally Power ARCH on the Canadian stock Market as well as making comparisons among

several ARCH-types models of different orders and assumptions on the distribution of residuals, which also will, hopefully, offer some suggestions to improve some aspects of the theories as well, for instance in preparation of data, on the residuals distribution, and also where taking into account of breaks.

The GARCH Models: Equations and Parameters

An ARCH stands for autoregressive conditional heteroskedasticity, which uses the conditional -that is depended on- variance of return on the recent returns information. Engle (1982) defines a stochastic process with conditional mean of zero and conditional variance that is calculated as a linear function of the past squared variables. The best-known ARCH specification is the GARCH (Generalised ARCH, 1986), and EGARCH (Exponential, generalised ARCH, 1991), and we are also tend to use GJR-GARCH (1993), Power ARCH (1993), FIGARCH (Fractionally Integrated GARCH), and FIEGARCH (Fractionally Integrated Exponential GARCH) (1996) for their strengths for our purposes.

The simplest **ARCH model (q)** is ARCH (1) with μ as the conditional mean for the return for period t and time varying conditional variance.

The ARCH (q) model can be expressed as:

$$\varepsilon_t = z_t \sigma_t ; z_t \sim i. i. d D(0, 1),$$

$$\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2, \quad \text{Eq. (19)}$$

, Where $D(\cdot)$ is a PDF with mean 0 and unit variance.

The conditional variance of ε_t is indeed an increasing function of the square of the shock that happened in $t - 1$. As the ARCH model can explain the volatility clustering, if the magnitude of ε_{t-1} is going to be large, then σ_t^2 and accordingly ε_t are likely to be large.

By adding the lagged variance in the conditional variance equation we get the **GARCH (q, p)** model:

$$\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^p \beta_j \sigma_{t-j}^2, \quad \text{Eq. (20)}$$

or using backshift operator $-L-$ we can write:

$$\sigma_t^2 = \omega + \alpha(L)\varepsilon_t^2 + \beta(L)\sigma_t^2, \quad \text{Eq. (21)}$$

, where L is the lag, with $\alpha(L) = \alpha_1L + \alpha_2L^2 + \dots + \alpha_qL^q$ and $\beta(L) = \beta_1L + \beta_2L^2 + \dots + \beta_pL^p$.

The GARCH model of Bollerslev (1986) allows reducing the number of estimated parameters.

The advantage is that we usually have fewer parameters to estimate for this model than the ARCH (q) model. This model provides a good explanation on the daily returns data.

ASYMMETRIC-GARCH MODELS

The conditional variance of future asset prices will not follow a changes in prices symmetrically when an X% increase of the price has a different influence on future volatility to an X% decrease of the price today. Nelson (1991) shows that the volatility of the next day of the US stock market will be influenced more by a decrease in the market than an increase, with the same magnitude.

EGARCH

Nelson (1991) proposed the EGARCH model, where he used Generalized Error Distribution (GED) in his work. Bollerslev and Mikkelsen (1996) reformed EGARCH model as:

$$\ln(\sigma_t^2) = \omega + [1 - \beta(L)]^{-1} + [1 + \alpha(L)] g(z_{t-1}), \quad \text{Eq. (21)}$$

, Where for skewed t-student distribution we have:

$$E(|z_t|) = \frac{4\xi^2}{\xi + \frac{1}{\xi}} \frac{\Gamma(\frac{1+\nu}{2})\sqrt{\nu-2}}{\sqrt{\pi}(\nu-1)\Gamma(\frac{\nu}{2})}, \quad \text{Eq. (23)}$$

, Where for the log-likelihood, ν is the degrees of freedom, $2 < \nu \leq \infty$ and $\Gamma(\cdot)$ is the gamma function. For symmetric Student we have $\xi = 1$.

GJR-GARCH

The GJR-GARCH model is introduced by Glosten, Jagannathan, and Runkle (1993). The generalized version of the model can be written as:

$$\sigma_t^2 = \omega + \sum_{i=1}^q (\alpha_i \varepsilon_{t-i}^2 + \gamma_i S_{t-i}^- \varepsilon_{t-i}^2) + \sum_{j=1}^p \beta_j \sigma_{t-j}^2, \quad \text{Eq. (24)}$$

, Where S_t^- is a dummy variable that take value of 1 (0) when γ_i is negative (positive). As this is an asymmetric model, it suggests that the conditional variance σ_t^2 is influences differently by the ε_t^2 , depends on whether ε_t is positive or negative.

APARCH

APARCH is presented by Ding, Granger, and Engle (1993). The APARCH (p, q) model can be written as follow:

$$\sigma_t^\delta = \omega + \sum_{i=1}^q \alpha_i (|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i}^2)^\delta + \sum_{j=1}^p \beta_j \sigma_{t-j}^\delta, \quad \text{Eq. (25)}$$

, Where $\delta > 0$ and $-1 < \gamma_i < 1$ ($i = 1, \dots, q$).

Considering Ding, Granger, and Engle (1993), when $\omega > 0$ and also $\sum_{i=1}^q \alpha_i E(|z| - \gamma_i z)^\delta + \sum_{j=1}^p \beta_j < 1$, a stationary solution is as:

$$E(\sigma_t^\delta) = \frac{\alpha_0}{1 - \sum_{i=1}^q \alpha_i (|z| - \gamma_i z)^\delta + \sum_{j=1}^p \beta_j}, \quad \text{Eq. (26)}$$

If $\gamma = 0$, $\delta = 2$ and z_t has zero mean and unit variance, stationarity condition of the GARCH (1,1) model, which is $\alpha_1 + \beta_1 < 1$, is met. Otherwise such a condition will depend on innovation process's assumption.

Long Memory-GARCH models

There are time series that their ACF decays slowly to zero at a polynomial rate, even after the lag increases. When there is such a long-range dependency in time series, these processes are called long-memory time series. One such example is the fractionally differenced process defined by

$$(1 - B)^d x_t = a_t; \quad -0.5 < d < 0.5 \quad \text{Eq. (27)}$$

, Where $\{a_t\}$ is a white noise series. Basis of Eq. (26) have been broadly discussed in the literature (e.g., see Hosking, 1981).

Ding, Granger, and Engle (1993) suggests that volatility usually changes gradually over a period of time and it takes a considerable period of time for a shock to decay, and as a result, there is persistence in the shocks. The models such as ARCH (q), and GARCH (q, p) are very powerful when dealing with series without a long-memory property.

However, the previously explained models, such as GARCH (q, p) model, could take into account of autocorrelations of both absolute returns and squared returns, they are incompatible to explain how the autocorrelations decay to zero. GARCH and EGARCH models have short-memory that the autocorrelations of conditional variance and squared returns are geometrically bounded.

We show the fractional integration parameter as: d , which is measured using the $(1 - L)^d$ filter.

When considering the fractional differentiating in an ARMA framework, we can think of an ARFIMA model. In the conditional mean, the ARFIMA specification has been proposed to fill the gap between short and complete persistence, so that the short-run behaviour of the time-series is captured by the ARMA parameters, while the fractional differencing parameter allows for modelling the long-run dependence.

FIGARCH

The FIGARCH model is proposed by Baillie, Bollerslev, and Mikkelsen (1996) as the Fractionally Integrated GARCH process by substituting the first difference operator of EGARCH by $(1-L)^d$.

The conditional variance of the FIGARCH (p, d, q) is provided by:

$$\sigma_t^2 = \omega [1 - \beta(L)]^{-1} + \{1 - [1 - \beta(L)]^{-1} \phi(L)(1-L)^d\} \varepsilon_t^2, \quad \text{Eq. (28)}$$

$$\sigma_t^2 = \omega^* + \sum_{i=1}^{\infty} \lambda_i L^i \varepsilon_t^2 = \omega^* + \lambda(L) \varepsilon_t^2, \quad \text{Eq. (29)}$$

, By replacing $\omega [1 - \beta(L)]^{-1}$ as ω^* and $\{1 - [1 - \beta(L)]^{-1} \phi(L)(1-L)^d\}$ as $\lambda(L)$, with $0 \leq d \leq 1$.

EGARCH (p, q) of Equation can be extended to account for long memory by factorizing the autoregressive polynomial $1 - B(L) = \phi(L)(1-L)^d$ where all the roots of $\phi(z) = 0$ lie outside the unit circle.

The **FIEGARCH** (p, d, q) is written as follows:

$$\ln(\sigma_t^2) = \omega + \phi(L)^{-1} (1-L)^{-d} + [1 + \alpha(L)] g(z_{t-1}), \quad \text{Eq. (30)}$$

By extending the fractional integration idea to the GARCH models, Bollerslev and Mikkelsen (1996) introduced the FIEGARCH model.

Basically the EGARCH (p, q) process can be defined as FIEGARCH (p, d, q) that captures the long memory by factorizing the autoregressive polynomial $1 - B(L) = \phi(L)(1-L)^d$, in which all the roots of $\phi(z) = 0$ lie outside the unit circle.

FIAPARCH

Similarly, the Fractionally Integrated APARCH (FIAPARCH) that is proposed by Tse (1998). FIAPARCH (p, d, q) model is specified as:

$$\sigma_t^\delta = \omega + \{1 - [1 - \beta(L)]^{-1} \phi(L)(1-L)^d\} (|\varepsilon_t| - \gamma \varepsilon_t)^\delta, \quad \text{Eq. (31)}$$

Dependency of the expected returns on volatility

The relationship between conditional mean μ and conditional variance h_t might be reasonable (mostly when the asset is an approximation for the market portfolio).

Therefore, the equation that defines μ includes h_t . We will discuss the in-mean variation of the models to address the potential existence of volatility in mean equation in our dataset.

Summary of the Steps

We will provide the summary of basic statistics such as minimum, maximum, mean and standard deviation, the value of the skewness and the kurtosis of the series, their t -tests and p -values. Moreover, the Jarque-Bera normality test will be also reported. We present details on the return series distribution.

The idea of our study is to check the conditions step by step and apply different assumptions and conditions to our observations and parameters of studies among the designated models. We take into account autocorrelation in our time-series, breaks, distributions, Arch effect, leverage effect or negative/positive shock, long-memory with calculations of d parameter and tend to fit the “best” model –that is parsimonious– by maximising the likelihood and by finding the model that minimises the Akaike and Schwartz criteria.

