The Hyperbolic Formal Affine Demazure Algebra

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Abstract

In this thesis, we extend the construction of the formal (affine) Demazure algebra due to Hoffnung, Malagón-López, Savage and Zainoulline in two directions. First, we introduce and study the notion of formal Demazure lattices of a Kac-Moody root system and show that the definitions and properties of the formal (affine) Demazure operators and algebras hold for such lattices. Second, we show that for the hyperbolic formal group law the formal Demazure algebra is isomorphic (after extending the coefficients) to the Hecke algebra.

Résumé

Dans cette thèse, nous généralisons dans deux directions la construction de l'algèbre formelle (affine) de Demazure par Hoffnung, Malagón-López, Savage et Zainoulline. Premièrement, nous introduisons et étudions la notion de réseaux formels de Demazure d'un système de racines de Kac-Moody et montrons que les définitions et les propriétés des opérateurs et algèbres formels (affins) de Demazure sont valables pour ces réseaux. Deuxièmement, nous montrons que pour la loi de groupe formel hyperbolique, l'algèbre formelle de Demazure est isomorphe (après l'extension des coefficients) à l'algèbre de Hecke.

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Contents

Chapter 1

Introduction

Let G be a semisimple linear algebraic group defined over an algebraically closed field and B a Borel subgroup of G. The cohomology of flag varieties G/B is a widely studied subject in modern mathematics. In the early 1970s, Demazure [Dem1] and Bernstein-Gelfand-Gelfand [BGG] approached the subject by introducing operators Δ_w corresponding to the elements w of the Weyl group W of the finite root system Φ associated with G. Let Λ be a lattice lying between the root lattice Λ_r and the weight lattice Λ_ω of Φ and let $S(\Lambda)$ be the symmetric algebra of Λ. Demazure showed that there exists a basis of $h(G/B)$, where h is Chow groups or Grothendieck groups, for which there is a correspondence between the Demazure operators Δ_w and the coefficients of the image of the characteristic map $c : S(\Lambda) \to h(G/B)$ with respect to that basis. Independently, Bernstein-Gelfand-Gelfand proved a correspondence between Δ_w and the classes of Schubert cells of $H(G/B)$ in the case of singular cohomology theory.

Later, Kostant and Kumar [KK1], [KK2] generalized the work from [BGG] for Kac-Moody groups G/P and equivariant cohomology theories. Let Q be the quotient field of the symmetric algebra of Λ_r. They defined an algebra Q_W, called the twisted group algebra, which is the smash product of the group algebra $\mathbb{Z}[W]$ and Q. Then they introduced a subalgebra D of Q_W, with generators satisfying the braid relations of a Hecke algebra and an idempotence relation.

More recently, in a series of papers [HMSZ], [CZZ1], [CZZ2], [CZZ3], Calmès, Hoffnung, Malagón-López, Savage, Zainoulline and Zhong generalized the Kostant and Kumar approach to oriented cohomology theories in the sense of Levine-Morel [LM]. Namely, they constructed an algebra $R[\![\Lambda]\!]_F$, called the formal group algebra, which depends on a commutative ring R, a lattice Λ lying between the root lattice and weight lattice of a finite root system Φ, and a formal group law F (see §2.1 for a review of formal group laws). By specialization to the additive formal group law, the formal group algebra is isomorphic to the symmetric algebra $S(\Lambda)$, which corresponds to the setting of Demazure. Then they defined a generalized version of the operators in [Dem1] and [BGG] acting on $R[\![\Lambda]\!]_F$, the formal Demazure/push-pull operators. Let W be the Weyl group of Φ and let Q^F be the quotient field of $R[\![\Lambda]\!]_F$. Analogous to the twisted group algebra Q_W of [KK1], they introduced the so-called twisted formal group algebra Q_W^F, which is the smash product of the group algebra $R[W]$ and Q^F. For the additive (resp. multiplicative) formal group law, they proved that the subring D_F of Q_W^F generated by the Demazure/push-pull elements is isomorphic to the 0-Hecke algebra (resp. nil-Hecke algebra). Finally, the subring $\mathbf{D}_F$ of Q_W^F generated by elements of D_F and $R[\![\Lambda]\!]_F$ is isomorphic to the affine 0-Hecke or affine nil-Hecke algebra, for the additive or multiplicative formal group law respectively. For related results in the topological context we refer to the papers [BE], [Co], [GR], [HHH] by Bressler, Cooper, Evans, Ganter, Harada, Henriques, Holm, and Ram.

In this thesis, we extend the construction of Calmès et al. to an arbitrary Kac-Moody root system and the hyperbolic formal group law

$$F_{\mu_1,\mu_2}(u,v) = \tfrac{u+v-\mu_1 uv}{1+\mu_2 uv}, \quad \mu_1, \mu_2 \in R.$$

The hyperbolic formal group law is a natural choice since both the additive (corresponding to usual cohomology) and the multiplicative (corresponding to K-theory) formal group laws can be obtained from it by specialization. It has been actively studied in the context of elliptic formal group laws by Buchstaber-Bunkova [BB1], [BB2] and has a rich topological background as it corresponds to the celebrated 2-parameter Todd genus introduced and studied by Hirzebruch in [Hirz]. Recently in [LZ1] and [LZ2], it was used to generalize the root polynomial approach of Billey-Graham-Willems to Schubert calculus.

The first step in our work is to define the analogue of the Demazure/BGG-operators in our setting. Since these operators are divided-difference operators, we need to determine which elements of $R[\![\Lambda]\!]_F$ are regular, in other words, not zero divisors. Observe that the elements of $R[\![\Lambda]\!]_F$ are power series in variables x_λ indexed by the elements of the lattice Λ. The action of the Weyl group on $R[\![\Lambda]\!]_F$ is given by $w(x_\lambda) = x_{w(\lambda)}$. Then we need to carefully choose the lattice Λ in order to define the formal Demazure operators. We want to have a lattice Λ lying between the root lattice Λ_r and the weight lattice Λ_ω of a Kac-Moody root system Φ such that, for a simple root α, the variable x_α is regular. With this in mind, we define a formal Demazure lattice Λ as a finitely generated subgroup of $\mathfrak{h}^*$, the dual of the Cartan subalgebra of the Kac-Moody algebra associated with Φ, such that every simple root can be extended to a basis of Λ. These lattices have some interesting properties. For example, they are stable under the action of the Weyl group and every real root can be extended to a basis of Λ. Therefore, we get that for any real root α in Φ, the variable x_α is regular in $R[\![\Lambda]\!]_F$ (see Lemma 5.1.3). This lemma combined with the fact that $u - s_\alpha(u)$ is divisible by x_α for any element $u \in R[\![\Lambda]\!]_F$ and any real root α allows us to define the formal Demazure operator Δ_α as

$$\Delta_\alpha(u) = \frac{u - s_\alpha(u)}{x_\alpha}.$$

Similarly, we define the formal push-pull operator and prove that both types of operators satisfy the same conditions as in the case of a finite root system.

Let Φ^{re} be the set of real roots of the Kac-Moody root system Φ. Since the elements x_α are regular for any real root α, we take Q^F to be the localization of $R[\![\Lambda]\!]_F$ at the multiplicative subset generated by these x_α's. Then we define the twisted formal group algebra Q^F_W as the smash product of $R[W]$ and Q^F_W. For any $w \in W$, we denote the elements of $R[W]$ corresponding to w by δ_w. For $\alpha \in \Phi^{re}$, we define the formal Demazure elements and formal push-pull element as

$$X_\alpha = \frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_\alpha \quad \text{and} \quad Y_\alpha = \frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_\alpha,$$

respectively. We define the formal Demazure algebra D_F as the R-subalgebra of Q^F_W generated by the formal Demazure elements X_α for all $\alpha \in \Phi^{re}$. The formal affine Demazure algebra $\mathbf{D}_F$ is the R-subalgebra of Q^F_W generated by the elements of the formal group algebra $R[\![\Lambda]\!]_F$ and by D_F. We proceed by checking that the properties of the formal elements and the formal (affine) Demazure algebra proven for finite root systems also hold in our setting. Roughly, the properties hold because the Weyl group of a Kac-Moody root system Φ is a crystallographic Coxeter group. Therefore, we get the following result (see Theorem 6.3.2).

Theorem. *Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots built into the construction of Φ. Let m_{ij} be the order of $s_{\alpha_i}s_{\alpha_j}$ for $i, j \in I, i \neq j$. The formal affine Demazure algebra $\mathbf{D}_F$ is generated as an R-algebra by $R[\![\Lambda]\!]_F$ and the formal Demazure elements X_{α_i}, and satisfies the following relations:*

(i) $\psi X_\alpha = X_\alpha s_\alpha(\psi) + \Delta_\alpha(\psi)$ *for all* $i \in I$ *and* $\psi \in R[\![\Lambda]\!]_F$ *(affine relation);*

(ii) $X_i^2 = X_i \kappa_i$ *for all* $i \in I$*;*

(iii) $X_i X_j = X_j X_i$ *for all* $i, j \in I$ *such that* $m_{ij} = 2$*;*

(iv) the braid relations of Proposition 6.2.7 for all $i, j \in I$ *such that* $m_{ij} = 3, 4, 6$*;*

These relations form a complete set of relations for $\mathbf{D}_F$.

We point out that there are no relations between X_i and X_j for $m_{ij} = \infty$. In fact, these relations also form a complete set of relations for D_F if we remove relation (i), the affine relation. Note that the formal (affine) Demazure algebra can also be generated by the formal push-pull elements and we obtain similar results for that set of generators (see Theorem 6.3.4).

Finally, for a ring R containing $\mathbb{Z}[t, t^{-1}]$, the Hecke algebra H associated with W is the R-algebra generated by elements $T_{s_{\alpha_i}}$ satisfying a quadratic relation and some braid relations. In the case where our formal group law is the hyperbolic formal group law with $\mu_1 = u(t + t^{-1})$ and $\mu_2 = -u^2$, for some $u \in R$, we get the following result (see Theorem 7.2.1).

Theorem. *The assignment $X_i \mapsto u(T_i + t)$ defines a morphism of R-algebras from the formal Demazure algebra D_F to the Hecke algebra H over R. If $u \in R^\times$, this morphism is an isomorphism.*

We also get similar results for the formal affine Demazure algebra $\mathbf{D}_F$ (see Corollaries 7.3.1, 7.3.2, and 7.3.4). However, we see that in that case the isomorphism depends on the choice of formal Demazure lattice Λ because of the additional affine relation.

The thesis is organized as follows. In Chapter 2 we present facts about formal group laws and introduce our main example, the hyperbolic formal group law. Morever, we recall basic properties of Kac-Moody root systems and prove some results about Coxeter groups necessary for the following chapters. In Chapter 3 we review the construction of the formal Demazure algebra in detail. We define the formal Demazure lattice in Chapter 4. We study the properties of these lattices and give several examples. Also, we compare the formal Demazure lattices to other similar lattices that have been introduced in the literature (see [Ku] and [MP]). In Chapters 5 and 6, we show that the definitions and some properties of the formal (affine) Demazure operators and algebras hold for the formal Demazure lattices, hence generalizing several results (by Calmès et al.). In Chapter 7, we prove that, for the hyperbolic formal group law and after extending the coefficients, the formal Demazure algebra is isomorphic to the Hecke algebra associated with the Weyl group of a Kac-Moody root system. The results in the last chapter are either new results or have been published recently in [L]. Finally, some of the computations in Chapter 6 were computer-aided; the Maple scripts can be found in the appendixes.

Chapter 2

Prerequesites

2.1 Formal group laws

Let R be a commutative unital ring. Let $R[\![x_1, ..., x_n]\!]$ be the ring of formal power series in the indeterminates $x_1, ..., x_n$.

Definition 2.1.1. [Haz, p. 1] A *one-dimensional commutative formal group law over a ring* R is a power series $F(u,v) \in R[\![u,v]\!]$ such that

$$\begin{aligned} F(u,0) &= u, \\ F(u,v) &= F(v,u), \\ F(F(u,v),w) &= F(u,F(v,w)). \end{aligned}$$

Remarks 2.1.2. (a) Let $F(u,v) = \sum_{i,j=0} a_{ij}u^i v^j \in R[\![u,v]\!]$ be a formal group law. Since $u = F(u,0) = a_{0,0} + a_{1,0}u + a_{2,0}u^2 + ...$, we have $a_{00} = 0 = a_{i,0}$ for $i > 1$. Similarly, $a_{0,j} = 0$ for all $j > 1$. Therefore, any formal group law is of the form

$$F(u,v) = u + v + \sum_{i,j \geq 1} a_{ij}u^i v^j. \tag{2.1.1}$$

(b) The expression $F(F(u,v),w)$ means that in $F(u,v)$ we replace u by $F(u,v)$ and v by w, and then expand to get an element in $R[\![u,v,w]\!]$. By (a), the substitution makes sense since there are no constant terms in $F(u,v)$.

(c) In general, a commutative formal group law of dimension n consists of n powers series in $2n$ variables satisfying the conditions above.

(d) We frequently use a different notation and write $u +_F v := F(u,v)$. Then we can express the conditions above as follows:

$$(u +_F v) +_F w = u +_F (v +_F w),$$

$$u +_F v = v +_F u,$$
$$u +_F 0 = u.$$

We will see in Corollary 2.1.7 that u is invertible with respect to $+_F$. Hence this justifies the terminology "formal group law".

Our central example is the following (see [BB1, Example 63], [BB2, Corollary 3.8])

Example 2.1.3. Let $\mu_1, \mu_2 \in R$. The *hyperbolic 2-parameter formal group law* is defined as

$$F_{\mu_1,\mu_2}(u,v) = \tfrac{u+v-\mu_1 uv}{1+\mu_2 uv} = (u+v-\mu_1 uv)\Big(\sum_{i\geq 0}(-\mu_2 uv)^i\Big).$$

Let R be a $\mathbb{Q}$-algebra. The *exponential* of a formal group law $F(u,v)$ over R is a function $f : R[\![u,v]\!] \to R[\![u,v]\!]$ such that $f(u+v) = F(f(u), f(v))$. We will see in Remark 2.1.11 that the exponential, if it exists, is unique. By [BB1, p. 35], the exponential of the hyperbolic formal group law is

$$\exp_{F_{\mu_1,\mu_2}}(u) = \tfrac{e^{\alpha u}-e^{\beta u}}{\alpha e^{\alpha u}-\beta e^{\beta u}}$$

where $\mu_1 = \alpha + \beta$, $\mu_2 = -\alpha\beta$, hence the name hyperbolic formal group law.

Examples 2.1.4. Here are some examples of one-dimensional commutative formal group laws:

(a) the *additive formal group law* $F_a(u,v) = u + v$,
(b) the *multiplicative (periodic) formal group law* $F_m(u,v) = u + v - \mu uv$, where $\mu \in R^\times$,
(c) the *Lorentz formal group law*, $F_L(u,v) = \frac{u+v}{1+\mu uv}$, where $\mu \in R^\times$.

We can easily see that the hyperbolic formal group law is a generalization of the 3 previous examples. If $\mu_1 = \mu_2 = 0$, we get the additive formal group law. If $\mu_2 = 0, \mu_1 = \mu$, we get the multiplicative (periodic) formal group law and if $\mu_1 = 0$ and $\mu_2 = \mu$, we get the Lorentz formal group law.

Definition 2.1.5. Let $F(u,v) \in R[\![u,v]\!]$ be a formal group law. The *inverse* of F is a power series $G(t) \in R[\![t]\!]$ such that $u +_F G(u) = 0$. We will see below in Corollary 2.1.7 that the inverse exists and is unique. We will therefore put $-_F u := G(u)$.

Proposition 2.1.6. (c.f. [Bo:A, IV, §4.7, Corollaire]) *Let $f(x) \in R[\![x]\!]$ be a formal power series without constant term. If the constant term of the formal power series $f'(x)$ is invertible in R, then there exist a unique power series $g(x) \in R[\![x]\!]$ such that $f(g(x)) = x = g(f(x))$. In other words, if $f(x) = \sum_{i=1}^{\infty} a_i x^i$, $a_0 = 0$ and $a_1 \in R^\times$ then there exist such a $g(x)$.*

Corollary 2.1.7. [Haz, Lemma 1.1.4] *A formal group law has a unique inverse.*

Proof: First, let us proof the existence of the inverse. Let $F(u,v) \in R[\![u,v]\!]$ be a formal group law. Let $H(u,v) = u - F(u,v)$. As a power series in the indeterminate v, the power series H has no constant term and $(\frac{\partial H(u,v)}{\partial v})_{v=0} = 1$. Then by Proposition 2.1.6 there exist $G(u,v)$ such that $H(u, G(u,v)) = v$. Therefore $F(u, G(u,v)) = u - v$ and $F(u, G(u,u)) = 0$.

Now, we can prove uniqueness. Let $G(u), G'(u) \in R[\![u]\!]$ such that $u +_F G(u) = u +_F G'(u) = 0$. Then,

$$G(u) = G(u) +_F 0 = G(u) +_F u +_F G'(u) = 0 +_F G'(u) = G'(u).$$

∎

Example 2.1.8. Let $F_{\mu_1,\mu_2}(u,v) = \frac{u+v-\mu_1 uv}{1+\mu_2 uv}$ be the hyperbolic formal group law. Then the inverse of $F_{\mu_1,\mu_2}(u,v)$ is the formal power series $-_F u = G(u) \in R[\![u]\!]$ such that

$$F_{\mu_1,\mu_2}(u, G(u)) = \frac{u + G(u) - \mu_1 u G(u)}{1 + \mu_2 u G(u)} = 0.$$

Therefore, we have

$$G(u) = -\frac{u}{1-\mu_1} = -u \sum_{n \geq 0} (-\mu_1 u)^n = -\sum_{n \geq 0} (-\mu_1)^n u^{n+1}.$$

Hence, we also have $-_F u = -\sum_{n\geq 0} (-\mu_1)^n u^{n+1}$ for the multiplicative (periodic) formal group law. For the Lorentz formal group law and the additive formal group law, we get $-_F u = -u$ since $\mu_1 = 0$.

Remark 2.1.9. Given an integer $m > 0$, by associativity of the formal group law F we can use the notation

$$m \cdot_F x = x +_F \cdots +_F x \text{ (m factors) and } (-m) \cdot_F x = -_F(m \cdot_F x).$$

Also, for any $m, n \in \mathbb{Z}$ we have

$$(m+n) \cdot_F x = (m \cdot_F x) +_F (n \cdot_F x).$$

Definition 2.1.10. [Haz, p. 3] Let $F(u,v), F'(u,v) \in R[\![u,v]\!]$ be two formal group laws. A *homomorphism of formal group laws* $f : F(u,v) \to F'(u,v)$ is a power series $f \in R[\![x]\!]$ without constant term such that

$$f\big(F(u,v)\big) = F'\big(f(u), f(v)\big)$$

or equivalently

$$f(u +_F v) = f(u) +_{F'} f(v).$$

A homomorphism $f : F(u,v) \to F'(u,v)$ is *invertible* if there exist $g : F'(u,v) \to F(u,v)$ such that $f\big(g(x)\big) = x = g\big(f(x)\big)$.

Claim: For $f(x) = a_1x + a_2x^2 + \cdots$ with $a_i \in R$, we claim that such a g exists if and only if $a_1 \in R^\times$. In this case it is an homomorphism of formal group laws.

Assume g exists, say $g(x) = b_0 + b_1x + b_2x^2 + \cdots$ for $b_i \in R$ such that $f\big(g(x)\big) = x = g\big(f(x)\big)$. Since

$$\begin{aligned} g\big(f(x)\big) &= g(a_1x + a_2x^2 + \cdots) \\ &= b_0 + b_1(a_1x + a_2x^2 + \cdots) + b_2(a_1x + a_2x^2 + \cdots)^2 + \cdots \\ &= b_0 + b_1a_1x + \text{ terms of degree} \geq 2 \ = x, \end{aligned}$$

we have $b_0 = 0$ and b_1 is invertible with inverse a_1. Then the power series $g : F'(u,v) \to F(u,v)$ is a homomorphism of formal group laws since

$$f\Big(F\big(g(u), g(v)\big)\Big) = F'\Big(f\big(g(u)\big), f\big(g(v)\big)\Big) = F'(u,v)$$

and by acting on both sides by g we get

$$F\big(g(u), g(v)\big) = (g \circ f)\Big(F\big(g(u), g(v)\big)\Big) = g\big(F'(u,v)\big).$$

Conversely by Proposition 2.1.6, if $a_1 \in R^\times$ then f is invertible. ∎

An invertible homomorphism of formal group law f is an *isomorphism*. An isomorphism of formal group law is called *strict* if $a_1 = 1$.

Remark 2.1.11. [BB1, p. 3] [Haz, p. 6] The exponential of a formal group law $F(u,v) \in R[\![u,v]\!]$ is the unique strict isomorphism $\exp : F_a(u,v) \to F(u,v)$ over the ring $R \otimes_{\mathbb{Z}} \mathbb{Q}$. In general, the exponential need not exist if $R \to R \otimes_{\mathbb{Z}} \mathbb{Q}$ is not injective, i.e., R has no additive torsions [Haz, p. 30].

Example 2.1.12. [Haz, Example 1.4.2] Let $R = \mathbb{Q}$. Let F_a and F_m be the additive and multiplicative formal group law of Example 2.1.4. Then $\exp(x) : F_a \to F_m$ and $\log(1+x) : F_m \to F_a$ defined as

$$\exp(x) = \sum_{n \geq 0} \frac{x^n}{n!}, \quad \log(1+x) = \sum_{n \geq 0} (-1)^{n+1} \frac{x^n}{n!}$$

are mutually inverse strict isomorphisms over $\mathbb{Q}$. This example justifies the names exponential and logarithm.

Theorem 2.1.13. [Haz, p. 6] *Every one dimensional commutative formal group law over a $\mathbb{Q}$-algebra R is strictly isomorphic over R to the additive formal group law over R.*

2.2 Coxeter groups

In this section, we review and prove some results about Coxeter groups. Most of these results can be found in [BjBr], [Deo1], [Deo2], [Hu2], or [S].

Definition 2.2.1. [Hu2, p. 105] We define a *Coxeter system* to be a pair (W, S) consisting of a group W and a set of generators $S \subseteq W$, subject only to relations of the form

$$(ss')^{m(s,s')} = 1,$$

where $m(s,s) = 1, m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in S. In case no relation occurs for a pair s, s', we make the convention that $m(s,s') = \infty$. Moreover, we may refer to W itself as a *Coxeter group*, when S and the presentation are understood.

