Topics related to tensorially absorbing inclusions and algebraic K-theory of C*-algebras

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Abstract

This thesis is split up into two parts: the first concerns certain applications of the de la Harpe-Skandalis determinant to K-theory of appropriately regular C*-algebras. The second is concerned with (unital) inclusions of C*-algebras which satisfy a strong tensorial absorption condition. The first chapter following the preliminary section is joint work with Aaron Tikuisis [ST23],[2] while the following chapters are independent. The penultimate chapter is [Sar23b] and the last chapter is essentially [Sar23a].

In the first chapter following the preliminaries, we examine the interplay between the algebraic K_1-group and the unitary algebraic K_1-group of a unital C*-algebra. We prove that for an abundance of unital C*-algebras, the algebraic K_1-group splits naturally as a direct sum of the unitary algebraic K_1-group and the space of continuous real-valued affine functions on the trace simplex. We further prove that if one considers Hausdorffized variants, then for any unital C*-algebra, there is a natural splitting of the Hausdorffized algebraic K_1-group in terms of the Hausdorffized unitary algebraic K_1-group and the space of continuous real-valued affine functions on the trace simplex. Moreover, this a splitting of topological groups.

The following chapter studies how certain group homomorphisms between unitary groups of C*-algebras induce maps on the trace simplex. In particular, we show that a contractive group homomorphism between unital C*-algebras which sends the circle to the circle, induces a map between their trace simplices. Under mild regularity conditions these further induce maps between Elliott invariants. As a consequence we show that certain inclusions of C*-algebras are in a correspondence with certain inclusions of unitary groups.

Finally we investigate what we call "$\mathcal{D}$-stable inclusions" of C*-algebras, where $\mathcal{D}$ is strongly self-absorbing. We give a systematic study and prove that such inclusions between unital, separable, $\mathcal{D}$-stable C*-algebras exist, are abundant, and are non-trivial.

[2]After the writing of [ST23], it was pointed out to the authors that George Elliott proved a variation of some of the main results in [Ell22, Theorem 5]. We use different techniques and exposition.

Dedications

This thesis is dedicated to my beautiful dog Bella. Naturally she would fail to understand this thesis, let alone this dedication, but she has given me immense joy over the years.

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Contents

Preface

This thesis has two main components: the first is related to several K-theoretic aspects of C*-algebras, and the second is concerned with certain inclusions of C*-algebras. Both components make frequent use of several regularity properties, such as $\mathcal{Z}$-stability, in order to obtain positive results.

For a unital C*-algebra A, polar decomposition of an invertible element realizes the unitary group $U(A)$ as a retract of the general linear group $GL(A)$, and consequently the fundamental groups $\pi_1(GL(A))$ and $\pi_1(U(A))$ are isomorphic, and there is a canonical bijection between sets of connected components $\pi_0(GL(A))$ and $\pi_0(U(A))$. This property passes to matrix amplifications and therefore to the "stabilized" groups $U_\infty(A)$ and $GL_\infty(A)$. Thus there is a canonical identification of $\pi_0(U_\infty(A))$ with $\pi_0(GL_\infty(A))$ via the map induced by the inclusion of $U_\infty(A)$ in $GL_\infty(A)$ (and in this case, both of these are the topological K_1-group of A), with inverse induced by taking an invertible element to the unitary part of its polar decomposition – effectively nullifying the effect of any positive invertible element. In the algebraic K_1-group, $\mathrm{K}_1^{\mathrm{alg}}(A)$, which is the abelianization of $GL_\infty(A)$, positive elements are non-trivial (which can even be witnessed for $A = \mathbb{C}$) and therefore taking the unitary part of the polar decomposition of an invertible element does not yield an isomorphism between $\mathrm{K}_1^{\mathrm{alg,u}}(A)$ and $\mathrm{K}_1^{\mathrm{alg}}(A)$, where $\mathrm{K}_1^{\mathrm{alg,u}}(A)$ is the unitary algebraic K_1-group defined as the abelianization of $U_\infty(A)$.

Based on joint work with Aaron Tikuisis [ST23], we prove in Chapter 2 that polar decomposition does in fact give a natural relationship between $\mathrm{K}_1^{\mathrm{alg}}(A)$ and $\mathrm{K}_1^{\mathrm{alg,u}}(A)$, although positive invertibles must be accounted for in terms of traces. Let $T(A)$ denote the set of all tracial states on a unital C*-algebra A, and $\mathrm{Aff}\, T(A)$ the real Banach space of continuous affine $\mathbb{R}$-valued functions on the trace simplex $T(A)$ (with the uniform norm).

Theorem A. *For an abundance of unital C*-algebras A, there is a natural direct sum decomposition*

$$\mathrm{K}_1^{alg}(A) \simeq \mathrm{K}_1^{alg,u}(A) \oplus \mathrm{Aff}\, T(A).$$

The class of C*-algebras within the scope of Theorem A contains the class of all unital, separable, simple, nuclear, $\mathcal{Z}$-stable C*-algebras satisfying the UCT – i.e., those that can be completely classified by K-theory and traces. The statement of the above theorem in its most general form says that we have a natural direct sum decomposition of

$$GL_\infty(A)/\ker\Delta \simeq U_\infty(A)/\ker\Delta|_{U^0_\infty(A)} \oplus \operatorname{Aff} T(A),$$

where $\Delta : GL^0_\infty(A) \to \operatorname{Aff} T(A)/\rho_A(K_0(A))$ is the de la Harpe-Skandalis determinant (here $\rho_A : K_0(A) \to \operatorname{Aff} T(A)$ is the pairing map). If one considers the Hausdorffized variants $\overline{\mathrm{K}}_1^{\mathrm{alg}}(A)$ and $\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)$, where we now mod out by the closure (with respect to the inductive limit topologies) of the derived groups $DU_\infty(A)$ and $DGL_\infty(A)$, respectively, then the analogous statement holds for all unital C*-algebras. Moreover, the direct sum decomposition is one of topological groups.

Theorem B. *Let A be a unital C*-algebra. There is a natural isomorphism of topological groups*

$$\overline{\mathrm{K}}_1^{alg}(A) \simeq \overline{\mathrm{K}}_1^{alg,u}(A) \oplus \operatorname{Aff} T(A).$$

The relationship being natural is perhaps the most interesting part of these theorems, in contrast to several unnatural descriptions of K-theoretic objects in terms of K-theory and traces. For example, Hausdorffized algebraic K-theory and Hausdorffized unitary algebraic K-theory can be described *unnaturally* in terms of the K-theory and traces of the C*-algebra.

In Chapter 3, we study how norm-continuous group homomorphisms between unitary groups of C*-algebras behave with respect to K-theory and traces. For a unital C*-algebra, we denote by $U^0(A)$ the connected component of the identity in the unitary group $U(A)$ of A. We prove that continuous group homomorphisms induce maps between spaces of continuous affine functions in a desirable way.

Theorem C. *Let A, B be unital C*-algebras. If $\theta : U^0(A) \to U^0(B)$ is a continuous group homomorphism, then there exists a bounded $\mathbb{R}$-linear map $\Lambda_\theta : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ such that*

$$\begin{array}{ccc} \pi_1(U^0(A)) & \xrightarrow{\tilde{\Delta}^1_A} & \operatorname{Aff} T(A) \\ {\scriptstyle \pi_1(\theta)}\downarrow & & \downarrow{\scriptstyle \Lambda_\theta} \\ \pi_1(U^0(B)) & \xrightarrow[\tilde{\Delta}^1_B]{} & \operatorname{Aff} T(B) \end{array}$$

commutes.

Here $\pi_1(\theta)$ is the map between fundamental groups induced by θ, and for a C*-algebra A, $\tilde{\Delta}^1_A$ is the *pre-determinant* (used in the definition of the de la Harpe-Skandalis determinant) that takes a continuous[3] path in $U^0(A)$ beginning at the unit to an element of $\operatorname{Aff} T(A)$. See Section 2.1 for details.

[3]As every continuous path of unitaries (resp. invertibles) is homotopic to a piece-wise smooth path of unitaries (resp. invertibles) – see Remark 2.1.2(1) – and $\tilde{\Delta}^1_A$ is homotopy-invariant, one can apply $\tilde{\Delta}^1_A$ to any path of unitaries (resp. invertibles).

Under stricter continuity assumptions (e.g., θ being contractive), along with the assumption that the homomorphism is injective and sends the circle to the circle, such a group homomorphism between unitary groups induces a map between the whole of K-theory and traces. Building on this and using the state of the art K-theoretic classification of morphisms [CGS+23], we see that embeddings of unitary groups actually give rise to embeddings of C*-algebras.

Theorem D. *Let A be a unital, separable, simple, nuclear C*-algebra satisfying the UCT which is either $\mathcal{Z}$-stable or has stable stable rank one, and B be a unital, separable, simple, nuclear $\mathcal{Z}$-stable C*-algebra. There is a contractive injective group homomorphism $U(A) \to U(B)$ which maps the circle to the circle if and only if there is a unital embedding $A \hookrightarrow B$.*

In Chapter 4, we study inclusions $B \subseteq A$ of C*-algebras which are isomorphic to the inclusion $B \otimes \mathcal{D} \subseteq A \otimes \mathcal{D}$, where $\mathcal{D}$ is a strongly self-absorbing C*-algebra. Tensorial absorption with a strongly self-absorbing C*-algebra is a necessary condition for classification by means of K-theory and traces and leads to many desirable properties. Many constructions and properties relating to strongly self-absorbing C*-algebras give rise to important C*-subalgebras. For example, in the proof that the Cuntz algebra $\mathcal{O}_2$ is self-absorbing, the canonical copy of the CAR algebra M_{2^∞}, as the fixed point algebra of the gauge action, plays an indispensable role. In this specific example, there are copies of M_2 in M_{2^∞} which approximately commute with the larger C*-algebra $\mathcal{O}_2$, and this gives rise to an isomorphism $\Phi : \mathcal{O}_2 \simeq \mathcal{O}_2 \otimes M_{2^\infty}$ such that $\Phi(M_{2^\infty}) = M_{2^\infty} \otimes M_{2^\infty}$. This means that there is an isomorphism which witnesses M_{2^∞}-stability of both the CAR algebra and the Cuntz algebra concurrently.

We systematically study such inclusions, observe various characterizations and permanence properties, and provide examples. In particular, we prove that such inclusions between unital, separable $\mathcal{D}$-stable C*-algebras always exist, are abundant and are non-trivial.

Theorem E. *Let $\mathcal{D}$ be a strongly self-absorbing C*-algebra and let A, B be unital, separable $\mathcal{D}$-stable C*-algebras.*

1. *The set of unital $\mathcal{D}$-stable embeddings $B \hookrightarrow A$ is point-norm dense in the set of all unital embeddings $B \hookrightarrow A$.*
2. *Every unital embedding $B \hookrightarrow A$ is approximately unitarily equivalent to a unital $\mathcal{D}$-stable embedding.*

We discuss $\mathcal{D}$-stability of intermediate algebras, and the existence of isomorphisms $\Phi : A \simeq A \otimes \mathcal{D}$ which realize $\mathcal{D}$-stability of many intermediate C*-algebras between B and A at once. Along the way, we prove a strengthened version of Elliott's intertwining argument that keeps track of countably many intermediate subalgebras at once.

Theorem F. *Let $B \subseteq A$ be a unital, $\mathcal{D}$-stable inclusion of separable C^*-algebras. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of C^*-algebras such that $B \subseteq C_n \subseteq A$ unitally for all n, then there exists an isomorphism $\Phi : A \simeq A \otimes \mathcal{D}$ such that*

1. $\Phi(B) = B \otimes \mathcal{D}$ *and*

2. $\Phi(C_n) = C_n \otimes \mathcal{D}$ *for all* $n \in \mathbb{N}$.

There are also some applications to non-commutative dynamical systems – namely to inclusions emerging from non-commutative dynamical systems with Rokhlin properties.

Chapter 1

Background and notation

1.1 Basic C*-algebra theory

We assume understanding of basic C*-algebra theory, although we briefly outline some basic notions that will be of use throughout this thesis. We use [Mur90] as a reference for basic C*-algebra theory, [BO08] as a reference for nuclearity and [RLL00] as a reference for topological K-theory. For general algebraic K-theory, [Wei13, Ros94] work as references.

Definition 1.1.1. *A C*-algebra A is a Banach algebra over $\mathbb{C}$, together with an adjoint $^*: A \to A$ such that whenever $a, b \in A$ and $\lambda \in \mathbb{C}$,*

1. $a^{**} = a$;
2. $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$;
3. $(ab)^* = b^* a^*$;
4. $\|a^* a\| = \|a\|^2$. *This is called the C*-identity.*

The canonical example of a C*-algebra is the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. In fact, by the Gelfand-Naimark-Segal construction [Mur90, Theorem 3.4.1], every abstract C*-algebra is *-isomorphic to a norm-closed *-subalgebra of $\mathcal{B}(\mathcal{H})$. Moreover, C*-algebras are the noncommutative analogue of the space of continuous functions of a (locally compact Hausdorff) topological space. This correspondence arises via Gelfand duality [Mur90, Theorem 2.1.10] – any abelian C*-algebra is of the form $C_0(X)$ for some locally compact Hausdorff space X, and this is functorial in a contravariant way.

First we note that C*-algebras, being normed spaces, are of course susceptible to approximation techniques. To make some approximations more readable we adopt the following notation. For two elements a, b in a C*-algebra, we will often write

$$a \approx_\varepsilon b \tag{1.1.1}$$

to mean that $\|a-b\|<\varepsilon$, whenever $\varepsilon>0$ is some prescribed error.

Nuclearity of C*-algebras is a desirable finite-dimensional approximation property that is sometimes thought of as the noncommutative analogue of compactness (in the unital case) for a topological space. To be more accurate, it is more akin to the noncommutative analogue of the existence of a partition of unity – this being a direct observation from the proof that abelian C*-algebras are nuclear, see [BO08, Proposition 2.4.2]. It says that the identity map can be factored through matrix algebras via completely positive maps.

We say that a linear map $\phi : A \to B$ between C*-algebras is contractive and completely positive (c.c.p. or c.p.c.) if it is a contraction and the matrix amplification $\phi \otimes \mathrm{id}_n : A \otimes M_n \to B \otimes M_n$ is positive for every $n \in \mathbb{N}$.

Definition 1.1.2. *A C*-algebra A is nuclear if for any $\varepsilon > 0$ and any finite set $\mathcal{F} \subseteq A$, there are $n \in \mathbb{N}$ and c.c.p. maps $\phi : A \to M_n$ and $\psi : M_n \to A$ such that $(\psi \circ \phi)(a) \approx_\varepsilon a$ for all $a \in \mathcal{F}$.*

This says that we have an *approximate factorization* of the identity map through matrix algebras which has a commutative diagram flavour in the following way. The C*-algebra A is nuclear if whenever $\varepsilon > 0$ and $\mathcal{F} \subseteq A$ is finite, there are an $n \in \mathbb{N}$ and c.c.p. maps ϕ, ψ which make the following diagram commute on $\mathcal{F}$ up to ε.

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ & {\scriptstyle\phi}\searrow \quad \nearrow{\scriptstyle\psi} & \\ & M_n & \end{array} \tag{1.1.2}$$

In the unital case, this approximation can be done with unital completely positive (u.c.p.) maps [BO08, Proposition 2.2.6]. We almost exclusively deal with unital C*-algebras in this thesis, so most approximate factorizations through matrix algebras will be via u.c.p. maps. We note that nuclearity is equivalent to every c.c.p. map (or u.c.p. map in the unital case) from another C*-algebra to A and from A to another C*-algebra having this approximate factorization through matrix algebras as in diagram (1.1.2) (indeed, pre- or post-compose any such map with the identity map) – we call such maps *nuclear*.

Nuclearity is a property intimately related to tensor products of C*-algebras. For C*-algebras A, B, let $A \odot B$ denote the algebraic tensor product of A and B (over $\mathbb{C}$). A C*-norm $\|\cdot\|_\alpha$ on $A \odot B$ is a norm on $A \odot B$ which is submultiplicative, preserves adjoints, and satisfies the C*-identity. The completion $A \otimes_\alpha B$ of $A \odot B$ with respect to any C*-norm $\|\cdot\|_\alpha$ is then a C*-algebra which canonically contains $A \odot B$ as a dense *-subalgebra. There are in general many C*-norms, but there are two that exist for any pair of C*-algebras A, B (although it is possible that they agree). The first, called the *min-norm*, arises from taking any pair of faithful representations $A \subseteq \mathcal{B}(\mathcal{H})$ and $B \subseteq \mathcal{B}(\mathcal{K})$ and considering the norm $\|\cdot\|_{\min}$ induced from the inclusion

$A \odot B \subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. This C*-norm can be shown to be independent of the faithful representations of A and B (and therefore only depends on the C*-algebraic structure of A and B), and it is the smallest C*-norm – that is, $\|x\|_{\min} \leq \|x\|_\alpha$ for any $x \in A \odot B$ whenever $\|\cdot\|_\alpha$ is a C*-norm on $A \odot B$ [BO08, Theorem 3.4.8]. We write $A \otimes B$ for $A \otimes_{\min} B$.

The other C*-norm $\|\cdot\|_{\max}$, called the *max-norm*, comes from taking the supremum over all C*-norms $\|\cdot\|_\alpha$ on $A \odot B$. This turns out to be a C*-norm and is unsurprisingly the greatest such. It is universal in the sense that whenever $\|\cdot\|_\alpha$ is a C*-norm on $A \odot B$, by [BO08, Corollary 3.4.9], there are surjective *-homomorphisms

$$A \otimes_{\max} B \to A \otimes_\alpha B \to A \otimes B. \tag{1.1.3}$$

Nuclearity is related to tensor products in the following way – see [BO08, Proposition 3.6.12].

Theorem 1.1.3. *Let A be a C*-algebra. Then A is nuclear if and only if for any C*-algebra B, the canonical surjection*

$$\pi : A \otimes_{\max} B \to A \otimes B. \tag{1.1.4}$$

is an isomorphism.

In particular, if A or B is nuclear, $A \otimes B$ can be written unambiguously because there is a unique C*-tensor product (although $A \otimes B$ will always denote the minimal tensor product completion of $A \odot B$ regardless).

We now discuss the polar decomposition of an invertible element in a unital C*-algebra A. Let $GL(A)$ denote the set of invertible elements of a unital C*-algebra A, and $U(A)$ the set of unitary elements – that is, the set of invertible elements $u \in A$ with $u^* = u^{-1}$. There is a general polar decomposition of any element in a C*-algebra A which allows one to write any element as the product of a partial isometry and a positive element. However, the partial isometry is often not in A and instead is in the double dual A^{**} (or any enveloping von Neumann algebra [Sak12, Theorem 1.12.1])[4]. However, restricting ourselves to invertible elements in a unital C*-algebra, the polar decomposition outputs a unitary element still residing in the C*-algebra. This will be an invaluable tool when discussing relationships between algebraic K_1-groups of C*-algebras. Recall that for an element a in a C*-algebra A, the absolute value $|a|$ of a is defined as $|a| := (a^*a)^{\frac{1}{2}}$ via continuous functional calculus.

Proposition 1.1.4 (Proposition 2.1.8 of [RLL00]). *Let A be a unital C*-algebra.*

[4]The double dual A^{**} is a C*-algebra which can be identified isometrically and algebraically with an enveloping von Neumann algebra of A. Indeed, if $\pi : A \to \mathcal{B}(\mathcal{H})$ is the universal representation (that is, the representation coming from a direct sum of equivalence classes of GNS representations corresponding to states), then $A^{**} \simeq \pi(A)''$. See [Bla06, Section III.5.2].

1. *If z is an invertible element in A, then so is $|z|$, and $\omega(z) := z|z|^{-1}$ belongs to $U(A)$. Clearly $z = \omega(z)|z|$.*
2. *The map $\omega : GL(A) \to U(A)$ defined in (1) is continuous, $\omega(u) = u$ for every $u \in U(A)$, and $\omega(z) \sim_h z$ in $GL(A)$ for every z in $GL(A)$.*
3. *If u, v are unitary elements in $U(A)$, and if $u \sim_h v$ in $GL(A)$, then $u \sim_h v$ in $U(A)$.*

Parts (2) and (3) above yield that $U(A)$ is a retract of $GL(A)$, and consequently the inclusion map induces an isomorphism of fundamental groups (with inverse induced by ω), as well as a bijection between the sets of connected components.

1.2 K-theoretic invariants of C*-algebras

The view of C*-algebras as "non-commutative topological spaces" is justified by the (contravariant) identification of abelian C*-algebras with (locally) compact Hausdorff spaces via Gelfand duality. Given the success of the Atiyah-Singer index theorem and consequences of understanding K-theoretic and K-homological data of a topological space, it is unsurprising that the topological K-theory is central to distinguishing isomorphism classes of C*-algebras. It can be argued, however, that the topological K-theory is an invariant more suitable to the noncommutative setting.

George Elliott classified AF (approximately finite) C*-algebras by their ordered K-theory in [Ell76], then in [Ell93] used K-theory to classify $A\mathbb{T}$-algebras (approximately circle algebras) of real rank zero. It was then conjectured that K-theory would be able to classify all unital, separable, simple, amenable C*-algebras. Although this is not true in its more general form, the conjecture has been continually modified over the past three decades to a point where K-theory, together with traces and an appropriate pairing between them, allows for the classification of a suitably large class of simple nuclear C*-algebras (see [CET+21, Corollary D] as well as [EGLN15, GLN20a, GLN20b] and the references therein) – that is, the class of (unital) separable, simple, nuclear $\mathcal{Z}$-stable C*-algebras satisfying the UCT can be classified by means of K-theory and traces. The C*-algebra $\mathcal{Z}$ is the Jiang-Su algebra [JS99], $\mathcal{Z}$-stability is the property of tensorially absorbing $\mathcal{Z}$, and the UCT is the Universal Coefficient Theorem of Rosenberg and Schochet [RS87]. A C*-algebra satisfies the UCT if it is in some sense weak homotopy equivalent (KK-equivalent) to an abelian C*-algebra.

We briefly outline the definition of topological K_0 and K_1 groups of a C*-algebra, and collect any results that we will be using throughout. We direct the reader to the wonderful and accessible blue book [RLL00] on the K-theory of operator algebras. We form several algebraic objects arising from direct limits of matrix algebras over a C*-algebra A. For a ring R, let $M_n(R)$ denote the $n \times n$-matrices over R, and let

$$M_\infty(R) := \varinjlim M_n(R) \tag{1.2.1}$$

be the (ring-theoretic) inductive limit of $M_n(R)$ with connecting maps $M_n(R) \to M_{n+1}(R)$ given by $r \mapsto r \oplus 0$, which takes an element $r \in M_n(R)$ and forms a block matrix by putting r into the upper left block. If A is a C*-algebra, we let $P_\infty(A) \subseteq M_\infty(A)$ be the subset of projections – that is, elements $p \in M_\infty(A)$ satisfying $p^2 = p^* = p$.

Definition 1.2.1. *Let A be a unital C*-algebra. We say that two projections $p, q \in P_\infty(A)$ are Murray-von Neumann equivalent, denoted $p \sim_{MvN} q$, if there exists $v \in M_\infty(A)$ such that $v^*v = p$ and $vv^* = q$.*

Modding out by Murray-von Neumann equivalence yields the Murray-von Neumann semigroup of projections, denoted by

$$V(A) := P_\infty(A)/\sim_{\mathrm{MvN}}, \tag{1.2.2}$$

which is an abelian semigroup with addition $[p] + [q] := [p \oplus q]$. The K_0-group of a unital C*-algebra is then the Grothendieck group of the Murray-von Neumann semigroup $V(A)$. For non-unital C*-algebras, one defines the K_0-group through a certain split exact sequence in terms of $\mathbb{Z}$ and the K_0-group of the unitization, although we omit the definition because we will be interested in unital C*-algebras almost exclusively.

If G is a group with underlying topology (not necessarily a topological group), we will denote by G^0 its connected component of the identity. The topological K_1-group of a unital C*-algebra A is the set of connected components of the ring $GL_\infty(A)$ defined as the inductive limit of $GL_n(A) := GL(M_n(A))$, which has the subspace topology of $M_n(A)$, with connecting maps $GL_n(A) \to GL_{n+1}(A)$ given by $x \mapsto x \oplus 1$. Here, $GL_\infty(A)$ can be equipped with either the inductive limit topology of the norm topology. With either topology, we have

$$K_1(A) := GL_\infty(A)/GL^0_\infty(A).^{5} \tag{1.2.3}$$

With addition in $K_1(A)$ defined by $[x] + [y] := [x \oplus y]$, this forms an abelian group because there are natural homotopies

$$\begin{pmatrix} x & \\ & y \end{pmatrix} \sim_h \begin{pmatrix} y & \\ & x \end{pmatrix} \sim_h \begin{pmatrix} xy & \\ & 1 \end{pmatrix} \sim_h \begin{pmatrix} yx & \\ & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & \\ & xy \end{pmatrix} \sim_h \begin{pmatrix} 1 & \\ & yx \end{pmatrix} \tag{1.2.4}$$

by Whitehead's lemma (see [RLL00, Lemma 2.5]). By Proposition 1.1.4, $K_1(A)$ is also equal to $U_\infty(A)/U^0_\infty(A)$. This can further be reflected in the fact that positive

[5] Whether one considers the inductive limit topology or the norm topology on $GL_\infty(A)$ (which usually differ, see Example 1.2.3), the connected component of the identity remains the same. This is because if $\xi : [0, 1] \to GL_\infty(A)$ is a continuous path, its image lies in $GL_n(A)$ for some $n \in \mathbb{N}$ by compactness of the interval.

invertibles are always connected to the identity. Indeed, if x is a positive invertible element, then the spectrum of x is contained in $(0, \infty)$, so x has a continuous logarithm. Then the path $t \mapsto e^{t \log x}$ is a continuous path from 1 to x.

We note that both $K_0(A)$ and $K_1(A)$ are functors from the category of unital C*-algebras with unital *-homomorphisms to the category of abelian group with group homomorphisms. Moreover, both K_0 and K_1 are invariant under approximate unitary equivalence of *-homomorphisms.

Finally we note that we can identify $K_0(A)$ with $\pi_1(U_\infty^0(A))$, the fundamental group of $U_\infty^0(A)$ (or $GL_\infty^0(A)$), and $K_1(A)$ with $\pi_0(U_\infty(A))$, the set of connected components of $U_\infty(A)$ (or $GL_\infty(A)$). In the former, we identify a projection $p \in M_n(A)$ with the loop $t \mapsto pe^{2\pi i t} + (1-p), t \in [0,1]$, and the latter identification is clear. See [RLL00, Chapter 11.4], for example.

We now define the algebraic K_1-group, along with the Hausdorffized variant. For a group G, let $DG = \langle ghg^{-1}h^{-1} \mid g,h \in G \rangle$ be the derived group – i.e., the group generated by commutators. For a unital ring R, the algebraic K_1 group, denoted $\mathrm{K}_1^{\mathrm{alg}}(R)$, is the stabilized invertible group mod the commutators. That is,

$$\mathrm{K}_1^{\mathrm{alg}}(R) := GL_\infty(R)/DGL_\infty(R). \tag{1.2.5}$$

If A is a unital C*-algebra, we can in turn form the unitary algebraic K_1 group, denoted $\mathrm{K}_1^{\mathrm{alg,u}}(A)$, which is the stabilized unitary group modulo the commutators:

$$\mathrm{K}_1^{\mathrm{alg,u}}(A) := U_\infty(A)/DU_\infty(A). \tag{1.2.6}$$

The objective of Chapter 2 is to show that the invariants $\mathrm{K}_1^{\mathrm{alg}}$ and $\mathrm{K}_1^{\mathrm{alg,u}}$ can be related via the polar decomposition of invertible elements for a large class of C*-algebras, by accounting for the positive elements in terms of tracial data. We can also form Hausdorffized variants of $\mathrm{K}_1^{\mathrm{alg}}$ and $\mathrm{K}_1^{\mathrm{alg,u}}$ respectively, by modding out by an appropriate closure of the respective derived groups.

We equip $GL_\infty(A)$ and $U_\infty(A)$ with the inductive limit topologies: that is, we equip $GL_\infty(A)$ with the finest topology making the inclusion maps $GL_n(A) \hookrightarrow GL_\infty(A)$ continuous for all $n \in \mathbb{N}$, and we equip $U_\infty(A)$ with the finest topology making the inclusion maps $U_n(A) \hookrightarrow U_\infty(A)$ continuous for all n. As such, a subset $S \subseteq GL_\infty(A)$ is open if and only if $S \cap GL_n(A)$ is open for each n, and the analogous statement holds for $U_\infty(A)$. We note that the inductive limit topology on $U_\infty(A)$ agrees with the subspace topology arising from the inclusion $U_\infty(A) \subseteq GL_\infty(A)$. There are two main observations:

1. the inductive limit topology on either $GL_\infty(A)$ or $U_\infty(A)$ does not coincide with the respective relative-norm topology in general.

2. The groups $GL_\infty(A)$ and $U_\infty(A)$ are not in general topological groups with their respective inductive limit topologies.

To see the first observation, we have the following lemma.

Lemma 1.2.2. *Let $X = \cup_n X_n$ be an increasing union of metric spaces X_n (that is, $X_n \subseteq X_{n+1}$ and the inclusion $X_n \hookrightarrow X_{n+1}$ is isometric and open onto its image), equipped with the inductive limit topology. Then whenever $(x_k)_{k\in\mathbb{N}}$ is a convergent sequence in X, there exist some $N, K \in \mathbb{N}$ such that $x_k \in X_N$ for all $k \geq K$.*

Proof. Suppose not. Then for all $N, K \in \mathbb{N}$, there are some $k \geq K$ such that $x_k \notin X_N$. By passing to a subsequence we can assume that at most finitely many x_k's are in each X_N for each $N \in \mathbb{N}$. Thus the set $S := \{x_k \mid k \in \mathbb{N}\} \subseteq X$ is closed since $S \cap X_N$ is finite, hence closed, for all $N \in \mathbb{N}$. Now if a sequence forms a closed set, then it must stabilize, meaning that there is some $N \in \mathbb{N}$ such that X_N contains infinitely many x_k's, which is a contradiction. □

We use the above lemma to produce a sequence which converges in the norm topology but not the inductive limit topology. This can even be done for the C*-algebra $\mathbb{C}$ of complex numbers.

Example 1.2.3. *Let $A = \mathbb{C}$ and consider $2 \in GL(\mathbb{C}) \subseteq GL_\infty(\mathbb{C})$. Let $\varepsilon > 0$ and let $\delta > 0$ be such that*

$$a \approx_\delta b \Rightarrow e^a \approx_{\frac{\varepsilon}{2}} e^b \tag{1.2.7}$$

whenever a, b lie in the same unital C-algebra. For $n \in \mathbb{N}$, we can approximate*

$$\begin{pmatrix} \log 2 & \\ & 0_n \end{pmatrix} \approx_{\frac{\log 2}{n}} \begin{pmatrix} \log 2 & \\ & -\frac{log 2}{n} \cdot 1_n \end{pmatrix} \tag{1.2.8}$$

where the right matrix has trace 0, hence is a commutator [Sho37, AM57]. Thus we can write the right matrix as $[a_n, b_n]$ for some $a_n, b_n \in M_{n+1}(\mathbb{C})$. Now, making use of the limit

$$e^{[a,b]} = \lim_n \left(e^{\frac{a}{n}} e^{\frac{b}{n}} e^{-\frac{a}{n}} e^{-\frac{b}{n}}\right)^n$$

whenever a, b are elements lying in the same unital C-algebra, we can in fact approximate $e^{[a_n,b_n]}$ by a multiplicative commutator (in norm), say*

$$e^{[a_n,b_n]} \approx_{\frac{\varepsilon}{2}} x_n \in DGL_{n+1}(\mathbb{C}). \tag{1.2.9}$$

Therefore whenever $N \in \mathbb{N}$ is such that $\frac{\log 2}{N} < \delta$, $n \geq N$ gives that

$$\begin{aligned} \begin{pmatrix} 2 & \\ & 1_n \end{pmatrix} &= \exp \begin{pmatrix} \log 2 & \\ & 0_n \end{pmatrix} \\ &\approx_{\frac{\varepsilon}{2}} e^{[a_n,b_n]} \\ &\approx_{\frac{\varepsilon}{2}} x_n. \end{aligned} \tag{1.2.10}$$

Letting $\varepsilon \to 0^+$, we can pick an appropriate sequence $(x_n)_{n\in\mathbb{N}}$ in $DGL_\infty(\mathbb{C})$ which converges to $2 \in GL(\mathbb{C}) \subseteq GL_\infty(\mathbb{C})$, but clearly the sequence does not stabilize to any level, so this sequence does not converge in the inductive limit topology by Lemma 1.2.2.

Therefore the norm topology on $GL_\infty(A)$ is in general weaker than the inductive limit topology. Moreover, in this example, we get that $\mathrm{K}_1^{\mathrm{alg}}(A) \simeq \mathbb{C}^\times$, so that algebraic K_1 does not in general agree with K_1 (which is 0 is in this case). That said, there are classes of C*-algebras whose algebraic K_1 coincides with the topological K_1. For example, stable C*-algebras have this property, as seen by Higson in [Hig88, Theorem 2.4.6].

The same argument above can be modified to show that the norm closure of $DGL_\infty(A)$ is in fact all of $GL_\infty^0(A)$.[6] Similarly, $U_\infty^0(A)$ is the norm closure of $DU_\infty^0(A)$ in $U_\infty(A)$. Thus taking the norm closure of the commutators and modding out just yields the classic topological K_1-group.

The second point, that $GL_\infty(A)$ and $U_\infty(A)$ are in general *not* topological groups, is discussed in a footnote in [CGS+23]. This essentially follows from work done in [TSH98]. The Hausdorffized algebraic K-theories $\overline{\mathrm{K}}_1^{\mathrm{alg}}(A)$ and $\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)$, however, are topological groups – as can be seen in [CGS+23, Remark 2.11]. There they show that if one considers the strongest topology on $U_\infty(A)$ making it into a topological group, then the quotient of the closure of $DU_\infty(A)$ with respect to this topology is topologically isomorphic to the quotient of $U_\infty(A)$ by the closure of $DU_\infty(A)$ in the inductive limit topology. The analogous result holds for general linear groups.

