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A CATEGORICAL SEMANTICS FOR TOPOLOGICAL QUANTUM COMPUTATION

By
Éric Oliver Paquette, B.A.
November 2004

A Thesis
submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Master of Science in Mathematics¹

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Abstract

The aim of this thesis is to develop an abstract categorical setup in order to show that \mathcal{C} -colored manifolds (i.e. compact closed manifolds with boundary where each boundary component is colored with an object of a semisimple strongly ribbon category) behaves basically in a similar manner as quantum circuits under the action of a unitary modular functor. There, the set of gates is composed only of braid operations, rotations and Dehn-twists.

We first introduce the basic mathematical structure of a quantum circuit. We then provide a complete development of a 2-dimensional CW-complex over an extended surface. Furthermore, we provide a complete development of the categorical framework in order to construct a \mathcal{C} -extended unitary modular functor (UMF) acting from the category of \mathcal{C} -colored surfaces and morphisms of \mathcal{C} -colored surfaces to the category of finite-dimensional vector spaces and linear isomorphisms.

We then conclude by giving a complete semantics for topological quantum computation including an abstract version of the inner product, basic data units, basic data transformations, projectors and the notion of topological invariance of the algorithms.

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Dedication

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Chapter 1

Introduction

As we know, quantum computing faces many major challenges in terms of implementation, including the problem of decoherence of quantum data. Indeed, a quantum computer containing more than a thousand qubits would be so unstable that it would essentially be impossible to compute anything on it. In the last few years, new ideas arose in order to rectify this major problem and topological quantum computing seems to be a promising candidate to solve it. The basic idea is to encode the information into global degrees of freedom of the system instead of local ones like the spin. In this thesis, we generalize some ideas that have been presented by Freedman, Kitaev, Larsen and Wang in [9] and [10]; we develop a general framework for topological quantum computation using a unitary modular functor of genus 0.

This thesis is organized as follows. In chapter 2, we provide a general introduction to quantum computing that is essentially mathematical. More specifically, we introduce the notions of Hilbert spaces and adjoint operators acting on such spaces, we provide a summary introduction to quantum computation with a few examples and finally, we provide a definition of the quantum circuit together with a few theorems in order to give a complete outline of the subject.

In chapter 3, we develop all the nomenclature essential to describing compact closed manifolds with or without boundary of genus 0 (that we will call extended surfaces)

and, we introduce the notion of marking on a surface as the analysis of the action of the mapping class group on such manifolds can be reduced to an analysis on its action on the marking. We then discuss how moves propagate in the sense of how a sequence of moves acts on the manifold and we axiomatize this concept. Next, we inspect the structure of a CW-complex on a given surface and define the notion of a maximal CW-complex as a CW-complex on an extended surface decomposed into elementary pieces with one, two or three punctures. Finally, we show the connectedness and the simple connectedness of the 2-dimensional CW-complex and the maximal CW-complex for a given extended surface.

In chapter 4, we incrementally build a semisimple ribbon category and provide a graphical language to interpret equivalence of morphisms in a ribbon category. That is, starting from an abelian category and presenting its properties, we inspect the notion of a semisimple abelian category. Next, we equip the latter with a monoidal structure, then with a braided monoidal structure. We inspect the notion of duality on objects and morphisms via the rigidity structure on a category; from there we develop the notion of ribbon category and, finally, we conclude by defining semisimplicity on ribbon categories.

In chapter 5, we introduce the notion of a UMF acting on \mathcal{C} -colored surfaces by first defining the notion of \mathcal{C} -extended modular functor. We define the Teichmüller groupoid in genus 0 as a category that encompasses most data introduced in chapter 3 and that is such that it presents strong similitudes with the structure of a modular functor. We then introduce the Moore-Sieberg data (MS-data) for semisimple abelian category. One can then build from any semisimple ribbon category a set of MS-data and, in addition, build a structure that is quite similar to the one of the Teichmüller groupoid. We finally build a \mathcal{C} -extended UMF in genus 0 using a given set of MS-data and show that this yields a representation of the Teichmüller groupoid.

In chapter 6, we introduce a categorical semantics for topological quantum computation based on the work of S. Abramsky and B. Coecke in [1] that we generalize

by adding a braided ribbon structure. In more details, we first show that some vector spaces obtained by the action of μ on \mathcal{C} -colored surfaces are trivial and we impose positivity on our UMF. We introduce an inner product on these vector spaces, adjoints of maps on this inner product and the notion of unitary maps. From there, we show that all the generators of the mapping class group of an e-surface (as well as their composition) are unitary, making the whole mapping class group unitary. We introduce the notion of projectors and topological invariance of the algorithms. We conclude with some remarks pertinent to the topological framework.

Finally, in chapter 7, we conclude and we give a set of open questions related to the present work.

Chapter 2

Quantum computing

We start this chapter by providing a short but complete introduction to quantum computing in two parts. First, we define Hilbert spaces as we will regard a qubit as a vector in a Hilbert space, and the abstract model for topological quantum computation that we will provide in chapter 6 is closely related to this type of structure. Then, we introduce a basic mathematical setup to quantum computing by means of comparison with classical computation. We end this chapter by providing a complete definition of the quantum circuit together with a few well-known theorems such as, for instance, the Solovay-Kitaev theorem. We refer the reader to [19] (from which this short introduction has been inspired) and [20] for more details on the subject.

2.1 Basic definitions and concepts

Here, we concentrate on the mathematical aspects of quantum computation. For a complete introduction to quantum mechanics, the reader is encouraged to look at the following texts: [6], [13] and [15].

2.1.1 Hilbert spaces

This introduction to Hilbert spaces closely follows the one given in section 3 of [2]. We have chosen to give that presentation as it is quite parallel to the one that we will

develop in chapter 6. We know that a complex vector space is a set V together with two extra structures

$$+ : V \times V \rightarrow V \quad \cdot : \mathbb{C} \times V \rightarrow V$$

which are respectively the addition of vectors and the scalar multiplication with the usual properties.

Now, an arrow $f : V \rightarrow V'$ is called a *linear operator* from vector space to vector space if it preserves the extra structures of the set V , i.e. given $\phi, \psi \in V$ and a scalar $c \in \mathbb{C}$ we have

$$f(\psi + \phi) = f(\psi) + f(\phi) \quad f(c\psi) = cf(\psi)$$

Definition 2.1.1 A Hilbert space H is a complex vector space together with an inner product

$$\langle | \rangle : H \times H \rightarrow \mathbb{C}$$

satisfying for all $\phi, \psi \in H$:

$$\langle \psi | \phi_1 + \phi_2 \rangle = \langle \psi | \phi_1 \rangle + \langle \psi | \phi_2 \rangle \quad (1)$$

$$\langle \psi | z\phi \rangle = z\langle \psi | \phi \rangle; \quad z \in \mathbb{C} \quad (2)$$

$$\langle \psi | \phi \rangle = \overline{\langle \phi | \psi \rangle} \quad (3)$$

$$\langle \psi | \psi \rangle \geq 0 \text{ and } \langle \psi | \psi \rangle = 0 \text{ iff } \psi = 0 \quad (4)$$

Finally, we require that H is complete with respect to the inner product given above.

Remark 2.1.2 In definition 2.1.1 we use the term complete in the sense that every Cauchy sequence converges in H . Hence, we can also say that all Hilbert spaces are Banach spaces (the converse is not true in general).

Furthermore, an arrow $f : H \rightarrow H'$ which preserves all the structures is called an *isometry*, i.e.

$$\langle f\psi | f\phi \rangle_{H'} = \langle \psi | \phi \rangle_H$$

In the case of Hilbert spaces, we can define the adjoint of an operator in the following manner:

Definition 2.1.3 *given an operator acting on Hilbert spaces, say $f : H \rightarrow H'$, the operator $f^\dagger : H' \rightarrow H$ is said to be the adjoint of f . The operator f^\dagger satisfies the following equation:*

$$\langle f^\dagger \psi | \phi \rangle_H = \langle \psi | f \phi \rangle_{H'}$$

Now, suppose that H coincides with H' and that we are given a $\phi \in H$ then, there is a unique operator $f_\phi : \mathbb{C} \rightarrow H$ (such that $f_\phi(1) = \phi$). In a similar manner, the action of $(\)^\dagger$ on an operator defined in the same manner from a given $\psi \in H$ yields a unique arrow (operator) $f_\psi^\dagger : H \rightarrow \mathbb{C}$. The composition of the two preceding operators is really $\langle \psi | \phi \rangle = f_\psi^\dagger \circ f_\phi : \mathbb{C} \rightarrow \mathbb{C}$ which is a linear operator from complex numbers to themselves. In quantum mechanics this is the amplitude of probability for the whole process to happen (in the sense that we see the preceding set of equations as the mathematical representation of a quantum system where a *process* occurs, such as the collapse of a wave-function into one of its eigenstates). Note that the converse is also true, if we are given an operator $f_\psi : \mathbb{C} \rightarrow H$ then, it gives rise to a vector $f_\psi(1) = \psi \in H$.

Remark 2.1.4 *Usually, we say that $\psi \in H$ is the state vector and H is the state space of the quantum system.*

Now, if we want to remain in the quantum mechanical case, having two different quantum states in their respective Hilbert spaces, H and H' . If we consider them to lie in a single system, this is the same as considering the Hilbert space $H \otimes H'$ (this is usually called a *compound system*). This tensor product has \mathbb{C} for identity.

Definition 2.1.5 *A self-adjoint operator is an operator $A : H \rightarrow H$ acting on Hilbert spaces that is such that $A = A^\dagger$*

In quantum mechanics, a measurement collapses the wavefunction into a particular state; the output of such a measurement is a value in the *spectrum* of a self-adjoint operator, i.e. one of its eigenvalues (we will assume that the spectrum of our operators

are always non-degenerate). Thus, given a self-adjoint operator we write it's *spectral decomposition* as

$$A = \sum_{i=1}^n \lambda_i P_i \quad (5)$$

where $n = \dim(H)$ and the λ_i are the eigenvalues of the operator A (i.e. the value of its spectrum). Now, the change induced by measurement is represented by the action of an operator which we now define:

Definition 2.1.6 *A projector is an arrow $P_i : H \rightarrow H$ arising in the spectral decomposition of a self-adjoint operator. It is idempotent and self-adjoint with respect to the composition, furthermore, the projectors are mutually orthogonal in the sense that $P_i \circ P_j = 0$ if $i \neq j$.*

Where the last definition and the following remark are stated along the lines of [1], section 2.

Remark 2.1.7 *Each P_i projects to a one-dimensional subspace of H (as we assumed that the spectrum of A was non-degenerate).*

The probability that a measurement yields a particular eigenvalue λ_i (assuming $\langle \psi | \psi \rangle$) is given by

$$\langle \psi | P_i \psi \rangle. \quad (6)$$

The later is called the Born rule.

2.1.2 Quantum computing

As classical computers have for basic unit the bit, which can take two values $\{0, 1\}$, a quantum computer have the *qubit* as basic unit. After a measure, the qubit can take two different values in the following set $\{|0\rangle, |1\rangle\}$ which is the computational basis for our quantum computation. Usually, if we speak of qubit, the information is considered to be encoded in a two-level quantum system where it is usually assumed that the information to be encoded is the spin of an electron or the polarization of a photon. The qubit is then represented by a state vector $\alpha|0\rangle + \beta|1\rangle$ for $\alpha, \beta \in \mathbb{C}$, it is

assumed that the state vector is normalized, that is $|\alpha|^2 + |\beta|^2 = 1$. The qubit is then a vector lying in a 2-dimensional complex Hilbert space denoted, for our purpose, by \mathbb{C}^2 . Furthermore, this last expression for the qubit tells us that the qubit is in fact a linear combination of $|0\rangle$ and $|1\rangle$. If we are computing with n qubits, the space where this computation take place is then n times the tensor product of \mathbb{C}^2 which will be denoted as $(\mathbb{C}^2)^{\otimes n}$.

More generally, the information can be encoded in a p -level system (where p is a positive integer that is, for our purpose greater or equal to 2) and then, we speak of *qupits*. In the latter case, the information is represented by a state vector $\sum_i \alpha_i |i\rangle$ lying in \mathbb{C}^p . If we are computing on a set of n qupits, the space where this computation take place is $(\mathbb{C}^p)^{\otimes n}$.

Given n state vectors representing qubits say $|\psi_0\rangle, \dots, |\psi_{n-1}\rangle$, then the global state vector $|\psi\rangle = \bigotimes_{i=0}^{n-1} |\psi_i\rangle \in (\mathbb{C}^2)^{\otimes n}$ is denoted by

$$|\psi\rangle = \sum_{i=0}^{2^n-1} \alpha_i |i\rangle \quad (7)$$

where the i in the ket (i.e. the vector denoted as $|i\rangle$) are typically written in binary notation so that we have a superposition of all the states between 0 and $2^n - 1$ inclusively. For instance, the ket $|010\rangle$ is the same as $|0\rangle \otimes |1\rangle \otimes |0\rangle$ and in this particular case, we can have any ket from $|000\rangle$ to $|111\rangle$ which correspond to 0 and $7 = 2^3 - 1$ respectively. This state vector is again submitted to the following usual normalization condition: $\sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1$.

If we compute with classical bits, we use a set of classical gates to act on data. For example, the logical gates **AND**, **OR** and **NOT** can be used to perform calculations on bits. Again, as in the case of the basic unit of computation, we can define quantum gates acting on qubits. These gates are usually elements of $U(2)$ (acting on one qubit) or elements of $U(4)$ (acting on a pair of qubits), where $U(n) = \{U \mid U \in \text{Mat}_n(\mathbb{C}), U^\dagger = U^{-1}\}$ is considered to be the Lie group of n by n unitary transformation that leaves invariant the Hermitian form $\langle | \rangle$ as defined in the previous section.

We will consider unitary gates to compute; we note that unitary gates have the following property: $UU^\dagger = 1$, where $U^\dagger = U^{-1}$ is the adjoint of U , as in definition 2.1.3, in other words, we have $\langle \psi | \phi \rangle = \langle U\psi | U\phi \rangle$.

We give a short list of quantum gates in the following example:

Example 2.1.8 *We list two quantum gates of $U(2)$ and one of $U(4)$:*

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad CNot = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Where $H, N \in U(2)$ are called the Hadamard and the Not gate respectively and $CNot \in U(4)$ is called the Controlled-Not gate. The names of N and $CNot$ are self-explanatory, the Hadamard gate takes a state of the form $|0\rangle$ or $|1\rangle$ and puts it in an equal superposition of these two states:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \quad (8)$$

Again, this can be generalized to qubits: instead of taking gates in $U(2)$ or $U(4)$, we take gates in $U(p)$ or $U(p^2)$.

After we have applied various gates to a qubit of our system, we can measure the qubits and the result will yield $|0\rangle$ or $|1\rangle$. Indeed, suppose we apply successively the gates $U_1, U_2, \dots, U_n \in U(2)$ to the qubit $q = \alpha|0\rangle + \beta|1\rangle$ then $(U_1 \circ U_2 \circ \dots \circ U_n)(q) = \alpha'|0\rangle + \beta'|1\rangle$. The probability to measure $|0\rangle$ is now $|\alpha'|^2$ and the probability to measure $|1\rangle$ is $|\beta'|^2$. In that sense, the complex numbers α and β can be viewed as complex amplitudes of probability and these amplitudes are affected by the application of the gates U_i ; $i \in \{1, \dots, n\}$ on the state vector.

2.2 The quantum circuit model

Before stating our model for the quantum circuits we give a few differences between the classical case and the quantum case.

First, it is important to note that since we are computing with unitary gates, any composition of unitary matrices is unitary and hence reversible. This tells us that quantum computation is always reversible and hence, gates like the classical **OR** gate which takes two boolean inputs and gives a boolean output have no counterparts in the quantum case as it is not reversible. Hence, there is no unitary matrix that could represent such an operation.

Second, the **FANOUT** gate that duplicates boolean data has no counterparts either in the quantum case, as the following result tells us:

Theorem 2.2.1 (No cloning) *Given an unknown (pure) quantum state $|\psi\rangle$, it is impossible to duplicate into two identical states, i.e. $|\psi\rangle \otimes |\psi\rangle$.*

Proof: we refer to [19] p. 532 for a proof of this theorem.

□

Third, quantum circuits are *acyclic* in the sense that we cannot send feedback from one part of the circuit to another part. Furthermore, all measurements are assumed to be postponed to the end of the computation.

We now generalize the concept of applying a unitary gate to a single qubit to many qubits in a formalism called the quantum circuit model.

Let $|\psi\rangle$ and $|\phi\rangle$ be quantum states. If we apply a unitary transformation U to ϕ only, we then apply the transformation $U \otimes 1$ to the tensor product of the two state vectors in the following manner:

$$(U \otimes 1)(\psi \otimes \phi) \tag{9}$$

Thus, given a set of quantum states $\{|\psi_i\rangle\}_{i=0,\dots,n-1}$. If we consider that all these states belong to the same quantum system then the state vector of the system is $|\psi_0\rangle \otimes \dots \otimes |\psi_{n-1}\rangle$. Let \bar{U}_i be a (unary) quantum gate that acts on the i th state

vector (i.e. a unitary transformation of the form $(\bigotimes_{j=0}^{i-1} I_2) \otimes U_i \otimes (\bigotimes_{k=i+1}^{n-1} I_2)$ where I_2 is the 2×2 identity matrix) and $\overline{U}_{i,i+1}$ is a (binary) quantum gate that acts on the i th and $(i + 1)$ th state vector (i.e. a unitary transformation of the form $(\bigotimes_{j=0}^{i-1} I_2) \otimes U_{i,i+1} \otimes (\bigotimes_{k=i+2}^{n-1} I_2)$) then, we apply successively some of these gates on a vector of the form:

$$\left(\bigotimes_{i=0}^{n-1} |\psi_i\rangle \right) \tag{10}$$

Note that the \overline{U} can be either non-trivial unitary transformations or the identity.

Now, with this generalization, we are ready to give our formulation of the quantum circuit.

Definition 2.2.2 *A quantum circuit is a unitary arrow $\mathbf{U} : (\mathbb{C}^p)^{\otimes n} \rightarrow (\mathbb{C}^p)^{\otimes n}$, that is an operator acting from the space of n qubits to itself together with two arrows $I : A \rightarrow (\mathbb{C}^p)^{\otimes n}$ and $O : (\mathbb{C}^p)^{\otimes n} \rightarrow A$ corresponding respectively to the input (from the set A) and the measure operation (or output to the set A). It is depicted by the following graph:*

$$A \xrightarrow{I} (\mathbb{C}^p)^{\otimes n} \xrightarrow{\mathbf{U}} (\mathbb{C}^p)^{\otimes n} \xrightarrow{O} A$$

The arrows I and O are there only to encompass the full process. During this thesis, we will essentially consider the quantum circuit to be the arrow \mathbf{U} itself.

It is important to note that it is possible to compute with a finite number of quantum gates. Before stating the Solovay-Kitaev theorem that formalizes the previous assertion we need to introduce some concepts. The following set of definitions together with the statement of the theorem are taken from [19], appendix 3, pp. 617-624. we give the notion of a dense set and an ϵ -net:

Definition 2.2.3 *A set $A \subset B$ (where B is equipped with a norm) is said to be dense in B if for any $b \in B$ and $\epsilon > 0$ there exists an $a \in A$ such that $|b - a| < \epsilon$.*

Definition 2.2.4 Let $A, C \subset B$, then A is said to be an ϵ -net for C (with $\epsilon > 0$), if for all $c \in C$ there exists an $a \in A$ such that $|c - a| < \epsilon$.

Theorem 2.2.5 (Solovay-Kitaev) Let \mathcal{U} be a finite set of elements of $SU(2)$ closed under taking the inverses, such that $\langle \mathcal{U} \rangle$ (the set of all words¹ of finite length) is dense in $SU(2)$. Let also $\epsilon > 0$ be given. Then \mathcal{U}_l (the set of all finite words of length l) is an ϵ -net in $SU(2)$ for $l = O(\log^c(1/\epsilon))$, where $c \approx 4$.

Proof: the proof is long and not very pertinent to the present exposition. We refer the reader to [19] pp. 617-624 for a full exposition on the subject.

□

Example 2.2.6 We provide an example of a dense set as the following triple of matrices together with their inverses:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Note that we might be able to generalize this theorem to $SU(p)$ as the previous theorem is proven using essentially the Lie group structure of $SU(2)$. Hence, there is no problem in considering only a finite set of quantum gates as the topological closure of \mathcal{U}_p (a finite set of elements in $SU(p)$) will always be $SU(p)$ itself.

¹where a word is considered to be the composition $U_l \circ \dots \circ U_1$ with $U_i \in \mathcal{U}$ for all $i \in \{1, \dots, l\}$

Chapter 3

Surfaces and markings

In this chapter we develop the nomenclature necessary to speak about e-surfaces of genus 0 and their morphisms. We start by a discussion about general concepts related to topology. We then formalize the notion of e-surfaces, parametrizations as isotopy classes of maps going from a given e-surface into the complex projective sphere with a finite number of holes matching the number of boundary components of the e-surface. We then develop the concept of a marking as a graph on an e-surface M which is a sufficient representation of the e-surface to analyse the action of the mapping class group on M . We then give a set of moves one can apply on a marking which encapsulates the action of the mapping class group and use this set of data to develop a 2-dimensional CW-complex, $\mathbb{M}(M)$ for an e-surface and the concept of a maximal CW-complex which is roughly a sub-complex of $\mathbb{M}(M)$ but where the surface is cut into elementary pieces with one, two or three holes, inducing a decomposition of the marking to match these elementary pieces. Finally, we show that this complex and the underlying maximal complex are connected and simply-connected. We closely follow the development of [3] sections 2 to 6 from which most definitions, lemmas and theorems are taken almost directly and furthermore, we add elements from [4] section 5.2.

3.1 Basic definitions and concepts

In this section we will give basic definitions related to extended surfaces (or e-surfaces) and their markings.

Before we begin with surfaces, we define a few concepts that will be used throughout the development. Note that for this set of definition, we assume X and Y to be topological spaces and all maps to be continuous maps.

Definition 3.1.1 *A map $f : X \rightarrow Y$ is said to be an homeomorphism if it is bijective and if f and f^{-1} are both continuous maps.*

Definition 3.1.2 *Let $f, f' : X \rightarrow Y$ be continuous maps, we say that f is homotopic to f' if there is a continuous map $F : X \times [0, 1] \rightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for all $x \in X$. Also we say that F is an homotopy of f and f' .*

The last definition is taken from [18], p. 323.

Definition 3.1.3 *A map $f : X \rightarrow Y$ is called an embedding if it an homeomorphism onto its image.*

Definition 3.1.4 *Two embeddings $f, f' : X \rightarrow Y$ are said to be isotopic if there exists an homotopy $F : X \times [0, 1] \rightarrow Y$ such that $F(x, t)$ is an embedding for all $t \in [0, 1]$.*

In other words, we do not accept singularities in the transition from f to f' .

We will use the concepts of genus and punctures in a surface at length; we therefore close this section by defining more formally these concepts.

Definition 3.1.5 *The connected sum of two manifolds, is the manifold obtained by identifying the boundary points of two disks removed from the manifold through an orientation reversal homeomorphism*

Theorem 3.1.6 *Any compact closed oriented connected surface without boundaries is either homeomorphic to a sphere or to a connected sum of tori.*

Proof: this theorem is well known, we refer the reader to [16] pp. 10-28 for the proof and a full exposition of the subject.

□

In our context, we will consider that our compact closed manifolds are either homeomorphic to the sphere or to the connected sum of tori. Now, the following set of definitions is derived from [16], pp. 33, and 37-43 respectively.

Definition 3.1.7 *A surface M is said to be of genus n if it is homeomorphic to the connected sum of n tori. In the case M is homeomorphic to a sphere, it is said to be of genus 0.*

Definition 3.1.8 *If a surface M is homeomorphic to a manifold with a finite number of open disks removed, we say that M is a manifold with boundaries. The boundaries will be called the punctures or holes of M .*

Definition 3.1.9 *A surface M with boundaries is said to be of genus n if \overline{M} (the surface obtained by gluing disks to each boundary components of M) is of genus n .*

3.2 Genus 0 surfaces

We first develop a nomenclature to describe genus 0 surfaces. These definitions are essential as they will become the atomic backbone of our study.

Definition 3.2.1 *An e-surface is defined as a 2-dimensional oriented compact closed manifold M with possible boundary components $(\partial M)_i$, with $i \in I$ where I is the label set of boundary components. We also assume that each $(\partial M)_i$ is marked by a distinguished point on the boundary.*

An example is provided below in figure 1.

