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NONLINEAR ANALYSIS OF AXISYMMETRIC SHELLS

by

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Abstract

The present study is concerned with nonlinear analysis of toroidal shells and storage tanks. The study serves to demonstrate the usefulness of the differential quadrature method in this area of computational mechanics.

The problem of the response of axisymmetric toroidal shells to uniform external pressure forms the first part of this study. Nonlinear thin shell theory, accounting for large displacements, is employed. The new differential quadrature method is used to obtain numerical results. For validation a bifurcation solution is found using the finite element method. The commercial code ADINA is used for this purpose. Finally the two methods are used to provide results for eight cases of toroidal shells.

An axisymmetric analysis of a liquid storage tank with a circular base plate resting on an elastic foundation forms the second part of this study. Nonlinear shell theory is used for the tank wall and base plate, while a linear model is used for the foundation. A convergence study is carried out to determine the appropriate analysis parameters for the method. Partial validation is obtained by comparison with previously published results. These results are compared with finite element method results. Additional results are presented covering a wide range of tank geometric parameters.

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Nomenclature

a, b, c, d	constant coefficients
A, B, A_1, A_2	Lame parameters
x, y, z	Rectangular coordinates
h, t, T	Thickness of a plate or a shell
q	Intensity of a continuously distributed load
p	Pressure
u, v, w, U, V, W	Components of displacements
E	Modulus of elasticity in tension and compression
C, J	Extensional rigidity
D	Flexural rigidity
K	Relative rigidity factor
[K]	Stiffness matrix
M_x, M_y	Bending moments per unit distance on x and y planes
M_r, M_θ	Radial and tangential moments per unit distance
N_x, N_y	Normal forces per unit distance on x and y planes
N_r, N_θ	Radial and tangential forces per unit distance
Q_x, Q_y	Shear forces per unit distance on x and y planes
Q_r, Q_θ	Radial and tangential shear forces per unit distance
R_1, R_2	Radii of curvature
L_i	Nonlinear differential operators

m_i	Coefficients of equations
h_{ij}	Coefficients of equations
H_{ij}	Coefficients of equations
$A_{ij}^{(r)}$	Weighting coefficients for r^{th} order derivative
DQM	Differential Quadrature Method
FEM	Finite Element Method
$(\ )^*$	Accumulated value during whole deformation process
κ_1, κ_2	Curvatures
r, θ	Polar coordinates
γ	Weight per unit volume
$\varepsilon_x, \varepsilon_y, \varepsilon_z$	Normal strains in x, y, and z directions
$\varepsilon_r, \varepsilon_\theta$	Radial and tangential normal strains
ν	Poisson's ratio

Chapter 1

Introduction

1.1 Introduction

The term shell is applied to bodies bounded by two curved surfaces, the distance between the surfaces being small in comparison with the other dimensions, and the thickness in most cases being constant. Shell structures are used in many branches of technology, such as building construction, mechanical engineering, shipbuilding, chemical engineering, aerospace engineering, and nuclear reactor engineering. Many components of modern engineering structures are made in the form of complicated thin shells of different shapes. The use of shells as structural elements makes it possible to select efficient parameters in regard to the strength and reliability of the structure. The primary function of a shell may be to transfer loads from one of its edges to another, to support a surface load, to provide a covering, to contain a fluid, or a combination of these.

If each bounding surface is generated by the rotation of a plane curve about a common axis, a shell of revolution is produced. The generating curve is called the meridian, and its plane is a meridional plane. In the particular case when the surface loadings are axisymmetrical, the deformation of the shell is independent of circumferential direction. In axisymmetrical problems involving shells of revolution, no shear forces exist and there are only two unknown membrane forces per unit length. Then the equations can become simpler. Both of the problems of this study concern axisymmetrical problems.

Toroidal and cylindrical shells are among the most widely used shell forms. The present study deals mainly with these two types of shells of revolution. The first part of this work is devoted to a complete toroidal shell. The second part deals with a cylindrical shell joined with a circular plate resting on an elastic foundation.

Toroidal shells are widely used structural elements. Their importance stems from their closed doubly curved shape that gives them enhanced rigidity. Applications of toroidal shells include aircraft and automobile tires, nuclear reactor vessels, space station structures, space vehicle liquid storage containers, and underwater structures. The analysis of this type of shell has attracted considerable attention including the nonlinear response to external pressure loading. In this study, the nonlinear response of axisymmetric toroidal shells to uniform external pressure is the first of two major problems considered.

Cylindrical shells are even more important structural elements. They are commonly used to store inert liquids such as water, or toxic fluids like gasoline and other chemicals. In this study the nonlinear behavior of a cylindrical shell together with a circular plate on an isotropic elastic soil medium is the second of two major problems considered.

1.2 Solution of nonlinear shell equations

The behavior of an elastic shell is said to be nonlinear if, under static conditions, the deflection of any point of the shell is not proportional to the magnitude of an applied load. Two sources of nonlinearity are often distinguished; geometric and material. For geometric nonlinear behavior, the strain-displacement relation is nonlinear but the stress-strain relations are linear. Most research work on nonlinear shell theory concerns this type of nonlinearlity. In this study as well, only geometric nonlinearity is considered.

A number of nonlinear theories dealing with thin shells have been presented beginning from the early 1900s. These include theories by Marguerre, Donnell, Flügge, Mushtari and Vlasov, Reissner, Novozhilov, Koiter, Sanders, and Budiansky (Teng and Hong, 1998).

In this study the Sanders theory is adopted. This theory has been applied widely to static, dynamic, and stability problems of various types of shells, and its accuracy has been accepted as being satisfactory for thin shells for all practical purposes.

In case loads are applied axisymmetrically to shells of revolutions like toroidal and cylindrical shells, axisymmetric analysis can be carried out. In such an analysis the problem reduces to one dimension mathematically, and the number of displacements and stress resultants is reduced. Then the equations are of much simpler form from those of nonsymmetric analysis.

To linearize the nonlinear differential equations $L(\Theta) = 0$, Newton's approach (Bert and Malik, 1996a) is adopted herein. Beginning with assumed variables (displacements) consistent with the boundary conditions, successively refined solutions are obtained through the following iteration scheme;

$$\Theta^{(n+1)} = \Theta^{(n)} + \theta^{(n)}$$

where θ is the variable's refinement and n is the iteration count. The refinement is determined by the solution of the following equation written in operator form

$$\theta L'(\Theta) + L(\Theta) = 0$$

where $L(\Theta)$ is the nonlinear operator and $L'(\Theta)$ is the Frechet derivative defined as

$$\theta L'(\Theta) = \frac{\partial}{\partial \theta} L(\Theta + \varepsilon\theta)\Big|_{\varepsilon=0}$$

The operator is differentiated partially with respect to ϵ and then, in the resulting derivative, ϵ is set equal to zero.

1.3 Differential Quadrature Method (DQM)

The differential quadrature method (DQM) is a general purpose approach to approximate and numerically solve boundary and initial value problems. The method has been applied to a large number of structures, such as beams, arches, and plates. Many researchers have shown that this method has the advantage of producing highly accurate solutions with moderate computational effort. Therefore it has the potential to become an alternative to conventional numerical methods in solving structural problems (Bert et al, 1988; Bert and Malik, 1996).

The basis of DQM is the representation of the derivatives of the function f(x) by a weighted sum of the function values at sampling points in the domain. The weighting coefficients are the basic factors paramount to the accuracy of differential quadrature solutions. Use of explicit formulas for these factors is essential because if they are found indirectly, ill-conditioning of the Vandermonde matrices involved leads to increasingly erroneous values with increasing higher order derivatives. With the primary objective of obtaining accurate weighting coefficients, Quan and Chang (1989) derived explicit formulae using polynomial test function for first and second order derivatives. Shu and Richards (1990) derived identical formulae and a general recurrence relationship which can generate weighting coefficients for any second and higher order derivatives.

The DQM was applied for the first time to a nonlinear one-dimensional boundary value problem by Civan (Civan and Sliepcevich, 1983). Another application of the DQM to a highly nonlinear problem was considered by Blick and Civan (1988) for the solution of the porous-media momentum equation. The DQM was employed for the nonlinear static flexure of thin circular plates subjected to uniform pressure and central point loads (Striz et al, 1988). Later, Bert et al (1989) analyzed the large deflection problems of thin isotropic and orthotropic rectangular plates. Several solutions were given to toroidal and cylindrical shells problems under various external or internal loads by Redekop et al (1999).

Most of the recent studies and researches concerning the DQM have been oriented to linear problems and a few limited very simple nonlinear problems. The DQM in this study is applied as a major numerical method for the highly and complicated nonlinear axisymmetric shell problems. The aims are to compare results with the those available from previous theoretical solutions and experiments. This approach employing the DQM for the nonlinear problems of shells forms a new area of research involving shells.

For the first part of this study, the one dimensional DQM is used with discretization in the meridional direction only, and harmonic test functions are used which account for the cyclic periodicity in this direction. For the second part, polynomial trial functions are selected and explicit formulas for weighting coefficients are employed. The governing equation is a fourth order differential equation, and therefore two boundary conditions for one sampling point should be applied. The so-called δ-technique proposed by Zang et al. (1989) is used. An additional point is chosen at a small distance ($\delta = 10^{-4}$ or $\delta = 10^{-5}$) from the boundary points.

1.4 Finite Element Method (FEM)

Since the development of the first 'general purpose' finite element (FE) computer programs in the 1960s, the finite element method (FEM) has become the main tool in computational structural mechanics. The success of the method is manifested by the number of program packages available, the number of FE users, and the amount of money spent on FEM worldwide.

Regardless of the program or the problem being solved, a finite element analysis (FEA) involves the following main steps:

(1) FE modeling including geometry definition, initial mesh definition, prescription of loads and boundary conditions, definition of element and material properties, problem setup and output requests.

(2) Assembly, which involves combining element contributions and solving a resulting set of linear equations.

(3) Result generation, which involves compilation and presentation of the results of the analysis in a form which is meaningful and provides insight to the analyst.

The first work on the extension of the FE procedure to geometrically nonlinear analysis of structures was reported by Turner et al.(1960) who showed that a new class of stiffness matrix has to be introduced if large deflection analysis is to be considered. This stiffness matrix depends on the state of stress existing in the element prior to the imposition of an additional disturbance, and is often called the initial stress stiffness. This matrix is to be superimposed on the conventional stiffness matrix to form the total stiffness matrix of the element. Derivations of such a matrix for axial force members and arbitrary in-plane finite elements were presented by Green (1960). The nonlinear behavior was therefore predicted by the use of a linearized incremental procedure. The accuracy of this procedure depends on the smallness of the load incrementation.

To overcome the drifting tendency of the linearized incremental method and the large amount of computational effort required by the Newton-Raphson iterative technique, Brebbia and Connor

(1969) suggested an economic form of analysis by employing a mixed procedure consisting of using a linearized incremental analysis for a limited number of load steps and then corrected with Newton-Raphson iterations. Many strategies have been proposed for geometrically nonlinear analyses of shells (Zienkiewicz, 1977; Bathe, 1982; Hughes and Liu, 1981), the most commonly used being the total and updated Lagrangian formulations.

For this study, the program ADINA (Automatic Dynamic Incremental Nonlinear Analysis) was adopted to carry out the FEM analysis (ADINA, 1999). In ADINA a nonlinear static analysis using axisymmetry elements is employed to make a model and the Newton-Raphson iteration scheme:

$$^{t+\Delta t}K^{(i-1)}\Delta U^{(i)} = {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)}$$
$$^{t+\Delta t}U^{(i)} = {}^{t+\Delta t}U^{(i-1)} + \Delta U^{(i)}$$

is used. Here K is the stiffness matrix, ΔU is the displacement increment, R is the vector of externally applied nodal loads, F is the force vector equivalent to the element stresses, t is the time step, and i is the load incremental step.

For the first part of this study an ATS (Automatic Time Stepping) algorithm is employed to find the buckling points. Sparse solvers are selected for both parts in this study to run the program. These solvers have been thoroughly tested on a broad range of problems and have been proven to yield drastic reductions in solution times.

1.5 Scope of the current study

In Chapter 1 thin shells and the nonlinear analysis of shells are introduced, and two numerical methods are briefly outlined. Chapter 2 gives a literature survey that depicts previous work done in the area of nonlinear theories of shells, nonlinear analysis of cylindrical shells and plates, differential quadrature method, and nonlinear FEM analysis. Chapters 3 and 4 form the two main parts of this study. They give detailed presentations and solutions for the nonlinear axisymmetric buckling analysis of toroidal shells, and for the nonlinear analysis of axisymmetric liquid storage tanks. These chapters contain the development of new approaches to solve the nonlinear problems of shells, as well as validation of the previous results using DQM and FEM. Chapter 5 concludes this study presenting the technical significance of the results by the two methods, as well as recommendations for further research.

Chapter 2

Literature Survey

2.1 Introduction

Shell theory originated historically with the consideration of elastic plates by Cauchy and Poisson. Modern shell technology may be traced back to the fundamental work of Love and Kirchhoff in the second half of the nineteenth century. But their theory was not entirely sufficient, adequate, or accurate. Galerkin tried to cover some of the deficiencies and obtained all the formulae of the theory of shells from the equations of the general theory of elasticity. Novozhilov represented the equations of the theory of shells in complex form, making them more compact. Vlasov and Rabotnov devoted efforts to the simplification of the equations of shell theory, applicable to a practical range of problems. Since then, considerable additional work and effort has been done on the theory, analysis, modelling and construction of shells (Teng and Hong, 1998).

Cylindrical tanks are commonly used to store liquids and form one of the major types of shells. Extensive literature is available for the analysis and design of cylindrical storage tanks by classical methods (Timoshenko and Woinowsky-Krieger, 1959). Yamaki (1984) organized the nonlinear theory of circular cylindrical shells and presented a number of experimental and theoretical results on the buckling of cylindrical shells. Kukreti et al (1993, 1997) solved the linear problem of a tank joined to a base plate using the energy method and the DQM respectively.

With the rapid developments of computing technology complicated nonlinear equations of shells can be solved using numerical analysis. The most popular and widely used method is the FEM. There are many recent studies (Mackerle, 1997) using this method dealing with nonlinear problems. The standard references on FEM are the books by Zienkiewicz, Bathe, and Hughes. Among them Bathe developed a special FEM code, ADINA, for the nonlinear problems. That code has been validated as an accurate nonlinear analysis for many shell problems.

The most recently developed method for the nonlinear structural problem is the DQM. This method has the potential to give very accurate results with less efforts and less computer running time than FEM. A well organized review paper was presented by Bert and Malik (1996) and many references are available for many branches of engineering problems. This method is now rapidly gaining importance in the solution of nonlinear problems.

2.2 Nonlinear theory of shells

A general nonlinear theory for thin shells was developed by Sanders (1963). He derived strain-displacement relations for thin shells valid for large displacements by making certain simplifying assumptions. The resulting equations are suitable as a basis for stability analysis or other problems in which the effects of deformation on equilibrium cannot be ignored. This theory is a first approximation theory in the sense that transverse shear and normal strains are neglected.

In 1965, Spier, Wilson and Slick presented a nonlinear analysis of thin toroidal shells of circular cross section. The differential equations which govern a nondimensional stress function and the rotation of a meridional tangent, were written in a convenient matrix form and solved by an iterative scheme.

Budiansky (1968) provided more developed notes on nonlinear shell theory. Exact tensor equations of equilibrium were derived for nonlinear membrane shell theory and small perturbations of pressurized membrane shells. Exact equations for the general nonlinear shell theory were also discussed, and approximate equations were derived for the small perturbations, buckling, and vibration of stressed shells. The emphasis of this theory is on exact formulations of the nonlinear and small-perturbation theories using tensor notation.

In 1994, Pai and Nayfeh (1994) presented a general geometrically nonlinear theory for the dynamics of laminated plates and shells undergoing large rotation and small-strain vibrations in 3-D space. The theory fully accounts for geometric nonlinearities by using the new concepts of local displacements and local engineering stress and strain measure, a new interpretation and manipulation of the virtual local rotations, an exact coordinate transformation, and the extended Hamilton principle. The theory is valid for any plate or shell geometry and contains most of the existing nonlinear and shear-deformable plate and shell theories as special cases.

Grigorenko (1996) used a refined formulation in shell theory. He examined some of the approaches used to solve unidimensional and two-dimensional boundary value problems for the deformation of shell elements of structures with variable geometric and mechanical parameters. The problems discussed are examined in linear and geometrically nonlinear formulations on the basis of classical and refined theories.

Bera (1998) derived a completely new set of uncoupled differential equations that govern the behavior of an elastic, unsymmetrical, doubly curved, heated sandwich shells whose face sheets are of unequal thickness and of different materials. The equations include nonlinear effects and can be used for the analysis of the movable as well as immovable edge conditions. The theory is verified by using numerical results for a rectangular unsymmetrical sandwich cylindrical shell under thermal loading.

Teng and Hong (1998) presented a new theory for numerical nonlinear and buckling analysis of thin shells. They re-examined the classical shell theories in the context of numerical nonlinear and buckling analysis by describing a set of nonlinear strain-displacement relations for thin shells of general form developed directly from the nonlinear theory of 3-D solids. All nonlinear terms are retained and additional complexity leads to only a small increase in computational effort. Analytical and numerical comparisons were carried out for thin shells of revolution. The results showed that the simplified Sander's theory with rotations about the normal omitted should not be used in finite element codes for complex shell structures.

2.3 Nonlinear analysis using FEM

A number of modern FEM shell analyses have been conducted which relate closely to the problems examined in this study. Many different studies using FEM to solve nonlinear problems of shells have been published from the 1960s. Among them is a well organized book by Bathe (1982) who presents detailed theories for nonlinear FEM.

Exeter and Henshell (1978) outlined an eigenvalue problem approach for predicting the axisymmetric pressure load which causes shells of revolution to buckle non-axisymmetrically. The method was based on a general shell approach rather than the usual thin shell one. The stability equations were far less complex, relative to those derived using a thin shell theory. Using this approach, the importance of the material and geometric nonlinear effects was demonstrated.

Lee and Hwang (1993) analyzed the dynamic response of composite cylindrical shell panels under initial in-plane stress impacted by a rigid ball. They used the FEM with a nonlinear shell theory based on Donnell's shell theory with von-Karman's large deflection assumptions. The dynamic transient response including deflections, stresses, strains, and contact forces were presented. In addition, the effect of curvature and initial stress in the shell on the impact responses, were also investigated. In 1995, Sabir and Djoudi (1995) investigated the large deflection, geometrically nonlinear behavior of shells using FEM. The Newton-Raphson iterative technique and linearised incrementation were employed. Strain-displacement relationship based on shallow shell formulation were used and applied to an element having curvatures. Complex load-deflection curves are obtained for cylindrical and spherical shells by incrementing loads as well as deflections.

Greer and Palazotto (1996) applied a total Lagrangian finite element scheme for arbitrarily large displacement and rotations to a composite beam, an isotropic deep arc, and a section of an isotropic circular toroidal shell. The scheme decomposes the motion into stretches and rigid body rotations, examining the deformed state with respect to an orthogonal, rigid translated and rotated triad located at the deformed point of interest. Large displacement and rotation analyses were undertaken using a local layer-wise assumed displacement field for modeling transverse behavior. The second Piola-Green measures were associated with the directions along the deformed edges of the element. A linear analysis of a toroidal shell section was taken to verify model validity, and the result showed very good agreement.

Meek and Wang (1998) presented a simple, effective finite element incremental formulation and procedure for geometrically nonlinear static and dynamic analysis of a shell structure with finite rotations based on both the total Lagrangian and updated Lagrangian description of motion. An incremental iterative method based on constant arc length method in conjunction with Newton-Raphson method was implemented for static and dynamic analysis.

To and Wang (1998) applied the FEM to laminated composite shell structures for the transient responses of geometrically nonlinear analysis. They employed a hybrid strain-based flat triangular laminated composite shell element. This element could be extended and modified to analyze problems with elastic-plastic deformations of finite strain and finite rotations.

2.4 Details of DQM

Since it's introduction by Bellman and Casti (1971) the DQM has been applied successfully to a variety of problems in structural mechanics (Bert et al., 1988; Jang et al., 1989). The main advantage of the DQM is that in many cases the computational efforts can be reduced. The first difficulty encountered in applying DQM to engineering problems involved an ill-conditioned Vandermonde matrix. This difficulty arose in the calculation of weighting coefficients. But Quan and Chang (1989) and Shu and Richards (1992) developed new explicit formulae to get the weighting coefficients matrix for the polynomial test functions.

In 1994, Striz, Chen and Bert (1994) extended the application of DQM from simple geometries and simple boundary conditions to more complicated geometries. To do this they proposed a domain decomposition technique for the DQM, where the whole structural domain is represented by a collection of simple element subdomains connected together at specific nodal points. This method was named the quadrature element method (QEM).

Malik and Bert (1994) applied the DQM to nonlinear steady-state incompressible and compressible lubrication problems. They proposed harmonic test functions for periodic field variables in one or more coordinate directions. Based on the work of Hamming (1973) they developed a new weighting coefficients matrix for harmonic test functions. By using Newton's approach the nonlinear differential equations were linearized. This study showed that the DQM can easily compete with the previous methods in the solutions of lubrication problems.

Chen, Striz, and Bert (1997) presented a new technique for treating boundary conditions, in which the boundary conditions were satisfied exactly. This approach eliminates the deficiencies of the δ-type grid arrangements, which represents an approximation. Two kinds of basis functions, Chebyshev and Lagrange, were used for demonstration. The results showed the Chebyshev polynomials to be more accurate than Lagrange's.

Explicit computation of weighting coefficients in the harmonic functions was developed by Shu and Xue (1997). Based on the Fourier series expansion and the same concept of explicit formulae for weighting coefficients by Shu and Richards (1992) they developed an explicit formulation for weighting coefficients of harmonic functions. The developed method was validated by its successful application to the free vibration analysis of a rectangular plate.

A new idea namely differential quadrature element method (DQEM) was proposed by Liu and Liew (1998). The basic idea is to divide the whole domain into several subdomains (elements) and to apply the DQM to each element. This method had been successfully applied to several bending problems of Reissner-Mindlin plates with different supports including mixing boundary conditions. The accuracy and applicability of this method was examined by comparing with existing solutions.

2.5 Analysis of storage tanks

Abundant literature is available for cylindrical storage tanks covering classical analysis and design techniques. In full form the tank problem involves the interaction of shell, base plate, foundation and fluid. Cheung and Zienkiewicz (1965) have applied the finite element method to the analysis of soil-structure interaction problems. Booker and Small (1983) have presented a flexibility-type analysis procedure, which accounts for the interaction between the tank wall and the plate foundation-soil system. Kukreti, Zaman, and Issa (1992) presented an analytical formulation that can be used to predict the linear flexural behavior of cylindrical storage tanks having circular plate foundations and resting on isotropic elastic half space. The formulation accounts for the interaction between the tank wall and the plate foundation as well as the interaction between the plate foundation and the elastic half space.

