Representation theory of Lie colour algebras and its connection with Brauer algebras

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Abstract

In this thesis, we study the representation theory of Lie colour algebras. Our strategy follows the work of G. Benkart, C. L. Shader and A. Ram in 1998, which is to use the Brauer algebras which appear as the commutant of the orthosymplectic Lie colour algebra when they act on a k-fold tensor product of the standard representation. We give a general combinatorial construction of highest weight vectors using tableaux, and compute characters of the irreducible summands in some borderline cases. Along the way, we prove the RSK-correspondence for tableaux and the PBW theorem for Lie colour algebras.

Dedications

To the ones that I love.

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Chapter 1

Introduction

One of the remarkable results in representation theory, now referred to as Schur-Weyl duality, connects the irreducible representations of the general linear group $GL_n(\mathbb{C})$ to the irreducible representations of the symmetric group S_k. Precisely, let V be an n-dimensional vector space. There is an action of $GL_n(\mathbb{C})$ on $V^{\otimes k}$ that commutes with the action of S_k on $V^{\otimes k}$. Moreover, the action of $GL_n(\mathbb{C})$ generates the full centralizer of the action of S_k, and vice versa. Thus Schur-Weyl duality says that there is a multiplicity free decomposition of $V^{\otimes k}$ as an $S_k \times GL_n(\mathbb{C})$-module, given by

$$V^{\otimes k} \cong \bigoplus_{\lambda} S^{\lambda} \otimes E^{\lambda}.$$

Here the sum is over all partitions λ of k with length at most n, and S^{λ} and E^{λ} are the irreducible representations of S_k and $GL_n(\mathbb{C})$ respectively parametrized by λ. From Schur-Weyl duality, one can compute the characters of these $GL_n(\mathbb{C})$-modules, which turn out to be Schur polynomials. These polynomials are certain symmetric polynomials, named after Issai Schur. Different variations of Schur polynomials appear as characters of representations of similar algebraic structures. For example the *hook*

Schur functions describe the characters of the irreducible representations of the Lie superalgebra $\mathfrak{gl}(m, n)$. See for example [BR87].

The main objective of this thesis is a generalization of Schur-Weyl duality due to G. Benkart, C. L. Shader and A. Ram to the setting of the orthosymplectic Lie colour algebras, $\mathfrak{spo}(V, \beta)$.

The general linear group $GL_n(\mathbb{C})$ admits several important subgroups, such as the orthogonal group $O(n)$ and the symplectic group $Sp(2n)$. In order to obtain a Schur-Weyl duality-like theorem in terms of $O(n)$, one has to find a larger algebra B whose action generates the full centralizer of the action of $O(n)$. Ideally, this algebra B should contain S_k as a subalgebra. In 1937, Richard Brauer introduced an algebra in [Bra37], now referred to as the Brauer algebra $B_k(\eta)$, where $\eta \in \mathbb{C}$. The Brauer algebra $B_k(n)$ (respectively $B_k(-2m)$) plays the same role as $\mathbb{C}[S_k]$ in Schur-Weyl duality when $GL_n(\mathbb{C})$ is replaced by $O(n)$ (respectively $Sp(2m)$). The Brauer algebra plays a principal role in the fundamental book, *The Classical Groups*, [Wey97] by Hermann Weyl.

The outline of the thesis is as follows. In Chapter 2, we study the orthosympletic Lie colour algebra, which is a family of Lie colour algebras which generalizes both the orthogonal and symplectic Lie algebras. In Chapter 3, we prove two versions of the Poincaré-Birkhoff-Witt (PBW) theorem in the Lie colour algebra setting. In Chapter 4, we discuss the commuting actions of the Brauer algebra and $\mathfrak{spo}(V, \beta)$ on a tensor space $V^{\otimes k}$. In Chapter 5, we construct highest weight vectors of $\mathfrak{spo}(V, \beta)$-submodules of $V^{\otimes k}$ following [BSR98]. In the last chapter, we calculate modules in a borderline case that is a case does not satisfy the hypothesis of [BSR98, Proposition 4.2]. We explore extra examples not covered by [BSR98, Proposition 4.2] or Theorem 5.4.1 which show how the conclusions can fail for a variety of reasons. Our main results are

Theorem 6.3.2 and Corollary 6.5.9. In Appendix A, we include a brief introduction to Schur polynomials and the RSK correspondence.

Our main original contributions in this thesis are

(i) We generalize the proof of the PBW theorem to the Lie colour algebra case. In fact, the PBW theorem for Lie algebras and Lie superalgebras are also deduced from our proof.

(ii) In [BSR98], Benkart et. al introduced a right action of the Brauer algebra $B_k(n-m)$ on $V^{\otimes k}$ which commutes with the left action of $\mathfrak{spo}(V,\beta)$ on $V^{\otimes k}$. We elaborate these commuting actions with detailed examples.

(iii) We give a detailed example in Section 5.5 to find the highest weight vectors of $V \otimes V$, and then find the submodules generated by the highest weight vectors.

(iv) Using this example for intuition, we extend one of the theorems [BSR98, Proposition 4.2] to a borderline case, which we do in Theorem 6.3.

(v) We compute the characters of the irreducible summands of $V \otimes V$ in this borderline case, and use them to show that the submodules we obtained in Theorem 6.3.2 coincide with those predicted by $\mathfrak{spo}(V,\beta) \times B_k$ duality.

In the future, it would be interesting to learn more about the combinatorial description of the characters of the $\mathfrak{spo}(V,\beta)$-submodules, in terms of variants of Young tableaux. Another unexplored avenue of research is the relation between the $\mathfrak{spo}(V,\beta)$-submodules generated by the highest weight vectors from Theorem 5.4.1 and the modules predicted by $\mathfrak{spo}(V,\beta) \times B_k$ duality. It is also interesting to extend [BSR98, Proposition 4.2] to other borderline cases beyond the ones we investigated here.

Chapter 2

Lie colour algebras and $\mathfrak{spo}(V, \beta)$

The goal of this chapter is to explore the structure theory of the orthosymplectic Lie colour algebra $\mathfrak{spo}(V, \beta)$. In Section 2.1, we give the definition of Lie colour algebras and analyze some of their basic properties. In Section 2.2, we define the modules of Lie colour algebras and give some examples such as the trivial module, the contragredient module and the tensor product of modules. Then we give the definition of $\mathfrak{g}$-module morphisms, and provide an important example, called the braiding morphism, in Proposition 2.2.10, which plays a vital role in the subsequent chapters. In Section 2.3, we describe the main object in this thesis, the othosymplectic Lie colour algebra $\mathfrak{spo}(V, \beta)$. In the last two sections, Section 2.4 and Section 2.5, we first give homogeneous bases of V, V^* and $\mathfrak{spo}(V, \beta)$ respectively, and then give an explicit description of the roots and root vectors in $\mathfrak{spo}(V, \beta)$. Moreover, we provide a basis of root vectors which will be used in the proof of one of the main theorems in our thesis, Theorem 5.4.1.

The material in this chapter comes from [BSR98] but the examples as well as the proofs are ours.

As of Section 2.4, in the rest of this thesis $\mathbb{F}$ will be assumed algebraically closed and $\operatorname{char} \mathbb{F} \neq 2$.

2.1 Lie colour algebras

Definition 2.1.1. *Let G be a finite abelian group with identity 1_G. A symmetric bicharacter on G over a field $\mathbb{F}$ is a map $\beta : G \times G \to \mathbb{F}^\times$ such that*

(i) $\beta(ab, c) = \beta(a, c)\beta(b, c)$, *for all* $a, b, c \in G$;

(ii) $\beta(a, bc) = \beta(a, b)\beta(a, c)$, *for all* $a, b, c \in G$; *and*

(iii) $\beta(a, b)\beta(b, a) = 1$, *for all* $a, b \in G$.

Notice that β is called a bicharacter because holding one variable fixed gives a character, which is a 1-dimensional representation in the other variable. In the rest of the thesis, β plays a very important role. We will define new categories of vector spaces, algebras and maps based on β.

Lemma 2.1.2. *Let β be a symmetric bicharacter on an abelian group G. Then we have the following:*

(i) $\beta(1_G, a) = \beta(a, 1_G) = 1$ *for all* a *in* G,

(ii) $\beta(a^{-1}, a) = \beta(a, a^{-1}) = \beta(a, a)^{-1}$ *for all* a *in* G,

(iii) $\beta(ab, ab) = \beta(a, a)\beta(b, b)$ *for all* a, b *in* G,

(iv) $\beta(a, a) \in \{-1, 1\}$ *for all* a *in* G.

Proof. Let $a, b \in G$. By Definition 2.1.1, we have

(i) $\beta(b,a) = \beta(1_G, a)\beta(b,a)$ which implies that $\beta(1_G, a) = 1$ for all a in G. Similarly we have $\beta(a, 1_G) = 1$ for all a in G.

(ii) Consequently we have $1 = \beta(a, 1_G) = \beta(a, aa^{-1}) = \beta(a,a)\beta(a,a^{-1})$. Hence $\beta(a, a^{-1}) = \beta(a,a)^{-1}$ for all a in G.

(iii) For all a, b in G, we have $\beta(ab, ab) = \beta(a,a)\beta(a,b)\beta(b,a)\beta(b,b) = \beta(a,a)\beta(b,b)$.

(iv) Since $\beta(a,b)\beta(b,a) = 1$, we have $\beta(a,a)\beta(a,a) = 1$ for all a in G. This implies $\beta(a,a) \in \{-1, 1\}$ for all a in G. □

Example 2.1.3. Let G be an abelian group. Then $\beta : G \times G \to \mathbb{F}^*$ such that $\beta(a,b) = 1$ for all a, b in G is a symmetric bicharacter.

Example 2.1.4. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\}$. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be in G. The group operation of G is $a + b \coloneqq (a_1 + b_1, a_2 + b_2)$. Let us verify that

$$\beta(a,b) = (-1)^{a \cdot b} \text{ where } a \cdot b = a_1 b_1 + a_2 b_2,\ a, b \in G$$

is a symmetric bicharacter.

Take $a = (a_1, a_2), b = (b_1, b_2)$ and $c = (c_1, c_2)$ in G. Then we have

$$\begin{aligned}\beta(a+b, c) = (-1)^{(a+b)\cdot c} &= (-1)^{(a_1+b_1)c_1 + (a_2+b_2)c_2} \\ &= (-1)^{a_1c_1 + a_2c_2}(-1)^{b_1c_1+b_2c_2} \\ &= \beta(a,c)\beta(b,c)\end{aligned}$$

which verifies the first condition of Definition 2.1.1. The second condition follows since

$a \cdot b = b \cdot a$. Now for the third condition, we have

$$\beta(a,b)\beta(b,a) = (-1)^{a \cdot b}(-1)^{b \cdot a} = (-1)^{2a \cdot b} = 1.$$

Therefore, the β we defined in Example 2.1.4 is a symmetric bicharacter.

Definition 2.1.5. *If an $\mathbb{F}$-vector space V has a direct sum decomposition*

$$V = \bigoplus_{a \in G} V_a,$$

where each V_a is subspace of V indexed by $a \in G$, then V is called a G-graded vector space. Moreover, if $v \in V_a$ for some $a \in G$, then v is called homogeneous of degree a. We say v has colour a.

Definition 2.1.6. *Given a symmetric bicharacter β on a group G, a Lie colour algebra $\mathfrak{g}$ is a G-graded vector space*

$$\mathfrak{g} = \bigoplus_{a \in G} \mathfrak{g}_a$$

together with an $\mathbb{F}$-bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

(i) $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{ab}$, *for all $a, b \in G$,*

(ii) $[x, y] = -\beta(b, a)[y, x]$, *for $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b$,*

(iii) For all $x \in \mathfrak{g}_a$, $y \in \mathfrak{g}_b$, and $z \in \mathfrak{g}_c$, the Jacobi identity for Lie colour algebras, given by

$$\beta(a,c)[x,[y,z]] + \beta(c,b)[z,[x,y]] + \beta(b,a)[y,[z,x]] = 0, \qquad (2.1.1)$$

is satisfied.

Lemma 2.1.7. *The Lie colour algebra Jacobi identity is equivalent to*

$$[x,[y,z]] = [[x,y],z] + \beta(b,a)[y,[x,z]], \text{ for all } x \in \mathfrak{g}_{\mathfrak{a}}, y \in \mathfrak{g}_{\mathfrak{b}} \text{ and } z \in \mathfrak{g}. \tag{2.1.2}$$

Proof. First suppose that $x \in \mathfrak{g}_a$, $y \in \mathfrak{g}_b$, and $z \in \mathfrak{g}_c$, for some $a, b, c \in G$. We swap z with $[x,y]$ in the second term of (2.1.1) and swap z with x in the third term of (2.1.1). Then (2.1.1) becomes

$$\beta(a,c)[x,[y,z]] - \beta(c,b)\beta(ab,c)[[x,y],z] - \beta(b,a)\beta(a,c)[y,[x,z]] = 0,$$

which can be simplified to

$$\beta(a,c)[x,[y,z]] - \beta(a,c)[[x,y],z] - \beta(b,a)\beta(a,c)[y,[x,z]] = 0.$$

Dividing both sides of the above equation by $\beta(a,c)$ and rearranging the equation gives

$$[x,[y,z]] = [[x,y],z] + \beta(b,a)[y,[x,z]].$$

The resulting equation is independent of the choice of c. Since (2.1.2) is linear in z, we thus conclude it holds for all $z \in \mathfrak{g}$. □

Example 2.1.8.

(i) Let $G = \{1_G\}$ be the trivial group. Then the Lie colour algebra $\mathfrak{g}$ is a Lie algebra in the classical sense.

(ii) Let $G = \mathbb{Z}_2 = \{0,1\}$, and $\beta(a,b) = (-1)^{ab}$ for all $a, b \in G$. Then the Lie colour algebra $\mathfrak{g}$ is a Lie superalgebra.

We can classify the elements a in G by the value of $\beta(a, a)$. According to Lemma 2.1.2, we know that $\beta(a, a) = \pm 1$. Therefore we define

$$G_{(0)} \coloneqq \{a \in G \mid \beta(a, a) = 1\} \text{ and } G_{(1)} \coloneqq \{a \in G \mid \beta(a, a) = -1\}. \tag{2.1.3}$$

We have $G = G_{(0)} \cup G_{(1)}$, and $G_{(0)}$ is a subgroup of G. Moreover, given a Lie colour algebra $\mathfrak{g}$, we define

$$\mathfrak{g}_{(0)} = \bigoplus_{a \in G_{(0)}} \mathfrak{g}_a \text{ and } \mathfrak{g}_{(1)} = \bigoplus_{a \in G_{(1)}} \mathfrak{g}_a.$$

This gives a decomposition of Lie colour algebras as vector spaces:

$$\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}.$$

Remark 2.1.9.

(i) $\mathfrak{g}_{(0)}$ is a Lie colour subalgebra of $\mathfrak{g}$.

(ii) $\mathfrak{g}_{(1)}$ is not subalgebra of $\mathfrak{g}$ unless $[\mathfrak{g}_{(1)}, \mathfrak{g}_{(1)}] = 0$.

The first part is clear since $G_{(0)}$ is a subgroup of G. Therefore $\mathfrak{g}_{(0)}$ is closed. For (ii): take $x \in \mathfrak{g}_a$ and $y \in \mathfrak{g}_b$ such that $\beta(a, a) = \beta(b, b) = -1$. So $x, y \in \mathfrak{g}_{(1)}$, and $[x, y] \in \mathfrak{g}_{ab}$. However, $\beta(ab, ab) = \beta(a, a)\beta(b, b) = 1$. Therefore $[x, y]$ is not in $\mathfrak{g}_{(1)}$ unless $[x, y] = 0$.

2.2 The category of modules of Lie colour algebras

In this section, as in classic Lie algebra textbooks, we discuss the concepts of Lie colour algebra modules and module morphisms. Moreover we define the braid morphism which will be a very important tool in later sections.

Definition 2.2.1. *Let $\mathfrak{g}$ be a Lie colour algebra. A $\mathfrak{g}$-module is a G-graded $\mathbb{F}$-vector space $V = \bigoplus_{a\in G} V_a$ together with a $\mathfrak{g}$-action*

$$\mathfrak{g} \times V \to V, \;\; (x, v) \mapsto xv$$

which is a bilinear map satisfying the following properties:

(i) If $x \in \mathfrak{g}_a$ and $v \in V_b$, then $xv \in V_{ab}$,

(ii) $[x, y]v = x(yv) - \beta(b, a)y(xv)$, for all $x \in \mathfrak{g}_a$, $y \in \mathfrak{g}_b$ and for all $v \in V$.

A $\mathfrak{g}$-module is also called a representation of $\mathfrak{g}$ over $\mathbb{F}$.

Example 2.2.2. Let $V = \mathfrak{g}$, with the action of $\mathfrak{g}$ on itself given by $(x, y) \mapsto [x, y]$ for all $x, y \in \mathfrak{g}$. Then together with this action, $\mathfrak{g}$ is a $\mathfrak{g}$-module by Definition 2.1.6 and Lemma 2.1.7. This $\mathfrak{g}$-module is called the adjoint representation.

Example 2.2.3. The trivial module of $\mathfrak{g}$ is the one-dimensional vector space V with grading $V = V_{1_G}$. The $\mathfrak{g}$-action is defined as $x \cdot v = 0$ for all $x \in \mathfrak{g}$ and $v \in V$.

Definition 2.2.4. *Let V be a $\mathfrak{g}$-module over the field $\mathbb{F}$. The contragredient module of V is the vector space $V^* := \{f : V \to \mathbb{F} \mid f$ is a linear functional $\}$, such that*

(i) the G-grading is defined by $(V^*)_a := \{f \in V^* \mid f(V_b) = 0$ if $a \neq b^{-1}\}$,

(ii) the $\mathfrak{g}$-action is defined on homogeneous elements by

$$(xf)(v) = -\beta(b, a)f(xv)$$

for all $x \in \mathfrak{g}_a, f \in (V^*)_b$ and $v \in V$.

Let $B = \{v_1, \ldots, v_n\}$ be a homogeneous basis of a $\mathfrak{g}$-module V. We denote $B^* = \{v^1, \ldots, v^n\}$ the dual basis of V^* defined by $v^i(v_j) = \delta_{i,j}$ for all $1 \leq i, j \leq n$. Notice that since B is a homogeneous basis of $V = \oplus_{a \in G} V_a$, it can be partitioned into bases for each V_a. Thus by the way we define v^i, we conclude that B^* is a homogeneous basis of V^*. Then we compute the colour of v^i by the following lemma.

Lemma 2.2.5. *Let $v_i \in B$ have colour c. Then v^i has colour c^{-1}.*

Proof. Let v_i have colour c. Since v^i is homogeneous, $v^i(v_i) = 1 \neq 0$ and $v^i(v_j) = 0$ for any v_j not in V_c implies that v^i has colour c^{-1} by Definition 2.2.4. □

Now that we have defined $\mathfrak{g}$-modules, we must define the morphisms between them.

Definition 2.2.6. *Let V and W be $\mathfrak{g}$-modules. A $\mathfrak{g}$-module morphism from V to W is an $\mathbb{F}$-linear map $\phi : V \to W$ satisfying*

(i) $\phi(xv) = x(\phi v)$ for all $x \in \mathfrak{g}$ and all $v \in V$, and

(ii) $\phi(V_a) \subseteq W_a$ for all $a \in G$.

The set of all $\mathfrak{g}$-module morphisms from V to W is denoted $\operatorname{Hom}_{\mathfrak{g}}(V, W)$.

Notice that If a $\mathfrak{g}$-module morphism has an inverse which is also a $\mathfrak{g}$-module morphism, then we call it $\mathfrak{g}$-module isomorphism. Moreover, if $V = W$ in Definition 2.2.6, then an element $\phi \in \operatorname{Hom}_{\mathfrak{g}}(V, V)$ is a graded operator on V which commutes with the action of $\mathfrak{g}$.

One frequently-used $\mathfrak{g}$-module example in this thesis is the tensor product given by the following definition.

Definition 2.2.7. *Suppose V and W are two $\mathfrak{g}$-modules. By definition V and W are G-graded vector spaces. Then $V \otimes W$ is again a G-graded vector space with respect to the G-grading*

$$(V \otimes W)_c = \bigoplus_{\substack{a,b \in G \\ ab=c}} V_a \otimes W_b.$$

We define the $\mathfrak{g}$-module structure on $V \otimes W$ by defining the $\mathfrak{g}$-action

$$x(v \otimes w) = xv \otimes w + \beta(b, a) v \otimes xw$$

for all $x \in \mathfrak{g}_a,\ v \in V_b$ and $w \in W$.

It is straightforward to verify that Definition 2.2.7 satisfies the conditions of Definition 2.2.1.

Example 2.2.8.

(i) Let V be the trivial module, as defined in Example 2.2.3. Then $(V \otimes W)_a = \mathbb{F} \otimes W_a \cong W_a$ for all $a \in G$. In particular, $V \otimes W \to W$ such that $a \otimes w \mapsto aw$ for all $a \in \mathbb{F}$ is a $\mathfrak{g}$-module isomorphism.

(ii) However, let $G = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ such that $V_{\bar{0}} = \{0\}$ and $V_{\bar{1}} = \mathbb{F}$. Let $W = W_{\bar{0}} \oplus W_{\bar{1}}$ such that $W_{\bar{0}} = \{0\}$ and $W_{\bar{1}} = W$. Let $\mathfrak{g}$ act trivially on V. Then $V \otimes W$ is isomorphic to W as vector spaces. However, the map $V_{\bar{1}} \otimes W_{\bar{1}} \to W_{\bar{1}}$ is not a $\mathfrak{g}$-module isomorphism since there does not exist a grading compatible homomorphism.

Lemma 2.2.9. *Let M, N, U, V be $\mathfrak{g}$-modules, and let $f : M \to U$ and $g : N \to V$ be two $\mathfrak{g}$-module morphisms. Then*

$$f \otimes g : M \otimes N \to U \otimes V$$

$$m \otimes n \mapsto f(m) \otimes g(n)$$

is again a $\mathfrak{g}$-module morphism.

Proof. We first verify the first condition of Definition 2.2.6. Take $x \in \mathfrak{g}_a, m \in M_b$ and $n \in N$. Then by Definition 2.2.7, $x(m \otimes n)$ is equal to

$$xm \otimes n + \beta(b, a)m \otimes xn. \tag{2.2.1}$$

Since $f(xm) = x(fm)$ and $g(xn) = x(gn)$, applying $f \otimes g$ to (2.2.1) gives

$$f(xm) \otimes g(n) + \beta(b, a)f(m) \otimes g(xn) = x(fm) \otimes g(n) + \beta(b, a)f(m) \otimes x(gn),$$

which is equal to

$$x\big(f(m) \otimes g(n)\big)$$

since f and g preserve colours.

Secondly, recall from Definition 2.2.7, we have

$$(M \otimes N)_c = \bigoplus_{\substack{a,b \in G \\ ab=c}} M_a \otimes N_b.$$

Then take any arbitrary homogeneous simple tensor $m_a \otimes n_b \in M_a \otimes N_b$. By 2.2.7, we have

$$(f \otimes g)(m_a \otimes n_b) = f(m_a) \otimes g(n_b) \in U_a \otimes V_b \subseteq (U \otimes V)_c.$$

By linearity, (ii) holds. □

Next we give some examples of $\mathfrak{g}$-module morphisms which play a vital role in

Chapter 4.

Proposition 2.2.10. *Let M, N be $\mathfrak{g}$-modules. Consider the linear map $\check{R}_{M,N}$ defined on homogeneous elements $m \in M_a$ and $n \in N_b$ by*

$$\begin{aligned}\check{R}_{M,N} : M \otimes N &\to N \otimes M \\ m \otimes n &\mapsto \beta(b,a) n \otimes m.\end{aligned}$$

Then $\check{R}_{M,N}$ is a $\mathfrak{g}$-module isomorphism called the braiding morphism.

Proof. First we verify that $\check{R}_{M,N}$ is a $\mathfrak{g}$-module morphism. The second condition of Definition 2.2.6 is satisfied since $m \otimes n$ has the same colour as $n \otimes m$. For the first condition, take $x \in \mathfrak{g}_a, m \in M_b$ and $n \in N_c$. Then we have

$$\begin{aligned}\check{R}_{M,N}\big(x(m \otimes n)\big) &= \check{R}_{M,N}(xm \otimes n + \beta(b,a) m \otimes xn) \\ &= \check{R}_{M,N}(xm \otimes n) + \beta(b,a)\check{R}_{M,N}(m \otimes xn) \\ &= \beta(c,ab) n \otimes xm + \beta(b,a)\beta(ac,b) xn \otimes m \\ &= \beta(c,a)\beta(c,b) n \otimes xm + \beta(b,a)\beta(a,b)\beta(c,b) xn \otimes m \\ &= \beta(c,a)\beta(c,b) n \otimes xm + \beta(c,b) xn \otimes m \\ &= \beta(c,b)\big(xn \otimes m + \beta(c,a) n \otimes xm\big) \\ &= \beta(c,b) x(n \otimes m) \\ &= x\check{R}_{M,N}(m \otimes n).\end{aligned}$$

Therefore by Definition 2.2.6, $\check{R}_{M,N}$ is a $\mathfrak{g}$-module morphism.

Moreover, since $\beta(a,b)\beta(b,a) = 1$ for all $a, b \in G$, we have that $\check{R}^2_{M,N} = 1_{M \otimes N}$. Thus, $\check{R}_{M,N}$ is an $\mathfrak{g}$-module isomorphism. $\square$

Let V be a $\mathfrak{g}$-module. The braid morphism $\check{R}_{V,V}$ is an isomorphism of $V \otimes V$ defined by swapping two factors, that is, sending $v_a \otimes v_b$ to $\beta(b,a)v_b \otimes v_a$ for all homogenous elements $v_a, v_b \in V$ with colour a and b respectively.

Then for all $k \in \mathbb{Z}_{\geq 2}$, we define $\check{R}_i = \check{R}_{(i,i+1)}$ on $V^{\otimes k}$ by

$$\check{R}_i = id^{\otimes(i-1)} \otimes (-\check{R}_{V,V}) \otimes id^{\otimes(k-i-1)}. \tag{2.2.2}$$

Notice that since $\check{R}_{V,V}$ is a $\mathfrak{g}$-module isomorphism, $-\check{R}_{V,V}$ is also a $\mathfrak{g}$-module isomorphism. The reason we introduce this minus sign is that in Chapter 4, we will define a right action of the Brauer algebra on $V^{\otimes k}$. Then the action of the generator s_i on $V^{\otimes k}$ on the right will be the same as $\check{R}_i$ acting on $V^{\otimes k}$ on the left.

Lemma 2.2.11. *The maps $\check{R}_i$ satisfy the same relations as the standard generators of the symmetric group. Namely we have*

(i) $\check{R}_i^2 = 1$,

(ii) $\check{R}_i\check{R}_j = \check{R}_j\check{R}_i$ *if* $|i-j| \geq 2$,

(iii) $\check{R}_i\check{R}_{i+1}\check{R}_i = \check{R}_{i+1}\check{R}_i\check{R}_{i+1}$.

Proof. The proof is a straightforward calculation from the definition of $\check{R}_i$'s and the property of the symmetric bicharacter. □

Consequently, for any permutation π, if we choose an expression $\pi = s_{i_1} \cdots s_{i_p}$ as a product of adjacent transpositions, where $s_{i_j} = (i_j, i_j + 1)$, then we can define $\check{R}_\pi$ as

$$\check{R}_\pi = \check{R}_{i_1}\check{R}_{i_2} \cdots \check{R}_{i_p}. \tag{2.2.3}$$

Lemma 2.2.11 implies that $\check{R}_\pi$ is independent of the choice of expression.

Lemma 2.2.12. *Let V be a $\mathfrak{g}$-module with homogeneous basis $B = \{v_1, \ldots, v_n\}$. Let V^* be the contragredient module of V with homogeneous basis $B^* = \{v^1, \ldots, v^n\}$ dual to B. Let V_{1_G} be the trivial $\mathfrak{g}$-module. Then the following maps*

(i) the map $pr_V : V_{1_G} \to V \otimes V^*$ such that $1 \mapsto \sum_{i=1}^n v_i \otimes v^i$, and

(ii) the evaluation map $ev_V : V^* \otimes V \to V_{1_G}$ such that $v^i \otimes v_j \mapsto \delta_{i,j}$ for all $1 \leq i, j \leq n$

are $\mathfrak{g}$-module morphisms.

Remark 2.2.13. Note that the map pr_V is also called the coevaluation map.

Proof. Note that $pr_V(x1) = pr_V(0) = 0$ for all $x \in \mathfrak{g}$. In order to prove that pr_V satisfies the first condition of Definition 2.2.6, it suffices to show that for all homogeneous elements $x \in \mathfrak{g}$, $xpr_V(1) = x(\sum_{i=1}^n v_i \otimes v^i) = 0$, that is to show $\Omega = \sum_{i=1}^n v_i \otimes v^i$ is a $\mathfrak{g}$-invariant tensor.

Let $B = \{v_1, \ldots, v_n\}$ be a homogeneous basis of V such that for all $1 \leq i \leq n$, v_i has colour $a_i \in G$. By Lemma 2.2.5, the dual basis vector v^i has colour a_i^{-1}. Let $x \in \mathfrak{g}$ be homogeneous with colour $b \in G$. We then write xv_i and xv^i explicitly. First notice that

$$xv_i = \sum_{k=1}^n c_{ik} v_k, \tag{2.2.4}$$

for some $c_{ik} \in \mathbb{F}$. Then for each $1 \leq j \leq n$, we have

$$\begin{aligned}(xv^i)(v_j) &= -\beta(a_i^{-1}, b)v^i(xv_j) \\ &= -\beta(a_i^{-1}, b)\sum_{\ell=1}^n c_{j\ell} v^i(v_\ell) \\ &= -\beta(a_i^{-1}, b)c_{ji}.\end{aligned}$$

Thus we deduce that

$$xv^i = \sum_{j=1}^{n} -\beta(c_i^{-1}, a)c_{ji}v^j. \tag{2.2.5}$$

Therefore we have

$$\begin{aligned} x\left(\sum_{i=1}^{n} v_i \otimes v^i\right) &= \sum_{i=1}^{n} xv_i \otimes v^i + \sum_{i=1}^{n} \beta(a_i, b)v_i \otimes xv^i \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} c_{ik}v_k \otimes v^i + \sum_{i=1}^{n}\sum_{j=1}^{n} -\beta(a_i, b)\beta(a_i^{-1}, b)c_{ji}v_i \otimes v^j \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} c_{ik}v_k \otimes v^i - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}v_i \otimes v^j = 0. \end{aligned}$$

Thus follows from the fact that both 1 and Ω have colour 1, pr_V is a $\mathfrak{g}$-module morphism. The proof of ev_V being a $\mathfrak{g}$-module morphism is similar. □

Notice that the maps in Lemma 2.2.12 are independent of the choice of basis. See for example [Kas95, Section II.3].

Now let V and W be two $\mathfrak{g}$-modules. We notice that $pr_{V\otimes W}$ can be obtained by the following composition of the maps:

$$V_{1_G} \xrightarrow{pr_V} V \otimes V^* \cong V \otimes V_{1_G} \otimes V^* \xrightarrow{\mathrm{id}\otimes pr_W \otimes \mathrm{id}} V \otimes W \otimes W^* \otimes V^*.$$

Similarly the map $ev_{V\otimes W}$ can be obtained by

$$W^* \otimes V^* \otimes V \otimes W^* \xrightarrow{\mathrm{id}\otimes ev_V \otimes \mathrm{id}} W^* \otimes V_{1_G} \otimes W \cong W \otimes W^* \xrightarrow{ev_W} V_{1_G}.$$

Thus inductively, we have the following lemma.

Lemma 2.2.14. *Let V be a $\mathfrak{g}$-module. Let $B = \{v_1, \ldots, v_n\}$ be a homogeneous basis of V. Let $\{v^1, \ldots, v^n\}$ be the dual of B such that $v^i(v_j) = \delta_{i,j}$ for all $1 \le i, j \le n$. Let*

$k \geq 1$. *Then both of the following maps are* $\mathfrak{g}$*-module morphisms*

(i) $\tilde{pr}_k : V_{1_G} \to V^{\otimes k} \otimes (V^*)^{\otimes k}$ *such that*

$$1 \mapsto \sum_{1\leq i_1,\ldots,i_k\leq n} v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v^{i_k} \otimes \cdots \otimes v^{i_1}, \text{ and}$$

(ii) $\tilde{ev}_k : (V^*)^{\otimes k} \otimes V^{\otimes k} \to V_{1_G}$ *such that*

$$\tilde{ev}_k(v^{i_1} \otimes \cdots \otimes v^{i_k} \otimes v_{j_k} \otimes \cdots \otimes v_{j_1}) \mapsto \delta_{i_k,j_k} \cdots \delta_{i_1,j_1} = \prod_{\ell=1}^{k} \delta_{i_\ell,j_\ell},$$

extended by linearity.

Now let us talk about the category of $\mathfrak{g}$-modules.

Definition 2.2.15. *Let* $\mathfrak{g}$ *be a finite dimensional Lie colour algebra over a field* $\mathbb{F}$*. We denote the category of* $\mathfrak{g}$*-modules by* $\mathcal{C}$*. Then the morphisms are the* $\mathfrak{g}$*-module morphisms.*

In particular, for any two objects V and W in $\mathcal{C}$, by Definition 2.2.7, $V \otimes W$ is a $\mathfrak{g}$-module. Therefore, $V \otimes W$ is in $\mathcal{C}$. With this extra tensor structure, $\mathcal{C}$ becomes a *monoidal category.*

We showed there was an identity object V_{1_G} such that $V \otimes V_{1_G} \cong V_{1_G} \otimes V \cong V$ for all $V \in \mathcal{C}$. Also, Proposition 2.2.10 provides us an $\mathfrak{g}$-module isomorphism to swap $V \otimes W$ to $W \otimes V$ with some β factor generated. With this morphism, $\mathcal{C}$ becomes a *braided monoidal category.*

Furthermore, for any object V in $\mathcal{C}$, the dual V^* of V is still an $\mathfrak{g}$-module. With the morphisms defined in Lemma 2.2.12, $V_{1_G} \to V \otimes V^*$, the category $\mathcal{C}$ becomes a *braided rigid monoidal category* of $\mathfrak{g}$-modules.

2.3 The orthosymplectic Lie colour algebra $\mathfrak{spo}(V,\beta)$

In this section, we first construct the general linear Lie colour algebra, $\mathfrak{gl}(V,\beta)$. Then we provide a particular subalgebra of $\mathfrak{gl}(V,\beta)$, the orthosymplectic Lie colour algebra $\mathfrak{spo}(V,\beta)$. The rest of this paper will mainly focus on this special case.

Let V be a graded vector space over a field $\mathbb{F}$, and let G be an abelian group. Let β be a symmetric bicharacter of G. Let $\mathrm{End}(V)$ be the vector space of all $\mathbb{F}$-linear maps from V to V. We let

$$\mathfrak{gl}(V,\beta)_a \coloneqq \{x \in \mathrm{End}(V) \ \mid xV_b \subseteq V_{ab} \text{ for all } b \in G\}. \tag{2.3.1}$$

Definition 2.3.1. *The general linear Lie colour algebra is the vector space*

$$\mathfrak{gl}(V,\beta) = \bigoplus_{a\in G} \mathfrak{gl}(V,\beta)_a$$

equipped with the Lie colour algebra bracket defined by

$$[x,y] = xy - \beta(b,a)yx$$

for all $x \in \mathfrak{gl}(V,\beta)_a$, $y \in \mathfrak{gl}(V,\beta)_b$.

Remark 2.3.2. Notice that as a vector space, $\mathfrak{gl}(V,\beta)$ is isomorphic to $\mathrm{End}(V)$.

Lemma 2.3.3. *Equipped with the Lie colour bracket defined above,* $\mathfrak{gl}(V,\beta)$ *is a Lie colour algebra.*

Proof. We need to show that $\mathfrak{gl}(V,\beta)$ satisfies Definition 2.1.6.

(i) Let $x \in \mathfrak{gl}(V,\beta)_a$, $y \in \mathfrak{gl}(V,\beta)_b$ and $v \in V_c$ for some a, b and $c \in G$. Then we have

$$[x,y]v = xyv - \beta(b,a)yxv.$$

For the first term, we have

$$xyv \in xyV_c \subseteq xV_{bc} \subseteq V_{abc}$$

and similarly for the second term we have

$$\beta(b,a)yxv \in yxV_c \subseteq yV_{ac} \subseteq V_{bac} = V_{abc}$$

since G is abelian. So we have $[x,y]V_c \subseteq V_{abc}$ holds for all $c \in G$. So $[x,y] \in V_{ab}$. Therefore we have shown that $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{ab}$.

(ii) Take $x \in \mathfrak{gl}(V,\beta)_a$ and $y \in \mathfrak{gl}(V,\beta)_b$. Then we have

$$-\beta(b,a)[y,x] = -\beta(b,a)yx + \beta(b,a)\beta(a,b)xy = [x,y].$$

(iii) The Lie colour Jacobi identity can be proved by a straightforward calculation by using the relations $\beta(a,b)\beta(b,a) = 1$ and $\beta(ab,c) = \beta(a,c)\beta(b,c)$ for all $a,b,c \in G$. □

Remark 2.3.4. We adopt the convention that if $v \in V_a$ and $u \in V_b$, then we write

$$\beta(v,u) = \beta(a,b), \text{ and } \mathfrak{gl}(V,\beta)_v = \mathfrak{gl}(V,\beta)_a.$$

That is, we use a vector itself to denote its colour and by extension, we write v^{-1} for a^{-1}.

We next construct a homogeneous basis for $\mathfrak{gl}(V,\beta)$ and discuss the colours of its basis vectors.

Let $B = \{v_1, \dots, v_n\}$ be a homogeneous basis of V. Recall that the elementary matrix $E_{v_i v_j}$ in $\mathrm{End}(V)$ is defined by $E_{v_i v_j} v_k = \delta_{jk} v_i$ for all $1 \le i, j, k \le n$, where δ_{jk} is the Kronecker delta.

Lemma 2.3.5. *The set $\{E_{v_i v_j} \mid 1 \le i, j \le n\}$ forms a homogeneous basis for $\mathfrak{gl}(V,\beta)$.*

Proof. By Remark 2.3.2, $\mathfrak{gl}(V,\beta)$ is isomorphic to $\mathrm{End}(V)$ as vector space. Thus the set $\{E_{v_i v_j} \mid 1 \le i, j \le n\}$ forms a basis for $\mathfrak{gl}(V,\beta)$. Then since $E_{v_i v_j}$ sends v_i to v_j for all $1 \le i, j \le n$, by (2.3.1) it has colour $v_i v_j^{-1}$. Therefore $E_{v_i v_j} \in \mathfrak{gl}(V,\beta)_{v_i v_j^{-1}}$ and thus it is homogeneous. □

Now we begin to construct the orthosymplectic Lie colour algebras.

Definition 2.3.6. *Let V be a G-graded vector space. An $\mathbb{F}$-bilinear map*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

is called a β-skew-symmetric bilinear form if it satisfies:

(i) $\langle \cdot, \cdot \rangle$ is nondegenerate;

(ii) $\langle V_a, V_b \rangle = 0$ whenever $b \neq a^{-1}$; and

(iii) $\langle v, w \rangle + \beta(b,a)\langle w, v \rangle = 0$, for all $v \in V_a$, $w \in V_b$.

Remark 2.3.7. From part *(ii)* of Definition 2.3.6, it follows that part *(iii)* holds trivially for $b \neq a^{-1}$. If however, $b = a^{-1}$, we have $\beta(b,a) = \beta(a^{-1},a) = \beta(a,a)^{-1} = \pm 1$. Therefore for all $v \in V_a$ and $w \in W_{a^{-1}}$, $\langle v, w \rangle + \beta(a^{-1},a)\langle w, v \rangle = 0$ implies $\langle v, w \rangle = \pm \langle w, v \rangle$.

We explore this form further in Section 2.4.

Definition 2.3.8. *The orthosymplectic Lie colour algebra*

$$\mathfrak{spo}(V,\beta) = \bigoplus_{a\in G} \mathfrak{spo}(V,\beta)_a$$

is the subalgebra of the Lie colour algebra $\mathfrak{gl}(V,\beta)$*, where for each* $a \in G$*, the graded component* $\mathfrak{spo}(V,\beta)_a$ *is defined as*

$$\mathfrak{spo}(V,\beta)_a = \{x \in \mathfrak{gl}(V,\beta)_a \ \mid \langle xw, v\rangle + \beta(b,a)\langle w, xv\rangle = 0,\ \forall w \in V_b,\ v \in V\}. \quad (2.3.2)$$

Lemma 2.3.9. *Equipped with the Lie colour bracket defined above,* $\mathfrak{spo}(V,\beta)$ *is a Lie colour algebra.*

Proof. Since for all $a \in G$, $\mathfrak{spo}(V,\beta)_a \subseteq \mathfrak{gl}(V,\beta)_a$. we have $\mathfrak{spo}(V,\beta) \subseteq \mathfrak{gl}(V,\beta)$ as G-graded vector spaces. Next we prove that $\mathfrak{spo}(V,\beta)$ is closed under the Lie colour bracket.

It suffices to show that for homogeneous elements x and y in $\mathfrak{spo}(V,\beta)$, we have $[x,y] \in \mathfrak{spo}(V,\beta)_{xy}$. Therefore it is enough to show that

$$\langle [x,y]w, v\rangle = -\beta(w,xy)\langle w, [x,y]v\rangle, \text{ for all } v \in V.$$

The method is to expand $[x,y] = xy - \beta(y,x)yx$ first, and use the linearity of $\langle\cdot,\cdot\rangle$. Therefore

$$\langle [x,y]w, v\rangle = \langle xyw, v\rangle - \beta(y,x)\langle yxw, v\rangle. \quad (2.3.3)$$

The first term of (2.3.3) can be written as

$$\begin{aligned}\langle xyw, v\rangle &= -\beta(yw, x)\langle yw, xv\rangle \\ &= \beta(yw, x)\beta(w, y)\langle w, yxv\rangle \\ &= \beta(w, xy)\beta(y, x)\langle w, yxv\rangle \\ &= \beta(w, xy)\langle w, \beta(y, x)yxv\rangle. \qquad (2.3.4)\end{aligned}$$

Similarly, the second term of (2.3.3) is $-\beta(y, x)\beta(xw, y)\beta(w, x)\langle w, xyv\rangle$ which can be simplified to

$$-\beta(w, xy)\langle w, xyv\rangle. \qquad (2.3.5)$$

Thus by adding (2.3.4) and (2.3.5) it follows that (2.3.5) is equal to

$$\beta(w, xy)\langle w, (\beta(y, x)yx - xy)v\rangle$$

which is $-\beta(w, xy)\langle w, [x, y]v\rangle$ as we claimed. □

2.4 Homogeneous bases for V, V^* and $\mathfrak{spo}(V, \beta)$

In this section, starting with a G-graded vector space carrying a β-skew-symmetric invariant bilinear form, we give homogeneous bases for a G-graded vector space V, its contragredient V^*, and $\mathfrak{spo}(V, \beta)$. Recall that from (2.1.3), we define

$$V_{(0)} = \bigoplus_{a \in G_{(0)}} V_a \qquad \text{and} \qquad V_{(1)} = \bigoplus_{a \in G_{(1)}} V_a.$$

Lemma 2.4.1. *Let $\langle \cdot, \cdot \rangle$ be as in Definition 2.3.6.*

(i) *The restricted form* $\langle \cdot, \cdot \rangle : V_{(0)} \times V_{(0)} \to \mathbb{F}$ *is a (nondegenerate) symplectic bilinear form.*

(ii) *The restricted form* $\langle \cdot, \cdot \rangle : V_{(1)} \times V_{(1)} \to \mathbb{F}$ *is a nondegenerate and symmetric bilinear form.*

Proof. Let $a, b \in G$. If $ab \neq 1$, then $\langle V_a, V_b \rangle = 0$. So we may assume that $b = a^{-1}$. Now if $a \in G_{(i)}$ for $i = 0, 1$, then by Lemma 2.1.2, $\beta(a^{-1}, a^{-1}) = \beta(a, a^{-1})^{-1} = \beta(a, a) = \pm 1$ implies $a^{-1} \in G_{(i)}$. Thus for all $v \in V_a$ and $w \in V_{a^{-1}}$, we have

$$\langle v, w \rangle = -\beta(a^{-1}, a)\langle w, v \rangle = \pm \langle w, v \rangle.$$

Thus $\langle \cdot, \cdot \rangle : V_{(0)} \times V_{(0)} \to \mathbb{F}$ is skew-symmetric, and $\langle \cdot, \cdot \rangle : V_{(1)} \times V_{(1)} \to \mathbb{F}$ is symmetric. Next we prove non-degeneracy. By Definition 2.3.6, $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ is a non-degenerate form. Then for each nonzero v in V, there exists some w in V such that $\langle v, w \rangle \neq 0$. Write $w = \sum_{b \in G} w_b$ as a sum of homogeneous vectors. By Definition 2.3.6 (ii), $\langle v, w \rangle = \langle v, w_b \rangle$ where $b = a^{-1}$. Without loss of generality, we can replace w with w_b. Now if $v \in V_{(i)}$ for $i = 0, 1$, then $\pm 1 = \beta(v, v) = \beta(w^{-1}, w^{-1}) = \beta(w, w)$. Therefore w is in $V_{(i)}$ as well. Therefore $\langle \cdot, \cdot \rangle : V_{(i)} \times V_{(i)} \to \mathbb{F}$ is a non-degenerate form for $i = 0, 1$. □

Let us fix our notation. From now on, we fix

$$\dim V_{(0)} = m \text{ and } \dim V_{(1)} = n.$$

Moreover since $\langle \cdot, \cdot \rangle : V_{(0)} \times V_{(0)} \to \mathbb{F}$ is symplectic, we have $m = 2r$ for some $r \in \mathbb{Z}_{\geq 0}$. But $V_{(1)}$ can have odd or even dimension. We define $s \in \mathbb{Z}_{\geq 0}$ by declaring $n = 2s$ or $n = 2s + 1$.

