PROBLEMS ON PROLONGATION OF GEODESICS

by

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BRIEF SUMMARY

Two-dimensional, rotationally symmetric, Riemannian spaces (R-spaces) are considered from the viewpoint of the geodesic-space (G-space) axioms, in particular from the viewpoint of the axiom of local prolongation of the geodesic lines.

Whereas a series of propositions show the existence of R-spaces which are not G-spaces, the main theorem excludes all possibility of existence of R-spaces with finite curvature which would not be G-spaces.

Nevertheless in the light of the counter-examples the definition of G-spaces given by Busemann seems to be tight.

LIST OF NOTATIONS

c	a parameter of the geodesics	page 3,4
$/c_n/$	a sequence of $c_0, c_1, c_2, \dots, c_n, \dots$	8
$f(u)$	the generator function of a surface of revolution	2,3
s	$s = s_c(u_0)$	5
$s_c(u)$	the length of a geodesic arc c, for $c \leq t \leq u$	4
u, v	surface - parameters	2
u_0		4,5
$v_c(u)$		4
α	the angle made by a geodesic and a meridian	4
α_0		4,5
		5
$\sigma(u)$	the length of a meridian arc $0 \leq t \leq u$	4
$\lambda(u) = \sqrt{1+f'^2(u)}$		3

I. INTRODUCTION AND THE BASIC CONCEPTS.

The discipline of "geometry of geodesics" wields synthetic tools as alternatives for the analytic ones. In fact, its axiomatic approach never makes use of any hypothesis concerning differentiability.

However interesting this comprehension of the differential geometry could be, the question arises that what kind of differentiability is equivalent to the G-space axioms, in particular, what Riemannian spaces are G-spaces and what are not.

G-spaces or *geodesic spaces* are defined by the following axioms, according to Busemann (Geometry of Geodesics, p. 37) :

1. The space is <u>metric</u>
2. The space is <u>finitely compact</u>
3. Given two distinct points x, z then a point y exists such that $xy + yz = xz$, where xy stands for the distance between x and y. In other words, the space is <u>convex</u>
4. To every point p of the space there corresponds a positive number ρ_p such that for any two distinct points x, y in the ρ_p-neighborhood of p a point z with $xy + yz = xz$ exists. In other words the <u>local prolongation of the segment</u> is possible in the space
5. If $xy + yz_1 = xz_1$ and $xy + yz_2 = xz_2$ with $yz_1 = yz_2$ then $z_1 = z_2$. In other words the prolongation of the segment is <u>unique</u>

The length of curves is defined as in elementary spaces. A curve is called *segment* if the distanse between the endpoints is equal to the length. The first three axioms secure the existence of the segment, so that its mention in the fourth and fifth may be justified.

A *geodesic* is a curve which extends indefinitely in both directions and behaves locally like a segment.

We are concerned here only with the fourth axiom, the axiom of prolongability of the segment. Counter-examples will be given to illustrate the strength of the assumptions included in the axioms. It will turn out that in a sense the present system of axioms is too strong, namely there are quite smooth Riemannian spaces which are not G-spaces because of the axiom of prolongability.

Our examples are two-dimensional Riemannian spaces, mostly surfaces of revolution. The intersection point of such a surface with the axis of rotation is called *vertex*. The intersection curve of the surface with any plane containing the axis is called *meridian*. All the meridians are congruent. The halves of a meridian separated by the vertex are geodesics. In the following examples the prolongation of the meridian through the vertex is the primary subject of the investigations. A given surface of revolution is a counter-example for the axiom of prolongability if a sequence of meridian-arcs can be found such that every arc contains the vertex and every segment joining the endpoints of an arc offers a shorter length than the arc, provided that the sequence of the distances of the endpoints tends to 0. In our examples usually these meridian arcs are symmetric, i.e. the vertex is the midpoint of the arcs. On the other hand, a surface of revolution satisfies the axiom of prolongability of the segment at the vertex if every sufficiently small meridian arc through the vertex is a segment.

The surfaces of revolution will be described in the following manner. The even function $f(u)$ with $f(0)=0$ is the *generator-function* of the surface, if its graph is a meridian. Every point (x,y,z) of the surface satisfies the equation

$$(1)\qquad \begin{cases} x(u,v) = u\cos v \\ y(u,v) = u\sin v \\ z(u,v) = f(u) \end{cases} \qquad \text{where } 0 \leq u\ ,\ 0 \leq v < 2\pi\ ,$$

by some value of the parameters (u,v).

The geometric meaning of u and v is clear, they are the polar coordinates in the plane perpendicular to the axis of rotation at the vertex, called *parameter-plane*, namely u is the radius vector and v is the polar angle. The mapping of the surface-points onto the parameter-plane indicated by the parametrization is an orthogonal projection.

As a matter of course the present discussion does not apply to a great number of surfaces. An essential requirement is, for instance, that the length of the meridian arc ought to exist. Thus all those functions $f(u)$ which are not rectifiable should be excluded from the class of the generator- functions. In fact, the following definition excludes somewhat more.