To explicitly define the Hausdorffized algebraic K_1-groups, we denote by $CGL_\infty(A)$ and $CU_\infty(A)$ the closures in the inductive limit topology of $DGL_\infty(A)$ and $DU_\infty(A)$ respectively. We then define

$$\overline{\mathrm{K}}_1^{\mathrm{alg}}(A) := GL_\infty(A)/CGL_\infty(A) \text{ and } \overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A) := U_\infty(A)/CU_\infty(A). \tag{1.2.11}$$

Next we discuss traces. A tracial functional $\tau : A \to \mathbb{C}$ on a C*-algebra is a bounded functional such that

$$\tau(ab) = \tau(ba) \text{ for all } a, b \in A. \tag{1.2.12}$$

The set of tracial states, denoted by $T(A)$, is canonically a compact convex subset of the dual space A^* (in the unital case) and forms a Choquet simplex [Sak12, Theorem 3.1.18].

The trace simplex has a contravariant identification with the continuous real-valued affine functions on it. For a compact convex subset X of a locally convex vector space, let Aff X denote the space of continuous affine functions $f : X \to \mathbb{R}$. We now

[6] We note that $DGL_\infty^0(A) = DGL_\infty(A)$ by Whitehead – see (2.2.25). Therefore the closure is contained in the connected component of the identity.

give exposition from [Goo86, Chapter 7] to discuss the one-to-one correspondence $X \leftrightarrow \operatorname{Aff} X$ between compact convex subsets of locally convex vector spaces and partially ordered real Banach spaces with order units. Clearly $\operatorname{Aff} X$ can be thought of as a partially ordered real Banach space with order unit (the ordering being the point-wise order coming from the inclusion $\operatorname{Aff} X \subseteq C_{\mathbb{R}}(X)$).

For $\mathfrak{X}$ a partially ordered abelian real Banach space with order unit, let

$$S(\mathfrak{X}) := \{f : \mathfrak{X} \to \mathbb{R} \mid f \text{ is unital and positive}\} \tag{1.2.13}$$

denote the *state space* of $\mathfrak{X}$. The space $S(\mathfrak{X})$ is canonically a compact convex subset of $\mathfrak{X}^*$ (it is compact with respect to the wk*-topology). The following is [Goo86, Theorem 7.1]

Theorem 1.2.4. *Let X be a compact convex subset of a locally convex Hausdorff space and $S = S(\operatorname{Aff}(X))$. Then the evaluation map $\psi : S \to X$ is an affine homeomorphism of X onto S.*

As unital positive maps $\mathfrak{X} \to \mathfrak{Y}$ between partially ordered real Banach spaces with order units induce continuous affine maps $S(\mathfrak{Y}) \to S(\mathfrak{X})$ between their state spaces (by evaluation), unital positive maps $\operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ induce affine maps $T(B) \to T(A)$ by the above theorem.

On the other hand, affine continuous maps $X \to Y$ between compact convex subsets of locally convex vector spaces clearly induce unital positive maps $\operatorname{Aff} Y \to \operatorname{Aff} X$ between spaces of affine functions (by evaluation).

For unital A, we define the pairing map $\rho_A : K_0(A) \to \operatorname{Aff} T(A)$ as follows: if $x \in K_0(A)$, we can write $x = [p] - [q]$ where $p, q \in M_n(A)$ are projections, and we then define

$$\rho_A(x)(\tau) := (\operatorname{tr}_n \otimes \tau)(p - q), \tau \in T(A), \tag{1.2.14}$$

where $\operatorname{tr}_n : M_n \to \mathbb{C}$ is the unnormalized trace. We describe the KT_u invariant as in [CGS+23]. Set, for a unital C*-algebra A,

$$KT_u(A) := (K_0(A), [1]_0, K_1(A), \operatorname{Aff} T(A), \rho_A), \tag{1.2.15}$$

which is the invariant consisting of the K_0-group, the K_0-class of the unit, the K_1-group, the continuous real-valued affine functions on the trace simplex, and the pairing map between K_0 and traces. For quintuples $(G, u, H, \mathfrak{X}, \rho)$ and $(G', u', H', \mathfrak{X}', \rho')$ consisting of abelian groups G, G', H, H', distinguished elements $u \in G, u' \in G'$, partially ordered real Banach spaces with order units $\mathfrak{X}, \mathfrak{X}'$ and group homomorphisms $\rho : G \to \mathfrak{X}, \rho' : G' \to \mathfrak{X}'$ such that $\rho(u) = 1_{\mathfrak{X}}$ and $\rho'(u') = 1_{\mathfrak{X}'}$, a KT_u-morphism is a triple

$$(\alpha, \beta, \gamma) : (G, u, H, \mathfrak{X}, \rho) \to (G', u', H', \mathfrak{X}', \rho') \tag{1.2.16}$$

where $\alpha : G \to G', \beta : H \to H'$ are group homomorphisms with $\alpha(u) = u'$, and $\gamma : \mathfrak{X} \to \mathfrak{X}'$ is a unital positive map such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G' \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho'} \\ \mathfrak{X} & \xrightarrow[\gamma]{} & \mathfrak{X}' \end{array} \tag{1.2.17}$$

commutes.

We note that in the case of a unital, separable, simple, nuclear, $\mathcal{Z}$-stable C*-algebra, $KT_u(A)$ completely determines the Elliott invariant

$$\mathrm{Ell}(A) := (K_0(A), K_0(A)^+, [1]_0, K_1(A), T(A), r_A), \tag{1.2.18}$$

where $K_0(A)^+ := \{[p] \mid p \in P_\infty(A)\} \subseteq K_0(A)$ is the positive cone, and

$$r_A : K_0(A) \times T(A) \to \mathbb{R} \tag{1.2.19}$$

is the pairing map, which is dual to ρ_A in the sense that

$$r_A(x, \tau) = \rho_A(x)(\tau) \text{ for } x \in K_0(A), \tau \in T(A). \tag{1.2.20}$$

Indeed, in the case when A is infinite [7], $K_0(A)^+ = K_0(A), T(A) = \emptyset$, and r_A is identically 0. If A is stably finite[8], has strict comparison (which is implied by $\mathcal{Z}$-stability) and is simple, then $\rho_A(x)(\tau) > 0$ for all τ if and only if $x \in K_0(A)^+$. Now for a KT_u-morphism (α, β, γ) between such algebras, γ will preserve the strict order $\gg$ on $\mathrm{Aff}\, T(A)$ given by $f \gg g$ if $f(\tau) > g(\tau)$ for all $\tau \in T(A)$ (since traces are automatically faithful by simplicity). Therefore the commutation of the pairing maps gives that the group homomorphism α is automatically positive.

1.3 Ultrapowers, central sequences and central sequence algebras

Fix a free ultrafilter $\omega \in \beta\mathbb{N}$. Throughout we will use ultrapowers to describe asymptotic behaviour. Alternatively one can use sequence algebras, although this comes down to a matter of taste and one can swap between the two if desired as we will provide local characterizations. This also means that all of what we do will be independent of the specific ultrafilter ω.

For a C*-algebra A, the ultrapower of A is the C*-algebra

$$A_\omega := \ell^\infty(A)/c_{0,\omega}(A), \tag{1.3.1}$$

[7] When A is unital, infinite means that there exists a proper isometry.

[8] This means whenever $n \in \mathbb{N}$, $M_n(A)$ has the property that every isometry is unitary.

where $c_{0,\omega} := \{(a_n)_{n\in\mathbb{N}} \in \ell^\infty(A) \mid \lim_{n\to\omega} \|a_n\| = 0\}$ is the ideal of ω-null sequences. We can embed A into A_ω canonically by means of constant sequences: we identify $a \in A$ with the equivalence class of the constant sequence $(a)_{n\in\mathbb{N}}$.

To ease notation, we will usually write elements of A_ω as sequences $(a_n)_{n\in\mathbb{N}}$, keeping in mind that these are equivalence classes without explicitly stating it every time. We note that the norm on A_ω is given by $\|(a_n)_{n\in\mathbb{N}}\| = \lim_{n\to\omega} \|a_n\|$.

Kirchberg's ε-test ([Kir06], Lemma A.1) is essentially the operator algebraists' Łoś' theorem without having to turn to (continuous) model theory. Heuristically, it says that if certain things can be done approximately in an ultrapower, then they can be done exactly in an ultrapower.

Lemma 1.3.1 (Kirchberg's ε-test)**.** *Let $(X_n)_n$ be a sequence of sets and suppose that for each n, there is a sequence $(f_n^{(k)})_{k\in\mathbb{N}}$ of functions $f_n^{(k)} : X_n \to [0,\infty)$. For $k \in \mathbb{N}$, let*

$$f_\omega^k(s_1, s_2, \dots) := \lim_{n\to\omega} f_n^{(k)}(s_n). \tag{1.3.2}$$

Suppose that for every $m \in \mathbb{N}$ and $\varepsilon > 0$, there is $s \in \prod_n X_n$ with $f_\omega^{(k)}(s) < \varepsilon$ for $k = 1, \dots, m$. Then there exists $t \in \prod_n X_n$ with $f_\omega^{(k)}(t) = 0$ for all $k \in \mathbb{N}$.

The above is useful, although if one so wishes, one can usually construct exact objects from approximate objects by using standard diagonalization arguments (under some separability assumptions). These sorts of arguments work in both the ultrapower setting and the sequence algebra setting.

Finally, if $\alpha \in \mathrm{Aut}(A)$ is an automorphism, there is an induced automorphism on A_ω, which we will denote by α_ω given by

$$\alpha_\omega((a_n)) := (\alpha(a_n)). \tag{1.3.3}$$

For a unital C*-algebra A, the C*-algebra of ω-central sequences is

$$A_\omega \cap A' := \{x \in A_\omega \mid [x, a] = 0 \text{ for all } a \in A\}, \tag{1.3.4}$$

where we are identifying $A \subseteq A_\omega$ with the constant sequences. Moreover, if $B \subseteq A$ is a unital inclusion and $S \subseteq A$ is a subset, we set

$$B_\omega \cap S' = \{b \in B_\omega \mid [b, s] = 0 \text{ for all } s \in S\}. \tag{1.3.5}$$

1.4 Strongly self-absorbing C*-algebras

A unital separable C*-algebra $\mathcal{D}$ is strongly self-absorbing if $\mathcal{D} \not\cong \mathbb{C}$ and there is an isomorphism $\phi : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the first factor embedding $d \mapsto d \otimes 1_{\mathcal{D}}$ (see [TW07]). All known strongly self-absorbing C*-algebras are: the Jiang-Su algebra $\mathcal{Z}$ [JS99], the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ [Cun77], UHF algebras of infinite type, and $\mathcal{O}_\infty$ tensor a UHF algebra of infinite type. Strongly self-absorbing C*-algebras have approximately inner flip, and therefore there are K-theoretic restrictions on strongly self-absorbing C*-algebras – see [Tik16, EST23]. They are also nuclear, simple, and have at most one tracial state [TW07].

Tensorial absorption with strongly self-absorbing C*-algebras gives rise to many regular properties, for example in terms of K-theory, traces, and the Cuntz semigroup [JS99, Rør91, Rør92, Rør04]. Of paramount interest is the Jiang-Su algebra $\mathcal{Z}$. An accumulation of work has successfully classified all (unital) separable, simple, nuclear, infinite-dimensional, $\mathcal{Z}$-stable C*-algebras satisfying the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet [RS87] by means of K-theory and traces. We describe how one might work with $\mathcal{Z}$-stability in terms of its standard building blocks. Recall that, for $n, m \geq 2$, the dimension drop algebras are

$$\mathcal{Z}_{n,m} := \{f \in C([0,1], M_n \otimes M_m) \mid f(0) \in M_n \otimes 1_{M_m}, f(1) \in 1_{M_n} \otimes M_m\}. \quad (1.4.1)$$

Such an algebra is a called a prime dimension drop algebra when n and m are coprime. The Jiang-Su algebra $\mathcal{Z}$ is the unique separable simple C*-algebra with unique tracial state which is an inductive limit of prime dimension drop algebras with unital connecting maps [JS99] (in fact, the dimension drop algebras can be chosen to have the form $\mathcal{Z}_{n,n+1}$). It is KK-equivalent to $\mathbb{C}$ and $\mathcal{Z}$-stability is often a necessary condition for K-theoretic classification.

By [RW10, Proposition 5.1] (or [Sat10, Proposition 2.1] for our desired formulation), $\mathcal{Z}_{n,n+1}$ is the universal C*-algebra generated by elements $c_1, \ldots, c_n$ and s such that

- $c_1 \geq 0$;
- $c_i c_j^* = \delta_{ij} c_1^2$;
- $s^*s + \sum_{i=1}^n c_i^* c_i = 1$;
- $c_1 s = s$.

If there are uniformly tracially large (in the sense of [TWW15, Definition 2.2]) order zero[9] c.p.c. maps $M_n \to A_\omega \cap A'$, these give rise to elements $c_1, \ldots, c_n \in A_\omega \cap A'$ with $c_1 \geq 0$ and $c_i c_j^* = \delta_{ij} c_1^2$, along with certain tracial information. If A has strict comparison, Matui and Sato used this tracial information to show that A has property (SI) [MS12], from which one can get an element $s \in A_\omega \cap A'$ such that $s^*s + \sum_{i=1}^n c_i^* c_i = 1$ and $c_1 s = s$. This gives a *-homomorphism $\mathcal{Z}_{n,n+1} \to A_\omega \cap A'$, which if can be done for each $n \in \mathbb{N}$, is enough to conclude that $\mathcal{Z} \hookrightarrow A_\omega \cap A'$ unitally and hence $A \simeq A \otimes \mathcal{Z}$ (see [TW08, Win11]). In fact, it suffices to show that $\mathcal{Z}_{2,3} \hookrightarrow A_\omega \cap A'$ (or $\mathcal{Z}_{n,n+1}$ for some $n \geq 2$), see [RW10, Theorem 3.4(ii)] and [Sch22, Theorem 5.15].

[9] order zero meaning orthogonality preserving: $\phi : A \to B$ is c.p.c. order zero if it is c.p.c. and $\phi(a)\phi(b) = 0$ whenever $a, b \in A$ are positive elements satisfying $ab = 0$.

Chapter 2

Polar decomposition in algebraic K-theory

The K-theory of C*-algebras is an invaluable tool for distinguishing isomorphism classes of C*-algebras. Further, for many C*-algebras, K-theory together with traces and appropriate pairing between them, forms a complete invariant: two "classifiable" C*-algebras (unital, separable, simple, nuclear, $\mathcal{Z}$-stable C*-algebra satisfying the UCT) are isomorphic if and only if they have the same K-theory and traces (see the references in Section 1.2).

However when it comes to the classification of morphisms, K-theory and traces can fail to distinguish two morphisms which are not approximately unitarily equivalent. Indeed, Nielsen and Thomsen produced automorphisms of a circle algebra which agree on topological K-theory but not on $\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)$, which is invariant under approximate unitary equivalence – see [NT96, Section 5].[10]

Oftentimes invariants which are necessary for the classification of morphisms (up to approximate unitary equivalence), such as $\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(\cdot)$, can be described in terms of the K-theory and traces of the C*-algebra – although necessarily in an unnatural way. For example, [Tho95, Theorem 3.2 and Corollary 3.3] give rise to the following exact sequence

$$0 \to \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} \to \overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A) \to K_1(A) \to 0, \tag{2.0.1}$$

which splits due to the fact that the left group can be identified with $U^0_\infty(A)/CU^0_\infty(A)$, which can be shown to be divisible (alternatively, $\operatorname{Aff} T(A)$ is divisible, as it is a real vector space, and any quotient of a divisible abelian group is divisible). Using similar arguments, one can describe $\overline{\mathrm{K}}_1^{\mathrm{alg}}(A)$ in terms of $K_1(A)$ and "complex-valued affine functions", and again this splitting will be unnatural.

[10] It was actually shown that there are homomorphisms which agree on K-theory and traces, but not on $U(A)/CU(A)$. However, this is (topologically) isomorphic to $\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)$ by the discussion in Section 2.3.

In this chapter, we will be interested in comparing $\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)$ with $\overline{\mathrm{K}}_1^{\mathrm{alg}}(A)$, along with finding relationships between their purely algebraic counterparts $\mathrm{K}_1^{\mathrm{alg,u}}(A)$ and $\mathrm{K}_1^{\mathrm{alg}}(A)$. Just considering topological K_1-group of a unital C*-algebra, this can be realized as the set of connected components of $GL_\infty(A)$ (which forms an abelian group in this case). Positive elements contribute nothing as they are all connected to the identity, so the K_1-class of an invertible is entirely determined by the unitary part in the polar decomposition (Proposition 1.1.4). This gives a canonical identification of $GL_\infty(A)/GL_\infty^0(A)$ and $U_\infty(A)/U_\infty^0(A)$. In the algebraic K-theoretic setting, positive elements often have non-trivial algebraic K_1-classes, both in the non-Hausdorffized and Hausdorffized settings (see Example 1.2.3). We show that if we keep track of positive invertible elements via tracial data, then in fact there will be natural relationships between the GL-variants and the U-variants of algebraic K_1-groups.

First we show that if Δ is the de la Harpe Skandalis determinant, originally defined in [dlHS84a], then we have the following natural splitting.

Theorem 2.A (Theorem 2.2.8)**.** *Let A be a unital C*-algebra. Then there is a natural isomorphism*

$$GL_\infty(A)/\ker\Delta \simeq U_\infty(A)/\ker\Delta|_{U_\infty^0(A)} \oplus \operatorname{Aff} T(A). \tag{2.0.2}$$

In particular, for C*-algebras where the left group in (2.0.2) is $\mathrm{K}_1^{\mathrm{alg}}(A)$ and the left direct summand of the right of (2.0.2) is $\mathrm{K}_1^{\mathrm{alg,u}}(A)$, we get a relationship between the general linear and unitary variants of algebraic K_1. Using [NR17], we have a relationship between the two for an abundance of C*-algebras.

Corollary 2.B (Corollary 2.2.12)**.** *Let A be a unital, separable, simple C*-algebra of stable rank one which is pure in the sense of [Win12, Definition 3.6] and such that every 2-quasitracial state on A is a trace (in particular, A can be any unital, separable, simple, finite, exact, $\mathcal{Z}$-stable C*-algebra). Then there is a natural isomorphism*

$$\mathrm{K}_1^{alg}(A) \simeq \mathrm{K}_1^{alg,u}(A) \oplus \operatorname{Aff} T(A). \tag{2.0.3}$$

Similar techniques can be applied with Thomsen's variant $\overline{\Delta}$ of the de la Harpe-Skandalis determinant in place of Δ to obtain the following.

Theorem 2.C (Theorem 2.2.15)**.** *Let A be a unital C*-algebra. Then there is a natural isomorphism of topological groups*

$$\overline{\mathrm{K}}_1^{alg}(A) \simeq \overline{\mathrm{K}}_1^{alg,u}(A) \oplus \operatorname{Aff} T(A). \tag{2.0.4}$$

This chapter is structured as follows. We introduce each of the variants of the de la Harpe–Skandalis determinant and discuss some relationships between kernels in Section 2.1. In Section 2.2, we prove the main results (Theorems 2.A and 2.C). In

Section 2.3 we look at non-stable analogues of the results in 2.2, under the hypothesis of certain K-theoretic regularity conditions.

Following the writing of [ST23], it was pointed out that George Elliott established a variation of the main results as [Ell22, Theorem 5]. We make use of different techniques and exposition.

2.1 The de la Harpe–Skandalis determinant

We recall the definition of the de la Harpe–Skandalis determinant [dlHS84a] (see [dlH13] for a more in-depth exposition). Let $\mathrm{Tr}_A : A \to A/\overline{[A,A]}$ be the quotient map from A to the quotient Banach space $A/\overline{[A,A]}$ where $\overline{[A,A]}$ is the closed linear span of additive commutators. We call Tr_A the universal trace on A and will usually omit the subscript when the C*-algebra is clear from context. For $n \in \mathbb{N}$, we canonically extend Tr to $M_n(A)$ by $\mathrm{Tr}((a_{ij})_{i,j}) = \sum_i \mathrm{Tr}(a_{ii})$ (the same can be done for any tracial map). For $n \in \mathbb{N} \cup \{\infty\}$ and a piece-wise smooth path $\xi : [0,1] \to GL_n(A)$, set

$$\tilde{\Delta}^n(\xi) := \frac{1}{2\pi i}\int_0^1 \mathrm{Tr}(\xi'(t)\xi(t)^{-1})dt.^{11} \tag{2.1.1}$$

We call the map $\tilde{\Delta}^n$ the *pre-determinant*. The following properties can be found as [dlHS84a, Lemme 1].

Proposition 2.1.1. *The map $\tilde{\Delta}^n$ which takes a piece-wise smooth path to an element in $A/\overline{[A,A]}$ has the following four properties:*

1. *it takes pointwise products to sums: if ξ_1, ξ_2 are two piece-wise smooth paths, then*
$$\tilde{\Delta}^n(\xi_1\xi_2) = \tilde{\Delta}^n(\xi_1) + \tilde{\Delta}^n(\xi_2), \tag{2.1.2}$$
where $\xi_1\xi_2$ is the piece-wise smooth path $t \mapsto \xi_1(t)\xi_2(t)$ from $\xi_1(0)\xi_2(0)$ to $\xi_1(1)\xi_2(1)$;
2. *if $\|\xi(t) - 1\| < 1$ for all $t \in [0,1]$, then*
$$2\pi i\tilde{\Delta}^n(\xi) = Tr\big(\log(\xi(1)) - \log\xi(0)\big); \tag{2.1.3}$$
3. *$\tilde{\Delta}^n(\xi)$ depends only on the homotopy[12] class of ξ;*
4. *if $p \in M_n(A)$ is an idempotent, then the path $\xi_p : [0,1] \to GL_n^0(A)$ given by $\xi_p(t) := pe^{2\pi it} + (1-p)$ satisfies $\tilde{\Delta}^n(p) = Tr(p)$.*

[11] We note that when $n = \infty$, by compactness, the image of ξ is contained in some $GL_m(A)$ for $m < \infty$, so we can take the trace.

[12] Here we allow continuous homotopies – the homotopies need not be piece-wise smooth.

The de la Harpe–Skandalis determinant (at the n^{th} level) is then the map

$$\Delta^n : GL^0_\infty(A) \to \left(A/\overline{[A,A]}\right)/\tilde{\Delta}^n(\pi_1(GL^0_n(A))) \tag{2.1.4}$$

given by $\Delta^n(x) := [\tilde{\Delta}^n(\xi_x)]$ where ξ_x is any piece-wise smooth path $\xi_x : [0,1] \to GL^0_n(A)$ from 1 to x. This is a group homomorphism to an abelian group, and therefore factors through the derived group, i.e., $DGL^0_n(A) \subseteq \ker \Delta^n$. For the case $n = \infty$, we just write $\tilde{\Delta}$ and Δ for $\tilde{\Delta}^\infty$ and Δ^∞ respectively. If the C*-algebra needs to be specified, we will write Δ^n_A or Δ_A.

Remark 2.1.2.

1. Every continuous path $[0,1] \to GL_n(A)$ is homotopic to a piece-wise smooth path (even a piece-wise smooth exponential path if we are in $GL^0_n(A)$), and as $\tilde{\Delta}^n$ is homotopy-invariant, it makes sense to apply $\tilde{\Delta}^n$ to any continuous path. Indeed, as in the proof of [dlHS84a, Lemme 3], take any continuous path $\xi : [0,1] \to GL_n(A)$ and choose k such that

$$\left\| \xi\left(\frac{j-1}{k}\right)^{-1} \xi\left(\frac{j}{k}\right) - 1 \right\| < 1 \text{ for all } j = 1, \dots, k. \tag{2.1.5}$$

Then taking $a_j := \frac{1}{2\pi i} \log\left(\xi(\frac{j-1}{k})^{-1}\xi(\frac{j}{k})\right), j = 1, \dots, k$, ξ will be homotopic to the path

$$\eta(t) = \xi\left(\frac{j-1}{k}\right) e^{(kt-j)a_j}, t \in \left[\frac{j-1}{k}, \frac{j}{k}\right]. \tag{2.1.6}$$

We note that if $a = \sum_{j=1}^k a_j$, then $\tilde{\Delta}^n(\xi) = \tilde{\Delta}^n(\eta) = \mathrm{Tr}(a)$. If ξ is a path of unitaries, then so is η and the a_j's are self-adjoint.

2. Unless we make any regularity assumptions, the maps Δ^n may have different codomains as n varies since the images $\tilde{\Delta}^n(\pi_1(GL^0_n(A)))$ may vary. We do however always have

$$\tilde{\Delta}^n(\pi_1(GL^0_n(A))) \subseteq \tilde{\Delta}^{n+1}(\pi_1(GL^0_{n+1}(A))) \tag{2.1.7}$$

since $\tilde{\Delta}^{n+1}(\xi \oplus 1) = \tilde{\Delta}^n(\xi)$ whenever ξ is a piece-wise smooth loop in $GL^0_n(A)$. However, when the canonical map $\pi_1(GL^0(A)) \to K_0(A)$ is surjective, we have that $\Delta^n = \Delta|_{GL^0_n(A)}$.

We note that $\pi_1(GL^0_\infty(A)) \simeq K_0(A)$ canonically via the map induced by $[\xi_p] \mapsto p$, where ξ_p is the path in property (4) above, and consequently Δ can be thought of as a map

$$GL^0_\infty(A) \to \left(A/\overline{[A,A]}\right)/\mathrm{Tr}(K_0(A)). \tag{2.1.8}$$

Let A_0 consist of elements $a \in A_{sa}$ satisfying $\tau(a) = 0$ for all $\tau \in T(A)$. This is a norm-closed real subspace of A_{sa} such that $A_0 \subseteq \overline{[A, A]}$ (as $\overline{[A, A]}$ is the subspace of all elements in A which vanish on every tracial state), and there is an isometric identification $A_{sa}/A_0 \simeq \operatorname{Aff} T(A)$ sending an element $[a]$ to $\widehat{a}$, where $\widehat{a}(\tau) := \tau(a)$. Indeed, it is not hard to see that the map $A_{sa}/A_0 \to \operatorname{Aff} T(A)$ given by $[a] \mapsto \hat{a}$ is a well-defined $\mathbb{R}$-linear map. Moreover [CP79, Theorem 2.9], together with a convexity argument, gives that this is isometric identification. To see surjectivity, we note that the image of this map contains constant functions and separates points, so [Goo86, Corollary 7.4] gives that the image is dense, and therefore all of $\operatorname{Aff} T(A)$ (since this is an isometry). We freely identify A_{sa}/A_0 with $\operatorname{Aff} T(A)$.

Lemma 2.1.3. *Let A be a unital C^*-algebra. The canonical map $\Theta : \operatorname{Aff} T(A) \to A/\overline{[A, A]}$ given by $\Theta(\hat{a}) := [a]$ is an $\mathbb{R}$-linear isometry.*

Proof. Identifying $\operatorname{Aff} T(A)$ with A_{sa}/A_0, Θ is the map $[a]_{A_{sa}/A_0} \mapsto [a]_{A/\overline{[A,A]}}$. Clearly this is $\mathbb{R}$-linear if it is well-defined, and it is well-defined since $A_0 \subseteq \overline{[A, A]}$. To show that it's isometric, we have the following chain of inequalities. For $a \in A_{sa}$,

$$\begin{aligned} \sup_{\tau \in T(A)} |\tau(a)| &\leq \sup_{\tau \text{ s.a, tracial, } \|\tau\|=1} |\tau(a)| \\ &\leq \sup_{\tau \text{ tracial, } \|\tau\|=1} |\tau(a)| \\ &= \inf_{x \in \overline{[A,A]}} \|a + x\| \\ &\leq \inf_{x \in A_0} \|a + x\| \\ &= \sup_{\tau \in T(A)} |\tau(a)|, \end{aligned} \tag{2.1.9}$$

where the inequalities are obvious, the first equality follows from a standard Hahn–Banach argument, and the last equality comes from our isometric identification $A_{sa}/A_0 \simeq \operatorname{Aff} T(A)$. □

Consequently, we can think of the map $\operatorname{Tr}|_{A_{sa}}$ as the map from $A_{sa} \to \operatorname{Aff} T(A)$ $(\simeq A_{sa}/A_0)$ given by $a \mapsto \hat{a}$.

Lemma 2.1.4. *Let A be a unital C^*-algebra. For $n \in \mathbb{N} \cup \{\infty\}$,*

$$\tilde{\Delta}^n(\pi_1(GL_n^0(A))) = \tilde{\Delta}^n(\pi_1(U_n^0(A))) \subseteq \Theta(\operatorname{Aff} T(A)). \tag{2.1.10}$$

Proof. As $U_n(A)$ is a retract of $GL_n(A)$, the first equality is clear by Proposition 2.1.1(3). Now suppose we have a piece-wise smooth loop $\xi : [0, 1] \to U_n^0(A)$. By [Phi92, Proposition 1.4], $\xi'(t)\xi(t)^{-1}$ is skew-adjoint so that $\frac{1}{2\pi i}\xi'(t)\xi(t)^{-1}$ is self-adjoint. Therefore $\operatorname{Tr}(\frac{1}{2\pi i}\xi'(t)\xi(t)^{-1}) \in \Theta(\operatorname{Aff} T(A))$ and, since $\Theta(\operatorname{Aff} T(A))$ is a closed real subspace,

$$\tilde{\Delta}^n(\xi) = \int_0^1 \mathrm{Tr}\left(\frac{1}{2\pi i}\xi'(t)\xi(t)^{-1}\right) dt \in \Theta(\mathrm{Aff}\, T(A)) \tag{2.1.11}$$

as well. □

Corollary 2.1.5. *Let A be a unital C^*-algebra. For $n \in \mathbb{N} \cup \{\infty\}$ and $u \in U_n^0(A)$,*

$$\Delta^n(u) \in \Theta(\mathrm{Aff}\, T(A))/\tilde{\Delta}^n(\pi_1(U_n^0(A))). \tag{2.1.12}$$

For $[x] \in A/\overline{[A,A]}$, we will write

$$\mathrm{Re}([x]) := [\mathrm{Re}(x)] = \Theta\left(\frac{\widehat{x+x^*}}{2}\right) \in \Theta(\mathrm{Aff}\, T(A)), \tag{2.1.13}$$

which is well-defined since $\overline{[A,A]}$ is closed under adjoints. Note that $\mathrm{Re}(i[x]) = 0$ if $[x] \in \Theta(A_{sa}/A_0)$, and therefore $\mathrm{Re}(i\Delta^n(\cdot)) : GL_n^0(A) \to A_{sa}/A_0$ is well-defined. With this, we have the following fact:

$$\mathrm{Re}(2\pi i\Delta^n(x)) = 2\pi i\Delta^n(|x|) = [\log|x|]. \tag{2.1.14}$$

To see this, let $\xi_0 : [0,1] \to U_n^0(A)$ be any path from 1 to u_x, the unitary part in the polar decomposition of x, and let $\xi_1 : [0,1] \to GL_n^0(A)$ be the path from 1 to $|x|$ given by $t \mapsto e^{2\pi i t \log|x|}$. Then $\Delta^n(x)$ is the class of $\tilde{\Delta}^n(\xi_0) + \tilde{\Delta}^n(\xi_1)$ mod $\tilde{\Delta}(\pi_1(GL_n^0(A)))$ (which is contained in $\Theta(\mathrm{Aff}\, T(A))$. As $\tilde{\Delta}^n(\xi_0) \in \Theta(\mathrm{Aff}\, T(A))$, $\mathrm{Re}(2\pi i\tilde{\Delta}^n(\xi_0)) = 0$, leaving $\mathrm{Re}(2\pi i\tilde{\Delta}^n(\xi_1)) = 2\pi i\tilde{\Delta}^n(\xi_1)$. Moreover, $2\pi i\tilde{\Delta}^n(\xi_1)$ is clearly equal to $\Theta(\widehat{\log|x|})$ by (2.1.1).

2.1.1 Thomsen's variant

Thomsen's variant of the de la Harpe–Skandalis determinant is the Hausdorffized version, taking into account the closure of the image of the homotopy groups. We consider the map

$$\bar{\Delta}^n : GL_n^0(A) \to \left(A/\overline{[A,A]}\right)/\overline{\tilde{\Delta}^n(\pi_1(GL_n^0(A)))}, \tag{2.1.15}$$

given by $\bar{\Delta}^n(x) := [\tilde{\Delta}^n(\xi_x)]$ where $\xi_x : [0,1] \to GL_n^0(A)$ is any piece-wise smooth path from 1 to $x \in GL_n^0(A)$. This is almost the same map as Δ^n, except the codomain is now the quotient by the closure of the image of the fundamental group under the pre-determinant (i.e., the Hausdorffization of the codomain). Unlike with Δ^n, [Tho95, Lemma 3.1] gives that the kernel of $\bar{\Delta}^n$ can be identified, without any regularity assumptions on the C*-algebra.

Lemma 2.1.6. *Let A be a unital C^*-algebra.*

1. $\ker \bar{\Delta}^n|_{U_n^0(A)} = CU_n^0(A)$;
2. $\ker \bar{\Delta}^n = CGL_n^0(A)$.

We note that [Tho95, Lemma 3.1] only proves (1) above. However working with exponentials e^a with $a \in A$ instead of e^{ia} for $a \in A_{sa}$ yields (2).

Two things follow for free here: the first is that $CGL_n^0(A) \cap U_n^0(A) = CU_n^0(A)$ (the inclusion $\supseteq$ is automatic, while $\subseteq$ follows from (1)). The second is that the canonical map $U_n^0(A)/CU_n^0(A) \to GL_n^0(A)/CGL_n^0(A)$ is an injection for $n \in \mathbb{N} \cup \{\infty\}$. Thomsen also gave the following unnatural direct sum decomposition of $\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)$ in terms of K-theory and traces (the sequence in (2.1.16) splits as one can show that $\mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}$ is a divisible abelian group).

Theorem 2.1.7 (Corollary 3.3, [Tho95])**.** *Let A be a unital C^*-algebra. There is an exact sequence*

$$0 \to \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))} \to \overline{\mathrm{K}}_1^{alg,u}(A) \to K_1(A) \to 0, \tag{2.1.16}$$

which splits unnaturally.

Indeed, the splitting above is necessarily unnatural as can be seen in [NT96, Section 5]. These give examples of morphisms which agree on K-theory and traces but disagree on $U(A)/CU(A)$.[13]

2.2 Polar decomposition

We produce direct sum decompositions of the algebraic K_1-group of a C*-algebra in terms of the unitary algebraic K_1-group and traces. We provide Hausdorffized versions as well. We motivate this with the example of the complex numbers.