Definition 3.2.2 *A morphism of e-surfaces is an homeomorphism $M \xrightarrow{\sim} M'$ which maps marked points to marked points and preserves the orientation of the surface. Every such morphism preserves the number of boundary components and then, induces the bijection $\{(\partial M)_i\}_{i \in I} \xrightarrow{\sim} \{(\partial M')_i\}_{i \in I}$.*

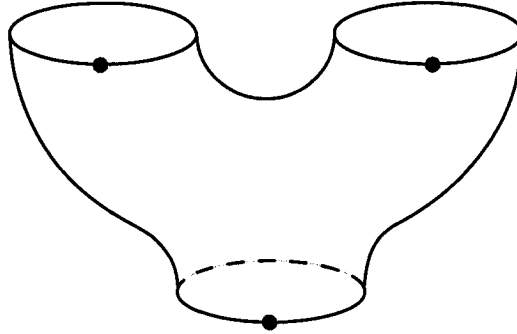


Figure 1: Example of an e-surface.

It is now important to define the set of all possible morphisms for a given e-surface M but before, we diverge a bit from the main source and we refine the notion of homotopy.

Definition 3.2.3 *Let $f, f' : X \rightarrow Y$ be continuous maps; we say that f' is boundary preserving homotopic to f if there is a continuous map $F : X \times [0, 1] \rightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$, and that $F(p_i, t) = f(p_i) = f'(p_i)$ for all t , where p_i is the distinguished point on the boundary component i .*

This definition is introduced in order to avoid some pathological cases in the following definition:

Definition 3.2.4 *Given an e-surface M , the mapping class group of M is the group of boundary preserving homotopy class of morphisms preserving the distinguished points from $M \xrightarrow{\sim} M$; it is denoted $\Gamma(M)$. Now, the pure mapping class group, $\text{P}\Gamma(M) \subset \Gamma(M)$, is the subgroup of morphisms whose action is trivial on $\partial(M)_i$ for all $i \in I$.*

As we will later study surfaces of genus 0, the following definition will be useful:

Definition 3.2.5 *The standard sphere is the Riemann sphere $\mathbb{P}\mathbb{C}$ (or the projective representation of the complex plane). We will work with $\mathbb{P}\mathbb{C}$ where n open disks has been removed. Such a sphere will be denoted $S_{0,n}$. If this sphere has n open disks*

removed, then for $z \in \mathbb{C}$, we have $|z - k| < 1/3$ for the disks removed and the marked points are set to be at $k - i/3$ for all $k \in \{1, \dots, n\}$; where $i = \sqrt{-1}$ and the number 1 to n are centers of disks. The mapping class group (resp. pure mapping class group) of the standard sphere will be denoted by $\Gamma_{0,n}$ (resp. $P\Gamma_{0,n}$). We also denote the elements of $\{\partial(S_{0,n})_i\}_{i=\{1,\dots,n\}}$ simply as $\{1, \dots, n\}$.

Figure 1 above, would be a representation of $S_{0,3}$.

Definition 3.2.6 Given a connected e-surface of genus 0, say M , we define a parametrization without cuts of M to be an isotopy equivalence class of morphisms of e-surfaces of the following form: $p : M \xrightarrow{\sim} S_{0,n}$.

The later can be found as definition 2.4 in [3] p. 3. We now introduce the concept of markings on a surface. These objects are viewed as graphs on the e-surface. The reasons to consider those objects is that they provide an easier way to understand and manipulate parametrizations of e-surfaces.

Let m_0 be the following graph on $S_{0,3}$:

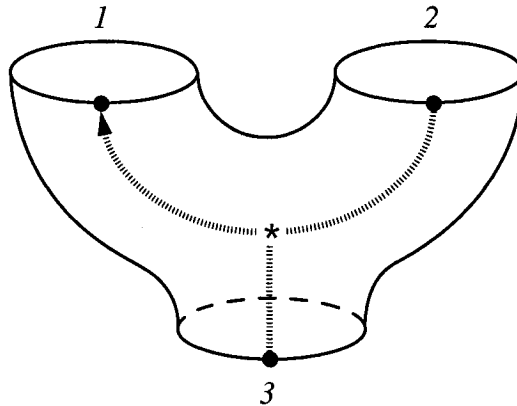


Figure 2: Example of a marking without cuts on $S_{0,3}$.

We note that this graph has a distinguished edge connecting $*$ with 1. In the above example m_0 is called standard marking without cuts of $S_{0,3}$. More generally, we have:

Definition 3.2.7 *On the standard sphere with n holes $S_{0,n}$, the standard marking without cut of $S_{0,n}$, denoted m_0 , is defined to be the graph connecting the vertex $*$ (the image of $-i$ in \mathbb{CP}) with the distinguished points of each components of $\{(\partial M)_i\}_{i \in I}$ through the shortest path on $S_{0,n}$ with respect to the standard norm in \mathbb{C} . This graph has a distinguished edge that connects $*$ with 1. If $n = 0$, then $m_0 = \emptyset$.*

We now link the previous definition to the one of parametrization without cut in the following way:

Definition 3.2.8 *Let M be a connected e-surface of genus 0, a marking without cut of M is the inverse image of the standard marking without cut on $S_{0,n}$ under $p : M \xrightarrow{\sim} S_{0,n}$, in other words, the marking without cut m of M is defined as:*

$$m = p^{-1}(m_0) \tag{11}$$

We now make some comments. If there is an homeomorphism $M \xrightarrow{\sim} M$ isotopic to the identity, then we consider the two marking graphs of M to be equivalent. There is a link between the parametrization without cuts and the set of all marking without cuts of a given e-surface M as the following proposition shows:

Proposition 3.2.9 *Given a connected e-surface of genus 0, say M , there is a bijection between the set of all parametrization without cuts of M and the set of all markings without cuts of M given by $p \mapsto p^{-1}(m_0)$.*

Note that the statement of this proposition is practically identical to the proposition 3.1 found in [3] p. 5.

Proof: We have to show that the assignment $p \mapsto p^{-1}(m_0)$ is injective and surjective.

- *Injectivity:* Assume $p, q : \xrightarrow{\sim} S_{0,n}$ are two parametrizations with $p^{-1}(m_0) = q^{-1}(m_0)$. Then, there exists a homeomorphism $f : M \xrightarrow{\sim} M$ isotopic to id such that $p = q \circ f$. It follows that $p = q \circ f \sim q \circ id = q$. Hence, p is isotopic to q and thus they are equal as parameterizations.

- *Surjectivity:* Let m be a marking without cut. By assumption (definition 3.2.7) $m = p^{-1}(m_0)$ for some parameterization p .

Hence, the assignment is bijective as required.

□

Both sets of the previous proposition will be denoted by $\mathbb{M}^\theta(M)$ indifferently later in the development.

Definition 3.2.10 *Let M be an e -surface of genus 0. A weak cut system, on M is a finite set of simple closed smooth curves lying on M (by ‘lying on M ’ we mean that those curves can be seen as subsets of points of M). Such a set will be denoted as C in what follows.*

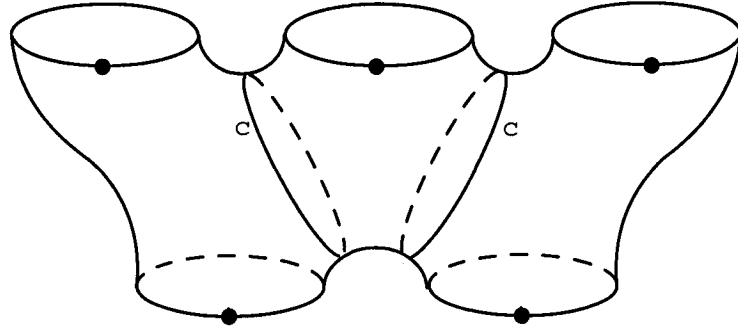
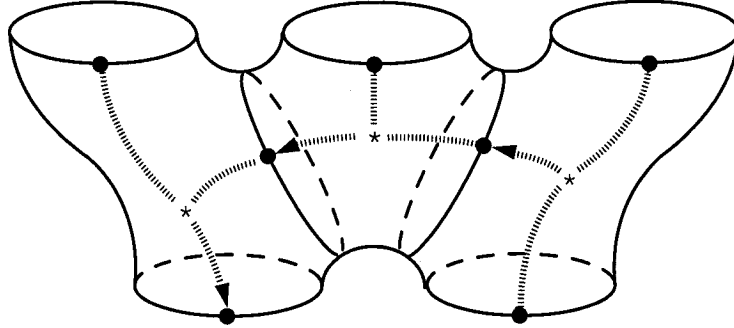


Figure 3: Example of a weak cut system on $S_{0,5}$.

Definition 3.2.11 *Given an e -surface M of genus 0 with a weak cut system C on it, denote the set of connected components of $M \setminus C$ by $\{M_j\}$, to which we glue a copy of $p^{-1}(S^1)$ where S^1 is the 1-sphere in place of the cut (we do that in order to make each of the connected components compact and then, e -surfaces of genus 0); this operation will be denoted $\overline{M \setminus C}$. Then, we define a parametrisation P of M as a set $(C, \{p_j\})$ where C is a weak cut system on M and the p_j are parametrizations without cuts defined on each of the connected components M_j of M , where $j \in J$ and J is the indexing set of these connected components.*

Figure 4: Example of a marked surface with $S_{0,5}$.

Remark 3.2.12 *In definition 3.2.11, we assume that if a cut $c \in C$ has a distinguished point p_c then, p_c is the same for the two connected components adjacent to c .*

Definition 3.2.13 *Let M be an e -surface. A marking of M is a pair (C, m) with C as in definition 3.2.11 and m is a marking graph on M so that it can be seen as the union of markings without cuts on each connected components of the set $\overline{M \setminus C}$. The set of all markings on M taken up to isotopy will be denoted by $\mathcal{M}(M)$.*

We can then generalize the proposition 3.2.9 in the following manner:

Proposition 3.2.14 *Given an e -surface of genus 0, say M , there is a bijection between the set of parametrization and the set of markings of M .*

Again, this is proposition 3.4 in [3] p.6.

Proof: The proof is immediate if we consider the bijection of submarkings m_j and the subparametrization p_j of each connected components M_j of $\overline{M \setminus C}$. Each subcomponent is without cuts and these local bijections induce a global bijection on M .

□

3.3 Operations on markings

3.3.1 Elementary moves

We know how $\Gamma(M)$ acts on M . Now, it will be useful to see how it acts on $\mathcal{M}(M)$. Thus, given a morphism $f \in \Gamma(M)$, it acts on an element $(C, m) \in \mathcal{M}(M)$ in the following manner: $f(C, m) = (f(C), f(m))$. The following operations are similar to those found in [3] p.6 except that here, we consider only surfaces of genus 0 for the gluing and its output:

Disjoint union: $\sqcup : \mathcal{M}(M_1) \times \mathcal{M}(M_2) \rightarrow \mathcal{M}(M_1 \sqcup M_2)$

Gluing: $\sqcup_{a,b} : \mathcal{M}(M) \sqcup \mathcal{M}(M') \rightarrow \mathcal{M}(M'')$ for $a \in \{(\partial M)_i\}_{i \in I}$ and $b \in \{(\partial M'_j)\}_{j \in J}$. In the previous, the image of a, b glued is a cut on M' . If the distinguished point of

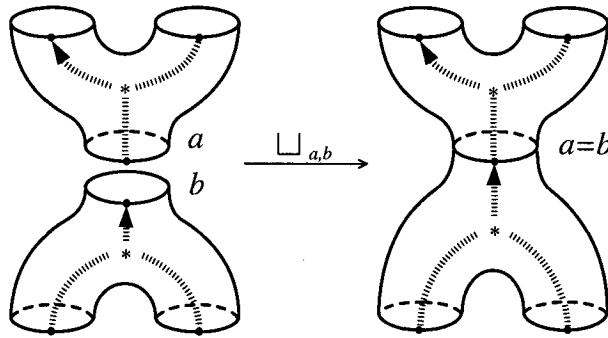


Figure 5: Example of gluing.

either a or b is the endpoint of an arrow of the marking before gluing, it is kept after the gluing operation. These elementary operations satisfy the natural associativity properties that will be listed under the propagation of moves below (i.e. section 3.3.2).

Before continuing, we need to define the notion of a CW-complex. The following definition is taken directly from [16] p. 215; we refer the reader to the same book pp. 211-218 for a full introduction to the subject. Furthermore, we use the concept

of open n -cells which is defined in [8] p. 11.

Definition 3.3.1 *A structure of CW-complex is defined on a Hausdorff space X by the prescription of an ascending sequence $X^0 \subset X^1 \subset X^2 \subset \dots$ of closed subspaces of X which satisfies the following conditions:*

- X^0 is a discrete space.
- For $n > 0$, X^n is obtained from X^{n-1} by the adjunction of a collection of open n -cells.
- $X = \bigcup_{n=1}^{\infty} X^n$.
- The space X and each of the subspaces X^n are equipped with weak topology.

We now define a 2-dimensional CW complex (i.e. a CW-complex where the cells are of at most dimension 2), denoted as $\mathbb{M}(M)$. There, set $\mathcal{M}(M)$ of all markings of M will be regarded as is the set of vertices, the edges of the complex are directed and will be called moves; they are presented below. As each arrow is invertible, the edges of the complex forms a groupoid. We will adopt the following notation or the moves: $E : M \rightsquigarrow M'$. Here is the set of all elementary genus 0 moves:

Z-move or rotation move: Given an e-surface M of genus 0, say M together with (\emptyset, m) a marking without cut on it. The Z -move is defined as $Z : (\emptyset, m) \rightsquigarrow (\emptyset, m')$ where m' is another marking graph on M which differs from m only by having a different distinguished edge (an example is provided in figure 6). As the surface is oriented, we will assume that Z induces a counterclockwise rotation of the distinguished edge. Under the parametrization p of the surface, this reduces to maps $n \mapsto 1$ if M is homeomorphic to $S_{0,n}$. In particular, this is illustrated in figure 6 in the case $M = S_{0,3}$.

F-move or fusion-move: Given an e-surface M of genus 0, together with $(\{c\}, m) \in \mathcal{M}(M)$ a marking with one cut c where we have the distinguished edge of one of the component points to c and it coincides with the last edge (i.e. the preimage of n under the action of $p : M \xrightarrow{\sim} S_{0,n}$ provided that M is homeomorphic to $S_{0,n}$) of the other

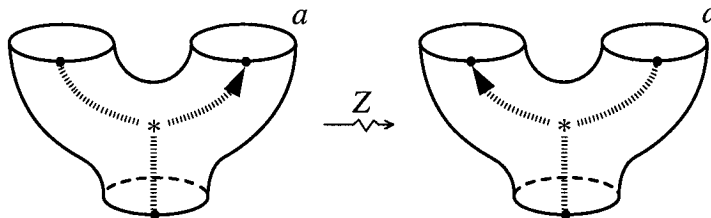


Figure 6: Z move on $S_{0,3}$.

component which also ends on the cut c . Then F is defined as $F_c : (\{c\}, m) \rightsquigarrow (\emptyset, m')$ where m' is the marking graph obtained by contracting the distinguished and the last edges those endpoints are at c . An example is provided below:

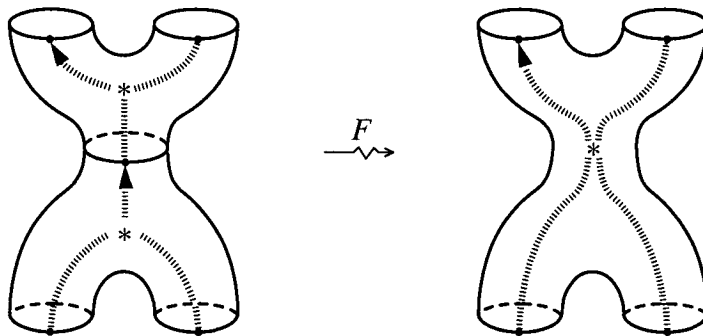


Figure 7: F move on two copies of $S_{0,3}$.

B-move or braid-move: Let $S_{0,3}$ be the trinion (i.e., the sphere with three punctures) with no cuts and with the standard marking m_0 on it as shown on the left hand side of figure 8. The braiding $B_{a,b}$ is defined as the move depicted on that figure. We can generalize that definition to an e-surface M in the following manner: for a given M , an extended surface of genus 0 with three punctures, let ϕ be the homomorphism such that $\phi : M \rightarrow S_{0,3}$. Then, we define the move

$B'_{a,b} : (\emptyset, \phi^{-1}(m_0)) \rightsquigarrow (\emptyset, \phi^{-1}(B_{a,b}(m_0)))$ in $\mathbb{M}(M)$.

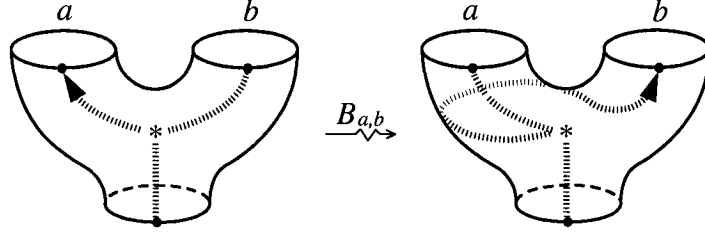


Figure 8: B-move $S_{0,3}$.

We now define generalized moves as some particular composition of the elementary moves above.

Generalized F-move: Let $M = M_1 \sqcup_c M_2$ be an e-surface with a fix order on boundary and $(\{c\}, m) \in \mathcal{M}(M)$ defined in the same way as in the definition of the F -move but where the distinguished edge is any edge of marking marking graph of M_1 and M_2 respectively. A *generalized F-move* is any composition of the form $Z^m F_c(Z^n \sqcup id)(id \sqcup Z^k)$ where m , n and k are integers. This generalized move is introduced in order to ensure that each F -move is well defined.

Generalized B-move: Let M be an e-surface of genus 0 together with m , a marking without cuts on M . Suppose also that the set of labels of M is divided into n non-trivial subsets denoted $\{L_i\}_{i=1, \dots, n}$. Roughly, the generalized B -move is acting on m by permuting two adjacent subset of punctures. More formally, it is given by the following sequence of moves: $B_{L_i, L_j} = F_{c_1} F_{c_2} F_{c_3} B_{c_1, c_2} F_{c_1}^{-1} F_{c_2}^{-1} F_{c_3}^{-1}$; $i \neq j$, $i, j \in \{1, \dots, n\}$. The sequence of F -moves creates a cut system on M where one of the edge shares a vertex with an edge having $*$ as endpoint and the two other vertices are the cuts c_1 and c_2 , lying at the basis of the markings ending on L_i and L_j . This produces a

marking with three vertices on which we apply the B -move. We provide an example below:

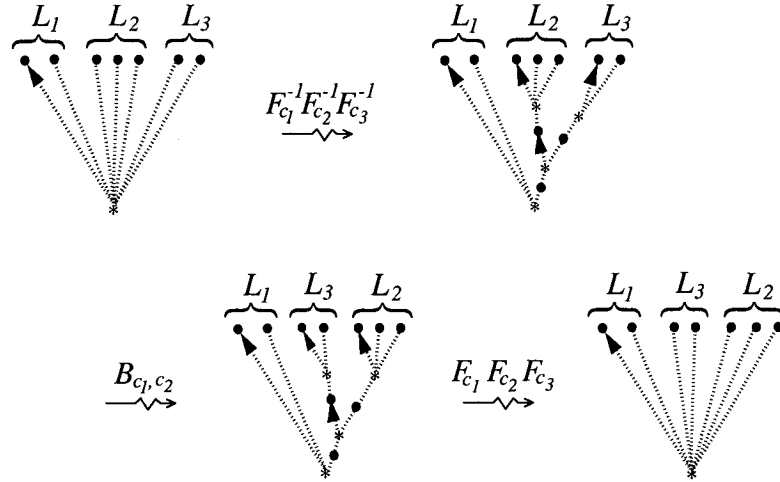


Figure 9: Example of a generalized B-move.

In what follows, we might use generalized moves without explicitly stating it.

Note that given an edge $E : m_1 \rightsquigarrow m_2; m_1, m_2 \in \mathbb{M}(M)$ and a morphism of e-surfaces $f : M \rightarrow M'$, we have $f(E) : f(m_1) \rightsquigarrow f(m_2)$.

We conclude this section by a theorem on the generators of the mapping class group of an e-surface:

Theorem 3.3.2 *The mapping class group $\Gamma_{0,n}$ is generated by braidings b_i and Dehn twists t_i .*

Proof: We refer the reader to [17] to see how $\Gamma_{0,n}$ is generated in such a manner.

□

It will be stated later how this theorem is connected with our definitions.

3.3.2 Propagation of moves

Now, the edges of $\mathcal{M}(M)$ will be obtained from the simple moves B , Z , F and their inverses in addition to the edges obtained from applying the operations of gluing and disjoint union defined below. This definition of the operations and relations of $\mathcal{M}(M)$ is almost identical to the one given in [3], p. 9, section 4.2.

Operations:

- *Disjoint Union:* Suppose $M = M_1 \sqcup M_2$ then, for every $E : m_1 \rightsquigarrow m'_1$ in $\mathbb{M}(M_1)$ and $m_2 \in \mathbb{M}(M_2)$ we add an edge $E \sqcup id_{m_2} : m_1 \sqcup m_2 \rightsquigarrow m'_1 \sqcup m_2$ in $\mathbb{M}(M)$.
- *Gluing:* Suppose $M_1 : \sqcup_{a,b} M$ then, for every $E : \sqcup_{a,b} M \rightsquigarrow \sqcup_{a,b} M'$ in $\mathbb{M}(M)$, we add an edge $\sqcup_{a,b} E : M_1 \rightsquigarrow M'_1$ where $M'_1 = \sqcup_{a,b} M'$.

Equivalence Relations:

- *Functoriality:* Let E, E' be edges in $\mathbb{M}(M_1)$ so that the composition EE' is defined, then $(E \sqcup id_{m_2})(E' \sqcup id_{m_2}) = (EE' \sqcup id_{m_2})$. We also have: $\sqcup_{a,b}(E, E') = (\sqcup_{a,b} E)(\sqcup_{a,b} E')$.
- *Associativity 1:* Let E be an edge in $\mathbb{M}(M_1)$ and let $m_2 \in \mathcal{M}(M_2), m_3 \in \mathcal{M}(M_3)$ be markings for M_2 and M_3 . We have: $(E \sqcup id_{m_2}) \sqcup id_{m_3} = E \sqcup id_{m_2 \sqcup m_3}$.
- *Associativity 2:* Let a, b, c and $d \in \{\partial(M)\}$ be four different boundary components of M and let $E : M \rightsquigarrow M'$ be edge, then $\sqcup_{a,b}(\sqcup_{c,d} E) = \sqcup_{c,d}(\sqcup_{a,b} E)$.
- *Associativity 3:* Let M be an e-surface so that $M = M_1 \sqcup M_2$, $a, b \in \{(\partial M)\}$ be boundary components of M_1 and M_2 respectively and $E : M \rightsquigarrow M'$ an edge in $\mathbb{M}(M_1)$, then $\sqcup_{a,b}(E \sqcup id) = \sqcup_{a,b}(E) \sqcup id$.

If we wish to describe a path in $\mathbb{M}(M)$, we need to have an initial marking m and then, if we are given a sequence of moves taken from $B^{\pm 1}$, $F^{\pm 1}$ and $Z^{\pm 1}$ thus defining a sequence of edge. With this in hand, we can define each vertex in the complex (seen as a marking) along that path by applying successively each move given in the sequence starting from the initial marking m .

Remark 3.3.3 Note that even if for all $f \in \Gamma(M)$ there is a corresponding edge $E \in \mathbb{M}(M)$, the converse is certainly not true because of the F-move.

We now impose the following relations among the moves. Again, the statement of the relations closely follows [3], p.11, section 4.7.

Genus 0 relations:

- *Rotation axiom:* Let M and m be like described in the definition of the Z -move, then $Z^n = id$ where $n = |\{(\partial M)\}|$.
- *Commutativity of disjoint union:* Let M be an e-surface such that $M = M_1 \sqcup M_2$ and let E_i ; $i \in \{1, 2\}$ be an edge in $\mathbb{M}(M_i)$, then $(E_1 \sqcup id)(id \sqcup E_2) = (id \sqcup E_2)(E_1 \sqcup id)$. They will both be denoted by $E_1 \sqcup E_2$.
- *Symmetry of F-move:* let M and m be like described in the definition of the F-move, then $Z^{n-1}F_c = F_c(Z^{-1} \sqcup Z)$, where $n = |\{(\partial M)\}|$.
- *Associativity of cuts:* Let M be a connected surface of genus 0 and $(C, m) \in \mathcal{M}(M)$ be a marking with two cuts say c_1 and c_2 . Then $F_{c_1}F_{c_2}(M) = F_{c_2}F_{c_1}$ where the F_{c_i} are generalized F-moves.
- *Cylinder axiom:* Let $S_{0,2}$ be a cylinder with boundary components a and b together with the standard marking on it (\emptyset, m_0) , M be e-surface together with $(c, m) \in \mathcal{M}(M)$ a marking on it and finally, let $a' \in \{(\partial M)\}$ be a boundary component of M . Then, for every move $E : M \rightsquigarrow M'$ in $\mathbb{M}(M)$, we require that the following square to commutes in $\mathbb{M}(M \sqcup_{a',a} S_{0,2})$:

$$\begin{array}{ccc}
 m \sqcup_{a',a} m_0 & \xrightarrow{E \sqcup_{a',a} id} & m' \sqcup_{a',a} m_0 \\
 \downarrow F'_a & & \downarrow F'_a \\
 m & \xrightarrow{E} & m'
 \end{array}$$

- *Braiding axiom:* Let M be an e-surface isomorphic to $S_{0,4}$ where $\{\partial(M)\} = \{a, b, c, d\}$ and m be a marking, then $B_{a,bd}(M) = B_{a,d}B_{a,b}(M)$ and $B_{ab,c}(M) = B_{a,c}B_{b,c}(M)$ where the B -moves are generalized.