Let the generator-function $f(u)$ be an even function with $f(0)=0$, defined in a certain interval $-K \leq u \leq K$, and let $f(u)$ be further

a) bounded

b) continuous

c) differentiable almost everywhere, i.e. the set of points where no derivative exists is at most countable

d) the derivative $f'(u)$ has no discontinuity of the second kind, i.e. the right and left-hand limits of $f'(u)$ exist and are finite everywhere.

The parametrization in (1) induces a metric on the surface:

$$ds^2 = (1+f'^2)\,du^2 + u^2\,dv^2 \tag{2}$$

The differential equation of the geodesic lines with respect to the arc length s of the geodesics can be written in the form (c.f. Struik, p. 134)

$$\frac{d^2v}{ds^2} + \frac{2}{u}\frac{du}{ds}\cdot\frac{dv}{ds} = 0 \tag{3}$$

The equation of geodesics, that is the general solution of (2) found by quadratures has the form

$$v(u) = \pm\int\frac{c}{t\sqrt{t^2-c^2}}\sqrt{1+f'^2(t)}\,dt + c^*$$

where $c \geq 0$ and c^* are the constants of integration. For the sake of brevity $\sqrt{1+f'^2(u)} = \lambda(u)$ will be used hereafter. To avoid ambiguity a definite integral

$$v(u) = \pm\int_c^u\frac{c\,\lambda(t)\,dt}{t\sqrt{t^2-c^2}} + \beta\,,\quad c>0\,,\ 0\leq\beta<\pi$$

replaces the indefinite one where the limits of integration have been chosen conveniently and the corresponding value of c^* has been denoted by β . The integral appearing in the former expression is improper nevertheless exists by virtue of the assumptions concerning $f(u)$, since $\lambda(u)>0$ is bounded and $\int_c^u\frac{c\,dt}{t\sqrt{t^2-c^2}}$ exists being equal to $\cos^{-1}\frac{c}{u}$.

To make the picture complete the meridians with equation

$$v = \pm\frac{\pi}{2} + \beta\,,\qquad 0\leq\beta<\pi$$

are added to the set of geodesics as the ones belonging to $c=0$. The equation of the meridians trivially satisfies (3).

The variation of either constant of integration while the other is being fixed yields a family of geodesics. First of all, fixed c and varying β lead to a family of congruent geodesics on account of the rotational symmetry. As a consequence it is sufficient to study the case when $\beta = 0$:

$$(4) \qquad v_c(u) = \pm \int_c^u \frac{c\,\lambda(t)\,dt}{t\sqrt{t^2 - c^2}} \,, \qquad c > 0$$

$$(4a) \qquad v = \pm \frac{\pi}{2} \,, \qquad c = 0$$

Secondly, varying c yields a family of geodesics each member of which is symmetric (and also perpendicular) to the meridian v = 0, that is to say if (u,v) satisfies (4) so does (u,-v).

The geometric meaning of c is obvious, it is the minimal value of u, it represents the shortest distance between the particular geodesic and the axis of rotation. Because of the inequality $0 \le c \le u$ the existence of an α with $\sin\alpha = \frac{c}{u}$, $0 \le \alpha \le \frac{\pi}{2}$

is guaranteed. A theorem of Clairaut says that α is the actual angle between the geodesic belonging to c and its intersecting meridian at the parameter-value u.

Another form of this theorem is that if α is the angle made by a fixed geodesic and a meridian on a surface of revolution, then the product u sin α is constant along the geodesic, where u is the radius of the intersection point. (C.f. Reference, Clairaut.)

In particular, if the intersecting meridian belongs to the family, i.e. it has the equation $v = \frac{\pi}{2}$, and the corresponding values of α and u are denoted by α_0 and u_0,

$$(5) \qquad \sin\alpha_0 = c/u_0 \,, \qquad 0 < \alpha_0 < \frac{\pi}{2}$$

The length of the arc $c \le t \le u$ on the geodesic line belonging to the parameter-value c is implicitly defined in equations (2) and (3); the explicit form can be obtained by quadratures:

$$(6) \qquad s_c(u) = \int_c^u \frac{t\,\lambda(t)\,dt}{\sqrt{t^2 - c^2}} \,, \qquad c > 0$$

$$(6a) \qquad \sigma(u) = \int_0^u \lambda(t)\,dt \,, \qquad c = 0 .$$

The improper integral appearing in (6) exists for $\lambda(u)$ is bounded and
$$\int_c^u \frac{t\,dt}{\sqrt{t^2-c^2}} = \sqrt{u^2-c^2}\,.$$
Likewise the integral in (6a) exists because the properties prescribed for f(u) make $\lambda(u)$ integrable. Both $s_c(u)$ and $\sigma(u)$ are supposed to be positive.

Let us mention one more formulation of Clairaut's theorem: for all the geodesic triangles one vertex of which coincides with that of the surface of revolution, the "absolute sine theorem" holds, i.e.
$$\phi(\sigma_1) : \phi(\sigma_2) = \sin\alpha_1 : \sin\alpha_2$$
where $\phi(\sigma(u))$ denotes the circumference of a circle of radius $\sigma(u)$, in the present case $\phi(\sigma(u)) = 2\pi u$. (The theorem is absolute in the sense that it holds in the hyperbolic and elliptic as well as in the euclidean geometry. Clairaut's theorem extends it to certain more general spaces with variable curvature.)