Example 2.2.1. *Let $A = \mathbb{C}$. Then we have isomorphisms*

$$\begin{aligned} \mathrm{K}_1^{alg}(A) &\simeq \mathbb{C}^\times, \quad \textit{and} \\ \mathrm{K}_1^{alg,u}(A) &\simeq \mathbb{T}, \end{aligned} \tag{2.2.1}$$

via the (usual) determinant map, and

$$\mathrm{Aff}\, T(A) \simeq \mathbb{R}, \tag{2.2.2}$$

since A has a unique trace. Hence we see that $\mathrm{K}_1^{alg}(A) \simeq \mathbb{T} \oplus \mathrm{Aff}\, T(A)$. The projection $\mathrm{K}_1^{alg}(A) \to \mathrm{Aff}\, T(A)$ is given by the canonical map $\log(|\cdot|) : \mathbb{C}^\times \to \mathbb{R}$, so it is the map

$$[x] \mapsto \log(|\det(x)|) = \log(\det(|x|)) = tr(\log(|x|)), \tag{2.2.3}$$

where tr is the unnormalized trace on $M_\infty(A)$ (which agrees with tr_n if $x \in M_n(A)$).

[13] $U(A)/CU(A)$ is isomorphic to $\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)$ as a topological group in this case since the C*-algebras in question satisfy certain K-theoretic regularity conditions - see Remark 2.3.5.

2.2.1 Non-Hausdorffized algebraic K-theory

We start by examining the structure of $U_n^0(A)/\ker \Delta^n|_{U_n^0(A)}$ and $GL_n^0(A)/\ker \Delta^n$. We will then apply these results to C*-algebras satisfying

$$DU_\infty^0(A) = \ker \Delta|_{U_\infty^0(A)} \text{ and } DGL_\infty^0(A) = \ker \Delta. \tag{2.2.4}$$

First we show that Δ^n is invariant under conjugation by elements of $GL_n(A)$. The following is not obvious from the fact that Δ^n is a homomorphism since $\Delta^n(s)$ is not defined when $s \notin GL_n^0(A)$.

Lemma 2.2.2. *Let A be a unital C*-algebra. For $x \in GL_n^0(A), \Delta^n(s^{-1}xs) = \Delta^n(x)$ for any $s \in GL_n(A)$.*

Proof. If $\xi : [0,1] \to GL_n^0(A)$ is a piece-wise smooth path from 1 to x, then $\eta := s^{-1}\xi(\cdot)s : [0,1] \to GL_n^0(A)$ is a piece-wise smooth path from 1 to $s^{-1}xs$ with $\eta'(t) = s^{-1}\xi'(t)s$ (whenever we can differentiate). Consequently (using (2.1.1)),

$$\tilde{\Delta}^n(\eta) = \tilde{\Delta}^n(\xi), \tag{2.2.5}$$

and the result follows. □

Lemma 2.2.3. *Let A be a unital C*-algebra. For $n \in \mathbb{N} \cup \{\infty\}, x, y \in GL_n(A)$,*

$$\Delta^n(|xy|) = \Delta^n(|x|) + \Delta^n(|y|). \tag{2.2.6}$$

Proof. Let $xy = u|xy|$, $x = u_x|x|$, and $y = u_y|y|$ be polar decompositions with $u, u_x, u_y \in U_n(A)$. Then $|u^*xy| = |xy| = u^*xy$, and in $GL_n(A)/DGL_n(A)$, we have

$$[|xy|] = [u^*xy] = [u^*u_x|x|u_y|y|] = [u^*u_xu_y] + [|x||y|]. \tag{2.2.7}$$

Hence by the previous lemma and using the fact (2.1.14) that $i\Delta^n(|z|) = \mathrm{Re}(i\Delta^n(z))$ for $z \in GL_n^0(A)$,

$$\begin{aligned} i\Delta^n(|xy|) &= \mathrm{Re}(i\Delta^n(|xy|)) \\ &= \mathrm{Re}\big(i\Delta^n(u^*u_xu_y) + \Delta^n(|x||y|)\big) \\ &= 0 + i\Delta^n(|x|) + i\Delta^n(|y|), \end{aligned} \tag{2.2.8}$$

as desired. □

Lemma 2.2.4. *Let A be a unital C*-algebra. The map*

$$\chi_n : GL_n(A)/\ker \Delta^n \to \mathrm{Aff}\, T(A) \tag{2.2.9}$$

defined by $[x] \mapsto \widehat{\log |x|}$ is a well-defined surjective group homomorphism.

Proof. With our identification $A_{sa}/A_0 = \operatorname{Aff} T(A)$, it is enough to show that

$$GL_n(A)/\ker\Delta^n \to A_{sa}/A_0 : x \mapsto [\log|x|] \tag{2.2.10}$$

is well-defined. Let $x, y \in GL_n(A)$ with $x = yz$ for some $z \in \ker\Delta^n$. Now we have

$$\begin{aligned}[\log|x|] &\overset{(2.1.14)}{=} 2\pi i\Delta^n(|x|)\\ &= 2\pi i\Delta^n(|yz|)\\ &\overset{(2.2.6)}{=} 2\pi i\Delta^n(|y|) + 2\pi i\Delta^n(|z|)\\ &\overset{(2.1.14)}{=} [\log|y|] + \operatorname{Re}(2\pi i\Delta^n(z))\\ &= [\log|y|].\end{aligned} \tag{2.2.11}$$

The fact that χ_n is additive also follows from the previous lemma. Finally, χ_n is surjective since, for $a \in A_{sa}$,

$$\chi_n([e^a]) = [a]. \tag{2.2.12}$$

□

We will write χ_n^0 for $\chi_n|_{GL_n^0(A)/\ker\Delta^n}$.

Lemma 2.2.5. *Let A be a unital C^*-algebra. The map*

$$s_n : \operatorname{Aff} T(A) \to GL_n^0(A)/\ker\Delta^n \tag{2.2.13}$$

given by $\hat{a} \mapsto [e^a]$ is a well-defined group homomorphism.

Proof. Again, we will identify $\operatorname{Aff} T(A) = A_{sa}/A_0$. If $[a] = [b]$ in A_{sa}/A_0, then

$$\Delta^n(e^a) = \frac{1}{2\pi i}\operatorname{Tr}(a) = \frac{1}{2\pi i}\operatorname{Tr}(b) = \Delta^n(e^b), \tag{2.2.14}$$

giving well-definedness. Moreover, for $[a], [b] \in A_{sa}/A_0$,

$$\begin{aligned}\Delta^n(e^{a+b}) &= \frac{1}{2\pi i}\operatorname{Tr}(a+b)\\ &= \frac{1}{2\pi i}\operatorname{Tr}(a) + \frac{1}{2\pi i}\operatorname{Tr}(b)\\ &= \Delta^n(e^a) + \Delta^n(e^b).\end{aligned} \tag{2.2.15}$$

Hence, $[e^{a+b}] = [e^a e^b]$ in $GL_n^0/\ker\Delta^n$. □

Remark 2.2.6. Now we explain why we are working with Tr instead of working with each tracial state concurrently. If we worked with Δ_τ, where τ ranges over $\tau \in T(A)$, the same arguments above will hold. However, unless one makes a separability assumption (more specifically, that $K_0(A)$ is countable), we don't necessarily have $\ker\Delta = \bigcap_{\tau\in T(A)} \ker\Delta_\tau$. Indeed, if we had a piece-wise smooth path ξ_τ from 1 to x with $\tilde{\Delta}_\tau(\xi_\tau) \in \tau(K_0(A))$ for all $\tau \in T(A)$, it is not necessarily true that we can find a single element of $x \in K_0(A)$ such that $\tilde{\Delta}_\tau(\xi_\tau) = \rho_A(x)(\tau)$ for all $\tau \in T(A)$. See Lemme 5 and Proposition 6 of [dlHS84a].

Proposition 2.2.7. *Let A be a unital C^*-algebra. The sequence*

$$0 \longrightarrow U_n^0(A)/ker\Delta^n|_{U_n^0(A)} \xrightarrow{\iota_n^0} GL_n^0(A)/\ker\Delta^n \underset{\curvearrowleft s_n}{\xrightarrow{\chi_n^0}} \operatorname{Aff} T(A) \longrightarrow 0 \tag{2.2.16}$$

is a split short exact sequence.

Proof. We know that ι_n^0 is injective and that $\chi_n^0 \circ s_n = \mathrm{id}_{\operatorname{Aff}(T(A))}$ (and hence χ_n^0 is surjective). We must show that

$$\ker \chi_n^0 = \operatorname{Im}(\iota_n^0). \tag{2.2.17}$$

The containment $\supseteq$ is trivial since a unitary has positive part equal to 1. For the reverse containment, suppose that $x \in GL_n^0(A)$ satisfies $\operatorname{Tr}(\log|x|) = 0$. Then, letting $x = u_x|x|$ be the polar decomposition,

$$\begin{aligned} \Delta^n(x) &= \Delta^n(u_x) + \Delta^n(|x|) \\ &= \Delta^n(u_x) + \frac{1}{2\pi i}\operatorname{Tr}(\log|x|) \\ &= \Delta^n(u_x) \in \iota_n^0\Big(U_n^0(A)/\ker\Delta^n|_{U_n^0(A)}\Big). \end{aligned} \tag{2.2.18}$$

□

We wish to remove the superscript 0 to get a sequence involving $U_n(A)/\ker\Delta^n|_{U_n^0(A)}$, and $GL_n(A)/\ker\Delta^n$, A_{sa}/A_0

Theorem 2.2.8. *Let A be a unital C^*-algebra. The sequence*

$$0 \longrightarrow U_n(A)/ker\Delta^n|_{U_n^0(A)} \xrightarrow{\iota_n} GL_n(A)/\ker\Delta^n \underset{\curvearrowleft s_n}{\xrightarrow{\chi_n}} \operatorname{Aff} T(A) \longrightarrow 0 \tag{2.2.19}$$

is a split short exact sequence.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_n^0(A)/\ker\Delta^n|_{U_n^0(A)} & \xrightarrow{\iota_n^0} & GL_n^0(A)/\ker\Delta^n & \xrightarrow{\chi_n^0} & \mathrm{Aff}\,T(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & U_n(A)/\ker\Delta^n|_{U_n^0(A)} & \xrightarrow{\iota_n} & GL_n(A)/\ker\Delta^n & \xrightarrow{\chi_n} & \mathrm{Aff}\,T(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_0(U_n(A)) & \xrightarrow{\simeq} & \pi_0(GL_n(A)) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.2.20}$$

where all the columns, as well as the 1st and 3rd rows are exact. As we have

$$\iota_n(U_n(A)/\ker\Delta^n|_{U_n^0(A)}) \subseteq \ker\chi_n, \tag{2.2.21}$$

it follows from [Mac67, Exercise II.5.2] (or a diagram chase) that the second row is also exact. It is easy to see that $s_n : \mathrm{Aff}(T) \to GL_n^0(A)/\ker\Delta^n \subseteq GL_n(A)/\ker\Delta^n$ is also a splitting for the second row as

$$\chi_n \circ s(\widehat{a}) = \chi_n([e^a]) = \widehat{\log|e^a|} = \widehat{a}. \tag{2.2.22}$$

□

Corollary 2.2.9. *Suppose that A is a unital C^*-algebra and let $n \in \mathbb{N}$. If $\ker\Delta^n|_{U_n^0(A)} = DU_n^0(A)$ and $\ker\Delta^n = DGL_n^0(A)$, then*

$$0 \longrightarrow U_n(A)/DU_n^0(A) \xrightarrow{\iota_n} GL_n(A)/DGL_n^0(A) \underset{\overset{s_n}{\curvearrowleft}}{\xrightarrow{\chi_n}} \mathrm{Aff}\,T(A) \longrightarrow 0 \tag{2.2.23}$$

is a split short exact sequence. In particular, for $n = \infty$, we have a natural split short exact sequence

$$0 \longrightarrow \mathrm{K}_1^{alg,u}(A) \xrightarrow{\iota} \mathrm{K}_1^{alg}(A) \underset{\overset{s}{\curvearrowleft}}{\xrightarrow{\chi_\infty}} \mathrm{Aff}\,T(A) \longrightarrow 0. \tag{2.2.24}$$

Proof. The first part follows from the above as $DU_n^0(A) = \ker \Delta^n|_{U_n^0(A)}$ and $DGL_n^0(A) = \ker \Delta^n$. For the last part, if $n = \infty$, then $DGL_\infty(A) = DGL_\infty^0(A)$ by Whitehead's lemma. Indeed, if $x \in GL_n(A)$ is a commutator, say $x = yzy^{-1}z^{-1}$, then $x \oplus 1 \oplus 1 \in GL_{3n}(A)$ can be written as a commutator as follows:

$$\begin{pmatrix} x & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} y & & \\ & y^{-1} & \\ & & 1 \end{pmatrix} \begin{pmatrix} z & & \\ & 1 & \\ & & z^{-1} \end{pmatrix} \begin{pmatrix} y^{-1} & & \\ & y & \\ & & 1 \end{pmatrix} \begin{pmatrix} z^{-1} & & \\ & 1 & \\ & & z \end{pmatrix}. \qquad (2.2.25)$$

The four matrices on the right are connected to the identity by Whitehead's lemma (see [RLL00, Lemma 2.1.5]). □

The above split exact sequence yields that

$$\mathrm{K}_1^{\mathrm{alg}}(A) \simeq \mathrm{K}_1^{\mathrm{alg,u}}(A) \oplus \mathrm{Aff}\, T(A) \qquad (2.2.26)$$

naturally via the isomorphism

$$[x] \mapsto [u_x] \oplus \widehat{\log |x|}. \qquad (2.2.27)$$

The following is an immediate consequence.

Corollary 2.2.10. *Let A, B be unital C^*-algebras such that*

$$DU_\infty^0(A) = \ker \Delta|_{U_\infty^0(A)} \text{ and } DGL_\infty^0(A) = \ker \Delta. \qquad (2.2.28)$$

If $x, y \in GL_\infty(A)$, the following are equivalent.

1. *$[u_x] = [u_y]$ in $\mathrm{K}_1^{alg,u}(A)$ and $\widehat{\log |x|} = \widehat{\log |y|}$ in $\mathrm{Aff}\, T(A)$;*
2. *$[x] = [y]$ in $\mathrm{K}_1^{alg}(A)$.*

For $\phi : A \to B$ a unital *-homomorphism between unital C*-algebras, denote by

1. $\mathrm{K}_1^{\mathrm{alg,u}}(\phi) : \mathrm{K}_1^{\mathrm{alg,u}}(A) \to \mathrm{K}_1^{\mathrm{alg,u}}(B)$;
2. $\mathrm{K}_1^{\mathrm{alg}}(\phi) : \mathrm{K}_1^{\mathrm{alg}}(A) \to \mathrm{K}_1^{\mathrm{alg}}(B)$;
3. $T(\phi) : T(B) \to T(A)$

the maps induced by ϕ.

Corollary 2.2.11. *Let A, B be unital C^*-algebras such that*

- *$DU_\infty^0(A) = \ker \Delta_A|_{U_\infty^0(A)}$ and $DGL_\infty^0(A) = \ker \Delta_A$;*
- *$DU_\infty^0(B) = \ker \Delta_B|_{U_\infty^0(B)}$ and $DGL_\infty^0(B) = \ker \Delta_B$.*

*Let $\phi, \psi : A \to B$ be unital *-homomorphisms. The following are equivalent.*

1. $\mathrm{K}_1^{alg,u}(\phi) = \mathrm{K}_1^{alg,u}(\psi)$ *and* $T(\phi) = T(\psi)$*;*
2. $\mathrm{K}_1^{alg}(\phi) = \mathrm{K}_1^{alg}(\psi)$.

There are many classes of unital C*-algebras satisfying the two hypotheses of the above corollary [dlHS84b, Tho93, Ng14, NR17, NR15]. In the penultimate listed reference, it is shown there that the hypotheses hold in the case that A is a unital, separable, simple, pure C*-algebra of stable rank one such that every 2-quasitracial state is a trace, and in the latter it is shown to hold for $M_3(A)$ whenever A is pure [Win12, Definition 3.6].

Corollary 2.2.12. *Let A be a unital, simple, separable, pure C*-algebra of stable rank one such that every 2-quasitrace is a trace. Then there is a natural isomorphism*

$$\mathrm{K}_1^{alg}(A) \simeq \mathrm{K}_1^{alg,u}(A) \oplus \mathrm{Aff}\, T(A). \tag{2.2.29}$$

In particular, (2.2.29) holds for all unital, separable, simple, nuclear, $\mathcal{Z}$-stable C-algebras.*

2.2.2 Hausdorffized algebraic K-theory

In the Hausdorffized setting, we obtain similar results by the same arguments. However, in this case, we have $\ker \bar{\Delta}^n|_{U_n^0(A)} = CU_n^0(A)$ and $\ker \bar{\Delta}^n = CGL_n^0(A)$ by Lemma 2.1.6. Let

$$\begin{aligned} \bar{\iota}_n &: U_n(A)/CU_n^0(A) \to GL_n(A)/CGL_n^0(A), \\ \bar{\chi}_n &: GL_n(A)/CGL_n^0(A) \to \mathrm{Aff}\, T(A), \quad \text{and} \\ \bar{s}_n &: \mathrm{Aff}\, T(A) \to GL_n(A)/CGL_n^0(A) \end{aligned} \tag{2.2.30}$$

be the variants of the maps ι_n, χ_n, s_n in the previous section (so our domains and codomains are now topological). Identifying $CU_n^0(A) = \ker \overline{\Delta}_n$ and applying the arguments from Section 2.2.1 gives that each of these maps are well-defined group homomorphisms for $n \in \mathbb{N} \cup \{\infty\}$. In the Hausdorffized setting, we show that these maps are also continuous. First a lemma to handle the $n = \infty$ case, as it is not necessarily true that an inductive limit of topological groups is a topological group.

Lemma 2.2.13. *Let $G = \cup_n G_n$ be an increasing union of topological groups and equip G with the inductive limit topology. Let $H \leq G$ be a subgroup such that the closure CH of H is also a subgroup of G. Then the quotient map $q : G \to G/CH$ is an open map.*

Proof. Let $S \subseteq G$ be open. As G/CH has the quotient topology, the set $q(S) \subseteq G/CH$ is open if and only if $q^{-1}(q(S)) \subseteq G$ is open in G. Thinking of G/CH as the space of CH-orbits of G where $CH \curvearrowright G$ by right translation, we have that

$$q^{-1}(q(S)) = \bigcup_{h \in CH} Sh \tag{2.2.31}$$

which is open if S is, since right translation still yields a homeomorphism in the inductive limit topology – see [TSH98, Proposition 1.1(ii)]. □

Proposition 2.2.14. *The maps in (2.2.30) are well-defined, continuous group homomorphisms. Moreover, $\overline{\iota}$ and $\overline{\chi}$ are open onto their images.*

Proof. A straightforward adaptation of the arguments of the previous section shows that these are well-defined group homomorphisms. We work with the $n = \infty$ case throughout, as the $n \in \mathbb{N}$ case is similar, and easier due to the fact that $GL_n(A)$ and $U_n(A)$ are topological groups.

Let us show that $\overline{\iota}$ is continuous. The diagram

$$\begin{array}{ccc} U_\infty(A) & \xrightarrow{\sigma} & GL_\infty(A) \\ {\scriptstyle q_U}\downarrow & & \downarrow{\scriptstyle q_{GL}} \\ \overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A) & \xrightarrow[\overline{\iota}]{} & \overline{\mathrm{K}}_1^{\mathrm{alg}}(A) \end{array} \tag{2.2.32}$$

commutes where the left and right maps are quotient maps and σ is the canonical inclusion. We note that for any subset $S \subseteq \overline{\mathrm{K}}_1^{\mathrm{alg}}(A)$ the commutation of the above diagram gives that

$$(q_U)^{-1}\left(\overline{\iota}^{-1}(S)\right) = \sigma^{-1}\left(q_{GL}^{-1}(S)\right). \tag{2.2.33}$$

Therefore if $S \subseteq \overline{\mathrm{K}}_1^{\mathrm{alg}}(A)$ is open, then

$$\begin{aligned} \overline{\iota}^{-1}(S) &= q_U\left(q_U^{-1}\left(\overline{\iota}^{-1}(S)\right)\right) \\ &= q_U\left(\sigma^{-1}\left(q_{GL}^{-1}(S)\right)\right), \end{aligned} \tag{2.2.34}$$

where $\sigma^{-1}\left(q_{GL}^{-1}(S)\right)$ is open because both q_{GL} and σ are continuous. As q_U is open by Lemma 2.2.13, it follows that $\overline{\iota}^{-1}(S)$ is open. This shows continuity.

Let us show that $\overline{\iota}$ is open onto its image. We note that taking the unitary part of the polar decomposition $\omega_n : GL_n(A) \to U_n(A) \subseteq U_\infty(A)$ is continuous for all n and therefore induces a continuous map $\omega : GL_\infty(A) \to U_\infty(A)$. Since $CGL_\infty(A) \cap U_\infty(A) = CU_\infty(A)$ by Lemma 2.1.6, we get an induced continuous map

$$\overline{\omega} : \overline{\mathrm{K}}_1^{\mathrm{alg}}(A) \to \overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A) \tag{2.2.35}$$

which clearly satisfies

$$\overline{\omega} \circ \overline{\iota} = \mathrm{id}_{\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)} \text{ and } \overline{\iota} \circ \overline{\omega}|_{\overline{\iota}\left(\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)\right)} = \mathrm{id}_{\overline{\iota}\left(\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)\right)}. \tag{2.2.36}$$

Now if $S \subseteq \overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)$ is open, then it is easily seen that

$$\overline{\iota}(S) = \overline{\iota}\left(\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)\right) \cap (\overline{\omega})^{-1}(S). \tag{2.2.37}$$

As $\overline{\omega}$ is continuous, $(\overline{\omega})^{-1}(S) \subseteq \overline{\mathrm{K}}_1^{\mathrm{alg}}(A)$ is open and so $\overline{\iota}(S) \subseteq \overline{\iota}\left(\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A)\right)$ is open with respect to the subspace topology. This shows that $\overline{\iota}$ is open onto its image.

For $\overline{\chi}$, let $g : GL_\infty(A) \to \mathrm{Aff}\, T(A)$ denote the map $g(x) := \widehat{\log|x|}$. The diagram

$$\begin{array}{ccc} GL_\infty(A) & \xrightarrow{\quad g \quad} & \mathrm{Aff}\, T(A) \\ & {\scriptstyle q_{GL}} \searrow \quad \nearrow {\scriptstyle \overline{\chi}} & \\ & \overline{\mathrm{K}}_1^{\mathrm{alg}}(A) & \end{array} \tag{2.2.38}$$

commutes, so we have that for $S \subseteq \mathrm{Aff}\, T(A)$

$$\begin{aligned} \overline{\chi}^{-1}(S) &= q_{GL}\left(q_{GL}^{-1}\left(\overline{\chi}^{-1}(S)\right)\right) \\ &= q_{GL}\left(g^{-1}(S)\right). \end{aligned} \tag{2.2.39}$$

Thus since we know that q_{GL} is open by Lemma 2.2.13, it suffices to show that g is continuous. But g is continuous if $g|_{GL_n(A)}$ is continuous for all n,[14] and this is true: indeed, $g|_{GL_n(A)}$ can be written as the composition

$$G_n(A) \xrightarrow{l|_{GL_n(A)}} A_{sa} \xrightarrow{\mathrm{Tr}|_{A_{sa}}} \mathrm{Aff}\, T(A) \tag{2.2.40}$$

where $l : GL_\infty(A) \to A_{sa}$ is the map given by $l(x) := \mathrm{tr} \log|x|$ where $\mathrm{tr} : M_\infty(A) \to A$ is the unnormalized trace. Seeing that $l|_{GL_n(A)}$ is continuous follows easily: if $x_n \to x$ in $GL_n(A)$, then $\mathrm{tr}\log|x_n| \to \mathrm{tr}\log|x|$.

To show that $\overline{\chi}$ is open, we again appeal to the diagram (2.2.38). It suffices to show that g is open – and to this end it suffices to show that $g|_{GL_n(A)}$ is open for each n.[15] For $GL_n(A)$, $\widehat{\log|x_0|} = \mathrm{tr}\,\widehat{\log|x_0|}$, where tr is the unnormalized trace, and so we

[14] If $X = \cup_n X_n$ is equipped with the inductive limit topology and $f : X \to Y$ is a function such that $f|_{X_n}$ is continuous for all n, then f is continuous. To see this, let $S \subseteq Y$ be open and note that $f^{-1}(S) = \cup_n f|_{X_n}^{-1}(S)$ is open.

[15] If $X = \cup_n X_n$ is an increasing union of topological spaces with the inductive limit topology and Y is another topological space, then for $S \subseteq X$, we have that $f(S) = \cup_n f(S \cap X_n)$.

can restrict to the case where $n = 1$. Let us without loss of generality work with open balls around the identity: let $\varepsilon > 0$ and consider

$$S := \{x \in GL(A) \mid \|x - 1\| < \varepsilon\}. \tag{2.2.41}$$

Looking at the image of S under $g_1 := g|_{GL(A)}$, we have

$$g_1(S) = \{\widehat{\log|x|} \mid \|x - 1\| < \varepsilon\}. \tag{2.2.42}$$

Let $x_0 \in GL(A)$ be such that $x_0 \approx_\varepsilon 1$ and let us show that there is an open ball around $\widehat{\log|x_0|}$ that is contained in $g_1(S)$. First note that for $\hat{h} \in \operatorname{Aff} T(A)$, with $h \in A_{sa}$, we have

$$\begin{aligned} \left\|\widehat{\log|x_0|} - \hat{h}\right\| &= \left\|\widehat{\log|x_0|} - \widehat{\log|e^h|}\right\| \\ &= \left\|\overline{\chi}([x_0]) - \overline{\chi}([e^h])\right\| \\ &= \left\|\overline{\chi}([x_0 e^{-h}])\right\| \\ &= \left\|\log\widehat{|x_0 e^{-h}|}\right\|. \end{aligned} \tag{2.2.43}$$

Now let $\delta > 0$ be such that whenever $a \in A$, we have $\|a\| < \delta$ implies that $\|e^a - 1\| < \varepsilon$. Then, for $\hat{h} \in \operatorname{Aff} T(A)$ with $\widehat{\log|x_0|} \approx_{\frac{\delta}{2}} \hat{h}$, we have by (2.2.43) that

$$\|\log\widehat{|x_0 e^{-h}|}\| < \frac{\delta}{2}. \tag{2.2.44}$$

Find a self-adjoint lift, say $k \in A_{sa}$, of $\log\widehat{|x_0 e^{-h}|}$ with $\|k\| < \delta$. Then we have that $\|e^k - 1\| < \varepsilon$ and

$$g_1(e^k) = \hat{k} = \log\widehat{|x_0 e^{-h}|}. \tag{2.2.45}$$

This shows that $B_{\frac{\delta}{2}}(\widehat{\log|x_0|}) \subseteq g_1(S)$.

Finally, let us show that $\overline{s}$ is continuous. We have that

$$\begin{array}{ccc} A_{sa} & \xrightarrow{\alpha} & GL_\infty(A) \\ {\scriptstyle \operatorname{Tr}|_{A_{sa}}}\downarrow & & \downarrow{\scriptstyle q_{GL}} \\ \operatorname{Aff} T(A) & \xrightarrow[\overline{s}]{} & \overline{\mathrm{K}}_1^{\mathrm{alg}}(A) \end{array} \tag{2.2.46}$$

commutes where $\alpha(a) := e^a$ – note that α is continuous and that the image of α is contained in $GL(A) \subseteq GL_\infty(A)$ (α is however *not* a homomorphism). Consequently if $S \subseteq \overline{\mathrm{K}}_1^{\mathrm{alg}}(A)$ is open, then since $\operatorname{Tr}|_{A_{sa}}$ is surjective,

$$\begin{aligned} \overline{s}^{-1}(S) &= \operatorname{Tr}|_{A_{sa}}\left(\operatorname{Tr}^{-1}_{A_{sa}}\left(\overline{s}^{-1}(S)\right)\right) \\ &= \alpha^{-1}\left(q_{GL}^{-1}(S)\right) \end{aligned} \tag{2.2.47}$$

is open as well since α and q_{GL} are continuous. □

Theorem 2.2.15. *For any unital C^*-algebra A and $n \in \mathbb{N} \cup \{\infty\}$, the sequence*

$$0 \longrightarrow U_n(A)/CU_n^0(A) \xrightarrow{\bar{\iota}_n} GL_n(A)/CGL_n^0(A) \xrightarrow{\bar{\chi}_n} \operatorname{Aff} T(A) \longrightarrow 0 \quad (\bar{s}_n : \operatorname{Aff} T(A) \to GL_n(A)/CGL_n^0(A)) \tag{2.2.48}$$

is a split short exact sequence of topological groups. In particular, we have the following split short exact sequence of topological groups:

$$0 \longrightarrow \overline{\mathrm{K}}_1^{alg,u}(A) \xrightarrow{\bar{\iota}} \overline{\mathrm{K}}_1^{alg}(A) \xrightarrow{\bar{\chi}_\infty} \operatorname{Aff} T(A) \longrightarrow 0 \quad (\bar{s} : \operatorname{Aff} T(A) \to \overline{\mathrm{K}}_1^{alg}(A)) \tag{2.2.49}$$

Proof. The same argument as in Theorem 2.2.8 gives an algebraic splitting. The fact that this is a splitting of topological groups follows as $\bar{\iota}_n, \overline{\chi}_n, \bar{s}_n$ are all continuous and $\bar{\iota}_n$ and $\overline{\chi}_n$ are open onto their images by Proposition 2.2.14. ☐

Corollary 2.2.16. *Let A, B be unital C^*-algebras, $x, y \in \overline{\mathrm{K}}_1^{alg}(A)$. The following are equivalent.*

1. $[u_x] = [u_y]$ *in* $\overline{\mathrm{K}}_1^{alg,u}(A)$ *and* $\widehat{\log|x|} = \widehat{\log|y|}$ *in* $\operatorname{Aff} T(A)$*;*
2. $[x] = [y]$ *in* $\overline{\mathrm{K}}_1^{alg}(A)$*.*

For A, B unital C*-algebras, $\phi : A \to B$ a unital *-homomorphism, denote by

1. $\overline{\mathrm{K}}_1^{\mathrm{alg,u}}(\phi) : \overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A) \to \overline{\mathrm{K}}_1^{\mathrm{alg,u}}(B)$;
2. $\overline{\mathrm{K}}_1^{\mathrm{alg}}(\phi) : \overline{\mathrm{K}}_1^{\mathrm{alg}}(A) \to \overline{\mathrm{K}}_1^{\mathrm{alg}}(B)$

the maps induced by ϕ.

Corollary 2.2.17. *Let A, B be unital C^*-algebras. Let $\phi, \psi : A \to B$ be unital *-homomorphisms. The following are equivalent.*

1. $\overline{\mathrm{K}}_1^{alg,u}(\phi) = \overline{\mathrm{K}}_1^{alg,u}(\psi)$ *and* $T(\phi) = T(\psi)$*;*
2. $\overline{\mathrm{K}}_1^{alg}(\phi) = \overline{\mathrm{K}}_1^{alg}(\psi)$*.*

2.3 Nonstable algebraic K-theory

Here we discuss some structure of the nonstable (both Hausdorffized and not) algebraic K_1-groups. In [Tho95, Theorem 3.2], Thomsen proved that the map

$$U^0_\infty(A)/CU^0_\infty \simeq \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))} \tag{2.3.1}$$

given by $[u] \mapsto \overline{\Delta}(u)$ is a homeomorphic isomorphism. It was noted that if $\pi(U^0(A)) \to K_0(A)$ is surjective, then of course we have that

$$U^0(A)/CU^0(A) \simeq U^0_n(A)/CU^0_n(A) \tag{2.3.2}$$

for all $n \in \mathbb{N} \cup \{\infty\}$. Indeed, if the canonical map $\pi_1(U^0(A)) \to K_0(A)$ is surjective, then the following diagram commutes

$$\begin{array}{ccc} U^0(A)/CU^0(A) & \xrightarrow{\ \bar{i}\ } & U^0_\infty(A)/CU^0_\infty(A) \\ {\scriptstyle \overline{D}_1}\big\downarrow & & \big\downarrow{\scriptstyle \overline{D}} \\ \mathrm{Aff}\, T(A)/\overline{\tilde{\Delta}(\pi_1(U^0(A)))} & \xrightarrow[\mathrm{id}]{} & \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))} \end{array} \tag{2.3.3}$$

where $\bar{i} : U^0(A)/CU^0(A) \to U^0_\infty(A)/CU^0_\infty(A)$ is the canonical map, $\overline{D}_1, \overline{D}$ are maps factoring Δ^1 and Δ through $CU^0(A)$ and $CU^0_\infty(A)$ respectively. As $\mathrm{id}, \overline{D}_1$ and $\overline{D}$ are all homeomorphic isomorphisms, it follows that the canonical map $\bar{i}$ is a homeomorphic isomorphism.

Remark 2.3.1. More generally one can study the question of when

$$U^0_n(A)/CU^0_n(A) \to U^0_m(A)/CU^0_m(A) \tag{2.3.4}$$

is an isomorphism for all $m \geq n$, even in the case where $\pi_1(U^0(A)) \to K_0(A)$ may not be surjective. See [GLX15] for details. One can of course get similar results using the general linear invariants, as well as the purely algebraic variants under the assumptions that $\ker \Delta_n|_{U^0_n(A)} = DU^0_n(A)$ or $\ker \Delta_n = DGL^0_n(A)$ for every n.

A similar argument gives the following in the algebraic setting.

Lemma 2.3.2. *Let A be a unital C^*-algebra and suppose that $\pi_1(U^0(A)) \to K_0(A)$ is surjective.*

1. *The canonical map $U^0(A)/\ker \Delta^1|_{U^0(A)} \to U^0_\infty(A)/\ker \Delta|_{U^0_\infty(A)}$ is an isomorphism.*
2. *The canonical map $GL^0(A)/\ker \Delta^1 \to GL^0_\infty(A)/\ker \Delta$ is an isomorphism.*

Proof. Writing out a similar diagram to (2.3.3), we have

$$\begin{array}{ccc} U^0(A)/\ker\Delta^1|_{U^0(A)} & \xrightarrow{\ i\ } & U^0_\infty(A)/\ker\Delta|_{U^0_\infty(A)} \\ {\scriptstyle D_1}\downarrow & & \downarrow{\scriptstyle D} \\ \operatorname{Aff}T(A)/\tilde{\Delta}(\pi_1(U^0(A))) & \xrightarrow[\text{id}]{} & \operatorname{Aff}T(A)/\rho_A(K_0(A)) \end{array} \tag{2.3.5}$$

The maps id, D_1, D are all group isomorphisms, so i must be as well (the maps i, D_1, D are the purely algebraic analogues of $\bar{i}, \overline{D}_1, \overline{D}$ above). We get a similar diagram in the GL setting with U replaced with GL and $\operatorname{Aff}T(A)$ replaced by $A/\overline{[A,A]}$. □

Using similar techniques to [GLX15], we have the following.

