INJECTIVE SHEAVES
WITH VALUES IN A CATEGORY

A thesis submitted

by

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to

the Faculty of Pure and Applied Science

of the University of Ottawa

in partial fulfillment of the requirements

for the degree of

Master of Science

in the subject of

Mathematics

May, 1964

ABSTRACT

In this thesis, we prove the following result: the category of sheaves with values in a Grothendieck category with a small generator and arbitrary products has enough injectives. The result remains valid if we replace the value-category by a Grothendieck category with a noetherian projective generator. The proofs are based on a theorem on transported structures which is due to J.M. Maranda.

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Section I

INTRODUCTION

The functors Ext^n and Tor_n are very powerful tools in the study of cohomology and homology theories of many algebraic systems and also of topological spaces. The construction of $Ext^n(A,B)$ for two objects in an abelian category (cf. II, 2.6) can be achieved by means of projective resolutions of A or injective coresolutions of B. An injective coresolution of B is an acyclic right complex over B in which all objects are injectives (cf. II, 2.1). In order that such injective coresolutions exist, we must require that the category has enough injectives, i.e. that every object can be embedded into an injective object. That is why we are interested in problems concerning the existence of enough injectives in a category.

The study of sheaves arose from Algebraic Geometry. Moreover they are used in proofs of several important isomorphism theorems in homology and in cohomology theories. We have pointed out that the functor Ext^n is a useful tool in studies of homology and cohomology theories; for instance, in [7, p. 176 and p. 189] it was shown that the Čech cohomology groups $\check{H}(X,F)$ of a paracompact topological space X with values in an abelian sheaf F (cf. II, 2.7) are isomorphic to $Ext_Z(Z,F)$, where Z is the ring of integers considered as a constant sheaf on X. Such results suggested the construction of $Ext^n(A,B)$ in the category of sheaves with values in an arbitrary category. Since it is very difficult to prove that such a category has enough projectives, (i.e., every sheaf can be written as a quotient sheaf of a projective sheaf), we therefore try to prove that it has enough injectives. (In fact, in some cases, for instance the category of abelian sheaves, the category actually does not have enough projectives). In this thesis, we prove that the category of sheaves has enough injectives when the value-category is the category of abelian groups, or has a small generator (cf. II, 2.5 and 2.8) and arbitrary direct products, or has a noetherian projective generator (cf. II, 2.1 and 2.8) and arbitrary direct products.

In section II of this thesis, we give a survey of basic definitions in the theory of categories and functors as well as several different definitions of sheaves. In section III, we prove some results concerning injective structures on which we base our proofs in section IV where we establish the results mentioned in the last paragraph. In section V, we prove a theorem about reflective subcategories of a special category; the result is a consequence of investigations made in section IV.

I wish to express my sincere thanks to Dr. H. Kleisli, who, as the thesis advisor, suggested this work, and who directed it with patience during its development.

Section II

FUNDAMENTALS OF THE THEORY OF CATEGORIES AND FUNCTORS

In this section we give the basic definitions in the theory of categories and functors which we need in the following sections. After a definition is introduced, unless it coincides with the usual definition for modules, we shall give some examples. When the article "the" appears in a definition, it should be understood that the "uniqueness" of the term defined up to isomorphisms can be easily verified.

2.1. Categories, monomorphisms, epimorphisms, isomorphisms, zero objects, injective objects, projective objects, subobjects and quotient objects.

A category $\mathbb{C}$ is a class of objects A, B, C, . . . with the following data:

(i) For any two objects A, B, a set $\mathrm{Hom}_{\mathbb{C}}(A,B)$ is given,

(ii) There is an associative composition law assigning to each $\alpha \in \mathrm{Hom}_{\mathbb{C}}(A,B)$ and each $\beta \in \mathrm{Hom}_{\mathbb{C}}(B,C)$ an element $\gamma = \beta\alpha \in \mathrm{Hom}_{\mathbb{C}}(A,C)$,

(iii) To each object A, there is an element $\iota_A \in \mathrm{Hom}_{\mathbb{C}}(A,A)$ which acts as a two-sided identity.

The elements α in $\mathrm{Hom}_{\mathbb{C}}(A,B)$ are called morphisms with domain A and range B. We sometimes write $\alpha : A \to B$ or $A \overset{\alpha}{\to} B$ instead of $\alpha \in \mathrm{Hom}_{\mathbb{C}}(A,B)$.

A morphism $\alpha: A \to B$ is a monomorphism if for any two morphisms $\beta,\gamma : C \to A$, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. Dually, a morphism $\alpha : A \to B$ is an epimorphism if for any two morphisms $\beta,\gamma : B \to C$, $\beta\alpha = \gamma\alpha$ implies

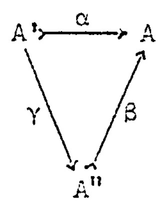
$\beta = \gamma$. It should be noted that an epimorphism is not necessarily surjective in "concrete" categories. For instance, the inclusion mapping from the set Q of all rational numbers into the set R of all real numbers is an epimorphism if R is considered as a topological space with the usual topology and Q is endowed with the induced topology. A morphism $\alpha: A \to B$ is an <u>isomorphism</u> if there exists a morphism $\beta : B \to A$ such that $\beta\alpha = \iota_A$ and $\alpha\beta = \iota_B$. It is clear that an isomorphism is at the same time a monomorphism and an epimorphism. We shall indicate monomorphisms and epimorphisms by $\rightarrowtail$ and $\twoheadrightarrow$ respectively.

An object $0 \in \mathbb{C}$ is called a <u>zero object</u> if for every $A \in \mathbb{C}$, $Hom_{\mathbb{C}}(A,0)$ and $Hom_{\mathbb{C}}(0,A)$ consist of exactly one morphism each. The morphism $A \to B$ which is the composition of the morphisms in $Hom_{\mathbb{C}}(A,0)$ and $Hom_{\mathbb{C}}(0,A)$ is called a <u>zero morphism</u> and is usually denoted by o.

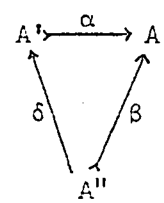
An object A is an <u>injective object</u> if for every monomorphism $\alpha : B \to C$, the induced mapping $\alpha^* : Hom_{\mathbb{C}}(C,A) \to Hom_{\mathbb{C}}(B,A)$, given by $\alpha^*(\beta) = \beta\alpha$ for every morphism $\beta : C \to A$, is surjective. Dually, an object A is a <u>projective object</u> if for every epimorphism $\alpha : B \to C$, the induced mapping $\alpha_* : Hom_{\mathbb{C}}(A,B) \to Hom_{\mathbb{C}}(A,C)$, given by $\alpha_*(\beta) = \alpha\beta$ for every morphism $\beta: A \to B$, is surjective. A category $\mathbb{C}$ is said to have <u>enough injectives</u> if for every object $A \in \mathbb{C}$ there exists a monomorphism $\alpha : A \to B$ with B injective. Dually, a category is said to have <u>enough projectives</u> if for every object $A \in \mathbb{C}$, there exists an epimorphism $\alpha : B \to A$ with B projective.

Examples: in the category of left R-modules over a ring R, every free module is projective and every divisible torsion-free module is injective. It is well-known that this category has enough projectives and injectives.

Let $\alpha : A' \rightarrowtail A$ and $\beta : A'' \rightarrowtail A$ be in $\mathbb{C}$. We say that α is related to β if there exist morphisms $\gamma : A' \to A''$ and $\delta : A'' \to A'$ such that the diagrams



and



are commutative. Clearly this relation is an equivalence relation. We will choose representatives of the equivalence classes and call [illegible] [illegible] subobjects of A. We sometimes also call the domain of the representative monomorphism a subobject of A. Quotient objects are defined dually.

2.2. Covariant functors, contravariant functors, bifunctors, and natural transformations.

Let $\mathbb{C}$ and $\mathbb{C}'$ be two categories. A covariant (contravariant) functor F from $\mathbb{C}$ to $\mathbb{C}'$ is a function which maps objects to objects and morphisms to morphisms such that

(i) for every $\alpha : A \to B$, $F\alpha : FA \to FB$ ($F\alpha : FB \to FA$)

(ii) if $\gamma = \beta\alpha$, then $F\gamma = F\beta F\alpha$ ($F\gamma = F\alpha F\beta$)

(iii) $F \iota_A = \iota_{FA}$

Given two categories $\mathbb{C}_1$ and $\mathbb{C}_2$, we can form a new category $\mathbb{C}_1 \times \mathbb{C}_2$. The objects of $\mathbb{C}_1 \times \mathbb{C}_2$ are pairs (A,B) with $A \in \mathbb{C}_1$, $B \in \mathbb{C}_2$, the morphisms are the pairs (α,β) with $\alpha \in \mathbb{C}_1, \beta \in \mathbb{C}_2$, and the composition of morphisms is defined componentwise. Define a mapping $\mathrm{Hom}_{\mathbb{C}_2}(F-,-) : \mathbb{C}_1 \times \mathbb{C}_2 \to \mathbb{S}$, $\mathbb{S}$ being the category of sets, as follows:

$$\mathrm{Hom}_{\mathbb{C}_2}(F-,-)\ (A_1,A_2) = \mathrm{Hom}_{\mathbb{C}_2}(FA_1,A_2)$$

and

$$\mathrm{Hom}_{\mathbb{C}_2}(F-,-)\ (\alpha_1,\alpha_2)\alpha = \alpha_2\alpha F\alpha_1 : FA_1 \to B_2$$

for all $(A_1,A_2) \in \mathbb{C}_1 \times \mathbb{C}_2$, $(\alpha_1,\alpha_2) : (A_1,A_2) \to (B_1,B_2) \in \mathbb{C}_1 \times \mathbb{C}_2$ and $\alpha : FB_1 \to A_2$ in $\mathbb{C}_2$. It is easily checked that when A_2 is held fixed the mapping is a contravariant functor from $\mathbb{C}_1$ into $\mathbb{S}$ and when A_1 is fixed, it becomes a covariant functor from $\mathbb{C}_2$ into $\mathbb{S}$. We denote these two functors by $\mathrm{Hom}_{\mathbb{C}_2}(F-,A_2)$ and $\mathrm{Hom}_{\mathbb{C}_2}(FA_1,-)$ respectively. A mapping of this type is called a <u>bifunctor</u>, contravariant in the first variable and covariant in the second variable.