Definition 2.2.2. [Hu2, p. 107-108] Since every generator has order 2 in W, every $w \in W$ can be written in the form $w = s_1 s_2 \cdots s_r$ for some $s_i \in S$ and $r \in \mathbb{N}$ ($r = 0$ implies $w = 1$). If r is as small as possible, we call r the *length* of w, written $l(w)$, and call any expression of w as a product of r elements of S a *reduced expression*. By convention, $l(1) = 0$.

Here are some elementary properties of the length function.

Lemma 2.2.3. [Hu2, p. 108] *Let $s \in S$ and $w \in W$. We have*

(L1) $l(w) = l(w^{-1})$,

(L2) $l(w) = 1$ *if and only if* $w \in S$,

(L3) $l(ww') \leq l(w) + l(w')$,

(L4) $l(ww') \geq l(w) - l(w')$,

(L5) $l(w) - 1 \leq l(ws) \leq l(w) + 1$.

Definition 2.2.4. [Hu2, p. 109] Let V be a vector space over $\mathbb{R}$, having a basis $\{\alpha_s | s \in S\}$ in one-to-one correspondence with S. We define a symmetric bilinear form B on V by

$$B(\alpha_s, \alpha_{s'}) = -\cos\left(\frac{\pi}{m(s,s')}\right).$$

By convention, $B(\alpha_s, \alpha_{s'}) = -1$ if $m(s,s') = \infty$. We have $B(\alpha_s, \alpha_s) = 1$ and $B(\alpha_s, \alpha_{s'}) \leq 0$ for $s \neq s'$. For each $s \in S$ we define a *reflection* $\sigma_s : V \to V$ by

$$\sigma_s(\lambda) = \lambda - 2B(\alpha_s, \lambda)\alpha_s.$$

One easily checks that σ_s is a reflection in α_s, i.e., $(\sigma_s)^2 = \mathrm{id}, \sigma_s(\alpha_s) = -\alpha_s$ and the fixed point set of σ_s has codimension 1.

Proposition 2.2.5. [Hu2, p. 110-113] *There is a unique homomorphism $\sigma : W \to \mathrm{GL}(V)$ sending s to σ_s. The group $\sigma(W)$ preserves the form B on V. Moreover, for each pair $s, s' \in S$, the order of ss' in W is precisely $m(s,s')$. The representation $\sigma : W \to \mathrm{GL}(V)$ is faithful.*

To simplify the notation we will write $w(\alpha_s)$ instead of $\sigma(w)(\alpha_s)$.

Definition 2.2.6. [Hu2, p. 111] We define the *root system* Φ of W to be the collection of vectors $w(\alpha_s)$, where $w \in W$ and $s \in S$. These are unit vectors because W preserves the form B on V. Note that $\Phi = -\Phi$ since $s(\alpha_s) = -\alpha_s$. The elements of Φ are called *roots*. Let $\Pi = \{\alpha_s | s \in S\}$ be the set of *simple roots* α_s. If α is any root, we can write it uniquely in the form

$$\alpha = \sum_{s \in S} c_s \alpha_s \quad (c_s \in \mathbb{R}).$$

Call a root α *positive* (resp. *negative*) and write $\alpha > 0$ (resp. $\alpha < 0$) if all $c_s \geq 0$ (resp. all $c_s \leq 0$). Write Φ^+ and Φ^- for the respective set of positive and negative roots. It follows from the next theorem that $\Phi = \Phi^+ \cup \Phi^-$.

For α represented as above, we call $\sum_{s \in S} c_s$ the *height* of α, abbreviated $\mathrm{ht}(\alpha)$. By definition, we have

$$\begin{aligned}
&\mathrm{ht}(\alpha) = 1 \text{ if } \alpha \in \Pi,\\
&\mathrm{ht}(\alpha) = -1 \text{ if } \alpha \in -\Pi,\\
&\mathrm{ht}(\alpha) > 1 \text{ if } \alpha \in \Phi^+ \setminus \Pi,\\
&\mathrm{ht}(\alpha) < -1 \text{ if } \alpha \in \Phi^- \setminus -\Pi.
\end{aligned}$$

Theorem 2.2.7. [Hu2, p. 111] *Let $w \in W$ and $s \in S$. If $l(ws) > l(w)$, then $w(\alpha_s) > 0$. If $l(ws) < l(w)$, then $w(\alpha_s) < 0$.*

Proposition 2.2.8. [S, 1.7-1.8] (a) *If $s \in S$, then s sends α_s to its negative, but permutes the remaining positive roots.*
(b) *For any $w \in W$, $l(w)$ equals the number of positive roots sent by w to negative roots.*

Lemma 2.2.9. [S, Proof of 1.8] *Let (W, S) be a Coxeter system. Given a reduced expression $w = s_1 \cdots s_r$ $(s_i \in S)$, set $\alpha_i = \alpha_{s_i}$ and $\beta_i := s_r s_{r-1} \cdots s_{i+1}(\alpha_i)$, interpreting β_r to be α_r. Then, the set of positive roots sent to negative roots by w consists of the r positive roots $\beta_1, ..., \beta_r$. In particular, the β_i's are distinct.*

By Lemma 2.2.9, we get the following corollary.

Corollary 2.2.10. *The set of negative roots sent to positive roots by w consists of the r distinct negative roots $\beta_i := s_r s_{r-1} \cdots s_{i+1}(-\alpha_i)$, where $\beta_r = -\alpha_r$ and $w = s_1 \cdots s_r$ is a reduced expression.*

Corollary 2.2.11. *Let $\beta = \alpha_1$ and let $v = s_{\alpha_1} \cdots s_{\alpha_l} = s_1 \cdots s_l = s_\beta s_2 \cdots s_l$ be a reduced expression, i.e. $l(v) = l$. Let $v' = s_\beta v$. Then $l(v') = l(v) - 1 = l - 1$ and we have*

$$\{\beta\} \cup s_\beta\big(v'(\Phi^-) \cap \Phi^+\big) = v(\Phi^-) \cap \Phi^+.$$

Proof: It is clear that $l(v') = l(v) - 1$ since $v' = s_\beta s_\beta s_2 \cdots s_l = s_2 \cdots s_l$ and this is a reduced expression because $s_1 \cdots s_l$ is a reduced expression.

By Corollary 2.2.10, the set of negative roots sent to positive roots by v consists of the l distinct negative roots $\beta_i := s_l s_{l-1} \cdots s_{i+1}(-\alpha_i)$, where $\beta_l = -\alpha_l$. Then

$$\begin{aligned} v(\Phi^-) \cap \Phi^+ &= \{v(\beta_i) | 1 \le i \le l\} \\ &= \{s_\beta \cdots s_l(s_l s_{l-1} \cdots s_{i+1}(-\alpha_i)) | 1 \le i \le l\} \\ &= \{s_\beta s_2 \cdots s_{i-1}(\alpha_i) | 1 \le i \le l\} \\ &= \{\beta\} \cup \{s_\beta s_2 \cdots s_{i-1}(\alpha_i) | 2 \le i \le l\} \\ &= \{\beta\} \cup s_\beta(\{s_2 \cdots s_{i-1}(\alpha_i) | 2 \le i \le l\}). \end{aligned}$$

Similarly, we have

$$v'(\Phi^-) \cap \Phi^+ = \{s_2 \cdots s_{i-1}(\alpha_i) | 2 \le i \le l\}.$$

Therefore, we have

$$v(\Phi^-) \cap \Phi^+ = \{\beta\} \cup s_\beta\big(v'(\Phi^-) \cap \Phi^+\big).$$

∎

Proposition 2.2.12. [S, 1.12] *Let W be finite. Then the following are equivalent for $w_0 \in W$:*

(1) w_0 has maximal length,

(2) w_0 maps Φ^+ to Φ^-,

(3) w_0 maps simple roots to negative simple roots.

Such an element exists, and it is unique.

Proof: The equivalence between (1) and (2), existence of w_0, and uniqueness is proven in [S]. It remains to prove that (2) and (3) are equivalent. Assume that w_0 sends positive roots to negative roots. Let α be a non-simple positive root. Then $\alpha = \sum_{\alpha_i \in \Pi} c_i \alpha_i$ where $c_i > 0$. We have $\operatorname{ht}(\alpha) > 1$, hence

$$\begin{aligned} \operatorname{ht}\big(w_0(\alpha)\big) = \operatorname{ht}\big(\sum c_i w_0(\alpha_i)\big) &= \sum c_i \operatorname{ht}\big(w_0(\alpha_i)\big) \\ &\leq -\sum c_i \text{ since } \operatorname{ht}\big(w_0(\alpha_i)\big) \leq -1 \\ &= -\operatorname{ht}(\alpha) < -1 \end{aligned}$$

Therefore, $w_0(\alpha)$ is a negative non-simple root. Thus, w_0 sends simple roots to negative simple roots. The converse is clear. ∎

Lemma 2.2.13. *Let (W, S) be a Coxeter system with $|S| = 2$, say $S = \{s_i, s_j\}$, and let $m_{ij} < \infty$. The longest element w_0 satisfies $w_0(\alpha_p) = -\alpha_{p'}$ for $p \in \{i, j\}$ where*

$$\begin{aligned} p = p' &\text{ if } m_{ij} \text{ even}, \\ i' = j, j' = i &\text{ if } m_{ij} \text{ odd}. \end{aligned}$$

Proof: Consider $V = \mathbb{R}\alpha_i \oplus \mathbb{R}\alpha_j$. As seen in [Hu2, p. 109], the form B is nondegenerate positive semidefinite, hence positive definite on V when $m_{ij} < \infty$ since for any $\lambda = a\alpha_i + b\alpha_j (a, b \in \mathbb{R})$,

$$B(\lambda, \lambda) = a^2 - 2ab\cos(\pi/m_{ij}) + b^2 = (a - b\cos(\pi/m_{ij}))^2 + b^2 \sin^2(\pi/m_{ij}) \geq 0.$$

Then we can look at $s_i s_j$ as an operator on V of order m_{ij}. We may view V as an Euclidian plane and s_i, s_j as orthogonal reflections.

For m_{ij} even, $w_0 = (s_is_j)^{\frac{m_{ij}}{2}} = (s_js_i)^{\frac{m_{ij}}{2}}$ and we know that s_is_j is a rotation by $\frac{2\pi}{m_{ij}}$, hence w_0 is a rotation by π, i.e. $w_0 = -\operatorname{id}$.

For m_{ij} odd, m_{ij} is a reflection sending simple roots to negative simple roots by Proposition 2.2.12. Since $l(w_0) = m_{ij} > 1$ and s_i is the only reflection sending α_i to $-\alpha_i$, w_0 must send α_i to α_j. Similarly, $w_0(\alpha_j) = -\alpha_i$. ∎

Let T denote the set of all reflections s_α, $\alpha \in \Phi$.

Theorem 2.2.14. [Hu2, p. 117] *(Strong Exchange Condition) Let $w = s_1 \cdots s_r (s_i \in S)$, not necessarily a reduced expression. Suppose a reflection $t \in T$ satisfies $l(wt) < l(w)$. Then there is an index i for which $wt = s_1 \cdots \hat{s}_i \cdots s_r$ (omitting s_i). If the expression for w is reduced, then i is unique.*

Definition 2.2.15. [Hu2, p. 118] Let $w, w' \in W$. Write $w' \to w$ if $w = w't$ for some reflection $t \in W$ with $l(w) > l(w')$. Then define $w' < w$ if there is a sequence $w' = w_0 \to w_1 \to \cdots \to w_m = w$ where $m > 0$. We write $w' \leq w$ if $w' < w$ or $w' = w$. The resulting relation $w' \leq w$ is a partial ordering of W with 1 as the unique minimal element. We call it the *Bruhat ordering.*

Lemma 2.2.16. [BjBr, Corollary 2.2.5] *Let $v, w \in W$. Then $w \leq v$ if and only if $w^{-1} \leq v^{-1}$.*

Proof: We include the proof of this lemma to recall a fact about root systems for later (see proof of Proposition 5.2.1). Assume $w \leq v$ so there exist a sequence $w = w_0 \to w_1 \to \cdots \to w_m = v$ where $w_{i+1} = w_it_{i+1}$ for some reflection $t_{i+1} \in W$. Then, $v = wt_1 \cdots t_m$ so $v^{-1} = t_m \cdots t_1 w^{-1}$. Let $t_i = s_{\beta_i}$ for some root $\beta_i \in \Phi$ and let $\gamma_i = w(\beta_i)$. For any $\delta \in \Phi$, we have

$$\begin{aligned} ws_{\beta_i}w^{-1}(\delta) &= w\big(s_{\beta_i}\big(w^{-1}(\delta)\big)\big) \\ &= w\big(w^{-1}(\delta) - 2B\big(w^{-1}(\delta), \beta_i\big)\beta_i\big) \\ &= w\big(w^{-1}(\delta) - 2B\big(\delta, w(\beta_i)\big)\beta_i\big) \\ &= \delta - 2B\big(\delta, w(\beta_i)\big)w(\beta_i) \\ &= s_{w(\beta_i)}(\delta) = s_{\gamma_i}(\delta). \end{aligned}$$

Therefore we get $s_{\beta_i}w^{-1} = w^{-1}s_{\gamma_i}$ for all i and

$$v^{-1} = s_{\beta_m} \cdots s_{\beta_1} w^{-1} = w^{-1} s_{\gamma_1} \cdots s_{\gamma_m}.$$

Thus, $v^{-1} \geq w^{-1}$. Similarly, we get the implication in the other direction by replacing v and w by v^{-1} and w^{-1}. ∎

We get the following corollary by replacing w by w^{-1}.

Corollary 2.2.17. *Let $v, w \in W$. Then $w^{-1} \leq v$ if and only if $w \leq v^{-1}$.*

Lemma 2.2.18. (cf. [Deo1, Thm 1.1, III, (ii)]) *Let $\beta = \alpha_1$ and let $v = s_{\alpha_1} \cdots s_{\alpha_l} = s_1 \cdots s_l = s_\beta s_2 \cdots s_l$ be a reduced expression. Let $v' = s_\beta v$. Then $w \leq v'$ implies $w \leq v$ and $s_\beta w \leq v$.*

Proof: We have $w \leq v'$ and $v' \leq v$ so $w \leq v$ by transitivity of the Bruhat ordering. To prove that $s_\beta w \leq v$ we have to look at two different cases as in the proof of [Hu2, Prop. 5.9]. Either $l(s_\beta w) = l(w) - 1$ or $l(s_\beta w) = l(w) + 1$.

If $l(s_\beta w) = l(w) - 1$, then $s_\beta w \leq w \leq v' \leq v$.

If $l(s_\beta w) = l(w) + 1$, we claim that $s_\beta w \leq s_\beta v' = v$. Without loss of generality, we can assume that $v' = tw$ for some reflection $t \in W$. Let $t' = s_\beta t s_\beta$. Then we have $t'(s_\beta w) = s_\beta v'$ so it is enough to show by the definition of Bruhat ordering that $l(s_\beta w) < l(s_\beta v')$ to prove $s_\beta w \leq s_\beta v'$. We will argue by contradiction. Suppose $l(s_\beta v') < l(s_\beta w)$. For any reduced expression $w = s_{i_1} \cdots s_{i_m}$, $s_\beta w = s_\beta s_{i_1} \cdots s_{i_m}$ is also reduced since $l(s_\beta w) = l(w) + 1 > l(w)$ by assumption. Then by the strong exchange condition 2.2.14, $s_\beta v' = t'(s_\beta w)$ is obtained from $s_\beta w$ by removing one factor in the reduced expression $s_\beta w = s_\beta s_{i_1} \cdots s_{i_m}$. This factor cannot be s_β since $s_\beta \neq t$. Therefore, $s_\beta v' = s_\beta s_{i_1} \cdots \hat{s_{i_j}} \cdots s_{i_m}$ for some i_j, or $v' = s_{i_1} \cdots \hat{s_{i_j}} \cdots s_{i_m}$ contradicting $l(v') > w$. ∎

2.3 Kac-Moody root systems

In this section, we will follow [Kac], [Ku], [MP], and [R]. We assume the reader is familiar with the basic concepts of Lie algebras. For an introduction to Lie algebras, we refer to [Hu].

Definition 2.3.1. Let $l \in \mathbb{N}$. A *realization* of a $l \times l$ matrix A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where $\mathfrak{h}$ is a vector space over a base field k ($\mathrm{char}(k) = 0$) of dimension $2l - \mathrm{rk}(A)$, $\Pi = \{\alpha_1, \ldots, \alpha_l\} \subseteq \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_l^\vee\} \subseteq \mathfrak{h}$ are linearly independent sets satisfying

$$\alpha_j(\alpha_i^\vee) = a_{ij}.$$

Two realizations $(\mathfrak{h}, \Pi, \Pi^\vee)$ and $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ are called *isomorphic* if there exists a vector space isomorphism $\phi : \mathfrak{h} \to \mathfrak{h}_1$ such that $\phi(\Pi^\vee) = \Phi_1^\vee$ and $\phi^*(\Pi_1) = \Pi$.

Proposition 2.3.2. [Kac, Prop. 1.1] *There exists a unique up to isomorphism realization for every $n \times n$ matrix A.*

Definition 2.3.3. Let $l \in \mathbb{N}$. A *generalized Cartan matrix* of size l is an $l \times l$ square matrix $A = (a_{ij})_{1 \le i,j \le l}$ with $a_{ij} \in \mathbb{Z}$ satisfying

(a) $a_{ii} = 2$,

(b) $a_{ij} \le 0$ for $i \neq j$,

(c) $a_{ij} = 0$ implies $a_{ji} = 0$.

A matrix A is called *decomposable* if after reordering of the indices it decomposes into a nontrivial direct sum, i.e., a block matrix with nontrivial submatrices on the diagonal and zeros everywhere else. Otherwise, it is called *indecomposable*.

Remark 2.3.4. For $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$, we will often denote $\lambda(h)$ by $\langle \lambda, h \rangle$.

For any element x of a Lie algebra L, we denote $\operatorname{ad} x$ the endomorphism of L sending y to $[x, y]$ for all $y \in L$.

Definition 2.3.5. Let A be a generalized Cartan matrix. The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$, in the sense of [Ku], [MP], and [R], is the Lie algebra over k, generated by $h \in \mathfrak{h}$ and symbols e_i and f_i $(1 \le i \le l)$, with the defining relations for all $1 \le i, j \le l$ and $h \in \mathfrak{h}$:

(1) $[\mathfrak{h}, \mathfrak{h}] = 0$,

(2) $[h, e_i] = \alpha_i(h)e_i$; $[h, f_i] = -\alpha_i(h)f_i$,

(3) $[e_i, f_j] = \delta_{ij}\alpha_i^\vee$,

(4) $(\operatorname{ad} e_i)^{1-a_{ij}}(e_j) = 0$,

(5) $(\operatorname{ad} f_i)^{1-a_{ij}}(f_j) = 0$ for $i \neq j$.

The relations $(4) - (5)$ are called the *Serre relations*.

Remark 2.3.6. This definition of a Kac-Moody algebra is equivalent to the definition of [Kac] if the generalized Cartan matrix is symmetrizable (see [Ku, Corollary 3.2.10]).

Let $\Lambda_r = \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i \subseteq \mathfrak{h}^*$ be the *root lattice* (the root lattice is denoted by Q in [Ku], here we follow the notation of [Tits]). Set $\Lambda_r^+ = \bigoplus_{i=1}^{l} \mathbb{Z}_+\alpha_i \subseteq \Lambda_r$, where $\mathbb{Z}_+$ is the set of nonnegative integers.

Theorem 2.3.7. [Kac, Theorem 1.2] [Ku, Theorem 1.2.1] *Let $\mathfrak{g}$ be a Kac-Moody algebra. We have a root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Lambda_r^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \Lambda_r^+} \mathfrak{g}_{-\alpha}$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$.

Definition 2.3.8. The set $\Phi = \{\alpha \in \Lambda_r \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ is called the *Kac-Moody root system* corresponding to $\mathfrak{g}$ with *simple roots* α_i and *simple coroots* $\alpha_i^\vee$.

By Theorem 2.3.7 we have that $\Phi = \Phi^+ \dot{\cup} \Phi^-$ where $\Phi^\pm = \Phi \cap \pm\Lambda_r^+$. We call Φ^+ (resp. Φ^-) the set of *positive* (resp. *negative*) *roots*.

Remark 2.3.9. The definition of root system of [Hu2], introduced in §2.2, differs from the more commonly used definition in Lie theory. We will say that a root system, as in Definition 2.2.6, is *crystallographic* if it also satisfies

$$\alpha(\beta^\vee) \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Phi.$$

In particular, a Kac-Moody root system is a crystallographic root system.

Definition 2.3.10. For any $1 \leq i \leq l$, let $s_i \in \mathrm{Aut}(\mathfrak{h}^*)$ be defined as $s_i(\phi) = \phi - \phi(\alpha_i^\vee)\alpha_i$ for $\phi \in \mathfrak{h}^*$. Let $W \subseteq \mathrm{Aut}(\mathfrak{h}^*)$ be the subgroup generated by $\{s_i \mid 1 \leq i \leq l\}$, called the *Weyl group* of $\mathfrak{g}$.

Proposition 2.3.11. [Kac, Prop. 3.13] [Ku, Prop. 1.3.21] *The pair $(W, \{s_i\}_{1\leq i\leq l})$ is a Coxeter group where the order m_{ij} of s_is_j $(1 \leq i \neq j \leq l)$ is given as follows:*

$a_{ij}a_{ji}$	*0*	*1*	*2*	*3*	≥ 4
m_{ij}	*2*	*3*	*4*	*6*	∞

Any Coxeter group with $m_{i,j} \in \{2, 3, 4, 6, \infty\}$ is called *crystallographic*. Thus the Weyl group of a Kac-Moody algebra is a crystallographic Coxeter group. Conversely, one can show that any crystallographic Coxeter group is the Weyl group of a (in general not unique) Kac-Moody algebra.

By a fundamental result of E.B. Vinberg [Vin], Kac-Moody root systems can be classified as follows. Namely, let A be an indecomposable generalized Cartan matrix. For a real column vector $u^t = (u_1, u_2, ...)$ we write $u > 0$ if all $u_i > 0$, and $u \geq 0$ if all $u_i \geq 0$. Then one and only one of the following three possibilities holds for A [Kac, p. 48]:

(Fin) $\det(A) \neq 0$; there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v \geq 0$.

(Aff) $\mathrm{corank}(A) = 1$; there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$.

(Ind) there exists $u > 0$ such that $Au < 0$; $Av \geq 0, v \geq 0$ imply $v = 0$.

