RECTILINEAR CROSSING NUMBER OF GRAPHS EXCLUDING A SINGLE-CROSSING GRAPH AS A MINOR

by

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Abstract

The crossing number of a graph G is the minimum number of crossings in any drawing of G in the plane. The rectilinear crossing number of G is the minimum number of crossings in any straight-line drawing of G.

The Fáry-Wagner theorem states that planar graphs have rectilinear crossing number zero. By Wagner's theorem, that is equivalent to stating that every graph that excludes K_5 and $K_{3,3}$ as minors has rectilinear crossing number 0. We are interested in discovering other proper minor-closed families of graphs which admit strong upper bounds on their rectilinear crossing numbers. Unfortunately, it is known that the crossing number of $K_{3,n}$, with $n \geq 1$, which excludes K_5 as a minor, is quadratic in n, more specifically $\Omega(n^2)$. Since every n-vertex graph in a proper minor closed family has $O(n)$ edges, the rectilinear crossing number of all such graphs is trivially $O(n^2)$. In fact, it is not hard to argue that $O(n)$ bound on the crossing number is the best one can hope for for general enough proper minor-closed families of graphs and that to achieve $O(n)$ bounds, one has to both exclude a minor and bound the maximum degree of the graphs in the family.

In this thesis, we do that for bounded degree graphs that exclude a single-crossing graph as a minor. A single-crossing graph is a graph whose crossing number is at most one. The main result of this thesis states that every graph G that does not contain a single-crossing graph as a minor has a rectilinear crossing number $O(\Delta n)$, where G has n vertices and maximum degree Δ. This dependence on n and Δ is best possible. Note that each planar graph is a single-crossing graph, as is the complete graph K_5 and the complete bipartite graph $K_{3,3}$. Thus, the result applies to K_5-minor-free graphs, $K_{3,3}$-minor free graphs, as well as to bounded treewidth graphs. In the case of bounded treewidth graphs, the result improves on the previous best known bound of $O(\Delta^2 \cdot n)$ by Wood and Telle [New York Journal of Mathematics, 2007]. In the

case of $K_{3,3}$-minor free graphs, our result generalizes the result of Dujmović, Kawarabayashi, Mohar and Wood [SCG 2008].

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1 Introduction

Before introducing the topic of this thesis, we must first establish important and relevant graph theory definitions.

1.1 *Definitions*

A graph G is defined as a pair $(V(G), E(G))$ where $V(G)$ is a set of elements called *vertices* and $E(G)$ is a (multi)set of unordered pairs of vertices. Elements of $E(G)$ are called *edges*. We denote by $|G|$ the number of vertices in G, that is $|V(G)|$, and by $||G||$ the number of edges of G, that is $|E(G)|$. An edge $e = \{v, w\}$ has two *endpoints*, v and w. For simplicity, we denote this edge vw. An edge e is *incident* to a vertex v if v is an endpoint of e. Two vertices are *adjacent* if they are both incident to the same edge. The *neighbours* of a vertex $v \in V(G)$ are the vertices adjacent to v in G. The set of neighbours of v in G is denoted $N_G(v)$. The *degree* of a vertex v is the number of edges incident to v. The maximum degree of a graph G, denoted by $\Delta(G)$, is the largest degree of a vertex in G.

A collection of two or more edges that have exactly the same pair of endpoints are called *parallel* edges. A *loop* is an edge whose endpoints are the same vertex. A graph is *simple* if it has no loops nor parallel edges. A graph is a *multigraph* if it has no loops, but may have parallel edges. A graph is *undirected* if the edges can be traversed in either direction.

In this thesis, all the graphs discussed are assumed to be simple, undirected and finite unless specified otherwise.

A *path* is a sequence of alternating vertices and edges with no repeated vertices. A graph is *connected* if there is a path between every pair of vertices. A *cycle* is a path with an edge joining the last vertex of the path to its first. A *tree* is a connected simple graph that has no cycles.

A graph $H = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and every edge in E' is also an edge in E. Furthermore, if every edge of G with both endpoints in V' is also an edge of H, then H is called an *induced subgraph* of G. We denote n induced subgraph of G with vertex set $V' \subseteq V$ by $G[V']$. $G[V']$ is commonly referred to as the graph induced in G by V'.

The *complete* graph is a graph with an edge between every pair of its vertices. K_n denotes the complete graph on n vertices. A *clique* C in a graph G is a subgraph of G that is a complete graph. An *independent set* in a graph G is any subset S of vertices of G with the property that no pair of vertices of S is adjacent in G.

The *complete bipartite graph* is a graph G with vertices that can be partitioned into sets A and B such that $G[A]$ is an independent set, $G[B]$ is an independent set, and for every pair of vertices $v \in A$ and $w \in B$, vw is an edge in G. If the size of A is denoted by a and the size of B by b, G is denoted as $K_{a,b}$.

A graph G is *isomorphic* to a graph H if there exists a one-to-one mapping f from the vertex set $V(G)$ of G to the vertex set $V(H)$ of H such that vw, with $v, w \in V(G)$, is an edge in G if and only if $f(v)f(w)$ is an edge in H.

A *drawing* of a graph $G = (V, E)$ is a representation of G where vertices of G are represented by distinct points in the plain, each edge of G is represented by a simple closed curve between its endpoints, no edge intersects vertices of G other than its own two endpoints, and no three edges intersect in one point unless all three share a common endpoint. A *rectilinear drawing* of a graph G is a drawing of G where each edge is represented by a straight line-segment. We say that a set of points P is in *general position* if no three points lie on one line and if no three line-segments between pairs of points in P intersect in one point unless all three share a common endpoint. For the ease of presentation we will add to the definition of rectilinear drawings a condition that all endpoints of G are in general position.

In Figure 1 below, we can see examples of drawings of simple, undirected and finite graphs.

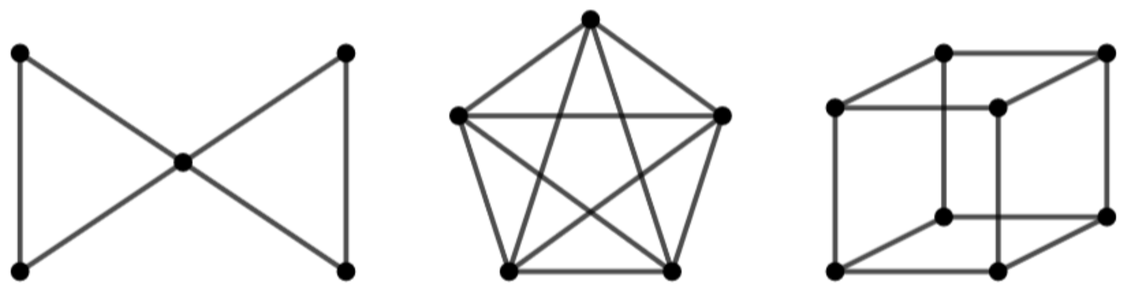


Figure 1: Examples of simple, undirected and finite graphs.

A pair of edges *cross* if they have a point in common other than their common endpoint. A drawing of a graph where no pair of edges cross is called a *crossing-free* drawing. A graph is *planar* if it has a crossing-free drawing. In a crossing-free drawing D, each connected region of $\Re^2 \setminus D$ is called a *face* of D. Each face is thus an open region in $\Re^2$, and its boundary is comprised of vertices and edges of D.

An *edge contraction* of an edge vw of a graph G is accomplished by merging the vertices v and w, deleting loops and replacing parallel edges by a single edge. A visualization of this process can be found in Figure 2.

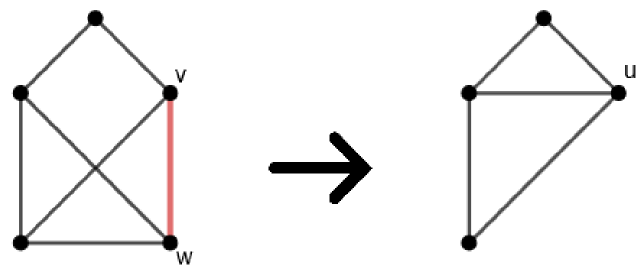


Figure 2: Example of an edge contraction of edge vw. The vertices v and w merge into one vertex u.

We finish the basic definitions with graph minors. A graph H is a *minor* of a graph G if it can be obtained from a subgraph of G by contracting edges. Otherwise, if H is not a minor of G, G is said to be *H-minor-free*, or equivalently that G *excludes* H as a minor.

As is common in graph theory, in this thesis we will often use a term *family of graphs* in place of a set of graphs. A family of graphs $\mathcal{F}$ is *minor-closed* if $G \in \mathcal{F}$ implies that every minor of G is in $\mathcal{F}$. We define $\mathcal{F}$ as *proper minor-closed* if it is not the family of all graphs. For example, consider a set of all planar graphs. Firstly, that family of graphs does not contain all graphs as there are graphs that do not have crossing-free drawings. Secondly, every minor of a planar graph is also planar. Hence, the family of all planar graphs is a proper minor-closed family. Wagner's theorem [1] states that a graph is planar if and only if it is K_5-minor-free and $K_{3,3}$-minor-free.

A famous theorem by Roberston and Seymour [2] declares that a similar statement is true for all proper minor-closed families of graphs. Specifically, they proved that for every proper minor-closed family $\mathcal{F}$, there is a finite set of graphs $H_1, \ldots, H_p$, for some positive integer p, such that every graph $G \in \mathcal{F}$ is H_i-minor-free for all $i \in \{1, \ldots, p\}$.

The *crossing number* of an undirected, simple and finite graph G, denoted by $cr(G)$, is the minimum number of edge crossings in any drawing of G in the plain. One of the earliest representations of this problem was in 1944 by Paul Turan, where bricks were to be brought from kilns to storage yards by rail [3]. At each crossing, there was a possibility of bricks or cars falling over, adding more hassle to the work. This problem was later translated to finding the crossing number of complete bipartite graphs. The exact bound on the crossing number of the complete bipartite graph $K_{a,b}$ is still unknown for most values of a and b [4]. The study of the crossing number of various graph families has a rich history (see for example [5] for a survey). The crossing number has various applications and has been been studied in VLSI circuit design [6], in visualization, in computational geometry [7], [8], and other fields in mathematics and theoretical computer science [9].

Computing the crossing number has been found to be NP-hard by Garey and Johnson [10], even in the case of cubic graphs [11]. On the other hand, Grohe [12] presented a quadratic-time algorithm that decides whether a given n-vertex graph has crossing number at most k for some fixed constant k. More specifically the running time of Grohe's algorithm is $O(f(k) \cdot n^2)$ for some function f. The running time has later been improved to linear in n, $O(f(k) \cdot n)$, by Kawarabayashi and Reed [13].