Let F and G be covariant functors from a category $\mathbb{C}$ to a category $\mathbb{C}'$. A function η is called a <u>natural transformation</u> from F to G if it assigns to each $A \in \mathbb{C}$, a morphism $\eta_A : FA \to GA$ in $\mathbb{C}'$ such that for any $\alpha : A \to B$ in $\mathbb{C}$, commutativity holds in the diagram

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FB & \xrightarrow{\eta_B} & GB \end{array}$$

A similar definition is made for natural transformations between contravariant functors.

It is seen that the class of all covariant functors from a category $\mathbb{C}$ into a category $\mathbb{C}'$ with natural transformations as morphisms forms a category, provided that the collection of all natural transformations between any two covariant functors is a set. This category is called a functor category. For this reason, sometimes we call natural transformations functor morphisms. When a functor morphism is an isomorphism, it is called a functor isomorphism or a natural equivalence. Natural transformations and equivalences between bifunctors are functions which, when either variable is held fixed, are natural transformations and equivalences between the resulting functors.

2.3. Adjoint functors.

The idea of adjoint functors is due to D. Kan [9] and is very fundamental in the modern development of algebra.

Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be two categories, $S : \mathbb{C}_1 \to \mathbb{C}_2$, and $T : \mathbb{C}_2 \to \mathbb{C}_1$ be covariant functors. Then S is said to be (left) adjoint to T if there exists a natural equivalence

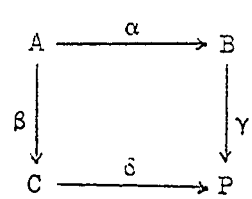
$$\eta : \mathrm{Hom}_{\mathbb{C}_2}(S-,-) \to \mathrm{Hom}_{\mathbb{C}_1}(-,T-)$$

where $\mathrm{Hom}_{\mathbb{C}_2}(S-,-)$ and $\mathrm{Hom}_{\mathbb{C}_1}(-,T-)$ are bifunctors, as described in 2.2.

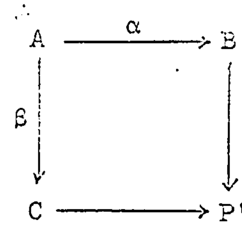
In the category $\mathbb{A}$ of abelian groups, the functors $A \otimes -$, which assigns to each $B \in \mathbb{A}$ the abelian group $A \otimes B$ and each homomorphism an obvious induced homomorphism, and the functor $\mathrm{Hom}_Z(A,-)$ are adjoint functors. For more examples, see [9].

2.4. Pushouts, pullbacks, difference kernels, kernels, difference cokernels, images and coimages.

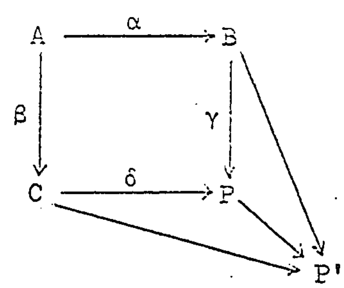
A commutative diagram,



is a pushout (diagram) if for any commutative diagram



there exists a unique morphism $P \to P'$ such that the diagram

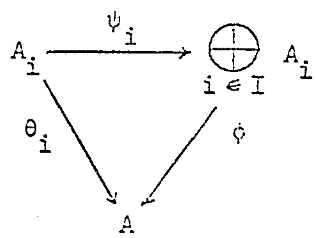


is commutative. If $C = B$ and $\delta = \gamma$; then γ is seen to be an epimorphism. Let γ be a representative of the equivalence class of all such epimorphisms. γ is called the difference cokernel of α and β. Sometimes P is also called the difference cokernel. In particular if β is a zero morphism, then P is the cokernel of α. If A is a subobject, P is usually written as B/A. Pullbacks, difference kernels and kernels are defined dually. These definitions coincide with the usual definitions in the category of R-modules over a ring R.

Let $\rho : B \to F$ be the cokernel of $\alpha : A \to B$. Then the image $\text{Im}\,\alpha$ of α is defined to be the kernel of $\rho\alpha$. Coimages are defined dually.

2.5. Direct sums, direct products, generators, cogenerators, intersections and unions of subobjects, direct limits and inverse limits.

Let $(A_i)_{i \in I}$ be a family of objects in $\mathbb{C}$. An object, denoted by $\bigoplus_{i \in I} A_i$, is called the <u>direct sum</u> of $(A_i)_{i \in I}$, if there exist morphisms $\psi_i : A_i \to \bigoplus_{i \in I} A_i$ for all $i \in I$, called <u>i-th injections</u>, such that for every $A \in \mathbb{C}$ and morphisms $\theta_i : A_i \to A, i \in I$, there exists a unique morphism $\phi : \bigoplus_{i \in I} A_i \to A$ such that the diagrams



are commutative for all $i \in I$. It is easy to see that if $\mathbb{C}$ has a zero object, all injections are monomorphisms. Direct products $\prod_{i \in I} A_i$ and projections $\pi_i : \prod_{i \in I} A_i \to A_i$ are defined dually. With these definitions, one sees that the direct sums of modules are the usual direct sums. They are set unions in the category of sets, tensor products in the category of commutative rings with identity elements and free products in the category of groups. The direct products in the above categories are introduced via cartesian products.

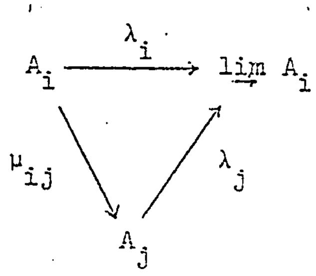
Let $\mathbb{C}$ be a category where for every subobject $A' \rightarrowtail A$ the quotient object A/A' exists. An object G is called a <u>generator</u> in $\mathbb{C}$, if for every proper subobject $A' \rightarrowtail A$ of A in $\mathbb{C}$, there exists a morphism $G \to A$ such that the composition $G \to A \to A/A'$ is not zero [7]. In the case where the direct sum of every family of objects in $\mathbb{C}$ exists, an object G is a generator if, and only if, each $A \in \mathbb{C}$ can be written as a quotient object of a direct sum of $(G_i)_{i \in I}$, $G_i = G$ for all $i \in I$ [7]. <u>Cogenerators</u> are defined dually.

Examples of generators are the ring R in the category of left R-modules and the direct sum of all cyclic groups of prime order in the category of torsion groups [cf. 6, p. 26]. A set consisting of a single element is an example of a cogenerator in the category of sets.

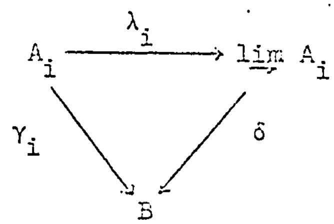
Let $(A_i)_{i \in I}$ be a family of subobjects of A. Then the epimorphisms from A to A/A_i determine a morphism $\mu : A \to \prod_{i \in I} A/A_i$ and the monomorphisms of A_i into A determine a morphism $\nu : \bigoplus_{i \in I} A_i \to A$. The <u>intersection</u> of $(A_i)_{i \in I}$ in A is the kernel of the morphism μ and the <u>union</u> of $(A_i)_{i \in I}$ in A is the image of the morphism ν. When I consists of two elements, the intersection of A_1, A_2 in A is isomorphic to the kernel of $A_1 \rightarrowtail A \twoheadrightarrow A/A_2$. Note that definitions of intersections and unions are not dual to each other.

Let I be a directed set and $\{A_i, \mu_{ij}\}_{i \in I}$ a direct family of objects over I in $\mathbb{C}$. An object, denoted by $\varinjlim \{A_i, \mu_{ij}\}$ or just simply by $\varinjlim A_i$, is called the <u>direct limit</u> of the direct family $\{A_i, \mu_{ij}\}_{i \in I}$, if

(i) there exists a family $(\lambda_i)_{i \in I}$ of morphisms, $\lambda_i : A_i \to \varinjlim A_i$, such that for every pair $(i,j) \in I \times I$, $i < j$, commutativity holds in the following diagram



(ii) for any object B and a family $(\gamma_i)_{i \in I}$ of morphisms, $\gamma_i : A_i \to B$, with the same property as the family $(\lambda_i)_{i \in I}$, there exists a unique morphism $\delta : \varinjlim A_i \to B$ such that for all $i \in I$, the following diagram

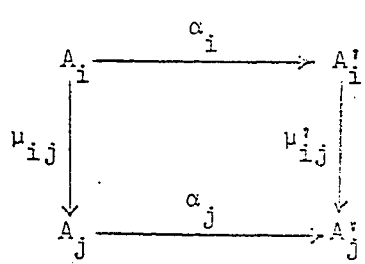


is commutative. Direct limits can actually be constructed explicitly if direct sums, difference kernels, cokernels and unions of subobjects exist.

The construction goes as follows: Let $\bigoplus_{i \in I} A_i$ be the direct sum of $(A_i)_{i \in I}$ and $\psi_j : A_j \to \bigoplus_{i \in I} A_i$, the j-th injections. Let $K_{ij}, j > i$, be the difference kernel of $A_i \overset{\mu_{ij}}{\to} A_j \overset{\psi_j}{\to} \bigoplus_{i \in I} A_i$ and $A_i \overset{\psi_i}{\to} \bigoplus_{i \in I} A_i$. Then K_{ij} can be regarded as a subobject of $\bigoplus_{i \in I} A_i$. Let Q be the union of all such K_{ij}. Then $\varinjlim A_i = \bigoplus_{i \in I} A_i \Big/ Q$.

<u>Inverse limits</u> $\varprojlim A_i$ are defined dually.

Let $\mathbb{D}$ be the collection of all direct families of objects in a category $\mathbb{C}$. Define a morphism between any two direct families $\{A_i, \mu_{ij}\}_{i \in I}$ and $\{A'_i, \mu'_{ij}\}_{i \in I}$ over the same directed set I to be a family of morphisms $\alpha_i : A_i \to A'_i$, for $i \in I$, such that for all $i \in I$ commutativity holds in the diagram



The composition of morphisms is defined componentwise. Then we see that $\mathbb{D}$ forms a category. It is easily checked that the mapping $\varinjlim$ which assigns to each direct family its direct limit is a covariant functor from $\mathbb{D}$ to $\mathbb{C}$. Similarly, $\varprojlim$ is a functor from the category of all inverse families to $\mathbb{C}$.