We claim that there exists a homogeneous basis

$$B_{(0)} = \{t_1, t_1^*, \ldots, t_r, t_r^*\}$$

of $V_{(0)}$ which decomposes $V_{(0)}$ into an orthogonal direct sum of hyperbolic planes $H_i = \text{Span}\{t_i, t_i^*\}$. Namely, let $t \in V_{(0)}$ be homogeneous and nonzero. Then as in the proof of Lemma 2.4.1, there exists a homogeneous $t' \in V_{(0)}$ such that $\langle t, t' \rangle \neq 0$. Also since $\operatorname{char} \mathbb{F} \neq 2$, we can assume $\langle t, t \rangle = \langle t', t' \rangle = 0$. Thus, replacing t' by a scalar multiple if necessary, this gives a hyperbolic pair $\{t, t'\}$. Repeating this process on the orthogonal complement of $\text{Span}\{t, t'\}$ completes the proof.

If $\mathbb{F}$ is algebraically closed, by a similar argument, there exists a homogeneous basis

$$B_{(1)} = \{u_1, u_1^*, \ldots, u_s, u_s^*, (u_{s+1})\}$$

of $V_{(1)}$ where each $\{u_i, u_i^*\}$ is a hyperbolic pair. The vector u_{s+1} is only included if the dimension of $V_{(1)}$ is odd, and we put a bracket around such u_{2s+1} to indicate this. In this case, we have $u_{s+1}^* = u_{s+1}$. When $\mathbb{F}$ is not algebraically closed, we add the additional hypothesis that $V_{(1)}$ admits a basis $B_{(1)}$ of the above form. Moreover, we make the convention that we extend the definition of * such that $v^{**} = v$ for all $v \in B$.

Therefore we have the following homogeneous basis for V:

$$B = B_{(0)} \cup B_{(1)} = \{t_1, t_1^*, \ldots, t_r, t_r^*, u_1, u_1^*, \ldots, u_s, u_s^*, (u_{s+1})\}. \tag{2.4.1}$$

Remark 2.4.2. In what follows, we frequently need the basis vectors with no $*$ signs.

Therefore for future reference, we let

$$B' = \{t_1, \cdots, t_r, u_1, \cdots, u_s, (u_{s+1})\} = B'_{(0)} \cup B'_{(1)}$$

where $B'_{(0)} = \{t_1, \cdots, t_r\}$ and $B'_{(1)} = \{u_1, \cdots, u_s, (u_{s+1})\}$.

Since $\langle \cdot, \cdot \rangle$ is a nondegenerate form, it induces an isomorphism $F : V \to V^*$ by $v \mapsto \langle v, \cdot \rangle$ for all $v \in V$. Next we prove in fact F is a $\mathfrak{g}$-module isomorphism between V and its contragredient.

Lemma 2.4.3. *The map*

$$\begin{aligned} F : V &\to V^* \\ v &\mapsto \langle v, \cdot \rangle \end{aligned}$$

is a $\mathfrak{g}$-module isomorphism.

Proof. It suffices to prove that for all homogeneous $x \in \mathfrak{g}$ and homogeneous $v, w \in V$ we have $F(xv)(w) = (xF(v))(w)$. Since $F(v) \in V^*$, by Definition 2.2.4, we have

$$(xF(v))(w) = -\beta(F(v), x)F(v)(xw) = -\beta(F(v), x)\langle v, xw \rangle.$$

By (2.3.2), we have the relation

$$\langle xv, w \rangle + \beta(F(v), x)\langle v, xw \rangle = 0$$

which implies that

$$F(xv)(w) = \langle xv, w \rangle = -\beta(F(v), x)\langle v, xw \rangle = (xF(v))(w)$$

for all $w \in V$.

Note that F is bijective, and let F^{-1} be the inverse linear map of F. Let $w \in V^*$ and notice that $F(x(F^{-1}w)) = (xF)(F^{-1}w) = xw$, thus applying F^{-1} to both sides we get $(xF^{-1})(w) = F^{-1}(xw)$ for all $x \in \mathfrak{g}$ and $w \in V^*$. Therefore F^{-1} is a $\mathfrak{g}$-module morphism. Thus, F is a $\mathfrak{g}$-module isomorphism. □

Lemma 2.4.4. *With the above setting, if $v_i \in B$ has colour c, then*

(i) $F(v_i)$ has colour c,

(ii) $F^{-1}(v^i)$ has colour c^{-1}.

Proof. The result is immediate by using the facts that both F and F^{-1} are $\mathfrak{g}$-module morphisms and v^i has opposite colour as v_i. □

We recall the following fact. Let $B = \{v_1, \ldots, v_n\}$ be an ordered basis of $(V, \langle\cdot,\cdot\rangle)$. Then denote by $\{v^1, \ldots, v^n\}$ the dual basis of V^* defined by $v^i(v_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$. With respect to these bases, the matrix F_B of the map $F : V \to V^*$ such that $v \mapsto \langle v, \cdot\rangle$ is given by $(F_B)_{i,j} = F_{i,j} = \langle v_i, v_j\rangle$. We follow [BSR98] to use $F^{-1}_{i,j}$ to denote the (i, j) entry of the inverse matrix F_B^{-1}.

Now based on the homogeneous basis B in (2.4.1), we can find the explicit matrix F_B. By the calculation above (2.4.1), for $i = 1, \ldots, r$ and $j = 1, \ldots, s, (s+1)$, we have

$$F_{t_i,t_i^*} = 1,\ F_{t_i^*,t_i} = -1,\ F_{u_j,u_j^*} = 1,\ F_{u_j^*,u_j} = 1,\ \text{and } F_{v,v'} = 0 \text{ otherwise.} \qquad (2.4.2)$$

The inverse of the matrix F_B is the matrix F_B^{-1} with

$$F^{-1}_{t_i,t_i^*} = -1,\ F^{-1}_{t_i^*,t_i} = 1,\ F^{-1}_{u_j,u_j^*} = 1,\ F^{-1}_{u_j^*,u_j} = 1,\ \text{and } F_{v,v'} = 0 \text{ otherwise.} \qquad (2.4.3)$$

Lemma 2.4.5. *For all $v, w \in B'$, we have $F_{v^*,w} = -\beta(w, v^*)F_{w,v^*}$.*

Proof. Since all $v, w \in B$ are homogeneous, we have

$$F_{v^*,w} = \langle v^*, w\rangle = -\beta(w, v^*)\langle w, v^*\rangle = -\beta(w, v^*)F_{w,v^*}. \quad \square$$

Theorem 2.4.6. *The following vectors form a basis for $\mathfrak{spo}(V, \beta)$, as v and w range over B' as indicated:*

(i) $Y_{v^*,w} = E_{v^*,w} + \beta(v, v)\beta(wv^*, w)E_{w^*,v}$, *where $v \neq w$ if $v, w \in B'_{(1)}$;*

(ii) $Y_{v,w^*} = E_{v,w^*} + \beta(v^*, w)E_{w,v^*}$, *where $v \neq w$ if $v, w \in B'_{(1)}$;*

(iii) $Y_{v,w} = E_{v,w} - \beta(w, w)\beta(v, w)E_{w^*,v^*}$.

Proof. First, it is straightforward that the vectors in Theorem 2.4.6 are linearly independent. Next, we first show that those vectors are in $\mathfrak{spo}(V, \beta)$. Then we show that they span $\mathfrak{spo}(V, \beta)$. It suffices to find all the homogeneous vectors $x \in \mathfrak{gl}(V, \beta)$ that satisfy (2.3.2). Now take a homogeneous $x \in \mathfrak{gl}(V, \beta)$. We have the relation $\langle xv, w\rangle + \beta(v, x)\langle v, xw\rangle = 0$, which is equivalent to

$$v^t x^t F_B w + \beta(v, x)v^t F_B x w = 0.$$

First, letting v and w run over all pairs of vectors in the set B', we can reduce the above matrix equation into equations of the form:

$$(x^t)_{v,w^*}F_{w^*,w} + \beta(v, x)F_{v,v^*}x_{v^*,w} = 0.$$

After taking the transpose, we have

$$x_{w^*,v}F_{w*,w} + \beta(v,x)F_{v,v^*}x_{v^*,w} = 0. \tag{2.4.4}$$

By Lemma 2.4.5, we can replace $F_{w^*,w}$ by $-\beta(w, w^*)F_{w,w^*}$, and the equation becomes

$$-x_{w^*,v}\beta(w, w^*)F_{w,w^*} + \beta(v, x)F_{v,v^*}x_{v^*,w} = 0,$$

which is equivalent to

$$F_{w,w^*}x_{w^*,v} = \beta(w, w^*)^{-1}\beta(v, x)F_{v,v^*}x_{v^*,w}.$$

By using the fact that $F_{v,v^*} = 1$ for all $v \in B'$, and the fact that $\beta(w, w^*)^{-1} = \beta(w, w)$ for all $w \in B'$, we have

$$x_{w^*,v} = \beta(w, w)\beta(v, x)x_{v^*,w}. \tag{2.4.5}$$

If $x_{v^*,w} \neq 0$, then since x is homogeneous in $\mathfrak{gl}(V, \beta)$, it must have the same colour as $E_{v^*,w}$ which is v^*w^*. Therefore (2.4.5) becomes $x_{w^*,v} = \beta(w, w)\beta(v, v^*w^*)x_{v^*,w}$, which (by using the fact $\beta(v, v^*) = \beta(v, v)$ for all $v \in B'$ and the fact $\beta(v^*, w) = \beta(v, w^*)$) is

$$x_{w^*,v} = \beta(w, w)\beta(v, v)\beta(v^*, w)x_{v^*,w}, \tag{2.4.6}$$

which also holds if $x_{v^*,w} = 0$.

Similarly, for all $v, w \in B'$, we have the relation

$$x_{w,v^*} = \beta(v^*, w)x_{vw^*}, \tag{2.4.7}$$

and for all $v, w \in B'$, we have

$$x_{w^*,v^*} = -\beta(w,w)\beta(v,w)x_{v,w}. \tag{2.4.8}$$

Note that when $v = w$ in $B'_{(1)}$, $\beta(v,v) = -1$. Therefore in these cases (2.4.7) and (2.4.6) imply $x_{v,v^*} = x_{v^*,v} = 0$.

Thus x is a homogeneous element of $\mathfrak{spo}(V,\beta)$ if and only if its entries satisfy (2.4.6), (2.4.7) and (2.4.8). Comparing these conditions with the list of vectors in the statement of the proposition yields:

(i) $Y_{v^*,w} = E_{v^*,w} + \beta(v,v)\beta(wv^*,w)E_{w^*,v} \in \mathfrak{spo}(V,\beta)_{v^*w^*}$, where $v \neq w$ if $v, w \in B'_{(1)}$;

(ii) $Y_{v,w^*} = E_{v,w^*} + \beta(v^*,w)E_{w,v^*} \in \mathfrak{spo}(V,\beta)_{vw}$, where $v \neq w$ if $v, w \in B'_{(1)}$;

(iii) $Y_{v,w} = E_{v,w} - \beta(w,w)\beta(v,w)E_{w^*,v^*} \in \mathfrak{spo}(V,\beta)_{vw^*}$

which proves that the vectors in Theorem 2.4.6 are all in $\mathfrak{spo}(V,\beta)$.

Now in order to show that these vectors span $\mathfrak{spo}(V,\beta)$, we first give an order to the basis B. Namely, we write

$$B = \{t_1 < t_1^* < \ldots < t_r < t_r^* < u_1 < u_!^* < \ldots < u_s < u_s^* < (u_{s+1})\}.$$

Take arbitrary $x \in \mathfrak{spo}(V,\beta)$. We let $x_{v,w}$ be the (v,w)-entry for all $v, w \in B$. Then we write $x = \sum_{v,w\in B} x_{v,w}E_{v,w}$ which can be regrouped as

$$x = \sum_{v,w\in B'} (x_{v,w}E_{v,w} + x_{w^*,v^*}E_{w^*,v^*}) + \sum_{\substack{v<w \\ v,w\in B'}} (x_{v^*,w}E_{v^*,w} + x_{w^*,v}E_{w^*,v})$$

$$+\sum_{\substack{v<w\\ v,w\in B'}}(x_{v,w^*}E_{v,w^*}+x_{w,v^*}E_{w,v^*})+\sum_{v\in B'_{(0)}}(x_{v^*,v}E_{v^*,v}+x_{v,v^*}E_{v,v^*}). \quad (2.4.9)$$

Applying the relation (2.4.8) to the first term yields

$$\sum_{v,w\in B'}x_{v,w}\left(E_{v,w}-\beta(w,w)\beta(v,w)E_{w^*,v^*}\right)=\sum_{v,w\in B'}x_{v,w}Y_{v,w}.$$

Similarly, applying relation (2.4.6) and (2.4.7) to the second and third terms of (2.4.9) yields

$$\sum_{\substack{v<w\\ v,w\in B'}}x_{v^*,w}Y_{v^*,w}\ \text{ and }\ \sum_{\substack{v<w\\ v,w\in B'}}x_{v,w^*}Y_{v,w^*}$$

respectively. For the fourth term of (2.4.9), notice that $Y_{v,v^*}=2E_{v,v^*}$ and $Y_{v^*v}=2E_{v^*,v}$. Thus the fourth term of (2.4.9) is

$$\sum_{v\in B'_{(0)}}\frac{1}{2}(x_{v^*,v}Y_{v^*,v}+x_{v,v^*}Y_{v,v^*}).$$

Therefore we can write any arbitrary $x\in\mathfrak{spo}(V,\beta)$ as

$$x=\sum_{v,w\in B'}x_{v,w}Y_{v,w}+\sum_{\substack{v<w\\ v,w\in B'}}x_{v^*,w}Y_{v^*,w}+\sum_{\substack{v<w\\ v,w\in B'}}x_{v,w^*}Y_{v,w^*}+\frac{1}{2}\sum_{v\in B'_{(0)}}(x_{v^*,v}Y_{v^*,v}+x_{v,v^*}Y_{v,v^*}).$$

Thus, the vectors in Theorem 2.4.6 span $\mathfrak{spo}(V,\beta)$, and in turn, they form a homogeneous basis of $\mathfrak{spo}(V,\beta)$. □

Remark 2.4.7. When $v=w\in B'$, the vector $Y_{v,w}$ simplifies to $H_v=E_{v,v}-E_{v^*,v^*}$. Notice that these elements always have colour 1, regardless of the grading on V.

2.5 Roots and root vectors in $\mathfrak{spo}(V,\beta)$

In this section, let $\mathfrak{g} = \mathfrak{spo}(V,\beta)$. We compute the weights of each basis vector in (2.4.1) of V under the standard action of $\mathfrak{g}$. Then we construct a root system of $\mathfrak{g}$ with respect to the homogeneous basis in Theorem 2.4.6. The first step is to find a Cartan subalgebra of $\mathfrak{g}$ and its dual.

Recall that we have a homogeneous basis

$$B = \{t_1, t_1^*, \ldots, t_r, t_r^*, u_1, u_1^*, \ldots, u_s, u_s^*, (u_{s+1})\}$$

of V, and the corresponding subset $B' = \{t_1, \ldots, t_r, u_1, \ldots, u_s, (u_{s+1})\}$ of V.

Definition 2.5.1. *Let $H_v = E_{v,v} - E_{v^*,v^*}$ as in Remark 2.4.7. We call*

$$\mathfrak{h} = \mathrm{Span}_{\mathbb{F}}\{H_v \mid v \in B'\}.$$

the standard Cartan subalgebra of $\mathfrak{g}$.

Lemma 2.5.2. *The set $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$.*

Proof. Take H_v and H_w in $\mathfrak{h}$. We have $[H_v, H_w] = H_vH_w - \beta(H_w, H_v)H_wH_v$ which is $H_vH_w - H_wH_v$ since H_v has colour 1_G. Also since H_v is diagonal matrix for all $v \in B'$, H_v commutes with H_w. Therefore we have $[H_v, H_w] = 0$ for all $v \in B'$. Thus $\mathfrak{h}$ is an abelian subalgebra. □

Definition 2.5.3. *An $\mathfrak{g}$-module W has a weight space decomposition with respect to $\mathfrak{h}^*$ if*

$$W = \bigoplus_{\alpha \in \mathfrak{h}^*} W_\alpha,$$

where $W_\alpha = \{w \in W \mid Hw = \alpha(H)w$ *for all* $H \in \mathfrak{h}\}$. *We call* α *a weight if the corresponding weight space* W_α *is nonzero, and the nonzero elements of a given weight space are called weight vectors.*

Definition 2.5.4. *The dual basis of* $\mathfrak{h}$ *is the set*

$$\{\alpha_v \in \mathfrak{h}^* \mid \alpha_v(H_w) = \delta_{vw} \text{ for all } v, w \in B'\}. \tag{2.5.1}$$

Let us prove that the α_v's in (2.5.1) are weights of the standard representation of $\mathfrak{spo}(V,\beta)$ on V.

Lemma 2.5.5. *Each basis vector* w *in* B *is a weight vector. For all* $w \in B'$, *the weight of* w *is* α_w *and the weight of* w^* *is* $-\alpha_w$. *In particular, the weight of* $u_{s+1} \in B$ *is* 0.

Proof. Let $v, w \in B'$ and take $H_v = E_{v,v} - E_{v^*,v^*}$ in $\mathfrak{h}$. Then we have

$$(E_{v,v} - E_{v^*,v^*})w = E_{v,v}w = \delta_{w,v}w = \alpha_w(H_v)w.$$

Therefore the weight of w is α_w. On the other hand, we have

$$(E_{v,v} - E_{v^*,v^*})w^* = -E_{v^*,v^*}w^* = -\delta_{w^*,v^*}w^* = -\delta_{w,v}w^* = -\alpha_w(H_v)w^*.$$

Thus w^* has weight $-\alpha_w$. $\square$

Lemma 2.5.6. *Let* $v = v_1 \otimes \cdots \otimes v_k$ *with each* $v_i \in B$. *Then the weight of* v *is given by*

$$\alpha_v = \sum_{i=1}^{k} \alpha_{v_i}.$$

Proof. Each $H \in \mathfrak{h}$ acts on $v_1 \otimes \cdots \otimes v_k$ diagonally as per Definition 2.2.7. Since the colour of H is 1, we have $\beta(H, v_i) = 1$ for all $v_i \in B$. Thus we have

$$\begin{aligned} H \cdot (v_1 \otimes \cdots \otimes v_k) &= \sum_{\ell=1}^{k} v_1 \otimes \cdots \otimes v_{\ell-1} \otimes (H \cdot v_\ell) \otimes v_{\ell+1} \otimes \cdots \otimes v_k \\ &= \sum_{\ell=1}^{k} v_1 \otimes \cdots \otimes v_{\ell-1} \otimes \alpha_{v_\ell}(H) v_\ell \otimes v_{\ell+1} \otimes \cdots \otimes v_k \\ &= \sum_{\ell=1}^{k} \alpha_{v_\ell}(H)(v_1 \otimes \cdots \otimes v_k). \end{aligned}$$

Therefore we have shown that the tensor $v_1 \otimes \cdots \otimes v_k$ is a weight vector with weight $\sum_{i=1}^{k} \alpha_{v_i}$. □

Next we provide a root system of $\mathfrak{g}$. First notice that the basis vectors defined in Theorem 2.4.6 are all weight vectors under the adjoint representation.

Lemma 2.5.7. *For vectors u, v, w in B', we have the following:*

(i) $[H_u, Y_{v,w^*}] = (\alpha_v + \alpha_w)(H_u) Y_{v,w^*}$,

(ii) $[H_u, Y_{v^*,w}] = (-\alpha_v - \alpha_w)(H_u) Y_{v^*,w}$,

(iii) $[H_u, Y_{v,w}] = (\alpha_v - \alpha_w)(H_u) Y_{v,w}$.

Proof. We only prove (i), and the rest can be calculated similarly. Since $Y_{v,w^*} = E_{v,w^*} + \beta(v^*, w) E_{w,v^*}$, notice that $E_{v,w^*} E_{u,u} = 0 = E_{u^*,u^*} E_{v,w^*}$ for all $u, v, w \in B'$. Thus we have $[H_u, Y_{v,w^*}] = E_{u,u} Y_{v,w^*} - Y_{v,w^*} E_{u^*,u^*}$. Therefore

$$\begin{aligned} [H_u, Y_{v,w^*}] &= E_{u,u} Y_{v,w^*} - Y_{v,w^*} E_{u^*,u^*} \\ &= (E_{u,u} E_{v,w^*} + \beta(v^*, w) E_{u,u} E_{w,v^*}) - (-E_{v,w^*} E_{u^*,u^*} - \beta(v^*, w) E_{w,v^*} E_{u^*,u^*}) \\ &= \delta_{uv} E_{u,w^*} + \delta_{u,w} \beta(v^*, w) E_{u,v^*} + \delta_{wu} E_{v,u^*} + \delta_{uv} \beta(v^*, w) E_{w,u^*} \end{aligned}$$

$$= \delta_{uv}(E_{u,w^*} + \beta(v^*,w)E_{w,u^*}) + \delta_{uw}(E_{v,u^*} + \beta(v^*,w)E_{u,v^*})$$
$$= (\alpha_v + \alpha_w)(H_u)(Y_{v,w^*}) \qquad \square$$

We now give definitions of roots and the root space decomposition of $\mathfrak{g}$.

Definition 2.5.8. *Let $\mathfrak{h} = \mathfrak{g}_0$ be the Cartan subalgebra of $\mathfrak{g}$ introduced in Definition 2.5.1. The root space decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$ is given by*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

where $\Phi \coloneqq \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ where $\mathfrak{g}_\alpha \coloneqq \{x \in \mathfrak{g} \mid [H,x] = \alpha(H)x,\ \forall H \in \mathfrak{h}\}$. We call α a root if $\alpha \in \Phi$ and call the associated $\mathfrak{g}_\alpha$ a root space.

All the vectors H_v lie in $\mathfrak{h} = \mathfrak{g}_0$. Moreover, by Lemma 2.5.7, for all $v, w \in B'$ the basis vectors Y_{v,w^*}, $Y_{v^*,w}$ and $Y_{v,w}$ are in the distinct nonzero root spaces $\mathfrak{g}_{\alpha_v+\alpha_w}$, $\mathfrak{g}_{-\alpha_v-\alpha_w}$ and $\mathfrak{g}_{\alpha_v-\alpha_w}$ respectively. Therefore we can conclude that the root spaces are each one-dimensional.

Remark 2.5.9. From the subscripts of our notation, we can read both the colour of a root vector, and the value of its corresponding root. For example, Y_{v,w^*} has colour vw and it lies in the space $\mathfrak{g}_{\alpha_v+\alpha_w}$. $Y_{v^*,w}$ has colour $v^{-1}w^{-1}$, and it is in the root space $\mathfrak{g}_{-\alpha_v-\alpha_w}$. $Y_{v,w}$ has colour vw^{-1}, and it is in the root space $\mathfrak{g}_{\alpha_v-\alpha_w}$.

In summary, as shorthand notation, we set $\varepsilon_i = \alpha_{t_i}$ and $\delta_j = \alpha_{u_j}$ for $1 \le i \le r$ and $1 \le j \le s$. The roots of $\mathfrak{spo}(V,\beta)$, are:

(i) When $n = 2s+1$,

$$\Phi_0 = \{\pm(\varepsilon_i \pm \varepsilon_j), \pm 2\varepsilon_i, \pm(\delta_k \pm \delta_l), \pm\delta_k \mid 1 \le i \neq j \le r, 1 \le k \neq l \le s\}$$
$$\Phi_1 = \{\pm(\varepsilon_i \pm \delta_j), \pm\varepsilon_i \mid 1 \le i \le r, 1 \le j \le s\};$$

(ii) When $n = 2s$

$$\Phi_0 = \{\pm(\varepsilon_i \pm \varepsilon_j), \pm 2\varepsilon_i, \pm(\delta_k \pm \delta_l) \mid 1 \leq i \neq j \leq r, 1 \leq k \neq l \leq s\}$$
$$\Phi_1 = \{\pm(\varepsilon_i \pm \delta_j) \mid 1 \leq i \leq r, 1 \leq j \leq s\}.$$

We denote the set of roots of $\mathfrak{spo}(V,\beta)$ by $\Phi = \Phi_0 \cup \Phi_1$. The root system we defined is not like the usual root system of Lie algebras. For example, when $n = 2s+1$, we can have both $\pm\varepsilon_i$ and $\pm 2\varepsilon_i$ as roots. However Φ_0 and Φ_1 are the set of even and odd roots for the Lie superalgebra $\mathfrak{spo}(2r|2s+1)$, see for example [FSS00, Table 6 and 9].

Lemma 2.5.10. *The union of the following sets*

(i) When $n = 2s+1$,

$$\Phi_0^+ = \{\varepsilon_i \pm \varepsilon_j, 2\varepsilon_1, 2\varepsilon_j, \delta_k \pm \delta_l, \delta_1, \delta_l \mid 1 \leq i < j \leq r, 1 \leq k < l \leq s\}$$
$$\Phi_1^+ = \{\varepsilon_i \pm \delta_j, \varepsilon_i \mid 1 \leq i \leq r, 1 \leq j \leq s\};$$

(ii) When $n = 2s$,

$$\Phi_0^+ = \{\varepsilon_i \pm \varepsilon_j, 2\varepsilon_1, 2\varepsilon_j, \delta_k \pm \delta_l \mid 1 \leq i < j \leq r, 1 \leq k < l \leq s\}$$
$$\Phi_1^+ = \{\varepsilon_i \pm \delta_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}.$$

are positive roots of $\mathfrak{spo}(V,\beta)$, denoted by Φ^+.

Proof. We see directly that $\Phi = \Phi^+ \cup -\Phi^+$, and that if two roots in Φ^+ are such that their sum is a root, then the sum is in Φ^+. Hence Φ^+ is a positive root system. □

Definition 2.5.11. *Given a set of positive roots, the corresponding set of simple roots Δ is a basis of V such that each $\alpha \in \Phi^+$ is a nonnegative integral linear combination of elements in Δ.*

Lemma 2.5.12. *With respect to the positive root system above, if $r=1$, $s=1$ and $n=2s$, the set of simple roots Δ is given by*

$$\Delta = \{\varepsilon_1+\delta_1, \varepsilon_1-\delta_1\}.$$

Otherwise, the set of simple roots $\Delta=\{\gamma_1,\dots,\gamma_{r+s}\}$ is given by

$$\gamma_i = \varepsilon_i-\varepsilon_{i+1},\ 1\le i\le r-1, \qquad \gamma_{r+j}=\delta_j-\delta_{j+1},\ 1\le j\le s-1$$

$$\gamma_r = \begin{cases} \varepsilon_r & \text{if } n=1,\\ \varepsilon_r-\delta_1 & \text{otherwise,}\end{cases} \qquad \gamma_{r+s} = \begin{cases} \delta_s & \text{if } n=2s+1,\\ \delta_{s-1}+\delta_s & \text{if } n=2s,\end{cases}$$

where we adopt the convention that $\varepsilon_i=\delta_i=0$ if $i\le 0$.

Proof. It can be checked that Δ satisfies Definition 2.5.11. □

Definition 2.5.13. *The set of simple root vectors Δ_Y is denoted $\{Y_\gamma|\gamma\in\Delta\}$. If $r=1, s=1$ and $n=2s$, then Δ_Y is the set*

$$\{E_{t_1,u_1}+\beta(t_1,u_1)E_{u_1^*,t_1^*}, \quad E_{t_1,u_1^*}+\beta(t_1^*,u_1)E_{u_1,t_1^*}\}.$$

Otherwise, the Y_γ's are explicitly given by:

(i) $Y_{\gamma_i} = E_{t_i,t_{i+1}} - \beta(t_i,t_{i+1})E_{t_{i+1}^*,t_i^*}$, *where* $1\le i\le r-1$,

(ii) $Y_{\gamma_r} = \begin{cases} E_{t_r,u_1}+\beta(t_r,u_1)E_{u_1,t_r^*}, & \text{if } n=1,\\ E_{t_r,u_1}+\beta(t_r,u_1^*)E_{u_1^*,t_r^*}, & \text{otherwise,}\end{cases}$

(iii) $Y_{\gamma_{r+j}} = E_{u_j,u_{j+1}}+\beta(u_j,u_{j+1})E_{u_{j+1}^*,u_j^*}$, *where* $1\le j\le s-1$,

(iv) $Y_{\gamma_{r+s}} = \begin{cases} E_{u_s,u_{s+1}}+\beta(u_s,u_{s+1})E_{u_{s+1},u_s^*}, & \text{if } n=2s+1,\\ E_{u_s,u_{s-1}^*}+\beta(u_s^*,u_{s-1})E_{u_{s-1},u_s^*}, & \text{if } n=2s.\end{cases}$

Lemma 2.5.14. *Let v be a weight vector with weight α. Let Y be a simple root vector in Δ_Y with weight λ. Then the weight of Yv is $\alpha + \lambda$.*

Proof. Notice that since $\beta(H, \cdot) = 1$, we have

$$H(Yv) = [H, Y]v + Y(Hv) = \lambda(H)Yv + Y\alpha(H)v,$$

which is $(\lambda + \alpha)(H)Yv$. □

Example 2.5.15. Take $r = 1, s = 2$, $m = 2r$, $n = 2s$. Then V has the basis

$$B = \{t_1, t_1^*, u_1, u_1^*, u_2, u_2^*\}.$$

An arbitrary H in $\mathfrak{h}$ is of the form $H = a_1 H_{t_1} + a_2 H_{u_1} + a_3 H_{u_2}$, which in matrix form is given with respect to the basis B by the following diagonal matrix:

$$H = \begin{pmatrix} a_1 & & & & & \\ & -a_1 & & & & \\ & & a_2 & & & \\ & & & -a_2 & & \\ & & & & a_3 & \\ & & & & & -a_3 \end{pmatrix}$$

where the off-diagonal entries are all 0's.

In the following matrix, we label each E_{vw} by the unique root, such that the corresponding root space has a nonzero projection onto $\mathrm{Span}\{E_{vw}\}$.

$$\begin{pmatrix} 0 & \varepsilon_1+\varepsilon_1 & \varepsilon_1-\delta_1 & \varepsilon_1+\delta_1 & \varepsilon_1-\delta_2 & \varepsilon_1+\delta_2 \\ -\varepsilon_1-\varepsilon_1 & 0 & -\varepsilon_1-\delta_1 & \delta_1-\varepsilon_1 & -\varepsilon_1-\delta_2 & -\varepsilon_1+\delta_2 \\ \delta_1-\varepsilon_1 & \delta_1+\varepsilon_1 & 0 & 0 & \delta_1-\delta_2 & \delta_1+\delta_2 \\ -\delta_1-\varepsilon_1 & \varepsilon_1-\delta_1 & 0 & 0 & -\delta_1-\delta_2 & -\delta_1+\delta_2 \\ \delta_2-\varepsilon_1 & \delta_2+\varepsilon_1 & \delta_2-\delta_1 & \delta_2+\delta_1 & 0 & 0 \\ -\delta_2-\varepsilon_1 & -\delta_2+\varepsilon_1 & -\delta_2-\delta_1 & -\delta_2+\delta_1 & 0 & 0 \end{pmatrix}.$$

The set of simple roots in this case is $\{\varepsilon_1-\delta_1, \delta_1-\delta_2, \delta_1+\delta_2\}$.

Finally we give the definition of highest weight vectors.

Definition 2.5.16. *Let W be an $\mathfrak{spo}(V,\beta)$-module. Let $\mathfrak{g}_+$ be the span of the positive root vectors. Then a vector w in W is a highest weight vector of weight $\lambda \in \mathfrak{h}^*$ if the following holds:*

(i) $Xw = 0$, *for all* $X \in \mathfrak{g}_+$,

(ii) $Hw = \lambda(H)w$ *for all* $H \in \mathfrak{h}$.

Chapter 3

The Poincaré-Birkhoff-Witt Theorem for Lie Colour Algebras

In this chapter, our goal is to state and prove the Poincaré-Birkhoff-Witt (PBW) theorem for Lie colour algebras. The statement and proof of the PBW theorem for Lie algebras in the usual sense can be obtained from our proof in this chapter by choosing β to be the trivial bicharacter of the trivial group. Our proof is inspired by [Hum72, Chapter 17] and [Car05, Chapter 9].

In Section 3.1, we give the definition of tensor algebra and symmetric algebra for any vector spaces, and briefly discuss their grading. In Section 3.2, we provide the definition of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of any arbitrary Lie colour algebra $\mathfrak{g}$. Then in Section 3.3, we provide a spanning set of $\mathfrak{U}(\mathfrak{g})$, which will be proved to be a basis of $\mathfrak{U}(\mathfrak{g})$ in Section 3.4. This existence of this basis is known as one of the versions of the PBW theorem. Thereafter, assuming Proposition 3.4.2, we prove another version of the PBW theorem which states that the associated graded algebra of $\mathfrak{U}(\mathfrak{g})$ is isomorphic to the symmetric algebra. Moreover we give an application of the PBW theorem at the end of this section. We give the technical details of the proof of Proposition 3.4.2 in Section 3.5.

3.1 Tensor algebras and symmetric algebras

Unlike Chapter 2, in this chapter, $\mathfrak{g}$ always denotes an arbitrary Lie colour algebra over a field $\mathbb{F}$. Nevertheless, all of the definitions in this section make sense if $\mathfrak{g}$ is just a G-graded vector space.

Definition 3.1.1. *For any $m \in \mathbb{Z}_+$, the m^{th} tensor product of $\mathfrak{g}$ is defined to be the vector space*

$$T^m(\mathfrak{g}) = \underbrace{\mathfrak{g} \otimes \mathfrak{g} \cdots \otimes \mathfrak{g}}_{m \ times}$$

that is, $T^m(\mathfrak{g})$ consists of all linear combinations of m-tensors on $\mathfrak{g}$.

Note that by convention, $T^0(\mathfrak{g}) = \mathbb{F}$.

Definition 3.1.2. *The tensor algebra $\mathcal{T}(\mathfrak{g})$ of $\mathfrak{g}$ is an associative algebra with unity, defined to be the direct sum of $T^m(\mathfrak{g})$ for all $m \in \mathbb{Z}_{\geq 0}$:*

$$\mathcal{T}(\mathfrak{g}) = \bigoplus_{i=0}^{\infty} T^i(\mathfrak{g}) = \mathbb{F} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots .$$

The multiplication on homogeneous generators of $\mathcal{T}(\mathfrak{g})$ is defined by the rule:

$$(v_1 \otimes \cdots \otimes v_k) \cdot (w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_m \in T^{k+m}(\mathfrak{g})$$

where $k, m \in \mathbb{Z}_+$, $v_1, \ldots, v_k, w_1, \ldots, w_m \in \mathfrak{g}$. Note that with this multiplication rule, $\mathcal{T}(\mathfrak{g})$ is a graded algebra.

Definition 3.1.3. *We say a G-graded algebra A is colour-commutative if for all homogeneous elements x, y in A, we have $xy = \beta(y,x)yx$.*

The tensor algebra is not colour-commutative if $\dim(\mathfrak{g}) > 1$. We construct a new algebra called the β-symmetric algebra, in which the multiplication is colour-commutative.

Definition 3.1.4. *The β-symmetric algebra of $\mathfrak{g}$ is defined to be the quotient algebra*

$$\mathcal{S}(\mathfrak{g}) = {}^{\mathcal{T}(\mathfrak{g})}\!\big/_{\mathcal{I}}$$

where $\mathcal{I}$ is the two sided ideal generated by $x \otimes y - \beta(y,x)y \otimes x$ for all homogeneous x, y in $\mathfrak{g}$.

Let us discuss more about $\mathcal{I}$ and $\mathcal{S}(\mathfrak{g})$. We first show that $\mathcal{I}$ is a homogeneous ideal. Since $\mathcal{I}$ is a two-sided ideal generated by $x \otimes y - \beta(y,x)y \otimes x$, $\mathcal{I}$ is spanned as a vector space by elements of the form $v \cdot (x \otimes y - \beta(y,x)y \otimes x) \cdot w$ for some homogeneous x, y in $\mathfrak{g}$ and $v, w \in \mathcal{T}(\mathfrak{g})$. Moreover, since $\mathcal{T}(\mathfrak{g}) = \oplus_{i \geq 0} T^i(\mathfrak{g})$ is a direct sum of homogeneous components, we can view $\mathcal{I}$ as the span of elements of the form $v \cdot (x \otimes y - \beta(y,x)y \otimes x) \cdot w$, for some $v \in T^m(\mathfrak{g})$ and $w \in T^n(\mathfrak{g})$ where $m, n \in \mathbb{Z}_+$. Therefore, in this case we have

$$v \cdot (x \otimes y - \beta(y,x)y \otimes x) \cdot w \in \mathcal{I} \cap T^{m+n+2}(\mathfrak{g}).$$

Therefore, since $\mathcal{I}$ is spanned by homogeneous elements, it is a homogeneous ideal and it admits a grading of the form

$$\mathcal{I} = \bigoplus_{j \geq 0} I^j \text{ where } I^j = \mathcal{I} \cap T^j(\mathfrak{g}).$$

Therefore $\mathcal{S}(\mathfrak{g})$ is also a graded algebra, with grading

$$\mathcal{S}(\mathfrak{g}) = \bigoplus_{i\geq 0} S^i(\mathfrak{g})$$

where $S^i(\mathfrak{g}) = {T^i(\mathfrak{g})}/{I^i} = {T^i(\mathfrak{g})}/{\mathcal{I}\cap T^i(\mathfrak{g})}$ for all $i \geq 0$.

Lemma 3.1.5. *Let $y_1, \ldots, y_n \in \mathfrak{g}$ be homogenous elements. Let $\pi \in S_n$. Then there is a scalar $\beta_\pi(y_1, \ldots, y_n) \in \mathbb{F}^\times$ such that in $\mathcal{S}(\mathfrak{g})$, we have*

$$y_1 \otimes \cdots \otimes y_n + \mathcal{I} = \beta_\pi(y_1, \ldots, y_n) y_{\pi(1)} \otimes \cdots \otimes y_{\pi(n)} + \mathcal{I}.$$

Proof. By the discussion above, for homogeneous elements $x, y \in \mathfrak{g}$, $v \in T^m(\mathfrak{g})$ and $w \in T^n(\mathfrak{g})$, we have

$$v \otimes x \otimes y \otimes w = \beta(y,x) v \otimes y \otimes x \otimes w \tag{3.1.1}$$

in $\mathcal{S}(\mathfrak{g})$. Therefore the result follows from the fact that the scalar β_π is obtained by writing π as a product of adjacent transpositions, and inductively applying relation (3.1.1). □

Remark 3.1.6. Indeed the function $\beta_\pi(y_1, \ldots, y_n)$ depends only on the colours of $y_1, \ldots, y_n$.

Definition 3.1.7. *Let*

$$T_m = \bigoplus_{i=0}^{m} T^i(\mathfrak{g}) \text{ and } S_m = \bigoplus_{i=0}^{m} S^i(\mathfrak{g})$$

be the m^{th} filtration subspaces of $\mathcal{T}(\mathfrak{g})$ and $\mathcal{S}(\mathfrak{g})$ respectively.

Now for any G-graded vector space V, we give a basis for $\mathcal{S}(\mathfrak{g})$. We first need the following lemma.

Lemma 3.1.8. *Let V be a G-graded vector space with G-graded decomposition $V = U \oplus W$. Then $\mathcal{S}(U \oplus W) \cong \mathcal{S}(U) \otimes \mathcal{S}(W)$.*

Proof. Let us consider Diagram (3.1.2).

$$\begin{array}{ccc} V = U \oplus W & \longrightarrow & \mathcal{S}(U \oplus W) \\ & \searrow^{\rho} & \downarrow \varphi \\ & & \mathcal{S}(U) \otimes \mathcal{S}(W) \end{array} \tag{3.1.2}$$

where $\rho(u) = u \otimes 1$ and $\rho(w) = 1 \otimes w$ for all $u \in U$ and $w \in W$. Then ρ extends to a map $\varphi : \mathcal{S}(U \oplus W) \to \mathcal{S}(U) \otimes \mathcal{S}(W)$. Here we use the universal property of the symmetric algebra. See for example [DF04, Chapter 11, Theorem 34].

It is trivial that φ is surjective. Now let ϕ be the embedding map from $\mathcal{S}(U) \otimes \mathcal{S}(W)$ into $\mathcal{S}(U \oplus W)$ generated by the inclusions $\mathcal{S}(U) \hookrightarrow \mathcal{S}(U \oplus W)$ and $\mathcal{S}(W) \hookrightarrow \mathcal{S}(U \oplus W)$. Then $\phi \circ \varphi|_V = \mathrm{id}$ which implies that φ is injective. $\square$

Corollary 3.1.9. *Let V be a G-graded vector space such that $V = \bigoplus_{a \in G} V_a$ where each V_a is one-dimensional vector space. Then $\mathcal{S}(V) = \bigotimes_{a \in G} \mathcal{S}(V_a)$.*

Remark 3.1.10. By Corollary 3.1.9, we can think of an element of $\mathcal{S}(V)$ as a product of elements of $\mathcal{S}(V_a)$.

Proposition 3.1.11. *Let V be a G-graded vector space. Let $\{x_1, \ldots, x_k\}$ be a homogeneous basis of V. Then the set*

$$\{x_1^{i_1} \cdots x_k^{i_k} | i_j \in \mathbb{Z}_{\geq 0} \text{ if } \beta(x_j, x_j) = 1, i_j \in \{0, 1\} \text{ if } \beta(x_j, x_j) = -1\} \tag{3.1.3}$$

is a basis of $\mathcal{S}(V)$.

Proof. By Corollary 3.1.9, we have $\mathcal{S}(V) = \bigotimes_{a\in G}\mathcal{S}(V_a)$. Since each V_a is one-dimensional for all $a \in G$, we have

$$\mathcal{S}(V_a) = {}^{\mathcal{T}(V_a)}\!\big/_{\langle x\otimes x - \beta(x,x)x\otimes x\rangle}$$

for all $x \in V_a$. Therefore, if $\beta(x,x) = \beta(a,a) = 1$, we have $\mathcal{S}(V_a) = \mathcal{T}(V_a)$ and if $\beta(x,x) = -1$, we have $\mathcal{S}(V_a) = {}^{\mathcal{T}(V_a)}\!\big/_{\langle x\otimes x\rangle}$. Thus the result follows by Remark 3.1.10. □

3.2 Universal enveloping algebras

Now we have enough tools to introduce the universal enveloping algebra. Recall that $\mathfrak{g}$ denotes a Lie colour algebra over a field $\mathbb{F}$.

Definition 3.2.1. *The β-universal enveloping algebra of $\mathfrak{g}$ is defined as*

$$\mathfrak{U}(\mathfrak{g}) = {}^{\mathcal{T}(\mathfrak{g})}\!\big/_{\mathcal{J}}$$

where $\mathcal{J}$ is the ideal generated by $x\otimes y - \beta(y,x)y\otimes x - [x,y]$ for all homogeneous $x, y \in \mathfrak{g}$.

Definition 3.2.2. *Let σ be the composite linear map*

$$\mathfrak{g} \hookrightarrow \mathcal{T}(\mathfrak{g}) \to {}^{\mathcal{T}(\mathfrak{g})}\!\big/_{\mathcal{J}} = \mathfrak{U}(\mathfrak{g})$$

such that $\sigma(x) = x + J$ for all $x \in \mathfrak{g}$.

Note that for all homogeneous $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} \sigma([x,y]) &= [x,y] + J \\ &= x \otimes y - \beta(y,x) y \otimes x + J \\ &= \sigma(x)\sigma(y) - \beta(y,x)\sigma(y)\sigma(x). \end{aligned} \tag{3.2.1}$$

Since $\mathcal{J}$ is not a homogeneous ideal, $\mathfrak{U}(\mathfrak{g})$ is not automatically graded in accordance with $\mathcal{T}(\mathfrak{g})$. But later in this chapter, we will consider a natural filtration of $\mathfrak{U}(\mathfrak{g})$ and construct the associated graded algebra of $\mathfrak{U}(\mathfrak{g})$, $\mathrm{gr}(\mathfrak{U}(\mathfrak{g}))$. Moreover the PBW theorem (Theorem (3.4.6)) states $\mathrm{gr}(\mathfrak{U}(\mathfrak{g})) \cong \mathcal{S}(\mathfrak{g})$.

We can also characterize the universal enveloping algebra by the following property, which says that representations of the non-associative, but finite-dimensional, algebra $\mathfrak{g}$ are in bijective correspondence with representations of the associative, but infinite-dimensional, algebra $\mathfrak{U}(\mathfrak{g})$.

Proposition 3.2.3. [Universal property] *Let $\mathcal{A}$ be an associative algebra with unity over $\mathbb{F}$, and let $\rho : \mathfrak{g} \to \mathcal{A}$ be an $\mathbb{F}$-linear map such that $\rho([x,y]) = \rho(x)\rho(y) - \beta(y,x)\rho(y)\rho(x)$. Then there exists a unique homomorphism of algebras $\varphi : \mathfrak{U}(\mathfrak{g}) \to \mathcal{A}$ (sending* 1 *to* 1*), such that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\sigma} & \mathfrak{U}(\mathfrak{g}) \\ & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle\varphi} \\ & & \mathcal{A} \end{array} \tag{3.2.2}$$

where σ is the map in (3.2.1)*.*

Remark 3.2.4. We can turn any associative algebra $\mathcal{A}$ into a Lie colour algebra by equipping $\mathcal{A}$ with the bracket $[a,b] = ab - \beta(y,x)ba$ for all $a, b \in \mathcal{A}$. Then ρ becomes a Lie colour algebra homomorphism.

