

A STUDY OF CERTAIN GENERALIZED HYPERGEOMETRIC
FUNCTIONS

A thesis submitted

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ABSTRACT

The thesis is concerned with the G- and H-functions. We apply the methods of integral transforms, fractional integration and representation theory of Lie group to study some properties of the G- and H-functions.

We apply a uniqueness theorem for the solution of Fredholm integral equation of the first kind to prove a necessary and sufficient condition for a pair of functions to be G₇ transforms of each other and for a pair of functions to be H-transforms of each other.

We solve the following integral equation

$$\int_0^{\infty} (xu)^{\mu-\frac{1}{2}} e^{-\frac{xu}{2}} W_{\lambda, \mu}(xu) f(u) du = g(x)$$

by reducing it to a Laplace transform by means of fractional integral operators. A more general integral equation whose kernel involves H-functions is briefly discussed. The problem of dual integral equations involving H-functions as kernels has been discussed too.

We also find integral representations of some H-functions by using group representation theory and then evaluate limiting values of some H-functions.

INTRODUCTION

Higher transcendental functions, also called special functions of mathematical physics, play an important role in the solutions of problems in theoretical physics. The core of special functions is the Gaussian hypergeometric function ${}_2F_1$ and its confluent forms, the confluent hypergeometric functions ${}_1F_1$ and ψ [31]. The confluent hypergeometric functions slightly modified are also known as Whittaker functions. The ${}_2F_1$ includes as special cases Legendre functions, the incomplete beta functions, the complete elliptic integrals of the first and second kinds, and most of the classical orthogonal polynomials. The confluent hypergeometric functions include as special cases Bessel functions, parabolic cylinder functions, and incomplete gamma functions. A natural generalization of the ${}_2F_1$ is the generalized hypergeometric function, the so-called ${}_pF_q$, which in turn is generalized by Meijer's G-function. The theory of the ${}_pF_q$ and the G-function is fundamental in the applications, since they contain as special cases all the commonly used functions of analysis. Further, these functions are the building blocks for many functions which are not members of the hypergeometric family. A special function known as H-function, a generalization of G-function, was recently introduced by Fox and occupied the attention of several authors.

These higher transcendental functions are analytic functions of their arguments and their properties are usually derived on the basis of their analytic character, using the methods of the theory of analytic functions, integral transforms, generating functions, differential equations, etc.,

On the other hand, many properties of the special functions can be derived from a unified point of view from the group representation property. The group-theoretic approach leads to a natural treatment of integral representations of special functions. In this connection operators of the representation take the form of integral operators whose kernels are expressed by means of special functions. This leads to various integral relations between special functions.

This thesis is devoted to a study of some properties of G- and H-functions.

From the point of view of approach, this thesis can be divided into two parts. Chapters I - III employ integral transform methods to study the properties of special functions. In Chapter IV we use group representation theory for the same purpose.

In the following a short summary will be given of the contents of the whole work.

In Chapter I, Mellin transform is introduced and a theorem due to Fox is given. Since the subsequent work heavily depends

upon this theorem, we also reproduce its proof. As an application of this theorem, we have a convolution theorem of Mellin transform in $L_2(0, \infty)$ which is similar to the result in $L(0, \infty)$.

The definitions of G- and H-functions are given in §1.2 and some of its elementary properties are listed in §1.3. Description of general integral transformations in $L_2(0, \infty)$ and Fourier kernels have occupied the space in the remaining part of Chapter I.

In Chapter II, the G- and H-transforms are introduced. We obtain a necessary and sufficient condition for two functions to be a pair of G-transforms by using a uniqueness theorem on solutions of Fredholm integral equation of the first kind which is proved in §2.2. The proof of the sufficiency of the condition is believed to be new in this thesis. Later we extend the result of G-transform to H-transform in §2.5.

In Chapter III, definitions of fractional integration are given and some of its properties summarized. We use the technique of fractional integration to solve the following integral equation

$$\int_0^{\infty} (xu)^{\mu-\frac{1}{2}} e^{-\frac{xu}{2}} W_{\lambda, \mu}(xu) f(u) du = g(x).$$

We show that the above integral equation may be reduced to a Laplace transform by means of fractional integral operators. Later the above integral equation has been generalized to a kernel involving H-functions. In the last section of this chapter the problem of dual integral equations involving H-functions as kernels has been

considered. We reduce the problem to that of solving a single integral equation by applying appropriate fractional integral operators.

In Chapter IV we construct a $(p+2)$ -dimensional real Lie group G . Then we apply Vilenkin's integral transform method to define a local representation of the group G and calculate the kernels of the operators of this representation. We find that the kernels which are associated to certain elements of G can be expressed in terms of H -functions. In §4.4 we obtain limiting values of some H -functions.

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CHAPTER I

DEFINITIONS AND PRELIMINARIES

1. In this chapter we shall collect definitions of some special functions and results on certain integral transforms which we shall need in the later chapters.

1.1 MELLIN TRANSFORM.

By

$$\text{l.i.m.}_{n \rightarrow \infty} F_n(x)$$

(limit in mean) we denote limit in $L_2(a, b)$ sense, i.e., a function $\phi(x)$ such that

$$\lim_{n \rightarrow \infty} \int_a^b |\phi(x) - F_n(x)|^2 dx = 0.$$

Let $\mathcal{M}\{f(z)\}$ denote the Mellin transform of $f(z)$. If $\mathcal{M}\{f(z)\} = F(s)$ we shall also write $f(z) = \mathcal{M}^{-1}\{F(s)\}$ and \mathcal{M}^{-1} will denote the inverse Mellin transform. Formally we have

$$(1.1.1) \quad F(s) = \mathcal{M}\{f(z)\} = \int_0^{\infty} f(z) z^{s-1} dz$$

and

$$(1.1.2) \quad f(z) = \mathcal{M}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\rho-1\infty}^{\rho+1\infty} F(s) z^{-s} ds.$$

The simplest conditions for (1.1.1) and (1.1.2) to hold are given in $L_2(0, \infty)$ space, which we shall use in this thesis.

If $f \in L_2(0, \infty)$ then

$$(1.1.3) \quad F(s) = \mathcal{M}\{f(z)\} = \text{l.i.m.}_{N \rightarrow \infty} \int_{1/N}^N f(z) z^{s-1} dz, \text{Re } s = \frac{1}{2}$$

and also $F \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. If $F \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ then

$$(1.1.4) \quad f(z) = \mathcal{M}^{-1}\{F(s)\} = \frac{1}{2\pi i} \text{l.i.m.}_{N \rightarrow \infty} \int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} F(s) z^{-s} ds, 0 < z < \infty$$

and also $f \in L_2(0, \infty)$ [40, p. 94, Theorem 71 with $k = \frac{1}{2}$]. Throughout this thesis the symbol \mathcal{M} will denote the Mellin transform.

If the integral of (1.1.1) and the l.i.m. of (1.1.3) both exist then they are equal [15, p. 458]. The same is true for (1.1.2) and (1.1.4).

If $x > 0$, from (1.1.3) or (1.1.4) it follows easily that if $f(xz)$ is considered as a function of z with x as a parameter then

$$(1.1.5) \quad \mathcal{M}\{f(xz)\} = x^{-s} \mathcal{M}\{f(z)\}.$$

The following result of Fox [15, p. 458] will be useful in our work.

Theorem 1.1.1 (Fox). If

- (i) f and g both belong to $L_2(0, \infty)$,
- (ii) $\mathcal{M}\{f(z)\} = F(s)$, $\mathcal{M}\{g(z)\} = G(s)$ and $G(s)$ is bounded on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$,

then

$$(1.1.6) \quad \int_0^{\infty} g(xz) f(z) dz \in L_2(0, \infty)$$

and

$$(1.1.7) \quad \mathcal{M}\left\{\int_0^{\infty} g(xz) f(z) dz\right\} = G(s) F(1-s).$$

Proof. Since $f \in L_2(0, \infty)$ we have $F(s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ by (1.1.3) and so, on writing $1-s$ for s , we also have $F(1-s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. From condition (ii), $G(s)$ is bounded on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$, i.e., on this line we have

$$|G(s) F(1-s)| \leq B|F(1-s)|,$$

where B is a finite positive constant. Hence

$G(s) F(1-s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and so, from (1.1.4), we have

$$(1.1.8) \quad \mathcal{M}^{-1}\{G(s)F(1-s)\} = \frac{1}{2\pi i} \lim_{N \rightarrow \infty} \int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} G(s)F(1-s)x^{-s} ds.$$

We now proceed to deduce from (1.1.8) the following result;

$$(1.1.9) \quad \mathcal{M}^{-1}\{G(s)F(1-s)\} = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} G(s)F(1-s)x^{-s} ds.$$

Denote the integral in (1.1.9) by I . From (i) and the statements accompanying (1.1.3) we see that $F(s)$, $G(s)$ and so $F(1-s)$, all belong to $L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence, on applying Schwarz's

inequality to I [39, p. 381], it follows that I converges absolutely. Consequently the value of I must be the same as the l.i.m. of (1.1.8); hence (1.1.9) is established.

One more step is required before we can establish the theorem. We may apply the Parseval Theorem for Mellin transforms [40, p. 95, Theorem 72] to I, an application justified by condition (i). From (1.1.9) we then obtain

$$(1.1.10) \quad \mathcal{M}^{-1}\{G(s)F(1-s)\} = \int_0^{\infty} g(xz)f(z)dz.$$

To establish (1.1.6), we have just proved that $G(s)F(1-s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and therefore, from the statement following (1.1.4), we have $\mathcal{M}^{-1}\{G(s)F(1-s)\} \in L_2(0, \infty)$. Hence the right-hand side of (1.1.10), belongs to $L_2(0, \infty)$ and (1.1.6) is established.

To establish (1.1.7) we note that both sides of (1.1.10) belong to $L_2(0, \infty)$ and so by (1.1.3) we may apply the operator \mathcal{M} to both sides. This application transforms (1.1.10) to (1.1.7) and so establishes (1.1.7).

Note that the condition (ii) in Theorem 1.1.1, viz. $G(s)$ is bounded on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$, may be changed to a weaker condition that $G(s)F(1-s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$.

If $f \in L_2(0, \infty)$, let $\mathcal{M}\{f(z)\} = F(s)$ then it can be easily shown that $\frac{1}{z} f(\frac{1}{z}) \in L_2(0, \infty)$ and $\mathcal{M}\{\frac{1}{z} f(\frac{1}{z})\} = F(1-s)$.

Let

$$(1.1.11) \quad (f * g)(x) = \int_0^{\infty} \frac{1}{z} f\left(\frac{x}{z}\right) g(z) dz = \int_0^{\infty} \frac{1}{z} f\left(\frac{1}{z}\right) g(xz) dz.$$

The function $f * g$ is called the convolution of f and g .

Simple calculations show that the convolution operation is commutative and associative whenever it is defined. Associated with Mellin transforms we have the following convolution theorem.

Theorem 1.1.2

(i) Let f and g both belong to $L_2(0, \infty)$,

(ii) $\mathcal{M}\{f(z)\} = F(s)$, $\mathcal{M}\{g(z)\} = G(s)$ and either $F(s)$

or $G(s)$ be bounded on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$,
then

$$(1.1.12) \quad (f * g)(x) = \int_0^{\infty} \frac{1}{z} f\left(\frac{x}{z}\right) g(z) dz \in L_2(0, \infty)$$

and

$$(1.1.13) \quad \mathcal{M}\{(f * g)(x)\} = F(s) G(s).$$

Proof. This follows directly from Theorem 1.1.1 and the remark just before (1.1.11).

1.2 DEFINITIONS OF G- AND H-FUNCTIONS.

In this thesis, we will be mainly concerned with establishing results involving G- and H-functions which are generalized hypergeometric functions in a sense described below.

The following notations will be used throughout this thesis [13, p. 757], [17, p. 444].

$$\Gamma_n(b_Q - t) = \prod_{j=n+1}^q \Gamma(b_j - t), \quad \Gamma(b_Q - t) = \Gamma_0(b_Q - t);$$

$$\Gamma_n(a_Q - \alpha_Q t) = \prod_{j=n+1}^q \Gamma(a_j - \alpha_j t), \quad \Gamma(a_Q - \alpha_Q t) = \Gamma_0(a_Q - \alpha_Q t);$$

$$\Gamma_n(k + a_Q - \alpha_Q t) = \left(\prod_{j=n+1}^q \Gamma(k + a_j - \alpha_j t) \right),$$

$$\Gamma(k + a_Q - \alpha_Q t) = \Gamma_0(k + a_Q - \alpha_Q t).$$

$(a_1 + b, \alpha_1)_{m,n}$, $n > m$, will stand for $n - m + 1$ pairs $(a_m + b, \alpha_m)$, $(a_{m+1} + b, \alpha_{m+1})$, ..., $(a_n + b, \alpha_n)$. Thus $(a_1, \alpha_1)_{1,n}$ will stand for n pairs (a_1, α_1) , (a_2, α_2) , ..., (a_n, α_n) and so on.

A G-function is written as on the left of (1.2.1) below and defined [31, p. 143] by the integral on the right

(1.2.1)

$$G_{p,q}^{m,n} \left(x \left| \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array} \right. \right) = \frac{1}{2\pi i} \int_L \frac{\Gamma(b_M - s) \Gamma(1 - a_N + s)}{\Gamma_m(1 - b_Q + s) \Gamma_n(a_P - s)} x^s ds,$$

where an empty product is interpreted as unity, $0 \leq m \leq q$, $0 \leq n \leq p$, parameters a_j and b_j are such that no pole of $\Gamma(b_j - s)$, $j = 1, \dots, m$, coincides with any pole of $\Gamma(1 - a_k + s)$, $k = 1, \dots, n$. Thus $a_k - b_j$ is not a positive integer. We retain these assumptions throughout. Also $x \neq 0$.

There are three different paths L of integration [31, p. 144]:

(i) L goes from $\sigma - i\infty$ to $\sigma + i\infty$ so that all poles of $\Gamma(b_j - s)$, $j = 1, \dots, m$, lie to the right, and all poles of $\Gamma(1 - a_k + s)$, $k = 1, \dots, n$, lie to the left of the path.

For the integral to converge, we need $\delta = m + n - \frac{1}{2}(p + q) > 0$,

$|\arg x| < \delta\pi$. If $|\arg x| = \delta\pi$, $\delta \geq 0$, the integral converges absolutely when $p = q$ if $\operatorname{Re}(v) < -1$; and when $p \neq q$, if with $s = \sigma + i\tau$, σ and τ real, σ is chosen so that for $\tau \rightarrow \pm\infty$,

$$(1.2.2) \quad (q - p)\sigma > \operatorname{Re}(v) + 1 - \frac{1}{2}(q - p),$$

where v is given by

$$(1.2.3) \quad v = \sum_{j=1}^q b_j - \sum_{j=1}^p a_j.$$

(ii) L is a loop beginning and ending at $+\infty$ and encircling all poles of $\Gamma(b_j - s)$, $j = 1, \dots, m$, once in the negative direction, but none of the poles of $\Gamma(1 - a_k + s)$, $k = 1, \dots, n$. The integral converges if $q \geq 1$ and either (a) $p < q$, or (b) $p = q$ and $|x| < 1$.

(iii) L is a loop beginning and ending at $-\infty$ and encircling all poles of $\Gamma(1 - a_k + s)$, $k = 1, \dots, n$, once in the positive direction, but none of the poles of $\Gamma(b_j - s)$, $j = 1, \dots, m$. The integral converges if $p \geq 1$ and either (a) $p > q$, or (b) $p = q$ and $|x| > 1$.

We shall always assume that the values of the parameters and the variable x are such that the definition with at least one of the contours (i), (ii), (iii), makes sense. In cases when more than one of these contours make sense, they lead to the same result so that there will be no ambiguity involved [31, p. 145].

Note that with $m = 1$, $p < q$, the G-function in (1.2.1) reduces to a single hypergeometric function ${}_pF_{q-1}$, namely [31, p. 146, eqn. (12)]

$$G_{p,q}^{l,n} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = \frac{\Gamma(1+b_1-a_N) x^{b_1}}{\Gamma_1(1+b_1-b_Q) \Gamma_n(a_p-b_1)} \times {}_pF_{q-1} \left(\begin{matrix} 1+b_1-a_1, \dots, 1+b_1-a_p \\ 1+b_1-b_2, \dots, 1+b_1-b_q \end{matrix} ; (-1)^{p-n-1} x \right).$$

It is for this reason that G-function is considered as a generalization of hypergeometric function. For more information on G-function one may refer to [31, Chapter 5].

Another function more general than the G-function, called the H-function, was first introduced by Fox [14, p. 408]. It is defined and denoted as follows [16, p. 142]

(1.2.5)

$$H_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) = \frac{1}{2\pi i} \int_L \frac{\Gamma(b_M - \beta_M s) \Gamma(1 - a_N + \alpha_N s)}{\Gamma_m(1 - b_Q + \beta_Q s) \Gamma_n(a_p - \alpha_p s)} x^s ds$$

where x may be real or complex but is not equal to zero and

empty product is interpreted as unity, m, n, p, q are integers satisfying

$$(1.2.6) \quad 0 \leq m \leq q, \quad 0 \leq n \leq p,$$

$\alpha_j, j = 1, \dots, p; \beta_h, h = 1, \dots, q,$ are positive numbers and $a_j, j = 1, \dots, p; b_h, h = 1, \dots, q,$ are complex numbers such that no pole of $\Gamma(b_h - \beta_h s), h = 1, \dots, m,$ coincides with any pole of $\Gamma(1 - a_j + \alpha_j s), j = 1, \dots, n,$ i.e.,

$$(1.2.7) \quad \alpha_j (b_h + v) \neq \beta_h (a_j - n - 1), \quad v, n = 0, 1, 2, \dots;$$

$h = 1, \dots, m; j = 1, \dots, n.$ We retain these assumptions throughout. L is a contour in the complex s -plane that goes from $\sigma - i\infty$ to $\sigma + i\infty$ such that the points

$$(1.2.8) \quad s = (b_j + v)/\beta_j, \quad j = 1, \dots, m; \quad v = 0, 1, 2, \dots,$$

which are poles of $\Gamma(b_j - \beta_j s), j = 1, \dots, m,$ lie to the right and the points

$$(1.2.9) \quad s = (a_j - 1 - n)/\alpha_j, \quad j = 1, \dots, n; \quad n = 0, 1, 2, \dots,$$

which are poles of $\Gamma(1 - a_j + \alpha_j s), j = 1, \dots, n,$ lie to the left of the contour $L.$ Such a contour is possible on account of (1.2.7). For the integral to converge [6, p. 83], we need

$$2\delta = \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j + \sum_{j=1}^n \alpha_j - \sum_{j=n+1}^p \alpha_j > 0, \quad |\arg x| < \delta\pi.$$

If $|\arg x| = \delta\pi, \delta \geq 0,$ the integral converges absolutely when

$$\sum_{j=1}^q \beta_j = \sum_{j=1}^p \alpha_j \text{ if } \frac{1}{2}(q - p) > 1 + \operatorname{Re}\left(\sum_{j=1}^q b_j - \sum_{j=1}^p a_j\right);$$

and when $\sum_{j=1}^q \beta_j \neq \sum_{j=1}^p \alpha_j$, if with $s = \sigma + i\tau$, σ, τ real, σ is chosen so that for $\tau \rightarrow \pm\infty$,

(1.2.10)

$$\sigma \left(\sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j \right) > \operatorname{Re} \left(\sum_{j=1}^q b_j - \sum_{j=1}^p a_j \right) + 1 - \frac{1}{2}(q - p).$$

Note that the H-function reduces to the G-function of (1.2.1) when all the α 's, β 's are equal to unity.

1.3 ELEMENTARY PROPERTIES OF G- AND H-FUNCTIONS.

From (1.2.1), G-function is symmetric in the parameters $a_1, \dots, a_n; a_{n+1}, \dots, a_p; b_1, \dots, b_m; \text{ and } b_{m+1}, \dots, b_q$. Thus, if one of the a_h 's, $h = 1, \dots, n$, is equal to one of the b_j 's, $j = m+1, \dots, q$, the G-function reduces to one of lower order [31, p. 149]. For example

$$(1.3.1) \quad G_{p,q}^{m,n} \left(x \left| \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_{q-1}, a_1 \end{array} \right. \right) = G_{p-1,q-1}^{m,n-1} \left(x \left| \begin{array}{c} a_2, \dots, a_p \\ b_1, \dots, b_{q-1} \end{array} \right. \right),$$

$n, p, q \geq 1$. Similarly, if one of the a_h 's, $h = n+1, \dots, p$, is equal to one of the b_j 's, $j = 1, \dots, m$, then the G-function reduces to one of lower order. For example,

$$(1.3.2) \quad G_{p,q}^{m,n} \left(x \left| \begin{array}{c} a_1, \dots, a_{p-1}, b_1 \\ b_1, \dots, b_q \end{array} \right. \right) = G_{p-1,q-1}^{m-1,n} \left(x \left| \begin{array}{c} a_1, \dots, a_{p-1} \\ b_2, \dots, b_q \end{array} \right. \right),$$

$m, p, q \geq 1$.

The following two results can be easily proved from

(1.2.1)

$$(1.3.3) \quad G_{p,q}^{m,n} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = G_{q,p}^{n,m} \left(\frac{1}{x} \left| \begin{matrix} 1-b_1, \dots, 1-b_q \\ 1-a_1, \dots, 1-a_p \end{matrix} \right. \right),$$

$$\arg \left(\frac{1}{x} \right) = -\arg x,$$

$$(1.3.4) \quad x^\sigma G_{p,q}^{m,n} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = G_{p,q}^{m,n} \left(x \left| \begin{matrix} a_1+\sigma, \dots, a_p+\sigma \\ b_1+\sigma, \dots, b_q+\sigma \end{matrix} \right. \right).$$

Similar results hold for H-function. For example

$$(1.3.5) \quad H_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q-1}, (a_1, \alpha_1) \end{matrix} \right. \right) = H_{p-1,q-1}^{m,n-1} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{2,p} \\ (b_1, \beta_1)_{1,q-1} \end{matrix} \right. \right),$$

$n, p, q \geq 1,$

$$(1.3.6) \quad H_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1,p-1}, (b_1, \beta_1) \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) = H_{p-1,q-1}^{m-1,n} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1,p-1} \\ (b_1, \beta_1)_{2,q} \end{matrix} \right. \right),$$

$m, p, q \geq 1,$

$$(1.3.7) \quad H_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) = H_{q,p}^{n,m} \left(\frac{1}{x} \left| \begin{matrix} (1-b_1, \beta_1)_{1,q} \\ (1-a_1, \alpha_1)_{1,p} \end{matrix} \right. \right),$$

$$\arg \left(\frac{1}{x} \right) = -\arg x,$$

and

$$(1.3.8) \quad x^\sigma H_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) = H_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_1+\sigma, \alpha_1)_{1,p} \\ (b_1+\sigma, \beta_1)_{1,q} \end{matrix} \right. \right).$$

From the definition of the G-function, (1.2.1), and the Mellin inversion formula (1.1.4) we have [10, p. 337]

$$(1.3.9) \quad \mathcal{M}\{G_{p,q}^{m,n} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)\} = \frac{\Gamma(b_M+s)\Gamma(1-a_N-s)}{\Gamma_m(1-b_Q-s)\Gamma_n(a_P+s)},$$

where $2(m+n) > p+q$, $-\min_{1 \leq j \leq m} \operatorname{Re}(b_j) < \operatorname{Re}(s) < 1 - \max_{1 \leq j \leq n} \operatorname{Re}(a_j)$.

Similar result holds for H-function, i.e.,

$$(1.3.10) \quad \mathcal{M}\{H_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right)\} = \frac{\Gamma(b_M+\beta_M s)\Gamma(1-a_N-\alpha_N s)}{\Gamma_m(1-b_Q-\beta_Q s)\Gamma_n(a_P+\alpha_P s)},$$

where

$$\sum_{j=1}^m \beta_j + \sum_{j=1}^n \alpha_j > \sum_{j=m+1}^q \beta_j + \sum_{j=n+1}^p \alpha_j,$$

$$-\min_{1 \leq j \leq m} \operatorname{Re}(b_j/\beta_j) < \operatorname{Re}(s) < \min_{1 \leq j \leq n} (1 - a_j)/\alpha_j.$$

1.4 UNITARY TRANSFORMATION IN $L_2(0, \infty)$.

Let T be an operator on $L_2(0, \infty)$ and $Tf = g$ then [1, pp. 152, 153, Theorems 77, 78] T is unitary if and only if there exist two unique (up to a set of measure zero) functions $k(a, x)$, $l(a, x)$ belonging to the class $L_2(0, \infty)$ for each a , $0 < a < \infty$, such that:

$$(1.4.1) \quad \int_0^a g(y)dy = \int_0^{\infty} k(a, x) f(x)dx,$$

$$(1.4.2) \quad \int_0^a f(x)dx = \int_0^{\infty} l(a, y) g(y)dy,$$

$$(1.4.3) \quad \int_0^{\infty} k(a,x) \bar{k}(b,x) dx = \int_0^{\infty} \ell(a,x) \bar{\ell}(b,x) dx = \min(a,b),$$

$$(1.4.4) \quad \int_0^b k(a,x) dx = \int_0^a \ell(b,x) dx.$$

hold.