As calculating the exact, or even an asymptotic, crossing number of a graph is hard, much of the research has been focused on deriving asymptotic bounds.

Planar graphs on n vertices are known to have at most $3n - 6$ edges. Consequently, there are graphs with at most $3n - 6$ edges that can be drawn with zero crossings. The following result, known as the *crossing lemma*, tells us that as soon as the graph has little bit more than $3n$ edges, it must have crossings in every drawing. Specifically, the crossing lemma, proposed by Erdos and Guy [14] and proved by Leighton [15] and Ajtai et al. [16], states the following.

Theorem 1. *For any $\epsilon > 0$ there exists a constant c_ϵ such that the following inequality is true for every graph G with $||G|| > 3 \cdot |G| + \epsilon$:*

$$cr(G) \geq c_\epsilon \frac{||G||^3}{|G|^2}$$

Let's consider some immediate consequences of this theorem. The theorem implies immediately that every n-vertex graph that has at least $3n + \epsilon$ edges has the crossing number at least $c_\epsilon \frac{(3n+\epsilon)^3}{n^2} \geq c_\epsilon \cdot n \in \Omega(n)$. Therefore, for almost all graphs, the best one can hope for is to have linear $O(n)$ crossing number. As soon as a graph has superlinear number of edges, it is implied by the crossing lemma that the crossing number cannot be $O(n)$. Thus, the only candidates for linear crossing number are graphs with linear number of edges. Some examples of families of graphs with linear number of edges are graphs of bounded degree and proper minor-closed families. We will elaborate later why these families by themselves are not good candidates to ensure a linear crossing number.

There have been fewer studies on the upper bounds on the crossing number of general families of graphs. It is evident that for every graph G, its crossing number is less than or equal to $\binom{||G||}{2}$. We say that a family of graphs has a *linear crossing number* if there exists a constant c such that every graph G in the family has $cr(G) \leq c|G|$.

Pach and Tóth [17] along with Böröczky et al. [18] presented the following theorem that showed that graphs of bounded Euler genus and bounded degree have a linear crossing number.[1]

Theorem 2 ([17], [18]). *For every integer $\gamma \geq 0$, there is a function f such that every graph G with Euler genus γ has crossing number*

$$cr(G) \leq f(\gamma) \cdot \sum_{v \in V(G)} deg(v)^2 \in O(f(\gamma) \cdot \Delta(G) \cdot |G|)$$

An improvement on the dependence on γ in Theorem 2 for orientable surfaces was shown by Djidjev and Vrtó [19], with $cr(G) \leq c \cdot \gamma \cdot \Delta(G) \cdot |G|$ for some constant c.

It is important to note that to obtain a linear upper bound, we must assume both bounded degree and some structural assumption (such as an excluded minor). Consider, for example, the graph $K_{3,n}$. It has linear number of edges and it excludes K_5 as a minor, yet it is known to have crossing number $\Omega(n^2)$ [20], [21]. Similarly, bounded degree graphs have linear number of edges, but that in itself does not guarantee a linear crossing number. For example, it is known that, for every big enough n, there are cubic graph on n vertices whose crossing number is $\Omega(n^2)$ [15], [22]–[24].

Wood and Telle [25] were the first to show that excluding a minor and bounding the maximum degree were sufficient to ensure a linear crossing number. Their result is stated in the following theorem.

[1] Let Σ be a surface. An *embedding* of a graph G in Σ is a crossing-free drawing of G in Σ. The *Euler genus* of Σ equals $2h$ if Σ is the sphere with h handles, and equals c if Σ is the sphere with c cross-caps. The *Euler genus* of a graph G is the minimum Euler genus of a surface in which there is an embedding of G.

Theorem 3 ([25]). *For every graph H, there is a constant $c := c(H)$ such that every H-minor-free graph G has crossing number*

$$cr(G) \leq c \cdot \Delta(G)^2 \cdot |G|$$

Theorem 3 was improved by Dujmović et al. [26] by reducing the quadratic dependence on $\Delta(G)$ to linear.

Theorem 4 ([26]). *For every graph H, there is a constant $c := c(H)$ such that every H-minor-free graph G has crossing number*

$$cr(G) \leq c \cdot \Delta(G) \cdot |G|$$

In addition, the result in Theorem 4 was shown to have the best possible dependence of $\Delta(G)$ and $|G|$. These results show that we know very strong, in fact best possible, bounds on the crossing number of all proper minor-closed families of graphs of bounded degree.

However, much less is known for the rectilinear crossing number. A *rectilinear crossing number* of a graph is the minimum number of crossings over all rectilinear drawings of G.

Fáry [27] and Wagner [28] proved independently that every planar graph has a straight-line drawing with no crossings. Hence, every planar graph G has the rectilinear crossing number 0, and thus for planar graphs G, $\overline{cr}(G) = cr(G)$.

One may be tempted to conjecture that the rectilinear crossing number and crossing number are tied. However, it has been shown that it is not the case. Bienstock and Dean [29] have demonstrated that even if the crossing number of a graph is small, its rectilinear crossing number may be unbounded. In particular, they proved that for every m and every $k \geq 4$, there exists a graph G with $cr(G) = k$, but $\overline{cr}(G) \geq m$. Therefore, Theorem 3 and Theorem 4 do not imply that bounded degree minor-closed families of graphs have linear rectilinear crossing number.

In fact, in addition to planar graphs, we are only aware of the following two results bounding the rectilinear crossing number of a bounded degree minor-closed family by $O(n)$.

Theorem 5 ([26])**.** *Every n-vertex graph G with no $K_{3,3}$ minor has rectilinear crossing number*

$$\overline{cr}(G) \leq \sum_{v \in V(G)} deg(v)^2 \leq 2 \cdot \Delta(G) \cdot ||G|| \in O(\Delta(G) \cdot n)$$

Rectilinear drawings where vertices are required to be in convex positions are called *convex drawings*. For a graph G, the minimum number of crossing over all convex drawings of G is called *convex crossing number* of G and is denoted by $cr^*(G)$. Clearly, for every G, $cr(G) \leq \overline{cr}(G) \leq cr^*(G)$. Wood and Telle prove the following result for graphs of treewidth k (see their Corollary 8.3).

Theorem 6 ([30])**.** *Every n-vertex graph G of treewidth k has convex crossing number*

$$cr^*(G) \in O(k^2 \cdot \Delta(G)^2 \cdot ||G||) \in O(k^3 \cdot \Delta(G)^2 \cdot n)$$

In the case of the rectilinear crossing number, Theorem 6 was improved further to $O(k \cdot \Delta(G) \cdot \sum_{v \in V(G)} deg(v)^2)$ by Dujmović *et al.* [26].

The goal of this thesis is to extend this result to much wider bounded degree minor-closed families of graphs.

A single-crossing graph is a graph whose crossing number is at most one. Note that a graph excluding a single-crossing graph as a minor may have arbitrarily large crossing number. For example, any n-vertex graph G, composed of disjoint union of $\frac{n}{6}$ copies of $K_{3,3}$, excludes K_5 as a minor (K_5 is a single-crossing graph) and yet the crossing number of G is $\Theta(n)$. The reason for that is that $K_{3,3}$ is not planar thus each copy of $K_{3,3}$ contributes 1 to the crossing number. As there are $\frac{n}{6}$ copies, G has $\Theta(n)$ crossing number.

The main result of this thesis is the following tight bound.

Theorem 7. *Let X be a single-crossing graph. There exists a constant $c := c(X)$, such that every X-minor-free graph G has a rectilinear crossing number of at most $c \cdot \Delta(G) \cdot |G|$.*

Since K_5 is a single-crossing graph, the following result is an immediate corollary of Theorem 7.

Corollary 1. *There exists a constant c such that every K_5-minor-free graph G has a rectilinear crossing number of at most $c \cdot \Delta(G) \cdot |G|$.*

It is known that the family of graphs of treewidth at most k excludes a planar grid of size k^c for some constant c [31]. Since each planar graph is a single-crossing graph, Theorem 7 implies that every n-vertex graph of bounded treewidth has a rectilinear crossing number $O(\Delta(G) \cdot n)$, thus improving the previous best known bound of Wood and Telle [30] stated in Theorem 6. See Theorem 14 for details on our result concerning bounded treewidth graphs.

In the next section, Section 2, we will introduce notions that will be helpful in proving Theorem 7. In Section 3 we will prove Theorem 7. We will conclude in Section 4.

2 Preliminaries

2.1 *Decompositions and Treewidth*

For graphs G and H, an *H-decomposition* of G is a collection $(B_x \subseteq V(G) : x \in V(H))$ of sets of vertices in G (called *bags*) indexed by the vertices of H, such that

1. for every edge vw of G, some bag B_x contains both v and w, and
2. for every vertex v of G, the set $\{x \in V(H) : v \in B_x\}$ induces a non-empty connected subgraph of H.

The *width* of a decomposition is the size of the largest bag minus 1. The *adhesion* of a decomposition is the size of the largest intersections between two bags that share an edge in H. If H is a tree, then an H-decomposition is called a *tree decomposition*. The *treewidth* of a graph G is the minimum width of any tree decomposition of G. Tree decomposition and treewidth are key concepts in graph minor structure theory and they have been extensively studied ever since their introduction by Halin [32] and independently by Robertson and Seymour [33]. For each constant k, the graphs of treewidth at most k form a proper minor-closed family. An example of a tree decomposition can be found in Figure 3.

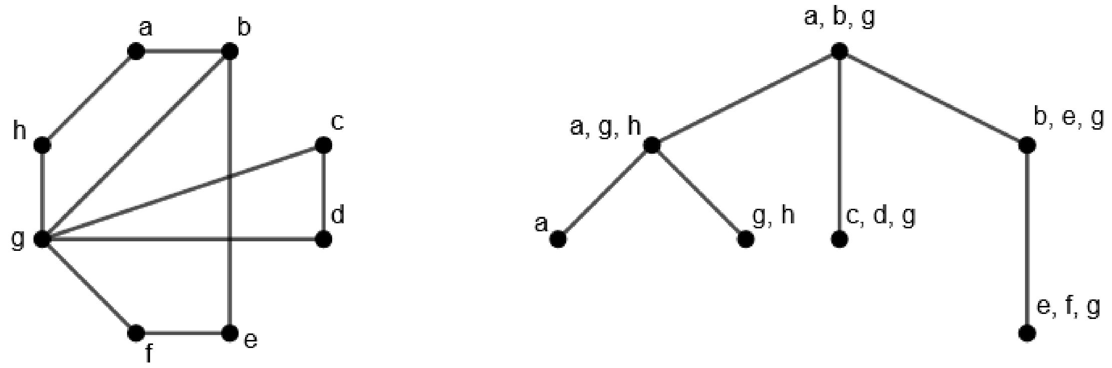


Figure 3: A graph (on the left) and its tree decomposition of width 2 (on the right).