We will say that a Kac-Moody root system is of *finite, affine or indefinite type* if the corresponding generalized Cartan matrix satisfies (Fin), (Aff) or (Ind) respectively.

Definition 2.3.12. Let A be a generalized Cartan matrix. We associate with A a graph $S(A)$, called the *Dynkin diagram* of A as follows. If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, the vertices i and j are connected by $|a_{ij}|$ lines, and these lines are equipped with an arrow pointing toward i is $|a_{ij}| > 1$. If $a_{ij}a_{ji} > 4$, the vertices i and j are connected by a bold-faced line equipped with an ordered pair of integers $|a_{ij}|, |a_{ji}|$.

The Kac-Moody root systems of finite type correspond to the root systems of finite-dimensional simple Lie algebras since these are the corresponding Kac-Moody algebras. Here is the complete list of Dynkin diagrams corresponding to Kac-Moody root systems of finite type.

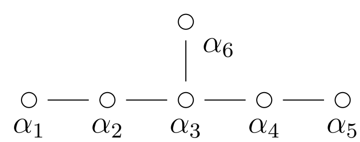
A_l

B_l

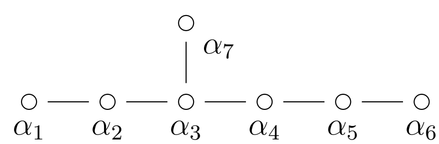
C_l

D_l

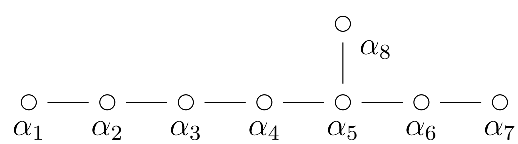
E_6



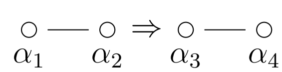
E_7



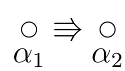
E_8



F_4

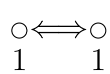


G_2

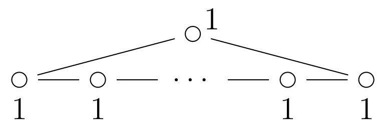


There is a complete classification of Kac-Moody root systems of affine type in terms of affine Dynkin diagrams [Kac, p. 54-55].

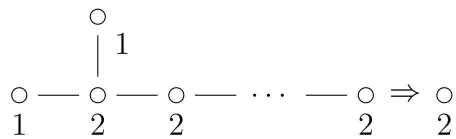
$A_1^{(1)}$



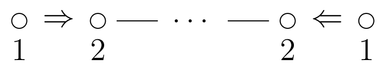
$A_l^{(1)}(l \geq 2)$



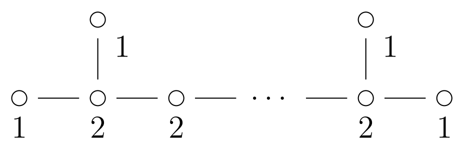
$B_l^{(1)}(l \geq 3)$



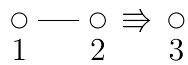
$C_l^{(1)}(l \geq 2)$



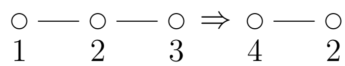
$D_l^{(1)}(l \geq 4)$



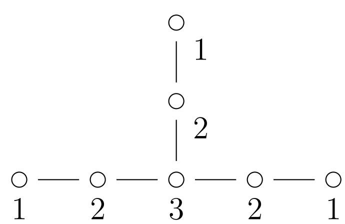
$G_2^{(1)}$



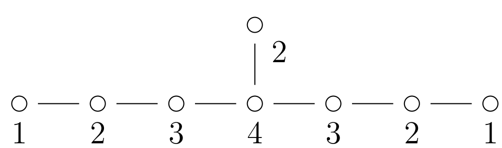
$F_4^{(1)}$



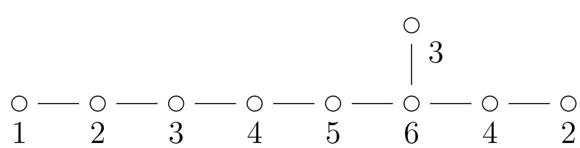
$E_6^{(1)}$



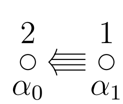
$E_7^{(1)}$



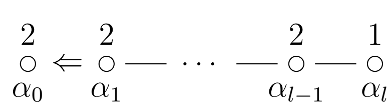
$E_8^{(1)}$



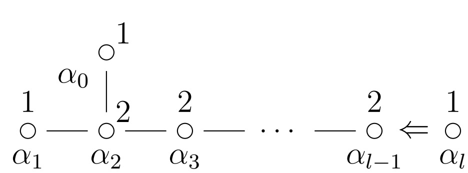
$A_2^{(2)}$



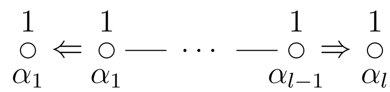
$A_{2l}^{(2)}(l \geq 2)$



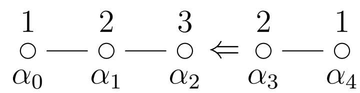
$A_{2l-1}^{(2)}(l \geq 3)$



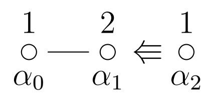
$D_{l+1}^{(2)} (l \geq 2)$



$E_6^{(2)}$



$D_4^{(3)}$



For a Kac-Moody root system of affine type, it is traditional to start numbering the simple roots at 0, i.e. the root basis is $\Pi = \{\alpha_i \mid i \in I\}$ with $I = \{0, ..., l\}$. In this case, there exists a unique element $\delta \in \Lambda_r$ called *null root* (which will play an important role in our arguments) defined as $\delta = \sum_{i=0}^{l} a_i \alpha_i$ where the a_i's are the numerical labels of each node in the Dynkin diagram. Note that α_0 is not arbitrary since $\alpha_0 = a_0^{-1}(\delta - \theta)$ where θ is the unique highest root of the underlying root system of finite type. By definition, for $i \in I$ we have

$$\langle \delta, \alpha_i^\vee \rangle = 0.$$

Moreover, there exists an element $d^* \in \mathfrak{h}^*$ (unique up to a summand proportional to δ) such that

$$\langle d^*, \alpha_0^\vee \rangle = 1, \quad \text{and } \langle d^*, \alpha_j^\vee \rangle = 0 \tag{2.3.1}$$

for $0 \leq i \leq l, 1 \leq j \leq l$.

Definition 2.3.13. A root $\alpha \in \Phi$ is called *real* if there exists $w \in W$ such that $\alpha = w(\alpha_i)$ for some simple root α_i. We denote the set of real roots by Φ^{re}.

For any $\alpha \in \Phi^{re}$, we have $\alpha = w(\alpha_i)$ for some $w \in W$, $\alpha_i \in \Pi$ and we define $\alpha^\vee := w(\alpha_i^\vee)$. One can show that this is well-defined (see [Kac, p. 59]). Then for any $\beta \in \Phi$ and $\alpha \in \Phi^{re}$ there exists $s_\alpha \in \mathrm{GL}(\mathfrak{h}^*)$ such that $s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$ and $s_\alpha \in W$ since $s_\alpha = w s_{\alpha_i} w^{-1}$.

A root α which is not a real root is called an *imaginary root*. We denote the set of imaginary roots Φ^{im} and by definition we have,

$$\Phi = \Phi^{re} \dot{\cup} \Phi^{im}.$$

Theorem 2.3.14. [Kac, p. 64] *Let A be an indecomposable generalized Cartan matrix.*

(a) If A is of finite type, then the set Φ^{im} is empty.

(b) If A is of affine type, then

$$\Phi^{im} = \{n\delta | n \in \mathbb{Z} \setminus \{0\}\},$$

where δ is the null root.

(c) If A is of indefinite type, then there exists a positive imaginary root $\alpha = \sum_i k_i\alpha_i$ such that $k_i > 0$ and $\langle \alpha, \alpha_i^\vee \rangle < 0$ for all $i = 1, ..., n$.

Definition 2.3.15. We call $\{\omega_i \mid i \in I\} \subseteq \mathfrak{h}^*$ a *set of fundamental weights* if

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}.$$

Since in general the $\alpha_i^\vee$ do not span $\mathfrak{h}$, a set of fundamental weights is not uniquely determined by the data defining Φ. For example, let Φ be of affine type, let δ be the null root (as seen in Theorem 2.3.14), and let $\{\omega_i \mid i \in I\}$ be a set of fundamental weights, then so is $\{\omega_i + n_i\delta \mid i \in I\}$ for any n_i in our base field k ($\mathrm{char}(k) = 0$) since

$$\langle \omega_i + n_i\delta, \alpha_j^\vee \rangle = \langle \omega_i, \alpha_j^\vee \rangle + n_i\langle \delta, \alpha_j^\vee \rangle = \delta_{ij} + n_i \cdot 0 = \delta_{ij}.$$

2.4 Topological rings

In this section, we review some facts about topological rings. We will follow [Bo:CA] and [Bo:GT].

We consider groups whose unit element will be denoted e. By an increasing sequence of groups we mean a sequence of groups such that $G_n \subseteq G_{n+1}$ for all $n \in \mathbb{Z}$.

Definition 2.4.1. [Bo:CA, Chap. 3, §2, no. 1, Def. 1] An increasing (resp. decreasing) sequence $(G_n)_{n\in\mathbb{Z}}$ of subgroups of a group G is called an *increasing* (resp. *decreasing*) *filtration on* G. A group with a filtration is called a *filtered group.*

If $(G_n)_{n\in\mathbb{Z}}$ is an increasing (resp. decreasing) filtration on a group G and we write $G'_n = G_{-n}$, then clearly $(G'_n)_{n\in\mathbb{Z}}$ is a decreasing (resp. increasing) filtration on G. Therefore, by convention, a filtration will always be a decreasing filtration.

Given a decreasing filtration $(G_n)_{n\in\mathbb{Z}}$ on a group G, clearly $\bigcap_{n\in\mathbb{Z}} G_n$ and $\bigcup_{n\in\mathbb{Z}} G_n$ are subgroups of G. The filtration is called *separated* if $\bigcap_{n\in\mathbb{Z}} G_n = \{e\}$, where e is the identity element of G, and *exhaustive* if $\bigcup_{n\in\mathbb{Z}} G_n = G$.

Definition 2.4.2. [Bo:CA, Chap. 3, §2, no. 1, Def. 2] Given a ring A, a filtration $(A_n)_{n\in\mathbb{Z}}$ of the additive group A is called *compatible* with the ring structure on A if

$$A_m A_n \subseteq A_{m+n}, \quad \text{for all } m, n \in \mathbb{Z}.$$

The ring A with a compatible filtration is called a *filtered ring.*

Example 2.4.3. [Bo:CA, Chap. 3, §2, no. 1, Example (3)] Let A be a ring and $\mathfrak{m}$ a two-sided ideal of A. Let us write $A_n = \mathfrak{m}^n$ for $n \geq 0$ and $A_n = A$ for $n < 0$. The filtration $(A_n)_{n\in\mathbb{Z}}$ is an exhaustive filtration on A called the $\mathfrak{m}$-*adic filtration.*

Definition 2.4.4. Let G and G' be two commutative groups and let $(G_n)_{n\in\mathbb{Z}}$ and $(G'_n)_{n\in\mathbb{Z}}$ be filtrations on G and G' respectively. A homomorphism $h : G \to G'$ is called *compatible with the filtration* on G and G' if $h(G_n) \subseteq G'_n$ for all $n \in \mathbb{Z}$.

We now need to review some facts of topology before introducing the topology defined by a filtration. We assume that the reader is familiar with some basic concepts of topology.

Definition 2.4.5. [Bo:GT, Chap. 1, §1, no. 3, Def. 5] In a topological space X, a *fundamental system of neighbourhoods* of a point x is any set $\mathfrak{G}$ of neighbourhoods of x such that for each neighbourhood V of x there is a neighbourhood $W \in \mathfrak{G}$ such that $W \subseteq V$.

Definition 2.4.6. [Bo:GT, Chap. 3, §6, no. 3, Def. 2] A *topological ring* is a set A which carries a ring structure and a topology satisfying the following axioms:

(AT_I) The addition map $(x, y) \mapsto x + y$ of $A \times A$ into A is continuous.

(AT_{II}) The inverse mapping $x \mapsto -x$ of A into A is continuous.

(AT_{II}) The multiplication map $(x, y) \mapsto xy$ of $A \times A$ into A is continuous.

We can now come back to filtered groups to introduce the topology defined by a filtration.

Let G be a group filtered by a family $(G_n)_{n\in\mathbb{Z}}$ of normal subgroups of G. There exists a unique topology on G which is compatible with the group structure and for which the G_n's constitute a fundamental system of neighbourhoods of the identity element e of G. It is called the *topology on G defined by the filtration.* When we mention the topology on a filtered group, we mean the topology defined by its filtration.

Proposition 2.4.7. [Bo:CA, Chap. 3, §2, no. 5] *For the topology on G to be Haussdorff it is necessary and sufficient that the filtration $(G_n)_{n\in\mathbb{Z}}$ be separated.*

Let G' be another filtered group and $u : G \to G'$ a homomorphism compatible with the filtrations, then u is continuous by definition of the topologies on G and G'.

Example 2.4.8. Let A be a ring and $\mathfrak{m}$ a two-sided ideal of A. The topology defined on A by the $\mathfrak{m}$-adic filtration (see Example 2.4.3) is called the $\mathfrak{m}$-*adic topology.* Since the $\mathfrak{m}$-adic filtration is exhaustive, A is a topological ring by [Bo:CA, Chap. 3, §2, no. 6, Corollary to Prop. 3].

The last concept we introduce in this section is the completion of a group. We refer to [AM].

Definition 2.4.9. [AM, p. 103] Consider a sequence of subgroups of G

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq \cdots$$

and homomorphisms

$$\theta_{n+1} : G_{n+1} \to G_n.$$

We call this an *inverse system*, and the group of all sequences (g_n), such that $g_n \in G_n$ and $\theta_{n+1} g_{n+1} = g_n$, is called the *inverse limit* of the system. We denote the inverse limit $\varprojlim G_n$, the homomorphisms θ_n being understood.

The *completion* $\widehat{G}$ of G is defined as

$$\widehat{G} = \varprojlim G/G_n.$$

Remark 2.4.10. A filtration $(G_n)_{n\in\mathbb{N}}$ on G induces a filtration $(G' \cap G_n)_{n\in\mathbb{N}}$ on a given subgroup G' and a filtration $\big((G' + G)/G'\big)_{n\in\mathbb{N}}$ on the quotient group G/G'.

By [AM, Corollary 10.3], we have that the completion preserves exact sequences. Since

$$0 \to G_n \to G \to G/G_n \to 0$$

is an exact sequence, we have the following exact sequence of completions

$$0 \to \widehat{G_n} \to \widehat{G} \to \widehat{G/G_n} \to 0.$$

Moreover, for $m \geq n$, the m-th term of the induced filtration of Remark 2.4.10 on G/G_n is

$$(G_n + G_m)/G_n = G_n/G_n = 0.$$

Therefore, we have

$$\widehat{G/G_n} = G/G_n.$$

Hence, we have the following result.

Lemma 2.4.11. [AM, Corollary 10.4] *Let $(G_n)_{n\in\mathbb{N}}$ be a filtration of a group G. The completion $\widehat{G_n}$ of G_n is a subgroup of the completion $\widehat{G}$ of G and*

$$\widehat{G}/\widehat{G_n} \cong G/G_n.$$

Definition 2.4.12. [AM, p. 105] A group G is *complete* if the map $\phi : G \to \widehat{G}$ is an isomorphism.

Remark 2.4.13. Note that by definition, a complete group is also Hausdorff since the kernel of the map $\phi : G \to \widehat{G}$ is $\bigcap_{n\in\mathbb{N}} G_n$.

By taking the inverse limit on each side of the isomorphism of Lemma 2.4.11, we get the following result.

Proposition 2.4.14. [AM, Prop. 10.5] *The map $\widehat{G} \to \widehat{\widehat{G}}$ is an isomorphism. In particular, the completion $\widehat{G}$ of G is complete.*

Example 2.4.15. [AM, p. 105] Let A be a ring and $\mathfrak{m}$ a two-sided ideal of A. Consider the $\mathfrak{m}$-adic topology on A as in Example 2.4.8. The completion $\widehat{A}$ of A is again a topological ring. The map $\phi : A \to \widehat{A}$ is a continuous ring homomorphism, whose kernel is $\bigcap_{n\in\mathbb{N}} \mathfrak{m}^n$.

Chapter 3

Formal group algebras

In this chapter, we will mostly follow the work of [CPZ], [CZZ1] and [HMSZ]. Throughout, R is a commutative unital ring and let $F(u, v)$ be a one-dimensional commutative formal group law. Let Λ be an abelian group. Let $R[x_\Lambda]$ be the ring of polynomials in the indeterminates x_λ for $\lambda \in \Lambda$.

Let $R = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \mathfrak{m}_2 \cdots$ be a sequence of ideals, i.e., a decreasing filtration. Recall from Definition 2.4.9 that the *completion* $\widehat{R}$ of R with respect to the ideals $\mathfrak{m}_i$ is the inverse limit of the rings $R/\mathfrak{m}_i$, which is by definition a subring of the direct product:

$$\begin{aligned}\widehat{R} &= \varprojlim R/\mathfrak{m}_i \\ &= \{g = (g_1, g_2, ...) \in \prod_{i \in \mathbb{N}} R/\mathfrak{m}_i \mid g_j \equiv g_i (\text{mod } \mathfrak{m}_i), \forall j > i\}.\end{aligned}$$

As in Example 2.4.3, a decreasing filtration of the form $\mathfrak{m}_i = \mathfrak{m}^i$ for some ideal $\mathfrak{m}$ of R is called a $\mathfrak{m}$-adic filtration of R. The completion of R with respect to $\mathfrak{m}$ is defined to be the completion with respect to the $\mathfrak{m}$-adic filtration. It is denoted $\widehat{R}_\mathfrak{m}$.

The ring $\widehat{R}$ has a filtration by ideals $\widehat{\mathfrak{m}_i}$ and by Lemma 2.4.11, we have $\widehat{R}/\hat{\mathfrak{m}}_i \cong R/\mathfrak{m}_i$.

Example 3.1.1. [Eis, p. 183] If $R = S[x_1, ..., x_n]$ is the polynomial ring over the ring S in n variables x_i, and $\mathfrak{m} = (x_1, ..., x_n)$, then the completion with respect to $\mathfrak{m}$ is the formal power series ring $\widehat{R}_\mathfrak{m} \cong S[\![x_1, ..., x_n]\!]$. Indeed, from the maps $S[\![x_1, ..., x_n]\!] \to R/\mathfrak{m}^i$ sending a formal power series f to $f + \mathfrak{m}^i$ we get a map

$$\begin{aligned}S[\![x_1, ..., x_n]\!] &\to \widehat{R}_\mathfrak{m}, \\ f &\mapsto (f + \mathfrak{m}, f + \mathfrak{m}^2, ...).\end{aligned}$$

The inverse map is given by sending $(f_1 + \mathfrak{m}, f_2 + \mathfrak{m}^2, ...) \in \widehat{R}_{\mathfrak{m}}$, where the f_i are polynomials and $f_i = f_j +$ monomials of total degree $> \min(i, j)$ to the power series $f_1 + (f_2 - f_1) + (f_3 - f_2) + \cdots$. This is a well-defined power series because the degree of $f_{i+1} - f_i$ is at least $i + 1$ and it is independent of the choice of f_i in $f_i + \mathfrak{m}^i$.

Remark 3.1.2. Note that for a monomial $X = rx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in R[\![x_1, ..., x_n]\!]$, where $a_1, ..., a_n \in \mathbb{N}$, the *total degree* of X is $a_1 + a_2 + \cdots + a_n$.

Definition 3.1.3. [CPZ, Def. 2.4] Let Λ be an abelian group and let $F(u, v)$ be a one-dimensional formal group law. Let $R[x_\Lambda]$ be the ring of polynomials with indeterminates x_λ for $\lambda \in \Lambda$. Let $\epsilon \colon R[x_\Lambda] \to R$, $x_\lambda \mapsto 0$ be the *augmentation map*. Let $R[\![x_\Lambda]\!] := \widehat{R}_{\ker(\epsilon)}$ be the completion of $R[x_\Lambda]$ with respect to the ideal $\ker(\epsilon)$. As in Example 2.4.15, the $\ker(\epsilon)$-adic topology extends to $R[\![x_\Lambda]\!]$. Let $\mathcal{J}_F$ be the closure in the $\ker(\epsilon)$-adic topology of the ideal of $R[\![x_\Lambda]\!]$ generated by x_0 and $x_{\lambda_1+\lambda_2} - (x_{\lambda_1} +_F x_{\lambda_2})$ for all $\lambda_1, \lambda_2 \in \Lambda$. The quotient algebra

$$R[\![\Lambda]\!]_F = R[\![x_\Lambda]\!]/\mathcal{J}_F$$

is called the *formal group algebra*.

Remark 3.1.4. Note here that we take the completion of $R[x_\Lambda]$ since in general the formal group law is a formal power series.

