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CONNECTEDNESS PROPERTIES OF THE SPACE OF CLOSED SUBSETS OF A TOPOLOGICAL SPACE

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by

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ABSTRACT

Given a topological space X we denote by $\mathfrak{F}(X)$ the class of all closed subsets of X and by $\mathfrak{F}_o(X)$ the class of all nonempty closed subsets of X. We place a topology on these sets called the topology of closed convergence τ_c.

In this thesis, we are mainly concerned with the connectedness properties of the spaces $\mathfrak{F}(X)$ and $\mathfrak{F}_o(X)$. Our work is divided into four chapters. In chapter I, we introduced a notion of convergence for a net of subsets and we proved some properties of the topological limits which are used throughout the thesis. We also defined the topology of closed convergence and proved that $(\mathfrak{F}(X), \tau_c)$ is compact and if X is locally compact then the topology of closed convergence induces the topological convergence of nets of sets, but if X is not locally compact, then there exists no topology on $\mathfrak{F}(X)$ which induces the topological convergence of nets of sets. In chapter II, we proved that $\mathfrak{F}$ and $\mathfrak{F}_o$ are functors from the category of compact Hausdorff spaces to itself which preserve the homotopy relation, we show that $\mathfrak{F}$ is not a functor if X is not compact; however $\mathfrak{F}_c^{\omega}(X)$ is a functor. In chapter III, we studied the connectedness properties of the spaces $\mathfrak{F}(X)$ and $\mathfrak{F}_o(X)$, and we proved that $\mathfrak{F}_o(C) \simeq C$ if C is the Cantor space and $\mathfrak{F}_o(X)$ is contractible if X is a finite connected simplicial complex. We also proved

that if X is connected then $\mathcal{F}_0(X)$ is connected and if X is not compact, $\mathcal{F}(X)$ is connected. In Chapter IV, we studied the inverse limit space $\mathcal{F}_0^\infty(X)$ and we proved that $\mathcal{F}_0(\varprojlim X_\lambda) \simeq \varprojlim \mathcal{F}_0(X_\lambda)$ for a net of compact Hausdorff spaces and $\mathcal{F}_0(X)$ is contractible if X is contractible, we also proved the continuity of the two maps $\phi(X)$: $X \to \mathcal{F}_0(X)$ and $\Psi(X): \mathcal{F}_0^2(X) \to \mathcal{F}_0(X)$, and finally we proved that $\mathcal{F}_0(\mathcal{F}_0^\infty(X)) \simeq \mathcal{F}_0^\infty(X)$.

CONTENTS

INTRODUCTION

Given a topological space X we denote by $\mathfrak{F}(X)$ the class of all closed subsets of X and by $\mathfrak{F}_o(X)$ the class of all nonempty closed subsets of X. We place a topology on these sets called the topology of closed convergence τ_c.

In this thesis, we are mainly concerned with the connectedness properties of the spaces $\mathfrak{F}(X)$ and $\mathfrak{F}_o(X)$. Our work is divided into four chapters. In chapter I, we introduced a notion of convergence for a net of subsets and we proved some properties of the topological limits which are used throughout the thesis. We also defined the topology of closed convergence and proved that $(\mathfrak{F}(X),\tau_c)$ is compact and if X is locally compact then the topology of closed convergence induces the topological convergence of nets of sets, but if X is not locally compact, then there exists no topology on $\mathfrak{F}(X)$ which induces the topological convergence of nets of sets. In chapter II, we proved that $\mathfrak{F}$ and $\mathfrak{F}_o$ are functors from the category of compact Hausdorff spaces to itself which preserve the homotopy relation, we show that $\mathfrak{F}$ is not a functor if X is not compact; however $\mathfrak{F}_c^\omega(X)$ is a functor. In chapter III, we studied the connectedness properties of the space $\mathfrak{F}(X)$ and $\mathfrak{F}_o(X)$, and we proved that $\mathfrak{F}_o(C) \approx C$ if C is the Cantor space and $\mathfrak{F}_o(X)$ is contractible if X is a finite connected simplicial complex. We also proved that if

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In Chapter II, III and IV the theorems were suggested by the supervisor and proved by the author, except the proofs of theorems 3.11 and 3.19 which were given by the supervisor.

When the proof was taken from the literature, an explicit reference was given.

CHAPTER I

THE TOPOLOGY OF CLOSED CONVERGENCE

On the convergence of nets of sets.

The topological convergence of a net of subsets of a topological space X may be defined in the same manner as the topological convergence of sequences of sets.

Definition (1.1). If $\{F_\lambda, \lambda\in\Lambda\}$ is a net of subsets of a topological space X, then the topological limit inferior $\operatorname{Li}_{\lambda\in\Lambda} F_\lambda$ is defined as the set of all $x\in X$ such that for every neighborhood U of x there is an element $\lambda_o\in\Lambda$ such that $F_\lambda\cap U\neq\phi$ for all $\lambda\geq\lambda_o$.

Definition (1.2). The topological limit superior $\operatorname{Ls}_{\lambda\in\Lambda} F_\lambda$ is defined as the set of all $x\in X$ such that for every neighborhood U of x, $F_\lambda\cap U\neq\phi$ for all λ in a cofinal subset of Λ.

Definition (1.3). A net $\{F_\lambda, \lambda\in\Lambda\}$ of subsets of a topological space X is said to be topologically convergent to F, if $\operatorname{Ls}_{\lambda\in\Lambda} F_\lambda = F = \operatorname{Li}_{\lambda\in\Lambda} F_\lambda$. We then write $\operatorname{Lim}_{\lambda\in\Lambda} F_\lambda = F$.

Here we state and prove some properties of topological limit, which will be of importance throughout this work.

1. For every net $\{F_\lambda, \lambda\in\Lambda\}$ the sets $\operatorname{Li}_{\lambda\in\Lambda} F_\lambda$ and $\operatorname{Ls}_{\lambda\in\Lambda} F_\lambda$ are closed and $\operatorname{Li}_{\lambda\in\Lambda} F_\lambda \subset \operatorname{Ls}_{\lambda\in\Lambda} F_\lambda$.

Proof: Since it is always true that $\operatorname{Li}F_\lambda \subset \overline{\operatorname{Li}F_\lambda}$ for any set then it remains to show that $\overline{\operatorname{Li}F_\lambda} \subset \operatorname{Li}F_\lambda$. For this let $x\in \overline{\operatorname{Li}F_\lambda}$, and U an open neighborhood of x. Then $(\operatorname{Li}F_\lambda)\cap U\neq\phi$.

This implies that there exists an element $x_o \in LiF_\lambda \cap U$. Then $x_o \in LiF_\lambda$ and U is a neighborhood of x_o; thus from the definition of LiF_λ there exists $\lambda_o \in \Lambda$ such that $F_\lambda \cap U \neq \phi$ for all $\lambda \geq \lambda_o$. But U was an arbitrary neighborhood of x. Then $x \in LiF_\lambda$ and hence $\overline{LiF_\lambda} \subset LiF_\lambda$. Thus $LiF_\lambda = \overline{LiF_\lambda}$ and hence LiF_λ is a closed subset. Now let $x \in \overline{LsF_\lambda}$, and U an open neighborhood of x. Then $LsF_\lambda \cap U \neq \phi$, so that there exists $x_o \in LsF_\lambda \cap U$. Then $x_o \in LsF_\lambda$ and U is a neighborhood of x_o; thus from the definition of LsF_λ, we have $F_\lambda \cap U \neq \phi$ for a cofinal subset of Λ. But U was an arbitrary neighborhood of x, then $x \in LsF_\lambda$ and this implies $\overline{LsF_\lambda} \subset LsF_\lambda$. Thus $\overline{LsF_\lambda} = LsF_\lambda$ and hence LsF_λ is a closed subset. It remains to prove that $LiF_\lambda \subset LsF_\lambda$, to do so let $x \in LiF_\lambda$, then for every neighborhood U of x there is an element $\lambda_o \in \Lambda$ such that $F_\lambda \cap U \neq \phi$ for all $\lambda \geq \lambda_o$. But the set $\{\lambda \in \Lambda : \lambda \geq \lambda_o\}$ is cofinal in Λ, so $F_\lambda \cap U \neq \phi$ for a cofinal subset in Λ and hence $x \in LsF_\lambda$. Thus $LiF_\lambda \subset LsF_\lambda$.

2. If F is closed and $F_\lambda \subset F$ for all $\lambda \in \Lambda$ then $LsF_\lambda \subset F$.

<u>Proof</u>: Let $x \in LsF_\lambda$; then for every neighborhood U of x, $F_\lambda \cap U \neq \phi$ for a cofinal subset of Λ. But $F_\lambda \subset F$ for all λ so that $F \cap U \neq \phi$ for every neighborhood U of x. Thus $x \in \overline{F} = F$ and hence $LsF_\lambda \subset F$.

3. $x \in LiF_\lambda$ if there is an element $\lambda_o \in \Lambda$ and a net $(x_\lambda)_{\lambda \in \Lambda}$ with $x_\lambda \in F_\lambda$, $\lambda \geq \lambda_o$ such that $\lim_\lambda x_\lambda = x$.

<u>proof</u>: Suppose there is an element $\lambda_o \in \Lambda$ and a net (x_λ) with $x_\lambda \in F_\lambda, \lambda \geq \lambda_o$ such that $\lim x_\lambda = x$. Then for each neighborhood U of x there exists $\lambda_1 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_1$. Then there exists $\lambda_2 \geq \lambda_1, \lambda_0$ such that $F_\lambda \cap U \neq \phi$ for all $\lambda \geq \lambda_2$ and then $x \in \mathrm{Li}F_\lambda$.

4. $x \in \mathrm{Ls}F_\lambda$ if and only if there is a subnet F_{λ_j} and for every j an element $x_{\lambda_j} \in F_{\lambda_j}$ such that $\lim x_{\lambda_j} = x$.

<u>Proof</u>: Let $x \in \mathrm{Ls}F_\lambda$. Then for every neighborhood U of x, $U \cap F_\lambda \neq \phi$ for a cofinal subset of Λ. Now define M as follows:

$M = \{(U,\lambda) : U$ is a neighborhood of x and $F_{\lambda'} \cap U \neq \phi$ for all $\lambda' \geq \lambda\}$. Order M as follows $(U,\lambda) \leq (V,\mu)$ if and only if $V \subset U$ and $\lambda \leq \mu$. Then we have:

(i) $(U,\lambda) \leq (U,\lambda)$ for each neighborhood U of x and each $\lambda \in \Lambda$.

(ii) If $(U,\lambda) \leq (V,\mu)$ and $(V,\mu) \leq (W,\nu)$ then $(U,\lambda) \leq (W,\nu)$

(iii) If (U,λ) and (V,μ) belongs to M, then $(U,\lambda) \leq (W,\nu)$ and $(V,\mu) \leq (W,\nu)$ where $W = U \cap V$ and $\nu \geq \lambda$, $\nu \geq \mu$.

Hence $(M,\leq)$ is a directed set. Now define a map $j : M \to \Lambda$ by $j(U,\lambda) = \lambda$, then we have:

(i) If $(U,\lambda) \leq (V,\mu)$ then $j(U,\lambda) \leq j(V,\mu)$.

(ii) Let $\lambda \in \Lambda$ and U a nhood of x, then there is $\lambda' \in \Lambda$ such that $F_{\lambda'} \cap U \neq \phi$ and $\lambda \leq \lambda'$,
hence $\lambda \leq j(U,\lambda')$. Then for each $\lambda \in \Lambda$ we can find (U,λ') in M such that $\lambda \leq j\ (U,\lambda')$. This defines a subnet $F_{\lambda_{(U,\mu)}}$ of F_λ and $F_{\lambda_{(U,\mu)}} \cap U \neq \phi$ for all (U,μ) in M. Let $(V,\nu) \geq (U,\mu)$, then for each (U,μ) there is an element $x_{\lambda_{(U,\mu)}} \in F_{\lambda_{(U,\mu)}} \cap U$, hence $x_{\lambda_{(V,\nu)}} \in U$ for all $(V,\nu) \geq (U,\mu)$ and since U was an arbitrary neighborhood of x, then $x_{\lambda_{(V,\nu)}} \to x$.

Conversely let F_{λ_j} be a subnet of F_λ and let $x_{\lambda_j} \in F_{\lambda_j}$ for all j, $x_{\lambda_j} \to x$. Let U be a neighborhood of x, then $x_{\lambda_j} \in U$ for all $j \geq j_o$ for some j_o. Thus $F_{\lambda_j} \cap U \neq \phi$ for all $j \geq j_o$, this implies that $F_{\lambda_j} \cap U \neq \phi$ for a cofinal subset of Λ since F_{λ_j} was a subnet of F_λ. Thus $x \in LsF_\lambda$.

5. If a net $(F_\lambda)_{\lambda \in \Lambda}$ is topologically convergent to F and M is a cofinal subset of Λ, then the net $(F_\lambda)_{\lambda \in M}$ is also topologically convergent and $\lim_{\lambda \in M} F_\lambda = F$.

Proof: Let $x \in F$. Then for every neighborhood U of x there is $\lambda_o \in \Lambda$ such that $F_\lambda \cap U \neq \phi$ for all $\lambda \geq \lambda_o$ and $\lambda \in \Lambda$. Since M is cofinal then there is $\lambda_1 \in M$ $\lambda_1 \geq \lambda_o$ such that $F_\lambda \cap U \neq \phi$ for all $\lambda \geq \lambda_1$ and $\lambda \in M$; hence $x \in \underset{\lambda \in M}{Li} F_\lambda$. Thus $F \subset \underset{\lambda \in M}{Li} F_\lambda$. Now let $x \notin F$ then there exists a nhood U_o of x and $\lambda_o \in \Lambda$ such that $F_\lambda \cap U_o = \phi$ for all $\lambda \geq \lambda_o$ and $\lambda \in \Lambda$; and since M is cofinal subset of Λ then $F_\lambda \cap U_o = \phi$ for all $\lambda \geq \lambda_o$ and $\lambda \in M$, and hence $x \notin \underset{\lambda \in M}{Ls} F_\lambda$. Thus $\underset{\lambda \in M}{Ls} F_\lambda \subset F$. From $F \subset \underset{\lambda \in M}{Li} F_\lambda$ and $F \subset \underset{\lambda \in M}{Ls} F_\lambda$ it follows that $(F_\lambda)_{\lambda \in M}$ is topologically convergent to F and $\lim_{\lambda \in M} F_\lambda = F$.

6. $Ls(F_\lambda \cup G_\lambda) = LsF_\lambda \cup LsG_\lambda$.