In 1992, using the nonlinear theory of shells, a study on the buckling of storage tanks was presented by Zhou, Zheng, and Zhang. Based on Sander's nonlinear thin shell theory and following the ring finite element discretization in the shell, the matrix equations of motion was derived. An iterative formulation for the solution of the equations was proposed. This study showed that effects of geometrical nonlinearity of the shell are of great significance on the axial and hoop membrane stresses near the buckle region.

Nonlinear behavior of the bottom plate in cylindrical liquid storage tanks was considered by Lau and Zeng (1995). The problem of axisymmetric uplift of the bottom plate was modelled by 1-D beam and 2-D plate models for both rigid and elastic foundation. The result showed that the support foundation flexibility might have significant effects on the uplift behavior of tanks.

Malhotra and Veletsos (1994) presented a refined and efficient method of analysis for a uniformly loaded, semi-infinite, prismatic beam resting on a rigid base and uplifted by a vertical force at on end. Motivated by the need for improved understanding of the uplifting resistance of the base plate in cylindrical liquid storage tanks, the method accounted for the effects of elastic end constraints, of the axial forces associated with large deflections, and of plastic yielding in the beam. In the same year, they developed another approach. Due consideration was given to the nonlinear effects associated with both membrane action and material yielding. Both axisymmetric and asymmetric conditions of uplifting were examined.

Kukreti and Siddiqi (1997) applied the DQM to the linear analysis of fluid storage tanks including foundation-superstructure interaction. In fact this study was based on the previous one in 1992 by Malik, Bert and Kukreti (1993). The results were compared with the FEM results given by Booker and Small (1983). The procedure accounted for the linear interaction between the tank wall and the plate foundation using slope and moment compatibility. Also using the contact stress equation for the elastic half space, the interaction between the plate foundation and the soil medium was accounted for. Another significance of this study was that the new technique of the DQM was applied to the tank problem. The domain was divided into sub-domains, and each sub-domain was normalized using geometric parameters to apply the DQM. By combining the sub-domains with boundary and compatibility conditions the domain decomposition technique was developed.

Another approach to the liquid storage tank by the DQM was proposed by Mirfakhraei and

Redekop (1999). Using an equivalent static loading the dynamic loading was represented. The nonlinear Flügge shell theory was used to set up buckling equations. The results were compared with that from FEM. The procedure of this analysis consists of two steps. In the first step, the pre-buckling analysis, the membrane and bending resultants are found for the hydrostatic and equivalent static lateral loadings using the Flügge equations. In the second step, the bucking load analysis, an eigenvalue problem is solved to find the critical load.

2.6 Summary

Although there are many researches of various theories and methods in the area of nonlinear analysis of shells beginning from early 1990's, there is rather limited literature on nonlinear numerical analysis of complicated shells using, for example, the DQM. Some numerical results are available for nonlinear analysis of shells which can serve to validate results from this study.

Chapter 3

Nonlinear Analysis of Toroidal Shells

3.1 Introduction

Axisymmetric toroidal shells and pressure vessels find extensive application in industry. Their present or intended use includes components of submersibles, offshore platform floatation cells, and rocket fuel tanks. The analysis of this type of shell has attracted considerable attention over the years (Zhang et al, 1998) including the nonlinear response to external pressure loading. Analysis using an incremental approach, or a linearized buckling approach have been reported by a number of authors including Spier et al (1965), Sobel and Fluegge (1967), Boedecker and Wempner (1972), Bushnell (1976), Gaidaichuk et al (1978), Panagiotopoulos (1985), Wang and Zhang (1991), and Galletly and Blachut (1995). The survey papers of Grigorenko (1996) and Teng and Hong (1998) indicate that analytical, FEM, and finite difference methods have been used to date.

In this study a solution based on the new DQM is presented for the nonlinear problem of a complete axisymmetric toroidal shells subjected to external pressure loading. The DQM has greater flexibility than analytical approaches, and has been reported by Mirfakhraei and Redekop (1999), Bert and Malik (1996), and Malik and Bert (1994) to have superior efficiency relative to purely numerical methods for engineering problems of 'regular geometry'.

The nonlinear shell theory outlined by Gaidaichuck et al (1978) serves as the basis for the DQM analysis. Results are also obtained using the commercial FEM program ADINA (1999). The two methods are used to analyze eight toroidal shell cases. Results obtained are compared with analytical and experimental work by Almroth et al (1969), Nordel and Crawford (1971), and Jordan (1973) and conclusions are drawn.

3.2 Geometry

The shell under consideration has a toroidal bend radius of R, a circular cross-sectional radius of r, and a thickness of h (Fig. 3.1a). The shell is complete in the circumferential (ϕ) and meridional ($\theta=x^1$) directions. As the shell is complete in both directions no boundary equations are required. The problem considered is clearly one of 'regular geometry', for which the DQM has been shown to be effective.

A uniform external pressure p is assumed to act on the surface of the shell. The level of the pressure is assumed to be such as to produce a nonlinear response in the shell. Previous studies

(Gaidaichuk et al, 1978) have indicated that the buckling mode under an uniform external pressure can be considered as axisymmetric with respect to the central axis. Experiments performed by Jordan (1973) for values of R/r from 1.19 to 7.94 also show that toroidal shells buckle in the axisymmetric mode, asymmetrically with respect to the equatorial plane. By taking account of this feature only a single meridional cross section of the shell needs to be considered in solving stability problems.

3.3 DQM for toroidal shell

The DQM (Mirfakhraei and Redekop, 1999; Bert and Malik, 1996a; Malik and Bert, 1994) is used herein to obtain numerical results. This method was introduced to problems of solid mechanics by Bert and Malik (1996a). As these authors provide a detailed account of the method, only an outline is given in the following.

The basis of the DQM is the representation of the derivatives of a function f(x) by a weighted sum of trial function values in the domain, i. e.

$$\left.\frac{d^r f}{dx^r}\right|_{x=x_i} = \sum_{j=1}^{M} A_{ij}^r f(x_j) \tag{3.1}$$

Here the A_{ij}^r are the unknown weighting coefficients of the r-th order derivative at the i-th sampling point in the domain, and M is the number of sampling points in the x direction. The weighting coefficients can be determined for all order derivatives for an appropriately chosen function (test function) by solution of a matrix equation, or by using specialized formulas (Bert and Malik, 1996a).

In the present problem of axisymmetry the one-dimensional DQM may be used, with discretization in the meridional direction only. Following Mirfakhraei and Redekop (1999) and Malik and Bert (1994), harmonic test functions are used which account for the cyclic periodicity in this direction. Continuity conditions across $\theta = 360°$ are then satisfied identically. The test functions are taken as

$$f(\theta) = \cos[2(k-1)\pi\theta]; \quad k = 1,2,3,\ldots,\frac{N}{2}+1$$
$$f(\theta) = \sin[2(k-N/2-1)\pi\theta]; \quad k = \frac{N}{2}+2, \frac{N}{2}+3,\ldots,N \tag{3.2}$$

where N is an even number. For equally spaced sampling points the weighting coefficients A_{ij}^{r} may be found explicitly (Mirfakhraei and Redekop, 1999) from the inverse of the Vandermonde matrix.

3.4 Shell theory

In 1978 Gaidaichuk et al (1978) gave a solution for the nonlinear response of isotropic axisymmetric toroidal shells. For completeness their development, with signs modified following the Budiansky (1968) theory, is presented herein.

An Euler representation with a "frozen-in" coordinate system is used. Resolvents of axisymmetric shells or shells of revolution of general form are assumed to take account of the first and second quadratic forms of the middle surface in the deformation process. It is assumed though that in the deformation process the rotation of the coordinate basis can be neglected, and the cosine of the angle of rotation can be taken as unity. Under this assumption the displacement vector can be projected onto the axes of the original basis throughout the loading process.

The equilibrium equations of the axisymmetric shell are given by

$$\frac{d}{dx^1}(BN_1) - \frac{dB}{dx^1}N_2 + k_1[\frac{d}{dx^1}(BM_1) - \frac{dB}{dx^1}M_2] = 0$$
$$\frac{d}{dx^1}\left\{\frac{1}{A}\left[\frac{d}{dx^1}(BM_1) - \frac{dB}{dx^1}M_2\right]\right\} - AB(k_1N_1 + k_2N_2) + pAB = 0 \tag{3.3}$$

where A and B are the Lame parameters, while k_1 and k_2 are the curvatures in the meridional and circumferential directions. The differentiation is with respect to the meridional coordinate $\theta = x^1$ only, due to the axisymmetry. N_1 and N_2 are the stress resultants, and M_1 and M_2 are the moment resultants. Finally p is the normal pressure.

The stress and moments resultants are given by

$$N_1 = \frac{Eh}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2)$$
$$N_2 = \frac{Eh}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1)$$
$$M_1 = \frac{Eh^3}{12(1-\nu^2)}(\chi_1 + \nu\chi_2) \tag{3.4}$$
$$M_2 = \frac{Eh^3}{12(1-\nu^2)}(\chi_2 + \nu\chi_1)$$

where E is the Young's modulus and ν is the Poisson ratio. The strains and curvatures present in these relations are given by

$$\varepsilon_1 = \frac{1}{A}\frac{du}{dx^1} + k_1 w + \frac{1}{2}\left(k_1 u - \frac{1}{A}\frac{dw}{dx^1}\right)^2$$
$$\varepsilon_2 = \frac{1}{AB}\frac{dB}{dx^1}u + k_2 w$$
$$\chi_1 = \frac{1}{A}\frac{d}{dx^1}\left(k_1 u - \frac{1}{A}\frac{dw}{dx^1}\right) \tag{3.5}$$
$$\chi_2 = \frac{1}{AB}\frac{dB}{dx^1}\left(k_1 u - \frac{1}{A}\frac{dw}{dx^1}\right)$$

where u and w are the meridional and normal displacement components.

The equations presented apply to arbitrary axisymmetric shells having geometric characteristics A, B, k_1, and k_2. These characteristics change as the shell deforms according to

$$
\begin{aligned}
A^*_{(n+1)} &= A^*_{(n)}(1+\varepsilon_1) \\
B^*_{(n+1)} &= B^*_{(n)}(1+\varepsilon_2) \\
k^*_{1(n+1)} &= k^*_{1(n)}(1-\varepsilon_1)+\chi_1 \\
k^*_{2(n+1)} &= k^*_{2(n)}(1-\varepsilon_2)+\chi_2
\end{aligned}
\tag{3.6}
$$

where n indicates the time step, i.e. the load level. Here and subsequently in this section quantities accumulated during the whole deformation process are marked with an asterisk, while quantities without an asterisk denote their increments during the corresponding change of the pressure p.

The Frechet differential is taken of the governing equations, and the derivative of the coefficients of the first quadratic form is neglected with respect to the load parameter p. The incremental equilibrium equations are then given by

$$
\begin{aligned}
&\frac{d}{dx^1}(B^*N_1)-\frac{dB^*}{dx^1}N_2+k_1^*[\frac{d}{dx^1}(B^*M_1)-\frac{dB^*}{dx^1}M_2] \\
&+(\chi_1-k_1^*\varepsilon_1)\left[\frac{d}{dx^1}(B^*M_1^*)-\frac{dB^*}{dx^1}M_2^*\right]=0 \\
\\
&\frac{d}{dx^1}\left\{\frac{1}{A^*}\left[\frac{d}{dx^1}(B^*M_1)-\frac{dB^*}{dx^1}M_2\right]\right\} \\
&-A^*B^*\left[(k_1^*N_1+k_2^*N_2)+(\chi_1-k_1^*\varepsilon_1)N_1^*+(\chi_2-k_2^*\varepsilon_2)N_2^*-(\varepsilon_1+\varepsilon_2)p^*-p\right]=0
\end{aligned}
\tag{3.7}
$$

Note that p* is the accumulated pressure while p is the incremental pressure. The incremental strains and curvatures are given by

$$
\begin{aligned}
\varepsilon_1 &= \frac{1}{A^*}\frac{du}{dx^1} + k_1^* w + k_1 w^* + \left[k_1^* u^* - \frac{1}{A^*}\frac{dw^*}{dx^1}\right]\left(k_1^* u + k_1 u^* - \frac{1}{A^*}\frac{dw}{dx_1}\right) \\
\varepsilon_2 &= \frac{1}{A^* B^*}\frac{dB^*}{dx^1} u + k_2^* w + k_2 w^* \\
\chi_1 &= \frac{1}{A^*}\frac{d}{dx^1}\left(-\frac{1}{A^*}\frac{dw}{dx^1} + k_1^* u + k_1 u^*\right) \\
\chi_2 &= \frac{1}{A^* B^*}\frac{dB^*}{dx^1}\left(-\frac{1}{A^*}\frac{dw}{dx^1} + k_1^* u + k_1 u^*\right)
\end{aligned}
\tag{3.8}
$$

The incremental displacements can be found from these latter two sets of equations for any value of the incremental pressure p.

Based on these two equation sets the DQM analog equations are developed as

$$
\begin{aligned}
&\left(H_{11}\sum A_{il}^{(2)} + H_{12}\sum A_{il}^{(1)}\right)u_j + H_{13}u_i \\
&+ \left(H_{14}\sum A_{il}^{(3)} + H_{15}\sum A_{il}^{(2)} + H_{16}\sum A_{il}^{(1)}\right)w_j + H_{17}w_i + H_{18} = 0 \\
&\left(H_{21}\sum A_{il}^{(3)} + H_{22}\sum A_{il}^{(2)} + H_{23}\sum A_{il}^{(1)}\right)u_j + H_{24}u_i \\
&+ \left(H_{25}\sum A_{il}^{(4)} + H_{26}\sum A_{il}^{(3)} + H_{27}\sum A_{il}^{(2)} + H_{28}\sum A_{il}^{(1)}\right) + H_{29}w_i + H_{30} = 0
\end{aligned}
\tag{3.9}
$$

where

$$
C = \frac{Eh}{1 - \nu^2}
$$

$$
D = \frac{Eh^3}{12(1 - \nu^2)}
$$

$$m_1 = \frac{1}{A^*};\ m_2 = k_1 w^*$$

$$m_3 = \left[k_1^* u^* - \frac{1}{A^*}\frac{dw^*}{dx}\right]$$

$$m_4 = -m_1 m_3;\ m_5 = k_1^* m_3;\ m_6 = m_2 + k_1 u^* m_3$$

$$m_7 = \frac{1}{A^* B^*}\frac{dB^*}{dx}$$

$$m_8 = k_2 w^*$$

$$m_9 = -m_1 \frac{dm_1}{dx}$$

$$m_{10} = m_1 k_1^*$$

$$m_{11} = m_1 \frac{dk_1^*}{dx} \tag{3.10}$$

$$m_{12} = m_1\left[\frac{dk_1}{dx}u^* + k_1 \frac{du^*}{dx}\right]$$

$$m_{13} = -m_1 m_7;\ m_{14} = m_7 k_1^*;\ m_{15} = m_7 k_1 u^*$$

$$m_{16} = Cm_1;\ m_{17} = Cm_4$$

$$m_{18} = C(m_5 + vm_7)$$

$$m_{19} = C(k_1^* + vk_2^*)$$

$$m_{20} = C(m_6 + vm_8)$$

$$m_{21} = Cvm_1;\ m_{22} = Cvm_4;\ m_{23} = C(m_7 + vm_5)$$

$$m_{24} = C(k_2^* + vk_1^*);\ m_{25} = C(m_8 + vm_6)$$

$$m_{26} = -Dm_1^*;\ m_{27} = D(m_9 + vm_{13})$$

$$m_{28} = Dm_{10};\ m_{29} = D(m_{11} + vm_{14});\ m_{30} = D(m_{12} + vm_{15})$$

$$m_{31} = -Dvm_1^2;\ m_{32} = D(m_{13} + vm_9)$$

$$m_{33} = Dvm_{10};\ m_{34} = D(m_{14} + vm_{11});\ m_{35} = D(m_{15} + vm_{12})$$

$$m_{36} = \left[\frac{d}{dx}\left(B^* M_1^*\right) - \frac{dB^*}{dx}M_2^*\right];\ m_{37} = A^* B^*$$

$$
\begin{aligned}
H_{11} &= B^{*}(m_{16} + k_1^{*} m_{28}) \\
H_{12} &= \frac{dB^{*}}{dx}\left[k_1^{*}\left(-m_{33} + m_{28}\right) + m_{16} - m_{21}\right] + B^{*}\left[k_1^{*}\left(m_{29} + \frac{dm_{28}}{dx}\right) + m_{18} + \frac{dm_{16}}{dx}\right] \\
H_{13} &= \frac{dB^{*}}{dx}\left[k_1^{*}\left(-m_{34} + m_{29}\right) + m_{18} - m_{23}\right] + B^{*}\left[k_1^{*}\frac{dm_{29}}{dx} + \frac{dm_{18}}{dx}\right] \\
H_{14} &= k_1^{*} B^{*} m_{26} \\
H_{15} &= \frac{dB^{*}}{dx}\left[k_1^{*}\left(-m_{31} + m_{26}\right)\right] + B^{*}\left[m_{17} + k_1^{*}\left(\frac{dm_{26}}{dx} + m_{27}\right)\right] \\
H_{16} &= \frac{dB^{*}}{dx}\left[k_1^{*}\left(-m_{32} + m_{27}\right) + m_{17} - m_{22}\right] + B^{*}\left[k_1^{*}\frac{dm_{27}}{dx} + m_{19} + \frac{dm_{17}}{dx}\right] \\
H_{17} &= \frac{dB^{*}}{dx}\left(m_{19} - m_{24}\right) + B^{*}\frac{dm_{19}}{dx} \\
H_{18} &= \frac{dB^{*}}{dx}\left[k_1^{*}\left(-m_{35} + m_{30}\right) + m_{20} - m_{25}\right] + B^{*}\left[k_1^{*}\frac{dm_{30}}{dx} + \frac{dm_{20}}{dx}\right]
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
H_{21} &= B^{*} m_1 m_{28} \\
H_{22} &= \frac{dB^{*}}{dx} m_1\left(2m_{28} - m_{33}\right) + B^{*}\left[m_1\left(m_{29} + 2\frac{dm_{28}}{dx}\right) + \frac{dm1}{dx} m_{28}\right] \\
H_{23} &= \frac{dB^{*}}{dx}\left[m_1\left(2m_{29} + 2\frac{dm_{28}}{dx} - m_{34} - \frac{dm_{33}}{dx}\right) + \frac{dm_1}{dx}\left(m_{28} - m_{33}\right)\right] \\
&\quad + B^{*}\left[m_1\left(\frac{d^2 m_{28}}{dx^2} + 2\frac{dm_{29}}{dx}\right) + \frac{dm_1}{dx}\left(\frac{dm_{28}}{dx} + m_{29}\right)\right] \\
&\quad + m_1\frac{d^2 B^{*}}{dx^2}(m_{28} - m_{33}) + m_{37}(m_1 p^{*} + k_1^{*} m_{16} + k_2^{*} m_{21} \\
H_{24} &= \frac{dB^{*}}{dx}\left[\frac{dm_1}{dx}\left(m_{29} - m_{34}\right) + m_1\left(2\frac{dm_{29}}{dx} - \frac{dm_{34}}{dx}\right)\right] \\
&\quad + B^{*}\left[m_1\frac{d^2 m_{29}}{dx^2} + \frac{dm_1}{dx}\frac{dm_{29}}{dx}\right] + m_1\frac{d^2 B^{*}}{dx^2}(m_{29} - m_{34}) \\
&\quad + m_{37}\left[p^{*}\left(m_5 + m_7\right) + k_1^{*} m_{18} + k_2^{*} m_{23}\right]
\end{aligned}
$$

$$H_{25} = B^{*} m_1 m_{26}$$

$$H_{26} = \frac{dB^{*}}{dx} m_1 \left(2m_{26} - m_{31}\right) + B^{*}\left[m_1\left(2m_{26} + m_{27}\right) + \frac{dm1}{dx} m_{26}\right]$$

$$H_{27} = \frac{dB^{*}}{dx}\left[m_1\left(2m_{27} + 2\frac{dm_{26}}{dx} - m_{32} - \frac{dm_{31}}{dx}\right) + \frac{dm_1}{dx}\left(m_{26} - m_{31}\right)\right]$$

$$+ B^{*}\left[m_1\left(\frac{d^2 m_{26}}{dx^2} + 2\frac{dm_{27}}{dx}\right) + \frac{dm_1}{dx}\left(\frac{dm_{26}}{dx} + m_{27}\right)\right] + m_1 \frac{d^2 B^{*}}{dx^2}(m_{26} - m_{31})$$

$$H_{28} = \frac{dB^{*}}{dx}\left[\frac{dm_1}{dx}\left(m_{27} - m_{32}\right) + m_1\left(2\frac{dm_{27}}{dx} - \frac{dm_{32}}{dx}\right)\right] + B^{*}\left[m_1 \frac{d^2 m_{27}}{dx^2} + \frac{dm_1}{dx}\frac{dm_{27}}{dx}\right]$$

$$+ m_1 \frac{d^2 B^{*}}{dx^2}(m_{27} - m_{32}) + m_{37}\left[p^{*} m_4 + k_1^{*} m_{17} + k_2^{*} m_{22}\right]$$

$$H_{29} = m_{37}\left[k_1^{*}\left(-p^{*} + m_{19}\right) + k_2^{*}\left(-p^{*} + m_{24}\right)\right]$$

$$H_{30} = \frac{dB^{*}}{dx}\left[m_1\left(2\frac{dm_{30}}{dx} - \frac{dm_{35}}{dx}\right) + \frac{dm_1}{dx}\left(m_{30} - m_{35}\right)\right] + B^{*}\left[m_1 \frac{d^2 m_{30}}{dx^2} + \frac{dm_1}{dx}\frac{dm_{30}}{dx}\right]$$

$$+ m_1 \frac{d^2 B^{*}}{dx^2}(m_{30} - m_{35}) + m_{37}\left(p^{*}\left(m_8 + m_6\right) + k_1^{*} m_{20} + k_2^{*} m_{25} + k_1 N_1^{*} + k_2 N_2^{*} + p\right)$$

The A^r_{ij} are known weighting coefficients, and the u_i, w_i are the unknown incremental displacement components at the sampling points. Both of these two analog equations are to be enforced at each of the sampling points. Fig. 3.2a gives a sample mesh.

Use of the Newton method (Bert and Malik, 1996a) leads to linear equations at each load step which may be solved for the incremental displacements. These equations have the form

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{Bmatrix} (\Delta_1) \\ (\Delta_2) \end{Bmatrix} = \begin{Bmatrix} (b_1) \\ (b_2) \end{Bmatrix} \tag{3.12}$$

where the vectors (Δ_1), (Δ_2) contain the unknown incremental displacements u_i, w_i and the vectors (b_1), (b_2) contain the equivalent loading components.