Chapter3

Data Analysis: Hypothesis, Dataset, and Data preparation

Dataset

We choose to work on S&P/TSX stock market index. We work on daily prices of the index since the beginning of the index in 1979 till the most recent data that covers all the historical prices available in the market, from Yahoo finance.

In the research analysis of the price dynamics, we use returns (log returns) rather than price series. The consequent prices could be correlated, but the changes in price could be a better tool to investigate for the asset prices dynamics. Prices are non-stationary and have trends; it is why we define returns as changes of the logarithmic prices.

We calculate the return in per cent:

$$r_t = 100[\text{Log}(P_t) - \text{Log}(P_{t-1})]$$

, When P_t is price in time t .

Using the return series graph, we will exhibit the volatility clustering. Also we study the presence of ARCH effect in the series.

We will exhibit the graph of the unconditional distribution of the returns and inspect if the distribution of it is normal or not. Indeed, we expect to see fat tails that indicate excess kurtosis⁸, to choose a proper distribution for our modeling purposes.

Additionally, we are using the methods to measure the correlation in a series itself, known as autocorrelation. When the error term is associated with the previous errors, it is supposed to be autocorrelated. Autocorrelation of order h is given as:

⁸ Kurtosis coefficient (KU) equals 3 for the normal distribution.

$$r_h = \frac{\text{cov}(y_t, y_{t-h})}{\sigma_{y,t} \sigma_{y,t-h}}$$

Plotting the autocorrelation function⁹ for multiple lags helps us to see the magnitude of the autocorrelation problem.

In addition to this inspection, we will show the autocorrelogram of the squared (or absolute) returns to underscore the existence of ARCH effects in the data.

In short we will show the price series plot, display the return series plot and the clustering effect. We will explain the models using different assumptions on the residual distribution, and investigate return, return square, absolute return series for autocorrelation of the series using autocorrelation Function (ACF), and partial autocorrelation function (PACF). Finally, we will choose a model that maximises the log-likelihood, and minimises the AIC, and BIC criteria.

List of Hypothesis

First Hypothesis: ON THE ARCH EFFECT

Engle's ARCH test assesses the null hypothesis that a series of residuals of (r_t) exhibits no conditional heteroskedasticity (ARCH effects), against the alternative that an ARCH(p) model describes the series.

Conduct the Ljung-Box statistic, is a function of accumulated sample autocorrelations. It is shown as Q-test that is to assess the autocorrelation on the squared residual series at different lags.

The null hypothesis (H_0) is no autocorrelation in the series, alternatively, the existence of volatility clustering in the residual series and autocorrelation.

⁹ Such a plot is known as “autocorrelogram”

The null hypothesis will be rejected if there is an ARCH effect, and consequently, there is significant volatility clustering in the residual series.

Second Hypothesis: ON THE LEVERAGE EFFECT

For an EGARCH (1,1) model, the GARCH and ARCH coefficients are expected to be positive, and the leverage coefficient is expected to be negative; large unanticipated downward shocks should increase the variance.

The GARCH model is nested in the GJR model. If all leverage coefficients are zero, then the GJR model reduces to the GARCH model. This means a GARCH model can be tested against a GJR model using the likelihood ratio test.

The sign effect is given by γ and the impact is significantly different from zero. Good news (positive errors) has an impact of $\alpha + \gamma$ while bad news (negative errors) has an $\alpha - \gamma$.

Importantly, if δ is significantly different from 2 but not significantly different from 1. This suggests that, instead of modeling the conditional variance (GARCH), it is more relevant in this case to model the conditional standard deviation. There would be more correlation between absolute returns than squared returns, a stylized fact of high frequency financial returns (often called ‘long memory’).

GJR (1,1) model, the asymmetry ratio is then defined as:

$$A = \frac{\alpha + \alpha^-}{\alpha},$$

The specific idea of the GJR-GARCH model is that the null hypothesis (H_0) of ‘No leverage effect’ is very simple to test. With $\gamma_1 = \dots = \gamma_q = 0$ one can conclude that the news impact curve is not symmetric, and hence, the previous negative and positive shocks have the same impact on the current volatility.

EGARCH (1) model, the asymmetry ratio is:

$$A = \frac{\gamma - \vartheta}{\gamma + \vartheta},$$

Third Hypothesis: ON THE LONG MEMORY IN VOLATILITY

Short memory ARCH models are typically special cases of long memory ARCH models. The special cases correspond to setting a long memory parameter d to zero.

The estimate of d rejects the null hypothesis of $d = 0$.

Long memory models are usually defined by applying the filter $(1 - L)^d$ to process followed by assuming the filtered process is stationary ARMA (p, q) process; and $0 < d < 1$.

Fourth Hypothesis: ON THE STABILITY OF THE RESIDUALS

The null hypothesis of the test is defined as the series under investigation is an i.i.d. process. BDS test has a good power against a wide class of data producing processes departing from the property of i.i.d, see Equation (18) for the test details.

There is some consideration on applications of the BDS test. For instance, in order to avoid committing type I error, the data should be a stationary process. Consequently, we may need to test for the unit root (Racicot, 2012).

Chapter4

Preparation of the Series and Initial Descriptive Statistics

Price series: S&P/TSX Composite Index

Figure (1)



Figure (1): the price series that seems as a nonstationary series

Figure (1) exhibits the price series of the S&P/TSX index from when the index price is recorded from 1979 to 2016. The stock shows an expected nonstationary process that exist in price series. In the following parts we show how to prepare the data to start the modelling process.

Return series in percentage

The whole sample includes the information on the S&P/TSX Composite Index available from 1979-06-29 to 2016-02-23 (count 9369 observations). Figure (2) presents the plot of the data on $r_t\%$.

Figure (2)

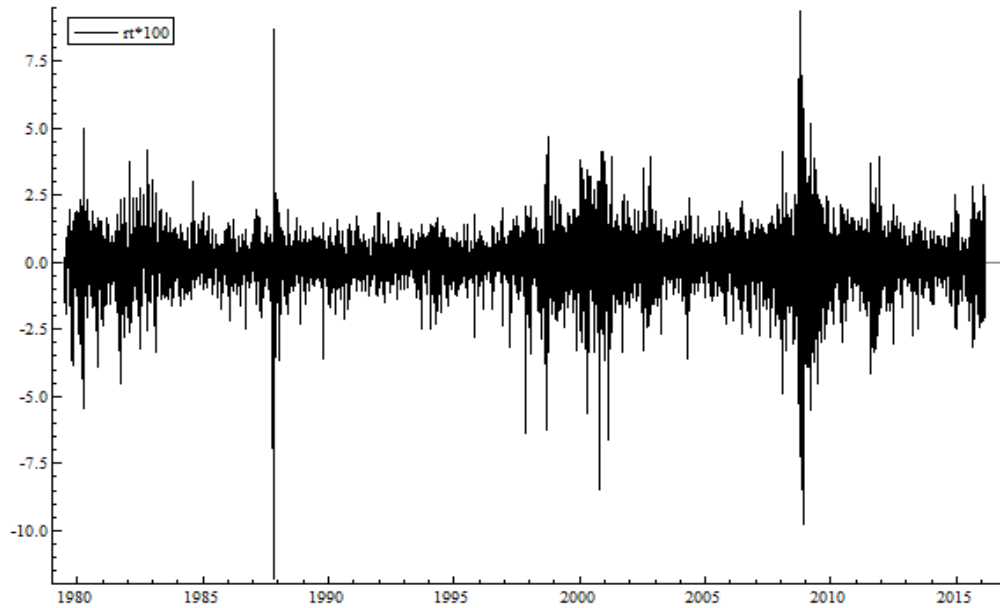


Fig. (2): r_t % for the whole sample size from 1979-2016

As we mentioned before, usually, we can observe the clustering effect in the return series, however, as the plot of the return series is provided over a very long time it is hard to recognise such effect. In the following part, we try to show the clustering effect for a part of the return series.

Show Clustering effect

We chose year 2010 as a sample to show the clustering effect. Figure (3) presents a better view of the changes of the return series. Figure (3) shows a zoomed in part of Figure (2), and provides more details on the changes of return series.

Figure (3)

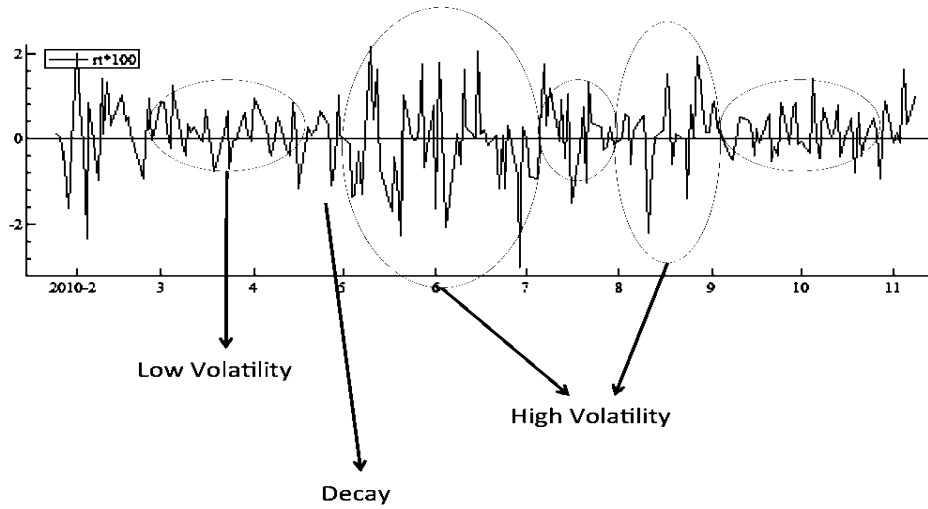


Fig. (3): Clustering effect

We provide the initial descriptive statistics of the return series that looks as a stationary process. It is important to recognise the distribution of the data, and precisely prepare the series for modelling purposes.

Histogram and Normality Tests

We present the histogram of the data in figure (4).