Lemma 2.3.3. *Let A be a unital C^*-algebra and suppose that $\pi_1(U^0(A)) \to K_0(A)$ is surjective.*

1. *If $\ker\Delta^1|_{U^0(A)} = DU^0(A)$, then $\ker\Delta^n|_{U^0_n(A)} = DU^0_n(A)$ for all $n \in \mathbb{N}\cup\{\infty\}$.*
2. *If $\ker\Delta^1 = DGL^0(A)$, then $\ker\Delta^n = DGL^0_n(A)$ for all $n \in \mathbb{N}\cup\{\infty\}$.*

Proof. We show (1) holds, (2) is similar. Suppose that $u \in \ker\Delta^n|_{U^0_n(A)}$. There are some $a \in A_{sa}$ such that $[u] = [e^{2\pi ia} \oplus 1_{n-1}]$ and a piece-wise smooth path $\xi : [0,1] \to U^0_n(A)$ with $\tilde{\Delta}(\xi) = \operatorname{Tr}(a)$ by Remark 2.1.2(1).

As $u \in \ker\Delta^n|_{U^0_n(A)}$, there is some piece-wise smooth loop $\eta : [0,1] \to U^0_n(A)$ with

$$\tilde{\Delta}^n(\xi) = \tilde{\Delta}^n(\eta). \tag{2.3.6}$$

As before, the surjectivity of $\pi_1(U^0(A)) \to K_0(A)$ implies that η is homotopic to $\eta_0 \oplus 1_{n-1}$ for some piece-wise smooth loop $\eta_0 : [0,1] \to U^0(A)$. Then $\eta_1(t) := e^{2\pi ita}\eta_0(t)^*$ defines a piece-wise smooth path in $U^0(A)$ from 1 to $e^{2\pi ia}$ such that $\tilde{\Delta}(\eta_1) = 0$. Therefore $e^{2\pi ia} \in \ker\Delta^1|_{U^0(A)} = DU^0(A)$ and consequently

$$[u] = [e^{2\pi ia} \oplus 1_{n-1}] = 0 \text{ in } U^0_n(A)/DU^0_n(A). \tag{2.3.7}$$

□

Now we finish by showing that we can work outside of the connected component.

Theorem 2.3.4. *Let A be a unital C^*-algebra such that*

1. *the canonical map $\pi_1(U^0(A)) \to K_0(A)$ is surjective;*
2. *the canonical map $U(A)/U^0(A) \to K_1(A)$ is an isomorphism.*

Then the following is true.

1. *If* $\ker \Delta^1|_{U^0(A)} = DU^0(A)$*, then* $U(A)/DU(A) \simeq \mathrm{K}_1^{alg,u}(A)$.
2. *If* $\ker \Delta^1 = DGL^0(A)$*, then* $GL(A)/DGL(A) \simeq \mathrm{K}_1^{alg}(A)$.

Proof. We show this in the unitary setting. First we again note that $DU^0_\infty(A) = DU_\infty(A)$ by (2.2.25), and so $\mathrm{K}_1^{\mathrm{alg,u}}(A) = U_\infty/DU^0_\infty(A)$. Moreover, (2) implies that A is K_1-injective, giving that $DU(A) = DU^0(A)$ and $DGL(A) = DGL^0(A)$ as well. Using property (1), together with the fact that $\ker \Delta^1|_{U^0(A)} = DU^0(A)$, gives that the canonical map

$$U^0(A)/DU^0(A) \simeq U^0_\infty(A)/DU^0_\infty(A) \tag{2.3.8}$$

is an isomorphism by combining Lemmas 2.3.2 and 2.3.3. Now combining (2) with (2.3.8) yields a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U^0(A)/DU(A) & \longrightarrow & U(A)/DU(A) & \longrightarrow & U(A)/U^0(A) & \longrightarrow & 0 \\ & & \downarrow{\simeq} & & \downarrow & & \downarrow{\simeq} & & \\ 0 & \longrightarrow & U^0_\infty(A)/DU_\infty(A) & \longrightarrow & \mathrm{K}_1^{\mathrm{alg,u}}(A) & \longrightarrow & K_1(A) & \longrightarrow & 0 \end{array} \tag{2.3.9}$$

where the left and right vertical maps are isomorphisms. Therefore the middle vertical map is an isomorphism by the Short Five Lemma [Mac67, Lemma I.3.1]. The argument in the general linear setting is the same with U replaced by GL and $\mathrm{K}_1^{\mathrm{alg,u}}$ replaced with $\mathrm{K}_1^{\mathrm{alg}}$. □

Remark 2.3.5. If A is unital, with the assumptions

1. $\pi_1(U^0(A)) \to K_0(A)$ is surjective and
2. $U(A)/U^0(A) \to K_1(A)$ is an isomorphism,

we also get that

$$U(A)/CU(A) \simeq \overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A) \text{ and } GL(A)/CGL(A) \simeq \overline{\mathrm{K}}_1^{\mathrm{alg}}(A), \tag{2.3.10}$$

even as topological groups. Indeed, looking at the unitary case for example, since the map

$$U^0(A)/CU^0(A) \simeq U^0_\infty(A)/CU^0_\infty(A) \tag{2.3.11}$$

is a topological group isomorphism by means of (2.3.3), it follows that the map

$$U(A)/CU^0(A) \to \overline{\mathrm{K}}_1^{\mathrm{alg,u}}(A) \tag{2.3.12}$$

is open as it will send open small neighbourhoods of the identity to open neighbourhoods (as sufficiently small neighbourhoods will be connected to the identity).

Again, as the map $U(A)/U^0(A) \to K_1(A)$ is injective, we also have that $DU(A) \subseteq U^0(A)$. Thus $DU^0(A) = DU(A)$ and $CU^0(A) = CU(A)$ in this case.

Finally we finish by stating that unital C*-algebras satisfying

1. $\pi_1(U^0(A)) \to K_0(A)$ is surjective and

2. $U(A)/U^0(A) \to K_1(A)$ is an isomorphism

are very common. Indeed, this includes the class of stable rank one C*-algebras [Rie87, Theorem 3.3], $\mathcal{Z}$-stable C*-algebras [Jia97, Theorem 3], and tensor products with coronas over σ-unital C*-algebras [Tho91, Theorem 4.9].

Chapter 3

Unitary group homomorphisms, traces, and K-theory

Unitary groups of C*-algebras have been long studied, and for many classes of operator algebras they form a complete invariant. In [Dye53], Dye studied the unitary group isomorphism problem between non-atomic W*-algebras, with the assumption of *weak bicontinuity* of the isomorphism. He later showed that the unitary group, this time as an algebraic object, determined the type of a factor [Dye55] (except for type I_{2n}) – here it was shown that such group isomorphisms were the restrictions of a *-isomorphism of conjugate linear *-isomorphism multiplied by a (possibly discontinuous – [Boo98, Appendix A] gives exposition) character. Sakai generalized Dye's results to show that any uniformly continuous unitary group isomorphism between AW*-factors comes from a *-isomorphism or conjugate linear *-isomorphism [Sak55] (see also [Yen56] for general AW*-algebras which have no component of type I_n).

Dye's method was generalized to large classes of real rank zero C*-algebras by Al-Rawashdeh, Booth and Giordano in [ARBG12], where they applied the method to obtain induced maps between K-theory, a general linear variant was done by Giordano and Sierakowski in [GS16]. The stably finite and purely infinite cases were handled separately. The unital, simple AH-algebras of slow dimension growth and of real rank zero were classified by the topological group isomorphism class of their unitary groups (or general linear groups), and the unital, simple, purely infinite UCT algebras were classified via the algebraic isomorphism classes of their unitary groups (or general linear groups). These results made use of the abundance of projections in real rank zero C*-algebras (at least to show there were isomorphic K_0-groups), and made use of the Dadarlat-Elliott-Gong [Dad95, Gon97] and Kirchberg-Phillips [Phi00] classification theorems respectively (see Theorems 3.3.1 and 8.4.1 of [Rør02] for each respective case).

In [Pat83], it was proven by Paterson that two unital C*-algebras are isomorphic if and only there is an isometric isomorphism of the unitary groups which acts as the identity on the circle. In a similar vein, the metric structure of the unitary group has also played a role in determining the Jordan *-algebra structure on C*-algebras. In [HM14], Hatori and Molnár showed that two unital C*-algebras are Jordan *-isomorphic if and only if their unitary groups are isometric as metric spaces, not taking into account any algebraic structure.

Chand and Robert have shown in [CR23] that if A and B are prime traceless C*-algebras with full square zero elements such that $U^0(A)$ is algebraically isomorphic to $U^0(B)$, then A is either isomorphic or anti-isomorphic to B. In fact, the group isomorphism is the restriction of a *-isomorphism or anti-*-isomorphism which follows from the fact that unitary groups associated to these C*-algebras have certain automatic continuity properties that allow one to use characterizations of *commutativity preserving maps* [Bre93] (see [AM03]). Chand and Robert also show that if A is a unital separable C*-algebra with at least one tracial state, then $U^0(A)$ admits discontinuous automorphisms. Thus the existence of traces is an obstruction to classification via algebraic structure on the unitary groups – at least an obstruction to unitary group homomorphisms being the restrictions of *-homomorphisms or anti-*-homomorphisms.

In this chapter, we show that uniformly continuous unitary group homomorphisms yield maps between traces with have several desirable K-theoretic properties, especially under stricter continuity assumptions. We state our main results.

Theorem 3.A (Corollary 3.1.6)**.** *Let A, B be unital C*-algebras. If $\theta : U^0(A) \to U^0(B)$ is a uniformly continuous group homomorphism, then there exists a bounded $\mathbb{R}$-linear map $\Lambda_\theta : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ such that*

$$\begin{array}{ccc} \pi_1(U^0(A)) & \xrightarrow{\tilde{\Delta}^1_A} & \operatorname{Aff} T(A) \\ {\scriptstyle \pi_1(\theta)}\downarrow & & \downarrow{\scriptstyle \Lambda_\theta} \\ \pi_1(U^0(B)) & \xrightarrow[\tilde{\Delta}^1_B]{} & \operatorname{Aff} T(B) \end{array} \tag{3.0.1}$$

commutes.

Recall that the K_0-group of a unital C*-algebra can be identified with the fundamental group $\pi_1(U^0_\infty(A))$. Restricting to C*-algebras with sufficient K_0-regularity – by this we mean C*-algebras whose K_0-group can be realized as loops in the connected component of its unitary group – we get a map between K_0-groups and a map between spaces of continuous real-valued affine functions which commute with the pairing.

Corollary 3.B (Corollary 3.1.6)**.** *Let A, B be unital C*-algebras such that the canonical maps*

$$\pi_1(U^0(A)) \to K_0(A) \text{ and } \pi_1(U^0(B)) \to K_0(B) \tag{3.0.2}$$

are isomorphisms. If $\theta : U^0(A) \to U^0(B)$ is a continuous group homomorphism then there exists a bounded linear map $\Lambda_\theta : \mathrm{Aff}\, T(A) \to \mathrm{Aff}\, T(B)$ such that

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \mathrm{Aff}\, T(A) \\ {\scriptstyle K_0(\theta)}\downarrow & & \downarrow{\scriptstyle \Lambda_\theta} \\ K_0(B) & \xrightarrow[\rho_B]{} & \mathrm{Aff}\, T(B) \end{array} \tag{3.0.3}$$

commutes, where $K_0(\theta)$ is the map induced by $\pi_1(\theta)$ together with the isomorphisms of (3.0.2).

C*-algebras satisfying the above hypothesis are quite common – for example C*-algebras having stable rank one [Rie87] or that are $\mathcal{Z}$-stable [Jia97] have this property. Viewing $\mathrm{Aff}\, T(A)$ and $\mathrm{Aff}\, T(B)$ as partially ordered real Banach spaces (under the uniform norm) with order units, it is not however true that Λ_θ is unital or positive (see Example 3.1.4). This is remedied by adding stricter continuity assumptions on the homomorphism θ (and possibly by replacing Λ_θ with $-\Lambda_\theta$).

When $\theta : U(A) \to U(B)$ is contractive, injective and sends the circle to the circle, then we show (Lemma 3.2.3) that either Λ_θ or $-\Lambda_\theta$ is unital and positive, and therefore θ induces a map between K-theory and traces in such a manner that respects the pairing (which in turn gives a map between Elliott invariants for certain simple C*-algebras). As a consequence, we can identify certain unitary subgroups with C*-subalgebras by using K-theoretic classification of embeddings [CGS+23].

Theorem 3.C (Corollary 3.2.13)**.** *Let A be a unital, separable, simple, nuclear C*-algebra satisfying the UCT which is either $\mathcal{Z}$-stable or has stable rank one, and B be a unital, separable, simple, nuclear $\mathcal{Z}$-stable C*-algebra. If there is a contractive injective group homomorphism $U(A) \to U(B)$ which maps the circle to the circle, then there is a unital embedding $A \hookrightarrow B$.*

This chapter is structured as follows. In Section 3.1 we use a continuous unitary group homomorphism to construct a map between spaces of continuous affine functions on the trace simplices, and use the de la Harpe-Skandalis determinant to show that this map has several desirable properties with respect to the map induced on the fundamental groups of the unitary groups. In Section 3.2 we discuss how our map between spaces of affine functions respects or flips the order under certain continuity assumptions on the unitary group homomorphism. In Section 3.3 we discuss general linear variants. We finish in Section 3.4 with some remarks concerning possible alternative methods, as well as some unanswered questions.

3.1 Continuous unitary group homomorphisms and traces

Throughout, A and B will be unital C*-algebras with non-empty compact trace simplices, and $\theta : U^0(A) \to U^0(B)$ will denote a uniformly continuous group homomorphism between the connected components of the unitary groups. We will specify any additional assumptions as we go along. As θ is a continuous group homomorphism, it sends commutators to commutators and limits of commutators to limits of commutators. Thus there are induced group homomorphisms

$$U^0(A)/CU^0(A) \to U^0(B)/CU^0(B) \text{ and } U^0(A)/DU^0(A) \to U^0(B)/DU^0(B). \quad (3.1.1)$$

Thomsen's isomorphism [Tho95, Theorem 3] then brings about maps between quotients of $\operatorname{Aff} T(A)$ and $\operatorname{Aff} T(B)$:

$$\begin{array}{ccc} U^0(A)/CU^0(A) & \xrightarrow{\simeq} & \operatorname{Aff} T(A)/\overline{\tilde{\Delta}^1_A(\pi_1(U^0(A)))} \\ \downarrow & & \downarrow \\ U^0(B)/CU^0(B) & \xrightarrow[\simeq]{} & \operatorname{Aff} T(B)/\overline{\tilde{\Delta}^1_B(\pi_1(U^0(B)))}. \end{array} \quad (3.1.2)$$

In a similar vein, by modding out by $\ker \Delta^1_A|_{U^0(A)}$ and $\ker \Delta^1_B|_{U^0(B)}$, respectively, instead of the closure of derived groups, there is a purely algebraic variant of the above diagram:

$$\begin{array}{ccc} U^0(A)/\ker \Delta^1_A|_{U^0(A)} & \xrightarrow{\simeq} & \operatorname{Aff} T(A)/\tilde{\Delta}^1_A(\pi_1(U^0(A))) \\ \downarrow & & \downarrow \\ U^0(B)/\ker \Delta^1_B|_{U^0(B)} & \xrightarrow[\simeq]{} & \operatorname{Aff} T(B)/\tilde{\Delta}^1_B(\pi_1(U^0(B))). \end{array} \quad (3.1.3)$$

In the special case where $\pi_1(U^0(A)) \to K_0(A)$ and $\pi_1(U^0(B)) \to K_0(B)$ are surjections, we have induced maps between quotients

$$\begin{aligned} \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} &\to \operatorname{Aff} T(B)/\overline{\rho_B(K_0(B))}, \\ \operatorname{Aff} T(A)/\rho_A(K_0(A)) &\to \operatorname{Aff} T(B)/\rho_B(K_0(B)) \end{aligned} \quad (3.1.4)$$

in the respective Hausdorffized and non-Hausdorffized settings.

One question to be answered is whether or not we can lift the maps on the right of (3.1.2) and (3.1.3) to maps $\operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$. These spaces have further structure as dimension groups with order units [Goo86, Chapter 7], so we would like to be able to alter the lift to get a map which is unital and positive. We show that we can always lift this map, and altering it to be unital and positive is possible under a certain continuity assumption on θ.

If we further assume that $K_0(A) \simeq \pi_1(U^0(A))$ and $K_0(B) \simeq \pi_1(U^0(B))$ in the canonical way (which is true in the presence of $\mathcal{Z}$-stability by [Jia97] or stable rank one [Rie87]), we would like this map to be compatible with the group homomorphism

$$K_0(\theta) : K_0(A) \to K_0(B) \tag{3.1.5}$$

arising from the diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\simeq} & \pi_1(U^0(A)) \\ {\scriptstyle K_0(\theta)}\downarrow & & \downarrow{\scriptstyle \pi_1(\theta)} \\ K_0(B) & \xrightarrow[\simeq]{} & \pi_1(U^0(B)). \end{array} \tag{3.1.6}$$

By compatible, we mean that

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \operatorname{Aff} T(A) \\ {\scriptstyle \pi_1(\theta)}\downarrow & & \downarrow \\ K_0(B) & \xrightarrow[\rho_B]{} & \operatorname{Aff} T(B) \end{array} \tag{3.1.7}$$

commutes, where the map on the right is the lift coming from the induced map on abelianizations as in (3.1.2) and (3.1.3). If our map between spaces of affine continuous functions is not unital and positive, but we can alter it accordingly, we must do the same to our map between K_0. We would still have a commuting diagram as above, but it would give that maps induced on $K_0(\cdot)$ and $\operatorname{Aff} T(\cdot)$ respect the pairing.

Stone's theorem [Con85, Section X.5] allows one to recover from a strongly continuous one parameter family $U(t)$ of unitaries a (possibly unbounded) self-adjoint operator X such that $U(t) = e^{itX}$ for all $t \in \mathbb{R}$. If it is a norm-continuous one parameter family of unitaries, one can recover a bounded self-adjoint operator X, and X will lie in the C*-algebra generated by the unitaries. The use of Stone's theorem to deduce that continuous group homomorphisms between unitary groups send exponentials to exponentials is not new. Sakai used it in the 1950's in order to show that a norm-continuous group isomorphism between unitary groups of AW*-algebras induces an *-isomorphism or anti-*-isomorphism between the algebras themselves [Sak55]. More recently, this sort of idea has been used to understand how the metric structure of the unitary groups can be related to the Jordan *-algebra structure of the algebras [HM14].

Lemma 3.1.1. *Let $a \in A_{sa}$ and represent $B \subseteq \mathcal{B}(\mathcal{H})$ faithfully and let $\theta : U^0(A) \to U^0(B)$ be a continuous group homomorphism. Then $(\theta(e^{2\pi ita}))_{t\in\mathbb{R}}$ is a one-parameter norm-continuous family of unitaries, and consequently is of the form $(e^{2\pi itb})_{t\in\mathbb{R}}$ for some $b \in B_{sa}$.*

Proof. Using the fact that θ is a norm-continuous homomorphism, $t \mapsto \theta(e^{2\pi ita})$ is a norm-continuous one-parameter family of unitaries. Stone's theorem gives that there is some self-adjoint $b \in \mathcal{B}(\mathcal{H})$ (the boundedness of b follows from norm-continuity). Since $\theta(e^{2\pi ita}) = e^{2\pi itb} \in B$ for all $t \in \mathbb{R}$, one can take t to be sufficiently small, then take a logarithm to get that $b \in B$ by continuous functional calculus. □

Let $S_\theta : A_{sa} \to B_{sa}$ be defined via the correspondence given above:

$$\theta(e^{2\pi ita}) = e^{2\pi it S_\theta(a)} \text{ for all } t \in \mathbb{R}. \tag{3.1.8}$$

Then S_θ is a bounded $\mathbb{R}$-linear map (see [Sak55, HM14], or note that it is easy to see that its kernel is closed). It is also easily checked to respect commutation, and that its canonical extension to a map from A to B actually sends commutators to commutators, although we won't explicitly use this. Recall that for a C*-algebra A, A_0 denotes the set of self-adjoint elements that vanish on every trace.

Lemma 3.1.2. *If $\theta : U^0(A) \to U^0(B)$ is a continuous group homomorphism, then S_θ is a bounded linear map and the following hold.*

1. *If θ is injective, then S_θ is injective.*
2. *If θ is a homeomorphism, then S_θ is bijective.*

Proof. As already remarked, S_θ is bounded. Assuming that θ is injective, suppose that $S_\theta(a) = S_\theta(b)$. Then

$$\theta(e^{2\pi ita}) = \theta(e^{2\pi itb}) \tag{3.1.9}$$

for all $t \in \mathbb{R}$. Injectivity of θ gives that that $e^{2\pi ita} = e^{2\pi itb}$ for all $t \in \mathbb{R}$. But this implies that $a = b$ since we can take t appropriately close to 0 and take logarithms.

Now if we further assume that θ is a homeomorphism, then $(\theta^{-1}(e^{2\pi itb}))_{t\in\mathbb{R}} \subseteq U^0(A)$ is a norm-continuous one-parameter family of unitaries which we can write as $(e^{2\pi ita})_{t\in\mathbb{R}}$ for some $a \in A_{sa}$. But then

$$\theta(e^{2\pi ita}) = \theta \circ \theta^{-1}(e^{\pi itb}) = e^{2\pi itb}. \tag{3.1.10}$$

Thus $S_\theta(a) = b$. □

We say that a linear map $\tau : A_{sa} \to E$, where E is a real Banach space, is a bounded trace is if it is a bounded $\mathbb{R}$-linear map such that $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$ (note that this is equivalent to $\tau \circ \mathrm{Ad}_u = \tau$ for all $u \in U(A)$).

Proposition 3.1.3. *Let $\theta : U^0(A) \to U^0(B)$ be a continuous group homomorphism, E be a real Banach space and $\tau : B_{sa} \to E$ a bounded trace. Then $\tau \circ S_\theta : A_{sa} \to E$ is a bounded trace. In particular, $S_\theta(A_0) \subseteq B_0$ and S_θ induces a bounded $\mathbb{R}$-linear map*

$$\Lambda_\theta : \mathrm{Aff}\, T(A) \to \mathrm{Aff}\, T(B). \tag{3.1.11}$$

Proof. Observe that, for $a \in A_{sa}$ and $u \in U(A)$, we have

$$\begin{aligned} e^{2\pi i t S_\theta(uau^*)} &= \theta(e^{2\pi i t uau^*}) \\ &= \theta(u e^{2\pi i t a} u^*) \\ &= \theta(u) e^{2\pi i t S_\theta(a)} \theta(u)^* \\ &= e^{2\pi i t \theta(u) S_\theta(a) \theta(u)^*} \end{aligned} \tag{3.1.12}$$

for all $t \in \mathbb{R}$. Therefore $S_\theta(uau^*) = \theta(u)S_\theta(a)\theta(u)^*$, and applying τ yields

$$\begin{aligned} \tau \circ S_\theta(uau^*) &= \tau\left(\theta(u)S_\theta(a)\theta(u)^*\right) \\ &= \tau(S_\theta(a)), \end{aligned} \tag{3.1.13}$$

i.e., $\tau \circ S_\theta$ is tracial.

Thus if $a \in A_0$, it vanishes on every tracial state (hence on every tracial functional), and so $\tau \circ S_\theta(a) = 0$ for all $\tau \in T(B)$. Therefore $S_\theta(A_0) \subseteq B_0$ and so S_θ factors through a map

$$\Lambda_\theta : \operatorname{Aff} T(A) \simeq A_{sa}/A_0 \to B_{sa}/B_0 \simeq \operatorname{Aff} T(B), \tag{3.1.14}$$

where we identify $A_{sa}/A_0 \simeq \operatorname{Aff} T(A)$ as described in Section 2.1. □

One cannot expect Λ_θ (or S_θ) to be unital or positive, as the following examples show.

Example 3.1.4.

1. *Consider a continuous homomorphism $\theta : \mathbb{T} \to \mathbb{T} = U^0(\mathbb{C}) = U(\mathbb{C})$. By Pontryagin duality, $\theta(z) = z^n$ for some $n \in \mathbb{Z}$. We then have that $S_\theta : \mathbb{R} \to \mathbb{R}$ is given by $S_\theta(x) = nx$. If $n \neq 1$, clearly S_θ is not unital. If $n < 0$, then S_θ is not positive since it sends 1 to $n < 0$. An important observation, however, is that if $n < 0$, $-S_\theta : \mathbb{R} \to \mathbb{R}$ is positive, and $-\frac{1}{n}S$ is unital and positive.*

2. *Consider $\theta : \mathbb{T}^3 \to \mathbb{T}$ given by $\theta(z, w, v) = \overline{z}wv$. The corresponding map $S_\theta : \mathbb{R}^3 \to \mathbb{R}$ is given by*
$$S_\theta(a, b, c) = -a + b + c. \tag{3.1.15}$$
Clearly $(1, 0, 0) \in \mathbb{C}^3$ is a positive element, however $S_\theta(1, 0, 0) = -1 < 0$. This map is however unital.

3. *Let $\theta : U_2 \to \mathbb{T}$ be defined by $\theta(u) = \det(u)$. Then $S_\theta : (M_2)_{sa} \to \mathbb{R}$ is defined by $S_\theta(A) = trA$, where tr is the unnormalized trace. Clearly this map is not unital, but it is positive.*

4. *Let $\theta : U_2 \to U_3$ be defined by $\theta(u) = u \oplus 1$. Then $S_\theta : (M_2)_{sa} \to (M_3)_{sa}$ is given by $S_\theta(A) = A \oplus 0$, which is again not unital, but is positive. The induced map $\Lambda_\theta : \mathbb{R} \to \mathbb{R}$ is given by $\Lambda_\theta(x) = \frac{2}{3}x$ for $x \in \mathbb{R}$.*

5. *Let* $\theta : \mathbb{T} \hookrightarrow U_2$ *be defined by*

$$\theta(\lambda) = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}. \tag{3.1.16}$$

Then S_θ *is a unital, positive isometry and* Λ_θ *gives rise to the identity map*

$$\mathbb{R} = \operatorname{Aff} T(\mathbb{C}) \to \operatorname{Aff} T(M_2) = \mathbb{R}. \tag{3.1.17}$$

The above examples are important. If $\theta(\mathbb{T}) \subseteq \mathbb{T}$, which is a moderate assumption (e.g., if θ was the restriction of a unital *-homomorphism or an anti-*-homomorphism), we can restrict the homomorphism to the circle to get a continuous group homomorphism $\mathbb{T} \to \mathbb{T}$. We understand such homomorphisms (for example, by Pontryagin duality [Fol16, Chapter 4]).

We now use (pre-)determinant techniques in order to show desirable relationships between our maps. In Section 2.1 we discussed the de la Harpe-Skandalis determinant, but there is also a canonical *determinant associated to a trace* which agrees with the de la Harpe-Skandalis determinant when the trace is the universal trace. We restrict ourselves to the unitary and self-adjoint setting. If τ is a bounded trace and $n \in \mathbb{N} \cup \{\infty\}$, then for a piecewise smooth path $\xi : [0,1] \to U_n^0(A)$, the pre-determinant is

$$\tilde{\Delta}_\tau^n(\xi) = \int_0^1 \tau\left(\frac{1}{2\pi i}\xi'(t)\xi(t)^{-1}\right) dt, \tag{3.1.18}$$

where the integral is just Riemann integration in the Banach space E. This is well-defined since $\frac{1}{2\pi i}\xi'(t)\xi(t)^{-1} \in A_{sa}$ by [Phi92, Proposition 1.4]. The determinant associated to τ (at the nth level) is then the map

$$\Delta_\tau^n : U_n^0(A) \to E/\tilde{\Delta}^n(\pi_1(U_n^0(A))) \tag{3.1.19}$$

given by $\Delta_\tau^n(u) = [\tilde{\Delta}_\tau^n(\xi)]$ where $\xi : [0,1] \to U_n^0(A)$ is any piecewise smooth path from 1 to u. All the same properties in Section 2.1 apply. There are of course general linear variants (for a bounded trace $\tau : A \to E$ for a complex Banach space E) and Hausdorffized variants where one mods out by the closure of the image of the fundamental group. We identify $\tilde{\Delta}_\tau^\infty(\pi_1(U_\infty^0(A)))$ with $\tau(K_0(A))$ in the natural way.

Proposition 3.1.5. *Let* E *be a real Banach space and* $\tau : B_{sa} \to E$ *a bounded trace.*

1. *Let* $\xi : [0,1] \to U^0(A)$ *be a piecewise smooth path with* $\xi(0) = 1$*. Then*

$$\tilde{\Delta}^1_{\tau \circ S_\theta}(\xi) = \tilde{\Delta}^1_\tau(\theta \circ \xi). \tag{3.1.20}$$

In particular, $\tilde{\Delta}^1_{\tau \circ S_\theta}(\pi_1(U^0(A))) \subseteq \tilde{\Delta}^1_\tau(\pi_1(U^0(B)))$.

2. The following diagram commutes:

$$\begin{array}{ccc} \pi_1(U^0(A)) & \xrightarrow{\tilde{\Delta}^1_{\tau\circ S_\theta}} & E \\ {\scriptstyle \pi_1(\theta)}\downarrow & & \downarrow{\scriptstyle id} \\ \pi_1(U^0(B)) & \xrightarrow[\tilde{\Delta}^1_\tau]{} & E. \end{array} \tag{3.1.21}$$

3. The following diagram commutes:

$$\begin{array}{ccc} U^0(A) & \xrightarrow{\Delta^1_{\tau\circ S_\theta}} & E/\tilde{\Delta}^1_{\tau\circ S_\theta}(\pi_1(U^0(A))) \\ {\scriptstyle \theta}\downarrow & & \downarrow \\ U^0(B) & \xrightarrow[\Delta^1_\tau]{} & E/\tilde{\Delta}^1_\tau(\pi_1(U^0(B))), \end{array} \tag{3.1.22}$$

where the map on the right is the canonical map induced from the inclusion $\tilde{\Delta}^1_{\tau\circ S_\theta}(\pi_1(U^0(A))) \subseteq \tilde{\Delta}^1_\tau(\pi_1(U^0(B)))$ coming from (1). The analogous diagram commutes if we consider Thomsen's variant of the de la Harpe-Skandalis determinant associated to τ and $\tau\circ S_\theta$ in (3.1.22).

Proof. By 2.1.2(1), we can find $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in A_{sa}$ such that ξ is homotopic to the path

$$\eta(t) = \left(\prod_{l=1}^{j-1} e^{2\pi i a_l}\right) e^{2\pi i(kt-j)a_j}, t \in \left[\frac{j-1}{k}, \frac{j}{k}\right], \tag{3.1.23}$$

with the convention that the product on the left is 1 for $j \leq 0$. Then whenever $\omega : A_{sa} \to F$ is a bounded trace to a real Banach space F, we have

$$\tilde{\Delta}^1_\omega(\xi) = \sum_{j=1}^k \omega(a_j). \tag{3.1.24}$$

Now $\theta\circ\xi$ is homotopic to $\theta\circ\eta$, which has the following form, for $t \in [\frac{j-1}{k}, \frac{j}{k}]$:

$$\begin{aligned} \theta\circ\eta(t) &= \left(\prod_{l=1}^{j} \theta(e^{2\pi i a_l})\right) \theta(e^{2\pi i(kt-j)a_j} \\ &= \left(\prod_{l=1}^{j} e^{2\pi i S_\theta(a_l)}\right) e^{2\pi i(kt-j)S_\theta(a_j)}. \end{aligned} \tag{3.1.25}$$

Of course, we then have that

$$\tilde{\Delta}^1_\tau(\theta\circ\xi) = \tilde{\Delta}^1_\tau(\theta\circ\eta) = \sum_{j=1}^k \tau(S_\theta(a_j)) = \tilde{\Delta}^1_{\tau\circ S_\theta}(\xi). \tag{3.1.26}$$

Part (2) follows from (1) and (3) follows from (2). The remark about Thomsen's variant is obvious. □

Corollary 3.1.6. *The following diagram commutes:*

$$\begin{array}{ccc} \pi_1(U^0(A)) & \xrightarrow{\tilde{\Delta}^1_A} & \operatorname{Aff} T(A) \\ {\scriptstyle \pi_1(\theta)}\downarrow & & \downarrow{\scriptstyle \Lambda_\theta} \\ \pi_1(U^0(B)) & \xrightarrow[\tilde{\Delta}^1_B]{} & \operatorname{Aff} T(B). \end{array} \tag{3.1.27}$$

In particular, when the canonical maps

$$\pi_1(U^0(A)) \to K_0(A) \text{ and } \pi_1(U^0(B)) \to K_0(B) \tag{3.1.28}$$

are isomorphisms, we have that

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \operatorname{Aff} T(A) \\ {\scriptstyle K_0(\theta)}\downarrow & & \downarrow{\scriptstyle \Lambda_\theta} \\ K_0(B) & \xrightarrow[\rho_B]{} & \operatorname{Aff} T(B) \end{array} \tag{3.1.29}$$

commutes, where $K_0(\theta) : K_0(A) \to K_0(B)$ is the map induced from the diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\simeq} & \pi_1(U^0(A)) \\ {\scriptstyle K_0(\theta)}\downarrow & & \downarrow{\scriptstyle \pi_1(\theta)} \\ K_0(B) & \xrightarrow[\simeq]{} & \pi_1(U^0(B)). \end{array} \tag{3.1.30}$$

Proof. The first part follows from Proposition 3.1.5(2) with the trace being the universal trace $\operatorname{Tr}_B|_{B_{sa}} : B_{sa} \to \operatorname{Aff} T(B)$. The second part follows since if ξ is a (piecewise smooth) path in $U_n(A)$ and $m > n$, then

$$\tilde{\Delta}^m_\tau(\xi \oplus 1_{m-n}) = \tilde{\Delta}^n_\tau(\xi). \tag{3.1.31}$$

□

Proposition 3.1.7. *Let A, B be unital C^*-algebras. Then $\Lambda_\theta : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ is a lift of the map*

$$\operatorname{Aff} T(A)/\tilde{\Delta}^1_A(\pi_1(U^0(A))) \to \operatorname{Aff} T(B)/\tilde{\Delta}^1_B(\pi_1(U^0(B))) \tag{3.1.32}$$

as described in (3.1.3).