The theory presented in this section was coded in a MATLAB program labelled nlaxx.m. The

DQM results presented below are based on this code.

3.5 FEM for toroidal shell

The commercial FEM program ADINA (1999) was used to provide a second solution for this shell problem. A number of advanced elements and techniques are available in this program for the solution of shell stress analysis and stability problems.

Account was made of the symmetry of the geometry and loading, and it was assumed that the buckling mode would also be symmetric. Thus it was necessary to model only the meridian of the shell and an axisymmetric element could be used. The element selected was a three-noded one with quadratic interpolation. Fig. 3.2b shows a sample mesh.

The basic algorithms of the Newton-Raphson iterations with and without line searches are:

$$\begin{aligned} {}^{t+\Delta t}K^{(i-1)}\Delta U^{(i)} &= {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)} \\ {}^{t+\Delta t}U^{(i)} &= {}^{t+\Delta t}U^{(i-1)} + \Delta U^{(i)} \end{aligned} \qquad \text{(Without line search)} \qquad (3.13)$$

$$\begin{aligned} {}^{t+\Delta t}K^{(i-1)}\Delta U^{(i)} &= {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)} \\ {}^{t+\Delta t}U^{(i)} &= {}^{t+\Delta t}U^{(i-1)} + \beta^{(i)}\Delta U^{(i)} \end{aligned} \qquad \text{(With line search)} \qquad (3.14)$$

where ${}^{t+\Delta t}K^{(i-1)}$ is the tangent stiffness matrix based on the solution calculated at the end of iteration $(i-1)$ at time $t+\Delta t$, R is the externally applied load vector, F is the consistent nodal force vector corresponding to the element stresses due to the displacement vector U.

The automatic step incrementation by the Automatic Time Stepping (ATS) method was employed to get the exact point of instability. The buckling load was determined using ATS as well as the corresponding mode shape.

3.6 Validation and results

Sample numerical results are presented in this section from the DQM solution and the ADINA FEM program. Eight cases of shells involving a wide range of geometric and material parameters were considered. A description of these cases is given in Table 3.1.

Cases 1-4 have relatively small R/r ratios, while cases 5-8 have moderate R/r ratios. Shells with different r/h ratios are represented for several of the R/r choices, thus providing an indication of the effect of thickness on the analyses. Tubes of large toroidal radii are not represented in these cases. In the last column of the table are given the previous experimental or analytical results taken from the literature.

Results obtained for the nonlinear response of toroidal shells under external pressure by the DQM are given in Table 3.2. Results are given for three meshes for the eight cases. The values given indicate the apparent buckling pressures, as indicated by load deflection (p-w) curves. These values can at best be found to two figure accuracy.

The DQM results indicate that the method leads to rapid convergence as the number of sampling points is increased. There is good agreement between the DQM predictions and the previous results. There are no large differences for any cases, either for small or for moderate R/r radius ratios, or for different thickness ratios.

Results obtained for the buckling pressures of toroidal shells by the ADINA FEM code are given in Table 3.3. Results are given for three meshes for each of the eight cases. These values were obtained automatically from the ADINA analysis using the automatic time stepping option.

The ADINA results indicate that the FEM converges rapidly as the number of nodes is increased. There is good agreement between the FEM and previous results, with no noticeable change with the R/r or r/h ratios.

Comparison of the last columns of Tables 3.2 and 3.3 in Table 3.4 indicates that there is very good agreement of the converged DQM and FEM results with the previously published values.

Nonlinear response curves obtained for the eight cases of Table 1 from the current DQM analysis are shown in Figs. 3.3a-3.10a. Corresponding buckling modes obtained using the FEM analysis are shown in Figs. 3.3b-3.10b.

For the p-w curves the displacement history of three points of the shell cross-section are shown. These points are located respectively at the extrados ($\theta=0°$), at the crown ($\theta=90°$), and at the intrados ($\theta=180°$). A sufficient number of loading points are included in each graph to include the region of nonlinearity around the buckling load. The deflection shown is in the radial direction, and it is observed that generally at the crown the deflection is inward and of relatively large magnitude. At the other two points the deflection is generally smaller.

The p-w curves are generally smooth. For all cases there is a sharp transition from the initial hard response to the subsequent soft response. In general for each of the cases the three curves indicate the same value for the buckling load.

The buckling modes of all eight cases are symmetrical about the equatorial plane of the torus. There is generally a large inward deflection at the crown. At the extrados and at the intrados the deflection is generally moderate. This agrees with the DQM deflection pattern near the buckling load. As well the FEM buckling modes agree well with ones previously published by Sobel and Flügge (1967) and Galletly and Blachut (1995).

3.7 Conclusion

Two solutions to the problem of the nonlinear response of symmetric toroidal shells subjected to external pressure have been presented. The first solution is obtained using the new differential quadrature method, and the second using the ADINA FEM code. Results obtained from the two methods show excellent agreement with previously published values. The study confirms the usefulness of the DQM for nonlinear analysis of shells.

Chapter 4

Nonlinear Analysis of Storage Tank

4.1 Introduction

The study of liquid-storage tanks continuous to be a very active area of engineering research (Rammerstorfer and Sharf (1990), Mirfakhraei et al (1995), Cho et al (1999)). Among the geometries of interest are cylindrical tanks resting on elastic foundations. These tanks are used for the storage of a variety of fluids such as water, wine, and oil. Due to their thinness they are susceptible to large structural deformations or buckling under extreme loading cases.

The literature on the deformation of tanks on elastic foundations is relatively sparse. Recent work (Wunderlich et al (1994), Kukreti and Siddiqi (1997), Malik et al (1993) and Lau and Zeng (1995)) has been carried out using the finite element, differential quadrature and analytical methods, and has dealt with interactions between fluid, structure, and foundation. Nonlinearity in response has been considered, but mostly in conjunction with a single element of the tank. To date no analytical study has appeared providing an analysis which considers possible non-linear large deformation response of both shell and base plate.

In this paper the differential quadrature method (DQM) is used to provide a theoretical solution to the nonlinear response problem of an axisymmetric liquid storage tank on an elastic foundation. The nonlinear theory developed considers moderately large deflections in the shell and plate and serves as a precursor for a full three-dimensional analysis. The accuracy of the solution is verified in the linear range by comparing results with ones obtained in previous studies, and in the nonlinear range by comparing results with ones found using the FEM. A detailed study is conducted of the convergence properties of the method. Numerical results are determined for several cases. The chapter ends with comments about the technical and mathematical significance of the solution.

4.2 Geometry and continuity conditions

The liquid storage tank consists of a thin circular cylindrical shell with length H, radius R. The thicknesses at the bottom and top respectively are denoted T_b , T_t, while the thickness at height x above the base by T . A typical point P on the shell mid-surface is defined by the geodetic coordinates x, y. The shell is attached integrally at its base to a thin circular plate of radius R and thickness T_p. A typical point Q on the plate mid-surface is defined by the cylindrical coordinates r, θ. The geometry of the problem is shown in Fig. 4.1.

As an axisymmetric analysis is to be undertaken it is sufficient mathematically to use the axial coordinate x for the shell, and the radial coordinate r for the plate. Displacements U_s, W_s and W_p, U_p are in the axial and radial directions for the shell and plate respectively. A radial pressure p is assumed to act on the shell and plate mid-surface. In the shell the pressure is given by $p = \gamma H(1 - x)$, while in the plate it is $p = \gamma H$. Conventions for the displacements, loading, and stress resultants are shown in Fig. 4.2.

The governing equation for bending of a thin axisymmetric shell or plate is a fourth order differential equation, and therefore two bending boundary conditions should be applied at each end. Implementation of the boundary conditions is not straightforward as a single DQM sampling only is available at each end. Jang et al. (1989) proposed using the so-called δ-technique wherein an additional point is chosen at a small distance $\delta = 10^{-5}$ from the actual boundary point. The δ-technique offers an adequate way for applying the double boundary conditions of shell and plate bending problems and has been applied quite successfully, for example, to problems involving simply supported and clamped boundary conditions.

As the shell is assumed free at the top, the boundary conditions

$$N_x = 0;\ Q_x = 0;\ M_x = 0; \tag{4.1}$$

must be satisfied at x = H. An integral connection is assumed at the junction of the shell with the base plate and thus the continuity conditions

$$\begin{aligned} &U_s = W_p;\ W_s = U_p;\ \frac{d}{dx}W_s = \frac{d}{dr}W_p \\ &N_x = Q_r;\ Q_x = N_r;\ M_x = M_r \end{aligned} \tag{4.2}$$

must be satisfied at x = 0, r = R. Finally the regularity conditions

$$Q_r = 0;\ M_r = 0;\ \frac{d}{dr}W_r = 0 \tag{4.3}$$

must be satisfied at the center of the plate, r = 0.

4.3 Modelling of the shell

To determine the geometric nonlinear response of the tank wall the axisymmetric form of the Sanders shell theory (Sanders ,1963; Yamaki, 1984) is employed. The relations for the strains and curvatures in terms of the displacements are taken as

$$\begin{aligned} &\varepsilon_x = \frac{d}{dx}U_s + \frac{1}{2}\left(\frac{d}{dx}W_s\right)^2;\quad \varepsilon_y = \frac{1}{R}W_s;\\ &\kappa_x = -\frac{d^2}{dx^2}W_s;\ \kappa_y = 0; \end{aligned} \tag{4.4}$$

The constitutive equations are given by

$$\begin{aligned} N_x &= J_s(\varepsilon_x + \nu\varepsilon_y);\quad N_y = J_s(\varepsilon_y + \nu\varepsilon_x)\\ M_x &= D_s(\kappa_x + \nu\kappa_y);\quad M_y = D_s(\kappa_y + \nu\kappa_x) \end{aligned} \tag{4.5}$$

where

$$J_s = \frac{E_s T}{1-\nu_s^2};\ D_s = \frac{E_s T^3}{12(1-\nu_s^2)};\ T = T_b + (T_t - T_b)\left(\frac{x}{H}\right) \tag{4.6}$$

and E_s, ν_s are the Young's modulus and Poisson ratio for the shell. The equilibrium equations are given by

$$\frac{d}{dx}N_x = 0$$
$$\frac{d^2}{dx^2}M_x + \frac{1}{R}N_y + N_x\frac{d^2}{dx^2}W_s + p = 0 \tag{4.7}$$

where $R = R_c + \frac{1}{2}T$ with R_c representing the inner radius of the shell. Combining eqns (4.4-4.5) the resultants can be expressed in terms of the displacements. The shear resultant is given in terms of the displacements through $Q_x = \frac{d}{dx}M_x$ and eqns (4.4-4.5). Inserting the resultants into the equilibrium equations gives

$$L_1 = \frac{d^2}{dx^2}U_s + m_1\frac{d}{dx}W_s\frac{d^2}{dx^2}W_s + m_2\frac{d}{dx}W_s = 0$$
$$L_2 = \frac{d^2}{dx^2}\left(D\frac{d^2}{dx^2}W_s\right) + m_3\frac{d^2}{dx^2}W_s\left(\frac{d}{dx}W_s\right)^2 + m_4\frac{d^2}{dx^2}W_s + m_5\left(\frac{d}{dx}W_s\right)^2 \tag{4.8}$$
$$+ m_6W_s + m_7\frac{d^2}{dx^2}W_s\frac{d}{dx}U_s + m_8\frac{d}{dx}U_s + m_9 = 0$$

where

$$\alpha = \frac{\gamma HR^2}{E_sT};\ m_1 = \frac{\alpha}{H};\ m_2 = -\frac{v_sH}{R};\ m_3 = -\frac{J_s\alpha^2}{2};\ m_4 = \frac{J_sv_sH^2\alpha}{R}$$
$$m_5 = -\frac{J_sv_sH^2\alpha}{2R};\ m_6 = \frac{J_sH^4}{R^2};\ m_7 = -J_sH\alpha;\ m_8 = -\frac{J_sv_sH^3}{R} \tag{4.9}$$
$$m_9 = -\frac{\gamma H^5(1-x)}{\alpha}$$

These are the two nonlinear equations which govern the displacement state for the shell.

The Newton method (Bert and Malik, 1996a) is applied to the nonlinear equations (4.8).

Beginning with an assumed displacement field $U(\xi)$ consistent with the boundary and continuity conditions one obtains successively refined solutions through an iteration scheme $U^{n+1} = U^n + dU^n$, where $dU = dU(\xi)$ is the displacement refinement and n is the iteration tally. The displacement refinement is found by solving the linear equation $dUL_i'(U) + L_i(U) = 0$, where the prime indicates the Frechet derivative of the operator. Thus the following linear relations are obtained for the displacement increments dU_s and dW_s

$$
\begin{aligned}
&\frac{d^2}{dx^2}dU_s + m_1\frac{d}{dx}W_s\frac{d^2}{dx^2}dW_s + h_{11}\frac{d}{dx}dW_s = -L_1 \\
&D\frac{d^4}{dx^4}dW_s + 2\frac{d}{dx}D\frac{d^3}{dx^3}dW_s + h_{21}\frac{d^2}{dx^2}dW_s \\
&+ h_{22}\frac{d}{dx}dW_s + h_{23}dW_s + h_{24}\frac{d}{dx}dU_s = -L_2
\end{aligned}
\tag{4.10}
$$

where

$$
\begin{aligned}
h_{11} &= m_1\frac{d^2}{dx^2}W_s + m_2 \\
h_{21} &= \frac{d^2}{dx^2}D + m_3\left(\frac{d}{dx}Ws\right)^2 + m_4Ws + m_7\frac{d}{dx}Us \\
h_{22} &= 2m_3\frac{d^2}{dx^2}W_s\frac{d}{dx}Ws + 2m_5\frac{d}{dx}Ws \\
h_{23} &= m_4\frac{d^2}{dx^2}W_s + m_6 \\
h_{24} &= m_7\frac{d^2}{dx^2}W_s + m_8
\end{aligned}
\tag{4.11}
$$

In Eqns (4.11) the expressions L_i on the RHS and the h_i are to be evaluated at the previous known displacement state. The Eqns (4.11) must be solved subject to the boundary and continuity conditions (4.1-4.2).

4.4 Modelling of circular plate

To determine the nonlinear geometric response of the plate the theory of Timoshenko and Woinowsky-Krieger (Malhotra and Veletsos (1994), Timoshenk and Woinowsky-Krieger (1984)) is employed. The relations for the strains and curvatures in terms of the displacements are taken as

$$\begin{aligned} \varepsilon_r &= \frac{d}{dr}U_p + \frac{1}{2}\left(\frac{d}{dr}W_p\right)^2; \quad \varepsilon_\theta = \frac{W_p}{r} \\ \kappa_r &= \frac{d^2}{dr^2}W_p; \quad \kappa_\theta = \frac{1}{r}\frac{d}{dr}W_p \end{aligned} \tag{4.12}$$

The constitutive equations are given by

$$\begin{aligned} N_r &= J_p(\varepsilon_r + \nu\varepsilon_\theta); \quad N_\theta = J_p(\varepsilon_\theta + \nu\varepsilon_r) \\ M_r &= D_p(\kappa_r + \nu\kappa_\theta); \quad M_\theta = D_p(\kappa_\theta + \nu\kappa_r) \end{aligned} \tag{4.13}$$

where

$$J_p = \frac{E_p T_p}{1-\nu_p^2}, \quad D_p = \frac{E_p T_p^3}{12(1-\nu_p^2)} \tag{4.14}$$

and E_p, ν_p are the Young's modulus and Poisson ratio for the plate. The equilibrium equations are given by

$$\begin{aligned} & N_r + r\frac{d}{dr}N_r - N_\theta = 0 \\ & \frac{d^2}{dr^2}M_r + \frac{2}{r}\frac{d}{dr}M_r - \frac{1}{r}\frac{d}{dr}M_\theta - \frac{1}{r}\frac{d}{dr}\left(rN_r\frac{d}{dr}W_p\right) = q - p \end{aligned} \tag{4.15}$$

where p is now the pressure on the plate ($p = \gamma H$) and q is the soil reaction arising from the elastic foundation. This reaction may be expressed (Kukreti and Siddiqi (1997), Malik et al (1993)) in terms of the plate's normal displacement W_p through

$$q(r) = \frac{1}{r}\frac{d}{dr}\int_r^1 \frac{t}{\sqrt{t^2 - r^2}}\left[\frac{d}{dt}\int_0^t \frac{rW_p}{\sqrt{t^2 - r^2}}dr\right]dt \tag{4.16}$$

Combining eqns (4.12-4.13) the resultants can be expressed in terms of the displacements. The shear resultant is given in terms of the displacements through

$$Q_r = -D_p\frac{d}{dr}\left[\frac{d}{dr}\left(r\frac{d}{dr}W_p\right)\right] + N_r\frac{d}{dr}W_p$$

Inserting the resultants into the equilibrium equations gives

$$\begin{aligned}
L_3 &= \frac{d}{dr}U_p + r\frac{d^2}{dr^2}U_p - \frac{1}{r}U_p + r\frac{d}{dr}W_p\frac{d^2}{dr^2}W_p \\
&\quad + \frac{1}{2}(1-\nu)\left(\frac{d}{dr}W_p\right)^2 = 0 \\
L_4 &= D_p\left[\frac{d^4}{dr^4}W_p + \frac{2}{r}\frac{d^3}{dr^3}W_p - \frac{1}{r^2}\frac{d^2}{dr^2}W_p + \frac{d}{dr}W_p\right] \\
&\quad - J_p\left[(1+\nu)\frac{1}{r}\frac{d}{dr}U_p\frac{d}{dr}W_p + \frac{\nu}{r}U_p\frac{d^2}{dr^2}W_p + \frac{1}{2r}\left(\frac{d}{dr}W_p\right)^2\right. \\
&\quad \left. + \frac{d^2}{dr^2}U_p\frac{d}{dr}W_p + \frac{3}{2}\frac{d^2}{dr^2}W_p\left(\frac{d}{dr}W_p\right)^2 + \frac{d}{dr}U_p\frac{d^2}{dr^2}W_p\right] \\
&\quad - q + p = 0
\end{aligned} \tag{4.17}$$

These are the two nonlinear equations which govern the displacement state for the plate.

Applying the Newton method (Bert and Malik (1996)) now to the equations (4.17), introducing the position variable t = r/R, normalizing and non-dimensionalizing, gives the following linear relations for the displacement increments dU_p and dW_p

$$
\begin{aligned}
& t^2 \frac{d^2}{dt^2} dU_p + t \frac{d}{dt} dU_p - dU_p + m_{10} \frac{d}{dt} W_p \frac{d^2}{dt^2} dW_p + h_{31} \frac{d}{dt} dW_p = -L_3 \\
& \sqrt{1-t^2}\left[t^3 \frac{d^4}{dt^4} dW_p + 2t^2 \frac{d^3}{dt^3} dW_p + h_{41} \frac{d^2}{dt^2} dW_p + h_{42} \frac{d}{dt} dW_p \right] \\
& + \sqrt{1-t^2}\left[h_{43} \frac{d^2}{dt^2} dU_p + h_{44} \frac{d}{dt} dU_p + h_{45} dU_p \right] - a_0 t^2 \frac{d^n}{dt^n} s(t) = -L_4
\end{aligned}
\tag{4.18}
$$

where

$$
a_0 = \frac{12(1-v_p^2)}{\pi K}; b_0 = \frac{12(1-v_p^2)}{K}; c_0 = \frac{-12}{T_p^2}\left(pR\phi_s\right)^2; d_0 = \frac{-12}{T_p^2}\left(pR^2\phi_s\right)
$$

$$
\begin{aligned}
& m_{10} = t^2 p\phi_s; \quad m_{11} = \frac{1}{2} t(1-v_p)p\phi \\
& h_{31} = m_{10} \frac{d^2}{dt^2} W_p + 2m_{11} \frac{d}{dt} W_p \\
& h_{41} = -t + \frac{3}{2} c_0 t^3 \left(\frac{d}{dt} W_p \right)^2 + d_0 t^3 \frac{d}{dt} U_p + d_0 v t^2 U_p \\
& h_{42} = 1 + \frac{3}{2} c_0 t^2 \left(\frac{d}{dt} W_p \right)^2 + 3c_0 t^3 \frac{d^2}{dt^2} W_p \frac{d}{dt} W_p + d_0 t^2 \frac{d}{dt} U_p \\
& \qquad + d_0 t^3 \frac{d^2}{dt^2} U_p + d_0 t^2 v \frac{d}{dt} U_p \\
& h_{43} = d_0 t^3 \frac{d}{dt} W_p; \; h_{44} = d_0 t^2 \frac{d}{dt} W_p + d_0 t^3 \frac{d^2}{dt^2} W_p + d_0 t^2 v \frac{d}{dt} W_p \\
& h_{45} = d_0 v t^2 \frac{d^2}{dt^2} W_p \\
& s(t) = t\sqrt{1-t^2}\, q(t)
\end{aligned}
\tag{4.19}
$$

Here $K = (1-v_f^2)\left(\frac{E_p}{E_f}\right)\left(\frac{T_p}{R}\right)^2$ is the relative rigidity factor, and E_f , v_f are the Young's modulus and Poisson ratio for the soil. The term $\frac{d^n}{dt^n}s(t)$ represents the n^{th} derivative of the function s(t) (Kukreti and Siddiqi (1997)). In eqn (4.18) the expressions L_i on the RHS are to be evaluated at the previous known displacement state. The eqn (4.18) must be solved subject to the boundary and compatibility conditions (4.2-4.3).

4.5 DQM for tank

As stated in chapter 3 the basis of the DQM is the representation in the domain of the derivatives of a function by a weighted sum of trial function values. Thus for a function $f(\xi)$ the derivatives are taken as

$$\left.\frac{d^r f(\xi)}{d\xi^r}\right|_{\xi} = \sum_{k=1}^{N} A_{ik}^{(r)} f(\xi_k) \tag{4.20}$$

Here the $A_{ik}^{(r)}$ are the weighting coefficients of the r-th order derivative at the i-th sampling point in the x direction, and N is the number of sampling points in the x direction.