Figure (4)

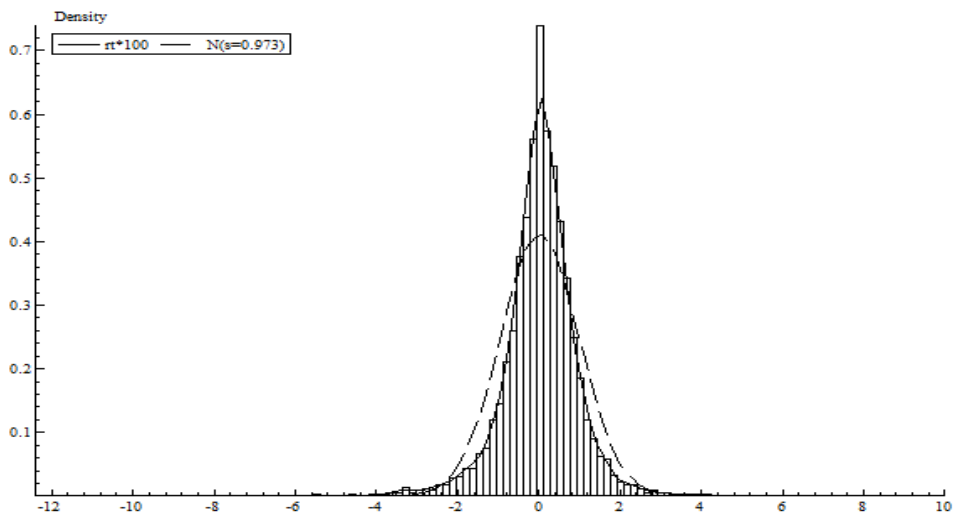


Fig. (4): Histogram on r_t % series

Fig. (4) shows the histogram of the return series with guidelines from the normal distribution (with the dash-line) that expresses a large excess kurtosis than that of a normal one. In the other words, in the return series we experience the fat-tail, which was expected.

As the distribution of the return series is shown above, it is “too simplistic” if someone assumes the normal distribution for the return series. In order to confirm the distribution is far from the normality assumptions, the normality test is provided below.

Normality Test

The skewness, extra kurtosis, and Jarque-Bera results are provided in Table (2).

Value	Statistic	t-Test	P-
Skewness	-0.84067	33.2	5.25
Excess Kurtosis	12.963	256.1	0.000
Jarque-Bera	66684.	.NaN	0.000

Table (1): Normality test

As Table (2) shows, excess kurtosis is very larger than the assumptions for the normality of the distribution. This is another presentation for the existence of the fat-tail.

Now we can start to consider the other properties of the series. The next inspection is on the homoscedasticity assumptions of the series. We aim to investigate for existence of heteroskedasticity in the series, which provides more detail on the first hypothesis of the paper.

Results on the First Hypothesis

The first step to take for existence of heteroskedasticity is to take the Ljung–Box statistics $Q(m)$ to the $\{a_t^2\}$ series; see (McLeod & Li, 1983). The null hypothesis is that the first m lags of ACF of the a_t^2 series are zero.

Q-test for number of lags	Test statistic For the $r_{t\%}$	Test statistic For the $r_{t\%}$ squared
5	75.7162 [0.000000]**	3422.42 [0.000000]**
10	105.050 [0.000000]**	5746.64 [0.000000]**
20	115.089 [0.000000]**	8547.52 [0.000000]**
50	205.123 [0.000000]**	13592.3 [0.000000]**

Table (2): Q (m) test on $r_{t\%}$ and its squared series with the significance of the p-value and the levels¹⁰

We write the null hypothesis of the test as: H_0 : No serial correlation and alternatively, the existence of the serial correlation in the series. As a result, we reject the null hypothesis for both series and conclude that both series are serially correlated.

ARCH Test for the return series

ARCH Tests on Lags	F-test
1-2	704.98 [0.0000]**
1-5	418.24 [0.0000]**
1-10	239.37 [0.0000]**

Table (2): ARCH test on the return series

¹⁰ The *, **, and *** show significance of the results for 10%, 5%, and 1% level, in respect.

As table (2) shows, the ARCH test for the return series is highly significant (on any lags), meaning that there is the existence of heteroskedasticity in the series. As a result, we reject the null of the series of residuals of (r_t) exhibits no conditional heteroskedasticity (ARCH effects), against the alternative that an ARCH (q) model describes the series. This test presents that there is an ARCH effect in our data.

Stationarity Condition of the Return Series

ADF Test.

The test that was chosen with 1 lag, show no intercept and no time trend for the return series. The unit root assumptions for the null hypothesis as below is written:

$H_0: r_{t\%}$ is I(1)

We calculated the ADF statistics: -67.97 that is significant for all the levels, meaning that we reject the null hypothesis and concluding that the series is a stationary process. Table (3) shows the critical value for the ADF on each level.

Test level (alpha)	Critical Value
10%	-1.61663
5%	-1.94093
1%	-2.56572

Table (3): ADF critical values

Autocorrelation Analysis

After running several tests on the return series, its squared and also the absolute series, and different combinations of the ARMA (p,q) orders, using the correlogram, the AR (1) gave us the best fit for the raw return series to make sure that we are ready starting for volatility modelling.

Consequently, AR (1) is selected for the return series, which is now ready for volatility modelling.

Breakpoints Test on the Return Series

In order to test for the breaks in the return series, we employ the QLR test to investigate the series. In order to do the tests for identifying potential breaks, we provide Quandt-Andrews variation of the test. Basically the test results for several Chow tests are summarizing as a single test statistic, in which a single Chow breakpoint test is achieved at each observation between two times, τ_1 , and τ_2 , and then all k test statistics from the calculated Chow tests are recorded into one test statistic. As we mentioned before, we run the test against the null hypothesis of no breakpoints between τ_1 and τ_2 .

We run sequential F-statistic determined breaks, and run the Quandt-Andrews test with 15% trimming. This test gives the following results in table (4):

Table (4)

Statistic	Value	Probability
Maximum LR F-statistic (Obs. 5419)	2.657440	0.6283
Maximum Wald F-statistic (Obs. 5419)	2.657440	0.6283
Expected LR F-statistic	0.307497	0.6338
Expected Wald F-statistic	0.307497	0.6338
Average LR F-statistic	0.523089	0.6268
Average Wald F-statistic	0.523089	0.6268

Table (5): Quandt-Andrews test with 15% trimming

To run the test, we regress our return series on a constant. The test aims at realising the potential changes in structure in the residual, which are the deviations of a series from the mean of the series. As a result, all three of the summary statistic

measures in the table (4) fail to reject the null hypothesis of no structural breaks at the 1% and 5% level within all possible dates tested. The maximum statistic was in 2.657440, and that is the most probable breakpoint location.

We also show the results for the multiple breakpoint test that supports the previous test results:

Table (5)

F stat	Critical Value for 5% level	Critical Value for 1% level
2.657440	8.58	12.29

Table (5): Multiple breakpoint tests, (Bai & Perron, 2003) tests of L+1 vs. L sequentially determined breaks using break test options of 15% trimming¹¹.

Figure (5) plots the F statistic on which the test is based, and the 1% and 5% critical F value of the test are well beyond the max F. The test also suggests that no significant breakpoint occurred on the return series for the sample duration, and thus there is no break in the series.

¹¹ the same number of observations are removed from the beginning of the estimation sample as from the end.

Figure (5)

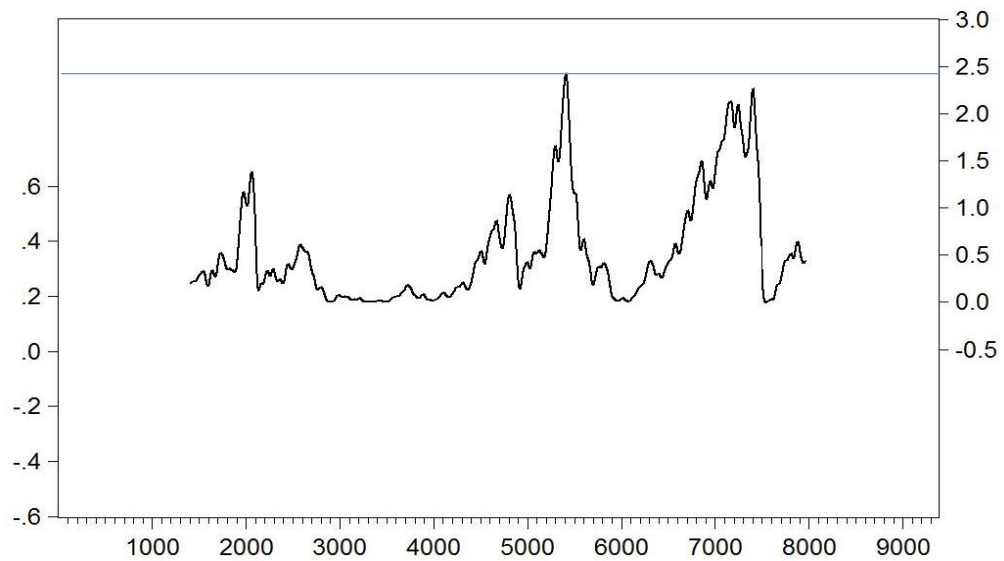


Figure (5): F max does not proceed that of the critical value for 1% or 5% level, used the Hodrick–Prescott filter and lambda of 14400.

Results on the Second Hypothesis

Estimation Results

Model1: AR(1)- GARCH (1,1); skewed t_student

We used the skewed Student distribution, with 6.73952 degrees of freedom.

The given maximum Log-likelihood reached among the different combinations of the GARCH (q, p) model was: -10616.5. The robust Standard Errors (Sandwich formula) were employed by the technology in the computations. Table (6) shows the results for the best-fit GHARCH (q, p):

Table (6)

Parameter	Coefficient	Std. Error	T-Value
Constant in Mean	0.036652	0.0080923	4.529***
AR (1)	0.142752	0.010641	13.42***

Constant in Variance	0.007975	0.0017671	4.513***
α	0.082049	0.0096930	8.465***
β	0.909441	0.010728	84.77***
Asymmetry	-0.100056	0.014126	- 7.083***

Table (6): AR (1)-GARCH (1,1) using the skewed t-student

Additionally, the constraint of $\alpha + \beta < 1$ is met as: $\alpha + \beta = 0.99149$, which makes no issues for stationarity conditions.

Below we provide the auto-correlogram results for the standardised residuals:

Table (7)

Number of lags	Test statistic on standardised residuals	Test statistic on squared standardised residuals
5	10.1777 [0.0375389]*	11.7882 [0.0081450]**
10	18.1265 [0.0337343]*	15.3296 [0.0530445]
20	24.7099 [0.1703146]	26.6233 [0.0863380]
50	59.1150 [0.1526345]	38.6631 [0.8298526]

Table (7): Q (m) test statistic on standardised residuals and squared standardised residuals

We cannot reject the null hypothesis of no serial correlation in the series. As a result, none of these series are serially correlated, showing that there is no issues with the residuals and the model is a suitable one.