Examples 3.1.5. (a) Let $\Lambda = \mathbb{Z}$. Let $F_a(u, v) = u + v$. Then, the ideal $\mathcal{J}_F$ is generated by x_0 and the elements $x_{i+j} - (x_i + x_j)$ for $i, j \in \mathbb{Z}$ and we have

$$R[\![\mathbb{Z}]\!]_{F_a} = R[\![..., x_{-1}, x_0, x_1, ...]\!]/\mathcal{J}_F.$$

Hence, in $R[\![\mathbb{Z}]\!]_{F_a}$, we have $x_0 = 0$ and $x_{i+j} = x_i + x_j$ for all $i, j \in \mathbb{Z}$. Therefore, $x_n = nx_1$ for all $n \in \mathbb{Z}$ and

$$R[\![\mathbb{Z}]\!]_{F_a} = R[\![x_1]\!].$$

(b) [CPZ, Example 2.18] In general, for any Λ and $F = F_a$, we have

$$\begin{aligned} R[\![\Lambda]\!]_F \quad &\cong \quad \widehat{S(\Lambda)} = \prod_{i \geq 0} S^i(\Lambda) \\ x_\lambda \quad &\mapsto \quad \lambda \in S^1(\Lambda) \end{aligned}$$

(c) Let $F = F_m$ and $\Lambda = \mathbb{Z}$. Let $R[\mathbb{Z}] = \{\sum_j r_j e^j \mid r_j \in R, j \in \mathbb{Z}\}$. Let $\mathrm{tr} : R[\mathbb{Z}] \to R, e^j \mapsto 1$ be the trace map and let $R[\mathbb{Z}]^\wedge$ be the completion of $R[\mathbb{Z}]$ with respect to $\ker(\mathrm{tr})$. Since $F_m(u, v) = u + v - \mu uv$, the ideal $\mathcal{J}_F$ is generated by x_0 and the elements $x_{i+j} - (x_i + x_j - \mu x_i x_j)$ for $i, j \in \mathbb{Z}$ and we have

$$R[\![\mathbb{Z}]\!]_{F_m} = R[\![..., x_{-1}, x_0, x_1, ...]\!]/\mathcal{J}_F.$$

Then, we have $\mu x_i x_j = x_i + x_j - x_{i+j}$ in $R[\![\mathbb{Z}]\!]_{F_m}$ and the map

$$\phi : R[\![\mathbb{Z}]\!]_{F_m} \to R[\mathbb{Z}]^\wedge, x_j \mapsto \mu^{-1}(1 - e^j)$$

is a homomorphism. It is in fact an isomorphism with inverse defined by

$$\phi^{-1} : R[\mathbb{Z}]^\wedge \to R[\![\mathbb{Z}]\!]_{F_m}, e^j \mapsto 1 - \mu x_j.$$

(d) [CPZ, Example 2.19] In general, for any Λ and $F = F_m$, we have

$$\begin{aligned} R[\![\Lambda]\!]_F &\simeq R[\Lambda]^\wedge = \{\sum_j r_j e^{\lambda_j} \mid r_j \in R, \lambda_j \in \Lambda\}, \\ x_\lambda &\mapsto \mu^{-1}(1 - e^{\lambda_j}). \end{aligned}$$

Note that a polynomial P is in $\ker(\epsilon)^i$ if and only if its valuation is at least i, i.e., the total degree of every monomial of P is at least i. Therefore, we have $\bigcap_i \ker(\epsilon)^i = \{0\}$ and the $\ker(\epsilon)$-adic topology is Hausdorff by Proposition 2.4.7. As in Example 2.4.15, $R[\![x_\Lambda]\!]$ is a complete Hausdorff ring with respect to the $\ker(\epsilon)$-adic topology. Since $\mathcal{J}_F$ is clearly contained in $\ker(\epsilon)$, we have an augmentation map $\epsilon' : R[\![\Lambda]\!]_F \to R$. Let $\mathcal{I}_F = \ker(\epsilon')$. Then, $R[\![\Lambda]\!]_F$ is a complete Hausdorff ring with respect to the $\mathcal{I}_F$-adic topology.

Let $f : R \to R'$ be a morphism of rings sending 1_R to $1_{R'}$ and respecting the formal group laws, i.e., for $F(u, v) = u + v + \sum_{i,j\geq 1} a_{ij} u^i v^j \in R[\![u, v]\!]$ and $F'(u, v) = u + v + \sum_{i,j\geq 1} b_{ij} u^i v^j \in R'[\![u, v]\!]$, we have $f(a_{ij}) = b_{ij}$. Then, for every abelian group Λ, f induces a ring homomorphism $f_* : R[\![\Lambda]\!]_F \to R'[\![\Lambda]\!]_{F'}$ sending $x_\lambda \in R[\![\Lambda]\!]_F$ to $x_\lambda \in R'[\![\Lambda]\!]_{F'}$ for every $\lambda \in \Lambda$. This morphism is continuous since it sends $\mathcal{I}_F$ to $\mathcal{I}_{F'}$, where $\mathcal{I}_{F'}$ is the kernel of the augmentation map $R'[\![\Lambda]\!]_{F'} \to R$. If $f' : R' \to R''$ is another such homomorphism respecting the formal group laws F' and F'', then $(f'f)_* = f'_*(f)_*$.

Let $f : \Lambda \to \Lambda'$ be a morphism of abelian groups. It induces a continuous ring homomorphism $\hat{f} : R[\![\Lambda]\!]_F \to R[\![\Lambda']\!]_F$ sending $x_\lambda \to x_{f(\lambda)}$. Moreover, if f is surjective, $\hat{f}$ is also surjective. If $f' : \Lambda' \to \Lambda''$ is another such homomorphism, then $\widehat{f'f} = \hat{f}'\hat{f}$.

Let $f : F \to F'$ be a morphism of formal group laws. Then, f induces a continuous ring homomorphism $f^* : R[\![\Lambda]\!]_{F'} \to R[\![\Lambda]\!]_F$ sending x_λ to $f(x_\lambda)$. This is a well-defined ring homomorphism since we have

$$\begin{aligned} f^*(x_{\lambda+\mu}) &= f(x_{\lambda+\mu}) \\ &= f(x_\lambda +_F x_\mu) \\ &= f(x_\lambda) +_{F'} f(x_\mu) \text{ since } f \text{ is a morphism of formal group laws} \end{aligned}$$

$$= f^*(x_\lambda) +_{F'} f^*(x_\mu)$$
$$= f^*(x_\lambda +_{F'} x_\mu).$$

The statements above prove the following lemma.

Lemma 3.1.6. [CPZ, Lemma 2.6] *Via the above constructions $(-)_*$, $(\hat{-})$ and $(-)^*$, the assignment taking (R, Λ, F) to the topological ring $R[\![\Lambda]\!]_F$ is covariant with respect to the ring morphisms $R \to R'$ and morphisms of abelian groups $\Lambda \to \Lambda'$, and is contravariant with respect to the morphisms of formal group laws $F \to F'$.*

Since $R[\![\Lambda]\!]_F$ is an R-algebra, we may view a formal group law $F(u, v) \in R[\![u, v]\!]$ as an element of $R[\![\Lambda]\!]_F[\![u, v]\!]$. Let $a, b \in \mathcal{I}_F$. We can specialize $F(a, b) \in R[\![\Lambda]\!]_F[\![a, b]\!]$ and we get a pairing

$$\boxplus : \mathcal{I}_F \times \mathcal{I}_F \to \mathcal{I}_F$$

defined by $a \boxplus b = F(a, b) = a +_F b$. Using the inverse of $F(a, b)$, we can define a map

$$\boxminus : \mathcal{I}_F \to \mathcal{I}_F$$

as $\boxminus a = -_F a$. Then for $n \in \mathbb{Z}_+$ we can define another map

$$\boxtimes : \mathbb{Z} \times \mathcal{I}_F \to \mathcal{I}_F$$

as $n \boxtimes a = a \boxplus \cdots \boxplus a$ (n factors) and $-n \boxtimes a = (\boxminus a) \boxplus \cdots \boxplus (\boxminus a)$ (n factors). By the continuity of the quotient map $R[\![x_\Lambda]\!] \to R[\![\Lambda]\!]_F$ and the properties of the formal group law F we have that for any $\lambda, \mu \in \Lambda$,

$$x_{\lambda+\mu} = x_\lambda \boxplus x_\mu$$

and

$$x_{-\lambda} = x_{-\lambda} \boxplus (x_\lambda \boxplus (\boxminus x_{-\lambda})) = (x_{-\lambda} \boxplus x_\lambda) \boxplus (\boxminus x_{-\lambda}) = 0 \boxplus (\boxminus x_\lambda) = \boxminus x_\lambda.$$

Given a ring homomorphism $f : R \to R'$, a formal group law F on R induces a formal group law on R' by replacing the coefficients of the formal group law $a_{ij} \in R$ by $f(a_{ij}) \in R'$. If Λ and Λ' are abelian groups, by taking the morphism $R \to R[\![\Lambda]\!]_F$, we can view F as a formal group law on $R[\![\Lambda]\!]_F$ and we can define $R[\![\Lambda]\!]_F[\![\Lambda']\!]_F$ by the usual construction of the formal group algebra. In particular, $R[\![\Lambda]\!]_F[\![\Lambda']\!]_F$ is a $R[\![\Lambda]\!]_F$-algebra. Moreover, we also have that $R[\![\Lambda \oplus \Lambda']\!]_F$ is a $R[\![\Lambda]\!]_F$-algebra since for the natural map $f : \Lambda \to \Lambda \oplus \Lambda'$ we get a ring homomorphism $\hat{f} : R[\![\Lambda]\!]_F \to R[\![\Lambda \oplus \Lambda']\!]_F$.

Theorem 3.1.7. [CPZ, Thm 2.10] *Let Λ and Λ' be abelian groups. Then we have an isomorphism of $R[\![\Lambda]\!]_F$-algebras*

$$R[\![\Lambda \oplus \Lambda']\!]_F \simeq R[\![\Lambda]\!]_F[\![\Lambda']\!]_F.$$

.

Proof: We will follow the proof in [CPZ]. We include the proof to emphasize that the isomorphism is not trivial (see Remark 3.1.11). First, we need to define maps $\phi : R[\![\Lambda \oplus \Lambda']\!]_F \to R[\![\Lambda]\!]_F[\![\Lambda']\!]_F$ and $\psi : R[\![\Lambda]\!]_F[\![\Lambda']\!]_F \to R[\![\Lambda \oplus \Lambda']\!]_F$. The map $R[x_{\Lambda\oplus\Lambda'}] \to R[\![x_\Lambda]\!][\![y_{\Lambda'}]\!]$ sending $x_{(\lambda,\gamma)} \to x_\lambda +_F y_\gamma$ extends, by the universal property of completion, to a continuous map

$$f : R[\![x_{\Lambda\oplus\Lambda'}]\!] \to R[\![x_\Lambda]\!][\![y_{\Lambda'}]\!].$$

Let

$$g : R[\![x_\Lambda]\!][\![y_{\Lambda'}]\!] \to R[\![\Lambda]\!]_F[\![y_{\Lambda'}]\!]$$

be the map sending $x_\lambda +_F x_\mu$ to $x_\lambda \boxplus x_\mu$ induced by the map $R[\![x_\Lambda]\!] \to R[\![\Lambda]\!]_F$.

Finally, let

$$h : R[\![\Lambda]\!]_F[\![y_{\Lambda'}]\!] \to R[\![\Lambda]\!]_F[\![\Lambda']\!]_F$$

be the quotient map. Taking the composition $h \circ g \circ f$, we get a map

$$R[\![x_{\Lambda\oplus\Lambda'}]\!] \to R[\![\Lambda]\!]_F[\![\Lambda']\!]_F.$$

For any $\lambda, \mu \in \Lambda$ and any $\gamma, \delta \in \Lambda'$, we have

$$\begin{aligned}
h \circ g \circ f(x_{(\lambda,\mu)} +_F x_{(\gamma,\delta)}) &= h \circ g((x_\lambda +_F y_\gamma) +_F (x_\mu +_F y_\delta)) \\
&\quad \text{by definition and continuity of } f \\
&= h \circ g((x_\lambda +_F x_\mu) +_F (y_\gamma +_F y_\delta)) \\
&\quad \text{by associativity and commutativity of } F \\
&= h((x_\lambda \boxplus x_\mu) +_F (y_\gamma +_F y_\delta)) \text{ by definition of } g \\
&= (x_\lambda \boxplus x_\mu) +_F (y_\gamma \boxplus y_\delta) \text{ by definition of } h \\
&= x_{\lambda+\mu} +_F y_{\gamma+\delta} \text{ by definition of } \boxplus \\
&= h(x_{\lambda+\mu} +_F y_{\gamma+\delta}) \text{ by definition of } h \\
&= h \circ g(x_{\lambda+\mu} +_F y_{\gamma+\delta}) \text{ by definition of } g \\
&= h \circ g \circ f(x_{(\lambda+\mu,\gamma+\delta)}) \\
&\quad \text{by definition and continuity of } f.
\end{aligned}$$

This shows that $h \circ g \circ f$ factors through $R[\![\Lambda \oplus \Lambda']\!]_F$ as

$$h \circ g \circ f : R[\![x_{\Lambda\oplus\Lambda'}]\!] \to R[\![\Lambda \oplus \Lambda']\!]_F \xrightarrow{\phi} R[\![\Lambda]\!]_F[\![\Lambda']\!]_F.$$

To define ψ, we extend the morphism $R[\![\Lambda]\!]_F \to R[\![\Lambda \oplus \Lambda']\!]_F$ to a morphism

$$R[\![\Lambda]\!]_F[y_{\Lambda'}] \to R[\![\Lambda \oplus \Lambda']\!]_F$$

by sending y_γ to $x_{(0,\gamma)}$. It induces a morphism

$$R[\![\Lambda]\!]_F[\![y_{\Lambda'}]\!] \to R[\![\Lambda \oplus \Lambda']\!]_F$$

on completions which factors through the continuous morphism $\psi : R[\![\Lambda]\!]_F[\![\Lambda']\!]_F \to R[\![\Lambda \oplus \Lambda']\!]_F$.

It remains to show that ϕ and ψ are inverses of each other. We have

$$\begin{aligned}\psi \circ \phi(x_{(\lambda,\gamma)}) = \psi(x_\lambda +_F y_\gamma) &= \psi(x_\lambda) +_F \psi(y_\gamma) \text{ by definition and continuity of } \psi \\ &= x_{(\lambda,0)} +_F y_{(0,\gamma)} \\ &= x_{(\lambda,\gamma)}.\end{aligned}$$

We can check the other composition only on y_γ since we ϕ and ψ are morphism of $R[\![\Lambda]\!]_F$-algebras and we have

$$\phi \circ \psi(y_\gamma) = \phi(x_{(0,\gamma)}) = x_0 +_F y_\gamma = 0 +_F y_\gamma = y_\gamma.$$

∎

Lemma 3.1.8. [CPZ, Lemma 2.11] *Let $\Lambda = \mathbb{Z}$. Then sending x_m to $m \cdot_F x$ defines a ring isomorphism*

$$R[\![\mathbb{Z}]\!]_F \simeq R[\![x]\!].$$

In particular, if Λ is a free abelian group of rank one, then $R[\![\Lambda]\!]_F$ is isomorphic to $R[\![x]\!]$.

Proof: The morphism $\phi : R[x_{\mathbb{Z}}] \to R[\![x]\!]$ sending x_n to $n \cdot_F x$ extends to a continuous morphism $\hat{\phi} : R[\![x_{\mathbb{Z}}]\!] \to R[\![x]\!]$. By continuity and uniqueness of completion, we have $\hat{\phi}(x_m +_F x_n) = \hat{\phi}(x_m) +_F \hat{\phi}(x_m)$ and $\hat{\phi}(-_F x_m) = -_F\hat{\phi}(x_m)$. Therefore, $\hat{\phi}(x_{m+n}) = \hat{\phi}(x_m +_F x_n)$ and $\hat{\phi}$ factors through a morphism

$$\bar{\phi} : R[\![\mathbb{Z}]\!]_F \to R[\![x]\!].$$

By the universal property of completion there exist a unique map

$$\bar{\psi} : R[\![x]\!] \to R[\![\mathbb{Z}]\!]_F$$

sending x to x_1. By continuity, we only need to check that $\bar{\phi} \circ \bar{\psi} = \mathrm{id}$ and $\bar{\psi} \circ \bar{\phi} = \mathrm{id}$ on generators. By definition, we have

$$\bar{\phi} \circ \bar{\psi}(x) = \bar{\phi}(x_1) = 1 \cdot_F x = x$$

and

$$\bar{\psi} \circ \bar{\phi}(x_n) = \bar{\psi}(n \cdot_F x) = n \boxtimes \bar{\psi}(x) = n \boxtimes x_1 = x_n.$$

∎

Example 3.1.9. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$. Then by sending x_m to $m \cdot_F x$ we get an isomorphism of ring

$$R[\![\mathbb{Z}/n\mathbb{Z}]\!]_F \simeq R[\![x]\!]/(n \cdot_F x).$$

Since $n \cdot x = nx + x^2 p$ where $p \in R[\![x]\!]$, if n is invertible in R, we have that $(n \cdot_F x) = (x)$ and

$$R[\![\mathbb{Z}/n\mathbb{Z}]\!]_F \simeq R.$$

Corollary 3.1.10. [CPZ, Cor. 2.12] *Let Λ be a free finitely generated abelian group of rank n. Then we have*

$$R[\![\Lambda]\!]_F \simeq R[\![x_1, ..., x_n]\!].$$

Proof: This follows by Theorem 3.1.7 and Lemma 3.1.8 and induction on n. ∎

Remark 3.1.11. The right hand side of the isomorphism does not depend on the formal group law F, but as seen in the proof Theorem 3.1.7 and Lemma 3.1.8, the isomorphism does depend on F.

Example 3.1.12. Let Λ be a finitely generated abelian group. Then,

$$\Lambda \simeq \Lambda_{\text{free}} \oplus \Lambda_{\text{tor}}$$

where Λ_{free} is the free subgroup of Λ and Λ_{tor} is its torsion subgroup. Suppose Λ is of rank n. Then by Corollary 3.1.10

$$R[\![\Lambda_{\text{free}}]\!]_F \simeq R[\![x_1, ..., x_n]\!].$$

Also, we have $\Lambda_{\text{tor}} \simeq \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_m\mathbb{Z}$ where $k_1, ..., k_m$ are powers of primes. Then if R is a $\mathbb{Q}$-algebra, by Example 3.1.9 we have

$$R[\![\Lambda_{\text{tor}}]\!]_F \simeq R[\![\mathbb{Z}/k_1\mathbb{Z}]\!]_F \cdots [\![\mathbb{Z}/k_m\mathbb{Z}]\!]_F \simeq R.$$

Therefore, by Theorem 3.1.7

$$R[\![\Lambda]\!]_F \simeq R[\![x_1, ..., x_n]\!].$$

Chapter 4

Formal Demazure lattices

4.1 Definition

Let $\Pi = \{\alpha_i \mid i \in I\}$ be the set of simple roots of a root system Φ. In the case of a finite root system with root lattice Λ_r and weight lattice

$$\Lambda_\omega = \{\Lambda \in \mathfrak{h}^* \mid \langle \Lambda, \Lambda_r^\vee \rangle \subseteq \mathbb{Z}\} = \{\Lambda \in \mathfrak{h}^* \mid \langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } i \in I\},$$

the intermediate lattices Λ, $\Lambda_r \subseteq \Lambda \subseteq \Lambda_\omega$, are the initial data in the construction of the equivariant oriented cohomology ring of flag varieties of [HMSZ] and [CZZ1]. As the main purpose of the thesis is to extend these results to the Kac-Moody case, a natural question arises of what should be taken as an analogue of the intermediate lattice Λ or, equivalently, to which lattices $\Lambda \supseteq \Lambda_r$ in the Kac-Moody case can one extend the calculus of formal Demazure and BGG operators of [HMSZ] and [CZZ1].

Let Φ be a Kac-Moody root system and let $\Pi = \{\alpha_i \mid i \in I\}$ be a set of simple roots. We use the notation introduced in §2.3. The following notion provides a candidate for the intermediate lattice.

Definition 4.1.1. [L, Def. 3.1] We call a finitely generated free subgroup Λ of $(\mathfrak{h}^*, +)$ a *formal Demazure lattice* if it has the following properties:

(FDL1) every simple root can be extended to a basis of Λ, in particular $\Lambda_r \subseteq \Lambda$,

(FDL2) $\langle \Lambda, \alpha_i^\vee \rangle \subseteq \mathbb{Z}, \forall \alpha_i \in \Pi$, i.e., $\Lambda \subseteq \Lambda_\omega$.

Remarks 4.1.2. (a) A formal Demazure lattice Λ is not necessarily a lattice of $\mathfrak{h}^*$ in the usual sense since a basis of Λ need not generate $\mathfrak{h}^*$. This small abuse of language is traditional.

(b) A vector $(k_1, ..., k_n) \in \mathbb{Z}^n$ is called a *unimodular vector* (or *unimodular row*) if there exist integers $(a_1, ..., a_n)$ such that $\sum_{i=1}^n k_i a_i = 1$. By [Bo:A, VII, §4.2, Lemme 1], cf. [CZZ1, Lemma 12.7], the condition (FDL1) is equivalent to any one of the conditions (FDL1′)-(FDL1‴):

(FDL1′) The coordinates of every simple root with respect to some $\mathbb{Z}$-basis of Λ form a unimodular vector.

(FDL1″) The coordinates of every simple root with respect to every $\mathbb{Z}$-basis of Λ form a unimodular vector.

(FDL1‴) For every $\alpha_i \in \Pi$, there exists a linear form $f \in \Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ such that $f(\alpha_i) = 1$.

(c) In Chapter 5 we use (FDL1) to define the Demazure operators. Therefore, the standard weight lattice Λ_ω may be too large for our purposes since in general, for some Kac-Moody root systems, not all simple roots can be extended to a basis of Λ_ω.

(d) Condition (FDL1‴) does not imply that every linear form f on Λ is integer-valued. In particular, we need the condition (FDL2) in the definition of the formal Demazure lattice so that $\langle -, \alpha_i^\vee \rangle$ is an integer-valued linear form on Λ and, as we will see later in Chapter 5, have W act on $R[\![\Lambda]\!]_F$.

(e) Obviously, the root lattice $\Lambda_r = \bigoplus_i \mathbb{Z}\alpha_i$ is a so-called *trivial* formal Demazure lattice. We are interested in bigger formal Demazure lattices. Examples will be given in 4.2.2 and 4.4.2.

Definition 4.1.3. Let Λ and Λ' be formal Demazure lattices. By definition, a *morphism* of formal Demazure lattices is a $\mathbb{Z}$-module homomorphism $\phi\colon \Lambda \to \Lambda'$ satisfying $\phi(\Lambda_r) \subseteq \Lambda_r$.

Together with this notion of a morphism, formal Demazure lattices form a category. In particular, we have the notion of an isomorphism of formal Demazure lattices.

Lemma 4.1.4. (c.f. [L, Lemma 3.3]) *A formal Demazure lattice Λ has the following properties:*

(i) *$w(\Lambda) = \Lambda$ for all $w \in W$ and the action of W on Λ/Λ_r is trivial,*

(ii) *$\langle \Lambda, \alpha^\vee \rangle \subseteq \mathbb{Z}$ for any real root $\alpha \in \Phi^{re}$,*

(iii) *every real root can be extended to a basis of Λ,*

(iv) *the automorphism group of Φ acts on the set of formal Demazure lattices.*

Proof: (i) Let $\lambda \in \Lambda$. Let $s_\alpha \in W$ be a simple reflection. Then $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha = \lambda - c\alpha$ for some $c \in \mathbb{Z}$ by (FDL2). Since by (FDL1) we have $\Lambda_r \subseteq \Lambda$, the root $\alpha \in \Lambda$ so $c\alpha \in \Lambda$ and $s_\alpha(\lambda) \in \Lambda$. Since W is generated by simple reflections and $s_\alpha^2 = \mathrm{id}$, we have $w(\Lambda) = \Lambda$ for any $w \in W$. This also shows that the action on Λ/Λ_r is trivial.