Proof: Let $x \in Ls(F_\lambda \cup G_\lambda)$, then for every neighborhood U of x, $(F_\lambda \cup G_\lambda) \cap U \neq \phi$ for a cofinal subset of Λ and thus $F_\lambda \cap U \neq \phi$ or $G_\lambda \cap U \neq \phi$ for a cofinal subset of Λ. Hence either $x \in LsF_\lambda$ or $x \in LsG_\lambda$ or both. Thus $x \in LsF_\lambda \cup LsG_\lambda$, hence $Ls(F_\lambda \cup G_\lambda) \subset LsF_\lambda \cup LsG_\lambda$. Conversely let $x \in LsF_\lambda \cup LsG_\lambda$, then $x \in LsF_\lambda$ or $x \in LsG_\lambda$ or both, hence for every neighborhood U of x, either $F_\lambda \cap U \neq \phi$ for a cofinal subset of Λ or $G_\lambda \cap U \neq \phi$ for a cofinal subset of Λ. Thus $(F_\lambda \cap U) \cup (G_\lambda \cap U) \neq \phi$ for a cofinal subset of Λ. Hence $(F_\lambda \cup G_\lambda) \cap U \neq \phi$ for a cofinal subset of Λ, hence $x \in Ls(F_\lambda \cup G_\lambda)$ and then $LsF_\lambda \cup LsG_\lambda \subset Ls(F_\lambda \cup G_\lambda)$ and we conclude that $Ls(F_\lambda \cup G_\lambda) = LsF_\lambda \cup LsG_\lambda$.

7. $Li(F_\lambda \cup G_\lambda) \supset LiF_\lambda \cup LiG_\lambda$.

Proof: If $F_\lambda \subset G_\lambda$ then $LiF_\lambda \subset LiG_\lambda$ since: Let $x \in LiF_\lambda$ then for every neighborhood U of x there exists λ_0 such that $F_\lambda \cap U \neq \phi$ for all $\lambda \geq \lambda_0$. Thus $G_\lambda \cap U \neq \phi$ for all $\lambda \geq \lambda_0$ and hence $x \in LiG_\lambda$. Since $F_\lambda \subset F_\lambda \cup G_\lambda$ then $LiF_\lambda \subset Li(F_\lambda \cup G_\lambda)$ and since $G_\lambda \subset F_\lambda \cup G_\lambda$ then $LiG_\lambda \subset Li(F_\lambda \cup G_\lambda)$. Thus $LiF_\lambda \cup LiG_\lambda \subset Li(F_\lambda \cup G_\lambda)$.

Definition (1.4). A net $(x_\lambda)_{\lambda \in \Lambda}$ in X is an ultranet (universal net) if and only if for each subset E of X, $(x_\lambda)_{\lambda \in \Lambda}$ is either residually in E or residually in X-E.

Theorem (1.1). Every net has a subnet which is an ultranet

Proof: see [7]

Theorem (1.2). If X is a topological space, any net $(F_\lambda)_{\lambda \in \Lambda}$ of subsets of X contains a converging subnet.

Proof: By the above theorem, assume that (F_λ) is an ultranet in 2^X, the set of all subsets of X. Then for every $E \subset 2^X$, (F_λ) is either residually in E or residually in $2^X - E$. Suppose that (F_λ) does not converge. Then $LsF_\lambda \neq LiF_\lambda$. So let $x \in LsF_\lambda - LiF_\lambda$, hence $x \in LsF_\lambda$ and $x \notin LiF_\lambda$. Then there is a neighborhood U of x such that for each $\lambda_0 \in \Lambda$ there exists $\lambda_1, \lambda_2 > \lambda_0$ such that $U \cap F_{\lambda_1} = \phi$ since $x \notin LiF_\lambda$ and $U \cap F_{\lambda_2} \neq \phi$

since $x \in LsF_\lambda$.

Now let $E=\{Y\subset X: Y\cap U=\phi\}\subset 2^X$. Since there exists $\lambda_2\geq\lambda_0$ such that $F_{\lambda_2}\cap U\neq\phi$ then (F_λ) is not residually in E, also since there exists $\lambda_1\geq\lambda_0$ such that $F_{\lambda_1}\cap U=\phi$ then (F_λ) is not residually in 2^X-E, hence (F_λ) is not an ultranet. This contradicts the assumption that (F_λ) is an ultranet. Then $LiF_\lambda=LsF_\lambda$ and (F_λ) must converge.

The topology of closed convergence.

Let (X,τ) be any topological space (no separation axiom being assumed) and let $\mathcal{F}(X)$ be the family of all closed subsets of X (including the empty set ϕ). Consider subsets of $\mathcal{F}(X)$ which are either of the form $[K,\mathcal{G}]$ or of the form $[K,\phi]$.

where $[K,\mathcal{G}]=\{F\in\mathcal{F}(X): F\ K=\phi \text{ and } F\cap G\neq\phi,\ G\in\mathcal{G}\}$.

and where K is a compact subset of X and $\mathcal{G}$ is a finite family of nonempty open subsets of X and $[K,\phi]=\{F\in\mathcal{F}(X): F\cap K=\phi\}$.

Then $[K,\mathcal{G}]\cap[K',\mathcal{G}']=\{F: F\cap K=\phi,\ F\cap K'=\phi,\ F\cap G\neq\phi, G\in\mathcal{G} \text{ and } F\cap G'\neq\phi, G'\in\mathcal{G}'\}$

$$=\{F: F\cap(K\cup K')=\phi \text{ and } F\cap G\neq\phi\ \ G\in\mathcal{G}\cup\mathcal{G}'\}$$

$$=[K\cup K',\mathcal{G}\cup\mathcal{G}']$$

also $[K,\mathcal{G}]\cap[K',\phi]=\{F: F\cap K=\phi,\ F\cap K'=\phi \text{ and } F\cap G\neq\phi\ \ G\in\mathcal{G}\}$

$$=\{F: F\cap(K\cup K')=\phi \text{ and } F\cap G\neq\phi\ \ G\in\mathcal{G}\}$$

$$=[K\cup K',\mathcal{G}].$$

and $[K,\phi]\cap[K',\phi]=\{F: F\cap K=\phi \text{ and } F\cap K'=\phi\}$

$$=\{F: F\cap(K\cup K')=\phi\}$$

$$=[K\cup K',\phi].$$

Then the class of all these sets is a base for a topology which is called the topology τ_c of closed convergence on $\mathcal{F}(X)$.

<u>Theorem (1.3)</u>. $(\mathcal{F}(X),\tau_c)$ is compact in the closed convergence topology.

<u>Proof</u>: Let $(F_\lambda)_{\lambda\in\Lambda}$ be an ultranet of elements of $\mathcal{F}(X)$ and let $\underset{\lambda\in\Lambda}{\mathrm{Li}}F_\lambda = F = \underset{\lambda\in\Lambda}{\mathrm{Ls}}F_\lambda$.

<u>Case 1</u>. If $F\neq\phi$. Let $[K,\mathcal{G}]$ be a basic nhood of F. Then $F\cap G\neq\phi$ for every $G\in\mathcal{G}$ and thus there exists $x\in F\cap G$. Then there exists $\lambda_0\in\Lambda$ such that for all $\lambda\geq\lambda_0$ and all $G\in\mathcal{G}$, $F_\lambda\cap G\neq\phi$. Suppose it were false that there exists λ_1, such that $F_\lambda\cap K=\phi$ for $\lambda\geq\lambda_1$. Then there is a cofinal subset M of Λ such that $F_\lambda\cap K\neq\phi$ for all $\lambda\in M$. Let $x_\lambda\in F_\lambda\cap K$ for all $\lambda\in M$. Since K is compact then there exists a subnet (x_{λ_i}) of $(x_\lambda)_{\lambda\in M}$ which converges to some $y\in K$; then $y\in\underset{\lambda\in\Lambda}{\mathrm{Ls}}F_\lambda = \underset{\lambda\in\Lambda}{\mathrm{Li}}F_\lambda = F$. So $F\cap K\neq\phi$ yields a contradiction since $[K,\mathcal{G}]$ was a basic nhood of F, and we conclude that $F_\lambda\cap K=\phi$ for all $\lambda\geq\lambda_1$, and hence $F_\lambda\in[K,\mathcal{G}]$ for all $\lambda\geq\lambda_2$, where $\lambda_2\geq\lambda_1,\lambda_0$ and then $F_\lambda\to F$ in the closed convergence topology.

<u>Case 2</u>. If $F=\phi$. Then $[K,\phi]$ will be a basic nhood of ϕ where $[K,\phi]=\{F\in\mathcal{F}(X): F\cap K=\phi\}$. Suppose that there is no λ_0 such that $\lambda\geq\lambda_0$ implies $F_\lambda\cap K=\phi$. Then there is a cofinal subset M of Λ such that $F_\lambda\cap K\neq\phi$ for all $\lambda\in M$. Then there exists $x_\lambda\in F_\lambda\cap K$ for all $\lambda\in M$. Since $x_\lambda\in K$ and K is compact then there is a subnet (x_{λ_i}) of $(x_\lambda)_{\lambda\in M}$ such that (x_{λ_i}) converges to a point

$y \in K$. Then $y \in LsF_\lambda$ which is equal to F. Hence $F \neq \phi, (F_\lambda) \to \phi$. Now since every ultranet converges then $(\mathcal{F}(X), \tau_c)$ is compact.

Theorem (1.4). If (X,τ) is locally compact space, then $(\mathcal{F}(X), \tau_c)$ is a compact Hausdorff space.

Proof: By the above theorem $(\mathcal{F}(X), \tau_c)$ is compact then it remains to prove that it is Hausdorff. Let $F_1, F_2 \in \mathcal{F}(X), F_2 \neq \phi$ $F_1 \neq F_2$ and let $x \in F_1 - F_2$. By local compactness there is a compact nhood K of x for which $K \cap F_2 = \phi$ then $F_2 \in [K, \{X\}]$, also $F_1 \in [\phi, \{K^o\}]$ where K^o is the interior of K. So we determined two disjoint nhoods of F_1 and F_2. Now set $F_2 = \phi$. Then $[K, \phi]$ and $[\phi, \{K^o\}]$ are two disjoint nhoods of F_2 and F_1 and hence $(\mathcal{F}(X), \tau_c)$ is Hausdorff.

Theorem (1.5). Let (X,τ) be a locally compact space, then the net $(F_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{F}(X)$ converges to F for the topology τ_c of closed convergence if and only if $LsF_\lambda \subset F \subset LiF_\lambda$. And local compactness is necessary.

Proof: First let $LsF_\lambda \subset F \subset LiF_\lambda$, $F \neq \phi$ and let $[K, \mathcal{G}]$ be a basic nhood of F in $\mathcal{F}(X)$. Suppose that $F_\lambda \cap K \neq \phi$ for every λ belonging to a cofinal subset M of Λ.

Then by property (1.5) $\lim_{\lambda \in M} F_\lambda = F$. So for every $\lambda \in M$ there exists $x_\lambda \in F_\lambda \cap K$. But K is compact then there is a subnet (x_{λ_j}) of (x_λ) such that $x_{\lambda_j} \to x_o$ and $x_o \in K$ hence $x_o \in LsF_\lambda = F$. Then $F \cap K \neq \phi$ which contradicts the fact that $[K, \mathcal{G}]$ was a basic nhood of F. Thus there exists λ_o such that for all $\lambda \geq \lambda_o$, $F_\lambda \cap K = \phi$.

Also suppose that $F_\lambda \cap G=\phi$, $G\in\mathcal{G}$ for every λ belonging to a cofinal subset M of Λ. Then by property (1.5) $\lim_{\lambda\in M} F_\lambda=F$. Since $F_\lambda\cap G=\phi$ for all $\lambda\in M$ then $F_\lambda \subset X-G$ for all $\lambda\in M$ and the set X-G is closed for all $G\in\mathcal{G}$. By property (1-2) $\mathrm{Ls}_{\lambda\in M} F_\lambda \subset X-G$, hence $F\subset X-G$ for all $G\in\mathcal{G}$. So that $F\cap G=\phi$ for all $G\in\mathcal{G}$ yields a contradiction. Hence there exists λ_1 such that $F_\lambda\cap G\neq\phi$ for all $\lambda\geq\lambda_1$ and $G\in\mathcal{G}$, so $F_\lambda\in[K,\mathcal{G}]$ for all $\lambda\geq\lambda_2$ where $\lambda_2\geq\lambda_1,\lambda_0$ and $F_\lambda\to F$ for the topology of closed convergence. Conversely let $(F_\lambda)_{\lambda\in\Lambda}$ be a net of closed nonempty subsets of X and let $(F_\lambda)_{\lambda\in\Lambda}$ converge to $F\neq\phi$ in the topology of closed convergence. Let $x\notin F$. Then by local compactness there exists a compact nhood K of x with $F\cap K=\phi$; then $F\in[K,\{X\}]$ and $[K,\{X\}]$ is a nhood of F. Since $F_\lambda\to F$ then there exists $\lambda_0\in\Lambda$ such that $F_\lambda\in[K,\{X\}]$ for all $\lambda\geq\lambda_0$. This implies that $F_\lambda\cap K=\phi$ for all $\lambda\geq\lambda_0$ hence $x\notin \mathrm{Ls}F_\lambda$ and thus $\mathrm{Ls}F_\lambda\subset F$.

Now let $x\in F$ and U an arbitrary open nhood of x. Then $F\in[\phi,\{U\}]$ and $[\phi,\{U\}]$ is a nhood of F. Again (F_λ) converges to F. Hence there exists $\lambda_0\in\Lambda$ such that $F_\lambda\in[\phi,\{U\}]$ for all $\lambda\geq\lambda_0$, and thus $F_\lambda\cap U\neq\phi$ for all $\lambda\geq\lambda_0$. Hence $x\in \mathrm{Li}F_\lambda$ and $F\subset \mathrm{Li}F_\lambda$ and we conclude that $\mathrm{Ls}F_\lambda\subset F\subset \mathrm{Li}F_\lambda$.

Second let $\mathrm{Ls}F_\lambda=\phi=\mathrm{Li}F_\lambda$ and let $[K,\phi]$ be a basic nhood of $F=\phi$. Suppose that $F_\lambda\cap K\neq\phi$ for a cofinal subset M of Λ. Then there exists $x_\lambda\in F_\lambda\cap K$ for every $\lambda\in M$. Since K is compact then there is a subnet (x_{λ_j}) of (x_λ) which converges to some $x_0 \in K$ and hence $x_0 \in LsF_\lambda$. Contradiction. Hence $F_\lambda \cap K = \phi$ for all $\lambda \geq \lambda_0$ and $F \in [K,\phi]$ for all $\lambda \geq \lambda_0$. hence F_λ converges to $F=\phi$ in the closed convergence. Conversely let F_λ converge to $F=\phi$ and let $[K,\phi]$ be a basic nhood of $F=\phi$. Then there exists $\lambda_0 \in \Lambda$ such that $F_\lambda \in [K,\phi]$ for all $\lambda \geq \lambda_0$, hence $F_\lambda \cap K = \phi$ for all $\lambda \geq \lambda_0$. Now assume that $LsF_\lambda \neq \phi$ and let $x \in LsF_\lambda$ and K be a compact nhood of x. Then there is a cofinal subset M of Λ such that $F_\lambda \cap K \neq \phi$ for all $\lambda \in M$, then $F_\lambda \notin [K,\phi]$ for a cofinal set of indices. This is impossible if F_λ converges to ϕ. Thus $LsF_\lambda = \phi = LiF_\lambda$. For necessity, Watson [14] has shown that if X is a separable metric space, then if X is not locally compact, then $\mathfrak{F}(X)$ is not a topological space, that means if $\mathcal{C} \in \mathfrak{F}(X)$ then $\bar{\mathcal{C}} \neq \mathcal{C}$, where $\bar{\mathcal{C}} = \{F : F = \lim F_\lambda$ where (F_λ) is a net in $\mathcal{C}\}$. And Mrowka [10] generalized this result and he has proved that if X is not locally compact, then there exists no topology in $\mathfrak{F}(X)$ which induces the topological convergence of nets of sets.