ARCH Test on the standardised residuals.

Table (8)

Number of Lags	F-test
1-2	5.7468 [0.0032]**
1-5	2.2942 [0.0429]*
1-10	1.5169 [0.1262]

Table (8): ARCH test on the standardised residuals of GARCH (q, p) model

As table (8) shows, the ARCH test for the standardised residuals is not significant, apart from some probability on lower lags, meaning that there is not a significant concern about the existence of heteroskedasticity in the residual series.

Figure (6)

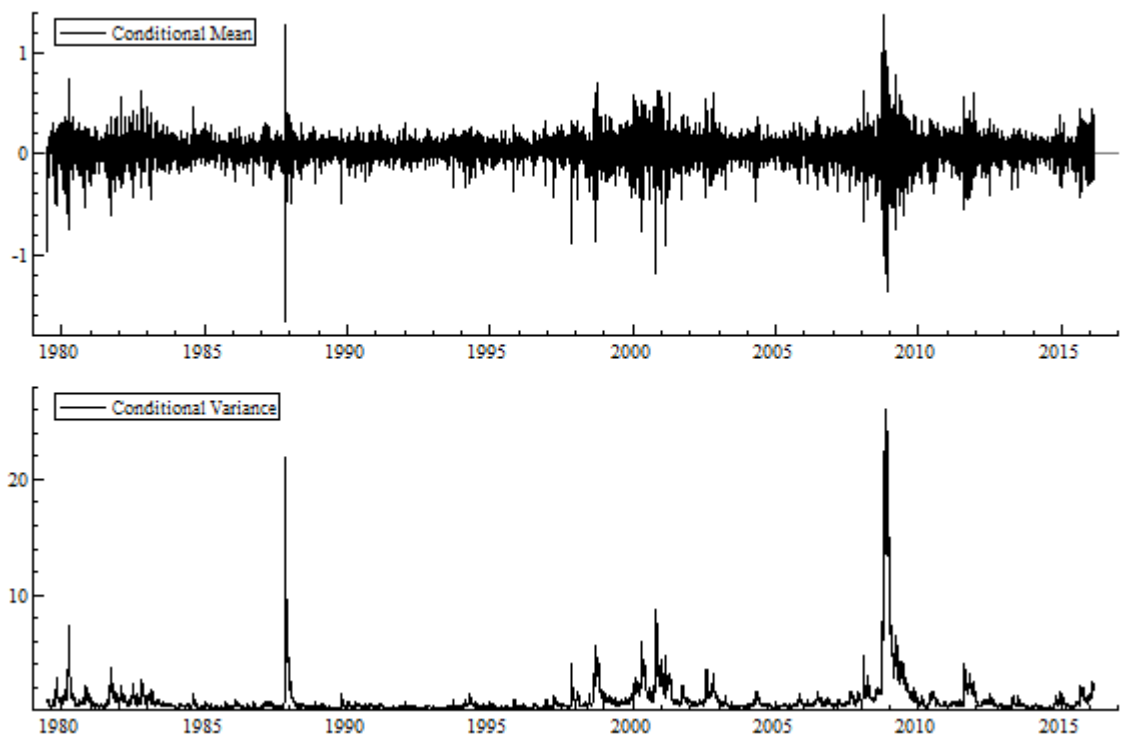


Figure (6): conditional mean and conditional variance for the AR(1)-GARCH(1,1) model

Model2: AR(1)- EGARCH (1,1); skewed t_student

We used the skewed Student distribution, with 6.87903 degrees of freedom.

The given maximum Log-likelihood reached among the different combinations of the EGARCH (q, p) model was: -10595.1. The robust Standard Errors (Sandwich

formula) were employed by the technology in the computations. Table (9) shows the results for the best-fit EGARCH (q, p):

Table (9)

Parameter	Coefficient	Std. Error	T-Value
Constant in Mean	0.029658	0.0081150	3.655***
AR (1)	0.148536	0.010019	13.42***
Constant in Variance	-2.340330	0.37103	- 6.308***
α	-0.258969	0.099233	-2.610**
β	0.989532	0.0022430	441.2***
θ_1	-0.058848	0.0093013	- 6.327***
θ_2	0.185188	0.019546	9.474***
Asymmetry	-0.099569	0.014074	- 7.075***

Table (9): AR (1)-EGARCH (1,1) using the skewed t-student

Below we provide the Q (m) test results for the standardised residuals:

Table (10)

Number of lags	Test statistic on standardised residuals	Test statistic on squared standardised residuals
5	7.06275 [0.1326093]	17.7558 [0.0004939]**
10	16.1462 [0.0638898]	22.0862 [0.0047587]**
20	22.3795 [0.2657789]	32.8537 [0.0173839]*

50	56.4994 [0.2151669]	45.7222 [0.5666617]
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Table (10): Q (m) test statistic on standardised residuals and squared standardised residuals

We cannot reject the null hypothesis of no serial correlation in the series. As a result, none of these series are serially correlated, showing that there are no issues with the residuals and the model is a suitable one. There are some minor significance results for higher levels that are not making any issues for the fit of the model.

ARCH Test on the standardised residuals.

Table (11)

Number of Lags	F-test
1-2	7.5444 [0.0005]**
1-5	3.3944 [0.0046]**
1-10	2.1500 [0.0180]*

Table (11): ARCH test on the standardised residuals of GARCH (q, p) model

As table (11) shows, the ARCH test for the standardised residuals is not significant, apart from some probability on lower lags, for higher test levels, meaning that there is not a significant concern about the existence of heteroskedasticity in the residual series.

Figure (7)

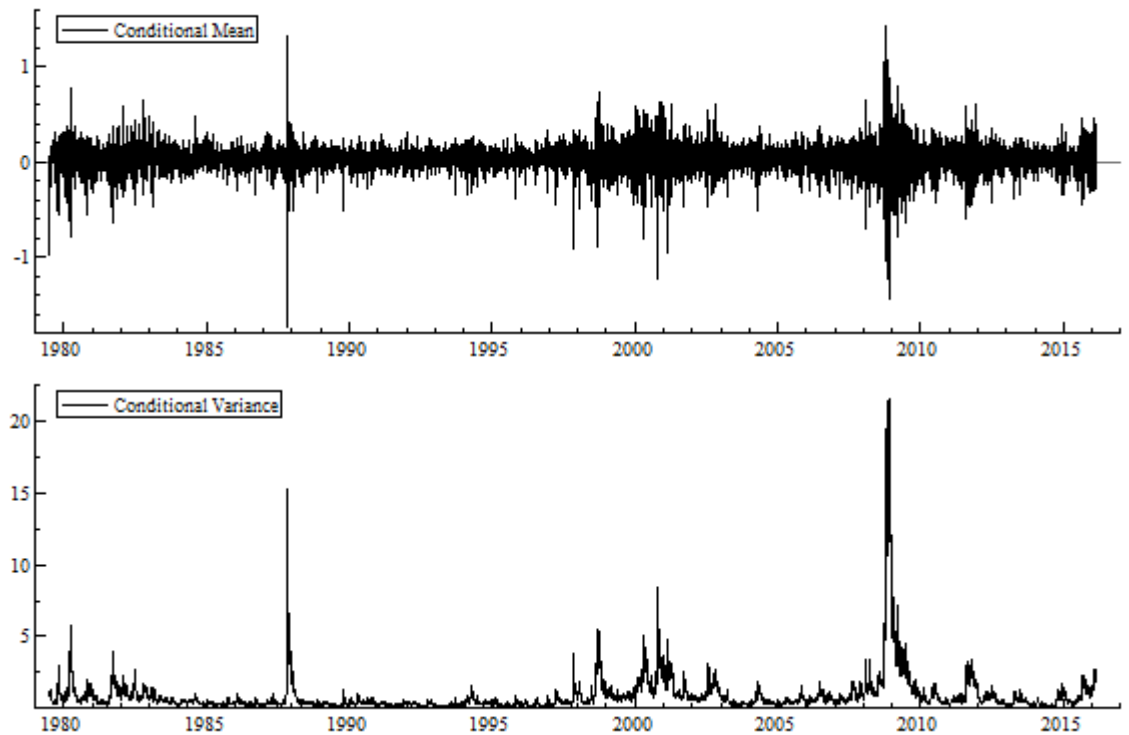


Figure (7): conditional mean and conditional variance for the AR(1)-EGARCH(1,1) model

Model3: AR (1)- GJR-GARCH (1,1); skewed t_student

We used the skewed Student distribution, with 6.85653 degrees of freedom.

The given maximum Log-likelihood reached among the different combinations of the GJR-GARCH (q, p) model was: -10598.6. The robust Standard Errors (Sandwich formula) were employed by the technology in the computations. Table (12) shows the results for the best-fit GJR-GHARCH (q, p):

Table (12)

Parameter	Coefficient	Std. Error	T-Value
Constant in Mean	0.028878	0.0082296	3.509***
AR (1)	0.146469	0.010609	13.81***
Constant in Variance	0.009229	0.0019901	4.638***

α	0.050157	0.0080233	6.251***
β	0.910313	0.011085	441.2***
γ_1	0.054376	0.010800	5.035***
Asymmetry	-0.102344	0.014136	- 7.240***

Table (12): AR (1)- GJR-GARCH (1,1) using the skewed t-student

Additionally, the constraint of $\alpha + \beta + k\gamma < 1$ is met as: $\alpha + \beta + \gamma = 0.990431$, with $k = 0.550994$, which makes no issues for stationarity conditions.

Below we provide the Q (m) test results for the standardised residuals:

Table (13)

Number of lags	Test statistic on standardised residuals	Test statistic on squared standardised residuals
5	8.19989 [0.0845245]	5.39723 [0.1449162]
10	16.6867 [0.0538537]	8.86230 [0.3540451]
20	23.4248 [0.2191487]	21.0038 [0.2792237]
50	57.6041 [0.1868940]	33.5287 [0.9439293]

Table (13): Q (m) test statistic on standardised residuals and squared standardised residuals

We cannot reject the null hypothesis of no serial correlation in the series. As a result none of these series are serially correlated, showing that there is no issues with the residuals and the model is a suitable one. There is no issue for the fit of the model.

ARCH Test on the standardised residuals.