- *Dehn twist axiom:* Let M be an e-surface isomorphic to $S_{0,2}$ where $\{(\partial M)\} = \{a, b\}$ and let (\emptyset, m) be a marking without cuts on it with the distinguished vertex a . Then, $ZB_{a,b}(M) = B_{a,b}Z(M)$ where the B -moves are generalized. This particular composition of Z and B -moves is called a Dehn twist and will be denoted T . Note that this move can also be generalized as for the the F and B -moves by applying F_c^{-1} aside a boundary component.

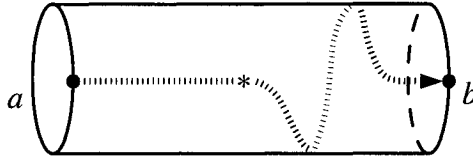


Figure 10: Example of a Dehn twist on $S_{0,2}$.

Note that all the previous relations describe closed paths in $\mathbb{M}(M)$. This can be verified explicitly by using graphical representation of the markings.

The above relations can again be generalized in the sense that they also apply to the operations of gluing and disjoint union. From [3], p.12, section 4.7, this is formalized as:

Propagation rules: For every relation $E = E'$ in $\mathbb{M}(M)$, we add the relations:

- i) $E \sqcup id = E' \sqcup id$ in $\mathbb{M}(M_1 \sqcup M_2)$ and,
- ii) $\sqcup_{a,b}(E) = \sqcup_{a,b}(E')$ in $\mathbb{M}(\sqcup_{a,b}(M))$.

which completes our characterization of the CW-complex $\mathbb{M}(M)$.

At the end of this chapter, we will prove that the CW-complex as we just defined it is connected and simply connected.

3.4 The complex $\mathbb{M}^{max}(M)$

The first thing to note here is that to any e-surface M , we can apply a cut system to M in such a way that M becomes a set of surfaces M_i (where i is the set of label of the connected components of M) that are homeomorphic to a sphere with 0, 1, 2 or 3 punctures. We will qualify such a system of *maximal* if we have cut the surface in such a way that it is represented as a minimal number of components with 0,1,2 or 3 punctures. Starting from our complex $\mathbb{M}(M)$, we can define a 2-dimensional CW-complex $\mathbb{M}^{max}(M)$ with vertices: $(C, m) \in \mathbb{M}(M)$ that are such that C is maximal. Although the edges of this complex are almost the same as in $\mathbb{M}(M)$, the fact that C must remain maximal induces a constraint on the F - and Z -moves but the B -move remains the same. We now give the remaining edges of the complex and their 2-cells.

B-move: As an edge, it remains the same as the B -move previously defined.

Z-move: As an edge, it remains almost the same as the Z -move described above, except that it is not defined on a sphere with more than three holes.

F-move: The edge F_c is defined in the same way as above, but we must note that it is not defined if the cut is placed where two trinions are glued or the result of the operation will be a sphere with 4 punctures and that is not defined within the complex $\mathbb{M}^{max}(M)$.

A-move: The A-move or associativity move is defined as following: Let M be a e-surface with four punctures labeled a , b , d and e respectively and a cut c that is such that $M = M_1 \sqcup_c M_2$ where M_1 and M_2 are two trinions where the punctures a and b appear on M_1 and the punctures d and e are on M_2 . Applying A on M will map c to c' in such a way that a and d will appear on the trinion M'_1 and the two others, b and e on M'_2 . Since the cuts are defined as non-intersecting curves on M , the two cuts cannot be present on m at the same time. We therefore suppose that the system is as follow (c, m) . The A-move is define as the move $A_{c',c} : (c, m) \rightsquigarrow (c', m)$.

In other word, it is a combination of the following F-moves: $A_{c',c} = F_{c'}^{-1}F_c$. The A-move is shown in figure 11 for a marking on $S_{0,4}$. For notation purpose we assume the upper-left submarking, i.e. the marking with vertices a, b and c , to be placed to the left of the symbol \sqcup that is, if we apply a transformation G to it, we apply $G \sqcup id$ to the (whole) marking.

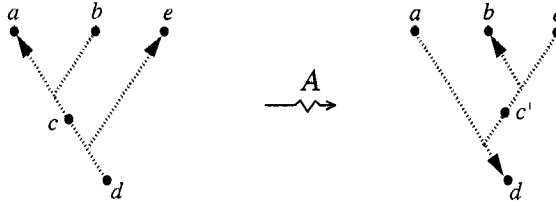


Figure 11: A-move on markings.

Again, we add all edges that are obtained from applying the disjoint union and the gluing operation and the relations together with the propagation rules still apply. We note that the complex $\mathbb{M}^{max}(M)$ is in fact a sub-complex of $\mathbb{M}(M)$ in the sense that the edge of the complex are those of $\mathbb{M}(M)$ which whose domain and targets are into $\mathbb{M}^{max}(M)$ modulo the fact that we relabel some edges in order to get A . Before getting to the equivalence relations, we note that we cannot have generalized B -move in $\mathbb{M}^{max}(M)$ except for the Dehn twist where it is defined in the same manner as above. We now give the set of equivalence relation of the complex $\mathbb{M}^{max}(M)$. This set of relations is presented almost exactly the same way as it is in [3], section 5.

Genus 0 relations in $\mathbb{M}^{max}(M)$

- **Weak associativity of cuts:** Let M be an e-surface of genus 0 and $(C, m) \in \mathbb{M}^{max}(M)$ be a marking on it with two cuts, say c_1 and c_2 which is such that $M = M_1 \sqcup_{c_1} M_2 \sqcup_{c_2} M_3$ and M_2 is a cylinder. We have: $F_{c_1} = F_{c_2}$.

- **Symmetry of F-move:** This relation is the same as in the genus 0 relations except that since we are in a maximal complex, one of M_1 or M_2 has at most three punctures while the other has at most two.
- **Rotation axiom:** This relation is the same as in the genus 0 relations except that, again, we restrict ourselves on an e-surface of at most three holes.
- **Commutativity of disjoint union:** This one is exactly the same as in the genus 0 relations.
- **Cylinder axiom:** This one is exactly the same as in the genus 0 relations.
- **Self-duality of the A-move:** Given M and $(\{c\}, m)$ like in the definition of the A-move, then $A^2(\{c\}, m) = id(\{c\}, m)$.
- **Triangle axiom:** The following diagram commutes.

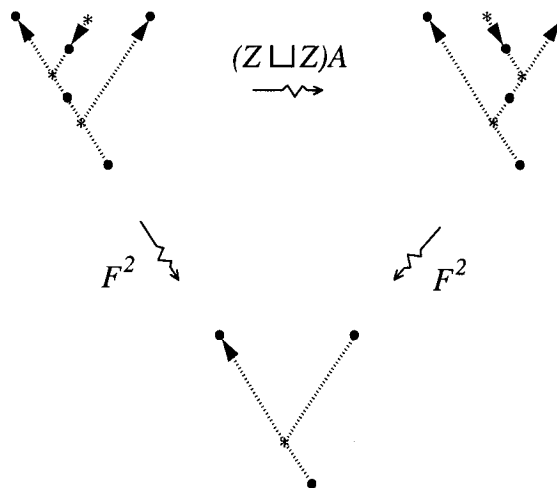


Figure 12: The triangle axiom.

This axiom is represented with the convention for the A-move above.

- **Pentagon axiom:** The diagram presented in figure 13 commutes, with arrows:

$$\begin{aligned}
 (1) &= (id \sqcup Z)A & (2) &= (Z \sqcup Z)A \\
 (4) &= (id \sqcup Z)A(Z^{-1} \sqcup id) & (3) &= (id \sqcup Z)A \\
 (5) &= (id \sqcup Z)A(id \sqcup Z)
 \end{aligned}$$

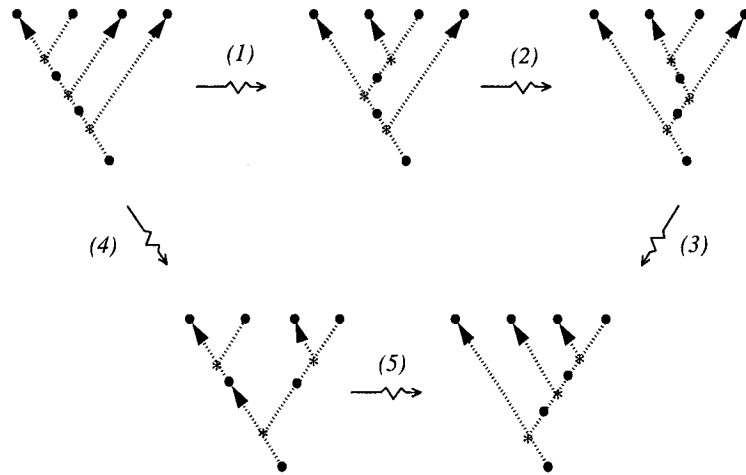


Figure 13: The pentagon axiom.

- **Hexagon axioms:** The diagram presented in figure 14 commutes, with arrows:

$$\begin{aligned}
 (1) &= (id \sqcup Z)A(id \sqcup Z) & (2) &= B_{a,bc} \\
 (4) &= B_{a,b} & (3) &= (id \sqcup Z)A(id \sqcup Z) \\
 (5) &= (id \sqcup Z)A(id \sqcup Z) & (6) &= B_{a,c}
 \end{aligned}$$

and with arrows:

$$\begin{aligned}
 (1) &= (id \sqcup Z)A(id \sqcup Z) & (2) &= B_{a,bc}^{-1} \\
 (4) &= B_{a,b}^{-1} & (3) &= (id \sqcup Z)A(id \sqcup Z) \\
 (5) &= (id \sqcup Z)A(id \sqcup Z) & (6) &= B_{a,c}^{-1}
 \end{aligned}$$

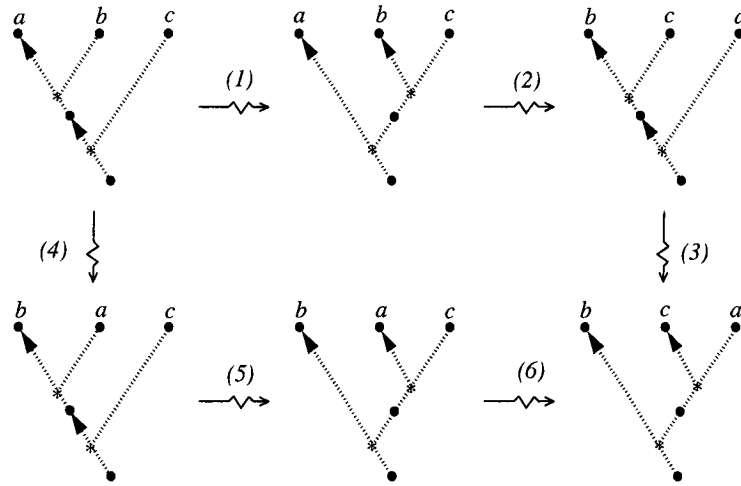


Figure 14: The hexagon axioms.

Note that for the previous axioms, in order to lighten somewhat the notation, the moves applied to the markings are restricted to the part of the marking that is affected (i.e. we don't add any *id* for the submarkings that remain fix).

Later, we will work in the context of a maximal complex. We would like to know if this complex is connected and simply connected in order to see that, starting from a given e-surface, we can apply any transformation we want on it in order to get to another e-surface. As the maximal complex $\mathbb{M}^{max}(M)$ is nothing but a subcomplex of \mathbb{M} we will work with the latter to show connectedness property and the inclusion above will yield the desired result.

3.5 Proof of the main theorem

The statement of the following theorem is found in [3] in subsection 4.9, p. 12. We closely follow the proof presented in section 6, pp. 20-23 and we fill the gaps.

Theorem 3.5.1 *The complex $\mathbb{M}(M)$ for a genus 0 surface M is connected and simply connected.*

Proof: We first construct the CW-complex $\overline{\mathbb{M}}(M)$ in the following manner: Starting from $\mathbb{M}(M)$, we add the edges $B_{A,B}$ (generalized braidings) and T_A (Dehn twists) to it as a composition of simple moves Z , B and F . Also, let us add all edges induced by the gluing and the disjoint union operations. As this involves only relabelling of the arrow, if we show that $\overline{\mathbb{M}}(M)$ is connected and simply connected, then this implies that it is also the case for $\mathbb{M}(M)$.

Let us also define the forgetful map $u : \mathcal{M}(M) \rightarrow \mathcal{C}(M)$; $(C, m) \mapsto C$ where $\mathcal{C}(M)$ is the set of all weak cut system on M .

Let $\mathbb{C}(M)$ be a CW-complex where the set of vertices is the set $\mathcal{C}(M)$ and the edges are $\overline{F}_c : C \rightarrow \emptyset$, that is fusion moves acting on a weak cut system of only one cut. Then, by disjoint union, we can extend this move to a set of cuts of arbitrary size. By definitions of a weak cut system, we also have $\overline{F}_c \overline{F}_{c'} = \overline{F}_{c'} \overline{F}_c$ provided that $c \neq c'$.

Note that the map u can be canonically extended to a map of CW-complexes $u' : \overline{\mathbb{M}}(M) \rightarrow \mathbb{C}(M)$ by defining $u(F) = \overline{F}$, $u(Z) = u(B) = id$.

Lemma 3.5.2 *The complex $\mathbb{C}(M)$ is connected and simply connected.*

Proof: Given two \overline{F} moves, then either $\overline{F}_c \overline{F}_{c'}^{-1} = \overline{F}_{c'}^{-1} \overline{F}_c$ or $\overline{F}_c \overline{F}_{c'}^{-1} = id$. This implies that every loop in this complex yields a composition of the form $\overline{F}_{c_1} \dots \overline{F}_{c_n} \overline{F}_{c_n}^{-1} \dots \overline{F}_{c_1}^{-1}$ by deformation. Since we are working with M as a surface of genus 0, starting from any weak cut system and applying a series of \overline{F} moves to each cuts yield the empty cut. Therefore starting at the empty cut, each composition of the form $\overline{F}_{c_1} \dots \overline{F}_{c_n} \overline{F}_{c_n}^{-1} \dots \overline{F}_{c_1}^{-1}$ is homotopic to the identity, which yields the desired result.

□

We would now like to verify that for every C of $\mathbb{C}(M)$, the preimage of u , i.e. $u^{-1}(C)$, is connected and simply connected in $\mathbb{M}(M)$. Let C be a vertex in $\mathbb{C}(M)$ and let

$\{M_i\}; i \in I$ be the set of connected components of $\overline{M} \setminus C$ where I is the indexing set. We have $u^{-1}(C) \subset \mathcal{M}(M)$ and this can be identified canonically with the product of $\mathcal{M}^\varnothing(M_i)$ over all i where $\mathcal{M}^\varnothing(M)$ is the set of all markings without cut of the component M_i . Therefore, it is enough to verify that that every $\overline{\mathcal{M}}^\varnothing(S_{0,n})$ is connected and simply-connected.

We already know (by proposition 3.2.9) that the set $\mathbb{M}^\varnothing(M)$ is in bijection with $\Gamma_{0,n}$. Consider:

- t_i : which is the Dehn twist around the i th puncture,
- b_i : which is the braiding of the i th and $(i+1)$ th puncture and,
- z : a homeomorphism that acts on the set of boundary component by $i \mapsto i+1$ and sends n to 1

as elements of $\Gamma_{0,n}$.

Lemma 3.5.3 *The group $\Gamma_{0,n}$ is generated by the elements b_i, t_i and z and are submitted to the following relations:*

$$\begin{aligned}
 b_i b_j &= b_j b_i & |i - j| > 1, \\
 b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}, \\
 b_i t_j &= t_j b_i & |i - j| \geq 1, \\
 b_i^{\pm 1} t_i &= t_i b_i^{\mp 1} \\
 t_i t_j &= t_j t_i, \\
 z^n &= 1, \\
 b_1 \dots b_{n-1} t_n &= z, \\
 z t_n &= t_1 z.
 \end{aligned}$$

Proof: This proof is given in [17] although it uses a massive amount of conformal field theory. For that reason, we do not reiterate the proof here.

□

We now link moves and homeomorphisms of M in the following manner. Let $f : M \xrightarrow{\sim} S_{0,n}$ be an homeomorphism that for a given set of labels $\{a_1, \dots, a_n\}$ of the boundary components induces an order $a_1 < \dots < a_n$ on it. This homeomorphism can be viewed as an element of $\mathcal{M}^\varnothing(M)$. Thus, it induces the following edges in $\overline{\mathbb{M}}^\varnothing(M)$:

- $B_{a_i, a_{i+1}} : f \rightsquigarrow b_i \circ f$
- $T_{a_i} : f \rightsquigarrow t_i \circ f$
- $Z : f \rightsquigarrow z \circ f$

This implies that each homeomorphism of the form $b_i \circ f$, $t_i \circ f$ and $z \circ f$ is an edge in $\overline{\mathbb{M}}^\varnothing$. Hence, they can be connected by a path in $\overline{\mathbb{M}}^\varnothing$ showing connectedness of that CW-complex.

In order to prove simple connectedness, we recall from generalized braiding that every path can be deformed to a path consisting of the composition of $B_{a,b}$ for neighboring (ordained) boundary components. Thus, any path can be contracted as composition of the moves $b_i \circ f$, $t_i \circ f$ and $z \circ f$, which implies that we must show that any closed loop of such a composition is now contractible. As these moves correspond to the generators of the previous lemma, this reduces to check that each loop induced from composition is contractible:

- $B_i B_j = B_j B_i$, $|i - j| > 1$: This one follows immediately from the commutativity of the disjoint union.
- $b_{i,i+1} b_{i+1,i+2} b_{i,i+1} = b_{i+1,i+2} b_{i,i+1} b_{i+1,i+2}$: Using the commutativity of the disjoint union and generalized braiding, we rewrite this expression as $B_{i,i+1} B_{i(i+1),i+2} = B_{(i+1)i,i+2} B_{i,i+1}$, which holds by commutativity of the disjoint union.
- $B_{i,i+1} T_j = T_j B_{i,i+1}$, $|i - j| \leq 1$: In the case that $|i - j| > 1$, the relation immediately holds by the commutativity of the disjoint union, in the case $|i - j| = 1$ we use first the cylinder axiom and then the commutativity of the disjoint union.

- $B_{i,i+1}^{\pm 1} T_i = T_{i+1} B_{i,i+1}^{\mp 1}$: Same as above except that we must also apply the cylinder axiom before and after the expression $B_{i,i+1} T_i$ in order to preserve the Dehn twist affecting the braid operation.
- $T_i T_j = T_j T_i$: Holds again by the commutativity of the disjoint union.
- $B_{1,2} \dots B_{n-1,n} T_n = Z$: We use generalized the braiding yielding $B_{1\dots n-1,n} T_n = Z$ by definition.
- $Z T_n = T_1 Z$: We check the case $n = 2$, we then have $Z T_2 = T_1 Z$, which is trivially true by the Dehn twist axiom. This is generally true for all n as the other boundary components are not affected by the moves.

Which shows that $\overline{\mathbb{M}}^{\partial}(M)$ is simply connected. Hence, we have that for u both the base and the fiber are connected and simply connected.

Lemma 3.5.4 *Given an edge in $\mathbb{C}(M)$ say $e : C_1 \rightsquigarrow C_2$ so that it lifts to both $e_1 : m_1 \rightsquigarrow m_2$ and $e_2 : m'_1 \rightsquigarrow m'_2$ belonging to $\mathbb{M}(M)$ then, we can choose two paths $e_3 : m_1 \rightsquigarrow m'_1$ and $e_4 : m_2 \rightsquigarrow m'_2$ in the same complex so that*

$$\begin{array}{ccc}
 m_1 & \xrightarrow{e_1} & m_2 \\
 e_3 \downarrow & & \downarrow e_4 \\
 m'_1 & \xrightarrow{e_2} & m'_2
 \end{array}$$

commutes.

Proof: Results of the connectedness of $\overline{\mathbb{M}}^{\partial}(M)$ show that two markings with the same cut system can be joined by a composition and products of simple moves Z and B applied to each components of $\overline{M \setminus C}$. Therefore, we consider only the two following elementary cases: let e_3 be either Z or B .

Z : The statement is verified following the symmetry of the F -move.

B : Let $M = S_{0,i} \sqcup_c S_{0,j}$ and let $a, b \in \{\partial(M)\}$, we need to show that there is a path e_4 such that

$$\begin{array}{ccc}
 m_1 & \xrightarrow{F_c} & m_2 \\
 B_{a,b} \downarrow & & \downarrow e_4 \\
 m'_1 & \xrightarrow{F_c} & m'_2
 \end{array}$$

commutes. We have two subcases:

- $a, b \neq c$: this is trivial, using $e_4 = B_{a,b}$.
- $b = c$: choose $e_4 = B_{a,S}$ where $S = \{\partial(S_{0,n})\}/C$.

Which concludes the proof of the lemma.

□

Finally,

Lemma 3.5.5 *given a two-cell in $\mathbb{C}(M)$, it's end-points can be lifted to a contractible loop in $\overline{\mathbb{M}}(M)$*

Proof: obviously, the only two-cells of $\mathbb{C}(M)$ are given by associativity, then they can be lifted to two-cells in $\overline{\mathbb{M}}(M)$ who also satisfies associativity.

□

Thence, as all the previous lemmas holds, we consider surjective the restriction of u' , $u'' : \mathbb{M}(M)^{(1)} \rightarrow \mathbb{C}(M)^{(1)}$ which maps the one-skeleton of $\mathbb{M}(M)$ to the one skeleton of $\mathbb{C}(M)^{(1)}$. Lemma 3.5.1 and our extension of lemma 3.5.2 tells us that the base and the fiber of u' are both connected and simply connected. Lemma 3.5.3 implies that this generalizes to the whole one-skeleton, and finally, the last lemma 3.5.4, that this generalizes to the 2-cells and then to u' . This yields the connectedness and the simply-connectedness of $\overline{\mathbb{M}}(M)$, and the same for $\mathbb{M}(M)$ by extension as required. This completes the proof of the main theorem.

□

Now, we prove that the same holds for the complex $\mathbb{M}^{max}(M)$ with the next result whose statement is taken from [3], p. 19.; we also follow closely the proof given there.

Corollary 3.5.6 *The complex $\mathbb{M}^{max}(M)$ for a genus 0 surface is connected and simply connected.*

Proof: If we show that $\mathbb{M}^{max}(M)$ and $\mathbb{M}(M)$ are homotopically equivalent (i.e. that each closed loop in $\mathbb{M}^{max}(M)$ can be extended to a closed loop in $\mathbb{M}(M)$ and the converse), then the result follows from the last theorem.

Let $m, m' \in \mathcal{M}(M)$. We say that m' is a *subdivision* of m if by applying a sequence of fusion moves one can obtain m from m' ; we will then write $m \subset m'$. We also define $Sub(m) = \{m' \in \mathbb{M}^{max}(M) | m \subset m'\}$.

Lemma 3.5.7 *Let $m', m'' \in Sub(m)$; we can then connect them by a path consisting of F , Z and A -moves lying in $Sub(m)$*

Proof: We consider $m = m_0$, the standard marking without cut of $S_{0,n}$. Apply a maximal cut system on it and then consider the subdivision $Sub(m)$ resulting from such a decomposition. The pentagon and triangle axiom are holding for submarkings containing the correct number of components. As these two axioms provide all the possible configurations of the markings and use only the F , Z and A -moves one can then join m' and $m'' \in Sub(m)$, and as the F -move in this context does not affect the maximality of the complex, we remain in $Sub(m)$ as required.

□

Lemma 3.5.8 *Every loop in $Sub(M)$ consisting of the same moves as in lemma 3.5.7 is contractible.*

Proof: This again follows from the commutativity of the triangle and the pentagon in the axioms of the same name.