To prove this result we need the following two lemmas.

<u>Lemma (1.6.)</u> For every two directed sets D_1 and D_2 there exists a directed set D and functions φ_1 and φ_2 such that φ_i maps D onto D_i and $\varphi_i(n) \leq \varphi_i(n')$ in D_i for every $n \leq n'$ in D.

Proof. Let $D = D_1 x D_2$ with the order $(n_1,n_2) \leq (m_1,m_2)$ if and only if $n_i \leq m_i$ $i = 1,2$ and $\varphi_i(n) = n_i$ for $n = (n_1,n_2) \in D$.

<u>Lemma (1.7)</u>. Let $\{y_k, k \in D_1\}$ be a net of points of a T_2 topological space X. If this net has no limit point and φ is a function

which maps a directed set D onto D_1 in such a way that $\varphi(n) \leq \varphi(n')$ in D_1 for every $n \leq n'$ in D, then the net $\{x_n, n \in D\}$ where $x_n = y_{\varphi(n)}$ also has no limit point.

Proof: See Mrowka [10].

Proof of the theorem. It is known that a convergence of nets of elements of a set X induced by some topology in X satisfies the following condition:

Let D be a directed set and suppose that for every $n \in D$ a directed set E_n is given, and let $E = \prod_{n \in D} E_n$. Suppose that to every pair (n,m), where $n \in D$, $m \in E_n$, an element $x_{n_m} \in X$ is assigned. If, for every $n \in D$, the net $\{x_{n_m}, m \in E_n\}$ is convergent to x_n and the net $\{x_n, n \in D\}$ is convergent to x, then the net $\{y_{(n,f)}, (n,f) \in D \times E\}$ where $y_{(n,f)} = x_{n_{f(n)}}$ is convergent to x (see Kelley [7], p. 69). Now we shall show that if X is not locally compact space, then the topological convergence of nets of closed subsets of X does not satisfy the above condition. Let x_o be a point of X which has no compact nhood and let D_1 be a basis of nhoods of x_o. Let us agree that $U \leq U'$ in D_1 if $U \supset U'$. Then D_1 is a directed set. Let x_1 be an arbitrary point of X which is different from x_o and let $\{y_k, k \in D_1\}$ be a net of elements of X which has no limit point. By lemma 1, there exists a directed set D and functions φ_1 and φ_2 such that φ_i maps D onto D_i and $\varphi_i(n) \leq \varphi_i(n')$ in D_i for every $n \leq n'$ in D. Let us set $U_n = \varphi_1(n)$ and

$x_n = y_{\varphi_2(n)}$ for every $n \in D$. Then $U_n \supset U_{n'}$ for $n \le n'$ in D. By lemma 2, the net $\{x_n, x_n \in D\}$ has no limit point. Since U_n is not compact, then there exists a net $\{x_m^{(n)}, m \in E_n\}$ of elements of U_n which has no limit point. Let us set $F_{n_m} = \{x_1\} \cup \{x_n\} \cup \{x_m^{(n)}\}$ for $n \in D$, $m \in E_n$. The net $\{F_{n_m}, m \in E_n\}$ is topologically convergent to $F_n = \{x_1\} \cup \{x_n\}$ for every $n \in D$ and the net $\{F_n, n \in D\}$ is topologically convergent to $F = \{x_1\}$. Let $E = \prod_{n \in D} E_n$ and let us consider the net $\{F_{(n,f)}, (n,f) \in D \times E\}$, where $F_{(n,f)} = F_{n_{f(n)}}$. If U is an arbitrary nhood of x_o and $U_{n_o} \subset U$ and f_o is an arbitrary element of E, then $F_{(n,f)} \cap U \neq \phi$ for every $(n,f) \geq (n_o, f_o)$ in DxE, hence $x_o \in \underset{(n,f) \in D \times E}{\mathrm{Li}} F_{(n,f)}$ and the net $\{F_{(n,f)}, (n,f) \in D \times E\}$ is not topologically convergent to F.

Definition (1.4). Hausdorff distance: For every two non-empty subsets E and F of the bounded metric space (M, ρ) one defines the Hausdorff distance d(E,F) "with respect to the metric ρ on M" by: $d(E,F) = \inf \{\varepsilon \in (0, \infty] : E \subset B_\varepsilon(F) \text{ and } F \subset B_\varepsilon(E)\}$ where $B_\varepsilon(E) = \{x \in M : \mathrm{dist}\,(x,E) < \varepsilon\}$ denotes the ε-nhood of E. Now we shall state some well known theorems which will be used in the proof of the next theorem.

Theorem (1.8). (Alexandroff) Any locally compact non compact Hausdorff space X can be embedded in a compact space X^* so that $X^* - X$ is a single point.

Proof. [2]

Theorem (1.9). If (M,ρ) is separable and locally compact metric space then the Alexandroff compactification M^* of M is metrizable.

Proof. [2]

Theorem (1.10). Let (M,ρ) be a separable metric space. For every sequence$(F_n)$of subsets of M there exists a converging subsequence with respect to the Hausdorff metric.

Proof. [6]

Theorem (1.11). Let (M,ρ) be a compact metric space, then the set $\mathcal{F}_o(M)$ of all nonempty closed subsets of M together with the Hausdorff distance d is a compact metric space.

Proof. [6]

Theorem (1.12). If (X,ρ) is locally compact, separable metric then $(\mathcal{F}(X),\tau_c)$ is metrizable "by the Hausdorff distance".

Proof. Let $X^*=X\cup\{p\}$ denote the Alexandroff compactification of (X,ρ), then by theorem (1.8) X^* is compact and by theorem (1.9) X^* is metrizable . Let ρ^* denote a metric on X^* and apply theorem (1.11) then $(\mathcal{F}_o(X^*),d)$ is a compact metric space, where d is the Hausdorff distance. To every closed subset F of X we associate the set $F^*=F\cup\{p\}$ which is a closed subset of X^*. In this way we establish a one-to-one correspondence between the set of all closed subsets of X and the set of all closed subsets of X^* which contains $\{p\}$.

Let $\mathcal{C}=\{F^*\in\mathcal{F}(X^*):P\in F^*\}$. If $F_1^*\in\mathcal{F}(X^*)$ and $P\notin F_1^*$ then $F_1^*\notin\mathcal{C}$. Let $\delta=d(F_1^*,P)>0$. Hence if $F_2^*\in B(F_1^*,\frac{\delta}{2})$ then $P\notin F_2^*$ thus $F_2^*\notin\mathcal{C}$, this implies that $\mathcal{F}(X^*)\mid\mathcal{C}$ is open hence $\mathcal{C}$ is closed and thus $\mathcal{F}(X)$ is closed subset of the compact metric space $(\mathcal{F}_o(X^*),d)$. Therefore for the induced topology τ_d, the set $\mathcal{F}(X)$ of closed subsets of X is a compact metric space with the Hausdorff distance. Now given $\varepsilon>0$, let $B(F^*,\varepsilon)\subset[K,\mathcal{G}]^*$. Since K is compact then $K\cap F^*=\phi$ in X^* where F^* is compact as a closed subset of a compact Hausdorff space. Then there exists $\delta_o\neq 0$ such that $\delta_o=\inf_{x\in K} d(x,F^*)$. Now

$[K,\mathcal{G}]$ is a basic open set in $\mathcal{F}(X)$ and $F\in[K,\mathcal{G}]$. For each i let $x_i\in F\cap G_i$, $G_i\in\mathcal{G}$ and let $B(x_i,\delta_i)\subset G_i$ be a ball of centre x_i and radious δ_i in G_i and take $\delta=\min\{\delta_o,\delta_1,\ldots,\delta_n\}$ and consider the ball $B(F^*,\frac{\delta}{2})\cap\mathcal{C}$ in $\mathcal{F}(X^*)$. We want to show that if $G^*\in B(F^*,\frac{\delta}{2})\cap\mathcal{C}$, then

$G\in[K,\mathcal{G}]$. First $G\cap K=\phi$: if not let $x\in G\cap K$; then we have $d(G^*,F^*)\geq d(x,F^*)\geq\delta_o$ which contradicts the fact that $G^*\in B(F^*,\frac{\delta}{12})\cap\mathcal{C}$. Second $G\cap G_i\neq\phi$ for all i: if not let $G\cap G_i=\phi$, so that $d(x_i,G)>\delta_i$. Since $x_i\in F$ for all i, this implies that $d(F_1^*)\geq\delta$ for all i contradicting the fact that $d(F^*,G^*)\leq\frac{\delta}{2}$ Hence $G\in[K,\mathcal{G}]$. Hence the topology τ_c of closed convergence in X is weaker than the induced topology τ_d. Therefore the identity map $i:(\mathcal{F}(X),\tau_d)\to(\mathcal{F}(X),\tau_c)$ is continuous, so that we have a one to one continuous function from a compact space $(\mathcal{F}(X),\tau_d)$ to a Hausdorff space $(\mathcal{F}(X),\tau_c)$. Then they are homeomorphic and hence $\tau_d=\tau_c$. A metric d on $\mathcal{F}(X)$ for the topology τ_c is given by the formula:

$$d(E,F)=\rho^*((E\cup\{P\}),(F\cup\{P\})).$$

and hence $(\mathcal{F}(X),\tau_c)$ is metrizable by the Hausdorff distance.

CHAPTER II

THE FUNCTORS $\mathfrak{F}$ AND $\mathfrak{F}_o$ AND SOME OF THEIR PROPERTIES.

Definition (2.1). Let B and C be categories. A functor F: B→C is an assignment of an object $F(X) \in B$ to each object $X \in B$ and a morphism $F(f) : F(X) \to F(Y)$ in C to each morphism $f: X \to Y$ in B, subject to the following conditions:

(1) preservation of composition. If fog is defined in B, then

$$F(fog) = F(f) \circ F(g).$$

(2) preservation of identities. For each $X \in B$ we have $F(1_X) = 1_{F(X)}$.

Theorem (2.1). Let f: X→Y be a continuous function from a compact Hausdorff space X to a locally compact Hausdorff space Y, then $F: \mathfrak{F}(X) \to \mathfrak{F}(Y)$ is continuous, where $F(C) = f(C)$. Thus F is a functor from the category of compact Hausdorff topological spaces to itself.

Proof. Let (C_n) be a net of closed subsets of X and $C_n \to C_o$ in $\mathfrak{F}(X)$. Let $y \in Ls\, F(C_n)$, and U a compact nhood of y, then there is a subnet $F(C_{n_i})$ and for every i an element $y_{n_i} \in F(C_{n_i})$ such that $y_{n_i} \to y$. Then there exists i_o such that $y_{n_i} \in U$ for all $i \geq i_o$, hence $F(C_{n_i}) \cap U \neq \phi$ for every $i \geq i_o$. Let $x_{n_i} \in C_{n_i}$ such that $f(x_{n_i}) = y_{n_i}$, since X is compact there is a subnet $(x_{n_{i_j}})$ of (x_{n_i}) such that $x_{n_{i_j}} \to x$, $x \in Ls\, C_{n_{i_j}} = C_o$. Hence $f(x_{n_{i_j}}) \to f(x)$ by the continuity of f, then $y_{n_{i_j}} \to f(x)$ and $f(x) \in f(C_o) = F(C_o)$.

By the uniqueness of limit $f(x)=y \in F(C_o)$ and thus $LS\, F(C_n) \subset F(C_o)$. Now let $y \in F(C_o)$, then $y = f(x)$ and $x \in C_o$. Let V be a nhood of $y=f(x)$. Then by the continuity of f there exists a nhood U of x such that $f(U) \subset V$. Since $C_n \to C_o$ and $x \in C_o$ then there is n_o such that $C_n \cap U \neq \phi$ for all $n \geq n_o$ and,

$$f(C_n \cap U) \subset f(C_n) \cap f(U) \subset f(C_n) \cap V = F(C_n) \cap V.$$

Hence $F(C_n) \cap V \neq \phi$ for all $n \geq n_o$, and $y \in Li\, F(C_n)$. Thus $F(C_o) \subset Li F(C_n)$. Hence $Lim\, F(C_n) = F(C_o)$ and F is continuous and this completes the proof.

In the above theorem let $F = \mathfrak{F}(f) : \mathfrak{F}(X) \to \mathfrak{F}(Y)$, then $\mathfrak{F}(f)(C) = f(C)$ and $\mathfrak{F}(fog)(C) = (fog)(C) = f(g(C) = f(\mathfrak{F}(g)(C))$ $= \mathfrak{F}(f)(\mathfrak{F}(g)(C)) = (\mathfrak{F}(f) o \mathfrak{F}(g))(C)$ also $\mathfrak{F}(1_X)(C) = 1_X(C) = C = 1_{\mathfrak{F}(X)}(C)$

Hence by definition (2.1) $\mathfrak{F}$ is a functor.

Since $\mathfrak{F}(f) : \mathfrak{F}(X) \to \mathfrak{F}(Y)$ is continuous and $\mathfrak{F}_o(X) \subset \mathfrak{F}(X)$ then $\mathfrak{F}_o(f) = \mathfrak{F}(f) | \mathfrak{F}_o(X) : \mathfrak{F}_o(X) \to \mathfrak{F}(Y)$ and since $\mathfrak{F}_o(Y) \subset \mathfrak{F}(Y)$ hence $\mathfrak{F}_o(f) : \mathfrak{F}_o(X) \to \mathfrak{F}_o(Y)$ is also continuous and hence $\mathfrak{F}_o$ is a functor.

<u>Theorem (2.2).</u> The functors $\mathfrak{F}$ and $\mathfrak{F}_o$ preserve the homotopy relation.

<u>Proof.</u> Let $h : f \simeq g$ where h is a homotopy between f and g, and f,g are continuous functions from X to Y then $h: X \times I \to Y$ is a continuous function such that $h(x,o) = f(x)$ and $h(x,1) = g(x)$ for all $x \in X$. Define $H : \mathfrak{F}(X) \times I \to \mathfrak{F}(Y)$ by $H(C,t) = h(C,t) = \{h(x,t) : x \in C\}$. Let (C_n) be a net of closed subsets

of X such that $C_n \to C$ in $\mathcal{F}(X)$, and let $t_n \to t$ in I. Then $(C_n, t_n) \to (C,t)$ in $\mathcal{F}(X) \times I$. We want to show that $H(C_n, t_n) \to H(C,t)$ or $h(C_n, t_n) \to h(C,t)$. Let $y = h(x,t) \in h(C,t)$, $x \in C$ and let U be a nhood of y. Then by the continuity of h we have $h(U \times W) \subset U$ for some nhood V of x in X and a nhood W of t in I. Now there exists n_1, such that $C_n \cap V \neq \phi$ for all $n \geq n_1$, and there exists n_2 such that $t_n \in W$ for all $n \geq n_2$. Let $x_n \in C_n \cap V$, then $x_n \in V$; hence $h(x_n, t_n) \in U$. But $h(x_n, t_n) \in h(C_n, t_n)$. Then there exists $n_o = \max(n_1, n_2)$ such that $h(C_n, t_n) \cap U \neq \phi$ for all $n \geq n_o$. Hence $y \in Li\ h(C_n, t_n)$ and thus $h(C,t) \subset Li\ h(C_n, t_n)$. Let $y \in Ls\ h(C_n, t_n)$ and let U be a closed nhood of y.