We now state some well-known results about treewidth that will be useful for our work. For more facts about treewidth, refer to the survey by Bodlaender [34].

A graph is *chordal* if every cycle of length four or more has a *chord*, which is an edge that is not part of the cycle but connects two vertices of the cycle.

Theorem 8. *A graph has treewidth at most k if and only if it is a subgraph of a chordal graph with maximum clique size at most* $k + 1$*.*

Accordingly, chordal graphs with maximum clique size $k + 1$ are edge maximal graphs of treewidth k.

A vertex in a graph G is *simplicial* if $G[N_G(v)]$ forms a clique in G, and it is *k-simplicial* if, in addition, $|N_G(v)| \leq k$. It is well known that any induced subgraph of a chordal graph has a simplicial vertex. That in turn implies that each chordal graph has a vertex ordering $v_1, \ldots, v_n$ such that for each vertex v_i, the neighbours of v_i in the graph induced by the first $i - 1$ vertices induces a clique.

Theorem 9. *A graph G has treewidth at most k if and only if it is a subgraph of a chordal graph H where each induced subgraph of H has a k-simplicial vertex.*

Our proof of the main result will require the use of multigraphs. Recall that a multigraph is a graph that may have parallel edges but no loops. For the remainder of this paper, we will always employ the word multigraph when parallel edges are allowed and will use the word graph when they are not allowed, specifically when the graph in question is simple.

The degree of a vertex v in a multigraph Q, denoted by $\deg_Q(v)$ is the number of edges of Q incident to v. However, unlike in simple graphs, $\deg_Q(v)$ is not necessarily equal to $|N_Q(v)|$.

A *rectilinear drawing* of a multigraph K represents vertices, $V(K)$, by a set of $|V(K)|$ points in the plain in general position and represents each edge by a line-segment between its endpoints The general position assumption implies the only vertices an edge intersects are its own endpoints, and no point in the drawing is in a 3 *distinct* line-segments (unless all 3 share a common endpoint). It should be noted that the parallel edges between a same pair of vertices in such a drawing overlap, as they are represented by the same line-segment. A *crossing-pair* is a pair of edges that do not have an endpoint in common and whose line-segments intersect at a common point. The number of crossings in a rectilinear drawing of a multigraph is the number of crossing-pairs in the drawing. The *rectilinear crossing number* of a multigraph K, denoted by $\overline{cr}(K)$, is the minimum number of crossings over all rectilinear drawings of K.

For example, suppose that in some rectilinear drawing of some multigraph K, the line-segment $\overline{vw}$, between vertices v and w, and the line-segment $\overline{xy}$, between vertices x and y, cross. Suppose further that in K, there are two parallel edges between v and w and three parallel edges between x and y. Then the crossing between segments $\overline{vw}$ and $\overline{xy}$ adds 6 to the total number of crossings in the rectilinear drawing of K, since at that crossing point, there are 6 crossing-pairs.

Note that by these definitions, a pair of overlapping edges in a rectilinear drawing of a multigraph is not considered a crossing-pair. That is due to the fact that in our main proof, we eventually replace overlapping edge-segments with edge-segments that have only one endpoint in common and such edges can never cross. Notice also that if one is allowed to replace line-segments by arcs in a rectilinear drawing of a multigraph K, then it is trivial to redraw K such that the resulting "arc" drawing of Q has no overlapping edges and has the same number of crossings as the rectilinear drawing of K. Finally, if K is a simple graph, these definitions of rectilinear drawing and rectilinear crossing number are equivalent to the earlier ones for simple graphs only.

2.3 *Rectilinear Drawings*

In the process of proving our main result, Theorem 7 in Section 3.1, we will construct drawings of graphs where at one stage we will replace some vertices in those drawings with disks that fulfill certain criteria. The following lemma will be helpful for that stage.

For any positive integer h, let $[h]$ denote the sequence of numbers $[1, \cdots, h]$.

When it is clear from the context, we will make no distinction between a vertex v of a graph and the point that represents it in a drawing. Specifically, we will refer to both as v when no confusion can arise. The same will be true of an edge e and the line-segment representing it in a drawing.

Lemma 1. *Let D be a rectilinear drawing of any graph G. Then for each vertex $w \in V(G)$, there exists a disk C_w of positive radius centered at w such that the following is true. Let $v_1, \ldots, v_d$ be the neighbours of w in G. Let P_w be* any *set of at most d points in C_w such that $V(G) \cup P_w$ is in general position. For each $i \in [d]$, replace the line-segment $\overline{wv_i}$ of D by a line-segment between v_i and any point in P_w. Denote that point by p_i. For any two v_i and v_j where $i \neq j$, p_i and p_j may not be distinct points. The resulting drawing D' (of the resulting graph G') has the following properties:*

1. *Any two edges in G, neither of which incident to w, cross in D' if and only if they cross in D.*
2. *For each $i \in [d]$, the edge wv_i and any edge xy where $\{x, y\} \subseteq V(G) - \{w, v_i\}$ cross in D if and only if $v_i p_i$ and xy cross in D'.*
3. *All the remaining crossings in D' are crossings between pairs of segments with distinct endpoints in P_w.*

It should be noted that if $|P_w| = 1$, that is if P_w has exactly one point, then D' is a rectilinear drawing of G where a pair of edges of G cross in D' if and only if they cross in D.

Proof. Start with a disk C centered at w such that the only parts of D that intersect C are w and the edges incident to w (see Figure 4). Then, for each $i \in [d]$, let S_i be the union of all possible line-segments from v_i to any point in C (see Figure 5). Let S denote the union of all S_i, $i \in [d]$.

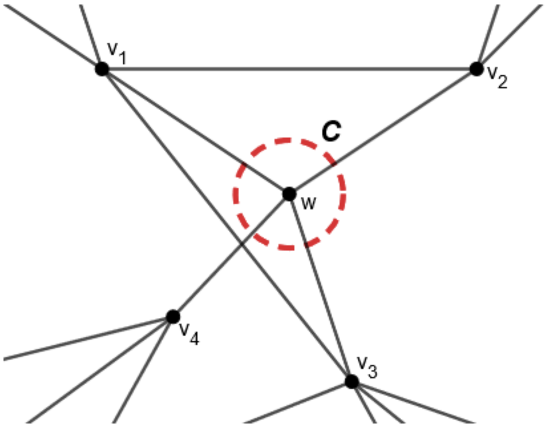


Figure 4: The small positive radius disk C centered at w.

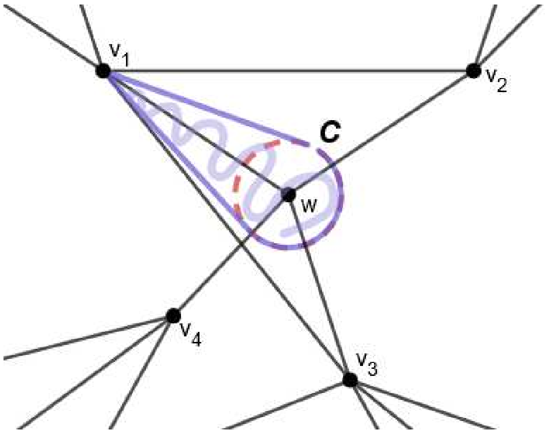


Figure 5: Union of all possible line segments from v_1 to any point in C, S_1, represented by the light purple outline.

By reducing the radius of C to some positive radius r and then redefining S accordingly, the following becomes true:

- No vertex of G is inside of S other than $w, v_1, \dots, v_d$.
- For each $i \in [d]$, the only vertices of G that are in S_i are v_i and w.
- For each $i \in [d]$, the only crossing points in S_i are crossings between wv_i and other edges of G.

- No segment between two crossings in D is fully contained in S, unless it lies on one of the edges wv_i, $i \in [d]$.

Such a positive radius r exists by continuity and the resulting disk meets the conditions imposed on C_w. □

3 Main Result

In order to prove our main result, Theorem 7, we will use as one of the tools the Robertson and Seymour's structure theorem for graphs that exclude a single-crossing graph as a minor [35]. This structure theorem uses the notion of clique-sum, thus we define it next.

Let G_1 and G_2 be two disjoint graphs. Let $C_1 = \{v_1, v_2, \cdots, v_k\}$ be a clique in G_1 and $C_2 = \{w_1, w_2, \cdots, w_k\}$ be a clique in G_2, each of size k, for some integer $k \geq 1$. Let G be a graph obtained from G_1 and G_2 by identifying v_i and w_i for each $i \in [k]$ and possibly deleting some of the edges $u_i u_j$ in the resulting clique $C = \{u_1, u_2, \cdots, u_k\}$ of G. Then we say that G is *obtained* by *k-clique-sums* of graphs G_1 and G_2 (at C_1 and C_2). A $(\leq k)$*-clique-sum* is an l-clique-sum for any $l \leq k$. See Figure 6 for visualization of this process.

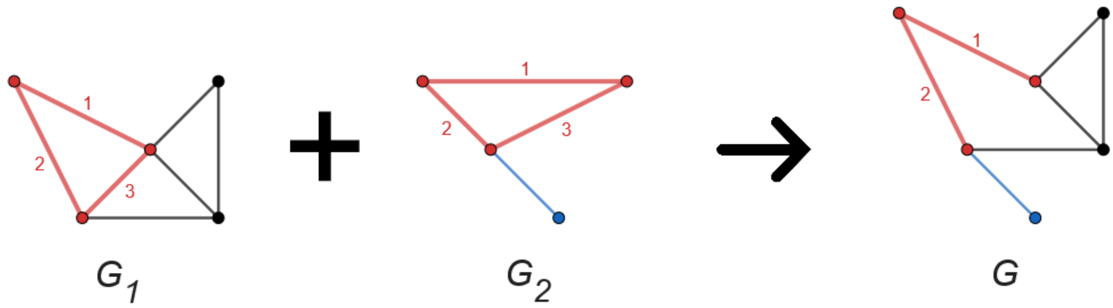


Figure 6: Example of a 3- clique-sum on two graphs G_1 and G_2, where the edge labelled 3 has been removed in the final graph G.