□

Now, given an $m \in \mathcal{M}$, we choose an element $\iota(m) \in Sub(m)$ which is such that M corresponding to m is cut in connected components of at most three holes in a maximal manner in the sense that we do not have cylinders between two trinions. We now extend $\iota : \mathcal{M}(M) \rightarrow \mathcal{M}^{max}(M)$ to a map of CW complexes. Given an edge B in $\mathbb{M}(M)$ which is a generalized braiding, we restrict it to a composition of B -moves

(not generalized) on trinions in $\mathbb{M}(M)$ which corresponds to the image of $\iota(B)$. Given an edge $F_c : M_1 \sqcup M_2 \rightsquigarrow M$, its image $\iota(F_c)$ becomes a path composed of Z , F and A -moves as in lemma 3.5.5. This path is unique up to homotopy by lemma 3.5.6. For $\iota(Z)$, it follows in the same manner. Hence, ι satisfies all relations in \mathbb{M} .

Conversely, we have $\iota' : \mathbb{M}^{max}(M) \hookrightarrow \mathbb{M}(M)$. The composition $\iota' \circ \iota$ is an equivalence of complexes which induces that every loop in \mathbb{M}^{max} is homotopic to a loop in $\mathbb{M}(M)$ where all loops are contractible, which in turn implies that each loop is contractible in $\mathbb{M}^{max}(M)$ as required.

□

These two results will allow us to build any possible algorithm made from topological maps when we will develop our semantics in chapter 6. Indeed, connectedness and simple-connectedness of the CW-complex tells us that any sequence of topological map is well defined.

Chapter 4

Ribbon categories

We now develop a categorical framework that is such that it finds some parallel with the mapping class group of e-surfaces developed in the previous chapter. This is a necessity in order to define a modular functor that will be as close as possible to that structure. We start by explaining the notions of abelian category and semisimple abelian category. We then develop the notions of monoidal category, which we then equip with a braid structure, a first similarity with the mapping class group of e-surfaces. Next, we continue by giving to the latter a rigid structure by inspecting the notion of duals that, as we shall see, parallels the notion of orientation of an e-surface and we equip the category with ribbon structure which then implies that the category is equipped with a twist which will find a counterpart to the Dehn-twist presented in the last section. Finally, we endow the ribbon category with a semisimple structure together with some constructions in that context and we conclude this chapter with a graphical calculus of morphisms for ribbon categories that will be useful to prove some lemma and theorems in the next chapters. The exposition follows many sources that will be cited in time, but the general orientation of it is taken from [4] chapters 1 and 2 and most extensions are from [14]. From now on we assume that we work with a field \mathbb{K} of characteristic 0 if needed.

4.1 Semisimple abelian categories

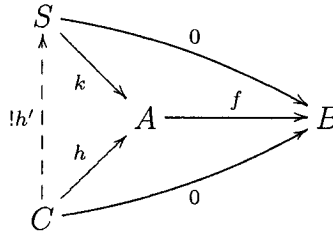
Before we begin to establish our categorical framework, we give a few definitions:

Definition 4.1.1 *In a category \mathcal{C} a zero object, denoted 0 , is an object that is both initial and terminal.*

In particular, this implies that for any pair of objects $A, B \in |\mathcal{C}|$ we can form the unique composition $0 : A \rightarrow 0 \rightarrow B$. This unique arrow 0 will be called the *zero-morphism*.

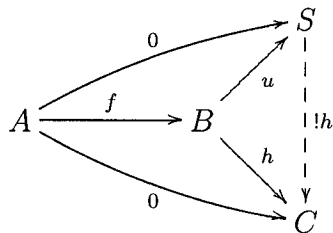
Now, the following notions are taken from [14] section VIII, pp. 191-193.

Definition 4.1.2 *Given an arrow $f : A \rightarrow B$ in $\text{Hom}_{\mathcal{C}}(A, B)$, we define the kernel f denoted $\ker(f) = k : S \rightarrow A$ as an arrow that is such that if $f \circ k = 0$, where 0 is the zero-morphism. Then, for every $h : C \rightarrow A$ such that $h \circ f = 0$, then h factors uniquely through k as $h = h' \circ k$ (with h' a unique arrow). This is depicted as the following diagram:*



Now, the notion of cokernel is exactly dual to the previous one, that is:

Definition 4.1.3 *Given an arrow $f : A \rightarrow B$ in $\text{Hom}_{\mathcal{C}}(A, B)$ we define the cokernel f denoted $\text{coker}(f) = u : B \rightarrow S$ such that $u \circ f = 0$ and if $h : B \rightarrow C$ is such that $h \circ f = 0$ then h factors through u as $h = u \circ h'$ (with h' a unique arrow). This is depicted as the following diagram:*



4.1.1 Category with biproducts

As we will consider only semisimple category for our semantics, it is essential that we define the notion of category with biproducts as it is part of the semisimple definition. We do so along the lines given in [1].

Definition 4.1.4 *Let $A_1, A_2 \in |\mathcal{C}|$, then, the biproduct of A and B is an object $(A \oplus B) \in |\mathcal{C}|$, for which we can choose the morphisms $\pi_i : A_1 \oplus A_2 \rightarrow A_i$ and $\iota_i : A_i \rightarrow A_1 \oplus A_2$ for $i = 1, 2$, which are called projections and injections respectively. They are such that the pair π_1, π_2 forms a product cone and the pair ι_1, ι_2 forms a coproduct cone. Hence, it makes the biproduct operation which is both a product and a coproduct. Furthermore, those morphisms are such that $\pi_i \circ \iota_j = \delta_{ij}$ where δ_{ij} is Kronecker's delta.*

Definition 4.1.5 *We say that $(\mathcal{C}, \oplus, 0)$ is said to be a category with finite biproducts provided that the biproduct of any pair of objects exists.*

We can extend the biproduct structure to the Hom -sets as an addition operation on morphisms. Given $f, g \in Hom_{\mathcal{C}}(A_1, A_2)$, we can define $f + g \in Hom_{\mathcal{C}}(A_1, A_2)$ as

$$f + g = A_1 \xrightarrow{\Delta} A_1 \oplus A_1 \xrightarrow{f \oplus g} A_2 \oplus A_2 \xrightarrow{\nabla} B. \quad (12)$$

Here, $\Delta = \langle id, id \rangle$ and $\nabla = [id, id]$ are respectively the diagonal and the codiagonal maps for the biproduct.

Remark 4.1.6 *Provided that it exists, the biproduct $A_1 \oplus A_2$ is unique up to an isomorphism and it is functorial both in A_1 and A_2 . Finally, it has the following structural properties:*

- i) $\pi_1 = [id_{A_1}, 0_{A_2, A_1}]$,*
- ii) $\pi_2 = [0_{A_1, A_2}, id_{A_2}]$,*
- iii) $\iota_1 = \langle id_{A_1}, 0_{A_1, A_2} \rangle$,*
- iv) $\iota_2 = \langle 0_{A_2, A_1}, id_{A_2} \rangle$ and*
- v) $\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = id_{A_1 \oplus A_2}$.*

This completes our introduction on categories with biproducts.

4.1.2 Additive and abelian categories

Definition 4.1.7 We say that a category \mathcal{C} with $U, V \in |\mathcal{C}|$ is additive over \mathbb{K} if:

i) All $\text{Hom}_{\mathcal{C}}(U, V)$ are \mathbb{K} -vector spaces and the compositions of morphisms are \mathbb{K} -bilinear for all $U, V \in |\mathcal{C}|$.

ii) The category \mathcal{C} has a zero object (denoted 0).

iii) Every finite family of objects in \mathcal{C} has a biproduct.

Furthermore, we say that an additive category is abelian if:

iv) All $f \in \text{Hom}_{\mathcal{C}}(U, V)$ have a kernel $\text{Ker}(f)$ and a cokernel $\text{Coker}(f)$ which are both morphisms of \mathcal{C} . The kernel and the cokernel are subject to the following relations: if $\text{Ker}(f) = 0$ (resp. $\text{Coker}(f) = 0$) then $f = \text{Ker}(\text{Coker}(f))$ (resp. $f = \text{Coker}(\text{Ker}(f))$).

Following [14], section VIII , p. 197, we define:

Definition 4.1.8 If \mathcal{B} and \mathcal{C} are abelian categories, a functor $F : \mathcal{B} \rightarrow \mathcal{C}$ is additive when every $F : \text{Hom}_{\mathcal{B}}(B, B') \rightarrow \text{Hom}_{\mathcal{C}}(FB, FB')$, with $B, B' \in |\mathcal{B}|$, is a homomorphism of abelian groups i.e. $F(f + f') = F(f) + F(f')$ for any pair f and f' .

We will assume that functors that map additive category to additive category are additive and \mathbb{K} -linear on Hom -spaces. Unless stated otherwise, we assume \mathcal{C} to be abelian.

We give a last relevant definition that will be useful later, the one of tensor product of abelian categories.

Definition 4.1.9 Given two abelian categories, say \mathcal{C}_1 and \mathcal{C}_2 with $A_i \in |\mathcal{C}_1|$ and $B_i \in |\mathcal{C}_2|$ respectively. We define the tensor product of \mathcal{C}_1 and \mathcal{C}_2 , denoted $\mathcal{C}_1 \boxtimes \mathcal{C}_2$, as the category with:

-Objects:

$$\bigoplus_i (A_i, B_i)$$

which are always finite sums, and

-Morphisms:

$$\text{Hom}_{\mathcal{C}_1 \boxtimes \mathcal{C}_2} \left(\bigoplus_i (A_i, B_i), \bigoplus_j (A'_j, B'_j) \right) = \bigoplus_{i,j} (\text{Hom}_{\mathcal{C}_1}(A_i, A'_i) \otimes \text{Hom}_{\mathcal{C}_2}(B_i, B'_i)).$$

Given $f_1, f_2 \in \bigoplus_i \text{Hom}_{\mathcal{C}_1}(A_i, A'_i)$ and $g_1, g_2 \in \bigoplus_j \text{Hom}_{\mathcal{C}_2}(B_j, B'_j)$ that are such that they compose in their respective categories, then the composition is given by $(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1)$ and extends by bilinearity to arbitrary morphisms. As we required that all the terms of the sum be zero but a finite number of them, this ensure the operators to be bounded. The identities are of the form $(id_A \otimes id_B)$ for any $A \in |\mathcal{C}_1|$ and $B \in |\mathcal{C}_2|$.

The tensor product of an abelian category \mathcal{C} n times with itself will be denoted $\mathcal{C}^{\boxtimes n}$ and, in a similar manner, given a finite index set A that will be used to label the objects $\{\mathcal{C}_a\}_{a \in A}$, we will denote the tensor product of those indexed objects as $\boxtimes_{a \in A} \mathcal{C}_a$. Finally, we define $\mathcal{C}^0 = \mathcal{C}^\emptyset = \text{Vect}_f \mathbb{K}$, the category of finite dimensional vector spaces over \mathbb{K} .

4.1.3 Semisimple categories

Suppose \mathcal{C} is an abelian category as described above and it is equipped with a biproduct \oplus then we have the following definitions:

Definition 4.1.10 *An object $V \in |\mathcal{C}|$ which is not isomorphic to 0 is said to be simple if for all $W \in |\mathcal{C}|$ and any injection $f : W \hookrightarrow V$, f is either the 0 morphism or an isomorphism.*

Definition 4.1.11 *An abelian category \mathcal{C} is semisimple if any $V \in |\mathcal{C}|$ is isomorphic to a biproduct of simple ones. That is:*

$$V \simeq \bigoplus_{i \in I} N_i V_i, \tag{13}$$

where the V_i are simple objects of \mathcal{C} , I is the set of isomorphism classes of simple objects and the N_i are positive integers such that only a finite number of them are non-zero.

Remark 4.1.12 Note that if $W \simeq \bigoplus_{i \in I} N_i V_i$, then we have $\text{Hom}_{\mathcal{C}}(V, W) \simeq \bigoplus_{i \in I} \text{Hom}_{\mathcal{C}}(V, N_i V_i)$

During this exposition we will assume that the semisimple categories considered are such that the set of endomorphism of simple objects is isomorphic to the base field \mathbb{K} .

4.2 Braided monoidal categories

4.2.1 Monoidal categories

We now need to consider monoidal structure on categories and check how associativity and commutativity behave under the monoidal structure.

A category \mathcal{C} is said to be monoidal if it is equipped with a tensor product (denoted \otimes) together with an identity for that operation. More formally, we have:

Definition 4.2.1 A strict monoidal category $\langle \mathcal{C}, \otimes, I \rangle$ is a category \mathcal{C} equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ such that for A, B and $C \in |\mathcal{C}|$ we have $A \otimes (B \otimes C) = (A \otimes B) \otimes C$. It is also equipped with an object $I \in |\mathcal{C}|$ acting as a right and a left identity for \otimes so that for any $A \in |\mathcal{C}|$ we have $A \otimes I = I \otimes A = A$.

Note that the associativity of the tensor product or even the equations $I \otimes V = V \otimes I = V$ does not hold in all category (associativity does not even hold in the category $\text{Vect}_f \mathbb{K}$, the category of finite dimensional \mathbb{K} -vector spaces). Thus we relax the strictness of the structure in order to change the equalities in natural isomorphisms.

Definition 4.2.2 A monoidal category $\langle \mathcal{C}, \otimes, I, \alpha, \lambda, \rho \rangle$ is a category \mathcal{C} equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, an object $I \in |\mathcal{C}|$, called the unit object, and a set of three natural isomorphisms α, λ and ρ . Explicitely, these natural isomorphisms are given as:

$$\alpha_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W), \quad (14)$$

$$\lambda_V : I \otimes V \rightarrow V \text{ and} \quad (15)$$

$$\rho_V : V \otimes I \rightarrow V. \quad (16)$$

Where α is just the associativity of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$. Note that all three are natural in $U, V, W \in |\mathcal{C}|$. Also, we require that they make the following diagrams commute for all U, V and $V_i \in |\mathcal{C}|$:

i) **Pentagon axiom:**

$$\begin{array}{ccc}
 & ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 & \\
 \alpha_{1,2,3} \otimes id \swarrow & & \searrow \alpha_{12,3,4} \\
 (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 & & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \\
 \alpha_{1,23,4} \downarrow & & \downarrow \alpha_{1,2,34} \\
 V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) & \xrightarrow{id \otimes \alpha_{2,3,4}} & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))
 \end{array}$$

ii) **Triangle axiom:**

$$\begin{array}{ccc}
 (U \otimes I) \otimes V & \xrightarrow{\alpha} & U \otimes (I \otimes V) \\
 \rho \otimes id \searrow & & \swarrow id \otimes \lambda \\
 & U \otimes V &
 \end{array}$$

This complete the definition of a monoidal category.

It is worth noting that every monoidal category is equivalent to a strict one (this theorem is due to MacLane, the interested reader might refer to [14] section XI.3) but as we study cases that are essentially not strict in essence, we will consider that all isomorphisms which are constructed via a composition of α , λ and ρ are considered canonical. Therefore, each such isomorphism that has the same domain and the same target will be considered as an equality between objects. Note also that, provided that \mathcal{C} is abelian, we require I to be a simple object and therefore we have $End(I) = \mathbb{K}$ by definition (see section 4.1.2).

4.2.2 Braided monoidal categories

We now inspect the case of a monoidal category equipped with a set of commutativity isomorphisms of the form $\sigma_{U,V} : U \otimes V \xrightarrow{\sim} V \otimes U$ and the requirement for these isomorphisms to be coherent with the natural isomorphisms α , λ and ρ listed in the

previous section for monoidal categories. We chose to inspect this problem by means of the braid group. The following definition is close to definition 1.2.1 given in [4], p. 15.

Definition 4.2.3 *An n -stranded braid is the union of n non-intersecting smooth curves (considered up to isotopy) in \mathbb{R}^3 going from $(i, 0, 0); i \in \{1, \dots, n\}$ to $(i, 0, 1)$ with $i \in \{1, \dots, n\}$ such that for all strands the third coordinate is strictly increasing.*

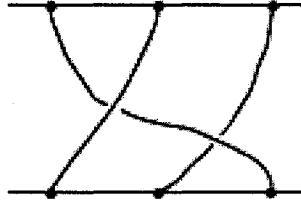
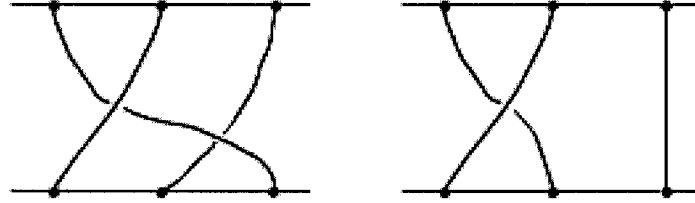


Figure 15: A braid in three strands.

We can define multiplication of two braids as a stacking operation over two elements of the set of braids. Suppose we have the following two braids a and b as described in figure 16, then we can form the product of a and b (denoted ab) by stacking a over b ; this product is depicted in figure 17 and we also depicted the identity of the 3-stranded braid (B_3) in figure 18.

This gives us a group structure for fixed n (the number of strands). Indeed, denote the set of braids in n strands B_n and its elements σ_i which twist the i th strand once *under* the $(i+1)$ th strand, the inverses are defined as σ_i^{-1} which twists the i th strand once *over* the $(i+1)$ th strand. These two elements are inverses one to the other as we are taking the graphs up to isotopy. Indeed, in B_3 , we have $\sigma_2\sigma_2^{-1} = 1$ as shown below:

Where \doteq means that we have equality up to isotopy of the strands.

Figure 16: Braid a (left) and b (right).

The following theorem is due to E. Artin and the group structure described above is often named the Artin Braid group after him.

Theorem 4.2.4 *The braid group B_n is generated by $\{\sigma_i\}$ for $i \in \{1, \dots, n-1\}$ and is subject to the following relations:*

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1 \quad (17)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (18)$$

Proof: This theorem is well known, we refer the reader to [5] for a complete exposition on the subject.

□

We now give the definition of a braided monoidal category in the same way as we did for monoidal category.

Definition 4.2.5 *A braided monoidal category $\langle \mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma \rangle$ is a monoidal category equipped with a natural isomorphism $\sigma : U \otimes V \xrightarrow{\sim} V \otimes U$. We require that the natural isomorphisms satisfy:*

- i) **Pentagon axiom:** See definition of a monoidal category, i.e. definition 4.2.2.*
- ii) **Triangle axiom:** Idem.*

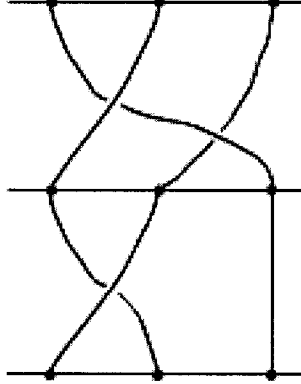


Figure 17: Product of a and b .

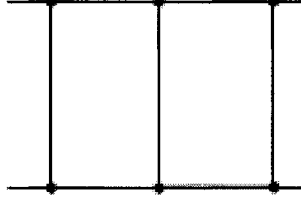
iii) **Hexagon axioms:**

a) we require that the following diagrams commute for all $U, V, W \in |\mathcal{C}|$:

$$\begin{array}{ccccc}
 & & U \otimes (V \otimes W) \xrightarrow{\sigma_{U,V \otimes W}} (V \otimes W) \otimes U & & \\
 & \nearrow \alpha_{U,V,W} & & \searrow \alpha_{V,W,U} & \\
 (U \otimes V) \otimes W & & & & V \otimes (W \otimes U) \\
 & \searrow \sigma_{U,V} \otimes id & & \nearrow id \otimes \sigma_{U,W} & \\
 & & (V \otimes U) \otimes W \xrightarrow{\alpha_{V,U,W}} V \otimes (U \otimes W) & &
 \end{array}$$

b) For any $U, V, W \in |\mathcal{C}|$ the following diagram commutes:

$$\begin{array}{ccccc}
 & & U \otimes (V \otimes W) \xrightarrow{\sigma_{U,V \otimes W}^{-1}} (V \otimes W) \otimes U & & \\
 & \nearrow \alpha_{U,V,W} & & \searrow \alpha_{V,W,U} & \\
 (U \otimes V) \otimes W & & & & V \otimes (W \otimes U) \\
 & \searrow \sigma_{U,V}^{-1} \otimes id & & \nearrow id \otimes \sigma_{U,W}^{-1} & \\
 & & (V \otimes U) \otimes W \xrightarrow{\alpha_{V,U,W}} V \otimes (U \otimes W) & &
 \end{array}$$

Figure 18: Braid identity for B_3 .

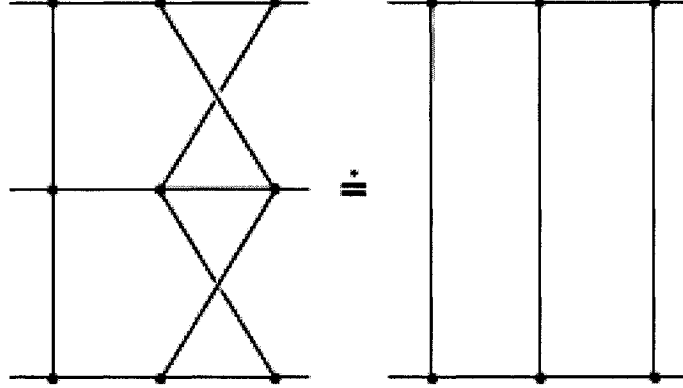
Remark 4.2.6 Following [4], pp. 16-17, we can also define a braided monoidal category as follows: We assign to each functorial isomorphism in the braided monoidal category an element of the braid group. To all natural isomorphisms inherent to the monoidal structure (that is α , λ and ρ), we assign the identity and to the natural isomorphism $\sigma_{V_k, V_{k+1}}$ that swaps two objects or blocks of tensored objects we assign the generator σ_k . Then, any composition of α , λ , ρ , σ and their inverses (call this composition f) acting on a (bracketed) tensored product of object depends only on the image of f in the braid group B_n . For instance suppose we have as an object $U \otimes (V \otimes W)$ and we apply the composition $\sigma_{U \otimes V} \circ \alpha$, in the braid group, we assign $\sigma_1 \otimes id$ as we can disregard parenthesis because of the pentagon axiom of the monoidal structure. This definition is coherent with definition 4.2.5.

Note that the naturality of σ means that for any two morphisms in \mathcal{C} say $f : V_1 \rightarrow V_2$ and $g : V_3 \rightarrow V_4$ with $V_i \in \mathcal{C}$ for all i we have the following relation:

$$\sigma_{V_2, V_4}(f \otimes g) = (g \otimes f)\sigma_{V_1, V_3} \quad (19)$$

Before concluding this section we give a last definition that take in account the case where σ is involutive:

Definition 4.2.7 A braided monoidal category is said to be symmetric if for all objects V and W , σ satisfies $\sigma_{V, W}\sigma_{W, V} = id_{V, W}$.

Figure 19: Product of σ_2 and σ_2^{-1} in B_3 .

4.3 Ribbon categories

4.3.1 Rigid monoidal categories

We now need to introduce the notion of duals in our monoidal category. Basically, the dualizing operation $(\)^*$ can be seen as a contravariant endofunctor over the monoidal category \mathcal{C} . We now introduce the notions of right and left duals to an object $V \in |\mathcal{C}|$ following [4], p. 30.

Definition 4.3.1 *Let $V \in |\mathcal{C}|$, where \mathcal{C} a monoidal category. A right dual to V is an object, denoted V^* , together with the following morphisms:*

$$e_V : V^* \otimes V \rightarrow I \quad (20)$$

$$i_V : I \rightarrow V \otimes V^* \quad (21)$$

that are such that the compositions $(id_V \otimes e_V) \circ (i_V \otimes id_V)$ and $(e_V \otimes id_{V^}) \circ (id_{V^*} \otimes i_V)$ are equal to id_V and id_{V^*} respectively. These two equalities are called the rigidity relations.*

In a very similar manner,

Definition 4.3.2 Let $V \in |\mathcal{C}|$, with \mathcal{C} a monoidal category. A left dual to V is an object (denoted *V) together with the following morphisms:

$$e'_V : V \otimes {}^*V \rightarrow I \quad (22)$$

$$i'_V : I \rightarrow {}^*V \otimes V \quad (23)$$

that are such that the compositions $(id_V \otimes e'_V) \circ (i'_V \otimes id_V)$ and $(e'_V \otimes id_{{}^*V}) \circ (id_{{}^*V} \otimes i_V)$ are equal to id_V and $id_{{}^*V}$ respectively.

Remark 4.3.3 The map e_V is often called the evaluation map in the category $Vect_f$.

We have:

Lemma 4.3.4 if $V \in |\mathcal{C}|$ has a right dual V^* then, it is unique up to an isomorphism coherent with e_V and i_V .