Then there exists a subnet $h(C_{n_i}, t_{n_i})$ and for every i an element $y_{n_i} = h(x_{n_i}, t_{n_i}) \in h(C_{n_i}, t_{n_i})$ such that $y_{n_i} \to y$; then there exists i_o such that $h(x_{n_i}, t_{n_i}) \in U$ for all $i \geq i_o$. Since X is compact then there exists a subnet $(x_{n_{i_j}})$ of (x_{n_i}) in $(C_{n_{i_j}})$ such that $x_{n_{i_j}} \to x \in Ls\ C_{n_{i_j}} = C$. Also by the compactness of I there exists a subnet $(t_{n_{i_j}})$ of (t_{n_i}) such that $t_{n_{i_j}} \to t$ in I. Then $h(x_{n_{i_j}}, t_{n_{i_j}}) \to h(x,t)$ by the continuity of h. Since U was a closed nhood and $h(x_{n_{i_j}}, t_{n_{i_j}})$ a net in U then $h(x, t) \in U$, also $h(x,t) \in h(C,t)$, $x \in C$ then $h(C, t) \cap U \neq \phi$ for each nhood U of y and hence $y \in \overline{h(C, t)} = h(C, t)$. Thus $Ls\ h(C_n, t_n) \subset h(C,t)$ and $h(C_n, t_n) \to$

$h(C, t)$, hence $H(C_n, t_n) \to H(C, t)$ and H is continuous.

$H(C, o) = h(C, o) = \{ h(x, o) : x \in C \} = \{ f(x), x \in C \} = f(C) = \mathfrak{F}(f)(C)$; $H(C, 1) = h(C, 1) = \{ h(x, 1), x \in C \} = \{ g(x), x \in C \} = g(C) = \mathfrak{F}(g)(C)$ and thus $H: \mathfrak{F}(X) \times I \to \mathfrak{F}(Y)$ is a homotopy between $\mathfrak{F}(f)$ and $\mathfrak{F}(g)$.

Theorem (2.3). If $A \subset X$ is closed then the inclusion map $j: \mathfrak{F}(A) \to \mathfrak{F}(X)$ is an embedding.

Proof. Define j such that $j(F)=F$; $F \in \mathfrak{F}(A)$. Then F is closed in X and j is one to one.

Let (F_λ) be a net of closed subsets of A such that $F_\lambda \to F_o$ in $\mathfrak{F}(A)$ and let $x_o \in F_o$ and U a nhood of x_o in X. Then $U \cap A$ is a nhood of x_o in A and hence there exists a λ_o such that $(U \cap A) \cap F_\lambda \neq \phi$ for all $\lambda \geq \lambda_o$. Hence $U \cap F_\lambda \neq \phi$ for all $\lambda \geq \lambda_o$ and $x_o \in Li\ F_\lambda$ in $\mathfrak{F}(X)$. Hence $F_o \subset Li\ F_\lambda$ in $\mathfrak{F}(X)$.

Let $x_o \in Ls\ F_\lambda$ in $\mathfrak{F}(X)$. Then for all nhoods U of x_o there exists a subnet (F_{λ_i}) of (F_λ) such that $U \cap F_{\lambda_i} \neq \phi$. Since $F_{\lambda_i} \subset A$ for all i, then $U \cap A \neq \phi$ and hence $x_o \in \overline{A} = A$. Then $U \cap A$ is a nhood of x_o in A and $(U \cap A) \cap F_{\lambda_i} \neq \phi$ for all i, hence $x_o \in Ls\ F_{\lambda_i}$ in $\mathfrak{F}(A)$ and then $x_o \in F_o$. Hence $Ls\ F_\lambda \subset F_o$ in $\mathfrak{F}(X)$ and we conclude that $F_\lambda \to F_o$ in $\mathfrak{F}(X)$ and j is continuous.

Now let (F_λ) be a net of closed subsets in $j(\mathfrak{F}(A))$ such that $F_\lambda \to F_o$ in $\mathcal{F}(X)$. Let $x_o \in F_o \subset A$ and let U be a nhood of x_o in A. Then $U = V \cap A$ where V is a nhood of x_o in X and there exists λ_o such that $V \cap F_\lambda \neq \phi$ for all $\lambda \geq \lambda_o$. But $F_\lambda \subset A$ for all λ so that $F_\lambda = F_\lambda \cap A$ for all λ. Hence $V \cap A \cap F_\lambda \neq \phi$ for all $\lambda \geq \lambda_o$ or $U \cap F_\lambda \neq \phi$ for all $\lambda \geq \lambda_o$, hence $x_o \in \text{Li } F_\lambda$ and $F_o \subset \text{Li } F_\lambda$ in $\mathcal{F}(A)$. Now let $x_o \in \text{Ls } F_\lambda$ in $\mathcal{F}(A)$ and let V be a nhood of x_o in X, then $V \cap A = U$ is a nhood of x_o in A, then there exists a subnet (F_{λ_i}) of (F_λ) such that $U \cap F_{\lambda_i} \neq \phi$ for all i, hence $V \cap A \cap F_{\lambda_i} \neq \phi$ for all i and thus $V \cap F_{\lambda_i} \neq \phi$ for all i, then $x_o \in \text{Ls } F_\lambda$ in $\mathcal{F}(X)$, hence $x_o \in F_o$ and thus $\text{Ls } F_\lambda \subset F_o$ in $\mathcal{F}(A)$ and we conclude that $\text{Ls } F_\lambda = F_o = \text{Li } F_\lambda$. Hence $F_\lambda \to F_o$ in $\mathcal{F}(A)$, therefore j is an embedding.

Theorem (2.4). If $U \subset X$ is open then the intersection map $f: \mathcal{F}(X) \to \mathcal{F}(U)$ such that $f(F) = F \cap U$ is continuous.

Proof. This map is well defined, since if F is closed subset of X then $F \cap U$ is closed subset of U with the relative topology. Let (F_λ) be a net of closed subsets of X such that $F_\lambda \to F_o$ in $\mathcal{F}(X)$. Then $f(F_\lambda) = F_\lambda \cap U$ for all λ and $f(F_o) = F_o \cap U$. Now let $x_o \in \text{Ls}(F_\lambda \cap U) \subset U$ since $F_\lambda \cap U \subset U$ for every λ. Then for every nhood V of x_o there is a subnet $(F_{\lambda_i} \cap U)$ of $(F_\lambda \cap U)$ such that $V \cap (F_{\lambda_i} \cap U) \neq \phi$ for all i, hence $(V \cap U) \cap F_{\lambda_i} \neq \phi$ for all i, then $V \cap F_{\lambda_i} \neq \phi$ for all i and $x_o \in \text{Ls } F_\lambda = F_o$. Hence $x_o \in F_o \cap U$ or $\text{Ls }(F_\lambda \cap U) \subset F_o \cap U$. Now let $x_o \in F_o \cap U$ then

$x_o \in F_o = \text{Li } F_\lambda$ and $x_o \in U$, then for every nhood V of x_o there exists λ_o such that

$(F_\lambda \cap V) \cap U \neq \phi$ for all $\lambda \geq \lambda_o$, hence $V \cap (U \cap F_\lambda) \neq \phi$ for all $\lambda \geq \lambda_o$ and $x_o \in \text{Li }(F_\lambda \cap U)$. Then $U \cap F_o \subset \text{Li }(F_\lambda \cap U)$ thus $f(F_\lambda) \to f(F_o)$. If $F_o \cap U = \phi$ and if $(F_\lambda \cap U)$ does not converge to ϕ, then $\text{Ls}(F_\lambda \cap U) \neq \phi$. Then by above $\text{Ls}(F_\lambda \cap U) \subset F_o \cap U$ a contradiction. Also if $\text{Ls }(F_\lambda \cap U) = \phi = \text{Li }(F_\lambda \cap U)$ then $F_\lambda \cap U \to \phi$. Thus if $F_o \cap U \neq \phi$ then by the first part of the proof $F_o \cap U \subset \text{Li}(F_\lambda \cap U)$. Contradiction so $F_o \cap U = \phi$ and in this case also $\text{Lim}(F_\lambda \cap U) = F_o \cap U$.

Hence $f: \mathcal{F}(X) \to \mathcal{F}(U)$ is a continuous function.

<u>Theorem (2.5)</u>. For each open set $U \subseteq R^n$ there is a continuous onto map $f: \mathcal{F}(R^n) \to \mathcal{F}(U)$. Also if $\{U_i\}_{i \in I}$ is a family of disjoint open subsets of R^n then there is a continuous onto map $g: \mathcal{F}(R^n) \to \prod_{i \in I} \mathcal{F}(U_i)$.

<u>Proof</u>. Let $\pi_i \circ g = f_i : \mathcal{F}(R^n) \to \mathcal{F}(U_i)$; $f_i(E) = E \cap U_i$. By theorem (2 . 4) the intersection map $f : \mathcal{F}(R^n) \to \mathcal{F}(U)$ such that $f(E) = E \cap U$ is continuous for every open subset U of R^n. Since for every closed subset F of U there is a closed subset E of R^n such that $E \cap U = F$ then f is an onto map. The projection map $\pi_i : \prod_{i \in I} \mathcal{F}(U_i) \to \mathcal{F}(U_i)$ is always continuous. Hence g is continuous. Let $F_i \in \mathcal{F}(U_i)$ then $V_i = U_i - F_i$ is open in U_i, then $V = \cup_{i \in I} V_i$ is open in R^n and $F = R^n - V \in \mathcal{F}(R^n)$ and $(\pi_i \circ g)(F) = F_i$ since U_i are disjoint, then g is an onto map.

<u>Theorem (2.6)</u>. Any nonempty closed set F of a topological space X is the limit of a

net of finite sets.

Proof. Let $J = \{j : j$ is a finite subset of $F\}$. Define $F_j = j$ and let $x \in F$, since $\{x\}$ is finite then take $j_o = \{x\}$ and order J by inclusion, that is to say $j \geq j'$ if $j' \subset j$. Let U be a nhood of x then $F_j \cap U \neq \phi$ for all $j \geq j_o$, hence $x \in \mathrm{Li}\, F_j$ and thus $F \subset \mathrm{Li}\, F_j$. On the other hand $F_j \subset F$ for all j and F is closed, then by property (1.2) $\mathrm{Ls}\, F_j \subset F$ and thus $\mathrm{Ls}\, F_j \subset F \subset \mathrm{Li}\, F_j$ and hence $F_j \to F$.

Corollary (2.7). The family $\mathfrak{F}_n(X)$ of all finite subsets of a T_1 space X is dense in $\mathfrak{F}_o(X)$.

Proof. Let $F \in \mathfrak{F}_o(X)$ then by the above theorem there exists a net (F_λ) of finite subsets of such that $F_\lambda \to F$, hence $\mathfrak{F}_n(X)$ is dense in $\mathfrak{F}_o(X)$.

Corollary (2.8). If $\mathfrak{F}_c(X) = \{C : C$ compact subset $X\}$ then $\mathfrak{F}_c(X)$ is dense in $\mathfrak{F}(X)$, where X is a Hausdorff space.

Proof. If $F \subset X$ is closed, then by the above theorem $F = \lim F_j$ where F_j is finite. But then F_j is compact, this implies that $F_j \in \mathfrak{F}_c(X)$ and hence $\mathfrak{F}_c(X)$ is dense in $\mathfrak{F}(X)$.

Theorem (2.9). If A, B are closed subsets of X, then the map $f: \mathfrak{F}(A) \times \mathfrak{F}(B) \to \mathfrak{F}(X)$ defined by $f(F, G) = F \cup G$, $F \in \mathfrak{F}(A)$, $G \in \mathfrak{F}(B)$ is continuous.

Proof. Let (F_λ) be a net of closed subsets of A such that $F_\lambda \to F_o$ in $\mathfrak{F}(A)$ and let (G_λ) be a net of closed subsets of B

such that $G_\lambda \to G_o$ in $\mathfrak{F}(B)$. Then $f(F_\lambda, G_\lambda) = F_\lambda \cup G_\lambda$ and $f(F_o, G_o) = F_o \cup G_o$. We want to prove that $F_\lambda \cup G_\lambda \to F_o \cup G_o$.

Let $x \in Ls(F_\lambda \cup G_\lambda)$, then by property (1.6) we have $x \in LsF_\lambda \cup LsG_\lambda = F_o \cup G_o$. Hence $Ls(F_\lambda \cup G_\lambda) \subset F_o \cup G_o$; Let $x \in F_o \cup G_o = LiF_\lambda \cup Li\, G_\lambda \subset Li(F_\lambda \cup G_\lambda)$ by property (1.7). Hence $F_o \cup G_o \subset Li(F_\lambda \cup G_\lambda)$ and then $F_\lambda \cup G_\lambda \to F_o \cup G_o$; thus f is continuous.

This can be generalized to a finite number of closed subsets $A_1, \ldots, A_n \subset X$.

<u>Theorem (2.10).</u> Let $A_1, A_2, \ldots, A_n$ be a finite number of closed subsets of X, then the map $f\colon \prod_{i=1}^{n} \mathfrak{F}(A_i) \to \mathfrak{F}(X)$ defined by $f(F_1, F_2, \ldots, F_n) = F_1 \cup F_2 \cup \ldots \cup F_n$ is continuous.

<u>Proof.</u> The proof is by induction on n. By theorem (2.3) and theorem (2.9) it is true for n=1, 2. Now assume that it is true for n=r, then $f\colon \prod_{i=1}^{r} \mathfrak{F}(A_i) \to \mathfrak{F}(X)$ is continuous and $f\colon \prod_{i=1}^{r} \mathfrak{F}(A_i) \times \mathfrak{F}(A_{r+1}) \to \mathfrak{F}(X)$ is continuous hence $f\colon \prod_{i=1}^{r+1} \mathfrak{F}(A_i) \to \mathfrak{F}(X)$ is continuous, hence it is true for all n, and thus $f\colon \prod_{i=1}^{n} \mathfrak{F}(A_i) \to \mathfrak{F}(X)$ is continuous.

Now we will generalize for a locally finite closed covering $\{A_i\}_{i\in I}$ of X.

<u>Theorem (2.11).</u> Let $\{A_i\}_{i\in I}$ be a locally finite closed covering of X, then the map $f\colon \prod_{i\in I} \mathfrak{F}(A_i) \to \mathfrak{F}(X)$ defined by $f(\{F_i\}) = \bigcup_{i\in I} F_i$ is continuous.