Table (14)

Number of Lags	F-test
1-2	2.6132 [0.0734]
1-5	1.0600 [0.3805]
1-10	0.88451 [0.5469]

Table (14): ARCH test on the standardised residuals of GJR-GARCH (q, p)

model

As table (14) shows, the ARCH test for the standardised residuals is not significant, apart from some probability on lower lags, for higher test levels, meaning that there is not a significant concern about the existence of heteroskedasticity in the residual series.

Figure (8)

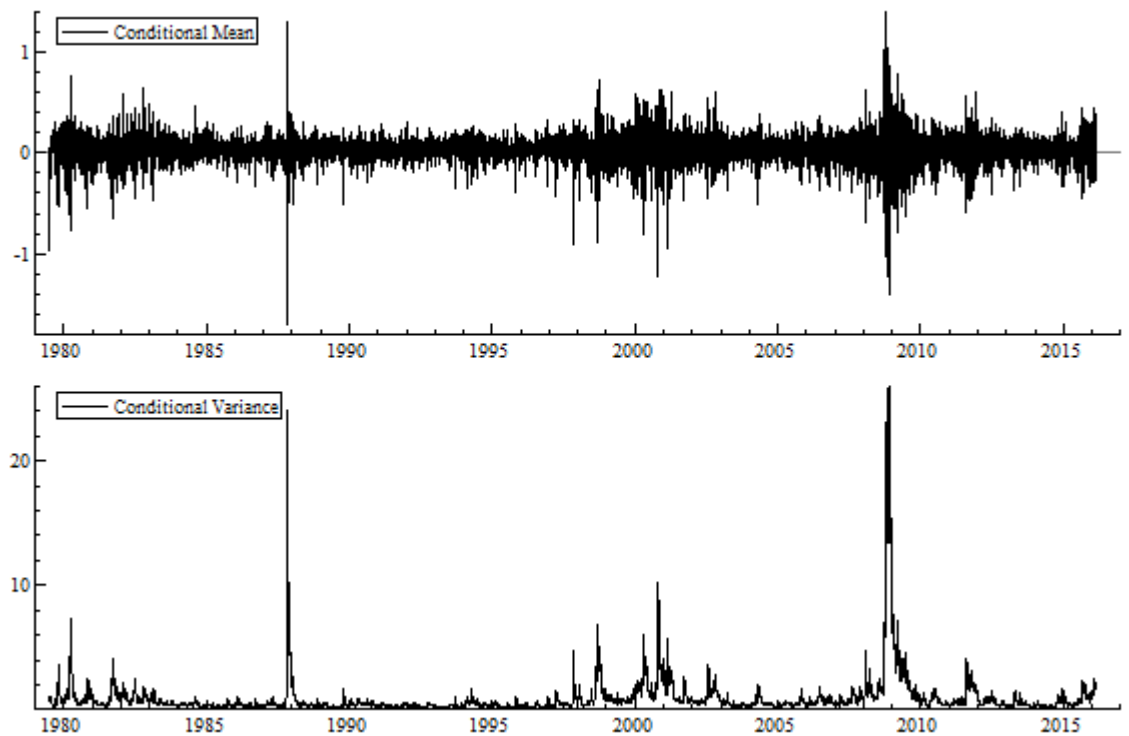


Figure (8): conditional mean and conditional variance for the AR(1)- GJR - GARCH(1,1) model

Model4: AR (1)- APARCH (1,1); skewed t_student

We used the skewed Student distribution, with 6.92483 degrees of freedom.

The given maximum Log-likelihood reached among the different combinations of the APARCH (q, p) model was: -10590.4. The robust Standard Errors (Sandwich formula) were employed by the technology in the computations. Table (15) shows the results for the best-fit APARCH (q, p):

Table (15)

Parameter	Coefficient	Std. Error	T-Value
Constant in Mean	0.028493	0.0083132	3.427***
AR (1)	0.146419	0.010699	13.69***

Constant in Variance	0.010206	0.0020763	4.915***
α	0.080012	0.0087800	9.113***
β	0.920697	0.0099302	92.72***
γ_1 (APARCH)	0.270292	0.049346	5.477***
δ	1.397313	0.13589	10.28***
Asymmetry	-0.10144	0.014192	- 7.148***

Table (15): AR (1)- APARCH (1,1) using the skewed t-student

Below we provide the Q (m) test results for the standardised residuals:

Table (16)

Number of lags	Test statistic on standardised residuals	Test statistic on squared standardised residuals
5	8.33073 [0.0801874]	13.2248 [0.0041748]**
10	16.9838 [0.0489697]*	17.0886 [0.0291995]*
20	23.4370 [0.2186407]	29.4502 [0.0431466]*
50	57.7105 [0.1843172]	42.0575 [0.7138545]

Table (16): Q (m) test statistic on standardised residuals and squared standardised residuals

We cannot reject the null hypothesis of no serial correlation in the series. As a result none of these series are serially correlated, showing that there is no issues with the residuals and the model is a suitable one. There is no issue for the fit of the model.

ARCH Test on the standardised residuals.

Table (17)

Number of Lags	F-test
1-2	6.2848 [0.0019]**
1-5	2.5476 [0.0260]*
1-10	1.6792 [0.0793]*

Table (17): ARCH test on the standardised residuals of APARCH (q, p) model

As table (17) shows, the ARCH test for the standardised residuals is not significant, apart from some probability on lower lags, for higher test levels, meaning that there is not a significant concern about the existence of heteroskedasticity in the residual series.

Figure (9)

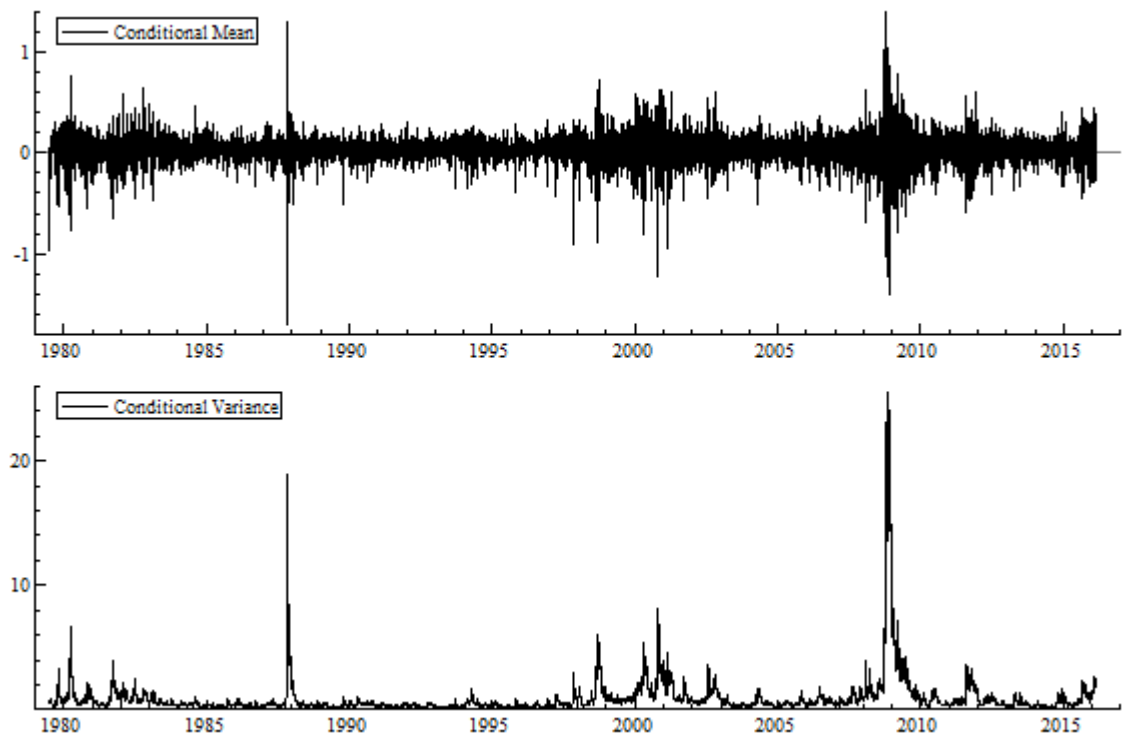


Figure (9): conditional mean and conditional variance for the AR(1)-APARCH(1,1) model

Models Comparison

So far we showed the results for the proposed model specification. We estimated GARCH (q, p), EGARCH (q, p), GJR-GARCH (q, p), and APARCH (q, p).

Now it is the time to make a comparison between the different variations of the GARCH family members.

In order to reach the best fit of each of the above models, we used log likelihood method to find the optimised number and combination of the p and q orders. Also, individually for each model, we used the AIC and BIC criteria to choose the most suitable variation of the several variations. Now using the AIC and BIC criteria, we make a comparison among the different models of the GARCH models:

Table (18)

	GARCH	EGARCH	GJR- GARCH	APARCH
AIC	2.268275	2.264135	2.264672	2.263133
BIC	2.273614	2.271000	2.270774	2.269998

Table (18): Model comparison- Information Criteria (to be minimized)

As table (18) exhibits, we can see a significant improvements of the results from our base GARCH (q, p) model to our asymmetric-GARCH models.

When comparing the three estimated asymmetric-models, we can see that the APARCH ($1, 1$) with the skewed student distribution, for the residual series, is providing the best fit.

This result also implies that the asymmetry in volatility series is a significant one. As a result, we showed that the “leverage-effect” is significant.

Such a result gives us the answer for the second hypothesis of our research. Therefore, we confirm the existence of the leverage effect in the volatility series, and thus the use of asymmetric-models is justified.

Results on the Third Hypothesis

On this part we present the results for the investigations on the long-memory properties of the volatility series.

Model5: AR (1)- FIEGARCH (1,0) BBM; GED

We used the GED distribution, with tail coefficient 1.35905. The given maximum Log-likelihood reached among the different combinations of the FIEGARCH (0, d, 1) model (Truncation order: 1850) model was: -10638.9. The robust Standard Errors

(Sandwich formula) were employed by the technology in the computations. Table (19) shows the results for the best-fit FIEGARCH (q, d, p):

Table (19)

Parameter	Coefficient	Std. Error	T-Value
Constant in Mean	0.040327	0.0072063	5.596***
AR (1)	0.146378	0.011595	12.62***
d	0.639248	0.033043	19.35***
α	--	--	--
β	0.4695	0.093384	5.029 ***
θ_1	-0.060685	0.011639	- 5.214***
θ_2	0.187078	0.022299	8.389***
GED (df)	1.359045	0.041679	32.61***

Table (19): AR (1)- FIEGARCH (0, d, 1) model using the GED

We report AIC = 2.273073 and BIC = 2.278412 for the best fit.