Proof: Suppose (V^*, e_V, i_V) and $(V^\times, \epsilon_V, \iota_V)$ are two duals for V . Then we can map V^* to V^\times with the following composition: $f := (e_V \otimes id) \circ (id \otimes \iota_V)$. Now, $g := (i_V \otimes id) \circ (id \otimes \epsilon_V)$ is an inverse for that map showing that $fg = id_{V^*}$ and $gf = id_{V^\times}$. Therefore f is an isomorphism. Finally, suppose that if we have two identities for the tensor product I and I' , then $I \simeq I \otimes I' \simeq I'$ and the rigidity relations provide that the following two diagrams commute.

$$\begin{array}{ccc} V^* \otimes V & \xrightarrow{f \otimes id} & V^\times \otimes V \\ & \searrow e_V & \swarrow \epsilon_V \\ & I & \end{array} \qquad \begin{array}{ccc} V \otimes V^* & \xrightarrow{id \otimes f} & V \otimes V^\times \\ & \swarrow i_V & \searrow \iota_V \\ & I & \end{array}$$

And hence, the isomorphism f is coherent with e_V and i_V .

The proof for a left dual is similar.

□

We also have the following well known result:

Lemma 4.3.5 *Let $V \in |\mathcal{C}|$ with V^* its dual and let $U, W \in |\mathcal{C}|$ then, there are canonical isomorphisms:*

$$\text{Hom}_{\mathcal{C}}(U \otimes V, W) \simeq \text{Hom}_{\mathcal{C}}(U, W \otimes V^*) \quad (24)$$

$$\text{Hom}_{\mathcal{C}}(U, V \otimes W) \simeq \text{Hom}_{\mathcal{C}}(V^* \otimes U, W) \quad (25)$$

Proof: Let $f_1 \in \text{Hom}_{\mathcal{C}}(U \otimes V, W)$ and F be such that $F(f_1) := (f_1 \otimes id) \circ (id \otimes i_V)$ which is in $\text{Hom}_{\mathcal{C}}(U, W \otimes V^*)$. Also, for an $f_2 \in \text{Hom}_{\mathcal{C}}(U, W \otimes V^*)$, let G be such that $G(f_2) := (id \otimes e_V) \circ (f_2 \otimes id)$ which is in $\text{Hom}_{\mathcal{C}}(U \otimes V, W)$. For $u \otimes v \in U \otimes V$ and $\bar{v} \in V^*$, we have:

$$\begin{aligned} (G(F(f_1)))(u \otimes v) &= (G((f_1 \otimes id) \circ (id \otimes i_V)))(u \otimes v) \\ &= ((id \otimes e_V) \circ ((f_1 \otimes id) \circ (id \otimes i_V)) \otimes id)(u \otimes v) \\ &= (id \otimes e_V)((f_1 \otimes id)(id \otimes i_V)(u) \otimes v) \\ &= (id \otimes e_V)((f_1 \otimes id)(u \otimes v \otimes \bar{v}) \otimes v) \\ &= (id \otimes e_V)(f_1(u \otimes v) \otimes \bar{v} \otimes v) \\ &= f_1(u \otimes v) \end{aligned}$$

And hence, $GFf_1 = id f_1$.

Conversely, let $f_2(u) = w \otimes \bar{v}$, we have:

$$\begin{aligned} F(G(f_2))(u) &= F((id \otimes e_V) \circ (f_2 \otimes id))(u) \\ &= (((id \otimes e_V) \circ (f_2 \otimes id)) \otimes id) \circ (id \otimes i_V)(u) \\ &= (((id \otimes e_V)(f_2 \otimes id)) \otimes id)(u \otimes v \otimes \bar{v}) \\ &= ((id \otimes e_V)(f_2(u) \otimes v)) \otimes \bar{v} \\ &= ((id \otimes e_V)((w \otimes \bar{v}) \otimes v)) \otimes \bar{v} \\ &= w \otimes \bar{v} \\ &= f_2(u) \end{aligned}$$

where the penultimate equality holds by rigidity. This shows that $FGf_2 = id f_2$. Thus F is an isomorphism. This isomorphism is natural as i_V and e_V are canonical

maps.

For the second identity, given an $g_1 \in \text{Hom}_{\mathcal{C}}(U, V \otimes W)$, set $F'(g_1) = (id \otimes g_1) \circ (e_V \otimes id)$ and for $g_2 \in \text{Hom}_{\mathcal{C}}(V^* \otimes U, W)$, set $G'(g_2) = (i_V \otimes id) \circ (id \otimes g_2)$. We can verify that F' and G' are inverses one to the other in a similar manner as above.

□

This result says that provided that the category \mathcal{C} has duals, it comes equipped with internal *Homs*. Indeed, the preceding result yields:

$$\text{Hom}_{\mathcal{C}}(U, V) = \text{Hom}_{\mathcal{C}}(I, V \otimes U^*) \quad (26)$$

This completes our study of the dualization on objects. For morphisms if the contravariant endofunctor $(\)^*$ act on a morphism $f \in \text{Hom}_{\mathcal{C}}(U, V)$ then, we will denote $(f)^*$ as f^* given by:

$$f^* : V^* \xrightarrow{id_{V^*} \otimes i_U} V^* \otimes U \otimes U^* \xrightarrow{id_{V^*} \otimes f \otimes id_{U^*}} V^* \otimes V \otimes U^* \xrightarrow{e_V \otimes id_{U^*}} U^*.$$

That is, if $f \in \text{Hom}_{\mathcal{C}}(U, V)$, then $f^* \in \text{Hom}_{\mathcal{C}}(V^*, U^*)$.

We are now ready to give the definition relevant to this section. That definition is derived from definition 2.1.2 in [4], p.30.

Definition 4.3.6 *Let \mathcal{C} be a braided monoidal category then \mathcal{C} is said to be rigid if every $V \in |\mathcal{C}|$ has left and right duals.*

Rigid braided monoidal categories have many properties induced by rigidity on the braided monoidal structure. They will be introduced in section 4.3.3 because they are proved using the pictorial technique introduced there. For now, we note that ${}^*I \simeq I \simeq I^*$ and $(V \otimes W)^* \simeq W^* \otimes V^*$. Those properties are derived from the rigidity relations and from the fact that if an object possesses a dual, it is unique.

4.3.2 Ribbon categories

We start by giving a definition of ribbon category¹ and after we derive a few properties.

¹Ribbon categories are sometime called tortile categories or even balanced rigid braided monoidal categories.

Definition 4.3.7 *let \mathcal{C} be a rigid braided monoidal category, then it is said to be a ribbon category if for all $V \in |\mathcal{C}|$ it comes equipped with a natural isomorphism*

$$\delta_V : V \rightarrow V^{**}, \quad (27)$$

subject to the following relations:

$$\delta_{V \otimes W} = \delta_V \otimes \delta_W \quad (28)$$

$$\delta_{V^*} = (\delta_V^*)^{-1} \quad (29)$$

$$\delta_I = id \quad (30)$$

Definition 4.3.8 *In any rigid braided monoidal category define the morphism $\gamma_V : V^{**} \rightarrow V$ as the composition:*

$$V^{**} \xrightarrow{(i_V \otimes id)} V \otimes V^* \otimes V^{**} \xrightarrow{id \otimes \sigma_{V^*, V^{**}}^{-1}} V \otimes V^{**} \otimes V^* \xrightarrow{id \otimes e_{V^*}} V$$

Note that, in general, $(V \otimes W)^{**} \neq V^{**} \otimes W^{**}$ (strictly speaking) in a rigid braided monoidal category because of the braid structure; of course, if \mathcal{C} is symmetric, the equality holds. Nonetheless, we do have the following results:

Lemma 4.3.9 *Let $U, V \in |\mathcal{C}|$ with \mathcal{C} a ribbon category, then we have:*

i) $\delta_{U \otimes V}$ given by

$$\begin{array}{ccc} (U \otimes V)^{**} & \xrightarrow{\gamma_{U \otimes V}} & U \otimes V \\ \downarrow \wr & & \uparrow \sigma_{V, U} \sigma_{V, U} \\ U^{**} \otimes V^{**} & \xrightarrow{\gamma_U \otimes \gamma_V} & U \otimes V \end{array}$$

ii) γ_{V^*} as

$$\begin{array}{ccc} V^{***} & \xrightarrow{\gamma_{V^*}} & V^* \\ \delta_V^* \downarrow & & \uparrow \delta_V^* \\ V^* & \xrightarrow{\gamma_V^*} & V^{***} \end{array}$$

iii) $\gamma_I = id$

Proof: The second and the third relations are immediate if we unfold the definitions. The proof of the first relation is postponed to the next section in order to give examples of the pictorial technique.

□

Definition 4.3.10 In any ribbon category \mathcal{C} and for any $V \in |\mathcal{C}|$, we can define the following natural isomorphism called twist:

$$\theta_V := \gamma_V \delta_V : V \rightarrow V, \quad (31)$$

Lemma 4.3.11 Let $U, V \in |\mathcal{C}|$ with \mathcal{C} is a ribbon category, then we have:

i) $\theta_{U \otimes V}$ given by

$$\begin{array}{ccc} U \otimes V & \xrightarrow{\theta_{U \otimes V}} & U \otimes V \\ & \searrow \theta_U \otimes \theta_V & \nearrow \sigma_{V,U} \sigma_{V,U} \\ & U \otimes V & \end{array}$$

ii) $\theta_{V^*} = (\theta_V)^*$

iii) $\theta_I = id$

Proof: The second and the third relations are immediate if we unfold the definitions. The proof of the first relation is postponed to the next section in order to give examples of the pictorial technique.

□

Note that if σ is involutive (yielding a rigid monoidal category which is symmetric) then we let $\delta_V = \gamma_V^{-1}$. This defines a ribbon structure on that category.

We now define the concepts of trace and dimension of an object.

Definition 4.3.12 Let $U \in |\mathcal{C}|$ where \mathcal{C} is a ribbon category let also be $f : U \rightarrow U$. The trace of f , denoted $tr(f) \in End(I)$ is defined as the composition:

$$I \xrightarrow{i_U} U \otimes U^* \xrightarrow{f \otimes id} U \otimes U^* \xrightarrow{\delta_U \otimes id} U^{**} \otimes U^* \xrightarrow{e_{U^*}} I \quad (32)$$

Example 4.3.13 Using the previous definition, and supposing that \mathcal{C} is an abelian category then, if $f = id_V$, then $tr(id_V) = dim(V) \in End(I) \simeq \mathbb{K}$, this scalar is called the dimension of V .

4.3.3 Semisimple ribbon categories

We start by giving the central definition of this subsection:

Definition 4.3.14 *A semisimple ribbon category is a semisimple category endowed with a ribbon structure where I (the tensor unit) is simple, the tensor product is bilinear and where the set of endomorphisms of simple objects is isomorphic to the base field \mathbb{K} .*

Recall from section 4.1 that each $V \in |\mathcal{C}|$ is isomorphic to a biproduct of simple ones (i.e. $V \simeq \bigoplus_{i \in I} N_i V_i$ with I the set of isomorphism classes of simple objects V_i). Hence $\{V_i\}_{i \in I}$ are the chosen representative of those classes. We give the following lemma (taken from [1]) in order to illustrate the relation between the biproduct, the tensor product induced by the monoidal structure and the operation $(\)^*$ induced by the rigidity structure.

Lemma 4.3.15 *Let \mathcal{C} be semisimple ribbon categories. Then:*

- i) $(U_1 \oplus U_2) \otimes U_3 \simeq (U_1 \otimes U_3) \oplus (U_2 \otimes U_3)$,*
- ii) $(U_1 \oplus U_2)^* \simeq U_1^* \oplus U_2^*$ and,*
- iii) $0^* \simeq 0$.*

Here, $U_i \in |\mathcal{C}|$ for $i = 1, 2$ and 3 and 0 is the zero object in \mathcal{C} .

Proof: We refer the reader to [1], section 5.

□

Now, as our category is rigid, it comes equipped with a contravariant endofunctor that dualizes the objects and the morphisms, denoted as $(\)^*$. Hence, given $V \in |\mathcal{C}|$, a simple object, we would like to know if V^* is also simple. This result is given in the following lemma:

Lemma 4.3.16 *Let V be a simple object in a semisimple ribbon category \mathcal{C} , then V^* is also simple.*

Proof: First note from the previous lemma that $(V \oplus W)^* \simeq V^* \oplus W^*$. Now suppose V is *not* simple, then $V \simeq U \oplus W$ for some non-trivial $U, W \in |\mathcal{C}|$ and hence, $V^* \simeq U^* \oplus W^*$. Therefore, V^* is not simple.

□

With that result in hand, define the involutive map $*$: $I \rightarrow I$ acting on the index set of simple objects in such a way that $V_{i^*} \simeq V_i^*$. In particular, we identify V_0 to I ; as I is assumed to be self-dual, this implies that $0^* = 0$. We now refine the multiplicity coefficients (the N_i above) in the following manner:

Definition 4.3.17 (fusion rule) *Let $V_i, V_j \in |\mathcal{C}|$, two simple objects in \mathcal{C} , a semisimple ribbon category. Consider their tensor product $V_i \otimes V_j$. This tensor product is isomorphic to a biproduct of simple objects in the following manner:*

$$V_i \otimes V_j \simeq \bigoplus_k N_{ij}^k V_k \quad (33)$$

Where the coefficients $N_{ij}^k = \dim(\text{Hom}(V_k, V_i \otimes V_j))$ are called the fusion coefficients.

Remark 4.3.18 *The later equality does not hold trivially, it comes from the fact that semisimple ribbon category possess a natural Grothendieck ring structure ([4] p. 32) with basis $\langle V_i \rangle$ which induces a structure of fusion algebra ([7] pp. 2-3). The coefficients N_{ij}^k are then the structure coefficients of that algebra.*

Note that by definition, the following equalities hold:

$$N_{ij}^k = N_{ji}^k = N_{ik^*}^{j^*} = N_{i^*j^*}^{k^*} \quad (34)$$

Furthermore $N_{ij}^0 = \delta_{ij^*}$ where δ_{ij} is Kronecker's delta. Of the previous set of equality only the equality $N_{ij}^k = N_{i^*j^*}^{k^*}$ is not totally obvious. Note first that $*$ is an involution (up to a natural isomorphism) inducing an automorphism on objects then, $(V_j \otimes V_i)^* \simeq V_i^* \otimes V_j^* \simeq V_{i^*} \otimes V_{j^*}$ which implies $V_{i^*} \otimes V_{j^*} \simeq \bigoplus_k N_{ij}^k V_{k^*}$ establishing the equality.

As we are working in a category where the set of endomorphisms of a simple object (say V_i) is equal to the base field \mathbb{K} and since $\theta_{V_i} \in \text{End}(V_i)$, we can introduce the following notation: Let $d_i, \theta_i \in \mathbb{K}$ be such that $\theta_{V_i} = \theta_i \text{id}_{V_i}$ and $d_i = \dim(V_i)$.

Lemma 4.3.19 *The scalars d_i and θ_i are subject to the following properties:*

- 1) $\theta_0 = 1$
- 2) $\theta_{i^*} = \theta_i$
- 3) $d_0 = 1$
- 4) $d_{i^*} = d_i$
- 5) $d_i d_j = \sum_k N_{ij}^k d_k$

Proof:

- 1) $\theta_I = id_I = 1id_I = \theta_0 id_I$ which shows that $1 = \theta_0$.
- 2) $d_0 = tr(id_I) = id_I = 1$ following from the fact that $End(I) \simeq \mathbb{K}$.
- 3) We know that $\theta_{i^*} id_{V_i} = \theta_{V_i^*} = (\theta_{V_i})^*$ taking the RHS and using definition of the dual of a morphism, we factor the scalar θ_i and only rigidity equation remains reducing to identity. Hence, the RHS is equal to $\theta_i id_{V_i^*}$ as required.
- 4) We know that $d_i = dim(V_i) = dim(V_i^*) = d_{i^*}$.
- 5) Using the fact that $dim(V \otimes W) = dim(V)dim(W)$ and the fusion rule, this follows immediately.

□

Now equipped with these definition, we note that in a semisimple ribbon category, any simple object is non-trivial. Indeed, we have:

Lemma 4.3.20 *Let $V_i \in |\mathcal{C}|$ with \mathcal{C} a semisimple ribbon category, then $d_i \neq 0$ for all $i \in I$, the index set of simple objects.*

Proof: As $N_{ii^*}^0 = N_{i0}^i = 1$, let $A \simeq V_0$ and B such that $Hom_{\mathcal{C}}(V_0, B) = 0$ then, the maps i_{V_i} and e_{V_i} coincide with the maps $V_0 \rightarrow A$ and $A \rightarrow V_0$ respectively. As both of these map are not trivial, their composition is not. Hence, by definition of dim , this implies that $d_i \neq 0$ as required.

□

We now introduce some notions that will be usefull later in chapter 5. From [4], p. 45, we define:

Definition 4.3.21 *Let \mathcal{C} be a semisimple ribbon category. Denote $\overline{\mathcal{C}^{\boxtimes 2}}$ the category whose objects are infinite sums of the form $\sum_i (A_i, B_i)$, where $A_i, B_i \in |\mathcal{C}|$ and (A_i, B_i) is a formal pairing.*

By infinite sum here we mean that we do not require that just a finite number of them be non-zero.

In particular rigidity of the ribbon category give rise to a subset of objects in the category defined above. These objects are of the form:

$$R = \bigoplus_{i \in I} (V_i, V_i^*) \quad (35)$$

Of course, these objects R are independent of the choice of the representatives of the isomorphisms classes of simple objects in the sense that if $V_i \simeq W_i$, this induces an isomorphism $R_{V_i} \simeq R_{W_i}$. More generally, we have, taken from [4], p. 45:

Definition 4.3.22 *an object $R \in \overline{\mathcal{C}^{\boxtimes 2}}$ is symmetric if we have an isomorphism $s : R \xrightarrow{\sim} R^{op}$ such that $ss^{op} = id$.*

In order to get a lighter notation, we might drop the \bigoplus in the notation and write R_1 to the left and R_2 to the right of an expression that contains a symmetric object.

4.3.4 Graphical calculus for morphisms

The purpose of this section is to give a pictorial technique that will be used later to prove some results that would have been much more difficult to prove axiomatically. In fact, this graphical calculus is nothing more than a representation of the morphisms in a ribbon category \mathcal{C} . We give here the representation essential to any ribbon categories \mathcal{C} (although, the associativity map α and the balancing map γ cannot be represented by this pictorial technique); thus we give a representation for each of the functorial and natural transformation in these categories and a representation for

morphisms. We consider U, V and $W \in |\mathcal{C}|$. The presentation here closely follows the one given in [4], section 2.3, pp. 35-43 except that our formulation of theorem 4.3.27 is slightly different and the proof is not given in that text.

We first need to assign a direction to each ribbon strand (i.e. instead of considering simple non-intersecting curves, we consider framed strands going in the direction of the z -axis, i.e. from bottom to top) by an object of a ribbon category. Before we develop the graphical calculus for morphisms, we give the formal definition of a ribbon tangle similarly as it is presented in [4], definition 2.3.4, p. 39:

Definition 4.3.23 *A n -ribbon tangle is the isotopy class of the union of n non-intersecting ribbons (i.e. framed tangles) in $\mathbb{R} \times 0 \times [0, 1]$ the sides of the ribbons are distinguishable in the sense they have an upward face (white) and a downward face (grey). The ribbons starts along $\mathbb{R} \times 0 \times 0$ and ends at $\mathbb{R} \times 0 \times 1$ with faces upwards.*

Definition 4.3.24 *Let \mathcal{C} be a ribbon category. Then, a \mathcal{C} -coloration of a ribbon tangle R_t is given by assigning an object of \mathcal{C} to each ribbon strand of R_t .*

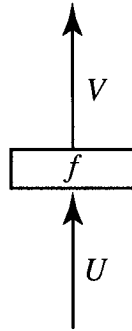
We now give the set of maps induced by the ribbon structure.

Identity morphism of I : It is represented by the empty picture.

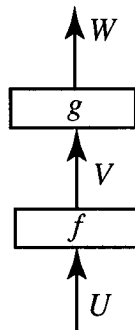
Identity morphism: The identity morphism $id \in Hom_{\mathcal{C}}(U, U)$ is just represented as an arrow labeled with U .



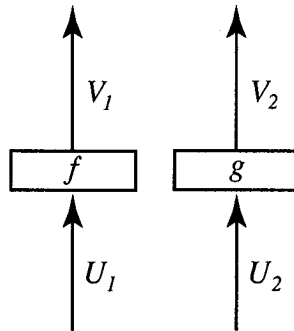
Morphism: Given an $f \in \text{Hom}_c(U, V)$ it is depicted by:



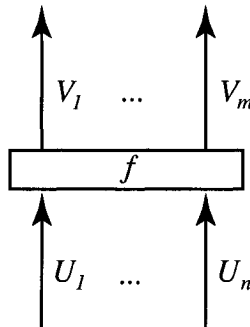
Composition: Given two morphism $f \in \text{Hom}_c(U, V)$ and $g \in \text{Hom}_c(V, W)$, their composition $g \circ f$ is represented by stacking one morphism above the other as in:



Tensor product of morphisms: Given two morphisms $f \in \text{Hom}_c(U_1, V_1)$ and $g \in \text{Hom}_c(U_2, V_2)$, their tensor product $f \otimes g \in \text{Hom}_c(U_1 \otimes U_2, V_1 \otimes V_2)$ is represented by putting the diagram of f aside the diagram for g as shown below:

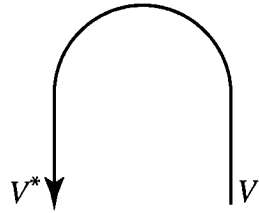


Morphism acting on tensor product of objects: Given a morphism $f \in \text{Hom}_{\mathcal{C}}(U_1 \otimes \dots \otimes U_n, V_1 \otimes \dots \otimes V_n)$ we have:

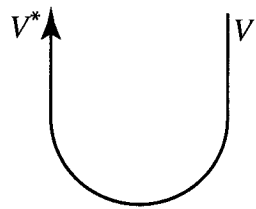


Now, given an object $V \in |\mathcal{C}|$, its dual V^* is represented by an arrow with the opposite orientation. With this we describe e_V and i_V .

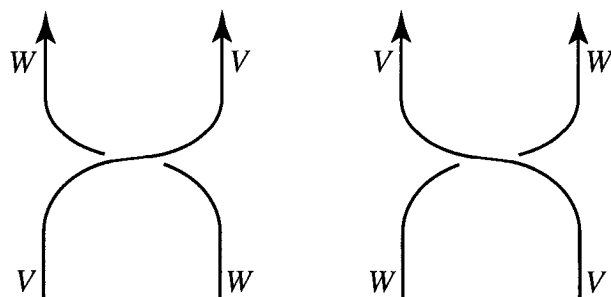
e_V : The map e_V then just flip the (graphical) orientation of a given object V as shown below:



i_V : Similarly, i_V flips the (graphical) orientation of a given object V but the other way around:



Braiding: The braid operation that flips objects $\sigma_{U,V}$ and its inverse $\sigma_{U,V}^{-1}$ will be represented as:



respectively.

From the graphical representations of i_V , e_V and those of the braidings, we can represent the twist as the following strand:



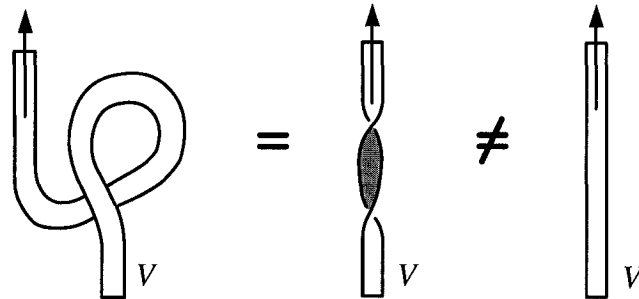
We would like to develop our pictorial technique in such way that given two terms representing morphisms in the language of a ribbon category we can show, by applying certain rules, whether or not these two morphisms are equal in ribbon categories. One of these operations is equality of the two pictorial representations up to isotopy. In general, the twist is not equal to the identity and this is the reason why we give the pictorial representation below.

Twist: The twist θ_V will be depicted as:



Remark 4.3.25 *the origin of the term ribbon comes from this failure of representing*

ribbon categories with simple tangles. Indeed, if we take a framed strands, the first representation of the twist yields:



which actually cancels the isotopy move that we don't want. That's the reason why those categories are called ribbon. In our notation, we prefer to use unframed strands in order to lighten the notation; we will use ribbons only if needed.

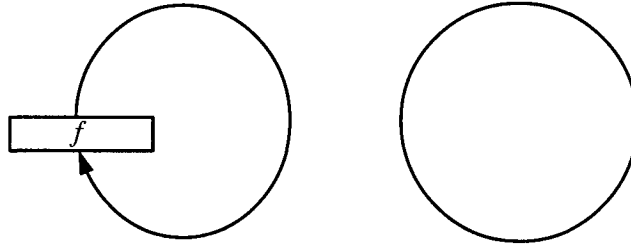
By the previous definition, we allow only complete twists of the ribbons in the sense that all the twists lying in the ribbon tangle must be a rotation of 2π of the ribbon in \mathbb{R}^3 . Then, π 'twists' of the form



is not allowed but all twists like



are (this is the inverse of the twist map given above). We finally give the pictorial representation for $tr(f)$ and $dim(V)$ as follows:



And this completes the set of maps inherent to the structure of ribbon categories.