Suppose that $k(a,x)$ has a partial derivative

$$(1.4.5) \quad \frac{\partial}{\partial a} k(a,x) = \bar{k}(a,x)$$

in the sense that there exists $\bar{k}(a,x)$ continuous in $0 \leq a < \infty$, $0 \leq x < \infty$, such that

$$(1.4.6) \quad k(a,x) = \int_0^a \bar{k}(\alpha, x) d\alpha.$$

Then (1.4.1) can also be written as [1, p. 155]

$$(1.4.7) \quad g(y) = \text{l.i.m.}_{n \rightarrow \infty} \int_0^{\infty} k(y,x) f(x) dx.$$

If there exist functions k_1, ℓ_1 defined everywhere in $0 < x < \infty$, such that

$$(1.4.8) \quad k(a,x) = \frac{k_1(ax)}{x}, \quad \ell(a,x) = \frac{\ell_1(ax)}{x},$$

then (1.4.4) gives

$$(1.4.9) \quad \int_0^{ab} \frac{\bar{k}_1(x)}{x} dx = \int_0^a \frac{\bar{\ell}_1(x)}{x} dx,$$

and therefore

$$(1.4.10) \quad \bar{\ell}_1(x) = \bar{k}_1(x) \text{ a.e.},$$

and (1.4.3) reduces to

$$(1.4.11) \quad \int_0^{\infty} \frac{k_1(ax)k_1(bx)}{x^2} dx = \min(a,b).$$

Conversely, starting with a function k_1 defined in $(0, \infty)$

such that $\frac{k_1(x)}{x} \in L_2(0, \infty)$, we define $k(a,x)$, $\ell(a,x)$ by

(1.4.8), (1.4.10). Then (1.4.11) implies (1.4.3) and (1.4.4).

Hence we have [1, p. 157, Theorem 79].

Theorem 1.4.1 Given $\frac{k_1(x)}{x} \in L_2(0, \infty)$ such that

$$(1.4.12) \quad \int_0^{\infty} \frac{k_1(ax)k_1(bx)}{x^2} dx = \min(a,b),$$

then there exists a transformation T in $L_2(0, \infty)$ which is unitary, and if $Tf = g$, then

$$(1.4.13) \quad \int_0^a g(x) dx = \int_0^{\infty} \frac{k_1(ay)}{y} f(y) dy,$$

$$(1.4.14) \quad \int_0^a f(x) dx = \int_0^{\infty} \frac{k_1(ay)}{y} g(y) dy.$$

This transformation is called a Watson transform [1, p. 157].

1.5 SYMMETRICAL FOURIER KERNELS.

If in Watson transforms, the function $k_1(x)$ is real-valued for $x \in (0, \infty)$, then

$$k_1(x) = \ell_1(x)$$

and (1.4.13) becomes

$$(1.5.1) \quad \int_0^a g(u)du = \int_0^{\infty} \frac{k_1(ay)}{y} f(y)dy.$$

In case we can differentiate with respect to a under the integral sign of (1.5.1), (1.4.14), we have

$$(1.5.2) \quad g(x) = \frac{d}{dx} \int_0^{\infty} \frac{k_1(xy)}{y} f(y)dy = \int_0^{\infty} k(xy)f(y)dy,$$

$$(1.5.3) \quad f(x) = \frac{d}{dx} \int_0^{\infty} \frac{k_1(xy)}{y} g(y)dy = \int_0^{\infty} k(xy)g(y)dy,$$

where $k(x) = \frac{d}{dx} k_1(x)$. Relations (1.5.2) and (1.5.3) may be combined into one as

$$(1.5.4) \quad f(x) = \int_0^{\infty} k(xu)du \int_0^{\infty} k(uy)f(y)dy.$$

A function k giving rise to a formula of the form (1.5.4) will be called a symmetrical Fourier kernel.

We shall give a few examples of symmetrical Fourier kernels.

(1) $k(x) = \sqrt{\frac{2}{\pi}} \cos x$ gives rise to Fourier Cosine transform [40, p. 3].

(2) $k(x) = \sqrt{\frac{2}{\pi}} \sin x$ gives rise to Fourier Sine transform [40, p. 4].

(3) $k(x) = \sqrt{x} J_{\nu}(x)$, $\nu \geq -\frac{1}{2}$ gives rise to Hankel transform [40, p. 240].

(4) $k(x) = \tilde{\omega}_{\mu, \nu}(x)$ gives rise to an integral transform studied by Watson [44, p. 308].

$$(5) \quad k(x) = \frac{1}{c} G_{2p, 2q}^{q, p} \left(x^{\frac{1}{c}} \left| \begin{matrix} a_1, \dots, a_p, 1-c-a_1, \dots, 1-c-a_p \\ b_1, \dots, b_q, 1-c-b_1, \dots, 1-c-b_q \end{matrix} \right. \right),$$

$c > 0$, where $G_{p, q}^{m, n}$ denotes the Meijer G-function, defined in

(1.2.1) gives rise to a Fourier type transform investigated by Fox [14, p. 396].

$$(6) \quad k(x) = \gamma \mu^{\frac{\gamma}{2}} x^{\frac{\gamma-1}{2}} G_{2p, 2q}^{q, p} \left((\mu x)^{\gamma} \left| \begin{matrix} a_1, \dots, a_p, -a_1, \dots, -a_p \\ b_1, \dots, b_q, -b_1, \dots, -b_q \end{matrix} \right. \right),$$

$\gamma, \mu > 0$, gives rise to a Fourier type transform investigated by Kesarwani [22, p. 298].

1.6 UNSYMMETRICAL FOURIER KERNELS.

Instead of (1.5.4), there are also formulae of the form

$$(1.6.1) \quad f(x) = \int_0^{\infty} k(xu) du \int_0^{\infty} h(uy) f(y) dy,$$

where $h \neq k$, i.e., k and h differ on a set of positive measure.

Pairs of functions k and h giving rise to a formula of the form

(1.6.1) are called unsymmetrical Fourier kernels.

Here we give some examples of unsymmetrical Fourier kernels.

$$(1) \quad k(x) = \sqrt{x} Y_{\nu}(x), \quad h(x) = \sqrt{x} H_{\nu}(x),$$

where Y_{ν} denotes the Bessel function of the second kind and

H_{ν} the Struve's function [45, pp. 64, 328], give rise to an

integral transform studied by Hardy and Titchmarsh [19, p. 119].

$$(2) \quad k(x) = \frac{1}{\sqrt{\pi}} (\cos x + \sin x + e^{-x}),$$

$$h(x) = \frac{1}{\sqrt{\pi}} (\cos x + \sin x - e^{-x}),$$

are unsymmetrical Fourier kernels obtained by Guinand [18, p. 192].

(3) The kernels

$$k(x) = A\sqrt{x} G_{p+q, m+n}^{m, p} \left(\frac{x^2}{4} \left| \begin{matrix} a_1, \dots, a_p, b_1, \dots, b_q \\ c_1, \dots, c_m, d_1, \dots, d_n \end{matrix} \right. \right),$$

$$h(x) = \frac{1}{A} \sqrt{x} G_{p+q, m+n}^{n, q} \left(\frac{x^2}{4} \left| \begin{matrix} -b_1, \dots, -b_q, -a_1, \dots, -a_p \\ -d_1, \dots, -d_n, -c_1, \dots, -c_m \end{matrix} \right. \right)$$

with certain restrictions on the parameters involved, give rise to an unsymmetrical Fourier type transform investigated by Kesarwani [23, p. 953].

(4) Functions of the form

$$(1.6.2) \quad k(x) = \frac{1}{2\pi i} \int_L \frac{\Gamma(c_M + \gamma_M(s - \frac{1}{2})) \Gamma(a_P - \alpha_P(s - \frac{1}{2}))}{\Gamma(d_N - \delta_N(s - \frac{1}{2})) \Gamma(b_Q + \beta_Q(s - \frac{1}{2}))} x^{-s} ds,$$

$$(1.6.3) \quad h(x) = \frac{1}{2\pi i} \int_L \frac{\Gamma(d_N + \delta_N(s - \frac{1}{2})) \Gamma(b_Q - \beta_Q(s - \frac{1}{2}))}{\Gamma(c_M - \gamma_M(s - \frac{1}{2})) \Gamma(a_P + \alpha_P(s - \frac{1}{2}))} x^{-s} ds,$$

which are a generalization of Meijer's G-function have recently occupied the attention of several authors. Under suitable conditions on the parameters involved, they have been proved to be a pair of unsymmetrical Fourier kernels [26, p. 362].

1.7 A NECESSARY CONDITION FOR TWO FUNCTIONS TO BE A PAIR OF FOURIER KERNELS.

In this section we shall establish heuristically a necessary condition for a given pair of functions to be Fourier kernels.

Let k and h be a pair of unsymmetrical Fourier kernels, i.e., if

$$(1.7.1) \quad g(x) = \int_0^{\infty} k(xy)f(y)dy$$

then

$$(1.7.2) \quad f(x) = \int_0^{\infty} h(xy)g(y)dy.$$

Let K , H , F and G denote the Mellin transforms of k , h , f and g respectively. On multiplying both sides of (1.7.1) by x^{s-1} and integrating over $(0, \infty)$, we obtain formally

$$\begin{aligned} G(s) &= \int_0^{\infty} x^{s-1}g(x)dx = \int_0^{\infty} x^{s-1}dx \int_0^{\infty} k(xy)f(y)dy \\ &= \int_0^{\infty} f(y)dy \int_0^{\infty} x^{s-1}k(yx)dx = \int_0^{\infty} y^{-s}f(y)dy \int_0^{\infty} w^{s-1}k(w)dw \\ &= F(1-s)K(s). \end{aligned}$$

Similarly, (1.7.2) gives

$$F(s) = G(1-s)H(s)$$

and, if we change s into $1-s$ in one of these equations, and multiply the corresponding sides, we deduce that

$$(1.7.3) \quad K(s)H(1-s) = 1.$$

Thus, a necessary condition for functions k, h to be a pair of unsymmetrical Fourier kernels is that their Mellin transforms satisfy the functional relation (1.7.3). In case of symmetrical Fourier kernel, the relation (1.7.3) reduces to

$$(1.7.4) \quad K(s)K(1-s) = 1.$$

Note also that, if k, h are a pair of unsymmetrical Fourier kernels, so are the following two pairs,

$$\sqrt{a} k(ax) \text{ and } \sqrt{a} h(ax);$$

$$\gamma x^{\frac{\gamma-1}{2}} k(x^\gamma) \text{ and } \gamma x^{\frac{\gamma-1}{2}} h(x^\gamma).$$

As an example, if we take $k(x) = \sqrt{x} J_\nu(x)$, it is readily found [10, pp. 326, 307]

$$K(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{4} + \frac{1}{2}\nu + \frac{s}{2})}{\Gamma(\frac{3}{4} + \frac{\nu}{2} - \frac{s}{2})}$$

which obviously satisfies the functional equation (1.7.4).

As another example, take

$$k(x) = \sqrt{x} H_\nu(x), \quad h(x) = \sqrt{x} Y_\nu(x),$$

then [10, pp. 335, 329]

$$K(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{4} + \frac{\nu}{2} + \frac{s}{2})}{\Gamma(\frac{3}{4} + \frac{\nu}{2} - \frac{s}{2})} \tan \pi(\frac{1}{4} + \frac{\nu}{2} + \frac{s}{2})$$

and

$$H(s) = -2^{s-\frac{1}{2}} \frac{1}{\pi} \Gamma(\frac{1}{4} + \frac{\nu}{2} + \frac{s}{2}) \Gamma(\frac{1}{4} - \frac{\nu}{2} + \frac{s}{2}) \cos \pi(\frac{1}{4} - \frac{\nu}{2} + \frac{s}{2}).$$

Now

$$\begin{aligned}
 K(s)H(1-s) &= 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{4}+\frac{v}{2}+\frac{s}{2})}{\Gamma(\frac{3}{4}+\frac{v}{2}-\frac{s}{2})} \tan \pi(\frac{1}{4}+\frac{v}{2}+\frac{s}{2}) \\
 &\quad \times \{-2^{\frac{1}{2}-s} \frac{1}{\pi} \Gamma(\frac{3}{4}+\frac{v}{2}-\frac{s}{2})\Gamma(\frac{3}{4}-\frac{v}{2}-\frac{s}{2}) \cos \pi(\frac{3}{4}-\frac{v}{2}-\frac{s}{2})\} \\
 &= -\csc \pi(\frac{1}{4}+\frac{v}{2}+\frac{s}{2}) \tan \pi(\frac{1}{4}+\frac{v}{2}+\frac{s}{2}) \cos \pi(\frac{3}{4}-\frac{v}{2}-\frac{s}{2}) \\
 &= 1,
 \end{aligned}$$

on using the relation [9, p. 3]

$$(1.7.5) \quad \Gamma(z)\Gamma(1-z) = \pi \csc \pi z.$$

Next take k, h as in example 4, §1.6, equations (1.6.2),

(1.6.3) respectively. Then

$$\begin{aligned}
 K(s) &= \frac{\Gamma(c_M + \gamma_M(s - \frac{1}{2}))\Gamma(a_P - \alpha_P(s - \frac{1}{2}))}{\Gamma(d_N - \delta_N(s - \frac{1}{2}))\Gamma(b_Q + \beta_Q(s - \frac{1}{2}))}, \\
 H(s) &= \frac{\Gamma(d_N + \delta_N(s - \frac{1}{2}))\Gamma(b_Q - \beta_Q(s - \frac{1}{2}))}{\Gamma(c_M - \gamma_M(s - \frac{1}{2}))\Gamma(a_P + \alpha_P(s - \frac{1}{2}))},
 \end{aligned}$$

which obviously satisfy the functional equation (1.7.3).

1.8 AN INVERSION THEOREM.

Relation (1.7.3) is the formal necessary condition for k and h to be unsymmetrical Fourier kernels. However, we need only assume the existence of the function $K(s)$ and $H(s)$ on the line $\sigma = \frac{1}{2}$, where $s = \sigma + i\tau$ so that (1.7.3) becomes

$$(1.8.1) \quad K(\frac{1}{2} + i\tau)H(\frac{1}{2} - i\tau) = 1.$$

We also assume that $K(\frac{1}{2} + i\tau)$ and $H(\frac{1}{2} + i\tau)$ are both bounded as $|\tau| \rightarrow \infty$.

Let $k_1(x)$ and $h_1(x)$ be so defined, that $\frac{k_1(x)}{x}$ and $\frac{h_1(x)}{x}$ are the inverse Mellin transforms of $\frac{K(\frac{1}{2} + i\tau)}{\frac{1}{2} - i\tau}$ and $\frac{H(\frac{1}{2} + i\tau)}{\frac{1}{2} - i\tau}$ respectively, that is

$$(1.8.2) \quad \frac{k_1(x)}{x} = \frac{1}{2\pi} \text{l.i.m.}_{N \rightarrow \infty} \int_{-N}^N \frac{K(\frac{1}{2} + i\tau)}{\frac{1}{2} - i\tau} x^{-\frac{1}{2} - i\tau} d\tau,$$

$$(1.8.2)' \quad \frac{h_1(x)}{x} = \frac{1}{2\pi} \text{l.i.m.}_{N \rightarrow \infty} \int_{-N}^N \frac{H(\frac{1}{2} + i\tau)}{\frac{1}{2} - i\tau} x^{-\frac{1}{2} - i\tau} d\tau.$$

The integrals are convergent in the L_2 -sense since on account

of the boundedness of $K(\frac{1}{2} + i\tau)$ and $H(\frac{1}{2} + i\tau)$, $\frac{K(\frac{1}{2} + i\tau)}{\frac{1}{2} - i\tau}$ and

$\frac{H(\frac{1}{2} + i\tau)}{\frac{1}{2} - i\tau}$ belong to $L_2(-\infty, \infty)$, and hence $\frac{k_1(x)}{x}$ and $\frac{h_1(x)}{x}$ belong

to $L_2(0, \infty)$. Analogous to results (1.5.2) and (1.5.3) for

symmetrical Fourier kernels, we have the following inversion

theorem for unsymmetrical Fourier kernels [40, p. 226, §8.9],

[25, p. 272].

Theorem 1.8.1 Let $K(\frac{1}{2} + i\tau)$ and $H(\frac{1}{2} + i\tau)$ be bounded functions of τ satisfying (1.8.1). Let k_1, h_1 be defined by (1.8.2), (1.8.2)' respectively, and f be any function belonging to $L_2(0, \infty)$. Then the formulae

$$(1.8.3) \quad g_k(x) = \frac{d}{dx} \int_0^{\infty} \frac{k_1(xu)}{u} f(u) du,$$

$$(1.8.4) \quad g_h(x) = \frac{d}{dx} \int_0^{\infty} \frac{h_1(xu)}{u} f(u) du,$$

define almost everywhere functions g_k, g_h respectively both belonging to $L_2(0, \infty)$. Also the reciprocal formulae

$$(1.8.5) \quad f(x) = \frac{d}{dx} \int_0^{\infty} \frac{h_1(xu)}{u} g_k(u) du,$$

$$(1.8.6) \quad f(x) = \frac{d}{dx} \int_0^{\infty} \frac{k_1(xu)}{u} g_h(u) du,$$

hold almost everywhere. And further,

$$(1.8.7) \quad \int_0^{\infty} \{f(x)\}^2 dx = \int_0^{\infty} g_k(x) g_h(x) dx.$$

Proof. Let f be any function of $L_2(0, \infty)$, and F its Mellin transform, so that $F(\frac{1}{2} + i\tau) \in L_2(-\infty, \infty)$. Since $|K(\frac{1}{2} + i\tau)|$ is bounded, $K(\frac{1}{2} + i\tau)F(\frac{1}{2} - i\tau)$ also belongs to $L_2(-\infty, \infty)$. Let g_k be its Mellin transform. Then $g_k \in L_2(0, \infty)$ and

$$(1.8.8) \quad \int_0^x g_k(u) du = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} K(s)F(1-s) \frac{x^{1-s}}{1-s} ds.$$

Now $\frac{K(s)}{1-s}$ is the Mellin transform of $\frac{k_1(x)}{x}$. Hence, by the Parseval formula for Mellin transforms [40, p. 95, Theorem 72]

$$(1.8.9) \quad \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{K(s)}{1-s} F(1-s) x^{-s} ds = \int_0^\infty \frac{k_1(xy)}{xy} f(y) dy.$$

Hence

$$(1.8.10) \quad \int_0^x g_k(u) du = \int_0^\infty \frac{k_1(xy)}{y} f(y) dy,$$

and (1.8.3) follows almost everywhere. Thus g_k , the k -transform of f , is the Mellin transform of $K(s)F(1-s)$, $\text{Re } s = \frac{1}{2}$. For the same reason (1.8.4) holds and $g_h \in L_2(0, \infty)$, the h -transform of f , is the Mellin transform of $H(s)F(1-s)$. By the same rule, the h -transform of g_k is the Mellin transform of

$$(1.8:11) \quad H(s)\{K(1-s)F(s)\} = F(s).$$

Thus the h -transform of g_k is f . Similarly, the k -transform of g_h is f . Thus (1.8.5), (1.8.6) hold. Finally on using (1.7.3) and the Parseval formula for Mellin transform, we obtain

$$(1.8.12) \int_0^{\infty} g_k(x)g_h(x)dx = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} K(s)F(1-s)H(1-s)F(s)ds$$
$$= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)F(1-s)ds = \int_0^{\infty} \{f(x)\}^2 dx.$$

CHAPTER II

ON FUNCTIONS SELF-RECIPROCAL IN G- AND H-TRANSFORMS

2. Methods connected with the use of integral transforms have gained wide importance in mathematical analysis. These methods have been successfully applied to the solution of differential and integral equations, the study of special functions, the evaluation of integrals and many other areas. See [5], [37], [38]. In this chapter, the integral transforms with G- and H-functions as kernels are introduced. The importance of G- and H-functions as kernels is due to their very general yet simple form from which many known as well as new kernels can be deduced as special cases. See [23, pp. 954-958], [29, pp. 86-91], [31, pp. 225-234]. We begin with obtaining a condition (Theorem 2.2.1) for the uniqueness of the solution in $L_2(0, \infty)$ of a Fredholm integral equation of the first kind, a particular case of which, namely Corollary 2.2.2, is essential for the subsequent work in this thesis. The main result of this chapter is contained in Theorem 2.3.1 which gives a necessary and sufficient condition for two functions in $L_2(0, \infty)$ space to be a pair of G-transforms. The necessity of the condition was proved by Kesarwani [28, p. 96]. It is the proof of the sufficiency of the condition that is believed to be new in this thesis. We also show that the kernel

$$G_{2p, 2q}^{q, p} \left(x \left| \begin{matrix} a_1, \dots, a_p, -a_1, \dots, -a_p \\ b_1, \dots, b_q, -b_1, \dots, -b_q \end{matrix} \right. \right), \quad q-1 > p \geq 0,$$

used in [28] may be extended to the case when $q-1 \geq p \geq 0$. In §2.5 we extend the results of §2.3 to the H-transforms and obtain Theorem 2.5.1 which is a generalization of Theorem 2.3.1.

2.1 INTEGRAL TRANSFORM INVOLVING G-FUNCTIONS.

The integral transform we discuss in this section arises from the integral formula

$$(2.1.1) \quad f(x) = \int_0^\infty k(xu) du \int_0^\infty k(ut) f(t) dt$$

where

$$(2.1.2) \quad k(x) = \gamma \mu^{\frac{\gamma}{2}} x^{\frac{\gamma-1}{2}} G_{2p, 2q}^{q, p} \left((\mu x)^\gamma \left| \begin{matrix} a_1, \dots, a_p, -a_1, \dots, -a_p \\ b_1, \dots, b_q, -b_1, \dots, -b_q \end{matrix} \right. \right).$$

Here μ and γ are positive real constants, p and q non-negative integers such that

$$(2.1.3) \quad q-1 \geq p \geq 0$$

and a_j , $j = 1, \dots, p$, and b_h , $h = 1, \dots, q$, are numbers satisfying

$$a_j - b_h \neq 1, 2, 3, \dots, \quad j = 1, \dots, p, \quad h = 1, \dots, q;$$

$$(2.1.4)$$

$$b_j - b_h \neq 0, \pm 1, \pm 2, \dots, \quad j = 1, \dots, q, \quad h = 1, \dots, q, \quad j \neq h$$

and obeying conditions mentioned in Theorem 2.1.1 given below due to Fox. This kernel has been introduced by Kesarwani [22, p. 298]. The conditions (2.1.4) ensure that the poles of $\Gamma(b_j - s)$ and $\Gamma(1 - a_j + s)$ lie on opposite sides of the contour used in the definition of the G-function and that they are simple poles.

The formula (2.1.1) gives rise to the reciprocal relations

$$(2.1.5) \quad F(x) = \int_0^{\infty} k(xu)f(u)du,$$

$$(2.1.6) \quad f(x) = \int_0^{\infty} k(xu)F(u)du,$$

connecting two functions f and F . We will call each of the two functions so related the G-transforms of each other. If, further, $F = f$ so that

$$(2.1.7) \quad f(x) = \int_0^{\infty} k(xu)f(u)du,$$

then following [40, p. 245] f is called self-reciprocal in the G-transform.

The formula (2.1.1), with the left-hand side replaced by $\{f(x + 0) + f(x - 0)\}/2$ at the points of discontinuity of $f(x)$, has been proved [14, p. 400] under the hypothesis that $t^\sigma f(t) \in L(0, \infty)$, with suitable σ , and that $f(t)$ is of bounded variation in the neighborhood of the point $t = x$. See [24,

p. 28] also. However, when we come to study the relations (2.1.5), (2.1.6) directly, we find that the theory of ordinary convergence is not always enough. A case in which a satisfactory theory can be developed is that in which $f \in L_2(0, \infty)$; even in this case, the integrals in (2.1.5), (2.1.6) do not generally exist, and we have to express the reciprocal relations in the form

$$(2.1.8) \quad F(x) = \frac{d}{dx} \int_0^{\infty} \frac{k_1(xu)}{u} f(u) du,$$

$$(2.1.9) \quad f(x) = \frac{d}{dx} \int_0^{\infty} \frac{k_1(xu)}{u} F(u) du,$$

where $k_1(x) = \int_0^x k(u) du$. In case we can differentiate with respect to x under the integral sign, the formulae (2.1.8), (2.1.9) reduce to (2.1.5), (2.1.6) respectively.