Below we provide the Q (m) test results for the standardised residuals:

Table (20)

Number of lags	Test statistic on standardised residuals	Test statistic on squared standardised residuals
5	10.2536 [0.0363656]*	2.37366 [0.6673931]
10	20.1395 [0.0170710]*	10.9426 [0.2796654]
20	25.4436 [0.1464524]	15.1043 [0.7159457]

50	60.8585 [0.1191785]	29.6104 [0.9871385]
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Table (20): Q (m) test statistic on standardised residuals and squared standardised residuals

We cannot reject the null hypothesis of no serial correlation in the series. As a result, none of these series are serially correlated, showing that there is no issues with the residuals and the model is a suitable one. There is no issue for the fit of the model.

ARCH Test on the standardised residuals.

Table (21)

Number of Lags	F-test
1-2	1.1312 [0.3227]
1-5	0.4705 [0.7984]
1-10	1.0769 [0.3760]

Table (21): ARCH test on the standardised residuals of FIEGARCH (0, d, 1)

model

As table (21) shows, the ARCH test for the standardised residuals is not significant, apart from some probability on lower lags, for higher test levels, meaning that there is not a significant concern about the existence of heteroskedasticity in the residual series.

Model6: AR (1)- FIAPARCH (1, d, 1) BBM; skewed t_student,

We used the skewed Student distribution, with 7.09721 degrees of freedom.

The given maximum Log-likelihood reached among the different combinations of the FIAPARCH (1, d, 1) model was: -10569.3. The robust Standard Errors

(Sandwich formula) were employed by the technology in the computations. Table (18)

shows the results for the best-fit FIAPARCH (p, d, q):

Table (22)

Parameter	Coefficient	Std. Error	T-Value
Constant in Mean	0.029595	0.0081867	3.615***
AR (1)	0.150407	0.010559	13.69***
Constant in Variance	0.035858	0.0073717	4.864***
d	0.426771	0.036505	11.69***
ϕ	0.220143	0.037257	5.909***
β	0.554898	0.052820	10.51***
γ_1 (APARCH)	0.276006	0.046597	5.923***
δ	1.626783	0.072713	22.37***
Asymmetry	-0.102014	0.014278	- 7.145***

Table (22): AR (1)- FIAPARCH (1, d , 1) using the skewed t-student

Information Criteria that are minimized are AIC = 2.258845 and BIC = 2.266473.

Below we provide the Q (m) test results for the standardised residuals:

Table (23)

Number of lags	Test statistic on standardised residuals	Test statistic on squared standardised residuals
5	8.11569 [0.0874310]	3.73785 [0.2911985]

10	17.4980 [0.0414657]*	9.57081 [0.2964500]
20	23.6981 [0.2079884]	13.5936 [0.7551718]
50	58.1483 [0.1739735]	27.7783 [0.9914623]

Table (23): Q (m) test statistic on standardised residuals and squared standardised residuals

We cannot reject the null hypothesis of no serial correlation in the series. As a result, none of these series are serially correlated, showing that there is no issues with the residuals and the model is a suitable one. There is no issue for the fit of the model.

ARCH Test on the standardised residuals.

Table (24)

Number of Lags	F-test
1-2	1.3733 [0.2533]
1-5	0.74085 [0.5928]
1-10	0.94471 [0.4903]

Table (24): ARCH test on the standardised residuals of FIAPARCH (1, d, 1)

model

As table (24) shows, the ARCH test for the standardised residuals is not significant, apart from some probability on lower lags, for higher test levels, meaning that there is not a significant concern about the existence of heteroskedasticity in the residual series.

Figure (10)

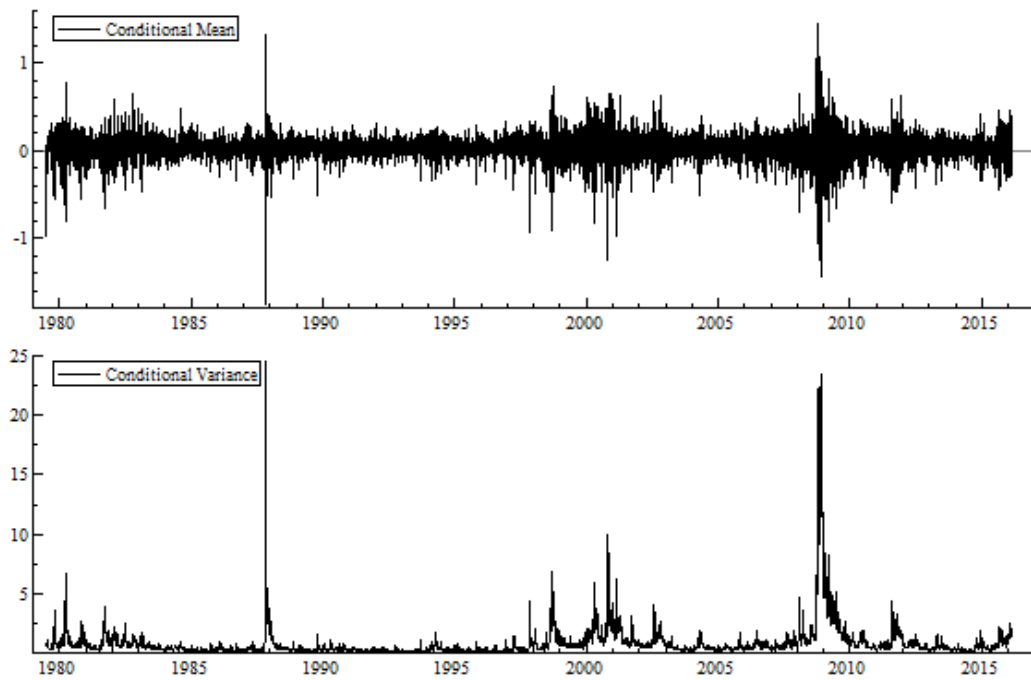


Figure (10): conditional mean and conditional variance for the AR (1)-FIARCH(1, d , 1) model

Model Fits Comparison

It is an important task to evaluate the fit of the models using the information criteria of AIC and BIC. Table (25) shows the result for such comparison:

Table (25)

	FIEGARCH	FIAPARCH
AIC	2.273073	2.258845
BIC	2.278412	2.266473

Table (25): Model comparison- Information Criteria (to be minimized)

As table (25) exhibits, we can see a significant improvements of the results from our all previously mentioned GARCH models.

When comparing all the models, by estimating the above FIAPARCH variation, we distinguished a considerable improvement of the fit. As a result, not only the long-

memory property of the series exists, but also we could get the best fit among all the variations with an asymmetric and long-memory class of the GARCH models.

As we were questioned the long-memory property of the volatility series, we shall mention that we successfully reported the fractional parameter d using the two specific long-memory models namely: FIEGARCH and FIAPARCH models.

We reported $d = 0.639248$ and $d = 0.426771$ as a fractional parameter of our fitted FIEGARCH and FIAPARCH models, respectively.

Results on the Fourth Hypothesis

BDS test results

In the following tables, the values of the BDS statistics for multiple dimensions for the designated models are provided. These values are not-significant at 5% or 1% levels, which implies the series can be considered as i.i.d. process.

Using the standardised residuals, the null hypothesis may be under-rejected (Maddala, Rao, & Vinod, 1993). Using the transformed version of the standardised residuals we still get non-significant results, confirming that the series are i.i.d. We reach the transformed residuals, using the natural logarithm of the square standardised residuals.

We used Eviews to compute the BDS test statistics.

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>	
2	0.000733	0.000796	0.920561	0.3573	
3	0.002092	0.001265	1.654099	0.0981	
4	0.002755	0.001506	1.829628	0.0673	
5	0.003353	0.001568	2.138113	0.0325	
6	0.003419	0.001512	2.261774	0.0237	
Raw epsilon		1.399518			
Pairs within epsilon		61858679	V-Statistic	0.705017	
Triples within epsilon		4.40E+11	V-Statistic	0.535574	
<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21827659	0.497709	30917157	0.704965	0.496976
3	15451538	0.352397	30909216	0.704935	0.350305
4	10945492	0.249683	30902184	0.704925	0.246928
5	7774241.	0.177380	30894183	0.704893	0.174026
6	5527629.	0.126147	30890026	0.704948	0.122728

Table (22): BDS test on the standardised residuals of the GARCH model

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>	
2	-0.001635	0.000825	-1.981138	0.0476	
3	-0.002427	0.001309	-1.853914	0.0638	
4	-0.002555	0.001557	-1.641261	0.1007	
5	-0.002687	0.001620	-1.659039	0.0971	
6	-0.002445	0.001559	-1.568454	0.1168	
Raw epsilon		3.118372			
Pairs within epsilon		61746606	V-Statistic	0.703890	
Triples within epsilon		4.40E+11	V-Statistic	0.535392	
<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21648610	0.493731	30860465	0.703823	0.495366
3	15177021	0.346210	30853513	0.703814	0.348638
4	10640508	0.242778	30845541	0.703783	0.245332
5	7452572.	0.170077	30842634	0.703867	0.172764
6	5220227.	0.119157	30836052	0.703867	0.121603

Table (23): BDS test on the transformed standardised residuals of the GARCH

model

From table (22) and (23) we can see that specifically for the lower dimensions, the BDS test on the standardised residual series may be a bit under-rejected. Using the transformed standardised residuals improves the results for large samples, as our study sample with a length more than 9000 data points.

The same logic applies to the rest of the models below. Following tables show the results for each model using standardised residuals and transformed standardised residuals, from table (24) through table (33).