Special attention is to be paid to the c - t r i a n g l e s, that is to say the doubly-right-angled triangles on the geodesic line c, which arise in connexion with (5), when $v = \frac{\pi}{2}$. The lengths of the sides of the c-triangle are: $s = s_c(u_0)$, $\sigma = \sigma(u_0)$ and $\sigma(c)$. If $s < \sigma$ for any small c on a surface, then we obtained a counter-example of the axiom of prolongability. For taking the supplementary c-triangle which consists of congruent sides on the continuation of the meridian $v = \frac{\pi}{2}$ and the geodesic line belonging to c beyond the plane of symmetry (= the plane of meridian v = 0), we get $2s < 2\sigma$. In other words, the arcs on the meridian with the surface-vertex as their midpoint are never segments however small their length 2σ is.

The definition of u_0 in (5) according to (4) is

$$\frac{\pi}{2} = \int_c^{u_0} \frac{c\,\lambda(t)\,dt}{t\sqrt{t^2-c^2}}\,, \qquad c > 0. \tag{7}$$

Another form of this expression can be obtained by using the integral mean $\lambda(\xi)$ of the function $\lambda(t)$ in the interval $c < t \le u_0$ and weighed by the function $\frac{c}{t\sqrt{t^2-c^2}}$; namely:
$$\frac{\pi}{2} = \lambda(\xi)\int_c^{u_0}\frac{c\,dt}{t\sqrt{t^2-c^2}} = \lambda(\xi)\left(\frac{\pi}{2} - \arcsin\frac{c}{u_0}\right)$$
which, in turn, by means of (5) gives a formula for α_0 :

$$(8) \qquad \alpha_0 = \frac{\lambda(\xi)-1}{\lambda(\xi)}\,\frac{\pi}{2}\ , \qquad c \le \xi < u_0 .$$

As far as the difference $s-\sigma$ is concerned, (6) and (6a) lead to the answer
$$s-\sigma = \int_c^{u_0}\left(\frac{t}{\sqrt{t^2-c^2}}-1\right)\lambda(t)\,dt - c - \int_0^c(\lambda(t)-1)\,dt .$$
Taking the integral mean value $\lambda(\xi^*)$ of the function $\lambda(t)$ over the interval $c < t \le u_0$ weighed by the function $\frac{t}{\sqrt{t^2-c^2}}-1$,
$$\int_c^{u_0}\left(\frac{t}{\sqrt{t^2-c^2}}-1\right)\lambda(t)\,dt = \lambda(\xi^*)\left(\sqrt{u_0^2-c^2}-(u_0-c)\right), \quad c \le \xi^* < u_0 ,$$
another form of the difference $s-\sigma$ is obtained:
$$s-\sigma = \left[\lambda(\xi^*)-1\right]c - \left(u_0-\sqrt{u_0^2-c^2}\right)\lambda(\xi^*) - \int_0^c(\lambda(t)-1)\,dt .$$
After dividing by $c > 0$ and considering (5):
$$(9) \qquad \frac{s-\sigma}{c} = \left(\lambda(\xi^*)-1\right) - \frac{1-\cos\alpha_0}{\sin\alpha_0}\lambda(\xi^*) - \frac{1}{c}\int_0^c(\lambda(t)-1)\,dt .$$

II. CONICAL SURFACES

The surface of revolution is said to be conical if its generator-function f(u) is not differentiable at u = 0, nevertheless the right-hand limit of the derivate f'(u) exists and positive there.

The simplest example in this category is that of the cone. The meridian, or what is the same, the generator=line of the cone as geodesic is not prolongable through the vertex, as it can be very easily seen. Rolling out the half-cone onto the plane geodesics will be mapped on straight lines, because the mapping is an isometry, so that every c-triangle becomes a euclidean right-triangle. Since its hypotenuse is always longer than any other side, the corresponding σ on the cone is longer than s.

An alternative proof follows from (9). In the case of the cone $f(u)=\gamma|u|$, where $\gamma > 0$, and $\lambda = \sqrt{1+\gamma^2} > 1$, const.. With this substitution (9) becomes
$$\frac{s-\sigma}{c} = (\lambda-1) - \frac{1-\cos\alpha}{\sin\alpha}\lambda - (\lambda-1) = -\frac{1-\cos\alpha}{\sin\alpha}\lambda ,$$
that is a negative value ~~if c is small enough~~. Let us observe that $\alpha_0 = \alpha$ is constant too, namely from (8)
$$\alpha = \frac{\pi}{2}\,\frac{\lambda-1}{\lambda} .$$

Furthermore, if c tends to 0, u_0 tends to 0 at the same rate since their ratio is constant, c.f. (5). -- Conical surfaces have similar properties:

Proposition 1.

Conical surfaces of revolution are not G-spaces.