✓ In [28, p. 96] Kesarwani obtained a necessary condition, given below as Theorem 2.3.1, so that a pair of functions f, F in $L_2(0, \infty)$ are G-transforms of each other. We will show in Theorem 2.3.1 that the condition is sufficient also. First we mention a few preliminary results.

2.1.1 Here we present a theorem due to Fox which provides a set of hypotheses for the validity of the reciprocity (2.1.8), (2.1.9). The parameters a_j and b_h have been slightly

changed from those in Fox's paper [14, p. 399] so as to make the theorem applicable to the kernel (2.1.2). See [25, p. 277] also.

Theorem 2.1.1 (Fox). If

- (i) $\gamma > 0, \mu > 0,$
- (ii) $\operatorname{Re}(\frac{1}{2} - a_j) > 0, j = 1, \dots, p,$
- (iii) $\operatorname{Re}(\frac{1}{2} + b_h) > 0, h = 1, \dots, q,$
- (iv) $f \in L_2(0, \infty),$
- (v) $k_1(x) = \int_0^x k(u)du,$

then the function

$$\frac{d}{dx} \int_0^{\infty} \frac{k_1(xu)}{u} f(u)du$$

is defined almost everywhere on $(0, \infty)$ and belongs to $L_2(0, \infty)$.

Denote this function by $F(x)$ as in (2.1.8). Then the reciprocal relation (2.1.9) also holds almost everywhere on $(0, \infty)$, and, further,

$$\int_0^{\infty} \{F(x)\}^2 dx = \int_0^{\infty} \{f(x)\}^2 dx.$$

2.1.2 Parseval's Theorem for G-transforms can be stated in the following form. If F and f are two functions satisfying the relations (2.1.5) and (2.1.6), and H and h are two functions similarly related, then [28, p. 95] we have the following result.

Theorem 2.1.2 If f and h belong to $L_2(0, \infty)$, then F and H also belong to $L_2(0, \infty)$ and

$$\int_0^{\infty} F(x)H(x)dx = \int_0^{\infty} f(x)h(x)dx,$$

both sides being L-integrals.

2.1.3 We reproduce the following results of Kesarwani [28, pp. 95-96] in a form that will be useful in proving our assertion.

It has been shown that

$$(2.1.10) \quad x^{\frac{\gamma-1}{2}} G_{p,q}^{q,p} \left((\sqrt{\mu x})^\gamma \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$$

is self-reciprocal in the G-transform under the conditions (2.1.4) and (ii), (iii) of Theorem 2.1.1. To verify this we have to prove that

$$(2.1.11) \quad \int_0^{\infty} k(xu)u^{\frac{\gamma-1}{2}} G_{p,q}^{q,p} \left((\sqrt{\mu u})^\gamma \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) du \\ = x^{\frac{\gamma-1}{2}} G_{p,q}^{q,p} \left((\sqrt{\mu x})^\gamma \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$$

Here $k(x)$ is defined in (2.1.2). The integral on the left-hand side is convergent under the above hypothesis if $q-1 \geq p \geq 0$. In [28, p. 95] Kesarwani showed that (2.1.11) is convergent only for $q-1 > p \geq 0$. We first consider the

case $q-1 \geq p \geq 1$. Changing the variable of integration by setting $(\sqrt{\mu}x)^\gamma = t$, the left-hand side of (2.1.11) equals

$$\begin{aligned}
 (2.1.12) \quad & x^{\frac{\gamma-1}{2}} \int_0^\infty G_{p,q} \left(t \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) x \\
 & G_{2p,2q}^{q,p} \left((\sqrt{\mu}x)^\gamma t \left| \begin{matrix} a_1, \dots, a_p, -a_1, \dots, -a_p \\ b_1, \dots, b_q, -b_1, \dots, -b_q \end{matrix} \right. \right) dt \\
 = & x^{\frac{\gamma-1}{2}} G_{2p+q,2q+p}^{p+q,p+q} \left((\sqrt{\mu}x)^\gamma \left| \begin{matrix} -b_1, \dots, -b_q, a_1, \dots, a_p, -a_1, \dots, -a_p \\ -a_1, \dots, -a_p, b_1, \dots, b_q, -b_1, \dots, -b_q \end{matrix} \right. \right) \\
 = & x^{\frac{\gamma-1}{2}} G_{p,q}^{q,p} \left((\sqrt{\mu}x)^\gamma \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right).
 \end{aligned}$$

Here we have made use of a known integral [31, p. 159, §5.6.2, case 2] involving the product of two G-functions and identities (1.3.1) and (1.3.2). For the case $q-1 \geq p = 0$

(2.1.13) . . .

$$\begin{aligned}
 & \int_0^\infty G_{0,q}^{q,0} \left(t \left| \begin{matrix} \\ b_1, \dots, b_q \end{matrix} \right. \right) G_{0,2q}^{q,0} \left((\sqrt{\mu}x)^\gamma t \left| \begin{matrix} \\ b_1, \dots, b_q, -b_1, \dots, -b_q \end{matrix} \right. \right) dt \\
 & = \frac{1}{2\pi i} \int_0^\infty G_{0,q}^{q,0} \left(t \left| \begin{matrix} \\ b_1, \dots, b_q \end{matrix} \right. \right) dt \int_{L_1} \frac{\Gamma(b_Q - s_1)}{\Gamma(1 + b_Q + s_1)} [(\sqrt{\mu}x)^\gamma t]^{s_1} ds_1.
 \end{aligned}$$

If the change of order of integration in (2.1.13) is permitted, then (2.1.12) holds. Let $s_1 = \sigma_1 + i\tau_1$. Then

(2.1.14) . . .

$$\int_{L_1} \left| \frac{\Gamma(b_Q - s_1)}{\Gamma(1 + b_Q + s_1)} (\sqrt{\mu x})^{\gamma s_1} \left| ds_1 \right| \int_0^\infty |t^{s_1} G_{0,q}^{q,0} \left(t \middle| b_1, \dots, b_q \right)| dt \right.$$

$$= \int_{L_1} \left| \frac{\Gamma(b_Q - s_1)}{\Gamma(1 + b_Q + s_1)} (\sqrt{\mu x})^{\gamma s_1} \left| ds_1 \right| \int_0^\infty t^{\sigma_1} \left| G_{0,q}^{q,0} \left(t \middle| b_1, \dots, b_q \right) \right| dt \right.$$

From the estimation given by Luke [31, p. 144, eqns. (5), (6)] the first integral on the right-hand side in (2.1.14) will be convergent if $\frac{1}{2} + \sigma_1 > \frac{1}{2q}$. From (1.2.2) we can always choose such σ_1 .

Now

$$(2.1.15) \quad \int_0^\infty t^{\sigma_1} \left| G_{0,q}^{q,0} \left(t \middle| b_1, \dots, b_q \right) \right| dt$$

$$= \int_0^1 t^{\sigma_1} \left| G_{0,q}^{q,0} \left(t \middle| b_1, \dots, b_q \right) \right| dt + \int_1^\infty t^{\sigma_1} \left| G_{0,q}^{q,0} \left(t \middle| b_1, \dots, b_q \right) \right| dt.$$

From [31, p. 145, eqn. (7)] the first integral on the right-hand side in (2.1.15) will be convergent if

$$(2.1.16) \quad \sigma_1 + b_j + 1 > 0, \quad j = 1, \dots, q.$$

Under condition (iii) of Theorem 2.1.1, then, (2.1.16) will be satisfied if

$$(2.1.17) \quad \frac{1}{2} + \sigma_1 > 0.$$

For the convergence of the second integral of the right-hand side in (2.1.15)

$$\begin{aligned}
 (2.1.18) \quad & \int_1^{\infty} t^{\sigma_1} |G_{0,q}^{q,0} \left(t \mid b_1, \dots, b_q \right)| dt \\
 &= \frac{1}{2\pi} \int_1^{\infty} t^{\sigma_1} dt \left| \int_{L_2} \Gamma(b_Q - s_2) t^{s_2} ds_2 \right| \\
 &\leq \frac{1}{2\pi} \int_1^{\infty} t^{\sigma_1} dt \int_{L_2} \left| \Gamma(b_Q - s_2) t^{s_2} \right| ds_2 \\
 &= \frac{1}{2\pi} \int_1^{\infty} t^{\sigma_1 + \sigma_2} dt \int_{L_2} \left| \Gamma(b_Q - s_2) \right| ds_2,
 \end{aligned}$$

where $s_2 = \sigma_2 + i\tau_2$ with $\sigma_2 < \text{Re}(b_j)$, $j = 1, \dots, q$. So the integral on the left-hand side in (2.1.18) will be convergent if

$$(2.1.19) \quad \sigma_1 + \sigma_2 + 1 < 0.$$

Since σ_2 is subjected only to the condition $\sigma_2 < \text{Re}(b_j)$, $j = 1, \dots, q$, it follows that for any given σ_1 , we can always choose σ_2 such that (2.1.19) holds.

From (2.1.14), (2.1.15) and (2.1.18) the order of integration in (2.1.13) may be changed. Hence (2.1.12) holds for $p = 0$ also.

As a particular case, note that when $p = 0$, $q = 1$, $b_1 = \nu/2$, $\gamma = 2$, $\mu = 1/2$, the kernel (2.1.2) reduces [27, p. 185] to $\sqrt{x} J_{\nu}(x)$ which is the kernel of Hankel transform [40, p. 245] of order ν , and the function (2.1.10) reduces to a constant

multiple of $x^{v+\frac{1}{2}} \exp(-\frac{x^2}{2})$ which is a well-known self-reciprocal function in the Hankel transform obtained by Tricomi [41, p. 286]. Note also that (2.1.10) is only a particular case of a more general self-reciprocal function obtained in [27, p. 184].

By making obvious changes of variables in (2.1.11), we obtain that if $y > 0$,

$$(yx)^{\frac{\gamma-1}{2}} G_{p,q}^{q,p} \left((\sqrt{\mu}yx)^{\gamma} \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) \text{ and}$$

(2.1.20)

$$\frac{1}{y} \left(\frac{x}{y}\right)^{\frac{\gamma-1}{2}} G_{p,q}^{q,p} \left(\left(\sqrt{\mu} \frac{x}{y}\right)^{\gamma} \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$$

form a pair of G-transforms. These functions belong to $L_2(0, \infty)$ whenever conditions (ii) and (iii) of Theorem 2.1.1 are fulfilled. See [31, §5.6.2].

2.2 UNIQUENESS THEOREM FOR THE SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND.

Theorem 2.2.1 Let

(i) $h \in L_2(0, \infty)$,

(ii) $\mathcal{H}\{h(x)\} = H(s)$, and $H(\frac{1}{2} + it)$ be bounded and

never vanish for all real t ,

(iii) $f \in L_2(0, \infty)$,

(iv) $\int_0^{\infty} h(xy)f(y)dy = 0$, for all $x > 0$.

Then $f(x) = 0$ almost everywhere on $(0, \infty)$.

Proof. By Theorem 1.1.1 both sides of

(2.2.1) $\int_0^{\infty} h(xy)f(y)dy = 0$

belong to $L_2(0, \infty)$. We obtain its Mellin transform so that

$0 = \mathcal{M}\left\{\int_0^{\infty} h(xy)f(y)dy\right\} = H(s)F(1-s)$, $s = \frac{1}{2} + it$, $-\infty < t < \infty$,

where $F(s) = \mathcal{M}\{f(x)\}$. By condition (ii) we have $F(s) = 0$,

$s = \frac{1}{2} + it$, $-\infty < t < \infty$. Hence by inverse Mellin transform

(1.1.4) we obtain $f(x) = 0$ almost everywhere on $(0, \infty)$.

Corollary 2.2.2 Let

(i) $\mu > 0$, $\gamma > 0$,

(ii) $\sum_{j=1}^q \beta_j + \sum_{j=1}^p \alpha_j > 0$,

(iii) $\text{Re}(1 - a_j) > \alpha_j/2$, $j = 1, \dots, p$,

(iv) $\text{Re}(b_j + \beta_j/2) > 0$, $j = 1, \dots, q$,

(v) $f \in L_2(0, \infty)$,

(vi) $\int_0^{\infty} f(t)(xt)^{\frac{\gamma-1}{2}} H_{p,q}^{q,p} \left((\sqrt{\mu}xt)^{\gamma} \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) dt = 0$

for all $x > 0$.

Then $f(x) = 0$ almost everywhere on $(0, \infty)$.

Proof. Let

$$(2.2.2) \quad h(x) = x^{\frac{\gamma-1}{2}} H_{p,q}^{q,p} \left((\sqrt{\mu}x)^\gamma \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) \\ = \mu^{(1-\gamma)/4} H_{p,q}^{q,p} \left((\sqrt{\mu}x)^\gamma \left| \begin{matrix} (a'_1, \alpha_1)_{1,p} \\ (b'_1, \beta_1)_{1,q} \end{matrix} \right. \right)$$

where

$$(2.2.3) \quad a'_j = a_j + \alpha_j \left(\frac{1}{2} - \frac{1}{2\gamma} \right), \quad j = 1, \dots, p,$$

$$b'_j = b_j + \beta_j \left(\frac{1}{2} - \frac{1}{2\gamma} \right), \quad j = 1, \dots, q.$$

By (1.3.10) & (1.1.5), the Mellin transform of $h(x)$ is obtained as

$$(2.2.4) \quad H(s) = \frac{1}{\gamma} \mu^{\frac{1-\gamma-s}{4}} \Gamma(b'_Q + \frac{\beta_Q s}{\gamma}) \Gamma(1 - a'_P - \frac{\alpha_P s}{\gamma}).$$

We will show that $H(s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and $H(s)$ is bounded

on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$ also. Now

(2.2.5)

$$H(\frac{1}{2} + it) = \frac{1}{\gamma} \mu^{-\frac{\gamma-1}{4} - \frac{it}{2}} \Gamma(b'_Q + \frac{\beta_Q}{2} + \frac{i\beta_Q t}{\gamma}) \Gamma(1 - \frac{\alpha_P}{2} - a'_P - \frac{i\alpha_P t}{\gamma}), \quad -\infty < t < \infty.$$

By conditions (iii) and (iv) the functions $\Gamma(b'_j + \frac{\beta_j}{2} + \frac{i\beta_j t}{\gamma})$,

$j = 1, \dots, q$; $\Gamma(1 - \frac{\alpha_j}{2} - a'_j - \frac{i\alpha_j t}{\gamma})$, $j = 1, \dots, p$, do not have

any singularities for real t . Using the asymptotic expansion

of gamma function [31, p. 33]

$$(2.2.6) \quad |\Gamma(x + iy)| \sim \sqrt{2\pi} |y|^{x-\frac{1}{2}} e^{-\frac{\pi}{2}|y|},$$

x, y real, x fixed, $|y| \rightarrow \infty$, we find that

$$(2.2.7) \quad |H(\frac{1}{2} + it)| \sim A \left| \frac{t}{\gamma} \right|^{\frac{p-q+\text{Re}B+C}{2}} \exp(-\frac{\pi}{2\gamma} D|t|), \quad |t| \rightarrow \infty,$$

where

$$A = \frac{1}{\gamma} \mu^{-\frac{\gamma}{4}} (2\pi)^{\frac{p+q}{2}} \prod_{j=1}^q \beta_j^{\frac{(\beta_j-1)}{2} + \text{Re}b_j} \prod_{j=1}^p \alpha_j^{\frac{(1-\alpha_j)}{2} - \text{Re}a_j},$$

$$B = \sum_{j=1}^q b_j - \sum_{j=1}^p a_j,$$

$$C = \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j,$$

$$D = \sum_{j=1}^q \beta_j + \sum_{j=1}^p \alpha_j.$$

Due to the factor $\exp(-\frac{\pi}{2\gamma} D|t|)$ and conditions (i) and (ii)

we know that $H(\frac{1}{2} + it)$ is bounded for all real t and $H(s)$ belongs to $L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence $h(x) = \mathcal{M}^{-1}\{H(s)\}$ belongs to $L_2(0, \infty)$. Since all the assumptions of Theorem 2.2.1 are fulfilled, the desired result follows.

A result analogous to Corollary 2.2.2 involving G-functions instead of H-functions follows on setting $\alpha_1 = 1, \beta_1 = 1$.

2.3 NECESSARY AND SUFFICIENT CONDITION FOR A PAIR OF FUNCTIONS TO BE G-TRANSFORMS.

Theorem 2.3.1 Let

- (i) $\mu > 0, \gamma > 0,$
- (ii) $q-1 \geq p \geq 0,$
- (iii) $\operatorname{Re}(\frac{1}{2} - a_j) > 0, j = 1, \dots, p,$
- (iv) $\operatorname{Re}(\frac{1}{2} + b_j) > 0, j = 1, \dots, q.$

Then a necessary and sufficient condition that functions F and f in $L_2(0, \infty)$ be a pair of G-transforms is that

$$(2.3.1) \quad \int_0^{\infty} f(t)(xt)^{\frac{\gamma-1}{2}} G_{p,q}^{q,p} \left((\sqrt{\mu}xt)^{\gamma} \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dt$$

$$= \frac{1}{x} \int_0^{\infty} F(t) \left(\frac{t}{x}\right)^{\frac{\gamma-1}{2}} G_{p,q}^{q,p} \left((\sqrt{\mu}\frac{t}{x})^{\gamma} \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dt$$

for all $x > 0$.

Proof. Necessity.

Since from (2.1.20) the functions

$$(xt)^{\frac{\gamma-1}{2}} G_{p,q}^{q,p} \left((\sqrt{\mu}xt)^{\gamma} \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$$

and

$$\frac{1}{x} \left(\frac{t}{x}\right)^{\frac{\gamma-1}{2}} G_{p,q}^{q,p} \left((\sqrt{\mu}\frac{t}{x})^{\gamma} \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right),$$

considered as functions of t , form a pair of G-transforms for all $x > 0$, we obtain (2.3.1) on applying Theorem 2.1.2.

Sufficiency.

Let $f_0 \in L_2(0, \infty)$ be the G-transform of F . By

the necessary condition we have

$$(2.3.2) \quad \int_0^{\infty} f_0(t)h(xt)dt = \frac{1}{x} \int_0^{\infty} F(t)h\left(\frac{t}{x}\right)dt,$$

for all $x > 0$, where

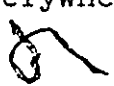
$$(2.3.3) \quad h(x) = x^{\frac{\gamma-1}{2}} G_{p,q}^{q,p} \left((\sqrt{\mu}x)^{\gamma} \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right).$$

But, by the hypothesis

$$(2.3.4) \quad \int_0^{\infty} f(t)h(xt)dt = \frac{1}{x} \int_0^{\infty} F(t)h\left(\frac{t}{x}\right)dt,$$

for all $x > 0$, also. Hence

$$(2.3.5) \quad \int_0^{\infty} \{f(t) - f_0(t)\}h(xt)dt = 0.$$

By Corollary 2.2.2 we must have $f(x) = f_0(x)$ almost everywhere on $(0, \infty)$, i.e., f and F are a pair of G-transforms. 

2.4 INTEGRAL TRANSFORM INVOLVING H-FUNCTIONS.

2.4.1 The function

$$(2.4.1) \quad H_{2p,2q}^{q,p} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1,p}, (1-a_1-\alpha_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q}, (1-b_1-\beta_1, \beta_1)_{1,q} \end{matrix} \right. \right)$$

and hence in view of the remark following (1.7.4), the function

(2.4.2)

$$k(x) = \gamma \mu^{\frac{\gamma}{2}} x^{\frac{\gamma-1}{2}} H_{2p, 2q}^{q, p} \left((\mu x)^{\gamma} \left| \begin{array}{c} (a_1, \alpha_1)_{1, p}, (1-a_1-\alpha_1, \alpha_1)_{1, p} \\ (b_1, \beta_1)_{1, q}, (1-b_1-\beta_1, \beta_1)_{1, q} \end{array} \right. \right),$$

$\mu, \gamma > 0,$

has been proved by Fox [14, p. 410, Theorem 5] to be a symmetrical Fourier kernel in $L_2(0, \infty)$ in the following sense.

Theorem 2.4.1 (Fox). If

(i) $\beta_i > 0, i = 1, \dots, q, \alpha_i > 0, i = 1, \dots, p,$ and

$$\sum_{i=1}^q \beta_i - \sum_{i=1}^p \alpha_i > 0,$$

(ii) $\operatorname{Re}(b_i + \beta_i/2) > 0, i = 1, \dots, q,$

(iii) $\operatorname{Re}(1 - a_i) > \alpha_i/2, i = 1, \dots, p,$

(iv) $\mu > 0, \gamma > 0,$

(v) $f \in L_2(0, \infty),$

(vi) $k_1(x) = \int_0^x k(u) du,$

then

$$F(x) = \frac{d}{dx} \int_0^{\infty} \frac{k_1(xu)}{u} f(u) du$$

defines, almost everywhere, a function $F(x) \in L_2(0, \infty),$

$$f(x) = \frac{d}{dx} \int_0^{\infty} \frac{k_1(xu)}{u} F(u) du$$

holds almost everywhere and

$$\int_0^{\infty} \{f(x)\}^2 dx = \int_0^{\infty} \{F(x)\}^2 dx.$$

The functions F , f which satisfy

$$(2.4.3) \quad F(x) = \int_0^{\infty} k(xu)f(u)du,$$

$$(2.4.4) \quad f(x) = \int_0^{\infty} k(xu)F(y)du$$

will be called H-transforms of each other. If, further, $F = f$ so that

$$f(x) = \int_0^{\infty} k(xu)f(u)du,$$

then f is called self-reciprocal in the H-transform.

2.4.2 Parseval's Theorem for H-transforms may be stated in the following form. If F and f are two functions satisfying the relations (2.4.3) and (2.4.4), and H and h are two functions similarly related, then

$$(2.4.5) \quad \int_0^{\infty} F(x)H(x)dx = \int_0^{\infty} f(x)h(x)dx.$$

Theorem 2.4.2 If f and h belong to $L_2(0, \infty)$ then F and H also belong to $L_2(0, \infty)$ and (2.4.5) is true, both sides being L-integrals.