Proof. Let us label the maps in the diagram (3.1.3):

$$\begin{array}{ccc} U^0(A)/\ker\Delta^1|_{U^0(A)} & \xrightarrow{\delta^1_A} & \operatorname{Aff} T(A)/\tilde{\Delta}^1_A(\pi_1(U^0(A))) \\ {\scriptstyle\tilde{\theta}}\downarrow & & \downarrow{\scriptstyle P} \\ U^0(B)/\ker\Delta^1|_{U^0(B)} & \xrightarrow[\delta^1_B]{} & \operatorname{Aff} T(B)/\tilde{\Delta}^1_B(\pi_1(U^0(B))) \end{array} \tag{3.1.33}$$

where $\tilde{\theta}([u]) := [\theta(u)]$, $\delta^1_A([e^{2\pi i a}]) := [\hat{a}]$, δ^1_B is defined similarly, and

$$P = \delta^1_B \circ \tilde{\theta} \circ (\delta^1_A)^{-1}. \tag{3.1.34}$$

But then we have that

$$\begin{aligned} P([a]) &= \delta^1_B \circ \tilde{\theta} \circ (\delta^1_A)^{-1}([\hat{a}]) \\ &= \delta^1_B \circ \tilde{\theta}([e^{2\pi i a}]) \\ &= \delta^1_B([\theta(e^{2\pi i a})]) \\ &= \delta^1_B([e^{2\pi i S_\theta(a)}]) \\ &= [\widehat{S_\theta(a)}] \\ &= [\Lambda_\theta(\widehat{a})] \end{aligned} \tag{3.1.35}$$

In particular, the diagram

$$\begin{array}{ccc} \operatorname{Aff} T(A) & \xrightarrow{q^1_A} & \operatorname{Aff} T(A)/\tilde{\Delta}^1_A(\pi_1(U^0(A))) \\ {\scriptstyle\Lambda_\theta}\downarrow & & \downarrow{\scriptstyle P} \\ \operatorname{Aff} T(B) & \xrightarrow[q^1_B]{} & \operatorname{Aff} T(B)/\tilde{\Delta}^1_B(\pi_1(U^0(B))) \end{array} \tag{3.1.36}$$

commutes, where q^1_A and q^1_B are the respective quotient maps. □

Proposition 3.1.8. *Let A, B be unital C^*-algebras and $\theta : U^0(A) \to U^0(B)$ be a continuous group homomorphism. Then $\Lambda_\theta : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ is a lift of the map*

$$\operatorname{Aff} T(A)/\overline{\tilde{\Delta}^1_A(\pi_1(U^0(A)))} \to \operatorname{Aff} T(B)/\overline{\tilde{\Delta}^1_B(\pi_1(U^0(B)))} \tag{3.1.37}$$

as described in (3.1.2).

Proof. One can mimic the proof above or apply the above result and appeal to the commuting diagram

$$\begin{array}{ccc} \operatorname{Aff} T(A)/\tilde{\Delta}^1_A(\pi_1(U^0(A))) & \longrightarrow & \operatorname{Aff} T(A)/\overline{\tilde{\Delta}^1_A(\pi_1(U^0(A)))} \\ \downarrow & & \downarrow \\ \operatorname{Aff} T(B)/\tilde{\Delta}^1_B(\pi_1(U^0(B))) & \longrightarrow & \operatorname{Aff} T(B)/\overline{\tilde{\Delta}^1_B(\pi_1(U^0(B)))}, \end{array} \tag{3.1.38}$$

where the vertical maps are defined via the diagrams from (3.1.3) and (3.1.2) respectively, and the horizontal maps are the canonical surjections. □

In particular, assuming some K_0-regularity gives that Λ_θ is a lift of a map between quotients of spaces of continuous real-valued affine functions by images of K_0.

Corollary 3.1.9. *Let A, B be unital C*-algebras such that the canonical maps*

$$\pi_1(U^0(A)) \to K_0(A) \text{ and } \pi_1(U^0(B)) \to K_0(B) \tag{3.1.39}$$

are surjections. If $\theta : U^0(A) \to U^0(B)$ is a continuous homomorphism, then Λ_θ is a lift of the right maps of the following two commutative diagrams:

$$\begin{array}{ccc} U^0(A)/CU^0(A) & \xrightarrow{\simeq} & \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} \\ \downarrow & & \downarrow \\ U^0(B)/CU^0(B) & \xrightarrow[\simeq]{} & \operatorname{Aff} T(B)/\overline{\rho_B(K_0(B))} \end{array} \tag{3.1.40}$$

and

$$\begin{array}{ccc} U^0(A)/\ker \Delta_A^1|_{U^0(A)} & \xrightarrow{\simeq} & \operatorname{Aff} T(A)/\rho_A(K_0(A)) \\ \downarrow & & \downarrow \\ U^0(B)/\ker \Delta_B^1|_{U^0(B)} & \xrightarrow[\simeq]{} & \operatorname{Aff} T(B)/\rho_B(K_0(B)). \end{array} \tag{3.1.41}$$

Further, if $\ker \Delta_A^1 = DU^0(A)$ and $\ker \Delta_B^1 = DU^0(B)$, then Λ_θ is a lift of the map induced by the diagram

$$\begin{array}{ccc} U^0(A)/DU^0(A) & \xrightarrow{\simeq} & \operatorname{Aff} T(A)/\rho_A(K_0(A)) \\ \downarrow & & \downarrow \\ U^0(B)/DU^0(B) & \xrightarrow[\simeq]{} & \operatorname{Aff} T(B)/\rho_B(K_0(B)). \end{array} \tag{3.1.42}$$

C*-algebras satisfying the last condition arise naturally – for example unital, separable, simple, pure C*-algebras of stable rank 1 such that every 2-quasitracial state on A is a trace [NR17] have this property.

3.2 The order on $\operatorname{Aff} T(\cdot)$

We now examine when the map induced on $\operatorname{Aff} T(\cdot)$ is positive in order to compare K-theory, traces, and the pairing. As we saw in Example 3.1.4, the map we get between spaces of affine functions on the trace simplices need not be positive nor unital in general. In this section, we will be able to use the map Λ_θ to construct a unital positive map, under some extra assumptions on θ.

We record the following results as they give us necessary and sufficient conditions for the Λ_θ to be positive. We use the C*-algebra-valued analogue of the fact that any unital, contractive linear functional is positive, along with the fact that completely positive maps are (completely) bounded with the norm determined by the image of the unit. Recall that an operator system is a self-adjoint unital subspace of a C*-algebra. We record some results about completely positive maps – these are a combination of Proposition 2.11, Theorem 3.9 and Proposition 3.6 in [Pau02] respectively.

Proposition 3.2.1. *Let $\mathcal{S}$ be an operator system and B a unital C*-algebra.*

1. *If $\phi : \mathcal{S} \to B$ is a unital contraction, then ϕ is positive.*
2. *If $B = C(X)$ and $\phi : \mathcal{S} \to B$ is positive, then it is bounded with $\|\phi\| = \|\phi(1)\|$.*

Lemma 3.2.2. *Let $\theta : U^0(A) \to U^0(B)$ be a continuous group homomorphism such that $\theta(\mathbb{T}) = \mathbb{T}$. If $\theta|_{\mathbb{T}}$ is injective, then $S_\theta(1) \in \{1, -1\}$.*

Proof. The restriction $\theta|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is a continuous group homomorphism, hence by Pontryagin duality is of the form $\theta(z) = z^n$ for some n. Injectivity implies that $n \in \{1, -1\}$. We then have that

$$e^{2\pi i S_\theta(1)t} = \theta(e^{2\pi it}) = e^{2\pi int} \tag{3.2.1}$$

for all $t \in \mathbb{R}$. This implies that $S_\theta(1) = n \cdot 1 \in \{1, -1\}$. □

Lemma 3.2.3. *Let $\theta : U^0(A) \to U^0(B)$ be continuous group homomorphism such that $\theta(\mathbb{T}) = \mathbb{T}$. If θ is injective, the following are equivalent.*

1. *one of Λ_θ or $-\Lambda_\theta$ is positive;*
2. *Λ_θ is contractive.*

Proof. By Lemma 3.2.2, we know that $S_\theta(1) \in \{1, -1\}$ and consequently $\Lambda_\theta(\hat{1}) \in \{\hat{1}, \widehat{-1}\}$ (where we recall that, for $a \in A_{sa}$, $\hat{a} \in \operatorname{Aff} T(A)$ is the affine function $\hat{a}(\tau) = \tau(a)$). By replacing Λ_θ with $-\Lambda_\theta$, we can without loss of generality assume that Λ_θ is unital. Using the fact that $\operatorname{Aff} T(A) + i \operatorname{Aff} T(A) \subseteq C(T(A))$ is an operator system and the canonical extension

$$\Lambda_\theta^{\mathbb{C}} : \operatorname{Aff} T(A) + i \operatorname{Aff} T(A) \to \operatorname{Aff} T(B) + i \operatorname{Aff} T(B) \subseteq C(T(B)) \tag{3.2.2}$$

is a unital linear map with abelian target algebra, this is an easy consequence of the two parts of Proposition 3.2.1. □

Lemma 3.2.4. *Let $\theta : U^0(A) \to U^0(B)$ be a continuous group homomorphism. If $K > 0$ is such that $\|\theta(u) - \theta(v)\| \leq K\|u - v\|$ for all $u, v \in U^0(A)$, then $\|S_\theta\| \leq K$ and $\|\Lambda_\theta\| \leq K$. If θ is isometric, then so is S_θ. If θ is a surjective isometry, then Λ_θ is a surjective isometry.*

Proof. The map S_θ being an isometry following from θ being an isometry can be seen in an argument in [HM14]; we exemplify said argument to show the bound condition. We use the observation that

$$\frac{e^{2\pi ita}-1}{t} \to 2\pi ia \tag{3.2.3}$$

as $t \to 0$. Since

$$\|e^{2\pi itS_\theta(a)}-1\| \leq K\|e^{2\pi ita}-1\| \tag{3.2.4}$$

for all $t \in \mathbb{R}$, we can divide both sides by $\frac{1}{2\pi}|t|$ and take $t \to 0$ to get that

$$\|S_\theta(a)\| \leq K\|a\|. \tag{3.2.5}$$

Now if θ is a surjective isometry, we identify $\operatorname{Aff} T(A) \simeq A_{sa}/A_0$ and $\operatorname{Aff} T(B) \simeq B_{sa}/B_0$ and note that $S_\theta(A_0) = B_0$ and that Λ_θ will preserve the quotient norms. $\square$

Corollary 3.2.5. *If $S_\theta(1) = n$ and $\|S_\theta\| = |n|$, then $\frac{1}{n}S_\theta$ is a unital contraction, hence positive. In particular, if $\theta(\mathbb{T}) = \mathbb{T}$ and $\theta|_{\mathbb{T}}$ is an injection, then either Λ_θ or $-\Lambda_\theta$ is unital and positive.*

Proof. The first part follows from the above lemma. If θ is an injection with $\theta(\mathbb{T}) = \mathbb{T}$, we have that $S_\theta(\hat{1}) \in \{\hat{1}, \widehat{-1}\}$ and that Λ_θ is contractive, so one of Λ_θ or $-\Lambda_\theta$ is a unital contraction, hence positive by part (1) of Proposition 3.2.1. $\square$

Theorem 3.2.6. *Suppose that $\theta : U^0(A) \to U^0(B)$ is a contractive injection such that $\theta(\mathbb{T}) = \mathbb{T}$. Then there is a continuous affine map $T_\theta : T(B) \to T(A)$.*

Proof. This follows from the fact that the induced map $\Lambda_\theta : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ will have the property that Λ_θ or $-\Lambda_\theta$ will be a unital positive map. Therefore by contravariant identification of compact convex sets (of locally convex Hausdorff linear spaces) with the state space of the space of continuous real-valued affine valued functions on them (Theorem 1.2.4), there exists a continuous affine map $T_\theta : T(B) \to T(A)$. $\square$

Theorem 3.2.7. *Let A, B be unital C^*-algebras, and $\theta : U^0(A) \to U^0(B)$ be a contractive topological group isomorphism such that $\theta(\mathbb{T}) = \mathbb{T}$. Then the map $T_\theta : T(B) \to T(A)$ induced by Λ_θ is an affine homeomorphism.*

Proof. As $\theta(\mathbb{T}) = \mathbb{T}$, $S_\theta(1) \in \{-1, 1\}$. Let $\pm\Lambda_\theta : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ be either Λ_θ or $-\Lambda_\theta$, depending on which is unital, positive, contractive and surjective by combining Lemmas 3.2.4, 3.2.3 and 3.1.2(2). By the duality of (compact) simplices and continuous affine functions on them, the map $T_\theta : T(B) \to T(A)$ is an affine homeomorphism. $\square$

Theorem 3.2.8. *Let $\theta : U(A) \to U(B)$ be a contractive injective homomorphism such that $\theta(\mathbb{T}) = \mathbb{T}$. If*

$$\pi_i(U(A)) \simeq K_{i-1}(A) \text{ and } \pi_i(U(B)) \simeq K_{i-1}(B) \text{ for } i = 0, 1, \tag{3.2.6}$$

via the canonical maps, then there is an induced map

$$KT_u(\theta) : KT_u(A) \to \mathrm{KT}_u(B). \tag{3.2.7}$$

Proof. Let

- $\Lambda := \Lambda_{\theta|_{U^0(A)}} : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$,
- $\theta_0 : \pi_1(U^0(A)) \to \pi_1(U^0(B))$ be the map induced on fundamental groups by $\theta|_{U^0(A)}$,
- $K_0(\theta)$ be the map induced on K_0 by θ_0 together with (3.2.6) for $i = 1$,
- $\theta_1 : \pi_0(U(A)) \to \pi_0(U^0(B))$ be the map induced by θ on connected components (so that $\theta_1([u]_{\sim_h}) = [\theta_1(u)]_{\sim_h}$) and
- $K_1(\theta)$ be the map induced by θ_1 together with (3.2.6) for $i = 0$.

Then

$$(\pm K_0(\theta), K_1(\theta), \pm\Lambda) : KT_u(A) \to KT_u(B) \tag{3.2.8}$$

is a KT_u-morphism, where $\pm\Lambda$ is either Λ or $-\Lambda$ depending on which one is unital and positive, and $\pm K_0(\theta)$ is either $K_0(\theta)$ if Λ is positive or $-K_0(\theta)$ if $-\Lambda$ is positive. Indeed, $\pm K_0(\theta), \theta_1, \pm\Lambda$ are all appropriate morphisms, and we have that

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \operatorname{Aff} T(A) \\ {\scriptstyle \pm K_0(\theta)}\downarrow & & \downarrow{\scriptstyle \pm\Lambda} \\ K_0(B) & \xrightarrow[\rho_B]{} & \operatorname{Aff} T(B) \end{array} \tag{3.2.9}$$

commutes[16] by Corollary 3.1.6. □

Corollary 3.2.9. *Let A, B be unital C^*-algebras, $\theta : U^0(A) \to U^0(B)$ is a contractive topological group isomorphism such that $\theta(\mathbb{T}) = \mathbb{T}$. If*

$$\pi_i(U(A)) \simeq K_{i-1}(A) \text{ and } \pi_i(U(B)) \simeq K_{i-1}(B) \text{ for } i = 0, 1, \tag{3.2.10}$$

via the canonical maps, then $KT_u(A) \simeq KT_u(B)$.

[16] The map $-K_0(\theta)$ will take a piece-wise smooth loop ξ to the loop $-\theta \circ \xi$ defined by $(-\theta \circ \xi)(t) = \theta(\xi(-t))$. From here its obvious that the diagram commutes.

Proof. By Corollary 3.2.8, we have an induced KT_u-morphism. This map is necessarily an isomorphism since θ is. $\square$

Corollary 3.2.10. *Let A, B be unital C*-algebras which are either $\mathcal{Z}$-stable or of stable rank one. Let $\theta : U(A) \to U(B)$ be a contractive injective homomorphism such that $\theta(\mathbb{T}) = \mathbb{T}$. Then there is an induced map*

$$KT_u(\theta) : KT_u(A) \to KT_u(B).$$

Proof. C*-algebras which are $\mathcal{Z}$-stable or have stable rank one satisfy the hypotheses of Theorem 3.2.8 by [Jia97] and [Rie87] respectively. So the theorem applies. $\square$

Remark 3.2.11. The strict ordering on $\operatorname{Aff} T(A)$ is given by $f \gg g$ if $f(\tau) > g(\tau)$ for all $\tau \in T(A)$. If A, B are unital and $\theta : U^0(A) \to U^0(B)$ is a contractive injective homomorphism such that $\theta(\mathbb{T}) = \mathbb{T}$, then $\pm\Lambda_\theta : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ is a unital positive contraction by Lemma 3.2.3 (again $\pm\Lambda_\theta$ is Λ_θ or $-\Lambda_\theta$ depending on which is positive). We moreover have that

$$\pm\Lambda_\theta(f) \gg \pm\Lambda_\theta(g) \iff f \gg g. \tag{3.2.11}$$

Indeed, let us show that $f \gg 0$ if and only if its image is $\gg 0$. As $\pm\Lambda_\theta$ has the form $\pm\Lambda_\theta(\hat{a}) = \widehat{\pm S_\theta(a)}$, it suffices to show that if $\tau(a) > 0$ for all $\tau \in T(A)$, then $\tau(\pm S_\theta(a)) > 0$ for all $\tau \in T(B)$. But this is trivial because $\tau \circ \pm S_\theta : A_{sa} \to \mathbb{R}$ extends canonically to a tracial state $A \to \mathbb{C}$, so evaluating it against a must give that it is strictly positive.

The above says the following: for certain C*-algebras, we can read off positivity in K_0, thinking of it as the fundamental group of the unitary group, from the strict positivity of the pre-determinant applied a the loop. Precisely, a non-zero element $x \in K_0(A)$, where A is a unital, simple C*-algebra with strict comparison, is in the positive cone if and only if the corresponding loop ξ_x satisfies $\tilde{\Delta}_\tau(\xi_x) > 0$ for all $\tau \in T(A)$.

Although the following is known, for example by very strong results in [AM03, Chapter 6] pertaining to certain prime C*-algebras, we give the following as a corollary by using K-theoretic classification results.

Corollary 3.2.12. *Let A, B be unital, separable, simple, nuclear $\mathcal{Z}$-stable C*-algebras satisfying the UCT. Then $A \simeq B$ if and only if there is a contractive isomorphism $U(A) \simeq U(B)$.*

Proof. Its clear that two isomorphic C*-algebras have isomorphic unitary groups. On the other hand, if $U(A) \simeq U(B)$, then since these C*-algebras are $\mathcal{Z}$-stable, Corollary 3.2.9 applies. As $KT_u(\cdot)$ recovers the Elliott invariant, which is a complete invariant for the C*-algebras as in the statement of the theorem (by [CET+21, Corollary D], [EGLN15, GLN20a, GLN20b] and the references therein), $A \simeq B$. $\square$

Using the state of the art classification of embeddings [CGS+23], there is an enlarged invariant of $KT_u(\cdot)$ which is able to classify morphisms between certain C*-algebras. Any KT_u-morphism automatically has a lift to this larger invariant, and so under the assumption that the KT_u-morphism is faithful (i.e., the map $T(B) \to T(A)$ induced by the map $\operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ sends traces on B to faithful traces on A), there is a *-homomorphism witnessing the KT_u-morphism. Therefore as a corollary of their main theorem, we have that for an abundance of C*-algebras, there is an (contractive) embedding of unitary groups if and only if there is an embedding of C*-algebras.

Corollary 3.2.13. *Let A be a unital, separable, simple nuclear C*-algebra satisfying the UCT which is $\mathcal{Z}$-stable or has stable rank one, and B be a unital, separable, simple, nuclear $\mathcal{Z}$-stable C*-algebra. If there is a contractive homomorphism $\theta : U(A) \to U(B)$ such that $\theta(\mathbb{T}) = \mathbb{T}$, then there is an embedding $A \hookrightarrow B$.*

Proof. Assuming such a θ exists, it gives rise to a KT_u-morphism

$$KT_u(\theta) : KT_u(A) \to KT_u(B). \tag{3.2.12}$$

As A, B are simple, the map $T_\theta : T(B) \to T(A)$ necessarily maps traces on B to faithful traces on A, and therefore the KT_u-morphism $KT_u(\theta)$ is "faithful". Therefore $KT_u(\theta)$ induces an embedding $A \hookrightarrow B$ by [CGS+23]. ☐

3.3 General linear variants

Here we briefly describe some general linear variants of the results above. There are natural analogues of the unitary algebraic K-theoretic results. In the presence of a continuous homomorphism $\theta : GL^0(A) \to GL^0(B)$, we have corresponding maps

$$\begin{array}{ccc} GL^0(A)/CGL^0(A) & \xrightarrow{\simeq} & \left(A/\overline{[A,A]}\right) / \overline{\tilde{\Delta}^1_A(\pi_1(U^0(A)))} \\ \downarrow & & \downarrow \\ GL^0(B)/CGL^0(B) & \xrightarrow[\simeq]{} & \left(A/\overline{[A,A]}\right) / \overline{\tilde{\Delta}^1_B(\pi_1(U^0(B)))}. \end{array} \tag{3.3.1}$$

Again, by modding out by $\ker \Delta^1_A$ and $\ker \Delta^1_B$ respectively instead of closures of derived groups, there is a purely algebraic variant of the above diagram:

$$\begin{array}{ccc} GL^0(A)/\ker \Delta^1_A & \xrightarrow{\simeq} & \left(A/\overline{[A,A]}\right) / \tilde{\Delta}^1_A(\pi_1(U^0(A))) \\ \downarrow & & \downarrow \\ GL^0(B)/\ker \Delta^1_B & \xrightarrow[\simeq]{} & \left(A/\overline{[A,A]}\right) / \tilde{\Delta}^1_B(\pi_1(U^0(B))). \end{array} \tag{3.3.2}$$

Thinking of $K_0(A)$ as the Grothendieck group of the semigroup of equivalence classes of idempotents and $K_0(A) \simeq \pi_1(GL^0_\infty(A))$, we would like to lift the maps on the right of (3.3.1) and (3.3.2) to a map

$$A/\overline{[A,A]} \to B/\overline{[B,B]}. \tag{3.3.3}$$

We will construct our map by first constructing a map between the Banach spaces of bounded traces and using duality. For a C*-algebra C denote by $\mathfrak{T}(C)$ the complex Banach space of bounded tracial functionals. Define $F_\theta : \mathfrak{T}(B) \to \mathfrak{T}(A)$ by

$$F_\theta(\tau)(a) := \lim_{r\to 0^+} \frac{1}{2\pi i r}\tau(\log\theta(e^{2\pi i r a})), \tau \in \mathfrak{T}(B), a \in A. \tag{3.3.4}$$

Proposition 3.3.1. *$F_\theta(\tau) : A \to \mathbb{C}$ is a well-defined tracial functional and $F_\theta : \mathfrak{T}(B) \to \mathfrak{T}(A)$ is a bounded linear map.*

Proof. We give an outline of the proof.

One can first convince oneself that $F_\theta(\tau)(a)$ is well-defined for every $a \in A$. To do this, one can use upper semi-continuity of the spectrum to convince oneself that $\theta(e^{2\pi i r a})$ is an exponential for small $r > 0$. Using the fact that θ is a continuous homomorphism, one can show that the sequence $(\frac{n}{2\pi i}\log\theta(e^{2\pi i \frac{a}{n}}))_n$ is eventually constant, from which it follows that $(\frac{1}{2\pi i r}\log\theta(e^{2\pi i r a}))_{r\to 0^+, r\in\mathbb{Q}_+}$ is eventually constant, and then it is easy to show that the limit $\lim_{r\to 0^+}\frac{1}{2\pi i r}\log\theta(e^{2\pi i r a})$ exists. Note that this gives a map $G_\theta : A \to B$ given by

$$G_\theta(a) := \lim_{r\to 0^+}\frac{1}{2\pi i r}\log\theta(e^{2\pi i r a}), \tag{3.3.5}$$

which can be shown to be bounded and linear. But then its clear that $F_\theta(\tau) = \tau \circ G : A \to \mathbb{C}$ is a well-defined bounded linear map. □

The Banach space $\mathfrak{T}(A)$ can be isometrically identified with $\left(A/\overline{[A,A]}\right)^*$ with duality $\langle \tau, [a]\rangle = \tau(a)$. We can use duality to define a map $\tilde{F}_\theta := F^*_\theta|_{A/\overline{[A,A]}} : A/\overline{[A,A]} \to B/\overline{[B,B]}$. We note that if G_θ is the map as in the proof above, we have that

$$\theta(e^a) = e^{G_\theta(a)} \tag{3.3.6}$$

and that

$$[G_\theta(a)] = \tilde{F}_\theta([a]). \tag{3.3.7}$$

Proposition 3.3.2. *The map $\tilde{F}_\theta$ is a lift of the maps in (3.3.1) and (3.3.2).*

Proof. This is essentially the same proof as Proposition 3.1.8. □

In a similar vein to the unitary case, contractive injections $GL(A) \to GL(B)$ which send the circle to the circle (or $\mathbb{C}^\times$ to $\mathbb{C}^\times$) give rise to KT_u-morphisms.

3.4 Final remarks and open questions

We list some alternate ways to go about things. Rather than using Stone's theorem to get a map between self-adjoint elements, one can define from a trace on B a trace on A directly. By this we mean that if E is a Banach space and $\tau : A_{sa} \to E$ is a bounded trace, the trace can be recovered from

$$\tau(a) = \lim_{r \to 0^+} \tau\left(\frac{1}{2\pi i r} \log e^{2\pi i r a}\right). \tag{3.4.1}$$

One applies the unitary group homomorphism θ to the only unitary in the above equation, and we can simply define

$$F(\tau)(a) = \lim_{r \to 0^+} \tau\left(\frac{1}{2\pi i r} \log \theta(e^{2\pi i r a})\right). \tag{3.4.2}$$

Of course one must check that this is a well-defined, bounded, tracial $\mathbb{R}$-linear map $A_{sa} \to E$, which is obvious if one uses Stone's theorem, but it can be done directly without it. Note that it is helpful to reformulate this in terms of $\tilde{\Delta}_\tau$ after picking appropriate paths in $U^0(A)$ and $U^0(B)$.

With this formulation one can define $F_\theta : T_s(B) \to T_s(A)$ using the above formula, where $T_s(\cdot)$ denotes the set of bounded, self-adjoint, tracial functionals on a C*-algebra, which is canonically isomorphic to the dual of $\operatorname{Aff} T(A) \simeq A_{sa}/A_0$. One can then use duality to define a map $\tilde{F}_\theta : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$, similar to how it was done in Section 3.3. It can be shown that $\tilde{F}_\theta = \Lambda_\theta$.

Now to list some open problems.

1. There are classes where topological isomorphisms between $U(A)$ and $U(B)$ (or even $U^0(A)$ and $U^0(B)$) come from *-isomorphisms or anti-*-isomorphisms. For example, if A, B are prime traceless C*-algebras containing full square zero elements, this is true by [CR23].

   If A is a unital, separable, nuclear C*-algebra satisfying the UCT and B is a unital simple separable nuclear $\mathcal{Z}$-stable C*-algebra, then unital embeddings $A \hookrightarrow B$ are classified by an invariant $\underline{K}T_u(\cdot)$ which is an enlargement of KT_u [CGS+23]. Thus any isometric unitary group homomorphism $U(A) \to U(B)$ will give a KT_u-morphism $KT_u(\theta)$ and therefore there will be an embedding $\phi : A \hookrightarrow B$ such that $KT_u(\phi) = KT_u(\theta)$. However it is not clear that ϕ satisfies $\phi|_{U(A)} = \theta$. More generally though – in the tracial setting – are there continuous group homomorphisms which do not have lifts to *-homomorphisms or anti-*-homomorphisms?

   Note that in [AM03, Chapter 6], Lie isomorphisms between certain C*-algebras are shown to be the sum of a Jordan *-isomorphism and a center-valued trace. Is there a result for general (injective) Lie homomorphisms between certain classes of C*-algebras?

2. This enlargement of KT_u discussed in [CGS+23] contains K-theory with coefficients (along with appropriate pairing maps – the Bosckstein maps discussed in [Sch84]). So we ask: do continuous group homomorphisms induce maps between K-theory with coefficients?

3. For a general continuous homomorphism $\theta : U^0(A) \to U^0(B)$, does the norm $\|S_\theta\|$ determine a Lipschitz constant for θ? We clearly have that

$$\|S_\theta\| \leq \inf\{K \mid \theta \text{ is } K\text{-Lipschitz}\} \tag{3.4.3}$$

   by Lemma 3.2.4. Is this equality?

4. For A simple (or prime), is it true that any continuous injective homomorphism $\theta : U^0(A) \to U^0(B)$ is isometric? Contractive? What if B is simple (or prime)?

Chapter 4

Tensorially absorbing inclusions

The study of inclusions of C*-algebras has been of recent interest. There is no short supply of research concerning inclusions relating to non-commutative dynamics [Pop00, Izu02, CS19, OT18, ER21], as well as inclusions of simple C*-algebras [Rør21]. We discuss inclusions from the lens of tensorially absorbing a strongly self-absorbing C*-algebra $\mathcal{D}$ [TW07].

When speaking of tensorial absorption with a strongly self-absorbing C*-algebra, central sequences play a role akin to McDuff's characterization of when a II_1 von Neumann algebra absorbs the unique hyperfinite II_1 factor $\mathcal{R}$ [McD69]. Central sequences have been studied since the inception of operator algebras as Murray and von Neumann used them to exhibit non-isomorphic II_1 factors by showing that $\mathcal{L}(\mathbb{F}_2)$ does not have property Γ [MvN43]. They were also used in Connes' theorem concerning the uniqueness of $\mathcal{R}$ [Con76], and the classification of auto-morphisms on hyperfinite factors [Con75, Con76]. In [Bis90, Bis94], Bisch considered the central sequence algebra $\mathcal{N}^\omega \cap \mathcal{M}'$ associated to an (irreducible) inclusion of II_1 factors $\mathcal{N} \subseteq \mathcal{M}$ and characterized when there was an isomorphism $\Phi : \mathcal{M} \simeq \mathcal{M}\overline{\otimes}\mathcal{R}$ such that $\Phi(\mathcal{N}) = \mathcal{N}\overline{\otimes}\mathcal{R}$ in terms of the existence of non-commuting sequences in $\mathcal{N}$ which asymptotically commute with the larger von Neumann algebra $\mathcal{M}$ (in the $\|\cdot\|_2$-norm). As pointed out by Izumi [Izu04], there are similar central sequence characterizations for unital inclusions of separable C*-algebras which tensorially absorb a strongly self-absorbing C*-algebra $\mathcal{D}$ (it was at least pointed out for $\mathcal{D}$ being one of $M_{n^\infty}, \mathcal{O}_2, \mathcal{O}_\infty$).

For a strongly self-absorbing C*-algebra $\mathcal{D}$ [TW07, Definition 1.3(iv)], we study $\mathcal{D}$-stable inclusions (see Section 4.2 for detailed definitions), analogous to Bisch's notion for an (irreducible) inclusion of II_1 factors [Bis90]. We say that an inclusion $B \subseteq A$ is $\mathcal{D}$-stable if there is an isomorphism $A \simeq A \otimes \mathcal{D}$ such that

$$\begin{array}{ccc} A & \xrightarrow{\simeq} & A \otimes \mathcal{D} \\ {\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle \iota\otimes\mathrm{id}_{\mathcal{D}}} \\ B & \xrightarrow{\simeq} & B \otimes \mathcal{D} \end{array} \tag{4.0.1}$$

commutes.

We study such inclusions systematically, discussing central sequence characterizations, permanence properties, and giving examples towards the end. We list some key findings here. The first is that $\mathcal{D}$-stable inclusions exist between $\mathcal{D}$-stable C*-algebras if there is any inclusion, and that the set of $\mathcal{D}$-stable inclusions is quite large. Moreover, as far as classification of embeddings up to approximate unitary equivalence (in particular by K-theory and traces), $\mathcal{D}$-stable embeddings are all that matter.

Theorem 4.A (Proposition 4.2.12, Corollary 4.2.13)**.** *Let A, B be unital, separable, $\mathcal{D}$-stable C*-algebras.*

1. *The set of $\mathcal{D}$-stable embeddings $B \hookrightarrow A$ is point-norm dense in the set of all embeddings $B \hookrightarrow A$.*
2. *Every embedding $B \hookrightarrow A$ is approximately unitarily equivalent to a $\mathcal{D}$-stable embedding.*

We note that this set is however not everything. We provide examples of non-$\mathcal{D}$-stable inclusions of $\mathcal{D}$-stable C*-algebras, namely by fitting a C*-algebra with perforated Cuntz semigroup or with higher stable rank (in particular non-$\mathcal{Z}$-stable C*-algebras) in between two $\mathcal{D}$-stable C*-algebras. The second useful tool is that a $\mathcal{D}$-stable inclusion allows one to find an appropriate isomorphism witnessing $\mathcal{D}$-stability of countably many intermediate subalgebras at once.

Theorem 4.B (Theorem 4.2.9)**.** *Let $B \subseteq A$ be a unital, $\mathcal{D}$-stable inclusion of separable C*-algebras. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of C*-algebras such that $B \subseteq C_n \subseteq A$ unitally for all n, then there exists a *-isomorphism $\Phi : A \simeq A \otimes \mathcal{D}$ such that*

1. *$\Phi(B) = B \otimes \mathcal{D}$ and*
2. *$\Phi(C_n) = C_n \otimes \mathcal{D}$ for all $n \in \mathbb{N}$.*

This is not a trivial condition, as it is not true that any such isomorphism sends every intermediate C*-algebra to its tensor product with $\mathcal{D}$ (see Example 4.2.7). In fact, one can always find an intermediate C*-algebra C between B and A and an isomorphism $A \simeq A \otimes \mathcal{D}$ sending B to $B \otimes \mathcal{D}$ which does not send C to $C \otimes \mathcal{D}$ (although, of course, we will still have $C \simeq C \otimes \mathcal{D}$).