* * *

First of all, the axiom of prolongability is not satisfied at the vertex. For according to (7) α_0 does not tend to 0, if c tends to 0. Namely if $\lambda' > 1$ is a fixed lower bound of the function $\lambda(t)$, then

$$\alpha_0 = \int_c^{u_0} \frac{c[\lambda(t)-1]\,dt}{t\sqrt{t^2-c^2}} > (\lambda'-1)\left(\frac{\pi}{2}-\alpha_c\right).$$

Here $\frac{\pi}{2}-\alpha_0 > 0$ and does not tend to 0, therefore α_c has a positive lower bound. Consequently $c/u_0 = \sin\alpha_0$ has a positive lower bound that can occur only if u_0 tends to 0 with c. Now, using the result (9) and dividing by $\lambda(\xi)-1$ obtained from (8), we have

$$\frac{s-\sigma}{c[\lambda(\xi)-1]} = \frac{\lambda(\xi^*)-1}{\lambda(\xi)-1} - \frac{\pi}{2}\,\frac{\frac{1-\cos\alpha_0}{\alpha_0^2}}{\frac{\sin\alpha_0}{\alpha_0}}\,\lambda(\xi^*) - \frac{\frac{1}{c}\int_0^c[\lambda(t)-1]\,dt}{\lambda(\xi)-1}, \text{ where } \begin{array}{l} c \le \xi < u_0, \\ c \le \xi^* < u_0. \end{array}$$

If c tends to 0, then u_0, ξ, ξ^* tend to 0, both $\lambda(\xi)-1$ and $\lambda(\xi^*)-1$ tend to the same value $\lambda-1$ different from 0, and their ratio tends to 1. The second term on the right-hand side of the equation tends to $-\frac{\pi}{4}\lambda$, whereas the third tends to 1 again so that

$$\lim_{c\to 0} \frac{s-\sigma}{c[\lambda(\xi)-1]} = -\frac{\pi}{4}\lambda.$$

In other words $s < \sigma$, no matter how small σ is, the axiom of prolongability does not hold at the vertex.

Furthermore, if the vertex of the surface has been deleted, this axiom may be satisfied; nevertheless, the space is not finitely compact any longer, in spite of the second axiom. This remark completes the proof of Proposition 1.

III. ALEKSANDROV - SURFACES

In his book: Die innere Geometrie der konvexen Flächen, Aleksandrov defines surfaces that are built of infinitely many truncated cones in a certain fashion. It has been conjectured that the Aleksandrov-surfaces are never G-spaces because they ought to inherit this property of the cone. This conjecture need not always be true. Nevertheless it will be proved here that with new assumptions the Aleksandrov-surfaces are really not G-spaces because their vertices are exceptional with reference to the fourth axiom. The surprising fact is that although the sequence of the slopes of the truncated cones $\gamma_1, \gamma_2, \dots, \gamma_n, \dots$ tends to 0, (that is, the surface is not conical, on the contrary, has a tangent plane at the vertex), the prolongation of the meridian as geodesic beyond the vertex is still not possible. However, as we shall see, if γ_n tends to 0 so rapidly that γ_{n+1}/γ_n tends to 0 too, or less strongly, γ_{n+1}/γ_n tends to a sufficiently small positive number, the former statement cannot be made, the Aleksandrov-surface may satisfy the axiom of prolongability at its vertex.

In order to prepare the definition of the Aleksandrov-surfaces we have to introduce a couple of sequences and functions.

Let $/c_n/$ be a sequence: $c_0, c_1, c_2, \dots c_n, \dots$, $c_n > 0$, and let $/c_n'/$ denote the sequence of the ratios of two consecutive numbers in $/c_n/$, i.e.: $c_0' = 1$, $c_1' = c_1$, $c_2' = c_2/c_1, \dots, c_n' = c_n/c_{n-1}, \dots$ also let $/c_n''/$ denote the sequence of the ratios of two consecutive numbers in $/c_n'/$, i.e.: $c_0'' = 1$, $c_1'' = c_1'$, $c_2'' = c_2'/c_1', \dots, c_n'' = c_n'/c_{n-1}', \dots$ By assumption sequence $/c_n/$ has the following properties:

A) strictly monotonically decreasing, or what is the same, $0 < c_n' < 1$, $n = \ , 1, 2, \dots$

B) c_n tends to 0 and so strongly that

C) c_n' tends to 0 .

On account of A) a sequence of numbers $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ with $0 < \alpha_n < \frac{\pi}{2}$ can be defined implicitly by $c_n' = \sin \alpha_n$. By means of $/\alpha_n/$ a new sequence $/\lambda_n/$ is introduced as follows:

$$\lambda_n - 1 = \frac{\alpha_n}{\frac{\pi}{2} - \alpha_n}, \quad n = 1, 2, \dots \quad \text{where} \quad \lambda_n > 1 .$$

Finally, $/\lambda_n/$ defines the sequence $/\gamma_n/$ through $\lambda_n = \sqrt{1+\gamma_n^2}$, $\gamma_n > 0$. Obviously

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \lim_{n\to\infty} (\lambda_n - 1) = 0 \quad \text{and} \quad \lim_{n\to\infty} \gamma_n = 0 .$$

The sequences $/c_n/$, $/\lambda_n/$ and $/\gamma_n/$ make it possible to introduce two functions:

$$\lambda(u) = \begin{cases} 1, & \text{if } u = 0 \\ \lambda_n, & \text{if } c_n \le u < c_{n-1} \end{cases} \qquad \gamma(u) = \begin{cases} 0, & \text{if } u = 0 \\ \gamma_n, & \text{if } c_n \le u < c_{n-1} . \end{cases}$$