Proof of Proposition 3.2.3. We first define an algebra homomorphism $\varphi' : \mathcal{T}(\mathfrak{g}) \to \mathcal{A}$ by

$$\varphi'(x_1 \otimes \cdots \otimes x_k) = \rho(x_1) \cdots \rho(x_k).$$

Since ρ is linear, the map φ' is well-defined, and it is straightforward to check φ' is a homomorphism. In particular, for all homogeneous x, y in $\mathfrak{g}$, we have

$$\varphi'(x \otimes y - \beta(y,x) y \otimes x - [x,y]) = \rho(x)\rho(y) - \beta(y,x)\rho(y)\rho(x) - \rho([x,y]),$$

which is 0 by the hypothesis of ρ. Thus φ' factors through to an algebra homomorphism $\varphi : \mathfrak{U}(\mathfrak{g}) \to \mathcal{A}$ such that the diagram (3.2.2) commutes.

To prove uniqueness, let $\varphi' : \mathfrak{U}(\mathfrak{g}) \to \mathcal{A}$ be another algebra homomorphism satisfying our assumption. Then for any $x \in \mathfrak{g}$, we have $\varphi'(\sigma(x)) = \rho(x) = \varphi(\sigma(x))$. Thus $\varphi' = \varphi$ on $\mathfrak{U}(\mathfrak{g})$ since $\sigma(\mathfrak{g})$ generates $\mathfrak{U}(\mathfrak{g})$. □

Proposition 3.2.5. *Let (ρ, V) be an $\mathfrak{g}$-module. Let v be an element of V. Then the subspace $\mathfrak{U}(\mathfrak{g}) \cdot v$ is a $\mathfrak{g}$-submodule of V.*

Proof. It is enough to show that for all $x \in \mathfrak{g}$ and $w \in \mathfrak{U}(\mathfrak{g}) \cdot v$ we have $\rho(x)w \in \mathfrak{U}(\mathfrak{g}) \cdot v$. We prove this proposition by using the universal property of $\mathfrak{U}(\mathfrak{g})$.

In Proposition 3.2.3, let $\mathcal{A} = \mathfrak{gl}(V, \beta)$, which is an associative algebra. Then $\rho : \mathfrak{g} \to \mathfrak{gl}(V, \beta)$ is a map satisfying the hypotheses of Proposition 3.2.3. There is an algebra homomorphism $\tilde{\rho}$ from $\mathfrak{U}(\mathfrak{g})$ to $\mathrm{End}(V, \beta)$ such that $\tilde{\rho} \circ \sigma = \rho$. For each $w \in \mathfrak{U}(\mathfrak{g}) \cdot v$, we have $w = \tilde{\rho}(u)(v)$ for some $u \in \mathfrak{U}(\mathfrak{g})$. Thus for all $x \in \mathfrak{g}$, we have

$$\rho(x)w = \tilde{\rho}(\sigma(x))\tilde{\rho}(u)(v)$$

$$= \tilde{\rho}(\sigma(x)u)(v) \in \mathfrak{U}(\mathfrak{g}) \cdot v,$$

which completes the proof. □

3.3 A spanning set of $\mathfrak{U}(\mathfrak{g})$

In order to understand more about the structure of the universal enveloping algebra, we give a spanning set of $\mathfrak{U}(\mathfrak{g})$ in this section. Then we state and prove the Poincaré-Birkhoff-Witt (PBW) theorem in the next section. First, we need to fix some general notation.

Let $\{x_1, \ldots, x_n\}$ be a homogeneous basis of $\mathfrak{g}$. We write $\sigma(x_i) = X_i$ in $\mathfrak{U}(\mathfrak{g})$ for all i in $\{1, \ldots, n\}$. Given a sequence $J = (j_1, \cdots, j_q)$, the length of J is defined by $\ell(J) = q$. Moreover we say J is *weakly increasing* if $j_i \leq j_k$ whenever $1 \leq i < k \leq q$. Let $X_J = X_{j_1} \cdots X_{j_q} \in \mathfrak{U}(\mathfrak{g})$ for any sequence J.

Definition 3.3.1. *Let $\mathfrak{U}_p(\mathfrak{g})$ be the subspace of $\mathfrak{U}(\mathfrak{g})$ spanned by the elements of the form X_J, where J is a sequence with $\ell(J) \leq p$.*

Remark 3.3.2. Notice that the subspaces $\mathfrak{U}_p(\mathfrak{g})$ give a filtration of $\mathfrak{U}(\mathfrak{g})$. That is, they give a chain of subspaces of $\mathfrak{U}(\mathfrak{g})$ such that $\mathbb{C} = \mathfrak{U}_0(\mathfrak{g}) \subseteq \mathfrak{U}_1(\mathfrak{g}) \subseteq \mathfrak{U}_2(\mathfrak{g}) \subseteq \cdots$ and $\mathfrak{U}(\mathfrak{g}) = \bigcup_{p \geq 0} \mathfrak{U}_p(\mathfrak{g})$.

Next we will use the following lemma to show that in fact $\mathfrak{U}_p(\mathfrak{g})$ in Definition 3.3.1 is spanned by elements of the form X_J with J weakly increasing and $\ell(J) \leq p$.

Lemma 3.3.3. *Let $y_1, \ldots, y_p$ be homogeneous elements of $\mathfrak{g}$. Let π be a permutation of $\{1, \ldots, p\}$. Then*

$$\sigma(y_1) \cdots \sigma(y_p) - \beta_\pi(y_1, \ldots, y_p)\sigma(y_{\pi(1)}) \cdots \sigma(y_{\pi(p)}) \in \mathfrak{U}_{p-1}(\mathfrak{g}) \tag{3.3.1}$$

where β_π is as defined in Lemma 3.1.5.

Proof. Suppose $\pi = (i, i+1)$ for some i in $\{1, \dots, p-1\}$. Let Y be the term

$$\sigma(y_1)\cdots\sigma(y_i)\sigma(y_{i+1})\cdots\sigma(y_p) - \beta(y_{i+1}, y_i)\sigma(y_1)\cdots\sigma(y_{i+1})\sigma(y_i)\cdots\sigma(y_p).$$

which is

$$Y = \sigma(y_1)\cdots\sigma(y_{i-1})\big(\sigma(y_i)\sigma(y_{i+1}) - \beta(y_{i+1}, y_i)\sigma(y_{i+1})\sigma(y_i)\big)\sigma(y_{i+2})\cdots\sigma(y_p).$$

By the definition of Lie colour algebra bracket, it becomes

$$Y = \sigma(y_1)\cdots\sigma(y_{i-1})\big([\sigma(y_i), \sigma(y_{i+1})]\big)\sigma(y_{i+2})\cdots\sigma(y_p)$$

which by the definition of $\mathfrak{U}(\mathfrak{g})$ is

$$\sigma(y_1)\cdots\sigma(y_{i-1})\sigma([y_i, y_{i+1}])\sigma(y_{i+2})\cdots\sigma(y_p). \tag{3.3.2}$$

Since $[y_i, y_{i+1}] \in \mathfrak{g}$, there are $p-1$ terms in (3.3.2). Thus, (3.3.2) is an element of $\mathfrak{U}_{p-1}(\mathfrak{g})$. Note that the scalar $\beta(y_{i+1}, y_i)$ coincides with $\beta_\pi(y_{i+1}, y_i)$ defined in Lemma 3.1.5. Since we have proved that the statement of Lemma 3.3.3 holds for any adjacent transpositions $(i, i+1)$, the statement also holds for any permutation π of $\{1, \dots, p\}$. □

Proposition 3.3.4. *Let $\mathfrak{C}_p = \{X_J \mid J$ weakly increasing $, \ell(J) \leq p\}$. Then the subspace $\mathfrak{U}_p(\mathfrak{g})$ is spanned by $\mathfrak{C}_p$.*

Proof. We prove this proposition by induction. The base case when $p = 1$ is trivial. Suppose that the statement holds for $p = n-1$. By Definition 3.3.1, $\mathfrak{U}_n(\mathfrak{g})$ is spanned by $X_J = \sigma(x_{j_1}) \cdots \sigma(x_{j_m})$ over all possible sequences $J = (j_1, \ldots, j_m)$ such that $m \leq n$.

Now fix such an X_J with $\ell(J) \leq n$. We prove that X_J is a linear combination of X_L's in $\mathfrak{C}_n$. Let π be a permutation such that $\pi(j_1) \leq \pi(j_2) \leq \cdots \leq \pi(j_m)$. Let $J_\pi = (\pi(j_1), \cdots, \pi(j_m))$. Then by Lemma 3.3.3, there is a nonzero scalar β_π and an element R of $\mathfrak{U}_{n-1}(\mathfrak{g})$ such that

$$X_J = \beta_\pi X_{J_\pi} + R,$$

where $X_{J_\pi} \in \mathfrak{C}_n$. By induction, R can be written as a linear combination of elements in $\mathfrak{C}_{n-1}$. Thus the result follows. □

Now we find a subset of $\mathfrak{C}_n$, which also spans $\mathfrak{U}_n(\mathfrak{g})$. Let J be a weakly increasing sequence satisfying

$$\text{each index } j \text{ such that } \beta(x_j, x_j) = -1 \text{ occurs at most once in } J. \tag{3.3.3}$$

Proposition 3.3.5. *Let*

$$\mathfrak{B}_p = \{X_J \mid J \textit{ weakly increasing }, \ell(J) \leq p \textit{ and } J \textit{ satisfies condition } (3.3.3)\}.$$

Then $\mathfrak{B}_p$ *is a spanning set of* $\mathfrak{U}_p(\mathfrak{g})$.

Proof. We prove this proposition by induction. The base case when $p = 1$ is trivial. Suppose that the statement holds for $p = n - 1$. First notice that if $X_J \in \mathfrak{C}_n$, then since J is weakly increasing, we have

$$X_J = X_i^{i_1} \cdots X_m^{i_m}, \text{ such that } \sum_{k=1}^{m} i_q \leq n.$$

We now show that X_J is a linear combination of elements in $\mathfrak{B}_p$. Assume that there exists an X_q in X_J such that $\beta(X_q, X_q) = \beta(x_q, x_q) = -1$ and the multiplicity i_q of X_q satisfies $i_q \geq 2$.

By the fact $\beta(X_q, X_q) = -1$ we have $X_q^2 = \frac{1}{2}[X_q, X_q]$. Also notice that since $[x_q, x_q] \in \mathfrak{g}$, we have $[x_q, x_q] = \sum_{i=1}^{n} c_i x_i$ for some scalar $c_i \in \mathbb{F}$. Thus $[X_q, X_q] = \sigma([x_q, x_q]) = \sum_{i=1}^{n} c_i \sigma(x_i) = \sum_{i=1}^{n} c_i X_i$. This implies that $X_q^2 = \frac{1}{2} \sum_{i=1}^{n} c_i X_i$. Therefore we deduce that if i_q is even, then $X_q^{i_q} = \left(\frac{1}{2} \sum_{i=1}^{n} c_i X_i\right)^{i_q/2}$. Thus we have

$$X_J = \left(\frac{1}{2}\right)^{i_q/2} X_1^{i_1} \cdots X_{q-1}^{i_{q-1}} \left(\sum_{i=1}^{n} c_i X_i\right)^{i_q/2} X_{q+1}^{i_{q+1}} \cdots X_m^{i_m},$$

where the sum of powers is less than n. Thus by induction hypothesis, we deduce that X_J is a linear combination of elements in $\mathfrak{B}_n$.

Similarly, if i_q is odd, then

$$X_J = \left(\frac{1}{2}\right)^{\frac{i_q-1}{2}} X_1^{i_1} \cdots X_{q-1}^{i_{q-1}} \left(\sum_{i=1}^{n} c_i X_i\right)^{\frac{i_q-1}{2}} X_q X_{q+1}^{i_{q+1}} \cdots X_m^{i_m},$$

which by induction is also a linear combination of elements in $\mathfrak{B}_n$. □

Corollary 3.3.6. *Let* $\mathfrak{B} = \bigcup_{p\geq 0} \mathfrak{B}_p$. *Then* $\mathfrak{B}$ *spans the universal enveloping algebra* $\mathfrak{U}(\mathfrak{g})$.

3.4 The PBW theorem

In this section, we will first show that the elements in $\mathfrak{B}$ are linearly independent. Therefore they form a basis for $\mathfrak{U}(\mathfrak{g})$. Then we state and prove the PBW theorem.

Let $\{x_1, \ldots, x_n\}$ be a homogeneous basis of $\mathfrak{g}$. For all $1 \leq i \leq n$, let $z_i = x_i + \mathcal{I}$ be the image of x_i under the canonical map $\mathfrak{g} \to \mathcal{S}(\mathfrak{g})$. Let $J = (j_1, \ldots, j_q)$ be a weakly increasing sequence satisfying condition (3.3.3). We define $z_J \coloneqq z_{j_1} \cdots z_{j_q} \in \mathcal{S}(\mathfrak{g})$. Let $\mathcal{B}_m = \{z_J \mid J$ weakly increasing and $\ell(J) \leq m$ such that J satisfies condition (3.3.3)$\}$. Then by Proposition 3.1.11, $\mathcal{B}_m$ is a basis of S_m and $\mathcal{B} = \bigcup_{m \geq 0} \mathcal{B}_m$ is a basis of $\mathcal{S}(\mathfrak{g})$. Thus if we can find a linear map that sends each $X_J = X_{j_1} \cdots X_{j_q}$ in $\mathfrak{B}$ to z_J in $\mathcal{B}$, we will be able to conclude that the X_J's are linearly independent.

So far, all we know about $\mathfrak{U}(\mathfrak{g})$ is that for any associative algebra $\mathcal{A}$, the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\sigma} & \mathfrak{U}(\mathfrak{g}) \\ & \searrow^{\rho} & \downarrow{\scriptstyle\varphi} \\ & & \mathcal{A} \end{array}$$

commutes (see Proposition 3.2.3). The most straightforward idea is to construct a Lie colour algebra homomorphism ρ from $\mathfrak{g}$ to $\mathcal{A} = \mathcal{S}(\mathfrak{g})$ first. However, if $\mathcal{A} = \mathcal{S}(\mathfrak{g})$, ρ being a Lie colour algebra homomorphism implies that $\rho([x, y]) = \rho(x)\rho(y) - \beta(y, x)\rho(y)\rho(x) = 0$ in $\mathcal{S}(\mathfrak{g})$ since $\mathcal{S}(\mathfrak{g})$ is colour-commutative. Hence the Lie colour bracket $[\rho(x), \rho(y)]$ will always vanish in $\mathcal{A}$, which makes us not able to analyze any structure of the Lie colour bracket.

Therefore instead of pursuing the above unsuccessful idea, we will use a variation of it. That is we construct a representation from $\mathfrak{g}$ to $\mathrm{End}(\mathcal{S}(\mathfrak{g}))$. Equivalently, we define a $\mathfrak{g}$-module structure on $\mathcal{S}(\mathfrak{g})$ with action $\rho : \mathfrak{g} \times \mathcal{S}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{g})$.

In order to construct such an action, we inductively define a family of bilinear maps $\rho_m : \mathfrak{g} \times S_m \to S_{m+1}$ such that $\rho_{m+1}\big|_{\mathfrak{g}\times S_m} = \rho_m$ for all m. Then we can extend all of the ρ_m to a ρ defined on $\mathfrak{g} \times \mathcal{S}(\mathfrak{g})$.

We give the following definition in order to introduce a natural action of $\mathfrak{g}$ on $\mathcal{S}(\mathfrak{g})$.

Definition 3.4.1. *Given a weakly increasing sequence $J = (j_1, \cdots, j_m)$ and $i \in \mathbb{Z}_{\geq 0}$, we write $i \leq J$ if $i \leq j_1$ or J is the empty sequence.*

Our idea is as follows. If there are some $x_i \in \mathfrak{g}$, $z_J \in \mathcal{B}_m$, then the sequence (i, J) is again weakly increasing and we get another basis element $z_i z_J \in \mathcal{B}_{m+1}$. Therefore we shall define

$$\rho_m(x_i)z_J = z_i z_J, \text{ if } i \leq J \text{ and } z_J \in \mathcal{B}_m. \tag{3.4.1}$$

Moreover, if $i \not\leq J$, all we can expect is $\rho_m(x_i)z_J \in S_{m+1}$. Therefore we should declare that

$$\rho_m(x_i)z_J - z_i z_J \in S_k \text{ when } \ell(J) = k \leq m. \tag{3.4.2}$$

Also, ρ_m should satisfy the Jacobi identity for all $m \geq 0$. Therefore we want the relation

$$\rho_m(x_i)\rho_m(x_j)z_L = \beta(x_j, x_i)\rho_m(x_j)\rho_m(x_i)z_L + \rho_m([x_i, x_j])z_L, \;\; \ell(L) \leq m-1 \tag{3.4.3}$$

to hold.

In summary, if there is a $\mathfrak{g}$-module homomorphism from $\mathfrak{g}$ to $\mathrm{End}(\mathcal{S}(\mathfrak{g}))$, then it must satisfy (3.4.1), (3.4.2) and (3.4.3). In Section 3.5, we will prove the following, even stronger, result.

Proposition 3.4.2. *There exists a unique linear map $\rho_m : \mathfrak{g} \to \mathrm{Hom}(S_m, S_{m+1})$ for each $m \geq 0$ such that* (3.4.1), (3.4.2) *and* (3.4.3) *hold.*

Assuming this proposition for now, let us deduce a few consequences. Let ρ be the unique map extending the ρ_m's. Let $(\rho, \mathcal{S}(\mathfrak{g}))$ be a $\mathfrak{g}$-module. Therefore by the universal enveloping property, there is a corresponding homomorphism $\tilde{\rho} : \mathfrak{U}(\mathfrak{g}) \to \mathrm{End}(\mathcal{S}(\mathfrak{g}))$ such that $\tilde{\rho} \circ \sigma = \rho$.

Corollary 3.4.3. *Let J be a weakly increasing sequence satisfying condition* (3.3.3)*. Then we have*

$$\tilde{\rho}(X_J) \cdot 1 = z_J.$$

Proof. Let $J = (j_1, \dots, j_q)$ be a weakly increasing sequence. Then

$$\begin{aligned} \tilde{\rho}(X_J) \cdot 1 &= \tilde{\rho}(X_{j_1}) \cdots \tilde{\rho}(X_{j_q}) \cdot 1 \\ &= \tilde{\rho}(\sigma(x_{j_1})) \cdots \tilde{\rho}(\sigma(x_{j_q})) \cdot 1 \\ &= \rho(x_{j_1}) \cdots \rho(x_{j_q}) \cdot 1 \\ &= z_{j_1} \cdots z_{j_q} = z_J, \end{aligned}$$

which completes the proof. □

Now together with this $\mathfrak{g}$-module structure and Proposition 3.2.3, we can prove the following Theorem.

Theorem 3.4.4 (PBW Theorem, first form)**.** *Let $\mathfrak{g}$ be a Lie colour algebra with ordered homogeneous basis $\{x_1, \dots, x_n\}$. Then a basis for the associated universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is given by*

$$\mathfrak{B} = \{X_1^{k_1} \cdots X_n^{k_n} \mid k_i \in \mathbb{Z}_{\geq 0} \text{ if } \beta(X_i, X_i) = 1 \text{ and } k_i \in \{0, 1\} \text{ if } \beta(X_i, X_i) = -1\}.$$

Proof. We first show that set $\mathfrak{B} = \cup_{p \geq 0} \mathfrak{B}_p$ is a basis of $\mathfrak{U}(\mathfrak{g})$. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathcal{S}(\mathfrak{g}))$

be the Lie colour algebra homomorphism guaranteed by Proposition 3.4.2. Let $J = (j_1, \ldots, j_q)$ be a weakly increasing sequence satisfying (3.3.3). By Corollary 3.4.3, we have

$$\tilde{\rho}(X_J) \cdot 1 = z_J.$$

Therefore the linear map

$$\varphi : \mathfrak{U}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{g}),\ X_J \mapsto \tilde{\rho}(X_J) \cdot 1$$

maps $\mathfrak{B}$ bijectively onto the set $\mathcal{B}$ which is a basis for $\mathcal{S}(\mathfrak{g})$. Therefore $\mathfrak{B}$ is a linearly independent set. Since $\mathfrak{B}$ spans $\mathfrak{U}(\mathfrak{g})$ by Proposition 3.3.4, $\mathfrak{B}$ is a basis of $\mathfrak{U}(\mathfrak{g})$. The result follows by noticing that

$$\begin{aligned} \mathfrak{B} &= \{X_J \mid J \text{ weakly increasing and } z_J \text{ satisfies condition (3.3.3)}\} \\ &= \{X_1^{k_1} \cdots X_n^{k_n} \mid k_i \in \mathbb{Z}_{\geq 0} \text{ if } \beta(X_i, X_i) = 1 \text{ and } k_i \in \{0, 1\} \text{ if } \beta(X_i, X_i) = -1\}. \square \end{aligned}$$

From the preceding discussion, we observe that there are profound connections between $\mathfrak{U}(\mathfrak{g})$ and $\mathcal{S}(\mathfrak{g})$. However they are not isomorphic to each other since $\mathfrak{U}(\mathfrak{g})$ is not necessarily colour-commutative and φ is not an algebra isomorphism. Our next goal is to construct the associated graded algebra of $\mathfrak{U}(\mathfrak{g})$, $gr(\mathfrak{U}(\mathfrak{g}))$. Then we prove that there is an isomorphism of algebras between $gr(\mathfrak{U}(\mathfrak{g}))$ and $\mathcal{S}(\mathfrak{g})$.

By Remark 3.3.1, we have filtration of $\mathfrak{U}(\mathfrak{g})$. Then we can construct the associated graded algebra $gr(\mathfrak{U}(\mathfrak{g}))$ of $\mathfrak{U}(\mathfrak{g})$ as

$$gr(\mathfrak{U}(\mathfrak{g})) = \bigoplus_{m \geq 0} G^m$$

where $G^m = {}^{\mathfrak{U}_m(\mathfrak{g})}\!\big/_{\mathfrak{U}_{m-1}(\mathfrak{g})}$ and $G^0 = \mathbb{F}$. The multiplication is defined by

$$(x_m + \mathfrak{U}_{m-1}(\mathfrak{g}))(x_n + \mathfrak{U}_{n-1}(\mathfrak{g})) = x_m x_n + \mathfrak{U}_{n+m-1}(\mathfrak{g})$$

for any $x_m \in \mathfrak{U}_m(\mathfrak{g})$ and $x_n \in \mathfrak{U}_n(\mathfrak{g})$.

Lemma 3.4.5. *With the multiplication defined above, $gr(\mathfrak{U}(\mathfrak{g}))$ is a colour-commutative algebra.*

Proof. Note that $gr(\mathfrak{U}(\mathfrak{g}))$ is generated as an algebra by $G^1 \cong \mathfrak{U}(\mathfrak{g}) \cong \mathfrak{g}$. Therefore it suffices to show that any two elements in G^1 colour-commute in $gr(\mathfrak{U}(\mathfrak{g}))$.

Take $x + \mathfrak{U}_0(\mathfrak{g})$ and $y + \mathfrak{U}_0(\mathfrak{g})$ in G^1 such that x and y are homogeneous. Then we have

$$(x + \mathfrak{U}_0(\mathfrak{g}))(y + \mathfrak{U}_0(\mathfrak{g})) = xy + \mathfrak{U}_1(\mathfrak{g}) \in G^2. \tag{3.4.4}$$

Whereas we have

$$\beta(y,x)(y + \mathfrak{U}_0(\mathfrak{g}))(x + \mathfrak{U}_0(\mathfrak{g})) = \beta(y,x)yx + \mathfrak{U}_1(\mathfrak{g}) \in G^2. \tag{3.4.5}$$

Since $xy - \beta(y,x)yx = [x,y] \in \mathfrak{U}_1(\mathfrak{g})$, the two classes (3.4.4) and (3.4.5) are equal. □

Theorem 3.4.6 (PBW theorem, second form)**.** *Let $\mathfrak{g}$ be a Lie colour algebra over a field $\mathbb{F}$. Then we have an isomorphism of algebras $gr(\mathfrak{U}(\mathfrak{g})) \cong \mathcal{S}(\mathfrak{g})$.*

Proof. Let $\{x_1, \ldots, x_n\}$ be an ordered homogeneous basis of $\mathfrak{g}$. By Theorem 3.4.4, the elements $X_1^{k_1} \cdots X_n^{k_n}$ such that $\sum_{j=1}^n k_j \leq m$ form a basis for $\mathfrak{U}_m(\mathfrak{g})$. Therefore the elements $X_1^{k_1} \cdots X_n^{k_n} + \mathfrak{U}_{m-1}(\mathfrak{g})$ such that $\sum_{j=1}^n k_j = m$ form a basis for ${}^{\mathfrak{U}_m(\mathfrak{g})}\!\big/_{\mathfrak{U}_{m-1}(\mathfrak{g})}$.

This gives us a linear isomorphism between the graded pieces $\alpha : G^m \to S^m$ such that

$$\alpha : X_1^{k_1} \cdots X_n^{k_n} + \mathfrak{U}_{m-1}(\mathfrak{g}) \mapsto z_1^{k_1} \cdots z_n^{k_n},$$

where $k_1 + \cdots + k_n = m$. Now we have

$$(X_1^{k_1} \cdots X_n^{k_n} + \mathfrak{U}_{m-1}(\mathfrak{g}))(X_1^{r_1} \cdots X_n^{r_n} + \mathfrak{U}_{n-1}(\mathfrak{g})) = X_1^{k_1} \cdots X_n^{k_n} X_1^{r_1} \cdots X_n^{r_n} + \mathfrak{U}_{n+m-1}(\mathfrak{g}).$$

Since $gr(\mathfrak{U}(\mathfrak{g}))$ is colour-commutative (Lemma 3.4.5), we can gather the $X_1, \ldots, X_n$ together. Thus the above element can be rearranged as

$$\beta_\pi X_1^{k_1+r_1} \cdots X_n^{k_n+r_n} + \mathfrak{U}_{n+m-1}(\mathfrak{g}),$$

where β_π is as defined in Lemma 3.1.5, which is the same scalar generated as we identify $z_1^{k_1} \cdots z_n^{k_n} z_1^{r_1} \cdots z_n^{r_n}$ with $\beta_\pi z_1^{k_1+r_1} \cdots z_n^{k_n+r_n}$ in $\mathcal{S}(\mathfrak{g})$. It follows that $\alpha(uv) = \alpha(u)\alpha(v)$ for all $u, v \in gr(\mathfrak{U}(\mathfrak{g}))$. Thus we have an algebra isomorphism $gr(\mathfrak{U}(\mathfrak{g})) \cong \mathcal{S}(\mathfrak{g})$. □

Now we give an application of the PBW theorem. Note that although the following discussion works for any Lie colour algebra, we state it in the context of $\mathfrak{spo}(V, \beta)$ in order to avoid re-defining related concepts.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{spo}(V, \beta)$. Let $\mathfrak{g}_+$ be the span of positive root vectors. Let Φ^- be the set of negative root vectors. Then recall that from Definition 2.5.16, if W is an $\mathfrak{spo}(V, \beta)$-module, a vector $w \in W$ is called a highest weight vector of weight $\lambda \in \mathfrak{h}^*$ if $Hw = \lambda(H)w$ for all $H \in \mathfrak{h}$ and $Xw = 0$ for all $X \in \mathfrak{g}_+$.

Corollary 3.4.7. *In the setting above, let w be a highest weight vector of W. The submodule $\mathfrak{U}(\mathfrak{g}) \cdot w$ is spanned by the vectors*

$$\{Y_{i_1} Y_{i_2} \cdots Y_{i_k} \cdot w \mid k \in \mathbb{N}, i_j \in -\Phi^+ \text{ for each } 1 \leq j \leq k\}.$$

Proof. Let $\mathfrak{g} = \mathfrak{spo}(V, \beta)$. Notice that by the PBW theorem (Theorem 3.4.6), by giving an order to the basis of $\mathfrak{g}$, we obtain an ordered basis of $\mathfrak{U}(\mathfrak{g})$. Then without loss of generality, the basis vectors of $\mathfrak{U}(\mathfrak{g})$ have the form of a product of negative roots vectors, $H \in \mathfrak{h}$, and positive root vectors. Since w is a highest weight vector, any positive root vector acts on w as 0, and H acts on it as scalar. Then the result follows since all that remains is to only consider a product of negative root vectors acting on w. □

3.5 Proof of Proposition 3.4.2

In the previous section, we proved the PBW theorem, modulo Proposition 3.4.2 which is about the existence of a certain $\mathfrak{g}$-module structure on $\mathcal{S}(\mathfrak{g})$. A terse proof Proposition 3.4.2 for Lie algebras in the classical sense can be found in [Hum72, Section 17.4]. In this section, in the more general setting of Lie colour algebras, we elaborate the proof of this proposition by induction. In order to apply induction, we relabel the three conditions which $\rho_m : \mathfrak{g} \to \mathrm{Hom}(S_m, S_{m+1})$ should satisfy by the following:

(A_m) $\rho_m(x_i)z_J = z_i z_J$ for all J with $\ell(J) \leq m$ and all $i \leq J$,

(B_m) $\rho_m(x_i)z_J - z_i z_J \in S_k$ for all J with $\ell(J) = k \leq m$ and all $1 \leq i \leq n$,

(C_m) $\rho_m(x_i)\rho_m(x_j)z_L = \beta(x_j, x_i)\rho_m(x_j)\rho_m(x_i)z_L + \rho_m([x_i, x_j])z_L$ for all L with $\ell(L) \leq m - 1$ and all $1 \leq i, j \leq n$.

Our goal is to define an action of $\mathfrak{g}$ on $\mathcal{S}(\mathfrak{g})$. It is enough to define the action on the basis. Recall that from Section 3.4, $\mathcal{B}_m = \{z_J \mid \ J \text{ weakly increasing and } \ell(J) \leq m\}$ is a basis of S_m and $\mathcal{B} = \bigcup_{m \geq 0} \mathcal{B}_m$ is a basis of $\mathcal{S}(\mathfrak{g})$.

Lemma 3.5.1. *Let $m \geq 1$. If there exists a linear map ρ_{m-1} which satisfies (A_{m-1}), (B_{m-1}) and (C_{m-1}), then there exists at most one linear map ρ_m extending ρ_{m-1}*

which satisfies (A_m), (B_m) and (C_m). Moreover for a weakly increasing sequence $J = (j_1, \dots, j_m)$, the map ρ_m must be recursively defined by

$$\rho_m(x_i)z_J = \begin{cases} z_i z_J, & \text{if } i \le J \\ z_i z_J + \beta(x_{j_1}, x_i)\rho_{m-1}(x_{j_1})w + \rho_{m-1}([x_i, x_{j_1}])z_{J'}, & \text{otherwise} \end{cases} \tag{3.5.1}$$

where $J' = (j_2, \dots, j_m)$ and $w = \rho_{m-1}(x_i)z_{J'} - z_i z_{J'} \in S_{m-1}$.

Proof. When $m = 0$, then in order to satisfy condition (A_0), there is only one way to define ρ_0, which is

$$\rho_0(x_i)1 = z_i$$

and conditions (B_0) and (C_0) are satisfied automatically. So ρ_0 exists and is unique.

Now suppose that we have a uniquely determined ρ_{m-1} satisfying conditions $(A_{m-1}), (B_{m-1})$ and (C_{m-1}). Then our goal is to construct ρ_m satisfying (A_m), (B_m) and (C_m) and extending ρ_{m-1}. In particular, we need to define the action of $\rho_m(x_i)$ on z_J for all x_i running over our basis of $\mathfrak{g}$ and J running over all weakly increasing sequence of length $\ell(J) \le m$.

If $i \le J$, then we have to define $\rho_m(x_i)z_J = z_i z_J$ in order to satisfy condition (A_m). If $i \not\le J$, then we have $j_1 < i$. Then we set $J = (j_1, J')$ and note that $j_1 \le J'$. Then

$$\rho_m(x_i)z_J = \rho_m(x_i)z_{j_1}z_{J'} = \rho_m(x_i)\rho_{m-1}(x_{j_1})z_{J'}$$

where the second equality uses the fact $z_{J'} \in \mathcal{B}_{m-1}$ and the induction hypothesis (A_{m-1}). Also since $\rho_{m-1} = \rho_m\big|_{\mathfrak{g}\times S_{m-1}}$ and in order to satisfy (C_m), $\rho_m(x_i)\rho_{m-1}(x_{j_1})z_{J'}$ must be equal to

$$\beta(x_{j_1}, x_i)\rho_m(x_{j_1})\rho_{m-1}(x_i)z_{J'} + \rho_{m-1}([x_i, x_{j_1}])z_{J'}. \tag{3.5.2}$$

The first term of 3.5.2 can be simplified. First notice that (B_{m-1}) implies

$$w = \rho_{m-1}(x_i)z_{J'} - z_i z_{J'} \in S_{m-1}.$$

Therefore by linearity we have

$$\begin{aligned}
&\beta(x_{j_1}, x_i)\rho_m(x_{j_1})\rho_{m-1}(x_i)z_{J'} \\
&\quad = \beta(x_{j_1}, x_i)\rho_m(x_{j_1})(z_i z_{J'} + w) \\
&\quad = \beta(x_{j_1}, x_i)(\rho_m(x_{j_1})z_i z_{J'} + \rho_{m-1}(x_{j_1})w) \quad \text{by } \rho_{m-1} = \rho_m\big|_{\mathfrak{g}\times S_{m-1}} \\
&\quad = \beta(x_{j_1}, x_i)(z_{j_1} z_i z_{J'} + \rho_{m-1}(x_{j_1})w) \quad \text{by } j_1 < i \text{ and } (A_m) \\
&\quad = \beta(x_{j_1}, x_i)(\beta(x_i, x_{j_1})z_i z_J + \rho_{m-1}(x_{j_1})w) \quad \text{by the colour-commutativity of } \mathcal{S}(\mathfrak{g})
\end{aligned}$$

which is $z_i z_J + \beta(x_{j_1}, x_i)\rho_{m-1}(x_{j_1})w$. Therefore in this case, if ρ_m extends ρ_{m-1} and satisfies (A_m), we must have

$$\rho_m(x_i)z_J = z_i z_J + \beta(x_{j_1}, x_i)\rho_{m-1}(x_{j_1})w + \rho_{m-1}([x_i, x_{j_1}])z_{J'}. \qquad \square$$

Lemma 3.5.1 gives us a recursive formula for a function ρ_m extending ρ_{m-1}, and by construction it satisfies (A_m). In the next two lemmas, we prove it satisfies (B_m) and (C_m).

From now on we write j instead of j_1 for simplicity. But otherwise retain the notation of Lemma 3.5.1.

Lemma 3.5.2. *The map that is uniquely defined by* (3.5.1) *satisfies* (B_m).

Proof. Let x_i be in the basis of $\mathfrak{g}$, $z_J \in \mathcal{B}_k$ for some $k \leq m$, and $i \not\leq J$. We need to prove that $\rho_m(x_i)z_J - z_i z_J \in S_k$. By the formula (3.5.1), we have

$$\rho_m(x_i)z_J = z_i z_J + \beta(x_j, x_i)\rho_{m-1}(x_j)w + \rho_{m-1}([x_i, x_j])z_{J'}$$

where $w \in S_{k-1}$ and thus, $\beta(x_j, x_i)\rho_{m-1}(x_j)w \in S_k$. Similarly, $z_{J'} \in S_{k-1}$ implies $\rho_{m-1}([x_i, x_j])z_{J'}$ in S_k. Therefore this proves $\rho_m(x_i)x_J - z_i z_J \in S_k$ so that (B_m) is satisfied. □

Now we are going prove that ρ_m satisfies (C_m) by case analysis.

Lemma 3.5.3. *Let L be a weakly increasing sequence with $\ell(L) = m - 1$. If either of the two conditions*

(i) $i \geq j$ and $j \leq L$, or

(ii) $i \leq j$ and $i \leq L$

hold, then ρ_m also satisfies (C_m) for these values of i and j.

Proof. Suppose $i \geq j$ and $j \leq L$. Then by (A_{m-1}), we know that $\rho_m(x_j)z_L = \rho_{m-1}(x_j)z_L = z_j z_L$. Therefore we apply the formula (3.5.1) to $\rho_m(x_i)z_j z_L$ to obtain

$$\rho_m(x_i)z_j z_L = z_i z_j z_L + \beta(x_j, x_i)\rho_{m-1}(x_j)w + \rho_{m-1}([x_i, x_j])z_L \tag{3.5.3}$$

where $w = \rho_{m-1}(x_i)z_L - z_i z_L \in S_{m-1}$ in this case. Since $w \in S_{m-1}$, we can write $\rho_{m-1}(x_j)w$ as $\rho_m(x_j)w$. Then we expand it as follows

$$\begin{aligned}\rho_m(x_j)w &= \rho_m(x_j)(\rho_{m-1}(x_i)z_L - z_i z_L) \\ &= \rho_m(x_j)\rho_{m-1}(x_i)z_L - \rho_m(x_j)z_i z_L \quad \text{by linearity}\end{aligned}$$

$$= \rho_m(x_j)\rho_{m-1}(x_i)z_L - z_j z_i z_L \qquad \text{since } j \leq i \text{ and } j \leq L$$
$$= \rho_m(x_j)\rho_{m-1}(x_i)z_L - \beta(x_i, x_j)z_i z_j z_L.$$

Since $z_L \in S_{m-1}$, we can replace ρ_{m-1} by ρ_m in this last term. Therefore if we substitute $\beta(x_j, x_i)\rho_m(x_j)w$ by $\beta(x_j, x_i)\rho_m(x_j)\rho_m(x_i)z_L - z_i z_j z_L$ in 3.5.3, we conclude

$$\rho_m(x_i)\rho_m(x_j)z_L = \beta(x_j, x_i)\rho_m(x_j)\rho_m(x_i)z_L + \rho_m([x_i, x_j])z_L, \tag{3.5.4}$$

as required.

Now we assume that $i \leq j$ and $i \leq L$. Then by a similar calculation in case (ii), we should conclude that

$$\rho_m(x_j)\rho_m(x_i)z_L = \beta(x_i, x_j)\rho_m(x_i)\rho_m(x_j)z_L + \rho_m([x_j, x_i])z_L. \tag{3.5.5}$$

After rearranging (3.5.5), we get

$$\beta(x_i, x_j)\rho_m(x_i)\rho_m(x_j)z_L = \rho_m(x_j)\rho_m(x_i)z_L - \rho_m([x_j, x_i])z_L;$$

since $[x_j, x_i] = -\beta(x_i, x_j)[x_i, x_j]$ and ρ_m is linear in each variable, this is equivalent to

$$\beta(x_i, x_j)\rho_m(x_i)\rho_m(x_j)z_L = \rho_m(x_j)\rho_m(x_i)z_L + \beta(x_i, x_j)\rho_m([x_i, x_j])z_L.$$

Multiplying by $\beta(x_j, x_i)$ on both sides gives

$$\rho_m(x_i)\rho_m(x_j)z_L = \beta(x_j, x_i)\rho_m(x_j)\rho_m(x_i)z_L + \rho_m([x_i, x_j])z_L.$$

This proves that the condition (C_m) holds in case (ii). $\square$

Now the only remaining case left to prove is when neither $i \leq L$ nor $j \leq L$, in which case we must have that $L = (\ell, L')$ with $\ell < i, j$.

Lemma 3.5.4. *Let L be a weakly increasing sequence with $\ell(L) = m-1$. Let i, j be such that neither $i \leq L$ nor $j \leq L$. Then ρ_m satisfies (C_m) for these values of i and j.*

By looking at (C_m), our strategy to prove this lemma is using Lemma 3.5.3 and the induction hypothesis to rewrite $\rho_m(x_i)\rho_m(x_j)z_L$ and $\beta(x_j, x_i)\rho_m(x_j)\rho_m(x_i)z_L$ explicitly. Then we prove that the difference between these two terms is $\rho_m([x_i, x_j])z_L$.

To save some space, we write $\beta(i, j) = \beta(x_i, x_j)$ for all $x_i, x_j \in \mathfrak{g}$. Also since $[x_i, x_j]$ has colour equal to the product of colours of x_i and x_j, we write ij for this colour. Moreover, instead of $\rho_m(x_i)z_L$, we write $x_i \cdot z_L$ in the proof.

Proof. First we consider the term $\rho_m(x_i)\rho_m(x_j)z_L = x_i \cdot x_j \cdot z_L$. Since $\ell \leq L = (\ell, L')$, we have $z_L = x_\ell \cdot z_{L'}$. Therefore

$$x_j \cdot z_L = x_j \cdot x_\ell \cdot z_{L'} \tag{3.5.6}$$

where $j > \ell$ and $\ell \leq L'$. Thus the right hand side of (3.5.6) satisfies our assumption of Lemma 3.5.3 (i). Therefore we have

$$x_j \cdot z_L = \beta(\ell, j)x_\ell \cdot x_j \cdot z_{L'} + [x_j, x_\ell] \cdot z_{L'}.$$

Therefore by linearity we have

$$x_i \cdot x_j \cdot z_L = \beta(\ell, j)x_i \cdot x_\ell \cdot x_j \cdot z_{L'} + x_i \cdot [x_j, x_\ell] \cdot z_{L'}. \tag{3.5.7}$$

Now we expand (3.5.7) even further.

For the first term, since L' has length $\leq m-2$, by (B_m), we can write $x_j \cdot z_{L'} = z_j z_{L'} + w$ for some $w \in S_{m-2}$. Therefore we can write $\beta(\ell, j) x_i \cdot x_\ell \cdot x_j \cdot z_{L'}$ as

$$\beta(\ell, j)(x_i \cdot x_\ell \cdot z_j z_{L'} + x_i \cdot x_\ell \cdot w). \tag{3.5.8}$$

Now since $\ell < i, j$ and $\ell \leq L'$, the first term of (3.5.8) satisfies the condition of Lemma 3.5.3 (i). Therefore we can expand it according to (C_m). Moreover, since $w \in S_{m-2}$, we can also expand the second term of (3.5.8) by the induction hypothesis. Therefore we can expand $\beta(\ell, j) x_i \cdot x_\ell \cdot x_j \cdot z_{L'}$ by considering $x_j \cdot z_{L'}$ as one term. Thus we have:

$$\beta(\ell, j) x_i \cdot x_\ell \cdot x_j \cdot z_{L'} = \beta(\ell, j)\big(\beta(\ell, i) x_\ell \cdot x_i \cdot x_j \cdot z_{L'} + [x_i, x_\ell] \cdot x_j \cdot z_{L'}\big).$$

Now for the second term of (3.5.7). Since L' has length $\leq m-2$, by the induction hypothesis, we can expand it as

$$\beta(j\ell, i)[x_j, x_\ell] \cdot x_i \cdot z_{L'} + [x_i, [x_j, x_\ell]] \cdot z_{L'}.$$

Therefore, in summary, we can expand (3.5.7) as

$$\begin{aligned}\beta(\ell, i)\beta(\ell, j) x_\ell \cdot x_i \cdot x_j \cdot z_{L'} + \underline{\beta(\ell, j)[x_i, x_\ell] \cdot x_j \cdot z_{L'}} \\ + \underline{\beta(j\ell, i)[x_j, x_\ell] \cdot x_i \cdot z_{L'}} + [x_i, [x_j, x_\ell]] \cdot z_{L'}\end{aligned} \tag{3.5.9}$$

where we have underlined some terms for easy reference. Similarly as before, if we swap the roles of i and j, we conclude that $\rho_m(x_j)\rho_m(x_i) z_L$ becomes

$$\beta(\ell, i)\beta(\ell, j) x_\ell \cdot x_j \cdot x_i \cdot z_{L'} + \beta(\ell, i)[x_j, x_\ell] \cdot x_i \cdot z_{L'}$$

$$+ \beta(i\ell, j)[x_i, x_\ell] \cdot x_j \cdot z_{L'} + [x_j, [x_i, x_\ell]] \cdot z_{L'}.$$

Therefore $\beta(j,i)\rho_m(x_j)\rho_m(x_i)z_L$ is

$$\beta(j,i)\beta(\ell,i)\beta(\ell,j)x_\ell \cdot x_j \cdot x_i \cdot z_{L'} + \underline{\beta(j,i)\beta(\ell,i)[x_j, x_\ell] \cdot x_i \cdot z_{L'}} + \underline{\beta(j,i)\beta(i\ell,j)[x_i, x_\ell] \cdot x_j \cdot z_{L'}} + \beta(j,i)[x_j, [x_i, x_\ell]] \cdot z_{L'}. \quad (3.5.10)$$

Subtract (3.5.10) from (3.5.9); the underlined terms will be cancelled, and the difference is:

$$\beta(\ell,i)\beta(\ell,j)x_\ell \cdot x_i \cdot x_j \cdot z_{L'} - \beta(j,i)\beta(\ell,i)\beta(\ell,j)x_\ell \cdot x_j \cdot x_i \cdot z_{L'} + [x_i, [x_j, x_\ell]] \cdot z_{L'} - \beta(j,i)[x_j, [x_i, x_\ell]] \cdot z_{L'} \quad (3.5.11)$$

The first two terms of (3.5.11) can be combined into one term as

$$\beta(\ell,i)\beta(\ell,j)x_\ell \cdot \big(\underline{x_i \cdot x_j \cdot z_{L'} - \beta(j,i)x_j \cdot x_i \cdot z_{L'}}\big).$$

Moreover since $\ell(L') \leq m-2$, by induction hypothesis, the underlined term can be simplified further as $[x_i, x_j] \cdot z_{L'}$ using (C_m). Therefore we have the reduced form

$$\beta(\ell,i)\beta(\ell,j)x_\ell \cdot [x_i, x_j] \cdot z_{L'}. \quad (3.5.12)$$

Using (C_m) again, (3.5.12) is equivalent to

$$\beta(\ell,i)\beta(\ell,j)\beta(ij,\ell)[x_i, x_j] \cdot x_\ell \cdot z_{L'} + \beta(\ell,i)\beta(\ell,j)[x_\ell, [x_i, x_j]] \cdot z_{L'}$$

which (by the fact that $\beta(\ell,i)\beta(\ell,j)\beta(ij,\ell)=1$) is

$$[x_i,x_j]\cdot x_\ell\cdot z_{L'}+\beta(\ell,i)\beta(\ell,j)[x_\ell,[x_i,x_j]]\cdot z_{L'}. \tag{3.5.13}$$

Thus by replacing the first two terms of (3.5.11) by (3.5.13), (3.5.11) becomes

$$[x_i,x_j]\cdot x_\ell\cdot z_{L'}+[x_i,[x_j,x_\ell]]\cdot z_{L'}+\beta(\ell,i)\beta(\ell,j)[x_\ell,[x_i,x_j]]\cdot z_{L'}-\beta(j,i)[x_j,[x_i,x_\ell]]\cdot z_{L'}. \tag{3.5.14}$$

Now based on the Jacobi identity, we need to do some manipulations of β to show that the sum of the last three terms in (3.5.14) is 0. More precisely, we show that

$$[x_i,[x_j,x_\ell]]+\beta(\ell,i)\beta(\ell,j)[x_\ell,[x_i,x_j]]-\beta(j,i)[x_j,[x_i,x_\ell]] \tag{3.5.15}$$

is 0. By Lemma 2.1.7, we have a different version of the Jacobi identity, namely in our case, we have

$$[x_i,[x_j,x_\ell]]-[[x_i,x_j],x_\ell]-\beta(j,i)[x_j,[x_i,x_\ell]]=0. \tag{3.5.16}$$

Notice that the first and third term in (3.5.15) and (3.5.16) are identical. Then if we swap $[x_i,x_j]$ with x_ℓ in $-[[x_i,x_j],x_\ell]$ (the second term of (3.5.16)), we obtain $\beta(\ell,ij)[x_\ell,[x_i,x_j]]$ which is exactly the same as the second term of (3.5.15). Thus, we conclude that (3.5.15) $=0$. Therefore (3.5.14) becomes $[x_i,x_j]\cdot x_\ell\cdot z_{L'}=[x_i,x_j]\cdot z_L$. This implies that (3.5.11) $=[x_i,x_j]\cdot z_L$ which is the difference between $x_i\cdot x_i\cdot z_L$ and $\beta(j,i)x_j\cdot x_i\cdot L$. Thus we have

$$x_i\cdot x_i\cdot z_L-\beta(j,i)x_j\cdot x_i\cdot z_L=[x_i,x_j]\cdot z_L$$

which is equivalent to (C_m). This completes the proof. □

In Lemma 3.5.1, we constructed the map $\rho_m : \mathfrak{g} \to \mathrm{Hom}(S_m, S_{m+1})$ and proved it satisfies (A_m). By Lemma 3.5.2, we proved ρ_m satisfies (B_m). By Lemma 3.5.3 and 3.5.4, we proved that ρ_m satisfies (C_m). Thus by the lemmas in this section, we proved Proposition 3.4.2, which guarantees the validities of the two versions of the PBW theorem, Theorem 3.4.4 and 3.4.6.