The above result, together with the Galois correspondence of Izumi [Izu02], allows us to get a result similar to the main theorem of [AGJP22]. There they prove that if $G \curvearrowright^{\alpha} A$ is an action of a finite group with the weak tracial Rokhlin property on a C*-algebra A with sufficient regularity conditions, then every C*-algebra between $A^{\alpha} \subseteq A$ and $A \subseteq A \rtimes_{\alpha} G$ is $\mathcal{Z}$-stable. Assuming we have a unital C*-algebra with the same regularity conditions, we show that we can witness $\mathcal{Z}$-stability of all such intermediate C*-algebras concurrently.

Corollary 4.C (Corollary 4.3.8). *Let A be a unital, simple, separable, nuclear $\mathcal{Z}$-stable C*-algebra and $G \curvearrowright^{\alpha} A$ be an action of a finite group with the weak tracial Rokhlin property. There exists an isomorphism $\Phi : A \rtimes_\alpha G \simeq (A \rtimes_\alpha G) \otimes \mathcal{Z}$ such that whenever C is a unital C*-algebra satisfying either*

1. $A^\alpha \subseteq C \subseteq A$ *or*
2. $A \subseteq C \subseteq A \rtimes_\alpha G$,

we have $\Phi(C) = C \otimes \mathcal{Z}$.

This chapter is structured as follows. We discuss various local properties in Section 4.1, and then formalize the notion of a $\mathcal{D}$-stable embedding in Section 4.2, examining several properties and consequences. In Section 4.3 we show how several examples arising from non-commutative dynamical systems fit into the framework of $\mathcal{D}$-stable inclusions. We finish with several examples in Section 4.4.

4.1 Approximately central approximate embeddings

Here we formalize some results on approximate embeddings. When $B \subseteq A$ is a unital inclusion of separable C*-algebras, this will yield local characterizations of nuclear subalgebras of $B_\omega \cap A'$, as defined in (1.3.5).

Definition 4.1.1. *Let $B \subseteq A$ be a unital inclusion of C*-algebras and let D be a unital, simple, nuclear C*-algebra. Let $\mathcal{F} \subseteq D, \mathcal{G} \subseteq A$ be finite sets and $\varepsilon > 0$. We say that a u.c.p. map $\phi : D \to B$ is an $(\mathcal{F}, \varepsilon)$-approximate embedding if*

1. $\phi(cd) \approx_\varepsilon \phi(c)\phi(d)$ *for all* $c, d \in \mathcal{F}$.

If ϕ additionally satisfies

2. $[\phi(c), a] \approx_\varepsilon 0$ *for all* $c \in \mathcal{F}$ *and* $a \in \mathcal{G}$,

then we say that ϕ is an $(\mathcal{F}, \varepsilon, \mathcal{G})$-approximately central approximate embedding.

We will usually write that ϕ is a $(\mathcal{F}, \varepsilon)$-embedding or $(\mathcal{F}, \varepsilon, \mathcal{G})$-embedding to mean that ϕ is an $(\mathcal{F}, \varepsilon)$-approximate embedding or $(\mathcal{F}, \varepsilon, \mathcal{G})$-approximately central approximate embedding respectively.

Remark 4.1.2. One can make a similar definition to the above if D is not simple or nuclear (or even unital). The aim is to discuss subalgebras of $B_\omega \cap A'$, and if $D \hookrightarrow B_\omega \cap A'$ is nuclear, then one can use Choi-Effros to lift the embedding to a sequence of u.c.p. maps which are approximately isometric, approximately multiplicative, and approximately commute with finite subsets of A. If D is simple, the approximate isometry condition follows for free since the embedding $D \hookrightarrow B_\omega \cap A'$ must be isometric.

If we loosen the simple and nuclear assumptions on D, we can still speak of bounded linear maps $\phi : D \to B$ (no longer necessarily u.c.p.) which are approximately isometric, approximately multiplicative, approximately adjoint-preserving, and approximately commute with a finite prescribed subset of A. This will allow one to discuss general subalgebras of $B_\omega \cap A'$. As we will only be interested in strongly self-absorbing subalgebras of $B_\omega \cap A'$, which are unital, separable, simple, and nuclear (see [TW07, Section 1.6]), we restrict ourselves to u.c.p. maps from unital, simple, nuclear C*-algebras which are approximately multiplicative and approximately commute with finite subsets of A.

Most of the work in this section can be done without assumptions of simplicity and nuclearity.

Lemma 4.1.3. *Suppose that A, B, D are unital C*-algebras with B separable and D simple, separable and nuclear. Suppose that $B \subseteq A$ and let $S \subseteq A$ be a separable subset. There are $(\mathcal{F}, \varepsilon, \mathcal{G})$-approximately central approximate embeddings $D \to B$ for all $\mathcal{F} \subseteq D, \mathcal{G} \subseteq S$ and $\varepsilon > 0$ if and only if there is a unital embedding $D \hookrightarrow B_\omega \cap S'$.*

Proof. Let (F_n) be an increasing sequence of finite subsets of D with dense union and let (G_n) be an increasing sequence of finite subsets of S with dense union. Let $\phi_n : D \to B$ be $(F_n, \frac{1}{n}, G_n)$-approximately central approximate embeddings. Let $\pi : \ell^\infty(B) \to B_\omega$ denote the quotient map and set

$$\psi := \pi \circ (\phi_n) : D \to B_\omega \tag{4.1.1}$$

which is a unital embedding such that $[\psi(d), a] = 0$ for all $d \in D$ and $a \in S$.

For the other direction, suppose that $\psi : D \to B_\omega \cap S'$ is a unital embedding, $\mathcal{F} \subseteq D, \mathcal{G} \subseteq S$ are finite and $\varepsilon > 0$. By the Choi-Effros lifting theorem (see, for example, [BO08, Theorem C.3]) there is a u.c.p. lift $\tilde{\psi} = (\tilde{\psi}_n) : D \to \ell^\infty(B)$ such that

- $\|\tilde{\psi}_n(cd) - \tilde{\psi}_n(c)\tilde{\psi}_n(d)\| \to^{n\to\omega} 0$,
- $\|[\tilde{\psi}_n(d), a]\| \to^{n\to\omega} 0$

for all $c, d \in D$ and $a \in A$. Take n large enough and set $\phi = \psi_n$, so that ϕ will be a $(\mathcal{F}, \varepsilon, \mathcal{G})$-approximately central approximate embedding. □

Corollary 4.1.4. *Let A, B, D be unital C*-algebras with B, D separable, simple and nuclear and $B \subseteq A$. Suppose that there are unital embeddings $\phi : D \to B_\omega$ and $\psi : B \to A_\omega$. Then there is a unital embedding $\xi : D \hookrightarrow A_\omega$. If $S \subseteq A_\omega$ is a separable subset with $\psi(B) \subseteq A_\omega \cap S'$, then ξ can be chosen with $\xi(C) \subseteq A_\omega \cap S'$.*

Proof. Let $\mathcal{F} \subseteq D$ be finite and $\varepsilon > 0$. Let $L := \max\{\max_{d\in\mathcal{F}} \|d\|, 1\}$. By the above lemma, there is an $(\mathcal{F}, \frac{\varepsilon}{2L})$-approximate embedding $\phi : D \to B$, so let $\mathcal{F}' = \phi(\mathcal{F})$. Now there is an $(\mathcal{F}', \frac{\varepsilon}{2L})$-approximate embedding $\psi : B \to A$. An easy calculation shows that $\psi \circ \phi : D \to A$ is an approximate $(\mathcal{F}, \varepsilon)$-embedding.

Appending the condition that $\psi : B \to A_\omega \cap S'$, then, for any finite subset $\mathcal{G} \subseteq S$, we can take $\psi : B \to A$ to be a $(\mathcal{F}', \frac{\varepsilon}{2L}, \mathcal{G})$-approximately central approximate embedding. This gives that $\psi \circ \phi : D \to A$ is be a $(\mathcal{F}, \varepsilon, \mathcal{G})$-approximately central approximate embedding. □

Corollary 4.1.5. *Let D be a C^*-algebra and $B \subseteq A$ be a unital inclusion of separable C^*-algebras such that B and D are unital, separable, simple and nuclear. Suppose that there is an embedding $\pi : A \hookrightarrow A_\omega \cap A'$ with $\pi(B) \subseteq B_\omega \cap A'$. If $D \hookrightarrow B_\omega$ unitally, then $D \hookrightarrow B_\omega \cap A'$ unitally.*

Proof. As $D \hookrightarrow B_\omega$ and $B \hookrightarrow B_\omega \cap A' \subseteq A_\omega \cap A'$, the above yields $D \hookrightarrow B_\omega \cap A'$. □

The following is useful for discussing $\mathcal{D}$-stability for some inclusions of fixed point subalgebras by certain automorphisms on UHF algebras. In particular, the following will work for automorphisms on UHF algebras of product-type, as well as tensor permutations (of finite tensor powers of UHF algebras).

Corollary 4.1.6. *Let $A = \bigotimes_{\mathbb{N}} B$ be an infinite tensor product of a unital, separable, nuclear C^*-algebra B and let D be unital, separable, simple, and nuclear. Let $\lambda \in \mathrm{End}(A)$ be the Bernoulli shift $\lambda(a) = 1 \otimes a$. If $\sigma \in \mathrm{Aut}(A)$ is such that $\lambda \circ \sigma = \sigma \circ \lambda$, and $D \hookrightarrow (A^\sigma)_\omega$ unitally, then $D \hookrightarrow (A^\sigma)_\omega \cap A'$ unitally.*

Proof. Note that $\pi = (\lambda^n)$ induces an embedding $A \hookrightarrow A_\omega \cap A'$. We just need to show that $\pi(A^\sigma) \subseteq (A^\sigma)_\omega \cap A'$. The hypothesis gives that $\lambda^n \circ \sigma = \sigma \circ \lambda^n$ for all n, hence $\pi(A^\sigma) \subseteq (A^\sigma)_\omega \cap A'$. The result now follows from the above. □

We note that if we have approximately central approximate embeddings $D \to B \subseteq A$, then we can also find approximately central approximate embedding $D \to u^*Bu \subseteq A$ for any $u \in U(A)$. In the separable setting, this just means $D \hookrightarrow B_\omega \cap A'$ implies that $D \hookrightarrow u^*B_\omega u \cap A'$ for any $u \in U(A)$.

Lemma 4.1.7. *Let $B \subseteq A$ be a unital inclusion of C^*-algebras and let D be a unital, separable, simple, nuclear C^*-algebra. Let $u \in U(A)$. If there are $(\mathcal{F}, \varepsilon, \mathcal{G})$-approximately central approximate embeddings $D \to B$ for all $\mathcal{F} \subseteq D, \mathcal{G} \subseteq A$ finite subsets and $\varepsilon > 0$, then there are $(\mathcal{F}, \varepsilon, \mathcal{G})$-approximately central approximate embeddings $D \to u^*Bu \subseteq A$ for all $\mathcal{F}, \varepsilon, \mathcal{G}$.*

Proof. Let $\mathcal{F} \subseteq D, \mathcal{G} \subseteq A$ be finite and $\varepsilon > 0$. Let $L = \max\{1, \max_{d\in\mathcal{F}} \|d\|\}$ and $\phi : D \to B$ be a $(\mathcal{F}, \frac{\varepsilon}{3L}, \mathcal{G} \cup \{u\})$-approximately central approximate embedding. Then $\psi = \mathrm{Ad}_u \circ \phi : D \to u^*Bu$ will be an $(\mathcal{F}, \varepsilon, \mathcal{G})$-embedding. □

We can also discuss existence of approximately central approximate embeddings in inductive limits (with injective connecting maps). This is an adaptation of [TW08, Proposition 2.2] to our setting.

Proposition 4.1.8. *Suppose that we have increasing sequences (B_n) and (A_n) of C*-algebras such that $B_n \subseteq A_n$ are unital inclusions. If $B = \overline{\cup_n B_n}, A = \overline{\cup_n A_n}$, and $D = \overline{\cup_n D_n}$ where (D_n) is an increasing sequence of unital, separable, simple, nuclear C*-algebras and there are $(\mathcal{F}, \varepsilon, \mathcal{G})$-embeddings $D_n \to B_n \subseteq A_n$ whenever $n \in \mathbb{N}, \mathcal{F} \subseteq D_n, \mathcal{G} \subseteq A_n$ are finite and $\varepsilon > 0$, then there are $(\mathcal{F}, \varepsilon, \mathcal{G})$-embeddings $D \to B \subseteq A$ for all $\mathcal{F} \subseteq D, \mathcal{G} \subseteq A$ finite and $\varepsilon > 0$.*

Proof. Let $\mathcal{F} \subseteq \mathcal{D}$ and $\mathcal{G} \subseteq A$ be finite sets and $\varepsilon > 0$. Let

$$L := \max\{1, \max_{d \in \mathcal{F}} \|d\|, \max_{a \in \mathcal{G}} \|a\|\} \tag{4.1.2}$$

and set $\delta := \frac{\varepsilon}{6L+5}$. Without loss of generality assume that $\varepsilon < 1$. Label $\mathcal{F} = \{d_1, \ldots, d_p\}$ and $\mathcal{G} = \{a_1, \ldots, a_q\}$ and find N large enough so that there are $d'_1, \ldots, d'_p \in D_N$ and $a'_1, \ldots, a'_q \in A_N$ with $d'_i \approx_\delta d_i$ and $a'_j \approx_\delta a_j$. Let $\mathcal{F}' := \{d'_1, \ldots, d'_p\}, \mathcal{G}' := \{a'_1, \ldots, a'_q\}$ and let $\phi : D_N \to B_N \subseteq A_N$ be an $(\mathcal{F}', \delta, \mathcal{G}')$-embedding. As D_N is nuclear, there are $k \in \mathbb{N}$ and u.c.p. maps $\rho : D_N \to M_k$ and $\eta : M_k \to B_N$ such that $\eta \circ \rho(d'_i) \approx_\delta \phi(d'_i)$ and $\eta \circ \rho(d'_i d'_j) \approx_\delta \phi(d'_i d'_j)$. Use Arveson's extension theorem (see [BO08, Section 1.6]) to extend ρ to a u.c.p. map $\tilde{\rho} : D \to M_k$ and let $\psi := \eta \circ \tilde{\rho} : D \to B_N$. As $B_N \subseteq B$, we can think of ψ as a map $\psi : D \to B$. Now for $i = 1, \ldots, p$, we have

$$\begin{aligned} \psi(d_i d_j) &\approx_{(2L+1)\delta} \psi(d'_i d'_j) \\ &= \eta \circ \rho(d'_i d'_j) \\ &\approx_\delta \phi(d'_i d'_j) \\ &\approx_\delta \phi(d'_i)\phi(d'_j) \\ &\approx_{2L\delta} \eta \circ \rho(d'_i)\eta \circ \rho(d'_j) \\ &= \psi(d'_i)\psi(d'_j) \\ &\approx_{(2L+1)\delta} \psi(d_i)\psi(d_j). \end{aligned} \tag{4.1.3}$$

Thus $\psi(d_i d_j) \approx_{(4+6L)\delta} \psi(d_i)\psi(d_j)$, and as $(4+6L)\delta \leq (6L+5)\delta = \varepsilon$, this implies that $\psi(d_i d_j) \approx_\varepsilon \psi(d_i)\psi(d_j)$. For approximate commutation with $\mathcal{G}$, we make use of the following two approximations: for a, a', a'', b, b' elements in a C*-algebra,

$$\begin{aligned} &\|[a, b]\| \leq (\|a\| + \|a'\|)\|b - b'\| + (\|b\| + \|b'\|)\|a - a'\| + \|[a', b']\|, \\ &\|[a', b']\| \leq 2\|b'\|\|a' - a''\| + \|[a'', b']\|. \end{aligned} \tag{4.1.4}$$

Note that for $a = \psi(d_i), a' = \psi(d_i'), a'' = \phi(d_i'), b = a_j, b' = a_j'$, we have that $\|a\|, \|b\| \leq L + 1$ and $\|a'\|, \|a''\|, \|b'\| \leq L$. Therefore from the above two inequalities we get

$$\begin{aligned} &\|[\psi(d_i), a_j]\| \leq 2L\|\psi(d_i) - \psi(c_i')\| + 2(L+1)\|a_j - a_j'\| + \|[\psi(d_i'), a_j]\|; \\ &\|[\psi(d_i'), a_j']\| \leq 2(L+1)\|\psi(d_i') - \phi(d_i')\| + \|[\phi(d_i'), a_j']\|. \end{aligned} \tag{4.1.5}$$

Using these approximations we have

$$\begin{aligned} \|[\psi(d_i), a_j]\| &\leq 2L\|\psi(d_i) - \psi(d_i')\| + 2(L+1)\|a_j - a_j'\| + \|[\psi(d_i'), a_j]\| \\ &< (4L+2)\delta + \|[\psi(d_i'), a_j]\| \\ &\leq (4L+2)\delta + 2(L+1)\|\psi(d_i') - \phi(d_i')\| + \|[\phi(c_i'), a_j']\| \\ &< (4L+2)\delta + 2(L+1)\delta + \delta \\ &= (6L+5)\delta = \varepsilon. \end{aligned}$$

□

The following will be useful to show that there are many $\mathcal{D}$-stable embeddings.

Lemma 4.1.9. *Let $\phi : B_0 \simeq B_1$ and $\psi : A_0 \simeq A_1$ be isomorphisms between unital C*-algebras and let D be a unital, simple, nuclear C*-algebra. Suppose that there is a *-homomorphism $\eta : B_1 \hookrightarrow A_1$ such that there are $(\mathcal{F}, \varepsilon, \mathcal{G})$-embeddings $D \to \eta(B_1) \subseteq A_1$ for all finite subsets $\mathcal{F} \subseteq D, \mathcal{G} \subseteq A_1$ and $\varepsilon > 0$. Let $\sigma = \psi^{-1} \circ \eta \circ \phi : B_0 \to A_0$. Then there are $(\mathcal{F}, \varepsilon, \mathcal{G})$-embeddings $D \to \sigma(B_0) \subseteq A_0$ for all $\mathcal{F} \subseteq D, \mathcal{G} \subseteq A_0$ finite and $\varepsilon > 0$.*

Proof. The diagram

$$\begin{array}{ccc} A_0 & \xrightarrow{\psi} & A_1 \\ \sigma\uparrow & & \uparrow\eta \\ B_0 & \xrightarrow[\phi]{} & B_1 \end{array} \tag{4.1.6}$$

commutes, and so if $\mathcal{F} \subseteq D, \mathcal{G} \subseteq A_0$ are finite, $\varepsilon > 0$ and $\xi : D \to \eta(B_1) \subseteq A_1$ is an $(\mathcal{F}, \varepsilon, \psi(\mathcal{G}))$-embedding, then $\psi^{-1} \circ \xi : D \to \psi^{-1}(\eta(B_1)) \subseteq \psi^{-1}(A_1) = A_0$ is an $(\mathcal{F}, \varepsilon, \mathcal{G})$ embedding. Moreover, from

$$\psi^{-1}(\eta(B_1)) = \psi^{-1}(\eta(\phi(B_0))) = \sigma(B_0), \tag{4.1.7}$$

it is clear that $\psi^{-1} \circ \xi$ is an $(\mathcal{F}, \varepsilon, \mathcal{G})$ embedding $D \to \sigma(B_0) \subseteq A_0$. □

4.2 Relative intertwinings and $\mathcal{D}$-stable embeddings

4.2.1 Relative intertwinings

It is well known that a strongly self-absorbing C*-algebra $\mathcal{D}$ embeds unitally into the central sequence algebra $(\mathcal{M}(A))_\omega \cap A'$ of a separable C*-algebra A if and only if

$A \simeq A \otimes \mathcal{D}$, where $\mathcal{M}(A)$ is the multiplier algebra of A (for example, [Rør02, Theorem 7.2.2(i)]). We alter the proof to keep track of a subalgebra in order to show that for a unital inclusion $B \subseteq A$ of separable C*-algebras, $\mathcal{D} \hookrightarrow B_\omega \cap A'$ if and only if there is an isomorphism $\Phi : A \to A \otimes \mathcal{D}$, which is approximately unitarily equivalent to the first factor embedding, and satisfies $\Phi(B) = B \otimes \mathcal{D}$. This was initially done for (irreducible) inclusions of II_1 factors in [Bis90] and commented on in [Izu04] for $\mathcal{D}$ being $M_{n^\infty}, \mathcal{O}_2, \mathcal{O}_\infty$. The proof we alter is Elliott's intertwining argument, which can be found as a combination of Proposition 2.3.5, Proposition 7.2.1 and Theorem 7.2.2 of [Rør02].

Proposition 4.2.1 (Relative intertwining). *Let A, B, C be unital, separable C*-algebras, and let $\phi : A \hookrightarrow C, \theta : B \to A, \psi : B \to C$ be unital *-homomorphisms such that $\phi \circ \theta(B) \subseteq \psi(B)$. Suppose there is a sequence (u_n) of unitaries in $\psi(B)_\omega \cap \phi(A)'$ such that*

- *$dist(v_n^* c v_n, \phi(A)_\omega) \to 0$ for all $c \in C$;*
- *$dist(v_n^* \psi(b) v_n, \phi \circ \theta(B)_\omega) \to 0$ for all $b \in B$.*

Then ϕ is approximately unitarily equivalent to an isomorphism $\Phi : A \simeq C$ such that $\Phi \circ \theta(B) = \psi(B)$.

Proof. Apply the below proposition with $B_m := B, \theta_m := \theta, \psi_m := \psi$ for all $m \in \mathbb{N}$. □

Proposition 4.2.2 (Countable relative intertwining). *Let A, B_m, C be unital, separable C*-algebras, $m \in \mathbb{N}$, and $\phi : A \hookrightarrow C, \theta_m : B_m \to A, \psi_m : B_m \to C$ be such that $\phi \circ \theta_m(B_m) \subseteq \psi_m(B_m)$ and $\psi_1(B_1) \subseteq \psi_m(B_m)$. Suppose there is a sequence $(v_n) \subseteq \psi_1(B_1)_\omega \cap \phi(A)'$ of unitaries such that*

- *$dist(v_n^* c v_n, \phi(A)_\omega) \to 0$ for all $c \in C$;*
- *$dist(v_n^* \psi_m(b) v_n, \phi \circ \theta_m(B_m)_\omega) \to 0$ for all $b \in B_m$.*

Then ϕ is approximately unitarily equivalent to an isomorphism $\Phi : A \simeq C$ such that $\Phi \circ \theta_m(B_m) = \psi_m(B_m)$ for all $m \in \mathbb{N}$.

Proof. We show that if there are unitaries $(v_n) \subseteq \psi_1(B_1)$ satisfying

- $[v_n, \phi(a)] \to 0$ for all $a \in A$;
- $\mathrm{dist}(v_n^* c v_n, \phi(A)) \to 0$ for all $c \in C$;
- $\mathrm{dist}(v_n^* \psi_m(b) v_n, \phi \circ \theta_m(B_m)) \to 0$ for all $b \in B_m$,

then the conclusion holds. Such unitaries can be found using Kirchberg's ε-test (Lemma 1.3.1).

Let $(a_n), (b_n^{(m)}), (c_n)$ be dense sequences of A, B_m, C respectively. We can inductively choose v_n, forming a subsequence (v_n) of the unitaries above (after re-indexing, we are still calling them v_n), such that there are $a_{jn} \in A, b_{jn}^{(m)} \in B_m$ with

- $v_n^* \cdots v_1^* c_j v_1 \cdots v_n \approx_{\frac{1}{n}} \phi(a_{jn})$;
- $v_n^* \cdots v_1^* \psi(b_j^{(m)}) v_1 \cdots v_n \approx_{\frac{1}{n}} \phi \circ \theta_m(b_{jn}^{(m)})$;
- $[v_n, \phi(a_j)] \approx_{\frac{1}{2^n}} 0$;
- $[v_n, \phi(a_{jl})] \approx_{\frac{1}{2^n}} 0$;
- $[v_n, \phi \circ \theta_m(b_j^{(m)})] \approx_{\frac{1}{2^n}} 0$;
- $[v_n, \phi \circ \theta_m(b_{jl}^{(m)})] \approx_{\frac{1}{2^n}} 0$,

where $j, m = 1, \ldots, n$ and $l = 1, \ldots, n-1$. Define, for $a \in \{a_n \mid n \in \mathbb{N}\}$,

$$\Phi(a) = \lim_n v_1 \cdots v_n \phi(a) v_n^* \cdots v_1^* \tag{4.2.1}$$

which extends to a *-isomorphism $\Phi : A \simeq C$, as in [Rør02, Proposition 2.3.5]. The proof also yields the following useful approximation:

$$\Phi \circ \theta_m(b_{jn}^{(m)}) \approx_{\frac{1}{2^n}} v_1 \cdots v_n \phi \circ \theta_m(b_{jn}^{(m)}) v_n^* \cdots v_1^* \tag{4.2.2}$$

for appropriate $n \geq j, m$.

We now need to check that $\Phi \circ \theta_m(B_m) = \psi_m(B_m)$. Approximate

$$\psi_m(b_j^{(m)}) \approx_{\frac{1}{n}} v_1 \cdots v_n \phi \circ \theta_m(b_{jn}^{(m)}) v_n^* \cdots v_1^* \approx_{\frac{1}{2^n}} \Phi \circ \theta_m(b_{jn}^{(m)}). \tag{4.2.3}$$

This yields $\psi_m(B_m) \subseteq \overline{\Phi \circ \theta_m(B_m)} = \Phi \circ \theta_m(B_m)$. On the other hand for any $\varepsilon > 0$ and $b \in B_m$, we can find n such that

$$\Phi \circ \theta_m(b) \approx_\varepsilon v_1 \cdots v_n \phi \circ \theta_m(b) v_n^* \cdots v_1^* \in \psi_m(B_m) \tag{4.2.4}$$

since $v_i \in \psi_1(B_1) \subseteq \psi_m(B_m)$ and $\phi \circ \theta_m(B_m) \subseteq \psi_m(B_m)$. Hence $\Phi \circ \theta_m(B_m) \subseteq \overline{\psi_m(B_m)} = \psi_m(B_m)$. $\square$

4.2.2 $\mathcal{D}$-stable embeddings

Definition 4.2.3. *Let $\iota : B \hookrightarrow A$ be an embedding and $\mathcal{D}$ be strongly self-absorbing. We say that ι is $\mathcal{D}$-stable (or $\mathcal{D}$-absorbing) if there exists an isomorphism $\Phi : A \simeq A \otimes \mathcal{D}$ such that $\Phi \circ \iota(B) = \iota(B) \otimes \mathcal{D}$.*

We will mostly have interest in the case where ι corresponds to the inclusion map and $B \subseteq A$ is a subalgebra. In this form, we will say $B \subseteq A$ is $\mathcal{D}$-stable (or $\mathcal{D}$-absorbing). Clearly ι being $\mathcal{D}$-stable is the same as $\iota(B) \subseteq A$ being $\mathcal{D}$-stable. We note that we can define the above for any *-homomorphism. Namely, a *-homomorphism $\phi : B \to A$ is $\mathcal{D}$-stable if $\phi(B) \subseteq A$ is.

Lemma 4.2.4. *If $\iota : B \hookrightarrow A$ is an embedding, then $\iota \otimes id_D : B \otimes \mathcal{D} \hookrightarrow A \otimes \mathcal{D}$ is $\mathcal{D}$-stable.*

Proof. Let $\phi : D \simeq D \otimes \mathcal{D}$ be an isomorphism. Then

$$\Phi := \mathrm{id}_A \otimes \phi : A \otimes \mathcal{D} \to A \otimes \mathcal{D} \otimes \mathcal{D} \tag{4.2.5}$$

is an isomorphism with

$$\Phi(\iota \otimes \mathrm{id}_{\mathcal{D}}(B \otimes \mathcal{D})) = (\iota \otimes \mathrm{id}_{\mathcal{D}}(B \otimes \mathcal{D})) \otimes \mathcal{D}. \tag{4.2.6}$$

□

We note that this is a strengthening of the notion of $\mathcal{D}$-stability for C*-algebras because if $\iota = \mathrm{id}_A : A \to A$, then ι is $\mathcal{D}$-stable if and only if A is $\mathcal{D}$-stable. This condition is different from the notion of $\mathcal{O}_2$ or $\mathcal{O}_\infty$-absorbing morphisms discussed in [BGSW22, Gab20, Gab19] – they require sequences from a larger algebra to commute with a smaller algebra, while we require sequences from a smaller algebra to commute with the larger algebra. In the former, neither of the algebras are required to be $\mathcal{D}$-stable, while the latter necessitates both to be $\mathcal{D}$-stable.

The following adapts [Rør02, Theorem 7.2.2].

Theorem 4.2.5. *Suppose that $B \subseteq A$ is a unital inclusion of separable C*-algebras. If $\mathcal{D}$ is strongly self-absorbing, then $B \subseteq A$ is $\mathcal{D}$-stable if and only if there is a unital inclusion $\mathcal{D} \hookrightarrow B_\omega \cap A'$.*

Proof. Let $\phi : A \to A \otimes \mathcal{D}$ be the first factor embedding $\phi(a) := a \otimes 1_{\mathcal{D}}$. First suppose that $\xi : \mathcal{D} \hookrightarrow B_\omega \cap A' \simeq (B \otimes 1_{\mathcal{D}})_\omega \cap (A \otimes 1_{\mathcal{D}})'$ is an embedding (so that $\phi(a)\xi(d) \in \phi(A)_\omega$ and $\phi(b)\xi(d) \in \phi(B)_\omega$). Let $\eta : \mathcal{D} \hookrightarrow (B \otimes \mathcal{D})_\omega \cap (A \otimes 1_{\mathcal{D}})'$ be given by $\eta(d) := (1 \otimes d)_n$ and notice that ξ, η have commuting ranges. As all endomorphisms of $\mathcal{D}$ are approximately unitarily equivalent by [TW07, Corollary

1.12], let $(v_n) \subseteq C^*(\xi(\mathcal{D}), \eta(\mathcal{D})) \simeq \mathcal{D} \otimes \mathcal{D}$ be such that $v_n^* \eta(d) v_n \to \xi(d)$ for $d \in \mathcal{D}$. For $b \in B$ and $d \in \mathcal{D}$, we have

$$\begin{aligned} v_n^*(b \otimes d) v_n &= v_n^*(b \otimes 1_{\mathcal{D}})(1_A \otimes d) v_n^* \\ &= v_n^* \phi(b) \eta(d) v_n \\ &= \phi(b) v_n^* \eta(d) v_n \\ &\to \phi(b) \xi(d) \in \phi(B)_\omega. \end{aligned} \tag{4.2.7}$$

Moreover the same argument shows that, for $a \in A$, we have

$$v_n^*(a \otimes d) v_n \to \phi(a) \xi(d) \in \phi(A)_\omega. \tag{4.2.8}$$

Now (v_n) satisfy the hypothesis of Proposition 4.2.1 with $C := A \otimes D$, ϕ being the first factor embedding, $\theta : B \to A$ being the inclusion and $\psi : B \simeq B \otimes \mathcal{D} \subseteq A \otimes \mathcal{D} = C$ (where this isomorphism exists since if $\mathcal{D} \hookrightarrow B_\omega \cap A'$, then clearly $\mathcal{D} \hookrightarrow B_\omega \cap B'$). From this we see that ϕ is approximately unitarily equivalent to an isomorphism $\Phi : A \simeq A \otimes \mathcal{D}$ such that $\Phi(B) = B \otimes \mathcal{D}$.

Conversely, if $B \subseteq A$ is $\mathcal{D}$-stable, let $\Phi : A \simeq A \otimes \mathcal{D}$ be an isomorphism such that $\Phi(B) = B \otimes \mathcal{D}$. By [TW07, Proposition 1.10(iv)], we can identify $\mathcal{D} \simeq \mathcal{D}^{\otimes\infty}$ and take $\xi : \mathcal{D} \hookrightarrow B_\omega \cap A'$ to be given by

$$\xi(d) = (\Phi^{-1}(1_A \otimes 1_{\mathcal{D}}^{\otimes n-1} \otimes d \otimes 1_{\mathcal{D}}^{\otimes\infty}))_n. \tag{4.2.9}$$

□

Corollary 4.2.6. *Let $\iota : B \hookrightarrow A$ be a unital embedding between separable C^*-algebras. If $\mathcal{D}$ is strongly self-absorbing and ι is $\mathcal{D}$-stable, then for every intermediate unital C^*-algebra C with $\iota(B) \subseteq C \subseteq A$, we have that $\iota(B) \subseteq C$ and $C \subseteq A$ are $\mathcal{D}$-stable. In particular, $C \simeq C \otimes \mathcal{D}$ for all such C.*

Proof. We have

$$\mathcal{D} \hookrightarrow B_\omega \cap A' \subseteq B_\omega \cap C' \tag{4.2.10}$$

and

$$\mathcal{D} \hookrightarrow B_\omega \cap A' \subseteq C_\omega \cap A'. \tag{4.2.11}$$

□

It is not however the case that any isomorphism $\Phi : A \simeq A \otimes \mathcal{D}$ with $\Phi(B) = B \otimes \mathcal{D}$ maps C to $C \otimes \mathcal{D}$.

Example 4.2.7. *Let $\mathcal{D}$ be strongly self-absorbing and consider*

$$\begin{aligned} A &:= \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D}, \\ C_1 &:= \mathcal{D} \otimes 1_{\mathcal{D}} \otimes \mathcal{D}, \\ C_2 &:= 1_{\mathcal{D}} \otimes \mathcal{D} \otimes \mathcal{D}, \\ B &:= 1_{\mathcal{D}} \otimes 1_{\mathcal{D}} \otimes \mathcal{D}. \end{aligned} \tag{4.2.12}$$

If $f : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ is the tensor flip and $\phi : \mathcal{D} \simeq \mathcal{D} \otimes \mathcal{D}$ is an isomorphism, let

$$\Phi := f \otimes \phi : A \simeq A \otimes \mathcal{D} \tag{4.2.13}$$

which satisfies $\Phi(B) = B \otimes \mathcal{D}$ (in particular $B \subseteq A$ is $\mathcal{D}$-stable). However,

$$\Phi(C_1) = C_2 \otimes \mathcal{D} \text{ and } \Phi(C_2) = C_1 \otimes \mathcal{D}. \tag{4.2.14}$$

In fact the above example can be generalized to show that for any $\mathcal{D}$-stable inclusion $B \subseteq A$, there are an isomorphism $\Phi : A \simeq A \otimes \mathcal{D}$ such that $\Phi(B) = B \otimes \mathcal{D}$ and an intermediate algebra $B \subseteq C \subseteq A$ with $\Phi(C) \neq C \otimes C \otimes \mathcal{D}$ (obviously we may still have that $\Phi(C) \simeq C \otimes \mathcal{D}$, but equality may not happen).