<u>Proof.</u> Let $\{C_{\lambda_i}\}$ be a net of closed subsets in $\prod_{i\in I}\mathcal{F}(A_i)$. Define $f(\{C_{\lambda_i}\}) = \bigcup_{i\in I} C_{\lambda_i}$. This map is well defined since the union of locally finite collection of closed subsets is closed, hence $\bigcup_{i\in I} C_{\lambda_i} \in \mathcal{F}(X)$. Let $\{C_{\lambda_i}\} \to \{C_{o_i}\}$ in $\prod_{i\in I}\mathcal{F}(A_i)$. since $f(\{C_{\lambda_i}\}) = \bigcup_{i\in I} C_{\lambda_i} = C_\lambda$ and $f(\{C_{o_i}\}) = \bigcup_{i\in I} C_{o_i} = C_o$, we want to prove that $C_\lambda \to C_o$ in $\mathcal{F}(X)$. Let $x\in LsC_\lambda$, U a nhood of x meeting only finitely many elements of $\{A_i\}_{i\in I}$. Then $C_\lambda \cap U \neq \phi$ for a cofinal subset of Λ or $(\bigcup_{i\in I} C_{\lambda_i}) \cap U \neq \phi$ for a cofinal subset of Λ. Since $\{A_i\}_{i\in I}$ is locally finite. Since $A_i \cap U = \phi$ except for a finite number, hence $U \cap (\bigcup_{i=1}^n C_{\lambda_i}) \neq \phi$, hence $x \in Ls\bigcup_{i=1}^n C_{\lambda_i}$ $= \bigcup_{i=1}^n LsC_{\lambda_i} = \bigcup_{i=1}^n C_{o_i}$, then $x \in \bigcup_{i\in I} C_{o_i} = C_o$. Thus $LsC_\lambda \subset C_o$.

Let $x \in C_o = \bigcup_{i\in I} C_{o_i}$, then $x \in C_{o_i}$ for some i and $x \in LiC_{\lambda_i}$. Let U be a nhood of x, then there exists λ_o such that $U \cap C_{\lambda_i}$ $\neq \phi$ for all $\lambda_i \geq \lambda_o$,

$$\bigcup_{i\in I}(U \cap C_{\lambda_i}) \neq \phi \text{ for all } \lambda_i \geq \lambda_o$$
$$U \cap (\bigcup_{i\in I} C_{\lambda_i}) \neq \phi \text{ for all } \lambda_i \geq \lambda_o.$$

Hence $x \in Li(\bigcup_{i\in I} C_{\lambda_i}) = LiC_\lambda$.

Then $LsC_\lambda \subset C_o \subset LiC_\lambda$ and hence $C_\lambda \to C_o$ and f is continuous.

<u>Definition (2.2).</u> Let X be a set, and $\mathcal{U} = \{A_i : i\in I\}$ be a family of subsets of X, with each A_i having a topology. Assume that for each $(i, j) \in I \times I$, both

(1) The topologies of A_i and A_j agree on $A_i \cap A_j$.

(2) Either (a) each $A_i \cap A_j$ is open in A_i and in A_j, or

(b) each $A_i \cap A_j$ is closed in A_i and in A_j.

The weak topology in X determined (or induced) by $\mathfrak{U}$ is $\mathcal{T}(\mathfrak{U}) = \{U \subset X$: for all i, $U \cap A_i$ is open in $A_i\}$.

Theorem (2.12). If X is a space with the weak topology induced by $\{A_i, i \in I\}$, then an $f: X \to Y$ is continuous if and only if $f|A_i : A_i \to Y$ is continuous.

Proof. The only proof required is that the continuity of each $f_i = f|A_i$ implies the continuity of f. Let $U \subset Y$ be open; then, $f^{-1}(U) \cap A_i = f_i^{-1}(U)$ is open in A_i for each i, and by definition of the weak topology, $f^{-1}(U)$ is therefore open in X and hence $f:X \to Y$ is continuous.

Theorem (2.13). Let $\mathcal{F}_c(X)$ be the set of compact subsets of a Hausdorff space X with the relative topology (as a subset of $\mathcal{F}(X)$) and let $\mathcal{F}_c^W(X)$ be the set of compact subsets with the weak topology induced by the family of subsets $\mathfrak{U} = \{\mathcal{F}(C) : C \subset X$ and C is compact$\}$. Then the identity map $i: \mathcal{F}_c^W(X) \to \mathcal{F}_c(X)$ is a continuous bijection which is not a homeomorphism, unless X is compact.

Proof. A map $g: \mathcal{F}_c^W(X) \to Z$ is continuous if and only if $g|\mathcal{F}(C): \mathcal{F}(C) \to Z$ is continuous for any compact subset C of X. But if $C \subset X$ is compact then C is closed since X is Hausdorff. Therefore the inclusion map $\mathcal{F}(C) \to \mathcal{F}(X)$ is continuous, and since it takes its image in $\mathcal{F}_c(X)$, we obtain that the identity map $i: \mathcal{F}_c^W(X) \to \mathcal{F}_c(X)$ is continuous and it is clear that this map is surjective and injective, hence i is a continuous bijective map. Now assume that X is locally compact but not compact, and let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X without a limit point. Then we can assume that all the x_λ's are distinct (otherwise, take a subnet with this property). Let $x_o \neq x_\lambda$ for all $\lambda \in \Lambda$. Define $F_\lambda = \{x_o, x_\lambda\}$, and let $\mathcal{C} = \{F_\lambda : \lambda\in\Lambda\} \subset \mathcal{F}_c^W(X)$. Then for all compact subsets $C \subset X$, $\mathcal{C} \cap \mathcal{F}(C)$ is a finite set. Indeed, if $\mathcal{C} \cap \mathcal{F}(C)$ is infinite, let $\{F_{\lambda_i} : i \in I\} = \mathcal{C} \cap \mathcal{F}(C)$. Then $F_{\lambda_i} \subset C$ for all $i \in I$ and this implies that there exists $x_{\lambda_i} \in C$ for all $i \in I$. Since C is compact, the net (x_{λ_i}) has a limit point if I is infinite. But this net has no limit point by assumption. This implies that I is a finite set. But then $\mathcal{C} \cap \mathcal{F}(C)$ is closed for all C compact in X, hence $\mathcal{C}$ is closed in $\mathcal{F}_c^W(X)$.

On the other hand, in $\mathcal{F}(X)$, $F_\lambda \to \{x_o\} \in \mathcal{F}_c(X)$: since $x_o \in F_\lambda$ for all λ then $x_o \in \mathrm{Li}F_\lambda$. Also if $x \in \mathrm{Ls}F_\lambda$, $x \neq x_o$, let U be a compact nhood of x such that $x_o \notin U$. Then there exists a subnet(F_{λ_i}) such that $F_{\lambda_i} \cap U \neq \phi$. This implies that $x_{\lambda_i} \in U$, and hence the subnet(x_{λ_i}) has a limit point in U, which is impossible by assumption. Thus there exist no $x \in \mathrm{Ls}F_\lambda$ such that $x \neq x_o$ hence $\mathrm{Ls}F_\lambda = \{x_o\} = \mathrm{Li}F_\lambda$ and $F_\lambda \to \{x_o\}$. Since $\{x_o\} \notin \mathcal{C}$ then $\mathcal{C}$ is not closed in $\mathcal{F}_c(X)$. Thus the spaces $\mathcal{F}_c^W(X)$ and $\mathcal{F}_c(X)$ are not homeomorphic when X is not compact.

<u>Theorem (2.14).</u> Given a continuous function $f:X \to Y$, where X and Y are Hausdorff spaces,

the induced map $F:\mathcal{F}_c(X) \to \mathcal{F}_c(Y)$ is not continuous in general, but the map $F:\mathcal{F}_c^W(X) \to \mathcal{F}_c^W(Y)$ is continuous.

Proof.

Now if $f:X \to Y$ is continuous and $C \subset X$ is compact, then $f|C : C \to f(C)$ is continuous and this induces a continuous map $F:\mathcal{F}(C) \to \mathcal{F}(f(C))$. Since $C \subset X$ is compact then $f(C) \subset Y$ is compact, then this induces a continuous map $\mathcal{F}(f(C)) \to \mathcal{F}_c^W(Y)$. Since the composition of continuous maps is continuous then $\mathcal{F}_c^W(X) \to \mathcal{F}_c^W(Y)$ is continuous.

Finally, let $X = (0, 1)$, $Y=[0,1]$ and let $f: X \to Y$ be the inclusion map. Then the induced map $F:\mathcal{F}_c(X) \to \mathcal{F}_c(Y)$ is not continuous, since if $F_n=\{\frac{1}{n}, \frac{1}{2}\}$, then $F_n \to \{\frac{1}{2}\}$ in $\mathcal{F}_c(X)$. But $(F(F_n))$ converges to $\{0, \frac{1}{2}\}$ in $\mathcal{F}_c(Y)$. Hence $\operatorname{Lim} F(F_n) \neq F(\operatorname{Lim} F_n)$ and hence F is not continuous. Hence $\mathcal{F}_c^W$ is a functor in general, but $\mathcal{F}_c$ is not a functor when X is not compact.

CHAPTER III

CONNECTEDNESS OF $\mathfrak{F}(X)$ AND $\mathfrak{F}_o(X)$

Definition (3.1). A subset A of X is a deformation retract of X if there is a retraction r: X→A and a homotopy H: X×I→X such that

$$H(x,o) = x \qquad x\in X$$

$$H(x,1) = r(x) \quad , \quad x\in X$$

$$H(a,t) = a \quad , \quad a\in A \quad , \ t\in I \ .$$

Definition (3.2). A space X is contractible if there exists a continuous function H: X×I→X such that, $H(x,o) = i(x) = x$; $H(x,1) = c(x) = x_o$. That is to say the identity function is homotopic to a constant map.

Definition (3.3). Two spaces X and Y are of the same homotopy type if there exist continuous maps (called homotopy equivalences) f: X→Y and g: Y→X such that $gf \simeq i_X$: X→X and $fg \simeq i_Y$: Y→Y.

Theorem (3.1). If X and Y are compact and have the same homotopy type, then $\mathfrak{F}_o(X)$ and $\mathfrak{F}_o(Y)$ have the same homotopy type. If X is not compact, $\mathfrak{F}$ does not preserve the homotopy type in general.

Proof. Since X and Y have the same homotpy type then there exist f: X→Y and g: Y→X such that $gf \simeq i_X$ and $fg \simeq i_Y$. But the functor $\mathfrak{F}_o$ preserve the homotopy relation between compact spaces, hence $\mathfrak{F}_o(gf) \simeq \mathfrak{F}_o(i_X)$

$$\mathfrak{F}_o(g) \circ \mathfrak{F}_o(f) \simeq i_{\mathfrak{F}_o(X)} \quad : \ \mathfrak{F}_o(X) \to \mathfrak{F}_o(X)$$

also $\mathfrak{F}_o(fg) \simeq \mathfrak{F}_o(i_Y)$

$$\mathfrak{F}_o(f) \circ \mathfrak{F}_o(g) \simeq i_{\mathfrak{F}_o(Y)} \quad : \ \mathfrak{F}_o(Y) \to \mathfrak{F}_o(Y)$$

Thus $\mathfrak{F}_o(X)$ and $\mathfrak{F}_o(Y)$ have the same homotopy type. If X is not compact, then let $X = \{0,2\}$ and $Y = [0,1) \cup [2,3)$. Then Y can be contracted to X, hence X and Y are of the same homotopy type. But let $Z = [0,3]$ then Y is an open subset of Z, hence $f: \mathcal{F}_0(Z) \to \mathcal{F}(Y)$ is continuous onto, where $f(F) = F \cap Y$ (Theorem 2.4).

But $\mathcal{F}_0(Z)$ is pathwise connected since Z is contractible Thus $\mathcal{F}(Y)$ is pathwise connected . But $\mathcal{F}(X) = \{\phi, \{0\}, \{2\}, X\}$ has four components, since X is a discrete space and finite. $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ are not of the same homotopy type, otherwise they would have the same number of components. Hence $\mathcal{F}$ does not preserve the homotopy type in general.

Theorem (3.2). If X is compact, then $\mathcal{F}_0(X)$ is a deformation retract of $\mathcal{F}_0(X \times I)$.

Proof. Since X is always a deformation retract of $X \times I$, then, there is a retraction $h: X \times I \to X$ and a homotopy $H: (X \times I) \times I \to X \times I$ such that $H: h \simeq i_{(X \times I)}$. But the functor $\mathcal{F}_0$ preserves the homotopy relation, hence there is a homotopy $\tilde{H}: \mathcal{F}_0(h) \simeq \mathcal{F}_0(i_{(X \times I)}) \overset{i}{=} \mathcal{F}_0(X \times I)$ between $\mathcal{F}_0(h)$ and $i_{\mathcal{F}_0(X \times I)}$ where $i_{\mathcal{F}_0(X \times I)}$ is the identity $i_{\mathcal{F}_0(X \times I)}: \mathcal{F}_0(X \times I) \to \mathcal{F}_0(X \times I)$. And $\mathcal{F}_0(h)$ is a retraction $\mathcal{F}_0(h): \mathcal{F}_0(X \times I) \to \mathcal{F}_0(X)$ since $h: X \times I \to X$ is retraction then h is continuous and $h(x, 0) = x$ for all $x \in X = X \times \{0\}$. Thus $\mathcal{F}_0(h)$ is continuous and $\mathcal{F}_0(h)(C) = h(C, 0) = \{h(x, 0) : x \in C\}$

$\mathcal{F}_0(h)(C): \{x: x \in C\} = C$ for all $C \in f_0(X)$.

Thus $\tilde{H}$ is a deformation retraction. Hence $\mathcal{F}_0(X)$ is a deformation retract of $\mathcal{F}_0(X \times I)$.

Theorem (3.3). If X is compact Hausdorff and contractible, then $\mathcal{F}_0(X)$ is contractible.

Proof. Since X is contractible then it is homotopically equivalent to a one point space $\{x\}$. Hence they are of the same homotopy type. But the functor $\mathcal{F}_o$ preserves the homotopy type relation. Hence $\mathcal{F}_o(X)$ is of the same homotopy type as the space $\mathcal{F}_o(\{x\}) = \{x\}$. Thus $\mathcal{F}_o(X)$ is contractible.

Corollary (3.4). Let $D^n = \{x \in R^n : \|x\| \leq 1\}$, the n-dimensional disk, then $\mathcal{F}_o(D^n)$ is contractible.

Proof. Since D^n is compact contractible then by the above theorem $\mathcal{F}_o(D^n)$ is contractible.

Theorem (3.5). The space $\mathcal{F}(R^n)$ is connected (pathwise connected).

Proof. Since $\mathcal{F}_o(D^n)$ is contractible, then it is connected (pathwise connected). But R^n is homeomorphic to the interior of the disk D^n, hence $R^n \subset D^n$ is an open subset of D^n. Then the intersection map $f: \mathcal{F}_o(D^n) \to \mathcal{F}(R^n)$ is continuous onto. Hence $\mathcal{F}(R^n)$ is connected (pathwise connected).

Theorem (3.6). If X is T_1 a connected space, $\mathcal{F}_o(X)$ is connected, and if X is non compact, $\mathcal{F}(X)$ is also connected.

Proof: Let $X_n \subset \mathcal{F}_o(X)$ be the set of closed subsets containing at most n points. Then $X_n \subset X_{n+1}$ and moreover, $\bigcup_{n=1}^{\infty} X_n$ is dense in $\mathcal{F}_o(X)$ by corollary (2.7).

It remains tobe proved that X_n is connected for all n.

Let $U_n : \prod_{k=1}^{n} \mathcal{F}_o(X) \to \mathcal{F}_o(X)$ be defined by

$$U_n(\{F_k\}) = \bigcup_{k=1}^{n} F_k$$

This map is continuous by theorem (2.10). Moreover, the map $\Phi(X) : X \to \mathcal{F}_0(X)$ given by $\Phi(X)(x) = \{x\}$ is continuous by theorem (4.4).