Chapter 4

The Brauer algebra action on $V^{\otimes k}$

The Brauer algebra was introduced by Richard Brauer in [Bra37]. Brauer's motivation was to obtain the centralizer of the action of the orthogonal group on tensor powers of its standard module. This idea has also been shown to work when the orthogonal group is replaced by the symplectic group (for example, see [BR99]). In Section 4.1, we discuss the Brauer algebra and in Section 4.2, following [BSR98], we define an action of the Brauer algebra on tensor powers of the standard $\mathfrak{spo}(V, \beta)$-module, which commutes with the action of $\mathfrak{spo}(V, \beta)$.

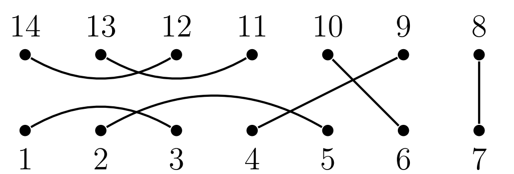
4.1 The Brauer algebra

In this section, for any positive integer k, we first give the definition of k-diagrams. These k-diagrams will span the Brauer algebra. In particular, we describe how to compose k-diagrams, resulting in the algebra structure. Moreover, we discuss a special subset of the k-diagrams so that we can define a subalgebra of the Brauer algebra which is isomorphic to the group algebra of S_k.

Definition 4.1.1. *Let $k \in \mathbb{Z}_+$. A k-diagram is a diagram with two parallel rows with k vertices in each row, and k edges such that each vertex is linked to precisely one edge. We number the bottom vertices from left to right by $1, 2, \cdots, k$ and the top vertices from right to left by $k+1, k+2, \cdots, 2k$. Note that loops are not allowed in a k-diagram.*

The set of all k-diagrams is denoted by $\mathcal{D}_k$.

Example 4.1.2. An example of a k-diagram is

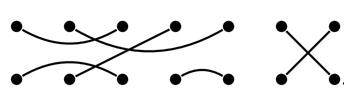


$\in \mathcal{D}_7$

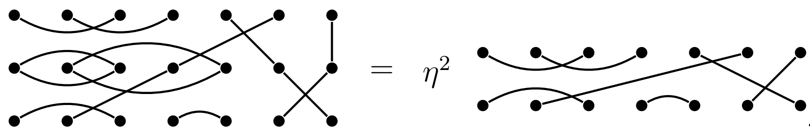
Remark 4.1.3. Let $k \in \mathbb{Z}_+$. There are $(2k-1)\cdot(2k-3)\cdots 5\cdot 3\cdot 1 = \frac{(2k)!}{2^k k!}$ k-diagrams since we have $2k-1$ possible ways to connect the first vertex with another vertex, then we have $2k-3$ possibilities to join the next unconnected vertex and so on.

Next we give an operation on $\mathcal{D}_k$ so that the vector space spanned by a basis corresponding to $\mathcal{D}_k$ becomes an algebra. This algebra is called the Brauer algebra. Let $\mathbb{F}$ be a field of characteristic 0 and let $\eta \in \mathbb{F}$. If $d_1, d_2 \in \mathcal{D}_k$, then we compose d_1 and d_2 by putting d_1 on top of d_2 identifying i and $2k-i+1$ in d_1 and d_2 respectively for all $1 \le i \le k$. Then after removing the middle line of vertices and any loops, we have another k-diagram d and we define $d_1 d_2 = \eta^c d$ where c is the number of loops.

Example 4.1.4. Let d_1 be the k-diagram we defined in Example 4.1.2 and

$d_2 =$ .

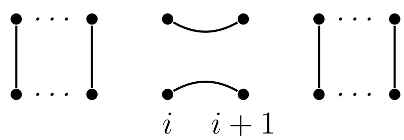
Then

$d_1 d_2 =$  .

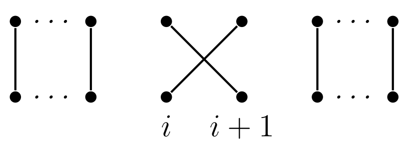
Definition 4.1.5. *The Brauer algebra $B_k(\eta)$ is the $\mathbb{F}$-span of the k-diagrams in $\mathcal{D}_k$ with multiplication induced by the composition of k-diagrams.*

By Remark 4.1.3, the dimension of $B_k(\eta)$ is $\frac{(2k)!}{2^k k!}$.

Proposition 4.1.6. *The Brauer algebra $B_k(\eta)$ is generated by the k-diagrams e_i and s_i, for $1 \leq i \leq k-1$, where*

$e_i =$ 

and

$s_i =$  ,

modulo the following relations.

$$
\begin{array}{llll}
s_i^2 = 1, & e_i^2 = \eta e_i, & e_i s_i = s_i e_i = e_i, & 1 \leq i \leq k-1, \\
s_i s_j = s_j s_i, & s_i e_j = e_j s_i, & e_i e_j = e_j e_i, & |i-j| \geq 1, \\
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & e_i e_{i+1} e_i = e_i, & e_{i+1} e_i e_{i+1} = e_{i+1}, & 1 \leq i \leq k-2, \\
s_i e_{i+1} s_i = s_{i+1} e_i, & e_{i+1} e_i s_{i+1} = e_{i+1} s_i, & & 1 \leq i \leq k-2.
\end{array}
$$

Proof. A proof can be found in [Wen88, Section 2]. □

Now our next plan is to give another presentation of $B_k(\eta)$ so that we can form a permutation corresponding to each k-diagram. For each k-diagram d, we can represent d as a sequence of pairs of linked vertices such that $d = \{(\ell_1, r_1), (\ell_2, r_2), \ldots, (\ell_k, r_k)\}$ where $\ell_i < r_i$ for all $\ell_i, r_i \in \{1, \ldots, 2k\}$.

Example 4.1.7. Let d be given in Example 4.1.2. We have

$$d = \{(1,3), (2,5), (4,9), (6,10), (7,8), (11,13), (12,14)\}.$$

There is a way to construct a permutation of $\{1, \ldots, 2k\}$ using $d \in \mathcal{D}_k$: let $d = \{(\ell_1, r_1), (\ell_2, r_2), \ldots, (\ell_k, r_k)\} \in \mathcal{D}_k$ with $\ell_1 < \cdots < \ell_k$. We construct the permutation π_d associated with d by letting

$$\pi_d = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & 2k \\ \ell_1 & \ell_2 & \cdots & \ell_k & r_k & r_{k-1} & \cdots & r_1 \end{pmatrix} \in S_{2k}.$$

Example 4.1.8. For the diagram $e_1 \in \mathcal{D}_2$, we have $e_1 = \{(1,2), (3,4)\}$, and the corresponding π_{e_1} is

$$\pi_{e_1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = (234) = (23)(34).$$

Example 4.1.9. For the diagram $s_1 \in \mathcal{D}_2$, we have $s_1 = \{(1,3), (2,4)\}$, and the corresponding π_{s_1} is

$$\pi_{s_1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = (34).$$

Let $d \in \mathcal{D}_k$ be any diagram such that each vertex in the lower row is connected to one in the upper row. Then $d = \{(1, r_1), (2, r_2), \ldots, (k, r_k)\}$. Therefore the corresponding permutation is

$$\pi_d = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & 2k \\ 1 & \cdots & k & r_k & \cdots & r_1 \end{pmatrix},$$

which is a permutation of $\{k+1, \ldots, 2k\}$. For example,

$$\pi_{s_i} = (2k - i, 2k - i + 1). \tag{4.1.1}$$

Remark 4.1.10. Identifying $\{k+1, \ldots, 2k\}$ with $\{1, \ldots, k\}$ by the transformation $i \mapsto 2k - i + 1$, we see that s_i goes to the transposition $(i, i+1)$. In general, any $d \in B_k(\eta)$ with only vertical edges can be written as $s_{i_1} \cdots s_{i_n}$, for some n and some $i_j \in \{1, \ldots, k-1\}$. Therefore, we can identify such d with the permutation $\sigma = (i_1, i_1 + 1) \cdots (i_n, i_n + 1) = s_{i_1} \cdots s_{i_n} \in S_k$, where s_{i_j} now denotes also the simple transpositions $(i_j, i_j + 1)$. In this way, we identify the subalgebra of $B_k(\eta)$ generated by the s_i with the group algebra $\mathbb{F}[S_k]$.

Note that this is the same identification as obtained by viewing the diagrams with only vertical edges as permutations of $\{1, \ldots, k\}$ in the natural way.

4.2 The commuting action of the Brauer algebra on the $\mathfrak{spo}(V, \beta)$-module $V^{\otimes k}$

In this section, let $\mathfrak{g} = \mathfrak{spo}(V, \beta)$. Let $B = \{v_1, \ldots, v_n\}$ be the ordered basis given in (2.4.1). Denote the dual basis by $\{v^1, \ldots, v^n\}$. For $k \in \mathbb{Z}_{\geq 0}$, we define an action of $\mathcal{D}_k$ on $V^{\otimes k}$, which is provided in [BSR98]. This action is designed to make the generator

s_i act by permutations, and the image of e_i on $V^{\otimes k}$ isomorphic to $V^{\otimes(k-2)}$. In order to define this action, we need the $\mathfrak{g}$-module morphisms $\check{R}_\pi$ in (2.2.3), $\tilde{pr}_k$ and $\tilde{ev}_k$ in Lemma 2.2.14. Also recall that from Section 2.3, we have the matrix form of our $\mathfrak{g}$-module isomorphism $F : V \to V^*$. Then we have the following definition.

Definition 4.2.1. *Let d be a k-diagram in $\mathcal{D}_k$, we define a map*

$$\Psi : \mathcal{D}_k \to \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes k}, V^{\otimes k})$$
$$d \mapsto \Psi_d$$

where Ψ_d is the composition

$$\begin{aligned}
\Psi_d : V^{\otimes k} \cong V_{1_G} \otimes V^{\otimes k} &\xrightarrow{\tilde{pr}_k \otimes id^{\otimes k}} V^{\otimes k} \otimes (V^*)^{\otimes k} \otimes V^{\otimes k} \\
&\xrightarrow{id^{\otimes k} \otimes (F^{-1})^{\otimes k} \otimes id^{\otimes k}} V^{\otimes 2k} \otimes V^{\otimes k} \\
&\xrightarrow{\check{R}_{\pi_d} \otimes id^{\otimes k}} V^{\otimes 2k} \otimes V^{\otimes k} \\
&\xrightarrow{id^{\otimes k} \otimes F^{\otimes k} \otimes id^{\otimes k}} V^{\otimes k} \otimes (V^*)^{\otimes k} \otimes V^{\otimes k} \\
&\xrightarrow{id^{\otimes k} \otimes \tilde{ev}_k} V^{\otimes k} \otimes V_{1_G} \cong V^{\otimes k}.
\end{aligned}$$

Notice that Ψ_d is a $\mathfrak{g}$-module morphism for all $d \in \mathcal{D}_k$ since the map Ψ_d is a composite of $\mathfrak{g}$-module morphisms.

Next we illustrate Definition 4.2.1 by evaluating Ψ on the generators of $B_2(\eta)$. We have the following proposition. Note that we write the action of $B_k(\eta)$ on the right, and we will justify this convention later.

Proposition 4.2.2. *Let $v_{i_1} \otimes v_{i_2} \in V \otimes V$ with $v_{i_1}, v_{i_2} \in B$. Then* *(i) the action of e_1 on $v_{i_1} \otimes v_{i_2}$ is given by*

$$(v_{i_1} \otimes v_{i_2})\Psi_{e_1} = F_{i_1,i_2} \sum_{j_1,j_2}^{n} F^{-1}_{j_1,j_2} v_{j_1} \otimes v_{j_2}, \ \textit{and}$$

(ii) the action of s_1 on $v_{i_1} \otimes v_{i_2}$ is given by

$$(v_{i_1} \otimes v_{i_2})\Psi_{s_1} = -\beta(v_{i_2}, v_{i_1}) v_{i_2} \otimes v_{i_1}.$$

Proof. Let $d = e_1 \in \mathcal{D}_2$. Then by Example 4.1.8, we have $\check{R}_{\pi_d} = \check{R}_{(23)}\check{R}_{(34)}$. Notice that when we apply $\check{R}_{(23)}\check{R}_{(34)}$ on $V^{\otimes k}$, we first apply $\check{R}_{(34)}$, and then apply $\check{R}_{(23)}$. By Lemma 2.4.4, $F^{-1}(v^i)$ and v_i have opposite colour. Again by Remark 2.3.4, we use a vector to represent its only colour and use the inverse v^{-1} to represent the opposite colour. Then the sequence of maps Ψ_{e_1} does the following:

$$\begin{aligned}
(v_{i_1} \otimes v_{i_2}) \to & \mathbf{1} \otimes v_{i_1} \otimes v_{i_2} \\
\xrightarrow{\tilde{pr}_k \otimes id^{\otimes 2}} & \sum_{j_1,j_2}^{n=1} v_{j_1} \otimes v_{j_2} \otimes v^{j_2} \otimes v^{j_1} \otimes v_{i_1} \otimes v_{i_2} \\
\xrightarrow{id^{\otimes 2} \otimes (F^{-1})^{\otimes 2} \otimes id^{\otimes 2}} & \sum_{j_1,j_2} v_{j_1} \otimes v_{j_2} \otimes F^{-1}(v^{j_2}) \otimes F^{-1}(v^{j_1}) \otimes v_{i_1} \otimes v_{i_2} \\
\xrightarrow{\check{R}_{(34)} \otimes id^{\otimes 2}} & \sum_{j_1,j_2} -\beta(v_{j_1}^{-1}, v_{j_2}^{-1}) v_{j_1} \otimes v_{j_2} \otimes F^{-1}(v^{j_1}) \otimes F^{-1}(v^{j_2}) \otimes v_{i_1} \otimes v_{i_2} \qquad (*) \\
\xrightarrow{\check{R}_{(23)} \otimes id^{\otimes 2}} & \sum_{j_1,j_2} \beta(v_{j_1}^{-1}, v_{j_2}^{-1}) \beta(v_{j_1}^{-1}, v_{j_2}) v_{j_1} \otimes F^{-1}(v^{j_1}) \otimes v_{j_2} \otimes F^{-1}(v^{j_2}) \otimes v_{i_1} \otimes v_{i_2} \\
\xrightarrow{id^{\otimes 2} \otimes F^{\otimes 2} \otimes id^{\otimes 2}} & \sum_{j_1,j_2} \beta(v_{j_1}^{-1}, 1) v_{j_1} \otimes F^{-1}(v^{j_1}) \otimes F(v_{j_2}) \otimes v^{j_2} \otimes v_{i_1} \otimes v_{i_2}. \qquad (4.2.1)
\end{aligned}$$

Notice that $\beta(v_{j_1}^{-1}, 1) = 1$ for any $v_{j_1} \in B$, and we will rewrite (4.2.1) using the matrix form of F and F^{-1} to complete the calculation. Therefore, we have

$$\begin{aligned}
&\sum_{j_1,j_2} v_{j_1} \otimes \sum_{p=1} F^{-1}_{j_1,p} v_p \otimes \sum_{q=1}^{n} F_{j_2,q} v^q \otimes v^{j_2} \otimes v_{i_1} \otimes v_{i_2} \\
\xrightarrow{id^{\otimes 2}\otimes \tilde{ev}_k} &\sum_{j_1,j_2} v_{j_1} \otimes \sum_{p=1}^{n} F^{-1}_{j_1,p} v_p \Big(v^{j_2}(v_{i_1}) \sum_{q=1}^{n} F_{j_2,q} v^q(v_{i_2}) \Big) \\
= &\sum_{j_1,j_2} v_{j_1} \otimes \sum_{p=1}^{n} F^{-1}_{j_1,p} v_p \Big(\delta_{j_2,i_1} \sum_{q=1}^{n} F_{j_2,q} \delta_{q,i_2} \Big) \\
= &\sum_{j_1} \sum_{\underline{j_2}} \Big(v_{j_1} \otimes \sum_{p=1}^{n} F^{-1}_{j_1,p} v_p \underline{\delta_{j_2,i_1}} F_{\underline{j_2},i_2} \Big).
\end{aligned}$$

Consider the underlined part. Since δ_{j_2,i_1} is nonzero only for $j_2 = i_1$, the sum over j_2 reduces to $j_2 = i_1$. Thus we have

$$\begin{aligned}
\sum_{j_1} v_{j_1} \otimes \sum_{p=1}^{n} F^{-1}_{j_1,p} v_p F_{i_1,i_2} &= F_{i_1,i_2} \sum_{j_1,p} F^{-1}_{j_1,p} v_{j_1} \otimes v_p \\
&= F_{i_1,i_2} \sum_{j_1,j_2} F^{-1}_{j_1,j_2} v_{j_1} \otimes v_{j_2}.
\end{aligned}$$

Therefore we have $(v_{i_1} \otimes v_{i_2})\Psi_{e_1} = F_{i_1,i_2} \sum_{j_1,j_2} F^{-1}_{j_1,j_2} v_{j_1} \otimes v_{j_2}$.

Now let $d = s_1$. In Example 4.1.9, we verified that $\pi_{u(s_1)} = (34)$. The first steps up until $(*)$ of the evaluation of $(v_{i_1} \otimes v_{i_2})\Psi_{s_1}$ is identical to that of $(v_{i_1} \otimes v_{i_2})\Psi_{e_1}$. Therefore we have

$$\begin{aligned}
(v_{i_1} \otimes v_{i_2}) \to &\sum_{j_1,j_2} -\beta(v^{-1}_{j_1}, v^{-1}_{j_2}) v_{j_1} \otimes v_{j_2} \otimes F^{-1}(v^{j_1}) \otimes F^{-1}(v^{j_2}) \otimes v_{i_1} \otimes v_{i_2} \qquad (*) \\
\xrightarrow{id^{\otimes 2}\otimes F^{\otimes 2} \otimes id^{\otimes 2}} &\sum_{j_1,j_2} -\beta(v_{j_1}, v_{j_2}) v_{j_1} \otimes v_{j_2} \otimes v^{j_1} \otimes v^{j_2} \otimes v_{i_1} \otimes v_{i_2} \\
\xrightarrow{id^{\otimes 2}\otimes \tilde{ev}_k} &\sum_{j_1,j_2} -\beta(v_{j_1}, v_{j_2}) v_{j_1} \otimes v_{j_2} \delta_{j_1,i_2} \delta_{j_2,i_1}
\end{aligned}$$

$$= -\,\beta(v_{i_2}, v_{i_1})v_{i_2} \otimes v_{i_1}.$$

Thus $(v_{i_1} \otimes v_{i_2})\Psi_{s_1} = -\beta(v_{i_2}, v_{i_1})v_{i_2} \otimes v_{i_1}$. □

Corollary 4.2.3. *For all $k \in \mathbb{Z}_{>0}$ and $1 \leq i \leq k-1$, we have*

$$\Psi_{e_i} = \mathrm{id}^{\otimes(i-1)} \otimes \Psi_{e_1} \otimes \mathrm{id}^{\otimes(k-i-1)},$$

and

$$\Psi_{s_i} = \mathrm{id}^{\otimes(i-1)} \otimes \Psi_{s_1} \otimes \mathrm{id}^{\otimes(k-i-1)}.$$

Proof. Although the proof is a time consuming process, it is similar to the proof of Proposition 4.2.2. □

By Remark 4.1.10, we identified the diagram s_i with the permutation $(i, i+1)$. Notice that for a simple tensor $v = v_1 \otimes \cdots \otimes v_k$, we have

$$v\Psi_{s_i} = -\beta(v_{i+1}, v_i)v_1 \otimes \cdots v_{i-1} \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k = \check{R}_i v.$$

In fact, for each $d \in \mathcal{D}_k$ with only vertical edges, by Remark 4.1.10, we view d as a permutation in S_k. Thus, if we write the permutation corresponding to d as $\pi = s_{i_1} \cdots s_{i_n}$ as a product of simple transpositions, then for all $v = v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$, we have

$$v\pi = v\Psi_d = v\Psi_{s_{i_1}} \cdots \Psi_{s_{i_n}} = \check{R}_{i_n} \cdots \check{R}_{i_1} v. \tag{4.2.2}$$

Moreover, we have

$$v\pi = (v_1 \otimes \cdots \otimes v_k)\pi = \beta_\pi v_{\pi(1)} \otimes \cdots \otimes v_{\pi(k)},$$

where β_π is the scalar defined in Lemma 3.1.5.

We will discuss more about the maps Ψ_{e_i} in Section 5.2.

Recall $n = \dim(V_{(0)})$ and $m = \dim(V_{(1)})$, and set $\eta = n - m$. We have seen for simple reflections that $v\Psi_{d_1 d_2} = v\Psi_{d_1}\Psi_{d_2}$. This justifies the reason we write the action of the Brauer algebra as a right action. It also suggests, as in the following theorem, that our action extends not to $B_k(n-m)$ but to $B_k(n-m)^{\mathrm{op}}$, where the algebra operation satisfies $d_1 \cdot d_2 \coloneqq d_2 d_1$.

Theorem 4.2.4. *The map* $\Psi : B_k(n-m)^{\mathrm{op}} \to \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes k}, V^{\otimes k})$ *is a homomorphism of algebras.*

Proof. First notice that since the k-diagrams span $B_k(n-m)$ (and hence span $B_k(n-m)^{\mathrm{op}}$), Definition 4.2.1 defines Ψ_b for any $b \in B_k(n-m)^{\mathrm{op}}$ by linearity. Thus $\Psi_b \in \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes k}, V^{\otimes k})$ for all $b \in B_k(n-m)^{\mathrm{op}}$. Next, in order to prove that it is an algebra homomorphism, one needs to verify that Ψ preserves each of the relations in Proposition 4.1.6. Most of the symmetric relations are proved in [BSR98]. We will prove one extra relation $\Psi_{s_1}\Psi_{e_2}\Psi_{e_1} = \Psi_{s_2}\Psi_{e_1}$ (corresponding to the relation $s_1 e_2 e_1 = s_2 e_1$ in $B_k(n-m)$ for all $k > 2$) since this relation is neither symmetric (and thus distinguishes between $B_k(n-m)$ and $B_k(n-m)^{\mathrm{op}}$) nor straightforward to deduce.

For all $k \geq 3$, by Corollary 4.2.3, since both $\Psi_{s_1}\Psi_{e_2}\Psi_{e_1}$ and $\Psi_{s_2}\Psi_{e_1}$ only act on the first three components of a tensor in $V^{\otimes k}$, it suffices to only consider a simple tensor $v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \in V^{\otimes 3}$ with $v_{i_1}, v_{i_2}, v_{i_3} \in B$. We prove that

$$(v_{i_1} \otimes v_{i_2} \otimes v_{i_3})\Psi_{s_1}\Psi_{e_2}\Psi_{e_1} = (v_{i_1} \otimes v_{i_2} \otimes v_{i_3})\Psi_{s_2}\Psi_{e_1}. \tag{4.2.3}$$

Recall that we use the vector itself to denote its colour. By Proposition 4.2.2, the right hand side of (4.2.3) is

$$\begin{aligned}(v_{i_1}\otimes v_{i_2}\otimes v_{i_3})\Psi_{s_2}\Psi_{e_1} &= -\beta(v_{i_3},v_{i_2})(v_{i_1}\otimes v_{i_3}\otimes v_{i_2})\Psi_{e_1}\\ &= -\beta(v_{i_3},v_{i_2})F_{i_1,i_3}\sum_{j_1,j_3}F^{-1}_{j_1,j_3}v_{j_1}\otimes v_{j_3}\otimes v_{i_2}. \qquad (4.2.4)\end{aligned}$$

The left hand side of (4.2.3) is

$$\begin{aligned}(v_{i_1}\otimes v_{i_2}\otimes v_{i_3})\Psi_{s_1}\Psi_{e_2}\Psi_{e_1} &= -\beta(v_{i_2},v_{i_1})(v_{i_2}\otimes v_{i_1}\otimes v_{i_3})\Psi_{e_2}\Psi_{e_1}\\ &= -\beta(v_{i_2},v_{i_1})\left(F_{i_1,i_3}\sum_{\ell_1,\ell_3}F^{-1}_{\ell_1,\ell_3}v_{i_2}\otimes v_{\ell_1}\otimes v_{\ell_3}\right)\Psi_{e_1},\end{aligned}$$

which is

$$-\beta(v_{i_2},v_{i_1})\left(F_{i_1,i_3}\sum_{\ell_1,\ell_3}\underline{F^{-1}_{\ell_1,\ell_3}F_{i_2,\ell_1}}\sum_{h_1,h_2}F^{-1}_{h_1,h_2}v_{h_1}\otimes v_{h_2}\otimes v_{\ell_3}\right). \qquad (4.2.5)$$

Now let us simplify (4.2.5). Consider the underlined term $F^{-1}_{\ell_1,\ell_3}F_{i_2,\ell_1}$. Since $F^{-1}_{\ell_1,\ell_3}\neq 0$ if and only if $v^*_{\ell_1}=v_{\ell_3}$, and $F_{i_2,\ell_1}\neq 0$ if and only if $v^*_{\ell_1}=v_{i_2}$, we deduce that this term is nonzero if and only if $v_{\ell_3}=v^*_{\ell_1}=v_{i_2}$. Therefore the underlined term is $F^{-1}_{\ell_1,\ell_1^*}F_{\ell_1^*,\ell_1}$ which is 1 for all $v_{\ell_1}\in B$ by the explicit matrix form of F we found in (2.4.2) and (2.4.3).

Thus (4.2.5) becomes

$$-\beta(v_{i_2},v_{i_1})F_{i_1,i_3}\sum_{h_1,h_2}F^{-1}_{h_1,h_2}v_{h_1}\otimes v_{h_2}\otimes v_{i_2}. \qquad (4.2.6)$$

Notice that $F_{i_1,i_3}\neq 0$ if and only if $v_{i_1}=v^*_{i_3}$, therefore by using the fact that

$\beta(v_{i_2}, v^*_{i_3}) = \beta(v_{i_3}, v_{i_2})$, (4.2.6) becomes

$$-\beta(v_{i_3}, v_{i_2})F_{i_1,i_3}\sum_{h_1,h_2} F^{-1}_{h_1,h_2} v_{h_1} \otimes v_{h_2} \otimes v_{i_2}.$$

By re-indexing h_i to j_i for all $i = 1, 2$, we conclude that the two sides of (4.2.3) are equal. □

Remark 4.2.5. Notice that if we wrote the action of the Brauer algebra on the left, the relation $\Psi_{s_1}\Psi_{e_2}\Psi_{e_1}(v_{i_1} \otimes v_{i_2} \otimes v_{i_3}) = \Psi_{s_2}\Psi_{e_1}(v_{i_1} \otimes v_{i_2} \otimes v_{i_3})$ would not hold.

Theorem 4.2.4 implies that for all $w \in V^{\otimes k}$, for all $X \in \mathfrak{g}$, and for all $b \in B_k(n-m)$, we have

$$(\rho(X)w)\Psi_b = \rho(X)(w\Psi_b),$$

where ρ denotes the $\mathfrak{g}$-module structure on $V^{\otimes k}$. Thus the right action of $B_k(n-m)$ on $V^{\otimes k}$ commutes with the left action of $\mathfrak{spo}(V, \beta)$ on $V^{\otimes k}$.

Remark 4.2.6. The Brauer action can be defined more systematically using the machinery of Brauer categories, introduced for example in [LZ16]. The advantage of the more elementary approach that we develop in this chapter, is that one obtains an explicit method to combinatorially compute the action of any element of the Brauer algebra.

Chapter 5

Highest weight vectors in $V^{\otimes k}$

In this chapter we construct highest weight vectors for tensor powers of the standard representation of $\mathfrak{spo}(V, \beta)$. The idea is to construct certain highest weight vectors for harmonic tensor spaces and to use the Brauer algebra. This technique was originally used in the purely even cases corresponding to orthogonal and symplectic Lie algebras; for a readable exposition, see [GW09]. This technique is generalized to the orthosymplectic Lie colour algebra $\mathfrak{spo}(V, \beta)$ by [BSR98]. In Section 5.1, we introduce (r, s)-hook tableaux and their associated simple tensors following [BSR98]. In Section 5.2 and 5.3, we introduce Young symmetrizers and contractions which are realized by elements of the Brauer algebra. We illustrate these with detailed examples. In Section 5.4, we will use these new concepts to construct highest weight vectors for $V^{\otimes k}$ where V is the standard $\mathfrak{spo}(V, \beta)$-module and $k \in \mathbb{Z}_{\geq 0}$. Furthermore, we analyze and justify how this construction works. We conclude in Section 5.5.3 with our own explicit example in $V \otimes V$, and construct submodules generated by the highest weight vectors we found.

5.1 (r, s)-hook tableaux and simple tensors

As demonstrated in [GW09, Chapter 9], there is a way to construct simple tensors v_T in $V^{\otimes k}$ from Young tableaux T. For the definition of Young tableaux, patitions and related concepts, please see Appendix A. The work of [BSR98] generalized this idea to form simple tensors from (r, s)-*hook shape tableaux*.

Let $\mathfrak{g} = \mathfrak{spo}(V, \beta)$. We construct simple tensors from the subset

$$B' = \{t_1, \ldots, t_r, u_1, \ldots, u_s, (u_{s+1})\}$$

of the basis B. The set of t_i comes from the set $B'_{(0)}$, and the set of u_j comes from the the set $B'_{(1)}$. Therefore we will put all t_i's together to form a smaller tableau $T_{(0)}$ within T, and similarly, we will gather all u_j's from B' to form another smaller tableau $T_{(1)}$ within T. In order to make this discussion more precise, we first give the following definitions.

Definition 5.1.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of $k \in \mathbb{Z}_{\geq 0}$. We say λ is a (r, s)-hook partition if $\lambda_{r+1} \leq s$. A standard tableau T of shape λ is called an (r, s)-hook tableau.*

Note that any tableaux which can be fitted in Figure 5.1 is called an (r, s)-hook tableau.

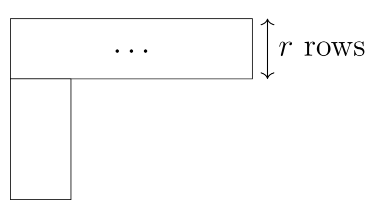


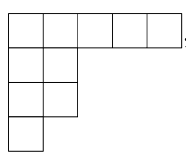
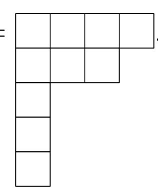
Figure 5.1: (r, s)-hook shape tableau frame

From now on, we let $\mathcal{K} = \{1, \dots, k\}$ for some $k \in \mathbb{Z}_{\geq 0}$. We denote the set of all (r, s)-hook tableaux with entries from $\mathcal{K}$ by $\Gamma_{r,s}(\mathcal{K})$.

Example 5.1.2. Let $\lambda = (17, 2, 2)$. Then λ is a $(1, 3)$-hook partition since $\lambda_{1+1} = 2 \leq 3$.

Definition 5.1.3. *Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash k$. Let $Y(\lambda)$ be the Young diagram of λ. Then the transpose of λ, denoted $\lambda' = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_n)$ is a partition of k whose parts are the lengths of the columns of $Y(\lambda)$. In particular, $Y(\lambda')$ has ℓ columns and $n = \lambda_1$ rows, and $\tilde{\lambda}_j = \mathrm{Card}\{i \mid \lambda_i - j \geq 0\}$, for all $1 \leq j \leq n$.*

Example 5.1.4. Let $\lambda = (5, 2, 2, 1)$. Then $\lambda' = (4, 3, 1, 1, 1)$. The corresponding Young diagrams are given by

$Y(\lambda) =$  , $Y(\lambda') =$  .

Definition 5.1.5. *Given a tableau T of shape λ, the transpose of T, denoted T' is the tableau of shape λ' in which the entries of the i^{th} row of T are the entries of the i^{th} column of T'.*

Note that if T is a standard Young tableau of shape λ, then T' is still a standard Young tableau of shape λ'.

Example 5.1.6. Let $T = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 6 & 8 & 9 \\ \hline 2 & 4 \\ \cline{1-2} 5 & 7 \\ \cline{1-2} 10 \\ \cline{1-1} \end{array}$. Then $T' = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 10 \\ \hline 3 & 4 & 7 \\ \cline{1-3} 6 \\ \cline{1-1} 8 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array}$.

Now for each (r, s)-hook tableau T, we separate it into two different tableaux $T_{(0)}$ and $T_{(1)}$: we take the first r rows to form a smaller tableau $T_{(0)}$. Then we take the transpose of the rest of T to form another smaller tableau $T_{(1)}$ which will have at most s rows.

Formally, let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an (r, s)-hook partition, and let T be a standard tableau of shape λ. The subtableau $T_{(0)}$ corresponds to the partition $\lambda_{(0)} = (\lambda_1, \ldots, \lambda_r)$. We form a smaller partition $(\lambda_{r+1}, \ldots, \lambda_\ell)$ by removing $\lambda_{(0)}$ from λ. Then $\lambda_{(1)} = (\lambda_{r+1}, \ldots, \lambda_\ell)' = (\tilde{\lambda}_{r+1}, \ldots, \tilde{\lambda}_{r+s})$ gives a new partition. The transpose of the subtableau of T obtained by removing $T_{(0)}$ is called $T_{(1)}$. Note that $T_{(1)}$ has shape $\lambda_{(1)}$. Note we start labelling the indices of $\lambda_{(1)}$ from $r + 1$ because $\lambda_{(1)}$ is constructed starting from the $(r + 1)^{\text{th}}$ row of T. By convention, if $\ell \leq r$, then $\tilde{\lambda}_{r+1} = \cdots = \tilde{\lambda}_{r+s} = 0$.

Example 5.1.7. Let $T = \begin{array}{|c|c|c|c|c|}\hline 1&3&6&8&9\\\hline 2&4\\\cline{1-2} 5&7\\\cline{1-2} 10\\\cline{1-1}\end{array}$. This is a $(2, 3)$-hook tableau and we have

$$T_{(0)} = \begin{array}{|c|c|c|c|c|}\hline 1&3&6&8&9\\\hline 2&4\\\cline{1-2}\end{array} \text{ and } T_{(1)} = \begin{array}{|c|c|}\hline 5&10\\\hline 7\\\cline{1-1}\end{array}.$$

The corresponding partitions are $\lambda_{(0)} = (5, 2)$ and $\lambda_{(1)} = (2, 1)$.

Now for each (r, s)-hook tableau, we construct a simple tensor with factors from B'.

Definition 5.1.8. *Let $T \in \Gamma_{r,s}(\mathcal{K})$. Then let $v_T = v_1 \otimes \cdots \otimes v_k$ be the simple tensor in $V^{\otimes k}$ where v_i is defined by*

$$v_i = \begin{cases} t_j & \text{if } i \text{ is in the } j^{\text{th}} \text{ row of } T_{(0)}, \\ u_j & \text{if } i \text{ is in the } j^{\text{th}} \text{ row of } T_{(1)}. \end{cases}$$

Example 5.1.9. In Example 5.1.7, the corresponding v_T is given by

$$v_T = t_1 \otimes t_2 \otimes t_1 \otimes t_2 \otimes u_1 \otimes t_1 \otimes u_2 \otimes t_1 \otimes t_1 \otimes u_1.$$

Lemma 5.1.10. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an (r,s)-hook partition. Let T be an (r,s)-hook tableau of shape λ in $\Gamma_{r,s}(\mathcal{K})$. Let $v_T = v_1 \otimes \cdots \otimes v_k$ be the simple tensor as in Definition 5.1.8 with $v_i \in B'$ for all $1 \le i \le k$. Then the weight of v_T is given by*

$$\lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r + \tilde{\lambda}_{r+1}\delta_1 + \cdots + \tilde{\lambda}_{r+s}\delta_s.$$

Proof. By Definition 5.1.8, the number of t_i in v_T is λ_i for all $1 \le i \le r$. The number of u_j in v_T is $\tilde{\lambda}_{r+j}$ for all $1 \le j \le s$. Therefore the result follows from Lemma 2.5.5. $\square$

Example 5.1.11. Let T be a $(2,3)$-hook tableau of shape $\lambda = (5,2,2,1)$ as in Example 5.1.7. We have $\lambda_{(0)} = (5,2)$ and $\lambda_{(1)} = (2,1)$. Then the corresponding v_T has weight $5\varepsilon_1 + 2\varepsilon_2 + 2\delta_1 + \delta_2$.

Lemma 5.1.12. *Let $v = v_1 \otimes \cdots \otimes v_k$ be a simple tensor in $V^{\otimes k}$ with $v_i \in B'$ for all $1 \le i \le k$. Let π be an element of S_k. Then*

(i) if there exists $j \in \{1, \ldots, r\}$ such that for all $i, i' \in \mathcal{K}$, $\pi(i) = i'$ and $i \neq i'$ implies $v_i = v_{i'} = t_j$, then $\check{R}_\pi v = \operatorname{sgn}(\pi) v$, and

(ii) if there exists $j \in \{1, \ldots, s\}$ such that for all $i, i' \in \mathcal{K}$, $\pi(i) = i'$ and $i \neq i'$ implies $v_i = v_{i'} = u_j$, then $\check{R}_\pi v = v$.

We will say that such a π permutes t_j with t_j or u_j with u_j respectively.

Proof. Since every permutation can be written as a product of transpositions, we prove this lemma for transpositions (i,j) by induction on $j - i$. Note that for transpositions, the conclusions of the lemma are $\check{R}_\pi v = -\beta(v_i, v_i)v$ in both cases.

The base case is $j-i=1$. Suppose that $\pi=(i,i+1)$, and $v_i=v_{i+1}$. Then by (2.2.2) and Proposition 2.2.10, $\check{R}_i v=-\beta(v_{i+1},v_i)v=-\beta(v_i,v_i)v$.

Now suppose that the result holds when $\pi=(i,j)$ with $j-i\geq 1$. We prove the result holds for $\pi=(i,j+1)$.

Let $\pi=(i,j+1)$ and suppose $v_i=v_{j+1}$. We have $\pi=(j,j+1)(i,j)(j,j+1)$. Then by (2.2.3), we have

$$\check{R}_\pi v=\check{R}_j\check{R}_{(i,j)}\check{R}_j v.$$

For easy reference, we use boldface to indicate the factors in v that have been swapped in each step. Therefore we have

$$\check{R}_j v=-\beta(v_{j+1},v_j)v_1\otimes\cdots\otimes v_{j-1}\otimes \boldsymbol{v_{j+1}}\otimes\boldsymbol{v_j}\otimes v_{j+2}\otimes\cdots\otimes v_k. \tag{5.1.1}$$

Using the inductive hypothesis, since the entries in the i^{th} and j^{th} positions are now equal, we have

$$\begin{aligned}\check{R}_j\check{R}_{(i,j)}v=\beta(v_i,v_i)\beta(v_{j+1},v_j)v_1\otimes\cdots\otimes v_{i-1}\otimes\boldsymbol{v_{j+1}}\otimes v_{i+1}\\ \otimes\cdots\otimes v_{j-1}\otimes\boldsymbol{v_i}\otimes v_j\otimes v_{j+2}\otimes\cdots\otimes v_k.\end{aligned} \tag{5.1.2}$$

Applying $\check{R}_j$ to (5.1.2) will permute v_i past v_j. Thus we have

$$\begin{aligned}\check{R}_\pi v=-\beta(v_i,v_i)\beta(v_{j+1},v_j)\beta(v_j,v_i)v_1\otimes\cdots\otimes v_{i-1}\otimes v_{j+1}\otimes v_{i+1}\\ \otimes\cdots\otimes v_{j-1}\otimes\boldsymbol{v_j}\otimes\boldsymbol{v_i}\otimes v_{j+2}\otimes\cdots\otimes v_k.\end{aligned} \tag{5.1.3}$$

Since $v_i=v_{j+1}$, the simple tensor in (5.1.3) is equal to v. We compute the coefficient

$$-\beta(v_i, v_i)\beta(v_{j+1}, v_j)\beta(v_j, v_i) = -\beta(v_i, v_i)\beta(v_i, v_j)\beta(v_j, v_i) = -\beta(v_i, v_i).$$

This completes the proof of the lemma. □

5.2 Young symmetrizers

Let us first briefly recall the role of Young symmetrizers in the classical representation theory.

Let S_k be the permutation group on k letters. Then when we consider the representation theory of the symmetric group S_k for some $k \in \mathbb{Z}_{\geq 0}$, an operator called a *Young symmetrizer* is used to construct irreducible S_k-modules which can be parametrized by partitions of k.

Definition 5.2.1. *Let λ be a partition of k. Let T be a standard Young tableau of shape λ. The row group $\mathcal{R}_T$ is the subgroup of S_k preserving the rows of T. The column group $\mathcal{C}_T$ is the subgroup of S_k preserving the columns of T.*

Example 5.2.2. Suppose that

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

Then $\mathcal{R}_T = \{\mathrm{id}, (12)\}$ and $\mathcal{C}_T = \{\mathrm{id}, (13)\}$, where we use id to denote the identity element in S_k.

Definition 5.2.3. *Let $\mathbb{F}$ be a field. Let $\mathbb{F}[S_k]$ be the group algebra of S_k. Let T be a standard Young tableau of shape $\lambda \vdash k$. The normalized Young symmetrizer is defined* *to be*

$$y_T = h(\lambda) \sum_{\substack{\psi \in \mathcal{R}_T \\ \gamma \in \mathcal{C}_T}} \operatorname{sgn}(\psi)\psi\gamma \in \mathbb{F}[S_k]$$

where $h(\lambda) \in \mathbb{Q}^+$ is chosen so that y_T is an idempotent.

Remark 5.2.4. In [GW09], Goodman and Wallach prove that $h(\lambda)=\frac{\dim S^\lambda}{k!}$ where S^λ is the irreducible S_k-module corresponding to the partition λ. In particular the dimension of S^λ is the number of standard Young tableaux of shape λ with entries from $\mathcal{K}$.

Example 5.2.5. Let T be the tableau in Example 5.2.2. There are only two standard Young tableaux of shape $(2,1)$. So by Remark 5.2.4, we have $h(\lambda) = \frac{2}{3!} = \frac{1}{3}$. Therefore we have

$$y_T = \frac{1}{3}(\mathrm{id} + (13) - (12) - (12)(13)).$$

Recall that the group algebra $\mathbb{F}[S_k]$ is a subalgebra of the Brauer algebra. Thus, the Young symmetrizer will be acting on $V^{\otimes k}$ on the right, so the following is with respect to a right action.

We first prove a useful lemma.

Lemma 5.2.6. *Let T be a standard tableau with entries from a subset $\mathcal{L} \subseteq \mathcal{K}$. Suppose $\sigma \in S_k$ is such that $\sigma(i) < \sigma(j)$ whenever $i < j$ and $i, j \in \mathcal{L}$. Define T^σ to be the tableau obtained from T by applying σ to each entry of T. Then T^σ is also a standard Young tableau, with entries from $\sigma(\mathcal{L})$. Moreover, $\sigma^{-1} y_T \sigma = y_{T^\sigma}$.*

Proof. It is clear that T^σ is a standard Young tableau. For $\psi \in \mathcal{R}_T$ we can write $\psi = \psi_{r_1} \cdots \psi_{r_\ell}$, where ψ_{r_j} is a permutation which only permutes entries in row j of T. Then we have

$$\sigma^{-1}\psi\sigma = (\sigma^{-1}\psi_{r_1}\sigma)\cdots(\sigma^{-1}\psi_{r_\ell}\sigma).$$

Therefore it suffices to prove that $\sigma^{-1}\psi_{r_j}\sigma$ is a permutation which only permutes the entries of row j of T^σ for all $1 \le j \le \ell$.