Corollary 4.2.8. *Let $B \subseteq A$ be a $\mathcal{D}$-stable inclusion. There exist a C^*-algebra C with $B \subseteq C \subseteq A$ and an isomorphism $\Phi : A \simeq A \otimes \mathcal{D}$ such that $\Phi(B) = B \otimes \mathcal{D}$ but $\Phi(C) \neq C \otimes \mathcal{D}$.*

Proof. We first claim that if $B \subseteq A$ is $\mathcal{D}$-stable, then we can identify $B \subseteq A$ with $B \otimes 1_{\mathcal{D}} \subseteq A \otimes \mathcal{D}$. If $\Psi : A \simeq A \otimes \mathcal{D}$ is such that $\Psi(B) = B \otimes \mathcal{D}$ and $f : \mathcal{D} \otimes \mathcal{D} \simeq \mathcal{D} \otimes \mathcal{D}$ is the tensor flip, we have

$$\Xi := (\mathrm{id}_A \otimes f) \circ (\Psi \otimes \mathrm{id}_{\mathcal{D}}) : A \otimes \mathcal{D} \simeq A \otimes \mathcal{D} \otimes \mathcal{D} \tag{4.2.15}$$

is such that $\Xi(B \otimes 1_{\mathcal{D}}) = B \otimes 1_{\mathcal{D}} \otimes \mathcal{D}$. This proves the claim.

Now by applying the claim twice, we can identify $B \subseteq A$ with the inclusion

$$B \otimes 1_{\mathcal{D}} \otimes 1_{\mathcal{D}} \otimes \mathcal{D} \subseteq A \otimes \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D}. \tag{4.2.16}$$

If $\phi : D \simeq \mathcal{D} \otimes \mathcal{D}$ is any isomorphism,

$$\Phi := \mathrm{id}_A \otimes f \otimes \phi : A \otimes \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \simeq A \otimes \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \tag{4.2.17}$$

is such that

$$\Phi(B \otimes 1_{\mathcal{D}} \otimes 1_{\mathcal{D}} \otimes \mathcal{D}) = B \otimes 1_{\mathcal{D}} \otimes 1_{\mathcal{D}} \otimes \mathcal{D} \otimes \mathcal{D}. \tag{4.2.18}$$

Taking C_1 and C_2 as in Example 4.2.7, we have that

$$\Phi(B \otimes C_1) = B \otimes C_2 \otimes \mathcal{D} \text{ and } \Phi(B \otimes C_2) = B \otimes C_1 \otimes \mathcal{D}. \tag{4.2.19}$$

□

However, we can always realize $\mathcal{D}$-stability for countably many intermediate C*-algebras at once using *some* isomorphism $A \simeq A \otimes \mathcal{D}$.

Theorem 4.2.9. *Suppose that $B_1 \subseteq B_m \subseteq A$ are unital inclusions of separable C^*-algebras (note that we are* **not** *asking for (B_m) to form a chain). If $\mathcal{D}$ is strongly self-absorbing and $\mathcal{D} \hookrightarrow (B_1)_\omega \cap A'$ unitally, there exists an isomorphism $\Phi : A \simeq A \otimes \mathcal{D}$ such that $\Phi(B_m) = B_m \otimes \mathcal{D}$ for all $m \in \mathbb{N}$.*

Proof. This is essentially the same proof as Theorem 4.2.5, except we use the countable relative intertwining (Proposition 4.2.2) in place of Proposition 4.2.1. Let ξ, η be as before and let $(v_n) \subseteq C^*(\xi(\mathcal{D}), \eta(\mathcal{D})) \simeq \mathcal{D} \otimes \mathcal{D}$ be such that $v_n^* \eta(d) v_n \to \xi(d)$ for $d \in \mathcal{D}$.

- If $a \in A, d \in \mathcal{D}, v_n^*(a \otimes d)v_n \to \phi(a)\xi(d) \in \phi(A)_\omega$;
- if $b \in B_m, v_n^*(b \otimes d)v_n \to \phi(b)\xi(d) \in \phi(B_m)_\omega$.

Now with $\phi : A \to A \otimes \mathcal{D}$ the first factor embedding, $\theta_m : B_m \to A$ the inclusion maps, and $\psi_m : B_m \simeq B_m \otimes \mathcal{D}$ (these exist since $\mathcal{D} \hookrightarrow (B_1)_\omega \cap A'$ implies that $\mathcal{D} \hookrightarrow (B_m)_\omega \cap B_m'$), our unitaries satisfy the hypothesis of Proposition 4.2.2 and therefore ϕ is approximately unitarily equivalent to a *-isomorphism $\Phi : A \simeq A \otimes \mathcal{D}$ such that $\Phi(B_m) = B_m \otimes \mathcal{D}$ for all m. □

The above works since norm ultrapowers have the property that unitaries lift to sequences of unitaries.[17] Tracial ultrapowers of II_1 von Neumann algebras also have this property.[18] Consequently if we work with the 2-norm $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$ where τ is the unique trace on a II_1 factor, all of the above arguments with the C*-norm replaced by $\|\cdot\|_2$ will yield back Bisch's result [Bis90, Theorem 3.1], provided we have the appropriate separability conditions.

Theorem 4.2.10. *Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of II_1 factors with separable preduals. Then $\mathcal{R} \hookrightarrow \mathcal{N}^\omega \cap \mathcal{M}'$ if and only if there exists an isomorphism $\Phi : \mathcal{M} \to \mathcal{M} \overline{\otimes} \mathcal{R}$ such that $\Phi(\mathcal{N}) = \mathcal{N} \overline{\otimes} \mathcal{R}$.*

4.2.3 Existence of $\mathcal{D}$-stable embeddings

We move to discuss the existence of $\mathcal{D}$-stable embeddings. First we show that each unital embedding of unital, separable $\mathcal{D}$-stable C*-algebras is approximately unitarily equivalent to a $\mathcal{D}$-stable embedding. From this it will follow that there are many $\mathcal{D}$-stable embeddings.

Lemma 4.2.11. *Let $\mathcal{D}$ be strongly self-absorbing. If $\iota : B \hookrightarrow A$ is a unital, $\mathcal{D}$-stable inclusion of separable C*-algebras and $u \in U(A)$, then $Ad_u \circ \iota : B \hookrightarrow A$ is $\mathcal{D}$-stable.*

Proof. Apply Lemma 4.1.7. □

[17]If $u = (u_n) \in A_\omega$ is unitary, then $\{n \in \mathbb{N} \mid \|u_n^* u_n - 1\|, \|u_n u_n^* - 1\| < 1\} \in \omega$. If n is in the set, replace u_n with the unitary part of its polar decomposition, and replace u_n with 1 otherwise.

[18]The tracial ultrapower of a II_1 von Neumann algebra is again a II_1 von Neumann algebra. Therefore if $u \in \mathcal{M}^\omega$ is unitary, it is of the form e^{ia} for some $a = a^* \in \mathcal{M}^\omega$. Lift a to a sequence (a_n) of self-adjoints in $\mathcal{M}$ and note that $u = (e^{ia_n})$, so that u has a unitary lift.

Proposition 4.2.12. *Let $\mathcal{D}$ be strongly self-absorbing, A, B be unital separable $\mathcal{D}$-stable C*-algebras and let $\iota : B \hookrightarrow A$ be an embedding. Then ι is approximately unitarily equivalent to a $\mathcal{D}$-stable embedding $B \hookrightarrow A$.*

Proof. As A, B are $\mathcal{D}$-stable, there are isomorphisms

$$\phi : B \simeq B \otimes \mathcal{D} \text{ and } \psi : A \simeq A \otimes \mathcal{D} \tag{4.2.20}$$

which are approximately unitarily equivalent to the first factor embeddings $b \mapsto b \otimes 1_{\mathcal{D}}, b \in B$ and $a \mapsto a \otimes 1_{\mathcal{D}}, a \in A$ respectively. As $\iota \otimes \mathrm{id}_{\mathcal{D}} : B \otimes \mathcal{D} \hookrightarrow A \otimes \mathcal{D}$ is $\mathcal{D}$-stable by Lemma 4.2.4,

$$\sigma := \psi^{-1} \circ (\iota \otimes \mathrm{id}_{\mathcal{D}}) \circ \phi : B \hookrightarrow A \tag{4.2.21}$$

is $\mathcal{D}$-stable by Lemma 4.1.9. Now we show that σ is approximately unitarily equivalent to ι. Let $\mathcal{F} \subseteq B$ be finite and $\varepsilon > 0$. Let $u \in U(B \otimes \mathcal{D})$ be such that $u^*(b \otimes 1_{\mathcal{D}})u \approx_{\frac{\varepsilon}{2}} \phi(b)$ for $b \in \mathcal{F}$ and $v \in U(A \otimes \mathcal{D})$ be such that $v^*(\iota(b) \otimes 1_{\mathcal{D}})v \approx_{\frac{\varepsilon}{2}} \psi \circ \iota(b)$ for $b \in \mathcal{F}$. Set $w = \psi^{-1}(\iota \otimes \mathrm{id}_{\mathcal{D}}(u))^* \psi^{-1}(v) \in U(A)$. Then for $b \in \mathcal{F}$,

$$\begin{aligned} w^*\sigma(b)w &= \psi^{-1}(v)^*\psi^{-1}(\iota \otimes \mathrm{id}_{\mathcal{D}}(u\phi(b)u^*))\psi^{-1}(v) \\ &\approx_{\frac{\varepsilon}{2}} \psi^{-1}(v)^*\psi^{-1}(\iota \otimes \mathrm{id}_{\mathcal{D}}(b \otimes 1_{\mathcal{D}}))\psi^{-1}(v) \\ &= \psi^{-1}(v)^*\psi^{-1}(\iota(b) \otimes 1_{\mathcal{D}})\psi^{-1}(v) \\ &\approx_{\frac{\varepsilon}{2}} \psi^{-1}(\psi(\iota(b))) \\ &= \iota(b). \end{aligned} \tag{4.2.22}$$

□

Corollary 4.2.13. *Let $\mathcal{D}$ be strongly self-absorbing. The set of $\mathcal{D}$-stable embeddings $B \hookrightarrow A$ of unital, separable, $\mathcal{D}$-stable C*-algebras is point-norm dense in the set of embeddings $B \hookrightarrow A$.*

Proof. Every embedding is approximately unitarily equivalent to a $\mathcal{D}$-stable embedding. As $\mathcal{D}$-stability of an embedding is preserved if one composes with Ad_u, it follows that every embedding is the point-norm limit of $\mathcal{D}$-stable embeddings. □

Remark 4.2.14. We note that it is not actually necessary that ι is an embedding. If $\pi : B \to A$ is any unital *-homomorphism between unital, separable, $\mathcal{D}$-stable C*-algebras, then π is approximately unitarily equivalent to a *-homomorphism $\pi' : B \to A$ such that $\pi'(B) \subseteq A$ is $\mathcal{D}$-stable. Consequently the set of unital *-homomorphisms $\pi : B \to A$ with $\pi(B) \subseteq A$ being $\mathcal{D}$-stable is in fact dense in the set of unital *-homomorphisms $B \to A$.

Later on, there will be some examples of non-$\mathcal{D}$-stable embeddings between $\mathcal{D}$-stable C*-algebras. Consequently, despite the fact $\mathcal{D}$-stable embeddings are point-norm dense, the set of $\mathcal{D}$-stable embeddings need not coincide with the set of all embeddings $B \hookrightarrow A$. Another clear consequence is that despite $\mathcal{D}$-stability of an embedding being closed under conjugation by a unitary, it is not true that it is preserved under approximate unitary equivalence (in fact, the examples in question show that $\mathcal{D}$-stability is not even preserved under asymptotic unitary equivalence). We finish with a corollary about embeddings into the Cuntz algebra $\mathcal{O}_2$ [Cun77].

Corollary 4.2.15. *Let B be a unital, separable, exact $\mathcal{D}$-stable C*-algebra, where $\mathcal{D}$ is strongly self-absorbing. Then there is a $\mathcal{D}$-stable embedding $B \hookrightarrow \mathcal{O}_2$.*

Proof. As $\mathcal{D}$ is unital, simple, separable and nuclear by [TW07, Section 1.6], $\mathcal{O}_2 \simeq \mathcal{O}_2 \otimes \mathcal{D}$ and $B \hookrightarrow \mathcal{O}_2$ unitally by Theorem 3.7 and Theorem 2.8 of [KP00] respectively. The above results then yield a $\mathcal{D}$-stable embedding $B \hookrightarrow \mathcal{O}_2$. □

We include this last result about the classification of morphisms via functors.

Theorem 4.2.16. *Let $\mathcal{D}$ be strongly self-absorbing and let F be a functor from a class of unital, separable, $\mathcal{D}$-stable C*-algebras satisfying the following.*

- *(E) If there exists a morphism $\Phi : F(B) \to F(A)$, then there exists a *-homomorphism $\phi : B \to A$ such that $F(\phi) = \Phi$.*
- *(U) If $\phi, \psi : B \to A$ are *-homomorphisms which are approximately unitarily equivalent, then*
$$F(\phi) = F(\psi). \tag{4.2.23}$$

Then whenever there is a morphism $\Phi : F(B) \to F(A)$, there exists $\phi : B \to A$ such that $F(\phi) = \Phi$ and $\phi(B) \subseteq A$ is $\mathcal{D}$-stable. Moreover, ϕ is unique up to approximate unitary equivalence.

Proof. By the existence (E), there exists a *-homomorphism $\phi : B \to A$. Now by Proposition 4.2.12 (Remark 4.2.14 allows us to work with general *-homomorphisms), there exists a *-homomorphism $\phi' : B \to A$ which is approximately unitarily equivalent to ϕ and $\phi'(B) \subseteq A$ is $\mathcal{D}$-stable. Uniqueness (U) gives that this is unique up to approximate unitary equivalence. □

4.2.4 Permanence properties

We now discuss some permanence properties.

Lemma 4.2.17. *Let $\mathcal{D}$ be strongly self-absorbing. Suppose that $\iota_j : B_j \hookrightarrow A_j, j = 1, 2$ are $\mathcal{D}$-stable inclusions. Then $\iota_1 \oplus \iota_2 : B_1 \oplus B_2 \hookrightarrow A_1 \oplus A_2$ is $\mathcal{D}$-stable.*

Proof. Let $\Phi_j : A_j \simeq A_j \otimes \mathcal{D}$ be isomorphisms such that $\Phi_j \circ \iota_j(B_j) = \iota_j(B_j) \otimes \mathcal{D}$ and consider

$$\Phi : A_1 \oplus A_2 \simeq (A_1 \oplus A_2) \otimes \mathcal{D} \tag{4.2.24}$$

given by the composition

$$A_1 \oplus A_2 \xrightarrow{\Phi_1 \oplus \Phi_2} (A_1 \otimes \mathcal{D}) \oplus (A_2 \otimes \mathcal{D}) \xrightarrow{\simeq} (A_1 \oplus A_2) \otimes \mathcal{D} \tag{4.2.25}$$

where the last isomorphism follows from (finite) distributivity of the min-tensor. Then we see that

$$\Phi(\iota_1(B_1) \oplus \iota_2(B_2)) = (\iota_1(B_1) \oplus \iota_2(B_2)) \otimes \mathcal{D}. \tag{4.2.26}$$

□

Lemma 4.2.18. *Let $\mathcal{D}$ be strongly self-absorbing. Suppose that $\iota_j : B_j \hookrightarrow A_j, j = 1, 2$ are inclusions and that at least one of ι_1 or ι_2 is $\mathcal{D}$-stable. Then $\iota_1 \otimes \iota_2 : B_1 \otimes B_2 \hookrightarrow A_1 \otimes A_2$ is $\mathcal{D}$-stable.*

Proof. We prove this if ι_2 is $\mathcal{D}$-stable, and a symmetric argument will yield the result if ι_1 is. Let $\Phi_2 : A_2 \simeq A_2 \otimes \mathcal{D}$ be such that $\Phi_2 \circ \iota_2(B_2) = \iota(B_2) \otimes \mathcal{D}$. Taking

$$\Phi := \mathrm{id}_{A_1} \otimes \Phi_2 : A_1 \otimes A_2 \simeq A_1 \otimes A_2 \otimes \mathcal{D}, \tag{4.2.27}$$

we have that

$$\Phi(\iota_1(B_1) \otimes \iota_2(B_2)) = \iota_1(B_1) \otimes \iota_2(B_2) \otimes \mathcal{D}. \tag{4.2.28}$$

□

Proposition 4.2.19. *Let $\mathcal{D}$ be strongly self-absorbing. Suppose that we have increasing sequences of unital separable C^*-algebras (B_n) and (A_n) such that $B_n \subseteq A_n$ unitally. Let $B = \overline{\cup_n B_n}$ and $A = \overline{\cup_n A_n}$. If $B_n \subseteq A_n$ is $\mathcal{D}$-stable for all n, then $B \subseteq A$ is $\mathcal{D}$-stable.*

Proof. This follows from Proposition 4.1.8, together with Lemma 4.1.3 and Theorem 4.2.5. □

Lastly we'll discuss unital inclusions $B \subseteq A$ of $C(X)$ algebras, where X is a compact Hausdorff space. We show that if X has finite covering dimension, then such an inclusion is $\mathcal{D}$-stable if and only if the inclusion $B_x \subseteq A_x$ along each fibre is $\mathcal{D}$-stable.

Lemma 4.2.20. *Let $\mathcal{D}$ be strongly self-absorbing. Suppose that $B_i \subseteq A_i$ are unital inclusions, for $i = 1, 2$, and $\psi : A_1 \to A_2$ is a surjective $*$-homomorphism such that $\psi(B_1) = B_2$. If $B_1 \subseteq A_1$ is $\mathcal{D}$-stable, then so is $B_2 \subseteq A_2$.*

Proof. We note that ψ induces a *-homomorphism

$$\tilde{\psi} : (B_1)_\omega \cap A_1' \to (B_2)_\omega \cap A_2' \tag{4.2.29}$$

and consequently if $\xi : \mathcal{D} \hookrightarrow (B_1)_\omega \cap A_1'$, we have a unital *-homomorphism

$$\eta := \tilde{\psi} \circ \xi : \mathcal{D} \to (B_2)_\omega \cap A_2'. \tag{4.2.30}$$

η is automatically injective since $\mathcal{D}$ is simple. □

Rephrasing the above in terms of commutative diagrams, it says that if we have a commutative diagram

$$\begin{array}{ccc} A_1 & \twoheadrightarrow & A_2 \\ \uparrow & & \uparrow \\ B_1 & \twoheadrightarrow & B_2 \end{array} \tag{4.2.31}$$

where the left inclusion is $\mathcal{D}$-stable, then the right inclusion is $\mathcal{D}$-stable as well.

Now we consider many of the results discussed in [HRW07, Section 4], except that we do it for inclusions of C*-algebras.

Definition 4.2.21. *Let X be a compact Hausdorff space. A $C(X)$-algebra is a C*-algebra A endowed with a unital *-homomorphism $C(X) \to \mathcal{Z}(\mathcal{M}(A))$, where $\mathcal{Z}(\mathcal{M}(A)$ is the center of the multiplier algebra $\mathcal{M}(A)$ of A.*

If $Y \subseteq X$ is a closed subset, we set $I_Y := C_0(X \setminus Y)A$, which is a closed two-sided ideal in A. We denote $A_Y := A/I_Y$ and the quotient map $A \to A_Y$ by π_Y. For an element $a \in A$, we write $a_Y := \pi_Y(a)$ and if Y consists of a single point x, we write A_x, I_x, π_x and a_x. We say that A_x is the fibre of A at x. We note that $A_X = A$.

If $B \subseteq A$ is a unital inclusion and $\theta_A : C(X) \to A, \theta_B : C(X) \to B$ are morphisms which witness A and B as $C(X)$-algebras, respectively, we say that $B \subseteq A$ is an inclusion of $C(X)$-algebras if

$$\begin{array}{ccc} B & \hookrightarrow & A \\ {\scriptstyle\theta_B}\uparrow & \nearrow{\scriptstyle\theta_A} & \\ C(X) & & \end{array} \tag{4.2.32}$$

commutes. Note that $\theta_B(C(X)) \subseteq \mathcal{Z}(A)$ and when discussing an inclusion of fibres $B_Y \subseteq A_Y$ we are considering $B_Y := \pi_Y^A(B) \subseteq \pi_Y^A(A) =: A_Y$, where $\pi_Y^A : A \to A_Y$ is the associated quotient map.

Remark 4.2.22 (Upper semi-continuity)**.** In [HRW07, Section 1.3], it was pointed out that the norm on a $C(X)$-algebra A is upper semi-continuous. This means that, fixing some $a \in A$, the function $x \mapsto \|a_x\|$ from X to $\mathbb{R}$ is upper semi-continuous (as it is the infimum of a family of continuous functions), and consequently the set $\{x \in X \mid \|a_x\| < \varepsilon\} \subseteq X$ is open for all $a \in A$ and $\varepsilon > 0$.

We note that Lemma 4.2.20 gives that if $B \subseteq A$ is $\mathcal{D}$-stable and $Y \subseteq X$ is closed, then $B_Y \subseteq A_Y$ is automatically $\mathcal{D}$-stable as well since we have the commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi_Y} & A_Y \\ \uparrow & & \uparrow \\ B & \xrightarrow[\pi_Y|_B]{} & B_Y. \end{array} \tag{4.2.33}$$

The converse needs a bit of work. This is the embedding-analogue of the beginning of [HRW07, Section 4]. We discuss how the proofs can be adapted and often omit approximations that were otherwise done there. We want a version of [HRW07, Lemma 4.5], which is a result about *gluing* c.c.p. maps together along fibres. In our setting, we are only interested in u.c.p. maps, and we want to show that if we *glue* two u.c.p. maps together whose images are contained in some $C(X)$-subaglebra B, then the *glued* map also has image contained in B. We borrow their Definition 4.2.

Definition 4.2.23. *Let A be a unital $C(X)$-algebra, for a compact Hausdorff space X, and let D be a unital C^*-algebra. Let $\phi : D \to A$ be a u.c.p. map and $Y \subseteq X$ a closed subset. If $\mathcal{F} \subseteq D, \mathcal{G} \subseteq A$ are finite and $\varepsilon > 0$, we say that ϕ is $(\mathcal{F}, \varepsilon, \mathcal{G})$-good for Y if*

1. $([\phi(d), a])_Y \approx_\varepsilon 0$ *and*
2. $\phi(dd')_Y \approx_\varepsilon \phi(d)_Y \phi(d')_Y$

whenever $d, d' \in \mathcal{F}$ and $a \in \mathcal{G}$. If $X = [0, 1]$, $Y \subseteq X$ is a closed interval, $\mathcal{F}' \supseteq \mathcal{F}$ is another finite set and $0 < \varepsilon' < \varepsilon$, we say that ϕ is $(\mathcal{F}, \varepsilon, \mathcal{G}; \mathcal{F}', \varepsilon')$-good for Y if ϕ is $(\mathcal{F}, \varepsilon, \mathcal{G})$-good for Y and there exists some closed neighbourhood V of the endpoints of Y such that ϕ is $(\mathcal{F}', \varepsilon', \mathcal{G})$-good for V.

First we need a lemma that follows as a consequence of $\mathcal{D}$-stability. It is the embedding analogue of [HRW07, Proposition 4.1].

Lemma 4.2.24. *Let $\mathcal{D}$ be strongly self-absorbing, and $B \subseteq A$ be a unital, $\mathcal{D}$-stable inclusion of separable C^*-algebras. Then for any $\mathcal{G} \subseteq A$ finite and $\varepsilon > 0$, there exist unital *-homomorphisms $\kappa : A \to A$ and $\mu : \mathcal{D} \to B$ such that*

1. $\kappa(B) \subseteq B$,
2. $[\kappa(A), \mu(\mathcal{D})] = 0$,
3. $\kappa(a) \approx_\varepsilon a$ *for all* $a \in \mathcal{G}$.

Proof. The proof is essentially the same as the proof of (a) $\Rightarrow$ (c) in [HRW07, Proposition 4.1]. As $B \subseteq A$ is $\mathcal{D}$-stable, let us identify $B \subseteq A$ with $B \otimes \mathcal{D} \subseteq A \otimes \mathcal{D}$. As $\mathcal{D}$ is strongly self-absorbing, [TW07, Theorem 2.3] gives a sequence (ϕ_n) of *-homomorphisms $\phi_n : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}$ such that

$$\phi_n(d \otimes 1_{\mathcal{D}}) \to d \text{ for all } d \in \mathcal{D}. \tag{4.2.34}$$

Define $\kappa_n : A \otimes \mathcal{D} \to A \otimes \mathcal{D}$ by

$$\kappa_n := (\mathrm{id}_A \otimes \phi) \circ (\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}), \tag{4.2.35}$$

and $\mu_n : \mathcal{D} \to B \otimes \mathcal{D}$ by

$$\mu_n := (\mathrm{id}_B \otimes \phi_n) \circ (1_A \otimes 1_{\mathcal{D}} \otimes \mathrm{id}_{\mathcal{D}}). \tag{4.2.36}$$

Then taking n large enough and letting κ and μ be κ_n and μ_n respectively, its clear that $\kappa(B \otimes \mathcal{D}) \subseteq B \otimes \mathcal{D}$, $[\kappa(A), \mu(\mathcal{D})] = 0$ and that $\kappa(a) \approx_\varepsilon a$ whenever a is in some prescribed finite subset $\mathcal{G} \subseteq A$ and $\varepsilon > 0$ is some prescribed error. □

Lemma 4.2.25. *Let $\mathcal{D}$ be strongly self-absorbing and A be a unital, separable $C([0,1])$-algebra. Suppose $\mathcal{F} \subseteq \mathcal{D}, \mathcal{G} \subseteq A$ are finite self-adjoint subsets of contractions with $1_{\mathcal{D}} \in \mathcal{F}$. Suppose that we have points $0 \leq r < s < t \leq 1$ and two u.c.p. maps $\rho, \sigma : \mathcal{D} \to A$ which are $(\mathcal{F}, \varepsilon, \mathcal{G})$-good for $[r,s], [s,t]$ respectively. Suppose that A_s is $\mathcal{D}$-stable.*

Then there are u.c.p. maps $\rho', \sigma' : \mathcal{D} \to A$ which are $(\mathcal{F}, \varepsilon, \mathcal{G})$-good for $[r,s], [s,t]$ respectively, and u.c.p. maps $\nu_{\rho'}, \nu_{\sigma'} : \mathcal{D} \to A, \mu_{\rho'}, \mu_{\sigma'} : \mathcal{D} \otimes \mathcal{D} \to A$ such that $\nu_{\rho'}, \nu_{\sigma'}$ are $(\mathcal{F}, 3\varepsilon, \mathcal{G})$-good for some interval $I \subseteq (r,t)$ containing s in its interior, and such that for any $a \in \mathcal{G}, d, d' \in \mathcal{F}$, we have

1. $([\rho'(d), \nu_{\rho'}(d')])_I \approx_{2\varepsilon} 0$
2. $([\sigma'(d), \nu_{\sigma'}(d')])_I \approx_{2\varepsilon} 0$
3. $\rho'(d)_I \nu_{\rho'}(d')_I \approx_\varepsilon \mu_{\rho'}(d \otimes d')_I$
4. $\sigma'(d)_I \nu_{\sigma'}(d')_I \approx_\varepsilon \mu_{\sigma'}(d \otimes d')_I$
5. $\nu_{\rho'}(d)_I \approx_{2\varepsilon} \nu_{\sigma'}(d)_I$.

If ρ, σ are $(\mathcal{F}, \varepsilon, \mathcal{G}; \mathcal{F}', \varepsilon)$-good for $[r,s], [s,t]$ respectively, for some finite $\mathcal{F}' \supseteq \mathcal{F}$ set of contractions and for some $0 < \varepsilon' < \varepsilon$, then we can arrange so that $\rho', \sigma', \nu_{\rho'}, \nu_{\sigma'}$ are $(\mathcal{F}', 3\varepsilon', \mathcal{G})$-good for the interval I, and that the above five conditions hold with ε' in place of ε and $\mathcal{F}'$ in place of $\mathcal{F}$.

Moreover, if $B \subseteq A$ is a unital inclusion of $C([0,1])$-algebras such that $\rho(\mathcal{D}) \subseteq B, \sigma(\mathcal{D}) \subseteq B$ and $B_s \subseteq A_s$ is $\mathcal{D}$-stable, then the images of all $\rho', \sigma', \mu_{\rho'}, \mu_{\sigma'}$ are contained in B (as are the images of $\nu_{\rho'}$ and $\nu_{\sigma'}$).

Proof. This is [HRW07, Lemma 4.4], except we've replaced c.c.p. maps with u.c.p. maps. One can easily check that the resulting maps are u.c.p. maps.

As for the "moreover" part, which is the only addition besides the unitality, we outline the definitions of these maps to show that the images of $\rho', \sigma', \mu_{\rho'}, \mu_{\sigma'}$ are contained in B. As $B_s \subseteq A_s$ is $\mathcal{D}$-stable, we can find $\kappa : A_s \to A_s$ and $\mu : \mathcal{D} \to B_s$ as in Lemma 4.2.24, where $\kappa(a_s) \approx a_s$ for an appropriate error whenever $a \in \mathcal{G}$. We use Choi-Effros to find u.c.p. lifts $\tilde{\rho}, \tilde{\sigma} : \mathcal{D} \to B$ for the maps $\kappa \circ \pi_s \circ \rho$ and $\kappa \circ \pi_s \circ \sigma$ respectively (note that $\kappa \circ \pi_s \circ \rho$ and $\kappa \circ \pi_s \circ \sigma$ lie in B_s, which is a *-homomorphism image of B). One then defines piece-wise linear functions $f, g : [0,1] \to [0,1]$ which attain both values 0 and 1 at the end points (their definition is not important to show the "moreover" part). Then ρ', σ' are defined as

$$\rho'(d) := (1-f) \cdot \rho(d) + f \cdot \tilde{\rho}(d) \text{ and } \sigma'(d) := (1-g) \cdot \sigma(d) + g \cdot \tilde{\sigma}(d) \tag{4.2.37}$$

Clearly ρ', σ' take values in B as $\rho, \tilde{\rho}, \sigma, \tilde{\sigma}$ all do and $(1-f), f, (1-g), g$ are in B. Now we define u.c.p. maps $\tilde{\mu}_{\rho'}, \tilde{\mu}_{\sigma'} : \mathcal{D} \otimes \mathcal{D} \to B_s$ by

$$\tilde{\mu}_{\rho'}(d \otimes d') := \rho'(d)_s \mu(d') \text{ and } \tilde{\mu}_{\sigma'}(d \otimes d') := \sigma'(d)_s \mu(d'). \tag{4.2.38}$$

Now by Choi-Effros, we can take u.c.p. lifts $\mu_{\rho'}$ and $\mu_{\sigma'}$ of $\tilde{\mu}_{\rho'}$ and $\mu_{\sigma'}$, respectively. As the images of $\tilde{\mu}_{\rho'}$ and $\mu_{\sigma'}$ lie in B_s, the images of $\mu_{\rho'}$ and $\mu_{\sigma'}$ will lie in B. □

Lemma 4.2.26. *Let A be a unital, separable $C([0,1])$-algebra. Suppose $\mathcal{F} \subseteq \mathcal{D}, \mathcal{G} \subseteq A$ are finite self-adjoint subsets with $1_{\mathcal{D}} \in \mathcal{F}$ and $\varepsilon > 0$. There exists $0 < \varepsilon' < \varepsilon$ and a finite subset $\mathcal{F}' \supseteq \mathcal{F}$ such that if $\rho, \sigma : \mathcal{D} \to A$ are u.p.c. maps and $0 \leq r < s < t \leq 1$ are points such that ρ is $(\mathcal{F}, \varepsilon, \mathcal{G}; \mathcal{F}', \varepsilon')$-good for $[r,s]$, σ is $(\mathcal{F}, \varepsilon, \mathcal{G}; \mathcal{F}', \varepsilon')$-good for $[s,t]$ and A_s is $\mathcal{D}$-stable, then there is a u.c.p. map $\psi : \mathcal{D} \to A$ which is $(\mathcal{F}, \varepsilon, \mathcal{G}; \mathcal{F}', \varepsilon')$-good for $[r,t]$.*

Moreover, if $B \subseteq A$ is a unital inclusion of $C([0,1])$-algebras such that $\rho(\mathcal{D}) \subseteq B$, $\sigma(\mathcal{D}) \subseteq B$ and $B_s \subseteq A_s$ is $\mathcal{D}$-stable, then $\psi(\mathcal{D}) \subseteq B$.

Proof. The first part is [HRW07, Lemma 4.5], except we've replaced c.c.p. maps with u.c.p. maps. One has to check that the resulting ψ is unital, but this follows easily if ρ and σ are.

We outline the construction of ψ to show unitality, as it will also be useful to show the "moreover" part, which is the only real addition. Let $u \in C([0,1], \mathcal{D} \otimes \mathcal{D})$ be a path of unitaries such that $u_0 = 1_{\mathcal{D} \otimes \mathcal{D}}$ and

$$u_1(d \otimes 1_{\mathcal{D}})u_1^* \approx_{\frac{\varepsilon}{4}} 1_{\mathcal{D}} \otimes d. \tag{4.2.39}$$

We replace ρ, σ with ρ', σ' as in the above lemma and this yields u.c.p. maps μ_ρ, μ_σ satisfying the hypotheses above for some interval $I \subseteq (r,t)$ with s in its interior. Define

$$\phi_\rho, \phi_\sigma : C([0,1]) \otimes \mathcal{D} \otimes \mathcal{D} \to A \tag{4.2.40}$$

by

$$\begin{aligned}\phi_\rho(f \otimes d \otimes d') &:= f \cdot \mu_\rho(d \otimes d') \\ \phi_\sigma(f \otimes d \otimes d') &:= f \cdot \mu_\sigma(d \otimes d').\end{aligned} \tag{4.2.41}$$

Note that these maps are unital. Take non-zero piece-wise linear functions

$$h_1, h_2, h_3, h_4 : [0,1] \to [0,1] \tag{4.2.42}$$

which sum to 1 (their specific form does not matter to show unitality of ψ nor the "moreover" part) and $g_\rho, g_\sigma : [0,1] \to [0,1]$ which sum to 1 (again, their specific form does not matter to show unitality of ψ nor the "moreover" part). Define unitaries $u_\rho, u_\sigma \in C([0,1]) \otimes \mathcal{D} \otimes \mathcal{D} \simeq C([0,1], \mathcal{D} \otimes \mathcal{D})$ by

$$u_{\rho x} := u_{g_\rho(x)} \text{ and } u_{\sigma x} := u_{g_\sigma(x)}. \tag{4.2.43}$$

Now define $\zeta_\rho, \zeta_\sigma : \mathcal{D} \to A$ by

$$\begin{aligned}\zeta_\rho(d) &:= \phi_\rho(u_\rho(1_{C([0,1])} \otimes d \otimes 1_\mathcal{D})u_\rho^*) \\ \zeta_\sigma(d) &:= \phi_\sigma(u_\sigma(1_{C([0,1])} \otimes d \otimes 1_\mathcal{D})u_\sigma^*),\end{aligned} \tag{4.2.44}$$

which are clearly unital. Finally the map $\psi : \mathcal{D} \to A$ is defined by

$$\psi(d) := h_1 \cdot \rho(d) + h_2 \cdot \zeta_\rho(d) + h_3 \cdot \zeta_\sigma(d) + h_4 \cdot \sigma(d). \tag{4.2.45}$$

Clearly ψ is unital.