Another notion we require is separating triangles. A cycle of length 3 in a planar graph G is called a *separating triangle* if there is no crossing-free drawing of G where that cycle bounds a face in the drawing (see Figure 7).

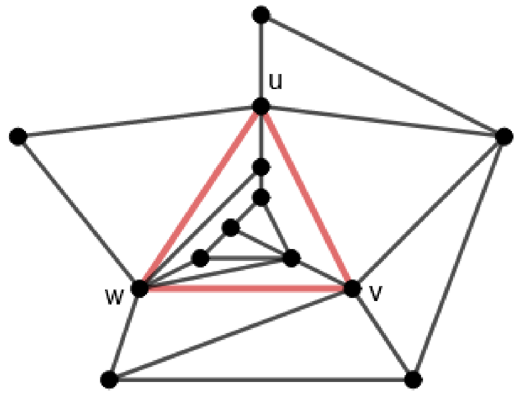


Figure 7: Example of a separating triangle $\{u, v, w\}$.

With these definitions in hand, we are ready to state the structure theorem, by Robertson and Seymour [36], for graphs that exclude a single-crossing graph as a minor.

Theorem 10 ([36])**.** *For every single-crossing graph X, there exists positive integers $t := t(X)$ and h with the following property. If G is an X-minor-free graph, then G can be obtained by (≤ 3)-clique-sums of graphs $G_1, \ldots, G_h$ such that for each $i \in [h]$, G_i is a planar graph (with no separating triangles) or the treewidth of G_i is at most t.*

The above theorem is equivalent to stating that every X-minor-free graph G has a tree decomposition of adhesion 3 such that each bag in the decomposition is either a planar graph (with no separating triangles) or a graph of treewidth at most t.

The graphs $G_1, \cdots, G_h$ in Theorem 10 are called the *pieces* of the decomposition. Note that the original statement of Theorem 10 by Robertson and Seymour [36] states that the planar pieces do not have separating triangles. The reason such a statement can be made is that any planar graph G containing a separating triangle can itself be obtained by 3-clique-sums of two strictly smaller planar graphs, G_1 and G_2, where the clique-sum is performed on that separating triangle. In particular, G_1 is the graph induced in G by the vertices of the separating triangle and the vertices outside of that triangle in an embedding of G. G_2 is the graph induced in G by the vertices of the separating triangle and the vertices inside of the separating triangle in that embedding of G.

Armed with these notions, we are now ready to state a more precise version of our main result, Theorem 7 below.

Theorem 11. *Let X be a single-crossing graph. Let G be an X-minor-free graph and t be the minimum integer such that G can be obtained by (≤ 3)-clique-sums of graphs $G_1, \ldots, G_h$, with $h \leq 1$, where, for each $i \in [h]$, G_i is a planar graph (with no separating triangles) or the treewidth of G_i is at most t. Then G has a rectilinear crossing number of at most $3 \cdot (t^2 + 2t + 2) \cdot \Delta(G) \cdot ||G||$.*

Theorem 11 is a strengthened version of Theorem 7. Hence, the remainder of this section will be dedicated to proving Theorem 11. To do so, one has to be able to produce rectilinear drawings of the pieces, $G_1, \ldots, G_h$, of the decomposition with the claimed number of crossings and then combine these drawings with clique-sums. The following is a sketch of the two main steps our proof will take.

Step 1. Foremost, Theorem 11 has to be true for the pieces G_i of the decomposition, namely the planar graphs and bounded treewidth graphs. By the Fáry-Wagner theorem [27], [28], we know that Theorem 11 is true for all planar graphs. In fact, it is true with bound zero for the rectilinear crossing number. On the contrary, if G_i is a bounded treewidth graph on n_i vertices, the required $O(\Delta(G_i) \cdot n_i)$ bound on its the rectilinear crossing was not known prior to our work. Thus, one of the goals of this thesis is to prove that bound for bounded treewidth graphs as one of the necessary steps in the proof of Theorem 11.

Step 2. Suppose now that for each piece, G_i of the decomposition, with $n_i := |G_i|$ we have already established the $O(\Delta(G_i) \cdot n_i)$ bound for the rectilinear crossing of G_i. The main goal then becomes demonstrating that the the rectilinear drawings of G_i–s can be joined by performing clique-sums without increasing the number of crossings in the final drawing of G by too much. In particular we need to join rectilinear drawings of G_i–s in such a way that the resulting number of crossings in the rectilinear drawing of G is $O(\Delta(G) \cdot n)$.

The main challenge for proving Theorem 11 is Step 2 above. To overcome that challenge, we introduce the notion of simplicial blowups of graphs. To ensure the proof of Theorem 11 is thorough, it is in fact not enough that in Step 1 we prove that the pieces of the decomposition have the required $O(\Delta(G_i) \cdot n)$ rectilinear crossing number. We must prove a stronger condition, namely that the simplicial blowups of the pieces have such a rectilinear crossing number.

In Section 3.1 we introduce simplicial blowups and demonstrate that Step 2 above can be done. In Section 3.2 we introduce graph partitions and present a helpful lemma for producing rectilinear drawings. In Section 3.3 and 3.4, we then prove that Step 1 above can be accomplished, or more precisely that simplicial blowups of planar graphs and bounded treewidth graphs have the desired rectilinear crossing number. Once those two steps have been achieved, we will conclude the proof of Theorem 11 in Section 3.5.

3.1 *Bound for Rectilinear Crossing Number Using Clique-Sums*

A multigraph K is called a $(\leq k)$-*simplicial blowup* of a graph G if K can be obtained from G by following these steps:

1. Add an independent set $S = \{u_1, u_2, \ldots, u_p\}$ of vertices to G, for some positive integer p.
2. For each $i \in [p]$, make u_i adjacent to all the vertices of some clique of size at most k of G.
3. Delete some edges from the cliques involved in Step 2 (optional).
4. Add parallel edges between vertices in S and its neighbours in G (optional).

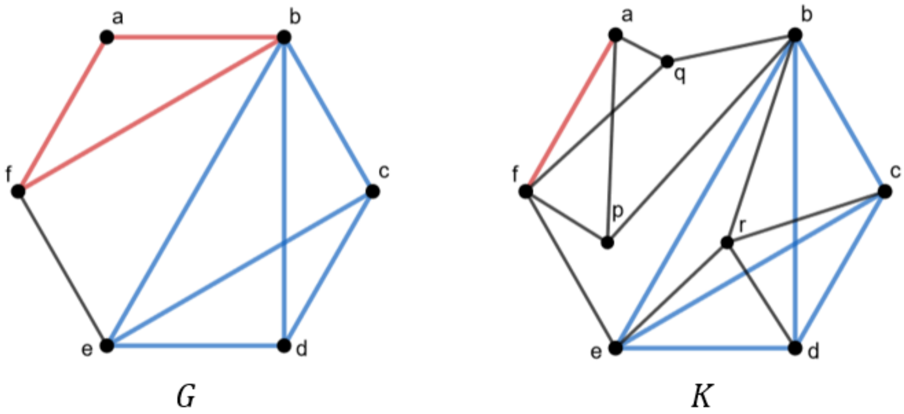


Figure 8: Example of a (≤ 4)-simplicial blowup K of a graph G.

Figure 8 illustrates an example of (≤ 4)-simplicial blowup. In particular, the graph on the right is a (≤ 4)-simplicial blowup of the graph on the left. In this example, we observe two cliques, $C_1 = \{a, b, f\}$ and $C_2 = \{b, c, d, e\}$ in G. We added the independent set $S = \{p, q, r\}$ to G and joined p and q to all the vertices in C_1 and r to all the vertices in C_2. Furthermore, edges ab and bf were removed during the process, resulting in the simplicial blowup K of G.

Theorem 12 is the key technical tool of this thesis. It shows how rectilinear drawings of simplicial blowups of the pieces of a decomposition can be combined into a rectilinear drawing of a graph obtained by clique-sums of the pieces, all while not increasing the final number of crossings by too much.

We say that a graph R is (k, c)-*agreeable* if for every induced subgraph R' of R and every ($\leq k$)-simplicial blowup R^* of R', $\overline{cr}(R^*) \leq c \cdot \Delta(R^*) \cdot ||R^*||$.

The following theorem is stated with a more general approach than we will require and as such may be useful in some future work. Specifically, the theorem does not require the pieces G_i of the decomposition to be planar or of bounded treewidth.

Theorem 12. *Let c be a positive number, k and h positive integers, and $G_1, \cdots, G_h$ graphs such that every G_i is (k,c)-agreeable. Then every graph G that can be obtained by $(\leq k)$-clique-sums of graphs $G_1, \cdots, G_h$ has rectilinear crossing number $\overline{cr}(G) \leq k \cdot (c+2) \cdot \Delta(G) \cdot ||G||$.*

Proof. We may assume that the indices $1, \ldots, h$ are such that for all $j \geq 2$, there exists a minimum i such that $i < j$ where G_i and G_j are joined at some clique C of G_i when constructing G. We define G_i to be the *parent* of G_j, with $P_j = V(C)$ being the *parent clique* of G_j. The parent clique of G_1 is the empty set.

Let T be a rooted tree with vertex set $\{1, \ldots, h\}$, where ij is an edge of T if and only if G_j is a child of G_i. Let T_i denote the subtree of T rooted at i and U_i be the set of the children of i in T.

For each $i \in [h]$, let $G_i^- = G_i - P_i$. Note that for each $v \in V(G)$, there is exactly one $i \in [h]$ such that v is in $V(G_i^-)$. Thus, $V(G_1^-), \cdots, V(G_h^-)$ is a partition of $V(G)$. We say that a vertex v of G *belongs* to vertex i of T if $v \in G_i^-$. For each $i \in [h]$, let $G[T_i]$ denote the graph induced in G by the vertices of G that belong to the vertices of T_i, that is the graph induced in G by $\bigcup\{V(G_j^-) : j \in T_i\}$.