Again, $\lim_{u\to 0}(\lambda(u)-1)=0$ and $\lim_{u\to 0}\gamma(u)=0$.
Lastly, there exists a continuous function f(u) in $0\leq u\leq c_0$ such that

$$f'(u)\equiv\gamma(u).$$

It is seen that f(u) is continuously differentiable everywhere save for the points $c_0, c_1, c_2, \ldots c_n, \ldots$ There is a choice for f(0), we assume that f(0) = 0. In this way the function f(u) has been defined uniquely, it is a monotonically increasing function, satisfies all the properties of the generator functions specified on page 2,3. Moreover, since $\lim_{u\to 0}\gamma(u)=0$, f(u) is differentiable at u = 0 and f'(0) = 0. Let us observe that having been given the sequence $/c_n/$ according to A), B) and C), there is one and only one function f(u).

The function f(u) generates a surface of revolution in the framework of (1). This surface is being called <u>Aleksandrov-surface</u>. (5), (7), (8) and (9) may be rewritten as follows:

$$\sin\alpha_n = c_n/c_{n-1} \tag{5*}$$

$$\pi/2 = \lambda_n(\pi/2-\alpha_n) \tag{7*}$$

$$\alpha_n = \frac{\lambda_n-1}{\lambda_n}\,\frac{\pi}{2} \tag{8*}$$

$$\frac{s_n-\sigma_n}{c_n} = (\lambda_n-1)-\frac{1-\cos\alpha_n}{\sin\alpha_n}\lambda_n-\frac{1}{c_n}\int_0^{c_n}[\lambda(t)-1]\,dt. \tag{9*}$$

The advantage of the Aleksandrov-surfaces is in the fact that a sequence of c-triangles can be given in such a way that each triangle is entirely on one truncated cone. But on a cone the handling of the geodesic triangles becomes rather easy.

Namely, in (9*) the last term is a convergent series:

$$A_n = \frac{1}{c_n}\int_0^{c_n}[\lambda(t)-1]\,dt = \frac{1}{c_n}\sum_{\nu=n+1}^{\infty}(\lambda_\nu-1)(c_{\nu-1}-c_\nu)$$

and obviously

$$\left(1-\frac{c_{n+1}}{c_n}\right)(\lambda_{n+1}-1) < A_n < \lambda_{n+1}-1 \quad , \quad n=1,2,\ldots$$

holds.

Divided by $\lambda_n - 1$ and passing to the limit, because of C), we have

$$\liminf_{n\to\infty} \frac{\lambda_{n+1}-1}{\lambda_n - 1} \leq \lim_{n\to\infty} \frac{A_n}{\lambda_n - 1} \leq \limsup_{n\to\infty} \frac{\lambda_{n+1}-1}{\lambda_n-1}$$

Consequently, for

$$\frac{\lambda_n}{\lambda_n - 1} = \frac{\pi}{2}\frac{1}{\alpha_n} \qquad \text{(c.f. (8*))}$$

and

$$\lim_{n\to\infty} \frac{\frac{1-\cos\alpha_n}{\alpha_n^2}}{\frac{\sin\alpha_n}{\alpha_n}} = \frac{1}{2}\ ,$$

the estimation

$$\left(1-\frac{\pi}{4}\right) - \cancel{\limsup} \frac{\lambda_{n+1}-1}{\lambda_n - 1} \leq \lim_{n\to\infty} \frac{s_n - \sigma_n}{c_n(\lambda_n - 1)} \leq \left(1-\frac{\pi}{4}\right) - \liminf_{n\to\infty} \frac{\lambda_{n+1}-1}{\lambda_n - 1}$$

holds.

This result of ours makes it clear that the several assumptions for the function f(u) still have not decided the local nature of the geodesic lines of the surface at the vertex inasmuch as the fourth axiom is concerned. We are free to prescribe the behaviour of the sequence $/\frac{\lambda_{n+1}-1}{\lambda_n - 1}/$ or what is the same, of the sequence $/c_n''/$, and by doing so, the behaviour of the sequence $/\frac{\gamma_{n+1}^2}{\gamma_n^2}/$ as well, since these three have the same values as limes superior and limes inferior respectively.

As the fourth assumption concerning $/c_n/$ we add

D$_1$) $/c_n''/$ is convergent and $\lim_{n\to\infty} c_n'' > 1 - \frac{\pi}{4}$.

In this case $\frac{s_n - \sigma_n}{c_n(\lambda_n - 1)}$ tends to a negative number and the surface is not a G-space. The sufficient condition included in D$_1$) can be expressed in terms of the slopes:

Proposition 2.

An Aleksandrov-surface does not satisfy the axiom of prolongability of geodesics at the vertex if the sequence of ratios of slopes of two consecutive truncated cones has a limit inf greater than $\sqrt{1-\frac{\pi}{4}}$:

$$\liminf_{n\to\infty} \gamma_{n+1}/\gamma_n > \sqrt{1-\pi/4}$$

Let us change now assumption D$_1$):

D$_2$) $/c_n''/$ is convergent and tends to 0.