We give a few definitions before stating the main result of this section which are needed to state theorem.

Definition 4.3.26 *A state of a \mathcal{C} -coloured ribbon tangle is a particular pictorial representation of that ribbon tangle.*

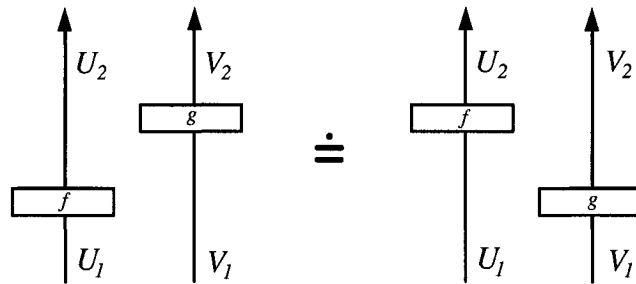
Now, we are ready to give our formulation Reshetikhin-Turaev theorem (see [4] p. 42) in order to formulate the relation \doteq that represents (graphically) the equivalence of morphisms in our ribbon category \mathcal{C} .

Theorem 4.3.27 *Given two \mathcal{C} -coloured tangles R_t^1 and R_t^2 that are such that they are coloured in the same way at $\mathbb{R} \times 0 \times 0$ and at $\mathbb{R} \times 0 \times 1$. If, by applying any finite set of elementary moves (isotopy of \mathbb{R}^2 , box sliding along a wire) on R_t^1 or on a*

subsequent state obtained from R_t^1 , we can transform R_t^1 to R_t^2 , then we say that the two ribbon are equivalent, it is denoted $R_t^1 \doteq R_t^2$. This equivalence of ribbon induces an equivalence on morphisms in any ribbon category \mathcal{C} in the sense that if $R_t^1 \doteq R_t^2$ then $F(R_t^1) = F(R_t^2)$ where F maps ribbon tangles to terms describing morphisms in \mathcal{C} .

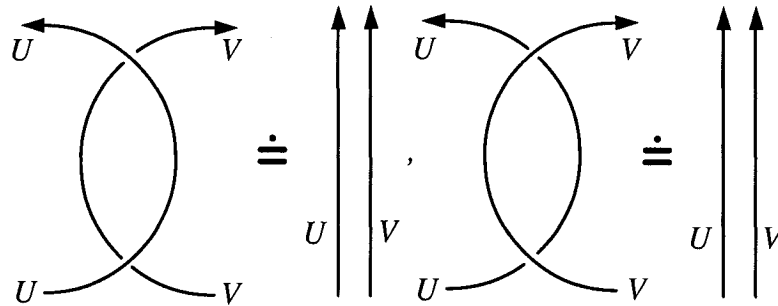
Proof: We list the set of elementary moves we can apply to a piece of a ribbon derived directly from the relations induced by the axiomatisation of the ribbon category. This is done modulo the natural isomorphisms $\alpha, \lambda, \rho, \delta$ and γ which are either lost in the formalism (which is the case for α, δ and γ but it does not matter as each ribbon category is equivalent to a strict one) or do not need to be represented (that is for λ and ρ as the arrow $I \rightarrow I$ is represented by the empty picture). We build the graphical representation of the terms via the graphical representation of the natural isomorphism; we shall see that from the axiomatisation of \mathcal{C} , isotopy of moves (analogue to the Reidemeister moves) and box sliding appears naturally.

-*Functoriality of \otimes :* Given two morphisms $f \in Hom_{\mathcal{C}}(U_1, U_2)$ and $g \in Hom_{\mathcal{C}}(V_1, V_2)$, functoriality of the tensor product $(f \otimes id) \circ (id \otimes g) = (id \otimes g) \circ (f \otimes id)$ is represented by:



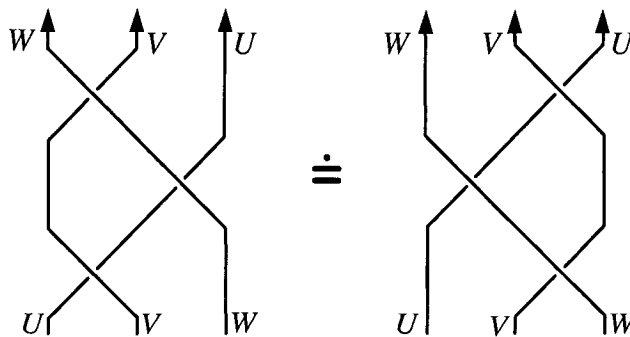
which says in other words that we can 'slide' the box of the morphisms along the ribbon tangle.

-*Braid cancellation*: the braid structure on \mathcal{C} gives us the following moves (which correspond to Reidemeister move II):



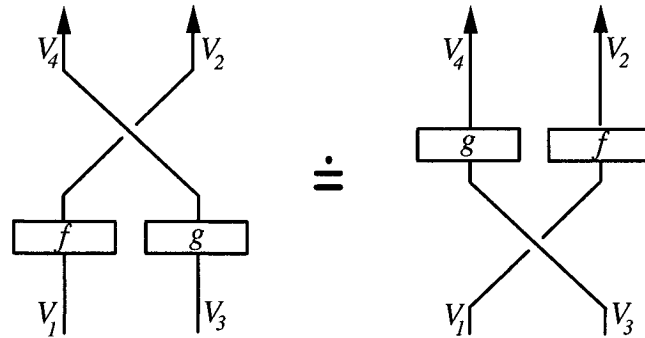
which are respectively the cancellation of $\sigma_{V,U}^{-1}\sigma_{V,U}$ and its converse.

The braid relations given by Artin's theorem are: $\sigma_{U_1,U_2}\sigma_{V_1,V_2} = \sigma_{V_1,V_2}\sigma_{U_1,U_2}$. This holds in a similar manner as the functoriality of \otimes as they do not affect the same ribbon tangles. The second relation (usually listed as Reidemeister move III) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is represented by (taking in account that we use \mathcal{C} -coloured ribbon tangles):



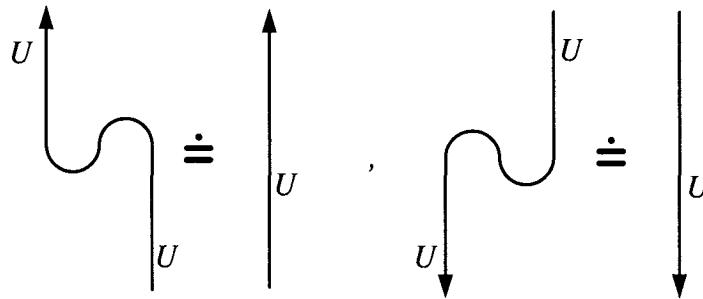
A similar relation holds for the σ_j^{-1} (listed as Reidemeister move IV).

Finally, the naturality of σ is represented by:



which completes the set of braid relations.

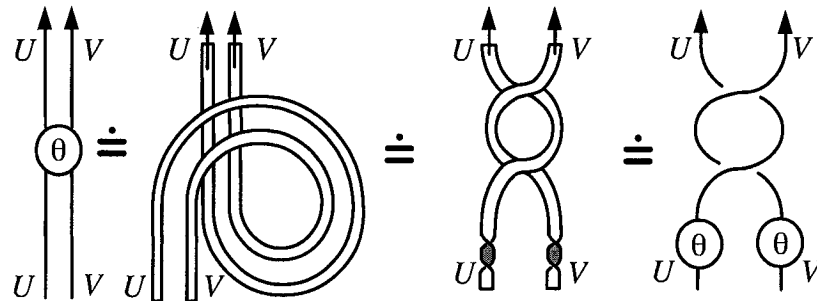
-*Rigidity relations*: They are depicted by the following picture (which are usually listed ‘move 0’ along Reidemeister moves):



-*Balancing relations*: Before we actually translate the balancing relation into the graphical language, we would like to note that it is important here to work with ribbons as unframed tangles cannot encapsulate correctly the topology of ribbon tangles. The equation represented by

$$\begin{array}{ccc}
 U \otimes V & \xrightarrow{\theta_{U \otimes V}} & U \otimes V \\
 & \searrow \theta_{U \otimes V} & \nearrow \sigma_{V,U} \sigma_{V,U} \\
 & U \otimes V &
 \end{array}$$

then becomes:



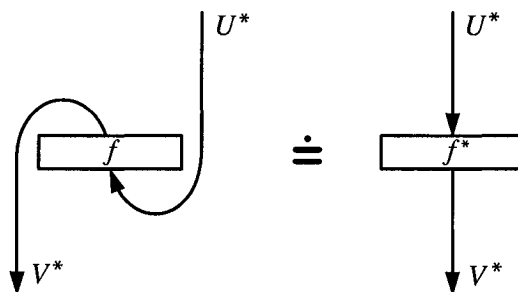
The third balancing relation $(\theta_{V^*} = (\theta_V)^*)$ should be represented but it's essentially the same as the dual for a morphism which is given in the example 4.3.28.

We have shown that applying simple moves to a piece of a ribbon tangle yields equality of equations representing morphisms in all ribbon categories by checking all relations of ribbon category. This completes the proof of the theorem.

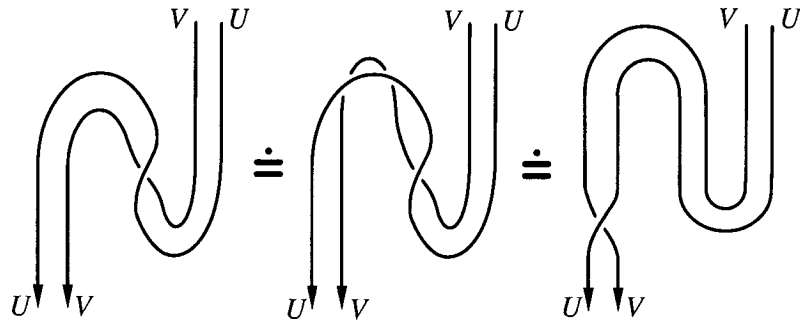
□

We now give some examples in order to illustrate the previous result.

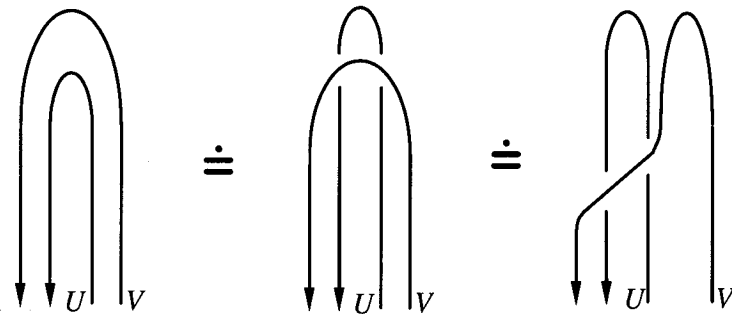
Example 4.3.28 First, the fact that given a morphism $f : U \rightarrow V$ in a rigid category \mathcal{C} we have $f^* : V^* \rightarrow U^*$ (see equation 26 and what follows) is represented by:



Example 4.3.29 We illustrate that $(\sigma_{UV})^* = \sigma_{U^*V^*}$ in the following manner:

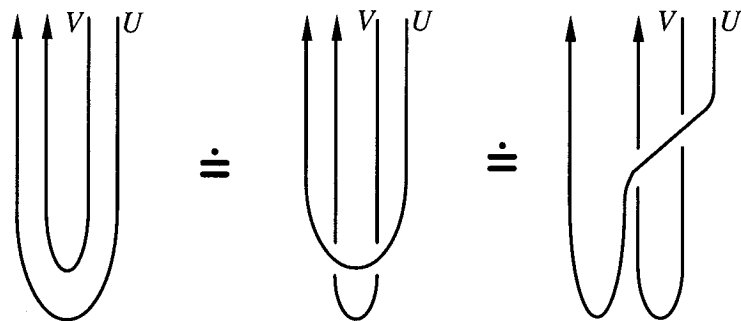


Example 4.3.30 The relation $e_{U \otimes V} = (e_U \otimes e_V)(\sigma_{V^*, U^* \otimes U} \otimes id)$ is depicted as:



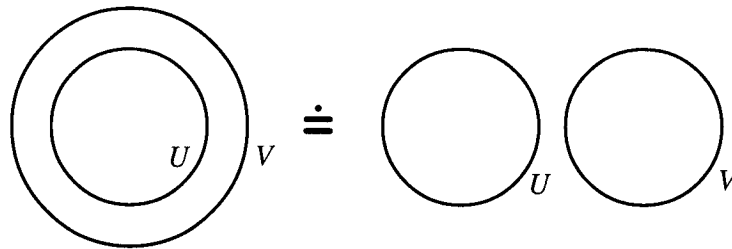
The same holds for σ^{-1} instead of σ for this relation and in the next.

Example 4.3.31 The relation $i_{U \otimes V} = (\sigma_{U^*, V \otimes V^*} \otimes id)(i_U \otimes i_V)$ is depicted as:



The proof of the first relation in lemma 4.3.9 *i*) is essentially the same as the graphical representation of the first balancing relation in the proof of the theorem 4.3.27 provided we use unframed tangles.

Example 4.3.32 *We finally show a pictorial proof of the well-known identity $\dim(U \otimes V) = \dim(U)\dim(V)$ as follows:*



This completes our presentation of the graphical calculus of morphisms.

Chapter 5

Genus 0 modular functors

In this chapter we develop the notion of \mathcal{C} -extended modular functor which is relevant for our exposition. We first refine some notions from chapter 2 in order to make them pertinent to this chapter. Then, we define a \mathcal{C} -extended modular functor acting on genus 0 surfaces. Next, we define the category of the Teichmüller groupoid in genus 0 ($Teich_0$) as a category that encapsulates all the data introduced in chapter 3 plus some additional structures that make it closer as a structure to a genus 0 \mathcal{C} -extended modular functor. Next, we define the notion of Moore-Sieberg data (MS-data) on a semisimple abelian category and show that one can build those starting from any ribbon category. We conclude the chapter by showing that one can build a genus 0 \mathcal{C} -extended UMF from a given set of MS-data which also yields a representation of the category $Teich_0$. For the exposition we closely follow [4], chapter 5, pp. 93-99 and pp. 108-119 .

5.1 Basic definition and concepts

Before actually speaking of modular functors (MF), we will redefine a few concepts borrowed from chapter 3 in order to give proper definitions of our MF.

Definition 5.1.1 *An extended surface (e-surface for short) is an oriented compact surface M with boundaries $\{(\partial M)_\alpha\}$ (where $\{\alpha\}$ is the set of labels of boundary components) together with a set of parametrizations $p_\alpha : (\partial M)_\alpha \rightarrow S^1$ where each p_α is*

an orientation-preserving homeomorphism.

Here, we consider S^1 to be the 1-sphere in the complex plane endowed with counter-clockwise orientation. Of course, homeomorphisms of e-surface must preserve the set of parametrizations $\{p_\alpha\}$.

Definition 5.1.2 We define an orientation reversal of an e-surface $(M, \{p_\alpha\})$ by the operation $(M, \{p_\alpha\}) \mapsto (\overline{M}, \{-\overline{p_\alpha}\})$. In other words, it reverses the orientation of the boundary component $p_\alpha^{-1}(S^1)$, the minus sign is there to reverse any possible tangent vectors assigned to a point of the boundary component.

Definition 5.1.3 A \mathcal{C} -coloured e-surface is an e-surface M together with an assignment $f : |\mathcal{C}| \rightarrow \{\partial(M)\}_\alpha$ that is such that it labels each boundary component of M an object of \mathcal{C} .

Definition 5.1.4 A morphism of \mathcal{C} -coloured e-surfaces is a morphism of e-surface that preserves the coloration of e-surface.

We now define a UMF of genus 0, but it will act on surfaces with a number of punctures that will be coloured by objects coming from a category \mathcal{C} . In that sense, this UMF will be qualified as \mathcal{C} -extended. This definition parallels the one given in definition 5.1.13 [4] p. 97.

Definition 5.1.5 Given an abelian category \mathcal{C} over \mathbb{K} with $V_\alpha \in \mathcal{C}$ and let $R = \bigoplus_i (V_i, V_i^*)$ be a symmetric object in $\overline{\mathcal{C}^{\boxtimes 2}}$, then a \mathcal{C} -extended modular functor in genus 0 is a rule μ such that assign to a \mathcal{C} -colored e-surface M of genus 0 in each connected component a linear functor $\mu(M) : \mathcal{C}^{\boxtimes\{\alpha\}} \rightarrow \text{Vect}_f$ that depends only on the coloring. Furthermore, to any morphism of \mathcal{C} -colored e-surface $f : M \xrightarrow{\sim} M'$, μ assign an isomorphism $\mu f : \mu(M; \{V_\alpha\}) \xrightarrow{\sim} \mu(M'; \{V_\alpha\})$, which depends of the isotopy class of f . It is also defined with the following natural isomorphisms:

$$\mu(\emptyset)(V) \simeq V \quad \mu(M \sqcup M')(V \otimes W) \simeq \mu(M)(V) \otimes \mu(M')(W) \quad (36)$$

and the following rule:

-Gluing isomorphism: Let $(\partial M)_\alpha$ and $(\partial M)_\beta$ be some boundary components of some different connected components of M and let p_α and p_β be their parametrizations respectively. We define a new surface $\sqcup_{\alpha,\beta}(M)$ by identifying the points $x \in (\partial M)_\alpha$ with $p_\beta^{-1}(\overline{-p_\alpha(x)}) \in (\partial M)_\beta$. Now, under the action of the functor, this becomes:

$$G_{\alpha,\beta} : \bigoplus_i \mu(M; \{V_\alpha\}, V_i, V_i^*) \xrightarrow{\sim} \mu(\sqcup_{\alpha,\beta} M; \{V_\alpha\}), \quad (37)$$

for a given symmetric object $R = \bigoplus (V_i, V_i^*)$. The preceding set of data is submitted to the following axioms:

Functoriality: $\mu(fg) = \mu(f)\mu(g)$

Naturality: The natural isomorphisms (eq. 36) and the gluing isomorphism (eq. 37) are natural in M .

Compatibility: The natural isomorphisms and the gluing isomorphism are coherent one with one other.

Symmetry of gluing: Given the isomorphism $R = \bigoplus_i (V_i^*, V_i) \simeq \bigoplus_i (V_i, V_i^*) = R^{op}$, this induces $G_{\alpha,\beta} = G_{\beta,\alpha}$.

Normalization: $\mu S^2 = \mathbb{K}$, where S^2 is the 2-sphere.

Furthermore, we say that μ is unitary provided that we also have the following natural isomorphisms $\mu(\overline{M}) \xrightarrow{\sim} \mu(M)^*$ and provided that these isomorphisms are also coherent with the isomorphisms μf , the gluing and the set of natural isomorphisms (eq. 36 and 37) that defines the modular functor. In particular, for the gluing this means that given the isomorphism $\mu(M) \simeq \mu(\overline{M})^*$, we define the pairing $(\ , \)_M : \mu(M) \otimes \mu(\overline{M}) \rightarrow \mathbb{K}$. Now let $f \in \mu(\sqcup_{\alpha,\beta} M)$ and $g \in \mu(\overline{\sqcup_{\alpha,\beta} M}) \simeq \mu(\sqcup_{\alpha,\beta} \overline{M})$. Define $G_{\alpha,\beta}^{-1}(f) = \sum_i f_i$ and $G_{\alpha,\beta}^{-1}(g) = \sum_i g_i$ where $f_i \in \mu(M; V_i, V_i^*)$ and $g_i \in \mu(\overline{M}; V_i^*, V_i)$. Then,

$$(f, g)_{\sqcup_{\alpha,\beta} M} = \sum_i c_i (f_i, g_i)_M ; c_i \in \mathbb{K} \quad (38)$$

where the c_i are constants depending on the choice of the symmetric object R .

Before introducing such a \mathcal{C} -extended modular functor, we need to introduce some constructions. This is done in the next few sections.

5.2 The Teichmüller groupoid

We now present a category whose structure encompasses the data presented in chapter 3. This type of category is known as a Teichmüller groupoid. The following definition is given along the lines of the one given in [4] pp.124-125.

Definition 5.2.1 *Let Teich_0 be the category whose objects are extended surfaces of genus 0, and morphisms are isotopy classes of morphisms of e-surfaces. Then*

- $\text{Hom}_{\text{Teich}_0}$ is a groupoid.
- The operation of the disjoint union of surface $\sqcup : \text{Teich}_0 \times \text{Teich}_0 \rightarrow \text{Teich}_0$ together with $\emptyset \in |\text{Teich}_0|$ equip Teich_0 with the structure of a symmetric monoidal category.
- There is a functor $S : \text{Teich}_0 \rightarrow \text{Sets}$ acting as $M \mapsto \{\partial(M)\}$ on object, i.e. it maps a given $M \in |\text{Teich}|$ to the set of its boundary components and maps the morphisms of Teich_0 to bijections in Sets . Note that S preserves the symmetric monoidal structure.
- Given $M = M_1 \sqcup M_2 \in |\text{Teich}_0|$, $a \in \{\partial(M_1)\}$ and $b \in \{\partial(M_2)\}$ we have a gluing operation $G_{a,b}(M_1 \sqcup M_2) = \sqcup_{a,b} M$. This operation satisfies:

a) Compatibility with S : $S(G_{a,b}(M_1 \sqcup M_2)) = S(M) \setminus \{a, b\}$.

b) Compatibility with the disjoint union: M_1, M_2, a and b be as in the definition of the gluing and let $M_3 \in |\text{Teich}_0|$, then $G_{a,b}(M_1 \sqcup M_2 \sqcup M_3) = (G_{a,b}(M_1 \sqcup M_2)) \sqcup M_3$.

c) Associativity: let $a \in S(M_1)$, $b \in S(M_2)$, $c \in S(M_3)$ and $d \in S(M_4)$, then $G_{a,b}G_{c,d}(M_1 \sqcup M_2 \sqcup M_3 \sqcup M_4) = G_{c,d}G_{a,b}(M_1 \sqcup M_2 \sqcup M_3 \sqcup M_4)$.

d) Naturality of G : let $f = f_1 \sqcup f_2 : M_1 \sqcup M_2 \rightarrow M'_1 \sqcup M'_2$ be a morphism of Teich_0 , then this induces an isomorphism $G_f : G_{a,b}(M_1 \sqcup M_2) \xrightarrow{\sim}$

$G_{a',b'}(M'_1 \sqcup M'_2)$ where $a' = Sf(a)$ and $b' = Sf(b)$. This set of isomorphisms is such that $G_{fg} = G_g G_h$ and $G_{id} = id$

We can see the strong simlutde between the definition of the functor and the previous definition. We will make that clearer later.

5.3 Moore-Sieberg data and semisimple ribbon categories

We now need to define some construction in order to show that the choice of a ribbon category fixes a choice of modular functors as there are some parallels between the two structures. The first construction that we introduced is called Moore-Sieberg data (MS data for short) and it will help us to build the structure of a ribbon category starting from an abelian category. This definition of the MS-data is similar to the definition 5.3.2 given in [4], pp. 109-110.

Definition 5.3.1 *Given a semisimple abelian category \mathcal{C} with a set of simple objects $\{V_i\}_{i \in I}$, one representative for each isomorphism class of simple objects, the MS data are defined as the following set of data:*

-Conformal blocks: *Given any $n \in \mathbb{N}$, there is a functor $\langle \cdot, \dots, \cdot \rangle : \mathcal{C}^{\boxtimes n} \rightarrow Vect_f$ such that given a set of objects $\{W_j\}_{j=1, \dots, n-1}$, $\langle V_i, W_1, \dots, W_{n-1} \rangle = 0$ for all but a finite number of i .*

-Rotations: *The set of natural isomorphisms of the form: $Z : \langle W_1, \dots, W_{n-1}, W_n \rangle \xrightarrow{\sim} \langle W_n, W_1, \dots, W_{n-1} \rangle$*

-Symmetric object: *$R \in \overline{\mathcal{C}^{\boxtimes 2}}$ not necessarily of the form $R = \bigoplus_{i \in J} (V_i, V_i^*)$, but $R = \bigoplus_j (R_j, R_j^*)$ for some R .*

-Gluing: *For any $j, k \in \mathbb{N}$, we have the natural isomorphisms $G : (\bigoplus_i (\langle U_1, \dots, U_j, R_i \rangle) \otimes (\langle R_i, W_1, \dots, W_k \rangle)) \mapsto \langle U_1, \dots, U_j, W_i, \dots, W_k \rangle$. And finally,*

-Commutativity: The natural isomorphisms $\sigma : \langle U_1, U_2, U_3 \rangle \xrightarrow{\sim} \langle U_1, U_3, U_2 \rangle$.