2.6. Abelian categories, exact sequences, exact functors, small categories, embeddings, full subcategories and Grothendieck categories.

A category $\mathbb{C}$ is abelian if

A 1) each of the disjoint sets $\mathrm{Hom}_{\mathbb{C}}(A,B)$ has an abelian group structure,

A 2) the composition law is a group homomorphism

$h : \mathrm{Hom}_{\mathbb{C}}(B,C) \otimes \mathrm{Hom}_{\mathbb{C}}(A,B) \to \mathrm{Hom}_{\mathbb{C}}(A,C)$,

written $h(\beta \otimes \alpha) = \beta\alpha$,

A 3) there is a zero object,

A 4) every pair of objects has a direct sum,

A 5) every morphism has a kernel and a cokernel,

A 6) every morphism α can be written as $\alpha = \lambda\sigma$ with λ monomorphic, σ epimorphic.

This definition is the original definition of an exact category given by Buchsbaum in the appendix of Cartan-Eilenberg's "Homological Algebra". The term "abelian category" was introduced by Grothendieck [7].

A standard example of abelian categories is the category of all R-modules over a ring R. The category of all right complexes of abelian groups with translations as morphisms is also an abelian category.

Categories satisfying A 1) - A 4) are sometimes called <u>additive</u> categories [11].

It is shown in [3] that in an abelian category every diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

can be completed to a pushout and its dual diagram to a pullback. Moreover each direct sum of a finite number of objects is isomorphic to the direct product of the same objects.

A sequence $A_1 \overset{\mu}{\to} A_2 \overset{\nu}{\to} A_3$ is <u>exact</u> if $\mathrm{Im}\mu = \mathrm{Ker}\nu$. A long sequence is exact if any subsequence of three consecutive objects is exact. When μ is monomorphic and ν is epimorphic, the exact sequence is sometimes called a <u>short exact sequence</u>.

A functor F which carries any short exact sequence $A_1 \rightarrowtail A_2 \twoheadrightarrow A_3$ to an exact sequence $FA_1 \overset{F\mu}{\rightarrowtail} FA_2 \overset{F\nu}{\to} FA_3$ is called a <u>left-exact</u> functor. <u>Right-exact functors</u> are defined dually. An <u>exact functor</u> is a functor which is both left-exact and right-exact.

Examples: $\mathrm{Hom}_R(A,-)$ is a left-exact functor from the category of left R-modules to the category of abelian groups. It is exact if, and only if, A is projective. On the other hand $A \otimes_R -$, A being an abelian group, is a right-exact functor from the category of abelian group to the category of left R-modules.

By a <u>small</u> category, we understand that the class of all morphisms in that category is a set, i.e., it possesses a cardinality.

A functor is <u>faithful</u> or is an <u>embedding</u> if it is injective on the class of all morphisms. A functor $F : \mathbb{C}_1 \to \mathbb{C}_2$ is <u>full</u> if for any $A,B \in \mathbb{C}_1$, the mapping $\mathrm{Hom}_{\mathbb{C}_1}(A,B) \to \mathrm{Hom}_{\mathbb{C}_2}(FA,FB)$ is surjective. A subcategory of a category is <u>full</u> if the embedding functor is full. The Mitchell-Freyd representation theorem says that every small abelian category can be embedded fully and exactly into a category of modules over a ring R [3, p. 152].

Let I be a directed set and $\{A_i,\mu_{ij}\}_{i\in I}, \{B_i,\nu_{ij}\}_{i\in I}, \{C_i,\rho_{ij}\}_{i\in I}$ direct families over I in an abelian category $\mathbb{C}$ with arbitrary direct sums. If $A_i \rightarrowtail B_i \twoheadrightarrow C_i$ is exact for all $i \in I$, in general we usually only have $\varinjlim A_i \to \varinjlim B_i \twoheadrightarrow \varinjlim C_i$ exact, i.e., $\varinjlim$ is a right-exact functor [cf. 13, p. 6]. If we postulate that $\varinjlim$ is also left-exact, then $\mathbb{C}$ is called a <u>Grothendieck</u> category. Such a category is also named in the literature an AB - 5 category or a right perfect category. Examples of Grothendieck categories are the category of left modules over a ring R, and the functor category of all covariant functors from a small abelian category into the category of abelian groups.

<u>Remark</u>: Grothendieck has shown that a Grothendieck category with a generator always has enough injectives [7, p. 135]. Thus we might think that to solve problems concerning the existence of enough injectives, it might be a good way to show that the category in question is a Grothendieck category and then to construct a generator. Heller and Rowe have succeeded in this way to prove that the category of sheaves (see below) with values in a Grothendieck category with arbitrary direct products and a projective generator has enough injectives [8]. However, their proof involves a rather lengthy con-

struction. Moreover, it is difficult to perform that construction in other categories of sheaves.

2.7. Abelian sheaves, sheaves with values in an abelian category.

Let X be a topological space and U,V, . . . open subsets of X. Then the topology T_X of X can be made into a category with inclusion mappings as morphisms. An abelian sheaf over X is a contravariant functor $F : T_X \to \mathbb{A}$, $\mathbb{A}$ being the category of abelian groups, satisfying the following axioms:

AS 1) Let $U = \bigcup_{i \in I} U_i$, $U_i \in T_X$ for all $i \in I$, and $s, t \in FU$. Let ρ_{UU_i} denote the homomorphism from FU to FU_i which is the image of the inclusion mapping from U_i to U under F. Then s = t if $\rho_{UU_i}(s) = \rho_{UU_i}(t)$, for all $i \in I$.

AS 2) Let U be as in AS 1). If $(s_i)_{i \in I}$ is a family of $s_i \in FU_i$, such that $\rho_{U_i U_i \cap U_j}(s_i) = \rho_{U_j U_i \cap U_j}(s_j)$ for all i, $j \in I$, then there exists an element $s \in FU$ such that $\rho_{UU_i}(s) = s_i$ for all $i \in I$.

A sheaf over X with values in an abelian category $\mathbb{C}$ is a contravariant functor $F : T_X \to \mathbb{C}$ such that for every $A \in \mathbb{C}$, the functor $\hom_{\mathbb{C}}(A, F-) : T_X \to \mathbb{A}$ is an abelian sheaf; in this case, $\mathbb{C}$ is called the value-category of F. There are other definitions for sheaves [cf. 6, 33]. To avoid lengthy description of terminology, we list only two:

(i) A contravariant functor $F : T_X \to \mathbb{C}$, $\mathbb{C}$ being an abelian category, is a sheaf if $FU = \varprojlim_I FU_i$, whenever $(U_i)_{i \in I}$ forms an inverse family and $U = \bigcup_{i \in I} U_i$;

(ii) If $\mathbb{C}$ possesses arbitrary direct products, then F is a sheaf if for every U, the horizontal sequence in the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & FU & \xrightarrow{\theta} & \prod FU_i & \xrightarrow{\psi} & \prod FU_i \cap U_j \\ & & & \searrow^{\rho_{UU_i}} & \downarrow \pi_i & \searrow^{\rho_{U_i U_i \cap U_j}\pi_i - \rho_{U_j U_i \cap U_j}\pi_j} & \downarrow \pi_{ij} \\ & & & & FU_i & & FU_i \cap U_j \end{array}$$

is exact.

Since there is no complete proof of their equivalence to be found in the literature, we shall give one here:

Taking the definition of abelian sheaves and the universal mapping property of inverse limits into account, it is easy to verify that our definition implies (i) and (ii). (In the verification, the fact that $\mathrm{Hom}_{\mathbb{C}}(A,-)$ commutes with inverse limits is also needed. A proof of the dual of this statement is given in [13, p. 46]). We now show that the converses are also true.

First, we show that (i) implies our definition. By (i), we have

$$\mathrm{Hom}_{\mathbb{C}}(A,FU) = \mathrm{Hom}_{\mathbb{C}}(A, \varprojlim_{I} FU_i) = \varprojlim_{I} \mathrm{Hom}_{\mathbb{C}}(A,FU_i),$$

whenever $(U_i)_{i \in I}$ forms an inverse family and $U = \bigcup_{i \in I} U_i$. Let $U = \bigcup_{i \in I'} U_i$ for any indexing set I' and $s,t \in \mathrm{Hom}_{\mathbb{C}}(A,FU)$ such that $(\rho_{UU_i})_*(s) = (\rho_{UU_i})_*(t)$, or $\rho_{UU_i} s = \rho_{UU_i} t$ for all $U_i, i \in I'$. Clearly $(U_i)_{i \in I'}$ can be made into an inverse family if we add all the finite intersections of U_i to it. [illegible]note this inverse family by $(U_i)_{i \in I}$.

Then $\rho_{UU_i \cap U_j} s = \rho_{U_i U_i \cap U_j} \rho_{UU_i} s = \rho_{U_i U_i \cap U_j} \rho_{UU_i} t = \rho_{UU_i \cap U_j} t$, i.e., $(\rho_{UU_i \cap U_j})_*(s) = (\rho_{UU_i \cap U_j})_*(t)$ for all members in $(U_i)_{i \in I}$. Hence it follows from the universal mapping property of the inverse limit that $s = t$. This proves AS 1) for $\mathrm{Hom}_{\mathbb{C}}(A, F-)$.

Before verifying AS 2) for $\mathrm{Hom}_{\mathbb{C}}(A, F-)$, we note that the inverse limit of an inverse family $\{A_i, \rho_{ij}\}_{i \in I}$ is isomorphic to the union of all difference kernels of any two morphisms $\pi_j : \prod_{i \in I} A_i \to A_j$ or [illegible] $: \prod_{i \in I} A_i \to A_j$. Now let $(s_i)_{i \in I}$, $s_i \in \mathrm{Hom}_{\mathbb{C}}(A, FU_i)$, be a family of morphisms such that $(\rho_{U_i U_i \cap U_j})_*(s_i) = (\rho_{U_j U_i \cap U_j})_*(s_j)$. Then $s = (s_i)_{i \in I}$ is in $\mathrm{Hom}_{\mathbb{C}}(A, FU)$ and $\rho_{UU_i}(s) = s_i$. This proves AS 2).