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>
2	-0.000410	0.000791	-0.518526	0.6041
3	0.000485	0.001256	0.386357	0.6992
4	0.001324	0.001492	0.887137	0.3750
5	0.002268	0.001553	1.461028	0.1440
6	0.002663	0.001494	1.782001	0.0747

Raw epsilon	1.397131			
Pairs within epsilon	61773579	V-Statistic	0.704047	
Triples within epsilon	4.39E+11	V-Statistic	0.533980	

<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21717588	0.495199	30874622	0.703995	0.495609
3	15317744	0.349346	30866678	0.703964	0.348861
4	10823415	0.246898	30859728	0.703956	0.245574
5	7674434.	0.175103	30851737	0.703924	0.172834
6	5450081.	0.124378	30847360	0.703975	0.121715

Table (24): BDS test on the standardised residuals of the EGARCH model

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>
2	-0.001332	0.000827	-1.609865	0.1074
3	-0.001925	0.001309	-1.470144	0.1415
4	-0.001913	0.001553	-1.231825	0.2180
5	-0.001901	0.001612	-1.179323	0.2383
6	-0.001685	0.001548	-1.088988	0.2762

Raw epsilon	3.096501			
Pairs within epsilon	61591466	V-Statistic	0.702121	
Triples within epsilon	4.38E+11	V-Statistic	0.533002	

<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21552832	0.491547	30782883	0.702053	0.492879
3	15084094	0.344091	30775961	0.702045	0.346016
4	10561000	0.240964	30768047	0.702015	0.242876
5	7392650.	0.168709	30765346	0.702103	0.170610
6	5173853.	0.118099	30758711	0.702102	0.119784

Table (25): BDS test on the transformed standardised residuals of the EGARCH model

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>	
2	0.000300	0.000793	0.378881	0.7048	
3	0.001273	0.001259	1.011585	0.3117	
4	0.001817	0.001496	1.213984	0.2248	
5	0.002433	0.001557	1.562259	0.1182	
6	0.002560	0.001499	1.707503	0.0877	
Raw epsilon		1.397230			
Pairs within epsilon		61790003	V-Statistic	0.704234	
Triples within epsilon		4.39E+11	V-Statistic	0.534322	
<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21760428	0.496176	30882905	0.704184	0.495875
3	15364638	0.350415	30874978	0.704154	0.349142
4	10856707	0.247658	30868111	0.704147	0.245841
5	7691913.	0.175501	30860111	0.704115	0.173069
6	5454719.	0.124483	30856188	0.704176	0.121924

Table (26): BDS test on the standardised residuals of the GJR-GARCH model

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>	
2	-0.001476	0.000826	-1.786652	0.0740	
3	-0.002189	0.001312	-1.667642	0.0954	
4	-0.002294	0.001562	-1.468631	0.1419	
5	-0.002373	0.001628	-1.458032	0.1448	
6	-0.002136	0.001569	-1.361595	0.1733	
Raw epsilon		3.118914			
Pairs within epsilon		61827776	V-Statistic	0.704815	
Triples within epsilon		4.41E+11	V-Statistic	0.536737	
<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21712741	0.495194	30901046	0.704748	0.496670
3	15247510	0.347818	30893850	0.704735	0.350007
4	10708382	0.244326	30885955	0.704705	0.246621
5	7515907.	0.171522	30882930	0.704787	0.173895
6	5275794.	0.120426	30876456	0.704789	0.122562

Table (27): BDS test on the transformed standardised residuals of the GJR-GARCH model

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>	
2	0.000515	0.000791	0.650349	0.5155	
3	0.001411	0.001256	1.124098	0.2610	
4	0.001904	0.001493	1.275672	0.2021	
5	0.002454	0.001553	1.580462	0.1140	
6	0.002525	0.001495	1.689029	0.0912	
Raw epsilon		1.398230			
Pairs within epsilon		61782647	V-Statistic	0.704150	
Triples within epsilon		4.39E+11	V-Statistic	0.534120	
<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21764576	0.496270	30879181	0.704099	0.495756
3	15365133	0.350427	30871233	0.704068	0.349015
4	10855231	0.247624	30864303	0.704061	0.245720
5	7688187.	0.175416	30856295	0.704028	0.172962
6	5448871.	0.124350	30852017	0.704081	0.121825

Table (28): BDS test on the standardised residuals of the APARCH model

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>	
2	-0.001341	0.000828	-1.619628	0.1053	
3	-0.001982	0.001310	-1.512364	0.1304	
4	-0.002026	0.001554	-1.303599	0.1924	
5	-0.002049	0.001613	-1.270704	0.2038	
6	-0.001815	0.001549	-1.171799	0.2413	
Raw epsilon		3.096852			
Pairs within epsilon		61586980	V-Statistic	0.702070	
Triples within epsilon		4.38E+11	V-Statistic	0.532967	
<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21549288	0.491466	30780650	0.702002	0.492807
3	15078161	0.343955	30773638	0.701992	0.345937
4	10552812	0.240777	30765711	0.701961	0.242803
5	7383075.	0.168491	30762831	0.702046	0.170540
6	5165647.	0.117912	30756232	0.702045	0.119726

Table (29): BDS test on the transformed standardised residuals of the APARCH model

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>	
2	-0.000376	0.000790	-0.476433	0.6338	
3	-0.001293	0.001252	-1.032478	0.3018	
4	-0.002116	0.001487	-1.422514	0.1549	
5	-0.002068	0.001546	-1.337643	0.1810	
6	-0.002005	0.001486	-1.348535	0.1775	
Raw epsilon		1.365485			
Pairs within epsilon		61720545	V-Statistic	0.703443	
Triples within epsilon		4.38E+11	V-Statistic	0.533059	
<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21682082	0.494389	30848334	0.703396	0.494766
3	15200826	0.346679	30840453	0.703366	0.347972
4	10636543	0.242635	30833831	0.703365	0.244751
5	7452648.	0.170042	30825836	0.703333	0.172110
6	5219728.	0.119121	30822414	0.703406	0.121125

Table (30): BDS test on the standardised residuals of the FIEGARCH model

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>	
2	-0.001505	0.000834	-1.803455	0.0713	
3	-0.002256	0.001321	-1.708152	0.0876	
4	-0.002329	0.001567	-1.486288	0.1372	
5	-0.002304	0.001627	-1.416081	0.1568	
6	-0.002045	0.001563	-1.308456	0.1907	
Raw epsilon		3.092773			
Pairs within epsilon		61610482	V-Statistic	0.702338	
Triples within epsilon		4.38E+11	V-Statistic	0.533645	
<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21558746	0.491682	30792488	0.702272	0.493186
3	15083175	0.344070	30785161	0.702255	0.346326
4	10555744	0.240844	30777431	0.702229	0.243173
5	7386099.	0.168560	30774500	0.702312	0.170864
6	5167779.	0.117960	30768173	0.702318	0.120006

Table (31): BDS test on the transformed standardised residuals of the EGARCH model

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>
2	-0.000809	0.000787	-1.027997	0.3040
3	-0.001067	0.001248	-0.855497	0.3923
4	-0.001649	0.001482	-1.112700	0.2658
5	-0.001655	0.001541	-1.073749	0.2829
6	-0.001786	0.001482	-1.204959	0.2282

Raw epsilon	1.401761		
Pairs within epsilon	61739187	V-Statistic	0.703655
Triples within epsilon	4.38E+11	V-Statistic	0.533210

<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21676201	0.494255	30857638	0.703608	0.495064
3	15224480	0.347219	30849726	0.703578	0.348286
4	10669778	0.243393	30843019	0.703575	0.245043
5	7481976.	0.170711	30835005	0.703543	0.172366
6	5238589.	0.119551	30831396	0.703610	0.121337

Table (32): BDS test on the standardised residuals of the FIAPARCH model

<u>Dimension</u>	<u>BDS Statistic</u>	<u>Std. Error</u>	<u>z-Statistic</u>	<u>Prob.</u>
2	-0.001308	0.000831	-1.573731	0.1155
3	-0.001914	0.001317	-1.453813	0.1460
4	-0.001939	0.001563	-1.240698	0.2147
5	-0.001966	0.001624	-1.210547	0.2261
6	-0.001771	0.001561	-1.134002	0.2568

Raw epsilon	3.094511		
Pairs within epsilon	61656622	V-Statistic	0.702864
Triples within epsilon	4.39E+11	V-Statistic	0.534227

<u>Dimension</u>	<u>C(m,n)</u>	<u>c(m,n)</u>	<u>C(1,n-(m-1))</u>	<u>c(1,n-(m-1))</u>	<u>c(1,n-(m-1))^k</u>
2	21599758	0.492617	30815537	0.702798	0.493925
3	15132249	0.345189	30808178	0.702780	0.347103
4	10604660	0.241960	30800390	0.702753	0.243899
5	7428946.	0.169538	30797515	0.702837	0.171504
6	5203416.	0.118774	30791154	0.702842	0.120544

Table (33): BDS test on the transformed standardised residuals of the FIAPARCH model

As the results show, the test statistics for the BDS tests are not significant, implying that we cannot reject the null hypothesis of the i.i.d. process for the series. This also confirms that the models that are fitted are suitable ones and we can be sure that the variations of the GARCH models that we used with the provided parameters

and orders are providing a good explanations of the data. It does not mean that these are the “best” descriptive models on the data, but considering the time and skills limitations of the author, we are hopeful that this research provides a basis for the further future work on the topic and the related data. We will talk more about the limitations in a following part of this paper.

Chapter5

Summary of Empirical Works and Their Importance

The statements that include sufficient detail to identify the probability distributions of future prices are asset price dynamics. Using the empirical methods that could explain the historical prices and discussing about how prices change, we investigate various important characteristics of the dynamics of asset prices for the S&P/TSX composite index. The volatility changes can explain very important facts about the asset returns. Volatility gauges the variability of prices over time. Focus of this research is on volatility modelling using conditional heteroscedasticity models.

As the ARCH-family (or GARCH) models are powerful tools in modelling on financial data, we employ some of the most popular variations of the GARCH models to capture different characteristics of the series under investigation.

We use the raw data: price series of the index, and process them to prepare our return series before start modelling volatility. We organise the empirical tests on basis of four main hypotheses, and provide the test results and comments, step-by-step.

In this paper we aim to detect the asymmetric volatility effect and investigate long memory properties in volatility in Canadian stock market index, using daily data. On one hand, we show, whether or not, there is a significant asymmetry in the volatility.

This is an important indication of how information impacts the market. We expect that asymmetry exists while studying the equity market indices, and if it exists, measure the magnitude of such asymmetry. We allocate the second hypothesis of the research on this topic. By estimating the EGARCH, GJR- GARCH, and APARCH variations, we show that the asymmetry in volatility exists, and we report the differences in the related parameters of the models.