These are submitted to the following list of axioms:

- Non-degeneracy: For all simple object V_i , there exist an object $U \in |\mathcal{C}|$ such that $\langle U, V_i \rangle \neq 0$.
- Normalization: $\langle \rangle : \mathcal{C}^{\boxtimes 0} \rightarrow \text{Vect}_f$ is the identity functor¹.
- Associativity of gluing: Given two copies of R , say $R^{(1)}$ and $R^{(2)}$, then $G_{R^{(1)}}G_{R^{(2)}} = G_{R^{(2)}}G_{R^{(1)}}$ that is $(G_{R^{(1)}} \otimes id) \circ (id \otimes G_{R^{(2)}}) = (id \otimes G_{R^{(2)}}) \circ (G_{R^{(1)}} \otimes id)$.
- Symmetry of gluing: For all $j, k \in \mathbb{N}$ define σ as the operation that rotates two tensor factors and the isomorphism $f : R^{op} \xrightarrow{\sim} R$. Then, the following diagram is commutative:

$$\begin{array}{ccc}
 \langle U_1, \dots, U_j, R_1 \rangle \otimes \langle R_2, W_1, \dots, W_k \rangle & \xrightarrow{G} & \langle U_1, \dots, U_j, W_1, \dots, W_k \rangle \\
 \downarrow (Z \otimes Z^{-1}) \circ \sigma & & \downarrow Z^k \\
 \langle W_1, \dots, W_k, R_2 \rangle \otimes \langle R_1, U_1, \dots, U_j \rangle & \xrightarrow{G \circ f} & \langle W_1, \dots, W_k, U_1, \dots, U_j \rangle
 \end{array}$$

- Rotation axiom: $id = Z^n : \langle U_1, \dots, U_n \rangle \xrightarrow{\sim} \langle U_1, \dots, U_n \rangle$.
- Hexagon axioms: Define $\sigma_{U_2, U_3 U_4}$ acting on $\langle U_1, U_2, U_3, U_4 \rangle$ as $Z^{-1}G(Z \otimes id) \circ \sigma \otimes id \circ G^{-1}$ and σ_{U_2, U_3} acting on the same 4-tuple as $Z^{-1}G \circ id \otimes \sigma \circ G^{-1}Z$ where the inverses of the two maps (G and Z) are defined in the obvious manner. Then, the following diagrams commute:

$$\begin{array}{ccc}
 \langle U_1, U_2, U_3, U_4 \rangle & \xrightarrow{\sigma_{U_2, U_3 U_4}} & \langle U_1, U_3, U_4, U_2 \rangle \\
 \searrow \sigma_{U_2, U_3} & & \nearrow \sigma_{U_2, U_4} \\
 & \langle U_1, U_3, U_2, U_4 \rangle &
 \end{array}$$

¹Note the we defined $\mathcal{C}^0 \stackrel{def}{=} \text{Vect}_f$.

$$\begin{array}{ccc}
\langle U_1, U_2, U_3, U_4 \rangle & \xrightarrow{\sigma_{U_2, U_3 U_4}^{-1}} & \langle U_1, U_3, U_4, U_2 \rangle \\
& \searrow \sigma_{U_2, U_3}^{-1} & \nearrow \sigma_{U_2, U_4}^{-1} \\
& \langle U_1, U_3, U_2, U_4 \rangle &
\end{array}$$

- Dehn twist axiom: Define σ_{U_1, U_2} as $G(\sigma \otimes id)G^{-1}$ then $Z\sigma_{U_1, U_2} = \sigma_{U_2, U_1}Z$.

This completes the definition.

In order to establish the parallel between MS-data and semisimple ribbon categories, let \mathcal{C} be a semisimple ribbon category, we give the following definition:

Definition 5.3.2 let $U, V, W \in |\mathcal{C}|$ with \mathcal{C} a semisimple ribbon category. Define the following set of data:

- The operation $\langle U_1, \dots, U_n \rangle := Hom_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n)$
- $R = \bigoplus_{i \in I} (V_i^*, V_i)$

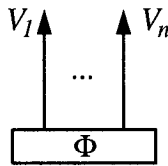
And the following set of natural isomorphisms:

- $Z : Hom_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n) \rightarrow Hom_{\mathcal{C}}(I, U_n \otimes U_1 \otimes \dots \otimes U_{n-1})$ to be the action of δ on the last element of the n -tuple $\langle U_1, \dots, U_n \rangle$ followed by the action of the rigidity isomorphisms.
- $G : (\bigoplus_{i \in I} Hom_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n \otimes V_i^*) \otimes Hom_{\mathcal{C}}(I, V_i \otimes W_1 \otimes \dots \otimes W_m)) \rightarrow Hom_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n \otimes W_1 \otimes \dots \otimes W_m)$ where V_i is a simple object. Again we use the action of $\delta : V \xrightarrow{\sim} V^{**}$ on the last element of the n -tuple $\langle U_1, \dots, U_n \rangle$ applied to the RHS of the tensor of Homs then, we use the rigidity isomorphisms and lastly we use the fact that in a semisimple category we have $Hom_{\mathcal{C}}(U, W) \simeq \bigoplus_{i \in I} Hom_{\mathcal{C}}(U, V_i) \otimes Hom_{\mathcal{C}}(V_i, W)$.
- $\sigma : Hom_{\mathcal{C}}(I, U \otimes V \otimes W) \rightarrow Hom_{\mathcal{C}}(I, U \otimes W \otimes V)$ which is obtained by applying twice the rigidity isomorphisms together with $\sigma_{V, W}$ as defined above.

Now, we link the two previous definitions in the following theorem whose formulation is taken from proposition 5.3.3 in [4], p. 110.

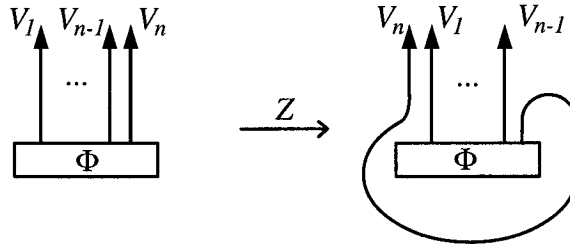
Proposition 5.3.3 *Let \mathcal{C} be a semisimple ribbon category, then the previous set of data together with the natural isomorphisms defined in definition 5.3.2 define MS-data.*

Proof: We use the graphical calculus for morphism presented in section 4.3.4. First, given a $\Phi \in \text{Hom}_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n)$, we represent it as:

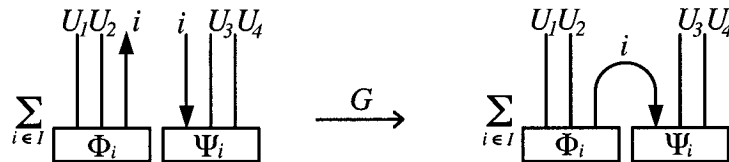


Now, with respect to the previous definitions, the natural isomorphisms presented there are depicted in the following manner:

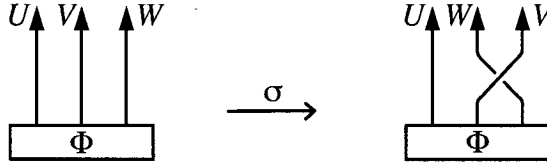
- Given a Φ as above, the natural isomorphism Z is represented by:



- Given Φ_i and Ψ_i as above where $i \in I$, the set of isomorphism classes of simple objects, G is given by:



- Finally, given a Φ as above, the natural isomorphism σ is depicted as:



These yield the MS-data's axioms in the following manner:

- *Non-degeneracy:* As each object in \mathcal{C} has a dual; this trivially holds.
- *Normalization:* This holds by definition.
- *Remaining axioms:* It is easy to see the remaining axioms hold using the pictorial representation of the natural isomorphisms given above.

□

This result will be crucial in our attempt to show that there is a parallel between ribbon categories and \mathcal{C} -extended unitary modular functors. The theorem that illustrates this assertion is given in the next section.

5.4 \mathcal{C} -extended unitary modular functors in genus 0 from a semisimple ribbon category

Equipped with the previous results, it is now possible to define a unitary \mathcal{C} -extended functor from any ribbon category. This is shown by the following result where the formulation of the theorem is given in theorem 5.4.1 in [3], p.116-117. We also closely follow the proof given in pp. 117-118 and we fill the gaps.

Theorem 5.4.1 *Given a semisimple ribbon category \mathcal{C} , we can define a non-degenerate \mathcal{C} -extended genus 0 unitary modular functor μ satisfying:*

- i) $\langle U_1, \dots, U_n \rangle = \text{Hom}_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n) = \mu(S_{0,n}; U_1, \dots, U_n)$.
ii) $R = \bigoplus_{i \in I} (V_i^*, V_i)$ and an isomorphism

$$F : R \xrightarrow{\sim} R^{op}; \bigoplus_{i \in I} (V_i^*, V_i) \mapsto \bigoplus_{i \in I} (V_i, V_i^*). \quad (39)$$

- iii) $\mu z = Z$ and $\mu b = \sigma$, where $z, b \in \Gamma(S_{0,n})$ defined in lemma 3.5.3.
iv) for every j, k, m and $n \in \mathbb{N}$, gluing is defined as the composition:

$$\begin{aligned} \mathcal{G} : \mu(S_{0,j+1}; U_1, \dots, U_m, R_1) \otimes \mu(S_{0,k+1}; R_2, W_1, \dots, W_n) \\ \xrightarrow{\mathcal{G}} \mu(S_{0,j+1} \sqcup_{j+1,1} S_{0,k+1}; U_1, \dots, U_m, W_1, \dots, W_n) \\ \xrightarrow{\bar{\alpha}} \mu(S_{0,j+k}; U_1, \dots, U_m, W_1, \dots, W_n) \end{aligned} \quad (40)$$

- v) let t_i denotes the Dehn twist around the i th puncture of $S_{0,n}$ then,

$$\mu t_i = \theta_{U_i} : \text{Hom}_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n) \rightarrow \text{Hom}_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n) \quad (41)$$

- vi) rigidity of \mathcal{C} induces unitarity of the MF with the pairing:

$$(\ , \)_{S_{0,n}} : \text{Hom}_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n) \otimes \text{Hom}_{\mathcal{C}}(I, U_n^* \otimes \dots \otimes U_1^*) \rightarrow \mathbb{K} \quad (42)$$

Explicitly we have:

$$(\phi, \psi)_{S_{0,n}} : I \simeq I \otimes I \rightarrow U_1 \otimes \dots \otimes U_n \otimes U_n^* \otimes \dots \otimes U_1^* \rightarrow I \quad (43)$$

with $d_i = \dim V_i$. Note also that $\mathbb{K} = \text{End}(I)$.

Remark 5.4.2 The fact that μ is unitary is induced by the fact that for $S_{0,n}$, there is a canonical isomorphism $\overline{S_{0,n}} \simeq S_{0,n}$ given by a reflection around the real axis which reverses the order of the punctures.

Proof: We show that given MS-data induced by a semisimple ribbon category (as in proposition 5.3.3), those MS-data defines a UTMF.

Given MS-data we construct a genus 0 unitary modular functor in the following manner:

1) For each pair of the form (M, m) with $m = (C, \{p_a\})$ is a parametrization of M we define a vector space $\mu(M, m)$ as follows: Let J be the set labels of the connected components of M where the latter are denoted M_j ; $j \in J$. For each cut $c \in C$ we take a copy of the symmetric object R_j ; $j \in J$ and following our notation, we assign to each boundary component $R_{c,k}$ where $k \in \{1, 2\}$ on each side of the cut. Note that since R is symmetric, the order in which we assign the $R_{c,k}$ does not matter but this induces a *positive* (the side where V_{i_c} is assigned) and a *negative* side (the side where we assign $V_{i_c}^*$) to each cut. We then have:

$$\mu(M, m) = \bigoplus_{i_c \in I, c \in C} \bigotimes_j \mu(S_{0, n_j}) \quad (44)$$

where I is the set of equivalence classes of simple objects in \mathcal{C} . Of course, this last formula is not unique but as we have $(V_i^*, V_i) \xrightarrow{\sim} (V_{i^*}, V_{i^*}^*)$ to identify the various equations.

2) Now, there are various ways to define a parametrization on a given e-surface M . Thus, for a given surface (M, m) define a system of isomorphisms $\phi_{m, m'} : \mu(M, m) \xrightarrow{\sim} \mu(M, m')$ that satisfies the compatibility relations $\phi_{m', m''} \circ \phi_{m, m'} = \phi_{m, m''}$ for all m, m' and m'' , this identifies all the vector spaces altogether and we define $\mu(M)$ as the canonical representative of all $\mu(M, m)$.

The previous system of isomorphisms induces a representation of $Teich_0$ as if we are given a $\psi : M_1 \xrightarrow{\sim} M_2$, an homeomorphism of e-surface and given a parametrization m_2 of M_2 then ψ gives rise to a parametrisation m_1 of M_1 in an obvious manner. Moreover, ψ identifies the cuts of M_1 and M_2 and their connected components in a one-to-one correspondence. We therefore have:

$$\mu(M_1, m_1) = \bigoplus_{i_c \in I, c \in C_1} \bigotimes_j \mu(S_{0, n_j}) \simeq \mu(M_2, m_2) \quad (45)$$

This last relation with the system of isomorphisms $\mu(M) \simeq \mu(M, m)$ gives rise to an isomorphism $\mu\psi : \mu(M_1) \xrightarrow{\sim} \mu(M_2)$ which does not depend on the choice of m_2

as each (M, m) is canonically represented under the action of μ . Now, given two isomorphisms ψ_1, ψ_2 defined as above, we start from a parametrization m'' of M'' and ψ_2 gives rise to a parametrization m' of M' ; in the same manner, ψ_1 gives rise to a parametrization m of M . We now look at the image of the e-surfaces under μ and we apply the system of isomorphisms $\mu(M) \simeq \mu(M, m)$, this is depicted as:

$$\begin{array}{ccccc}
 & & \psi_2\psi_1 & & \\
 & \searrow & \curvearrowright & \swarrow & \\
 (M, m) & \xrightarrow{\psi_1} & (M', m') & \xrightarrow{\psi_2} & (M'', m'') \\
 \downarrow \mu & & \downarrow \mu & & \downarrow \mu \\
 \mu(M, m) & & \mu(M', m') & & \mu(M'', m'') \\
 \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 \mu(M) & \xrightarrow{\mu\psi_1} & \mu(M') & \xrightarrow{\mu\psi_2} & \mu(M'') \\
 & \searrow & \curvearrowleft & \swarrow & \\
 & & \mu(\psi_2\psi_1) & &
 \end{array}$$

The fact that the $\mu(\psi_i)$ for $i = 1, 2$ does not depend on the parametrizations but that the ψ_i maps on the same surfaces implies that $\mu(\psi_2 \circ \psi_1) = \mu(\psi_2) \circ \mu(\psi_1)$. For the same consideration we have $\mu(id) = id$ if we take $\psi_2 = \psi_1^{-1}$. The way we defined the decomposition of the e-surface M above clearly tells us that μ satisfies the gluing axiom.

3) By the main theorem of chapter 2, we know that every two parametrizations of M can be joined by a sequence of simple moves Z, B and F . Identify the Z -move with the Z map, the B -move with σ and the F -move with G in the construction for semisimple ribbon categories given in the previous section, this yields an identification of the axiom of $Teich_0$ together with the MS data.

4) The fact the μ is unitary is induced via the defined pairing as by hypothesis the MS-data are build from a ribbon category and together with the remark given before the proof.

Therefore, each MS-data from a ribbon category \mathcal{C} yield a genus 0 UMF.

□

We have shown that if we choose a semisimple ribbon category \mathcal{C} to color our e-surfaces, then, this also induces the choice of a UMF μ .

This completes the introduction of the theoretical background necessary to develop our semantics for topological quantum computing.

Chapter 6

A categorical semantics for TQC

This chapter seeks to unify all the theoretical framework presented in the previous chapters in a coherent categorical semantics for topological quantum computation (TQC). We first introduce an extra categorical structure that will be used later which is the notion of strong ribbon categories. After, starting from the set data obtained from the unitary modular functor μ acting on a \mathcal{C} -colored surface, we equip the vector space with an inner product. With respect to that inner product, we define the notion adjoints and unitary maps. We investigate the notion of topological qubit as the basic data unit for topological quantum computation. Then, we prove that all the elements of the mapping class group of a given e-surface are unitary with respect to the inner product previously defined. We introduce the notion of projectors and we speak of the notion of topological invariance of the algorithms. Finally, we give some comments on the relation between this semantics and topological quantum computing.

6.1 Semisimple strongly ribbon categories

In this section, we generalize the work of B. Coecke and S. Abramsky in [1] in order to define the notion of strong ribbon categories that will be needed for what follows. In [1], only the symmetric case is studied.

Definition 6.1.1 *A dagger ribbon category $\langle \mathcal{C}, \dagger \rangle$ is a ribbon category together with a functor $\dagger : \mathcal{C} \rightarrow \mathcal{C}^{op}$ which acts trivially on objects and which coherently preserves*

the ribbon structure.

To every morphism $f \in \text{Hom}_{\mathcal{C}}(U, V)$ we associate a morphism $f^\dagger \in \text{Hom}_{\mathcal{C}^{\text{op}}}(V, U)$, called the adjoint of f , such that for all $f \in \text{Hom}_{\mathcal{C}}(U_1, U_2)$, $g \in \text{Hom}_{\mathcal{C}}(U_2, U_3)$ and $h \in \text{Hom}_{\mathcal{C}}(U_3, U_4)$, we have:

$$id_{U_1}^\dagger = id_{U_1} : U_1 \rightarrow U_1, \quad (46)$$

$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger : U_3 \rightarrow U_1, \quad (47)$$

$$(f \otimes h)^\dagger = f^\dagger \otimes h^\dagger : U_2 \otimes U_4 \rightarrow U_1 \otimes U_3, \quad (48)$$

$$f^{\dagger\dagger} = f : U_1 \rightarrow U_2, \quad (49)$$

$$\alpha_{U_1, U_2, U_3}^\dagger = \alpha_{U_1, U_2, U_3}^{-1} : U_1 \otimes (U_2 \otimes U_3) \rightarrow (U_1 \otimes U_2) \otimes U_3, \quad (50)$$

$$\rho_U^\dagger = \rho_U^{-1} : U \otimes I \rightarrow U, \quad (51)$$

$$\lambda_U^\dagger = \lambda_U^{-1} : I \otimes U \rightarrow U, \quad (52)$$

$$\sigma_{U_1, U_2}^\dagger = \sigma_{U_1, U_2}^{-1} : U_2 \otimes U_1 \rightarrow U_1 \otimes U_2, \quad (53)$$

$$\theta_U^\dagger = \theta_U^{-1} : U \rightarrow U. \quad (54)$$

Here α , ρ , λ are the structural maps from the monoidal structure, σ is from the braided monoidal structure and θ is from the ribbon structure.

Remark 6.1.2 As z , the counterpart of the rotation map (Z -move) defined in chapter 2, can be deduced from σ and θ , we did not include it in the previous definition, but the dagger of z is, again, its inverse as will be shown later in section 6.4.

Definition 6.1.3 A strongly ribbon category is a ribbon category endowed with a dagger structure, which is such that the following diagram commutes for all $U \in |\mathcal{C}|$:

$$\begin{array}{ccc} I & \xrightarrow{e_U^\dagger} & U \otimes U^* \\ \downarrow i_U & & \downarrow \sigma_{U, U^*} \\ U^* \otimes U & \xrightarrow{id_{U^*} \otimes \theta_U} & U^* \otimes U \end{array}$$

Definition 6.1.4 A semisimple strongly ribbon category is a strongly ribbon category endowed with a dagger and a semisimple biproduct structure, such that $\pi_i^\dagger = \iota_i : U_i \rightarrow U_1 \oplus U_2$, for all $U_1, U_2 \in |\mathcal{C}|$ and $i = 1, 2$.

From now on, we will only consider categories that are semisimple strongly ribbon.

6.2 Inner product

6.2.1 Existence of the inner product

We now assume that we work in categories where \mathbb{K} is algebraically closed in addition to be of characteristic 0. We also add to the axiomatization of our semisimple strongly ribbon category that the following composition

$$I \xrightarrow{\psi} V \xrightarrow{\psi^\dagger} I \geq 0 \quad (55)$$

always holds. It makes sense as $End_{\mathcal{C}}(I) \simeq \mathbb{K}$.

We start this section by showing that the vector spaces generated by the modular functor μ might be trivial, which is not very convenient for quantum computation.

Proposition 6.2.1 *Let $\langle U_1, \dots, U_n \rangle$ be a conformal block so that it yields the vector space $Hom_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n)$ under the action of μ . Then $Hom_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n) \simeq Hom_{\mathcal{C}}(I, \bigoplus_{N_0} I)$, where I is the tensor unit.*

Proof: First, \mathcal{C} being a semisimple ribbon category by hypothesis, objects of the form $U_1 \otimes \dots \otimes U_n$ are isomorphic to a biproduct of simple objects. Denote this biproduct by $\bigoplus_{i \in I} N_i V_i$. Now, we have:

$$Hom_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n) \simeq Hom_{\mathcal{C}}(I, \bigoplus_{i \in I} N_i V_i) \quad (56)$$

$$= \bigoplus_{i \in I} Hom_{\mathcal{C}}(I, N_i V_i) \quad (57)$$

$$= \bigoplus_{i \in I} \bigoplus_{j \in J} Hom_{\mathcal{C}}(I, V_{ij}) \quad (58)$$

where J is an index set so that $|J| = N_i$ for a fixed $i \in I$.

The following is a version of Schur's lemma:

Lemma 6.2.2 *An arrow of the form $\psi : I \rightarrow V$, where $V \in |\mathcal{C}|$ is simple is either 0 or an isomorphism.*

Proof: Let $\psi \in \text{Hom}_{\mathcal{C}}(I, V)$ with V a simple object and consider:

$$\text{Ker}(\psi) \xrightarrow{f} I \xrightarrow{\psi} V \quad (59)$$

Then as f is an injection and I is a simple object, either f is the 0 arrow or it is an isomorphism. Suppose first that it is the 0 arrow, then ψ is also an injection and as V is a simple object, this implies that ψ is either the 0 arrow or an isomorphism. If f is an isomorphism then, ψ is automatically the 0 arrow. Thus, we have that all the ψ that do not have $V_0 \simeq I$ for codomain are 0.

□

Hence, we have:

$$\text{Hom}_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n) \simeq \text{Hom}_{\mathcal{C}}(I, \bigoplus_{i \in I} N_i V_i) \quad (60)$$

$$= \bigoplus_{N_0} \text{Hom}_{\mathcal{C}}(I, I) \quad (61)$$

$$= \text{Hom}_{\mathcal{C}}(I, \bigoplus_{N_0} I) \quad (62)$$

as required.

□

Corollary 6.2.3 *If $N_0 = 0$, then the vector space $\text{Hom}_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n)$ obtained under the same assumptions as above is the zero vector space.*

Proof: The result is obvious by proposition 6.2.1

□

We would like to avoid such pathological cases. Thus, we will assume that μ is always positive i.e. the coloring chosen for the colored e-surface always yield a non-trivial vector space.

Definition 6.2.4 We define the operation $(\)_*$ on the elements of a vector space of the form $\text{Hom}_{\mathcal{C}}(I, U_1 \otimes \dots \otimes U_n)$ as the covariant extension of the operation $(\)^*$. Hence, given a $\psi : I \rightarrow U_1 \otimes \dots \otimes U_n$ we obtain $(\psi^\dagger)^* = \psi_* : I^* \rightarrow U_n^* \otimes \dots \otimes U_1^*$.

Remark 6.2.5 Note that given a colored e -surface M and its vector space $\mu(M)$, the vectors of the form ψ_* lie in the vector space $\mu(\overline{M}) = \mu(M)^*$, where \overline{M} is the colored e -surface obtained from M via a reflexion over the real axis inducing a change of orientation of the e -surface. Note that for all μ , we have an orientation reversal application $M \rightarrow \overline{M}$ inducing the isomorphism $\mu(\overline{M}) \xrightarrow{\sim} \mu(M)^*$, the later isomorphism is not linear but skew-linear, thus inducing the skew-linearity of the form $\langle \ , \ \rangle_{\mathcal{C}}$.

From now on, as $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, we will denote objects of the form $V_1 \otimes \dots \otimes V_n$ simply by V .

Definition 6.2.6 In our context, a (\mathcal{C} -extended) ket ψ is given by a map:

$$\psi : I \rightarrow V \quad (63)$$

It is denoted by $|\psi\rangle_{\mathcal{C}}$. While a (\mathcal{C} -extended) bra corresponding to ψ is given by a map:

$$\psi^\dagger : V \rightarrow I \quad (64)$$

Which will be denoted by ${}_{\mathcal{C}}\langle\psi|$.