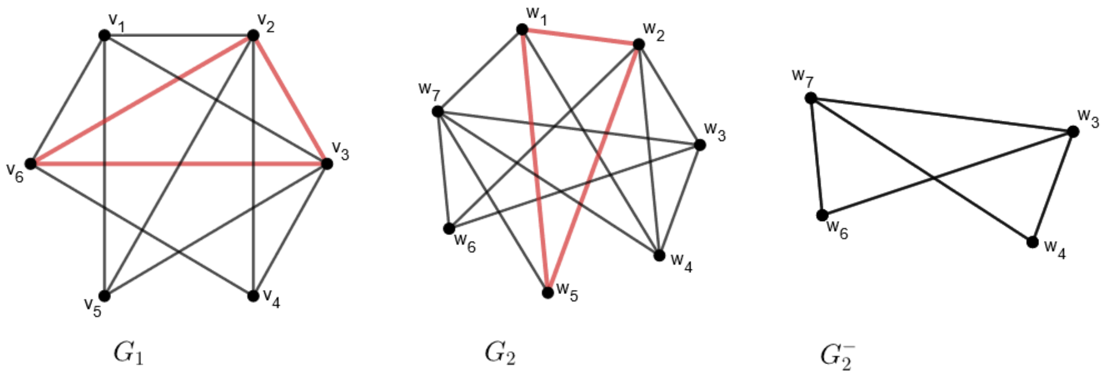


Figure 9: Example of two graphs that will be joined on a clique C (red edges), with G_1 being the parent of G_2.

In the example found in Figure 9, G_1 and G_2 will be joined at the respective cliques $C_1 = \{v_2, v_3, v_6\}$ and $C_2 = \{w_1, w_2, w_5\}$. Then, we have that P_2 is C_1 in G_1 and C_2 in G_2.

Defining the $(\leq$ k$)$-simplicial blowups of pieces. To prove the theorem, we will define a specific $(\leq k)$-simplicial blowup, denoted Q_i^-, for each G_i^-, $i \in [h]$. To define Q_i^-, start with G_i^-. For each child G_j of G_i, add a new vertex c_j to G_i^-. We call c_j a *dummy* vertex and say that c_j *represents* G_j in Q_i^-.

Note that for all $j \in [2, \cdots, h]$, there is exactly one $i < j$ such that Q_i^- has a vertex that represents G_j (namely, the vertex c_j). For each edge $vw \in E(G)$, where $v \in V(G_i^-) \cap P_j$ and $w \in G_\ell^-$, where $\ell \in V(T_j)$, connect v to c_j by an edge. Label that edge with the triple $(v, w, \mathcal{P}_{vw})$, where $\mathcal{P}_{vw}$ is the path in T from i to ℓ. We call the edge labelled $(v, w, \mathcal{P}_{vw})$ in Q_i^- an *isthmus* edge, as it represents the edge vw in the final drawing of G.

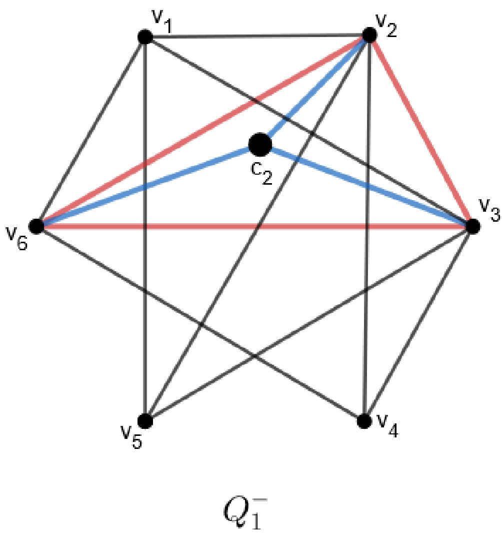


Figure 10: Example of a simplicial blowup Q_1^- for graph G_1 with the isthmus edges coloured in blue (v_2c_2, v_3c_2, and v_6c_2).

Continuing the example from Figure 9, we have the dummy vertex c_2 representing G_2 (Figure 10). Say we identified the vertices v_2 and w_1. The neighbours of w_1 in $G_2 \setminus P_2$ are w_4 and w_7. Then, two (parallel) edges from v_2 to c_2 will be drawn with the respective labels $(v_2, w_4, \mathcal{P}_{v_2 w_4})$ and $(v_2, w_7, \mathcal{P}_{v_2 w_7})$ (Figure 11).

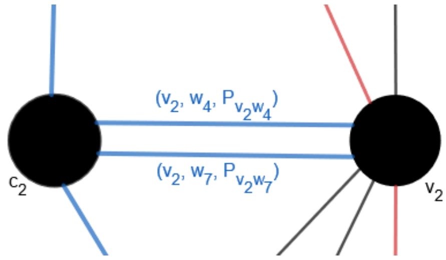


Figure 11: Close-up of the isthmus edges between v_2 and c_2.

We say two isthmus edges are *siblings* if they are adjacent to the same dummy vertex. We note that the vertices of P_j are involved in clique-sums in Q_i^-. Additionally, each isthmus edge has an endpoint in G_i^- that is involved in some clique-sums in Q_i^-. We finally remove from Q_i^- the edges in P_j that are not in G. We set the resulting graph to be Q_i^-.

Notice that Q_i^- is a $(\leq k)$-simplicial blowup of G_i^-. Since G_i is (k, c)-agreeable and since G_i^- is an induced subgraph of G_i, it follows that $\overline{cr}(Q_i^-) \leq c \cdot \Delta(Q_i^-) \cdot ||Q_i^-||$.

For $i \in [h]$, consider a rectilinear drawing of Q_i^- with at most $c \cdot \Delta(Q_i) \cdot ||Q_i||$ crossings. We will construct the desired rectilinear drawing of G by joining these rectilinear drawings of Q_i^-. How to join and modify these drawings without exceeding the desired the number of crossings is one of the key aspects to our proof to follow. However, consider for a moment solely a disjoint union of these rectilinear drawings. The resulting rectilinear drawing of the disjoint union thus has most $\sum_{i \in [h]} c \cdot \Delta(Q_i^-) \cdot ||Q_i^-||$ crossings.

Notice that there is one-to-one mapping between the edges of G and the edges in the union of all Q_i^-–s, that is $\bigcup_{i\in[h]} E(Q_i^-)$. Thus, $||G|| = \sum_{i\in[h]} ||Q_i^-||$. Hence, if for all $i \in [h]$, $\Delta(Q_i^-) \leq k \cdot \Delta(G)$, the above sum would be upper bounded by $c \cdot k \cdot \Delta(G) \cdot \sum_{i\in[h]} ||Q_i^-|| = c \cdot k \cdot \Delta(G) \cdot ||G||$. This is akin to the upper bound that we want on the rectilinear crossing number of G. Thus, we want to first bound the degree of each vertex in Q_i^- by $k \cdot \Delta(G)$. This is not completely obvious due to the dummy vertices as well as the clique-sums allowing for edge deletions.

Claim 1. For every $i \in [h]$ and every $v \in Q_i^-$, $deg_{Q_i^-}(v) \leq k \cdot \Delta(G)$.

Proof. There are three cases to consider.

CASE 1. *v is a dummy vertex of Q_i^-.*

By construction, for some $j \in U_i$, v represents some G_j and is adjacent to at most k vertices of the parent clique P_j in G_i. Each edge between a vertex $u \in P_j$ and v corresponds to an edge in G adjacent to u. Thus, v is incident to at most $k \cdot \Delta(G)$ edges, giving $deg_{Q_i^-}(v) \leq k \cdot \Delta(G)$.

CASE 2. *v is in G_i^- (and thus not a dummy vertex) and v is not involved in any clique-sums.*

Then, it follows that $deg_{Q_i^-}(v) \leq deg_G(v)$.

CASE 3. *v is in G_i^- (and thus not a dummy vertex) and is involved in at least one clique-sum.*

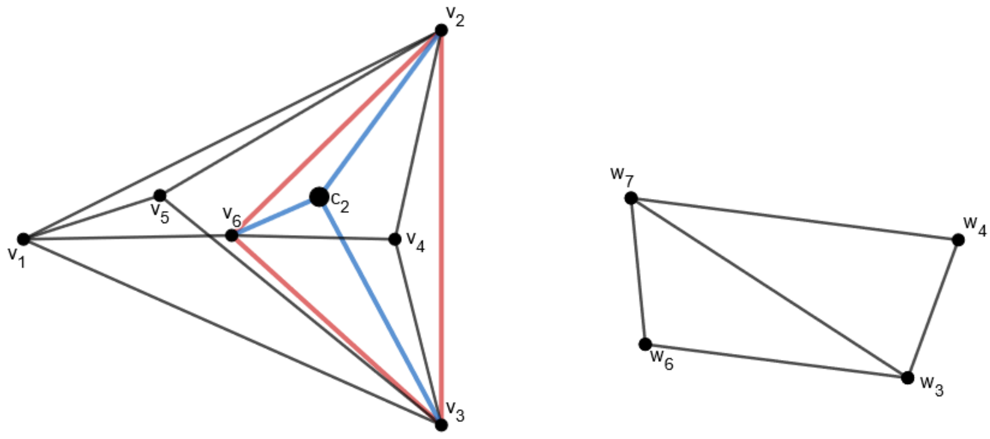
Consider every $j \in U_i$ such that $v \in P_j$. Then v has a least one neighbour in $G[T_j]$ and thus at least one edge connecting it to c_j, otherwise the clique-sum could have omitted v. Additionally, there exists a one-to-one mapping between the set of edges in G between v and its neighbours in $G[T_j]$ and the set of (parallel) edges between v and c_j in Q_i^-. In other words, there is a one-to-one mapping between the isthmus edges incident to v in Q_i^- and the isthmus edges incident to v in G. Finally, consider the non-isthmus edges incident to v in G_i^-. Each edge of G_i that has been removed in the construction of Q_i^- (namely the edges removed from P_j) was also removed in G, thus $deg_{Q_i^-}(v) \leq deg_G(v)$. □

With degrees of the vertices of Q_i^- sorted out, we are ready to describe how to construct a rectilinear drawing of G from the rectilinear drawings of Q_i^-–s

Constructing the rectilinear drawing of G from rectilinear drawings of $\mathbf{Q_i^-}$–s. By the earlier discussion, we know that for each $i \in [h]$, $\overline{cr}(Q_i^-) \leq c \cdot \Delta(Q_i^-) \cdot ||Q_i^-||$. By Claim 1, $\overline{cr}(Q_i^-) \leq c \cdot k \cdot \Delta(G) \cdot ||Q_i^-||$. Let $D(Q_i^-)$ denote a rectilinear drawing of Q_i^- with at most at most $c \cdot k \cdot \Delta(G) \cdot ||Q_i^-||$ crossings. For the remainder of the proof, we will show how to construct a rectilinear drawing, $D(G)$, of G by combining the rectilinear drawings $D(Q_i^-)$ of Q_i^-, $i \in [h]$, such that the resulting number of crossings in $D(G)$ is as claimed in the theorem.