Thus $F_n = \prod_{k=1}^{n} \Phi(X) : \prod_{k=1}^{n} X \to \prod_{k=1}^{n} \mathcal{F}_0(X)$ where

$$F_n(x_1,\dots,x_n) = (\{x_1\},\dots,\{x_n\}) \text{ is continuous}$$

Then $U_n \circ F_n : \prod_{k=1}^{n} X \to \mathcal{F}_0(X)$ is such that $U_n \circ F_n(\prod_{k=1}^{n} X) = X_n$ also continuous. Since X is connected then $\prod_{k=1}^{n} X$ is connected as a finite product of connected spaces, therefore X_n is connected, as the image of a connected set. Then $\bigcup_{n=1}^{\infty} X_n$ is also connected since $\bigcap_{k=1}^{\infty} X_n \neq \phi$. Hence its closure $\mathcal{F}_0(X)$ is connected. However, $\bigcup_1^{\infty} X_n$ is dense in $\mathcal{F}_0(X)$ and moreover, it is dense in $\mathcal{F}(X)$ if X is not compact (Take an ultranet (x_λ) in X which does not converge in X. Then $(\{x_\lambda\})$ is a net in $\bigcup_{n=1}^{\infty} X_n$ converging to ϕ in $\mathcal{F}(X)$.

<u>Theorem (3.7).</u> Let A and B be closed subsets of X and $X = A \cup B$. If $\mathcal{F}(A)$ and $\mathcal{F}(B)$ are pathwise connected, then so is $\mathcal{F}(X)$.

<u>Proof.</u> The function $f: \mathcal{F}(A) \times \mathcal{F}(B) \to \mathcal{F}(X)$ such that $f(F,G) = F \cup G$ is continuous, since $X = A \cup B$. Then $f(A,B) = X$ and hence f is onto. $\mathcal{F}(A) \times \mathcal{F}(B)$ is pathwise connected since $\mathcal{F}(A)$ and $\mathcal{F}(B)$ are pathwise connected. The continuous image of a pathwise connected space is pathwise connected, thus $\mathcal{F}(X)$ is pathwise connected

<u>Theorem (3.8).</u> Let A and B be closed subsets of a T_1-space X, $X = A \cup B$

and $A \cap B \neq \phi$. Then if $\mathcal{F}_0(A)$ and $\mathcal{F}_0(B)$ are pathwise connected so is $\mathcal{F}_0(X)$.

Proof. Define the map $f: \mathcal{F}_0(A) \times \mathcal{F}_0(B) \to \mathcal{F}_0(X)$ by $f(C,D) = C \cup D$. Then f is continuous. Hence the image of f is pathwise connected.

Now let C be a nonempty closed subset of X then $C \cap A$ is closed subset of A and $C \cap B$ is closed subset of B and hence $(C \cap A) \cup (C \cap B) \in \mathcal{F}_0(X)$. If $C \cap B \neq \phi$ then $C \cap B \in \mathcal{F}_0(B)$ and if $C \cap A \neq \phi$ then $C \cap A \in \mathcal{F}_0(A)$. But if both are not empty then $(C \cap B) \cup (C \cap A) \in \text{Imf}$. Thus $\mathcal{F}_0(X) = \text{Imf} \cup \mathcal{F}_0(B) \cup \mathcal{F}_0(A)$, and all the three are pathwise connected. Since $A \cap B \neq \phi$, then there exists $x \in A \cap B$. Hence $\{x\} \in \mathcal{F}_0(A)$ and $\{x\} \in \mathcal{F}_0(B)$ and also $\{x\} \in \text{Imf}$. Thus $\text{Imf} \cap \mathcal{F}_0(A) \cap \mathcal{F}_0(B) \neq \phi$. Hence $\mathcal{F}_0(X)$ is pathwise connected.

Definition (3.4). A space X is totally disconnected if the components in X are the points. Equivalently then, X is totally disconnected if and only if the only nonempty connected subsets of X are the one point sets.

Definition (3.5). A set A in a space X is perfect in X if and only if A is closed and dense in itself; that is to say, each point of A is an accumulation point of A.

Theorem (3.9). A compact Hausdorff space X is totally disconnected if and only if whenever $x \neq y$ in X, there is an open-closed set in X containing x and not y.

Proof. See [15]

Corollary (3.10). The Cantor set is the only totally disconnected, perfect compact metric space (Up to homeomorphism).

Proof. See [15].

Theorem (3.11). If C is the Cantor space, then $\mathcal{F}_o(C)$ is homeomorphic to C.

Proof.

Claim 1. $\mathcal{F}_o(C)$ is totally disconnected. Let C_1 and C_2 be two distinct nonempty closed sets in C. Let $x \in C_2 \setminus C_1$ and $y \in C_1$. Since C is totally disconnected then there exists an open-closed set $U_y \subset C$ such that $x \in U_y$ and $y \notin U_y$, hence $y \in V_y = C \setminus U_y$. Then V_y is also open-closed. Do this for all $y \in C_1$. Since C_1 is compact then there exists $y_1, y_2, \ldots, y_n$ such that, $C_1 \subset V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n} = V$. Let $U = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_n}$, then U is both open and closed, since all U_{y_i} are open and closed, hence U is compact. Also V is open and closed and hence is compact. Moreover $x \in U$, since $x \in U_{y_i}$ for all i, and $C_1 \subset V$ and $U \cap V = \phi$. Then $C_1 \cap U = \phi$ and $C_1 \cap V \neq \phi$ hence $C_1 \in [U, \{V\}]$. Since $x \in C_2$ and $x \in U$ then $C_2 \cap U \neq \phi$, then $C_2 \notin [U, \{V\}]$. It remains to show that $[U, \{V\}]$ is also closed. But

$$\mathcal{F}_o(C) \setminus [U,\{V\}] = \{F \in \mathcal{F}_o(C) : F \notin [U,\{V\}]\}$$

$$= \{F \in \mathcal{F}_o(C) : F \cap U \neq \phi \text{ or } F \cap V = \phi\}$$

$$= [\phi, \{U\}] \cup [V, \{C\}].$$

and both of these sets are open so that $\mathcal{F}_o(C) \setminus [U,\{V\}]$ is open. Hence $[U,\{V\}]$ is closed in $\mathcal{F}_o(C)$. Thus $\mathcal{F}_o(C)$

is totally disconnected.

Claim 2. $\mathcal{F}_o(C)$ is perfect. Let $F \subset C$ be a closed nonempty set and let $x_o \in F$. Since C is perfect, $x_o \in \overline{C \setminus \{x_o\}} = C$; that is to say, there is a sequence $x_n \in C \cap B(x_o, \frac{1}{n})$ such that $x_n \to x_o$ and $x_n \neq x_o$ for all n. Let $F_n = [F \setminus B(x_o, \frac{1}{n})] \cup \{x_n\}$. Since $x_n \to x_o$, then $F_n \to F$ in $\mathcal{F}_o(C)$. Since $F_n \neq F$ for all n, then $\mathcal{F}_o(C)$ is perfect. But $\mathcal{F}_o(C)$ is compact Hausdorff and metrizable. Thus $\mathcal{F}_o(C)$ is totally disconnected, perfect compact metric space and hence $\mathcal{F}_o(C)$ is homeomorphic to C.

Definition (3.6). A Peano space is a compact, connected, locally connected metric space.

Theorem (3.12). (Hahn and Mazurkiewicz) A Hausdorff space X is a continuous image of the unit interval I if and only if it is a Peano space.

Proof. See [15].

In May 1972 R. Schori and E. West proved the following :

Theorem (3.13). $\mathcal{F}_o(I)$ is homeomorphic to the Hilbert cube Q, where $I = [0,1]$.

Proof. See [13].

And in September 1974 D.W. Curtis and R.M. Schori generalized this result in the following

Theorem (3.14). $\mathcal{F}_o(X) \approx Q$, the Hilbert cube, if and only if X is a nondegenerate Peano space (locally connected metric continuum).

Proof. See [1].

Corollary (3.15). If X is a Peano space, then $\mathcal{F}_o(X)$ is a Peano space.

Proof. This follows by the above theorem since Q is a Peano space.

Now we want to prove that $\mathcal{F}_o(X)$ is contractible for a finite connected simplicial complex. To do so we need the following preliminary. In the following four definitions X is a metric space

Definition (3.7). Let P be the partition $0 = t_o < t_1 < \ldots < t_n = 1$ of $[0,1]$ and $f: [0,1] \to X$ be a continuous map. Then $L_P(f) = d(f(t_o),f(t_1)) + d(f(t_1),f(t_2)) + \ldots + d(f(t_{n-1}),f(t_n))$. and the length of the path $L(f) = \sup\{L_P(f): P$ is a partition of $[0,1]\}$.

Definition (3.8). X is finitely pathwise connected if and only if for all $x, y \in X$ there exists a path of finite length between x and y.

Definition (3.9). X is a space without detours if for all $x \in X$ and for all $\varepsilon > o$ there exists $\delta > o$ such that for all y such that $d(x,y) < \delta$ then x and y can be linked by a path of length less than ε.

Definition (3.10). The inner metric d_i on X is defined as follows, $d_i(x,y) = \inf\{L(f): f$ is a path between x and y$\}$.

Theorem (3.16). d_i is a metric equivalent to the metric d if and only if X is finitely pathwise connected without detours.

Proof. See [12] pp. 119.

Theorem (3.17). The nhoods $B_\varepsilon(x) = \{y: d_i(x,y)<\varepsilon\}$ are all finitely pathwise connected.

Proof. See [12] pp. 121.

Theorem (3.18) If X is compact and $x \neq y$ then there exists a path $f: [0,1]\to X$ such that $f(0) = x$ and $f(1) = y$ and $L(f) = d_i(x,y)$. This path is called the shortest path.

Proof See [12] pp. 141.

Definition (3.11). A simplicial complex K is a subspace of some Euclidean space, consisting of a finite number of Euclidean simplexes such that the intersection of any two is a face of each.

Theorem (3.19). If X is a finite connected simplicial complex, then $\mathfrak{F}_o(X)$ is contractible.

Proof. Since X is a finite simplicial complex then for every two points $x,y \in X$ there exists a path of finite length between x and y. Hence X is finitely pathwise connected. It is clear that X is a space without detours. Thus by the above theorem the inner metric d_i is equivalent to the metric d on X. Hence we can assume that X is endowed with the inner metric. Define

$$H : \mathfrak{F}_o(X) \times [0,M] \to \mathfrak{F}_o(X), \text{ where } M = \text{diameter of } X.$$

by $H(C,t) = C_t = \{y : d(y,C)\leq t\}$.

Let (C_n) be a sequence in $\mathfrak{F}_o(X)$ such that $C_n\to C$ and let (t_n) be a sequence in $[0,M]$ such that $t_n\to t$. Then $H(C_n,t_n) = C_{n_{t_n}}$.

Let $x \in \text{Ls } C_{n_{t_n}}$, then there exists a subsequence $x_i \in C_{n_{i_{t_{n_i}}}}$ such that $x_i \to x$. This implies that $d(x_i, C_{n_i}) \le t_{n_i}$ hence there exists $y_i \in C_{n_i}$ such that $d(x_i, y_i) \le t_{n_i}$. But y_i is a sequence in a compact space, then there exists a subsequence (y_{i_j}) of (y_i) such that $y_{i_j} \to y$ and $y \in \text{Ls } C_{n_{i_j}} = C$. Also there exists corresponding subsequences (x_{i_j}) and $(t_{n_{i_j}})$ of (x_i) and (t_{n_i}) respectively such that $x_{i_j} \to x$ and $t_{n_{i_j}} \to t$. Thus $d(x,y) \le t$ and hence $x \in C_t$. Hence $\text{Ls } C_{n_{t_n}} \subset C_t$.

Now let $x \in C_t$, then $d(x,C) \le t$. Then there exists $y \in C$ such that $d(x,y) \le t$. Take a path of minimal length from x to y, call it f and take $\varepsilon > 0$. Since $y \in C = \text{Li } C_n$, then there exists n_o such that $C_n \cap B_{\varepsilon/4}(y) \ne \phi$ for all $n \ge n_o$. Also there exists n_1 such that for $n \ge n_1$, $|t_n - t| \le \varepsilon/4$ since $t_n \to t$. Let $z_n = f(s)$ such that $d(f(s), x) = \varepsilon/2$ be on the shortest path between x and y. This is possible by the intermediate value theorem. Then $d(z_n, y) \le d(x,y) - \varepsilon/2$. And by the triangle inequality, taking $y_n \in C_n \cap B_{\varepsilon/4}(y)$ for $n \ge n_o$.

$$\begin{aligned} d(z_n,y_n) &\le d(z_n,y) + d(y,y_n) \\ &\le d(x,y) - \varepsilon/2 + \varepsilon/4 \\ &< t - \varepsilon/4 \\ &< t_n \quad \text{if } n \ge n_1 \end{aligned}$$

Hence $z_n \in C_{n_{t_n}}$. By construction $d(z_n, x) = \varepsilon/2 < \varepsilon$, then $z_n \in B_\varepsilon(x)$ for all $n \ge \max\{n_o, n_1\} = n_2$. Thus $C_{n_{t_n}} \cap B_\varepsilon(x) \ne \phi$ for all $n \ge n_2$.

Hence $x \in Li\, C_{n_{t_n}}$ and thus $C_t \subset Li\, C_{n_{t_n}}$. Hence H is a continuous function. And

$$H(C,0) = C_o = \{y: d(y,C) = 0\} = C$$

$$H(C,M) = C_M = \{y: d(y,C) \leq M\} = X$$

Thus $\mathcal{F}_o(X)$ is contractible and hence pathwise connected.

CHAPTER IV

THE SPACE $\mathfrak{F}_0^{\infty}(X)$

Definition (4.1).

Let A be any directed set and suppose X_α is a topological space for each $\alpha\in A$. For each α and β with $\alpha\le\beta$, let $f_{\beta\alpha}:X_\beta\to X_\alpha$ be a continuous map. The collection of spaces X_α and maps $f_{\beta\alpha}$ will be called an inverse limit spectrum, denoted $<X_\alpha; f_{\beta\alpha}>$, provided the following condition is satisfied: if $\alpha\le\beta\le\gamma$, then $f_{\gamma\alpha}= f_{\beta x}\circ f_{\gamma\beta}$.

The inverse limit space of an inverse limit spectrum $<X_\alpha;f_{\beta\alpha}>$ is the set $X_\infty=\varprojlim X_\alpha=\{x\in\Pi X_\alpha \mid \text{ whenever } \alpha\le\beta; x_\alpha= f_{\beta\alpha}(x_\beta)\}$ with the subspace topology.

Theorem (4.1). If $\mathcal{U}=\{U_i\}_{i\in I}$ is a base of open sets of a locally compact Hausdorff space X then $\mathfrak{F}(X)\approx\varprojlim\mathfrak{F}(U_i)$.

Proof. Define an order on I by $i\ge j$ if and only if $U_i\supset U_j$, then the inverse limit space $\varprojlim\mathfrak{F}(U_i)$ can be described as follows.

$\varprojlim\mathfrak{F}(U_i)=\{\{C_i\}$: C_i is closed subset of U_i and whenever

$$i\ge j \qquad f_{ij}(C_i)= C_i\cap U_j = C_j\}.$$

Define the map $\varphi(c):\mathfrak{F}(X)\to\Pi\mathfrak{F}(U_i)$ as follow $\varphi_i(C)=\{C\cap U_i\}=\{C_i\}$.