Polynomial trial functions are selected in this study, i.e. $f(\xi) = 1, \xi, \xi^2, \ldots, \xi^{N-1}$ for the shell and plate. For these functions explicit formulas for the weighting coefficients are available (Bert et al., 1996). For the first order derivative the formulas are

$$\begin{aligned} A_{ij}^{(1)} &= \frac{\pi(\xi_i)}{(\xi_i - \xi_j)\pi(\xi_j)};\ i, j = 1, 2, \ldots, N;\ i \neq j \\ \pi(\xi_j) &= \prod_{j=1}^{N} (\xi_i - \xi_j);\ i \neq j \end{aligned} \tag{4.21}$$

while for the higher order derivatives the formulas are

$$A_{ij}^{(r)} = r\left[A_{ii}^{(r-1)} A_{ij}^{(1)} - \frac{A_{ij}^{(r-1)}}{\xi_i - \xi_j} \right];\ i,j = 1,2,\ldots,N;\ i \neq j;\ 2 \leq r \leq (N-1)$$

$$A_{ij}^{(r)} = A_{ii}^{(r)} = -\sum_{k=1}^{N} A_{ik}^{(r)};\ i,j = 1,2,\ldots,N;\ i \neq j;\ 1 \leq r \leq (N-1)$$

For the sampling points in the shell and plate the Chebyshev-Gauss-Lobatto spacing (Bert and Malik, 1996; Shin and Redekop, 2000) is used. In this scheme the coordinates for the shell are taken as $\xi_i = \zeta_i (H/R)$ where the ζ_i are given by

$$\zeta_1 = 0;\ \zeta_2 = \delta;\ \zeta_{N-1} = 1 - \delta;\ \zeta_N = 1$$
$$\zeta_i = \frac{1 - \cos\left(\dfrac{\pi(i-2)}{N-3}\right)}{2};\ 2 < i < (N-1) \tag{4.22}$$

Similarly sampling points are defined in the plate. At each sampling point either the DQM analogue of a governing equation for the domain or a condition for the boundary support is represented. For each boundary point there are three conditions, while for each domain point there are only two governing equations. It is necessary to enforce one of the boundary conditions at an interior point. This point, a 'δ point', is taken a short distance ($\delta = 10^{-4}$ or $\delta = 10^{-5}$ on a unit domain) from the boundary point.

Use in the governing equations (4.10,4.19) of the quadrature rules for the derivatives (4.20-4.22) leads to the transformed DQM domain equations which have the form

$$\sum A_{ij}^{(2)} dU_{s_j} + (m_1 \frac{d}{dx} W_p \sum A_{ij}^{(2)} + h_{11} \sum A_{ij}^{(1)}) dW_{s_j} = -L_{1i}$$

$$h_{24} \sum A_{ij}^{(1)} dU_{s_j} + \left[D_s \sum A_{ij}^{(4)} + 2\frac{d}{dx} D_s \sum A_{ij}^{(3)} + h_{21} \sum A_{ij}^{(2)} + h_{22} \sum A_{ij}^{(1)} \right] dW_{s_j}$$

$$+ h_{23} dW_{s_i} = -L_{2i} \tag{4.23}$$

$$\left[t^2 \sum A_{ij}^{(2)} + t \sum A_{ij}^{(1)} \right] dU_{p_j} - dU_{p_i} + \left[t m_{10} W_p \sum A_{ij}^{(2)} + h_{31} \sum A_{ij}^{(1)} \right] dW_{p_j} = -L_{3i}$$

$$\sqrt{1-t^2} \left[t^3 \sum A_{ij}^{(4)} + 2t^2 \sum A_{ij}^{(3)} + h_{41} \sum A_{ij}^{(2)} + h_{42} \sum A_{ij}^{(1)} \right] dW_{p_j} - a_0 t^2 A_{sij}$$

$$+ \sqrt{1-t^2} \left[h_{43} \sum A_{ij}^{(2)} + h_{44} \sum A_{ij}^{(1)} \right] dU_{p_j} + \sqrt{1-t^2}\, h_{45} dU_{p_i} = -L_{4i}$$

Here the dU_p, dW_p, dU_s, dW_s are unknown displacement increments at the sampling points. The expressions L_i are to be evaluated at the previous known displacement state. The expressions involving the boundary conditions on the edges and the compatibility conditions may similarly be transformed.

The assembly of the domain and boundary equations yields a matrix equation of the form

$$[K]\{\Delta\} = \{Q\} \tag{4.24}$$

where [K] is the tangent stiffness matrix, $\{\Delta\}$ is a vector containing the displacement increments at the sampling points, while vector {Q} contains known constants relating to the previous time step. The form of the matrix [K] is shown in Fig. 4.3. Equation (4.24) forms the core in the Newton method for the solution of the nonlinear response problem of the tank. An incremental approach is used until the final load level is reached.

The theory for the nonlinear analysis presented in the preceding was programmed in a MATLAB code labelled tanknl.m. A linear version of the procedure was programmed also, in a code labelled tankl.m. Results from these programs are given in the following.

4.6 FEM for tank

The commercial FEM program ADINA (1998) was used to provide a comprehensive verification of the DQM theoretical results. An axisymmetric six degree-of-freedom shell element is available in this program for the solution of nonlinear shell response problems.

The shell and plate were modeled in each case, as well as the elastic foundation (Fig 4.4 gives a sample mesh). Results were found for several mesh sizes to indicate the convergence of the solution. For the final analysis a mesh containing 11394 degrees of freedom was employed for the whole model.

4.7 Validation

Numerical results are presented in this section for three storage tanks. The tanks are respectively of concrete, steel, and aluminum as described in Table 4.1. Dimensions in this table are given in metres while moduli in MPa. The size and thickness of the tanks differ, but the soil material is the same in each case.

Results for the concrete tank are given in Fig 4.5. Normal displacements and moments obtained by the linear and nonlinear DQM program are shown for the shell and plate. In these and subsequent figures the origin of the coordinate system for the shell is at the bottom, while the origin for the plate is at the center. As a thick concrete tank is involved the two sets of results virtually coincide. The current results also virtually coincide with the previous results of Kukreti and Siddiqi (1997) obtained using a linear theory.

Results for a thin steel tank are given in Table 4.2 and Fig 4.6. The results in Table 4.2 indicate that there is rapid convergence both in the DQM and FEM solutions. There is close agreement in maximum displacement values, with a difference less than 2%. The curves in Fig 4.6 for normal displacements in the shell indicate close agreement in results from the two methods. The curves for the normal displacement in the plate indicate a small difference in centre-to-edge relative displacement from the two methods. For moments only DQM results are shown, and these indicate small differences between the linear and non-linear analyses.

Results for a thin aluminum tank are given in Fig 4.7. The curves for normal displacements again indicate close agreement between DQM and FEM results. For moments the DQM results given indicate small differences between the linear and nonlinear programs. The results for these three tanks indicate the validity of the current theory and algorithm.

4.8 Results

Calculations were carried out using the linear and nonlinear DQM programs to determine the influence of the H/R ratio and shell-plate thickness on the maximum normal displacement in the shell. Fig 4.8 gives the results for H/R ratios of 0.5, 1.0 and 2.0. The horizontal ordinate gives the thickness of shell and plate. No results were available from the nonlinear program for a thickness less than 1 mm for the H/R = 0.5 case due to numerical instability. Overall there are no significant differences in the DQM linear and nonlinear results. All three sets of results indicate a strongly nonlinear relation between maximum displacement and shell-plate thickness.

Figs 4.9 and 4.10 give additional results for aluminum tanks, now covering shell-plate thicknesses of 4mm and 3mm. Results given for displacements and moments for the 4mm thickness by the linear and nonlinear DQM show close agreement. Results for the 3mm thickness show considerable differences in displacement in the shell, and in the moments at the outer edge of the plate. Recognizing that good results were obtained for the 4mm and 5mm tanks it is concluded that the DQM is not accurate for very thin tanks.

Results for maximum displacements in a steel tank having a uniformly varying shell thickness are given in Fig 4.11. Curves are given for T_t / T_b ratio of 0.1, 0.25, 0.5, 0.75, and 1.0. It is seen that the tank with constant thickness has the smallest displacement, while the tank with the greatest variation in thickness the largest displacement. The maximum displacement values going from the 0.1 to 1.0 thickness ratio is of the order of 100%.

Nonlinearity of the deflection of shell was checked by the DQM and FEM. Results for a thin steel tank are given in Fig 4.12. The results indicate the maximum normal displacements in shell at each load step. The results from the DQM and FEM show close agreement and indicate that there is little nonlinearity. The reason for this can be explained by the coefficients in Eqns (4.8-4.9). The coefficients of nonlinear terms are so much smaller than that of linear terms that the effect of nonlinear terms are almost neglected.

4.9 Conclusions

A solution for the nonlinear response problem of an axisymmetric liquid storage tank has been presented. Results obtained using the nonlinear DQM agree very closely with results obtained using the FEM and the linear DQM. A study of the convergence of the solution has shown that there is considerable flexibility in the choice of sampling point spacing. Overall the study has demonstrated the accuracy and versatility with regard to boundary conditions of the differential quadrature method in the solution of geometric nonlinear shell problems.

Chapter 5

Conclusions

5.1 Buckling of toroidal shells

Based on the current work on buckling of toroidal shells the following conclusions are drawn:

- results for buckling loads and buckling modes from the DQM and FEM agree well with each other and with previously presented experimental and theoretical results.
- the usefulness of the differential quadrature method for nonlinear buckling analysis of shells was demonstrated.
- the buckling modes of all cases are symmetrical about the equatorial plane of the torus.

- corresponding to each critical load there is a unique mode shape.

- there is generally a large inward deflection at the crown in the buckled shell.

5.2 Nonlinear analysis of tank

Based on the work on nonlinear analysis of tanks the following conclusions are drawn:

- results obtained using the DQM agree very closely with results obtained using the FEM.

- a new approach of the DQM for nonlinear analysis of shells is demonstrated.

- normal displacements from linear and nonlinear analysis using the DQM vary only slightly for this tank problem.

- decrease of shell thickness leads to larger normal displacements and after a certain critical thickness the displacements increase very sharply.

- decrease of the ratio of top over bottom thickness of shell leads to a sharply increase of normal displacements.

- bending moments at the interaction point of shell and plate from nonlinear analysis are larger than those from linear analysis.

5.3 Methods of analyses

Two numerical methods are used in this study, the DQM and the FEM. Generally the results obtained from the two methods agree very closely with each other, and with available results presented in the literature.

The DQM approach for the complicated nonlinear differential equations of shells has been demonstrated through this study. It has been shown that it can provide an accurate and efficient means of solutions even in the case of high order variable coefficients as shown in the 2nd part of this study.

The FEM is the secondary method in this study. It is very useful in formulating the models in case of simple geometry such as axisymmetry elements. It takes a little bit more computer time but less time for development. Conducting nonlinear analysis is straight forward using the ADINA in this study, and the results are so accurate that it can be used as a means of validation.

5.4 Suggestions for further research

A new approach of the DQM to the complicated nonlinear problems of shells is presented in this study for the simple geometry and loading conditions using axisymmetry analysis which is a 1-dimensional problem considering 3 degree of freedom at each sampling point or node. The logical next step could be to expand this approach to 3-D problems of complicated geometry. Several loading cases with complicated boundary conditions could be considered.

In both parts in this study only nonlinear static problems are considered. Nonlinear dynamic problems of complicated shells using the DQM could form the subject of further research. A new version of the DQM, namely the quadrature element method could be applied to the more complicated geometry. This would entail dividing the whole domain into multiple sub-domains similar to a FEM mesh.

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Table 3.1: Toroidal shell cases and published buckling pressure

Case	R/r	r/h	E (MPa)	Previous results (MPa)
1	2	50	3447.5	0.044 (Almroth et al, 1969)
2	2	100	210000	0.563 (Wang and Zhang, 1991)
3	3	23.3	3496	0.217 (Nordell and Crawford, 1971)
4	4	100	210000	0.363 (Sobel and Fluegge, 1967)
5	6.32	20	117000	-
6	6.32	70	117000	0.393 (Sobel and Fluegge, 1967)
7	7.94	23.3	2165	0.058 (Galletly and Blachut, 1995)
8	8.04	71.5	210000	0.486 (Sobel and Fluegge, 1967)

Table 3.2: DQM buckling pressure (MPa)

Case	Node numbers		
	n=24	n=48	n=96
1	0.045	0.045	0.045
2	0.560	0.530	0.530
3	0.230	0.220	0.220
4	0.360	0.330	0.330
5	5.000	5.000	5.000
6	0.375	0.350	0.350
7	0.053	0.053	0.050
8	0.480	0.480	0.480

Table 3.3: FEM buckling pressure (MPa)

Case	Mesh size (nodes)		
	m=24	m=48	m=96
1	0.046	0.046	0.046
2	0.277	0.532	0.531
3	0.207	0.206	0.205
4	0.310	0.332	0.332
5	4.563	5.280	5.300
6	0.315	0.310	0.310
7	0.070	0.067	0.066
8	0.462	0.452	0.454

Table 3.4: Comparison of results

Case	Methods (MPa)		
	Previous results	FEM	DQM
1	0.044	0.046	0.045
2	0.563	0.531	0.530
3	0.217	0.205	0.220
4	0.363	0.332	0.330
5	-	5.300	5.000
6	0.393	0.310	0.350
7	0.058	0.067	0.050
8	0.486	0.454	0.480

Table 4.1: Description of tank cases

Case	Material	H (m)	R (m)	$T_t = T_b$ (m)	T_p (m)	$E_s = E_p$ (kPa)	$v_s = v_p$	E_f (kPa)	v_f
1	concrete	7.5	9.0	.36	.36	1.4e7	.0	20e3	.4
2	steel	3.0	1.5	.005	.005	200e6	.3	20e3	.4
3	aluminum	1.52	1.8	.005	.005	70e6	.3	20e3	.4

Table 4.2: Convergence of displacements for steel tank

	Number of nodes in shell - W_s max in shell				
FEM	16	32	40	64	80
	1.02E-04	9.66E-05	8.71E-05	8.44E-05	8.31E-05
DQM	19	20	21	23	25
	8.04E-05	8.16E-05	8.11E-05	8.22E-05	8.21E-05