(ii) Let $\lambda \in \Lambda$ and let $\alpha \in \Phi^{re}$. Then there exist $\alpha_i \in \Pi$ and $w \in W$ such that $w(\alpha_i) = \alpha$. Therefore, by (FDL2), $\langle \lambda, \alpha^\vee \rangle = \langle \lambda, w(\alpha_i^\vee) \rangle = \langle w^{-1}(\lambda), \alpha_i^\vee \rangle \in \mathbb{Z}$ by (i).

(iii) Let $\alpha \in \Phi^{re}$. As before, $\alpha = w(\alpha_i)$ for some $w \in W$ and some simple root $\alpha_i \in \Pi$. By (FDL1), the simple root α_i can be completed to a basis B of Λ. Then $w(B)$ is a basis of Λ containing α and $w(B) \subseteq \Lambda$ by (i).

(iv) The group of k-linear automorphisms of Φ, $\mathrm{Aut}(\Phi)$, stabilizes Φ^{re} and Φ^{im}. For any $\sigma \in \mathrm{Aut}(\Phi)$ and any formal Demazure lattice Λ, we need to check that $\sigma(\Lambda)$ is also a formal Demazure lattice. For any simple root α_i, the root $\sigma^{-1}(\alpha_i)$ is a real root, hence it can be extended to a basis of Λ, say $\{\sigma^{-1}(\alpha_i), \beta_1, ..., \beta_m\}$, by (iii). Therefore, the set $\{\alpha_i, \sigma(\beta_1), ..., \sigma(\beta_m)\}$ is a basis of $\sigma(\Lambda)$. Also, we have $\langle \sigma(\Lambda), \alpha_j^\vee \rangle = \langle \Lambda, \sigma^{-1}(\alpha_j^\vee) \rangle \in \mathbb{Z}$ by (ii).

∎

4.2 Formal Demazure lattices of rank 2 root systems

We want to describe formal Demazure lattices which are lattices (in the usual sense) in $\bigoplus_{i \in I} \mathbb{Q}\alpha_i$. We will start by working with root systems of rank 2. In the next sections we will extend the results to any root system of rank 3 and then for arbitrary rank. Here we will use α_0 and α_1 as our simple roots since our example will be an affine root system. Any lattice Λ of $\mathbb{Q}\alpha_0 \oplus \mathbb{Q}\alpha_1$ has the form $\mathbb{Z}\lambda_0 \oplus \mathbb{Z}\lambda_1$ for $\mathbb{Q}$-linearly independent λ_i. Hence there exists an invertible matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$$

such that $\lambda_0 = a\alpha_0 + c\alpha_1$ and $\lambda_1 = b\alpha_0 + d\alpha_1$.

Lemma 4.2.1. *For such a lattice Λ the condition $\alpha_i \in \Lambda$, which is part of* (FDL1)*, holds if and only if $B^{-1} \in \mathrm{Mat}_2(\mathbb{Z})$, i.e.*

$$B = \tfrac{1}{\Delta'} B', \quad \textit{where } B' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \textit{ and } \Delta' = \det(B'). \tag{4.2.1}$$

Since the coordinates of α_0 and α_1 with respect to the basis $\{\lambda_0, \lambda_1\}$ of Λ are $(d' \;\; -c')$ and $(-b' \;\; a')$ respectively, it follows that

$$\text{(FDL1)} \qquad \Longleftrightarrow \qquad (a' \;\; b') \text{ and } (c' \;\; d') \text{ are unimodular.} \tag{4.2.2}$$

Proof: The condition $\alpha_i \in \Lambda$ means that there exists integers m, n, p, q such that

$$\begin{aligned} \alpha_0 &= m\lambda_0 + n\lambda_1 = (ma + nb)\alpha_0 + (mc + nd)\alpha_1 \quad \text{and} \\ \alpha_1 &= p\lambda_0 + q\lambda_1 = (pa + qb)\alpha_0 + (pc + qd)\alpha_1. \end{aligned}$$

Equivalently

$$B\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad B\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

or with $\Delta = \det(B)$ we have

$$\begin{aligned} \begin{pmatrix} m \\ n \end{pmatrix} &= B^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d \\ -c \end{pmatrix}, \\ \begin{pmatrix} p \\ q \end{pmatrix} &= B^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} -b \\ a \end{pmatrix}. \end{aligned}$$

Thus letting

$$\begin{aligned} a' &= \frac{1}{\Delta}a = q & \qquad b' &= \frac{1}{\Delta}b = -p \\ c' &= \frac{1}{\Delta}c = -n & \qquad d' &= \frac{1}{\Delta}d = m \end{aligned}$$

we get $B = \Delta B'$ for B' as claimed and $\Delta = \det(B) = \Delta^2 \det(B')$ hence $\Delta\Delta' = 1$. That conversely any B satisfying the given condition leads to a lattice with $\alpha_i \in \Lambda$ follows by retracing our steps above. The equivalence (4.2.2) follows from the equivalent condition (FDL1′) for (FDL1). ∎

The condition (FDL2) depends on the generalized Cartan matrix corresponding to the root system so we may now look at some specific root systems to find examples of formal Demazure lattices.

Example 4.2.2. Let Φ be the affine root system $\mathrm{A}_1^{(1)}$, associated with the generalized Cartan matrix $A = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ of affine type. We have, using the notation above,

$$\langle \lambda_0, \alpha_0^\vee \rangle = a\langle \alpha_0, \alpha_0^\vee \rangle + c\langle \alpha_1, \alpha_0^\vee \rangle = 2(a - c) = \tfrac{2}{\Delta'}(a' - c') = -\langle \lambda_0, \alpha_1 \rangle$$

and similarly, $\langle \lambda_1, \alpha_0^\vee \rangle = 2(b-d) = \frac{2}{\Delta'}(b'-d') = -\langle \lambda_1, \alpha_1^\vee \rangle$. Thus

$$\text{(FDL2)} \quad \iff \quad a'-c' \in \tfrac{\Delta'}{2}\mathbb{Z} \quad \text{and} \quad d'-b' \in \tfrac{\Delta'}{2}\mathbb{Z}. \tag{4.2.3}$$

For example, a matrix B' with $c'=0$ satisfies the conditions (4.2.3) and (4.2.2) if and only if there exists $m \in \mathbb{Z}$ such that

$$B' = \begin{pmatrix} a' & \epsilon(1-\frac{a'}{2}m) \\ 0 & \epsilon \end{pmatrix}, \quad \epsilon = \pm 1, \quad \text{and} \quad (a',\ 1-\tfrac{a'}{2}m) \text{ is unimodular.}$$

In particular, we need a' or m to be even in order for $1-\frac{a'}{2}m$ to be an integer. Then there are 3 different cases.

If $a' = 2n, n \in \mathbb{Z}$ and $m = 2r, r \in \mathbb{Z}$, we have

$$B = \tfrac{1}{2n\epsilon}\begin{pmatrix} 2n & \epsilon(1-2nr) \\ 0 & \epsilon \end{pmatrix}$$

and $(2n, 1-2nr)$ is unimodular since $2n(r)+(1-2nr)(1) = 1$, hence we have a formal Demazure lattice

$$\Lambda = \mathbb{Z}\alpha_0 + \mathbb{Z}\left(\left(\tfrac{1}{2n}-r\right)\alpha_0 + \tfrac{1}{2n}\alpha_1\right) = \mathbb{Z}\alpha_0 + \mathbb{Z}\tfrac{1}{2n}(\alpha_0+\alpha_1) = \mathbb{Z}\alpha_0 + \mathbb{Z}\tfrac{1}{2n}\delta.$$

If a' is even and m is odd, then a' must be divisible by 4 for $(a', 1-\frac{a'}{2}m)$ to be unimodular. Then, we have $a' = 4n, n \in \mathbb{Z}$ and $m = 2r+1, r \in \mathbb{Z}$ and

$$B = \tfrac{1}{4n\epsilon}\begin{pmatrix} 4n & \epsilon(1-4nr-2n) \\ 0 & \epsilon \end{pmatrix}$$

so we have a formal Demazure lattice

$$\Lambda = \mathbb{Z}\alpha_0 + \mathbb{Z}\left(\left(\tfrac{1}{4n}-r-\tfrac{1}{2}\right)\alpha_0 + \tfrac{1}{4n}\alpha_1\right) = \mathbb{Z}\alpha_0 + \mathbb{Z}\left(\left(\tfrac{1}{4n}+\tfrac{1}{2}\right)\alpha_0 + \tfrac{1}{4n}\alpha_1\right).$$

If $a' = 2n+1, n \in \mathbb{Z}$ and $m = 2r, r \in \mathbb{Z}$, then

$$B = \tfrac{1}{(2n+1)\epsilon}\begin{pmatrix} 2n+1 & \epsilon(1-2nr-r) \\ 0 & \epsilon \end{pmatrix}.$$

We have that $(2n+1, 1-2nr-r)$ is unimodular so we get a formal Demazure lattice of the form

$$\Lambda = \mathbb{Z}\alpha_0 + \mathbb{Z}\frac{1}{2n+1}\delta.$$

Therefore, if $c = 0$, we either get

$$\Lambda = \mathbb{Z}\alpha_0 + \mathbb{Z}\tfrac{1}{n}\delta \tag{4.2.4}$$

or

$$\Lambda_n := \mathbb{Z}\alpha_0 + \mathbb{Z}\left(\left(\tfrac{1}{4n}+\tfrac{1}{2}\right)\alpha_0 + \tfrac{1}{4n}\alpha_1\right) \tag{4.2.5}$$

for any $n \in \mathbb{Z}$ as formal Demazure lattices.

We will study these formal Demazure lattices in the following lemma and see that we get many non-isomorphic formal Demazure lattices from this example.

Lemma 4.2.3. [L, Lemma 3.5] *Let $n, m \in \mathbb{Z} \setminus \{0\}$ and let Λ_n and Λ_m be defined by* (4.2.5).

(a) *Then $\Lambda_m \subseteq \Lambda_n$ if and only if $\frac{n}{m}$ is an odd integer. Hence we have an infinite chain of formal Demazure lattices, e.g.*

$$\Lambda_1 \subsetneq \Lambda_3 \subsetneq \Lambda_9 \subsetneq \cdots .$$

(b) *$\Lambda_n \cong \Lambda_m$ if and only if $n = m$.*

Proof: (a) We have

$$\begin{aligned}(\tfrac{1}{4m} + \tfrac{1}{2})\alpha_0 + \tfrac{1}{4m}\alpha_1 \in \Lambda_n \Leftrightarrow \exists a, b \in \mathbb{Z} \text{ such that } & a\alpha_0 + b((\tfrac{1}{4n} + \tfrac{1}{2})\alpha_0 + \tfrac{1}{4n}\alpha_1)) \\ &= (\tfrac{1}{4m} + \tfrac{1}{2})\alpha_0 + \tfrac{1}{4m}\alpha_1 \\ \Leftrightarrow\ & a + b(\tfrac{1}{4n}) = \tfrac{1}{4m} + \tfrac{1}{2} \text{ and } b(\tfrac{1}{4n}) = \tfrac{1}{4m} \\ \Leftrightarrow\ & \tfrac{n}{m} \in \mathbb{Z} \text{ and } 2 \mid (1 - \tfrac{n}{m})\end{aligned}$$

which implies the result.

(b) Since $(\frac{1}{4n} + \frac{1}{2})\alpha_0 + \frac{1}{4n}\alpha_1$ is of order $4n$ in Λ_n/Λ_r, we have $\Lambda_n/\Lambda_r \cong \mathbb{Z}_{4n}$ for any $n \in \mathbb{Z}$. Since isomorphisms of formal Demazure lattices preserve the root lattice, we have $\Lambda_n \cong \Lambda_m$ if and only if $n = m$. ∎

Remarks 4.2.4. (1) If $\Lambda \subseteq k\alpha_0 \oplus k\alpha_1$ is a formal Demazure lattice, then so is $\mathbb{Z}d^* \oplus \Lambda$. Indeed, since $\mathfrak{h}^* = kd^* \oplus k\alpha_0 \oplus k\alpha_1$, this follows immediately from the definition and (2.3.1).

(2) If we do not assume that $c = 0$ in Example 4.2.2, the computations to check if a lattice is a formal Demazure lattice become more complicated, hence making the task of finding a classification of all formal Demazure lattices difficult. As we will see in Corollaries 7.3.1, 7.3.2, and 7.3.4, different formal Demazure lattices give us different algebras. Therefore, finding a classification of the formal Demazure lattices would be an interesting problem for future research.

4.3 Formal Demazure lattices of rank 3 root systems

We can now mimic the proof of Lemma 4.2.1 for a rank 3 root system.

Lemma 4.3.1. *Let Φ be any root system of rank 3. Any lattice Λ of $\mathbb{Q}\alpha_0 \oplus \mathbb{Q}\alpha_1 \oplus \mathbb{Q}\alpha_2$ has the form $\mathbb{Z}\lambda_0 \oplus \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ for $\mathbb{Q}$-linearly independent λ_i's. Hence there exists an invertible 3×3 matrix*

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Q})$$

such that

$$\begin{aligned} \lambda_0 &= a\alpha_0 + d\alpha_1 + g\alpha_2, \\ \lambda_1 &= b\alpha_0 + e\alpha_1 + h\alpha_2, \\ \lambda_2 &= c\alpha_0 + f\alpha_1 + i\alpha_2. \end{aligned}$$

For such a lattice Λ, we have

$$\text{(FDL1)} \iff \left(\tfrac{1}{\det(B)}C_{1,j}, \tfrac{1}{\det(B)}C_{2,j}, \tfrac{1}{\det(B)}C_{3,j}\right) \in \mathbb{Z}^3 \text{ are unimodular for all } 1 \leq j \leq 3,$$

where C_{ij} are the cofactors of B.

Proof: We start by proving that $\alpha_k \in \Lambda$ for $0 \leq k \leq 2$ if and only if $\frac{1}{\det(B)}C_{i,j} \in \mathbb{Z}$ for $1 \leq i, j \leq 3$. The condition $\alpha_k \in \Lambda$ means that there exists integers $m, n, p, q, r, s, t, u, v$ such that

$$\begin{aligned} \alpha_0 &= m\lambda_0 + n\lambda_1 + p\lambda_2, \\ \alpha_1 &= q\lambda_0 + r\lambda_1 + s\lambda_2, \\ \alpha_2 &= t\lambda_0 + u\lambda_1 + v\lambda_2. \end{aligned}$$

Equivalently

$$B\begin{pmatrix} m \\ n \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad B\begin{pmatrix} q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad B\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

or with $\Delta = \det(B)$ and $C_{i,j}$ being the cofactors of B we have

$$\begin{aligned} \begin{pmatrix} m \\ n \\ p \end{pmatrix} &= B^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \end{pmatrix}, \\ \begin{pmatrix} q \\ r \\ s \end{pmatrix} &= B^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} C_{1,2} \\ C_{2,2} \\ C_{3,2} \end{pmatrix}, \\ \begin{pmatrix} t \\ u \\ c \end{pmatrix} &= B^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} C_{1,3} \\ C_{2,3} \\ C_{3,3} \end{pmatrix}. \end{aligned}$$

Thus we have $\frac{1}{\det(B)}C_{i,j} \in \mathbb{Z}$. The converse follows by retracting our steps. The equivalence

$$\text{(FDL1)} \iff (\tfrac{1}{\det(B)}C_{1,j}, \tfrac{1}{\det(B)}C_{2,j}, \tfrac{1}{\det(B)}C_{3,j}) \in \mathbb{Z}^3 \text{ are unimodular for all } 1 \leq j \leq 3$$

follows from the equivalent condition (FDL1′) for (FDL1).

∎

Example 4.3.2. Let $\phi = A_2^{(1)}$ be an affine root system of rank 3. We can use Lemma 4.3.1 and condition (FDL2) to find formal Demazure lattices corresponding to this root system. The generalized Cartan matrix of $A_2^{(1)}$ is

$$A := \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

We are looking for a lattice $\mathbb{Z}\lambda_0 \oplus \mathbb{Z}\lambda_1 \oplus \lambda_2$ as in Lemma 4.3.1. We may look at lattices of the form $\mathbb{Z}\alpha_0 \oplus \mathbb{Z}(b\alpha_0 + \alpha_1) \oplus \mathbb{Z}\lambda_2$, i.e. we have the matrix of coefficients

$$B = \begin{pmatrix} 1 & b & c \\ 0 & 1 & f \\ 0 & 0 & i \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Q}).$$

Let $\Delta := \det(B)$. Then since $\frac{1}{\Delta}C_{i,j} \in \mathbb{Z}$ for any cofactor $C_{i,j}$, we have $\frac{1}{\Delta}C_{2,1} = b \in \mathbb{Z}$. Therefore, we may assume $b = 0$ because $\mathbb{Z}\alpha_0 \oplus \mathbb{Z}(b\alpha_0 + \alpha_1) \oplus \mathbb{Z}\lambda_2 = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\lambda_2$ for any $b \in \mathbb{Z}$. By Lemma 4.3.1 we also need $(\frac{f-c}{i}, \frac{f}{i}, \frac{1}{i})$ to be unimodular, hence each coordinate must be an integer. Recall that for (FDL2) to hold, we need

$$\langle \lambda_2, \alpha_j^\vee \rangle = \langle c\alpha_0 + f\alpha_1 + i\alpha_2, \alpha_j^\vee \rangle \in \mathbb{Z},$$

for all $0 \leq j \leq 2$. For some integers x, y, z we have the equation $A(c\ f\ i)^t = (x\ y\ z)^t$ and we can write c and f in terms of i, y, z as long as $x = \frac{-1}{4}(y + z) \in \mathbb{Z}$, i.e. $y + z \in 4\mathbb{Z}$. Then we need

$$(\tfrac{f-c}{i}, \tfrac{f}{i}, \tfrac{1}{i}) = (-1 + \tfrac{y}{24i} + \tfrac{5z}{24i}, 1 + \tfrac{y}{6i} - \tfrac{z}{6i}, \tfrac{1}{i}) \tag{4.3.1}$$

to be unimodular.

For example, we may take $i = \frac{1}{6}$ and $y + z = 4$. Then we get

$$(-1 + \tfrac{y}{24i} + \tfrac{5z}{24i}, 1 + \tfrac{y}{6i} - \tfrac{z}{6i}, \tfrac{1}{i}) = (z, 5, 6)$$

which is unimodular for any $z \in \mathbb{Z}$. Therefore,

$$\Lambda = \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1 + \mathbb{Z}(\tfrac{-z}{6}\alpha_0 + \tfrac{5-2z}{6}\alpha_1 + \tfrac{1}{6}\alpha_2)$$

are formal Demazure lattices for any $z \in \mathbb{Z}$.

Remark 4.3.3. One could find more examples by taking different values of $\frac{1}{i} \in \mathbb{Z}$ and $y + z \in 4\mathbb{Z}$ such that (4.3.1) is unimodular. Also, this approach can also be applied to any root system of rank 3 by using the appropriate generalized Cartan matrix A.

4.4 More examples and facts about formal Demazure lattices

We can generalize Lemma 4.3.1 to any root system of rank n.

Lemma 4.4.1. *Let Φ be any root system of rank n and let $I = \{1, ..., n\}$. Any lattice Λ of $\bigoplus_{i \in I} \mathbb{Q}\alpha_i$ has the form $\bigoplus_{i \in I} \mathbb{Z}\lambda_i$ for $\mathbb{Q}$-linearly independent λ_i. Hence there exists an invertible matrix $B = (b_{ij})_{i,j \in I} \in \mathrm{GL}_n(\mathbb{Q})$ such that $\lambda_i = \sum_{j \in I} b_{ji}\alpha_j$. For such a lattice Λ, we have*

$$\text{(FDL1)} \iff \left(\tfrac{1}{\det(B)} C_{i_1,j}, \tfrac{1}{\det(B)} C_{i_2,j}, \ldots, \tfrac{1}{\det(B)} C_{i_n,j}\right) \textit{ are unimodular } \forall j \in I,$$

where C_{ij} are the cofactors of B.

Proof: The proof is a straightforward generalization of the proof of Lemma 4.3.1 and will be omitted. ∎

Examples 4.4.2. (i) If $\Lambda \subseteq \bigoplus_{i \in I} k\alpha_i$ is a formal Demazure lattice, then so is $\mathbb{Z}d^* \oplus \Lambda$. Indeed, since $\mathfrak{h}^* = kd \oplus \left(\bigoplus_{i \in I} k\alpha_i\right)$, this follows immediately from the definition and (2.3.1).

(ii) We can now find non-trivial formal Demazure lattices for all affine root systems Φ. Recall the null root $\delta = \sum_{i=0}^{l} a_i\alpha_i$ where the a_i's are the numerical labels of each node in the Dynkin diagrams of the affine root systems (see §2.3). If

$$\text{there exists a pair } i, j, i \neq j \text{ such that } a_i = a_j = 1, \tag{4.4.1}$$

then for any $m \in \mathbb{Z} \setminus \{0\}$

$$\Lambda = \mathbb{Z}d^* + \mathbb{Z}\alpha_0 + \cdots + \mathbb{Z}\alpha_{i-1} + \mathbb{Z}\tfrac{1}{m}\delta + \mathbb{Z}\alpha_{i+1} + \cdots + \mathbb{Z}\alpha_l$$

is a formal Demazure lattice since

$$\alpha_i = 0 \cdot d^* + (-a_0)\alpha_0 + \cdots + (-a_l)\alpha_l + m(\tfrac{1}{m}\delta)$$

and $(0, -a_0, \ldots, -a_l, m)$ is a unimodular vector. This is a unimodular vector because we have a vector $(0, 0, ..., 0, -1, 0, ..., 0)$, where -1 is in the j-th position, such that

$$0(0) + (-a_0)(0) + \cdots + (-a_{j-1})(0) + (-a_j)(-1) + (-a_{j+1})(0) + \cdots + m(0) = a_j = 1.$$

The condition (4.4.1) is fulfilled for Φ of type $A_l^{(1)}$, $B_l^{(1)}$, $C_l^{(1)}$, $D_l^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $A_{2l-1}^{(2)}$, $D_{l+1}^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$. This covers all affine root systems except $G_2^{(1)}$, $F_4^{(1)}$, $E_8^{(1)}$, $A_2^{(2)}$, and $A_{2l}^{(2)}$ (see table in §2.3). These remaining cases will be treated now.