Next, we show that if $\mathbb{C}$ possesses arbitrary direct products, (ii) implies our definition. Since $\mathrm{Hom}_{\mathbb{C}}(A, -)$ is a left-exact functor, it follows that the horizontal sequence in the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathbb{C}}(A, FU) & \xrightarrow{\theta_*} & \mathrm{Hom}_{\mathbb{C}}(A, \prod_{U_i \subseteq U} FU_i) & \xrightarrow{\psi_*} & \mathrm{Hom}_{\mathbb{C}}(A, \prod_{U_i, U_j \subseteq U} FU_i \cap U_j) \\
 & \searrow^{(\rho_{UU_i})_*} & \downarrow (\pi_i)_* & \searrow^{(\rho_{U_i U_i \cap U_j} \pi_i)_* - (\rho_{U_j U_i \cap U_j} \pi_j)_*} & \downarrow (\pi_{ij})_* \\
 & & \mathrm{Hom}_{\mathbb{C}}(A, FU_i) & & \mathrm{Hom}_{\mathbb{C}}(A, FU_i \cap U_j)
\end{array}
$$

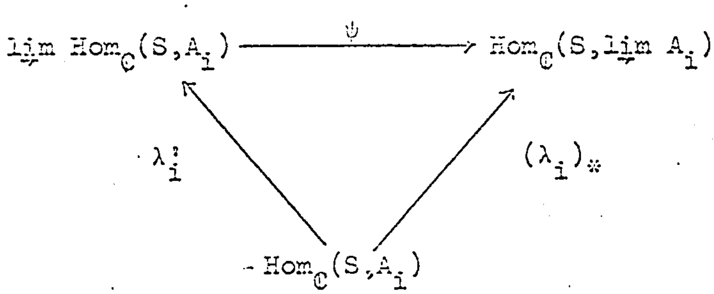
is exact. Let $s, t \in \mathrm{Hom}_{\mathbb{C}}(A, FU)$ such that $(\rho_{UU_i})_*(s) = (\rho_{UU_i})_*(t)$, for all $U_i \subseteq U$. It follows that $\pi_i \theta s = \pi_i \theta t$ for all π_i, hence $\theta s = \theta t$. Since θ is monomorphic, $s = t$. Let now (s_i) be a set of morphisms such that $s_i \in \mathrm{Hom}_{\mathbb{C}}(A, FU_i)$ and such that $(\rho_{U_i U_i \cap U_j})_* s_i = (\rho_{U_j U_i \cap U_j})_* s_j$. Then clearly $(s_i)_{i \in I}$ can be regarded as an element in $\prod_{U_i \subseteq U} \mathrm{Hom}_{\mathbb{C}}(A, FU_i)$ $= \mathrm{Hom}_{\mathbb{C}}(A, \prod_{U_i \subseteq U} FU_i)$. This element is easily seen to be mapped to zero and hence it belongs to the kernel $\mathrm{Hom}_{\mathbb{C}}(A, FU)$. This proves AS 2) for $\mathrm{Hom}_{\mathbb{C}}(A, F-)$. Hence $\mathrm{Hom}_{\mathbb{C}}(A, F-)$ is a sheaf.

2.8. <u>Small objects, cosmall objects, noetherian objects and artinian objects.</u>

Let $\{A_i, \mu_{ij}\}_{i \in I}$ be a direct family of objects in an abelian category $\mathbb{C}$. Then for any object $S \in \mathbb{C}$, the μ_{ij} induce a family of group homomorphisms $(\mu_{ij})_* : \mathrm{Hom}_{\mathbb{C}}(S, A_i) \to \mathrm{Hom}_{\mathbb{C}}(S, A_j)$. The system $\mathrm{Hom}_{\mathbb{C}}(S, A_i)$ together with all such $(\mu_{ij})_*$ is clearly a direct family. Thus there exists a unique homomorphism

$$\psi : \varinjlim \mathrm{Hom}_{\mathbb{C}}(S, A_i) \to \mathrm{Hom}_{\mathbb{C}}(S, \varinjlim A_i)$$

making the following diagrams



commutative. (The λ_i' are the morphisms associated to the direct limit $\varinjlim \mathrm{Hom}_{\mathbb{C}}(S, A_i)$ of the system $\{\mathrm{Hom}_{\mathbb{C}}(S, A_i), (\mu_{ij})_*\}$ in $\mathbb{A}$). If ψ is a monomorphism for any direct family $\{A_i, \mu_{ij}\}_{i \in I}$, S is called a <u>small</u> object [6, p. 26]. <u>Cosmall</u> objects are defined dually, i.e., for any inverse family $\{A_i, \mu_{ij}\}_{i \in I}$, the homomorphism

$$\theta : \mathrm{Hom}_{\mathbb{C}}(S, \varprojlim A_i) \to \varprojlim \mathrm{Hom}_{\mathbb{C}}(S, A_i)$$

which is obtained in a similar manner is an epimorphism.

Examples of small objects are: the ring R in the category $\mathbb{M}_R$ of R-modules and $\bigoplus_p C_p$, C_p being cyclic groups of prime order, in the category of torsion abelian groups. A set consisting of a single element is an example of a cosmall object in the category of sets.

In an abelian category $\mathbb{C}$, an object N is called a noetherian object if the family of proper subobjects of N satisfies the ascending chain condition. An artinian object is an object whose family of proper subobjects satisfies the descending chain condition.

Section III

INJECTIVE STRUCTURES

In this section the concepts of an injective structure and a projective structure are introduced, and we will prove some results which we will make use of in Section IV.

3.1. Injective Structures.

Let $\mathbb{C}$ be a category, $\mathcal{M}$ a class of morphisms and $\mathcal{J}$ a class of objects in $\mathbb{C}$. An object $C \in \mathbb{C}$ is called <u>$\mathcal{M}$ - injective</u> if, for every $\alpha : A \to B$ in $\mathcal{M}$, the induced mapping

$$\alpha^* : \mathrm{Hom}_{\mathbb{C}}(B,C) \to \mathrm{Hom}_{\mathbb{C}}(A,C)$$

is surjective. A morphism $\alpha : A \to B$ is called <u>$\mathcal{J}$ - proper</u> if, for every object $C \in \mathcal{J}$, the induced mapping α^* is surjective.

A couple $(\mathcal{M}, \mathcal{J})$ consisting of a class $\mathcal{M}$ of morphisms and a class $\mathcal{J}$ of objects in $\mathbb{C}$ is called an <u>injective structure</u> [12] in $\mathbb{C}$ if

(1) $\mathcal{J}$ is the class of all $\mathcal{M}$ - injective objects,

(2) $\mathcal{M}$ is the class of all $\mathcal{J}$ - proper morphisms,

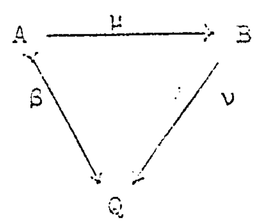
and

(3) for every object $A \in \mathbb{C}$, there exist $Q \in \mathcal{J}$ and $\alpha : A \to Q$ in $\mathcal{M}$.

An object A is called a <u>retract</u> of B if there exist $\alpha : A \to B$ and $\beta : B \to A$ such that $\beta\alpha = \iota_A$. Let the class of all retracts of a class $\mathcal{J}$ of objects be denoted by $R\mathcal{J}$.

<u>Proposition 1</u>: A category $\mathbb{C}$ has enough injectives if, and only if, the couple $(\mathcal{M}, \mathcal{J})$, where $\mathcal{M}$ is the class of all monomorphisms, $\mathcal{J}$ the class of all injectives, is an injective structure.

<u>Proof</u>: Assume that $\mathbb{C}$ has enough injectives. Let $\mathcal{M}$ be the class of all monomorphisms and $\mathcal{J}$ the class of all injective objects. Clearly (1) is satisfied. Furthermore $\mathcal{M}$ is clearly contained in the class of all $\mathcal{J}$- proper morphisms. Now let $\mu : A \to B$ be any $\mathcal{J}$- proper morphism. Since $\mathbb{C}$ has enough injectives, there exist $Q \in \mathcal{J}$ and $\beta : A \rightarrowtail Q$. By the injectivity of Q, there is a morphism $\nu : B \to Q$ making the following diagram



commutative. This proves that μ is monomorphic and hence $\mathcal{M}$ contains the class of all $\mathcal{J}$- proper morphisms. (3) is clearly true. Thus $(\mathcal{M}, \mathcal{J})$ is an injective structure. The converse is obvious.

Thus we see that the concept of a category with an injective structure is actually a generalization of the usual concept of a category with enough injective objects. Other examples of injective structures are: the couple $(\mathcal{M}, \mathcal{J})$ where $\mathcal{M}$ is the class of all morphisms and $\mathcal{J}$ the class consisting of all zero objects in $\mathbb{C}$, or where $\mathcal{M}$ is the class of all <u>retraction</u> morphisms, (a morphism $\alpha : A \to B$ is a retraction morphism if there exists a morphism $\beta : B \to A$ such that $\beta\alpha = \iota_A$), and $\mathcal{J}$ the class of all objects in $\mathbb{C}$.

<u>Proposition 2</u>: A couple $(\mathcal{M}, R\mathcal{I})$, where $R\mathcal{I}$ is the class of all retracts of objects in $\mathcal{I}$, is an injective structure in $\mathbb{C}$ if, and only if,

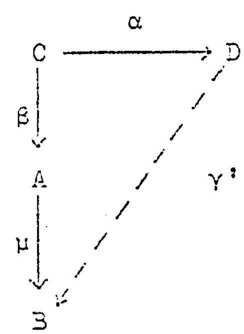
(1') $\mathcal{I}$ is contained in the class of all $\mathcal{M}$- injective objects,

(2') $\mathcal{M}$ contains the class of all $\mathcal{I}$ - proper morphisms,

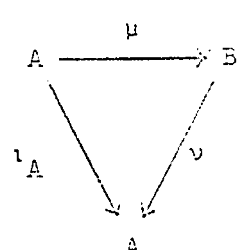
and

(3') for every $A \in \mathbb{C}$, there exists an $\alpha : A \to Q$ in $\mathcal{M}$, where $Q \in \mathcal{I}$.