2.4.3 Analogous to (2.1.10) we will show that the function

$$(2.4.6) \quad x^{\frac{\gamma-1}{2}} H_{p,q}^{q,p} \left((\sqrt{\mu} x)^{\gamma} \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right)$$

is self-reciprocal in the H-transform. To verify this we have to prove that

$$(2.4.7) \quad \int_0^{\infty} k(xu) u^{\frac{\gamma-1}{2}} H_{p,q}^{q,p} \left((\sqrt{\mu} u)^{\gamma} \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) du \\ = x^{\frac{\gamma-1}{2}} H_{p,q}^{q,p} \left((\sqrt{\mu} x)^{\gamma} \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right)$$

The integral on the left-hand side is convergent under the condition (i) of Theorem 2.4.1. In fact, changing the variable of integration by setting $(\sqrt{\mu} u)^{\gamma} = t$, the left-hand side of (2.4.7) equals

$$(2.4.8) \quad x^{\frac{\gamma-1}{2}} \int_0^{\infty} H_{p,q}^{q,p} \left(t \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) \times \\ H_{2p,2q}^{q,p} \left((\sqrt{\mu} x)^{\gamma} t \left| \begin{matrix} (a_1, \alpha_1)_{1,p}, (1-a_1-\alpha_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q}, (1-b_1-\beta_1, \beta_1)_{1,q} \end{matrix} \right. \right) dt \\ = x^{\frac{\gamma-1}{2}} \frac{1}{2\pi i} \int_0^{\infty} H_{p,q}^{q,p} \left(t \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) dt \\ \times \int_{L_1} \frac{\Gamma(b_Q - \beta_Q s_1) \Gamma(1 - a_P + \alpha_P s_1)}{\Gamma(b_Q + \beta_Q + \beta_Q s_1) \Gamma(1 - a_P - \alpha_P - \alpha_P s_1)} (\sqrt{\mu} x)^{\gamma s_1} t^{s_1} ds_1$$

$$\begin{aligned}
 &= x^{\frac{\gamma-1}{2}} \frac{1}{2\pi i} \int_{L_1} \frac{\Gamma(b_Q - \beta_Q s_1) \Gamma(1 - a_P + \alpha_P s_1)}{\Gamma(b_Q + \beta_Q + \beta_Q s_1) \Gamma(1 - a_P - \alpha_P - \alpha_P s_1)} (\sqrt{\mu} x)^{\gamma s_1} ds_1 \\
 &\quad \times \int_0^\infty t^{s_1} H_{p,q}^{q,p} \left(t \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) dt \\
 &= x^{\frac{\gamma-1}{2}} \frac{1}{2\pi i} \int_{L_1} \Gamma(b_Q - \beta_Q s_1) \Gamma(1 - a_P + \alpha_P s_1) (\sqrt{\mu} x)^{\gamma s_1} ds_1 \\
 &= x^{\frac{\gamma-1}{2}} H_{p,q}^{q,p} \left((\sqrt{\mu} x)^\gamma \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right).
 \end{aligned}$$

To justify the change of order of integration, we have to show that the following integral

$$\begin{aligned}
 (2.4.9) \quad & \int_{L_1} \left| \frac{\Gamma(b_Q - \beta_Q s_1) \Gamma(1 - a_P + \alpha_P s_1)}{\Gamma(b_Q + \beta_Q + \beta_Q s_1) \Gamma(1 - a_P - \alpha_P - \alpha_P s_1)} (\sqrt{\mu} x)^{\gamma s_1} \right| |ds_1| \\
 & \times \int_0^\infty \left| H_{p,q}^{q,p} \left(t \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) t^{s_1} \right| dt
 \end{aligned}$$

is convergent. Let $s_1 = \sigma_1 + i\tau_1$, σ_1, τ_1 real. By using

(2.2.6) the first integral in (2.4.9) will be convergent if

$$(2.4.10) \quad (2\sigma_1 + 1) \left(\sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j \right) > 1.$$

To estimate the second integral in (2.4.9), we have

$$\begin{aligned}
 (2.4.11) \quad & \int_0^{\infty} |H_{p,q}^{q,p} \left(t \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. t^{\sigma_1} \right) | dt \\
 & \leq \int_0^{\infty} t^{\sigma_1} |H_{p,q}^{q,p} \left(t \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) | dt \\
 & = \int_0^1 t^{\sigma_1} |H_{p,q}^{q,p} \left(t \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) | dt + \int_1^{\infty} t^{\sigma_1} |H_{p,q}^{q,p} \left(t \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) | dt.
 \end{aligned}$$

From [2, p. 279, eqn. (6.5)] we know that

$$(2.4.12) \quad H_{p,q}^{q,p} \left(t \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) = O(|t|^\alpha) \text{ for small } t,$$

where

$$(2.4.13) \quad \alpha = \min. \operatorname{Re}(b_j/\beta_j), \quad j = 1, \dots, p.$$

Therefore the first integral on the right-hand side of (2.4.11) will be convergent if

$$(2.4.14) \quad 1 + \alpha + \sigma_1 > 0.$$

We may always choose σ_1 to satisfy the conditions (2.4.10) and (2.4.14) due to (1.2.10). To estimate the second integral on the right-hand side of (2.4.11) we have

$$\begin{aligned}
 (2.4.15) \quad & \int_1^{\infty} t^{\sigma_1} |H_{p,q}^{q,p} \left(t \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) | dt \\
 & \leq \frac{1}{2\pi} \int_1^{\infty} t^{\sigma_1} dt \int_{L_2} |\Gamma(b_Q - \beta_Q s_2) \Gamma(1 - a_p + \alpha_p s_2) t^{s_2}| |ds_2|
 \end{aligned}$$

$$\leq \frac{1}{2\pi} \int_1^{\infty} t^{\sigma_1 + \sigma_2} dt \int_{L_2} |\Gamma(b_Q - \beta_Q s_2) \Gamma(1 - a_p + \alpha_p s_2)| |ds_2|$$

where $s_2 = \sigma_2 + i\tau_2$. Hence the left-hand side of (2.4.15)

is convergent if

$$(2.4.16) \quad \sigma_1 + \sigma_2 + 1 < 0.$$

For given σ_1 , σ_2 can always be chosen so that condition

(2.4.16) holds. Therefore (2.4.9) is convergent. This justifies the change of order of integration in (2.4.8).

By making obvious changes of variables in (2.4.7), we obtain that if $y > 0$,

$$(2.4.17) \quad (yx)^{\frac{\gamma-1}{2}} H_{p,q}^{q,p} \left((\sqrt{\mu} yx)^{\gamma} \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) \text{ and}$$

$$\frac{1}{y} \left(\frac{x}{y}\right)^{\frac{\gamma-1}{2}} H_{p,q}^{q,p} \left((\sqrt{\mu} \frac{x}{y})^{\gamma} \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right)$$

form a pair of H-transforms.

2.5 NECESSARY AND SUFFICIENT CONDITION FOR A PAIR OF FUNCTIONS TO BE H-TRANSFORMS.

Analogous to the G-transforms, we have

Theorem 2.5.1 Under the hypotheses (i), (ii), (iii), (iv) of Theorem 2.4.1, a necessary and sufficient condition that functions f and F in $L_2(0, \infty)$ be a pair of H-transforms is that

$$\begin{aligned}
 (2.5.1) \quad & \int_0^{\infty} f(t)(xt)^{\frac{\gamma-1}{2}} H_{p,q}^{\alpha,p} \left((\sqrt{\mu} xt)^{\gamma} \middle| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right) dt \\
 &= \frac{1}{x} \int_0^{\infty} F(t) \left(\frac{t}{x}\right)^{\frac{\gamma-1}{2}} H_{p,q}^{\alpha,p} \left((\sqrt{\mu} \frac{t}{x})^{\gamma} \middle| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right) dt
 \end{aligned}$$

for all $x > 0$.

Proof. Necessity.

Since from (2.4.17) the functions

$$(xt)^{\frac{\gamma-1}{2}} H_{p,q}^{\alpha,p} \left((\sqrt{\mu} xt)^{\gamma} \middle| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right)$$

and

$$\frac{1}{x} \left(\frac{t}{x}\right)^{\frac{\gamma-1}{2}} H_{p,q}^{\alpha,p} \left((\sqrt{\mu} \frac{t}{x})^{\gamma} \middle| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right)$$

considered as functions of t , form a pair of H-transforms for all $x > 0$, we obtain (2.5.1) by applying Theorem 2.4.2.

Sufficiency.

Let $f_0 \in L_2(0, \infty)$ be the H-transform of F . By the necessary condition we have

$$\int_0^{\infty} f_0(t)h(xt)dt = \frac{1}{x} \int_0^{\infty} F(t)h\left(\frac{t}{x}\right)dt,$$

for all $x > 0$, where

$$h(x) = x^{\frac{\gamma-1}{2}} H_{p,q}^{\alpha,p} \left((\sqrt{\mu} x)^{\gamma} \middle| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right)$$

By the hypothesis

$$\int_0^{\infty} f(t)h(xt)dt = \frac{1}{x} \int_0^{\infty} F(t)h\left(\frac{t}{x}\right)dt, \quad x > 0,$$

also. Hence

$$\int_0^{\infty} \{f(t) - f_0(t)\}h(xt)dt = 0.$$

By Corollary 2.2.2 we must have $f(x) = f_0(x)$ a.e. on $(0, \infty)$,

i.e., f and F are a pair of H-transforms.

CHAPTER III

FRACTIONAL INTEGRATION AND $W_{\lambda, \mu}$ -TRANSFORM

3. Integration and differentiation of fractional order appear in various branches of mathematical analysis and its applications. While working with certain integral transforms we find that fractional integration and differentiation can profitably be developed into a useful tool to obtain interesting results. Such a development would entail standardized definitions, a good notation, and collection of the basic properties of fractional derivatives and integrals in a form suitable for the intended project. Short tables of fractional integrals already exist [11, Chapter 13] and could be usefully expanded.

In 1940 Kober defined certain operators of fractional differentiation and integration which were generalized by Erdélyi in 1951. These operators are very useful in connexion with the work in hand. In this thesis we consider these operators defined on the function space $L_2(0, \infty)$. The present chapter is devoted to the application of fractional integral operators to certain integral transforms.

In §3.1, we collect some properties of fractional integration and in §3.2, we solve the integral equation

$$(3.1) \quad \int_0^{\infty} (xu)^{\mu-\frac{1}{2}} e^{-\frac{xu}{2}} W_{\lambda, \mu}(xu) f(u) du = g(x), \quad x > 0,$$

where $W_{\lambda, \mu}(x)$ is the Whittaker's confluent hypergeometric function [36, p. 14]. We show that the above integral equation may be reduced to an equation representing a Laplace transform by means of fractional integration. See Theorem 3.2.4. We give some examples to illustrate the application of this theorem. In §3.3 we solve an integral equation whose kernel is an H-function.

In §3.4 the problem of dual integral equations involving H-functions as kernels has been considered. We reduce the problem to that of solving a single integral equation by applying appropriate fractional integral operator which, in turn, can be solved by the method given in §3.3.

3.1 FRACTIONAL INTEGRATION.

Various definitions have been given of fractional integration. We use here the definitions given by Kober [30] and generalized by Erdélyi [8]. These authors define four types of fractional integral operators but of these four types the last two are little more than variants of the first two. For this reason we shall deal with the first two types only.

Let $f \in L_2(0, \infty)$, $\mathcal{W}\{f(x)\} = F(s)$ and $F_1(t) = F(\frac{1}{2} + it)$. We shall denote by $L_2^{(\alpha)}(0, \infty)$ or $L_2^{(\alpha)}$ the set of functions f belonging to $L_2(0, \infty)$ for which $|t|^{\max(0, -\text{Re}\alpha)} F_1(t)$ belongs to $L_2(-\infty, \infty)$. Obviously $L_2^{(\alpha)}(0, \infty)$ is identical with $L_2(0, \infty)$ when

$\operatorname{Re} \alpha \geq 0$, and $L_2^{(\alpha)}(0, \infty)$ is a proper subset of $L_2(0, \infty)$ when $\operatorname{Re} \alpha < 0$ [7, p. 300].

The operators $I_x^{\eta, \alpha}$ and $K_x^{\eta, \alpha}$ are defined by the formulae [30, p. 193]

$$I_x^{\eta, \alpha} f(x) = \frac{1}{\Gamma(\alpha)} x^{-\eta-\alpha} \int_0^x (x-t)^{\alpha-1} t^\eta f(t) dt, x > 0, \quad (3.1.1)$$

$$K_x^{\eta, \alpha} f(x) = \frac{1}{\Gamma(\alpha)} x^\eta \int_x^\infty (t-x)^{\alpha-1} t^{-\eta-\alpha} f(t) dt, x > 0.$$

By [30, p. 199, Theorem 2] if $f \in L_2(0, \infty)$, $\operatorname{Re} \eta > -\frac{1}{2}$,

$\operatorname{Re} \alpha > 0$ then $I_x^{\eta, \alpha} f(x)$ and $K_x^{\eta, \alpha} f(x)$ exist almost everywhere in $(0, \infty)$ and belong to $L_2^{(\alpha)}(0, \infty)$. For simplicity we assume

η, α the super-scripts of $I_x^{\eta, \alpha}$, $K_x^{\eta, \alpha}$ to be real from now on.

Also when no confusion can result, we write $I^{\eta, \alpha}$, $K^{\eta, \alpha}$ for

$I_x^{\eta, \alpha}$, $K_x^{\eta, \alpha}$ respectively. Here it will be sufficient to note

that the elements of $L_2^{(-\alpha)}(0, \infty)$ are $[\alpha]$ times continuously

differentiable [7, p. 301], where $[\alpha]$ denotes the largest integer $\leq \alpha$.

For $\alpha = 0$

$$(3.1.2) \quad I^{\eta, \alpha} f(x) = f(x), \quad K^{\eta, \alpha} f(x) = f(x)$$

and for $\alpha < 0$, we define $g(x) = I^{\eta, \alpha} f(x)$ and $h(x) = K^{\eta, \alpha} f(x)$

as the solutions in $L_2(0, \infty)$ (if any) of the integral equations

$$(3.1.3) \quad f(x) = I^{\eta+\alpha, -\alpha} g(x), \quad f(x) = K^{\eta+\alpha, -\alpha} h(x),$$

these solutions being unique [30, p. 205]. These solutions exist if $\eta + \alpha > -\frac{1}{2}$. We see that for real α and under the condition $\eta > -\frac{1}{2} + \max(0, -\alpha)$, $I^{\eta, \alpha} f$, $K^{\eta, \alpha} f$ are defined and belong to $L_2^{(-\alpha)}(0, \infty)$ provided that $f \in L_2^{(\alpha)}(0, \infty)$ [7, p. 300, Theorem 7]: For the operators so extended we have [12, p. 688]

$$(3.1.4) \quad \begin{aligned} I^{\eta, \alpha} I^{\eta+\alpha, \beta} &= I^{\eta+\alpha, \beta} I^{\eta, \alpha} = I^{\eta, \alpha+\beta}, \\ K^{\eta, \alpha} K^{\eta+\alpha, \beta} &= K^{\eta+\alpha, \beta} K^{\eta, \alpha} = K^{\eta, \alpha+\beta}, \\ (I^{\eta, \alpha})^{-1} &= I^{\eta+\alpha, -\alpha}, \quad (K^{\eta, \alpha})^{-1} = K^{\eta+\alpha, -\alpha} \end{aligned}$$

provided all operations make sense.

Let n be a positive integer. Then [12, p. 688]

$$(3.1.5) \quad \begin{aligned} I^{\eta, -n} f(x) &= x^{-\eta+n} \frac{d^n}{dx^n} \{x^\eta f(x)\}, \\ K^{\eta, -n} f(x) &= (-1)^n x^\eta \frac{d^n}{dx^n} \{x^{-\eta+n} f(x)\} \end{aligned}$$

by explicit computation, and this result in combination with (3.1.4) leads to explicit expressions for $I^{\eta, \alpha} f$, $K^{\eta, \alpha} f$. When $\alpha+n \geq 0$, where n is a positive integer, $I^{\eta, \alpha} = I^{\eta+\alpha+n, -n} I^{\eta, \alpha+n}$ and similarly for K , so that

$$(3.1.6) \quad \begin{aligned} I^{\eta, \alpha} f(x) &= x^{-n-\alpha} \frac{d^n}{dx^n} \{x^{\eta+\alpha+n} I^{\eta, \alpha+n} f(x)\}, \\ K^{\eta, \alpha} f(x) &= (-1)^n x^{\eta+\alpha+n} \frac{d^n}{dx^n} \{x^{-n-\alpha} K^{\eta, \alpha+n} f(x)\}. \end{aligned}$$

By [30, p. 203, Theorem 5(a)] for $\eta > -\frac{1}{2}$, $\alpha > 0$

$$\mathcal{M}\{I^{\eta, \alpha} f(x)\} = \frac{\Gamma(1+\eta-s)}{\Gamma(1+\eta+\alpha-s)} \mathcal{M}\{f(x)\},$$

(3.1.7)

$$\mathcal{M}\{K^{\eta, \alpha} f(x)\} = \frac{\Gamma(\eta+s)}{\Gamma(\eta+\alpha+s)} \mathcal{M}\{f(x)\}.$$

Fractional integral operators with respect to x^A , $A > 0$, may be defined by similar formulae by replacing x by x^A [8, p. 220]. Thus we write

$$I_{x^A}^{\eta, \alpha} f(x) = \frac{1}{\Gamma(\alpha)} x^{-A\eta-A\alpha} \int_0^x (x^A - t^A)^{\alpha-1} t^{A\eta} f(t) d(t^A),$$

(3.1.8)

$$K_{x^A}^{\zeta, \alpha} f(x) = \frac{1}{\Gamma(\alpha)} x^{A\zeta} \int_x^\infty (t^A - x^A)^{\alpha-1} t^{-A\zeta-A\alpha} f(t) d(t^A).$$

When no confusion can result, we write $I_A^{\eta, \alpha}$, $K_A^{\zeta, \alpha}$ for $I_{x^A}^{\eta, \alpha}$,

$K_{x^A}^{\zeta, \alpha}$ respectively.

Let U be the Heaviside unit step function, i.e.,

$$(3.1.9) \quad U(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0, \end{cases}$$

and for $A > 0$, define functions

$$I^{\eta, \alpha, A}(x) = \frac{A}{\Gamma(\alpha)} (x^A - 1)^{\alpha-1} x^{-A\eta-A\alpha} U(x - 1),$$

(3.1.10)

$$K^{\zeta, \alpha, A}(x) = \frac{A}{\Gamma(\alpha)} (1 - x^A)^{\alpha-1} x^{A\zeta} U(1 - x).$$

Then $I^{\eta, \alpha, A}$ and $K^{\zeta, \alpha, A}$ belong to $L_p(0, \infty)$ if $\alpha > 1 - \frac{1}{p}$,

$\zeta > -\frac{1}{pA}$, $\eta > \frac{1}{pA} - 1$, $1 \leq p < \infty$. For $p = 2$, we have [4, p. 101]

$$\mathcal{M}\{I^{\eta, \alpha, A}(x)\} = \frac{\Gamma(1+\eta-\frac{s}{A})}{\Gamma(1+\eta+\alpha-\frac{s}{A})}, \quad \alpha > \frac{1}{2}, \eta > \frac{1}{2A} - 1, \operatorname{Re} s \leq \frac{1}{2},$$

(3.1.11)

$$\mathcal{M}\{K^{\zeta, \alpha, A}(x)\} = \frac{\Gamma(\zeta+\frac{s}{A})}{\Gamma(\zeta+\alpha+\frac{s}{A})}, \quad \alpha > \frac{1}{2}, \zeta > -\frac{1}{2A}, \operatorname{Re} s \geq \frac{1}{2},$$

and $\mathcal{M}\{I^{\eta, \alpha, A}(x)\}$, $\mathcal{M}\{K^{\zeta, \alpha, A}(x)\}$ belong to $L_2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ and are bounded on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$. Hence if $A > 0$, $\alpha > \frac{1}{2}$, $\eta > \frac{1}{2A} - 1$, and $\zeta > -\frac{1}{2A}$, the fractional integral operators (3.1.8) can be written in the form of the convolution (1.1.11), that is, if $f \in L_2(0, \infty)$,

$$(3.1.12) \quad I_A^{\eta, \alpha} f(x) = (I^{\eta, \alpha, A} * f)(x), \quad K_A^{\zeta, \alpha} f(x) = (K^{\zeta, \alpha, A} * f)(x).$$

It has been pointed out by Fox [15, p. 460] that the Kober's proofs [30, pp. 203 - 204] for the case $A = 1$ may apply, with perhaps some minor modifications, to the case $A > 0$. Hence, from [30, p. 203], a set of sufficient conditions for the existence of (3.1.8) are

$$(3.1.13) \quad \alpha > 0, \eta > \frac{1}{2A} - 1, \zeta > -\frac{1}{2A} \text{ and } f \in L_2(0, \infty).$$

Further,

$$\mathcal{M}\{I_A^{\eta, \alpha} f(x)\} = \frac{\Gamma(1+\eta-\frac{s}{A})}{\Gamma(1+\eta+\alpha-\frac{s}{A})} \mathcal{M}\{f(x)\}, \quad \alpha > 0, A > 0, \eta > \frac{1}{2A} - 1, \operatorname{Re} s \leq \frac{1}{2},$$

(3.1.14)

$$\mathcal{M}\{K_A^{\zeta, \alpha} f(x)\} = \frac{\Gamma(\zeta+\frac{s}{A})}{\Gamma(\zeta+\alpha+\frac{s}{A})} \mathcal{M}\{f(x)\}, \quad \alpha > 0, \zeta > -\frac{1}{2A}, \operatorname{Re} s \geq \frac{1}{2}.$$

As an application of (3.1.14), let

$$(3.1.15) \quad f(x) = H_{p,q}^{m,n} \left((cx)^B \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right),$$

where $\sum_{j=1}^n \alpha_j + \sum_{j=1}^m \beta_j > \sum_{j=n+1}^p \alpha_j + \sum_{j=m+1}^q \beta_j$, $B > 0$, $c > 0$.

Then by the asymptotic expansion of gamma function (2.2.6) and [40, p. 94, Theorem 71], $f \in L_2(0, \infty)$ and hence

$$(3.1.16) \quad \mathcal{M}_A^{\eta, \alpha} \{K_A^{\eta, \alpha} f(x)\} = \frac{c^{-s} \Gamma(\eta + \frac{s}{A}) \Gamma(b_M + \frac{\beta_M s}{B}) \Gamma(1 - a_N - \frac{\alpha_N s}{B})}{\Gamma(\eta + \alpha + \frac{s}{A}) \Gamma_m(1 - b_Q - \frac{\beta_Q s}{B}) \Gamma_n(a_P + \frac{\alpha_P s}{B})}$$

This implies

(3.1.17)

$$K_A^{\eta, \alpha} \{H_{p,q}^{m,n} \left((cx)^B \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right)\} = H_{p+1,q+1}^{m+1,n} \left((cx)^B \left| \begin{matrix} (a_1, \alpha_1)_{1,p}, (\eta + \alpha, \frac{B}{A}) \\ (\eta, \frac{B}{A}), (b_1, \beta_1)_{1,q} \end{matrix} \right. \right)$$

where $\eta > -\frac{1}{2A}$, $\alpha > 0$, $A > 0$, $B > 0$, $c > 0$,

$\sum_{j=1}^n \alpha_j + \sum_{j=1}^m \beta_j > \sum_{j=n+1}^p \alpha_j + \sum_{j=m+1}^q \beta_j$. Hence from (3.1.8),

after an obvious change of variables, we obtain

$$(3.1.18) \quad \frac{A}{\Gamma(\alpha)} \int_1^\infty y^{-A\eta - A\alpha + A - 1} (y^A - 1)^{\alpha - 1} H_{p,q}^{m,n} \left(xy \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) dy$$

$$= H_{p+1,q+1}^{m+1,n} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1,p}, (\eta + \alpha, \frac{1}{A}) \\ (\eta, \frac{1}{A}), (b_1, \beta_1)_{1,q} \end{matrix} \right. \right), \quad x > 0,$$

where $\eta > -\frac{1}{2A}$, $\alpha > 0$, $A > 0$,

$$\sum_{j=1}^n \alpha_j + \sum_{j=1}^m \beta_j > \sum_{j=n+1}^p \alpha_j + \sum_{j=m+1}^q \beta_j.$$

Similarly we have,

$$(3.1.19) \quad I_A^{\eta, \alpha} \left\{ H_{p, q}^{m, n} \left((cx)^B \left| \begin{matrix} (a_1, \alpha_1)_{1, p} \\ (b_1, \beta_1)_{1, q} \end{matrix} \right. \right) \right\} \\ = H_{p+1, q+1}^{m, n+1} \left((cx)^B \left| \begin{matrix} (-n, \frac{B}{A}), (a_1, \alpha_1)_{1, p} \\ (b_1, \beta_1)_{1, q}, (-n-\alpha, \frac{B}{A}) \end{matrix} \right. \right),$$

where $\eta > \frac{1}{2A} - 1$, $A > 0$, $B > 0$, $c > 0$, $\alpha > 0$,

$$\sum_{j=1}^n \alpha_j + \sum_{j=1}^m \beta_j > \sum_{j=n+1}^p \alpha_j + \sum_{j=m+1}^q \beta_j;$$

$$(3.1.20) \quad \frac{A}{\Gamma(\alpha)} \int_0^1 y^{A\eta+A-1} (1-y^A)^{\alpha-1} H_{p, q}^{m, n} \left(xy \left| \begin{matrix} (a_1, \alpha_1)_{1, p} \\ (b_1, \beta_1)_{1, q} \end{matrix} \right. \right) dy \\ = H_{p+1, q+1}^{m, n+1} \left(x \left| \begin{matrix} (-n, \frac{1}{A}), (a_1, \alpha_1)_{1, p} \\ (b_1, \beta_1)_{1, q}, (-n-\alpha, \frac{1}{A}) \end{matrix} \right. \right),$$

where $A > 0$, $\alpha > 0$, $\eta > \frac{1}{2A} - 1$,

$$\sum_{j=1}^n \alpha_j + \sum_{j=1}^m \beta_j > \sum_{j=n+1}^p \alpha_j + \sum_{j=m+1}^q \beta_j.$$

We use the convolution property to derive some interesting results on H-functions. We have

(3.1.21)

$$\mathcal{M}\{K^{\zeta, \beta, B}(x)\} = \frac{\Gamma(\zeta + \frac{s}{B})}{\Gamma(\zeta + \beta + \frac{s}{B})}, \quad \beta > \frac{1}{2}, \quad \zeta > -\frac{1}{2B}, \quad B > 0, \quad \text{Res } s \geq \frac{1}{2},$$

which means

$$(3.1.22) \quad \frac{1}{2\pi i} \int_L \frac{\Gamma(\zeta + \frac{s}{B})}{\Gamma(\zeta + \beta + \frac{s}{B})} x^{-s} ds = K^{\zeta, \beta, B}(x).$$