In this case the sequence $\frac{s_n - \sigma_n}{c_n(\lambda_n - 1)}$ has a positive limit. Whether all such sequences have, is an open question. At any rate, the Aleksandrov surfaces cannot be made "too smooth", otherwise they are not obvious counter-examples if at all.

To realize what have been said let $/c_n/$ be a geometrical sequense of the second order with $q = \frac{1}{2}$, i.e.

$$c_0 = 1,\ c_1 = q,\ c_2 = q^3,\ c_3 = q^6, \dots,\ c_n = q^{\frac{n(n+1)}{2}}, \dots$$

Conditions A), B), C) and D_1) are fulfilled. The surface determined by this sequence furnishes Proposition 2. with a concrete example.

If $/c_n/$ is a geometrical sequence of the third order with $q=\frac{1}{2}$, i.e.

$$c_0 = 1,\ c_1 = q,\ c_2 = q^4,\ c_3 = q^{10}, \dots,\ c_n = q^{\frac{n(n+1)(n+2)}{6}}, \dots$$

then conditions A), B), C) and D_2) are satisfied and the corresponding surface is already too smooth.

Naturally $/c_n''/$ need not be convergent at all. In order to illustrate the variety of the possible structures of the Aleksandrov-surfaces which are in a sense only the simplest of all surfaces of revolution, one more example is outlined here. The situation can be made highly complicated by constructing a sequence $/c_n/$ with divergent $/c_n''/$. One instance of this is: $c''_{2n-1} = q_1,\ c''_{2n} = q_2,\ 0 < q_1 < 1-\pi/4 < q_2 < 1,$

$$c'_0 = 1,\quad c'_{2n-1} = q_1 c'_{2n-2},\quad c'_{2n} = q_2 c'_{2n-1}$$

$$c_0 = 1,\quad c_n = c'_n \cdot c_{n-1},\quad n = 1, 2, \dots$$

The present sequence $/c_n/$ defines an Aleksandrov-surface which does not satisfy the axiom of prolongability. Namely the sequence $\left/\frac{s_{2n} - \sigma_{2n}}{c_{2n}(\lambda_{2n} - 1)}\right/$ has a negative limit. It is remarkable, however, that the other sequence $\left/\frac{s_{2n+1} - \sigma_{2n+1}}{c_{2n+1}(\lambda_{2n+1} - 1)}\right/$ tends to a positive number. This fact emphasizes the necessity of investigating every possible sequence $s_{n_i} - \sigma_{n_i}$ if the fulfilment of the axiom of prolongability is to be proved, while on the other hand one negative sequence suffices to refute.

If $0 < q_1 < q_2 < 1-\frac{\pi}{4}$, $/c_n''/$ is divergent again, but all the convergent subsequences have limits less than the critical value $1-\frac{\pi}{4}$, therefore the convergent subsequences of $\left/\frac{s_n - \sigma_n}{c_n(\lambda_n - 1)}\right/$ cannot tend to a negative number. However, it cannot be decided by simple means whether the fourth axiom holds or not.

IV. SMOOTH SURFACES WITHOUT CURVATURE

All the Aleksandrov-survaces we saw had a tangent plane at the vertex, although they were not smooth because in each case the vertex was an accumulation point of surface-points without tangent plane. There are, however, surfaces with continuous tangent plane everywhere in a neighborhood of the vertex, whereas they still do not satisfy the axiom of prolongability. The surface of revolution discussed in the present chapter possesses even a Gaussian curvature everywhere except for the vertex.

Let us consider the function $f(u) = \int_0^u \sqrt{t^2+2t}\, dt$ if $u \geq 0$,
$f(-u) = f(u)$

as generator of a surface. Then

(a) $$\sqrt{1+f'^2(u)} - 1 = \lambda(u) - 1 = u$$

and (7) has the form

(b) $$\alpha_0 = c \cosh^{-1} \frac{u_0}{c} .$$

On the basis of the identity

$$\frac{t}{\sqrt{t^2-c^2}} = \frac{c^2}{t\sqrt{t^2-c^2}} + \frac{\sqrt{t^2-c^2}}{t}$$

and on account of (a)

$$\int_c^{u_0} \frac{t}{\sqrt{t^2-c^2}}[\lambda(t)-1]\,dt = c\int_c^{u_0} \frac{c\,dt}{\sqrt{t^2-c^2}} + \int_c^{u_0} \sqrt{t^2-c^2}\,dt$$

can be written. Carrying out the integrations on the right-hand side and taking (6) into account on the left-hand side:

$$s - \sqrt{u_0^2-c^2} = c^2 \cosh^{-1}\frac{u_0}{c} + \frac{1}{2}\left[u_0 \cdot \sqrt{u_0^2-c^2} - c^2\cosh^{-1}\frac{u_0}{c}\right]$$

or, what is the same,

$$s = \frac{1}{2} c^2 \cosh^{-1}\frac{u_0}{c} + \left(1+\frac{1}{2}u_0\right)\sqrt{u_0^2-c^2}$$

is obtained. On the other hand,

$$\sigma = \int_0^{u_0} \lambda(t)\,dt = \int_0^{u_0}(1+t)\,dt = \left(1+\frac{1}{2}u_0\right)u_0 .$$