Note that removing dummy vertices (and their incident isthmus edges) from $D(Q_i^-)$ gives a rectilinear drawing of G_i^-. Denote these rectilinear drawings by $D(G_i^-)$. In the final drawing, $D(G)$, the drawing of each G_i^- will be $D(G_i^-)$, possibly scaled and/or rotated. In other words, in $D(G)$, the implied rectilinear drawing of the disjoint union of G_i^--s will be a disjoint union of $D(G_i^-)$–s. Only the isthmus edges will be redrawn in this construction.

We will join the rectilinear drawings $D(Q_i^-)$, $i \in [h]$ in the order of their indices. For $\ell \in [h]$, D_ℓ will be a rectilinear drawing joining $D(Q_1^-), D(Q_2^-), \ldots, D(Q_\ell^-)$ (joining is detailed below). The rectilinear drawing D_h will be our desired rectilinear drawing of G. While joining these drawings, we will maintain the invariant that for each $j > \ell$ such that the parent of G_j is some G_i with $i \in [\ell]$, the rectilinear drawing D_ℓ has the representative dummy vertex of each G_j. Furthermore we maintain that D_ℓ minus the dummy vertices (that is $D_\ell - \cup_{j>\ell} c_j$) is isomorphic to $G - \cup_{j>\ell} V(G_j^-)$.



$D(Q_1^-)$ $D(Q_2^-)$

Figure 12: Good rectilinear drawings of Q_1^- and Q_2^-.

We start by defining $D_1 = D(Q_1^-)$. D_1 satisfies the above invariant. For $j \in [2, \cdots, h]$ we construct D_ℓ from $D_{\ell-1}$ and $D(Q_\ell^-)$ as follows. By the invariant, $D_{\ell-1}$ has a dummy vertex c_ℓ representing G_ℓ. Let C_ℓ be a disk centered in the point c_ℓ in $D_{\ell-1}$ that meets the conditions of Lemma 1. Let $v_1, v_2, \ldots, v_d$ be the neighbours of c_ℓ in $D_{\ell-1}$. Construct D_ℓ by following steps (see Figure 13 for an illustration).

1. Remove c_ℓ and its incident (isthmus) edges.

2. Scale down $D(Q_\ell^-)$. Place it inside C_ℓ and rotate it such that all the vertices of D_ℓ are in general position.

3. For each isthmus edge labelled with (x, y, P_{xy}) that was incident to c_ℓ, (re)draw it as a line-segment from x to y if y in Q_ℓ^-. Otherwise, by construction, $D(Q_\ell^-)$ has a point c_j, $j > \ell$ and $y \in G[T_j]$. In that case, draw a

line-segment between x and c_j. Note that, by Lemma 1, the only new crossings (pairs) that this introduces are crossings between a pair of (re)drawn sibling isthmus edges (that were both incident to c_ℓ) and one such isthmus edge (incident to c_ℓ) and edges strictly inside the disk C_ℓ.

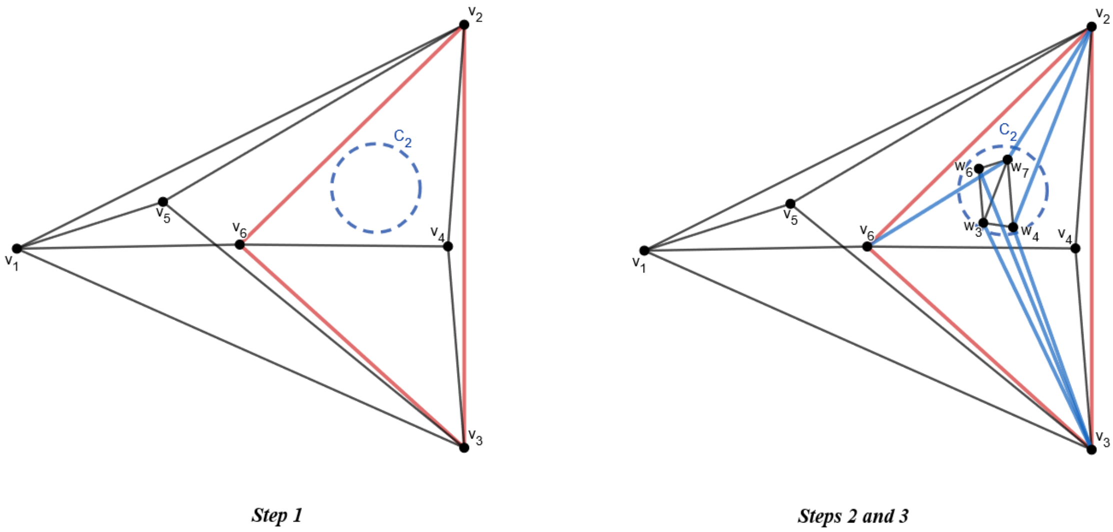


Figure 13: Illustration of the steps when joining the drawings of G_1^- and G_2^-. The drawing on the right represents D_2.

The resulting drawing D_ℓ satisfies the invariant. Note that at the end of this process, when $\ell = h$, there are no more dummy vertices and each edge labelled (x, y, P_{xy}) in D_h is an actual line-segment connecting vertex x and y in G and thus actually represents the isthmus edge xy of G. The final drawing D_h is a rectilinear drawing $D(G)$ of G. It remains to prove that $D(G)$ has the claimed number of crossings.

Before joining the drawings $D(Q_i^-)$, $i \in [h]$, the total number of crossings in the disjoint union of all drawings was at most $c \cdot k \cdot \Delta(G) \cdot ||G||$, as argued in Claim 1. We name this quantity the *initial sum*. We now prove that joining these drawings into a drawing of G does not increase the initial sum by much. Specifically, we will show that all new crossings can be charged to the edges of G such that each edge is charged at most $O(\Delta(G))$ new crossings, which will complete the proof.

By the construction, the new crossings must involve at least one isthmus edge. Consider such an isthmus edge e labelled (v, w, P_{vw}), where $v \in Q_i^-$ and $w \in Q_p^-$, $i < p$ and $w \in G[T_j]$ where $j \in U_i$ (and thus $p \in T_j$). There are four cases to consider.

CASE 1. Consider first a crossing in $D(G)$ between e and a non-isthmus edge e' in Q_i^-. That crossing is already accounted for in the initial sum by the crossing in $D(Q_i^-)$ between e' and vc_j.

CASE 2. Consider next a crossing between e and an isthmus edge e' labelled (x, y, P_{xy}), where $x \in Q_i^-$ and $y \in G[T_\ell]$, with $\ell \in U_i$ but $\ell \neq j$ (so e and e' are not sibling isthmus edges). That crossing was accounted for as well in the initial sum by the crossing in $D(Q_i^-)$ between vc_j and xc_ℓ.

CASE 3. Consider next a crossing between e and an isthmus edge e' labelled (x, y, P_{xy}), where $x \in Q_i^-$ and $y \in G[T_j]$ (thus e and e' are sibling isthmus edges). It must be that $v \neq x$ as otherwise e and e' cannot cross. By construction both w and y are in the disk C_ℓ. In the construction of G, $G[T_j]$ is added via a $(\leq k)$-clique-sum to G_i (with parent clique P_j). Thus, at most $k \cdot \Delta(G)$ (isthmus) edges cross the cycle bounding C_ℓ. Thus e can be crossed by at most $k\Delta(G)$ such edges e'. We charge these $k \cdot \Delta(G)$ crossings to e.

CASE 4. Finally, consider a crossing between e and any edge e' where both endpoints of e' are in $G[T_j]$. The endpoints of e' are thus in Q_a^- and Q_b^- where $j \leq a \leq b$. We charge the crossing to e'. (Think of that crossing being charged to e' in Q_a^-). As argued above, at most $k \cdot \Delta(G)$ (isthmus) edges cross the cycle C_a that replaced the dummy vertex c_a thus each such edge e' is charged at most $k \cdot \Delta(G)$ new crossings.

By the arguments above, each edge of G is charged at most $2 \cdot k \cdot \Delta(G)$ new crossings. Thus together with the initial sum that gives the total number of crossings of at most $(c+2) \cdot k \cdot \Delta(G) \cdot ||G||$. □

3.2 *Rectilinear Drawings of Multigraphs via Graph Partitions*

As mentioned previously, in order to prove our main result, Theorem 11, we will use as a main tool the theorem that we have just proved, Theorem 12. Theorem 10 tells us that in order to use Theorem 12, we need to show that planar graphs and bounded treewidth are $(3, c)$-agreeable for some constant c. In this section, we define graph partitions and prove a lemma that will be helpful in proving that planar graphs are $(3, c)$-agreebale in Section 3.3 and that bounded treewidth graphs are (k, c)-agreeable in Section 3.4.

An *H-partition* of a (multi)graph G is comprised of a graph H and a partition of vertices of G such that

- each vertex of H is a non-empty set of vertices of G (called a *bag*),
- every vertex of G is in exactly one bag of H, and
- distinct bags A and B are adjacent in H if and only if some edge of G has one endpoint in A and the other endpoint in B.

An illustration of such a partition can be found in Figure 14.

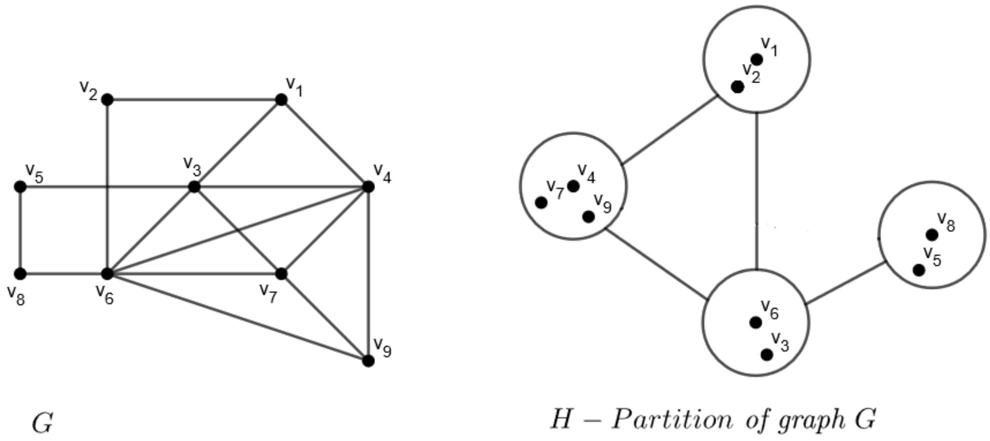


Figure 14: Example of a graph G and an H-partition of G.