Let $S = \{i_1, \dots, i_m\}$ be the set of entries of row j of T. Then the entries of row j of T^σ are $S\sigma = \{\sigma(i_1), \dots, \sigma(i_m)\}$. From the definition of the right action, we know that $\sigma^{-1}\psi_{r_j}\sigma$ permutes the elements $S\sigma$ the same way that ψ_{r_j} permutes the entries of the S. It follows that the map $\psi \mapsto \sigma^{-1}\psi\sigma$ is a bijection from $\mathcal{R}_T$ onto $\mathcal{R}_{T^\sigma}$. Similarly, the map $\gamma \mapsto \sigma^{-1}\gamma\sigma$ is a bijection from $\mathcal{C}_T$ to $\mathcal{C}_{T^\sigma}$.Therefore by Definition 5.2.3, we have

$$\begin{aligned}
\sigma^{-1}y_T\sigma &= \sigma^{-1}h(\lambda)\sum_{\substack{\psi\in\mathcal{R}_T\\ \gamma\in\mathcal{C}_T}} \operatorname{sgn}(\psi)\psi\gamma\sigma \\
&= h(\lambda)\sum_{\substack{\psi\in\mathcal{R}_T\\ \gamma\in\mathcal{C}_T}} \operatorname{sgn}(\psi)(\sigma^{-1}\psi\sigma)(\sigma^{-1}\gamma\sigma) \\
&= h(\lambda)\sum_{\substack{\psi^\sigma\in\mathcal{R}_{T^\sigma}\\ \gamma^\sigma\in\mathcal{C}_{T^\sigma}}} \operatorname{sgn}(\psi^\sigma)\psi^\sigma\gamma^\sigma \\
&= y_{T^\sigma}.
\end{aligned}$$

□

5.3 Contraction maps on $V^{\otimes k}$

In classical representation theory, let W be a representation of a Lie algebra $\mathfrak{l}$. Then for all $1 \le i, j \le k$, *contractions* $C^{i,j}$ are linear maps from $W^{\otimes k}$ to $W^{\otimes(k-2)}$ which contract the i^{th} and j^{th} factors of the tensor into a scalar. For example, $C^{1,2}(a \otimes b \otimes c \otimes d) = \omega(a,b)c \otimes d$ where $\omega(a,b)$ is a scalar valued bilinear map depending on a and b. Thus, if $\dim(W) = n$, then $\dim(\mathrm{Ker}(C^{i,j}))$=$n^k - n^{k-2}$. Typically, when we have a contraction map from $V^{\otimes k}$ to $V^{\otimes(k-2)}$, we tensor an invariant tensor to get a map from $V^{\otimes k}$ to $V^{\otimes k}$. Some of the facts concerning contraction maps are discussed in [BBL90] and

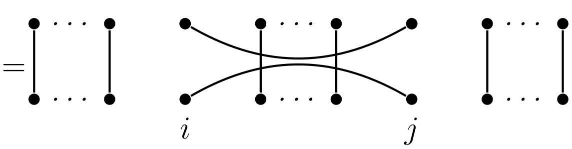
[Bro55].

The vectors in $W^{\otimes k}$ which are annihilated by any composition of contraction maps $C^{i,j}$ were first called *traceless tensors* in [Wey97]. The set of all such tensors is called the *harmonic space of k-tensors* in [GW09], which is an $\mathfrak{l}$-submodule of $W^{\otimes k}$.

In [BBL90], Benkart et. al. described an algorithm to find highest weight vectors for all classical Lie algebras, which is then generalized to the algorithm given in [BSR98] for Lie colour algebras.

We begin by constructing contraction maps $C_{i,j} : V^{\otimes k} \to V^{\otimes k}$.

Definition 5.3.1. *Let $c_{i,j}$ be the diagram*

$c_{i,j} =$ 

in the Brauer algebra $B_k(n-m)$. Then the contraction map $C_{i,j} \in \mathrm{End}(V^{\otimes k})$ is given by $C_{i,j} = \Psi_{c_{i,j}}$.

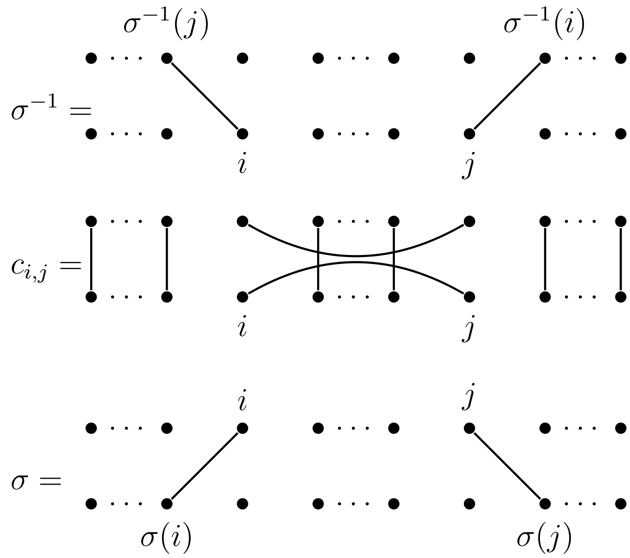
Since for all $i \neq j$, $c_{i,j} = c_{j,i}$ in the Brauer algebra, we have that $C_{i,j} = C_{j,i}$. Thus, we can restrict ourselves to contraction maps $C_{i,j}$ such that $1 \leq i < j \leq k$.

Example 5.3.2. Since $c_{i,i+1} = e_i$, we have $C_{i,i+1} = \Psi_{e_i}$.

Lemma 5.3.3. *Let $c_{i,j}$ be the element of the Brauer algebra $B_k(\eta)$ as defined in Definition 5.3.1. Let σ be a permutation in S_k. Then we have $\sigma^{-1}c_{i,j}\sigma = c_{\sigma(i),\sigma(j)}$.*

Proof. Let us consider the product of $\sigma^{-1}c_{i,j}\sigma$ in $B_k(\eta)$ which we visualize in Figure 5.2. Then the result follows. □

Lemma 5.3.4. *Let $C_{i,j}$ be a contraction map on $V^{\otimes k}$, where $1 \leq i < j \leq k$. Then for any $\sigma \in S_k$, we have $\sigma^{-1}C_{i,j}\sigma = C_{\sigma(i),\sigma(j)}$.*



which is

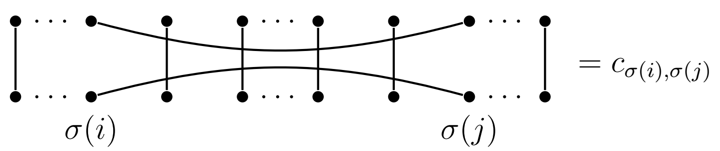


Figure 5.2: The composition $\sigma^{-1}c_{i,j}\sigma$ as diagrams in the Brauer algebra, and the resulting diagram $c_{\sigma(i),\sigma(j)}$. We suppress most of the vertical edges in both σ and σ^{-1}. We only keep the vertical edges linking i, j with $\sigma(i), \sigma(j)$ respectively in σ, and the vertical edges linking i, j with $\sigma^{-1}(i), \sigma^{-1}(j)$ respectively in σ^{-1}, and we label both top and bottom rows by $1, \ldots, k$ for simplicity.

Proof. By Theorem 4.2.4, Ψ is a homomorphism from $B_k(n-m)^{\mathrm{op}}$ to $\mathrm{Hom}_{\mathfrak{g}}(V^{\otimes k}, V^{\otimes k})$. Thus, by Lemma 5.3.3, we have (as a right action)

$$C_{\sigma(i),\sigma(j)} = \Psi_{c_{\sigma(i),\sigma(j)}} = \Psi_{\sigma^{-1}}\Psi_{c_{i,j}}\Psi_{\sigma} = \sigma^{-1}\Psi_{c_{i,j}}\sigma = \sigma^{-1}C_{i,j}\sigma$$

as claimed. $\square$

Lemma 5.3.5. *Let V be an $\mathfrak{spo}(V,\beta)$-module. The image of $C_{1,2} = \Psi_{e_1}$ in $V^{\otimes k}$ is isomorphic to $V^{\otimes(k-2)}$.*

Proof. It is enough to prove that the image of $C_{1,2}$ in $V^{\otimes 2}$ is isomorphic to the trivial module of $\mathfrak{spo}(V,\beta)$.

Apply Corollary 4.2.2 to our homogenous basis B, we have that for any $u, v \in B$,

$$(u \otimes v)\Psi_{e_1} = F_{u,v} \sum_{s,t \in B} F^{-1}_{s,t} s \otimes t = \Omega. \tag{5.3.1}$$

Recall that from Lemma 2.2.12, we have a $\mathfrak{g}$-module morphism pr_V such that in our case, if we relabel the homogeneous basis $B = \{v_1, \dots, v_{n+m}\}$, then $pr_V(1) = \sum_{i=1}^{n+m} v_i \otimes v^i$ is a $\mathfrak{g}$-invariant tensor. Also recall from Lemma 2.4.3, $F^{-1} : V^* \to V$ is a $\mathfrak{g}$-module isomorphism. Then we have

$$\begin{aligned}(\mathrm{id} \otimes F^{-1}) \circ pr_V(1) &= \sum_{i=1}^{m+n} v_i \otimes F^{-1}(v^i) \\ &= \sum_{i=1}^{m+n} F^{-1}_{i,j} v_i \otimes v_j = \Omega.\end{aligned}$$

Thus Ω is a $\mathfrak{g}$ invariant tensor since $x(\mathrm{id} \otimes F^{-1})(pr_V(1)) = (\mathrm{id} \otimes F^{-1})(x pr_V(1)) = 0$ for all $x \in \mathfrak{g}$. Therefore that the image of $C_{1,2}$ on $V \otimes V$ is isomorphic to the trivial module and in turn, we conclude that $V^{\otimes k} C_{1,2} \cong V^{\otimes (k-2)}$. $\square$

Notice that the values of $F^{-1}_{s,t}$ are given explicitly in (2.4.3), and we conclude that (5.3.1) is

$$\Omega = \sum_{i=1}^{r} (t_i^* \otimes t_i - t_i \otimes t_i^*) + \sum_{j=1}^{s} (u_j \otimes u_j^* + u_j^* \otimes u_j). \tag{5.3.2}$$

More generally, using the relation $C_{i,j} = \sigma^{-1} C_{1,2} \sigma$ for some permutation $\sigma \in S_k$, we find the following. Let $w = w_1 \otimes \cdots \otimes w_k$ with $w_\ell \in B$, then for $1 \le i < j \le k$, the vector $wC_{i,j}$ is a nonzero scalar multiple of

$$F_{w_i,w_j} \sum_{s,t\in B} F_{s,t}^{-1} w_1 \otimes \cdots \otimes w_{i-1} \otimes s \otimes w_{i+1} \otimes \cdots \otimes w_{j-1} \otimes t \otimes w_{j+1} \otimes \cdots \otimes w_k. \quad (5.3.3)$$

In particular, this implies that $V^{\otimes k}C_{i,j} \cong V^{\otimes(k-2)}$, for all $1 \leq i < j \leq k$.

As in the classical sense, we want to compose multiple contraction maps.

Definition 5.3.6. *Let $\underline{p} = \{p_1, \ldots, p_j\}$ and $\underline{q} = \{q_1, \ldots, q_j\}$ be two ordered disjoint subsets of $\mathcal{K}$ such that $p_\ell < q_\ell$ for all $1 \leq \ell \leq j$. Then we define*

$$C_{\underline{p},\underline{q}} \coloneqq C_{p_1,q_1} \cdots C_{p_j,q_j}.$$

We also call $C_{\underline{p},\underline{q}}$ a contraction map.

We pair each $p_\ell \in \underline{p}$ and $q_\ell \in \underline{q}$ together to define a set of pairs

$$(\underline{p}, \underline{q}) = \{(p_1, q_1), \ldots, (p_j, q_j) \mid p_\ell < q_\ell \text{ for } 1 \leq \ell \leq j\}. \quad (5.3.4)$$

For each $j = 1, \ldots, \lfloor k/2 \rfloor$, we define the set of all such $(\underline{p}, \underline{q})$'s by

$$\mathcal{P}(j) \coloneqq \{(\underline{p}, \underline{q}) \mid \text{ the cardinality of } \underline{p} \text{ and } \underline{q} \text{ are both } j\},$$

and the set of all possible indices of contraction maps on $V^{\otimes k}$ is given by

$$\mathcal{P} = \bigcup_{j=0}^{[k/2]} \mathcal{P}(j). \quad (5.3.5)$$

We give the following definitions of harmonic tensor spaces before we construct highest weight vectors.

Definition 5.3.7. *The harmonic tensor space $\mathcal{H}(V^{\otimes k}, C_{\underline{p},\underline{q}})$ of $V^{\otimes k}$ is the kernel of the contraction map $C_{\underline{p},\underline{q}}$ acting on $V^{\otimes k}$.*

Lemma 5.3.8. *The harmonic tensor space $\mathcal{H}(V^{\otimes k}, C_{\underline{p},\underline{q}})$ is an $\mathfrak{spo}(V,\beta)$-submodule of $V^{\otimes k}$.*

Proof. Let $\mathfrak{g} = \mathfrak{spo}(V,\beta)$. We have that $v \in \mathcal{H}(V^{\otimes k}, C_{\underline{p},\underline{q}})$ if and only if $vC_{\underline{p},\underline{q}} = 0$. Since the action of $\mathfrak{g} = \mathfrak{spo}(V,\beta)$ on $V^{\otimes k}$ commutes with the action of $B_k(n-m)$, we have for all $X \in \mathfrak{g}$, $(Xv)C_{\underline{p},\underline{q}} = X(vC_{\underline{p},\underline{q}}) = X(0) = 0$, which implies $\mathfrak{g} \cdot v \subseteq \mathcal{H}(V^{\otimes k}, C_{\underline{p},\underline{q}})$. $\square$

Example 5.3.9. Suppose that $r = 1$, $s = 0$ and $B = \{t_1, t_1^*\}$. Consider the $\mathfrak{spo}(V,\beta)$-module $V^{\otimes 2} = \mathrm{Span}\{v \otimes w \mid v, w \in B\}$. Using (5.3.1), $u \otimes v \in \mathrm{Ker}(C_{1,2})$ if $u \neq v^*$. With a little more work, we can see that $\mathcal{H}(V^{\otimes 2}, C_{1,2})$ is the span of

$$\{t_1 \otimes t_1, t_1 \otimes t_1^* + t_1^* \otimes t_1, t_1^* \otimes t_1^*\}.$$

Definition 5.3.10. *Let $\underline{s}$ and $\underline{t}$ be disjoint subsets of $\mathcal{K} - (\underline{p} \cup \underline{q})$. Then the harmonic tensor space $\mathcal{H}(V^{\otimes k}C_{\underline{p},\underline{q}}, C_{\underline{s},\underline{t}})$ is the kernel of the contraction map $C_{\underline{s},\underline{t}} : V^{\otimes k}C_{\underline{p},\underline{q}} \to V^{\otimes k}C_{\underline{p},\underline{q}}$.*

Notice that $\mathcal{H}(V^{\otimes k}C_{\underline{p},\underline{q}}, C_{\underline{s},\underline{t}})$ is also an $\mathfrak{spo}(V,\beta)$-submodule.

5.4 Construction of highest weight vectors

In this section, we construct highest weight vectors of several submodules of the $\mathfrak{spo}(V,\beta)$-module $V^{\otimes k}$ following the algorithm presented by Benkart et. al. in [BSR98]. This result is stated in Theorem 5.4.1, and proven in the course of several lemmas.

Let us first motivate the choice of $w = w_{T,\underline{p},\underline{q}}$ appearing in the theorem. The idea is to choose some special simple tensor $w = w_1 \otimes \cdots \otimes w_k \in V^{\otimes k}$, and then transform it into a highest weight vector by applying contraction maps and normalized Young symmetrizers on the right.

Let us first investigate the contraction maps. By Lemma 5.3.4 and Lemma 5.3.5, $C_{i,j}$ will contract two factors w_i and w_j of w into a $\mathfrak{g}$-invariant tensor, namely $F_{w_i,w_j}\Omega$. Since $w_i, w_j \in B$, we have $F_{w_i,w_j} \neq 0$ if and only if $w_i = w_j^*$. Therefore, $wC_{i,j}$ is nonzero if and only if $w_i = w_j^*$. Thus, we can start with vectors $w_i = t_1$ and $w_j = t_1^*$ in the position that we will apply contractions, which will be $i \in \underline{p}$ and $j \in \underline{q}$.

Now for $i \notin (\underline{p} \cup \underline{q})$, we can construct an (r, s)-hook tableau with entries from $\mathcal{K} - (\underline{p} \cup \underline{q})$. Then we choose each w_i by using Definition 5.1.8 for all $i \in \mathcal{K} - (\underline{p} \cup \underline{q})$.

Theorem 5.4.1. *Let λ be an (r, s)-hook partition of $k - 2j$, for all $0 \le j \le [k/2]$. Let $(\underline{p}, \underline{q}) \in \mathcal{P}(j)$ and fix $T \in \Gamma_{r,s}(\mathcal{K} - (\underline{p} \cup \underline{q}))$ of shape λ. Let $T_{(0)}$ and $T_{(1)}$ be the corresponding subtableaux of shape $\lambda_{(0)} = (\lambda_1, \cdots, \lambda_r)$ and $\lambda_{(1)} = (\tilde{\lambda}_{r+1}, \cdots, \tilde{\lambda}_{r+s})$ respectively. Let $w_{T,\underline{p},\underline{q}} = w_1 \otimes \cdots \otimes w_k$ be the simple tensor defined by*

$$w_i = \begin{cases} t_1 & \text{if } \ i \in \underline{p}, \\ t_1^* & \text{if } \ i \in \underline{q}, \\ t_\ell & \text{if } \ i \in \mathcal{K} - (\underline{p} \cup \underline{q}) \text{ and } i \text{ is in } \ell^{\text{th}} \text{ row of } T_{(0)}, \\ u_\ell & \text{if } \ i \in \mathcal{K} - (\underline{p} \cup \underline{q}) \text{ and } i \text{ is in } \ell^{\text{th}} \text{ row of } T_{(1)}. \end{cases}$$

Then $v = w_{T,\underline{p},\underline{q}} C_{\underline{p},\underline{q}} y_T$ is a highest weight vector of $V^{\otimes k}$. For any $\underline{s}, \underline{t} \subset \mathcal{K} - (\underline{p} \cup \underline{q})$ such that $\underline{s} \cap \underline{t} = \varnothing$, v is an element of $\mathcal{H}(V^{\otimes k} C_{\underline{p},\underline{q}}, C_{\underline{s},\underline{t}}) y_T$.

Moreover, if $n = 2s$ and if there are s rows in $T_{(1)}$, then there is extra highest weight vector $\overline{v} = \overline{w}_{T,\underline{p},\underline{q}} C_{\underline{p},\underline{q}} y_T$, where $\overline{w}_{T,\underline{p},\underline{q}}$ is the simple tensor obtained from $w_{T,\underline{p},\underline{q}}$ by replacing each occurrence of u_s with u_s^*.

We will prove Theorem 5.4.1 by means of a sequence of lemmas. For simplicity, we write w for $w_{T,\underline{p},\underline{q}}$, $\overline{w}$ for $\overline{w}_{T,\underline{p},\underline{q}}$ and $\mathrm{wt}(u)$ for the weight of a vector u. We begin by determining the weights of v and $\overline{v}$.

Lemma 5.4.2. *Let w and $C_{\underline{p},\underline{q}}$ be defined as in Theorem 5.4.1. Then*

(i) $wC_{\underline{p},\underline{q}}$ is annihilated by all $C_{s,t}$ such that $s,t \in \mathcal{K} - (\underline{p} \cup \underline{q})$,

(ii) the weight of v equals $\mathrm{wt}(w)$ which is

$$\lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r + \tilde{\lambda}_{r+1}\delta_1 + \cdots + \tilde{\lambda}_{r+s}\delta_s, \text{ and}$$

(iii) the weight of $\overline{v}$ is

$$\mathrm{wt}(w) - 2\tilde{\lambda}_{r+s}\delta_s. \tag{5.4.1}$$

Proof.

(i) If s and t are both in $\mathcal{K} - (\underline{p} \cup \underline{q})$, then the corresponding w_s and w_t will be in the set $B' = \{t_1, \ldots, t_r, u_1, \ldots, u_s, (u_{s+1})\}$. Therefore $C_{s,t}$ will contract w_s and w_t into $F_{w_s,w_t}\Omega$ which is 0 since $F_{w_s,w_t} = 0$ for all $w_s, w_t \in B'$.

(ii) By the construction of w, the numbers of t_1's and t_1^*'s coming from $\underline{p}$ and $\underline{q}$ are equal in w, and we have $\mathrm{wt}(t_1) = -\mathrm{wt}(t_1^*)$. Therefore the weight of w is determined by its factors w_i such that $i \in \mathcal{K} - (\underline{p} \cup \underline{q})$. Therefore Lemma 5.1.10 implies that

$$\mathrm{wt}(w) = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r + \tilde{\lambda}_{r+1}\delta_1 + \cdots + \tilde{\lambda}_{r+s}\delta_s. \tag{5.4.2}$$

Moreover, $C_{\underline{p},\underline{q}}$ will contract pairs of t_1 and t_1^* of w into a $\mathfrak{g}$-invariant tensor which also has weight 0, we have $\mathrm{wt}(wC_{\underline{p},\underline{q}}) = \mathrm{wt}(w)$. Therefore the equation $\mathrm{wt}(v) = \mathrm{wt}(wC_{\underline{p},\underline{q}}y_T) = \mathrm{wt}(w)$ holds since y_T preserves weights.

(iii) The result follows from (ii) and the fact that $\overline{w}$ is obtained from w by changing u_s to u_s^*. □

Now we verify that $v \neq 0$. This follows from Lemma 5.4.3 and Proposition 5.4.4. The same process is also valid for the vector $\overline{v}$.

First, let $\mathcal{E}$ be the subspace of $V^{\otimes k}$ spanned by the simple tensors of the form $\underline{w}_1 \otimes \cdots \otimes \underline{w}_k$ such that $\underline{w}_j \in B$ for all $1 \leq j \leq k$, and $\underline{w}_i \neq w_i$ for some $1 \leq i \leq k$. Clearly $w \notin \mathcal{E}$. Then we identify a key property of the action of y_T on $wC_{\underline{p},\underline{q}}$ by the following lemma.

Lemma 5.4.3. *Let $\psi \in \mathcal{R}_T$ and $\sigma \in \mathcal{C}_T$. Then $w\,\mathrm{sgn}(\psi)\psi\sigma = cw$ for some scalar c if and only if $\psi \in \mathcal{R}_{T_{(0)}}$ and $\sigma \in \mathcal{R}_{T_{(1)}}$. In this case, $c = 1$.*

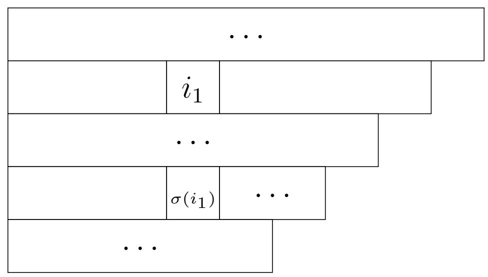
Proof. Take $\psi \in \mathcal{R}_{T_{(0)}}$. Then ψ permutes elements within the i^{th} row of $T_{(0)}$ for $1 \leq i \leq r$. By the definition of w, the elements in the i^{th} row of $T_{(0)}$ will be recorded as t_i in w. Therefore ψ will only permute t_i with t_i in w. By Lemma 5.1.12, we have $w\psi = \mathrm{sgn}(\psi)w$.

Now take $\sigma \in \mathcal{C}_T$. If $\sigma \in \mathcal{R}_{T_{(1)}}$, then σ will only permute u_j with u_j for all $1 \leq j \leq s$. Again by Lemma 5.1.12, we have $w\sigma = w$. Thus we have $w\,\mathrm{sgn}(\psi)\psi\sigma = w$ if $\psi \in \mathcal{R}_{T_{(0)}}$ and $\sigma \in \mathcal{R}_{T_{(1)}}$.

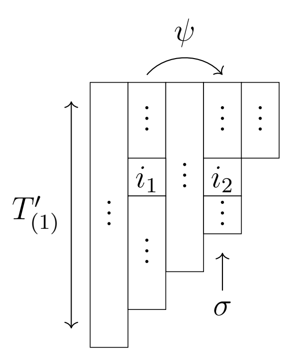
In order to prove the other direction is true, it is enough to prove the following statement:

"If either $\sigma \notin \mathcal{R}_{T_{(1)}}$ or $\psi \notin \mathcal{R}_{T_{(0)}}$, then $w\psi\sigma \in \mathcal{E}$. "

If $\sigma \notin \mathcal{R}_{T_{(1)}}$, without loss of generality, we may assume that $\sigma \in \mathcal{C}_T \setminus \mathcal{R}_{T_{(1)}}$. Then there exists i_1 appearing in the j^{th} row of $T_{(0)}$ such that $\sigma(i_1) \neq i_1$. We have $w_{i_1} = t_j$. If $\sigma(i_1)$ appears in $T_{(0)}$, then $\sigma(i_1)$ is in a different row than i_1. So $w_{\sigma(i_1)} \neq w_{i_1} = t_j$. Otherwise, $w_{\sigma(i_1)} = u_\ell$ for some ℓ, and again $w_{i_1} \neq w_{\sigma(i_1)}$. Moreover, for any $\psi \in \mathcal{R}_T$, the vector in the i_1^{th} position of $w\psi$ is still t_j. This implies that the vector in the i_1^{th} position of $w\psi\sigma$ is not t_j. So $w\psi\sigma \in \mathcal{E}$. This argument can be visualized by the following figure; notice in this case, i_1 and $\sigma(i_1)$ are in the same column but different rows.



Now let us consider that $\psi \notin \mathcal{R}_{T_{(0)}}$. Then without loss of generality, we assume that $\psi \in \mathcal{R}_T \setminus \mathcal{R}_{T_{(0)}}$. Then there exists $i_1 \neq i_2$ in the same row of T but not in $T_{(0)}$ such that $\psi(i_1) = i_2$. If i_ℓ is in column j_ℓ of T for $\ell = 1, 2$, then $w_{i_1} = u_{j_1} \neq u_{j_2} = w_{i_2}$. Therefore the vector in the i_1^{th} position of $w\psi$ is $w_{\psi(i_1)} = w_{i_2} = u_{j_2}$. Now for any $\sigma \in \mathcal{C}_T$, $\sigma(i_2)$ is still in column j_2 of T. If $\sigma(i_2)$ appears in $T_{(0)}$, then $w_{\sigma\psi(i_1)} = w_{\sigma(i_2)} = t_j$ for some j. Otherwise, $\sigma(i_2)$ appears in $T_{(1)}$. Then it will also be in row j_2 of $T_{(1)}$, so $w_{\sigma\psi(i_1)} = u_{j_2}$. In either case, the vector in the i_1^{th} position of $w\psi\sigma$ is not $u_{j_1} = w_{i_1}$. Thus if $\sigma \in \mathcal{C}_T$ but $\psi \in \mathcal{R}_T \setminus \mathcal{R}_{T_{(0)}}$, then $w\psi\sigma \in \mathcal{E}$. This argument can be visualized by the following figure; notice that in our case, we draw the transpose of $T_{(1)}$, and σ permutes entries in the same column of i_2.



□

Proposition 5.4.4. *The vector* $wC_{\underline{p},\underline{q}}$ *can be written as* $\beta_{\underline{p},\underline{q}} w + R$*, where* $R \in \mathcal{E}$ *and* $\beta_{\underline{p},\underline{q}} \neq 0$*.*

Proof. We prove this proposition by first writing $wC_{i,j}$ explicitly for some $1 \leq i < j \leq k$. Take $w = w_1 \otimes \cdots \otimes w_k$ which is constructed according to Theorem 5.4.1. Then for each pair $(i,j) \in (\underline{p}, \underline{q})$, we have $w_i = t_1$ and $w_j = t_1^*$.

Therefore by (5.3.3) and the fact that $F_{t_1,t_1^*} = 1$, we have (up to a nonzero scalar multiple, indicated by $\sim$)

$$wC_{i,j} \sim \sum_{u,v \in B} F_{u,v}^{-1} w_1 \otimes \cdots \otimes w_{i-1} \otimes u \otimes w_{i+1} \otimes \cdots \otimes w_{j-1} \otimes v \otimes w_{j+1} \otimes \cdots \otimes w_k. \quad (5.4.3)$$

Note that if $u = t_1$ and $v = t_1^*$, we have $F_{u,v}^{-1} = -1$. Thus, $wC_{i,j} \sim w + R_{i,j}$, where

$$R_{i,j} = \sum_{\substack{u,v \in B \\ u \neq t_1, v \neq t_1^*}} F_{u,v}^{-1} w_1 \otimes \cdots \otimes w_{i-1} \otimes u \otimes w_{i+1} \otimes \cdots \otimes w_{j-1} \otimes v \otimes w_{j+1} \otimes \cdots \otimes w_k.$$

Note that the vector in the i^{th} position in w is t_1 but the vector in the same position in any simple summand in $R_{i,j}$ cannot be t_1. Thus $R_{i,j} \in \mathcal{E}$.

If we apply a contractions $C_{s,t}$ with $1 \leq s < t \leq k$ and $\{s,t\} \neq \{i,j\}$ on $R_{i,j}$, the vectors in the i^{th} and j^{th} positions of $R_{i,j}$ will remain unchanged. Thus every simple tensor appearing in the expression for $R_{i,j}C_{s,t}$ is an element of $\mathcal{E}$. Since $wC_{s,t} \sim w + R_{s,t}$, it follows by induction that $wC_{\underline{p},\underline{q}} \sim w + R$, where $R \in \mathcal{E}$. □

Corollary 5.4.5. *If we write* $v = wC_{\underline{p},\underline{q}} y_T$ *as a linear combination of simple tensors in the basis* B*, then the coefficient of* w *in* v *is given by* $\beta_{\underline{p},\underline{q}} h(\lambda) \left|\mathcal{R}_{T_{(0)}}\right| \left|\mathcal{R}_{T_{(1)}}\right|$*. In particular,* $v \neq 0$*.*

Proof. By Proposition 5.4.4, we have $wC_{\underline{p},\underline{q}} = \beta_{\underline{p},\underline{q}} w + R$, where $R \in \mathcal{E}$. Then we consider the action of normalized Young symmetrizer y_T on w and R separately. We first show that Ry_T is in $\mathcal{E}$.

By the definition of y_T and the way we construct w (Theorem 5.4.1), y_T only permutes the vectors in R at the positions which will not be contracted by $C_{\underline{p},\underline{q}}$. For each $i \in \underline{p}$, $w_i = t_1$. By the proof of Proposition 5.4.4, for each simple tensor in R, there exists $i \in \underline{p}$ such that the vector in its i^{th} position is not t_1. Therefore every summand of Ry_T is an element of $\mathcal{E}$, which implies $Ry_T \in \mathcal{E}$.

Moreover, we have

$$wy_T = h(\lambda) \sum_{\psi \in \mathcal{R}_T, \sigma \in \mathcal{C}_T} w \operatorname{sgn}(\psi)\psi\sigma.$$

By Lemma 5.4.3, $w \operatorname{sgn}(\psi)\psi\sigma = w$ if and only if $\psi \in \mathcal{R}_{T_{(0)}}$ and $\sigma \in \mathcal{R}_{T_{(1)}}$. This implies that if either $\psi \notin \mathcal{R}_{T_{(0)}}$ or $\sigma \notin \mathcal{R}_{T_{(1)}}$, then $w \operatorname{sgn}(\psi)\psi\sigma \in \mathcal{E}$. Thus we can write wy_T as a linear combination of w and $R'' \in \mathcal{E}$. Namely, we have

$$wy_T = h(\lambda) \sum_{\substack{\psi \in \mathcal{R}_{T_{(0)}} \\ \sigma \in \mathcal{R}_{T_{(1)}}}} w + R''.$$

Therefore

$$wC_{\underline{p},\underline{q}} y_T = \beta_{\underline{p},\underline{q}} h(\lambda) \sum_{\substack{\psi \in \mathcal{R}_{T_{(0)}} \\ \sigma \in \mathcal{R}_{T_{(1)}}}} w + R'' + Ry_T,$$

where $R'', Ry_T \in \mathcal{E}$. Thus, the coefficient of w in v is given by $\beta_{\underline{p},\underline{q}} h(\lambda) \left|\mathcal{R}_{T_{(0)}}\right| \left|\mathcal{R}_{T_{(1)}}\right|$ which is not 0. Therefore v is not a zero vector. $\square$

Our ultimate goal is to prove that v and $\overline{v}$ are highest weight vectors. By the previous discussion we have proved that they are nonzero weight vector with weight as given in (5.4.2) and (5.4.1) respectively. Therefore it remains to check whether the simple root vectors Y defined in Definition 2.5.13 acting on v and $\overline{v}$ will give 0. To do so, we use the following lemmas which helps us to view $v = w_{T,\underline{p},\underline{q}} C_{\underline{p},\underline{q}} y_T$ more concretely. The same process also works for $\overline{v}$ as we outline at the end of this section.

Definition 5.4.6. *For all $1 \leq j \leq \lfloor k/2 \rfloor$, we let $C_{\underline{o},\underline{e}} = C_{1,2} C_{3,4} \cdots C_{2j-1,2j}$.*

Lemma 5.4.7. *Let $\sigma \in S_k$ be the permutation such that*

(i) $\sigma(p_t) = 2t - 1$ and $\sigma(q_t) = 2t$ for all $1 \leq t \leq j$, and

(ii) $\sigma(i_1) > \sigma(i_2)$ for all $i_1 > i_2$ and $i_1, i_2 \in \mathcal{K} - (\underline{p} \cup \underline{q})$.

Then for any contraction maps $C_{\underline{p},\underline{q}} = C_{p_1,q_1} \ldots C_{p_j,q_j}$, we have $\sigma C_{\underline{o},\underline{e}} \sigma^{-1} = C_{\underline{p},\underline{q}}$.

Proof. The result follows from the fact $C_{\underline{o},\underline{e}} = C_{1,2} C_{3,4} \cdots C_{2j-1,2j}$ and Lemma 5.3.4 □

Lemma 5.4.8. *Let $\sigma \in S_k$ be the permutation defined in Lemma 5.4.7. Let T^σ be the standard tableau defined in Lemma 5.2.6 with entries in $\mathcal{K} - (\underline{p} \cup \underline{q})$. We have*

$$w_{T,\underline{p},\underline{q}} \sigma = \beta_\sigma(w_1, \ldots, w_k) w_{T^\sigma,\underline{o},\underline{e}},$$

where $\beta_\sigma(w_1, \ldots, w_k)$ is the same scalar defined in Lemma 3.1.5.

Proof. By the way we defined σ, we have

$$\begin{aligned} w_{T,\underline{p},\underline{q}} \sigma &= (w_1 \otimes \cdots \otimes w_k)\sigma \\ &= \beta_\sigma(w_1, \ldots, w_k) w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)}. \end{aligned}$$

Let $v_i = w_{\sigma(i)}$ for all $1 \leq i \leq k$. In particular, we have

$$w_{\sigma(i)} = \begin{cases} t_1 & \text{if } \sigma(i) \in \underline{p}, \\ t_1^* & \text{if } \sigma(i) \in \underline{q}, \\ t_\ell & \text{if } \sigma(i) \text{ is in the } \ell^{\text{th}} \text{ row of } T_{(0)}, \\ u_\ell & \text{if } \sigma(i) \text{ is in the } \ell^{\text{th}}\text{th row of } T_{(1)}. \end{cases}$$

Recalling the definition of T^σ from Lemma 5.2.6, we deduce that this is equivalent to

$$v_i = \begin{cases} t_1 & \text{if } i \in \underline{o}, \\ t_1^* & \text{if } i \in \underline{e}, \\ t_\ell & \text{if } i \text{ is in the } \ell^{\text{th}} \text{ row of } T^\sigma_{(0)}, \\ u_\ell & \text{if } i \text{ is in the } \ell^{\text{th}} \text{ row of } T^\sigma_{(1)}. \end{cases}$$

Since this defines $w_{T^\sigma,\underline{o},\underline{e}}$, we have $w_{T,\underline{p},\underline{q}}\sigma = \beta_\sigma(w_1, \ldots, w_k) w_{T^\sigma,\underline{o},\underline{e}}$. □

For simplicity, we write β_σ for $\beta_\sigma(w_1, \ldots, w_k)$.

Corollary 5.4.9. *Retain the notation in Lemma 5.4.7 and 5.4.8. We have*

$$w_{T,\underline{p},\underline{q}} C_{\underline{p},\underline{q}} y_T = \beta_\sigma w_{T^\sigma,\underline{o},\underline{e}} C_{\underline{o},\underline{e}} y_{T^\sigma} \sigma^{-1}.$$

Proof. First notice that $w_{T,\underline{p},\underline{q}} C_{\underline{p},\underline{q}} y_T$ is equal to

$$(w_{T,\underline{p},\underline{q}}\sigma)(\sigma^{-1} C_{\underline{p},\underline{q}}\sigma)(\sigma^{-1} y_T \sigma)\sigma^{-1}. \tag{5.4.4}$$

Then by Lemma 5.4.8, 5.4.7 and 5.2.6, (5.4.4) becomes

$$\beta_\sigma w_{T^\sigma,\underline{o},\underline{e}} C_{\underline{o},\underline{e}} y_{T^\sigma} \sigma^{-1},$$

which completes the proof. □

Corollary 5.4.10. *The vector* $w_{T,\underline{p},\underline{q}}C_{\underline{p},\underline{q}}y_T$ *is equal to*

$$\beta_\sigma(\Omega \otimes \cdots \otimes \Omega \otimes v_{2j+1} \otimes \cdots \otimes v_k)y_{T^\sigma}\sigma^{-1} \tag{5.4.5}$$

where Ω *is the invariant tensor defined in* (5.3.2)*, and* $v_\ell = w_{\sigma(\ell)}$*.*

Proof. Notice that

$$w_{T^\sigma,\underline{o},\underline{e}} = t_1 \otimes t_1^* \otimes \cdots \otimes t_1 \otimes t_1^* \otimes v_{2j+1} \otimes \cdots \otimes v_k, \tag{5.4.6}$$

where the first $2j$ terms of (5.4.6) are j pairs of $t_1 \otimes t_1^*$. Therefore by Lemma 5.3.5 $C_{\underline{o},\underline{e}}$ acting on (5.4.6) will contract the first $2j$ terms into j times of Ω. Thus the result follows from Corollary 5.4.9. □

Now in order to prove that $v = w_{T,\underline{p},\underline{q}}C_{\underline{p},\underline{q}}y_T$ is a highest weight vector, it suffices to prove that the simple root vectors Y act on (5.4.5) as 0.

Since Ω is a $\mathfrak{g}$-invariant tensor, Y acts on each factor Ω as 0. Therefore let

$$\theta_{Y,i} = \Omega \otimes \cdots \otimes \Omega \otimes v_{2j+1} \otimes \cdots \otimes (Yv_{2j+i}) \otimes \cdots \otimes v_k. \tag{5.4.7}$$

We have

$$Y \cdot (\Omega \otimes \cdots \otimes \Omega \otimes v_{2j+1} \otimes \cdots \otimes v_k) = \sum_{i=1}^{k-2j} c_{Y,i}\theta_{Y,i}$$

for some scalars $c_{Y,i}$.

Recall that the v_i's are all in B'. Each simple root vector (Definition 2.5.13) acts by zero on all but one vector in B'. We record how the simple root vectors act on $b \in B'$ in Table 5.1.

$Y \in \Delta_Y$	$b \in B'$	Yb	Conditions
Y_{γ_i}	t_{i+1}	t_i	$1 \leq i \leq r-1$
Y_{γ_r}	u_1	t_r	
$Y_{\gamma_{i+j}}$	u_{j+1}	u_j	$1 \leq j \leq s-1$
$Y_{\gamma_{r+s}}$	u_{s+1}	u_s	$n = 2s+1$
Y	v	0	for all other cases

Table 5.1: The action of simple root vectors from Definition 2.5.13 on $v \in B'$

Therefore, for all pairs (Y, v_{2j+i}) in the fifth row of Table 5.1, we have $\theta_{Y,i} = 0$. Now we claim that $\theta_{Y,i} y_{T^\sigma} = 0$ for all $Y \in \Delta_Y$ and for all $1 \leq i \leq 2k - j$. From Table 5.1, we observe that Y acting on v_{2j+i} will either lower the subscript by one, or change u_1 into t_r. Let us first consider an example.

Example 5.4.11. Let $Y = E_{t_2,t_3} - \beta(t_2,t_3)E_{t_3^*,t_2^*}$. Then Y acting on $\Omega \otimes t_3 \otimes t_1 \otimes t_3$ gives

$$\Omega \otimes t_2 \otimes t_1 \otimes t_3 + \beta(t_3 t_1, t_2 t_3^*)\Omega \otimes t_3 \otimes t_1 \otimes t_2.$$

Then $\theta_{Y,1} = \Omega \otimes t_2 \otimes t_1 \otimes t_3$, $\theta_{Y,2} = 0$ and $\theta_{Y,3} = \Omega \otimes t_3 \otimes t_1 \otimes t_2$.

Lemma 5.4.12. *Let $\theta = \theta_{Y,i} \neq 0$ for some $Y = Y_{\gamma_i} \in \Delta_Y$ and such that $v_{2j+i} = t_{\ell+1}$. Then for all $\psi \in \mathcal{R}_{T^\sigma}$ there exists some permutation π in $\mathcal{C}_{T^\sigma}$ such that $\theta\psi\pi = -\theta\psi$.*

Proof. By the construction of $w_\sigma = w_{T^\sigma,\underline{o},\underline{e}} = t_1 \otimes t_1^* \otimes \cdots \otimes t_1 \otimes t_1^* \otimes v_{2j+1} \otimes \cdots \otimes v_k$, we know that for all $i \in \{1, \ldots, k-2j\}$, if $v_{2j+i} = t_{\ell+1}$ for some $1 \leq \ell \leq r-1$, then $2j+i$ is in the $(\ell+1)^{\text{th}}$ row of $T^\sigma_{(0)}$. In particular, the tensor w_σ looks like

$$\cdots \otimes \underbrace{t_{\ell+1}}_{(2j+i)^{\text{th}} \textit{ position}} \otimes \cdots \tag{5.4.8}$$

and so the $(\ell+1)^{\text{th}}$ row of the tableau T^σ looks like

By hypothesis θ has the form

$$\cdots \otimes \underbrace{t_\ell}_{(2j+i)^{\text{th}} \textit{ position}} \otimes \cdots .$$

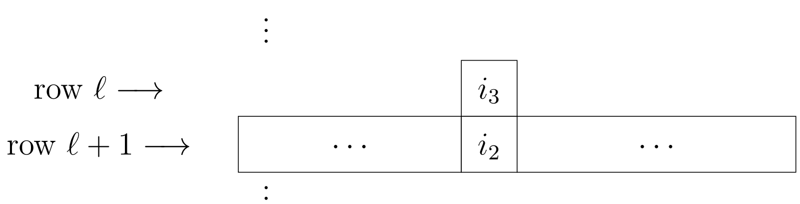
First suppose that $\psi \in \mathcal{R}_{T^\sigma}$ is such that $\psi(2j+i) = 2j+i$. Let i_1 be the value in the box directly above $2j+i$ in T^σ, as in:

Therefore we have $v_{i_1} = t_\ell$. Hence the permutation $(i_1, 2j+i) \in \mathcal{C}_{T^\sigma}$ will permute t_j with t_j in $\theta\psi$. Thus we have $\theta\psi(i_1, 2j+i) = -\theta\psi$ by Lemma 5.1.12.

Next, suppose instead that $\psi \in \mathcal{R}_{T^\sigma}$ such that $\psi(2j+i) = i_2$ for some $i_2 \neq 2j+i$ in the $(\ell+1)^{\text{th}}$ row of T^σ. Then $\theta\psi$ is the tensor

$$\cdots \otimes \underbrace{t_\ell}_{i_2^{\text{th}} \textit{ position}} \otimes \cdots \otimes \underbrace{t_{\ell+1}}_{(2j+i)^{\text{th}} \textit{ position}} \otimes \cdots .$$

Let i_3 be the entry of the tableau T^σ in row ℓ directly above i_2, as in the diagram:



In this case, we have $v_{i_3} = t_\ell$. Hence the permutation $(i_2, i_3) \in \mathcal{C}_{T^\sigma}$ will permute the corresponding t_ℓ and t_ℓ in $\theta\psi$. Therefore, we have $\theta\psi(i_2, i_3) = -\theta\psi$. Thus (i_2, i_3) is the desired permutation.

We have proved that if Y acting on $v_{2j+i} = t_{\ell+1}$ for some ℓ lowers the index $\ell + 1$ to ℓ, then for all $\psi \in \mathcal{R}_{T^\sigma}$, there exists some permutation $\pi \in \mathcal{C}_{T^\sigma}$ such that $\theta\psi\pi = -\theta\psi$. Note that the case when Y changes u_1 into t_r can be proved similarly. $\square$

Lemma 5.4.13. *Retain the set up in Lemma 5.4.12, we have $\theta y_{T^\sigma} = 0$.*

Proof. By Lemma 5.4.12, for all $\psi \in \mathcal{R}_{T^\sigma}$, there exists some $\pi \in \mathcal{C}_{T^\sigma}$ such that $\theta\psi\pi = -\theta\psi$. Thus suppose $\psi \in \mathcal{R}_{T^\sigma}$, there exists a choice of $\pi \in \mathcal{C}_{T^\sigma}$ such that

$$\sum_{\gamma \in \mathcal{C}_{T^\sigma}} \operatorname{sgn}(\psi)(\theta\psi)\gamma = \sum_{\gamma \in \mathcal{C}_{T^\sigma}} \operatorname{sgn}(\psi)(\theta\psi)\pi\gamma = -\sum_{\gamma \in \mathcal{C}_{T^\sigma}} \operatorname{sgn}(\psi)(\theta\psi)\gamma,$$

which implies that $\sum_{\gamma \in \mathcal{C}_{T^\sigma}} \operatorname{sgn}(\psi)(\theta\psi)\gamma = 0$ for each $\psi \in \mathcal{R}_{T^\sigma}$ and therefore $\theta y_{T^\sigma} = 0$. $\square$

Now let us consider the case when Y changes $u_{\ell+1}$ into u_ℓ for all $1 \le \ell \le s-1$.