Or, in other words

$$(3.1.23) \quad H_{1,0}^{1,1} \left(x \left| \begin{matrix} (\zeta + \beta, \frac{1}{B}) \\ (\zeta, \frac{1}{B}) \end{matrix} \right. \right) = K^{\zeta, \beta, B}(x).$$

$$= \begin{cases} \frac{B}{\Gamma(\beta)} (1-x)^B \beta^{-1} x^{B\zeta}, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$

By repeated application of convolution Theorem 1.1.2, and using the fact that convolution operation is associative, we have

$$(3.1.24) \quad \mathcal{M}\{K^{\xi_1, \beta_1, B_1} * \dots * K^{\xi_n, \beta_n, B_n}(x)\} = \frac{\Gamma(\xi_N + \frac{s}{B_N})}{\Gamma(\xi_N + \beta_N + \frac{s}{B_N})},$$

or

$$(3.1.25) \quad K^{\xi_1, \beta_1, B_1} * \dots * K^{\xi_n, \beta_n, B_n}(x) = \frac{1}{2\pi i} \int_L \frac{\Gamma(\xi_N + \frac{s}{B_N})}{\Gamma(\xi_N + \beta_N + \frac{s}{B_N})} x^{-s} ds,$$

where $B_j > 0$, $\xi_j > -\frac{1}{2B_j}$, $\beta_j > \frac{1}{2}$, $j = 1, \dots, n$. In particular,

$$(3.1.26) \quad K^{\zeta, \beta, B} * K^{\xi, \gamma, C}(x) = \frac{1}{2\pi i} \int_L \frac{\Gamma(\zeta + \frac{s}{B}) \Gamma(\xi + \frac{s}{C})}{\Gamma(\zeta + \beta + \frac{s}{B}) \Gamma(\xi + \gamma + \frac{s}{C})} x^{-s} ds$$

$$= H_{2,0} \left(x \left| \begin{matrix} (\zeta + \beta, \frac{1}{B}), (\xi + \gamma, \frac{1}{C}) \\ (\zeta, \frac{1}{B}), (\xi, \frac{1}{C}) \end{matrix} \right. \right)$$

On the other hand

$$(3.1.27) \quad K^{\zeta, \beta, B} * K^{\xi, \gamma, C}(x) = \int_0^{\infty} \frac{1}{t} K^{\zeta, \beta, B}(\frac{x}{t}) K^{\xi, \gamma, C}(t) dt$$

$$= \frac{BC}{\Gamma(\beta)\Gamma(\gamma)} x^{B\zeta} \int_0^{\infty} [1 - (\frac{x}{t})^B]^{\beta-1} (1-t)^{\gamma-1} t^{C\xi - B\zeta - 1} U(1 - \frac{x}{t}) U(1-t) dt$$

$$= \frac{BC}{\Gamma(\beta)\Gamma(\gamma)} x^{B\zeta} \int_x^{\infty} [1 - (\frac{x}{t})^B]^{\beta-1} (1-t)^{\gamma-1} t^{C\xi - B\zeta - 1} U(1-t) dt$$

$$= \begin{cases} \frac{BC}{\Gamma(\beta)\Gamma(\gamma)} x^{B\zeta} \int_x^1 [1 - (\frac{x}{t})^B]^{\beta-1} (1-t)^{\gamma-1} t^{C\xi - B\zeta - 1} dt, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$

Therefore

$$(3.1.28) \quad H_{2,0} \left(x \left| \begin{matrix} (\zeta + \beta, \frac{1}{B}), (\xi + \gamma, \frac{1}{C}) \\ (\zeta, \frac{1}{B}), (\xi, \frac{1}{C}) \end{matrix} \right. \right)$$

$$= \frac{BC}{\Gamma(\beta)\Gamma(\gamma)} x^{B\zeta} \int_x^{\infty} [1 - (\frac{x}{t})^B]^{\beta-1} (1-t)^{\gamma-1} t^{C\xi - B\zeta - 1} U(1-t) dt,$$

and

$$(3.1.29) \quad K^{\zeta, \beta, B} * K^{\xi, \gamma, C}(x) = 0 \text{ for } x \geq 1.$$

By induction and the fact that convolution operation is associative we obtain

$$(3.1.30) \quad K^{\xi_1, \beta_1, B_1} * \dots * K^{\xi_n, \beta_n, B_n}(x) = 0, \text{ for } x \geq 1,$$

where $B_j > 0, \xi_j > \frac{1}{2B_j}, \beta_j > \frac{1}{2}, j = 1, \dots, n$. This implies

$$(3.1.31) \quad \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(\zeta_N + B_N s)}{\Gamma(\zeta_N + \beta_N + B_N s)} x^{-s} ds = 0, \text{ for } x \geq 1,$$

with $\zeta_j > -\frac{B_j}{2}, \beta_j > \frac{1}{2}, B_j > 0, j = 1, \dots, n$. By analytic continuation the restriction on $\beta_j, j = 1, \dots, n$, may be

replaced by $\sum_{j=1}^n \beta_j > 1$ if $n \geq 2$. Thus, for $x \geq 1$,

$$(3.1.32) \quad H_{n,0}^{n,n} \left(x \left| \begin{matrix} (b_1 + a_1, \beta_1)_{1,n} \\ (b_1, \beta_1)_{1,n} \end{matrix} \right. \right) = 0, \sum_{j=1}^n a_j > 1, n \geq 2.$$

If we replace x by $1/x$, then the above becomes

$$(3.1.33) \quad H_{n,n}^{0,n} \left(x \left| \begin{matrix} (b_1, \beta_1)_{1,n} \\ (b_1 + a_1, \beta_1)_{1,n} \end{matrix} \right. \right) = 0, \sum_{j=1}^n a_j < -1, n \geq 2,$$

for $0 < x \leq 1$. Following the method of derivation of (3.1.28) and using the functions $I^{\eta, \alpha, A}(x)$ and $I^{\zeta, \beta, B}(x)$ instead of $K^{\zeta, \beta, B}(x)$ and $K^{\xi, \gamma, C}(x)$ we obtain

$$(3.1.34) \quad H_{2,2}^{0,2} \left(x \left| \begin{matrix} (-\eta, \frac{1}{A}), (-\zeta, \frac{1}{B}) \\ (-\eta - \alpha, \frac{1}{A}), (-\zeta - \beta, \frac{1}{B}) \end{matrix} \right. \right) \\ = \frac{AB}{\Gamma(\alpha)\Gamma(\beta)} x^{-\eta - \alpha} \int_0^x \left[\left(\frac{x}{t} \right)^A - 1 \right]^{\alpha-1} (t^B - 1)^{\beta-1} t^{\eta + \alpha - B\zeta - B\beta - 1} U(t-1) dt,$$

where $\alpha + \beta > 1$, $\eta > \frac{1}{2A} - 1$, $\zeta > \frac{1}{2B} - 1$, $A > 0$, $B > 0$, $\alpha > 0$, $\beta > 0$.

Next, since

$$(3.1.35) \quad \frac{\Gamma(1+\eta-\frac{s}{A})\Gamma(\zeta+\frac{s}{B})}{\Gamma(1+\eta+\alpha-\frac{s}{A})\Gamma(\zeta+\beta+\frac{s}{B})} = \mathcal{M}\{I^{\eta,\alpha,A} * K^{\zeta,\beta,B}(x)\},$$

where $\zeta > -\frac{1}{2B}$, $\beta > \frac{1}{2}$, $B > 0$, $\alpha > \frac{1}{2}$, $\eta > \frac{1}{2A} - 1$, $A > 0$,

$\text{Re } s = \frac{1}{2}$, we obtain

$$(3.1.36) \quad \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(1+\eta-\frac{s}{A})\Gamma(\zeta+\frac{s}{B})}{\Gamma(1+\eta+\alpha-\frac{s}{A})\Gamma(\zeta+\beta+\frac{s}{B})} x^{-s} ds = I^{\eta,\alpha,A} * K^{\zeta,\beta,B}(x)$$

$$= \frac{AB}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\infty} \frac{1}{t} \left[\left(\frac{x}{t}\right)^{A-1} \right]^{\alpha-1} \left(\frac{x}{t}\right)^{-A\eta-A\alpha} (1-t^B)^{\beta-1} t^{B\zeta} U\left(\frac{x}{t}-1\right) U(1-t) dt,$$

or

$$(3.1.37) \quad H_{2,2}^{1,1} \left(x \left| \begin{matrix} (-\eta, \frac{1}{A}), (\zeta+\beta, \frac{1}{B}) \\ (\zeta, \frac{1}{B}), (-\eta-\alpha, \frac{1}{A}) \end{matrix} \right. \right)$$

$$= \frac{AB}{\Gamma(\alpha)\Gamma(\beta)} x^{-A\eta-A\alpha} \int_0^x \left[\left(\frac{x}{t}\right)^A - 1 \right]^{\alpha-1} (1-t^B)^{\beta-1} t^{A\eta+A\alpha+B\zeta-1} U(1-t) dt.$$

3.2 $W_{\lambda,\mu}$ -TRANSFORM.

The integral equation (3.1) can be treated as an integral

transform whose kernel is $x^{\mu-\frac{1}{2}} e^{-\frac{x}{2}} W_{\lambda,\mu}(x)$. This transform

introduced by Varma [42, p. 209] is a generalization of Laplace transform. In the present section we apply fractional integration to solve (3.1). First we prove a few lemmas.

3.2.1 Lemma 3.2.1 If $\text{Re } \mu > -\frac{1}{4}$, $2\mu \neq 0, 1, 2, \dots$, then

$$k(x) = x^{\mu - \frac{1}{2}} e^{-\frac{x}{2}} W_{\lambda, \mu}(x) \in L_2(0, \infty)$$

and

$$K(s) = \mathcal{N}\{k(x)\} = \frac{\Gamma(s)\Gamma(2\mu+s)}{\Gamma(\frac{1}{2}+\mu-\lambda+s)}, \text{Re } s > 0$$

is bounded on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$, and is exponentially small as $|t| \rightarrow \infty$.

Proof. First we note that k is continuous in $(0, \infty)$ and [31, p. 134]

$$(3.2.1) \quad k(x) = x^{2\mu} e^{-x} \Psi\left(\frac{1}{2} - \lambda + \mu, 2\mu + 1; x\right)$$

and hence the asymptotic behaviour of $k(x)$ as $|x| \rightarrow \infty$ is given [31, p. 127] by

$$(3.2.2) \quad k(x) = x^{\mu+\lambda-\frac{1}{2}} e^{-x} \{1 + o(\frac{1}{|x|})\} \sim x^{\mu+\lambda-\frac{1}{2}} e^{-x},$$

$|\arg x| \leq \pi - \delta < \pi$. To obtain asymptotic behaviour for $k(x)$ as $|x| \rightarrow 0$, we note that [36, p. 14]

(3.2.3)

$$k(x) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\lambda-\mu)} x^{2\mu} e^{-x} {}_1F_1\left(\frac{1}{2}-\lambda+\mu, 2\mu+1; x\right) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}-\lambda+\mu)} e^{-x} {}_1F_1\left(\frac{1}{2}-\lambda-\mu, 1-2\mu; x\right)$$

from which it follows that as $|x| \rightarrow 0$

$$(3.2.4) \quad k(x) \sim \begin{cases} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\lambda-\mu)} x^{2\mu} & \text{if } \operatorname{Re} \mu < 0, 2\mu \neq -1, -2, \dots, \\ \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\lambda-\mu)} x^{2\mu} + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}-\lambda+\mu)} & \text{if } \operatorname{Re} \mu = 0, \text{ and } \mu \neq 0, \\ \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}-\lambda+\mu)} & \text{if } \operatorname{Re} \mu > 0, 2\mu \neq 1, 2, \dots \end{cases}$$

Therefore, $k(x)$ is bounded on an interval $(0, \delta)$ in case $\operatorname{Re} \mu \geq 0$ and $\mu \neq 0$. In case $0 > \operatorname{Re} \mu > -\frac{1}{4}$ we have

$$\int_0^\delta |k(x)|^2 dx = \left| \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\lambda-\mu)} \right|^2 \int_0^\delta x^{4\operatorname{Re} \mu} dx < \infty.$$

Thus

$$(3.2.5) \quad \int_0^\delta |k(x)|^2 dx < \infty$$

whenever $\operatorname{Re} \mu > -\frac{1}{4}$ and $2\mu \neq 0, 1, 2, \dots$

Further,

$$(3.2.6) \quad \int_N^\infty |k(x)|^2 dx \sim \int_N^\infty e^{-2x} x^{2\operatorname{Re}(\mu+\lambda)-1} dx < \infty.$$

The estimates (3.2.5) and (3.2.6) imply that

$$k(x) = x^{\mu-\frac{1}{2}} e^{-\frac{x}{2}} W_{\lambda, \mu}(x) \in L_2(0, \infty) \text{ if } \operatorname{Re} \mu > -\frac{1}{4} \text{ and } 2\mu \neq 0, 1, 2, \dots$$

For the Mellin transform $K(s)$ of $k(x)$, we have

$$(3.2.7) \quad \begin{aligned} K(s) &= \int_0^\infty k(x) x^{s-1} dx = \int_0^\infty x^{s+\mu-\frac{3}{2}} e^{-\frac{x}{2}} W_{\lambda, \mu}(x) dx \\ &= \frac{\Gamma(s) \Gamma(2\mu+s)}{\Gamma(\frac{1}{2}+\mu-\lambda+s)}, \quad s = \frac{1}{2} + it, \quad -\infty < t < \infty, \end{aligned}$$

on using a known integral [36, p. 53, eqn. (3.6.14)]. Since [31, p. 33, (12)]

$$(3.2.8) \quad \frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b} \{1 + o(\frac{1}{|z|})\}, \quad |\arg(z+a)| \leq \pi - \delta < \pi,$$

we obtain, with the help of relation (2.2.6),

$$(3.2.9) \quad |K(\frac{1}{2}+it)| \sim A|t|^{\operatorname{Re}(\mu+\lambda)-\frac{1}{2}} e^{-\frac{\pi}{2}|t|} \rightarrow 0 \text{ as } |t| \rightarrow \infty,$$

where A is some constant. Hence K(s) is bounded for $s = \frac{1}{2}+it$,

$-\infty < t < \infty$, if $\operatorname{Re} \mu > -\frac{1}{4}$. This completes the proof of the lemma.

For convenience we use the following language. If g is of the form of (3.1), we say that g is the $W_{\lambda, \mu}$ -transform of f and denote it by $g(x) = W_{\lambda, \mu}\{f(x)\}$. It may be noted that the result in the following lemma is an improvement over a similar result of Saxena [49, p. 469] since the condition on μ is less restrictive.

Lemma 3.2.2 Let

- (i) $\operatorname{Re} \mu > -\frac{1}{4}$, $2\mu \neq 0, 1, 2, \dots$,
- (ii) $f \in L_2(0, \infty)$ and $\mathcal{M}\{f(x)\} = F(s)$.

Then

$$(3.2.10) \quad g(x) = W_{\lambda, \mu}\{f(x)\} \in L_2(0, \infty)$$

and

$$(3.2.11) \quad \mathcal{M}[W_{\lambda, \mu}\{f(x)\}] = \frac{\Gamma(2\mu+s)\Gamma(s)}{\Gamma(\frac{1}{2}+\mu-\lambda+s)} F(1-s).$$

Proof. This lemma follows immediately from Lemma 3.2.1 and Theorem 1.1.1.

Theorem 3.2.3 Let

$$(i) \quad \mu - \lambda > -1, \mu + \lambda \geq \frac{1}{2}, 2\mu \neq 0, 1, 2, \dots,$$

$$(ii) \quad m > -\frac{1}{4}, \frac{1}{2} - m - k \geq 0, 2m \neq 0, 1, 2, \dots,$$

$$(iii) \quad f \in L_2(0, \infty).$$

Then

$$(3.2.12) \quad K^{2m, \frac{1}{2}-m-k} K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}} [W_{\lambda, \mu} \{f(x)\}] \\ = W_{k, m} \{f(x)\}, \text{ a.e.},$$

and in particular

$$(3.2.13) \quad K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}} [W_{\lambda, \mu} \{f(x)\}] \\ = \int_0^{\infty} e^{-xu} f(u) du = \mathcal{L} \{f(x)\}, \text{ a.e.},$$

where \mathcal{L} denotes the Laplace transform [10, p. 127].

Proof. In Lemma 3.2.2, we proved that under condition (i) the function

$$(3.2.14) \quad g(x) = W_{\lambda, \mu} \{f(x)\}$$

belongs to $L_2(0, \infty)$. So by [30, p. 199, Theorem 2]

$$h(x) = K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}} g(x) \in L_2(0, \infty).$$

By the same reason $K^{2m, \frac{1}{2}-m-k} h(x) \in L_2(0, \infty)$. Hence by (3.1.7) and (3.2.11),

$$(3.2.15) \quad \mathcal{M} \left\{ K^{2m, \frac{1}{2}-m-k} K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}} [W_{\lambda, \mu} \{f(x)\}] \right\} \\ = \frac{\Gamma(2m+s)\Gamma(s)}{\Gamma(m-k+\frac{1}{2}+s)} F(1-s),$$

where $F(s) = \mathcal{M} \{f(x)\}$. By (3.2.11) again

$$(3.2.16) \quad \mathcal{M} \left\{ K^{2m, \frac{1}{2}-m-k} K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}} [W_{\lambda, \mu} \{f(x)\}] \right\} \\ = \mathcal{M} \{W_{k, m} \{f(x)\}\}.$$

On taking inverse Mellin transform of both sides of (3.2.16) we obtain (3.2.12). In particular, if we take $m+k = \frac{1}{2}$, we have (3.2.13), [32, p. 305].

Theorem 3.2.4 (Saxena). Let

(i) $\mu - \lambda > -1, \mu + \lambda \geq \frac{1}{2}, 2\mu \neq 0, 1, 2, \dots,$

(ii) $f \in L_2(0, \infty)$ be a solution of

$$(3.2.17) \quad g(x) = W_{\lambda, \mu} \{f(x)\}.$$

Then [49, p. 470]

$$(3.2.18) \quad f(x) = \mathcal{L}^{-1}\{K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}} g(x)\}, \text{ a.e.},$$

where \mathcal{L}^{-1} denotes the inverse Laplace transform.

Proof. By Lemma 3.2.2, $g \in L_2(0, \infty)$. Apply the operator $K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}}$ to both sides of (3.2.17) then by (3.2.13) of Theorem 3.2.3

$$(3.2.19) \quad K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}} g(x) = \int_0^{\infty} e^{-xu} f(u) du = \mathcal{L}\{f(x)\}.$$

Apply the inverse Laplace operator to both sides of (3.2.19) to obtain (3.2.18).

3.2.2 Illustrations. In this section we assume $\mu - \lambda > -1$,

$\mu + \lambda \geq \frac{1}{2}$, $2\mu \neq 0, 1, 2, \dots$

Example 1. Let $W_{\lambda, \mu}\{f(x)\} = g(x) = e^{-ax}$, $a > 0$. Then

$$\begin{aligned} K^{\mu-\lambda+\frac{1}{2}, \mu+\lambda-\frac{1}{2}} g(x) &= \frac{1}{\Gamma(\mu+\lambda-\frac{1}{2})} x^{\mu-\lambda+\frac{1}{2}} \int_x^{\infty} (t-x)^{\mu+\lambda-\frac{3}{2}} t^{-2\mu} e^{-at} dt \\ &= \frac{1}{\Gamma(\mu+\lambda-\frac{1}{2})} \int_1^{\infty} (y-1)^{\mu+\lambda-\frac{3}{2}} y^{-2\mu} e^{-axy} dy \\ &= \frac{1}{\Gamma(\mu+\lambda-\frac{1}{2})} \int_0^{\infty} (y-1)^{\mu+\lambda-\frac{3}{2}} y^{-2\mu} U(y-1) e^{-axy} dy \end{aligned}$$

$$= \frac{a^{\mu-\lambda+\frac{1}{2}}}{\Gamma(\mu+\lambda-\frac{1}{2})} \int_0^{\infty} e^{-xt} (t-a)^{\mu+\lambda-\frac{3}{2}} t^{-2\mu} U\left(\frac{t}{a}-1\right) dt.$$

Here U is the unit step function as defined in (3.1.9).

Hence by Theorem 3.2.4, it follows that the last expression is $\mathcal{L}\{f(x)\}$ and then

$$f(x) = \frac{a^{\mu-\lambda+\frac{1}{2}}}{\Gamma(\mu+\lambda-\frac{1}{2})} (x-a)^{\mu+\lambda-\frac{3}{2}} x^{-2\mu} U\left(\frac{x}{a}-1\right).$$

It may be interesting to verify the result of this example. We have [11, p. 211; (73); 32, p. 305]

$$\begin{aligned} W_{\lambda, \mu}\{f(x)\} &= \frac{a^{\mu-\lambda+\frac{1}{2}}}{\Gamma(\mu+\lambda-\frac{1}{2})} \int_0^{\infty} (xu)^{\mu-\frac{1}{2}} e^{-\frac{xu}{2}} W_{\lambda, \mu}(xu)(u-a)^{\mu+\lambda-\frac{3}{2}} u^{-2\mu} U\left(\frac{u}{a}-1\right) du \\ &= \frac{a^{\mu-\lambda+\frac{1}{2}}}{\Gamma(\mu+\lambda-\frac{1}{2})} x^{\mu-\frac{1}{2}} \int_a^{\infty} u^{-\mu-\frac{1}{2}} e^{-\frac{xu}{2}} W_{\lambda, \mu}(xu)(u-a)^{\mu+\lambda-\frac{3}{2}} du \\ &= (ax)^{\mu-\frac{1}{2}} e^{-\frac{ax}{2}} W_{\frac{1}{2}-\mu, \mu}(ax) = e^{-ax} \end{aligned}$$

Example 2. Let $W_{\lambda, \mu}\{f(x)\} = g(x) = G_{\substack{m, n \\ p, q}} \left(x \left| \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array} \right. \right),$

where $2(m+n) > p+q$, $0 < \operatorname{Re}(a_j - 1) < \mu - \lambda + \frac{1}{2}$, $j = 1, \dots, n$.

Then by Theorem 3.2.4,

$$\mathcal{L}\{f(x)\} = K a^{\mu-\lambda+\frac{1}{2}, \mu+\lambda-\frac{1}{2}} G_{\substack{m, n \\ p, q}} \left(x \left| \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array} \right. \right)$$

$$\begin{aligned}
 &= \frac{1}{\Gamma(\mu+\lambda-\frac{1}{2})} x^{\mu-\lambda+\frac{1}{2}} \int_x^\infty (t-x)^{\mu+\lambda-\frac{3}{2}} t^{-2\mu} G_{p,q}^{m,n} \left(t \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dt \\
 &= \frac{1}{\Gamma(\mu+\lambda-\frac{1}{2})} \int_1^\infty y^{-2\mu} (y-1)^{\mu+\lambda-\frac{3}{2}} G_{p,q}^{m,n} \left(xy \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dy \\
 &= G_{p+1,q+1}^{m+1,n} \left(x \left| \begin{matrix} a_1, \dots, a_p, 2\mu \\ \mu-\lambda+\frac{1}{2}, b_1, \dots, b_q \end{matrix} \right. \right) = G_{q+2,p+2}^{n,m+2} \left(\frac{1}{x} \left| \begin{matrix} 1, \lambda-\mu+\frac{1}{2}, 1-b_1, \dots, 1-b_q \\ 1-a_1, \dots, 1-a_p, 1-2\mu, 1 \end{matrix} \right. \right)
 \end{aligned}$$

on using a known integral [31, p. 170, (6)] and (1.3.3), (1.3.1).

Hence by [10, p. 222, (34)]

$$\begin{aligned}
 f(x) &= \frac{1}{x} G_{q+1,p+2}^{n,m+1} \left(x \left| \begin{matrix} \lambda-\mu+\frac{1}{2}, 1-b_1, \dots, 1-b_q \\ 1-a_1, \dots, 1-a_p, 1-2\mu, 1 \end{matrix} \right. \right) \\
 &= \frac{1}{x} G_{p+2,q+1}^{m+1,n} \left(\frac{1}{x} \left| \begin{matrix} a_1, \dots, a_p, 2\mu, 0 \\ \mu-\lambda+\frac{1}{2}, b_1, \dots, b_q \end{matrix} \right. \right)
 \end{aligned}$$

Before we give another example, we apply Theorem 1.1.1

to calculate the Laplace transform of $x^{-\sigma} H_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right)$.