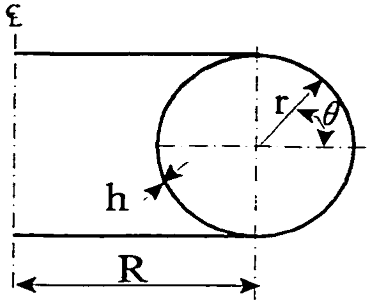

Figure 3-1a: Geometry

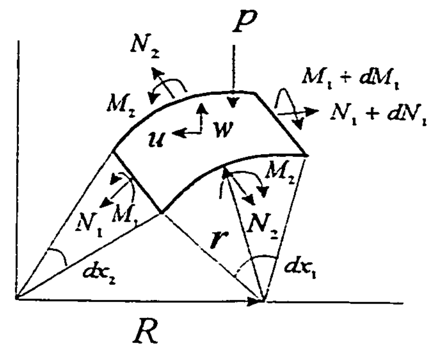

Figure 3-2a: Resultants

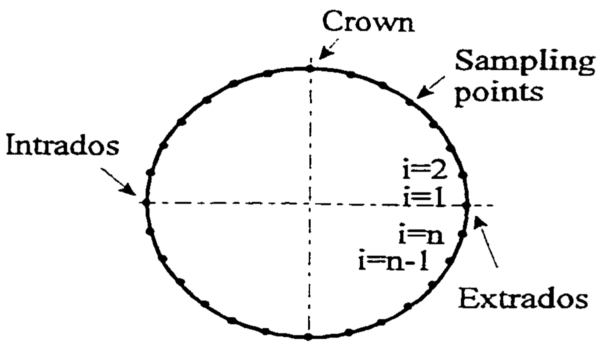

Figure 3-2a: Sample DQM mesh

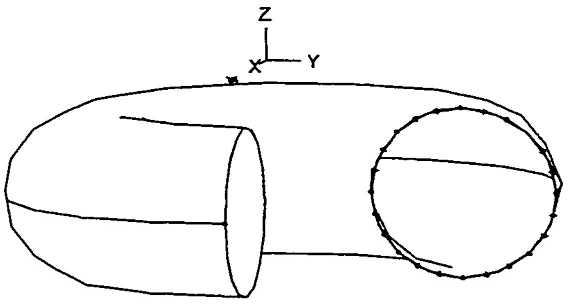

Figure 3-2b: Sample FEM mesh

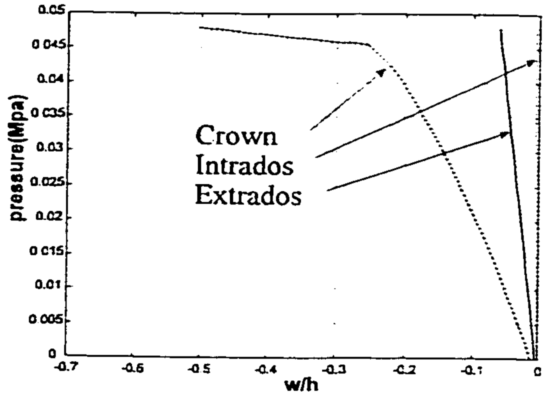

Figure 3-3a : Case 1 - DQM w-p curve

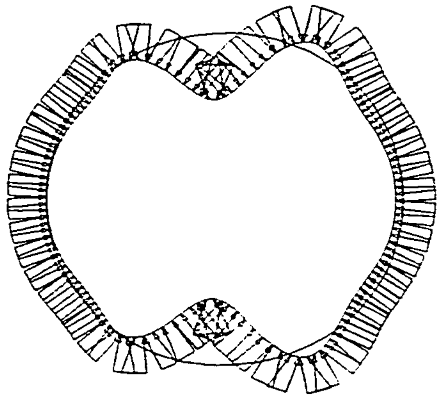

Figure 3-3b : Case 1 - FEM buckled cross-section

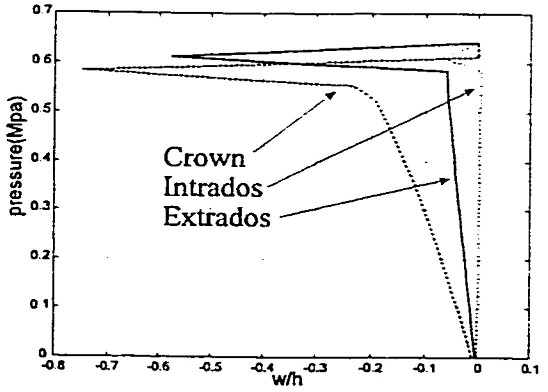

Figure 3-4a: Case 2 - DQM w-p curve

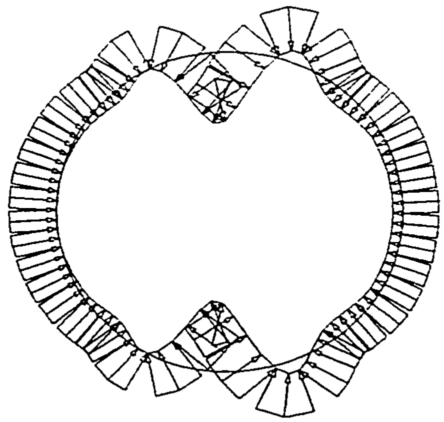

Figure 3-4b: Case 2 - FEM buckled cross-section

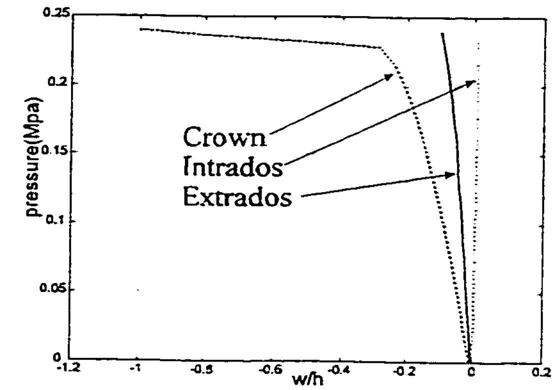

Figure 3-5a: Case 3 - DQM w-p curve

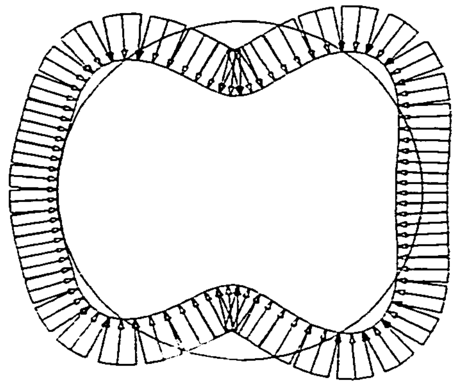

Figure 3-5b: Case 3- FEM buckled cross-section

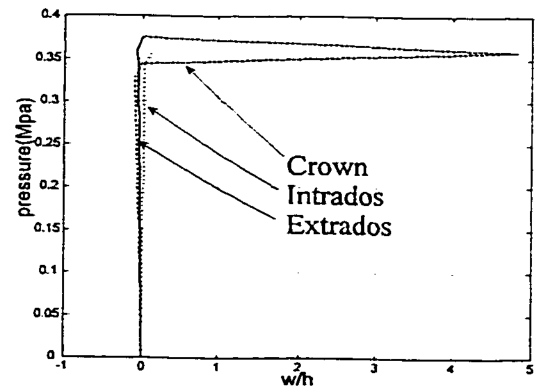

Figure 3-6a: Case 4- DQM w-p curve

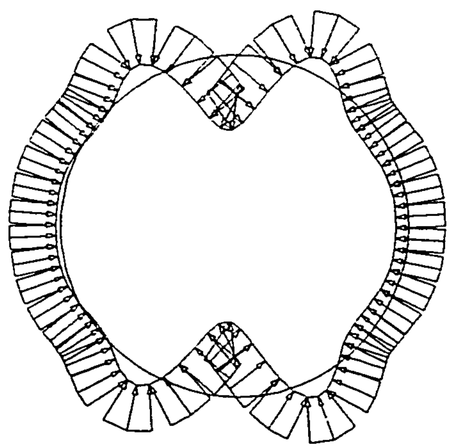

Figure 3-6b: Case 4-FEM buckled cross-section

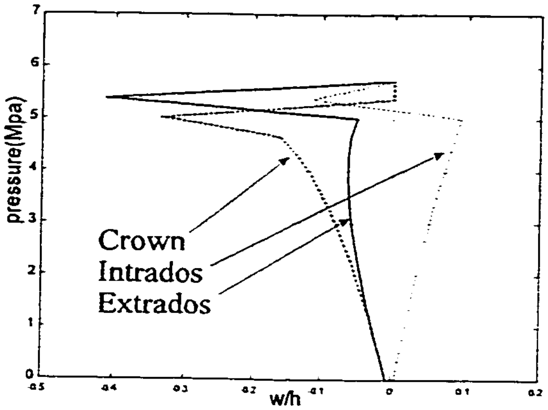

Figure 3-7a: Case 5 - DQM w-p curve

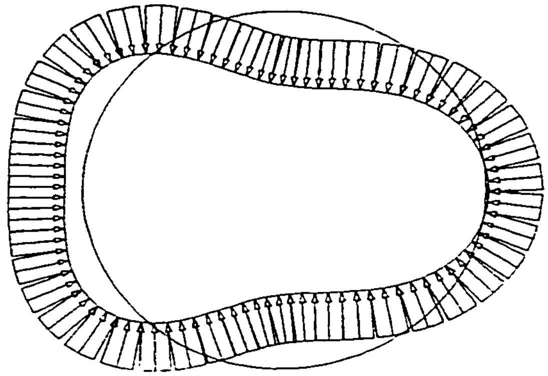

Figure 3-7b: Case 5 - FEM buckled cross-section

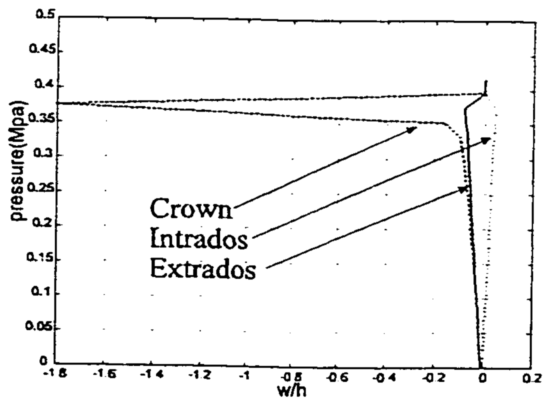

Figure 3-8a: Case 6 - DQM w-p curve

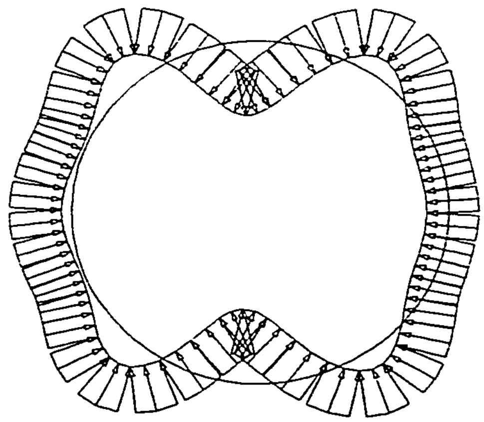

Figure 3-8b: Case 6 - FEM buckled cross-section

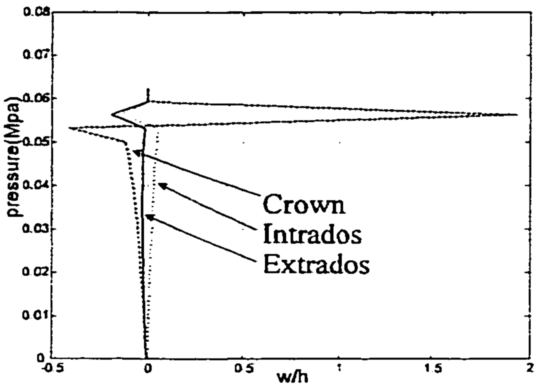

Figure 3-9a: Case 7 - DQM w-p curve

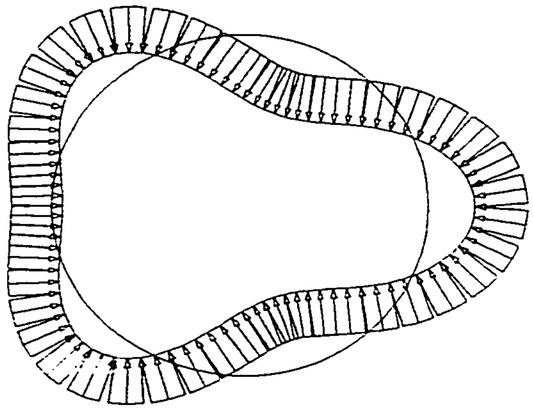

Figure 3-9b: Case 7 - FEM buckled cross-section

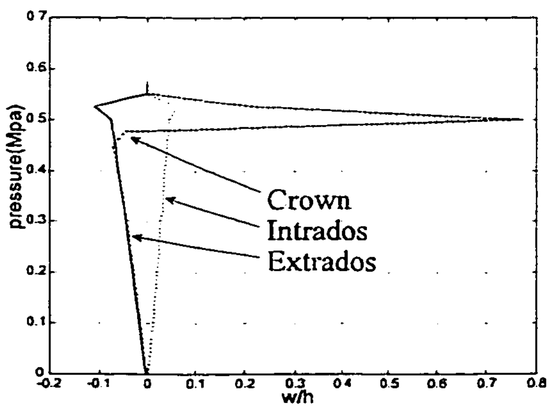

Figure 3-10a: Case 8 - DQM w-p curve

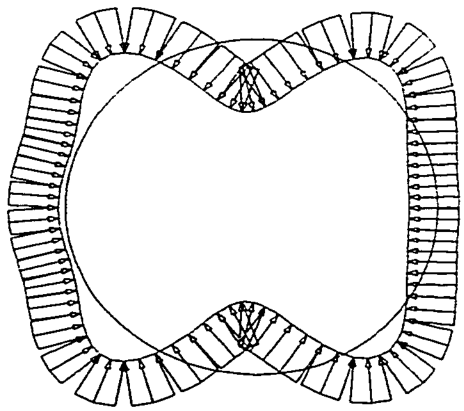

Figure 3-10b: Case 8 - FEM buckled cross-section

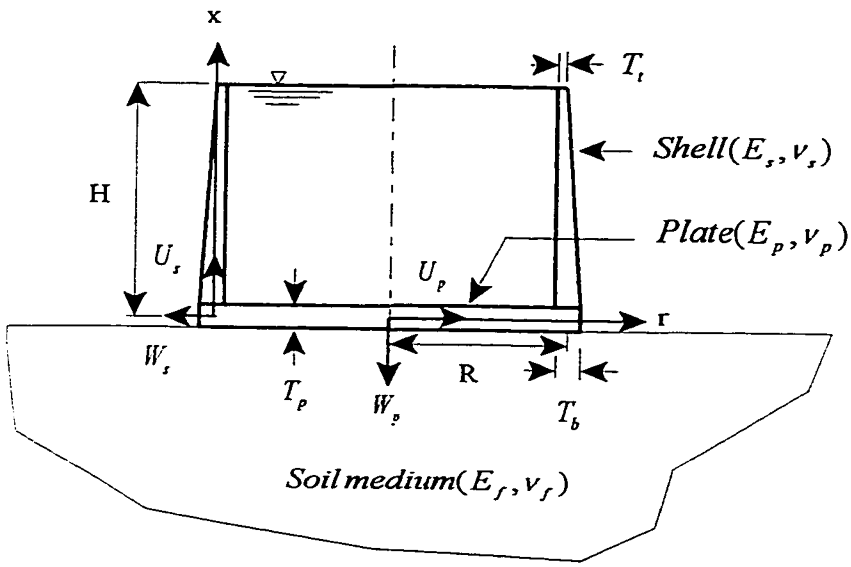

Figure 4-1: Geometry

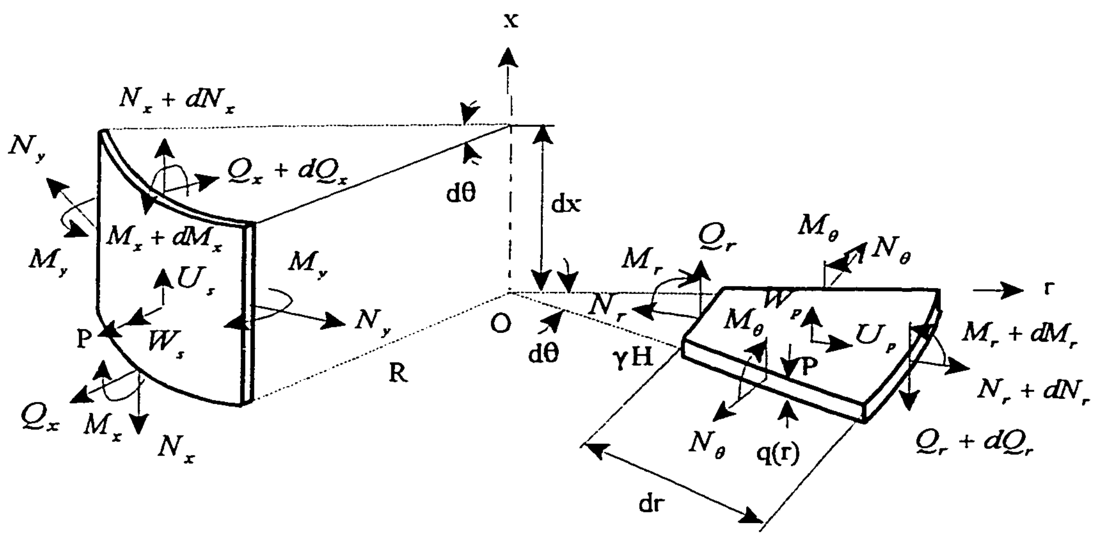

Figure 4-2: Resultants

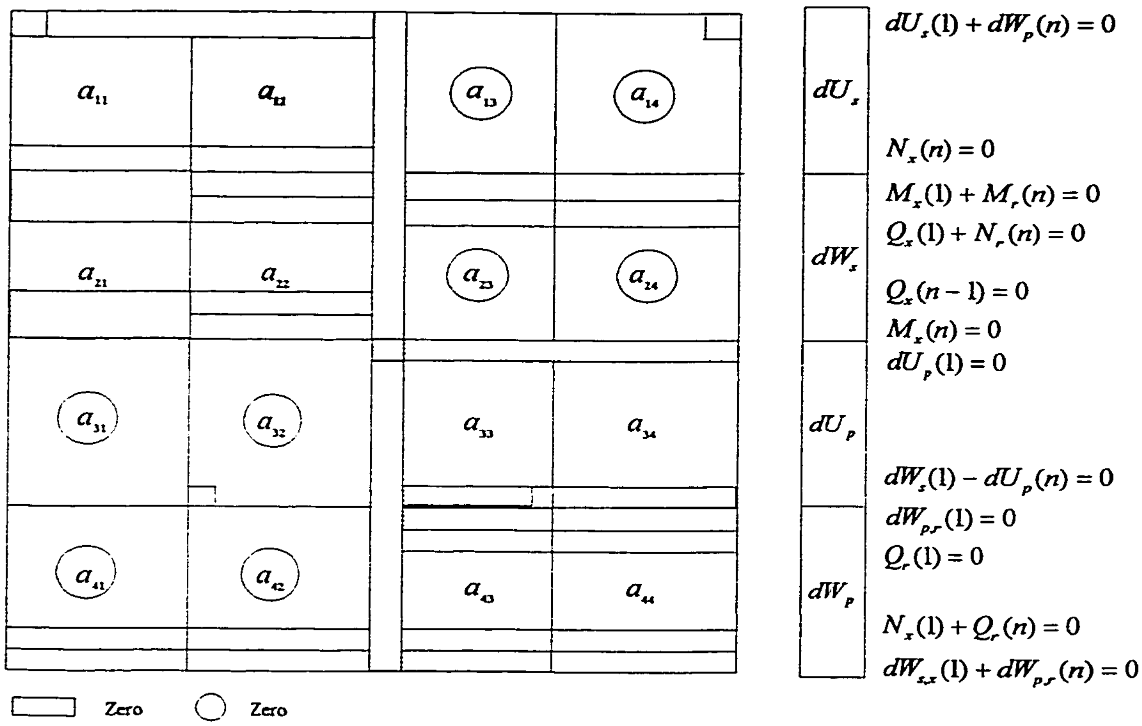

Figure 4-3: Configuration of the stiffness matrix

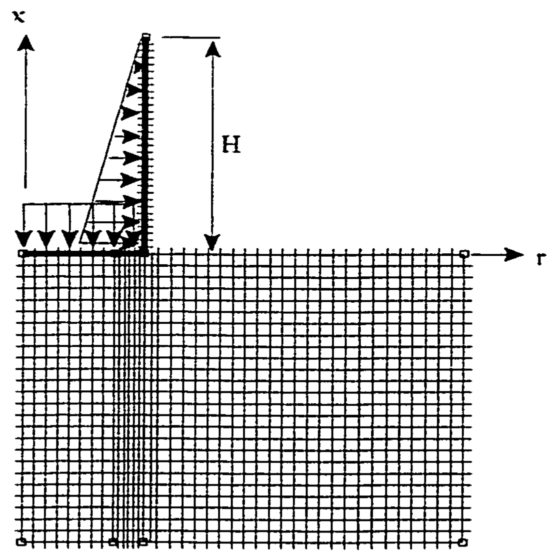

Figure 4-4: Sample model (FEM)

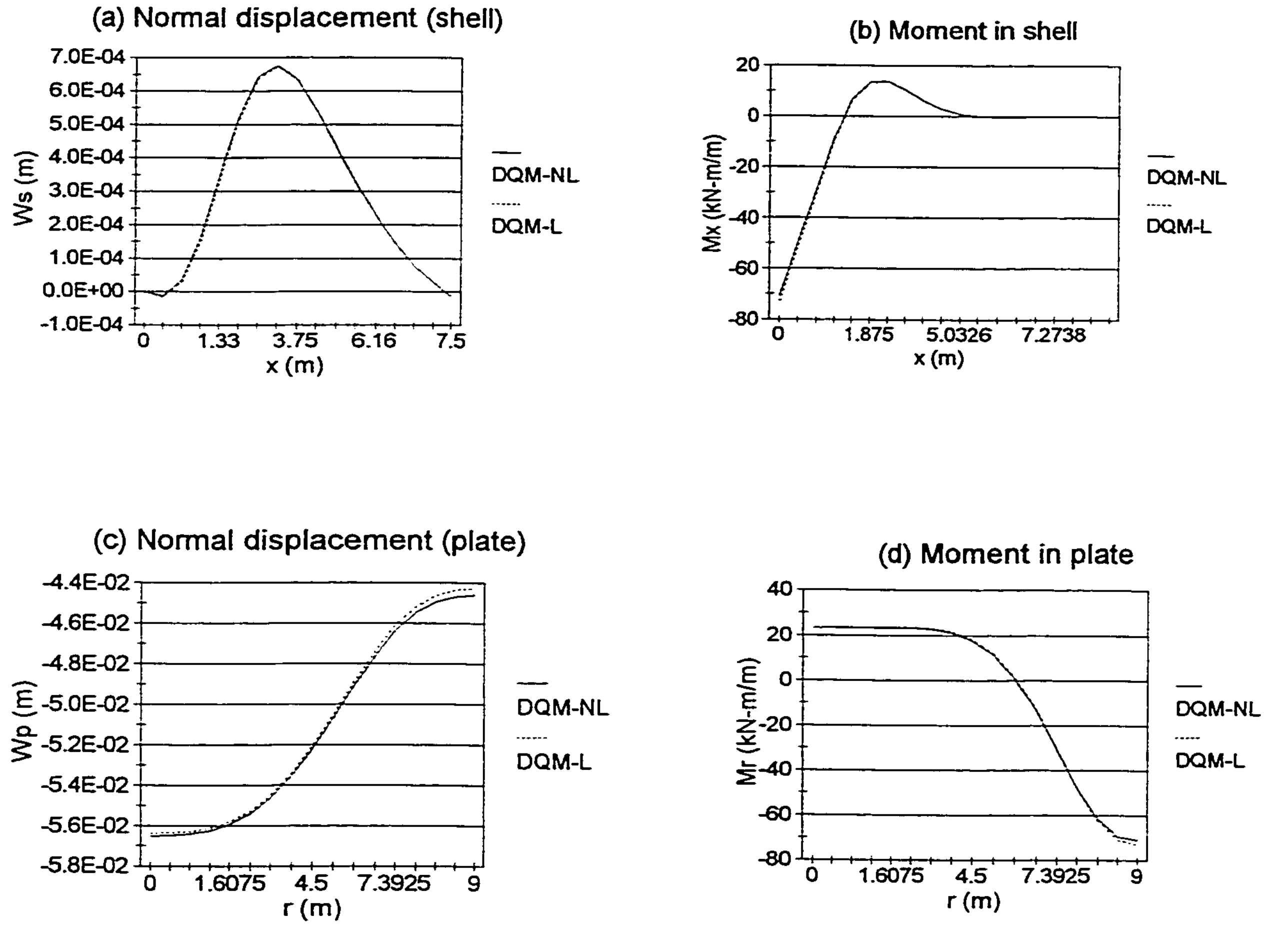

Figure 4-5: Check against Kukreti et al (1997) - concrete tank

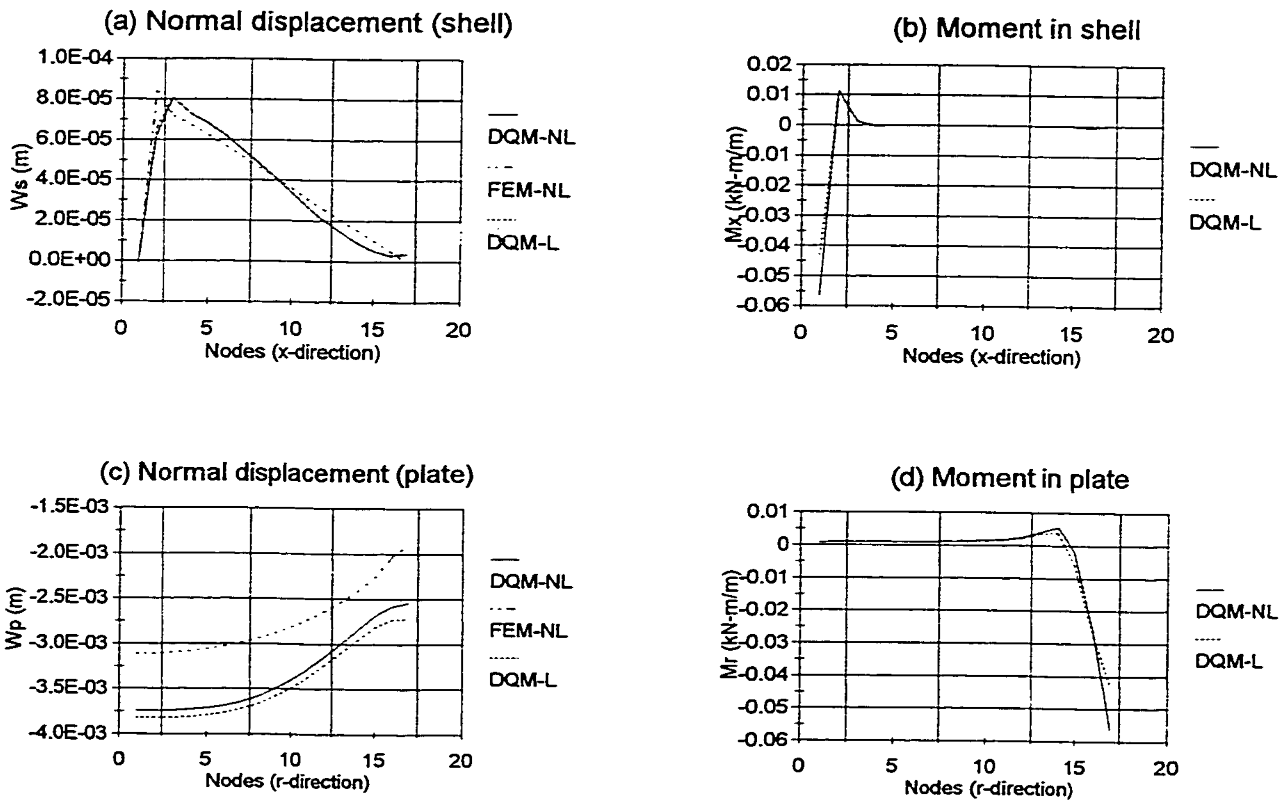

Figure 4-6: Check against FEM - steel tank
H=3 (m); R=1.5 (m); T=4 (mm)

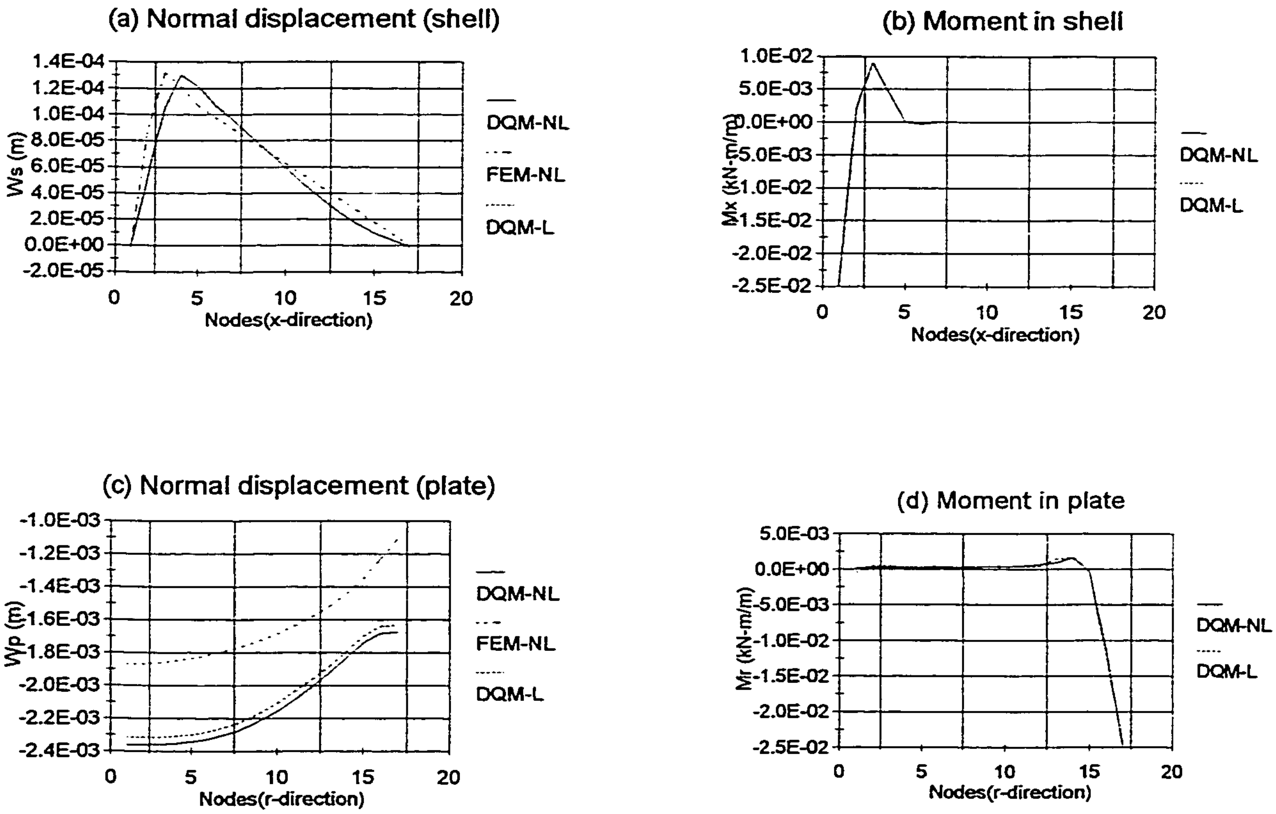

Figure 4-7: Check against FEM - aluminum tank
H=1.52 (m); R=1.8 (m); T=5 (mm)