The *width* of a partition is the maximum number of vertices in a bag. The *density* of a bag of an H-partition is the number of edges of G with at least one endpoint in that bag. The *density* of an H-partition is the maximum density over all bags of H. A bag is said to be *solitary* if it contains exactly one vertex of G.

The proof of the following lemma is a slight modification of a similar result by Wood and Arne Telle [37].

Lemma 2. *Let K be a multigraph and H a simple graph such that K has an H-partition with width w and density d. Let X be the set of all vertices of K that are not in solitary bags of H. Then we have the following.*

1. $\overline{cr}(K) \leq \overline{cr}(H) \cdot w^2 \cdot \Delta(K)^2 + (w-1) \cdot \sum_{v \in X} deg_K(v)^2$
2. *if H is planar, then*
   (a) there exists a rectilinear drawing of K with a most $2 \cdot d$ crossings per edge.
   (b) if in addition, the non-solitary bags of H form an independent set, then there is a straight-line drawing of K with at most d crossings per edge.

Proof. We start with a rectilinear drawing $D(H)$ of H with $\overline{cr}(H)$ crossings. Consider any vertex (bag) B of H. Let $C_\epsilon(B)$ be a disk of radius $\epsilon > 0$ centered at B in $D(H)$. For each edge AB of H, let $C_\epsilon(AB)$ be the union of all the line-segments with one endpoint in $C_\epsilon(B)$ and the other in $C_\epsilon(A)$. Note that there exists an ϵ small enough such that:

- $C_\epsilon(B) \cap C_\epsilon(A) = \emptyset$ for all distinct bags B and A of H;
- $C_\epsilon(AB) \cap C_\epsilon(PQ) = \emptyset$ for every pair of edges AB and PQ of H that have no endpoints in common and do not cross in $D(H)$;
- $C_\epsilon(AB) \cap C_\epsilon(Q) = \emptyset$ for every triple of distinct bags A, B, Q of H where AB is an edge of H; and,
- For each crossing pair of edges AB and PQ in $D(H)$, $C_\epsilon(AB) \cap C_\epsilon(PQ)$ is non-empty. We call such a region of the plain *busy* region of pair AB and PQ. Finally, the busy regions of distinct pair of edges are pairwise disjoint.

For each vertex v of K such that $v \in B$, draw v as a point in $C_\epsilon(B)$ such that the final set of points representing $V(K)$ is in general position. Draw every edge of K straight. This defines a rectilinear drawing $D(K)$ of K, since no edge in $D(K)$ contains a vertex other than its own endpoints and no three edges of $D(K)$ cross at one point.

We first prove that the number of crossings in $D(K)$ is at most $\overline{cr}(H) \cdot w^2 \cdot \Delta(K)^2 + (w-1) \cdot \sum_{v \in X} deg_K(v)^2$ which will prove the first part of the theorem. Consider two crossing edges e and f in $D(K)$. There are two cases to consider (based on two types of crossings that can occur in $D(K)$).

- Case 1: there is bag B of H that has at least one endpoint of e and at least one endpoint of f. Order all the vertices of $B = \{v_1, v_2, \cdots, v_\ell\}$, $l \leq w$ such that $deg_K(v_1) \leq \cdots \leq deg_K(v_\ell)$. Let v_i be an endpoint of e and v_j and endpoint of v_j, $i < j$. We charge the crossing between e and f to v_j.

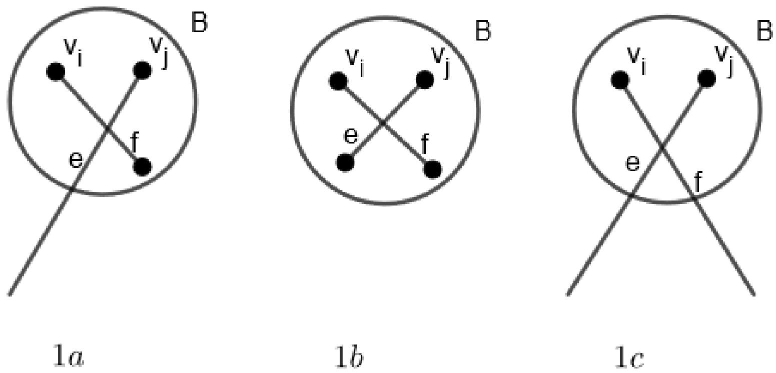


Figure 15: The possible drawings of the crossing between e and f for Case 1.

Thus, the number of crossings charged to v_j is at most

$$\sum_{i<j} deg_K(v_i) \cdot deg_K(v_j) \leq \sum_{i<j} deg_K(v_j)^2 \leq (\ell - 1) \cdot deg_K(v_j)^2 \leq (w-1) \cdot deg_K(v_j)^2$$

The vertices in the solitary bags of H are charged 0 crossings, rendering the total number of crossings of the first type at most $(w-1)\sum_{v \in X} deg_K(v)^2$.

- Case 2: there is no bag of H that has both an endpoint of e and an endpoint of f. This implies that four endpoints of e and f are in four distinct bags, A, B, P, Q of H. Let $e \in C_\epsilon(AB)$ and $f \in C_\epsilon(PQ)$. Since e and f cross, their crossing point must be the busy region of AB and PQ. Denote that region by R. There are at most $\Delta(K) \cdot w$ edges of K drawn inside $C_\epsilon(AB)$ and at most $\Delta(K) \cdot w$ edges of K drawn inside $C_\epsilon(PQ)$. We charge the crossings between these pairs of edges to the busy region R.

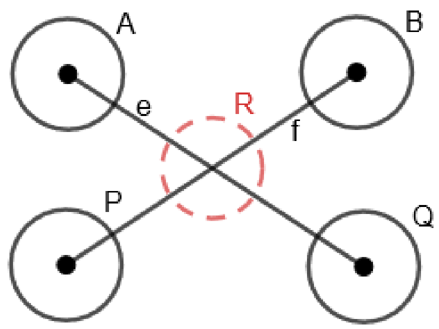


Figure 16: The drawing of the crossing between e and f for Case 2.

Thus, the number of crossings charged to R is at most $w\Delta(K) \cdot w\Delta(K) = w^2 \cdot \Delta(K)^2$. Since $D(H)$ has $\overline{cr}(H)$ crossings, there are exactly $\overline{cr}(H)$ busy regions determined by crossing edges in $D(H)$. Thus the total number of crossings of the second type is at most $\overline{cr}(H) \cdot w^2 \cdot \Delta(K)^2$.

Thus, $\overline{cr}(K) \leq \overline{cr}(H) \cdot w^2 \cdot \Delta(K)^2 + (w-1) \cdot \sum_{v \in X} deg_K(v)^2$ as stated in part 1.

We now prove the second part of the theorem. In this case, H is planar. By the Fáry-Wagner theorem [27], [28], there is a rectilinear drawing $D(H)$ of H with no crossings. Starting with such crossing-free drawing $D(H)$, we produce a rectilinear drawing $D(K)$ of K using the algorithm described above. Let e be an edge of K with an endpoint in some bag A of H. We now prove that the number of crossings on e in $D(K)$ is at most $2d$ as claimed in part 2a. There are two cases to consider:

- Case 1: both endpoints of e are in A. Then, e is only crossed by edges that have at least one endpoint in A. As there are at most d such edges, there is at most d crossings on e in $D(K)$.
- Case 2: the other endpoint of e is in a bag B of K distinct from A. Then, since $D(H)$ is crossing-free, e can only be crossed by the edges that have at least

one endpoint in A or in B. There is at most $2d$ such edges, thus there is at most $2d$ crossings on e in $D(K)$.

In either case, e is crossed by at most $2d$ edges in $D(K)$ as required by part 2a.

Finally, consider the case when the non-solitary bags of H form an independent set in H. Let e be an edge of K. If two endpoints of e are in two distinct solitary bags of H then no edge of K crosses e since $D(H)$ is crossing-free. Therefore, in that case, trivially, there are at most d crossings on e in $D(K)$. Thus, we may assume that at least one endpoint of e is in non-solitary bag of H. Let A denote that bag. If the other endpoint of e is also in A, the result follows from Case 1 above. Therefore, we may assume that the other endpoint, v, of e is in a bag B of H distinct from A. B is then a solitary bag (by the independent set assumption). Since the edges incident to the same vertex (v in this case) cannot cross, the only edges that can cross e are those with an endpoint in A. There is at most d edges with endpoints in A and thus there are at most d crossings on e in $D(K)$. □

3.3 *Rectilinear Crossing Number of Simplicial Blowups of Planar Graphs*

Theorem 10 tells us that in order to use Theorem 12, it is enough to consider (≤ 3)-simplicial blowups of planar graphs with no separating triangles. In other words, it is enough to prove that they are $(3, c)$-agreeable for some constant c. The next lemma achieves that.

Lemma 3. *Every planar graph that has no separating triangles is $(3, 3)$-agreeable.*

Proof. Let G be a planar graph that has no separating triangles. Since every induced subgraph of G is also planar and with no separating triangles, it is enough to show that every (≤ 3)-simplicial blowup Q of G has rectilinear crossing number $\overline{cr}(Q) \leq 3 \cdot \Delta(Q) \cdot ||Q||$.

Let $S = V(Q) - V(G)$. Since adding a 1-simplicial or 2-simplicial vertex to a planar graph results in a planar graph, we may assume that each vertex in S has exactly 3 neighbours in G. We now define an H-partition of Q. To start, we make H isomorphic to G and put each $v \in V(G)$ in the bag B_v in H. Currently, all the bags in H are solitary bags. Since G, and therefore the current H, has no separating triangles and as S is an independent set in Q, we have that for each $v \in S$, $N_Q(v)$ induces a face in an embedding of G and thus it is a face in the equivalent embedding of H. For each vertex set $\{x, y, z\}$ in H that forms such a face, we add a bag B_{xyz} adjacent to x, y and z. The resulting graph H is simple and planar. For each vertex $v \in S$ adjacent to x, y and z in Q, add v to the corresponding bag B_{xyz} in H. Thus, the defined graph H and the assignment of vertices of Q to its bags defines an H-partition of Q. An example of such a construction can be seen in Figure 17.