Now for the "moreover" part. If $\rho(\mathcal{D}) \subseteq B$ and $\sigma(\mathcal{D}) \subseteq B$, clearly the first and fourth terms in the definition of ψ will lie in B. So it suffices to show that $\zeta_\rho(\mathcal{D}) \subseteq B$ and $\zeta_\sigma(\mathcal{D}) \subseteq B$, and for this it suffices to show that $\mu_\rho(\mathcal{D} \otimes \mathcal{D}) \subseteq B$ and $\mu_\sigma(\mathcal{D} \otimes \mathcal{D}) \subseteq B$ (since h_1, h_2, h_3, h_4 all lie in B). But this follows from the "moreover" part of the previous lemma. □

With this, we get the analogue of their Theorem 4.6, the proof being essentially the same as well, except we insist that the our u.c.p. maps commute with a prescribed finite subset of A.

Theorem 4.2.27. *Let $\mathcal{D}$ be strongly self-absorbing, and X be a compact Hausdorff space with finite covering dimension. Suppose that $B \subseteq A$ is a unital inclusion of $C(X)$-algebras. Then $B_x \subseteq A_x$ is $\mathcal{D}$-stable for all $x \in X$ if and only if $B \subseteq A$ is $\mathcal{D}$-stable.*

Proof. As previously mentioned, if $B \subseteq A$ is $\mathcal{D}$-stable, then $B_x \subseteq A_x$ is $\mathcal{D}$-stable for all x.

For the converse, the proof is essentially the same as [HRW07, Theorem 4.6]. Using the arguments there, one can simplify to the case where we can argue this for $C([0,1])$-algebras (by using [HW41, Theorem V.3], which says that a compact space of dimension $\leq n$ is homeomorphic to a subset of $[0,1]^{2n+1}$, and then working component-wise). Now for $\mathcal{F} \subseteq \mathcal{D}, \mathcal{G} \subseteq A$ and $\varepsilon > 0$, let $\mathcal{G}_x := \{a_x \mid a \in \mathcal{G}\}$. Without loss of generality suppose that $\mathcal{F}^* = \mathcal{F}, \mathcal{G}^* = \mathcal{G}$ and that $1_{\mathcal{D}} \in \mathcal{F}$. Let $\mathcal{F}', \varepsilon'$ be as in Lemma 4.2.26.

By $\mathcal{D}$-stability of the inclusion $B_x \subseteq A_x$ there are u.c.p. $(\mathcal{F}', \varepsilon', \mathcal{G}_x)$-embeddings $\psi_x : \mathcal{D} \to B_x \subseteq A_x$ which lift by Choi-Effros to u.c.p. maps $\psi'_x : \mathcal{D} \to B$. The norm is upper semi-continuous (Remark 4.2.22), and this yields intervals $I_x \subseteq [0,1]$ such that ψ'_x is $(\mathcal{F}', \varepsilon', \mathcal{G})$-good for $\overline{I_x}$. Note that ψ'_x being $(\mathcal{F}', \varepsilon', \mathcal{G})$-good for the whole of I_x implies that it is $(\mathcal{F}, \varepsilon, \mathcal{G}; \mathcal{F}', \varepsilon')$-good for $\overline{I_x}$. Compactness then allows us to split the interval as

$$0 = t_0 < t_1 < \cdots < t_n = 1 \tag{4.2.46}$$

and to take $\psi_i : \mathcal{D} \to B$ u.c.p. which are $(\mathcal{F}, \varepsilon, \mathcal{G}; \mathcal{F}', \varepsilon')$-good for $[t_{i-1}, t_i]$ for $i = 1, \ldots, n$ ($\psi_i = \psi'_x$ for some $x \in [0,1]$). Now by repeatedly using the gluing lemma (Lemma 4.2.26) to glue these maps together, we can find a u.c.p. map $\psi : \mathcal{D} \to B$ which is an $(\mathcal{F}, \varepsilon, \mathcal{G})$-embedding. □

4.3 Crossed products

In this section we discuss how inclusions coming from non-commutative dynamics fit into the framework of a tensorially absorbing inclusions. We'll shortly discuss group actions $G \curvearrowright^{\alpha} A$ with Rokhlin properties and consider the inclusion of a C*-algebra in its crossed product $A \subseteq A \rtimes_{\alpha} G$, as well as the inclusion of the fixed point subalgebra of the action in the C*-algebra $A^{\alpha} \subseteq A$. We then discuss diagonal inclusions associated to certain group actions.

This first result says that if we have an isomorphism $A \simeq A \otimes \mathcal{D}$ which is G-equivariant with respect to an action point-wise fixing the right tensor factor, then the corresponding inclusion $A \subseteq A \rtimes_{r,\alpha} G$ is $\mathcal{D}$-stable.

Proposition 4.3.1. *Let $G \curvearrowright^{\alpha} A$ be an action of a countable discrete group on a unital separable C*-algebra. Suppose that $\alpha \simeq \alpha \otimes id_{\mathcal{D}}$, that is, there is an isomorphism $\Phi : A \simeq A \otimes \mathcal{D}$ such that*

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & A \otimes \mathcal{D} \\ \alpha_g \big\downarrow & & \big\downarrow \alpha_g \otimes id_{\mathcal{D}} \\ A & \xrightarrow[\Phi]{} & A \otimes \mathcal{D} \end{array} \tag{4.3.1}$$

commutes for all $g \in G$. Then $A \subseteq A \rtimes_{r,\alpha} G$ is $\mathcal{D}$-stable.

Proof. Let $\psi : \mathcal{D} \simeq \mathcal{D}^{\otimes \infty}$ and let $\phi_n : \mathcal{D} \to \mathcal{D}^{\otimes \infty}$ be the nth factor embedding:

$$\phi_n(d) := 1_{\mathcal{D}}^{\otimes n-1} \otimes d \otimes 1_{\mathcal{D}}^{\otimes \infty}. \tag{4.3.2}$$

We claim that $\xi(d) := (\Phi^{-1}(1_A \otimes \psi^{-1} \circ \phi_n(d)))_n : \mathcal{D} \to A_\omega$ is an embedding such that $\xi(\mathcal{D}) \subseteq A_\omega \cap A'$ and $(\alpha_g)_\omega \circ \xi = \xi$ for all $g \in G$ – that is, ξ is an embedding $\mathcal{D} \hookrightarrow A_\omega \cap (A \rtimes_{r,\alpha} G)'$. The first claim is obvious, so we prove the second. We have

$$\begin{aligned}
&\|\alpha_g(\Phi^{-1}(1_A \otimes \psi^{-1}(\phi_n(d)))) - \Phi^{-1}(1_A \otimes \psi^{-1}(\phi_n(d)))\| \\
&\qquad = \|\Phi \circ \alpha_g(\Phi^{-1}(1_A \otimes \psi^{-1}(\phi_n(d)))) - \Phi(\Phi^{-1}(1_A \otimes \psi^{-1}(\phi_n(d))))\| \\
&\qquad = \|\alpha_g \otimes \mathrm{id}_{\mathcal{D}}(1_A \otimes \psi^{-1}(\phi_n(d))) - 1_A \otimes \psi^{-1}(\phi_n(d)))\| \\
&\qquad = 0.
\end{aligned} \tag{4.3.3}$$

□

The next lemma of note is the following.

Lemma 4.3.2. *Suppose that $G \curvearrowright^\alpha A$ is an action of a finite group on a unital separable C*-algebra A such that $A \subseteq A \rtimes_\alpha G$ is $\mathcal{D}$-stable. Then $A^\alpha \subseteq A \rtimes_\alpha G$ is $\mathcal{D}$-stable. In particular, if $A \subseteq A \rtimes_\alpha G$ is $\mathcal{D}$-stable, then $C \simeq C \otimes \mathcal{D}$ whenever $A^\alpha \subseteq C \subseteq A \rtimes_\alpha \mathcal{D}$.*

Proof. For an element $(x_n) \in A_\omega \cap (A \rtimes_\alpha G)'$, an easy averaging argument shows that

$$(x_n) = \left(\frac{1}{|G|} \sum_{g \in G} \alpha_g(x_n) \right) \tag{4.3.4}$$

in A_ω, and the right is clearly point-wise fixed by α_g for all $g \in G$. So $A_\omega \cap (A \rtimes_\alpha G)'$ is actually equal to $(A^\alpha)_\omega \cap (A \rtimes_\alpha G)'$, and the existence of a unital embedding of $\mathcal{D}$ in $A_\omega \cap (A \rtimes_\alpha G)'$ is in fact equivalent to the existence of a unital embedding of $\mathcal{D}$ into $(A^\alpha)_\omega \cap (A \rtimes_\alpha G)'$. The result follows. □

The Galois correspondence of Izumi [Izu02] yields the following.

Theorem 4.3.3. *Let A be a unital, simple, separable C*-algebra and let $G \curvearrowright^\alpha A$ be an action of a finite group by outer automorphisms. If $A \subseteq A \rtimes_\alpha \mathcal{D}$ is $\mathcal{D}$-stable, then there exists an isomorphism $\Phi : A \rtimes_\alpha G \simeq (A \rtimes_\alpha G) \otimes \mathcal{D}$ such that whenever C is a unital C*-algebra satisfying either*

1. *$A^\alpha \subseteq C \subseteq A$ or*
2. *$A \subseteq C \subseteq A \rtimes_\alpha G$,*

we have $\Phi(C) = C \otimes \mathcal{D}$.

Proof. Applying [Izu02, Corollary 6.6] gives the following two correspondences:

1. there is a one-to-one correspondence between subgroups of G with intermediate C*-algebras $A^\alpha \subseteq C \subseteq A$ given by

$$H \leftrightarrow A^{\alpha_H}; \tag{4.3.5}$$

2. there is a one-to-one correspondence between subgroups of G and intermediate C*-algebras $A \subseteq C \subseteq A \rtimes_\alpha G$ given by

$$H \leftrightarrow A \rtimes_{\alpha|_H} H. \tag{4.3.6}$$

In particular, there are only finitely many C*-algebras C between either $A^\alpha \subseteq A$ or $A \subseteq A \rtimes_\alpha G$. As all such lie between the $\mathcal{D}$-stable inclusion $A^\alpha \subseteq A \rtimes G$, Theorem 4.2.9 yields the desired isomorphism. □

4.3.1 (Tracial) Rokhlin properties

Here we will restrict ourselves to finite groups for simplicity, although many results hold more generally (see [HW07, HO13, GH18]).

Definition 4.3.4. *Let A be a unital, separable C*-algebra. We say that a finite group action $G \curvearrowright^\alpha A$ has the Rokhlin property if there are pairwise orthogonal projections $(p_g)_{g\in G} \subseteq A_\omega \cap A'$ summing to 1_{A_ω} such that $(\alpha_g)_\omega(p_h) = p_{gh}$ for $g, h \in G$.*

Proposition 4.3.5. *Let A be a unital, separable $\mathcal{D}$-stable C*-algebra. If $G \curvearrowright^\alpha A$ is an action of a finite group with the Rokhlin property, then $A^\alpha \subseteq A \rtimes_\alpha G$ is $\mathcal{D}$-stable.*

Proof. This follows from [HW07, Theorem 3.3], together with Lemma 4.3.2. □

Definition 4.3.6. *Let A be a unital, separable C*-algebra. We say that a finite group action $G \curvearrowright^\alpha A$ has the weak tracial Rokhlin property if for all $\mathcal{F} \subseteq A$ finite, $\varepsilon > 0$ and $0 \neq a \in A_+$, there are pairwise orthogonal normalized positive contractions $(e_g)_{g\in G} \subseteq A$ such that*

1. $1 - \sum_g e_g \precsim a$;[19]
2. $[e_g, x] \approx_\varepsilon 0$ *for all* $x \in \mathcal{F}, g \in G$;
3. $\alpha_g(e_h) \approx_\varepsilon e_{gh}$ *for all* $g, h \in G$.

We note that both Rokhlin and weak tracial Rokhlin actions are necessarily outer.

[19]For two positive elements x, y in a C*-algebra, we write $x \precsim y$ to mean that x is Cuntz-subequivalent to y. That is, there are (r_n) in the C*-algebra such that $r_n^* y r_n \to x$. See [HO13, Section 2].

Proposition 4.3.7. *Let A be a unital, simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebra. If $G \curvearrowright^{\alpha} A$ is an action of a finite group with the weak tracial Rokhlin property, then $A^{\alpha} \subseteq A \rtimes_{\alpha} G$ is $\mathcal{Z}$-stable.*

Proof. Let $k \in \mathbb{N}$. By [HO13, Theorem 5.6] $A \rtimes_{\alpha} G$ is tracially $\mathcal{Z}$-absorbing, meaning there are tracially large (in the sense of [TWW15]) c.p.c. order zero maps $\phi : M_k \to (A \rtimes_{\alpha} G)_{\omega} \cap (A \rtimes_{\alpha} G)'$, which can be chosen to be c.p.c. order zero maps $\phi : M_k \to A_{\omega} \cap (A \rtimes_{\alpha} G)'$ by the proof of [HO13, Lemma 5.5]. These tracially large c.p.c. order zero maps yield sequences of positive contractions $c_1 = (c_{1n}), \dots, c_k = (c_{kn}) \in A_{\omega} \cap (A \rtimes_{\alpha} G)'$ such that if $(e_n) = e := 1 - \sum_i c_i^* c_i$, we have

$$\lim_{n\to\omega} \max_{\tau \in T(A)} \tau(e_n) = 0, \inf_m \lim_{n\to\omega} \min_{\tau\in T(A)} \tau(c_{1n}^m) > 0 \tag{4.3.7}$$

and $c_i c_j^* = \delta_{ij} c_1^2$. By [GH18, Proposition 4.11] (which is much more general, applicable to all countable amenable groups), $A \subseteq A \rtimes_{\alpha} G$ has equivariant property (SI) since A has property (SI).[20] Consequently there exists $s \in A_{\omega} \cap (A \rtimes_{\alpha} G)'$ such that $s^* s = 1 - \sum_i c_i^* c_i$ and $c_1 s = s$. Altogether,

- $c_1 \geq 0$;
- $c_i c_j^* = \delta_{ij} c_1^2$;
- $s^* s + \sum_i c_i^* c_i = 1$;
- $c_1 s = s$.

As mentioned in the proof of $(iv) \Rightarrow (i)$ of [MS12], $\mathcal{Z}_{n,n+1}$ is the universal C*-algebra generated by $n+1$ elements satisfying the above four relations (see [RW10, Proposition 5.1] and [Sat10, Proposition 2.1]), and consequently we have a unital *-homomorphism $\mathcal{Z}_{n,n+1} \to A_{\omega} \cap (A \rtimes_{\alpha} G)'$. Therefore $\mathcal{Z} \hookrightarrow A_{\omega} \cap (A \rtimes_{\alpha} G)'$, giving that the desired inclusion is $\mathcal{Z}$-stable by Lemma 4.3.2. □

Corollary 4.3.8. *Let A be a unital, simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebra and $G \curvearrowright^{\alpha} A$ be an action of a finite group with the weak tracial Rokhlin property. There exists an isomorphism $\Phi : A \rtimes_{\alpha} G \simeq (A \rtimes_{\alpha} G) \otimes \mathcal{Z}$ such that whenever C is a unital C*-algebra satisfying either*

1. *$A^{\alpha} \subseteq C \subseteq A$ or*
2. *$A \subseteq C \subseteq A \rtimes_{\alpha} G$,*

we have $\Phi(C) = C \otimes \mathcal{Z}$.

Proof. This results from combining Proposition 4.3.7 together with Theorem 4.3.3, making note that this is an outer action. □

[20] A unital, separable, simple, nuclear, $\mathcal{Z}$-stable C*-algebra has property (SI) as in [MS12]

4.3.2 The diagonal inclusion associated to a group action

In the von Neumann setting, a certain diagonal inclusion associated to several automorphisms was considered in [Pop89, Kaw99, Bur10], and they play a role in subfactor theory. Here we consider a unital C*-algebraic inclusion of the same form.

Definition 4.3.9. *Let A be a C*-algebra, $\alpha_1, \dots, \alpha_n \in \mathrm{Aut}(A)$. The diagonal inclusion associated to $\alpha_1, \dots, \alpha_n$ is*

$$B(\alpha_1, \dots, \alpha_n) = \left\{ \bigoplus_{i=1}^{n} \alpha_i(a) \mid a \in A \right\} \subseteq M_n(A). \tag{4.3.8}$$

If $G \curvearrowright^{\alpha} A$ is an action of a finite group, we'll write

$$B(\alpha) = \left\{ \bigoplus_{g \in G} \alpha_g(a) \mid a \in A \right\} \subseteq M_{|G|}(A). \tag{4.3.9}$$

We note that a diagonal $B(\alpha) \subseteq M_{|G|}(A)$ is unique up to unitary conjugation (by permutation unitaries). As $\mathcal{D}$-stability of an inclusion is preserved under unitary conjugation, there is no ambiguity in speaking of $\mathcal{D}$-stability of the inclusion $B(\alpha) \subseteq M_{|G|}(A)$.

Proposition 4.3.10. *Let $G \curvearrowright^{\alpha} A$ be an action of a countable discrete group on a unital, separable C*-algebra. If $G = \langle g_1, \dots, g_n \rangle$, then $A \subseteq A \rtimes_\alpha G$ is $\mathcal{D}$-stable if and only if*

$$B(id_A, \alpha_{g_1}, \dots, \alpha_{g_n}) \subseteq M_{n+1}(A) \tag{4.3.10}$$

is $\mathcal{D}$-stable.

Proof. First suppose that $A \subseteq A \rtimes_\alpha G$ is $\mathcal{D}$-stable. Let $\mathcal{F} \subseteq \mathcal{D}, \mathcal{G} \subseteq M_{n+1}(A)$ be finite and $\varepsilon > 0$. Let $\mathcal{G}' \subseteq A$ be the set of matrix coefficients of elements of $\mathcal{G}$, together with the identity of A, and let $L := \max\{1, \max_{a \in \mathcal{G}'} \|a\|\}$. Relabel $\mathrm{id}_A, \alpha_{g_1}, \dots, \alpha_{g_n}$ as $\alpha_1, \dots, \alpha_{n+1}$. Let

$$\delta := \frac{\varepsilon}{(4L+1)(n+1)^2} \tag{4.3.11}$$

and let $\psi : \mathcal{D} \to A$ be a u.c.p. $(\mathcal{F}, \delta, \mathcal{G}' \cup \{u_{g_i}\}_{i=1}^n)$-embedding, where (u_g) are the implementing unitaries for α. Let $\phi : D \to B(\alpha) \subseteq M_{|G|}(A)$ be given by

$$\phi(d) := \bigoplus_{i=1}^{n+1} (\alpha_i \circ \psi)(d). \tag{4.3.12}$$

Clearly ϕ will be $(\mathcal{F},\delta)$-multiplicative since each component is the composition of a *-homomorphism (which are contractive) with a map which is $(\mathcal{F},\delta)$-multiplicative. Now for $d \in \mathcal{F}$ and $a = (a_{ij}) \in \mathcal{G}$, we have

$$\begin{aligned}
\|[\phi(d),(a_{ij})]\| &\leq \sum_{i,j=1}^{n+1} \|\alpha_i(\psi(d))a_{ij} - a_{ij}\alpha_j(\psi(d))\| \\
&\leq \sum_{i,j=1}^{n+1} \|\alpha_i(\psi(d))a_{ij} - \psi(d)a_{ij}\| \\
&\quad + \|\psi(d)a_{ij} - a_{ij}\psi(d)\| + \|a_{ij}\psi(d) - a_{ij}\alpha_j(\psi(d))\| \\
&\leq \sum_{i,j=1}^{n+1} \|a_{ij}\| \left(\|\alpha_i(\psi(d)) - \psi(d)\| + \|\psi(d) - \alpha_j(\psi(d))\|\right) \\
&\quad + \|[\psi(d), a_{i,j}]\| \\
&< (n+1)^2(2L(\delta+\delta)+\delta) \\
&= (n+1)^2(4L+1)\delta = \varepsilon.
\end{aligned} \tag{4.3.13}$$

Conversely if the associated diagonal inclusion is $\mathcal{D}$-stable we note that if $(x_k) \subseteq B(\mathrm{id}_A, \alpha_{g_1}, \ldots, \alpha_{g_n})$ is central for $M_{n+1}(A)$, writing

$$x_k = \bigoplus_{i=1}^{n+1} \alpha_i(a_k) \tag{4.3.14}$$

yields that $(a_k) \subseteq A$ is central for A and is asymptotically fixed by $\alpha_{g_i}, i = 1, \ldots, n$. In particular if $\mathcal{D} \hookrightarrow B(\mathrm{id}_A, \alpha_{g_1}, \ldots, \alpha_{g_n})_\omega \cap (M_{n+1}(A))'$, then $\mathcal{D} \hookrightarrow A_\omega \cap (A \rtimes_\alpha G)'$. □

Corollary 4.3.11. *Let $G \curvearrowright^\alpha A$ be an action of a finite group on a unital, separable C*-algebra. Then $A \subseteq A \rtimes_\alpha G$ is $\mathcal{D}$-stable if and only if*

$$B(\alpha) \subseteq M_{|G|}(A) \tag{4.3.15}$$

is $\mathcal{D}$-stable.

4.4 Examples

4.4.1 Non-examples

We first start with some non-examples. Villadsen's C*-algebras with perforation will be useful (see [TW09] for good exposition). Let $\mathcal{Q} = \bigotimes_n M_n$ denote the universal UHF C*-algebra.

Theorem 4.4.1 ([Vil98, Tom08b]). *There exists a unital, simple, separable, nuclear C*-algebra C satisfying the UCT such that $C \not\simeq C \otimes \mathcal{Z}$ and C contains the universal UHF algebra unitally. Moreover C is tracial and can be chosen to be AH with*

$$(K_0(A), K_0(A)^+, [1]_0, K_1(A)) = (\mathbb{Q}, \mathbb{Q}_+, 1, 0). \tag{4.4.1}$$

Corollary 4.4.2. *There exists an embedding $\mathcal{Q} \hookrightarrow \mathcal{Q}$ which is not $\mathcal{Z}$-stable. In particular, it is not $\mathcal{Q}$-stable.*

Proof. Let C be as above. Note that $\mathcal{Q} \subseteq C$ so we must find an embedding $C \hookrightarrow \mathcal{Q}$. As C is unital, separable, exact, satisfies the UCT and has a faithful amenable trace (it has traces, and every such trace will be faithful and amenable since C is nuclear and simple) and there is clearly a morphism between K_0-groups, [Sch20, Theorem D] gives an embedding $C \hookrightarrow \mathcal{Q}$. Consequently there is an embedding

$$\mathcal{Q} \hookrightarrow C \hookrightarrow \mathcal{Q} \tag{4.4.2}$$

which is not $\mathcal{Q}$-stable since there is an intermediate C*-algebra C with $C \not\simeq C \otimes \mathcal{Z}$. □

Corollary 4.4.3. *There is an embedding $\mathcal{Z} \hookrightarrow \mathcal{Q}$ which is not $\mathcal{Z}$-stable.*

Proof. Take C as above and take the chain of embeddings (noting that $\mathcal{Q}$ is $\mathcal{Z}$-stable)

$$\mathcal{Z} \hookrightarrow \mathcal{Q} \otimes \mathcal{Z} \simeq \mathcal{Q} \hookrightarrow C \hookrightarrow \mathcal{Q}. \tag{4.4.3}$$

□

Corollary 4.4.4. *There is an embedding $\mathcal{Z} \hookrightarrow \mathcal{O}_2$ which is not $\mathcal{Z}$-stable.*

Proof. Just take the same embedding as above together with an embedding $\mathcal{Q} \hookrightarrow \mathcal{O}_2$. □

Remark 4.4.5. All *-homomorphisms between strongly self-absorbing C*-algebras are approximately unitarily equivalent by [TW07, Corollary 1.12], or even asymptotically unitarily equivalent by [DW09, Theorem 2.2]. Therefore $\mathcal{D}$-stability is not closed under these equivalences (nor homotopy, see [DW09, Corollary 3.1]).

The only method we have used to show that an inclusion is not $\mathcal{D}$-stable is by finding an intermediate algebra which is not $\mathcal{D}$-stable. There are plenty of examples of stably finite C*-algebras with perforation or higher-stable rank (in particular non-$\mathcal{Z}$-stable C*-algebras [Rør04]) [Vil98, Vil99, EV00, Tom05, HRW07, Tom08a, Tom08b, TW09, Mor09, Tik12]. This gives rise to the following two questions.

1. Is there a unital inclusion $B \subseteq A$ of separable C*-algebras such that whenever C is such that $B \subseteq C \subseteq A$, we have $C \simeq C \otimes \mathcal{D}$ but $B \subseteq A$ is not $\mathcal{D}$-stable? Is $\mathcal{D}$-stability equivalent to every intermediate C*-algebra being $\mathcal{D}$-stable?

2. To get non-examples we use stably finite C*-algebras with perforation in between sufficiently regular C*-algebras. Is there a way to do this for purely infinite C*-algebras, or is finiteness the only obstruction? Thus we can ask: if $\mathcal{D}$ is a purely infinite strongly self-absorbing C*-algebra, is every embedding of $\mathcal{D}$ into itself $\mathcal{D}$-stable? More specifically, if $B \subseteq A$ is a unital inclusion of simple, separable, purely infinite C*-algebras, is the inclusion $\mathcal{O}_\infty$-stable?

Our third question asks if we can get non-examples arising from dynamical systems.

3. Is there a unital, separable $\mathcal{D}$-stable C*-algebra and a (finite) group action $G \curvearrowright^\alpha A$ such that $A \rtimes_\alpha G$ is $\mathcal{D}$-stable, but the inclusion is not? One would need $A \rtimes_\alpha G$ to be $\mathcal{D}$-stable for non-dynamical reasons.

4.4.2 Cyclicly permuting tensor powers

Here we give a dynamical example to illustrate the discussion in Section 4.3. In particular, we can look at a consequence of Corollary 4.1.6.

Example 4.4.6. *Let $p, q \in \mathbb{N}$ be coprime and consider the qth tensor power of the UHF algebra $A = M_{p^\infty}^{\otimes q}$. Let us examine the action $\mathbb{Z}_q \curvearrowright^\sigma A$ given by cyclically permuting the tensors:*

$$\sigma(a_1 \otimes \cdots \otimes a_q) = a_2 \otimes \cdots \otimes a_q \otimes a_1. \tag{4.4.4}$$

One can prove directly or use [HO13] or [AGJP22] in order to conclude that this action has the weak tracial Rokhlin property and consequently that $A^\sigma \subseteq A \rtimes_\sigma \mathbb{Z}_q$ is $\mathcal{Z}$-stable.

Alternatively, one can use techniques similar to [HR84], [HR85], or [HW07] in order to compute the K-theory of the fixed point algebra A^σ to be

$$K_0((M_{p^\infty}^{\otimes q})^\sigma) \simeq \lim_{\to} \left(\mathbb{Z}^q, \begin{pmatrix} p + \frac{p^q - p}{q} & \frac{p^q-p}{q} & \cdots & \frac{p^q-p}{q} \\ \frac{p^q-p}{q} & p + \frac{p^q-p}{q} & \cdots & \frac{p^q-p}{q} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p^q-p}{q} & \frac{p^q-p}{q} & \cdots & p + \frac{p^q-p}{q} \end{pmatrix} \right), \tag{4.4.5}$$

from which one can show that $K_0(A^\sigma)$ is p-divisible.Then using the fact that $K_0(A^\sigma)$ is p-divisible and A^σ is AF, it follows that A^σ is M_{p^∞}-stable. Using Corollary 4.1.6, we then see that $M_{p^\infty} \hookrightarrow (A^\sigma)_\omega \cap A'$. In particular, we have that $A^\sigma \subseteq A \rtimes_\sigma \mathbb{Z}_q$ is M_{p^∞}-stable (since clearly if this embedding is fixed by $\mathbb{Z}_q$, it will commute with the implementing unitaries as well).

Example 4.4.7. *Following up on the previous example, if we consider the embedding*

$$B := \left\{ \begin{pmatrix} x & & & \\ & \sigma(x) & & \\ & & \ddots & \\ & & & \sigma^{q-1}(x) \end{pmatrix} \mid x \in M_{p^\infty}^{\otimes q} \right\} \subseteq M_q(M_{p^\infty}^{\otimes q}) := A, \tag{4.4.6}$$

then $B \subseteq A$ is M_{p^∞}-stable by Proposition 4.3.10.

4.4.3 The canonical inclusion of the CAR algebra in $\mathcal{O}_2$

Example 4.4.8. *Let $\mathcal{O}_2 = C^*(s_1, s_2)$ be the Cuntz algebra generated by two isometries [Cun77], and consider the inclusion*

$$M_{2^\infty} \simeq \overline{span}\{s_\mu s_\nu^* \mid |\mu| = |\nu|\} \subseteq \mathcal{O}_2, \tag{4.4.7}$$

where for a word $\mu = \{i_1, \ldots, i_p\} \in \{1,2\}^p$, $s_\mu = s_{i_1} \cdots s_{i_p}$. This copy of the CAR algebra is precisely the fixed point subalgebra of the gauge action (see [Rae05]). Consider the endomorphism $\lambda : \mathcal{O}_2 \to \mathcal{O}_2$ given by

$$\lambda(x) := s_1 x s_1^* + s_2 x s_2^*. \tag{4.4.8}$$

We note that a sequence (x_n) is ω-asymptotically central for $\mathcal{O}_2$ if and only if it is ω-asymptotically fixed by λ. Indeed, if (x_n) is central, then $\|\lambda(x_n) - x_n\| \to^{n\to\omega} 0$ since $[x_n, s_i] \to 0$ for $i = 1, 2$. On the other hand if (x_n) is asymptotically fixed by λ, then the inequalities

$$\begin{aligned} \|s_i x_n - x_n s_i\| &= \|s_1 x_n s_1^* s_i + s_2 x_n s_2^* s_i - x_n s_i\| \le \|\lambda(x_n) - x_n\|\|s_i\| \\ \|s_i^* x_n - x_n s_i^*\| &= \|s_i^* x_n - s_i^* s_1 s_1^* - s_i^* s_2 x_n s_2^*\| \le \|s_i^*\|\|\lambda(x_n) - x_n\| \end{aligned} \tag{4.4.9}$$

imply that (x_n) is asymptotically central.

We note that $\lambda|_{M_{2^\infty}}$ is the forward tensor shift if we identify $M_{2^\infty} = \bigotimes_{\mathbb{N}} M_2$ (see for example [Dav96, Section V.4]). Now [BSKR93] gives an embedding $\xi : M_2 \hookrightarrow (M_{2^\infty})_\omega$ such that $\lambda_\omega \circ \xi = \xi$. In particular $M_{2^\infty} \hookrightarrow (M_{2_\infty})_\omega \cap \mathcal{O}_2'$ so that this inclusion is M_{2^∞}-stable.

Thinking of $\mathcal{O}_2$ as the semigroup crossed product $\mathcal{O}_2 \simeq M_{2^\infty} \rtimes_\lambda \mathbb{N}$ (see [Rør95, Rør21]), any intermediate C*-algebra is automatically CAR stable. Consequently each intermediate subalgebra $M_{2^\infty} \rtimes d\mathbb{N} = C^*(M_{2^\infty}, s_1^d)$ is M_{2^∞}-stable. We can do this all concurrently.

Corollary 4.4.9. *There exists an isomorphism $\Phi : \mathcal{O}_2 \simeq \mathcal{O}_2 \otimes M_{2^\infty}$ such that*

$$\Phi(C^*(M_{2^\infty}, s_1^d)) \simeq C^*(M_{2^\infty}, s_1^d) \otimes M_{2^\infty} \tag{4.4.10}$$

for all $d \in \mathbb{N}$. The same holds if we replace M_{2^∞} by $\mathcal{Z}$.

Now let us play with some diagonal inclusions associated to powers of the Bernoulli shift λ on $\mathcal{O}_2$ above. This will be similar to what was discussed in Section 4.3.2, except we allow endomorphisms.

Example 4.4.10. *Consider, for $n \in \mathbb{N}$, the diagonal inclusion*

$$B_n := \left\{ \begin{pmatrix} x & & & \\ & \lambda(x) & & \\ & & \ddots & \\ & & & \lambda^{n-1}(x) \end{pmatrix} \mid x \in \mathcal{O}_2 \right\} \subseteq M_n(\mathcal{O}_2) =: A_n. \tag{4.4.11}$$

Note that both A_n and B_n are isomorphic to $\mathcal{O}_2$, and in fact this gives a non-trivial inclusion of $\mathcal{O}_2$ into itself which is $\mathcal{O}_2$-stable. This is $\mathcal{O}_2$-stable since a sequence is asymptotically fixed by λ if and only if it asymptotically commutes with the algebra. A similar argument to that of Proposition 4.3.10 will yield that this inclusion is $\mathcal{O}_2$-stable.

One can even restrict the diagonal to elements of the CAR algebra $M_{2^\infty} \subseteq \mathcal{O}_2$ sitting as the fixed point subalgebra of the gauge action as above.

Example 4.4.11. *Consider*

$$B_n^{(2)} := \left\{ \begin{pmatrix} x & & & \\ & \lambda(x) & & \\ & & \ddots & \\ & & & \lambda^{n-1}(x) \end{pmatrix} \mid x \in M_{2^\infty} \right\} \subseteq M_n(\mathcal{O}_2) = A_n. \tag{4.4.12}$$

This is M_{2^∞}-stable for the same reasons as above. This gives another inclusion $M_{2^\infty} \simeq B_n^{(2)} \subseteq M_n(\mathcal{O}_2) \simeq \mathcal{O}_2$ which is CAR-stable.

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