Lemma 5.4.14. *Let $\theta = \theta_{Y,i} \neq 0$ for some $Y = Y_{\gamma_{r+\ell}} \in \Delta_Y$ and such that $v_{2j+i} = u_{\ell+1}$. Then there exists some permutation $\pi \in \mathcal{R}_{T^\sigma}$ such that $\theta\pi = \theta$ and $\operatorname{sgn}(\pi) = -1$.*

Proof. The proof is similar to the proof of Lemma 5.4.13. Noticing that if $v_{2j+i} = u_{\ell+1}$ for some $1 \le \ell \le s-1$, then $2j+1$ is in the $(\ell+1)^{\text{th}}$ column of $(T^\sigma_{(1)})'$ (or equivalently, the $(\ell+1)^{\text{th}}$ row of $T^\sigma_{(1)}$). Therefore there exists a permutation $\pi \in \mathcal{R}_{T^\sigma}$ such that π permutes $2j+i$ and the entry in the box directly to the left of the box of $2j+i$, say i_1. Note that $v_{i_1} = u_\ell$ since i_1 is in the ℓ^{th} column of $(T^\sigma_{(1)})'$. Moreover, since $Y_{\gamma_{r+\ell}} v_{2j+i} = u_\ell$, the $(2j+i)^{\text{th}}$ position of θ is also u_ℓ. In this way, we simply choose

$\pi = (i_1, 2j+i)$, and thus, $\mathrm{sgn}(\pi) = -1$. Then π permutes u_ℓ with u_ℓ in θ, which implies $\theta\pi = \theta$. $\square$

Lemma 5.4.15. *Retain the set-up in Lemma 5.4.14, we have $\theta y_{T^\sigma} = 0$.*

Proof. By Lemma 5.4.14, there exists some permutation $\pi \in \mathcal{R}_{T^\sigma}$ such that $\theta\pi = \theta$ and $\mathrm{sgn}(\pi) = -1$. Therefore we have

$$\sum_{\psi \in \mathcal{R}_{T^\sigma}} \mathrm{sgn}(\psi)\theta\psi = \sum_{\psi \in \mathcal{R}_{T^\sigma}} \mathrm{sgn}(\psi)\mathrm{sgn}(\pi)(\theta\pi)\psi = -\sum_{\psi \in \mathcal{R}_{T^\sigma}} \mathrm{sgn}(\psi)(\theta\psi)$$

which implies that $\sum_{\psi \in \mathcal{R}_{T^\sigma}} \mathrm{sgn}(\psi)\theta\psi = 0$ and therefore $\theta y_{T^\sigma} = 0$. $\square$

Proposition 5.4.16. *The vector $w_{T,\underline{p},\underline{q}} C_{\underline{p},\underline{q}} y_T$ we defined in Theorem 5.4.1 is a highest weight vector.*

Proof. By Lemma 5.4.13 and 5.4.15, every nonzero $\theta_{Y,i}$ will be transformed into 0 by y_{T^σ}. Thus $Y w_{T^\sigma \underline{o},\underline{e}} C_{\underline{o},\underline{e}} y_{T^\sigma} = 0$, and in turn, $Y w_{T,\underline{p},\underline{q}} C_{\underline{p},\underline{q}} y_T = 0$ which implies that $w_{T,\underline{p},\underline{q}} C_{\underline{p},\underline{q}} y_T$ is a highest weight vector. $\square$

Notice that the whole process (from Definition 5.4.6 to Proposition 5.4.16) of proving v is a highest weight vector is also valid for proving $\overline{v}$ is a highest weight vector. The only change we need to make is to notice that we have one extra case in Table 5.1, that is, if $n = 2s$, $T_{(1)}$ has s rows, and $b = u_s^*$, then $Y_{\gamma_{r+s}} u_s^* = u_{s-1}$. Therefore we have one extra case to analyze, that is, the simple root vector acting on v_{2j+i} changes u_s^* to u_{s-1}. However, the proof of this case is the same as the one we provided in the proof of Lemma 5.4.14 by changing $u_{\ell+1}$ into u_s^* and changing u_ℓ into u_{s-1} in Lemma 5.4.14.

Thus, we conclude that the vectors produced by Theorem 5.4.1 are highest weight vectors.

5.5 An illustration of Theorem 5.4.1

We have proved that the vectors in Theorem 5.4.1 are highest weight vectors. Next, we illustrate how to find these vectors in Section 5.5.1 by expanding Example 2.5.15. Then we verify these vectors are highest weight vectors in Section 5.5.2. In Section 5.5.3, we find $\mathfrak{spo}(V, \beta)$-submodules generated by these highest weight vectors. Then we deduce that that $V \otimes V$ can be written as a direct sum of these submodules. Moreover, we will extend the result in this section to a more generalized version in Section 6.3.

5.5.1 The highest weight vectors

Retain the setup in Example 2.5.15. Let $V = V_{(0)} \oplus V_{(1)}$ be the standard $\mathfrak{spo}(V, \beta)$-module such that $\dim(V_{(0)}) = 2r = 2$ and $\dim(V_{(1)}) = 2s = 4$. Thus V has a homogeneous basis $B = \{t_1, t_1^*, u_1, u_1^*, u_2, u_2^*\}$. Elements of the Cartan subalgebra of $\mathfrak{g}$ are given by $H = \mathrm{diag}(a_1, -a_1, a_2, -a_2, a_3, -a_3)$. Furthermore, we consider $V^{\otimes k} = V^{\otimes 2}$.

In order to find the vectors $v = w_{T,\underline{p},\underline{q}} C_{\underline{p},\underline{q}} y_T$, we first find all possible $(\underline{p}, \underline{q}) \in \mathcal{P}(j)$. Since $k = 2$, we have $0 \leq j \leq 1 = \lfloor k/2 \rfloor$. Therefore, by (5.3.5), the set of all indices of all possible contractions are

$$\mathcal{P} = \bigcup_{j=0}^{1} \mathcal{P}(j) = \mathcal{P}(0) \cup \mathcal{P}(1) = \{(\varnothing, \varnothing)\} \cup \{(1,2)\} = \{(\varnothing, \varnothing), (1,2)\},$$

where we use $\varnothing$ to denote the empty sequence in order to distinguish from the empty tableau $\varnothing$.

Let us first consider $(\underline{p}, \underline{q}) = (1, 2)$. We have $C_{\underline{p},\underline{q}} = C_{1,2}$. The only possible

$(1,2)$-hook tableau T in $\Gamma_{1,2}(\oslash)$ is the empty tableau $\varnothing$. In this case we have

$$T = \varnothing \text{ with } T_{(0)} = \varnothing \text{ and } T_{(1)} = \varnothing.$$

Since $1 \in \underline{p}$ and $2 \in \underline{q}$, we have $w_{\varnothing,\underline{1},\underline{2}} = t_1 \otimes t_1^*$. Additionally, we have $y_T = \mathrm{id}$ since $T = \varnothing$. Therefore the highest weight vector in this case is

$$v_1 = (t_1 \otimes t_1^*)C_{1,2}\mathrm{id}. \tag{5.5.1}$$

In Proposition 4.2.2, we already computed that

$$v_1 = -t_1 \otimes t_1^* + t_1^* \otimes t_1 + u_1 \otimes u_1^* + u_1^* \otimes u_1 + u_2 \otimes u_2^* + u_2^* \otimes u_2 = \Omega. \tag{v_1}$$

Next we consider the case when $(\underline{p}, \underline{q}) = (\oslash, \oslash)$. We have $C_{\underline{p},\underline{q}} = C_{\oslash,\oslash} = \Psi_{\mathrm{id}}$. In addition, there are two possible $(1,2)$-hook tableaux in $\Gamma_{1,2}(\mathcal{K})$. In particular, we have

$$T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \text{ with } T_{(0)} = \begin{array}{|c|}\hline 1 \\ \hline \end{array} \text{ and } T_{(1)} = \begin{array}{|c|}\hline 2 \\ \hline \end{array}, \text{ or} \tag{5.5.2}$$

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \text{ with } T_{(0)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \text{ and } T_{(1)} = \varnothing. \tag{5.5.3}$$

Let T be given as in (5.5.2). We have $y_T = \frac{1}{2}(\mathrm{id} + (12))$. In addition, since 1 is in the 1$^{\text{st}}$ row of $T_{(0)}$, and 2 is in the 1$^{\text{st}}$ row of $T_{(1)}$, we have $w_{T,\varnothing,\varnothing} = t_1 \otimes u_1$. Thus the highest weight vector in this case is

$$v_2 = \frac{1}{2}(t_1 \otimes u_1)\Psi_{\mathrm{id}}(\mathrm{id} + (12)).$$

Then we write v_2 explicitly. By Remark 4.1.10, the action of $\mathrm{id} + (12)$ on $V \otimes V$ is given by the map $\Psi_{\mathrm{id}} + \Psi_{s_1}$. Therefore we have

$$v_2 = \frac{1}{2}(t_1 \otimes u_1)\Psi_{\mathrm{id}} + \frac{1}{2}(t_1 \otimes u_1)\Psi_{s_1}$$

which (by Proposition 4.2.2) is

$$v_2 = \frac{1}{2}(t_1 \otimes u_1) - \frac{1}{2}\beta(u_1, t_1)(u_1 \otimes t_1). \tag{v_2}$$

Finally, let T be defined as in (5.5.3). We have $y_T = \frac{1}{2}(\mathrm{id} - (12))$. Moreover, since both 1 and 2 are in the 1^{st} row of $T_{(0)}$, we obtain $w_{T,\varnothing,\varnothing} = t_1 \otimes t_1$. Therefore in this case, the highest weight vector is

$$\begin{aligned} v_3 &= \frac{1}{2}(t_1 \otimes t_1)\Psi_{\mathrm{id}}(\mathrm{id} - (12)) \\ &= \frac{1}{2}(t_1 \otimes t_1 - (-\beta(t_1, t_1)t_1 \otimes t_1)) \\ &= t_1 \otimes t_1. \end{aligned} \tag{v_3}$$

5.5.2 Verification that v_1, v_2 and v_3 are highest weight vectors

Recall that in order to check if v_1, v_2 and v_3 are highest weight vectors, it suffices to check that H acts on these three vectors by scalars, and then verify that for all positive root vectors X of $\mathfrak{spo}(V, \beta)$, X acts on these three vectors as 0.

The vector v_1 is a highest weight vector with weight 0 since we already proved that it is an $\mathfrak{spo}(V, \beta)$-invariant vector in Lemma 5.3.5.

Notice that the possible weights of vectors in $V \otimes V$ are in the set

$$\Phi = \{\pm 2\varepsilon_1, \pm\varepsilon_1 \pm \delta_1, \pm\varepsilon_1 \pm \delta_2, \pm 2\delta_1, \pm 2\delta_2\}.$$

Therefore by Lemma 2.5.14, in order to prove that v_2 is a highest weight vector, it suffices to only consider the simple root vectors $Y \in \Delta_Y$ such that $\mathrm{wt}(Yv_2) \in \Phi$. Since $\mathrm{wt}(v_2) = \varepsilon_1 + \delta_1$, the only Y that satisfies this condition is

$$Y = E_{t_1,u_1} + \beta(t_1, u_1)E_{u_1^*,t_1^*}.$$

First notice that E_{t_1,u_1} acting on t_1 gives 0, and $E_{u_1^*,t_1^*}$ acting on either t_1 or u_1 gives 0. Thus we have

$$\begin{aligned} Yv_2 &= \beta(t_1, t_1u_1^*)t_1 \otimes E_{t_1,u_1}u_1 - \beta(u_1, t_1)E_{t_1,u_1}u_1 \otimes t_1 \\ &= \beta(t_1, t_1)\beta(t_1, u_1^*)t_1 \otimes t_1 - \beta(u_1, t_1)t_1 \otimes t_1 \\ &= \beta(t_1, u_1^*)t_1 \otimes t_1 - \beta(u_1, t_1)t_1 \otimes t_1 \end{aligned}$$

which is 0 by the fact that $\beta(t_1, u_1^*) = \beta(u_1, t_1)$. Therefore v_2 is a highest weight vector. By a similar but easier argument, v_3 is also a highest weight vector.

5.5.3 $\mathfrak{spo}(V, \beta)$-submodules generated by the highest weight vectors

We apply Corollary 3.4.7 to find the corresponding $\mathfrak{spo}(V, \beta)$-submodules generated by the highest weight vectors we found. We first summarize the possible negative root vectors in $\mathfrak{spo}(V, \beta)$ with the corresponding weights in Table 6.1.

Using Corollary 3.4.7, we apply the negative root vectors in Table 6.1 to v_1, v_2 and v_3 respectively to find a for the submodules generated by v_1, v_2 and v_3 respectively.

First of all, we know that v_1 generates the trivial module, denoted as $W_\varnothing$. Then let us consider the case which the highest weight vector is v_2. We use Table 5.3 to

notation	negative root vector
$Y_{-2\varepsilon_1}$	$E_{t_1^*,t_1}$
$Y_{\delta_1-\varepsilon_1}$	$E_{u_1,t_1} - \beta(u_1,t_1)E_{t_1^*,u_1^*}$
$Y_{-\delta_1-\varepsilon_1}$	$E_{u_1^*,t_1} - \beta(u_1^*,t_1)E_{t_1^*,u_1^*}$
$Y_{\delta_2-\varepsilon_1}$	$E_{u_2,t_1} - \beta(u_2,t_1)E_{t_1^*,u_2^*}$
$Y_{-\delta_2-\varepsilon_1}$	$E_{u_2^*,t_1} - \beta(u_2^*,t_1)E_{t_1^*,u_2}$
$Y_{-\delta_1+\delta_2}$	$E_{u_2,u_1} - \beta(u_2,u_1)E_{u_1^*,u_2^*}$
$Y_{-\delta_1-\delta_2}$	$E_{u_1^*,u_2} - \beta(u_1^*,u_1)E_{u_2^*,u_1}$

Table 5.2: Negative root vectors of $\mathfrak{spo}(V,\beta)$ with $m=2$ and $n=4$.

record a basis of $\mathfrak{U}(\mathfrak{g})\cdot v_2 = W_{\square}$, and the weights of these basis vectors.

weights	actions	vectors
$\varepsilon_1+\delta_1$	v_2	$t_1 \otimes u_1 - \beta(u_1,t_1)u_1 \otimes t_1$
$\varepsilon_1-\delta_1$	$Y_{-\delta_1+\delta_2}\cdot(Y_{-\delta_1-\delta_2}\cdot v_2)$	$t_1 \otimes u_1^* - \beta(u_1^*,t_1)u_1^* \otimes t_1$
$\varepsilon_1+\delta_2$	$Y_{-\delta_1+\delta_2}\cdot v_2$	$t_1 \otimes u_2 - \beta(u_2,t_1)u_2 \otimes t_1$
$\varepsilon_1-\delta_2$	$Y_{-\delta_1-\delta_2}\cdot v_2$	$t_1 \otimes u_2^* - \beta(u_2^*,t_1)u_2^* \otimes t_1$
$-\varepsilon_1+\delta_1$	$Y_{-2\varepsilon_1}\cdot v_2$	$t_1^* \otimes u_1 - \beta(u_1,t_1^*)u_1 \otimes t_1^*$
$-\varepsilon_1-\delta_1$	$Y_{-\delta_1-\delta_2}\cdot(Y_{-\delta_1-\varepsilon_1}\cdot(Y_{\delta_2-\varepsilon_1}\cdot v_2))$	$t_1^* \otimes u_1^* - \beta(u_1^*,t_1^*)u_1^* \otimes t_1^*$
$-\varepsilon_1+\delta_2$	$Y_{-\delta_1-\varepsilon_1}\cdot(Y_{\delta_2-\varepsilon_1}\cdot v_2)$	$t_1^* \otimes u_2 - \beta(u_2,t_1^*)u_2 \otimes t_1^*$
$-\varepsilon_1-\delta_2$	$Y_{-\delta_1-\varepsilon_1}\cdot(Y_{-\delta_2-\varepsilon_1}\cdot v_2)$	$t_1^* \otimes u_2^* - \beta(u_2^*,t_1^*)u_2^* \otimes t_1^*$
$\delta_1+\delta_2$	$Y_{\delta_2-\varepsilon_1}\cdot v_2$	$u_1 \otimes u_2 - \beta(u_2,u_1)u_2 \otimes u_1$
$\delta_1-\delta_2$	$Y_{-\delta_2-\varepsilon_1}\cdot v_2$	$u_1 \otimes u_2^* - \beta(u_2^*,u_1)u_2^* \otimes u_1$
$-\delta_1+\delta_2$	$Y_{-\delta_1+\delta_2}\cdot(Y_{-\delta_1-\varepsilon_1}\cdot v_2)$	$u_1^* \otimes u_2 - \beta(u_2,u_1^*)u_2 \otimes u_1^*$
$-\delta_1-\delta_2$	$Y_{-\delta_1-\delta_2}\cdot(Y_{-\delta_1-\varepsilon_1}\cdot v_2)$	$u_1^* \otimes u_2^* - \beta(u_2^*,u_1^*)u_2^* \otimes u_1^*$
$2\delta_1$	$Y_{\delta_1-\varepsilon_1}\cdot v_2$	$u_1 \otimes u_1$
$-2\delta_1$	$Y_{-\delta_1-\delta_2}\cdot(Y_{-\delta_1+\delta_2}\cdot(Y_{-\delta_1-\varepsilon_1}\cdot v_2))$	$u_1^* \otimes u_1^*$
$2\delta_2$	$Y_{-\delta_1+\delta_2}\cdot(Y_{\delta_2-\varepsilon_1}\cdot v_2)$	$u_2 \otimes u_2$
$-2\delta_2$	$Y_{-\delta_1-\delta_2}\cdot(Y_{-\delta_2-\varepsilon_1}\cdot v_2)$	$u_2^* \otimes u_2^*$
0	$Y_{-\delta_1-\delta_2}\cdot(Y_{\delta_2-\varepsilon_1}\cdot v_2)$	$u_1^* \otimes u_1 + u_1 \otimes u_1^* - u_2 \otimes u_2^* - u_2^* \otimes u_2$
0	$Y_{-\delta_1-\varepsilon_1}\cdot v_2$	$-t_1 \otimes t_1^* + t_1^* \otimes t_1 + u_1^* \otimes u_1 + u_1 \otimes u_1^*$

Table 5.3: A basis of $\mathfrak{spo}(V,\beta)$-submodule generated by the highest weight vector v_2. The third column records all possible vectors obtained by acting negative root vectors on v_2. The first column records the corresponding weights of the vectors in the third column, and the second column records how the negative root vectors act on v_2 in order to produce the corresponding vectors in the third column.

It is routine to verify that the vectors we found in Table 5.3 are linearly independent. We have also verified that for each $w \in W_{\square}$, there exists $X \in \mathfrak{U}(\mathfrak{g})$ such that Xw is the highest weight vector, whence $W_{\square}$ is irreducible. Moreover, by counting the number of vectors in Table 5.4, we have $\dim(W_{\square}) = 17$.

We then draw Figure 5.3, which represents a three-dimensional space with ε_1-axis, δ_1-axis and δ_2-axis so that we can see the symmetry of the weights. In particular, we use two circles around the origin $(0, 0, 0)$ to indicate that there are 2 linearly independent zero weight vectors.

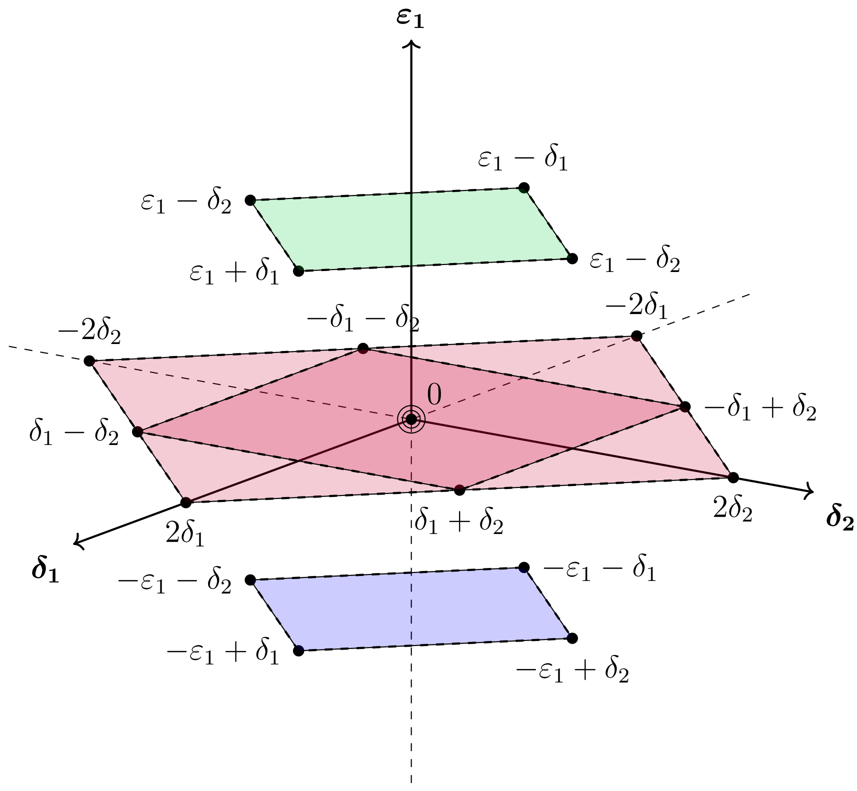


Figure 5.3: Weights in the submodule generated by v_2, where different colours represent different hyperplanes.

Finally let us consider the case of v_3. Again Table 5.4 records a basis for $\mathfrak{U}(\mathfrak{g})v_3 = W_{\square\square}$, the weights of these basis vectors, and the actions of negative root vectors on v_3 in order to produce the resulting basis vectors. Figure 5.4 shows the weights of the basis vectors of the $\mathfrak{spo}(V, \beta)$-module $W_{\square\square}$.

weights	actions	vectors
$2\varepsilon_1$	v_3	$t_1 \otimes t_1$
$-2\varepsilon_1$	$Y_{-2\varepsilon_1} \cdot Y_{-2\varepsilon_1} \cdot v_3$	$t_1^* \otimes t_1^*$
$\varepsilon_1 + \delta_1$	$Y_{\delta_1 - \varepsilon_1} \cdot v_3$	$t_1 \otimes u_1 + \beta(u_1, t_1) u_1 \otimes t_1$
$\varepsilon_1 - \delta_1$	$Y_{-\delta_1 - \varepsilon_1} \cdot v_3$	$t_1 \otimes u_1^* + \beta(u_1^*, t_1) u_1^* \otimes t_1$
$\varepsilon_1 + \delta_2$	$Y_{\delta_2 - \varepsilon_1} \cdot v_3$	$t_1 \otimes u_2 + \beta(u_2, t_1) u_2 \otimes t_1$
$\varepsilon_1 - \delta_2$	$Y_{-\delta_2 - \varepsilon_1} \cdot v_3$	$t_1 \otimes u_2^* + \beta(u_2^*, t_1) u_2^* \otimes t_1$
$-\varepsilon_1 + \delta_1$	$Y_{\delta_1 - \varepsilon_1} \cdot (Y_{-2\varepsilon_1} \cdot v_3)$	$t_1^* \otimes u_1 + \beta(u_1, t_1^*) u_1 \otimes t_1^*$
$-\varepsilon_1 - \delta_1$	$Y_{-\delta_1 - \varepsilon_1} \cdot (Y_{-2\varepsilon_1} \cdot v_3)$	$t_1^* \otimes u_1^* + \beta(u_1^*, t_1^*) u_1^* \otimes t_1^*$
$-\varepsilon_1 + \delta_2$	$Y_{\delta_2 - \varepsilon_1} \cdot (Y_{-2\varepsilon_1} \cdot v_3)$	$t_1^* \otimes u_2 + \beta(u_2, t_1^*) u_2 \otimes t_1^*$
$-\varepsilon_1 - \delta_2$	$Y_{-\delta_2 - \varepsilon_1} \cdot (Y_{-2\varepsilon_1} \cdot v_3)$	$t_1^* \otimes u_2^* + \beta(u_2^*, t_1^*) u_2^* \otimes t_1^*$
$\delta_1 + \delta_2$	$Y_{\delta_2 - \varepsilon_1} \cdot (Y_{\delta_1 - \varepsilon_1} \cdot v_3)$	$u_1 \otimes u_2 + \beta(u_2, u_1) u_2 \otimes u_1$
$\delta_1 - \delta_2$	$Y_{-\delta_2 - \varepsilon_1} \cdot (Y_{-\delta_1 - \varepsilon_1} \cdot v_3)$	$u_1 \otimes u_2^* + \beta(u_2^*, u_1) u_2^* \otimes u_1$
$-\delta_1 + \delta_2$	$Y_{-\delta_1 + \delta_2} \cdot (Y_{-\delta_1 - \varepsilon_1} \cdot (Y_{\delta_1 - \varepsilon_1} \cdot v_3))$	$u_1^* \otimes u_2 + \beta(u_2, u_1^*) u_2 \otimes u_1^*$
$-\delta_1 - \delta_2$	$Y_{-\delta_1 - \delta_2} \cdot (Y_{-\delta_1 - \varepsilon_1} \cdot (Y_{\delta_1 - \varepsilon_1} \cdot v_3))$	$u_1^* \otimes u_2^* + \beta(u_2^*, u_1^*) u_2^* \otimes u_1^*$
0	$Y_{-2\varepsilon_1} \cdot v_3$	$t_1^* \otimes t_1 + t_1 \otimes t_1^*$
0	$Y_{-\delta_1 - \varepsilon_1} \cdot (Y_{\delta_1 - \varepsilon_1} \cdot v_3)$	$u_1^* \otimes u_1 - u_1 \otimes u_1^* - t_1 \otimes t_1^* - t_1^* \otimes t_1$
0	$Y_{-\delta_2 - \varepsilon_1} \cdot (Y_{\delta_2 - \varepsilon_1} \cdot v_3)$	$u_2^* \otimes u_2 - u_2 \otimes u_2^* - t_1 \otimes t_1^* - t_1^* \otimes t_1$

Table 5.4: the $\mathfrak{spo}(V, \beta)$-submodule generated by the highest weight vector v_3

5.5.4 Summary: a decomposition of $V \otimes V$ as $\mathfrak{spo}(V, \beta)$-modules when $n = 2, m = 4$ and $k = 2$

From the calculations, we see that $W_{\varnothing}$ is the trivial modules. Moreoever, $W_{\substack{\square\\\square}}$ is spanned by the vectors which are linear combination of vectors of the form

$$x \otimes y - \beta(y, x) y \otimes x$$

for all $x, y \in B$, and $W_{\square\square}$ is spanned by the vectors of the form

$$x \otimes y + \beta(y, x) y \otimes x$$

for all $x, y \in B$. Thus $W_{\varnothing}$, $W_{\substack{\square\\\square}}$ and $W_{\square\square}$ have pairwise trivial intersection. The dimension of $W_{\varnothing}$, $W_{\substack{\square\\\square}}$ and $W_{\square\square}$ are 1, 18 and 17 respectively. Therefore we have $V \otimes V$ decomposes into a direct sum of irreducible $\mathfrak{spo}(V, \beta)$-modules

$$V \otimes V \cong W_{\varnothing} \oplus W_{\substack{\square\\\square}} \oplus W_{\square\square},$$

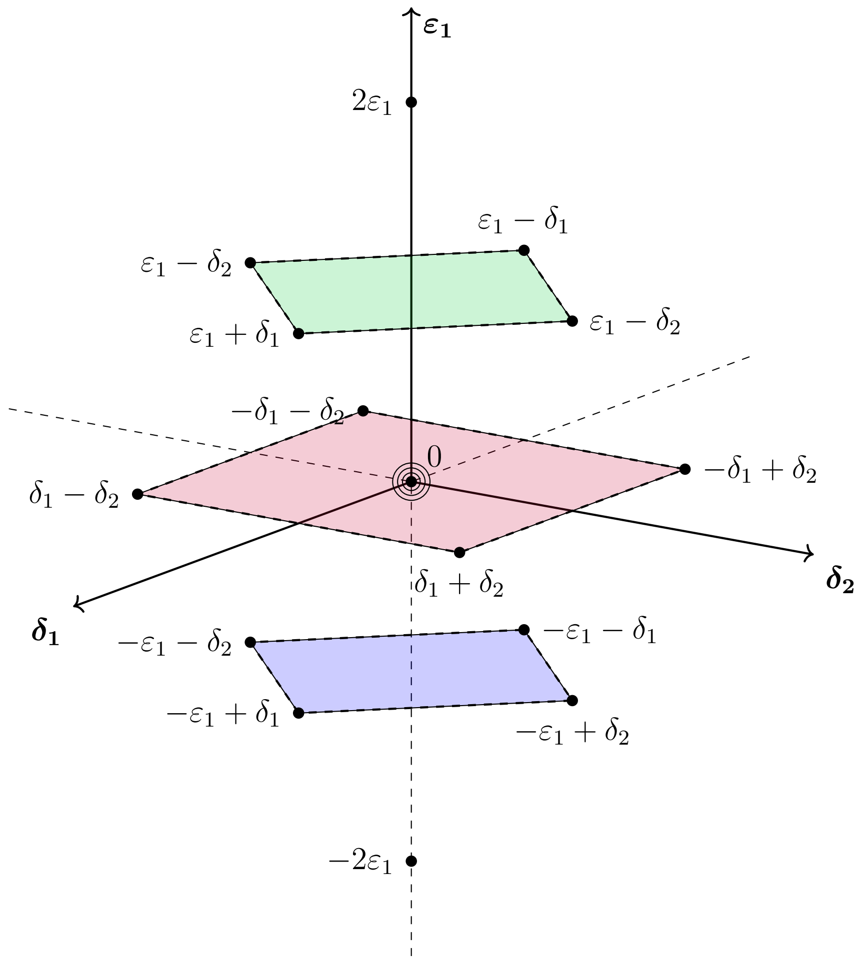


Figure 5.4: Weights of the basis vectors of the submodule generated by v_3.

We will extend the discussion in this section further to a more general case when $|n - m| = k = 2$. We will analyze this decomposition further with respect to the Brauer algebra action in Section 6.1.

Chapter 6

Formulas for the characters of $\mathfrak{spo}(V,\beta)$-modules in some borderline cases

In this chapter, we first state the $\mathfrak{spo}(V,\beta) \times B_k$ decomposition provided in [BSR98]. Then we examine reducibility properties of some borderline cases thoroughly. We then compute the characters of the irreducible summands. Besides the results stated in Section 6.1, Definition 6.5.3 and Definition 6.5.5, the results in this chapter are our own.

6.1 Schur-Weyl duality-like decomposition of $V^{\otimes k}$

By [AB95, Section 13, Theorem 18], a finite dimensional algebra A over $\mathbb{C}$ is semisimple if it is isomorphic to a direct sum of matrix algebras over $\mathbb{C}$. In [BSR98, Section 4], Benkart et. al. stated that the Brauer algebra $B_k(n-m)$ is semisimple if $|n-m| > k$.

When the Brauer algebra $B_k(n-m)$ is semisimple, by the work of [Wen88], it can be decomposed into a direct sum of matrix algebras such that the simple summands are indexed by partitions. In particular, there exists an isomorphism of algebras

$$\Theta : B_k(n-m) \to \bigoplus_{\lambda \in P_k} M_{d_\lambda}(\mathbb{F}), \tag{6.1.1}$$

where

$$P_k := \{\lambda \vdash k - 2h \mid h = 0, 1, \ldots, \lfloor k/2 \rfloor\}, \tag{6.1.2}$$

and $M_{d_\lambda}(\mathbb{F})$ is the set of $d_\lambda \times d_\lambda$ matrix with entries in $\mathbb{F}$ for some nonnegative integers d_λ depending on $\lambda \in P_k$.

For each $\lambda \in P_k$, and $1 \leq i, j \leq d_\lambda$, let $E_{\lambda,i,j}$ be the matrix in the λ^{th} block of $\bigoplus_{\lambda \in P_k} M_{d_\lambda}(\mathbb{F})$ with 1 at the (i, j) entry and 0 elsewhere. We denote $e_{\lambda,i,j} = \Theta^{-1}(E_{\lambda,i,j})$. Then $e_{\lambda,i,j}$ is an element in $B_k(n-m)$. We construct a $\mathfrak{spo}(V, \beta)$-module by the following lemma.

Lemma 6.1.1 ([BSR98, Equation (4.1)]). *Let $V^{\otimes k}$ be the tensor product of standard $\mathfrak{spo}(V, \beta)$-modules. Let $\lambda \in P_k$, and let λ' be the transpose of λ. Then*

$$U^{Y(\lambda')} := V^{\otimes k} e_{\lambda,i,j}$$

is an $\mathfrak{spo}(V, \beta)$-module.

Proof. Take $x \in \mathfrak{spo}(V, \beta)$. Since the action of the Brauer algebra $B_k(n-m)$ commutes with the action of $\mathfrak{spo}(V, \beta)$ on $V^{\otimes k}$, we have

$$xU^{Y(\lambda')} = x(V^{\otimes k} e_{\lambda,i,j}) = (xV^{\otimes k}) e_{\lambda,i,j} \subseteq V^{\otimes k} e_{\lambda,i,j} = U^{Y(\lambda')}.$$

Therefore $U^{Y(\lambda')}$ is an $\mathfrak{spo}(V,\beta)$-module. $\square$

Note that $U^{Y(\lambda')}$ can be the trivial vector space. We will give an example such that $U^{Y(\lambda')} = \{0\}$ in Example 6.4.1.

Theorem 6.1.2 ([BSR98, Proposition 4.2])**.** *Let $|n-m| > k$. Then we have*

(i) The $\mathfrak{spo}(V,\beta)$-module $U^{Y(\lambda')}$ is independent of the choice of i, j.

(ii) As an $\mathfrak{spo}(V,\beta) \times B_k(n-m)$ module, we have

$$V^{\otimes k} \cong \bigoplus_{\lambda \in P_k} U^{Y(\lambda')} \otimes B_\lambda,$$

where B_λ is the irreducible $B_k(n-m)$-module labelled by the partition $\lambda \in P_k$.

However, Benkart et. al. did not claim or show anything about the irreducibility of $U^{Y(\lambda')}$. In fact in [BSR98, Section 0.2(d)], they mention that $U^{Y(\lambda')}$ is in general not necessarily irreducible.

Moreover, the relation between $U^{Y(\lambda')}$ and the highest weight vectors produced in Theorem 5.4.1 is not clear from [BSR98]. They did not indicate whether or not the highest weight vectors $w_{T,\underline{p},\underline{q}} C_{\underline{p},\underline{q}} y_T$, with T of shape λ' lie in the appropriate submodules $U^{Y(\lambda')}$.

In the next section, we extend the discussion in Section 5.5 to a more general case where $|n-m| = k = 2$ for all possible $n, m > 1$. Then we verify that in this special case, Theorem 6.1.2 still holds, and relate the highest weight modules from Theorem 5.4.1 to these $U^{Y(\lambda)}$'s.

6.2 The Brauer algebra $B_2(\eta)$

In this section, we first show that $B_2(\eta)$ is semisimple whenever $\eta \neq 0$ by showing that $B_2(\eta)$ is abelian and there are exactly three irreducible 1-dimensional modules. We then explicitly show that $B_2(0)$ is not semisimple by giving an indecomposable but not irreducible module. We give a matrix realization of $B_2(\eta)$ for all η.

Recall from Definition 4.1.5 that for all $\eta \in \mathbb{F}$, $B_2(\eta)$ has dimension 3. Let ι be the identity element in $B_2(\eta)$. Then we have

$$B_2(\eta) = \text{span}\{\iota, e_1, s_1\}.$$

Also recall from Proposition 4.1.6, the relations determining $B_2(\eta)$ are given by

$$e_1 s_1 = s_1 e_1 = e_1, \qquad s_1 s_1 = \iota, \qquad e_1 e_1 = \eta e_1.$$

We deduce that $B_2(\eta)$ is an abelian algebra. Therefore $B_2(\eta)$ is semisimple if and only if it has exactly three (irreducible) 1-dimensional modules.

Now suppose that $\eta \neq 0$. We find all three 1-dimensional $B_2(\eta)$-modules. Let (ϕ, V) be an irreducible $B_2(\eta)$-module. Then since $B_2(\eta)$ is abelian, we can check that both the image and kernel of $\phi(e_1)$ are invariant under $B_2(\eta)$. Thus, since ϕ is irreducible, we deduce that either $\phi(e_1)$ is surjective or $\phi(e_1)$ is the zero map.

(i) If $\phi(e_1)$ is surjective, then every element of V can be written as ve_1 for some $v \in V$. Thus, the relation $e_1 s_1 = e_1$ implies that s_1 acts by 1. The relation $e_1^2 = \eta e_1$ implies that e_1 acts by η. Moreover, since ι is the identity diagram, ι

acts by 1. Thus as an irreducible $B_2(\eta)$-module, we have

$$v\iota = v, \qquad vs_1 = v, \qquad ve_1 = \eta e_1. \tag{6.2.1}$$

(ii) Now if $\phi(e_1)$ is the zero map, the only case we need to consider is how s_1 acts. By the relation $s_1^2 = \iota$, we deduce that s_1 acts by either 1 or -1. Thus we have the following irreducible $B_2(\eta)$-modules.

$$v\iota = v, \qquad vs_1 = v, \qquad ve_1 = 0. \tag{6.2.2}$$

$$v\iota = v, \qquad vs_1 = -v, \qquad ve_1 = 0. \tag{6.2.3}$$

Notice that the modules in (6.2.2) and (6.2.3) are the same as the trivial module (sometimes denoted as $S_{\square\square}$) and the sign module (sometimes denoted as $S_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}$) of S_2 respectively.

We denote the modules we defined in (6.2.1), (6.2.2) and (6.2.3) as

$$B_{\varnothing}, \; B_{\square\square} \; \text{ and } \; B_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \tag{6.2.4}$$

respectively.

Since $B_2(\eta)$ is semisimple for all $\eta \neq 0$, it can be decomposed into a direct sum of these three 1-dimensional subalgebras. Let M be the set of 3×3 diagonal matrices.

Using the modules we found above, the map $\Theta : B_2(\eta) \to M$ such that

$$\Theta(\iota) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \Theta(e_1) = \begin{pmatrix} \eta & & \\ & 0 & \\ & & 0 \end{pmatrix}, \Theta(s_1) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

is an algebra isomorphism.

In particular, using this matrix realization, we determine that $e_{\varnothing,1,1} = \frac{1}{\eta}e_1$, $e_{(2),1,1} = -\frac{1}{\eta}e_1 + \frac{1}{2}(s_1 + id)$ and $e_{(1,1),1,1} = \frac{1}{2}(id - s_1)$.

Now we consider the Brauer algebra $B_2(0)$ which is a 3-dimensional but not semisimple algebra. In this case, (6.2.3) still gives an irreducible 1-dimensional module, $B_{\substack{\square\\\square}}$. However, (6.2.1) and (6.2.2) are the same. In fact, we have a 2-dimensional indecomposable $B_2(0)$-module, which we will denote $B_{\varnothing'}$ with basis $\{v, w\}$ such that

$$\begin{aligned} &v\iota = v, && vs_1 = v, && ve_1 = w. \\ &w\iota = w, && ws_1 = w, && we_1 = 0. \end{aligned} \tag{6.2.5}$$

Note that $B_{\varnothing'}$ contains $B_\varnothing = B_{\square\square}$ as a submodule. The corresponding isomorphism $\Theta : B_2(0) \to M$ is given by

$$\Theta(\iota) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \Theta(e_1) = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}, \Theta(s_1) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

6.3 The decomposition of $V \otimes V$

We are now ready to extend the discussion in Section 5.5. Let $\mathfrak{g} = \mathfrak{spo}(V, \beta)$ and $V = V_{(0)} \oplus V_{(1)}$. Let $\dim(V_{(0)}) = m$ and $\dim(V_{(1)}) = n$ such that $|n - m| = 2 = k$. Note that this is a case which is not covered by the hypothesis of Theorem 6.1.2. We modify our calculation in Section 5.5 to find the highest weight vectors in $V \otimes V$ and the submodules generated by them. We prove that we can decompose $V \otimes V$ into a direct sum of three irreducible $\mathfrak{spo}(V, \beta) \times B_2(n - m)$-submodules when $|n - m| = 2$ and $n, m > 1$.

Theorem 6.3.1. *Let $r, s \geq 1$. Let $m = 2r$ and $n = 2s$ be such that $|n - m| = 2$. Then there are three linearly independent highest weight vectors in $V \otimes V$.*

Proof. First notice that if $r = 1$, then $s = 2$. We have already discussed this case in Section 5.5. Therefore we can restrict ourselves to consider the case when $r > 1$.

Since $k = 2$, the set of all possible indices $(\underline{p}, \underline{q})$ of contraction maps is given by

$$\mathcal{P} = \{(\varnothing, \varnothing), (1, 2)\}.$$

By a similar argument as in Section 5.5, when $(\underline{p}, \underline{q}) = (1, 2)$, the highest weight vector is given by

$$v_1 = \sum_{i=1}^{r} (t_i^* \otimes t_i - t_i \otimes t_i^*) + \sum_{j=1}^{s} (u_j \otimes u_j^* + u_j^* \otimes u_j) = \Omega. \qquad (6.3.1)$$

Let us now consider $(\underline{p}, \underline{q}) = (\varnothing, \varnothing)$. We have $C_{\underline{p},\underline{q}} = \Psi_{id}$. Moreover, the set of possible T's in $\Gamma_{r,s}(\{1,2\})$ is

$$T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \text{ with } T_{(0)} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \text{ and } T_{(1)} = \varnothing, \text{ or} \tag{6.3.2}$$

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \text{ with } T_{(0)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \text{ and } T_{(1)} = \varnothing. \tag{6.3.3}$$

Let T be defined in (6.3.2). In this case, since 1 is in the first row of $T_{(0)}$, and 2 is in the second row of $T_{(0)}$, we have $w_{T,\varnothing,\varnothing} = t_1 \otimes t_2$. Therefore the highest weight vector is $(t_1 \otimes t_2)\Psi_{id}(\Psi_{id} + \Psi_{s_1})$ which is

$$t_1 \otimes t_2 - \beta(t_2, t_1) t_2 \otimes t_1.$$

Similarly, for T defined in (6.3.3), the highest weight vector is $t_1 \otimes t_1$.

In summary, if $r = 1$, we have three linearly independent highest weight vectors as we discussed in Section 5.5. If $r > 1$, the linearly independent highest weight vectors are given by

$$v_1 = \Omega, \tag{v_1}$$

$$v_2 = t_1 \otimes t_2 - \beta(t_2, t_1) t_2 \otimes t_1, \tag{v_2}$$

$$v_3 = t_1 \otimes t_1. \tag{v_3}$$

Thus in either case, we have three independent highest weight vectors in total. □

Similarly as in Section 5.5, we generate bases for the corresponding modules via an application of the PBW theorem (Corollary 3.4.7). Let us define $W^{\varnothing} = \mathfrak{U}(\mathfrak{g}) v_1$, $W^{\substack{\square\\\square}} = \mathfrak{U}(\mathfrak{g}) v_2$ and $W^{\square\square} = \mathfrak{U}(\mathfrak{g}) v_3$.

We first summarize the list of all of the negative root vectors in Table 6.1. Then we use extra tables to record the actions of the negative root vectors on each of the highest weight vectors, and the weights of the resulting vectors.

notation	negative root vector	conditions
$Y_{-2\varepsilon_i}$	$E_{t_i^*,t_i}$	$1 \le i \le r$
$Y_{-\varepsilon_i+\varepsilon_j}$	$E_{t_j,t_i} - \beta(t_j,t_i)E_{t_i^*,t_j^*}$	$1 \le i < j \le r$
$Y_{-\varepsilon_i-\varepsilon_j}$	$E_{t_i^*,t_j} + \beta(t_j^*,t_i)E_{t_j^*,t_i}$	$1 \le i < j \le r$
$Y_{-\varepsilon_i+\delta_j}$	$E_{u_j,t_i} + \beta(u_j,t_i)E_{t_i^*,u_j^*}$	$1 \le i \le r, 1 \le j \le s$
$Y_{-\varepsilon_i-\delta_j}$	$E_{u_j^*,t_i} - \beta(u_j^*,t_i)E_{t_i^*,u_j}$	$1 \le i \le r, 1 \le j \le s$
$Y_{-\delta_k+\delta_\ell}$	$E_{u_\ell,u_k} + \beta(u_\ell,u_k)E_{u_k^*,u_\ell^*}$	$1 \le k < \ell \le s$
$Y_{-\delta_k-\delta_\ell}$	$E_{u_k^*,u_\ell} + \beta(u_k^*,u_\ell)E_{u_\ell^*,u_k}$	$1 \le k < \ell \le s$

Table 6.1: The negative root vectors when $m = 2r > 2$ and $n = 2s \ge 2$. Notice that if $s = 1$, then we do not have the last two rows of this table.