For case $G_2^{(1)}$, $\Lambda = \mathbb{Z}\frac{1}{m}\delta + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ is a formal Demazure lattice for any $m \in \mathbb{Z}$ since $\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2$ and $(m, -2, -3)$ is unimodular because

$$m(0) + (-2)(1) + (-3)(-1) = -2 + 3 = 1.$$

Similarly, for case $F_4^{(1)}$ and $E_8^{(1)}$, $\Lambda = \mathbb{Z}\frac{1}{m}\delta + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \cdots$ is a formal Demazure lattice since $a_1 = 2, a_2 = 3$ and $(m, -2, -3, \ldots)$ is unimodular.

For the two remaining cases, $A_2^{(2)}$ and $A_{2l}^{(2)}$, $\Lambda = \mathbb{Z}\alpha_0 + \cdots + \mathbb{Z}\frac{1}{m}\delta$ is a formal Demazure lattice if m is an odd integer. Indeed, we have

$$\alpha_l = -2\alpha_0 - 2\alpha_1 - \cdots - 2\alpha_{l-1} + m(\tfrac{1}{m}\delta)$$

since $\delta = 2\alpha_0 + 2\alpha_1 + \cdots 2\alpha_{l-1} + \alpha_l$ and $(-2, -2, ..., -2, m)$ is a unimodular vector if m is an odd integer.

(iii) Let Φ be an arbitrary Kac-Moody root system and let $\{\omega_1, \ldots, \omega_l\}$ be a set of fundamental weights. Then $\Lambda = \sum_{i=1}^{l} \mathbb{Z}\omega_i$ is a formal Demazure lattice unless $\alpha_i(\alpha_j^\vee) \in 2\mathbb{Z}$ for some j and for all $i = 1, \ldots, l$ which happens if and only if we have a column of even integers in the generalized Cartan matrix corresponding to Φ.

It is a formal Demazure lattice since $\alpha_i = \sum_{j=1}^{l} \alpha_i(\alpha_j^\vee)\omega_j$ and $\big(\alpha_i(\alpha_1^\vee), \ldots, \alpha_i(\alpha_l^\vee)\big)$ is a unimodular vector because $\alpha_i(\alpha_j^\vee) = -(2k+1)$ for some $j \neq i, k \in \mathbb{Z}_+$. Then we have a row of integers $(b_1, ..., b_l)$, where $b_i = -k$, $b_j = -1$ and $b_r = 0$ for all $r \neq i, j$, such that

$$\begin{aligned}\sum_{s=1}^{l} \alpha_i(\alpha_s^\vee) \cdot b_s = \alpha_i(\alpha_i^\vee) \cdot b_i + \alpha_i(\alpha_j^\vee) \cdot b_j &= 2(-k) + (-(2k+1))(-1) \\ &= -2k + 2k + 1 = 1.\end{aligned}$$

(iv) We can also have formal Demazure lattices for root systems of indefinite type. For example, let $A = \left(\begin{smallmatrix} 2 & -4 \\ -4 & 2 \end{smallmatrix}\right)$. Then $\Lambda = \mathbb{Z}\alpha_1 + \mathbb{Z}(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2)$ is a formal Demazure lattice since $\alpha_2 = -1(\alpha_1) + 2(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2)$ and $(-1, 2)$ is a unimodular vector.

(v) If Φ is a finite root system, i.e., the generalized Cartan matrix is a Cartan matrix, any abelian group Λ satisfying $\Lambda_r \subseteq \Lambda \subseteq \Lambda_\omega$ is free of finite rank and satisfies (FDL2). It is shown in [CZZ1, Lemma 2.1] that any such Λ satisfies (FDL1) and hence is a formal Demazure lattice, except when $\Lambda = \Lambda_\omega$ and Φ is of type C_l, $l \geq 1$. This is explained by the fact that the only finite root systems with a column of even integers in their Cartan matrix are those of type C_l, $l \geq 1$.

(vi) Let Φ be a Kac-Moody root system whose associated generalized Cartan matrix A is invertible. Then Λ_ω/Λ_r is finite, namely $|\Lambda_\omega/\Lambda_r| = |\det(A)|$. Indeed, let $f : \Lambda_r \to \Lambda_\omega$ be the injection map. We have bases $B_r = \{\alpha_i : i \in I\}$ of Λ_r and $B_w = \{\omega_i : i \in I\}$ of Λ_ω (the ω_i's form a basis of Λ_ω since A is invertible and $\dim(\mathfrak{h}) = 2n - \mathrm{rk}(A) = 2n - n = n$). The matrix of f with respect to these bases is A since if $f(\alpha_i) = \alpha_i = \sum_j m_{ji}\omega_j$ then $m_{ji} = \langle \alpha_i, \alpha_j^\vee \rangle = a_{ji}$. On the other side, by [Bo:A, VII, §4.3, Th. 1], there exists a basis $e_1, \ldots, e_n$ of Λ_ω and integers $m_i \in \mathbb{Z}$ such that $(m_i e_i)$ is a basis of Λ_r. The matrix of f with respect to these bases is the diagonal matrix $(m_1, \ldots, m_n)$ whence has determinant $\prod_i m_i = \det(A)$. Moreover, $\Lambda_\omega/\Lambda_r = (\mathbb{Z}/m_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/m_n\mathbb{Z})$ whence $|\Lambda_\omega/\Lambda_r| = |m_1 \cdots m_n| = |\det(A)|$. Hence there are only finitely many formal Demazure lattices in this case.

We can compare our notion of formal Demazure lattices to similar subgroups of $\mathfrak{h}^*$ appearing in the literature.

Remark 4.4.3. In this remark, let $k = \mathbb{C}$. Following [Ku, Section 6.1.6], we define an *integral Cartan subalgebra* $\mathfrak{h}_\mathbb{Z}$ of $\mathfrak{g}$ as a finitely generated $\mathbb{Z}$-submodule of $\mathfrak{h}$ satisfying the following conditions:

(C1) $\mathfrak{h}_\mathbb{Z}$ is an integral form of $\mathfrak{h}$, i.e., the natural map $\mathfrak{h}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C} \to \mathfrak{h}$ is an isomorphism,

(C2) all simple coroots $\alpha_i^\vee \in \mathfrak{h}_\mathbb{Z}$,

(C3) $\mathfrak{h}_\mathbb{Z}^* := \mathrm{Hom}_\mathbb{Z}(\mathfrak{h}_\mathbb{Z}, \mathbb{Z}) \subseteq \mathfrak{h}^*$ contains all simple roots α_i, and

(C4) $\mathfrak{h}_\mathbb{Z}/\sum_{i\in I} \mathbb{Z}\alpha_i^\vee$ is torsion free.

However, for an integral Cartan subalgebra $\mathfrak{h}_\mathbb{Z}$, $\mathfrak{h}_\mathbb{Z}^*$ is not necessarily a formal Demazure lattice. For example, if $\Phi = A_1^{(1)}$ and $\mathfrak{h}_\mathbb{Z} = \mathbb{Z}d \oplus \mathbb{Z}\alpha_0^\vee \oplus \mathbb{Z}\alpha_1^\vee$, then since $\alpha_1(d) = 0, \alpha_1(\alpha_0^\vee) = -2$, and $\alpha_1(\alpha_1^\vee) = 2$ we have that $\frac{1}{2}\alpha_1 \in \mathfrak{h}_\mathbb{Z}^*$, hence α_1 cannot be extended to a basis of $\mathfrak{h}_\mathbb{Z}^*$. This situation occurs only if the Cartan matrix of Φ has a column of even integers.

Example 4.4.4. Let $\Phi = A_1^{(1)}$. We claim that the $\mathbb{Z}$-submodule $\mathfrak{h}_\mathbb{Z} = \mathbb{Z}(2d + \frac{1}{2}\alpha_1^\vee) + \mathbb{Z}\alpha_0^\vee + \mathbb{Z}\alpha_1^\vee$ of $\mathfrak{h} = \mathbb{Z}d + \mathbb{Z}\alpha_0^\vee + \mathbb{Z}\alpha_1^\vee$ is an integral Cartan subalgebra such that $\mathfrak{h}_\mathbb{Z}^*$ is a formal Demazure lattice.

Proof: Clearly $\mathfrak{h}_\mathbb{Z}$ satisfies (C1) and (C2). It also satisfies (C3) since $\langle \alpha_0, 2d + \frac{1}{2}\alpha_1^\vee \rangle = 2 - 2(1/2) = 1 \in \mathbb{Z}$ and $\langle \alpha_1, 2d + \frac{1}{2}\alpha_1^\vee \rangle = 2(1/2) = 1 \in \mathbb{Z}$. Moreover, we have $n(2d + \frac{1}{2}\alpha_i^\vee) \notin \mathbb{Z}\alpha_0^\vee + \mathbb{Z}\alpha_1^\vee$ for any $n \in \mathbb{Z}$, hence (C4) holds and $\mathfrak{h}_\mathbb{Z}$ is an integral Cartan subalgebra. It remains to show that $\mathfrak{h}_\mathbb{Z}^*$ is a formal Demazure lattice. Let $x\alpha_0 + y\alpha_1 + zd^* \in \mathfrak{h}_\mathbb{Z}^*$. Then

$$\langle x\alpha_0 + y\alpha_1 + zd^*, 2d + \frac{1}{2}\alpha_i^\vee \rangle = x + y \in \mathbb{Z},$$

$$\langle x\alpha_0 + y\alpha_1 + zd^*, \alpha_0^\vee \rangle = 2(x-y) + z \in \mathbb{Z},$$

and

$$\langle x\alpha_0 + y\alpha_1 + zd^*, \alpha_1^\vee \rangle = -2(x-y) \in \mathbb{Z}.$$

Thus, $z \in \mathbb{Z}$, $x+y \in \mathbb{Z}$, and $x-y \in \frac{1}{2}\mathbb{Z}$. Let $x+y = n$ for some $n \in \mathbb{Z}$ and $x-y = \frac{1}{2}m$ for some $m \in \mathbb{Z}$. Then

$$x - y = (n-y) - y = n - 2y = m/2$$

so

$$y = \frac{1}{2}n - \frac{1}{4}m \quad \text{and} \quad x = \frac{1}{2}n + \frac{1}{4}m.$$

Therefore, $\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}d^* + \mathbb{Z}\frac{1}{2}\delta + \mathbb{Z}\frac{1}{4}(\alpha_0 - \alpha_1)$. To show that it is a formal Demazure lattice, we need to check that all simple roots can be extended to a basis of $\mathfrak{h}_{\mathbb{Z}}^*$. We have

$$\alpha_0 = 0(d^*) + 1(\frac{1}{2}\delta) + 2(\frac{1}{4}(\alpha_0 - \alpha_1))$$

and

$$\alpha_1 = 0(d^*) + 1(\frac{1}{2}\delta) + (-2)(\frac{1}{4}(\alpha_0 - \alpha_1)).$$

Since $(0,1,2)$ and $(0,1,-2)$ are unimodular rows, all simple roots can be extended to a basis of $\mathfrak{h}_{\mathbb{Z}}^*$, hence it is a formal Demazure lattice. ∎

For an intergal Cartan subalgebra $\mathfrak{h}_{\mathbb{Z}}$, condition (C1) means that the rank of $\mathfrak{h}_{\mathbb{Z}}$ as a $\mathbb{Z}$-module is equal to the rank of $\mathfrak{h}$. Therefore, a formal Demazure lattice which is of rank, as a $\mathbb{Z}$-module, less than the rank of $\mathfrak{h}$ is clearly not the dual of an integral Cartan subalgebra. The following example shows that even if a formal Demazure lattice is of full rank, it is not necessarily the dual of an integral Cartan subalgebra.

Example 4.4.5. Let $\Phi = A_1^{(1)}$. Let $\Lambda = \mathbb{Z}d^* + \Lambda_n = \mathbb{Z}d^* + \mathbb{Z}\alpha_0 + \mathbb{Z}\frac{1}{n}\delta$ be a formal Demazure lattice (see (4.2.4)). We claim that there exist a $\mathbb{Z}$-submodule $\mathfrak{h}'$ of $\mathfrak{h} = \mathbb{Z}d + \mathbb{Z}\alpha_0^\vee + \mathbb{Z}\alpha_1^\vee$ which is not an integral Cartan subalgebra but such that $(\mathfrak{h}')^* = \mathrm{Hom}(\mathfrak{h}', \mathbb{Z}) = \Lambda$.

Proof: Let $\Lambda_* = \{h \in \mathfrak{h} | \langle \lambda, h \rangle \in \mathbb{Z}, \forall \lambda \in \Lambda\} = \{h \in \mathfrak{h} | \langle \lambda, h \rangle \in \mathbb{Z}, \lambda = d^*, \alpha_0, \frac{1}{n}\delta\}$ be the pre-dual of Λ. Let $ad + b\alpha_0^\vee + c\alpha_1^\vee \in \Lambda_*$. We have

$$\begin{aligned}
\langle d^*, ad + b\alpha_0^\vee + c\alpha_1^\vee \rangle &= a(0) + b(1) + c(0) = b \in \mathbb{Z}, \\
\langle \alpha_0, ad + b\alpha_0^\vee + c\alpha_1^\vee \rangle &= a(1) + b(2) + c(-2) = a + 2(b-c) \in \mathbb{Z}, \\
\langle \tfrac{1}{n}\delta, ad + b\alpha_0^\vee + c\alpha_1^\vee \rangle &= a(\tfrac{1}{n}) + b(0) + c(0) = \tfrac{a}{n} \in \mathbb{Z}.
\end{aligned}$$

Then $a \in n\mathbb{Z}$, $b \in \mathbb{Z}$, and $b - c \in 2\mathbb{Z}$ which implies that $c = b - \frac{m}{2}$ for some $m \in \mathbb{Z}$. Therefore,

$$\Lambda_* = \mathbb{Z}(nd) + \mathbb{Z}(\alpha_0^\vee + \alpha_1^\vee) + \mathbb{Z}\tfrac{1}{2}\alpha_1^\vee.$$

Now, let $ad^* + b\alpha_0 + c\alpha_1 \in (\Lambda_*)^*$. We have

$$\begin{aligned}
\langle ad^* + b\alpha_0 + c\alpha_1, nd\rangle &= a(0) + b(n) + c(0) = nb \in \mathbb{Z},\\
\langle ad^* + b\alpha_0 + c\alpha_1, \alpha_0^\vee + \alpha_1^\vee\rangle &= a(1) + b(0) + c(0) = a \in \mathbb{Z},\\
\langle ad^* + b\alpha_0 + c\alpha_1, \tfrac{1}{2}\alpha_1^\vee\rangle &= a(0) + b(-1) + c(1) = -b + c \in \mathbb{Z}.
\end{aligned}$$

Then $a \in \mathbb{Z}$, $b = \frac{m}{n}$ for some $m \in \mathbb{Z}$, and $c = p + \frac{m}{n}$ for some $p \in \mathbb{Z}$. Therefore,

$$(\Lambda_*)^* = \mathbb{Z}d^* + \mathbb{Z}\alpha_1 + \mathbb{Z}\tfrac{1}{n}(\alpha_0 + \alpha_1) = \mathbb{Z}d^* + \mathbb{Z}\alpha_1 + \mathbb{Z}\tfrac{1}{n}\delta.$$

Since $k\alpha_0 = k \cdot n(\frac{1}{n}\delta) - k\alpha_1 \in (\Lambda_*)^*$ for all $k \in \mathbb{Z}$, we have $\Lambda \subseteq (\Lambda_*)^*$. Also, we have $l\alpha_1 = l \cdot n(\frac{1}{n}\delta) - l\alpha_0 \in \Lambda$ for all $l \in \mathbb{Z}$. Hence $(\Lambda_*)^* = \Lambda$ and $\Lambda_* = \mathfrak{h}'$.

Moreover, $\mathfrak{h}' = \mathbb{Z}(nd) + \mathbb{Z}(\alpha_0^\vee + \alpha_1^\vee) + \mathbb{Z}\frac{1}{2}\alpha_1^\vee$ is not an integral Cartan subalgebra since $\mathfrak{h}'/\sum_i \mathbb{Z}\alpha_i^\vee$ is not torsion free. ∎

Remark 4.4.6. Following [MP, Section 6.1], we can define a *restricted weight lattice* as a subgroup Λ of $\mathfrak{h}^*$ satisfying the following properties:

(RWL1) $\Lambda \subseteq \Lambda_\omega$,

(RWL2) $\langle \Lambda, \alpha_i^\vee\rangle \subseteq \mathbb{Z}, \forall i \in I$,

(RWL3) there exists a minimal regular weight $\rho \in \Lambda$, i.e. $\langle \rho, \alpha_i^\vee\rangle = 1, \forall i \in I$,

(RWL4) Λ is a free $\mathbb{Z}$-module with a basis B consisting of k-linearly independent elements and such that B contains a set of fundamental weights.

Again, we have that a restricted weight lattice is not necessarily a formal Demazure lattice. For example, if $\Phi = A_1^{(1)}$, $\Lambda = \mathbb{Z}d^* + \mathbb{Z}\frac{1}{2}\alpha_1 + \mathbb{Z}\delta$ is a restricted weight lattice because $\{\omega_0 = d^*, \omega_1 = d^* + \frac{1}{2}\alpha_1\}$ is a set of fundamental weights. However, it is not a formal Demazure lattice since $\frac{1}{2}\alpha_1$ cannot be extended to a basis of Λ. Also, any set of fundamental weights is of the form $\{\omega_0 + m\delta, \omega_1 + n\delta\}$ for some $m, n \in \mathbb{Z}$. So we have that $\Lambda = \mathbb{Z}d^* + \mathbb{Z}\alpha_0 + \frac{1}{2}\delta$ is a formal Demazure lattice which is not a restricted weight lattice since $\omega_1 + n\delta \notin \Lambda$ for any $n \in \mathbb{Z}$.

Chapter 5

Formal Demazure operators

5.1 Definition

Let Φ be a Kac-Moody root system of any type with rank n, let $\Pi = \{\alpha_1, ..., \alpha_n\}$ be the set of simple roots, let W be the Weyl group of Φ, and let Λ a formal Demazure lattice of Φ. Let F be a one-dimensional formal group law.

Definition 5.1.1. For $\alpha \in \Phi$, we will say that an element x_α of $R[\![\Lambda]\!]_F$ if *regular* is x_α is not a zero divisor.

We want to know what are the conditions on $\alpha \in \Phi$ for x_α to be regular. First we will need some facts about commutative rings.

Lemma 5.1.2. [CZZ1, Lemma 12.3] *Let $f \in R[\![x_1, ..., x_n]\!]$ such that $f = a_1x_1 + \cdots + a_nx_n +$ terms of total degree ≥ 2.*

(a) If a_i is regular in R, i.e., not a zero divisor, for some i, then f is regular in $R[\![x_1, ..., x_n]\!]$.

(b) If the vector of coefficients $(a_1, ..., a_n)$ can be completed to a basis of R^n, then f is regular in $R[\![x_1, ..., x_n]\!]$.

The following lemma is a genralization of [CZZ1, Lemma 2.2].

Lemma 5.1.3. [L, Lemma 4.2] *For $\alpha \in \Phi^{re}$, x_α is regular in $R[\![\Lambda]\!]_F$.*

Proof: Since Λ is a formal Demazure lattice and $\alpha \in \Phi^{re}$, the real root α can be completed to a basis of Λ by Lemma 4.1.4. Also, by Corollary 3.1.10 we have $R[\![\Lambda]\!]_F \cong R[\![x_1, \ldots, x_n]\!]$ with $x_1 = x_\alpha$. Therefore, x_α is regular by Lemma 5.1.2. ∎

Remark 5.1.4. Let $\Lambda = \mathfrak{h}_{\mathbb{Z}}^*$ where $\mathfrak{h}_{\mathbb{Z}}$ is any integral Cartan subalgebra of $\mathfrak{g}$. Suppose the Cartan matrix of Φ has no column of even integers. Then by an argument similar to the proof of [CZZ1, Lemma 2.2], x_α is regular in $R[\![\Lambda]\!]_F$ for any real root $\alpha \in \Phi^{re}$. If the Cartan matrix of Φ has a column of even integers, x_α is regular if 2 is regular in R. The same situation occurs if Λ is a restricted weight lattice.

Definition 5.1.5. We define a continuous linear operator $\widetilde{\Delta_\alpha}$ on $R[\![\Lambda]\!]$ as follows

$$\widetilde{\Delta_\alpha}(u) = u - s_\alpha(u), \forall u \in R[\![\Lambda]\!].$$

Lemma 5.1.6. *For any $\alpha, \lambda, \mu \in \Lambda$, $s_\alpha(x_\lambda +_F x_\mu) = s_\alpha(x_\lambda) +_F s_\alpha(x_\mu)$.*

Proof: By definition of the action of s_α on $R[\![\Lambda]\!]$, we have

$$\begin{aligned} s_\alpha(x_\lambda +_F x_\mu) &= s_\alpha(x_\lambda + x_\mu + \sum_{i,j\geq 1} a_{ij} x_\lambda^i x_\mu^j) \\ &= s_\alpha(x_\lambda) + s_\alpha(x_\mu) + \sum_{i,j\geq 1} a_{ij} s_\alpha(x_\lambda)^i s_\alpha(x_\mu)^j \\ &= s_\alpha(x_\lambda) +_F s_\alpha(x_\mu). \end{aligned}$$

∎

Lemma 5.1.7. *The linear operator $\widetilde{\Delta_\alpha}$ leaves the ideal $\mathcal{J}_F$ invariant, i.e. $\widetilde{\Delta_\alpha}(\mathcal{J}_F) \subseteq \mathcal{J}_F$, and the closure is preserved.*

Proof: First, we have

$$\widetilde{\Delta_\alpha}(x_0) = x_0 - s_\alpha(x_0) = x_0 - x_{s_\alpha(0)} = x_0 - x_0 = 0.$$

Then, we have to determine how the operator acts on the other generators of $\mathcal{J}_F$. We have

$$\begin{aligned} \widetilde{\Delta_\alpha}(x_{\lambda+\mu} - (x_\lambda +_F x_\mu)) &= x_{\lambda+\mu} - (x_\lambda +_F x_\mu) - s_\alpha(x_{\lambda+\mu} - (x_\lambda +_F x_\mu)) \\ &= x_{\lambda+\mu} - (x_\lambda +_F x_\mu) - (s_\alpha(x_{\lambda+\mu}) - (s_\alpha(x_\lambda) +_F s_\alpha(x_\mu))) \\ &= x_{\lambda+\mu} - (x_\lambda +_F x_\mu) - (x_{s_\alpha(\lambda)+s_\alpha(\mu)} - (x_{s_\alpha(\lambda)} +_F x_{s_\alpha(\mu)})) \in \mathcal{J}_F. \end{aligned}$$

Therefore, $\widetilde{\Delta_\alpha}(\mathcal{J}_F) \subseteq \mathcal{J}_F$ and the closure is preserved since the operator is continuous. ∎

In particular, we see in the proof of the lemma above that $s_\alpha(\mathcal{J}_F) \subseteq \mathcal{J}_F$ for any reflection $s_\alpha \in W$. Therefore, $w(\mathcal{J}_F) \subseteq \mathcal{J}_F$ for any $w \in W$ and we get that W acts by R-linear automorphisms on $R[\![\Lambda]\!]_F$ as follows:

$$\begin{array}{rcl} R[\![\Lambda]\!]_F & \to & R[\![\Lambda]\!]_F \\ \sum\limits_{i=(i_1,\ldots,i_n)\in\mathbb{N}^n} a_i x_{\lambda_1}^{i_1}\ldots x_{\lambda_n}^{i_n} & \mapsto & \sum\limits_{i=(i_1,\ldots,i_n)\in\mathbb{N}^n} a_i x_{w(\lambda_1)}^{i_1}\ldots x_{w(\lambda_n)}^{i_n}. \end{array}$$

In particular, for any $w \in W$ and $x_\alpha \in R[\![\Lambda]\!]_F$, we have

$$w(x_\alpha) = x_{w(\alpha)}.$$

The previous lemma also helps us define a linear operator on the quotient $R[\![\Lambda]\!]/\mathcal{J}_F$.