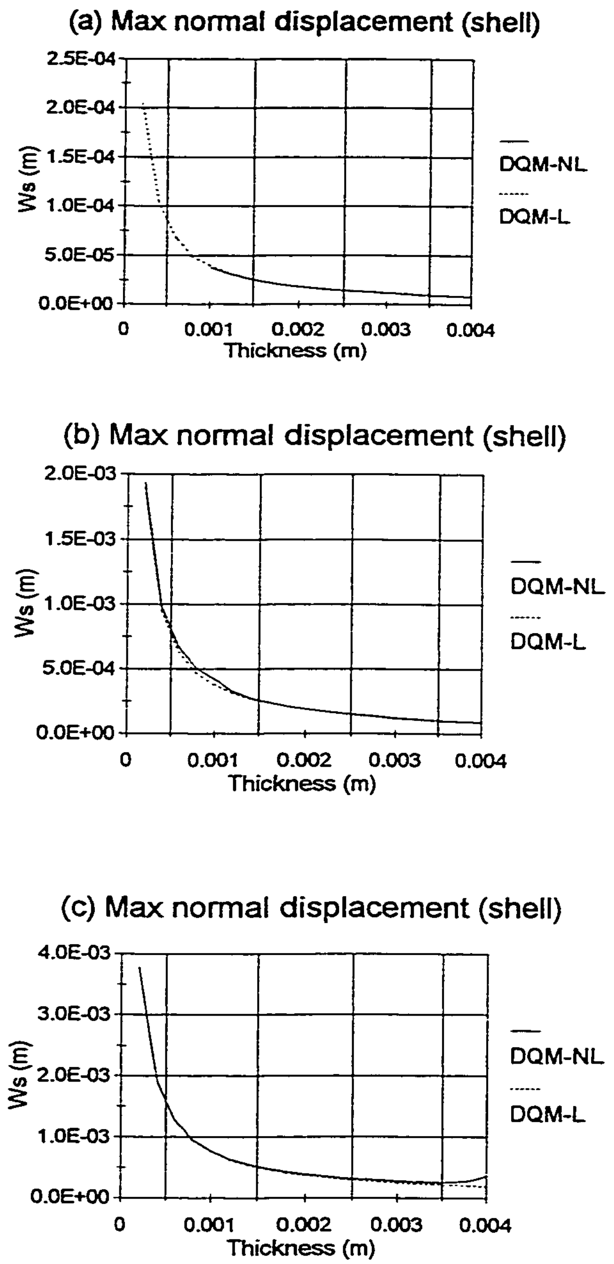

Figure 4-8: Checking linear vs nonlinear DQM - steel tanks
(a) H/R = 0.5; (b) H/R = 1.0 ; (c) H/R = 2.0

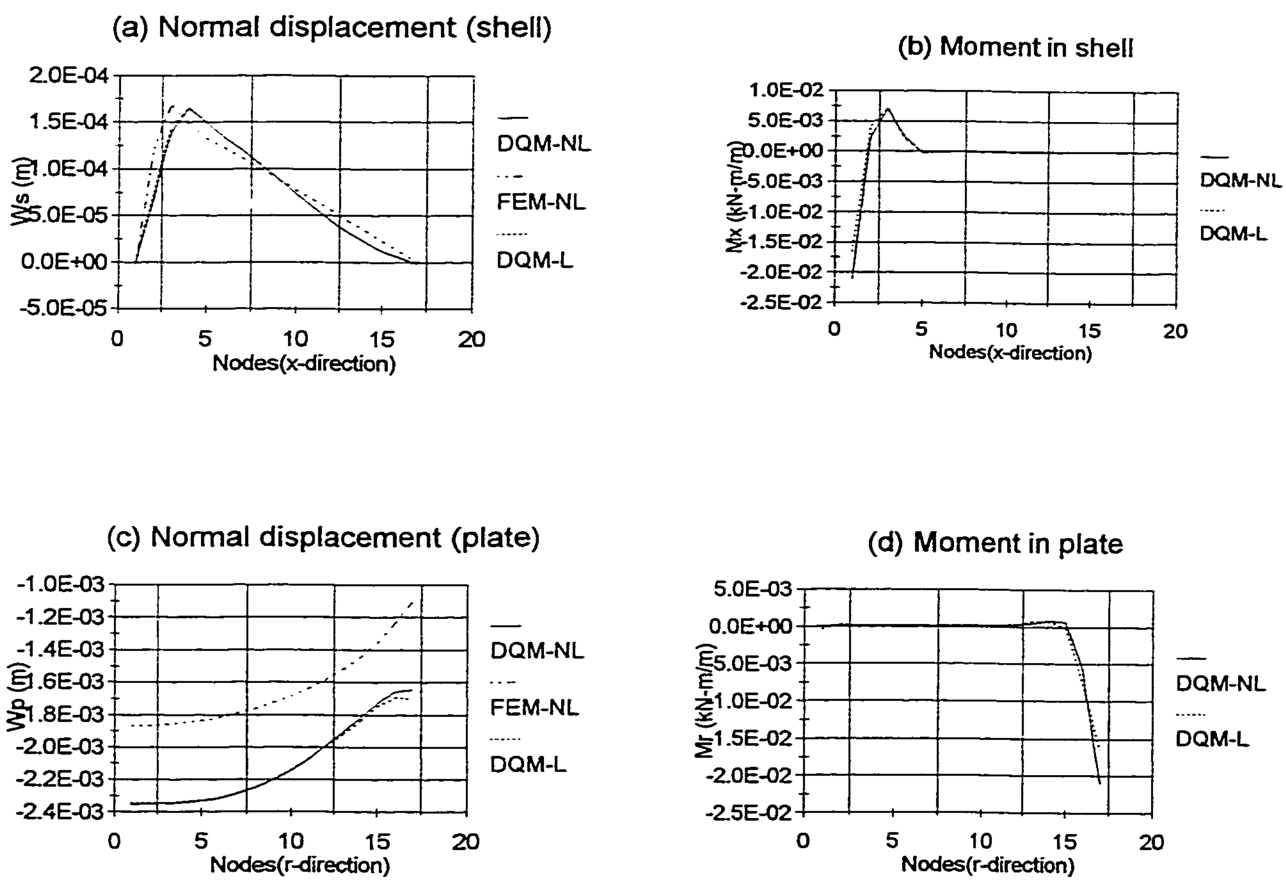

Figure 4-9: Check against FEM - aluminum tank
H=1.52 (m); R=1.8 (m); T=4 (mm)

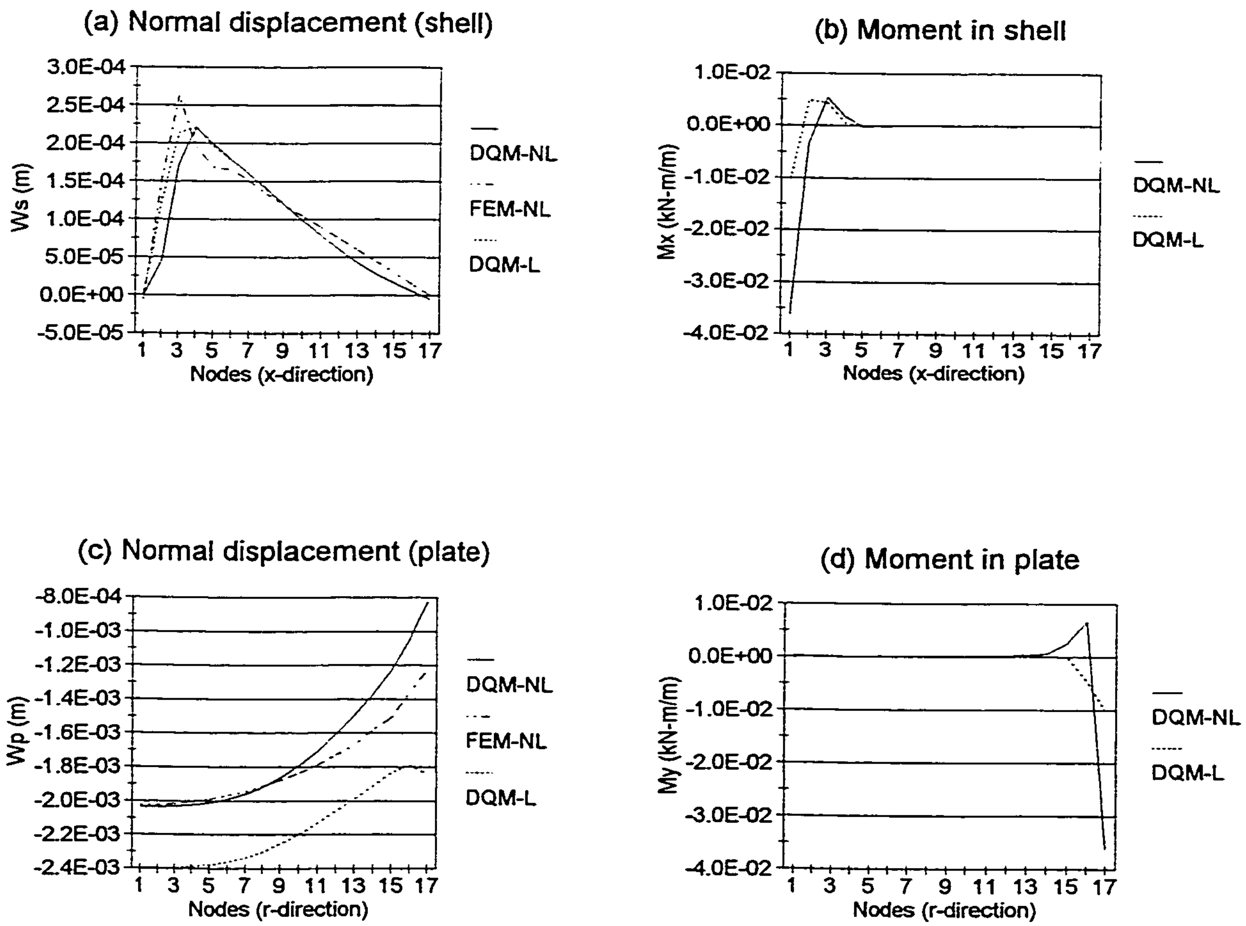

Figure 4-10: Check against FEM - aluminum tank
H=1.52 (m); R=1.8 (m); T=3 (mm)