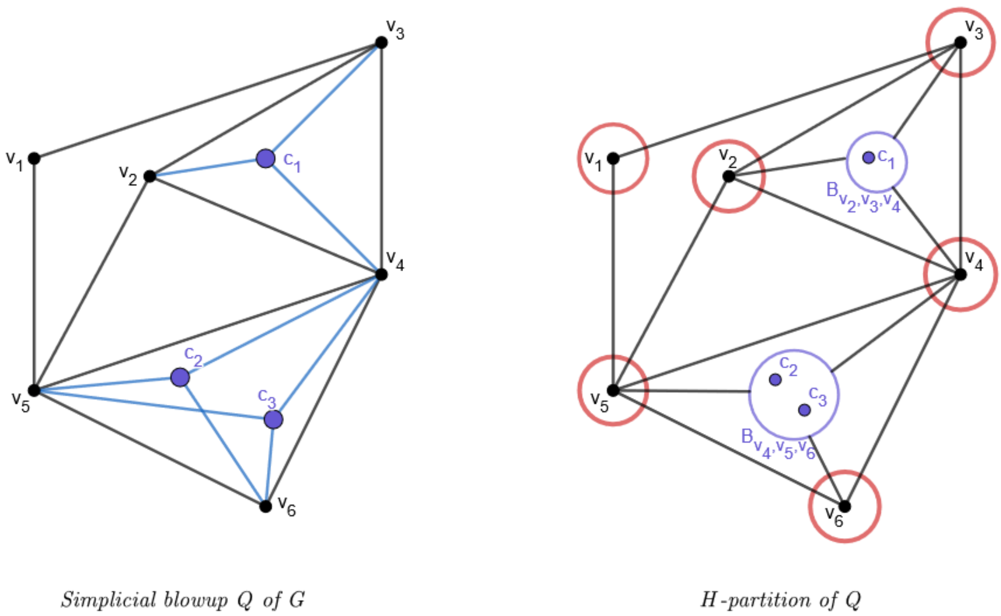


Figure 17: Illustration of a simplicial blowup Q of a graph G, with $V(G) = \{v_1, \cdots, v_6\}$ and $S = \{c_1, c_2, c_3\}$, and its H-partition.

As every vertex of Q in bag B_{xyz} is adjacent to all vertices in $\{x, y, z\}$, the maximum number of edges of Q with an endpoint in one non-solitary bag B_{xyz} is at most $\deg_Q(x) + \deg_Q(y) + \deg_Q(z) \leq 3\Delta(Q)$. The maximum number of edges of Q with an endpoint in one solitary bag is clearly $\Delta(Q)$. Thus, the density of the H-partition is at most $3\Delta(Q)$. Additionally, the non-solitary bags of H form an independent set which, by Lemma 2, signifies that Q has a rectilinear drawing with at most $3\Delta(Q)$ crossings per edge, giving the desired result, $\overline{cr}(Q) \leq 3 \cdot \Delta(Q) \cdot ||Q||$. □

In this section, we prove that bounded treewidth graphs are (k, c)-agreeable for some constants k and c. We start with following trivial bound applicable to all graphs.

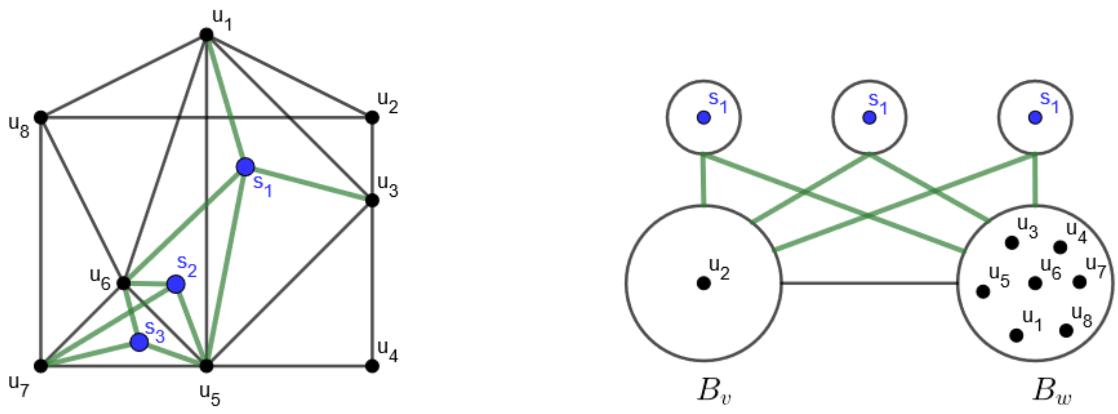
Lemma 4. *Every graph G is $(|G|,\ |G| - 1)$-agreeable.*

Proof. If $|G| = 1$, the statement is trivial since every 1-simplicial blowup of G is a star thus the crossing number of every such blow-up is zero.

Assume now that $|G| \geq 2$. Since every induced subgraph of G is also in the class of all graphs, it is enough to show that every $(\leq |G|)$-simplicial blowup Q of G has rectilinear crossing number $\overline{cr}(Q) \leq (|G| - 1) \cdot \Delta(Q) \cdot ||Q||$.

Let $S = V(Q) - V(G)$. We build an H-partition of Q as follows. Start with $H := K_2$ with $V(H) = \{v, w\}$. Place one vertex of G in B_v and all the remaining vertices of G in B_w. Add a independent set of $|S|$ of vertices to H and make each connected to v and w. It is simple to verify that H is a simple planar graph.[2] Place each vertex of S in a new vertex (bag) of H. That defines an H-partition of Q where H is a simple planar graph and where all but one bag of H are solitary. See Figure 18 for an illustration of the result.

[2] More specifically $H = K^*_{2,|S|}$, where $K^*_{a,b}$ denotes a graph obtained from $K_{a,b}$ by modifying the part with the a vertices by adding all pairs of edges between those a vertices.



Simplicial blowup Q of G *H-partition of Q*

Figure 18: Illustration of a simplicial blowup Q of a graph G, with $V(G) = \{u_1, \cdots, v_8\}$ and $S = \{s_1, s_2, s_3\}$, and its H-partition.

The one non-solitary bag B_w thus forms an independent set in H. The density of H is then at most $(|G| - 1) \cdot \Delta(Q)$. By Lemma 2, we obtain the desired result, $\overline{cr}(Q) \leq (|G| - 1) \cdot \Delta(Q) \cdot ||Q||$. □

The following result, obtained by setting $|G| = t$, is an immediate corollary of Lemma 4.

Corollary 2. *The complete graph, K_t, is $(t,\ t-1)$-agreeable.*

In order to prove the desired result for bounded treewidth graphs, we will use the following well known fact, which is equivalent to Theorem 9. For more basic facts about treewidth, see the survey by Bodlaender [34].

Theorem 13. *Any graph of treewidth at most t can be obtained by $(\leq t)$-clique-sums on graphs $G_1, \cdots, G_h$, where each G_i, $i \in [h]$, is the complete graph K_{t+1}.*

Armed with Lemma 2, Theorem 13 and our main tool, Theorem 12, we are now ready to prove that every bounded treewidth graph G has $\overline{cr}(G) \in O(\Delta(G) \cdot |G|)$.

Theorem 14. *For $k \geq 1$, let $\mathcal{G}$ denote a family of graphs of treewidth at most k. For every graph $G \in \mathcal{G}$, $\overline{cr}(G) \leq k \cdot (k+2) \cdot \Delta(G)||G||$.*

Proof. By Theorem 13, let $G_1, \cdots, G_h$ be the complete graphs K_{k+1} that are pieces in the decomposition of G. By Corollary 2 we have that for each i, $i \in [h]$, G_i is $(k+1,\ k)$-agreeable and thus $(k,\ k)$-agreeable. This fulfills the sole condition of Theorem 12. Thus, $\overline{cr}(G) \leq k \cdot (k+2) \cdot \Delta(G)||G||$. □

As stated in the introduction, this result improves on the previous, $O(k^2 \cdot \Delta(G)^2 \cdot ||G||)$, best known bound of Wood and Telle [30] stated in Theorem 6.

Since every k-simplicial blowup of any graph of treewidth at most k itself has treewidth at most k, we get the following immediate corollary of Theorem 14.

Lemma 5. *For every positive integer k, every graph of treewidth at most k is $(k,\ k \cdot (k+2))$-agreeable.*

3.5 *Proof of Theorem 11*

Recall the statement of Theorem 11. Lemma 3 states that every planar piece G_i of the decomposition is $(3,3)$-agreeable. Consider the non-planar pieces of the decomposition. By Theorem 10 of Robertson and Seymour, they have treewidth at most t, where $t \geq 3$, as graphs of treewidth at most 2 are planar [38]. Lemma 5 states that every piece of treewidth at most t is $(t,\ t \cdot (t+2))$-agreeable and thus since $t \geq 3$, they are $(3,\ t \cdot (t+2))$-agreeable. Since $t \geq 1$ for all pieces of the decomposition, if we choose $c := t \cdot (t+2)$ all the pieces of the decomposition are $(3, c)$-agreeable. Theorem 12 (and Theorem 10 by Robertson and Seymour) then implies that G has rectilinear crossing number at most $3 \cdot (t \cdot (t+2)+2) \cdot \Delta(G) \cdot ||G|| = 3 \cdot (t^2+2t+2) \cdot \Delta(G) \cdot ||G||$.

4 Conclusion

In this thesis, we proved that n-vertex bounded degree single-crossing minor-free graphs have $O(n)$ rectilinear crossing number. More strongly, we proved that for any fixed single-crossing graph X, every n-vertex X-minor-free graph G has rectilinear crossing number at most $O(\Delta(G) \cdot n)$. The result represents a strong improvement over the previous state of the art for single-crossing minor-free graphs, as argued in the introduction.

The ultimate goal for future work would be to obtain the above result for any fixed graph X. Note that not even an $O(f(\Delta) \cdot n)$ bound is known for such families for any function f. In fact, the best known bound on the rectilinear crossing number of such families is $O(n \log n)$ [39].

In order to attempt to prove an $O(\Delta(G) \cdot n)$ or $O(f(\Delta) \cdot n)$ bound on the rectilinear crossing number of all proper minor-closed families of graphs of bounded degree, Robertson and Seymour's graph minors theory tells us that it is necessary to prove the result for all bounded Euler genus graphs. That would be akin to Theorem 2 but with the crossing number replaced by the rectilinear crossing number. However, such a result is not even known for all bounded degree toroidal graphs, which are graphs that have an embedding on the torus surface. Hence, studying bounds on the rectilinear crossing number of toroidal graphs of bounded degree should be the starting point for this investigation.

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