<u>Proof</u>: Suppose (1') - (3') are satisfied. Let $A \in R\mathcal{I}$. Then there exist $B \in \mathcal{I}$ which is $\mathcal{M}$- injective and $\mu : A \to B$, $\nu : B \to A$ such that $\nu\mu = \iota_A$. Let $\alpha \in \mathcal{M}$. There exists γ' such that the following diagram



is commutative. Let $\gamma = \nu\gamma'$. Then we have $\beta = \nu\mu\beta = \nu\gamma'\alpha = \gamma\alpha = \alpha^*(\gamma)$. This proves α^* is surjective. Hence $R\mathcal{I}$ is contained in the class of all $\mathcal{M}$- injectives. Conversely, Let B be any $\mathcal{M}$ injective object. Then by (3'), there exists in $\mathcal{M}$ a $\mu : A \to B$ with B in $\mathcal{I}$. Since B is $\mathcal{M}$ - injective, there exists $\nu : B \to A$ such that commutativity holds in the following diagram

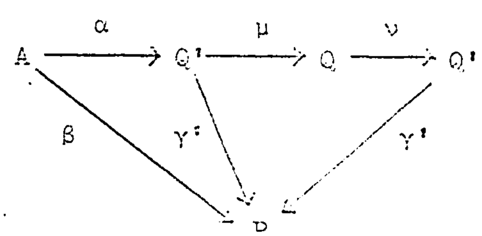


Hence A is a retract of B, i.e., $A \in R\mathcal{J}$. So $R\mathcal{J}$ is actually the class of all $\mathcal{M}$- injective objects. This proves (1).

Next, let $\alpha : C \to D$ be in $\mathcal{M}$ and A in $R\mathcal{J}$. In exactly the same way as in the last paragraph, we see that α^* is surjective. This shows that $\mathcal{M}$ is contained in the class of all $R\mathcal{J}$ - proper morphisms. The converse is also true for every $R\mathcal{J}$ - proper morphism is clearly an $\mathcal{J}$ - proper morphism. This proves (2).

Now let $A \in \mathfrak{C}$. Then there exist $Q \in \mathcal{J}$ and $\alpha : A \to Q$ in $\mathcal{M}$. Clearly $Q \in R\mathcal{J}$. This proves (3).

Conversely, suppose $(\mathcal{M}, R\mathcal{J})$ is an injective structure. (1') and (2') are clearly satisfied. Let $A \in \mathfrak{C}$. There exist an $\alpha : A \to Q'$ in $\mathcal{M}$, and $Q' \in R\mathcal{J}$. Hence there exist $\mu : Q' \to Q$, $\nu : Q \to Q'$, $Q \in \mathcal{J}$, such that $\nu\mu = \iota_Q$. It remains to be shown that $\mu\alpha \in \mathcal{M}$. Let $P \in R\mathcal{J}$ and $\beta : A \to P$. Then there exists a morphism γ' such that commutativity holds in the following diagram



Define $\gamma = \gamma'\nu$. Thus we have $\beta = \gamma'\nu\mu\alpha = \gamma\mu\alpha = (\mu\alpha)^*(\gamma)$.

<u>Theorem 3</u> (Transported structure theorem): Let S be a covariant functor from $\mathfrak{C}'$ to $\mathfrak{C}$, T a covariant functor from $\mathfrak{C}$ to $\mathfrak{C}'$, and $(\mathcal{M}, R\mathcal{J})$ an injective structure in $\mathfrak{C}$. If there is a surjective mapping,

$$\phi_{B',A} : \mathrm{Hom}_{\mathfrak{C}}(SB', A) \to \mathrm{Hom}_{\mathfrak{C}'}(B', TA)$$

for every $B' \in \mathfrak{C}'$ and every $A \in \mathcal{J}$, which is natural in both variables, then the couple $(\mathcal{M}', R\mathcal{J}')$ given by

$$\mathcal{M}' = S^{-1}\mathcal{M}, \quad \mathcal{J}' = T\mathcal{J}$$

is an injective structure in $\mathfrak{C}'$.

It is clear that when S is adjoint to T, all the conditions on S and T in the theorem are satisfied. We shall use this fact in the following section.

<u>Proof</u>: Let $A' \in \mathcal{J}'$. Then $A' = TA$ for some $A \in \mathcal{J}$. Suppose $\alpha : B' \to C'$ is in $\mathcal{M}'$. The following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathfrak{C}}(SB', A) & \xrightarrow{\ \sigma_{B',A}\ } & \mathrm{Hom}_{\mathfrak{C}'}(B', TA = A') \\
\downarrow (S\alpha)^{*} & & \downarrow \alpha^{*} \\
\mathrm{Hom}_{\mathfrak{C}}(SC', A) & \xrightarrow{\ \sigma_{C',A}\ } & \mathrm{Hom}_{\mathfrak{C}'}(C', TA = A')
\end{array}$$

shows that α^{*} is surjective. Thus $\mathcal{J}'$ is contained in the class of all $\mathcal{M}'$ - injective objects. The same diagram also shows that condition (2') is satisfied.

Let $A' \in \mathfrak{C}'$. Then $SA' \in \mathfrak{C}$, and therefore there exist $Q \in \mathcal{J}$ and $\mu : SA' \to Q \in \mathcal{M}$. Denote the image of μ under $\sigma_{A',Q}$ by $\mu' : A' \to TQ = Q'$. Clearly, $Q' \in \mathcal{J}'$. It remains to be shown that μ' is in $\mathcal{M}'$. For this purpose, let $C' \in \mathcal{J}'$ and $\alpha' : A' \to C'$. Then $C' = TC$ for some $C \in \mathcal{J}$. Since $\sigma_{A',C}$ is epimorphic, there exists $\alpha : SA' \to C$ such that $\sigma_{A',C}\alpha = \alpha'$. Thus there exists $\beta: Q \to C$ such that the following diagram

$$\begin{array}{ccc} SA' & \xrightarrow{\ \mu\ } & Q \\ & {\scriptstyle\alpha}\searrow \qquad \swarrow{\scriptstyle\beta} & \\ & C & \end{array}$$

is commutative. Hence the diagram

$$\begin{array}{ccc} A' & \xrightarrow{\ \mu'\ } & Q' \\ & {\scriptstyle\alpha'}\searrow \qquad \swarrow{\scriptstyle T\beta} & \\ & C' & \end{array}$$

commutes. Indeed, from the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}}(SA',Q) & \xrightarrow{\ \sigma_{A',Q}\ } & \mathrm{Hom}_{\mathbb{C}'}(A',TQ) \\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle (T\beta)_*} \\ \mathrm{Hom}_{\mathbb{C}}(SA'C) & \xrightarrow[\ \sigma_{A',C}\ ]{} & \mathrm{Hom}_{\mathbb{C}'}(A',TC) \end{array}$$

we have

$$T\beta\mu' = T\beta\, \sigma_{A',Q}\mu = \sigma_{A',C}\beta\mu = \sigma_{A',C}\alpha = \alpha'$$

<u>Proposition 4</u>: Let $(\mathcal{M}_i, R\mathcal{J}_i)_{i\in I}$ be a family of injective structures in $\mathbb{C}$. If $\mathbb{C}$ possesses infinite direct products, then the couple $(\mathcal{M}, R\mathcal{J})$ given by

$$\mathcal{M} = \bigcap_{i\in I}\mathcal{M}_i \quad , \quad \mathcal{J} = \{\prod_{i\in I} A_i \mid A_i \in \mathcal{J}_i\}$$

is another injective structure in $\mathbb{C}$.

Note that $\mathcal{M}$ is not empty for it contains at least all isomorphisms in $\mathbb{C}$.

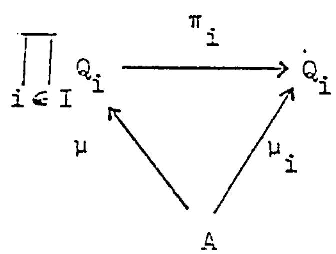
<u>Proof</u>: Let $A \in \mathcal{J}$. Then $A = \prod_{i\in I} A_i$, $A_i \in \mathcal{J}_i$. Let $\mu : B \to C \in \mathcal{M}$ Then, since $\mu : B \to C \in \mathcal{M}_i$ for every $i \in I$, $\mu^* : \mathrm{Hom}_{\mathbb{C}}(C,A_i) \to \mathrm{Hom}_{\mathbb{C}}(B,A_i)$,

is surjective. So is

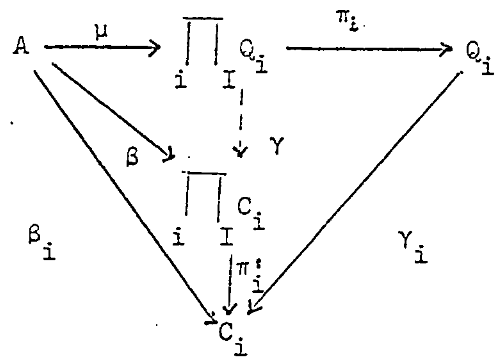
$$\mu^* : \mathrm{Hom}_{\mathbb{C}}(C, \prod_{i\in I} A_i) = \prod_{i\in I} \mathrm{Hom}_{\mathbb{C}}(C,A_i) \to \mathrm{Hom}_{\mathbb{C}}(B, \prod_{i\in I} A_i)$$

$$= \prod_{i\in I} \mathrm{Hom}_{\mathbb{C}}(B,A_i).$$

Thus (1') and (2') of proposition 2 are satisfied. Let $A \in C$. Then there exist $Q_i \in \mathcal{J}_i$ and $\mu_i : A \to Q_i$ in $\mathcal{M}_i$, for each $i \in I$. Thus there exists a unique morphism $\mu : A \to \prod_{i\in I} Q_i$ determined by the commutative diagrams



$i \in I$. It remains to be shown that μ is in $\mathcal{M}$. So let $C \in \mathcal{J}$ and $\beta : A \to C$ in $\mathbb{C}$. Then $C = \prod_{i\in I} C_i$. As all C_i are $\mathcal{M}$- injective, there exist $\gamma_i, i \in I$, making the following diagram



where π_i, π_i' are the projections, commutative. Thus there exists a unique morphism $\gamma : \prod_{i\in I} Q_i \to \prod_{i\in I} C_i$ making the triangle on the right hand side commutative. Since $\pi_i'\beta = \beta_i = \gamma_i\pi_i\mu = \pi_i'\gamma\mu$ for all $i \in I$, it follows that $\beta = \gamma\mu = \mu^*(\gamma)$ because of the universal mapping property of $\prod_{i\in I} C_i$. Hence μ is in $\mathcal{M}$.