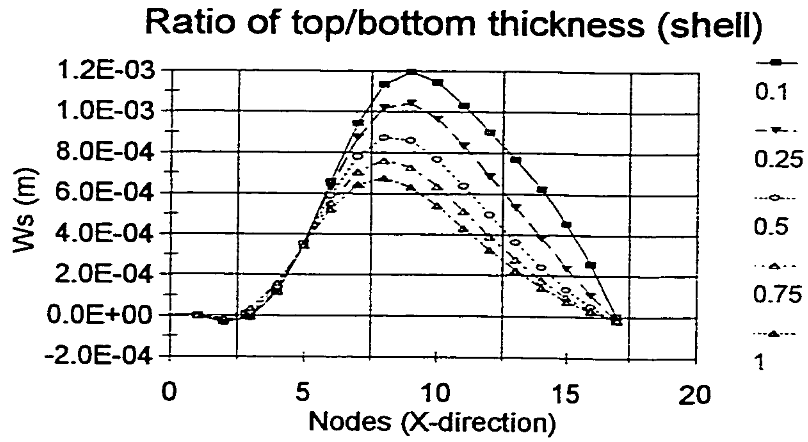

Figure 4-11: Effect of thickness ratio on shell normal displacement

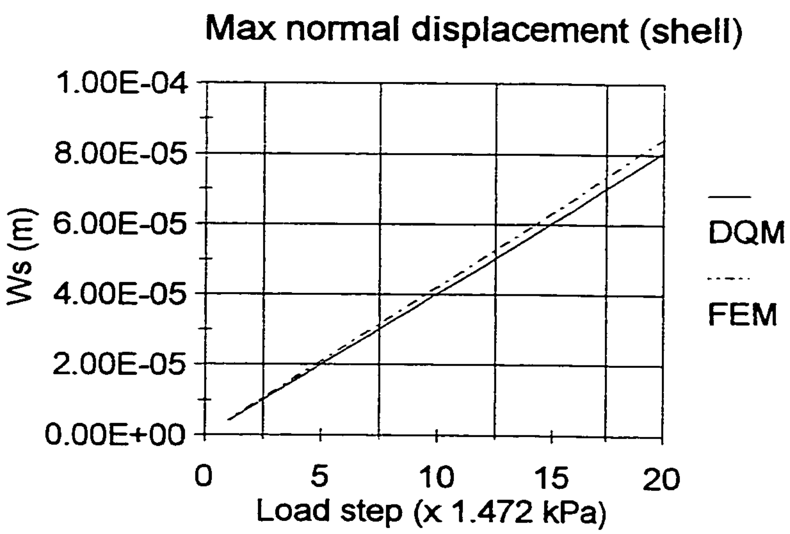

Figure 4-12: Nonlinearity of shell displacement

Program nlaxx.m

```
% non-linear response of axisymmetric toroidal shell using DQM
% original theory by Gaidaichuk Sov Appl Mech v14 1978 pp 931-937
% perfect circular section subjected to uniform external pressure
% incremental analysis using Frechet derivatives
% --------------------------------------------------------------
% section input - input the problem parameters
% n=Number of stations in theta direction (even)
% ey=youngs modulus (MPa), pr=poisson ratio
% rarr=bend radius/rs, rathic=shell thick/rs
%
f1=fopen('C:\nlaxg1.dat','w');
f2=fopen('C:\nlaxg2.dat','w');
format long
n=24
etol=0.01;
% shell geometry and material - units MPa (N/mm2) and mm
%
% Case 1 almroth sobel hunter expt pc=0.044MPa use khi=50 pfact=.001(n=96)
rs=63.5; rarr=2.0; rathic=.02;
ey=3447.5; pr=.365;
khi=20; pfact=.0024;iter=3;
%
% Case 2 Wang and Zhang tab4 pc=ey*2.68e-6=0.563Mpa
rs=63.5; rarr=2.0; rathic=0.01;
ey=210000.; pr=.3;
khi=25; pfact=.028;iter=3;
%
% Case 3 nordell and crawford expt pc=0.217MPa use khi=110 pfact=.0025
% rs=50.8; rarr=3.0; rathic=1/23.3;
% ey=3496.; pr=.365;
% khi=20; pfact=.012;iter=3;
%
% Case 4 sobel Tab4 model-3 Pc= ey*1.73e-6 = 0.363Mpa use khi=50 pfact=.0001
% rs=50.8; rarr=4.0; rathic=1/100;
% ey=210000; pr=.3;
% khi=25; pfact=.015;iter=3;
%
% Case 5 DQM(n=24)
% rs=100; rarr=6.32; rathic=1/20;
% ey=117000.; pr=.3;
% khi=20; pfact=.34;iter=4;
```

```
%
% Case 6 sobel Tab5 model-4 expt 0.392MPa, FDM 0.333Mpa use khi=80 pfact=.005
%rs=100; rarr=6.32; rathic=1/70;
%ey=117000.; pr=.3;
%khi=25; pfact=.02;iter=3;
%
% Case 7 galletly Tab4 model-1 expt 0.0577MPa(0.076Mpa)use khi=50 pfact=.002
%rs=50; rarr=7.94; rathic=1/23.3;
%ey=2165.; pr=.4;
%khi=24; pfact=.003;iter=4;
%
% Case 8 sobel Tab5 model 1 expt 0.486MPa use khi=50 pfact=.01
%rs=50; rarr=8.04; rathic=1/71.5;
%ey=210000; pr=.3;
%khi=25; pfact=.024;iter=3;
%
fprintf(f1,'ey=%10.4f pr=%5.3f r=%5d rarr=%5.3f rathic=%5.3f\n',ey,pr,rs,rarr,rathic);
%
ra=rs*rarr; thic=rs*rathic; rs2=rs*rs; gam=1/rarr;
cee=(ey*thic)/(1-pr*pr); dee=cee*thic*thic/12;
rad=pi/180; xlo=.0;
%-------------------------------------------------------------------
% section wcmd - find the weight coeffts for the meridional direction
% sampling points are equally spaced in theta direction
% output - b1(i,j),b2(i,j),b3(i,j),b4(i,j) for 1st to 4th deriv
n2=n/2;
%
for i=1:n
tet(i)=2*pi*(i-1)/n; ib(i,1)=.5; ib(i,n2+1)=.5;
end
%
for i=2:2:n; ib(i,n2+1)=-0.5; end
%
for i=2:n2; ib(1,i)=1; end
%
for i=n2+2:n; ib(1,i)=0; end
%
for i=2:n; for j=n2+2:n
ib(i,j)=sin(2*pi*(i-1)*(j-1-n2)/n);
end; end
%
for i=2:n; for j=2:n2
ib(i,j)=cos(2*pi*(i-1)*(j-1)/n);
```

```
end; end
ib=ib/n2;
%
for i=1:n; for j=1:n2+1
rr1(j)=(1-j)*sin((j-1)*tet(i));
rr2(j)=-1*((j-1)^2)*cos((j-1)*tet(i));
rr3(j)=((j-1)^3)*sin((j-1)*tet(i));
rr4(j)=((j-1)^4)*cos((j-1)*tet(i));
end
%
for j=n2+2:n
rr1(j)=(j-1-n2)*cos((j-1-n2)*tet(i));
rr2(j)=-1*((j-1-n2)^2)*sin((j-1-n2)*tet(i));
rr3(j)=-1*((j-1-n2)^3)*cos((j-1-n2)*tet(i));
rr4(j)=((j-1-n2)^4)*sin((j-1-n2)*tet(i));
end
%
b1(i,:)=(ib*rr1')';
b2(i,:)=(ib*rr2')';
b3(i,:)=(ib*rr3')';
b4(i,:)=(ib*rr4')';
end
double(b1);double(b2);double(b3);double(b4);
%
%-------------------------------------------------------------------
% Definition and initialization for the variables.
wpl=zeros(khi,1);wql=zeros(khi,1);whl=zeros(khi,1);
u=zeros(n,1);w=zeros(n,1);us=zeros(n,1);ws=zeros(n,1);wsh=zeros(n,1);
du=zeros(n,1);dus=zeros(n,1);dw=zeros(n,1);ddw=zeros(n,1);dws=zeros(n,1);
dbs=zeros(n,1);ddbs=zeros(n,1);dk1=zeros(n,1);dk1s=zeros(n,1);
e1=zeros(n,1);e2=zeros(n,1);z1=zeros(n,1);z2=zeros(n,1);
nn1=zeros(n,1);nn2=zeros(n,1);mm1=zeros(n,1);mm2=zeros(n,1);
nn1s=zeros(n,1);nn2s=zeros(n,1);mm1s=zeros(n,1);mm2s=zeros(n,1);
n1s=zeros(n,1);n2s=zeros(n,1);
m1=zeros(n,1);m2=zeros(n,1);m3=zeros(n,1);m4=zeros(n,1);m5=zeros(n,1);
m6=zeros(n,1);m7=zeros(n,1);m8=zeros(n,1);m9=zeros(n,1);m10=zeros(n,1);
m11=zeros(n,1);m12=zeros(n,1);m13=zeros(n,1);m14=zeros(n,1);m15=zeros(n,1);
m16=zeros(n,1);m17=zeros(n,1);m18=zeros(n,1);m19=zeros(n,1);m20=zeros(n,1);
m21=zeros(n,1);m22=zeros(n,1);m23=zeros(n,1);m24=zeros(n,1);m25=zeros(n,1);
m26=zeros(n,1);m27=zeros(n,1);m28=zeros(n,1);m29=zeros(n,1);m30=zeros(n,1);
m31=zeros(n,1);m32=zeros(n,1);m33=zeros(n,1);m34=zeros(n,1);m35=zeros(n,1);
m36=zeros(n,1);m37=zeros(n,1);m38=zeros(n,1);dm1=zeros(n,1);dm16=zeros(n,1);
dm17=zeros(n,1);dm18=zeros(n,1);dm19=zeros(n,1);dm20=zeros(n,1);
```

```
dm21=zeros(n,1);dm22=zeros(n,1);dm23=zeros(n,1);dm24=zeros(n,1);
dm25=zeros(n,1);dm26=zeros(n,1);dm27=zeros(n,1);dm28=zeros(n,1);
dm29=zeros(n,1);dm30=zeros(n,1);dm31=zeros(n,1);dm32=zeros(n,1);
dm33=zeros(n,1);dm34=zeros(n,1);dm35=zeros(n,1);
ddm26=zeros(n,1);ddm27=zeros(n,1);ddm28=zeros(n,1);ddm29=zeros(n,1);
ddm30=zeros(n,1);dmm1s=zeros(n,1);
h11=zeros(n,1);h12=zeros(n,1);h13=zeros(n,1);h14=zeros(n,1);h15=zeros(n,1);
h16=zeros(n,1);h17=zeros(n,1);h18=zeros(n,1);
h21=zeros(n,1);h22=zeros(n,1);h23=zeros(n,1);h24=zeros(n,1);h25=zeros(n,1);
h26=zeros(n,1);h27=zeros(n,1);h28=zeros(n,1);h29=zeros(n,1);h30=zeros(n,1);
dinc=zeros(2*n-1,1);rinc=zeros(2*n-1,1);
%-------------------------------------------------------------------
% Set-up the curvatures (k1,k2) and the Lame parameters (as,bs)
k1=zeros(n,1); k2=zeros(n,1); k1s=(1/rs)*ones(n,1); k2s=zeros(n,1);
as=rs*ones(n,1); bs=zeros(n,1);
for i=1:n
x=tet(i);
zz=1+gam*cos(x);
rx=rs*zz;
bs(i)=rx;
k2s(i)=gam*cos(x)/rx;
end;
dk1=b1*k1;
das=b1*as;
dbs=b1*bs;
ddbs=b2*bs;
dk1s=b1*k1s;
ps=0.;p=0.;
k=0.;
mwh=0.; mwh1=0.;mwh2=0.;mwh3=0.;del=0.;
% start loop over load steps
% ===================================================================
while k<khi
k=k+1
p=-pfact;
ps=ps+p
kk=0.;
fact=1.;
%
u=zeros(n,1);w=zeros(n,1);norm=zeros(2*n-1,1);
%
while (fact>etol)&(kk<iter)
kk=kk+1;
```

```
for i=1:n;
m1(i)=1/as(i);dm1(i)=-1/as(i)^2*das(i);
m2(i)=k1(i)*ws(i);
m3(i)=k1s(i)*us(i)-m1(i)*dws(i);
m4(i)=-m1(i)*m3(i);
m5(i)=m3(i)*k1s(i);
m6(i)=m2(i)+m3(i)*k1(i)*us(i);
m7(i)=1/as(i)/bs(i)*dbs(i);
m8(i)=k2(i)*ws(i);
m9(i)=-m1(i)*dm1(i);
m10(i)=m1(i)*k1s(i);
m11(i)=m1(i)*dk1s(i);
m12(i)=m1(i)*(dk1(i)*us(i)+k1(i)*dus(i));
m13(i)=-m1(i)*m7(i);
m14(i)=m7(i)*k1s(i);
m15(i)=m7(i)*k1(i)*us(i);
%
m16(i)=cee*m1(i);
m17(i)=cee*m4(i);
m18(i)=cee*(m5(i)+pr*m7(i));
m19(i)=cee*(k1s(i)+pr*k2s(i));
m20(i)=cee*(m6(i)+pr*m8(i));
m21(i)=pr*m16(i);
m22(i)=pr*m17(i);
m23(i)=cee*(m7(i)+pr*m5(i));
m24(i)=cee*(k2s(i)+pr*k1s(i));
m25(i)=cee*(m8(i)+pr*m6(i));
m26(i)=-dee*m1(i)^2;
m27(i)=dee*(m9(i)+pr*m13(i));
m28(i)=dee*m10(i);
m29(i)=dee*(m11(i)+pr*m14(i));
m30(i)=dee*(m12(i)+pr*m15(i));
m31(i)=-dee*pr*m1(i)^2;
m32(i)=dee*(m13(i)+pr*m9(i));
m33(i)=pr*m28(i);
m34(i)=dee*(m14(i)+pr*m11(i));
m35(i)=dee*(m15(i)+pr*m12(i));
m37(i)=-as(i)*bs(i);
m38(i)=dbs(i)*(mm1s(i)-mm2s(i))+bs(i)*dmm1s(i);
end;
dm16=b1*m16;
dm17=b1*m17;
dm18=b1*m18;
```

```
dm19=b1*m19;
dm20=b1*m20;
dm26=b1*m26;ddm26=b2*m26;
dm27=b1*m27;ddm27=b2*m27;
dm28=b1*m28;ddm28=b2*m28;
dm29=b1*m29;ddm29=b2*m29;
dm30=b1*m30;ddm30=b2*m30;
dm31=b1*m31;
dm32=b1*m32;
dm33=b1*m33;
dm34=b1*m34;
dm35=b1*m35;
for i=1:n;
h11(i)=bs(i)*(m16(i)+k1s(i)*m28(i));
h12(i)=dbs(i)*(k1s(i)*(-m33(i)+m28(i))+m16(i)-m21(i));
h12(i)=h12(i)+bs(i)*(k1s(i)*(dm28(i)+m29(i))+m18(i)+dm16(i))+m36(i)*(-k1s(i)*m1(i)+m10(i));
h13(i)=dbs(i)*(m18(i)-m23(i)+k1s(i)*(-m34(i)+m29(i)));
h13(i)=h13(i)+bs(i)*(dm18(i)+k1s(i)*dm29(i))+m36(i)*(-k1s(i)*m5(i)+m11(i));
h14(i)=k1s(i)*bs(i)*m26(i);
h15(i)=k1s(i)*dbs(i)*(-m31(i)+m26(i));
h15(i)=h15(i)+bs(i)*(m17(i)+k1s(i)*(dm26(i)+m27(i)))-m36(i)*m1(i)^2;
h16(i)=dbs(i)*(m17(i)-m22(i)+k1s(i)*(-m32(i)+m27(i)));
h16(i)=h16(i)+bs(i)*(m19(i)+dm17(i)+k1s(i)*dm27(i))+m36(i)*(-k1s(i)*m4(i)+m9(i));
h17(i)=dbs(i)*(m19(i)-m24(i))+bs(i)*dm19(i)-m36(i)*k1s(i)^2;
h18(i)=dbs(i)*(k1s(i)*(-m35(i)+m30(i))+m20(i)-m25(i))+k1(i)*m38(i);
h18(i)=h18(i)+bs(i)*(dm20(i)+k1s(i)*dm30(i))+m36(i)*(-k1s(i)*m6(i)+m12(i));
h21(i)=m1(i)*bs(i)*m28(i);
h22(i)=m1(i)*dbs(i)*(2*m28(i)-m33(i))+bs(i)*(m1(i)*(m29(i)+2*dm28(i))+dm1(i)*m28(i));
h23(i)=dbs(i)*(m1(i)*(2*m29(i)+2*dm28(i)-m34(i)-dm33(i))+dm1(i)*(m28(i)-m33(i)));
h23(i)=h23(i)+bs(i)*(m1(i)*(2*dm29(i)+ddm28(i))+dm1(i)*(dm28(i)+m29(i)));
h23(i)=h23(i)+m1(i)*ddbs(i)*(m28(i)-m33(i));
h23(i)=h23(i)+m37(i)*(nn1s(i)*m10(i)-m1(i)*ps+k1s(i)*m16(i)-
nn1s(i)*k1s(i)*m1(i)+k2s(i)*m21(i));
h24(i)=dbs(i)*(dm1(i)*(m29(i)-m34(i))+m1(i)*(2*dm29(i)-dm34(i)));
h24(i)=h24(i)+m1(i)*ddbs(i)*(m29(i)-m34(i))+bs(i)*(dm1(i)*dm29(i)+m1(i)*ddm29(i));
h24(i)=h24(i)+m37(i)*(-ps*(m5(i)+m7(i))+k1s(i)*(m18(i)-nn1s(i)*m5(i)));
h24(i)=h24(i)+m37(i)*(k2s(i)*(m23(i)-nn2s(i)*m7(i))+nn2s(i)*m14(i)+nn1s(i)*m11(i));
h25(i)=m1(i)*bs(i)*m26(i);
h26(i)=m1(i)*dbs(i)*(2*m26(i)-m31(i))+bs(i)*(m1(i)*(2*dm26(i)+m27(i))+dm1(i)*m26(i));
h27(i)=dbs(i)*(m1(i)*(2*(dm26(i)+m27(i))-m32(i)-dm31(i))+dm1(i)*(m26(i)-m31(i)));
h27(i)=h27(i)+bs(i)*(m1(i)*(2*dm27(i)+ddm26(i))+dm1(i)*(dm26(i)+m27(i)));
h27(i)=h27(i)+m1(i)*ddbs(i)*(m26(i)-m31(i))+m37(i)*nn1s(i)*m1(i)^2;
h28(i)=dbs(i)*(dm1(i)*(m27(i)-m32(i))+m1(i)*(2*dm27(i)-dm32(i)));
```

```
h28(i)=h28(i)+bs(i)*(dm1(i)*dm27(i)+m1(i)*ddm27(i))+m1(i)*ddbs(i)*(m27(i)-m32(i));
h28(i)=h28(i)+m37(i)*(k1s(i)*(m17(i)-nn1s(i)*m4(i))-
ps*m4(i)+k2s(i)*m22(i)+nn1s(i)*m9(i)+nn2s(i)*m13(i));
h29(i)=m37(i)*(k2s(i)*(k2s(i)*nn2s(i)-ps+m24(i))+k1s(i)*(k1s(i)*nn1s(i)-ps+m19(i)));
h30(i)=dbs(i)*(m1(i)*(2*dm30(i)-dm35(i))+dm1(i)*(m30(i)-m35(i)));
h30(i)=h30(i)+bs(i)*(m1(i)*ddm30(i)+dm1(i)*dm30(i))+m1(i)*ddbs(i)*(m30(i)-m35(i));
h30(i)=h30(i)+m37(i)*(k2s(i)*(m25(i)-nn2s(i)*m8(i))+k1s(i)*(m20(i)-nn1s(i)*m6(i)));
h30(i)=h30(i)+m37(i)*(nn2s(i)*m15(i)+nn1s(i)*m12(i)-
ps*(m8(i)+m6(i))+k1(i)*n1s(i)+k2(i)*n2s(i)-p);
end;
%--------------------------------------------------------------------
% set up and solve the dqm equation
s11(:,:)=zeros(n); s12(:,:)=zeros(n);
s21(:,:)=zeros(n); s22(:,:)=zeros(n);
sa(:,:)=zeros(2*n-1);sb=zeros(2*n-1,1);rhs=zeros(2*n-1,1);
% terms which have no derivatives
for i=1:n;
  s11(i,i)=h13(i); s12(i,i)=h17(i);
  s21(i,i)=h24(i); s22(i,i)=h29(i);
% terms which have derivatives
for j=1:n;
  s11(i,j)=s11(i,j)+h11(i)*b2(i,j)+h12(i)*b1(i,j);
  s12(i,j)=s12(i,j)+h14(i)*b3(i,j)+h15(i)*b2(i,j)+h16(i)*b1(i,j);
  s21(i,j)=s21(i,j)+h21(i)*b3(i,j)+h22(i)*b2(i,j)+h23(i)*b1(i,j);
  s22(i,j)=s22(i,j)+h25(i)*b4(i,j)+h26(i)*b3(i,j)+h27(i)*b2(i,j)+h28(i)*b1(i,j);
end;
end;
for i=1:n-1;for j=1:n-1
   sa(i,j)=s11(i+1,j+1);
end;end;
for i=1:n-1;for j=1:n;
   sa(i,j+n-1)=s12(i+1,j);
end;end
for i=1:n;for j=1:n-1;
   sa(i+n-1,j)=s21(i,j+1);
end;end
for i=1:n;for j=1:n;
   sa(i+n-1,j+n-1)=s22(i,j);
end;end
for i=1:n-1
  rhs(i)=-h18(i+1);
end;
for i=1:n
```

```
  rhs(i+n-1)=-h30(i);
end;
sb=(inv(sa))*rhs;
for i=2:n
  u(i)=sb(i-1);
end;
for i=1:n
  w(i)=sb(i+n-1);
end;
norm=norm+sb;
fact=transpose(sb)/transpose(norm);
clear sa;clear sb;clear rhs;
% ---------------------------------------------------------------
% calculate results for elastic response
du=b1*u;
dw=b1*w;
ddw=b2*w;
us=us+u;
ws=ws+w;
dus=b1*us;
dws=b1*ws;
for i=1:n
  e1(i)=m1(i)*du(i)+k1(i)*ws(i)+k1s(i)*w(i);
  e1(i)=e1(i)+(k1s(i)*us(i)-m1(i)*dws(i))*(k1(i)*us(i)+k1s(i)*u(i)-m1(i)*dw(i));
  e2(i)=m7(i)*u(i)+k2s(i)*w(i)+k2(i)*ws(i);
  z1(i)=-m1(i)^2*ddw(i)+m9(i)*dw(i)+m10(i)*du(i)+m11(i)*u(i);
  z1(i)=z1(i)+m1(i)*(dk1(i)*us(i)+k1(i)*dus(i));
  z2(i)=m13(i)*dw(i)+m14(i)*u(i)+m7(i)*k1(i)*us(i);
  %
  as(i)=as(i)*(1+e1(i));
  bs(i)=bs(i)*(1+e2(i));
  %
  k1(i)=-k1s(i)*e1(i)+z1(i);
  k2(i)=-k2s(i)*e2(i)+z2(i);
  k1s(i)=k1s(i)+k1(i);
  k2s(i)=k2s(i)+k2(i);
end;
nn1=cee*(e1+pr*e2);
nn2=cee*(e2+pr*e1);
mm1=dee*(z1+pr*z2);
mm2=dee*(z2+pr*z1);
n1s=n1s+nn1;
n2s=n2s+nn2;
```

```
mm1s=mm1s+mm1;
mm2s=mm2s+mm2;
dmm1s=b1*mm1s;
%
das=b1*as;
dbs=b1*bs;
ddbs=b2*bs;
dk1s=b1*k1s;
end;
% End of iteration loop
%
wsh=ws/thic;
mwh=wsh(1);
mwh1=wsh(n/4+1);
mwh3=wsh(n/2+1);
del=(mwh-mwh2)/mwh
mwh2=mwh;
fprintf(f1,'------------------------------------------------------------------------\n');
fprintf(f1,'pressure(N/mm2)=%6.4f ps(N/mm2)=%8.5f k=%2d\n',p,ps,k);
fprintf(f1,'------------------------------------------------------------------------\n');
fprintf(f2,'------------------------------------------------------------------------\n');
fprintf(f2,'k=%2d ps=%8.5f wsh(n2)=%15.10f\n',k,ps,wsh(n2));
for i=1:n
  fprintf(f1,'u(%3d)=%15.10f ',i,u(i));
  fprintf(f1,'w(%3d)=%15.10f ',i,w(i));
  fprintf(f1,'wsh(%3d)=%15.10f\n',i,wsh(i));
end
if abs(mwh)>5; mwh=0.;end;
if abs(mwh1)>5; mwh1=0.;end;
if abs(mwh3)>5; mwh3=0.;end;
wpl(k,1)=mwh;
wql(k,1)=mwh1;
whl(k,1)=mwh3;
end;
% end loop over load steps
% ==========================================================================
l=pfact*khi; nstep=khi; y=linspace(0,l,nstep);
plot(wpl,y,'k-');xlabel('w/h');ylabel('pressure(Mpa)');grid on;hold on;
plot(wql,y,'b--');hold on;plot(whl,y,'r:');hold off;
status=fclose(f1);
status=fclose(f2);
```

Program tankl.m

```
% non-linear analysis of axisymmetric circular cylindrical tank - using DQM
% Delta-technic is applied for weighting coefficients
% Newton method is applied for the iteration scheme.
%-----------------------------------------------------------------
% section input - input the problem parameters
% n=Number of stations in domain [0,1] (odd)
% et,ep,es=youngs modulus (MPa), prt,prp,prs=poisson ratio
%
format short e
n=19
%
%-----------------------------------------------------------------
% section wccd - find weighting coeffts for circumferential direction
% unequal spacing with one delta point at each end for eta direction
% output - a1(i,j),a2(i,j),a3(i,j),a4(i,j) - 1st to 4th derivatives
x=zeros(n,1);y=zeros(n,1);
%
for i=1:n; tt=(pi*(i-1))/(n-1);y(i)=(1-cos(tt))/2; end
y(2)=2e-3; y(n-1)=1-y(2);
x=y;
%
for i=1:n; pai(i)=1; for j=1:n
if i~=j; pai(i)=(x(i)-x(j))*pai(i); end
end;end
%
for i=1:n; for j=1:n
if i~=j; a1(i,j)=pai(i)/((x(i)-x(j))*pai(j)); end
end;end
%
for i=1:n; a1(i,i)=0; for j=1:n
if i~=j; a1(i,i)=a1(i,i)-a1(i,j); end
end;end
%
for i=1:n; for j=1:n
if i~=j; a2(i,j)=2*(a1(i,i)*a1(i,j)-a1(i,j)/(x(i)-x(j))); end
end;end
%
for i=1:n; a2(i,i)=0; for j=1:n
if i~=j; a2(i,i)=a2(i,i)-a2(i,j); end
end;end
```

```
%
for i=1:n; for j=1:n
if i~=j; a3(i,j)=3*(a2(i,i)*a1(i,j)-a2(i,j)/(x(i)-x(j))); end
end;end
%
for i=1:n; a3(i,i)=0; for j=1:n
if i~=j; a3(i,i)=a3(i,i)-a3(i,j); end
end;end
%
for i=1:n; for j=1:n
if i~=j; a4(i,j)=4*(a3(i,i)*a1(i,j)-a3(i,j)/(x(i)-x(j))); end
end;end
%
for i=1:n; a4(i,i)=0; for j=1:n
if i~=j; a4(i,i)=a4(i,i)-a4(i,j); end
end;end
%------------------------------------------------------------------
% Set-up the inverse Vandermonde Matrix;
c(:,:)=zeros(n); vi(:,:)=zeros(n);
p=-x;
for i=1:n;
  s=zeros(n,1); ss(:,:)=zeros(n);
  c(i,1)=1;
  c(i,n)=1;
  for j=1:n; if j~=i;
      c(i,1)=c(i,1)*p(j);
      c(i,n-1)=c(i,n-1)+p(j);end;
  end;
        %
  p(i)=0.;
  for j=1:n; for jj=j+1:n;
    s(j)=s(j)+p(jj);
  end;
        end;
  for j=1:n; for jj=j+1:n;
    ss(1,j)=ss(1,j)+p(jj)*s(jj);
  end;end;
        for k=2:n;
        for j=1:n; for jj=j+1:n;
    ss(k,j)=ss(k,j)+p(jj)*ss(k-1,jj);
  end;end;
        end;
        %
```

```
    for j=1:n;
      c(i,n-2)=c(i,n-2)+p(j)*s(j);
    end;
    if n>4;
    k=0.;
    while k<(n-4)
    k=k+1;
    for j=1:n;
      c(i,n-(k+2))=c(i,n-(k+2))+p(j)*ss(k,j);
    end;end;end;
          p=-x;
end;
%
for i=1:n; for j=1:n;
    vi(i,j)=c(i,j)/pai(i);
end;end;
vi=vi';
%
%-------------------------------------------------------------------
inc=0.2;
mt=4
ndisp=zeros(mt/inc,4);nthic=zeros(mt/inc,1);
gam=9.81;
rp=1.2;
% property of tank wall
h=0.5*rp;
del=0.;
kk=0.;
while (del<mt-inc/2)&(abs(ndisp(:,1))<10)&kk<1
tb=(mt-del)*1e-3
tt=0.5*tb;
kk=kk+1
del=del+inc;
et=2e8;
prt=0.3;
% property of base plate
tp=tb;
ep=et;
prp=prt;
% property of soil medium
es=20e3;
prs=.4;
%
```

```
ct=et/(12*(1-prt^2));
rt=tt-tb;
rb=rt/tb;
ri=rp-tb/2;
phs=(1-prs^2)/es;
key=(1-prs^2)*(ep/es)*(tp/rp)^3;
cp=key/12/(1-prp^2);
cpp=1/cp/pi;
a0=12*(1-prp^2)/(pi*key);
b0=12*(1-prp^2)/key;
%
rad=pi/180; xlo=.0;
%
%----------------------------------------------------------------------
% Definition and initialization for the variables.
w=zeros(n,1);v=zeros(n,1);
rx=zeros(n,1);d=zeros(n,1);ax=zeros(n,1);
tx=zeros(n,1);dx=zeros(n,1);dxx=zeros(n,1);vs=zeros(n,1);
wx=zeros(n,1);wxx=zeros(n,1);w3x=zeros(n,1);w4x=zeros(n,1);
vy=zeros(n,1);vyy=zeros(n,1);v3y=zeros(n,1);v4y=zeros(n,1);
qx=zeros(n,1);qy=zeros(n,1);mx=zeros(n,1);my=zeros(n,1);
ux=zeros(n,1);uxx=zeros(n,1);c1x=zeros(n,1);c2x=zeros(n,1);
ai=zeros(n,1);el(:,:)=zeros(n);jei(:,:)=zeros(n);
as(:,:)=zeros(n);yk(:,:)=zeros(n);yr(:,:)=zeros(n);
%----------------------------------------------------------------------
% Set-up the weighting coefficient matrix for s(r);
sy=zeros(n,1);
for i=1:n
sy(i)=sqrt(1-y(i)^2);
end;
y(1)=1e-6;
ai(1)=1;
ai(2)=pi/4;
for k=3:n;
ai(k)=(1-1/k)*ai(k-2);
end;
for i=1:n
el(i,1)=sy(i);
jei(i,1)=-y(i);
el(i,2)=1/2*(sy(i)+y(i)^2*log((1+sy(i))/y(i)));
jei(i,2)=-y(i)+y(i)*sy(i)*log((1+sy(i))/y(i));
for k=3:n;
el(i,k)=sy(i)/k+(1-1/k)*y(i)^2*el(i,k-2);
```

```
jei(i,k)=-y(i)/k+(1-1/k)*y(i)^2*jei(i,k-2)+2*(1-1/k)*y(i)*sy(i)*el(i,k-2);
end;end
%
for j=1:n
for k=1:n
  yr(j,k)=k*ai(k)*jei(j,k);
end;end
as=yr*vi;
y(1)=0.;
%----------------------------------------------------------------
% Sep-up the parameters;
for i=1:n;
tx(i)=tb+rt*x(i);
rx(i)=ri+tx(i)/2;
d(i)=ct*tb^3*(1+3*rb*x(i)+3*rb^2*x(i)^2+rb^3*x(i)^3);
dx(i)=ct*tb^3*(3*rb+6*rb^2*x(i)+3*rb^3*x(i)^2);
dxx(i)=ct*tb^3*(6*rb^2+6*rb^3*x(i));
ax(i)=-(gam*h*rx(i)^2)/et/tx(i);
%ax(i)=-gam*h*rp*phs;
h36(i)=sy(i)*y(i)^3*b0;
end;
%
%------------set up dqm equation------------------------------------
a11(:,:)=zeros(n); a12(:,:)=zeros(n);
a21(:,:)=zeros(n); a22(:,:)=zeros(n);
% terms which have no derivatives
for i=1:n;
  a22(i,i)=et*tx(i)*h^4/rx(i)^2;
% terms which have derivatives
for j=1:n;
  a11(i,j)=sy(i)*(y(i)^3*a4(i,j)+2*y(i)^2*a3(i,j)-y(i)*a2(i,j)+a1(i,j));
  a11(i,j)=a11(i,j)-a0*y(i)^2*as(i,j);
  a22(i,j)=a22(i,j)+d(i)*a4(i,j)+2*dx(i)*a3(i,j)+dxx(i)*a2(i,j);
end;
end;
%
%----------------------------------------------------------------
% section loadv - set up the load vector for the right hand side
% output - rhs(i) - load vector for stiffness equation
%----------------------------------------------------------------
rhs=zeros(2*n,1);
for i=1:n;
rhs(i)=-h36(i);
```

```
rhs(n+i)=gam*h^5*(1-x(i))/ax(i);
end;
rhs(1)=0.;
rhs(n-1)=0.;
rhs(n)=0.;
rhs(n+1)=0.;
rhs(n+2)=0.;
rhs(2*n-1)=0.;
rhs(2*n)=0.;
%------------------------------------------------------------
% enforce the boundary and compatible conditions
%------------------------------------------------------------
for j=1:n;
a11(1,j)=a1(1,j);
a11(n-1,j)=cp*(a2(n-1,j)+prp/y(n-1)*a1(n-1,j));
a11(n,j)=cp*(a3(n,j)+a2(n,j)-a1(n,j));
a12(n-1,j)=ax(1)/gam/h^3/rp^2*d(1)*a2(1,j);
a22(1,j)=0.;
a21(2,j)=gam*h*phs*a1(n,j);
a22(2,j)=ax(2)/h*a1(2,j);
a22(n-1,j)=ax(n-1)*d(n-1)*a2(n-1,j);
a22(n,j)=ax(n)*(d(n)*a3(n,j)+dx(n)*a2(n,j));
end;
for i=1:n;
a12(i,1)=0.;
a22(i,1)=0.;
end;
a22(1,1)=1.;
%------------------------------------------------------------
% enforce the stiffness matrix
%------------------------------------------------------------
sa(:,:)=zeros(2*n);dd=zeros(2*n,1);
for i=1:n
for j=1:n
   sa(i,j)=a11(i,j);
   sa(i,n+j)=a12(i,j);
   sa(n+i,j)=a21(i,j);
   sa(n+i,n+j)=a22(i,j);
end;
end
%
%------------------------------------------------------------
% section solu - solve for displacement coeffts and find results
```

```
% find the vector of displacement components rslt
%-----------------------------------------------------------------
%
dd=inv(sa)*rhs;
for i=1:n
  v(i)=dd(i);
  w(i)=dd(n+i);
end;
if max(abs(w))<1;
ndisp(kk,1)=-abs(max(w))*ax(3);
ndisp(kk,2)=-w(4)*ax(4);
ndisp(kk,3)=-w(6)*ax(6);
ndisp(kk,4)=-w(12)*ax(12);
nthic(kk)=tb;
end;end;
%%--------------------End of outer while loop;
plot(ndisp,nthic)
%
x=x*h;
y=y*rp;
wx=a1*w;
wxx=a2*w;
w3x=a3*w;
w4x=a4*w;
vs=as*v;
vy=a1*v;
vyy=a2*v;
v3y=a3*v;
v4y=a4*v;
for i=1:n
mx(i)=-ax(i)/gam/h^3/rp^2*d(i)*wxx(i);
qx(i)=-ax(i)/gam/h^4/rp*(d(i)*w3x(i)+dx(i)*wxx(i));
my(i)=cp*vyy(i);
end;
v=v*gam*h*rp*phs;
for i=1:n
  w(i)=w(i)*ax(i);
  vp(i)=-(v(n)-v(i));
  nt(i)=w(i)*et*tb/rp;
end
mx=mx*gam*h*rp^2;
qx=qx*gam*h*rp;
my=my*gam*h*rp^2;
```

```
%plot(x,mx);hold on;plot(y,my);hold off;
```