To ensure that the function $x^{-\sigma} H_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right)$ belongs to

$L_2(0, \infty)$, we assume that

$$\sum_{j=1}^n \alpha_j + \sum_{j=1}^m \beta_j > \sum_{j=n+1}^p \alpha_j + \sum_{j=m+1}^q \beta_j,$$

and

$$-\min_{1 \leq j \leq m} \operatorname{Re}\left(\frac{b_j}{\beta_j}\right) < -\frac{1}{2} - \operatorname{Re} \sigma < \min_{1 \leq j \leq n} \operatorname{Re}\left(\frac{1-a_j}{\alpha_j}\right).$$

Since the function $f(x) = e^{-x}$ belongs to $L_2(0, \infty)$ and its Mellin transform $\mathcal{M}\{f(x)\} = \Gamma(s)$ is clearly bounded on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$, by Theorem 1.1.1, we have

$$\begin{aligned} (3.2.20) \quad & \mathcal{M}\left\{\mathcal{L}\left(x^{-\sigma} H_{p,q}^{m,n} \left(x \left| \begin{array}{l} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right. \right)\right)\right\} \\ &= \mathcal{M}\left\{\mathcal{L}\left(H_{p,q}^{m,n} \left(x \left| \begin{array}{l} (a_1 - \sigma \alpha_1, \alpha_1)_{1,p} \\ (b_1 - \sigma \beta_1, \beta_1)_{1,q} \end{array} \right. \right)\right)\right\} \\ &= \mathcal{M}\left\{\int_0^\infty e^{-xy} H_{p,q}^{m,n} \left(y \left| \begin{array}{l} (a_1 - \sigma \alpha_1, \alpha_1)_{1,p} \\ (b_1 - \sigma \beta_1, \beta_1)_{1,q} \end{array} \right. \right) dy\right\} \\ &= \Gamma(s) \frac{\Gamma(b_M - \sigma \beta_M + \beta_M(1-s)) \Gamma(1-a_N + \sigma \alpha_N - \alpha_N(1-s))}{\Gamma_m(1-b_Q + \sigma \beta_Q - \beta_Q(1-s)) \Gamma_n(a_P - \sigma \alpha_P + \alpha_P(1-s))} \\ &= \Gamma(s) \frac{\Gamma(b_M + \beta_M - \sigma \beta_M - \beta_M s) \Gamma(1-a_N - \alpha_N + \sigma \alpha_N + \alpha_N s)}{\Gamma_m(1-b_Q - \beta_Q + \sigma \beta_Q + \beta_Q s) \Gamma_n(a_P + \alpha_P - \sigma \alpha_P - \alpha_P s)}. \end{aligned}$$

This implies that

$$\begin{aligned} (3.2.21) \quad & \mathcal{L}\left\{x^{-\sigma} H_{p,q}^{m,n} \left(x \left| \begin{array}{l} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right. \right)\right\} \\ &= H_{q,p+1}^{n+1,m} \left(x \left| \begin{array}{l} (1-b_1 - \beta_1 + \sigma \beta_1, \beta_1)_{1,q} \\ (0, 1), (1-a_1 - \alpha_1 + \sigma \alpha_1, \alpha_1)_{1,p} \end{array} \right. \right) \end{aligned}$$

$$= x^{\sigma-1} H_{p+1,q}^{m,n+1} \left(\frac{1}{x} \left| \begin{array}{l} (\sigma, 1), (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right. \right)$$

This result has been obtained by Jain [21, p. 374] also.

Example 3. Let $W_{\lambda,\mu}\{f(x)\} = g(x) = H_{p,q}^{m,n} \left(x \left| \begin{array}{l} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right. \right)$,

with $\sum_{j=1}^n \alpha_j + \sum_{j=1}^m \beta_j > \sum_{j=n+1}^p \alpha_j + \sum_{j=m+1}^q \beta_j$.

By Theorem 3.2.4

$$\begin{aligned} \mathcal{L}\{f(x)\} &= K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}} H_{p,q}^{m,n} \left(x \left| \begin{array}{l} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right. \right) \\ &= H_{p+1,q+1}^{m+1,n} \left(x \left| \begin{array}{l} (a_1, \alpha_1)_{1,p}, (2\mu, 1) \\ (\frac{1}{2}+\mu-\lambda, 1), (b_1, \beta_1)_{1,q} \end{array} \right. \right) \\ &= \frac{1}{x} H_{q+1,p+1}^{n,m+1} \left(\frac{1}{x} \left| \begin{array}{l} (\lambda-\mu-\frac{1}{2}, 1), (1-b_1-\beta_1, \beta_1)_{1,q} \\ (1-a_1-\alpha_1, \alpha_1)_{1,p}, (-2\mu, 1) \end{array} \right. \right) \\ &= \frac{1}{x} H_{q+2,p+2}^{n,m+2} \left(\frac{1}{x} \left| \begin{array}{l} (0, 1), (\lambda-\mu-\frac{1}{2}, 1), (1-b_1-\beta_1, \beta_1)_{1,q} \\ (1-a_1-\alpha_1, \alpha_1)_{1,p}, (-2\mu, 1), (0, 1) \end{array} \right. \right) \end{aligned}$$

Therefore by (3.2.21) and (1.3.8), (1.3.7)

$$\begin{aligned} f(x) &= \mathcal{L}^{-1} \left\{ \frac{1}{x} H_{q+2,p+2}^{n,m+2} \left(\frac{1}{x} \left| \begin{array}{l} (0, 1), (\lambda-\mu-\frac{1}{2}, 1), (1-b_1-\beta_1, \beta_1)_{1,q} \\ (1-a_1-\alpha_1, \alpha_1)_{1,p}, (-2\mu, 1), (0, 1) \end{array} \right. \right) \right\} \\ &= H_{q+1,p+2}^{n,m+1} \left(x \left| \begin{array}{l} (\lambda-\mu-\frac{1}{2}, 1), (1-b_1-\beta_1, \beta_1)_{1,q} \\ (1-a_1-\alpha_1, \alpha_1)_{1,p}, (-2\mu, 1), (0, 1) \end{array} \right. \right) \end{aligned}$$

$$= \frac{1}{x} H_{p+2, q+1}^{m+1, n} \left(\frac{1}{x} \left| \begin{array}{l} (a_1, \alpha_1)_{1, p}, (2\mu, 1), (0, 1) \\ (\frac{1}{2} + \mu - \lambda, 1), (b_1, \beta_1)_{1, q} \end{array} \right. \right)$$

3.2.3 Theorem 3.2.5. Let

- (i) $m > -\frac{1}{4}, \frac{1}{2} - m - k \geq 0, 2m \neq 0, 1, 2, \dots,$
- (ii) $h(x) = \mathcal{L}\{f(x)\},$
- (iii) $f \in L_2(0, \infty).$

Then $K^{2m, \frac{1}{2} - m - k} h(x)$ is the $W_{k, m}$ -transform of $f(x)$.

Proof. By (3.2.13) of Lemma 3.2.3

(3.2.22)

$$h(x) = \mathcal{L}\{f(x)\} = K^{\frac{1}{2} + \mu - \lambda, \mu + \lambda - \frac{1}{2}} [W_{\lambda, \mu}\{f(x)\}],$$

where λ, μ be such that $\mu - \lambda > -1, \mu + \lambda \geq \frac{1}{2}$ and $2\mu \neq 0, 1, 2, \dots$

Apply the operator $K^{2m, \frac{1}{2} - m - k}$ to both sides of (3.2.22) and with the help of (3.2.12) we obtain the desired result.

Lemma 3.2.6. [36, p. 25, eqn. (2.4.16)]. Let n be a positive integer. Then

$$\begin{aligned} (3.2.23) \quad & (-1)^n x^{n+2\mu} \frac{d^n}{dx^n} \{x^{-2\mu}(xu)^{\mu - \frac{1}{2}} e^{-\frac{xu}{2}} W_{\lambda, \mu}(xu)\} \\ & = (xu)^{\mu + \frac{n}{2} - \frac{1}{2}} e^{-\frac{xu}{2}} W_{\lambda + \frac{n}{2}, \mu + \frac{n}{2}}(xu). \end{aligned}$$

Lemma 3.2.7. If

- (i) $K(x,t)$ has x -partial derivative continuous in $a \leq x \leq b$ and $t \geq 0$,
- (ii) $\int_0^\infty |K(x,t)|^2 dt$ is convergent in $a \leq x \leq b$,
- (iii) $\int_0^\infty \left| \frac{\partial}{\partial x} K(x,t) \right|^2 dt$ is convergent uniformly in $a \leq x \leq b$,
- (iv) $f \in L_2(0, \infty)$,

then

$$(3.2.24) \quad F(x) = \int_0^\infty K(x,t)f(t)dt$$

is differentiable in $a < x < b$ and

$$(3.2.25) \quad F'(x) = \int_0^\infty \frac{\partial}{\partial x} K(x,t)f(t)dt.$$

Proof. From the conditions (ii) and (iv) we know that

$$(3.2.26) \quad \int_0^\infty K(x,t)f(t)dt$$

defines a function $F(x)$. For, from Schwarz's inequality [39, p. 381]

$$(3.2.27) \quad \left| \int_N^\infty K(x,t)f(t)dt \right| \leq \left\{ \int_N^\infty |K(x,t)|^2 dt \int_N^\infty |f(t)|^2 dt \right\}^{\frac{1}{2}} \rightarrow 0.$$

Similarly

$$(3.2.28) \quad \int_0^{\infty} \frac{\partial}{\partial x} K(x,t) f(t) dt$$

defines a function $G(x)$.

Now by the Mean Value Theorem and Schwarz's inequality

$$(3.2.29) \quad \lim_{h \rightarrow 0} \left| \frac{F(x+h) - F(x)}{h} - G(x) \right|^2$$

$$= \lim_{h \rightarrow 0} \left| \int_0^{\infty} \left[\frac{K(x+h,t) - K(x,t)}{h} - \frac{\partial}{\partial x} K(x,t) \right] f(t) dt \right|^2$$

$$= \lim_{\xi \rightarrow 0} \left| \int_0^{\infty} \left[\frac{\partial}{\partial x} K(x+\xi,t) - \frac{\partial}{\partial x} K(x,t) \right] f(t) dt \right|^2, \quad 0 < \xi < h$$

$$\leq \lim_{\xi \rightarrow 0} \int_0^{\infty} \left| \frac{\partial}{\partial x} K(x+\xi,t) - \frac{\partial}{\partial x} K(x,t) \right|^2 dt \int_0^{\infty} |f(t)|^2 dt$$

$$= \lim_{\xi \rightarrow 0} \|f\|^2 \left\{ \int_0^N \left| \frac{\partial}{\partial x} K(x+\xi,t) - \frac{\partial}{\partial x} K(x,t) \right|^2 dt \right.$$

$$\left. + \int_N^{\infty} \left| \frac{\partial}{\partial x} K(x+\xi,t) - \frac{\partial}{\partial x} K(x,t) \right|^2 dt \right\}$$

$$\leq \lim_{\xi \rightarrow 0} \|f\|^2 \left\{ \int_0^N \left| \frac{\partial}{\partial x} K(x+\xi,t) - \frac{\partial}{\partial x} K(x,t) \right|^2 dt \right.$$

$$\left. + 2 \int_N^{\infty} \left| \frac{\partial}{\partial x} K(x+\xi,t) \right|^2 dt + 2 \int_N^{\infty} \left| \frac{\partial}{\partial x} K(x,t) \right|^2 dt \right\},$$

where

$$(3.2.30) \quad \|f\|^2 = \int_0^{\infty} |f(t)|^2 dt.$$

From (iii), for any given $\epsilon > 0$, there exists a $N(\epsilon)$ independent on x such that

$$(3.2.31) \quad \int_N^\infty \left| \frac{\partial}{\partial x} K(x,t) \right|^2 dt \leq \frac{\epsilon^2}{8 \|f\|^2} \text{ for all } N \geq N(\epsilon).$$

By (i) $\frac{\partial}{\partial x} K(x,t)$ exists and continuous in $a \leq x \leq b$, hence it is uniformly continuous in $a \leq x \leq b, 0 \leq t \leq N$. Therefore for the given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$(3.2.32) \quad \left| \frac{\partial}{\partial x} K(x+\xi,t) - \frac{\partial}{\partial x} K(x,t) \right|^2 < \frac{\epsilon^2}{2N \|f\|^2}$$

whenever $|\xi| < \delta, 0 \leq t \leq N$. Therefore

$$(3.2.33) \quad \lim_{h \rightarrow 0} \left| \frac{F(x+h) - F(x)}{h} - G(x) \right|^2$$

$$\leq \lim_{\xi \rightarrow 0} \|f\|^2 \left\{ \frac{\epsilon^2}{2N \|f\|^2} \int_0^N dt + \frac{2\epsilon^2}{8 \|f\|^2} + \frac{2\epsilon^2}{8 \|f\|^2} \right\} = \epsilon^2.$$

This is equivalent to

$$F'(x) = G(x) = \int_0^\infty \frac{\partial}{\partial x} K(x,t) f(t) dt.$$

Lemma 3.2.8. Let

- (i) n be a positive integer,
- (ii) $x > 0$,
- (iii) $\mu > -\frac{1}{4}, 2\mu \neq 0, 1, 2, \dots$,
- (iv) $f \in L_2(0, \infty)$.

Then

$$(3.2.34) \quad (-1)^n x^{n+2\mu} \frac{d^n}{dx^n} [x^{-2\mu} W_{\lambda, \mu} \{f(x)\}] = W_{\lambda + \frac{n}{2}, \mu + \frac{n}{2}} \{f(x)\}.$$

Proof. If the order of differentiation and integration involved on the left-hand side of (3.2.34) can be interchanged then the desired result would follow immediately from Lemma 3.2.6. Now at each stage of this differentiation we have to differentiate expressions of the type $x^{-a}I$, where $a > \frac{1}{2}$, $a \neq 0, 1, 2, \dots$, I is given by

$$(3.2.35) \quad I = \int_0^{\infty} (xu)^{\mu+r-\frac{1}{2}} e^{-\frac{xu}{2}} W_{\lambda+\frac{r}{2}, \mu+\frac{r}{2}}(xu) f(u) du$$

and r is a positive integer. Consequently by Lemma 3.2.7 the validity of the desired interchange of order will be established if we can prove that I is uniformly convergent with respect to $x \geq \delta > 0$. From Schwarz's inequality [39, p. 381] we have

$$(3.2.36) \quad \left| \int_N^{\infty} (xu)^{\mu+\frac{r}{2}-\frac{1}{2}} e^{-\frac{xu}{2}} W_{\lambda+\frac{r}{2}, \mu+\frac{r}{2}}(xu) f(u) du \right|^2 \\ \leq \int_N^{\infty} \left| (xu)^{\mu+\frac{r}{2}-\frac{1}{2}} e^{-\frac{xu}{2}} W_{\lambda+\frac{r}{2}, \mu+\frac{r}{2}}(xu) \right|^2 du \int_N^{\infty} |f(u)|^2 du \\ = \frac{1}{x} \int_{xN}^{\infty} |v|^{\mu+\frac{r}{2}-\frac{1}{2}} e^{-\frac{v}{2}} W_{\lambda+\frac{r}{2}, \mu+\frac{r}{2}}(v) |f(u)|^2 dv \int_N^{\infty} |f(u)|^2 du.$$

From (3.2.2), we obtain

$$(3.2.37) \quad v^{\mu+\frac{r}{2}-\frac{1}{2}} e^{-\frac{v}{2}} W_{\lambda+\frac{r}{2}, \mu+\frac{r}{2}}(v) \sim v^{\mu+\lambda+r-\frac{1}{2}} e^{-v}, \quad |v| \rightarrow \infty.$$

It then follows that the first integral on the right of (3.2.36) converges and is bounded for all positive (and zero) values of N and $x (\geq \delta > 0)$. Let B denote the upper bound of this integral. Since $f \in L_2(0, \infty)$, given $\epsilon > 0$, we can choose N_0 dependent on ϵ and independent of $x (\geq \delta > 0)$ so that

$$(3.2.38) \quad \int_N^\infty |f(u)|^2 du < \frac{\delta \epsilon^2}{B}, \quad N > N_0.$$

Hence

$$\left| \int_N^\infty (xu)^{\mu + \frac{r}{2}} e^{-\frac{xu}{2}} W_{\lambda + \frac{r}{2}, \mu + \frac{r}{2}}(xu) f(u) du \right| < \epsilon, \quad N > N_0,$$

where N_0 is independent of $x (\geq \delta > 0)$. This establishes the uniform convergence of (3.2.35) with respect to x and so justifies the change in the order of differentiation and integration involved in (3.2.34).

Theorem 3.2.9. Let

- (i) $\mu - \lambda > -1, \mu + \lambda \geq \frac{1}{2}, 2\mu \neq 0, 1, 2, \dots;$
- (ii) $f \in L_2(0, \infty),$
- (iii) $h(x) = \mathcal{L}\{f(x)\}.$

Then $K^{2\mu, \frac{1}{2} - \mu - \lambda} h(x)$ is the $W_{\lambda, \mu}$ -transform of $f(x)$.

Proof: By (iii) and (3.2.13) of Lemma 3.2.3, $h(x)$ may be written as $h(x) = K^{\frac{1}{2} + \mu - \lambda, \mu + \lambda - \frac{1}{2}} W_{\lambda, \mu}\{f(x)\}.$ Therefore

$K^{2\mu, \frac{1}{2}-\mu-\lambda}$ $h(x)$ is meaningful. Hence by (3.1.4) we have

$$(3.2.40) \quad K^{2\mu, \frac{1}{2}-\mu-\lambda} h(x) = (K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}})^{-1} h(x) \\ = (K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}})^{-1} K^{\frac{1}{2}+\mu-\lambda, \mu+\lambda-\frac{1}{2}} W_{\lambda, \mu} \{f(x)\} = W_{\lambda, \mu} \{f(x)\}.$$

This theorem, which is the converse of Theorem 3.2.4, enables us to construct tables of $W_{\lambda, \mu}$ -transforms from tables of Laplace transforms and fractional integrals when $\mu - \lambda > -1$, $\mu + \lambda \geq \frac{1}{2}$; and $2\mu \neq 0, 1, 2, \dots$

3.3 A GENERALIZATION.

In this section we consider very briefly the integral equation

$$(3.3.1) \quad \int_0^{\infty} H(xu)f(u)du = g(x)$$

where $f \in L_2(0, \infty)$ is to be determined, $g \in L_2(0, \infty)$ is given, H is an H-function of the form

$$(3.3.2) \quad H(x) = H_{m+n, m+n+1}^{m+1, n} \left(x \left| \begin{matrix} (a_1, \alpha_1)_{1, n}, (d_1, \beta_1)_{1, m} \\ (0, 1), (b_1, \beta_1)_{1, m}, (c_1, \alpha_1)_{1, n} \end{matrix} \right. \right) \\ = \frac{1}{2\pi i} \int_L \Gamma(s) \frac{\Gamma(b_M + \beta_M s) \Gamma(1 - a_N - \alpha_N s)}{\Gamma(1 - c_N - \alpha_N s) \Gamma(d_M + \beta_M s)} x^{-s} ds,$$

where all α 's, β 's are positive. Note that the function given in (3.3.2) belongs to $L_2(0, \infty)$. By Theorem 2.2.1 if f exists then it is unique. When $n = 0$, $m = 1$, $b_1 = 2\mu$, $d_1 = \mu - \lambda + \frac{1}{2}$,

$\beta_1 = 1$, we have [31, p. 226] $H(x) = x^{\mu-\frac{1}{2}} e^{-\frac{x}{2}} W_{\lambda, \mu}(x)$.

Theorem 3.3.1. Let

- (i) $A > 0, \alpha > 0,$
- (ii) $\zeta > -\frac{1}{2A}, \eta > \frac{1}{2A} - 1,$
- (iii) $f \in L_2(0, \infty),$
- (iv) the parameters be such that

$$H_{p,q}^{m,n} \left(x \left| \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right. \right) \in L_2(0, \infty)$$

and

$$\mathcal{M} \left\{ H_{p,q}^{m,n} \left(x \left| \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right. \right) \right\} = \mathcal{H}(s)$$

is bounded on the line $s = \frac{1}{2} + it, -\infty < t < \infty.$

Then

$$(3.3.3) \quad I_{x^A}^{\eta, \alpha} \int_0^\infty H_{p,q}^{m,n} \left(xy \left| \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right. \right) f(y) dy$$

$$= \int_0^\infty I_{x^A}^{\eta, \alpha} H_{p,q}^{m,n} \left(xy \left| \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right. \right) f(y) dy, \text{ a.e.,}$$

$$(3.3.4) \quad K_{x^A}^{\zeta, \alpha} \int_0^\infty H_{p,q}^{m,n} \left(xy \left| \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right. \right) f(y) dy$$

$$= \int_0^\infty K_{x^A}^{\zeta, \alpha} H_{p,q}^{m,n} \left(xy \left| \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right. \right) f(y) dy, \text{ a.e.}$$

Proof. By Theorem 1.1.1 and conditions (iii) and (iv) we know that

$$(3.3.5) \quad \int_0^{\infty} H_{p,q}^{m,n} \left(xy \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) f(x) dy \in L_2(0, \infty)$$

and

$$(3.3.6) \quad \mathcal{M} \left\{ \int_0^{\infty} H_{p,q}^{m,n} \left(xy \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) f(y) dy \right\} = \mathcal{H}(s) F(1-s),$$

where $F(s) = \mathcal{M}\{f(x)\}$. Hence by (3.1.14) and (3.1.19) we have

$$\begin{aligned} & \mathcal{M} \left\{ I_{x^A}^{\eta, \alpha} \int_0^{\infty} H_{p,q}^{m,n} \left(xy \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) f(y) dy \right\} \\ &= \frac{\Gamma(1+\eta-\frac{s}{A})}{\Gamma(1+\eta+\alpha-\frac{s}{A})} \mathcal{M} \left\{ \int_0^{\infty} H_{p,q}^{m,n} \left(xy \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) f(y) dy \right\} \\ &= \frac{\Gamma(1+\eta-\frac{s}{A})}{\Gamma(1+\eta+\alpha-\frac{s}{A})} \frac{\Gamma(b_M+\beta_M s) \Gamma(1-a_N-\alpha_N s)}{\Gamma_m(1-b_Q-\beta_Q s) \Gamma_n(a_P+\alpha_P s)} F(1-s) \\ &= \mathcal{M} \left\{ \int_0^{\infty} H_{p+1, q+1}^{m, n+1} \left(xy \left| \begin{matrix} (-\eta, \frac{1}{A}), (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q}, (-\eta-\alpha, \frac{1}{A}) \end{matrix} \right. \right) f(y) dy \right\} \\ &= \mathcal{M} \left\{ \int_0^{\infty} I_{x^A}^{\eta, \alpha} H_{p,q}^{m,n} \left(xy \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) f(y) dy \right\}. \end{aligned}$$

This is equivalent to (3.3.3). Similarly for the operator $K_{x^A}^{\zeta, \alpha}$.

Thus the proof is complete.

Now, in (3.3.2) if

$$(3.3.8) \quad a_j < c_j < 1 - \frac{\alpha_j}{2}, \quad j = 1, \dots, n,$$

$$b_j > d_j > -\frac{\beta_j}{2}, \quad j = 1, \dots, m,$$

then from (3.3.1) we have

$$(3.3.9) \quad \prod_{j=1}^n I_{\frac{1}{\alpha_j}}^{-c_j, c_j - a_j} \prod_{j=1}^m K_{\frac{1}{\beta_j}}^{d_j, b_j - d_j} g(x)$$

$$= \prod_{j=1}^n I_{\frac{1}{\alpha_j}}^{-c_j, c_j - a_j} \prod_{j=1}^m K_{\frac{1}{\beta_j}}^{d_j, b_j - d_j}$$

$$\times \int_0^{\infty} H_{m+n, m+n+1}^{m+1, n} \left(xu \left| \begin{matrix} (a_1, \alpha_1)_{1, n}, (d_1, \beta_1)_{1, m} \\ (0, 1), (b_1, \beta_1)_{1, m}, (c_1, \alpha_1)_{1, n} \end{matrix} \right. \right) f(u) du$$

$$= \int_0^{\infty} \prod_{j=1}^n I_{\frac{1}{\alpha_j}}^{-c_j, c_j - a_j} \prod_{j=1}^m K_{\frac{1}{\beta_j}}^{d_j, b_j - d_j}$$

$$\times H_{m+n, m+n+1}^{m+1, n} \left(xu \left| \begin{matrix} (a_1, \alpha_1)_{1, n}, (d_1, \beta_1)_{1, m} \\ (0, 1), (b_1, \beta_1)_{1, m}, (c_1, \alpha_1)_{1, n} \end{matrix} \right. \right) f(u) du$$

$$= \int_0^{\infty} e^{-xu} f(u) du = \mathcal{L}\{f(x)\}.$$

Hence

$$(3.3:10) \quad f(x) = \mathcal{L}^{-1} \left\{ \prod_{j=1}^n I_{\frac{1}{\alpha_j}}^{-c_j, c_j - a_j} \prod_{j=1}^m K_{\frac{1}{\beta_j}}^{d_j, b_j - d_j} g(x) \right\}.$$

3.4 DUAL INTEGRAL EQUATIONS.

In the analysis of mixed boundary value problems we often encounter pairs of dual integral equations. A well known example is

$$(3.4.1) \quad \int_0^{\infty} y^{\rho} J_{\mu}(xy) f(y) dy = g(x), \quad 0 < x < 1,$$

$$(3.4.2) \quad \int_0^{\infty} y^{\sigma} J_{\nu}(xy) f(y) dy = h(x), \quad 1 < x < \infty,$$

where J_{λ} is the usual Bessel function, g and h are given and f is to be determined. Recently, Buschman [4] indicated that by using fractional integral operators, the dual integral equations of the type (3.4.1) and (3.4.2) can be reduced to a single integral equation of the type

$$(3.4.3) \quad \int_0^{\infty} y^{\tau} J_{\lambda}(xy) f(y) dy = F(x), \quad 0 < x < \infty,$$

where τ and λ are related to ρ , σ , μ , ν and F to g and h .