Subtracting the latter from the former:

$$s - \sigma = \frac{1}{2} c^2 \cosh^{-1}\frac{u_0}{c} - \left(1+\frac{1}{2}u_0\right)\left(u_0 - \sqrt{u_0^2-c^2}\right) .$$

Making use of (b) and dividing by c :

$$\frac{s-\sigma}{c} = \frac{1}{2}\alpha_0 - \left(1+\frac{1}{2}u_0\right)\frac{1-\cos\alpha_0}{\sin\alpha_0} .$$

Since $\dfrac{1-\cos\alpha_0}{\alpha_0 \sin\alpha_0}$ tends to $\frac{1}{2}$, if α_0 tends to 0,

$$\lim_{c\to 0}\frac{s-\sigma}{c\alpha_0} = 0$$

so that even finer approach is needed. Let us observe that if α_0 is small enough, then

$$\frac{1}{2}\alpha_0 < \frac{1-\cos\alpha_0}{\sin\alpha_0}$$

because the comparison of the series of $\frac{1}{2}\alpha_0 \sin\alpha_0$ and series of $1-\cos\alpha_0$ shows that although the first terms are identical, the second is less in the series of $\frac{1}{2}\alpha_0\sin\alpha_0$. This means that

$$\frac{s-\sigma}{c\alpha_0} \to 0$$

~~but~~ through the negative numbers, the surface of revolution generated by f(u) does not satisfy the axiom of prolongability.

The tangent plane of the surface is continuous because

$$f'(u) = \sqrt{u^2+2u}$$

is continuous. The Gaussian curvature of the surface exists and is continuous everywhere but at the vertex because

$$f''(u) = \frac{u+1}{\sqrt{u^2+2u}} \quad , \quad u \neq 0$$

exists and continuous everywhere but at u = 0.

This example leads us to

<u>Proposition 3</u>.

There are surfaces of revolution having a continuous tangent plane everywhere, also a continuous Gaussian curvature everywhere but at the vertex, which do not satisfy the axiom of prolongability of geodesics and therefore are not G-spaces.

* * *

The smoothness of a surface of revolution cannot be improved further, provided that the vertex remains a point of no prolongation.

In the next chapter it will become clear that a surface of revolution with Gaussian curvature at the vertex is a G-space. As yet it is an open question, however, whether there are surfaces of revolution other than those with curvature at the vertex, which are G-spaces. The conjecture is that there are; those surfaces generated by functions $g(u)$ where $g(u)$ tends to 0 more rapidly than $f(u)$ in the present example when u tends to 0, are G-spaces.

V. SURFACES WITH CURVATURE. THE MAIN THEOREM

To prove a positive statement concerning the possibility of the prolongation of geodesics under certain conditions involves more difficulties. It is to be established that every small enough meridian arc is the shortest arc on the surface between its endpoints. In order to do this in the case of surfaces having a Gaussian curvature at the vertex we mention a few lemmas in advance.

Lemma 1.

On a surface of revolution any segment one endpoint of which coincides with the vertex essentially lies on a meridian.

* * *

This is a simple consequence of the facts that every meridian is a geodesic, and through the vertex in any direction there passes one and only one meridian, one and only one geodesic line.

Lemma 2.

Prolongation of the meridian at the vertex is possible if and only if $\sigma < s$ in the c-triangle for sufficiently small c.

* * *

Clearly the condition is necessary. The outline of the proof of the sufficiency is indicated here.

The geodesic arc belonging to a total polar angle π has the length S, its chord which is a meridian arc has the length Σ. On the same geodesic line the c-triangle can be found with sides s and σ, where the Greek letter stands for the length measured along the meridian again. It is to be shown that whenever $S - \Sigma < 0$, then also $s - \sigma < 0$ on the same geodesic line.

$$S - \Sigma = \int_c^{u_1} \left(\frac{t}{\sqrt{t^2-c^2}} - 1\right)\lambda(t)\,dt + \int_c^{u_2} \left(\frac{t}{\sqrt{t^2-c^2}} - 1\right)\lambda(t)\,dt - 2\int_0^c \lambda(t)\,dt =$$

$$= 2(s-\sigma) + \left\{ \int_{u_0}^{u_2} \left(\frac{t}{\sqrt{t^2-c^2}} - 1\right)\lambda(t)\,dt - \int_{u_1}^{u_0} \left(\frac{t}{\sqrt{t^2-c^2}} - 1\right)\lambda(t)\,dt \right\},$$

where $0 < c < u_1 < u_0 < u_2$,

and $\displaystyle\int_c^{u_0} \frac{c\,\lambda(t)\,dt}{t\sqrt{t^2-c^2}} = \frac{\pi}{2}$, $\displaystyle\int_c^{u_1} \frac{c\,\lambda(t)\,dt}{t\sqrt{t^2-c^2}} + \int_c^{u_2} \frac{c\,\lambda(t)\,dt}{t\sqrt{t^2-c^2}} = \pi$.

But $\displaystyle\int_{u_0}^{u_2} \left(\frac{t}{\sqrt{t^2-c^2}} - 1\right)\lambda(t)\,dt$ happens to be greater than $\displaystyle\int_{u_1}^{u_0} \left(\frac{t}{\sqrt{t^2-c^2}} - 1\right)\lambda(t)\,dt$

therefore $s - \sigma$ must be negative if $S - \Sigma$ is negative.