Program tankn.m

```
% non-linear analysis of axisymmetric circular cylindrical tank - using DQM
% Chebyshev spaces and delta technic are applied for weighting coefficients
% Newton method is applied for the iteration scheme.
%----------------------------------------------------------------
% section input - input the problem parameters
% n=Number of stations in domain [0,1] (odd)
% et,ep,es=youngs modulus (MPa), prt,prp,prs=poisson ratio
%
format long
n=19
%
gam=9.81;
% property of tank wall
h=2;
tb=4e-3; tt=4e-3;
et=2e8;
prt=0.3;
% property of base plate
rp=4;
tp=5e-3;
ep=2e8;
prp=0.3;
% property of soil medium
es=20e3;
prs=.4;
%
ct=et/(12*(1-prt^2));
jt=et*tb/(1-prt^2);
dp=ep*tp^3/(12*(1-prp^2));
jp=ep*tp/(1-prp^2);
rt=tt-tb;
rb=rt/tb;
ri=rp-tb/2;
phs=(1-prs^2)/es;
key=(1-prs^2)*(ep/es)*(tp/rp)^3;
cp=key/12/(1-prp^2);
a0=12*(1-prp^2)/pi/key;
b0=12*(1-prp^2)/key;
c0=-12/tp^2*(gam*h*rp*phs)^2;
d0=-12/tp^2*gam*h*rp^2*phs;
```

```
n1=-dp*gam*h*phs/rp^2;
n2=jp*(gam*h*phs)^2;
n3=jp/2*(gam*h*phs)^3;
n4=prp*n2;
%
rad=pi/180; xlo=.0;
%-------------------------------------------------------------------
% section wccd - find weighting coeffts for circumferential direction
% unequal spacing with one delta point at each end for eta direction
% output - a1(i,j),a2(i,j),a3(i,j),a4(i,j) - 1st to 4th derivatives
x=zeros(n,1);y=zeros(n,1);
%
for i=1:n; tt=(pi*(i-1))/(n-1);y(i)=(1-cos(tt))/2; end
y(2)=2e-4; y(n-1)=1-2e-4;
x=y;
%
for i=1:n; pai(i)=1; for j=1:n
if i~=j; pai(i)=(x(i)-x(j))*pai(i); end
end;end
%
for i=1:n; for j=1:n
if i~=j; a1(i,j)=pai(i)/((x(i)-x(j))*pai(j)); end
end;end
%
for i=1:n; a1(i,i)=0; for j=1:n
if i~=j; a1(i,i)=a1(i,i)-a1(i,j); end
end;end
%
for i=1:n; for j=1:n
if i~=j; a2(i,j)=2*(a1(i,i)*a1(i,j)-a1(i,j)/(x(i)-x(j))); end
end;end
%
for i=1:n; a2(i,i)=0; for j=1:n
if i~=j; a2(i,i)=a2(i,i)-a2(i,j); end
end;end
%
for i=1:n; for j=1:n
if i~=j; a3(i,j)=3*(a2(i,i)*a1(i,j)-a2(i,j)/(x(i)-x(j))); end
end;end
%
for i=1:n; a3(i,i)=0; for j=1:n
if i~=j; a3(i,i)=a3(i,i)-a3(i,j); end
end;end
```

```
%
for i=1:n; for j=1:n
if i~=j; a4(i,j)=4*(a3(i,i)*a1(i,j)-a3(i,j)/(x(i)-x(j))); end
end;end
%
for i=1:n; a4(i,i)=0; for j=1:n
if i~=j; a4(i,i)=a4(i,i)-a4(i,j); end
end;end
%-------------------------------------------------------------------
% Set-up the inverse Vandermonde Matrix;
c(:,:)=zeros(n); vi(:,:)=zeros(n);
p=-x;
for i=1:n;
  s=zeros(n,1); ss(:,:)=zeros(n);
  c(i,1)=1;
  c(i,n)=1;
  for j=1:n; if j~=i;
      c(i,1)=c(i,1)*p(j);
      c(i,n-1)=c(i,n-1)+p(j);end;
  end;
        %
  p(i)=0.;
  for j=1:n; for jj=j+1:n;
    s(j)=s(j)+p(jj);
  end;
        end;
  for j=1:n; for jj=j+1:n;
    ss(1,j)=ss(1,j)+p(jj)*s(jj);
  end;end;
        for k=2:n;
        for j=1:n; for jj=j+1:n;
    ss(k,j)=ss(k,j)+p(jj)*ss(k-1,jj);
  end;end;
        end;
        %
  for j=1:n;
    c(i,n-2)=c(i,n-2)+p(j)*s(j);
  end;
  if n>4;
  k=0.;
  while k<(n-4)
  k=k+1;
  for j=1:n;
```

```
    c(i,n-(k+2))=c(i,n-(k+2))+p(j)*ss(k,j);
   end;end;end;
        p=-x;
 end;
 %
 for i=1:n; for j=1:n;
     vi(i,j)=c(i,j)/pai(i);
 end;end;
 vi=vi';
 %------------------------------------------------------------------
 % Definition and initialization for the variables.
 t=zeros(n,1);u=zeros(n,1);v=zeros(n,1);w=zeros(n,1);
 dt=zeros(n,1);du=zeros(n,1);dv=zeros(n,1);dw=zeros(n,1);
 ty=zeros(n,1);tyy=zeros(n,1);ux=zeros(n,1);uxx=zeros(n,1);
 vy=zeros(n,1);vyy=zeros(n,1);v3y=zeros(n,1);v4y=zeros(n,1);
 wx=zeros(n,1);wxx=zeros(n,1);w3x=zeros(n,1);w4x=zeros(n,1);
 m1=zeros(n,1);m2=zeros(n,1);m3=zeros(n,1);m4=zeros(n,1);m5=zeros(n,1);
 m6=zeros(n,1);m7=zeros(n,1);m8=zeros(n,1);m9=zeros(n,1);m10=zeros(n,1);
 m11=zeros(n,1);h11=zeros(n,1);h21=zeros(n,1);h22=zeros(n,1);h23=zeros(n,1);
 h24=zeros(n,1);h31=zeros(n,1);h41=zeros(n,1);h42=zeros(n,1);h43=zeros(n,1);
 h44=zeros(n,1);h45=zeros(n,1);bc1=zeros(n,1);bc2=zeros(n,1);bc3=zeros(n,1);
 bc4=zeros(n,1);bc5=zeros(n,1);bc6=zeros(n,1);j11=zeros(n,1);
 l1=zeros(n,1);l2=zeros(n,1);l3=zeros(n,1);l4=zeros(n,1);
 rx=zeros(n,1);ax=zeros(n,1);tx=zeros(n,1);
 d=zeros(n,1);dx=zeros(n,1);dxx=zeros(n,1);vs=zeros(n,1);
 qx=zeros(n,1);qy=zeros(n,1);mx=zeros(n,1);my=zeros(n,1);
 ai=zeros(n,1);el(:,:)=zeros(n);jei(:,:)=zeros(n);
 as(:,:)=zeros(n);yk(:,:)=zeros(n);yr(:,:)=zeros(n);
 %------------------------------------------------------------------
 % Set-up the weighting coefficient matrix for s(r);
 sy=zeros(n,1);
 for i=1:n
 sy(i)=sqrt(1-y(i)^2);
 end;
 y(1)=1e-10;
 ai(1)=1;
 ai(2)=pi/4;
 for k=3:n;
 ai(k)=(1-1/k)*ai(k-2);
 end;
 for i=1:n
 el(i,1)=sy(i);
 jei(i,1)=-y(i);
```

```
el(i,2)=1/2*(sy(i)+y(i)^2*log((1+sy(i))/y(i)));
jei(i,2)=-y(i)+y(i)*sy(i)*log((1+sy(i))/y(i));
for k=3:n;
el(i,k)=sy(i)/k+(1-1/k)*y(i)^2*el(i,k-2);
jei(i,k)=-y(i)/k+(1-1/k)*y(i)^2*jei(i,k-2)+2*(1-1/k)*y(i)*sy(i)*el(i,k-2);
end;end
%
for j=1:n
for k=1:n
  yr(j,k)=k*ai(k)*jei(j,k);
end;end
as=yr*vi;
y(1)=0.;
%-------------------------------------------------------------------
%
for i=1:n;
tx(i)=tb+rt*x(i);
rx(i)=ri+tx(i)/2;
d(i)=ct*tb^3*(1+3*rb*x(i)+3*rb^2*x(i)^2+rb^3*x(i)^3);
dx(i)=ct*tb^3*(3*rb+6*rb^2*x(i)+3*rb^3*x(i)^2);
dxx(i)=ct*tb^3*(6*rb^2+6*rb^3*x(i));
ax(i)=-gam*h*rp*phs;
m1(i)=ax(i)/h;
m2(i)=-prt*h/rx(i);
m3(i)=-jt*ax(i)^2/2;
m4(i)=jt*prt*h^2*ax(i)/rx(i);
m5(i)=-jt*prt*h^2*ax(i)/2/rx(i);
m6(i)=jt*h^4/rx(i)^2;
m7(i)=-jt*h*ax(i);
m8(i)=-jt*prt*h^3/rx(i);
m9(i)=-gam*h^5*(1-x(i))/ax(i);
m10(i)=y(i)^2*gam*h*phs;
m11(i)=1/2*y(i)*(1-prp)*gam*h*phs;
bc1(i)=-ax(i)*d(i)/gam/h^3/rp^2;
bc2(i)=-ax(i)/gam/h^4/rp;
bc3(i)=jt/gam/h/rp;
bc4(i)=jp/gam/h/rp;
bc5(i)=-key/12/(1-prp^2);
bc6(i)=1/gam/h/rp;
end;
k=0.;
rsw=1;
%---------------------------------------------------------------//
```

```
% Start of loop;
while (rsw>1e-6)&(k<15)
k=k+1
% set-up coefficients
for i=1:n;
h11(i)=m1(i)*wxx(i)+m2(i);
h21(i)=dxx(i)+m3(i)*wx(i)^2+m4(i)*w(i)+m7(i)*ux(i);
h22(i)=2*m3(i)*wxx(i)*wx(i)+2*m5(i)*wx(i);
h23(i)=m4(i)*wxx(i)+m6(i);
h24(i)=m7(i)*wxx(i)+m8(i);
h31(i)=m10(i)*vyy(i)+2*m11(i)*vy(i);
h41(i)=-y(i)+3/2*c0*y(i)^3*vy(i)^2+d0*y(i)^3*ty(i)+d0*prp*y(i)^2*t(i);
h42(i)=1+3/2*c0*y(i)^2*vy(i)^2+3*c0*y(i)^3*vyy(i)*vy(i);
h42(i)=h42(i)+d0*(y(i)^2*ty(i)+y(i)^3*tyy(i)+y(i)^2*prp*ty(i));
h43(i)=d0*y(i)^3*vy(i);
h44(i)=d0*y(i)^2*(vy(i)+y(i)*vyy(i)+prp*vy(i));
h45(i)=d0*prp*y(i)^2*vyy(i);
l1(i)=uxx(i)+m1(i)*wx(i)*wxx(i)+m2(i)*wx(i);
l2(i)=d(i)*w4x(i)+2*dx(i)*w3x(i)+dxx(i)*wxx(i)+m3(i)*wxx(i)*wx(i)^2;
l2(i)=l2(i)+m4(i)*wxx(i)*w(i)+m5(i)*wx(i)^2+m6(i)*w(i)+m7(i)*wxx(i)*ux(i);
l2(i)=l2(i)+m8(i)*ux(i)+m9(i);
l3(i)=y(i)^2*tyy(i)+y(i)*ty(i)-t(i)+m10(i)*vy(i)*vyy(i)+m11(i)*vy(i)^2;
l4(i)=sy(i)*(y(i)^3*v4y(i)+2*y(i)^2*v3y(i)-y(i)*vyy(i)+vy(i));
l4(i)=l4(i)+sy(i)*c0*(1/2*y(i)^2*vy(i)^3+3/2*y(i)^3*vyy(i)*vy(i)^2);
l4(i)=l4(i)+sy(i)*d0*((1+prp)*y(i)^2*ty(i)*vy(i)+prp*y(i)^2*t(i)*vyy(i));
l4(i)=l4(i)+sy(i)*d0*(y(i)^3*tyy(i)*vy(i)+y(i)^3*ty(i)*vyy(i));
l4(i)=l4(i)-a0*y(i)^2*vs(i)+b0*y(i)^3*sy(i);
if i>1;
j11(i)=n2*ty(i)-n1/y(i)^2+3*n3*vy(i)^2+n4/y(i)*t(i);end;
end;
%
%------------set up dqm equation-------------------------------------
a11(:,:)=zeros(n); a12(:,:)=zeros(n); a13(:,:)=zeros(n); a14(:,:)=zeros(n);
a21(:,:)=zeros(n); a22(:,:)=zeros(n); a23(:,:)=zeros(n); a24(:,:)=zeros(n);
a31(:,:)=zeros(n); a32(:,:)=zeros(n); a33(:,:)=zeros(n); a34(:,:)=zeros(n);
a41(:,:)=zeros(n); a42(:,:)=zeros(n); a43(:,:)=zeros(n); a44(:,:)=zeros(n);
% terms which have no derivatives
for i=1:n;
  a22(i,i)=h23(i);
  a33(i,i)=-1;
  a43(i,i)=sy(i)*h45(i);
% terms which have derivatives
for j=1:n;
```

```
    a11(i,j)=a2(i,j);
    a12(i,j)=m1(i)*wx(i)*a2(i,j)+h11(i)*a1(i,j);
    a21(i,j)=h24(i)*a1(i,j);
    a22(i,j)=a22(i,j)+d(i)*a4(i,j)+2*dx(i)*a3(i,j)+h21(i)*a2(i,j)+h22(i)*a1(i,j);
    a33(i,j)=a33(i,j)+y(i)^2*a2(i,j)+y(i)*a1(i,j);
    a34(i,j)=m10(i)*vy(i)*a2(i,j)+h31(i)*a1(i,j);
    a43(i,j)=a43(i,j)+sy(i)*(h43(i)*a2(i,j)+h44(i)*a1(i,j));
    a44(i,j)=sy(i)*(y(i)^3*a4(i,j)+2*y(i)^2*a3(i,j)+h41(i)*a2(i,j)+h42(i)*a1(i,j));
    a44(i,j)=a44(i,j)-a0*y(i)^2*as(i,j);
end;
end;
%
%----------------------------------------------------------------
% section loadv - set up the load vector for the right hand side
% output - rhs(i) - load vector for stiffness equation
%----------------------------------------------------------------
rhs=zeros(4*n,1);
for i=1:n;
rhs(i)=-l1(i);
rhs(n+i)=-l2(i);
rhs(2*n+i)=-l3(i);
rhs(3*n+i)=-l4(i);
end;
rhs(1)=0.;
rhs(n)=0.;
rhs(n+1)=0.;
rhs(n+2)=0.;
rhs(2*n-1)=0.;
rhs(2*n)=0.;
rhs(2*n+1)=0.;
rhs(3*n)=0.;
rhs(3*n+1)=0.;
rhs(3*n+2)=0.;
rhs(4*n-1)=0.;
rhs(4*n)=0.;
%----------------------------------------------------------------
% enforce the boundary and compatible conditions
%----------------------------------------------------------------
for j=1:n;
  a11(1,j)=0.;
  a12(1,j)=0.;
  a11(n,j)=bc3(n)*ax(n)/h*a1(n,j);
  a12(n,j)=bc3(n)*(ax(n)/h)^2*wx(n)*a1(n,j);
```

```
  a21(1,j)=0.; a21(2,j)=0.; a21(n-1,j)=0.; a21(n,j)=0.;
  a22(1,j)=bc1(1)*a2(1,j);
  a22(2,j)=bc2(2)*(dx(2)*a2(2,j)+d(2)*a3(2,j));
  a22(n-1,j)=bc2(n-1)*(dx(n-1)*a2(n-1,j)+d(n-1)*a3(n-1,j));
  a22(n,j)=bc1(n)*a2(n,j);
  a23(2,j)=bc4(n)*gam*h*phs*a1(n,j);
  a24(1,j)=bc5(n)*(a2(n,j)+prp/y(n)*a1(n,j));
  a24(2,j)=bc4(n)*(gam*h*phs)^2*vy(n)*a1(n,j);
  a33(1,j)=0.; a33(n,j)=0.;
  a34(1,j)=0.; a34(n,j)=0.;
  a41(n-1,j)=bc3(1)*ax(1)/h*a1(1,j);
  a42(n-1,j)=bc3(1)*(ax(1)/h)^2*wx(1)*a1(1,j);
  a42(n,j)=-ax(1)/h*a1(1,j);
  a44(n,j)=gam*h*phs*a1(n,j);
  a43(1,j)=0.; a43(n,j)=0.;
  a44(1,j)=a1(1,j);
  a43(2,j)=bc6(2)*n2*vy(2)*a1(2,j);
  a44(2,j)=bc6(2)*(n1*a3(2,j)+n1/y(2)*a2(2,j)+j11(2)*a1(2,j));
  a43(n-1,j)=bc6(n-1)*n2*vy(n-1)*a1(n-1,j);
  a44(n-1,j)=bc6(n-1)*(n1*a3(n-1,j)+n1/y(n-1)*a2(n-1,j)+j11(n-1)*a1(n-1,j));
end;
a11(1,1)=-ax(1);
a14(1,n)=gam*h*rp*phs;
a12(n,n)=a12(n,n)-bc3(n)*prt*ax(n)/rx(n);
a23(2,n)=a23(2,n)+bc4(n)+prp*gam*h*phs/y(n);
a32(n,1)=-ax(1);
a33(n,n)=gam*h*rp*phs;
a42(n-1,1)=a42(n-1,1)-bc3(1)*prt*ax(1)/rx(1);
a43(2,2)=a43(2,2)+bc6(2)*n4/y(2)*vy(2);
a43(n-1,n-1)=a43(n-1,n-1)+bc6(n-1)*n4/y(n-1)*vy(n-1);
for i=1:n;
        a23(i,1)=0.;
        a33(i,1)=0.;
        a43(i,1)=0.;
end;
a33(1,1)=1.;
%-----------------------------------------------------------------
% enforce the stiffness matrix
%-----------------------------------------------------------------
sa(:,:)=zeros(4*n);dd=zeros(4*n,1);
for i=1:n
for j=1:n
  sa(i,j)=a11(i,j); sa(i,n+j)=a12(i,j);
```

```
    sa(i,2*n+j)=a13(i,j); sa(i,3*n+j)=a14(i,j);
    sa(n+i,j)=a21(i,j); sa(n+i,n+j)=a22(i,j);
    sa(n+i,2*n+j)=a23(i,j); sa(n+i,3*n+j)=a24(i,j);
    sa(2*n+i,j)=a31(i,j); sa(2*n+i,n+j)=a32(i,j);
    sa(2*n+i,2*n+j)=a33(i,j); sa(2*n+i,3*n+j)=a34(i,j);
    sa(3*n+i,j)=a41(i,j); sa(3*n+i,n+j)=a42(i,j);
    sa(3*n+i,2*n+j)=a43(i,j); sa(3*n+i,3*n+j)=a44(i,j);
 end;
 end
 %
 %-------------------------------------------------------------------
 % section solu - solve for displacement coeffts and find results
 % find the vector of displacement components rslt
 %-------------------------------------------------------------------
 %
 sdw=0.; sw=0.;
 dd=inv(sa)*rhs;
 for i=1:n
   du(i)=dd(i);
   dw(i)=dd(n+i);
   dt(i)=dd(2*n+i);
   dv(i)=dd(3*n+i);
 end;
 t=t+dt;
 u=u+du;
 v=v+dv;
 w=w+dw;
 ty=a1*t; tyy=a2*t; ux=a1*u;  uxx=a2*u;
 vy=a1*v; vyy=a2*v; v3y=a3*v; v4y=a4*v;
 wx=a1*w; wxx=a2*w; w3x=a3*w; w4x=a4*w;
 vs=as*v;
 for i=2:n-1
 sdw=sdw+dw(i)^2;
 sw=sw+w(i)^2;
 end;
 rsw=sdw/sw
 end;
 %%--------------------End of while loop;
 %plot(x,-w)
 x=x*h;
 y=y*rp;
 for i=1:n
 mx(i)=bc1(i)*wxx(i);
```

```
qx(i)=bc2(i)*(dx(i)*wxx(i)+d(i)*w3x(i));
my(i)=-bc5(i)*vyy(i);
end;
t=t*gam*h*rp*phs;
v=v*gam*h*rp*phs;
for i=1:n
  u(i)=u(i)*ax(i);
  w(i)=w(i)*ax(i);
  wp(i)=w(1)-w(i);
  vp(i)=-(v(n)-v(i));
end
mx=mx*gam*h*rp^2;
qx=qx*gam*h*rp;
my=my*gam*h*rp^2;
plot(x,mx);hold on;plot(y,my);hold off;
```