This map is well defined, since if $i\ge j$ then $U_j\subset U_i$, $C_i=U_i\cap C$ is a closed subset of U_i and $C_j= U_j\cap C =U_j\cap U_i\cap C =U_j\cap C_i=f_{ij}(C_i)$.

Hence $\{C_i\}\in\varprojlim\mathfrak{F}(U_i)$ and φ is well defined function from $\mathfrak{F}(X)$ into $\varprojlim\mathfrak{F}(U_i)$.

φ is a one to one function: Since if $C\ne D$ let $x\in C$ and $x\notin D$.

Take U_i such that $x\epsilon U_i$. Then $U_i \cap C \neq U_i \cap D$ or $\varphi(C) \neq \varphi(D)$.

φ is an onto function : Let $C_i \subset U_i$ be a closed subset of U_i for every i such that if $U_j \subset U_i$, $C_i \cap U_j = C_j$ and define $C = \bigcup_{i\epsilon I} C_i$. Then C is closed in X: Let $x\epsilon\overline{C}$ and take k such that $x\epsilon U_k$. Then for every nhood V of x and $V\subset U_k$ we have $V\cap C\neq\phi$. But if $V\subset U_k$ then $V\cap U_k \cap C\neq\phi$ implies $V\cap C_k \neq \phi$. Then $x\epsilon\overline{C}_k$ in U_k. Hence $x\epsilon C_k$ since C_k is closed in U_k and $x\epsilon C = \bigcup_{i\epsilon I} C_i$. Hence $\overline{C}\subset C$ and thus C is closed subset of X.

$C\cap U_i = C_i$. If $C = \bigcup_{j\epsilon I} C_j$, then $C\cap U_i = \bigcup_{j\epsilon I} C_j \cap U_i$,

since $C_i \cap U_i = C_i$ then $C\cap U_i \supset C_i$.

Now let $x\epsilon C_j \cap U_i$ and $i\neq j$. Since $C_j \subset U_j$ then $U_i \cap U_j \supset U_i \cap C_j$ then $x\epsilon U_i \cap C_j \subset U_i \cap U_j$. But $\{U_i\}_{i\epsilon I}$ is a base for the topology of X so that there exists $k\epsilon I$ such that $x\epsilon U_k \subset U_i \cap U_j$.

Then $x\epsilon C_j \cap U_i \cap U_k = C_k \cap U_i = C_k$, since $C_k \subset U_k \subset U_i$.

Thus $x\epsilon C_k$. But $C_k = C_i \cap U_k$. This implies that $x\epsilon C_i$.

Hence $C_j \cap U_i \subset C_i$ for all j. Thus $\bigcup_{j\epsilon I} C_j \cap U_i \subset C_i$

Hence $C\cap U_i \subset C_i$, and we conclude that $C\cap U_i = C_i$.

Then φ is an onto function.

φ is continuous. Since $f: \mathcal{F}(X) \to \mathcal{F}(U_i)$ is continuous for every i, by theorem(2.4)then $\varphi: \mathcal{F}(X) \to \prod_{i\epsilon I} \mathcal{F}(U_i)$ is continuous. Since $\mathcal{F}(U_i)$ is a compact Hausdorff space for every i, then $\varprojlim \mathcal{F}(U_i)$ is compact Hausdorff by theorem 29.11 [15]. Since $\mathcal{F}(X)$ is compact and $\varprojlim \mathcal{F}(U_i)$ is Hausdorff space as a subspace of the Hausdorff space $\prod_{i\epsilon I} \mathcal{F}(U_i)$. Then by theorem (17.14) [15] is a homeomorphism.

Thus $\mathfrak{F}(X) \approx \varprojlim \mathfrak{F}(U_i)$.

Theorem (4.2). If $\ldots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \to \ldots \to A_1 \xrightarrow{\alpha_1} A_o$ is a sequence of compact Hausdorff spaces and continuous maps, and if $X = \varprojlim A_n$, then $\mathfrak{F}_o(X) \approx \varprojlim \mathfrak{F}_o(A_n)$.

proof. Let $Y = \varprojlim \mathfrak{F}_o(A_n) = \{\{C_n\} : C_n \in \mathfrak{F}_o(A_n)$ and $\alpha_n(C_n) = C_{n-1}$ for all $n\}$.

Let $\gamma : \mathfrak{F}_o(X) \to Y$ be the map $\gamma(C) = \{\zeta_n(C)\}$, where $\zeta_n : X \to A_n$ is the restriction of the projection map $\Pi A_n \to A_n$.

Claim 1. γ is a surjection: Let $\{C_n\}_{n=0,1,\ldots} \in Y$. Then $C_n \subset A_n$ is a closed subset of A_n and $\alpha_n(C_n) = C_{n-1}$.

Consider the sequence $\ldots \to C_n \xrightarrow{\alpha_n} C_{n-1} \to \ldots \to C_1 \xrightarrow{\alpha_1} C_o$.

Let $C = \varprojlim C_n = \{\{x_n\} : x_n \in C_n$ and $\alpha_n(x_n) = x_{n-1}$ for all $n\}$ and let $\{x_n\} \in C$. Then $x_n \in C_n$ and $\alpha_n(x_n) = x_{n-1}$ for all n. But $C_n \subset A_n$. Then $x_n \in A_n$ and $\alpha_n(x_n) = x_{n-1}$ for all n; then $\{x_n\} \in X$ and hence $C \subset X$. But C is compact Hausdorff and nonempty since each C_n is. Hence C is closed in X and hence $C \in \mathfrak{F}_o(X)$. Let $x_n \in C_n$ and let $x_{n-1} = \alpha_n(x_n)$, $x_{n-2} = \alpha_{n-1}(x_{n-1}), \ldots$ and choose $x_{n+1} \in \alpha_{n+1}^{-1}(x_n)$, $x_{n+2} \in \alpha_{n+2}^{-1}(x_{n+1}), \ldots$ Then $\{x_k\} \in C \subset X$ and $\zeta_n(\{x_k\}) = x_n$ and hence the map $\zeta_n : C \to C_n$ is surjective and thus $\zeta_n(C) = C_n$ for all n. Hence $\gamma(C) = \{\zeta_n(C)\} = \{C_n\}$ and γ is a surjective map.

Claim 2. γ is injective. To prove this we need the following:

Lemma (4.3). If $C \subset X$ is closed then $C = \varprojlim \zeta_n(C)$.

Proof. Let $C_n=\zeta_n(C)$. Then $\alpha_n(C_n)=\alpha_n(\zeta_n(C))=\zeta_{n-1}(C)=C_{n-1}$

Let $x\in\varprojlim C_n\subset X$. Then $x=\{a_n\}$ where $a_n\in C_n$ and $\alpha_n(a_n)=a_{n-1}$. Then $a_n\in C_n=\zeta_n(C)$ hence $a_n=\zeta_n(c_n)$ for some $c_n\in C$. Since C is compact and $c_n\in C$ for all n, then there exists $c\in C$, an accumulation point of the sequence, and hence also a subsequence (c_{n_i}) of (c_n) converging to c.

Fix n_o, then $\zeta_{n_o}(c_{n_o})=a_{n_o}$.

Let i be such that $n_i>n_o$. Then

$$\alpha_{n_o+1}\circ\alpha_{n_1+2}\circ\cdots\circ\alpha_{n_i-1}\circ\zeta_{n_i}(c_{n_i})=\alpha_{n_o+1}\circ\cdots\alpha_{n_i}(a_{n_i})=a_{n_o}.$$

Then $\zeta_{n_o}(c_{n_i})=a_{n_o}$ for all $n_i>n_o$ since $\zeta_{n_o}=\alpha_{n_o+1}\circ\alpha_{n_o+2}\circ\alpha_{n_i-1}\circ\zeta_{n_i}$.

Then $\lim_{i\to\infty}\zeta_{n_o}(c_{n_i})=a_{n_o}$. Thus $\zeta_{n_o}(c)=a_{n_o}$ for all n_o, and hence $C=\{a_n\}=x$. Hence $\varprojlim C_n\subset C$.

On the other hand if $x=\{a_n\}\in C$, then $a_n=\zeta_n(x)\in\zeta_n(C)$ for all n, $a_n\in C_n$ and hence $x\in\varprojlim C_n$. Thus $C\subset\varprojlim C_n=\varprojlim\zeta_n(C)$. And we conclude that $C=\varprojlim\zeta_n(C)$. To complete the proof of the above theorem, suppose that $\gamma(C_1)=\gamma(C_2)$. Then $\zeta_n(C_1)=\zeta_n(C_2)$ for all n. By the lemma $C_1=\varprojlim\zeta_n(C_1)$, and $C_2=\varprojlim\zeta_n(C_2)$. Then if $\zeta_n(C_1)=\zeta_n(C_1)$ for all n, it is clear that $C_1=C_2$ and hence γ is injective.

Claim 3. γ is continuous. But γ is continuous if and only if the composition with the projections $\mathcal{F}_o(X)\overset{\gamma}{\to}Y\overset{p_n}{\to}\mathcal{F}_o(A_n)$ is continuous for all n. But this composition $p_n\circ\gamma$ is just the map $\mathcal{F}_o(\zeta_n)$, which is already known to be continuous, (cf theorem 2.1). Thus γ is a continuous bijection. Hence it is a homeomorphism since $\mathcal{F}_o(X)$ is compact and Y is Hausdorff.

<u>Theorem (4.4)</u>. If X is Hausdorff then the map $\Phi(X):X\to\mathfrak{F}_o(X)$ such that $\Phi(x)=\{x\}$ is continuous.

<u>Proof</u>. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net of elements of X such that $x_\lambda\to x$ in X. Let $y\in Ls\{x_\lambda\}$ and U a nhood of y, there exists a subnet $\{x_{\lambda_i}\}$ of $\{x_\lambda\}$ and for every i an element $x_{\lambda_i}\in\{x_{\lambda_i}\}$ such that $x_{\lambda_i}\to y$. By uniqueness of limit $y=x\in\{x\}$.

Hence $Ls\{x_\lambda\}\subset\{x\}$.

Since $x_\lambda\to x$, then for every nhood U of x there exists λ_o such that $x_\lambda\in U$ for all $\lambda\geq\lambda_o$. Hence $\{x_\lambda\}\cap U\neq\phi$ for all $\lambda\geq\lambda_o$ and thus $x\in Li\{x_\lambda\}$ or $\{x\}\subset Li\{x_\lambda\}$. Hence $Ls\{x_\lambda\}\subset\{x\}\subset Li\{x_\lambda\}$ and hence $\Phi(x_\lambda)\to\Phi(x)$, and $\Phi(x)$ is a continuous function.

<u>Theorem (4.5)</u>. If X is locally compact and compact then there is a continuous function $\Psi(X):\mathfrak{F}_o^2(X)\to\mathfrak{F}_o(X)$ defined as $\Psi(X)\ (\mathcal{C})=\cup\{C:C\in\mathcal{C}\}$.

<u>Proof</u>. Let $\mathcal{C}\in\mathfrak{F}_o^2(X)$, then $\mathcal{C}$ is closed subset of $\mathfrak{F}_o(X)$, that is to say $\mathcal{C}$ is a family of closed subsets of X, such that if $F_\lambda\in\mathcal{C}$ and $F_\lambda\to F_o$ then $F_o\in\mathcal{C}$.

Define $\Psi(\mathcal{C})=\bigcup_{C\in\mathcal{C}} C$.

<u>Claim 1</u>. $\Psi(X)$ is well defined. Let $x\in\overline{\bigcup_{C\in\mathcal{C}} C}$. Then there exists a net $x_\lambda\in\bigcup_{C\in\mathcal{C}} C$ such that $x_\lambda\to x$.

For every λ choose $C_\lambda\in\mathcal{C}$ such that $x_\lambda\in C_\lambda$. Then $\{C_\lambda\}$ is a net in $\mathfrak{F}_o(X)$. Since $\mathfrak{F}_o(X)$ is compact then there exists a convergent subnet (C_{λ_i}) such that $C_{\lambda_i}\to C_o$. Then $C_o\in\mathcal{C}$ since $\mathcal{C}$ is closed subset of $\mathfrak{F}_o(X)$.

Since $x_{\lambda_i} \to x$ then (x_{λ_i}) is eventually in every nhood of x. Hence any nhood of x meets C_{λ_i} thus $x \in LsC_{\lambda_i}$. But $C_{\lambda_i} \to C_o$. Then $LsC_{\lambda_i} = C_{o_i}$, hence $x \in C_o \in \mathcal{C}$. Thus $x \in \bigcup_{C \in \mathcal{C}} C$. Hence $\bigcup_{C \in \mathcal{C}} C$ is a closed subset of X and then $\bigcup_{C \in \mathcal{C}} C \in \mathcal{F}_o(X)$ and $\Psi(X)$ is well defined.

<u>Claim 2</u>. $\Psi(X)$ is continuous. Let $\mathcal{C}_\lambda$ be a net in $\mathcal{F}_o^2(X)$ such that $\mathcal{C}_\lambda \to \mathcal{C}$ in $\mathcal{F}_o^2(X)$. We want to prove that $\Psi(X)(\mathcal{C}_\lambda) \to \Psi(X)(\mathcal{C})$. Let $x \in \bigcup_{C \in \mathcal{C}} C$. $x \in C$ for some $C \in \mathcal{C}$. Let U be a nhood of x then $C \cap U \neq \phi$. Hence $[\phi,\{U\}] = \{F : F \cap U \neq \phi\}$ is a nhood of C. Since $C \in \mathcal{C}$ then $C \in Li\mathcal{C}_\lambda$ because $\mathcal{C}_\lambda \to \mathcal{C}$. Thus there exists λ_o such that $[\phi,\{U\}] \cap \mathcal{C}_\lambda \neq \phi$ for all $\lambda \geq \lambda_o$, this implies that there exists $C_\lambda \in \mathcal{C}_\lambda$ such that $C_\lambda \in [\phi,\{U\}]$ for all $\lambda \geq \lambda_o$, i.e. $C_\lambda \cap U \neq \phi$ for all $\lambda \geq \lambda_o$. So $(\bigcup_{C_\lambda \in \mathcal{C}_\lambda} C_\lambda) \cap U \neq \phi$ for all $\lambda \geq \lambda_o$. Thus $x \in Li(\bigcup_{C_\lambda \in \mathcal{C}_\lambda} C_\lambda)$ and hence $\bigcup_{C \in \mathcal{C}} C \subset Li(\bigcup_{C_\lambda \in \mathcal{C}_\lambda} C_\lambda)$. Let $x \in Ls(\bigcup_{C_\lambda \in \mathcal{C}_\lambda} C_\lambda)$ and let U be a compact nhood of x. This implies that there exists a subnet $\bigcup_{C_{\lambda_i} \in \mathcal{C}_{\lambda_i}} C_{\lambda_i}$ such that $\bigcup_{C_{\lambda_i} \in \mathcal{C}_i} C_{\lambda_i} \cap U \neq \phi$ for all λ_i. Then for each λ_i there exists $C_{\lambda_i} \in \mathcal{C}_{\lambda_i}$ such that $C_{\lambda_i} \cap U \neq \phi$. Hence there exists $x_{\lambda_i} \in C_{\lambda_i} \cap U$. Then (x_{λ_i}) is a net in U. But U is a compact nhood. Then there exists a convergent subnet $x_{\lambda_{i_j}} \in C_{\lambda_{i_j}} \cap U$ such that $x_{\lambda_{i_j}} \to x$ in U. Now $(C_{\lambda_{i_j}})$ is a net in $\mathcal{F}_o(X)$ which is compact.