First, since Ω is $\mathfrak{spo}(V,\beta)$-invariant, v_1 generates the trivial module, $W^{\varnothing}$, and we have

$$\dim(W^{\varnothing}) = 1. \tag{6.3.4}$$

Let us now consider the case when the highest weight vector is v_2. We group different weights vectors by different tables (Table 6.2 to 6.6).

weights	actions	vectors	conditions
$\varepsilon_1+\varepsilon_2$	v_2	$t_1 \otimes t_2 - \beta(t_2,t_1)t_2 \otimes t_1$	
$\varepsilon_i+\varepsilon_j$	$Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_2+\varepsilon_j} \cdot v_2$	$t_j \otimes t_i - \beta(t_i,t_j)t_i \otimes t_j$	$1 < j \le r$
$\varepsilon_1+\varepsilon_j$	$Y_{-\varepsilon_2+\varepsilon_j} \cdot v_2$	$t_j \otimes t_1 - \beta(t_1,t_j)t_1 \otimes t_j$	$1 < j \le r$
$\varepsilon_1-\varepsilon_j$	$Y_{-\varepsilon_2-\varepsilon_j} \cdot v_2$	$t_j^* \otimes t_1 - \beta(t_1,t_j^*)t_1 \otimes t_j^*$	$1 < j \le r$
$\varepsilon_i-\varepsilon_j$	$Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_2-\varepsilon_j} \cdot v_2$	$t_j^* \otimes t_i - \beta(t_i,t_j^*)t_i \otimes t_j^*$	$1 < i < j \le r$
$-\varepsilon_1+\varepsilon_j$	$Y_{-2\varepsilon_1} \cdot Y_{-\varepsilon_2+\varepsilon_j} \cdot v_2$	$t_j \otimes t_1^* - \beta(t_1^*,t_j)t_1^* \otimes t_j$	$1 < j \le r$
$-\varepsilon_i+\varepsilon_j$	$Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-2\varepsilon_1} \cdot Y_{-\varepsilon_2+\varepsilon_j} \cdot v_2$	$t_j \otimes t_i^* - \beta(t_i^*,t_j)t_i^* \otimes t_j$	$1 \le i < j \le r$
$-\varepsilon_1-\varepsilon_j$	$Y_{-2\varepsilon_1} \cdot Y_{-\varepsilon_2-\varepsilon_j} \cdot v_2$	$t_j^* \otimes t_1^* - \beta(t_1^*,t_j^*)t_1^* \otimes t_j^*$	$1 < j \le r$
$-\varepsilon_i-\varepsilon_j$	$Y_{-\varepsilon_1-\varepsilon_i} \cdot Y_{-\varepsilon_2-\varepsilon_j} \cdot v_2$	$t_j^* \otimes t_i^* - \beta(t_i^*,t_j^*)t_i^* \otimes t_j^*$	$1 < i < j \le r$

Table 6.2: The vectors of weights $\pm\varepsilon_i \pm \varepsilon_j$ for all $1 \le i < j \le r$ in $\mathfrak{U}(\mathfrak{g})v_2$.

weights	actions	vectors	conditions
$2\delta_\ell$	$Y_{-\varepsilon_2+\delta_l} \cdot Y_{-\varepsilon_1+\delta_l} \cdot v_2$	$u_\ell \otimes u_\ell$	$1 \le \ell \le s$
$-2\delta_\ell$	$Y_{-\varepsilon_2-\delta_l} \cdot Y_{-\varepsilon_1-\delta_l} \cdot v_2$	$u_\ell^* \otimes u_\ell^*$	$1 \le \ell \le s$

Table 6.3: The vectors of weights $\pm 2\delta_\ell$ for all $1 \le \ell \le r$ in $\mathfrak{U}(\mathfrak{g})v_2$.

weights	actions	vectors	conditions
$\varepsilon_1+\delta_\ell$	$Y_{-\varepsilon_2+\delta_l}\cdot v_2$	$t_1\otimes u_\ell-\beta(u_\ell,t_1)u_\ell\otimes t_1$	$1\le i\le r, 1\le \ell\le s$
$\varepsilon_i+\delta_\ell$	$Y_{-\varepsilon_1+\varepsilon_i}\cdot Y_{-\varepsilon_2+\delta_l}\cdot v_2$	$t_i\otimes u_\ell-\beta(u_\ell,t_i)u_\ell\otimes t_i$	$1\le i\le r, 1\le \ell\le s$
$\varepsilon_1-\delta_\ell$	$Y_{-\varepsilon_2-\delta_l}\cdot v_2$	$t_1\otimes u_\ell^*-\beta(u_\ell^*,t_1)u_\ell^*\otimes t_1$	$1\le i\le r, 1\le \ell\le s$
$\varepsilon_i+\delta_\ell$	$Y_{-\varepsilon_1+\varepsilon_i}\cdot Y_{-\varepsilon_2-\delta_l}\cdot v_2$	$t_i\otimes u_\ell^*-\beta(u_\ell^*,t_i)u_\ell^*\otimes t_i$	$1\le i\le r, 1\le \ell\le s$
$-\varepsilon_i+\delta_\ell$	$Y_{-\varepsilon_i-\delta_l}\cdot Y_{-\varepsilon_2+\delta_l}\cdot Y_{-\varepsilon_1+\delta_l}\cdot v_2$	$t_i^*\otimes u_\ell-\beta(u_\ell,t_i^*)u_\ell\otimes t_i^*$	$1\le i\le r, 1\le \ell\le s$
$-\varepsilon_i-\delta_\ell$	$Y_{-\varepsilon_i+\delta_l}\cdot Y_{-\varepsilon_2-\delta_l}\cdot Y_{-\varepsilon_1-\delta_l}\cdot v_2$	$t_i^*\otimes u_\ell^*-\beta(u_\ell^*,t_i^*)u_\ell^*\otimes t_i^*$	$1\le i\le r, 1\le \ell\le s$

Table 6.4: The vectors of weights $\pm\varepsilon_i\pm\delta_\ell$ for all $1\le i\le r, 1\le \ell\le s$ in $\mathfrak{U}(\mathfrak{g})v_2$.

weights	actions	vectors	conditions
$\delta_k+\delta_\ell$	$Y_{-\varepsilon_i+\delta_k}\cdot Y_{-\varepsilon_1+\varepsilon_i}\cdot Y_{-\varepsilon_2+\delta_l}\cdot v_2$	$u_k\otimes u_\ell-\beta(u_\ell,u_k)u_\ell\otimes u_k$	$1\le k<\ell\le s$
$-\delta_k-\delta_\ell$	$Y_{-\varepsilon_2-\delta_k}\cdot Y_{-\varepsilon_1-\delta_l}\cdot v_2$	$u_k^*\otimes u_\ell^*-\beta(u_\ell^*,u_k^*)u_\ell^*\otimes u_k^*$	$1\le k<\ell\le s$
$-\delta_k+\delta_\ell$	$Y_{-\varepsilon_i-\delta_k}\cdot Y_{-\varepsilon_1+\varepsilon_i}\cdot Y_{-\varepsilon_2+\delta_l}\cdot v_2$	$u_k^*\otimes u_\ell-\beta(u_\ell,u_k^*)u_\ell\otimes u_k^*$	$1\le k\ne\ell\le s$
$\delta_k-\delta_\ell$	$Y_{-\varepsilon_1+\delta_k}\cdot Y_{-\varepsilon_2-\delta_l}\cdot v_2$	$u_k\otimes u_\ell^*-\beta(u_\ell^*,u_k)u_\ell^*\otimes u_k$	$1\le k\ne\ell\le s$

Table 6.5: The vectors of weights $\pm\delta_k\pm\delta_\ell$ for all $1\le k<\ell\le r$ in $\mathfrak{U}(\mathfrak{g})v_2$.

weights	actions	vectors	conditions
0	$Y_{-\varepsilon_i-\delta_l}\cdot Y_{-\varepsilon_1+\varepsilon_i}\cdot Y_{-\varepsilon_2+\delta_l}\cdot v_2$	$u_\ell^*\otimes u_\ell+u_\ell\otimes u_\ell^*-t_i\otimes t_i^*+t_i^*\otimes t_i$	$1\le i\le r$, $1\le \ell\le s$
0	$Y_{-\delta_k-\delta_l}\cdot Y_{-\varepsilon_i+\delta_k}\cdot Y_{-\varepsilon_1+\varepsilon_i}\cdot Y_{-\varepsilon_2+\delta_l}\cdot v_2$	$u_\ell^*\otimes u_\ell+u_\ell\otimes u_\ell^*-u_k^*\otimes u_k-u_k\otimes u_k^*$	$1\le k<\ell\le s$

Table 6.6: The vectors of weight 0 in $\mathfrak{U}(\mathfrak{g})v_2$.

If we apply negative root vectors to the vectors in Table 6.2 to 6.6, we will get another vectors in these tables or 0. Thus we deduce that $\mathfrak{U}(\mathfrak{g})v_2$ is spanned by the vactors in Table 6.2 to 6.6. Moreover we deduce that each nonzero weight space has dimension 1. The vectors of weight 0 in Table 6.6 are linearly dependent, but we find a linearly independent subset and conclude that the 0 weight space is spanned by the vectors

$$u_\ell^*\otimes u_\ell+u_\ell\otimes u_\ell^*-u_1^*\otimes u_1-u_1\otimes u_1^*, \tag{6.3.5}$$

for $1\le \ell\le s$ and the vectors

$$-t_i\otimes t_i^*+t_i^*\otimes t_i+u_1^*\otimes u_1+u_1\otimes u_1^* \tag{6.3.6}$$

for $1\le i\le r$. Therefore the 0 weight space has dimension $r+s-1$. By summarizing the weights appearing in Tables 6.2 to 6.6, we find that the nonzero weights are

$$\{\pm 2\varepsilon_i \mid 1 \leq i \leq r\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq r\}$$
$$\cup \{\pm\delta_k \pm \delta_\ell \mid 1 \leq k, \ell \leq s\} \cup \{\pm\varepsilon_i \pm \delta_\ell \mid 1 \leq i \leq r, 1 \leq \ell \leq s\}. \quad (6.3.7)$$

Thus by counting, we have

$$\dim(W^{\square}) = m^2 + mn + n^2 + \frac{n-m}{2} - 1. \quad (6.3.8)$$

Finally, let us consider the highest weight vector v_3. In summary, we put the necessary information in Tables 6.7 to 6.11.

weights	actions	vectors	conditions
$2\varepsilon_1$	v_1	$t_1 \otimes t_1$	
$-2\varepsilon_1$	$Y_{-2\varepsilon_1} \cdot Y_{-2\varepsilon_1} \cdot v_1$	$t_1^* \otimes t_1^*$	
$2\varepsilon_i$	$Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$t_i \otimes t_i$	$1 < i \leq r$
$-2\varepsilon_i$	$Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_1-\varepsilon_i} \cdot v_1$	$t_i^* \otimes t_i^*$	$1 < i \leq r$

Table 6.7: The vectors of weights $\pm 2\varepsilon_i$ for all $1 \leq i \leq r$ in $\mathfrak{U}(\mathfrak{g})v_3$.

weights	actions	vectors	conditions
$\varepsilon_i + \varepsilon_j$	$Y_{-\varepsilon_i+\varepsilon_j} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$t_i \otimes t_j + \beta(t_j, t_i)t_i \otimes t_j$	$1 \leq i < j \leq r$
$\varepsilon_i - \varepsilon_j$	$Y_{-\varepsilon_i-\varepsilon_j} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$t_i \otimes t_j^* + \beta(t_j^*, t_i)t_j^* \otimes t_i$	$1 \leq i < j \leq r$
$-\varepsilon_i + \varepsilon_j$	$Y_{-\varepsilon_i+\varepsilon_j} \cdot Y_{-2\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$t_i^* \otimes t_j + \beta(t_j, t_i^*)t_j \otimes t_i^*$	$1 \leq i < j \leq r$
$-\varepsilon_i - \varepsilon_j$	$Y_{-\varepsilon_i-\varepsilon_j} \cdot Y_{-2\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$t_i^* \otimes t_j^* + \beta(t_j^*, t_i^*)t_j^* \otimes t_i^*$	$1 \leq i < j \leq r$

Table 6.8: The vectors of weights $\pm\varepsilon_i \pm \varepsilon_j$ for all $1 \leq i < j \leq r$ in $\mathfrak{U}(\mathfrak{g})v_3$.

weights	actions	vectors	conditions
$\varepsilon_i + \delta_\ell$	$Y_{-\varepsilon_i+\delta_l} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$t_i \otimes u_\ell + \beta(u_\ell, t_i)u_\ell \otimes t_i$	$1 \leq i \leq r, 1 \leq \ell \leq s$
$\varepsilon_i - \delta_\ell$	$Y_{-\varepsilon_i-\delta_l} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$t_i \otimes u_\ell^* + \beta(u_\ell^*, t_i)u_\ell^* \otimes t_i$	$1 \leq i \leq r, 1 \leq \ell \leq s$
$-\varepsilon_i + \delta_\ell$	$Y_{-\varepsilon_i+\delta_l} \cdot Y_{-2\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$t_i^* \otimes u_\ell + \beta(u_\ell, t_i^*)u_\ell \otimes t_i^*$	$1 \leq i \leq r, 1 \leq \ell \leq s$
$-\varepsilon_i - \delta_\ell$	$Y_{-\varepsilon_i-\delta_l} \cdot Y_{-2\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$t_i^* \otimes u_\ell^* + \beta(u_\ell^*, t_i^*)u_\ell^* \otimes t_i^*$	$1 \leq i \leq r, 1 \leq \ell \leq s$

Table 6.9: The vectors of weights $\pm\varepsilon_i \pm \delta_j$ for all $1 \leq i \leq r, 1 \leq \ell \leq s$ in $\mathfrak{U}(\mathfrak{g})v_3$.

weights	actions	vectors	conditions
$\delta_k + \delta_\ell$	$Y_{-\varepsilon_i+\delta_k} \cdot Y_{-\varepsilon_i+\delta_l} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$u_k \otimes u_\ell + \beta(u_\ell, u_k)u_\ell \otimes u_k$	$1 \leq k < l \leq s$
$\delta_k - \delta_\ell$	$Y_{-\varepsilon_i+\delta_k} \cdot Y_{-\varepsilon_i-\delta_l} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$u_k \otimes u_\ell^* + \beta(u_\ell^*, u_k)u_\ell^* \otimes u_k$	$1 \leq k < l \leq s$
$-\delta_k + \delta_\ell$	$Y_{-\delta_k+\delta_l} \cdot Y_{-\varepsilon_i+\delta_l} \cdot Y_{-\varepsilon_i-\delta_l} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$u_k^* \otimes u_\ell + \beta(u_\ell, u_k^*)u_\ell \otimes u_k^*$	$1 \leq k < l \leq s$
$-\delta_k - \delta_\ell$	$Y_{-\delta_k-\delta_l} \cdot Y_{-\varepsilon_i+\delta_l} \cdot Y_{-\varepsilon_i-\delta_l} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot Y_{-\varepsilon_1+\varepsilon_i} \cdot v_1$	$u_k^* \otimes u_\ell^* + \beta(u_\ell^*, u_k^*)u_\ell^* \otimes u_k^*$	$1 \leq k < l \leq s$

Table 6.10: The vectors of weights $\pm\delta_k \pm \delta_\ell$ for all $1 \leq i < j \leq r$ in $\mathfrak{U}(\mathfrak{g})v_3$.

weights	actions	vectors	conditions
0	$Y_{-2\varepsilon_i}\cdot Y_{-\varepsilon_1+\varepsilon_i}\cdot Y_{-\varepsilon_1+\varepsilon_i}\cdot v_1$	$t_i\otimes t_i^* + t_i^*\otimes t_i$	$1\le i\le r$
0	$Y_{-\varepsilon_i+\delta_l}\cdot Y_{-\varepsilon_i-\delta_l}\cdot Y_{-\varepsilon_1+\varepsilon_i}\cdot Y_{-\varepsilon_1+\varepsilon_i}\cdot v_1$	$u_\ell^*\otimes u_\ell - u_\ell\otimes u_\ell^* - t_i\otimes t_i^* - t_i^*\otimes t_i$	$1\le \ell\le s$

Table 6.11: The vectors of weight 0 in $\mathfrak{U}(\mathfrak{g})v_3$.

From the above calculation, each non-zero weight space is spanned by one vector, and the zero weight space is spanned by

$$\{t_i\otimes t_i^* + t_i\otimes t_i^*,\, u_\ell^*\otimes u_\ell - u_\ell\otimes u_\ell^* \mid 1\le i\le r, 1\le l\le s\}\,. \tag{6.3.9}$$

Therefore the dimension of the zero weight space is $r+s$. The set of nonzero weights is

$$\{\pm 2\delta_\ell \mid 1\le \ell\le s\}\cup\{\pm\varepsilon_i\pm\varepsilon_j \mid 1\le i<j\le r\}$$
$$\cup\,\{\pm\delta_k\pm\delta_\ell \mid 1\le k,\ell\le s\}\cup\{\pm\varepsilon_i\pm\delta_\ell \mid 1\le i\le r, 1\le \ell\le s\}. \tag{6.3.10}$$

Thus we have

$$\dim(W^{\square\square}) = m^2+mn+n^2-\frac{n-m}{2}. \tag{6.3.11}$$

Notice that $W^{\varnothing}$, $W^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ and $W^{\square\square}$ have pairwise trivial intersection, and the sum of their dimensions (6.3.4)+(6.3.8)+(6.3.11) is $(m+n)^2$ which is $\dim(V\otimes V)$. Therefore $V\otimes V = W^{\varnothing}\oplus W^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\oplus W^{\square\square}$.

Again as in Section 5.5, in each of the cases, we have checked that for each $w\in W^{Y(\lambda)}$ with $\lambda\in\{\varnothing,(1,1),(2)\}$, there exists $X\in\mathfrak{U}(\mathfrak{g})$ such that Xw is the highest weight vector, whence these modules are irreducible. Since this gives a decomposition of $V\otimes V$ into a direct sum of inequivalent $\mathfrak{spo}(V,\beta)$-submodules, each of the summands must be $B_2(n-m)$ invariant.

Next we determine the action of the Brauer algebra $B_2(n-m)$ on each of these summands by applying the generators to the basis vectors of $W^{\varnothing}$,$W^{\substack{\square\\\square}}$ and $W^{\square\square}$ respectively.

First notice that the basis vectors of $W^{\square\square}$ are linear combinations of vectors of the form

$$\omega_{x,y} = x \otimes y + \beta(y,x) y \otimes x \text{ for all } x, y \in B.$$

Since ι always acts by 1, it suffices to check how e_1 and s_1 act on the spanning vectors.

First consider $\omega_{x,y} s_1 = \omega_{x,y} \Psi_{s_1}$. We have

$$\omega_{x,y} s_1 = -\beta(y \otimes x) y \otimes x - \beta(y,x)\beta(x,y) x \otimes y = -\omega_{x,y}.$$

Then notice that each of the nonzero weight spaces has basis $\omega_{x,y}$ for a particular choice of $x, y \in B$ and $x \neq y^*$, and in this case we have $\omega_{x,y} e_1 = 0$. Now for the basis vectors in the zero weight space, a basis is given in (6.3.9). In this case, we can also check that

$$(t_1 \otimes t_1^* + t_1^* \otimes t_1) e_1 = F_{t_1, t_1^*} \Omega + F_{t_1^*, t_1} \Omega = 0,$$

and similarly

$$(u_1 \otimes u_1^* - u_1^* \otimes u_1) e_1 = \Omega - \Omega = 0.$$

Thus, in this case, the Brauer algebra $B_2(n-m)$ acts on $W^{\square\square}$ as a direct sum of modules $B^{\substack{\square\\\square}}$.

Similarly, the action of $B_2(n-m)$ on $W^{\varnothing}$ is by $B_{\varnothing}$, and on $W^{\substack{\square\\\square}}$ is by $B_{\square\square}$. Thus, we have the following theorem.

Theorem 6.3.2. *Let $r, s \geq 1$ and $|n-m| = k = 2$. As an $\mathfrak{spo}(V,\beta) \times B_2(n-m)$-*

module, we can decompose $V \otimes V$ as

$$V \otimes V = \bigoplus_{\lambda \in P_2} W^{Y(\lambda')} \otimes B_{Y(\lambda)}.$$

Proof. We proved that $V \otimes V = W^{\varnothing} \oplus W^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus W^{\square\square}$ as $\mathfrak{spo}(V,\beta)$-modules. By the discussion above, as an $\mathfrak{spo}(V,\beta) \times B_2(n-m)$-module, this is exactly

$$V \otimes V = (W^{\varnothing} \otimes B_{\varnothing}) \oplus (W^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \otimes B_{\square\square}) \oplus (W^{\square\square} \otimes B_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}).$$

This is consistent with the Schur-Weyl duality-like decomposition. Note also that the parametrizing set of $B_2(n-m)$-modules is $P_2 = \{\lambda \vdash 2-2h \mid h = 0,1\} = \{\varnothing, (1,1), (2)\}$. $\square$

We have shown that when $|m-n| = k = 2$ and $r, s \geq 1$, we can decompose $V \otimes V$ into a direct sum of three irreducible $\mathfrak{spo}(V,\beta) \times B_2(n-m)$-submodules, and that the highest weight vectors are those given by Benkart et. al.'s formula (Theorem 5.4.1). This extends [BSR98, Proposition 4.2].

6.4 More examples on borderline cases

In this section, we give examples to analyze Theorem 5.4.1 more by considering some additional borderline cases. In particular, in Example 6.4.1, we show that if $m = 2r = 2$, $n = 2s = 0$ and $k = 2$, we can decompose $V \otimes V$ into a direct sum of only two irreducible $\mathfrak{spo}(V,\beta) \times B_2(n-m)$-modules. In Example 6.4.2, under the assumption $m = 2r = 0$, $n = 2s = 2$ and $k = 2$, we show that Theorem 5.4.1 does not always give the full list of highest weight vectors. In Example 6.4.3, we show that when $m = n = 2$ and $k = 2$, we cannot decompose $V \otimes V$ into a direct sum of $\mathfrak{spo}(V,\beta)$-modules. We omit most of the calculations and only give the necessary information.

Example 6.4.1. Assume that $m = 2r = 2$, $n = 2s = 0$ and $k = 2$. In this case the only $(1,0)$-hook tableaux are $T = \varnothing$ and $T = \square\square$. Notice that, for $T = \square\square$, we have

$$T_{(0)} = \square\square \text{ and } T_{(1)} = \varnothing.$$

Thus by Theorem 5.4.1, we have two highest weight vectors

$$v_1 = \Omega = -t_1 \otimes t_1^* + t_1^* \otimes t_1 \text{ and } v_2 = t_1 \otimes t_1.$$

Then v_1 generates the trivial module $W^{\varnothing}$, and the submodule $W^{\square\square} = \mathfrak{U}(\mathfrak{g})v_2$ (which is irreducible) of $V \otimes V$ has a basis

$$\{t_1 \otimes t_1,\ t_1 \otimes t_1^* + t_1^* \otimes t_1,\ t_1^* \otimes t_1^*\}.$$

Notice that in this case, $W^{\square\square}$ coincides with $\mathcal{H}(V^{\otimes 2}, C_{1,2})$ in Example 5.3.9.

We calculate the $B_2(-2)$ action on these modules to deduce that $V \otimes V$ decomposes into irreducible $\mathfrak{spo}(V,\beta) \times B_2(-2)$ modules as

$$V \otimes V = (W^{\varnothing} \otimes B_{\varnothing}) \oplus (W^{\square\square} \otimes B_{\substack{\square\\\square}}),$$

where $B_{\varnothing}$ and $B_{\substack{\square\\\square}}$ are given by (6.2.1) and (6.2.3) respectively.

Notice that in Example 6.4.1, we only have two summands. By Theorem 6.1.1, the third summand would be $U^{\substack{\square\\\square}} = V^{\otimes 2} e_{(2),i,j}$, corresponding to the Brauer module

$B_{\square\square}$, but we can calculate that $U^{\substack{\square\\\square}}$ is the trivial vector space.

Example 6.4.2. Assume that $m = 2r = 0$, $n = 2s = 2$ and $k = 2$. In this case the only $(0,1)$-hook tableaux are $T = \varnothing$ and $T = \substack{\square\\\square}$. Notice that, for $T = \substack{\square\\\square}$, we have

$$T_{(0)} = \varnothing \text{ and } T_{(1)} = \square\square.$$

Since in this case $n = 2s$ and there is $s = 1$ row of $T_{(1)}$, by Theorem 5.4.1 we have three highest weight vectors

$$v_1 = \Omega = u_1 \otimes u_1^* + u_1^* \otimes u_1,$$
$$v_2 = u_1 \otimes u_1, \text{ and}$$
$$\overline{v_2} = u_1^* \otimes u_1^*.$$

Each of the above highest weight vectors generates a 1-dimensional submodule. Let $W^{\varnothing} = \mathfrak{U}(\mathfrak{g})v_1$, $W_1^{\substack{\square\\\square}} = \mathfrak{U}(\mathfrak{g})v_2$ and $W_2^{\substack{\square\\\square}} = \mathfrak{U}(\mathfrak{g})\overline{v_2}$. Therefore, the highest weight vectors produced in Theorem 5.4.1 only generate a 3-dimensional submodule of $V \otimes V$.

Now let us decompose $V \otimes V$ under the action of $B_2(2)$. We calculate that

$$(V \otimes V)e_{\varnothing,1,1} = \text{Span}\{\Omega\} = W^{\varnothing}, \tag{6.4.1}$$

$$(V \otimes V)e_{(2),1,1} = \text{Span}\{u_1 \otimes u_1, u_1^* \otimes u_1^*\} = W_1^{\substack{\square\\\square}} \oplus W_2^{\substack{\square\\\square}}, \tag{6.4.2}$$

$$(V \otimes V)e_{(1,1),1,1} = \text{Span}\{u_1 \otimes u_1^* - u_1^* \otimes u_1\}. \tag{6.4.3}$$

Thus let $v_3 = u_1 \otimes u_1^* - u_1^* \otimes u_1$. In fact, $\mathfrak{spo}(V,\beta)$ is just the orthogonal Lie colour algebra of rank 1, which is a colour-commutative algebra. Thus every weight vector, in particular v_3, is a highest weight vector. Let $W^{\square\square} = \mathfrak{U}(\mathfrak{g})v_3$.

We have shown that $V \otimes V$ decomposes into $\mathfrak{spo}(V,\beta) \times B_2(n-m)$ modules as

$$V \otimes V \cong (W^{\varnothing} \otimes B_{\varnothing}) \oplus ((W_1^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus W_2^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}) \otimes B_{\square\square}) \oplus (W^{\square\square} \otimes B_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}).$$

Example 6.4.2 shows that the highest weight vectors found in Theorem 5.4.1 are not always all of the highest weight vectors. Moreover, in this example, $U^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = W_1^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus W_2^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ is not irreducible.

Example 6.4.3. Consider $m = 2r = 2, n = 2s = 2$ and $k = 2$. This is a case not covered by the hypothesis of Theorem 6.1.2 (since $B_2(0)$ is not semisimple). Therefore the conclusion is not expected to hold.

In this case, the $(1,1)$-hook tableaux are $T_1 = \varnothing$, $T_2 = \square\square$ and $T_3 = \begin{smallmatrix}\square\\\square\end{smallmatrix}$.

For $T_1 = \varnothing$, the highest weight vector is

$$\Omega = -t_1 \otimes t_1^* + t_1^* \otimes t_1 + u_1 \otimes u_1^* + u_1^* \otimes u_1,$$

and it generates the trivial submodule $W^{\varnothing}$.

For $T_2 = \square\square$, the highest weight vector is $t_1 \otimes t_1$. The submodule $W^{\square\square} = \mathfrak{U}(\mathfrak{g})(t_1 \otimes t_1)$ is 8 dimensional, and its basis is recorded in Table 6.12.

basis vectors	weights
$t_1 \otimes t_1$	$2\varepsilon_1$
$t_1^* \otimes t_1^*$	$-2\varepsilon_1$
$u_1 \otimes t_1 + \beta(t_1, u_1) t_1 \otimes u_1$	$\varepsilon_1 + \delta_1$
$u_1^* \otimes t_1 + \beta(t_1, u_1^*) t_1 \otimes u_1^*$	$\varepsilon_1 - \delta_1$
$u_1 \otimes t_1^* + \beta(t_1^*, u_1) t_1^* \otimes u_1$	$-\varepsilon_1 + \delta_1$
$u_1^* \otimes t_1^* + \beta(t_1^*, u_1^*) t_1^* \otimes u_1^*$	$-\varepsilon_1 - \delta_1$
$u_1^* \otimes u_1 - u_1 \otimes u_1^*$	0
$t_1^* \otimes t_1 + t_1 \otimes t_1^*$	0

Table 6.12: A basis for $\mathfrak{U}(\mathfrak{g})(t_1 \otimes t_1)$.

For $T_3 = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$, we have two different highest weight vectors

$$v_1 = t_1 \otimes u_1 - \beta(u_1, t_1) u_1 \otimes t_1 \tag{6.4.4}$$

$$\overline{v_1} = t_1 \otimes u_1^* - \beta(u_1^*, t_1) u_1^* \otimes t_1. \tag{6.4.5}$$

The submodule $W_1^{\substack{\square\\\square}}$ generated by (6.4.4) is 4 dimensional, and it has basis

$$\{t_1 \otimes u_1 - \beta(u_1, t_1) u_1 \otimes t_1,\ u_1 \otimes t_1^* - \beta(t_1^*, u_1) t_1^* \otimes u_1,\ u_1 \otimes u_1,\ \Omega\}.$$

The submodule $W_2^{\substack{\square\\\square}}$ generated by (6.4.5) is also 4 dimensional, and it has basis

$$\{t_1 \otimes u_1^* - \beta(u_1^*, t_1) u_1^* \otimes t_1,\ u_1^* \otimes t_1^* - \beta(t_1^*, u_1^*) t_1^* \otimes u_1^*,\ u_1^* \otimes u_1^*,\ \Omega\}.$$

Notice that $\mathfrak{spo}(V,\beta)$-modules $W_1^{\substack{\square\\\square}}$ and $W_2^{\substack{\square\\\square}}$ are both 4-dimensional, but have different weights, so they are not isomorphic.

Notice also that since Ω appears in three different submodules ($W^{\varnothing}, W_1^{\substack{\square\\\square}}$ and $W_2^{\substack{\square\\\square}}$), we do not have a decomposition of $V \otimes V$ into a direct sum of irreducible highest weight modules. The sum of the modules we found is 15-dimensional, and there is no 1-dimensional invariant complement.

Now let us consider the Brauer action. We calculate that $B_2(0)$ acts on $W^{\square\square}$ as a direct sum of modules $B_{\substack{\square\\\square}}$. On the subspace spanned by $W_1^{\substack{\square\\\square}}$ and $W_2^{\substack{\square\\\square}}$, the Brauer algebra acts by $B_{\varnothing} = B_{\square\square}$. However if we set $v = \frac{1}{2}(u_1 \otimes u_1^* + u_1^* \otimes u_1)$ and $w = \Omega$, then $\mathrm{Span}\{v, w\}$ is a $B_2(0)$-submodule isomorphic to $B_{\varnothing'}$. Therefore $V \otimes V$ is a sum of $B_2(0)$-submodules, but the summands are not necessarily $\mathfrak{spo}(V,\beta)$-invariant.

6.5 The characters of $W^{Y(\lambda)}$ and $U^{Y(\lambda)}$

In this section, let $r > 0$, $s \geq 1$ and $|n-m| = k = 2$. We compute the characters of $W^{\varnothing}$, $W^{\boxminus}$ and $W^{\square\square}$. Then we compare our results with the character formula for $U^{Y(\lambda)}$ given in [BSR98, Theorem 4.24(e)].

Let $B = \{t_1, t_1^*, \ldots, t_r, t_r^*, u_1, u_1^*, \ldots, u_s, u_s^*, (u_{s+1})\}$ be a homogeneous basis of V as in (2.4.1). We define a corresponding ordered set of commuting variables

$$\mathcal{Z} = \{z_{t_1} < z_{t_1^*} < \cdots < z_{t_r} < z_{t_r^*} < z_{u_1} < z_{u_1^*} < \cdots < z_{u_s} < z_{u_s^*} < (z_{u_{s+1}})\} \qquad (6.5.1)$$

such that $z_b z_{b^*} = 1$ for all $b \in B$.

Definition 6.5.1. *Let W be an irreducible $\mathfrak{spo}(V,\beta)$-submodule of $V^{\otimes k}$. The character of W, $\chi(W)$, is a polynomial in $\mathcal{Z}$ with the property that the coefficient of $z_{b_1}^{i_1} \ldots z_{b_{m+n}}^{i_{m+n}}$ is the dimension of the $\sum_{j=1}^{m+n} i_j \mathrm{wt}(b_j)$-weight space, where $\sum_{p=1}^{m+n} i_p = k$ and $b_j \in B$.*

Lemma 6.5.2. *Let $r > 0$, $s \geq 0$ and $|n-m| = k = 2$. The characters of $W^{\varnothing}$, $W^{\boxminus}$ and $W^{\square\square}$ from Theorem 6.3.2 are given explicitly by*

(i) $\chi(W^{\varnothing}) = 1$.

(ii) $$\chi(W^{\square\square}) = \sum_{b \in B_{(0)}} z_b^2 + \sum_{\substack{b,b' \in B \\ z_b < z_{b'} \\ z_b z_{b'} \neq 1}} z_b z_{b'} + (r+s).$$

(iii) $$\chi(W^{\boxminus}) = \sum_{b \in B_{(1)}} z_b^2 + \sum_{\substack{b,b' \in B \\ z_b < z_{b'} \\ z_b z_{b'} \neq 1}} z_b z_{b'} + (r+s-1).$$

Proof. Since $W^{\varnothing}$ is the trivial module, the first case follows. We prove the second case.

Let $W = W^{\square\square}$. Then by (6.3.9), the nonzero weight spaces all have dimension 1, and the zero weight space has dimension $r+s$. Moreover, the nonzero weights in W are given by the following set

$$\{\pm 2\varepsilon_i \mid 1 \le i \le r\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le r\} \\ \cup \{\pm\delta_i \pm \delta_j \mid 1 \le i, j \le s\} \cup \{\pm\varepsilon_i \pm \delta_j \mid 1 \le i \le r, 1 \le j \le s\} \quad (6.5.2)$$

Thus, the character $\chi(W)$ is

$$\sum_{b \in B_{(0)}} z_b^2 + \sum_{\substack{b,b' \in B_{(0)} \\ z_b < z_{b'} \\ z_b z_{b'} \neq 1}} z_b z_{b'} + \sum_{\substack{b,b' \in B_{(1)} \\ z_b < z_{b'} \\ z_b z_{b'} \neq 1}} z_b z_{b'} + \sum_{\substack{b \in B_{(0)} \\ b' \in B_{(1)}}} z_b z_{b'} + r + s \quad (6.5.3)$$

which is $\sum_{b \in B_{(0)}} z_b^2 + \sum_{\substack{b,b' \in B \\ z_b < z_{b'} \\ z_b z_{b'} \neq 1}} z_b z_{b'} + (r+s)$ as we claimed.

The last case can be proved similarly by changing the first term of (6.5.3) into $\sum_{b \in B_{(1)}} z_b^2$, and by changing $r+s$ into $r+s-1$. □

We now cite [BSR98, Theorem 4.24(e)] to give a combinatorial description of the character of $\mathfrak{spo}(V,\beta)$-submodules by using the *hook Schur functions*. First we need to define *bi-tableaux*.

Definition 6.5.3. *Let λ be a partition. Let $\mathcal{Z}$ be defined in* (6.5.1)*. A bi-tableau T of shape λ with entries from $\mathcal{Z}$ is a tableau T such that*

(i) the part of T filled with z_t's and z_{t^*}'s is a semi-standard Young tableau of shape $\mu \subseteq \lambda$.

(ii) The z_u's and z_{u^*}'s are weakly increasing along columns and strictly increasing along rows.

The set of all bi-tableaux of shape λ with entries from $\mathcal{Z}$ is denoted as $\mathcal{T}(\mathcal{Z},\lambda)$.

Example 6.5.4.

z_{t_1}	z_{t_1}	$z_{t_1^*}$
z_{u_1}	z_{u_2}	
z_{u_1}		

is a bi-tableau of shape $\lambda = (3,2,1)$.

Definition 6.5.5. *Let T be a bi-tableau of shape λ. Let z^T be the product of all entries of T, which is a monomial in $\mathcal{Z}$. The hook Schur function $sh_\lambda(\mathcal{Z})$ of shape λ is defined as*

$$sh_\lambda(\mathcal{Z}) = \sum_{T\in\mathcal{T}(\mathcal{Z},\lambda)} z^T. \tag{6.5.4}$$

By convention, if λ is a partition of 0, we define $sh_\lambda(\mathcal{Z}) = 1$. By convention, we define $sh_{(-n)}(\mathcal{Z}) = 0$ for all $n > 0$.

Notice that in Definition 6.5.5, if $B = \{t_1, t_1^*, \ldots, t_r, t_r^*\}$ and we do not assume $z_b z_{b^*} = 1$ for any $b \in B'$, then $sh_\lambda(\mathcal{Z})$ is a Schur polynomial as defined in Definition A.5.5.

Example 6.5.6. Let $\lambda = (1)$. Then

$$sh_\lambda(\mathcal{Z}) = \sum_{b\in B} z_b.$$

Theorem 6.5.7 ([BSR98] Theorem 4.24(e))**.** *Let $U^{Y(\lambda)}$ be the $\mathfrak{spo}(V,\beta)$-submodule defined in Lemma 6.1.1 The character of $U^{Y(\lambda)}$ is given by*

$$\chi(U^{Y(\lambda)}) = \frac{1}{2}\det\left(sh_{(\lambda_i-i-j+2)}(\mathcal{Z}) + sh_{(\lambda_i-i+j)}(\mathcal{Z}).\right) \tag{6.5.5}$$

Now we compare the character we found in Lemma 6.5.2 with the character computed based on Theorem 6.5.7.

Proposition 6.5.8. *Let $r > 0$ and $s \geq 0$. Let $|n-m| = 2 = k$. Let λ be an (r,s)-hook tableau in the set $\left\{\varnothing, \ydiagram{1,1}, \ydiagram{2}\right\}$. The character of $U^{Y(\lambda)}$ coincides with the character of $W^{Y(\lambda)}$.*

Proof. Let us first consider $\lambda = (1,1)$. We have $Y(\lambda) = \ydiagram{1,1}$. Therefore we have

$$\chi(U^{Y(\lambda)}) = \frac{1}{2}\det\begin{pmatrix} sh_{(1)}(\mathcal{Z}) + sh_{(1)}(\mathcal{Z}) & sh_{(0)}(\mathcal{Z}) + sh_{(2)}(\mathcal{Z}) \\ sh_{(0)}(\mathcal{Z}) + sh_{(0)}(\mathcal{Z}) & sh_{(-1)}(\mathcal{Z}) + sh_{(1)}(\mathcal{Z}) \end{pmatrix}$$

which is (by the fact that $sh_0(\mathcal{Z}) = 1$ and $sh_{-1}(\mathcal{Z}) = 0$) equal to

$$\frac{1}{2}\det\begin{pmatrix} 2sh_{(1)}(\mathcal{Z}) & 1 + sh_{(2)}(\mathcal{Z}) \\ 2 & sh_{(1)}(\mathcal{Z}) \end{pmatrix}$$

which is also equal to

$$sh^2_{(1)}(\mathcal{Z}) - sh_{(2)}(\mathcal{Z}) - 1. \tag{6.5.6}$$

Now the first term of (6.5.6) is $(\sum_{b\in B} z_b)^2$ which can be rewritten as

$$sh^2_{(1)}(\mathcal{Z}) = \sum_{b\in B} z_b^2 + \sum_{\substack{b,b'\in B \\ z_b < z_{b'}}} 2z_b z_{b'}. \tag{6.5.7}$$

The second term of (6.5.6) is

$$sh_{(2)}(\mathcal{Z}) = \sum_{b\in B_{(0)}} z_b^2 + \sum_{\substack{b,b'\in B \\ z_b < z_{b'}}} z_b z_{b'}. \tag{6.5.8}$$

Therefore 6.5.6 becomes

$$\sum_{b\in B} z_b^2 + \sum_{\substack{b,b'\in B\\ z_b<z_{b'}}} 2z_b z_{b'} - (\sum_{b\in B_{(0)}} z_b^2 + \sum_{\substack{b,b'\in B\\ z_b<z_{b'}}} z_b z_{b'}) - 1$$

which is

$$\sum_{b\in B_{(1)}} z_b^2 + \sum_{\substack{b,b'\in B\\ z_b<z_{b'}}} z_b z_{b'} - 1. \tag{6.5.9}$$

For second term of (6.5.9), if $b^* = b'$, we have $z_b z_{b'} = 1$. In addition, we have $r+s$ pairs of such b and b'. Therefore (6.5.9) becomes

$$\sum_{b\in B_{(1)}} z_b^2 + \sum_{\substack{b,b'\in B\\ z_b<z_{b'}\\ z_b z_{b'}\neq 1}} z_b z_{b'} + (r+s-1)$$

which is $\chi(W^{\square})$ in Lemma 6.5.2.

Now let us consider $\lambda = (2)$. We have $Y(\lambda) = \square\square$. Therefore we have

$$\chi(U^{Y(\lambda)}) = \frac{1}{2}\det\begin{pmatrix} sh_{(2)}(\mathcal{Z}) + sh_{(2)}(\mathcal{Z}) & sh_{(1)}(\mathcal{Z}) + sh_{(-1)}(\mathcal{Z}) \\ sh_{(-1)}(\mathcal{Z}) + sh_{(-1)}(\mathcal{Z}) & sh_{(-2)}(\mathcal{Z}) + sh_{(0)}(\mathcal{Z}) \end{pmatrix}$$

which is

$$\frac{1}{2}\det\begin{pmatrix} 2sh_{(2)}(\mathcal{Z}) & sh_{(1)}(\mathcal{Z}) \\ 0 & 1 \end{pmatrix} = sh_{(2)}(\mathcal{Z}).$$

By a similar argument, we have

$$sh_{(2)}(\mathcal{Z}) = \sum_{b\in B_{(0)}} z_b^2 + \sum_{\substack{b,b'\in B\\ z_b<z_{b'}\\ z_b z_{b'}\neq 1}} z_b z_{b'} + (r+s) = \chi(W^{\square\square})$$

as we claimed. $\square$

Although [BSR98] did not provide any evidence that the submodules generated by highest weight vectors are the same as $U^{Y(\lambda)}$, we proved this is true in the case $|n - m| = k = 2$ and $r > 0$. Thus we have the following corollary.

Corollary 6.5.9. *Let $m = 2r$ and $n = 2s$ with $r > 0$, $s \geq 0$ and $|n - m| = 2 = k$. Then*

(i) there are exactly three highest weight vectors (two if $s = 0$),

(ii) $V \otimes V = (W^{\varnothing} \otimes B_{\varnothing}) \oplus (W^{\square\square} \otimes B_{\substack{\square\\\square}}) \oplus (W^{\substack{\square\\\square}} \otimes B_{\square\square})$ is a direct sum of irreducible $\mathfrak{spo}(V, \beta) \times B_2(2)$-submodules (set $W^{\substack{\square\\\square}} = \{0\}$ if $s = 0$), and

(iii) for all (r, s)-hook partition, $\lambda \in \{\varnothing, (2), (1, 1)\}$ we have $\chi(W^{Y(\lambda)}) = \chi(U^{Y(\lambda)})$, and in particular, we have $W^{Y(\lambda)} \cong U^{Y(\lambda)}$.

In fact we have a similar result for $r = 0$ and $n = 2s = 2$.

Corollary 6.5.10. *Let $m = 0$, $n = 2$ and $k = 2$. Then*

(i) every nonzero vector in $V \otimes V$ is a highest weight vector;

(ii) there is a decomposition $V \otimes V = (W^{\varnothing} \otimes B_{\varnothing}) \oplus (W^{\square\square} \otimes B_{\substack{\square\\\square}}) \oplus (W^{\substack{\square\\\square}} \otimes B_{\square\square})$ into a direct sum of $\mathfrak{spo}(V, \beta) \times B_2(2)$-submodules, where

$$W^{\substack{\square\\\square}} := W_1^{\substack{\square\\\square}} \oplus W_2^{\substack{\square\\\square}},$$

and

(iii) for all (r, s)-hook partition, $\lambda \in \{\varnothing, (2), (1, 1)\}$ we have $\chi(W^{Y(\lambda)}) = \chi(U^{Y(\lambda)})$, and in particular, we have $W^{Y(\lambda)} \cong U^{Y(\lambda)}$.

We have extended Theorem 6.1.2 to a borderline case when $|n-m|=k=2$. Also, we have proved that in this case the $\mathfrak{spo}(V,\beta)$-submodule $U^{Y(\lambda)}$ coincides with the $\mathfrak{spo}(V,\beta)$-submodule $W^{Y(\lambda)}$ generated by highest weight vector $w_{T,\underline{p},\underline{q}}C_{\underline{p},\underline{q}}y_T$. However, the relation between $U^{Y(\lambda)}$ and $W^{Y(\lambda)}$ in general case remains unclear.

Moreover, in the future, we hope to study another combinatorial description of the character of $U^{Y(\lambda)}$, which can be derived from the formula given in [BSR98, Theorem 5.1]:

$$\chi(U^{Y(\lambda)}) = \sum_T z^T, \tag{6.5.10}$$

where T runs over all $\mathfrak{spo}(m,n)$-tableaux of shape λ, and the $\mathfrak{spo}(m,n)$-tableau is a restricted version of Definition 6.5.3.

Appendix A

Schur polynomials and the RSK-Correspondence

Schur polynomials and Young tableaux play vital roles in the representation theory of the general linear groups and the symmetric groups, and they are the fundamental tools we used in our thesis. Since they are also interesting in their own right, we provide this appendix to give a closer look at these concepts.

In this appendix, we first give a definition of Schur polynomials derived from a quotient of determinants (Definition A.1.3). We then define Young tableaux and describe the RSK-correspondence in the following sections. Using this, we give a combinatorial definition of Schur polynomials (see Definition A.5.5), and we prove that the two definitions of Schur polynomials are equivalent. Then we use this new definition to give a different proof of the fact that Schur polynomials are symmetric.

A.1 Symmetric functions and Schur polynomials

In this section, we briefly discuss the definition of symmetric polynomials and in turn, we give an important example of symmetric polynomials, Schur polynomials. In representation theory, Schur polynomials describe the characters of finite-dimensional irreducible representations of the general linear groups. See for example [Sag01]. Further identities and combinatorial properties about Schur polynomials can be found in [Mac15].

Let $\mathbb{C}[x_1, \dots x_n]$ be the ring of complex n-variable polynomials. Then consider the action of the symmetric group S_n on $\mathbb{C}[x_1, \dots x_n]$ given by permuting variables. We say a polynomial is *symmetric* if it is invariant under this action.