3.2. Projective structures.

Projective structures are obtained from injective structures by dualization. Let $\mathfrak{C}$ be a category, $\mathcal{M}$ a class of morphisms in $\mathfrak{C}$, and $\mathcal{P}$ a class of objects in $\mathfrak{C}$. Then an object $C \in \mathfrak{C}$ is called $\mathcal{M}$-projective if for every $\mu : A \to B$ in $\mathcal{M}$, the induced mapping

$$\mu_* : \mathrm{Hom}_{\mathfrak{C}}(C,A) \to \mathrm{Hom}_{\mathfrak{C}}(C,B)$$

is surjective. A morphism $\mu : A \to B$ in $\mathfrak{C}$ is called $\mathcal{P}$- proper if for every $C \in \mathcal{P}$, μ_* is surjective.

A couple $(\mathcal{M}, \mathcal{P})$, $\mathcal{M}$ being a class of morphisms, $\mathcal{P}$ a class of objects in $\mathfrak{C}$, is a projective structure in $\mathfrak{C}$ if they satisfy the following conditions:

$(1)^*$ $\mathcal{P}$ is the class of all $\mathcal{M}$- projective objects,

$(2)^*$ $\mathcal{M}$ is the class of all $\mathcal{P}$- proper morphisms,

and

$(3)^*$ for every object $A \in \mathfrak{C}$, there exist $F \in \mathcal{P}$ and a $\mu : F \to A$ in $\mathcal{M}$.

Proposition 1^*: $\mathfrak{C}$ has enough projectives if, and only if, the couple $(\mathcal{M}, \mathcal{P})$ where $\mathcal{M}$ is the class of all epimorphisms, $\mathcal{P}$ the class of all projective objects, is a projective structure.

Proposition 2^*: $(\mathcal{M}, R\mathcal{P})$ is a projective structure iff

$(1')^*$ $\mathcal{P}$ is contained in the class of all $\mathcal{M}$- projective objects,

$(2')^*$ $\mathcal{M}$ contains the class of all $\mathcal{P}$- proper morphisms,

and

$(3')^*$ for every $A \in \mathfrak{C}$, there exists $\mu : F \to A$ in $\mathcal{M}$, with $F \in \mathcal{P}$.

Theorem 3^*: Let S be a covariant functor from $\mathfrak{C}'$ to $\mathfrak{C}$, T a covariant functor from $\mathfrak{C}$ to $\mathfrak{C}'$, and $(\mathcal{M}', R\mathcal{P}')$ a projective structure in

$\mathbb{C}'$. If there is a surjective mapping,

$$\eta_{A',B} : \mathrm{Hom}_{\mathbb{C}'}(A',TB) \to \mathrm{Hom}_{\mathbb{C}}(SA',B)$$

for each $A' \in \mathcal{P}'$ and each $B \in \mathbb{C}$, which is natural in both variables, then the couple $(\mathcal{M}, R\mathcal{P})$ given by

$$\mathcal{M} = T^{-1}\mathcal{M}' \quad , \quad \mathcal{P} = S\mathcal{P}',$$

is a projective structure in $\mathbb{C}$.

Proposition 4^* : Let $(\mathcal{M}_i, R\mathcal{P}_i)_{i\in I}$ be a family of projective structures in $\mathbb{C}$. If $\mathbb{C}$ possesses infinite direct sums, then the couple $(\mathcal{M}, R\mathcal{P})$ given by $\mathcal{M} = \bigcap_{i\in I} \mathcal{M}_i$, $\mathcal{P} = \{\bigoplus_{i\in I} A_i \mid A_i \in \mathcal{P}_i\}$,

is another projective structure.

Note that there are two more dualizations of theorem 3, viz., if in theorem 3, we replace T by a contravariant functor T^* then we can transport projective structures into injective structures, and if, in theorem 3^*, we replace S by a contravariant functor S^*, then one can transport injective structures into projective structures.

Remark: Definitions and results of this section were set forth in [10], perhaps with slight differences. Theorem 3 is just a reformulation of the crucial idea of the Eckmann-Schopf proof for the existence of enough modules [2].

Section IV

EXISTENCE OF ENOUGH INJECTIVE SHEAVES

In this section using the results of section III we want to show that certain categories of sheaves have enough injectives.

Lemma 1 [1, p. 221]. Let F be an abelian sheaf over a topological space X. If $\varinjlim_{U \ni x} FU = 0$, and $s \in FU$, $s \neq 0$, then there exists an open neighbourhood V of x, $V \subset U$, such that $\rho_{UV}(s) = 0$.

Proof: Recall $\varinjlim_{U \ni x} FU = \bigoplus_{U \ni x} FU / Q = 0$, where Q is the smallest subgroup of $\bigoplus_{U \ni x} FU$ containing all elements of $\bigoplus_{U \ni x} FU$ of the form

$$\psi_W \rho_{UW}(t) - \psi_U(t), \qquad t \in FU,\ U \supset W \supset \{x\},$$

ψ_W being the w-th injection. As the image of $s \in FU$ in $\varinjlim_{U \ni x} FU$ is zero, $\psi_U(s)$ can be expressed in the form

$$\sum_{W \supset Z} (\psi_Z \rho_{WZ}(t) - \psi_W(t)) .$$

Let V be an open neighbourhood of x contained in all W,Z appearing in this sum. We have

$$\psi_V \rho_{UV}(s) = (\psi_V \rho_{UV}(s) - \psi_U(s)) + \psi_U(s)$$

and $\psi_Z \rho_{WZ}(t) - \psi_W(t) = \psi_V \rho_{WV}(t) - \psi_W(t) + \psi_V \rho_{ZV}(-\rho_{WZ}(t)) - \psi_Z(-\rho_{WZ}(t))$.

Hence $\psi_V \rho_{UV}(s)$ can be expressed as a sum

$$\psi_V \rho_{UV}(s) = \sum_W (\psi_V \rho_{WV}(t) - \psi_W(t)).$$

Since relations in $\bigoplus_{U \ni x}$ FU result in individual summands, it follows for $W \neq V$, $\psi_W(t) = 0$; hence $t = 0$ since ψ_W is monomorphic, and hence $\psi_V \rho_{WV}(t) = 0$. For $W = V$, it is clear that $\psi_V \rho_{VV}(t) - \psi_V(t) = 0$. So $\psi_V \rho_{UV}(s) = 0$. Hence $\rho_{UV}(s) = 0$ because ψ_V is monomorphic.

Lemma 2: Let F be as in lemma 1. If $\varinjlim_{U \ni x} FU = 0$ for every $x \in X$, then $FU = 0$ for all open subsets U of X.

Proof: Assume the contrary, i.e., there exists a U such that $FU \neq 0$. Let $s \in FU$, $s \neq 0$. By lemma 1, for every $x \in U$, there exists an open neighbourhood V(x) of x contained in U such that $\rho_{UV(x)}(s) = 0$. Clearly $U = \bigcup_{x \ni U} V(x)$ and $\rho_{UV(x)}(s) = \rho_{UV(x)}(0) = 0$ for all V(x). This contradicts AS 1). So $FU = 0$.

Theorem 3: The category $\mathbb{F}$ of abelian sheaves has enough injectives.

Proof: For every $x \in X$, define $S_x : \mathbb{F} \to \mathbb{A}$ to be the functor given by $S_x F = \varinjlim_{U \ni x} FU$, $F \in \mathbb{F}$. This functor is adjoint to the embedding functor $T_x : \mathbb{A} \to \mathbb{F}$ which assigns to every $A \in \mathbb{A}$, the sheaf F_A such that $F_A U = A$ if $x \in U$, $F_A U = 0$ if $x \notin U$. The natural equivalence

$$\eta_{F,A} : \mathrm{Hom}_{\mathbb{A}}(S_x F, A) \to \mathrm{Hom}_{\mathbb{F}}(F, F_A)$$

is given by $\eta_{F,A}(\alpha) = \begin{cases} \alpha\lambda_U & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$ for every $\alpha \in \mathrm{Hom}_{\mathbb{A}}(S_x F, A)$. (Here λ_U denotes the morphism from FU to $\varinjlim$ FU as in section II). Since $\mathbb{A}$ has enough injectives [11, p. 93] the class $\mathcal{J}$ of all injectives and the class $\mathcal{M}$ of all monomorphisms form an injective structure (Proposition III.1). So does $(\mathcal{M}, R\mathcal{J})$. By theorem III.3, $(\mathcal{M}_x, R\mathcal{J}_x)$ given by $\mathcal{M}_x = S_x^{-1}\mathcal{M}$, $\mathcal{J}_x = T_x\mathcal{J}$, is an injective structure in $\mathbb{F}$. Hence by Proposition III.4, $(\mathcal{M}', R\mathcal{J}')$, where $\mathcal{M}' = \bigcap_{x \in X} \mathcal{M}_x$, $\mathcal{J}' = \{\prod F_x \mid F_x \in \mathcal{J}_x\}$, is another injective structure in $\mathbb{F}$.

Let $\mu : F_1 \to F_2$ be a monomorphism in $\mathbb{F}$. Then, $\mathbb{A}$ being a Grothendieck category implies that $S_x\mu : S_xF_1 \to S_xF_2$ is a monomorphism for all $x \in X$. Conversely, suppose $S_x\mu : S_xF_1 \to S_xF_2$ is monomorphic for all $x \in X$. Let K be the kernel of $\mu : F_1 \to F_2$. K is a sheaf. Since $\varinjlim$ is exact $S_xK \to S_xF_1 \xrightarrow{S_x\mu} S_xF_2$ is exact. Thus $S_xK = 0$ for all $x \in X$. So by lemma 2, $K = 0$ and hence μ is a monomorphism. Therefore $\mathcal{M}'$ consists of all monomorphisms of $\mathbb{F}$ and the assertion is thus proved.

<u>Theorem 4</u>: The category $\mathbb{F}$ of sheaves over a topological space X with values in a Grothendieck category $\mathbb{C}$ which has a small generator G and arbitrary direct products has enough injectives.

<u>Proof</u>: The same adjoint functors are defined as in the last proof and in the same way we obtain the injective structure $(\mathcal{M}', R\mathcal{J}')$. It then remains to be shown that $\mathcal{M}'$ actually consists of all monomorphisms in $\mathbb{F}$. So let $\mu : F_1 \to F_2$ be a monomorphism in $\mathbb{F}$. It is clear that the exactness of $\varinjlim$ implies that $S_x\mu$ is monomorphic for all $x \in X$. Conversely, let $S_x\mu : S_xF_1 \to S_xF_2$ be monomorphic for all $x \in X$.