Lemma 3.

On the sphere u_0 is constant and equal to the radius.

* * *

This trivial statement follows from the definition of u_0 and from the fact that the geodesics of a sphere are circles of the same radius.

The next lemma reveals a basic difference between surfaces with and without curvature in connexion with the loval behaviour of the geodesics.

Lemma 4.

On a surface of revolution $u_0(c)$ has a positive lower bound if the generator function $f(u)$ possesses a second derivative at $u = 0$, furthermore $u_0(c)$ tends to 0 with c if $f''(0)$ does not exist.

* * *

Let us consider the family of geodesics all perpendicular to a fixed meridian. A certain meridian, the one which is perpendicular to the fixed one, belongs to the family. Now Lemma 4 asserts that the members of the family locally intersect the meridian which belongs to the family if and only if the Gaussian curvature of the surface exists at the vertex.

We prove first that $0 < r < u_0(c)$ with a fixed r when $f''(0)$ and thus the Gaussian curvature of the surface at the vertex exist. If $f''(0)$ exists, then the graph of $f(u)$ possesses an osculating circle at $u = 0$. Consequently it is possible to attach a tangent sphere of radius r to the surface in such a way that the sphere is entirely on the inner side of the surface. It suffices to choose r less than the radius of the osculating circle and also less than the radius of that neighborhood of $u = 0$ where $f'(u) \leq f^{*\prime}(u)$, if $f^*(u)$ is the generator-function of the tangent sphere, i.e.

$$f^*(u) = r - \sqrt{r^2 - u^2} .$$

The equation of a surface-geodesic $v = v_c(u)$ represents a curve in the parameter-plane. The equation of a sphere-geodesic $v = v_c^*(u)$ represents an ellipse there. Let that surface-geodesic and that sphere-geodesic correspond to one-another which belong to the same value of c. It will be shown that the curve $v = v_c(u)$ lies entirely outside the ellipse $v = v_c^*(u)$.

Since

$$f'(u) \leq f^{*\prime}(u) , \quad 0 \leq u \leq u_0 ,$$

on account of (4)

$$v_c(u) < v_c^*(u) , \quad c \leq u \leq u_0 .$$

Both $v_c(u)$ and $v_c^*(u)$ are strictly monotonic so that the inverse functions

$$u = u_c(v) \qquad \text{and} \qquad u = u_c^*(v)$$

are single-valued and the reversed inequality

$$u_c(v) > u_c^*(v) , \quad 0 < v \leq \pi/2$$

holds. In particular,

$$u_0 = u_c(\tfrac{\pi}{2}) > u_c^*(\tfrac{\pi}{2}).$$

But according to Lemma 3 $u_c^*(\frac{\pi}{2}) = r$ independently of c therefore $u_0(c) > r$ however small r may be, thus the first part of Lemma 4 has been proved.

Conversely, if $f''(0)$ dees not exist, $u_0(c)$ tends to 0 when c tends to 0. The same construction will not yield a sphere entirely on the inner side of the surface this time, because however small c might be, for a certain neighborhood of $u = 0$ $f'(u) \geqq f^{*\prime}(u)$.
On this basis it can be shown that there is a c such that $u_c(\frac{\pi}{2}) < u_c^*(\frac{\pi}{2}) = r$ no matter how small r has been chosen. Namely, let us suppose that $f'(u) > f^{*\prime}(u)$ holds for $0 < u < k$, $k < r$. Given $\varepsilon > 0$, there corresponds a c with $v_c^*(k) = \frac{\pi}{2} - \varepsilon$. If

$$\varepsilon < \int_{k/2}^{k} \frac{c}{t\sqrt{t^2-c^2}} \left(\sqrt{1+f'^2} - \sqrt{1+f^{*\prime 2}}\right) dt ,$$

then $u_0 = u_c(\frac{\pi}{2}) < k < r$ already where r is arbitrarily small and this completes the proof of Lemma 4.

Classical differential geometry teaches the elementary theorem that if an arc of geodesic can be imbedded in a field of geodesics, then it offers the shortest distance between any two of its points as compared to all other curves in the region for which the field is defined. (C.f. Struik, p. 143)

This theorem combined with Lemma 1 and 2 and also the first statement of Lemma 4 leads to the

M A I N T H E O R E M.

The meridian of a surface of revolution is locally prolongable through the vertex as a geodesic line if the surface has a finite Gaussian curvature there.

* * *

Therefore all the surfaces of revolution generated by twice differentiable functions may be considered as G-spaces in so far as the axiom of prolongability of geodesics is concerned.

Of course, the rotational symmetry is a very strong requirement in addition to the existence of the second derivative. An interesting example of H.G.Helfenstein shows that almost nothing can be saved from the present theorem for more general surfaces. This example constitutes a surface possessing a finite Gaussian curvature at a point, although a geodesic line can be given through that point such that no sub-arc of the geodesic containing that point is a segment.

These facts let us suspect that the G-space-axioms may be too strong, they exclude quite a few spaces which are worth while studying in the framework of a general theory.