In order to define Schur polynomials, we need *the Vandermonde matrix*, V_n, with entries $V_{i,j} = x_i^{j-1}$ for all $1 \le i, j \le n$. The determinant of V_n is given by

$$\Delta = \prod_{1 \le i < j \le n} (x_i - x_j)$$

Definition A.1.1. *By a partition of n, we mean a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_k)$ satisfying $|\lambda| = \sum_i^k \lambda_i = n$. We write $\lambda \vdash n$ if $|\lambda| = n$. Moreover, we define the length of λ, $\ell(\lambda)$ to be the index of the last nonzero entry of λ.*

Example A.1.2. $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (4, 3, 2, 0, 0)$ is a partition of 9 with $\ell(\lambda) = 3$.

Definition A.1.3. *Let λ be a partition of length $\le n$. We define the Schur polynomial* *in* $\mathbb{C}[x_1, \dots x_n]$ *by*

$$s_\lambda = s_\lambda(x_1, \cdots, x_n) = \frac{\det\left(x_j^{\lambda_i+n-i}\right)_{1\le i,j\le n}}{\Delta}.$$

Example A.1.4. Let $\lambda = (2)$ and consider $s_\lambda = s_{(2)}(x_1, x_2, x_3)$. Then we have

$$s_\lambda = \det \begin{pmatrix} x_1^4 & x_2^4 & x_3^4 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{pmatrix} \Big/ (x_1-x_2)(x_1-x_3)(x_2-x_3) = x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3.$$

Proposition A.1.5. *The Schur polynomial* $s_\lambda(x_1, \cdots, x_n)$ *is symmetric.*

Proof. Let $\Delta = \prod_{1\le i<j\le n}(x_j - x_i)$ and $q(x_1, \dots, x_n) = \det\left(x_j^{\lambda_i+n-i}\right)_{1\le i,j\le n}$. Then $q(x_1, \dots, x_n)$ is an alternating polynomial since it is a determinant. That is, we have

$$q(x_1, \dots, x_i, \dots, x_j, \dots, x_n) = -q(x_1, \dots, x_j, \dots, x_i, \dots, x_n)$$

which implies $q(x_1, \cdots, x_n)|_{x_i=x_j} = 0$. Therefore we have $(x_i - x_j)$ divides q.

By the above argument the product of the factors $(x_i - x_j)$ divides q. It follows that Δ divides q, so that q/Δ is a polynomial.

Moreover, since Δ is also an alternating polynomial, q/Δ is a symmetric polynomial. □

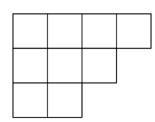
A.2 Young tableaux

In this section, we first give the definition of Young tableaux. Then we provide an algorithm to insert a letter into a tableau (Algorithm 1) with its reverse algorithm

(Algorithm 2). With these algorithms, we define an operation on Young tableaux, called a product. Then we give the definition of words in some alphabet, and give an algorithm to transform a word into a Young tableau. At the end this section, we deduce that the tableau product corresponds to the concatenation of words. We synthesized this material from [Ful97].

Definition A.2.1. *Given a partition* $\lambda = (\lambda_1, \cdots, \lambda_k)$*, the Young diagram of shape* λ*, denoted* $Y(\lambda)$*, is a top-left-aligned diagram with at most* k *rows of boxes, and* λ_i *boxes in row number* i*.*

Example A.2.2. The Young diagram for $\lambda = (4, 3, 2, 0, 0) \vdash 9$ is



An *alphabet* A is a well ordered set. From now on, unless stated otherwise, we use the set of positive integers $\mathbb{Z}_{>0}$ as an alphabet.

Definition A.2.3. *Let* λ *be a partition of* n*. A semi-standard Young tableau is a Young diagram of shape* λ *with entries from* A *in each box such that the entries are:*

(i) weakly increasing along each row from left to right;

(ii) strictly increasing along each column from top to bottom.

We call a Young tableau a standard Young tableau if it is a semi-standard Young tableau with strictly increasing entries in each row.

We use $\varnothing$ to denote the diagram or tableau associated to the empty partition, $\lambda = (0)$.

Example A.2.4. Let $\lambda = (4, 3, 2)$. Then

(i) $\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 4 \\ \cline{1-2}\end{array}$ is a semi-standard Young tableau of shape λ.

(ii) $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 9 & 10 \\ \cline{1-2}\end{array}$ is a standard Young tableau of shape λ.

Given a tableau T with n boxes, and a new entry x in A, in order to construct a new tableau from T and x, we introduce the Row-Insertion Algorithm (See Algorithm 1). The resulting new tableau is denoted $T \leftarrow x$ and consists of $n + 1$ boxes, and the entries of $T \leftarrow x$ are x together with all of the entries of T.

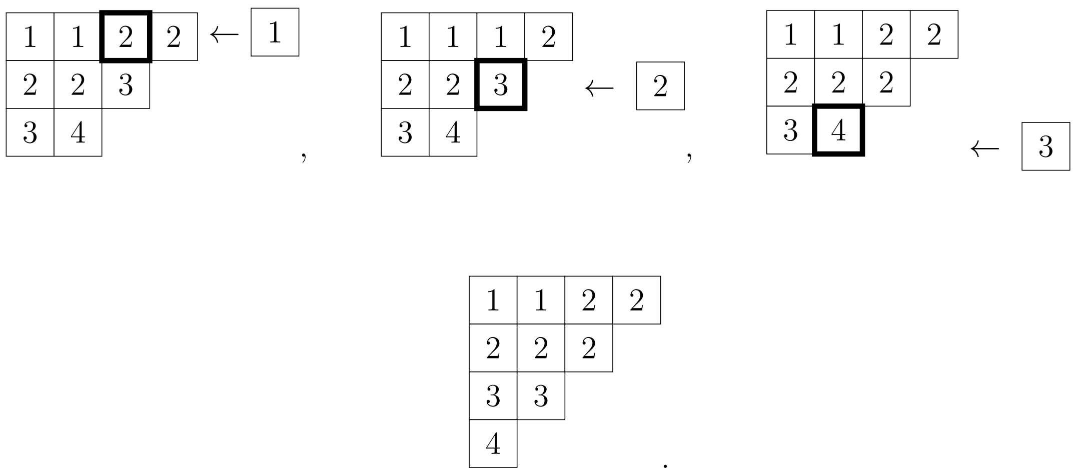
Algorithm 1 Row-insertion Algorithm

Given a tableau T and an entry x. Set $Entry := x$, $Row\_Number := 1$
while $Entry \neq \varnothing$, **do**
  if $Entry$ is greater than or equal to all the entries in the Row_Number row of T, **then** put a new box at the end of the Row_Number row with $Entry$ in it.
    $Entry := \varnothing$
  else Find the left-most entry, y, in the Row_Number row that is strictly larger than $Entry$, and replace y by $Entry$.
    $Entry := y$
  end if
  $Row\_Number := Row\_Number + 1$
end while

Example A.2.5. Let $x = 1$ and row insert x into the tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 4 \\ \cline{1-2}\end{array}$$

Using Algorithm 1, 1 will replace the first 2 in the first row, and 2 then replaces 3 in the second row, which replaces 4 in the third row. Then 4 will generate a new row at the bottom of the diagram. We illustrate this process as the following sequence of tableau insertions.



Lemma A.2.6. *If T is a semi-standard Young tableau and x is an element in A, then $T \leftarrow x$ is again a semi-standard Young tableau.*

Proof. $T \leftarrow x$ is clearly weakly increasing along rows by the way we insert x. We then prove that $T \leftarrow x$ is strictly increasing along columns. Now suppose that some element y is replaced by x. We have $y > x$. Then we denote the entry immediately below y (if it exists) by z. Then we have $z > y > x$. We claim that y will not replace entries which are in a position on the right side of z. This is because if y replaces some entry on the right of z, we should have $z \leq y$, which is a contradiction.

From the argument above, y replaces an entry below and to the left of the position of x, say u. Then $u \leq x < y$. Thus the resulting tableau is still column strict. Hence $T \leftarrow x$ is semi-standard by induction. □

Definition A.2.7. *Given two tableaux T and U, the product tableau $T \bullet U$ is obtained by row-inserting the entries of U into T from bottom to top and from left to right.*

Example A.2.8. Let T be a tableau. Then $T \bullet \begin{array}{|c|c|}\hline x & z \\ \hline y \\ \cline{1-1}\end{array} = T \leftarrow y \leftarrow x \leftarrow z$.

For future reference, we note that Algorithm 1 is reversible if we know the resulting tableau and the position of the added box.

Theorem A.2.9. *Suppose that U is a tableau such that $U = T \leftarrow x$ for some tableau T and positive integer x. Then T and x are uniquely determined from U and the position of the added box to T in U. Furthermore, there is an algorithm for computing T from U and the position of the added box.*

Algorithm 2 is the algorithm promised in Theorem A.2.9. Since it precisely reverses Algorithm 1, the theorem follows.

Algorithm 2 Backwards Row-Insertion Algorithm

Given the entry in the added box, $Entry := y$, and the row number of y, $Row\_Number := i$.
while $Row\_Number > 0$ **do** Find the right-most entry in row Row_Number which is strictly less than $Entry$, say z, replace z by $Entry$.
    $Entry := z$
    $Row\_Number := Row\_Number - 1$
end while
Output: $x := Entry$.

Next we are going to give the definition of words, and then relate words to tableaux.

Definition A.2.10. *A word is a finite sequence of elements of A. We usually express words using concatenation.*

Example A.2.11. $w = bac$ is a word in the alphabet $\{a, b, c, \ldots\}$.

Definition A.2.12. *Let A be an alphabet. Given a word $w = x_1 x_2 \cdots x_n$ in A, the* *tableau of w is the tableau obtained by row inserting the letters of w from x_1 to x_n :*

$$T(w) = \Big(\Big(\boxed{x_1} \leftarrow x_2)\Big) \leftarrow x_3)\Big) \leftarrow \cdots \leftarrow x_n).$$

Lemma A.2.13. *Given two arbitrary words w and $w' = x_1 \dots x_n$, then*

$$T(ww') = T(w) \leftarrow w' := (T(w) \leftarrow x_1) \leftarrow \cdots \leftarrow x_n.$$

This lemma follows directly from the definition.

Definition A.2.14. *Given a semi-standard Young tableau T, we can construct the corresponding word of T, denoted as $\mathcal{W}(T)$, by reading the entries of T from bottom to top and from left to right.*

Example A.2.15. The word of the tableau T in Example A.2.4 is 342231122.

A tableau T can also be recovered from its word $\mathcal{W}(T)$: simply break the word wherever one entry is *strictly greater* than the next, and the pieces are the rows of $T(\mathcal{W}(T))$, read from bottom to top, left to right.

Example A.2.16. In alphabet $\mathbb{Z}_{>0}$, the word 4332221122 breaks into 4 | 33 | 222 | 1122 and the corresponding tableau is the one we obtained in Example A.2.5.

Definition A.2.17. *We call a word w a tableau word if $w = \mathcal{W}(T)$ for some tableau T.*

Lemma A.2.18. *Let T and U be two tableaux. The product of two tableaux $T \bullet U$ is equivalent to row-inserting $\mathcal{W}(U)$ into T from left to right.*

Proof. The result follows from the definition of $T \bullet U$ and the definition of $\mathcal{W}(U)$. $\square$

A.3 Knuth Equivalence

In this section, an equivalence relation between words called *Knuth equivalence* will be introduced and in turn, we show that two words are Knuth equivalent if and only if they have the same tableaux. Moreover, we prove that the product of tableaux (Definition A.2.7) is associative. The materials are mainly from [Ful97], but the important proofs of Proposition A.3.6, A.3.9 and A.3.10 are our own.

Definition A.3.1. *There are two rules of words in an ordered alphabet, named elementary Knuth transformations, which are:*

$$y\ z\ x \mapsto y\ x\ z \qquad \text{if} \qquad x < y \leq z \tag{K'}$$

$$x\ z\ y \mapsto z\ x\ y \qquad \text{if} \qquad x \leq y < z \tag{K''}$$

Note that K' and K'' are not inverse to each other.

Definition A.3.2. *We say two words are Knuth equivalent if they can be transformed into each other by a sequence of elementary Knuth transformations and their inverses. If words w and w' are Knuth equivalent, then we write $w \overset{k}{\equiv} w'$.*

Example A.3.3. 12124 is Knuth equivalent to 21124 by using K' on the first three letters of 12124.

Proposition A.3.4. *Consider the tableau $T \leftarrow x$. We have*

$$\mathcal{W}(T \leftarrow x) \overset{k}{\equiv} \mathcal{W}(T)x.$$

Proof. Notice that an elementary Knuth transformation is always going to interchange the largest and smallest in three successive letters when the largest letter is in the middle position. The row-insertion of an element x into a tableau T is equivalent to the following process:

We first try to test x against the last entry of the first row of T, say z_q. If $x \geq z_q$, then we put a new box at the end of the first row with x in it. If $x < z_q$ and if the entry z_{q-1} immediate before z_q is also strictly larger than x, then we swap x and z_q. Notice that in this case, we have $x < z_{q-1} \leq z_q$ which is the condition of K'. We repeat the process by testing the relation among x, z_{q-1} and z_{q-2} until we reach some y_p and x' in the first row such that y_p is the entry immediate before x' and the relation $x < y_p$ no longer holds, which means we have $y_p \leq x < x'$. Moreover, this process can be viewed as:

$$\begin{aligned} y_p x' z_1 \cdots z_{q-1} z_q x &\mapsto y_p x' z_1 \cdots z_{q-1} x z_q && (x < z_{q-1} \leq z_q) \\ &\mapsto y_p x' z_1 \cdots z_{q-2} x z_{q-1} z_q && (x < z_{q-2} \leq z_{q-1}) \\ &\cdots \\ &\mapsto y_p x' z_1 x z_2 \cdots z_q && (x < z_1 \leq z_2) \\ &\mapsto y_p x' x z_1 \cdots z_q && (x < x' \leq z_1). \end{aligned}$$

Each of the above transformation is K'. Now we have $y_p \leq x < x'$ which is the condition of K''. Then we can swap x' with y_p, and continue the process with x' replaced by x. Thus by repeated application of K'' we have:

$$\begin{aligned} y_1 \cdots y_p x' x z_1 &\mapsto y_1 \cdots y_{p-1} x' y_p x z_1 && (y_p \leq x < x') \\ &\mapsto y_1 \cdots y_{p-2} x' y_{p-1} y_p x z_1 && (y_{p-1} \leq y_p < x') \\ &\cdots \\ &\mapsto y_1 x' y_2 y_3 \cdots y_p x z_1 && (y_2 \leq y_3 < x') \end{aligned}$$

$$\mapsto x'y_1y_2y_3\cdots y_pxz_1 \qquad\qquad (y_1 \leq y_2 < x').$$

Each of the transformations is K'', and this process indicates that x' is now effectively in the second row of the tableau. We can now continue by testing x' with the entries in the second row of T, and finish the row-insertion process once a box has to be added.

Therefore, the row-insertion process is about swapping consecutive three letters by using the rules K' and K'' accordingly. Thus we have $\mathcal{W}(T \leftarrow x) \overset{k}{\equiv} \mathcal{W}(T)x$. □

Corollary A.3.5. *For all words w, $\mathcal{W}\big(T(w)\big) \overset{k}{\equiv} w$.*

Proof. Let $w = w_1 \cdots w_r$, where $w_1, \ldots, w_r$ are in the alphabet A. Then we have $T(w) = \varnothing \leftarrow w_1 \leftarrow \cdots \leftarrow w_r$. Therefore by using Lemma A.3.4, we have

$$\mathcal{W}(T(w)) \overset{k}{\equiv} \mathcal{W}(\varnothing \leftarrow w_1 \leftarrow \cdots \leftarrow w_{r-1})w_r \overset{k}{\equiv} \cdots \overset{k}{\equiv} w_1 \cdots w_r. \quad \square$$

Proposition A.3.6. *For any two words w_1 and w_2, we have if $T(w_1) = T(w_2)$, then $w_1 \overset{k}{\equiv} w_2$.*

Proof. By Corollary A.3.5, we know $w \overset{k}{\equiv} \mathcal{W}\big(T(w)\big)$. Therefore, if $T(w_1) = T(w_2)$, then

$$w_1 \overset{k}{\equiv} \mathcal{W}\big(T(w_1)\big) = \mathcal{W}\big(T(w_2)\big) \overset{k}{\equiv} w_2.$$

Hence we have $w_1 \overset{k}{\equiv} w_2$. □

Corollary A.3.7. *For any two words w and w', we have*

$$T(ww') = T(w) \bullet T(w').$$

Proof. The result follows from Corollary A.3.5 and Proposition A.3.6. $\square$

Next we will prove that if two words are Knuth equivalent, then they have the same tableau. Let us first consider two straightforward examples.

Lemma A.3.8. *Let x, y, z be characters in an alphabet A. Then*

(i) if $w_1 = xzy$ and $w_2 = zxy$, with $x \leq y < z$, we have

$$T(w_1) = T(w_2) = \begin{array}{|c|c|}\hline x & y \\ \hline z & \\ \cline{1-1}\end{array}, \text{ and}$$

(ii) if $w_1 = yzx$ and $w_2 = yxz$, with $x < y \leq z$, we have

$$T(w_1) = T(w_2) = \begin{array}{|c|c|}\hline x & z \\ \hline y & \\ \cline{1-1}\end{array}.$$

Proof. The result follows by using the row-insertion algorithm on w_1 and w_2. $\square$

Notice that in the first and second case, w_1 and w_2 are Knuth equivalent by K'' and K' respectively.

Proposition A.3.9. *For two words w_1 and w_2, if $w_1 \overset{k}{\equiv} w_2$, then $T(w_1) = T(w_2)$.*

Proof. Without loss of generality, we assume that there is only a single Knuth transformation, for example $wzxyw' \overset{k}{\equiv} wxzyw'$. Then by Lemma A.2.13, it suffices to show $T(wxzy) = T(wzxy)$. Therefore we assume that the Knuth transformation happens at the end of a word. Then we proceed by induction on the number of rows of $T(w)$. That is we show that for all $xzy \overset{k}{\equiv} zxy$ with $x \leq y < z$, we have $T(wxzy) = T(wzxy)$, and for all $yzx \overset{k}{\equiv} yxz$ with $x < y \leq z$, we have $T(wyzx) = T(wyxz)$. The base case (when $T(w) = \varnothing$) was proven in Lemma A.3.8. Suppose that if $T(w)$ has $n-1$ rows, the two implications still hold. Now we prove that if $T(w)$ has n rows, the equality still holds.

Our strategy is the following: we insert x, y, z in the given order into the first row of $T(w)$ and keep track of what is replaced. Then we show that the first row of $T(wzxy)$ is the same as the first row of $T(wxzy)$ (similar for $T(wyzx)$ and $T(wyxz)$). The letters which have been displaced from the first row of $T(w)$, are therefore inserted into the second row of $T(w)$, or equivalently, are inserted into the tableau formed from removing the first row of $T(w)$, denoted as T', which has $n-1$ rows. Thus once we show that sequences of letters replaced are Knuth equivalent, the result follows by induction hypothesis.

Let the first row of $T(w)$ be

x_1	x_2	....	x_d

for some $d \in \mathbb{N}$. For the purpose of the following argument, we make the convention that if $r > d$, then x_r does not exist, and should be ignored. If z replaces such an x_r, this means z is appended to the end of the row.

Let us first consider that xzy and zxy are Knuth equivalent by K''. That is, we have $x \leq y < z$. Then in this case, we have the following two scenarios .

Scenario 1: There exists some r with $2 \leq r \leq d+1$ such that

$$x_{r-1} \leq x < z < x_r \leq x_{r+1}. \tag{A.3.1}$$

Then inserting zxy gives the following: z replaces x_r, then x replaces z, and then y replaces x_{r+1}. Therefore the box of x_r is now contains x, and the box of x_{r+1} contains y. The replaced letters (in order) are $x_r z x_{r+1}$.

On the other hand, inserting xzy gives the following: x replaces x_r, then z replaces x_{r+1}, and then y replaces z. The box of x_r contains x, and the box of x_{r+1} contains y. The replaced letters (in order) are $x_r x_{r+1} z$.

From the above discussion, the first row of $T(wzxy)$ is the same as the first row of $T(wxzy)$. Moreover, by (A.3.1), we have $z < x_r \leq x_{r+1}$ which implies $x_r z x_{r+1} \overset{k}{\equiv} x_r x_{r+1} z$. Thus, since we are now inserting $x_r z x_{r+1}$ and $x_r x_{r+1} z$ into T' separately, we conclude that $T(wzxy) = T(wxzy)$ by induction hypothesis.

Scenario 2: There exists r, s with $1 \leq r < s \leq d$ such that

$$x_{r-1} \leq x < x_r \leq z < x_s. \tag{A.3.2}$$

Then inserting xzy gives the following: x replaces x_r, then z replaces x_s, then y will replaces some y' where y' belongs to a box to right of the position of x_r. Therefore the replaced letters are $x_r x_s y'$ and $x_{r+1} \leq y' \leq z < x_s$.

Inserting zxy gives the following: z replaces x_s, then x replaces x_r, then y will replace the same y' as we discussed above. Therefore the replaced letters are $x_s x_r y'$

In this case, it is straightforward that the first row of $T(wzxy)$ is the same as the first row of $T(wxzy)$. By (A.3.2), $x_r \leq y' < x_s$ implies that $x_r x_s y' \overset{k}{\equiv} x_s x_r y'$. Thus the result follows by induction hypothesis.

The case when xzy and zxy are Knuth equivalent by K' can be proved similarly. We proved this proposition when w_1 and w_2 can be changed by applying a sequence which consists of K' or K'' only, and this implies that it is also true for a sequence of K' and K'' and their inverses. Thus we conclude that for any Knuth equivalent words, they have the same tableaux. □

Proposition A.3.10. *The tableau product of Definition A.2.7 is associative.*

Proof. Let U and V be two tableaux. Then by Lemma A.2.18,

$$U \bullet V = U \leftarrow \mathcal{W}(V) = \varnothing \leftarrow \mathcal{W}(U) \leftarrow \mathcal{W}(V) = T(\mathcal{W}(U)\mathcal{W}(V)).$$

Therefore by Corollary A.3.5 and Proposition A.3.6, we have

$$\mathcal{W}(U \bullet V) \equiv \mathcal{W}\big(T(\mathcal{W}(U)\mathcal{W}(V))\big) \equiv \mathcal{W}(U)\mathcal{W}(V).$$

Let W be another tableau. Then we have

$$\begin{aligned}\mathcal{W}\big((U \bullet V) \bullet W\big) &\stackrel{k}{\equiv} \mathcal{W}(U \bullet V)\mathcal{W}(W)\\ &\stackrel{k}{\equiv} \mathcal{W}(U)\mathcal{W}(V)\mathcal{W}(W)\\ &\stackrel{k}{\equiv} \mathcal{W}(U)\mathcal{W}(V \bullet W)\\ &\stackrel{k}{\equiv} \mathcal{W}\big(U \bullet (V \bullet W)\big).\end{aligned}$$

Then by Proposition A.3.9, we have $U \bullet (V \bullet W) = (U \bullet V) \bullet W$.

□

A.4 The RSK-Correspondence

In this section, following [Ful97, Chapter 4], we construct a matrix with nonnegative integer entries from an ordered two-rowed array, and thereafter we obtain a one-to-one correspondence between a matrix and a pair of tableaux (P, Q), which is known as the Robinson-Schensted-Knuth correspondence, also referred to as the RSK-Correspondence.

Definition A.4.1. *An ordered two-rowed array is a* $2 \times n$ *matrix*

$$\mathcal{A} = \begin{pmatrix} u_1 & u_2 & \dots & u_r \\ v_1 & v_2 & \dots & v_r \end{pmatrix}$$

such that

$$u_i \leq u_j, \ \forall 1 \leq i < j \leq r, \ \text{and } u_i = u_j \text{ implies } v_i \leq v_j, \ \forall 1 \leq i < j \leq n.$$

Definition A.4.2. *We associate an* $m \times n$ *matrix* $\mathcal{M}$ *to an ordered two-rowed array such that the* (i, j) *entry of* $\mathcal{M}$ *is the number of times of* $\begin{pmatrix} i \\ j \end{pmatrix}$ *occurs in the array.*

Example A.4.3. For the two-rowed array $\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 2 \end{pmatrix}$, the corresponding matrix is

$$\mathcal{M} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}.$$

Next given an arbitrary two-rowed array, $\begin{pmatrix} u_1 & u_2 & \dots & u_r \\ v_1 & v_2 & \dots & v_r \end{pmatrix}$, we construct a pair of tableaux (T, Q) of the same shape by using the row-insertion algorithm:

Start with $T_1 = \boxed{v_1}$, and $Q_1 = \boxed{u_1}$. In order to construct (T_2, Q_2) from (T_1, Q_1), row-insert v_2 in T_1, getting T_2. Then add a box to Q_1 in the position of the new box in T_2, and place u_2 in it. Repeat the process of inserting (u_k, v_k) for all $2 \leq k \leq r$ until there are no more unused entries in the two-rowed array.

We call Q the *recording tableau* because we use it to record the process of row-insertion.

Example A.4.4. Using the two-rowed array in Example A.4.3, we obtain a pair of tableaux as:

$$(T, Q) = \left(\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 2 \\ \cline{1-3} \end{array}, \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} \end{array} \right).$$

For any matrix $\mathcal{M}$ with nonnegative integer entries, we have an associated two-rowed array $\mathcal{A}$. We write the corresponding pair of tableaux of $\mathcal{M}$ as $(T(\mathcal{M}), Q(\mathcal{M}))$. Viewing the second row of $\mathcal{A}$ as a word w, we deduce that $T(\mathcal{M}) = T(w)$.

Corollary A.4.5. *Let $\mathcal{M}$ be an $n \times m$ matrix with nonnegative integer entries. Write $\mathcal{M} = \begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}$ where $\mathcal{M}_1$ consists of the first k rows of $\mathcal{M}$ for all $1 \leq k \leq n$, and $\mathcal{M}_2$ consists of the rest rows. Then $T(\mathcal{M}) = T(\mathcal{M}_1) \bullet T(\mathcal{M}_2)$.*

Proof. The result follows by the construction of a two-rowed array from $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2$ and Corollary A.3.7, we have □

Remark A.4.6. Let $\mathcal{A}$ be an two-rowed array with first row $\mathcal{A}_1$ and second row $\mathcal{A}_2$. Let $\mathcal{M}$ be the associated $m \times n$ matrix of $\mathcal{A}$, with entries $\mathcal{M}_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then

(i) the row sum of the i^{th} row of $\mathcal{M}$ is the same as the number of occurrences of i in $\mathcal{A}_1$, and thus is the same as the number of occurrences of i in Q.

(ii) The sum of the entries in column j of $\mathcal{M}$ is the same as the number of occurrences of j in $\mathcal{A}_2$, which is the same as the number of occurrence of j in T.

Similar to the preceding algorithms, this transformation from an ordered two-rowed array to a pair of tableaux is also reversible. Given a pair of tableaux (P, Q), in order to construct (P_{k-1}, Q_{k-1}) from (P_k, Q_k), we do the following:

(i) First look for the largest number in Q_k and find the right-most, top-most box with this largest number in it.

(ii) Then find the corresponding box in P_k and apply the backwards row-insert algorithm (Algorithm 2) to P_k with that box. The resulting tableau is P_{k-1}, and the element is removed from the top row of P_k is the k^{th} element of the second row of the two-rowed array.

(iii) Remove the box we found in Q_k to form Q_{k-1}, and the box removed contains the k^{th} element of the first row of the two-rowed array.

By repeating the process until there are no boxes left in the tableaux, we get a two-rowed array.

By the preceding discussion in this section, we have the following theorem:

Theorem A.4.7 (the RSK-correspondence)**.** *With the above setting, there is a one-to-one correspondence between matrices with nonnegative integer entries and pairs of tableaux.*

A.5 An application of the RSK-Correspondence

In this section, we give an application of the RSK-correspondence, which implies that a certain n-variable polynomial $Sch_\lambda(x_1, x_2, \cdots, x_n)$ (see Definition A.5.5) is symmetric. Then at the end of this section, we generalize the proof provided in [Pro89] to show that for each λ this polynomial is equal to the Schur polynomial we defined in Definition A.1.3.

For a tableau T with entries from alphabet $\{1, \dots, n\}$, let m_i be the number of occurrences of i in T. Let $m(T) = (m_1, \dots, m_n)$. We call $m(T)$ the *type* of T. Note that m_ℓ can be 0 for some $\ell \in \{1, \dots, n\}$.

Example A.5.1. Let $T =$

1	1	2	2
2	2	3	
3	5		

. Then the type of T is $m(T) = (2, 4, 2, 0, 1)$.

Proposition A.5.2. *Given a tableau T_0 of shape λ, then the number of tableaux T of shape λ having $m(T) = m(T_0)$ is the same as the number of tableaux T of shape λ with $m(T) = (m_{\sigma(1)}, \dots, m_{\sigma(n)})$, where σ is a permutation of $\{1, \dots, n\}$.*

Remark A.5.3. Fix any tableau T_0 of shape λ. Then by Theorem A.4.7, for any tableaux Q of shape λ, there is a one-to-one correspondence between the pairs of tableaux (T_0, Q) and matrices $\mathcal{M}$ such that $T(\mathcal{M}) = T_0$. From Remark A.4.6 it follows that Proposition A.5.2 is equivalent to showing that the two sets

$$E_1 = \{\mathcal{M} \ : \ T(\mathcal{M}) = T_0 \text{ and } \mathcal{M} \text{ has row sums } m_1, \ m_2, \cdots, m_n\}$$

and

$$E_2 = \{\mathcal{M} \ : \ T(\mathcal{M}) = T_0 \text{ and } \mathcal{M} \text{ has row sums } m_{\sigma(1)}, \ m_{\sigma(2)}, \cdots, m_{\sigma(n)}\}$$

have the same cardinality.

Before we prove Proposition A.5.2, we need the following lemma.

Lemma A.5.4. *Let C be a matrix with only two rows with integer entries and corresponding pair of tableaux $\big(T(C), Q(C)\big)$. Then there is a unique two-rowed matrix $\tilde{C}$ with corresponding pair of tableaux $\big(T(\tilde{C}), Q(\tilde{C})\big)$ such that $\tilde{C}$ has swapped row sums and $T(\tilde{C}) = T(C)$.*

Proof. By Remark A.4.6, since C has only two rows, the row sum of the i^{th} row is the number of occurrences of i in $Q(C)$ for $i = 1, 2$. Thus $Q(C)$ only has 1 and 2 as entries. Then by Remark A.5.3, it suffices to prove that with a particular $Q(C)$ and $T(C) = T(\tilde{C})$, there is only one corresponding possibility for $Q(\tilde{C})$.

Now suppose that $Q(C)$ is given by:

1	$\cdots$	1	1	$\cdots$	1	2	$\cdots$	2
2	$\cdots$	2		$\longleftrightarrow$ s			$\longleftrightarrow$ t	
	$\longleftrightarrow$ r							

which means the row sum for C is $r + s$ and $r + t$. Therefore let $\tilde{C}$ have row sum $r + t$ and $r + s$. Thus $Q(\tilde{C})$ must be

1	$\cdots$	1	1	$\cdots$	1	2	$\cdots$	2
2	$\cdots$	2		$\longleftrightarrow$ t			$\longleftrightarrow$ s	
	$\longleftrightarrow$ r							

Since there are $(r + t)$ 1's and $(r + s)$ 2's in $Q(\tilde{C})$, this is the only way to get such a semi-standard Young tableau. Thus $Q(\tilde{C})$ is uniquely determined. $\square$

Proof of Proposition A.5.2. In order to prove Proposition A.5.2 holds for all permutations, it suffices to prove it when σ is the adjacent transposition of $(k, k + 1)$. Given

A in E_1, we can write A as a block matrix

$$A = \begin{bmatrix} B \\ C \\ D \end{bmatrix}$$

where B consists of the first $k-1$ rows, C consists of next two rows and D consists of the rest.

By Corollary A.4.5, we have

$$T(A) = T(B) \bullet T(C) \bullet T(D).$$

Then by Lemma A.5.4, if the matrix C has row sums m_k and m_{k-1}, then there is a unique corresponding $\tilde{C}$ with row sums m_{k-1} and m_k such that $T(C) = T(\tilde{C})$.

Therefore, for each A in E_1, there is a unique corresponding matrix $\tilde{A} = \begin{bmatrix} B \\ \tilde{C} \\ D \end{bmatrix}$ in E_2. Thus the sets E_1 and E_2 are in bijection, which proves the proposition by Remark A.4.6. □

Now we introduce a new polynomial defined in terms of semi-standard Young tableaux and in turn, prove it is a symmetric polynomial by Proposition A.5.2.

Definition A.5.5. *Let λ be a partition of any nonnegative integer. We define an n-variable polynomial Sch_λ associated to λ by*

$$Sch_\lambda(x_1, x_2, \cdots, x_n) = \sum_T x^{m(T)} = \sum_T x_1^{m_1} \cdots x_n^{m_n}.$$

where the summation is over all semi-standard Young tableaux T of shape λ with entries from the alphabet $\{1, \ldots, n\}$, and $(m_1, \ldots, m_n) = m(T)$.

Example A.5.6. Let $n = 3$, $\lambda = (2)$. Then the possible tableaux are

$$\boxed{1 \mid 1}, \quad \boxed{1 \mid 2}, \quad \boxed{1 \mid 3}, \quad \boxed{2 \mid 2}, \quad \boxed{2 \mid 3}, \quad \boxed{3 \mid 3},$$

and

$$Sch_{(2)}(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3.$$

Remark A.5.7. By Definition A.5.5, the coefficients of $Sch_\lambda(x_1, x_2, \cdots, x_n)$ are always integers.

Proposition A.5.8. *The polynomial Sch_λ defined in Definition A.5.5 is symmetric.*

Proof. By Proposition A.5.2, the coefficient of $x_1^{m_1} \cdots x_n^{m_n}$ is the same as the coefficient of $x_1^{m_{\sigma(1)}} \cdots x_n^{m_{\sigma(n)}}$ for all permutation σ of $\{1, \ldots, n\}$. Thus the result follows. $\square$

Recall the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ in Definition A.1.3. It is clear that the polynomials in Example A.1.4 and A.5.6 are identical. Thus, in this case, we have

$$s_{(2)}(x_1, x_2, x_3) = Sch_{(2)}(x_1, x_2, x_3).$$

Next we prove that for any arbitrary partition λ, and $n \in \mathbb{Z}_{\geq 0}$, we have

$$s_\lambda(x_1, \ldots, x_n) = Sch_\lambda(x_1, \cdots, x_n).$$

We first need the following lemma.

Lemma A.5.9. *We have*

$$s_\lambda(x_1, \dots x_n) = \sum_\mu s_\mu(x_1, \dots, x_{n-1}) x_n^{|\lambda|-|\mu|}$$

where $|\lambda| = \sum_i \lambda_i$ *and the sum is over all partitions* μ *such that*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n.$$

Proof. Using Definition A.1.3 of $s_\lambda(x_1, \dots, x_n)$, we see that

$$s_\lambda(x_1, \dots, x_{n-1}, 1) = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \cdots & x_{n-1}^{\lambda_1+n-1} & 1 \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \cdots & x_{n-1}^{\lambda_2+n-2} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1^{\lambda_{n-1}-1} & x_2^{\lambda_{n-1}-1} & \cdots & x_{n-1}^{\lambda_{n-1}-1} & 1 \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \cdots & x_{n-1}^{\lambda_n} & 1 \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_{n-1}^{n-1} & 1 \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_{n-1}^{n-2} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1 & x_2 & \cdots & x_{n-1} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}}.$$

Then if we subtract the last column from each of the other columns and keep the last column unchanged, the determinant does not change, and we have:

$$s_\lambda(x_1,\ldots,x_{n-1},1) = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1}-1 & x_2^{\lambda_1+n-1}-1 & \cdots & x_{n-1}^{\lambda_1+n-1}-1 & 1 \\ x_1^{\lambda_2+n-2}-1 & x_2^{\lambda_2+n-2}-1 & \cdots & x_{n-1}^{\lambda_2+n-2}-1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1^{\lambda_{n-1}-1}-1 & x_2^{\lambda_{n-1}-1}-1 & \cdots & x_{n-1}^{\lambda_{n-1}-1}-1 & 1 \\ x_1^{\lambda_n}-1 & x_2^{\lambda_n}-1 & \cdots & x_{n-1}^{\lambda_n}-1 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^{n-1}-1 & x_2^{n-1}-1 & \cdots & x_{n-1}^{n-1}-1 & 1 \\ x_1^{n-2}-1 & x_2^{n-2}-1 & \cdots & x_{n-1}^{n-2}-1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1-1 & x_2-1 & \cdots & x_{n-1}-1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix}}.$$

For all $1 \le i \le n-1$, we factor out $x_i - 1$ from the i^{th} column of each of the determinants in the numerator and denominator to get the following expression for the right hand side of the above.

$$\frac{\begin{vmatrix} x_1^{\lambda_1+n-2}+x_1^{\lambda_1+n-3}+\cdots+x_1+1 & \cdots & x_{n-1}^{\lambda_1+n-2}+x_{n-1}^{\lambda_1+n-3}+\cdots+x_{n-1}+1 & 1 \\ x_1^{\lambda_2+n-3}+x_1^{\lambda_2+n-4}+\cdots+x_1+1 & \cdots & x_{n-1}^{\lambda_2+n-3}+x_{n-1}^{\lambda_2+n-4}+\cdots+x_{n-1}+1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_1^{\lambda_{n-1}-2}+x_1^{\lambda_{n-1}-3}+\cdots+x_1+1 & \cdots & x_{n-1}^{\lambda_{n-1}-2}+x_{n-1}^{\lambda_{n-1}-3}+\cdots+x_{n-1}+1 & 1 \\ x_1^{\lambda_n-1}+x_1^{\lambda_n-2}+\cdots+x_1+1 & \cdots & x_{n-1}^{\lambda_n-1}+x_{n-1}^{\lambda_n-2}+\cdots+x_{n-1}+1 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^{n-2}+x_1^{n-3}+\cdots+x_1+1 & \cdots & x_{n-1}^{n-2}+x_{n-1}^{n-3}+\cdots+x_2+1 & 1 \\ x_1^{n-3}+x_1^{n-4}+\cdots+x_1+1 & \cdots & x_{n-1}^{n-3}+x_{n-1}^{n-4}+\cdots+x_{n-1}+1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{vmatrix}}.$$

Recursively from $i = 1$ to $n-1$, subtracting the $(i+1)^{\text{th}}$ row from the i^{th} and keeping the last row unchanged, the above expression becomes equal to:

$$\frac{\begin{vmatrix} x_1^{\lambda_1+n-2}+\cdots+x_1^{\lambda_2+n-2} & \cdots & x_{n-1}^{\lambda_1+n-2}+\cdots+x_{n-1}^{\lambda_2+n-2} & 0 \\ x_1^{\lambda_2+n-3}+\cdots+x_1^{\lambda_3+n-3} & \cdots & x_{n-1}^{\lambda_2+n-3}+\cdots+x_{n-1}^{\lambda_3+n-3} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ x_1^{\lambda_i+n-(i+1)}+\cdots+x_1^{\lambda_{i+1}+n-(i+1)} & \cdots & x_{n-1}^{\lambda_i+n-(i+1)}+\cdots+x_{n-1}^{\lambda_{i+1}+n-(i+1)} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ x_1^{\lambda_{n-1}-2}+\cdots+x_1^{\lambda_n} & \cdots & x_{n-1}^{\lambda_{n-1}-2}+\cdots+x_{n-1}^{\lambda_n} & 0 \\ x_1^{\lambda_n-1}+\cdots+x_1+1 & \cdots & x_{n-1}^{\lambda_n-1}+\cdots+x_{n-1}+1 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^{n-2} & \cdots & x_{n-1}^{n-2} & 0 \\ x_1^{n-3} & \cdots & x_{n-1}^{n-3} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{vmatrix}}.$$

Then we use cofactor expansion for the numerator and denominator along the last column to get

$$\frac{\begin{vmatrix} x_1^{\lambda_1+n-2}+\cdots+x_1^{\lambda_2+n-2} & \cdots & x_{n-1}^{\lambda_1+n-2}+\cdots+x_{n-1}^{\lambda_2+n-2} \\ x_1^{\lambda_2+n-3}+\cdots+x_1^{\lambda_3+n-3} & \cdots & x_{n-1}^{\lambda_2+n-3}+\cdots+x_{n-1}^{\lambda_3+n-3} \\ \vdots & \ddots & \vdots \\ x_1^{\lambda_i+n-(i+1)}+\cdots+x_1^{\lambda_{i+1}+n-(i+1)} & \cdots & x_{n-1}^{\lambda_i+n-(i+1)}+\cdots+x_{n-1}^{\lambda_{i+1}+n-(i+1)} \\ \vdots & \ddots & \vdots \\ x_1^{\lambda_{n-1}-2}+\cdots+x_1^{\lambda_n} & \cdots & x_{n-1}^{\lambda_{n-1}-2}+\cdots+x_{n-1}^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-2} & \cdots & x_{n-1}^{n-2} \\ x_1^{n-3} & \cdots & x_{n-1}^{n-3} \\ \vdots & \ddots & \vdots \\ x_1^{0} & \cdots & x_{n-1}^{0} \end{vmatrix}}.$$

Notice that the (i,j) entry of the matrix in the numerator is of the form

$$\sum_{\mu_i=\lambda_i}^{\lambda_{i+1}} x_j^{\mu_i+(n-1)-i},$$

where $1 \leq j \leq n-1$. Thus preceding quotient is

$$\frac{\begin{vmatrix} \sum_{\mu_1=\lambda_1}^{\lambda_2} x_1^{\mu_1+n-2} & \cdots & \sum_{\mu_1=\lambda_1}^{\lambda_2} x_{n-1}^{\mu_1+n-2} \\ \vdots & \ddots & \vdots \\ \sum_{\mu_i=\lambda_i}^{\lambda_{i+1}} x_1^{\mu_i+(n-1)-i} & \cdots & \sum_{\mu_i=\lambda_i}^{\lambda_{i+1}} x_{n-1}^{\mu_i+(n-1)-i} \\ \vdots & \ddots & \vdots \\ \sum_{\mu_{n-1}=\lambda_{n-1}}^{\lambda_n} x_1^{\mu_{n-1}} & \cdots & \sum_{\mu_{n-1}=\lambda_{n-1}}^{\lambda_n} x_{n-1}^{\mu_{n-1}} \end{vmatrix}}{\begin{vmatrix} x_1^{n-2} & \cdots & x_{n-1}^{n-2} \\ x_1^{n-3} & \cdots & x_{n-1}^{n-3} \\ \vdots & \ddots & \vdots \\ x_1^0 & \cdots & x_{n-1}^0 \end{vmatrix}}.$$

Thus by using the fact that the determinant is linear in each row, we conclude that the preceding quotient of determinants is $\sum_\mu s_\mu(x_1, \cdots, x_{n-1})$, for all partition μ such that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$.

In order to homogenize the sum $\sum_\mu s_\mu(x_1, \ldots, x_{n-1})$ to recover $s_\lambda(x_1, \ldots, x_n)$, we should multiply each of the above terms by $x_n^{|\lambda|-|\mu|}$. Therefore we have

$$s_\lambda(x_1, \cdots x_n) = \sum_\mu s_\mu(x_1, \cdots, x_{n-1}) x_n^{|\lambda|-|\mu|}$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_n$ and the sum is over all partitions μ such that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n. \quad \square$$

Proposition A.5.10. *Definition A.1.3 is equivalent to Definition A.5.5. In other words,*

$$s_\lambda(x_1, \cdots x_n) = Sch_\lambda(x_1, \cdots x_n).$$

Proof. It suffices to prove that we can express $s_\lambda(x_1, \ldots, x_n)$ as a sum of monomials, and each monomial corresponds to a semi-standard Young tableau of shape λ.

Also notice that by Definition A.5.5, the power of x_n in a monomial $x_1^{t_1} \cdots x_n^{t_n}$ corresponds to the number of n's in the corresponding semi-standard Young tableau. By removing the n's, we get a new monomial associated to some partition μ. Therefore we will construct a collection of tableaux by repeatly using Lemma A.5.9.

Start with a shape $\lambda^0 = \lambda$ and apply Lemma A.5.9 once. For each resulting $\lambda^1 = \mu$, we construct a tableau by placing an n in each box where the boxes are in λ^0 but not in λ^1. Then for each $\lambda^1 = \mu$ we obtained, apply Lemma A.5.9 again. We therefore place $n-1$ in each box which is outside of the resulting shape λ^2 but inside λ^1. Now we have constructed some tableaux with empty boxes and boxes with n and $n-1$.

Repeat this procedure $n-2$ more times with $n-2$'s, $n-3$'s, $\cdots$ and 1s. After n steps we are left with $\lambda^n = \varnothing$, so we can no longer apply Lemma A.5.9 again. We finish the procedure by noticing that $s_\varnothing(x_1, \ldots, x_n) = 1$.

Therefore, we have expressed $s_\lambda(x_1, \cdots x_n,)$ as a sum of monomials which are indexed by a set of tableaux.

Moreover these tableaux we constructed are semi-standard. Because first of all the entries obviously weakly increase across the rows and down the columns. Also $\lambda_i^{k+1} \geq \lambda_{i+1}^k$ and $\lambda_{n-k+1}^k = 0$ imply that two $n-k$'s will not be placed in the same column at step $k = 0, 1, \cdots, n-1$. This implies that the entries are strictly increasing along columns.

Thus we have assigned a semi-standard Young tableau of shape λ to each summands appearing in $s_\lambda(x_1, \ldots, x_n)$. Also on the other hand, by the way we construct induction, we are able to assign a summand in $s_\lambda(x_1, \ldots, x_n)$ to each semi-standard Young tableaux of shape λ appearing in $Sch_\lambda(x_1, \ldots, x_n)$. Thus there is a bijection between $s_\lambda(x_1, \ldots, x_n)$ and $Sch_\lambda(x_1, \ldots, x_n)$, which completes the proof. $\square$

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