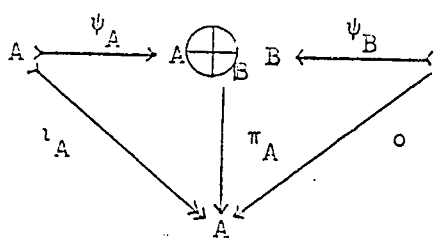
Let $K = \operatorname{Ker}\mu$. K is a sheaf. Then $S_x K = \operatorname{Ker} S_x\mu$ is zero and hence $\operatorname{Hom}_{\mathbb{C}}(G, S_x K) = 0$ for every $x \in X$. Since G is small we have $\varinjlim_{U \ni x} \operatorname{Hom}_{\mathbb{C}}(G, KU) = 0$ for all $x \in X$. This implies that $\operatorname{Hom}_{\mathbb{C}}(G, KU) = 0$ and hence $KU = 0$ because G is a generator. Thus μ is monomorphic and $\mathcal{M}'$ is the class of all monomorphisms.

Corollary 5: The category of sheaves with values in the category $\mathbb{M}_R^o$ of all unitary right R-modules has enough injectives.

Proof: R is a small generator and $\mathbb{M}_R^o$ has enough injectives [10].

Lemma 6: Let $\mathbb{B}$ be a full abelian subcategory of the category $\mathbb{M}_R$ of left R-modules. Then the finite direct sums formed in $\mathbb{B}$ are the same as those formed in $\mathbb{M}_R$.

Proof: It suffices to prove this for the direct sum of two left R-modules. Let $A, B \in \mathbb{B}$, $A \bigoplus_{\mathbb{B}} B$ their direct sum in $\mathbb{B}$ and ψ_A, ψ_B the injections of A, B into $A \bigoplus_{\mathbb{B}} B$. Since $\mathbb{B}$ is abelian, $A \bigoplus_{\mathbb{B}} B$ is isomorphic to the direct product of $A \times_{\mathbb{B}} B$. The projections π_A, π_B are determined by the following commutative diagrams:



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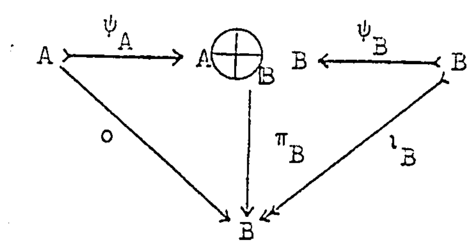
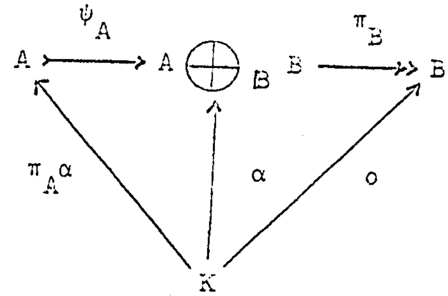
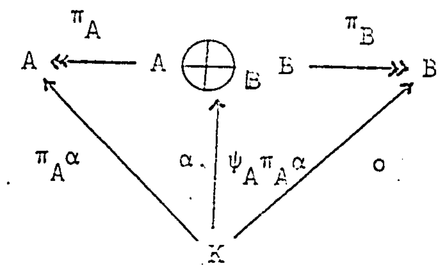


We want to show first that $\psi_A : A \rightarrowtail A \bigoplus_{\mathbb{B}} B$ is the kernel of π_B: that $A \xrightarrow{\psi_A} A \bigoplus_{\mathbb{B}} B \xrightarrow{\pi_B} B$ is zero follows from the second diagram above. Let $\alpha : K \to A \bigoplus_{B} B$ such that $\pi_B \alpha = 0$. Then $\pi_A \alpha : K \to A$ can make the following diagram

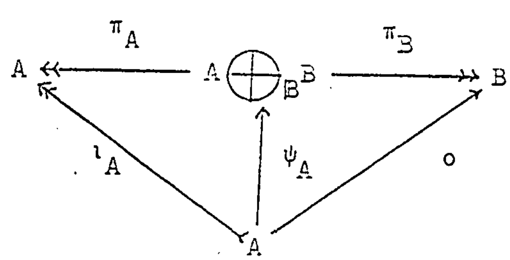


commutative. Indeed $\psi_A \pi_A \alpha$, α both make the following diagram

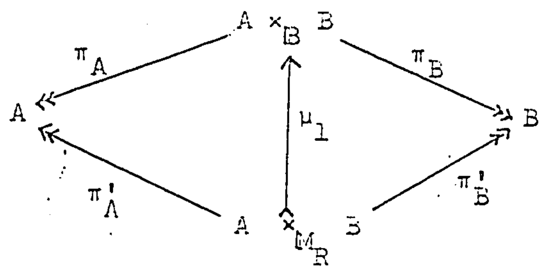


commutative. The uniqueness of $\pi_A \alpha$ is clear for if $\alpha' : K \to A$ is another morphism such that $\psi_A \alpha' = \alpha$, then $\alpha' = \pi_A \psi_A \alpha' = \pi_A \alpha$. Thus $\psi_A : A \to A \bigoplus_B B$ is the kernel of π_B.

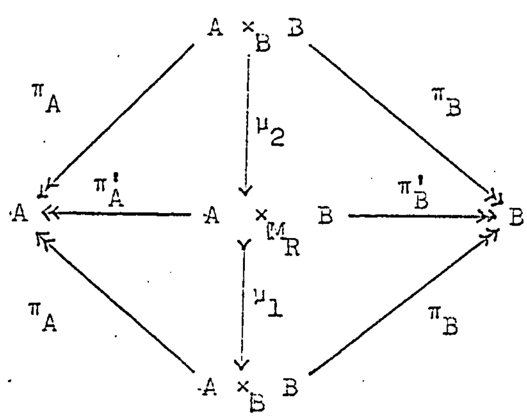
Since ψ_A, ψ_B are monomorphisms, we can consider A,B as submodules of $A \bigoplus_B B$. Let C be the <u>set theoretic</u> intersection of A,B in $A \bigoplus_B B$. As element chasing is possible in B, it is easily checked that C is isomorphic to the categorical intersection which is the kernel of the composition homorphism $A \xrightarrow{\psi_A} A \bigoplus_B B \to A \bigoplus_B B / B$ (cf. p. 10). Hence $C \in B$. Now assume that $C \neq 0$. Consider the following commutative diagram



Let $c \in C \subset A$, $c \neq 0$. Then since $\pi_B \psi_A(c) = 0$, $\psi_A(c) = c'$ is in the kernel of π_B, i.e., belongs to A. Hence $c' = c$ and $\pi_A(c') = c$. The same reasoning shows $\pi_B(c') = c$. This contradicts the commutativity of the diagram. Hence $C = 0$. Define $\mu_1 : A \times_{\mathbb{M}_R} B \to A \times_{\mathbb{B}} B$ by $\mu_1(a,b) = a + b$ for $(a,b) \in A \times_{\mathbb{M}_R} B$. Clearly commutativity holds in the diagram



where π_A', π_B' are the projections from $A \times_{\mathbb{M}_R} B$, the direct product formed in $\mathbb{M}_R$, onto A,B respectively. Now consider the commutative diagram



where μ_2 is the homomorphism determined by π_A, π_B. By the universal mapping property of the direct product, it follows that $\mu_1 \mu_2 = \iota_{A \times_{\mathbb{B}} B}$.

Thus, μ_1 is epimorphic and hence isomorphic.

Note that in the proof we did not use the fact that the set theoretic intersection is in $\mathcal{B}$. We stated this fact in the proof because we need it later in lemma 9.

Lemma 7 [13, p. 16]: A Grothendieck category $\mathbb{C}$ with a projective generator G can be embedded fully and exactly into the category M_R^o of right R-modules by the functor $Hom_{\mathbb{C}}(G,-)$, where $R = Hom_{\mathbb{C}}(G,G)$.

This lemma is due to Mitchell. In fact, this is merely a part of his representation theorem (loc. cit.).

Lemma 8 [4, p. 355]: If $\mathbb{C}$ is a Grothendieck category with a generator, A a noetherian object of $\mathbb{C}$, and $(B_i)_{i \in I}$ a direct family of subobjects of $\mathcal{B}$, then the canonical homomorphism

$$\psi_A : \bigcup_{i \in I} Hom_{\mathbb{C}}(A,B_i) \to Hom_{\mathbb{C}}(A, \bigcup_{i \in I} B_i)$$

is an isomorphism.

The above lemma is due to Gabriel. (loc. cit.)

Lemma 9: If $\mathbb{C}$ is a Grothendieck category with a noetherian projective generator G, then the functor $Hom_{\mathbb{C}}(G,-)$ is a full exact embedding of $\mathbb{C}$ onto M_R^o.

Proof: In view of lemma 7, it remains to be shown that $Hom_{\mathbb{C}}(G,-)$ is surjective on the class of objects in $\mathbb{C}$. Consider A_i, $i \in I$, as subobjects of $\bigoplus_{i \in I} A_i$, we have

$$\bigcup_{i\in I} \mathrm{Hom}_{\mathbb{C}}(G,A_i) \rightarrowtail\!\!\twoheadrightarrow \mathrm{Hom}_{\mathbb{C}}(G, \bigoplus_{i\in I} A_i)$$

by lemma 8. As in the proof of lemma 6, we know that the groups $\mathrm{Hom}_{\mathbb{C}}(G,A_i)$, considered as subobjects in $\mathrm{Hom}_{\mathbb{C}}(G, \bigoplus_{i\in I} A_i)$, intersect each other trivially in $\mathrm{Hom}_{\mathbb{C}}(G, \bigoplus_{i\in I} A_i)$. Thus $\bigcup_{i\in I} \mathrm{Hom}_{\mathbb{C}}(G,A_i)$, the image of the homomorphism $\bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(G,A_i) \to \mathrm{Hom}_{\mathbb{C}}(G, \bigoplus_{i\in I} A_i)$, is isomorphic to $\bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(G,A_i)$.