Then there exists a subnet$(C_{\lambda_{i_{j_k}}})$ of$(C_{\lambda_{i_j}})$ such that $C_{\lambda_{i_{j_k}}} \to C_o$ in $\mathcal{F}_o(X)$. Moreover $C_{\lambda_{i_{j_k}}} \in \mathcal{C}_{\lambda_{i_{j_k}}}$ hence any nhood of C_o meets $\mathcal{C}_{\lambda_{i_{j_k}}}$, then $C_o \in Ls\mathcal{C}_{\lambda_{i_j}}$. But $\mathcal{C}_{\lambda_{i_j}} \to \mathcal{C}$, then $C_o \in \mathcal{C}$. Also $x_{\lambda_{i_j}} \to x$ and $x_{\lambda_{i_j}} \in C_{i_j}$ which is convergent to C_o, so that $x \in LsC_{\lambda_{i_j}} = C_o$

Then $x \in C_o \in \mathcal{C}$.

Hence $x \in \cup_{C\in\mathcal{C}} C$ and thus $Ls \cup_{C_\lambda \in \mathcal{C}_\lambda} C_\lambda \subset \cup_{C\in\mathcal{C}} C$ and we conclude that $\text{Lim} \cup_{C_\lambda \in \mathcal{C}_\lambda} C_\lambda = \cup_{C\in\mathcal{C}} C$ or $\Psi(X)(\mathcal{C}_\lambda) \to \Psi(X)(\mathcal{C})$ and thus $\Psi(X): \mathcal{F}_0^2(X) \to \mathcal{F}_o(X)$ is a continuous function.

<u>Theorem (4.6)</u>. If $f: X \to Y$ is a continuous map between compact Hausdorff spaces, then the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\Phi(X)} & \mathcal{F}_o(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{F}_o(f)} \\ Y & \xrightarrow{\Phi(Y)} & \mathcal{F}_o(Y) \end{array}$$

<u>Proof</u>: Take $x \in X$, then $(\mathcal{F}_o(f) o\, \Phi(X))\ (x) = \mathcal{F}_o(f)(\Phi(X)(x))$
$= \mathcal{F}_o(f)(\{x\}) = \{f(x)\}$.

and $[\Phi(Y) o f]\ (x) = \Phi(Y)(f(x)) = \{f(x)\}$.

Thus $\mathcal{F}_o(f) o \Phi(X) = \Phi(Y) o\ f$ and the diagram is commutative.

Theorem (4.7). For each compact Hausdorff space X,

$$\Psi(X) \circ \Phi(\mathfrak{F}_0(X)) = \text{identify map } i: \mathfrak{F}_0(X) \to \mathfrak{F}_0(X).$$

Proof: Take $C \in \mathfrak{F}_0(X)$ then $(\Psi(X) \circ \Phi(\mathfrak{F}_0(X))(C) = \Psi(X)(\Phi(\mathfrak{F}_0(X))(C))$

$$= \Psi(x)(\{C\}) = \bigcup_{C \in \{C\}} C = C.$$

Thus $\Psi(X) \circ \Phi(\mathfrak{F}_0(X)) = i$.

Theorem (4.8). The following diagram is commutative for each compact Hausdorff space X:

$$\begin{array}{ccc} \mathfrak{F}_0^2(X) & \xrightarrow{\Psi(X)} & \mathfrak{F}_0(X) \\ \big\downarrow{\scriptstyle \mathfrak{F}_0^2(f)} & & \big\downarrow{\scriptstyle \mathfrak{F}_0(f)} \\ \mathfrak{F}_0^2(Y) & \xrightarrow{\Psi(Y)} & \mathfrak{F}_0(Y) \end{array}$$

Proof: Let $\mathcal{C} \in \mathfrak{F}_0^2(X)$ then

$$(\mathfrak{F}_0(f) \circ \Psi(X))(\mathcal{C}) = \mathfrak{F}_0(f)(\Psi(X)(\mathcal{C})) = \mathfrak{F}_0(f)(\bigcup_{C \in \mathcal{C}} C) = \bigcup_{C \in \mathcal{C}} f(C)$$

and $(\Psi(Y) \circ \mathfrak{F}_0^2(f))(\mathcal{C}) = \Psi(Y)(\mathfrak{F}_0^2(f)(\mathcal{C})) = \Psi(Y)(f(\mathcal{C}))$ where $f(\mathcal{C}) = \{f(C), C \in \mathcal{C}\}$ $= \bigcup_{C \in \mathcal{C}} f(C)$

Thus $\mathfrak{F}_0(f) \circ \Psi(X) = \Psi(Y) \circ \mathfrak{F}_0^2(f)$ and the diagram is commutative.

Theorem (4.9). The following diagram is commutative.

$$\begin{array}{ccc} \mathfrak{F}_0(X) & \xrightarrow{\Phi(\mathfrak{F}_0(X))} & \mathfrak{F}_0^2(X) \\ \big\downarrow{\scriptstyle \mathfrak{F}_0(\Phi(X))} & & \big\downarrow{\scriptstyle \Psi(X)} \\ \mathfrak{F}_0^2(X) & \xrightarrow{\Psi(X)} & \mathfrak{F}_0(X) \end{array}$$

Proof: Let $C \in \mathcal{F}_0(X)$, then

$$(\Psi(X) \circ \Phi(\mathcal{F}_0(X)))(C) = \Psi(X)\ (\Phi(\mathcal{F}_0(X))(C)) = \Psi(X)\ (\{C\})$$
$$= \bigcup_{C \in \{C\}} C = C \quad \text{by theorem (4.7)}$$

and $(\Psi(X) \circ \mathcal{F}_0(\Phi(X)))\ (C) = \Psi(X)\ (\mathcal{F}_0(\Phi(X))(C)) = \Psi(X)(\{C\})$

$$= \bigcup_{C \in \{C\}} C = C$$

and the diagram is commutative.

Theorem (4.10). The following diagram is commutative .

$$\begin{array}{ccc}
\mathcal{F}_0^3(X) & \xrightarrow{\Psi(\mathcal{F}_0(X))} & \mathcal{F}_0^2(X) \\
\mathcal{F}_0(\Psi(X)) \downarrow & & \downarrow \Psi(X) \\
\mathcal{F}_0^2(X) & \xrightarrow[\Psi(X)]{} & \mathcal{F}_0(X)
\end{array}$$

Proof: Let $\mathcal{D} \in \mathcal{F}_0^3(X)$, then $\mathcal{D}$ is a closed subset of $\mathcal{F}_0^2(X)$ and hence a collection of families of closed subsets of X.

Then,

$$(\Psi(X) \circ \mathcal{F}_0(\Psi(X)))(\mathcal{D}) = \Psi(X)\ (\mathcal{F}_0(\Psi(X))(\mathcal{D}))$$
$$= \Psi(X)\ \{\bigcup_{C \in \mathcal{C}} C : \mathcal{C} \in \mathcal{D}\}$$

$$= \bigcup_{\mathcal{C} \in \mathcal{D}} (\bigcup_{C \in \mathcal{C}} C)$$

and also

$$(\Psi(X) \circ \Psi(\mathcal{F}_0(X)))(\mathcal{D}) = \Psi(X)\ (\Psi(\mathcal{F}_0(X))(\mathcal{D}))$$
$$= \Psi(X)\ (\bigcup_{\mathcal{C} \in \mathcal{D}} \mathcal{C})$$
$$= \bigcup_{C \in \bigcup_{\mathcal{C} \in \mathcal{D}} \mathcal{C}} C$$
$$= \bigcup_{\mathcal{C} \in \mathcal{D}} (\bigcup_{C \in \mathcal{C}} C) \quad \text{by associativity of the union.}$$

Thus the diagram is commutative.

Now let X be a contractible compact Hausdorff space and let $h: X \times I \to X$ be a contraction, then $h(x,0)=x$ for all x and $h(x,1)= x_o$ for all x.

Let $H^1: \mathcal{F}_o(X)\times I \to \mathcal{F}_o(X)$ be defined by $H^1(C,t)= h(C,t)$. Then H^1 is continuous by theorem (1.2) and $H^1(C,0)=h(C,0)=C$ and $H^1(C,1)= h(C,1)=\{x_o\}$. Define by induction $H^n: \mathcal{F}_o^n(X)\times I \to \mathcal{F}_o^n(X)$ as follows $H^n(\mathcal{C},t)=\{\mathcal{C}'= H^{n-1}(\mathcal{C}'',t), \mathcal{C}'' \in \mathcal{C}\}$.

<u>Lemma (4.11)</u>. The following diagram is commutative.

$$\begin{array}{ccc}
\mathcal{F}_o^{n+1}(X) & \xrightarrow{\ \mathcal{F}_o^{n-1}\Psi(X)\ } & \mathcal{F}_o^{n}(X) \\
\uparrow H^{n+1} & & \uparrow H^{n} \\
\mathcal{F}_o^{n+1}(X)\times I & \xrightarrow[\ \mathcal{F}_o^{n-1}(X)\times 1_I\ ]{} & \mathcal{F}_o^{n}(X)\times I
\end{array}$$

where 1_I is the identity map of I

<u>Proof</u>: We have to show that for all $\mathcal{C} \in \mathcal{F}_o^{n+1}(X)$ and for all $t \in I$

$$H^n(\mathcal{F}_o^{n-1}\Psi(X)(\mathcal{C}),t) = \mathcal{F}_o^{n-1}\Psi(X)(H^{n+1}(\mathcal{C},t)).$$

or $$H_t^n(\mathcal{F}_o^{n-1}\Psi(X)(\mathcal{C})) = \mathcal{F}_o^{n-1}\Psi(X)\ H_t^{n+1}(\mathcal{C}).$$ Where $H_t^i : \mathcal{F}_o^i(X)\to\mathcal{F}_o^i(X)$ is the map $H_t^i(\mathcal{C}) = H^i(\mathcal{C},t)$ where $h_t(x)=h(x,t)$.

Now we have $H_t^1(C) = H^1(C,t)= h(C,t) = \mathcal{F}_o(h_t)(C)$,

$H_t^2 = \mathcal{F}_o^2(h_t) = \mathcal{F}_o(\mathcal{F}_o(h_t)) = \mathcal{F}_o(H^1)$ and by induction we have $H_t^i = \mathcal{F}_o(H_t^{i-1})$. Thus $H_t^n = \mathcal{F}_o^n(h_t)$ and $H_t^{n+1} = \mathcal{F}_o^{n+1}(h_t)$.

Thus $\mathcal{F}_o^{n-1}\Psi(X)\ H_t^{n+1} = \mathcal{F}_o^{n-1}\Psi(X) \circ \mathcal{F}_o^{n+1}(h_t)$

$$= \mathfrak{F}_o^{n-1}\Psi(X) \circ \mathfrak{F}_o^{n-1} \circ \mathfrak{F}_o^{2} \quad (h_t)$$

$$= \mathfrak{F}_o^{n-1}(\Psi(X) \circ \mathfrak{F}_o^{2}(h_t)) \text{ Since } \mathfrak{F}_o^{n-1} \text{ is a functor}$$

$$= \mathfrak{F}_o^{n-1}(\mathfrak{F}_o(h_t)\,\Psi(X)) \text{ by theorem (4.8)}$$

$$= \mathfrak{F}_o^{n-1}(\mathfrak{F}_o(h_t) \circ \mathfrak{F}_o^{n-1}(\Psi(X))$$

$$= \mathfrak{F}_o^{n}(h_t) \circ \mathfrak{F}_o^{n-1}(\Psi(X))$$

$= H_t^n \circ \mathfrak{F}_o^{n-1}(\Psi(X))$ and this is what we set to prove.

Now $\cdots \to \mathfrak{F}_o^{n+1}(X) \xrightarrow{\mathfrak{F}_o^{n-1}\Psi(X)} \mathfrak{F}_o^{n}(X) \cdots \xrightarrow{\mathfrak{F}_o\,\Psi(X)} \mathfrak{F}_o^{2}(X) \xrightarrow{\Psi(X)} \mathfrak{F}_o(X)$ is a sequence of compact Hausdorff spaces and continuous maps. We define $\mathfrak{F}_o^{\infty}(X) = \varprojlim \mathfrak{F}_o^{n}(X)$, the inverse limit space of the above sequence. We have the following .

<u>Theorem (4.12)</u>.If X is compact Hausdorff space and contractible, then $\mathfrak{F}_o^{\infty}(X)$ contractible

<u>Proof</u>: Consider the following commutative diagram

$$\begin{array}{ccccc}
\mathfrak{F}_o^{\infty}(X) & \cdots & \mathfrak{F}_o^{n+1}(X) & \xrightarrow{\mathfrak{F}_o^{n-1}\Psi(X)} & \mathfrak{F}_o^{n}(X) \\
\uparrow H^{\infty} & & \uparrow H^{n+1} & & \uparrow H^{n} \\
\mathfrak{F}_o^{\infty}(X)\times I & \cdots & \mathfrak{F}_o^{n+1}(X)\times I & \xrightarrow[\mathfrak{F}_o^{n-1}\Psi(X)\times]{} & \mathfrak{F}_o^{n}(X)\times I
\end{array}$$

This induces a map H^{∞} between the inverse limit spaces of the two sequences, H^{∞} is continuous since each H^{i} is. But the inverse limit of the bottom sequence is $\mathcal{F}_o^\infty(X)\times I$. Thus we have H^∞: $\mathcal{F}_o^\infty(X)\times I\to\mathcal{F}_o^\infty(X)$, also H_o^∞ is the inverse limit map of the maps H_o^n which are all identity maps, then H_o^∞ is the identity, and H_1^∞ is the inverse limit map of the maps H_1^n which are all constant , then H_1^∞ is a constant map. Thus H^∞: $\mathcal{F}_o^\infty(X)\times I\to\mathcal{F}_o^\infty(X)$ is a contraction map and hence $\mathcal{F}_o^\infty(X)$ is a contractible space.

Corollary (4.13). If X is compact Hausdorff and contractible $\mathcal{F}_o^\infty(X)$ is a pathwise connected space such that $\mathcal{F}_o(\mathcal{F}_o^\infty(X))\approx\mathcal{F}_o^\infty(X)$.

Proof: By the above theorem $\mathcal{F}_o^\infty(X)$ is contractible and hence pathwise connected. Now consider the following sequence of compact Hausdorff spaces. $\ldots \xrightarrow{\mathcal{F}_o^2\Psi(X)} \mathcal{F}_o^3(X) \xrightarrow{\mathcal{F}_o\Psi(X)} \mathcal{F}_o^2(X) \xrightarrow{\Psi(X)} \mathcal{F}_o(X)$

its inverse limit space is $\mathcal{F}_o^\infty(X)$. Thus by theorem (4.2)

$$\mathcal{F}_o(\mathcal{F}_o^\infty(X)) \approx \varprojlim \mathcal{F}_o(\mathcal{F}_o^n(X)) \approx \mathcal{F}_o^\infty(X).$$

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