Definition 5.1.8. For all $\alpha \in \Lambda$, we define a linear operator $\overline{\Delta_\alpha}$ on $R[\![\Lambda]\!]_F = R[\![\Lambda]\!]/\mathcal{J}_F$ as follows

$$\overline{\Delta_\alpha}(u + \mathcal{J}_F) = \widetilde{\Delta_\alpha}(u) + \mathcal{J}_F.$$

In $R[\![\Lambda]\!]_F$, we have the following equality:

$$s_\alpha(x_\lambda) = x_{s_\alpha(\lambda)} = x_{\lambda - \langle\lambda,\alpha^\vee\rangle\alpha} = x_\lambda +_F (-\langle\lambda,\alpha^\vee\rangle \cdot_F x_\alpha), \forall\alpha,\lambda \in \Lambda.$$

Lemma 5.1.9. [CPZ, Lemma 3.3] *For any $n \in \mathbb{Z}$, the element $x - x +_F (n \cdot_F y)$ is divisible by y in $R[\![x, y]\!]$.*

Proof: Let $G(x, y) \in R[\![x, y]\!]$ be any power series, $G(x, y) = \sum_{i,j\geq 0} a_{ij}x^i y^j$. Then,

$$G(x, 0) - G(x, y) = \sum_{i\geq 0} a_{i0}x^i - \sum_{i,j\geq 0} a_{ij}x^i y^j = \sum_{i\geq 0, j\geq 1} a_{ij}x^i y^j$$

is divisible by y. Let $G(x, y) = x +_F (n \cdot_F y)$. Then,

$$G(x, 0) - G(x, y) = x +_F (n \cdot_F 0) - x +_F (n \cdot_F y) = x - x +_F (n \cdot_F y)$$

is divisible by y. ∎

The next result is a generalization of [CPZ, Corollary 3.4].

Corollary 5.1.10. *For any $u \in R[\![\Lambda]\!]_F$ and any $\alpha \in \Phi^{re}$, $u - s_\alpha(u)$ is uniquely divisible by x_α.*

Proof: We have that for any $\lambda \in \Lambda, x_\lambda - s_\alpha(x_\lambda) = x_\lambda - x_{s_\alpha(\lambda)} = x_\lambda - x_{\lambda - \alpha^\vee(\lambda)\alpha} = x_\lambda - x_\lambda +_F (-\alpha^\vee(\lambda) \cdot_F x_\alpha)$ which, by Lemma 5.1.9, is divisible by x_α. Since for any $\lambda, \mu \in \Lambda$,

$$x_\lambda x_\mu - s_\alpha(x_\lambda x_\mu) = (x_\lambda - s_\alpha(x_\lambda))x_\mu + x_\lambda(x_\mu - s_\alpha(x_\mu)) - (x_\lambda - s_\alpha(x_\lambda))(x_\mu - s_\alpha(x_\mu)) \tag{5.1.1}$$

it follows by induction that for any monomial v, $v - s_\alpha(v)$ is divisible x_α. Since u is the limit of a sum of monomials and x_α is regular by Lemma 5.1.3, $u - s_\alpha(u)$ is uniquely divisible by x_α. ∎

Definition 5.1.11. For any $\alpha \in \Phi^{re}$, we define a R-linear operator on $R[\![\Lambda]\!]_F$ as

$$\Delta_\alpha(u) = \frac{\overline{\Delta_\alpha}(u)}{x_\alpha} = \frac{u - s_\alpha(u)}{x_\alpha}$$

for all $u \in R[\![\Lambda]\!]$. We call the Δ_α's the *formal Demazure operators.*

Remark 5.1.12. By Corollary 5.1.10, we have that for any $u \in R[\![\Lambda]\!]_F$, $\overline{\Delta_\alpha}(u) = u - s_\alpha(u)$ is divisible by x_α and it follows that $\Delta_\alpha(u) \in R[\![\Lambda]\!]_F$. We will sometimes write Δ_α^F to emphasize the formal group law F over R.

5.2 Properties of the formal Demazure operators

The following list of relations is a generalization of [CPZ, Corollary 3.4].

Proposition 5.2.1. *For any $u, v \in R[\![\Lambda]\!]_F$ and any $w \in W$, we have the following formulas for $\Delta_\alpha, \alpha \in \Phi^{re}$.*

(a) $\Delta_\alpha(1) = 0, \quad \Delta_\alpha(u)x_\alpha = u - s_\alpha(u), \quad \Delta_\alpha(u)x_\alpha = \Delta_{-\alpha}(u)x_{-\alpha},$

(b) $\Delta_\alpha^2(u)x_\alpha = \Delta_\alpha(u) + \Delta_{-\alpha}(u),$

(c) $s_\alpha\Delta_\alpha(u) = -\Delta_{-\alpha}(u), \quad \Delta_\alpha s_\alpha(u) = -\Delta_\alpha(u),$

(d) $\Delta_\alpha(uv) = \Delta_\alpha(u)v + u\Delta_\alpha(v) - \Delta_\alpha(u)\Delta_\alpha(v)x_\alpha = \Delta_\alpha(u)v + s_\alpha\Delta_\alpha(v),$

(e) $w\Delta_\alpha w^{-1}(u) = \Delta_{w(\alpha)}(u).$

Proof: We follow the proof in [CPZ] and complete the missing parts.

(a) We have

$$\Delta_\alpha(1) = \frac{1 - s_\alpha(1)}{x_\alpha} = \frac{1-1}{x_\alpha} = 0$$

and $\Delta_\alpha(u)x_\alpha = u - s_\alpha$ holds by multiplication by x_α. Also, we have

$$\Delta_\alpha(u)x_\alpha = u - s_\alpha(u) = u - s_{-\alpha}(u) = \Delta_{-\alpha}(u)x_{-\alpha}.$$

(b) We have

$$\begin{aligned}
\Delta_\alpha\big(\Delta_\alpha(u)\big)x_\alpha &= \Delta_\alpha\left(\frac{u - s_\alpha(u)}{x_\alpha}\right) = \frac{1}{x_\alpha}\left(\frac{u - s_\alpha(u)}{x_\alpha} - s_\alpha\left(\frac{u - s_\alpha(u)}{x_\alpha}\right)\right)x_\alpha \\
&= \Delta_\alpha(u) - \frac{s_\alpha(u) - s_\alpha^2(u)}{s_\alpha(x_\alpha)} = \Delta_\alpha(u) - \frac{s_\alpha(u) - u}{x_{-\alpha}} \\
&= \Delta_\alpha(u) + \frac{u - s_\alpha(u)}{x_{-\alpha}} = \Delta_\alpha(u) + \frac{u - s_{-\alpha}(u)}{x_{-\alpha}} \\
&= \Delta_\alpha(u) + \Delta_{-\alpha}(u).
\end{aligned}$$

(c) We have

$$s_\alpha\big(\Delta_\alpha(u)\big) = s_\alpha\left(\frac{u - s_\alpha(u)}{x_\alpha}\right) = \frac{s_\alpha(u) - u}{x_{-\alpha}} = -\Delta_{-\alpha}(u)$$

and

$$\Delta_\alpha\big(s_\alpha(u)\big) = \frac{s_\alpha(u) - s_\alpha^2(u)}{x_\alpha} = \frac{s_\alpha(u) - u}{x_\alpha} = -\Delta_\alpha(u).$$

(d) It is easy to show by opening the brackets on the right hand side that

$$uv - s_\alpha(uv) = \big(u - s_\alpha(u)\big)v + u\big(v - s_\alpha(v)\big) - \big(u - s_\alpha(u)\big)\big(v - s_\alpha(v)\big).$$

Then, we have

$$\begin{aligned}
\Delta_\alpha(uv) &= \frac{uv - s_\alpha(uv)}{x_\alpha} \\
&= \frac{1}{x_\alpha}\Big(\big(u - s_\alpha(u)\big)v + u\big(v - s_\alpha(v)\big) - \big(u - s_\alpha(u)\big)\big(v - s_\alpha(v)\big)\Big) \\
&= \left(\frac{u - s_\alpha(u)}{x_\alpha}\right)v + u\left(\frac{v - s_\alpha(v)}{x_\alpha}\right) - \left(\frac{u - s_\alpha(u)}{x_\alpha}\right)\left(\frac{v - s_\alpha(v)}{x_\alpha}\right)x_\alpha \\
&= \Delta_\alpha(u)v + u\Delta_\alpha(v) - \Delta_\alpha(u)\Delta_\alpha(v)x_\alpha = \Delta_\alpha(u)v + \big(u - \Delta_\alpha(u)x_\alpha\big)\Delta_\alpha(v) \\
&= \Delta_\alpha(u)v + \big(u - \big(u - s_\alpha(u)\big)\big)\Delta_\alpha(v) = \Delta_\alpha(u)v + s_\alpha(u)\Delta_\alpha(v).
\end{aligned}$$

(e) Recall from the proof of Lemma 2.2.16 that for any $w \in W$ and any $\gamma \in \Phi$, we have

$$ws_\alpha w^{-1}(\gamma) = s_{w(\alpha)}(\gamma).$$

Therefore, we have

$$\begin{aligned} w\Delta_\alpha w^{-1}(u) = w\left(\frac{w^{-1}(u) - s_\alpha w^{-1}(u)}{x_\alpha}\right) &= \frac{u - ws_\alpha w^{-1}(u)}{x_{w(\alpha)}} \\ &= \frac{u - s_{w(\alpha)}(u)}{x_{w(\alpha)}} \\ &= \Delta_{w(\alpha)}(u). \end{aligned}$$

∎

Example 5.2.2. Let $F = F_{\mu_1,\mu_2}$ be the hyperbolic formal group law with $\mu_1, \mu_2 \in R$ and let $\alpha \in \Phi^{re}$. We have

$$F_{\mu_1,\mu_2}(x_\alpha, x_{-\alpha}) = \frac{x_\alpha + x_{-\alpha} - \mu_1 x_\alpha x_{-\alpha}}{1 + \mu_2 x_\alpha x_{-\alpha}} = 0.$$

Hence, $x_\alpha + x_{-\alpha} - \mu_1 x_\alpha x_{-\alpha} = 0$ and we have

$$\frac{x_{-\alpha}}{x_\alpha} = \frac{1}{\mu_1 x_\alpha - 1}.$$

Therefore,

$$\Delta_\alpha(x_\alpha) = \frac{x_\alpha - s_\alpha(x_\alpha)}{x_\alpha} = 1 - \frac{x_{-\alpha}}{x_\alpha} = 1 - \frac{1}{\mu_1 x_\alpha - 1}.$$

Let I denote a sequence $(i_1, ..., i_l)$ of l integers in $[n] := \{1, ..., n\}$ and let $w_I = s_{i_1} \cdots s_{i_l}$, where $s_{i_j} = s_{\alpha_{i_j}}$, be the corresponding product of simple reflections. The decomposition I is reduced if w_I has length l. We define the linear operator Δ_I as

$$\Delta_I = \Delta_{\alpha_{i_1}} \circ \cdots \circ \Delta_{\alpha_{i_l}}.$$

Theorem 5.2.3. [CPZ, Theorem 3.9], c.f. [Dem2, Theorem 2 p. 86], c.f. [BE, Theorem 3.7] *Let F be a formal group law of the form $F(u, v) = u + v - \mu uv$ for some $\mu \in R$. Let I and I' be two reduced decompositions of w in simple reflections. Then*

$$\Delta_I = \Delta_{I'}.$$

Moreover, the independence of the decomposition holds only if F is of this form.

We introduce the following operators to determine the action of Δ_I on uv for any $u, v \in R[\![\Lambda]\!]_F$.

Definition 5.2.4. (cf. [CZZ1, Def. 4.7]) We define R-linear operators $B_i^{(j)} : R[\![\Lambda]\!]_F \to R[\![\Lambda]\!]_F$, where $j \in \{-1, 0, 1\}$ and $i \in [n]$, as

$$B_i^{(-1)} := \Delta_{\alpha_i}, \quad B_i^{(0)} := s_{\alpha_i}, \quad \text{and} \quad B_i^{(1)} := \text{ multiplication by } (-x_{\alpha_i}).$$

** Is it ok if I put cf. when citing a reference for a definition instead of writing a sentence saying that the following definition is a generalization of ... **

For $m > 0$, we have that

$$B_i^{(j)}(\mathcal{I}_F^m) \subseteq \mathcal{I}_F^{m+j} \text{ for all } j \in \{-1, 0, 1\}.$$

Indeed, $B_i^{(1)}(\mathcal{I}_F^m) \subseteq \mathcal{I}_F^{m+1}$ since you multiply by $-x_{\alpha_i} \in \mathcal{I}_F$, $B_i^{(0)}(\mathcal{I}_F^m) \subseteq \mathcal{I}_F^m$ since a reflection s_β sends $x_\alpha \in \mathcal{I}_F$ to $x_{s_\beta(\alpha)} \in \mathcal{I}_F$, and $B_i^{(-1)}(\mathcal{I}_F^m) \subseteq \mathcal{I}_F^{m-1}$ since you divide by $x_{\alpha_i} \in \mathcal{I}_F$.

Let $I = (i_1, ..., i_l)$ be a sequence of length l, and let E be a subset of $[l]$. We denote by $I_{|E}$ the subsequence of I consisting of all i_j's with $j \in E$. Here is a generalization of [CZZ1, Lemma 4.8].

Lemma 5.2.5. *Let $I = (i_1, ..., i_l)$. Then for any $u, v \in R[\![\Lambda]\!]_F$ we have*

$$\Delta_I(uv) = \sum_{E_1, E_2 \subseteq [l]} p^I_{E_1,E_2} \Delta_{I_{|E_1}}(u) \Delta_{I_{|E_2}}(v),$$

where $p^I_{E_1,E_2} = B_1 \circ \cdots \circ B_l(1) \in \mathcal{I}_F^{|E_1|+|E_2|-l}$ with the operator $B_j : R[\![\Lambda]\!]_F \to R[\![\Lambda]\!]_F$ defined as

$$B_j = \begin{cases} B_{i_j}^{(1)} \circ B_{i_j}^{(0)}, & \text{if } j \in E_1 \cap E_2, \\ B_{i_j}^{(-1)}, & \text{if } j \notin E_1 \cup E_2, \\ B_{i_j}^{(0)}, & \text{otherwise.} \end{cases}$$

Proof: We follow the proof of [CZZ1] and add some details. We work by induction on the length l of I. If $l = 1$, $I = (i_1)$ and the only possibilities for E_1 and E_2 are both empty, both equal to $\{1\}$ or one empty and one equal to $\{1\}$. If $E_1 = E_2 = \emptyset$, then $1 \notin E_1 \cup E_2$ so $B_1 = B_{i_1}^{(-1)} = \Delta_{i_1}$. If $E_1 = \emptyset$ and $E_2 = \{1\}$ or $E_1 = \{1\}$ and $E_2 = \emptyset$, we have $1 \in E_1 \cup E_2$ and $1 \notin E_1 \cap E_2$, so $B_1 = B_{i_1}^{(0)} = s_{i_1}$. Finally, if $E_1 = E_2 = \{1\}$, we have $1 \in E_1 \cap E_2$ and $B_1 = B_{i_1}^{(1)} = -x_{i_1}$. Therefore, we have

$$\Delta_{i_1}(uv) = \Delta_{i_1}(u)v + u\Delta_{i_1}(v) - x_{i_1}\Delta_{i_1}(u)\Delta_{i_1}(v) \text{ by Proposition 5.2.1(d)}$$

$$
\begin{aligned}
&= -\Delta_{i_1}(1)uv + s_{i_1}(1)u\Delta_{i_1}(v) + s_{i_1}(1)\Delta_{i_1}(u)v - x_{i_1}\Delta_{i_1}(u)\Delta_{i_1}(v) \\
&= p^I_{\emptyset,\emptyset}uv + p^I_{\emptyset,\{1\}}u\Delta_{i_1}(v) + p^I_{\{1\},\emptyset}\Delta_{i_1}(u)v - p^I_{\{1\},\{1\}}\Delta_{i_1}(u)\Delta_{i_1}(v) \\
&= \sum_{E_1,E_2\subseteq[1]} p^I_{E_1,E_2}\Delta_{I_{|E_1}}(u)\Delta_{I_{|E_2}}(v)
\end{aligned}
$$

since $\Delta_{|\emptyset} = \mathrm{id}$, $\Delta_{i_1}(1) = 0$ by Proposition 5.2.1(a) and $s_{i_1}(1) = 1$.

Let $I' = (i_2, ..., i_l)$ for $l > 1$. By induction, we get

$$
\Delta_I(uv) = \Delta_{i_1}\big(\Delta_{I'}(uv)\big) = \Delta_{i_1}\left(\sum_{E_1,E_2\subseteq[l-1]} p^{I'}_{E_1,E_2}\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)\right).
$$

For fixed $E_1, E_2 \subseteq [l-1]$, by taking $u = p^{I'}_{E_1,E_2}$ and $v = \Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)$ in Proposition 5.2.1(d) we have

$$
\begin{aligned}
\Delta_{i_1}\left(p^{I'}_{E_1,E_2}\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)\right) = {}&\Delta_{i_1}(p^{I'}_{E_1,E_2})\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v) \\
&+ s_{i_1}(p^{I'}_{E_1,E_2})\Delta_{i_1}\left(\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)\right).
\end{aligned}
$$

Similarly, by taking $u = \Delta_{I'_{|E_1}}(u)$ and $v = \Delta_{I'_{|E_2}}(v)$ in Proposition 5.2.1(d) we have

$$
\begin{aligned}
\Delta_{i_1}\left(\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)\right) = {}&\left(\Delta_{i_1}\circ\Delta_{I'_{|E_1}}\right)(u)\Delta_{I'_{|E_2}}(v) + \Delta_{I'_{|E_1}}(u)\left(\Delta_{i_1}\circ\Delta_{I'_{|E_2}}\right)(v) \\
&- x_{i_1}\left(\Delta_{i_1}\circ\Delta_{I'_{|E_1}}\right)(u)\left(\Delta_{i_1}\circ\Delta_{I'_{|E_2}}\right)(v).
\end{aligned}
$$

If we combine these two equations, we get

$$
\begin{aligned}
\Delta_{i_1}\cdot\left(p^{I'}_{E_1,E_2}\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)\right) &= \Delta_{i_1}(p^{I'}_{E_1,E_2})\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v) \\
&+ s_{i_1}(p^{I'}_{E_1,E_2})\cdot\left[\left(\Delta_{i_1}\circ\Delta_{I'_{|E_1}}\right)(u)\Delta_{I'_{|E_2}}(v) + \Delta_{I'_{|E_1}}(u)\left(\Delta_{i_1}\circ\Delta_{I'_{|E_2}}\right)(v)\right] \\
&- x_{i_1}s_{i_1}(p^{I'}_{E_1,E_2})\cdot\left(\Delta_{i_1}\circ\Delta_{I'_{|E_1}}\right)(u)\left(\Delta_{i_1}\circ\Delta_{I'_{|E_2}}\right)(v).
\end{aligned}
$$

Recall that

$$
B^{(-1)}_{i_1}(p^{I'}_{E_1,E_2}) = \Delta_{i_1}(p^{I'}_{E_1,E_2}), \quad B^{(0)}_{i_1}(p^{I'}_{E_1,E_2}) = s_{i_1}(p^{I'}_{E_1,E_2}), \quad B^{(1)}_{i_1}\circ B^{(0)}_{i_1} = -x_{i_1}s_{i_1}(p^{I'}_{E_1,E_2}).
$$

Then we have

$$
\begin{aligned}
\Delta_{i_1}\cdot\left(p^{I'}_{E_1,E_2}\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v)\right) &= B^{(-1)}_{i_1}(p^{I'}_{E_1,E_2})\cdot\Delta_{I'_{|E_1}}(u)\Delta_{I'_{|E_2}}(v) \\
&+ B^{(0)}_{i_1}(p^{I'}_{E_1,E_2})\cdot\left[(\Delta_{i_1}\circ\Delta_{I'_{|E_1}})(u)\Delta_{I'_{|E_2}}(v) + \Delta_{I'_{|E_1}}(u)(\Delta_{i_1}\circ\Delta_{I'_{|E_2}})(v)\right]
\end{aligned}
$$

$$+\left(B_{i_1}^{(1)} \circ B_{i_1}^{(0)}\right)(p_{E_1,E_2}^{I'}) \cdot (\Delta_{i_1} \circ \Delta_{I'_{|E_1}})(u)(\Delta_{i_1} \circ \Delta_{I'_{|E_2}})(v).$$

By using the notation $E+1 = \{e+1 | e \in E\}$ for a set of integers E, we have

$$\Delta_{I'_{|E_j}} = \Delta_{I_{|E_j+1}}, \quad \Delta_{i_1} \circ \Delta_{I_{|\{1\}\cup E_j+1}}$$

for $j \in \{1,2\}$. Therefore, we have

$$\begin{aligned}\Delta_{i_1}\left(p_{E_1,E_2}^{I'}\Delta_{I_{|E_1+1}}(u)\Delta_{I_{|E_2+1}}(v)\right) &= B_{i_1}^{(-1)}(p_{E_1,E_2}^{I'}) \cdot \Delta_{I_{|E_1+1}}(u)\Delta_{I_{|E_2+1}}(v)\\ &+ B_{i_1}^{(0)}(p_{E_1,E_2}^{I'}) \cdot \left[\Delta_{I_{|\{1\}\cup(E_1+1)}}(u)\Delta_{I_{|E_2+1}}(v) + \Delta_{I_{|E_1+1}}(u)\Delta_{I_{|\{1\}\cup(E_2+1)}}(v)\right]\\ &+ \left(B_{i_1}^{(1)} \circ B_{i_1}^{(0)}\right)(p_{E_1,E_2}^{I'}) \cdot \Delta_{I_{|\{1\}\cup(E_1+1)}}(u)\Delta_{I_{|\{1\}\cup(E_2+1)}}(v).\end{aligned}$$

Since any subset of $[l]$ is of the form $E+1$ or $\{1\} \cup (E+1)$ with $E \subseteq [l-1]$, the claim follows. ∎

Definition 5.2.6. cf. [HMSZ, Def. 4.2] Consider the power series $g^F(u,v)$ defined by $u +_F v = u + v - uvg^F(u,v)$. For $\alpha \in \Phi^{re}$, we have

$$0 = x_\alpha +_F x_{-\alpha} = x_\alpha + x_{-\alpha} - x_\alpha x_{-\alpha} g^F(x_\alpha, x_{-\alpha}).$$

Then, we have

$$g^F(x_\alpha, x_{-\alpha}) = \frac{x_\alpha + x_{-\alpha}}{x_\alpha x_{-\alpha}} = \frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}} \in R[\![\Lambda]\!]_F.$$

We denote

$$\kappa_\alpha^F := g^F(x_\alpha, x_{-\alpha}) = \frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}.$$

We will only write κ_α when the formal group law F is understood.