Definition 6.2.7 With definition 2.1.3 and given $\phi, \psi : I \rightarrow V$ we define the inner product as:

$$\langle\psi, \phi\rangle_{\mathcal{C}} = \psi^\dagger \circ \phi \quad (65)$$

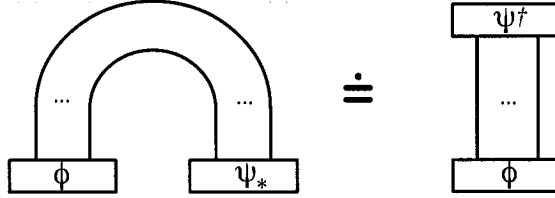
which is an endomorphism of I and therefore a scalar as $\text{End}_{\mathcal{C}}(I) \simeq \mathbb{K}$.

Proposition 6.2.8 Given two kets, $|\psi\rangle_{\mathcal{C}}$ and $|\phi\rangle_{\mathcal{C}}$, obtained from $\mu\langle V \rangle$ then:

$$e_V \circ (\phi \otimes \psi_*) = \psi^\dagger \circ \phi = \langle\psi, \phi\rangle_{\mathcal{C}} \quad (66)$$

Where ψ_* is issued from the conformal block $\langle V \rangle$ with opposite orientation under the action of μ .

Proof: We give a pictorial proof.



□

We now extend Dirac's notation to our setup in the same manner as in [1], section 7.2.

That is given $\phi : I \rightarrow V$, $\psi : I \rightarrow W$ and $f : V \rightarrow W$, we get:

$$\langle \phi | f | \psi \rangle_c = \phi^\dagger \circ f \circ \psi \quad (67)$$

Note also that we have:

$$\langle \phi, f \circ \psi \rangle_c = \phi^\dagger \circ f \circ \psi = (f^\dagger \circ \phi)^\dagger \circ \psi = \langle f^\dagger \circ \phi, \psi \rangle_c \quad (68)$$

6.2.2 Unitary and self-adjoint maps

With our definition of the inner product as defined in the previous subsection, we can define unitary maps. We do this with the following definition:

Definition 6.2.9 A unitary map is a map $U : V \rightarrow W$ that is such that $U^\dagger = U^{-1}$ i.e. it is an isometry for the inner product $\langle \cdot, \cdot \rangle_c$ so that for all $\phi, \psi : I \rightarrow V$, we have $\langle U\phi, U\psi \rangle_c = \langle \phi, \psi \rangle_c$ for all $\psi, \phi \in \text{Hom}_c(I, V)$.

We also have:

Definition 6.2.10 A self-adjoint map is a map $f : V \rightarrow V$ that is such that $f = f^\dagger$.

We now adapt the notion of spectral decomposition given in [1], section 7.4, to our context:

Definition 6.2.11 *The spectral decomposition of V is a unitary isomorphism U defined to be the map induced by the biproduct decomposition under the fusion rule.*

$$U : V \rightarrow \bigoplus_{i \in I} N_i V_i \quad (69)$$

The spectrum is then the set J so that $|J| = N_0$ as we have shown in the previous subsection that $N_i = 0$ for all $i \neq 0$.

6.3 Basic data units

Our theory lacks of fundamental data unit. We then introduce the notion of topological qubit but we need first to introduce some concepts. Following [16], p.9 we have:

Definition 6.3.1 *let M_1 and M_2 be disjoint e-surfaces. The connected sum of M_1 and M_2 , denoted $M_1 \# M_2$ is the surface obtained by identifying the boundary points of two disks removed from the surface through an orientation reversal homeomorphism. The surface thus obtained is also an e-surface.*

Remark 6.3.2 $S_{0,m} \# S_{0,n} = S_{0,m+n}$

From [1], section 8, we define:

Definition 6.3.3 *A state space is represented by an object $V \in |\mathcal{C}|$.*

In the previous definition, V is just the coloring of an e-surface in our context, this becomes the codomain for the kets $\psi \in \text{Hom}_{\mathcal{C}}(I, V)$. We then adapt the notion of basic variable given in [1], section 8 to our purpose as:

Definition 6.3.4 *A topological qubit is a state space V together with a unitary isomorphism: $U : \bigoplus_{i \in I} N_i V_i \rightarrow V$ where the V_i are the simple objects of \mathcal{C} .*

Of course, in that context, the ‘p’ of qubit is equal to $|J|$ or N_0 .

Lemma 6.3.5 *Given a set of n \mathcal{C} -colored surfaces $\{M_i\}$ that are all homeomorphic to $S_{0,1}$ and with coloring $\{V_i\}$ where i vary from 1 to n . Then under the action of μ the vector space obtained on the disjoint union of the M_i coincide with the vector space obtained in the same way on $M_1\#\dots\#M_n$.*

Proof: we have:

$$\mu(M_1 \sqcup \dots \sqcup M_n) = \mu(M_1) \otimes \dots \otimes \mu(M_n) \quad (70)$$

$$= \langle V_1 \rangle \otimes \dots \otimes \langle V_n \rangle \quad (71)$$

$$= \text{Hom}_{\mathcal{C}}(I, V_1) \otimes \dots \otimes \text{Hom}_{\mathcal{C}}(I, V_n) \quad (72)$$

$$= \text{Hom}_{\mathcal{C}}(I, V_1 \otimes \dots \otimes V_n) \quad (73)$$

$$= \langle V_1, \dots, V_n \rangle \quad (74)$$

$$= \mu(M_1\#\dots\#M_n) \quad (75)$$

Yielding the result. □

Definition 6.3.6 *A compound system is given by the tensor product of two objects V_1 and V_2 issued from the application of μ on the conformal blocks $\langle V_1 \rangle$ and $\langle V_2 \rangle$ respectively.*

We can also initialize a topological qubit, we characterize such a process in the following manner (which is almost identical to the notion of preparation given in [1] section 8):

Definition 6.3.7 *An initialization of a topological qubit in the state space V is a morphism $\psi : I \rightarrow V$ for which there is a unitary transformation $U : I \oplus W \rightarrow V$ such that the following diagram commutes:*

$$\begin{array}{ccc} I & \xrightarrow{\psi} & V \\ \iota_1 \downarrow & \nearrow U & \\ I \oplus W & & \end{array}$$

6.4 Basic data transformation as unitary maps

We now need to give a notion of unitary maps in order to parallel regular quantum computing.

Remark 6.4.1 *Given $\phi : I \rightarrow V$ and $\psi : I \rightarrow W$, a unitary map $U : W \rightarrow V$ provides that the scalar product is well defined as:*

$$\langle \phi | U | \psi \rangle_c = \langle \phi, U \psi \rangle_c = \langle U^\dagger \phi, \psi \rangle_c \quad (76)$$

Now, note that the braidings are only defined on conformal blocs of three objects. If we wish to compute with conformal block with more than three objects, we need to define generalized braiding so that we are able to braid objects with each other without restriction.

Definition 6.4.2 *Let $S_{0,n}$ be the standard sphere with n punctures, a strong cut system on it is defined as a cut system C which is such that if n is even each pair of punctures is jointly bounded by a simple closed curve $c \in C$, if n is odd then each pair of punctures except that one puncture is jointly bounded by a simple closed curve $c \in C$.*

Thus, suppose we start with $(S_{0,n}, m_0)$ where m_0 defines a strong cut system C on $S_{0,n}$ so that i is paired with $i + 1$, then we can attain another strong cut system applying a sequence of $n - 1$ A moves only so that the label $i + 1$ is now paired with $i + 2$ (of course, the additions $i + j$; $j = \{1, 2\}$ are taken mod n). As the A move is defined from successive application of F moves, this can be represented in the case of a modular functor using G . We will denote this sequence of A moves from C to C' by \mathcal{A} . This sequence of moves gives us the possibility to define a generalized braiding on M having a representation under the action of the UMF as in the representation of the vector space we are allowed to braid only two adjacent objects provided that we slice the vector space in sets of three objects via gluing. We provide an example for $S_{0,6}$ below:

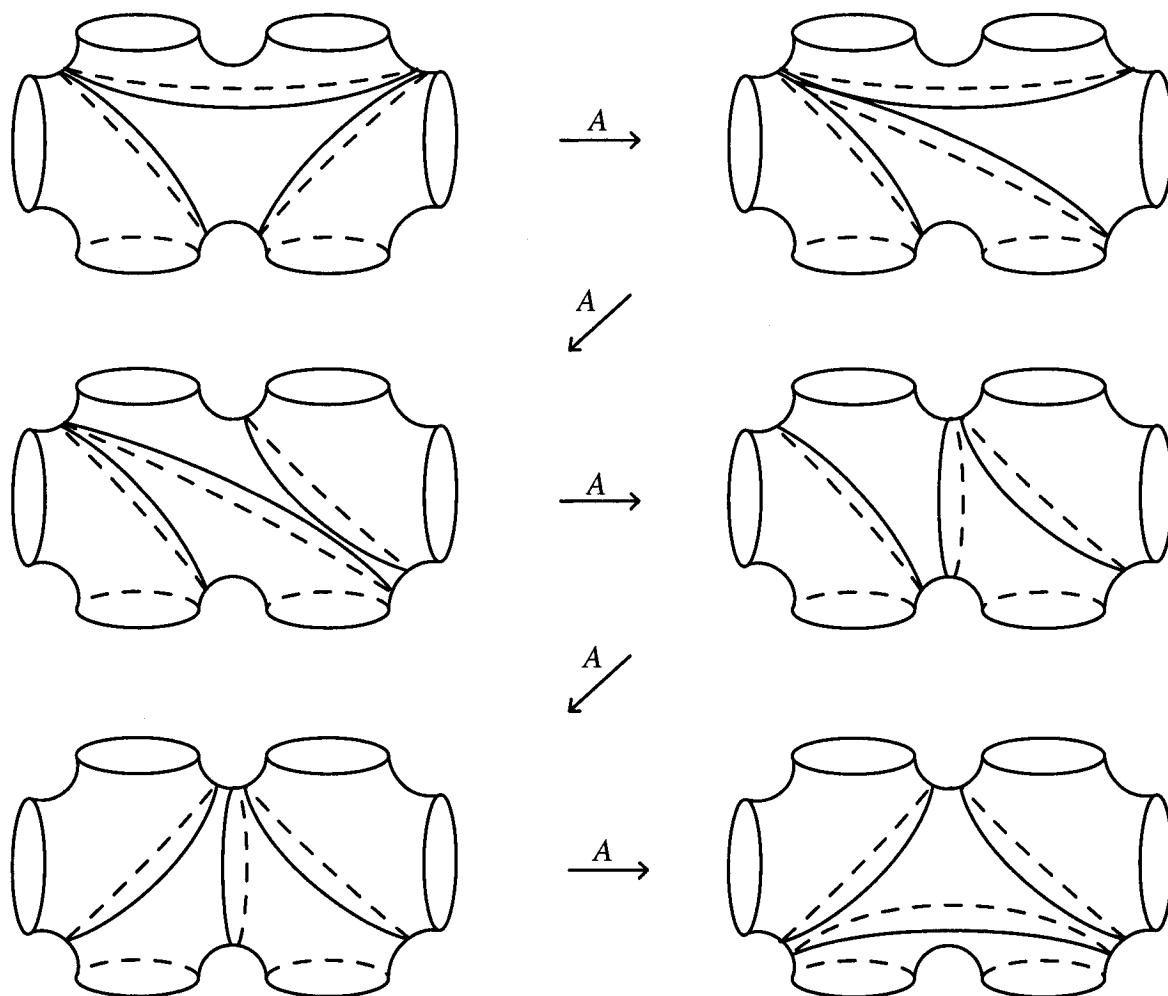


Figure 20: Example of an \mathcal{A} -move on $S_{0,6}$

Example 6.4.3 Given a conformal block of 5 objects, $\langle V_1, V_2, V_3, V_4, V_5 \rangle$ and suppose that we have a strong cut system on it that jointly bounds V_1 and V_2 by a simple closed curve and the same for V_3 and V_4 and that V_5 is left alone. The generalized braiding $B_{V_2V_3V_4, V_5}$ is given by:

$$B_{V_2V_3V_4, V_5} = \mathcal{A}B_{V_4, V_5} \mathcal{A}B_{V_3, V_5} \mathcal{A}B_{V_2, V_5} \tag{77}$$

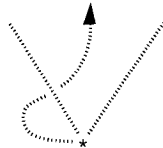
Remark 6.4.4 Note that given a morphism of \mathcal{C} -colored surface $f : M \rightarrow M'$ where $M' = \mathcal{A}(M')$, then f provides a one-to-one correspondence with between their cuts c . Denote their respective strong cut systems by C and C' and label the connected components by the index set J , Then, f give rise to an identification $\mu(M) = \bigoplus_{i_c \in I, c \in C} \bigotimes_{j \in J} \mu(S_{0, n_j}) = \mu(M')$ where j vary through all possible values of J . We then have that μ is blind to all the possible cut decompositions. We therefore suppress \mathcal{A} from our notation.

We now ready to give the central result of this section.

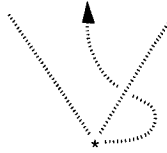
Theorem 6.4.5 Given a colored e -surface M , any elements of the mapping class group $\Gamma(M)$ is unitary under μ with respect to the inner product.

Proof: We first show that each of the generator of mapping class group is unitary and then, we show that composition of unitary operation is unitary.

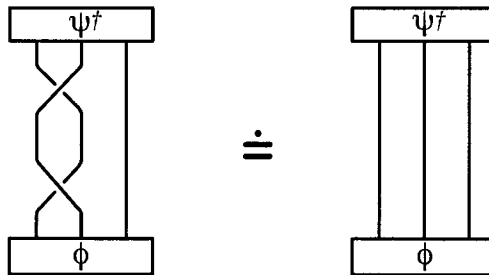
Braids: Without loss of generality, we work on $S_{0,3}$, the complex projective sphere with three holes. On the standard marking of $S_{0,3}$, the operation $b_{1,2}$ is represented by:



Now, on $\overline{S_{0,3}}$, the complex projective sphere with three holes and opposite orientation, the same operation on its marking yields:

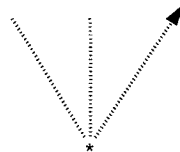


Translating that in the graphical calculus for morphisms yields:

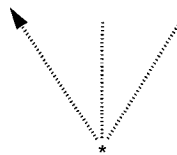


showing that B is an isometry for the inner product.

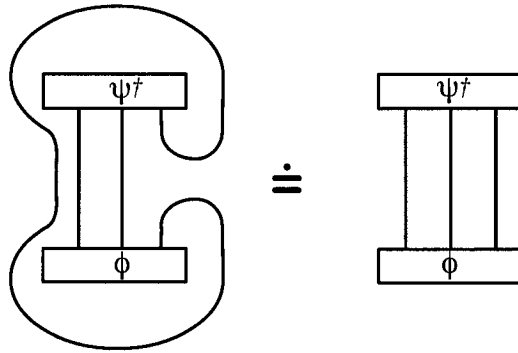
Rotations: Still working on $S_{0,3}$, the operation z on the standard marking yields (assuming that the distinguished arrow is initially on left leg of the marking):



On the same sphere but with opposite orientation we get (where the distinguished arrow is initially on the right leg of the marking):

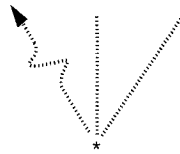


Finally, in the graphical calculus for morphisms, we have:

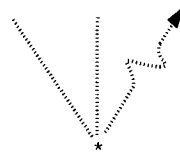


Hence, z is an isometry for the inner product.

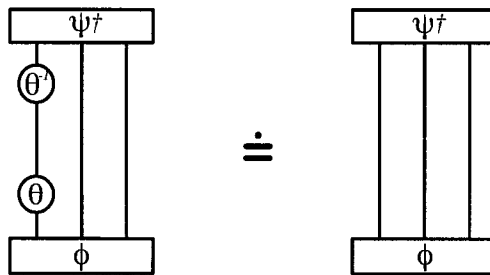
Denh-twists: Applying t on the standard marking, we have:



On the sphere with opposite orientation we obtain:



And in the graphical calculus of morphisms:



Showing that t is unitary.

Finally, it remains to show that composition of unitary operation is unitary. This trivially holds as composition of maps is just stacking operation along the wires in our graphical representation of the scalar product. As the mapping class group $\Gamma(M)$ for a given surface M is closed under composition, this yields the desired result. \square

6.5 Projectors

This section is added only for completeness reasons. It is taken from [1] section 7.4 to which we refer the reader for a more complete exposition.

Lemma 6.5.1 *As \mathcal{C} is semisimple, it has biproduct. Then, we can choose a set of projections $\{\pi_i\}$ and injections $\{\iota_i\}$ for each $\bigoplus_{k=1}^n V_i$ so that*

$$\pi_i \circ \iota_j = \delta_{ij} \quad \sum_{k=1}^n \iota_k \circ \pi_k = 1_{\bigoplus_k V_k} \quad (78)$$

where δ_{ij} is Kronecker's delta.

Definition 6.5.2 *We define the following maps:*

$$i_j := U^\dagger \circ \iota_j : V_j \rightarrow V \quad (79)$$

$$p_j := (\iota_j)^\dagger = \pi_j \circ U : V \rightarrow V_j \quad (80)$$

Definition 6.5.3 *A projector is the map:*

$$P_j := i_j \circ p_j : V \rightarrow V \quad (81)$$

which is self-ajoint, orthogonal and provides a resolution of the identity.

6.6 Topological equivalence of the algorithms

As the fundamental data transformation are of topological nature, the 2-cells of the CW-complex defined in chapter 3 are inducing a topological equivalence between the algorithm.

Definition 6.6.1 *Given an e -colored surface and algorithms:*

$$\mathfrak{A} = m_j \circ m_{j-1} \circ \dots \circ m_1 \quad (82)$$

$$\mathfrak{B} = m'_k \circ m'_{k-1} \circ \dots \circ m'_1 \quad (83)$$

where the m_i are b , z or t moves and $j, k \in \mathbb{N}$. Then, we say that \mathfrak{A} is equivalent to \mathfrak{B} if either $m_i = m'_i$ for all i or by applying one or many of the following equalities:

$$b_i b_j = b_j b_i \quad |i - j| > 1,$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1},$$

$$b_i t_j = t_j b_i \quad |i - j| \geq 1,$$

$$b_i^{\pm 1} t_i = t_i b_i^{\mp 1}$$

$$t_i t_j = t_j t_i,$$

$$z^n = 1,$$

$$b_1 \dots b_{n-1} t_n = z,$$

$$z t_n = t_1 z,$$

we can transform the sequence of moves of \mathfrak{A} to the sequence of moves of \mathfrak{B} .

6.7 Further comments

We conclude this chapter by a short comment on the nature of the generators of the mapping class group when regarded under the action of μ .

If we look at the maps z , t and b under the action of μ , these would correspond to a swap map, phase shifts and universal R-matrices respectively. It is worth noting that L. Kauffman and S. J. Lomonaco Jr. showed in [12] that braiding operators, when seen as R-matrix were universal quantum gates and hence, this implies that the generators of the mapping class group also form a universal set.

Note also that as we have in the axiomatization of the monoidal structure of \mathcal{C} that $I \otimes A \simeq A$ for all $A \in |\mathcal{C}|$, if I appears in the coloring of M , it does not change

the structure of the vector space under the action of μ . As we are computing on the element of the tensor product of objects of \mathcal{C} and that no 'natural' decomposition of that tensor product is available in terms of the basis of the computational space (this basis is induced by the skew-linear form that we defined in section 6.2), the system is always entangled.

Chapter 7

Conclusions and future work

During the last chapter of this thesis, we have developed a categorical semantics for topological quantum computation in showing that given a \mathcal{C} -colored e-surface M and considering the action of a unitary modular functor on it, one can define an inner product on the vector space issued from the action of the modular functor thus endowing it with a Hilbert space structure. We then defined the basic data unit and shown that the whole mapping class group of an arbitrary M was unitary under the action of μ with respect to the inner product previously defined. Finally, we provided a set of equivalences derived from the relations of the CW-complex $\mathbb{M}(M)$ that might be applied to a given topological algorithm applied to see whether or not this algorithm is equivalent to another given algorithm in terms of computation.

Our categorical semantics contrasts with the one provided by B. Coecke and S. Abramsky in [1], in the sense that it applies to quantum systems that present topological properties. Furthermore, another important difference is that the topological systems system are always totally entangled.

It also contrast with the work of Freedman and al. (see, for instance, [9] and [10] and various other papers) in the sense that they developed their theory for physical systems that models anyons essentially, i.e. exotic particles that have fractionnal statistics. Our theory is more general in the sense that, provided a quantum system

that has topological properties, we can adapt it by adding constraints to our axiomatization. For instance, in the case of anyons, we need to have complete braiding (i.e. of the form $\sigma_{V,U}\sigma_{U,V}$) in order to induce a phase shift in the state vector. It would also be an interesting question to see how well our semantics adapts to their system.

The first question that one may ask is how well we may develop some quantum protocols with the set of gates proposed.

We may also wish to know if the set of gates that generates the mapping class group is dense in the sense of the Solovay-Kitaev theorem (given in this work as theorem 2.2.5).

From our point of view, in this thesis, we only worked with genus 0 unitary modular functors. Preliminary investigations seem to show that all the results presented in chapter 6 generalize to modular functors that act on e-surfaces of arbitrary genera. In fact, the only problem lies in extra moves that need to be added to the CW-complex of a given e-surface; as these moves are only compositions of the generators already presented, they are more than likely to be unitary. We might also need to add more topological invariance relations.

Finally, we might like to develop a functional language base on this categorical framework and investigate the possible links with linear logic or other mathematical constructions. Concerning this subject, the interested reader might want to have a look at [21].

Bibliography

- [1] S. Abramsky and B. Coecke, *A categorical semantics of quantum protocols*, in Proceedings of the 19th IEEE Conference on Logic in Computer Science (LiCS'04): 415-425, 2004.
- [2] J. C. Baez, *Quantum Quandaries: A Category-Theoretic Perspective*, to appear in Structural Foundations of Quantum Gravity, eds. Steven French, Dean Rickles and Juha Saatsi, Oxford U. Press, 2004. Also available at <http://math.ucr.edu/home/baez/quantum/quantum.html>.
- [3] B. Bakalov, A. Kirillov, *On the Lego-Teichmüller game*, *Tranf. Groups* **5** (3): 7-44, 2000.
- [4] B. Bakalov, A. Kirillov, *Lectures on Tensor Categories and Modular Functors*, American Mathematical Society in University Lecture Series, 2001.
- [5] J. S. Birman, *Braids, Links, and the Mapping Class Group* in *Annals of Mathematical Studies*, **82**, Princeton University Press, 1976.
- [6] C. Cohen-Tannoudji, B. Diu, F. Laloe, *Quantum Mechanics (Vol. 1)*, Wiley-Interscience, 1978.
- [7] A. J. Feingold, *Fusion Rules for Affine Kac-Moody Algebra*, in Proceedings of the Ramanujan International Symposium on Kac-Moody Lie Algebras and Applications, *Contemporary Mathematics* **343**, N. Sthanumoorthy and Kailash C. Misra eds., American Mathematical Society, 2002.

- [8] R. Fritsch, R. A. Piccinini, *Cellular Structures in Topology*, Cambridge Studies in Advanced Mathematics, **19**, 1990.
- [9] M. H. Freedman, A. Kitaev, Z. Wang, *Simulation of Topological Field Theories by Quantum Computers*, Comm. Math. Phys. **227** (3): 587-603, 2002.
- [10] M. Freedman, M. Larsen, and Z. Wang, *A modular functor which is universal for quantum computation*, Springer-Verlag in Comm. Math. Phys., **227** (4): 605-622, 2001.
- [11] A. Joyal and R. Street, *Braided tensor categories*. Advances in Math. **102**: 20-78, 1993.
- [12] L. H. Kauffman and S. J. Lomonaco Jr., *Braiding Operators are Universal Quantum Gates*, 2004. Available at arXiv:quant-ph/0401090 v2.
- [13] R. L. Liboff, *Quantum Mechanics*, Addison Wesley, 1997.
- [14] S. Mac Lane, *Categories for the Working Mathematician*, Springer Graduate Texts in Mathematics, **5**, 2nd edition, 1998.
- [15] L. Marchildon, *Mécanique quantique*, De Boek Université, 2000.
- [16] W. S. Massey, *Algebraic Topology: An Introduction*, Springer Graduate Texts in Mathematics, **57**, 1977.
- [17] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, Comm. Math. Phys. **123**: 177-254, 1989.
- [18] J. R. Munkres, *Topology*, Prentice Hall, 2000.
- [19] M. A. Nielsen, I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 2000.
- [20] J. Preskill, *Lecture notes for physics 229 : Quantum information and computation*, 1999. Available at:
<http://www.theory.caltech.edu/people/preskill/ph229/>.

- [21] P. Selinger, *Towards a Quantum Programming Language*, *Mathematical Structures in Computer Science* **14(4)**: 527-586, 2004.