In this section we consider the dual integral equations

$$(3.4.4) \quad \int_0^{\infty} H_{m+1,n}^{m+n,m+n+1} \left(xy \left| \begin{matrix} (a_1, \alpha_1)_{1,n}, (d_1, \beta_1)_{1,m} \\ (0, 1), (b_1, \beta_1)_{1,m}, (c_1, \alpha_1)_{1,n} \end{matrix} \right. \right) f(y) dy = g(x),$$

$$0 < x < 1,$$

$$(3.4.5) \quad \int_0^{\infty} H_{m+1,n}^{m+n,m+n+1} \left(xy \left| \begin{matrix} (e_1, \gamma_1)_{1,n}, (h_1, \delta_1)_{1,m} \\ (0, 1), (f_1, \delta_1)_{1,m}, (g_1, \gamma_1)_{1,n} \end{matrix} \right. \right) f(y) dy = h(x),$$

$$1 < x < \infty.$$

Of course, we assume that the parameters involved in the H-functions satisfy the conditions required on page 9 so as to make these functions meaningful. We do not write any power of y before the H-functions in (3.4.4), (3.4.5) since the identity

$$(3.4.6) \quad z^\sigma H_{p,q}^{m,n} \left(z \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) = H_{p,q}^{m,n} \left(z \left| \begin{matrix} a_1 + \sigma \alpha_1, \alpha_1 \\ (b_1 + \sigma \beta_1, \beta_1)_{1,q} \end{matrix} \right. \right)$$

allows any power factor to be absorbed beforehand.

From (3.1.19) we know that

$$(3.4.7) \quad I_A^{\eta, \alpha} \left\{ H_{p,q}^{m,n} \left(cx \left| \begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \right. \right) \right\} \\ = H_{p+1, q+1}^{m, n+1} \left(cx \left| \begin{matrix} (-\eta, \frac{1}{A}), (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q}, (-\eta - \alpha, \frac{1}{A}) \end{matrix} \right. \right)$$

In particular (formally)

$$(3.4.8) \quad I_{\frac{1}{\gamma_1}}^{-e_1, e_1 - g_1} I_{\frac{1}{\alpha_1}}^{-c_1, c_1 - a_1} \\ \times H_{m+n, m+n+1}^{m+1, n} \left(cx \left| \begin{matrix} (a_1, \alpha_1)_{1, n}, (d_1, \beta_1)_{1, m} \\ (0, 1), (b_1, \beta_1)_{1, m}, (c_1, \alpha_1)_{1, n} \end{matrix} \right. \right) \\ = H_{m+n+2, m+n+3}^{m+1, n+2} \left(cx \left| \begin{matrix} (e_1, \gamma_1), (c_1, \alpha_1), (a_1, \alpha_1)_{1, n}, (d_1, \beta_1)_{1, m} \\ (0, 1), (b_1, \beta_1)_{1, m}, (c_1, \alpha_1)_{1, n}, (a_1, \alpha_1), (g_1, \gamma_1) \end{matrix} \right. \right)$$

$$= H_{m+n, m+n+1}^{m+1, n} \left(\text{cx} \left| \begin{array}{l} (e_1, \gamma_1), (a_1, \alpha_1)_{2, n}, (d_1, \beta_1)_{1, m} \\ (0, 1), (b_1, \beta_1)_{1, m}, (c_1, \alpha_1)_{2, n}, (g_1, \gamma_1) \end{array} \right. \right).$$

Similarly from (3.1.17)

(3.4.9)

$$K_{\frac{1}{\beta_1}}^{b_1, d_1 - b_1} K_{\frac{1}{\delta_1}}^{h_1, f_1 - h_1} H_{m+n, m+n+1}^{m+1, n} \left(\text{cx} \left| \begin{array}{l} (e_1, \gamma_1)_{1, n}, (h_1, \delta_1)_{1, m} \\ (0, 1), (f_1, \delta_1)_{1, m}, (g_1, \gamma_1)_{1, n} \end{array} \right. \right)$$

$$= H_{m+n, m+n+1}^{m+1, n} \left(\text{cx} \left| \begin{array}{l} (e_1, \gamma_1)_{1, n}, (h_1, \delta_1)_{2, m}, (d_1, \beta_1) \\ (0, 1), (b_1, \beta_1), (f_1, \delta_1)_{2, m}, (g_1, \gamma_1)_{1, n} \end{array} \right. \right)$$

Now, if f is a solution of the dual integral equation

(3.4.4), (3.4.5) which belongs to $L_2(0, \infty)$ then

$$(3.4.10) \quad T g(x) \equiv \prod_{j=1}^n \frac{-e_j, e_j - g_j}{\gamma_j} \frac{-c_j, c_j - a_j}{\alpha_j} g(x)$$

$$= T \int_0^{\infty} H_{m+n, m+n+1}^{m+1, n} \left(xy \left| \begin{array}{l} (a_1, \alpha_1)_{1, n}, (d_1, \beta_1)_{1, m} \\ (0, 1), (b_1, \beta_1)_{1, m}, (c_1, \alpha_1)_{1, n} \end{array} \right. \right) f(y) dy$$

$$= \int_0^{\infty} T H_{m+n, m+n+1}^{m+1, n} \left(xy \left| \begin{array}{l} (a_1, \alpha_1)_{1, n}, (d_1, \beta_1)_{1, m} \\ (0, 1), (b_1, \beta_1)_{1, m}, (c_1, \alpha_1)_{1, n} \end{array} \right. \right) f(y) dy$$

$$= \int_0^{\infty} H_{m+n, m+n+1}^{m+1, n} \left(xy \left| \begin{array}{l} (e_1, \gamma_1)_{1, n}, (d_1, \beta_1)_{1, m} \\ (0, 1), (b_1, \beta_1)_{1, m}, (g_1, \gamma_1)_{1, n} \end{array} \right. \right) f(y) dy;$$

$$(3.4.11) \quad S h(x) \equiv \prod_{j=1}^m \frac{b_j, d_j - b_j}{\beta_j} K_{\frac{1}{\delta_j}}^{h_j, f_j - h_j} h(x)$$

$$\begin{aligned}
 &= S \int_0^{\infty} H_{m+n, m+n+1}^{m+1, n} \left(xy \left| \begin{array}{c} (e_1, \gamma_1)_{1, n}, (h_1, \delta_1)_{1, m} \\ (0, 1), (f_1, \delta_1)_{1, m}, (g_1, \gamma_1)_{1, n} \end{array} \right. \right) f(y) dy \\
 &= \int_0^{\infty} S H_{m+n, m+n+1}^{m+1, n} \left(xy \left| \begin{array}{c} (e_1, \gamma_1)_{1, n}, (h_1, \delta_1)_{1, m} \\ (0, 1), (f_1, \delta_1)_{1, m}, (g_1, \gamma_1)_{1, n} \end{array} \right. \right) f(y) dy \\
 &= \int_0^{\infty} H_{m+n, m+n+1}^{m+1, n} \left(xy \left| \begin{array}{c} (e_1, \gamma_1)_{1, n}, (d_1, \beta_1)_{1, m} \\ (0, 1), (b_1, \beta_1)_{1, m}, (g_1, \gamma_1)_{1, n} \end{array} \right. \right) f(y) dy,
 \end{aligned}$$

whenever

$$1 - \frac{\gamma_j}{2} > e_j > g_j, \quad 1 - \frac{\alpha_j}{2} > c_j > a_j, \quad j = 1, \dots, n;$$

(3.4.12)

$$d_k > b_k > -\frac{\beta_k}{2}, \quad f_k > h_k > -\frac{\delta_k}{2}, \quad k = 1, \dots, m.$$

Thus we may write (3.4.4) and (3.4.5) in the form of a single integral equation under conditions (3.4.12)

$$(3.4.13) \quad \int_0^{\infty} H_{m+n, m+n+1}^{m+1, n} \left(xy \left| \begin{array}{c} (e_1, \gamma_1)_{1, n}, (d_1, \beta_1)_{1, m} \\ (0, 1), (b_1, \beta_1)_{1, m}, (g_1, \gamma_1)_{1, n} \end{array} \right. \right) f(y) dy = F(x),$$

$$0 < x < \infty,$$

where

$$\begin{aligned}
 (3.4.14) \quad F(x) = & U(1-x) \prod_{j=1}^n I_{\frac{1}{\gamma_j}}^{-e_j, e_j - g_j} \prod_{j=1}^n I_{\frac{1}{\alpha_j}}^{-c_j, c_j - a_j} g(x) \\
 & + U(x-1) \prod_{j=1}^m K_{\frac{1}{\beta_j}}^{b_j, d_j - b_j} \prod_{j=1}^m K_{\frac{1}{\delta_j}}^{h_j, f_j - h_j} h(x).
 \end{aligned}$$

The problem of solving a single integral equation of the form (3.4.13) has been discussed in the previous section.

CHAPTER IV

GROUP REPRESENTATION* AND CERTAIN H-FUNCTIONS

4. At first sight, the theory of special functions appears as a chaotic collection of formulae: apart from the fact that there exists an immense aggregate of the special functions themselves, for each of them there are, at present, all sorts of differential equations, integral representations, recurrence formulae, composition theorems and so on. To establish some kind of order in this chaos of formulae seems to be a difficult task.

However, the development of the theory of group representations has now made it possible to comprehend the theory of certain classes of special functions from a single point of view. We can apply it to the hypergeometric functions and its various special and degenerate cases, the functions of Bessel and Legendre, the orthogonal polynomials of Jacobi, Chebyshev, Laguerre and Hermite etc. [33; 43].

In this chapter, we apply Vilenkin's integral transform method as set forth in [43] to obtain some identities involving H-functions. We first construct a $(p+2)$ -dimensional real Lie group G . Then we apply Vilenkin's integral transform method to define a local representation of the group G and calculate the kernels of the operators of this representation. We find that the

* For the definition of group representation see [43; p. 8].

kernels which are associated to certain elements of G can be expressed in terms of H-functions. In the last section of this chapter we obtain limiting values of some H-functions.

4.1 THE LIE GROUP G .

Let G denote the $(p+2)$ -dimensional real Lie group[†] whose elements are $(p+2)$ -tuples $g(c; a_0, a_1, \dots, a_p)$ satisfying the multiplication law

$$(4.1.1) \quad g(c; a_0, a_1, \dots, a_p)g(c'; a'_0, a'_1, \dots, a'_p) = g(c + c'; \phi_0, \phi_1, \dots, \phi_p)$$

where

$$(4.1.2) \quad \phi_\ell = a_\ell + \sum_{k=\ell}^p \binom{k}{\ell} c^{k-\ell} a'_k, \quad 0 \leq \ell \leq p,$$

$\binom{k}{\ell} = \frac{k!}{\ell!(k-\ell)!}$, the binomial coefficient. It is straightforward,

though tedious, to verify that the multiplication is associative. Furthermore, $g(0; 0, 0, \dots, 0)$ is the identity element and $g(-c; b_0, b_1, \dots, b_p)$ is the unique inverse of the element $g(c; a_0, a_1, \dots, a_p)$, where

$$(4.1.3) \quad b_\ell = - \sum_{k=0}^{p-\ell} (-1)^k \binom{\ell+k}{\ell} c^k a_{\ell+k}, \quad 0 \leq \ell \leq p.$$

4.2 A REPRESENTATION OF G .

Let \mathcal{D} be the subspace of $L_2(0, \infty)$ whose elements are infinitely differentiable functions with compact support. Then

[†] For the definition of Lie group see [20, p. 88].

\mathcal{D} is dense in $L_2(0, \infty)$. [46, p. 101; 43, p. 87]. For

every $f \in \mathcal{D}$, let $[\alpha_f, \beta_f]$ be the smallest closed interval

containing the support of f . Now let $g = g(c; a_0, a_1, \dots, a_p) \in G$.

Then for $|c|$ sufficiently small and $f \in \mathcal{D}$, we associate the operator $T(g)$ which is defined by

$$(4.2.1) \quad [T(g)f](x) = \exp\left(-\sum_{\ell=0}^p a_\ell x^\ell\right) \left(1 + \frac{c}{x}\right)^\omega f(x+c)$$

where ω is fixed and not an integer. It is easy to verify that

$$(4.2.2) \quad T(g'g)f = T(g')[T(g)f].$$

The function $f^{(g)} \equiv T(g)f$ will belong to \mathcal{D} if $0 < c < \alpha_f$. Moreover, let

$$(4.2.3) \quad 2\alpha = \min.(\alpha_f, \alpha_{f^{(g)}}).$$

Then $T(g'g)f \in \mathcal{D}$ if $c' < \alpha$, $c < \alpha$.

If $f \in \mathcal{D}$, we define the Mellin transform F of f by

$$(4.2.4) \quad F(\lambda) = \mathcal{M}\{f(x)\} = \int_0^\infty x^{\lambda-1} f(x) dx.$$

It is well known that [43, pp. 94-95]

$$(4.2.5) \quad f(x) = \frac{1}{2\pi i} \int_{\rho-1-i\infty}^{\rho+1-i\infty} x^{-\lambda} F(\lambda) d\lambda, \quad \rho \text{ any real number.}$$

The operators $T(g)$ induce operators $R(g)$ on the space of Mellin transforms, defined by

$$(4.2.6) \quad R(g)\mathcal{M}f = \mathcal{M}T(g)f.$$

It is easy to show that for given $f \in \mathcal{D}$

$$(4.2.7) \quad R(g'g)\mathcal{M}f = R(g')R(g)\mathcal{M}f$$

where $g', g \in G$ be such that

$$(4.2.8) \quad c' < \alpha, c < \alpha,$$

$$(4.2.9) \quad 2\alpha = \min.(\alpha_f, \alpha_{f(g)}).$$

Thus,

$$(4.2.10) \quad [R(g)F](\lambda) = \int_0^{\infty} x^{\lambda-1} f(g)(x) dx.$$

$$= \int_0^{\infty} x^{\lambda-1} \exp\left(-\sum_{\ell=0}^p a_{\ell} x^{\ell}\right) \left(1+\frac{c}{x}\right)^{\omega} f(x+c) dx$$

$$= \int_c^{\infty} \exp\left(-\sum_{\ell=0}^p a_{\ell} (y-c)^{\ell}\right) \left(1+\frac{c}{y-c}\right)^{\omega} (y-c)^{\lambda-1} f(y) dy$$

$$= \int_c^{\infty} e^{-\sum_{\ell=0}^p a_{\ell} (y-c)^{\ell}} y^{\omega} (y-c)^{\lambda-\omega-1} dy \frac{1}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} F(v) y^{-v} dv.$$

If the iterated integral is absolutely convergent, we can interchange the order of integration and write

$$(4.2.11) \quad [R(g)F](\lambda) = \frac{1}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} F(v) dv \int_c^{\infty} \exp\left(-\sum_{\ell=0}^p a_{\ell} (y-c)^{\ell}\right) \times (y-c)^{\lambda-\omega-1} y^{\omega-v} dy.$$

$$= \int_{\rho-i\infty}^{\rho+i\infty} K(\lambda, v; g) F(v) dv,$$

where

$$(4.2.12) \quad K(\lambda, v; g) = \frac{1}{2\pi i} \int_c^{\infty} \exp\left(-\sum_{\ell=0}^p a_{\ell} (y-c)^{\ell}\right) y^{\omega-v} (y-c)^{\lambda-\omega-1} dy$$

$$= \frac{1}{2\pi i} \int_0^{\infty} \exp\left(-\sum_{\ell=0}^p a_{\ell} x^{\ell}\right) (x+c)^{\omega-v} x^{\lambda-\omega-1} dx,$$

$g = g(c; a_0, a_1, \dots, a_p) \in G$. In particular the interchange of order of integration is valid if the parameters

c, a_0, a_1, \dots, a_p and ω satisfy any of the following conditions:

(1) $a_j \geq 0, j = 1, \dots, p$, but at least one of $a_j > 0, c > 0$,

$\operatorname{Re} \lambda > \operatorname{Re} \omega$.

(2) $a_j \geq 0, j = 1, \dots, p$, but at least one of $a_j > 0, c = 0$,

$\operatorname{Re} \lambda > \operatorname{Re} \nu$.

(3) $a_j = 0, j = 1, \dots, p, c > 0, \operatorname{Re} \nu > \operatorname{Re} \lambda > \operatorname{Re} \omega$.

In this chapter, we shall always assume that at least one of these three conditions holds. Moreover, if at least one of $a_j > 0, j = 1, \dots, p$, then $K(\lambda, \nu; g)$ is an analytic function of $c (\geq 0)$.

In the remaining part of this chapter, we shall use the following notations for brevity. For $c > 0, a_j > 0, a_k > 0, a_l > 0$,

$$g(c; 0, \dots, 0) \equiv g(c; 0),$$

$$g(0; 0, \dots, 0, a_l, 0, \dots, 0) \equiv g(0; a_l),$$

$$g(c; 0, \dots, 0, a_l, 0, \dots, 0) \equiv g(c; a_l),$$

$$g(0; 0, \dots, 0, a_k, 0, \dots, 0, a_l, 0, \dots, 0) \equiv g(0; a_k, a_l),$$

$$g(c; 0, \dots, 0, a_k, 0, \dots, 0, a_l, 0, \dots, 0) \equiv g(c; a_k, a_l),$$

$$g(0; 0, \dots, 0, a_j, 0, \dots, 0, a_k, 0, \dots, 0, a_l, 0, \dots, 0) \equiv g(0; a_j, a_k, a_l).$$

Let $g = g(c; 0)$. Then

$$\begin{aligned}
 (4.2.13) \quad K(\lambda, \nu; g(c; 0)) &= \frac{1}{2\pi i} \int_0^{\infty} (x+c)^{\omega-\nu} x^{\lambda-\omega-1} dx \\
 &= \frac{c^{\lambda-\nu}}{2\pi i} \int_0^{\infty} y^{\lambda-\omega-1} (1+y)^{\omega-\nu} dy \\
 &= \frac{c^{\lambda-\nu}}{2\pi i} \frac{\Gamma(\lambda-\omega)\Gamma(\nu-\lambda)}{\Gamma(\nu-\omega)}.
 \end{aligned}$$

Similarly let $g = g(0; a_\ell)$, then

$$\begin{aligned}
 (4.2.14) \quad K(\lambda, \nu; g(0; a_\ell)) \\
 = \frac{1}{2\pi i} \int_0^{\infty} e^{-a_\ell x^\ell} x^{\lambda-\nu-1} dx = \frac{1}{2\pi i} \frac{1}{\ell} a_\ell^{\frac{\nu-\lambda}{\ell}} \Gamma\left(\frac{\lambda-\nu}{\ell}\right), \quad \ell=1, 2, \dots, p.
 \end{aligned}$$

4.3 INTEGRALS CONNECTED WITH SOME H-FUNCTIONS.

Let $g_1 = g(c; a_0, a_1, \dots, a_p)$, $g_2 = g(d; b_0, b_1, \dots, b_p)$,

$f^{(g_2)}(x) = [T(g_2)f](x)$, $F(\lambda) = (\mathcal{M}f)(\lambda)$. Since

$$(4.3.1) \quad R(g_1 g_2)F = R(g_1)R(g_2)F,$$

we write this equality by means of the kernels of the operators

$R(g_1)$, $R(g_2)$, $R(g_1 g_2)$. We find that for each given $F = \mathcal{M}f \in \mathcal{M}(\mathcal{D})$

one has the relation

$$\begin{aligned}
 (4.3.2) \quad \int_{\alpha-1\infty}^{\alpha+1\infty} K(\lambda, \nu; g_1 g_2)F(\nu) d\nu &= [R(g_1 g_2)F](\lambda) \\
 &= [R(g_1)(R(g_2)F)](\lambda) \\
 &= \int_{\rho-1\infty}^{\rho+1\infty} K(\lambda, \sigma; g_1)[R(g_2)F](\sigma) d\sigma \\
 &= \int_{\rho-1\infty}^{\rho+1\infty} K(\lambda, \sigma; g_1) d\sigma \int_{\alpha-1\infty}^{\alpha+1\infty} K(\sigma, \nu; g_2)F(\nu) d\nu.
 \end{aligned}$$

The above relation holds only for the case, $c < \alpha$, $d < \alpha$, where $2\alpha = \min.(\alpha_f, \alpha_f(g_2))$. By analytic continuation if at least one of $a_j > 0$, $j = 1, \dots, p$ in g_1 and at least one of $b_j > 0$, $j = 1, \dots, p$ in g_2 , then the above relation holds for all c, d greater than or equal to zero. Hence, if one can justify the change of order of integration, the kernel $K(\lambda, \nu; g)$ must satisfy the relation [34, p. 41]

$$(4.3.3) \quad K(\lambda, \nu; g_1 g_2) = \int_{\rho-1\infty}^{\rho+1\infty} K(\lambda, \sigma; g_1) K(\sigma, \nu; g_2) d\sigma,$$

the value of ρ must be such that the integrals considered are absolutely convergent for $\sigma = \rho + it$.

Choosing g_1 and g_2 in a special way, we obtain various identities.

(1) Let $g_1 = g(0; a_k)$, $g_2 = g(0; a_\ell)$. Then $g_1 g_2 = g_2 g_1 = g(0; a_k, a_\ell)$, where for definiteness we assume $k < \ell$. For

$k = \ell$, $g_1 g_2 = g(0; a_k + a_\ell)$. In both cases

$$(4.3.4) \quad K(\lambda, \nu; g(0; a_k, a_\ell)) = K(\lambda, \nu; g_1 g_2)$$

$$= \int_{\rho-1\infty}^{\rho+1\infty} K(\lambda, \sigma; g(0; a_k)) K(\sigma, \nu; g(0; a_\ell)) d\sigma$$

$$= \left(\frac{1}{2\pi i}\right)^2 \frac{1}{k \cdot \ell} a_k^{-\frac{\lambda}{k}} a_\ell^{\frac{\nu}{\ell}} \int_{\rho-1\infty}^{\rho+1\infty} a_k^{\frac{\sigma}{k}} a_\ell^{-\frac{\sigma}{\ell}} \Gamma\left(\frac{\lambda-\sigma}{k}\right) \Gamma\left(\frac{\sigma-\nu}{\ell}\right) d\sigma$$

$$= \frac{1}{2\pi i} \frac{1}{k \cdot \ell} a_k^{-\frac{\lambda}{k}} a_\ell^{\frac{\nu}{\ell}} H_{1,1}^{1,1} \left(\begin{matrix} a_k \frac{1}{k} \\ a_\ell \frac{1}{\ell} \end{matrix} \middle| \begin{matrix} (1+\frac{\nu}{\ell}, \frac{1}{\ell}) \\ (\frac{\lambda}{k}, \frac{1}{k}) \end{matrix} \right)$$

On the other hand, from (4.2.12) we have

$$(4.3.5) \quad K(\lambda, \nu; g(0; a_k, a_\ell)) = \frac{1}{2\pi i} \int_0^\infty e^{-a_k x^k - a_\ell x^\ell} x^{\lambda-\nu-1} dx.$$

Equating the right hand sides of (4.3.4) and (4.3.5) and writing a for a_k , b for a_ℓ , we obtain

$$(4.3.6) \quad k \cdot \ell \int_0^\infty e^{-ax^k - bx^\ell} x^{\lambda-\nu-1} dx \\ = a^{-\frac{\lambda}{k}} b^{\frac{\nu}{\ell}} H_{1,1}^{1,1} \left(\frac{a}{b} \left| \begin{matrix} (1+\frac{\nu}{\ell}, \frac{1}{\ell}) \\ (\frac{\lambda}{k}, \frac{1}{k}) \end{matrix} \right. \right), \operatorname{Re}(\lambda - \nu) > 0,$$

k, ℓ any positive integers. By analytic continuation (4.3.6) holds for $|\arg a| < \frac{\pi}{2}$, $|\arg b| < \frac{\pi}{2}$, k, ℓ any positive reals. When $k = 1$, $\ell = 2$, (4.3.6) reduces to a known result [10, p. 146, (24)].