Let M be any right R-module. There exists an epimorphism $\theta_{M,J} : \bigoplus_{i\in J} \mathrm{Hom}_{\mathbb{C}}(G,G) \to M$. Let Q be the kernel of $\theta_{M,J}$. There exists again an epimorphism $\theta_{Q,K} : \bigoplus_{i\in K} \mathrm{Hom}_{\mathbb{C}}(G,G) \to Q$. $\theta_{Q,K}$ can be extended in an obvious way to

$$\overline{\theta}_{Q,K} : \bigoplus_{i\in K} \mathrm{Hom}_{\mathbb{C}}(G,G) = \mathrm{Hom}_{\mathbb{C}}(G, \bigoplus_{i\in K} G) \to \bigoplus_{i\in J} \mathrm{Hom}_{\mathbb{C}}(G,G) = \mathrm{Hom}_{\mathbb{C}}(G, \bigoplus_{i\in J} G).$$

Thus Q, being the image of $\overline{\theta}_{Q,K}$, belongs to the image of $\mathbb{C}$ under $\mathrm{Hom}_{\mathbb{C}}(G,-)$. This image of $\mathbb{C}$ can be easily checked to be a subcategory of $M_R o$. Hence M has an isomorphic copy in this subcategory. Thus the proof is complete.

From lemma 9 together with Corollary 5, we have:

<u>Theorem 10</u>: The category of sheaves with values in a Grothendieck category with a noetherian projective generator has enough injectives.

<u>Corollary 11</u>: A Grothendieck category with a noetherian projective generator possesses arbitrary direct products.

<u>Corollary 12</u>: In a Grothendieck category a noetherian projective generator is a small generator.

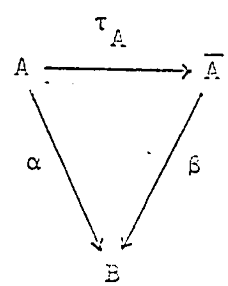
<u>Corollary 13</u>: Any full Grothendieck subcategory of the category $\mathbb{M}_R o$ of right R-modules over a noetherian ring R containing the ring R is equivalent to $\mathbb{M}_R o$, i.e., every right R-module in $\mathbb{M}_R o$ has an isomorphic copy in the subcategory.

Section V

REFLECTIVE SUBCATEGORIES

The theorem contained in this section was obtained when we were looking for a proof for Heller and Rowe's theorem. It reduces the important question of whether a full subcategory is reflective to a question concerning the finitely generated objects only.

[illegible] $\mathbb{C}'$ be a subcategory of [illegible] category $\mathbb{C}$, and [illegible] object of $\mathbb{C}$. A pair $(\overline{A},\tau_A)$, $\overline{A}\in\mathbb{C}'$ and $\tau_A : A \to \overline{A}$, is called a <u>reflection</u> of A in $\mathbb{C}'$ if $(\overline{A},\tau_A)$ satisfies the universal mapping property in $\mathbb{C}'$, i.e., for any $B\in\mathbb{C}'$, and $\alpha : A \to B$, there exists a unique morphism β in $\mathbb{C}'$ such that commutativity holds in the diagram



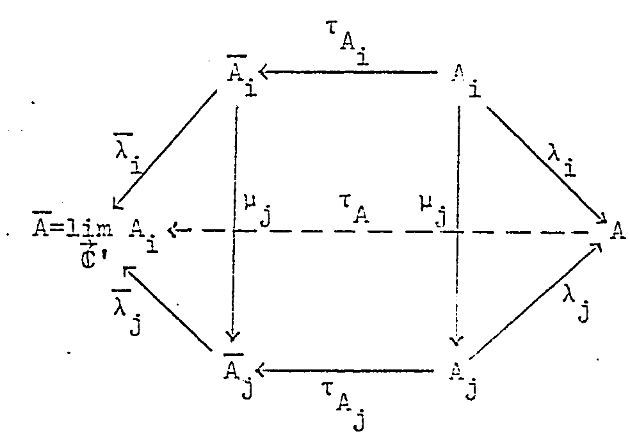
If every object in $\mathbb{C}$ has a reflection (which is easily seen to be unique up to isomorphisms) in $\mathbb{C}'$, then $\mathbb{C}'$ is called a <u>reflective subcategory</u> of $\mathbb{C}$. It is easy to check that the function, which assigns to every $A\in\mathbb{C}$ the object $\overline{A}$ of its reflection $(\overline{A},\tau_A)$ and to each morphism the induced morphism, is a covariant functor, usually called a <u>reflector</u> of $\mathbb{C}$ with respect to $\mathbb{C}'$. Moreover, this functor is adjoint to the embedding from $\mathbb{C}'$ to $\mathbb{C}$. <u>Coreflections</u> of objects, <u>coreflective subcategories</u> and <u>coreflectors</u> of categories are defined dually.

An object in an abelian category with a generator G is <u>finitely generated</u> if it is a quotient object of $\bigoplus_{i \in I} G$ for some finite indexing set I.

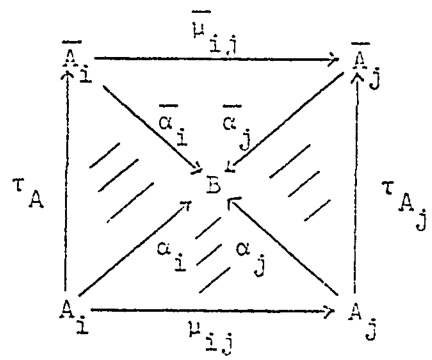
<u>Lemma 1</u> [13 p. 13]. Let $\mathbb{C}$ be a Grothendieck category with a projective generator G. Then every object is the direct limit of the family of its subobjects directed by inclusion.

<u>Theorem 2</u>: Let $\mathbb{C}$ be a Grothendieck category with a projective [illegible] G. If $\mathbb{C}'$ is any full subcategory of $\mathbb{C}$ with direct limits such that every finitely generated object in $\mathbb{C}$ has a reflection in $\mathbb{C}'$, then $\mathbb{C}'$ is reflective in $\mathbb{C}$.

<u>Proof</u>: For every finitely generated object A in $\mathbb{C}$, we denote its reflection to be the pair $(\overline{A}, \tau_A)$. Let A be any object of $\mathbb{C}$. Then by lemma 1, we have $A = \varinjlim A_i, (A_i)_{i \in I}$ being the family of its finitely generated subobjects. In the following diagram



where $\varinjlim_{\mathbb{C}'} A_i$ means that the limit is taken in $\mathbb{C}'$, we see that there exists a unique morphism from $\tau_A : A \to \overline{A}$ such that the divided diagram is commutative in the upper part and the lower part. We want to show that the couple $(\overline{A}, \tau_A)$ is the reflection of A. To this end, let B be any object in $\mathbb{C}'$ and $\alpha : A \to B$ a morphism. By setting $\alpha_i = \alpha\lambda_i$, we have the following diagram

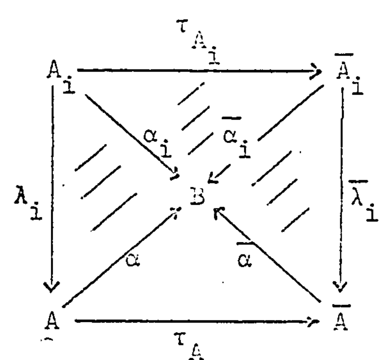


where $\overline{\mu}_{ij}$, $\overline{\alpha}_i$, $\overline{\alpha}_j$ are the induced morphisms in $\mathbb{C}'$. Thus the diagram is commutative in the shaded regions. Taking commutativity of the square into account, we have

$$\overline{\alpha}_i \tau_{A_i} = \alpha_j \mu_{ij} = \overline{\alpha}_j \tau_{A_j} \mu_{ij} = \overline{\alpha}_j \mu_{ij} \tau_{A_i}.$$

By the uniqueness of $\overline{\alpha}_i$, we have $\overline{\alpha}_i = \overline{\alpha}_j \overline{\mu}_{ij}$ proving the commutivity in the unshaded region.

To complete the proof we need to show that commutativity in the unshaded region in the diagram



follows from the commutativity in the shaded regions. Indeed, we have

$$\alpha\lambda_i = \alpha_i = \overline{\alpha}\tau_{A_i} = \overline{\alpha\lambda_i} \quad \tau_{A_i} = \overline{\alpha}\tau_A\lambda_i$$

which together with the uniqueness of α gives $\alpha = \bar{\alpha}\tau_A$.

The uniqueness of $\bar{\alpha}$ can be verified in a similar way. Let $\beta\tau_A = \alpha$. From the last diagram, we have

$$\bar{\alpha}_i \tau_{A_i} = \alpha_i = \alpha\lambda_i = \beta\tau_A\lambda_i = \beta\bar{\lambda}_i\tau_{A_i}.$$

By the uniqueness of $\bar{\alpha}_i$, we then have $\bar{\alpha}_i = \beta\bar{\lambda}_i$ and again by the universal mapping property of direct limits, we have $\beta = \bar{\alpha}$.

SUMMARY

We have proved the results mentioned in the introduction by using the concept of an injective structure and a theorem on transportation of such structures from one category into another (Theorem 3 in section III). This theorem is due to J.M. Maranda [12]; however we have given here a new and very simple proof (cf. also [10]). The first result (Theorem 4 in section IV) is new; the second (Theorem 10 in section IV) is contained as a special case in a theorem of Heller and Rowe [8] (cf. below, also remark on p. 14).

The "transported structure" theorem can be also applied to problems other than those in section IV. For instance, it leads to a very simple proof for that the category of modules over a sheaf of rings has enough injectives [10].

The result of Theorem 3 in section IV is well-known but the algebraic proof of Lemma 2 is new. One might conjecture that such a proof can be carried over to any abelian category but this seems to be very difficult. Therefore we had to impose some additional conditions on the value-category to obtain Theorem IV. 4. However, the question of deciding the necessary condition for Lemma 2 to hold is yet open.

Heller and Rowe have proved that the category of sheaves with values in a Grothendieck category possessing arbitrary direct products

and a projective generator has enough injectives. We tried in vain to give a proof using the "transported structure" theorem. The failure makes us think that there are probably cases where the category of sheaves and the value-category both have enough injectives but no functors exist as that in the "transported structure" theorem.

In the effort to produce a proof for Heller and Rowe's theorem, we obtained Lemma 6 of section IV and Theorem 2 of section V. We did not succeed, since we were not able to obtain information on infinite direct sums formed in a full subcategory without additional condition such as the one in Lemma 9 of section IV. However, Theorem 2 seems to be of importance in axiomatizing in categorical terms the category of modules. We did not pursue here this question any further.

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