On the other hand, we investigate for the long-range dependency in volatility and discuss how the shocks are persistence. By using the long-memory- GARCH models, we not only can successfully take into account both short and long memory, but also we can compute the d parameter that explains the fractional decay of the series and the pace that a shock dies out. We report d parameter by fitting FIEGARCH and FIAPARCH variation of the long-memory models. This is the focus of the tests regarding the third hypothesis of our research.

By considering the breaks in the residuals of the fitted models we confirm that there is no issue in the process of the GARCH-modelling. We calculate BDS test statistics for the six models that are estimated, and the results support that the fitted models are suitable ones. This is the main task under investigation on the fourth hypothesis.

Research Limitations

We design this research and its methods with regards to the time limitations that we had and complexity level of the whole project. As this research is a part to fulfil a master studies, we could not go much beyond the presented work, and our methods and data was limited under such circumstance.

For instance, another investigation using the multivariate-GARCH models can be added to this research and, additionally, use of the sectors data and US market data are suggested. In this research we try a good step toward working on the volatility modelling literature on Canadian data, however, the mentioned level and depth of study would take more time and diligence effort that is beyond this work and may be achievable in a further future research.

Further Future Research

We should mention that this work can be extended to the market data on the sectors and subsectors for analysing a specific policy or simply to investigate on the associations of the different sectors of the index. There are some related works in the literature on the association between the energy sector and the Canadian market performance that can be improved using the more sophisticated models, such as: FIEGARCH, FIEGARCH-M, and FIAPARCH, to give more information on such association, where more parameters and conditions are controlled and some better practical applications can be achieved.

In a similar way as comparison among the sectors, the data among different markets can be compared for any potential associations of the market return or spill-over of the volatility and local versus global factors. For instance, a study of the US and Canadian equity markets can be a fruitful research for investigating a specific case for policy makers, or financial regulations of a central bank.

References

- Andersen, T. G., & Bollerslev, T. (1997). Heterogeneous Information Arrivals and Return Volatility Dynamics: Uncovering the Long-Run in High Frequency Returns. *American Finance Association, Journal of Finance*, 975- 1005.
- Andersen, T. G., Bollerslev, T., & Lange, S. (1999). Forecasting financial market volatility: Sample frequency vis-à-vis forecast horizon. *Journal of Empirical Finance*, 457– 477.
- Anderson, T. G., Bollerslev, T., & Das, A. (1998). Testing for Market Microstructure Effects in Intraday Volatility: A Reassessment of the Tokyo FX Experiment. *NBER Working Paper*.
- Andrews, D. W. (1993). Tests for Parameter Instability and Structural Change With Unknown Change Point. *Econometrica*, 821–856.
- Bai, & Perron. (2003). Computation and Analysis of Multiple Structural Change Models. *Journal of Applied Econometric*, 72- 78.
- Baillie, R. T., & DeGennaro, R. P. (1990). Stock Returns and Volatility. *Journal of Financial and Quantitative Analysis*, 203- 214.
- Baillie, R. T., Bollerslev, T., & Mikkelsen, H. (1996). Fractionally integrated generalized autoregressive conditional. *Journal of Econometrics*, 3- 30.
- Baillie, R. T., Chung, C.-F., & Tieslau, M. A. (1996). Analysing Inflation by the Fractionally Integrated Arfima--Garch Model. *Journal of Applied Econometrics*, 23- 40.
- Baillie, R., & Bollerslev, T. (1993). Cointegration, Fractional Cointegration, and Exchange Rate Dynamics. *Econometrics and Economic Theory*.
- Baillie, R., & Chung, C.-F. (1993). Small Sample Bias in Conditional Sum-of-Squares Estimators of Fractionally Integrated ARMA Models. *Empirical Economics*, 791- 806.
- Barberis, N., & Thaler, R. (2003). Chapter 18 A survey of behavioral finance . In B. L. Gardner, & G. C. Rausser, *Financial Markets and Asset Pricing* (pp. 1053-1128). NORTH-HOLLAND.
- Black, F. (1976). Studies of Stock Market Volatility Changes. *Proceedings of the American Statistical Association, Business and Economic Statistics Section*, 177–181.
- Black, F., & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 637– 654.
- Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 307- 321.

- Bollerslev, T., & Mikkelsen, H. (1996). Modeling and pricing long memory in stock market volatility. *Journal of Econometrics*, 151- 184.
- Bollerslev, T., Chou, R. Y., & Kroner, K. F. (1992). ARCH modeling in finance : A review of the theory and empirical evidence. *Elsevier, Journal of Econometrics*, 5-59.
- Bollerslev, T., Engle, R. F., & Nelson, D. B. (1994). *ARCH models*. Handbook of econometrics, Elsevier.
- Brock, W., Scheinkman, J., Dechert, W., & LeBaron, B. (1996). A Test for Independence Based on the Correlation Dimension. *Econometric Reviews*, 197-237.
- Chow, G. (1960). Tests of Equality Between Sets of Coefficients in two Linear Regressions. *Econometrica*, 591- 605.
- Christensen, B. J., & Nielsen, M. Ø. (2007). The effect of long memory in volatility on stock market fluctuations. *Review of Economics and Statistics*., 684– 700.
- Christensen, B. J., Nielsen, M. O., & Zhu, J. (2010). Long memory in stock market volatility and the volatility-in-mean effect: The FIEGARCH-M Model. *Journal of Empirical Finance*, 460– 470.
- Christensen, B., Nielsen, M., & Zhu, J. (2015). The Impact of Financial Crises on the Risk-Return Tradeoff and the Leverage Effect. *Economic Modelling*, 407-418.
- Christie, A. A. (1982). The stochastic behavior of common stock variances: Value, leverage and interest rate effects. *Journal of financial Economics, Elsevier*.
- de Lima, P. J. (1996). Nuisance Parameter Free Properties of Correlation Integral Based Statistics. *Econometric Reviews*, 237- 259.
- Ding, Z., Granger, C. W., & Engle, R. F. (1993). A Long Memory Property of Stock Market Returns and a New Model. *Journal of Empirical Finance*, 83–106.
- Engle, R. F. (1982). •Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica*, 987–1007.
- Engle, R. F., & Bollerslev, T. (1986). Modeling the Persistence of Conditional Variances. *Econometric Reviews*.
- Engle, R. F., & Bollerslev, T. (1986). Modeling the Persistence of Conditional Variances. *Econometric Reviews*, 1-50.
- Fama, E. F. (1965). The Behavior of Stock Market Prices. *The Journal of Business*., 34-105.
- Fama, E. F. (1970). Efficient Capital Markets: A Review of Theory and Empirical Work. *The Journal of Finance*.

- Fisher, I. (1930). *THE THEORY OF INTEREST, AS DETERMINED BY IMPATIENCE TO SPEND INCOME AND OPPORTUNITY TO INVEST IT*. Macmillan Company.
- Glosten, L. R., Jagannathan, R., & Runkle, D. E. (1993). On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *The Journal of Finance*.
- Granger, C. W. (1980). Long Memory Relationships and the Aggregation of Dynamic Models. *Journal of Econometrics*, 227–238.
- Granger, C. W., & Joyeux, R. (1980). An Introduction to Long-Memory Time Series Models and Fractional Differencing. *Journal of Time Series Analysis*, 15–29.
- Han, Y. W. (2003). Long Memory Property and Central Bank Intervention during the Currency Crisis in the Daily Korean won-US dollar Exchange Rates. *THE JOURNAL OF THE KOREAN ECONOMY*, 93- 116.
- Hosking, J. R. (1981). Fractional Differencing. *Biometrika*, 165- 176.
- Hurst, H. E. (1951). Long-term Storage Capacity of Reservoirs . *Trans. Amer. Soc. Civil Eng*, 770- 808 .
- Lintner, J. (1965). The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets,. *Review of Economics and Statistics*, 13– 37.
- Lo, A. W. (1991). Long-Term Memory in Stock Market Prices. *Econometrica*, 1279- 1313.
- Maddala, G. S., Rao, C. R., & Vinod, H. D. (1993). *Hand Book of Statistics 11*. North-Holland.
- Mandelbort, B. B., & Van Ness, J. W. (1968). Fractional Brownian Motions, Fractional Noises and Applications. *SIAM Review*.
- Mandelbrot, B. (1963). The Variation of Certain Speculative Prices. *University of Chicago Press, The Journal of Business*, 394.
- McLeod, A., & Li, W. (1983). Diagnostic checking ARMA time series models using squared-residual autocorrelations. *Journal of Time Series Analysis*, 269–273.
- Nelson, D. B. (1991). Conditional Heteroskedasticity in Asset Returns: a New Approach. *Econometrica*, 349–370.
- Quandt, R. (1960). Tests of the Hypothesis That a Linear Regression System Oobeys two Separate Regimes. *Journal of American Statistical Association*, 320- 330.
- Racicot, F.-É. (2012). Notes on Nonlinear Dynamics. *Aestimatio, The ieb International Journal of Finance*, 162- 221.
- Racicot, F.-É., & Théoret, R. (2016). Macroeconomic Shocks, Forward-Looking Dynamics, and the Behavior of Hedge Funds. *Journal of Banking & Finance*, 41-61.

- Rambaccussing, D. (2014). A Simple Test of Volatility Models: the Case of U.K. and Romania . *Journal of Business Management and Applied Economics*.
- Sharpe, W. F. (1964). CAPITAL ASSET PRICES: A THEORY OF MARKET EQUILIBRIUM UNDER CONDITIONS OF RISK. *Journal of Finance*, 425– 442 .
- Shleifer, A., & Vishny, R. W. (1986). Large shareholders and corporate control. *The Journal of Political Economy*, 461- 488.
- Sivakumar, P. B., & Mohandas, V. P. (2009). Modeling and predicting stock returns using the ARFIMA-FIGARCH. *Nature & Biologically Inspired Computing*.
- Statman, M. (2014). Behavioral finance: Finance with normal people. *Borsa Istanbul Review*, 65- 73.
- Stock, J., & Watson, M. (2011). Introduction to Econometrics. *Addison-Wesley*.
- Taylor, S. J. (2005). *Asset Price Dynamics, Volatility, and Prediction*. Princeton University Press.
- Tse, Y. K. (1998). The Conditional Heteroscedasticity of the Yen-Dollar Exchange Rate. *Journal of Applied Econometrics, John Wiley & Sons, Ltd*.
- Williams, J. B. (1938). *The Theory of Investment Value*. Harvard University Press- Fraser Publishing.