(2) Take $g_1 = g(0; a_\ell)$, $g_2 = g(c; 0)$. Then $g_1 g_2 = g(c; a_\ell)$

$$(4.3.7) \quad K(\lambda, \nu; g(c; a_\ell)) = \int_{\rho-1\infty}^{\rho+1\infty} K(\lambda, \sigma; g(0; a_\ell)) K(\sigma, \nu; g(c; 0)) d\sigma$$

$$= \left(\frac{1}{2\pi i}\right)^2 \frac{1}{\ell} a_\ell^{-\frac{\lambda}{\ell}} c^{-\nu} \int_{\rho-1\infty}^{\rho+1\infty} a_\ell^{\frac{\sigma}{\ell}} c^\sigma \frac{\Gamma(\frac{\lambda-\sigma}{\ell}) \Gamma(\sigma-\omega) \Gamma(\nu-\sigma)}{\Gamma(\nu-\omega)} d\sigma$$

$$= \left(\frac{1}{2\pi i}\right)^2 a_\ell^{\frac{\nu-\lambda}{\ell}} \frac{1}{\Gamma(\nu-\omega)} \int_{\rho'-1\infty}^{\rho'+1\infty} a_\ell^\sigma c^{\ell\sigma} \frac{\Gamma(\frac{\lambda-\nu}{\ell}-\sigma) \Gamma(-\ell\sigma) \Gamma(\nu-\omega+\ell\sigma)}{\Gamma(\nu-\omega)} d\sigma$$

$$= \frac{1}{2\pi i} a_\ell^{\frac{\nu-\lambda}{\ell}} \frac{1}{\Gamma(\nu-\omega)} H_{1,2}^{2,1} \left(a_\ell c^\ell \left| \begin{matrix} (1+\omega-\nu, \ell) \\ (\frac{\lambda-\nu}{\ell}, 1), (0, \ell) \end{matrix} \right. \right)$$

On the other hand from (4.2.12) we have

$$(4.3.8) \quad K(\lambda, \nu; g(c; a_\ell)) = \frac{1}{2\pi i} \int_0^\infty e^{-a_\ell x^\ell} (x+c)^{\omega-\nu} x^{\lambda-\omega-1} dx.$$

From (4.3.7), (4.3.8), after replacing $\omega-\nu$ by ω , $\lambda-\nu$ by λ , we obtain

$$(4.3.9) \quad \int_0^\infty e^{-ax^\ell} (x+c)^\omega x^{\lambda-\omega-1} dx = \frac{a^{-\frac{\lambda}{\ell}}}{\Gamma(-\omega)} H_{1,2}^{2,1} \left(ac^\ell \left| \begin{matrix} (1+\omega, \ell) \\ (\frac{\lambda}{\ell}, 1), (0, \ell) \end{matrix} \right. \right),$$

$\text{Re}(\lambda - \omega) > 0$, ω not a non-negative integer, $a > 0$, $c > 0$, ℓ positive integers. By analytic continuation, the parameters a , c , ℓ may be released to $\text{Re } a > 0$, $|\arg c| < \pi$, $|\arg ac^\ell| < (2\ell+1)\frac{\pi}{2}$, ℓ any positive real.

When $\ell = 1$, $\lambda = \frac{1}{2}$, $\text{Re } \omega = \frac{1}{2} - \nu < \frac{1}{2}$, (4.3.9) reduces to [10, p. 139, (20)]. When $\ell = 1$, $\lambda = -\frac{1}{2}$, $\text{Re } \omega = -\nu - \frac{1}{2} < -\frac{1}{2}$, (4.3.9) reduces to [10, p. 139, (21)]. When $a = \ell = 1$, $\lambda = \mu$, $\omega = \frac{\mu-1}{2} + \lambda$, (4.3.9) reduces to [3, p. 63, (5)] with the aid of [31, p. 231, (12)].

(3) Let $g_1 = g(c; 0)$, $g_2 = g(0; a_\ell)$. Then $g_1 g_2 = g(c; \phi_0, \phi_1, \dots, \phi_\ell, 0, \dots, 0)$, where

$$(4.3.10) \quad \phi_j = \binom{\ell}{j} c^{\ell-j} a_\ell, \quad 0 \leq j \leq \ell.$$

$$(4.3.11) \quad K(\lambda, \nu; g_1 g_2) = \int_{\rho-1}^{\rho+1} K(\lambda, \sigma; g(c; 0)) K(\sigma, \nu; g(0; a_\ell)) d\sigma$$

$$\begin{aligned}
 &= \left(\frac{1}{2\pi i}\right)^2 c^\lambda a_\ell^{\frac{v}{\ell}} \Gamma(\lambda - \omega) \frac{1}{\ell} \int_{\rho-1-i\infty}^{\rho+1-i\infty} c^{-\sigma} a_\ell^{-\frac{\sigma}{\ell}} \frac{\Gamma(\sigma-\lambda) \Gamma(\frac{\sigma-v}{\ell})}{\Gamma(\sigma-\omega)} d\sigma \\
 &= \frac{1}{\ell} \frac{1}{2\pi i} c^\lambda a_\ell^{\frac{v}{\ell}} \Gamma(\lambda - \omega) H_{1,2}^{2,0} \left(ca_\ell^{\frac{1}{\ell}} \left| \begin{matrix} (-\omega, 1) \\ (-\lambda, 1), (-\frac{v}{\ell}, \frac{1}{\ell}) \end{matrix} \right. \right)
 \end{aligned}$$

Combining (4.3.11), (4.2.12) we have

$$\begin{aligned}
 (4.3.12) \quad &\frac{1}{2\pi i} \int_0^\infty e^{-\sum_{j=0}^{\ell} \phi_j x^j} (x+c)^{\omega-v} x^{\lambda-\omega-1} dx \\
 &= \frac{1}{2\pi i} \frac{1}{\ell} c^\lambda a_\ell^{\frac{v}{\ell}} \Gamma(\lambda-\omega) H_{1,2}^{2,0} \left(ca_\ell^{\frac{1}{\ell}} \left| \begin{matrix} (-\omega, 1) \\ (-\lambda, 1), (-\frac{v}{\ell}, \frac{1}{\ell}) \end{matrix} \right. \right)
 \end{aligned}$$

But by (4.3.10) we know

$$(4.3.13) \quad \sum_{j=0}^{\ell} \phi_j x^j = \sum_{j=0}^{\ell} \binom{\ell}{j} c^{\ell-j} a_\ell x^j = a_\ell (x+c)^\ell$$

Therefore after replacing a_ℓ by a , $\omega - v$ by ω , $\lambda - v$ by λ and with the help of (1.3.8) we obtain

$$\begin{aligned}
 (4.3.14) \quad &\ell \int_0^\infty e^{-a(x+c)^\ell} (x+c)^\omega x^{\lambda-\omega-1} dx \\
 &= c^\lambda \Gamma(\lambda - \omega) H_{1,2}^{2,0} \left(ca^{\frac{1}{\ell}} \left| \begin{matrix} (-\omega, 1) \\ (-\lambda, 1), (0, \frac{1}{\ell}) \end{matrix} \right. \right)
 \end{aligned}$$

$\text{Re}(\lambda - \omega) > 0$, $a > 0$, $c > 0$, ℓ any positive integers. By analytic continuation (4.3.14) holds for $\text{Re } a > 0$, $|\arg c| < \pi$, $\ell > 0$, $|\arg ca^{1/\ell}| < \frac{\pi}{2\ell}$, $\text{Re}(\lambda - \omega) > 0$. When $\ell = 1$, (4.3.14) reduces to [3, p. 63, (5)] again with the aid of [31, p. 231, (11)].

(4) Let $g_1 = g(0; a_k)$, $g_2 = g(0; b_k, b_\ell)$. Then

$g_1 g_2 = g(0; a_k + b_k, b_\ell)$. By using (4.3.3), (4.2.14) and

(4.3.4) we have

$$(4.3.15) \quad K(\lambda, \nu; g(0; a_k + b_k, b_\ell))$$

$$= \int_{\rho-1\infty}^{\rho+1\infty} K(\lambda, \sigma; g(0; a_k)) K(\sigma, \nu; g(0; b_k, b_\ell)) d\sigma$$

$$= \left(\frac{1}{2\pi i}\right)^2 \frac{1}{k^{2 \cdot \ell}} a_k^{\frac{\lambda}{k}} b_\ell^{\frac{\nu}{\ell}} \int_{\rho-1\infty}^{\rho+1\infty} a_k^{\frac{\sigma}{k}} b_k^{-\frac{\sigma}{k}} \Gamma\left(\frac{\lambda-\sigma}{k}\right) H_{1,1}^{1,1} \left(\frac{b_k \frac{1}{k} \left(1 + \frac{\nu}{\ell}, \frac{1}{\ell}\right)}{b_\ell \frac{1}{\ell} \left(\frac{\sigma}{k}, \frac{1}{k}\right)} \right) d\sigma$$

$$= \left(\frac{1}{2\pi i}\right)^2 \frac{1}{k \cdot \ell} a_k^{\frac{\lambda}{k}} b_\ell^{\frac{\nu}{\ell}} \int_{\rho'-1\infty}^{\rho'+1\infty} a_k^{\frac{\sigma}{k}} b_k^{-\frac{\sigma}{k}} \Gamma\left(\frac{\lambda-\sigma}{k}\right) H_{1,1}^{1,1} \left(\frac{\frac{1}{k} \left(1 + \frac{\nu}{\ell}, \frac{1}{\ell}\right)}{b_\ell \frac{1}{\ell} \left(\sigma, \frac{1}{k}\right)} \right) d\sigma$$

On the other hand from (4.3.4) we have

$$(4.3.16) \quad K(\lambda, \nu; g(0; a_k + b_k, b_\ell))$$

$$= \frac{1}{2\pi i} \frac{1}{k \cdot \ell} (a_k + b_k)^{\frac{\lambda}{k}} b_\ell^{\frac{\nu}{\ell}} H_{1,1}^{1,1} \left(\frac{(a_k + b_k)^{\frac{1}{k}} \left(1 + \frac{\nu}{\ell}, \frac{1}{\ell}\right)}{b_\ell \frac{1}{\ell} \left(\frac{\lambda}{k}, \frac{1}{k}\right)} \right)$$

After writing $a_k = a$, $b_k = b$, $b_\ell = c$, and replacing $\frac{\nu}{\ell}$ by ν ,

$\frac{\lambda}{k}$ by λ , from (4.3.15) and (4.3.16) we obtain

$$(4.3.17) \quad (a+b)^{-\lambda} H_{1,1}^{1,1} \left(\frac{c(a+b)^{\frac{1}{k}} \left(1 + \nu, \frac{1}{\ell}\right)}{(\lambda, \frac{1}{k})} \right)$$

$$= a^{-\lambda} \frac{1}{2\pi i} \int_{\rho-1\infty}^{\rho+1\infty} a^\sigma b^{-\sigma} \Gamma(\lambda-\sigma) H_{1,1}^{1,1} \left(\frac{cb^{\frac{1}{k}} \left(1 + \nu, \frac{1}{\ell}\right)}{(\sigma, \frac{1}{k})} \right) d\sigma$$

$a > 0, b > 0, c > 0, \ell, k$ positive integers. By analytic continuation (4.3.17) holds for $k > 0, \ell > 0$, i.e.,

$$(4.3.18) \quad a^\lambda (a+b)^{-\lambda} H_{1,1}^{1,1} \left(c(a+b)^k \middle| \begin{matrix} (1+\nu, \ell) \\ (\lambda, k) \end{matrix} \right) \\ = \frac{1}{2\pi i} \int_{\rho-1-i\infty}^{\rho+1-i\infty} a^\sigma b^{-\sigma} \Gamma(\lambda-\sigma) H_{1,1}^{1,1} \left(cb^k \middle| \begin{matrix} (1+\nu, \ell) \\ (\sigma, k) \end{matrix} \right) d\sigma,$$

$a > 0, b > 0, c > 0, k > 0, \ell > 0$.

(5) Let $g_1 = g(0; a_\ell)$, $g_2 = g(0; b_k, b_\ell)$. Then the relation $g_1 g_2 = g(0; b_k, a_\ell + b_\ell)$ leads to

$$(4.3.19) \quad \ell c^{-\lambda} (a+b)^\nu H_{1,1}^{1,1} \left(c(a+b)^{\frac{1}{\ell}} \middle| \begin{matrix} (1+\nu, \frac{1}{\ell}) \\ (\frac{\lambda}{k}, \frac{1}{k}) \end{matrix} \right) \\ = a^{\frac{\lambda}{\ell}} b^\nu \frac{1}{2\pi i} \int_{\rho-1-i\infty}^{\rho+1-i\infty} a^{\frac{\sigma}{\ell}} c^{-\sigma} \Gamma\left(\frac{\lambda-\sigma}{\ell}\right) H_{1,1}^{1,1} \left(cb^{\frac{1}{\ell}} \middle| \begin{matrix} (1+\nu, \frac{1}{\ell}) \\ (\frac{\sigma}{k}, \frac{1}{k}) \end{matrix} \right) d\sigma,$$

$a > 0, b > 0, c > 0, k > 0, \ell > 0$.

(6) Let $g_1 = g(0; a_j)$, $g_2 = g(0; a_k, a_\ell)$. Then

$g_1 g_2 = g(0; a_j, a_k, a_\ell)$ and

$$(4.3.20) \quad K(\lambda, \nu; g(0; a_j, a_k, a_\ell))$$

$$= \int_{\rho-1-i\infty}^{\rho+1-i\infty} K(\lambda, \sigma; g(0; a_j)) K(\sigma, \nu; g(0; a_k, a_\ell)) d\sigma$$

$$= \left(\frac{1}{2\pi i}\right)^2 \frac{1}{j \cdot k \cdot \ell} a_j^{\frac{\lambda}{j}} a_\ell^{\frac{\nu}{\ell}} \int_{\rho-1-i\infty}^{\rho+1-i\infty} a_j^{\frac{\sigma}{j}} a_k^{\frac{-\sigma}{k}} \Gamma\left(\frac{\lambda-\sigma}{j}\right) H_{1,1}^{1,1} \left(\frac{a_k}{a_\ell} \middle| \begin{matrix} (1+\nu, \frac{1}{\ell}) \\ (\frac{\sigma}{k}, \frac{1}{k}) \end{matrix} \right) d\sigma.$$

On the other hand

$$(4.3.21) \quad K(\lambda, \nu; g(0; a_j, a_k, a_\ell)) = \frac{1}{2\pi i} \int_0^\infty e^{-a_j x^j - a_k x^k - a_\ell x^\ell} x^{\lambda - \nu - 1} dx.$$

Comparing (4.3.20) and (4.3.21) and by analytic continuation, we obtain

$$(4.3.22) \quad \int_0^\infty e^{-ax^j - bx^k - dx^\ell} x^{\lambda - \nu - 1} dx = \frac{1}{j \cdot k \cdot \ell} a^{-\frac{\lambda}{j}} d^{\frac{\nu}{\ell}} \int_{\rho-1}^{\rho+1} a^{\frac{\sigma}{j}} b^{-\frac{\sigma}{k}} \Gamma\left(\frac{\lambda-\sigma}{j}\right) H_{1,1}^{1,1} \left(\frac{b}{d} \left| \begin{matrix} (1+\frac{\nu}{\ell}, \frac{1}{\ell}) \\ (\frac{\sigma}{k}, \frac{1}{k}) \end{matrix} \right. \right) d\sigma,$$

$a > 0, b > 0, d > 0, j > 0, k > 0, \ell > 0.$

(7) Let $g_1 = g(0; a_\ell), g_2 = g(c; b_\ell)$. Then $g_1 g_2 = g(c; a_\ell + b_\ell)$

and proceeding as in previous examples, we obtain

$$(4.3.23) \quad \frac{a^\lambda}{(a+b)^\lambda} H_{1,2}^{2,1} \left(c(a+b) \left| \begin{matrix} (\omega, \ell) \\ (\lambda, 1), (0, \ell) \end{matrix} \right. \right) = \frac{1}{2\pi i} \int_{\rho-1}^{\rho+1} \left(\frac{a}{b}\right)^\sigma \Gamma(\lambda-\sigma) H_{1,2}^{2,1} \left(bc \left| \begin{matrix} (\omega, \ell) \\ (\sigma, 1), (0, \ell) \end{matrix} \right. \right) d\sigma,$$

where a, b, c, ℓ are positive reals.

(8) Let $g_1 = g(0; a_k), g_2 = g(c; a_\ell)$. Then the relation $g_1 g_2 = g(c; a_k, a_\ell)$ leads to

$$(4.3.24) \quad k \int_0^\infty e^{-ax^k - bx^\ell} (x+c)^\omega x^{\lambda-\omega-1} dx = \frac{a^{-\frac{\lambda}{k}}}{\Gamma(-\omega)} \frac{1}{2\pi i} \int_{\rho-1}^{\rho+1} a^{\frac{\sigma}{k}} b^{-\frac{\sigma}{\ell}} \Gamma\left(\frac{\lambda-\sigma}{k}\right) H_{1,2}^{2,1} \left(bc^\ell \left| \begin{matrix} (1+\omega, \ell) \\ (\frac{\sigma}{\ell}, 1), (0, \ell) \end{matrix} \right. \right) d\sigma,$$

where a, b, c, k, ℓ positive reals, ω not an integer.

(9) Let $g_1 = g(c; 0)$, $g_2 = g(d; a_\ell)$. Then the relation

$$g_1 g_2 = g(c+d; \phi_0, \phi_1, \dots, \phi_\ell, 0, \dots, 0), \quad \phi_j = \binom{\ell}{j} c^{\ell-j} a_\ell, \\ j = 0, 1, \dots, \ell,$$

implies

$$(4.3.25) \quad \int_0^\infty e^{-a(x+c)} (x+c+d)^\omega x^{\lambda-\omega-1} dx \\ = c^\lambda \frac{\Gamma(\lambda-\omega)}{\Gamma(-\omega)} \frac{1}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} c^{-\sigma} a^{-\frac{\sigma}{\ell}} \frac{\Gamma(\sigma-\lambda)}{\Gamma(\sigma-\omega)} H_{1,2}^{2,1} \left(\begin{matrix} ad^\ell \\ (\frac{\sigma}{\ell}, 1), (0, \ell) \end{matrix} \middle| \begin{matrix} (1+\omega, \ell) \end{matrix} \right) d\sigma,$$

where a, c, d positive real, ℓ positive integer, ω not an integer; $\text{Re}(\lambda - \omega) > 0$.

4.4 SOME LIMITING VALUES.

From (4.2.12) we have

$$(4.4.1) \quad K(\lambda, \nu; g(c; a_\ell)) = \frac{1}{2\pi i} \int_0^\infty e^{-a_\ell x^\ell} (x+c)^{\omega-\nu} x^{\lambda-\omega-1} dx,$$

where the right-hand side is absolutely convergent if $a_\ell > 0$,

$\text{Re } \lambda > \text{Re } \omega$. Then

$$(4.4.2) \quad \lim_{a_\ell \rightarrow 0^+} K(\lambda, \nu; g(c; a_\ell)) = \frac{1}{2\pi i} \lim_{a_\ell \rightarrow 0^+} \int_0^\infty e^{-a_\ell x^\ell} (x+c)^{\omega-\nu} x^{\lambda-\omega-1} dx \\ = \frac{1}{2\pi i} \int_0^\infty (x+c)^{\omega-\nu} x^{\lambda-\omega-1} dx \\ = K(\lambda, \nu; g(c; 0))$$

and from (4.3.7) and (4.2.13) we obtain

$$(4.4.3) \quad \lim_{a \rightarrow 0^+} a^{-\nu} H_{1,2}^{2,1} \left(ac \left| \begin{matrix} (1+\omega, \ell) \\ (\nu, 1), (0, \ell) \end{matrix} \right. \right) = c^{\nu} \Gamma(\ell\nu - \omega) \Gamma(-\ell\nu),$$

where $c > 0$, $\ell > 0$, $\text{Re } \omega < \text{Re } \ell\nu < 0$.

Also for $a_{\ell} > 0$ we have

$$(4.4.4) \quad \begin{aligned} \lim_{c \rightarrow 0^+} K(\lambda, \nu; g(c; a_{\ell})) &= \frac{1}{2\pi i} \lim_{c \rightarrow 0^+} \int_0^{\infty} e^{-a_{\ell} x^{\ell}} (x+c)^{\omega-\nu} x^{\lambda-\omega-1} dx \\ &= \frac{1}{2\pi i} \int_0^{\infty} e^{-a_{\ell} x^{\ell}} x^{\lambda-\nu-1} dx \\ &= K(\lambda, \nu; g(0; a_{\ell})). \end{aligned}$$

Therefore from (4.3.7) and (4.2.14) we obtain

$$(4.4.5) \quad \lim_{c \rightarrow 0^+} \frac{1}{\Gamma(-\omega)} H_{1,2}^{2,1} \left(ac \left| \begin{matrix} (1+\omega, 1) \\ (\lambda, \ell), (0, 1) \end{matrix} \right. \right) = \Gamma(\lambda),$$

$\text{Re } \lambda > 0$, $a > 0$, $\ell > 0$. In particular, let $\ell = 1$; $a = 1$. Then

[31, p. 231, (12)]

$$(4.4.6) \quad \begin{aligned} \Gamma(\lambda) &= \lim_{c \rightarrow 0^+} \frac{1}{\Gamma(-\omega)} H_{1,2}^{2,1} \left(c \left| \begin{matrix} (1+\omega, 1) \\ (\lambda, 1), (0, 1) \end{matrix} \right. \right) \\ &= \lim_{c \rightarrow 0^+} \frac{1}{\Gamma(-\omega)} G_{1,2}^{2,1} \left(c \left| \begin{matrix} 1+\omega \\ \lambda, 0 \end{matrix} \right. \right) \\ &= \lim_{x \rightarrow 0^+} \Gamma(\lambda-\omega) x^{\frac{\lambda-1}{2}} W_{\frac{1}{2}(1-\lambda+2\omega), \frac{\lambda}{2}}(x), \quad \text{Re } \lambda > 0. \end{aligned}$$

Similarly

$$\lim_{a_k \rightarrow 0^+} K(\lambda, \nu; g(0; a_k, a_{\ell})) = K(\lambda, \nu; g(0; a_{\ell})).$$

From (4.3.4) and (4.2.14) we obtain

$$(4.4.7) \quad \lim_{b \rightarrow 0^+} b^{-\lambda} H_{1,1}^{1,1} \left(ab \left| \begin{matrix} (1+\frac{\nu}{\ell}, \frac{1}{\ell}) \\ (\frac{\lambda}{k}, \frac{1}{k}) \end{matrix} \right. \right) = k a^{\lambda} \Gamma\left(\frac{\lambda-\nu}{\ell}\right),$$

$a > 0, k > 0, \ell > 0, \operatorname{Re}(\lambda - \nu) > 0$. From (4.3.14), we obtain

$$(4.4.8) \quad \lim_{c \rightarrow 0^+} c^{\lambda} \Gamma(\lambda - \omega) H_{1,2}^{2,0} \left(ca^{\frac{1}{\ell}} \left| \begin{matrix} (-\omega, 1) \\ (-\lambda, 1), (0, \frac{1}{\ell}) \end{matrix} \right. \right) \\ = \ell \lim_{c \rightarrow 0^+} \int_0^{\infty} e^{-a(x+c)^{\ell}} (x+c)^{\omega} x^{\lambda-\omega-1} dx \\ = \ell \int_0^{\infty} e^{-ax^{\ell}} x^{\lambda-1} dx = a^{-\frac{\lambda}{\ell}} \Gamma\left(\frac{\lambda}{\ell}\right),$$

or

$$(4.4.9) \quad \lim_{c \rightarrow 0^+} c^{\lambda} H_{1,2}^{2,0} \left(c \left| \begin{matrix} (\omega, 1) \\ (-\lambda, 1), (0, \ell) \end{matrix} \right. \right) = \frac{\Gamma(\ell\lambda)}{\Gamma(\lambda+\omega)},$$

$\ell > 0, \ell\lambda \neq 0, -1, -2, \dots$

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