Remark 5.2.7. The elements $1/x_\alpha$ and $1/x_{-\alpha}$ do not lie in $R[\![\Lambda]\!]_F$. However, the power series $g^F(x_\alpha, x_{-\alpha})$ does. We write κ_α as a sum of fractions to simplify the notation.

Examples 5.2.8. Let us compute κ_α^F for the additive, multiplicative, Lorentz, and hyperbolic formal group law.

(i) For the additive formal group law F_a, we have

$$F_a(x_\alpha, x_{-\alpha}) = x_\alpha + x_{-\alpha}$$

so $\kappa_\alpha^{F_a} = g^{F_a}(x_\alpha, x_{-\alpha}) = 0$.

(ii) For the multiplicative formal group law F_m, we have

$$F_m(x_\alpha, x_{-\alpha}) = x_\alpha + x_{-\alpha} - \mu x_\alpha x_{-\alpha}$$

so $\kappa_\alpha^{F_m} = g^{F_m}(x_\alpha, x_{-\alpha}) = \mu$.

(iii) For the Lorentz formal group law F_L, we have

$$F_L(x_\alpha, x_{-\alpha}) = \frac{x_\alpha + x_{-\alpha}}{1 + \mu x_\alpha x_{-\alpha}} = 0.$$

Hence $x_\alpha + x_{-\alpha} = 0$ and $\kappa_\alpha^{F_L} = 0$.

(iv) For the hyperbolic formal group law F_{μ_1,μ_2}, we have

$$F_{\mu_1,\mu_2}(x_\alpha, x_{-\alpha}) = \frac{x_\alpha + x_{-\alpha} - \mu_1 x_\alpha x_{-\alpha}}{1 + \mu_2 x_\alpha x_{-\alpha}} = 0.$$

Therefore,

$$\frac{x_\alpha + x_{-\alpha}}{x_\alpha x_{-\alpha}} = \mu_1$$

and $\kappa_\alpha^{F_{\mu_1,\mu_2}} = \mu_1$.

5.3 Formal push-pull operators and their properties

In this section we introduce a second type of operators, called the formal push-pull operators. In a geometric context, they correspond to the composition of push-forward and pull-back defined by the map $G/B \to G/P_i$, where B is a Borel subgroup of G and P_i is the minimal parabolic subgroup corresponding to the simple root α_i (see [BE, §5]).

Definition 5.3.1. (cf. [CPZ, Def. 3.11]) We define an R-linear operator C_α^F on $R[\![\Lambda]\!]_F$ by

$$C_\alpha^F(u) := u\kappa_\alpha^F - \Delta_\alpha(u), \text{ where } u \in R[\![\Lambda]\!]_F.$$

We call the C_α's the *formal push-pull operators*. We will only write C_α when the formal group law F is understood.

The next proposition is a generalization of [CPZ, Prop. 3.12].

Proposition 5.3.2. *The following formulas hold for any $u, v \in R[\![\Lambda]\!]_F, \lambda \in \Lambda$ and $w \in W$.*

(1) $C_\alpha(1)=\kappa_\alpha, \quad C_\alpha(x_{-\alpha})=2,$

(2) $C_\alpha(u)x_\alpha x_{-\alpha}=ux_\alpha+s_\alpha(u)x_\alpha, \quad C_\alpha(ux_{-\alpha})=u+s_\alpha(u),$

(3) $C_\alpha s_\alpha(u)=C_{-\alpha}(u), \quad s_\alpha C_\alpha(u)=C_\alpha(u),$

(4) $C_\alpha(uv)=C_\alpha(u)v+s_\alpha(u)C_\alpha(v)-s_\alpha(u)v\kappa_\alpha=C_\alpha(u)v-s_\alpha(u)\Delta_\alpha(v),$

(5) $wC_\alpha w^{-1}(u)=C_{w(\alpha)}(u),$

(6) $C_\alpha\Delta_\alpha=\Delta_\alpha C_\alpha=\Delta_\alpha C_{-\alpha}=0.$

Proof: We complete the proof of [CPZ, Prop. 3.12]. By definition of C_α,

$$C_\alpha(u)=u\kappa_\alpha-\Delta_\alpha(u)=\frac{u}{x_\alpha}+\frac{u}{x_{-\alpha}}-\left(\frac{u-s_\alpha(u)}{x_\alpha}\right)=\frac{u}{x_{-\alpha}}+\frac{s_\alpha(u)}{x_\alpha}.$$

We will use this equation and Proposition 5.2.1 to prove properties (1)-(6).

(1) We have

$$C_\alpha(1)=\frac{1}{x_{-\alpha}}+\frac{s_\alpha(1)}{x_\alpha}=\frac{1}{x_{-\alpha}}+\frac{1}{x_\alpha}=\kappa_\alpha$$

and

$$C_\alpha(x_{-\alpha})=\frac{x_{-\alpha}}{x_{-\alpha}}+\frac{s_\alpha(x_\alpha)}{x_\alpha}=\frac{x_{-\alpha}}{x_{-\alpha}}+\frac{x_\alpha}{x_\alpha}=1+1=2.$$

(2) We have

$$C_\alpha(u)x_\alpha x_{-\alpha}=\frac{ux_\alpha x_{-\alpha}}{x_{-\alpha}}+\frac{s_\alpha(u)x_\alpha x_{-\alpha}}{x_\alpha}=ux_\alpha+s_\alpha(u)x_{-\alpha}$$

and

$$C_\alpha(ux_{-\alpha})=\frac{ux_{-\alpha}}{x_{-\alpha}}+\frac{s_\alpha(ux_{-\alpha})}{x_\alpha}=u+\frac{s_\alpha(u)x_\alpha}{x_\alpha}=u+s_\alpha(u).$$

(3) We have

$$C_\alpha\big(s_\alpha(u)\big)=\frac{s_\alpha(u)}{x_{-\alpha}}+\frac{s_\alpha\big(s_\alpha(u)\big)}{x_\alpha}=\frac{s_\alpha(u)}{x_{-\alpha}}+\frac{u}{x_\alpha}=\frac{u}{x_{-(-\alpha)}}+\frac{s_{-\alpha}(u)}{x_{(-\alpha)}}=C_{-\alpha}(u)$$

and

$$s_\alpha C_\alpha(u)=s_\alpha\left(\frac{u}{x_{-\alpha}}+\frac{s_\alpha(u)}{x_\alpha}\right)=\frac{s_\alpha(u)}{x_\alpha}+\frac{u}{x_{-\alpha}}=C_\alpha(u).$$

(4) We have

$$C_\alpha(uv)=\frac{uv}{x_{-\alpha}}+\frac{s_\alpha(uv)}{x_\alpha}$$

$$
\begin{aligned}
&= \frac{uv}{x_{-\alpha}} + \frac{s_\alpha(u)v}{x_\alpha} + \frac{s_\alpha(u)v}{x_{-\alpha}} + \frac{s_\alpha(uv)}{x_\alpha} - \frac{s_\alpha(u)v}{x_\alpha} - \frac{s_\alpha(u)v}{x_{-\alpha}} \\
&= \left(\frac{u}{x_{-\alpha}} + \frac{s_\alpha(u)}{x_\alpha}\right) v + s_\alpha(u)\left(\frac{v}{x_{-\alpha}} + \frac{s_\alpha(v)}{x_\alpha}\right) - s_\alpha(u)v\left(\frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}\right) \\
&= C_\alpha(u)v + s_\alpha(u)C_\alpha(v) - s_\alpha(u)v\kappa_\alpha \\
&= C_\alpha(u)v + s_\alpha(u)\big(C_\alpha(v) - v\kappa_\alpha\big) = C_\alpha(u)v - s_\alpha(u)\Delta_\alpha(v)
\end{aligned}
$$

since $C_\alpha(v) - v\kappa_\alpha = -\Delta_\alpha(v)$ by definition of $C_\alpha(v)$.

(5) Since we know $ws_\alpha w^{-1} = s_{w(\alpha)}$, we have

$$
\begin{aligned}
wC_\alpha w^{-1}(u) = w\left(\frac{w^{-1}(u)}{x_{-\alpha}} + \frac{s_\alpha(w^{-1}(u))}{x_\alpha}\right) &= \frac{u}{x_{-w(\alpha)}} + \frac{ws_\alpha w^{-1}(u)}{x_{w(\alpha)}} \\
&= \frac{u}{x_{-w(\alpha)}} + \frac{s_{w(\alpha)}(u)}{x_{w(\alpha)}} = C_{w(\alpha)}(u).
\end{aligned}
$$

(6) Note that $\kappa_\alpha = \kappa_{-\alpha}$ and

$$
s_\alpha(\kappa_\alpha) = s_\alpha\left(\frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}\right) = \frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha} = \kappa_\alpha.
$$

Then we have

$$
\begin{aligned}
C_\alpha\Delta_\alpha(u) &= C_\alpha\left(\frac{u - s_\alpha(u)}{x_\alpha}\right) = \left(\frac{u - s_\alpha(u)}{x_\alpha}\right)\kappa_\alpha - \Delta_\alpha^2(u) \\
&= \frac{u\kappa_\alpha - s_\alpha(u\kappa_\alpha)}{x_\alpha} - \Delta_\alpha^2(u) = \Delta_\alpha\big(u\kappa_\alpha - \Delta_\alpha(u)\big) = \Delta_\alpha C_\alpha(u).
\end{aligned}
$$

Similarly, we have $C_\alpha\Delta_\alpha = \Delta_\alpha C_\alpha = \Delta_\alpha C_{-\alpha}$ since

$$
\begin{aligned}
\Delta_\alpha C_{-\alpha}(u) = \Delta_\alpha(u\kappa_{-\alpha} - \Delta_{-\alpha}(u)) &= \frac{u\kappa_{-\alpha} - s_\alpha(u\kappa_{-\alpha})}{x_\alpha} - \Delta_\alpha\Delta_{-\alpha}(u) \\
&= \left(\frac{u - s_\alpha(u)}{x_\alpha}\right)\kappa_{-\alpha} + \Delta_\alpha s_\alpha\Delta_\alpha(u) \text{ by Proposition 5.2.1(c)} \\
&= \left(\frac{u - s_\alpha(u)}{x_\alpha}\right)\kappa_\alpha - \Delta_\alpha^2(u) \text{ by Proposition 5.2.1(c).}
\end{aligned}
$$

By Proposition 5.2.1(b), we have

$$
\Delta_\alpha^2(u) = \frac{\Delta_\alpha(u)}{x_\alpha} + \frac{\Delta_{-\alpha}(u)}{x_\alpha} = \frac{\Delta_\alpha(u)}{x_\alpha} + \frac{\Delta_\alpha(u)}{x_{-\alpha}}.
$$

Then

$$
C_\alpha\Delta_\alpha(u) = \Delta_\alpha(u)\kappa_\alpha - \Delta_\alpha^2(u) = \frac{\Delta_\alpha(u)}{x_\alpha} + \frac{\Delta_\alpha(u)}{x_{-\alpha}} - \frac{\Delta_\alpha(u)}{x_\alpha} - \frac{\Delta_\alpha(u)}{x_{-\alpha}} = 0.
$$

∎

Example 5.3.3. Let $F = F_{\mu_1,\mu_2}$ be the hyperbolic formal group law and let $\alpha \in \Phi^{re}$. By Example 5.2.2, we have

$$\Delta_\alpha(x_\alpha) = 1 - \frac{x_{-\alpha}}{x_\alpha} = 1 - \frac{1}{\mu_1 x_\alpha - 1}.$$

Therefore,

$$C_\alpha(x_\alpha) = \kappa_\alpha x_\alpha - \Delta_\alpha(x_\alpha) = \mu_1 x_\alpha - 1 + \frac{1}{\mu_1 x_\alpha - 1}$$

since we know that $\kappa_\alpha = \mu_1$ by Example 5.2.8.

Chapter 6

Formal affine Demazure algebras

6.1 Formal Demazure/push-pull elements

Let Λ be a formal Demazure lattice of a Kac-Moody root system of any type. Let Q^F be the localization of $R[\![\Lambda]\!]_F$ at the multiplicative subset generated by $\{x_\alpha \mid \alpha \in \Phi^{re}\}$. Since W preserves the set of real roots, the action of W on $R[\![\Lambda]\!]_F$ extends to an action on Q^F.

Most of the results in this chapter are generalizations of results in [CZZ1], [CZZ2], and [HMSZ].

Definition 6.1.1. (cf. [HMSZ, Def. 6.1]) We define the *twisted formal group algebra* to be the R-module $Q_W^F := R[W] \otimes_R Q^F$ with multiplication defined as

$$(\delta_{w'}\psi')(\delta_w\psi) = \delta_{w'w}w^{-1}(\psi')\psi, \forall w, w' \in W, \psi, \psi' \in Q^F$$

where δ_w is the element in $R[W]$ corresponding to $w \in W$ and $\delta_1 = 1 \in R[W]$.

Remarks 6.1.2. (1) By definition of the multiplication, we have $\delta_{w'}\delta_w = \delta_{w'w}$ for all $w, w' \in W$.

(2) Note that Q_W^F is a free right Q^F-module with basis $\{\delta_w\}_{w\in W}$. Also, Q_W^F is not a Q^F-algebra since $\delta_e Q^F = Q^F\delta_e$ is not central in Q_W^F.

Definition 6.1.3. For each $\alpha \in \Phi^{re}$ we define the *formal Demazure element* as

$$X_\alpha := \frac{1}{x_\alpha}(1 - \delta_{s_\alpha}) = \frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha} = \frac{1}{x_\alpha} - \frac{\delta_{s_\alpha}}{x_{-\alpha}} \in Q_W^F$$

and the *formal push-pull element* by

$$Y_\alpha := \kappa_\alpha - X_\alpha = \frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_{s_\alpha} \in Q_W^F.$$

Remark 6.1.4. When we write quotients in Q_W^F we mean multiplication by the left by the numerator, e.g. $\frac{\delta_{s_\alpha}}{x_\alpha} = \delta_{s_\alpha}\frac{1}{x_\alpha}$.

The lemma below is a generalization of [CZZ2, §3] and [ZZ, Lemma 2.10].

Lemma 6.1.5. *The formal Demazure and push-pull operators satisfy the following relations:*

(1) $\delta_w X_\alpha \delta_{w^{-1}} = X_{w(\alpha)}$ *and* $\delta_w Y_\alpha \delta_{w^{-1}} = Y_{w(\alpha)}$*,*

(2) $X_\alpha \delta_{s_\alpha} = -X_\alpha$*,* $\delta_{s_\alpha} X_\alpha = X_\alpha + \kappa_\alpha \delta_{s_\alpha} - \kappa_\alpha$*,*

(3) $Y_\alpha \delta_{s_\alpha} = -Y_\alpha + \kappa_\alpha \delta_{s_\alpha} + \kappa_\alpha$*,* $\delta_{s_\alpha} Y_\alpha = Y_\alpha$*,*

(4) $X_\alpha^2 = \kappa_\alpha X_\alpha = X_\alpha \kappa_\alpha, Y_\alpha^2 = \kappa_\alpha Y_\alpha = Y_\alpha \kappa_\alpha$*,*

(5) $X_\alpha Y_\alpha = Y_\alpha X_\alpha = 0$*.*

Proof: We provide a proof since the proofs are not complete in the references. We will show the proofs of (1),(2),(4), and (5) for X_α. The remaining proofs for Y_α are similar. For (1), since $ws_\alpha w^{-1} = s_{w(\alpha)}$, we have

$$\begin{aligned}\delta_w X_\alpha \delta_{w^{-1}} &= \delta_w \left(\frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\right)\delta_{w^{-1}} = \left(\frac{1}{x_{w(\alpha)}}\delta_w - \frac{1}{x_{w(\alpha)}}\delta_w\delta_{s_\alpha}\right)\delta_{w^{-1}} \\ &= \frac{1}{x_{w(\alpha)}}\delta_w\delta_{w^{-1}} - \frac{1}{x_{w(\alpha)}}\delta_w\delta_{s_\alpha}\delta_{w^{-1}} = \frac{1}{x_{w(\alpha)}} - \frac{1}{x_{w(\alpha)}}\delta_{s_{w(\alpha)}} = X_{w(\alpha)}.\end{aligned}$$

For (2), we have

$$X_\alpha\delta_{s_\alpha} = \left(\frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\right)\delta_{s_\alpha} = \frac{1}{x_\alpha}\delta_{s_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\delta_{s_\alpha} = \frac{1}{x_\alpha}\delta_{s_\alpha} - \frac{1}{x_\alpha} = -X_\alpha$$

and

$$\begin{aligned}\delta_{s_\alpha} X_\alpha &= \delta_{s_\alpha}\left(\frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\right) = \frac{1}{x_{-\alpha}}\delta_{s_\alpha} - \frac{1}{x_{-\alpha}} \\ &= \frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha} + \frac{1}{x_\alpha}\delta_{s_\alpha} + \frac{1}{x_{-\alpha}}\delta_{s_\alpha} - \frac{1}{x_\alpha} - \frac{1}{x_{-\alpha}} = X_\alpha + \kappa_\alpha\delta_{s_\alpha} - \kappa_\alpha.\end{aligned}$$

For (4), we have

$$\begin{aligned}X_\alpha^2 = \left(\frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\right)\left(\frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\right) &= \frac{1}{x_\alpha^2} - \frac{1}{x_\alpha x_{-\alpha}}\delta_{s_\alpha} - \frac{1}{x_\alpha^2}\delta_{s_\alpha} + \frac{1}{x_\alpha x_{-\alpha}} \\ &= \left(\frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}\right)\left(\frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\right) = \kappa_\alpha X_\alpha\end{aligned}$$

$$= \left(\frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\right)\left(\frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}\right) = X_\alpha \kappa_\alpha.$$

For (5), we have

$$\begin{aligned} X_\alpha Y_\alpha &= \left(\frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\right)\left(\frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_{s_\alpha}\right) = \frac{1}{x_\alpha}\frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha^2}\delta_{s_\alpha} - \frac{1}{x_\alpha^2}\delta_{s_\alpha} - \frac{1}{x_\alpha}\frac{1}{x_{-\alpha}} \\ &= 0 = \frac{1}{x_{-\alpha}}\frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}\frac{1}{x_\alpha}\delta_{s_\alpha} + \frac{1}{x_\alpha}\frac{1}{x_{-\alpha}}\delta_{s_\alpha} - \frac{1}{x_\alpha}\frac{1}{x_{-\alpha}} \\ &= \left(\frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_{s_\alpha}\right)\left(\frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\right) = Y_\alpha X_\alpha. \end{aligned}$$

∎

Let $X_{\alpha_i} =: X_i$ and $Y_{\alpha_i} =: Y_i$ for any simple root α_i. For any sequence $I = (i_1, ..., i_l)$ we denote

$$\delta_I := \delta_{s_{i_1}\dots s_{i_l}}, X_I := X_{i_1}\cdots X_{i_l} \text{ and } Y_I := Y_{i_1}\cdots Y_{i_l}.$$

The following result is a generalization of[CZZ1, Lemma 5.4] and [CZZ1, Cor. 5.6].

Lemma 6.1.6. *Given a reduced sequence I_v of $v \in W$ of length l let*

$$X_{I_v} = \sum_{w\in W} a_{v,w}\delta_w = \sum_{w\in W} \delta_w a'_{v,w}$$

for some $a_{v,w}, a'_{v,w} \in Q^F$. Then

(a) *$a_{v,w} = 0$ unless $w \leq v$ with respect to the Bruhat order on W,*

(b) *$a_{v,v} = (-1)^l \prod_{\alpha\in v(\Phi_-)\cap\Phi_+} x_\alpha^{-1} = a'_{v,v^{-1}}$,*

(c) *$a'_{v,w} = 0$ unless $w \leq v^{-1}$.*

Moreover, we have $\delta_{I_v} = \sum_{w\leq v} b^X_{v,w} X_w$ for some $b^X_{v,w} \in R[\![\Lambda]\!]_F$. Furthermore, $b_{w,w} = \prod_{\alpha\in v(\Phi_-)\cap\Phi_+}(-1)^l x_\alpha$.

Proof: We mimic the proof of [CZZ1]. We will prove the first part of the lemma by induction on the length l of v. The lemma holds for $l = 1$ since if $v = s_i$ for some $i \in \{1, ..., \mathrm{rk}(\Phi)\}$, $v(\Phi_-) \cap \Phi_+ = \alpha_i$ and $X_i = \frac{1}{x_i} - \frac{1}{x_i}\delta_{s_i}$. Therefore, (a)-(c) hold because $a_{s_i,s_i} = -x_i^{-1} = (-1)^1 x_i^{-1}$ and $a_{s_i,w} = 0 = a'_{s_i,w}$ for $w \geq v$.

Let $I_v = (i_1, ..., i_l)$ be a reduced sequence of v and let $\beta = \alpha_{i_1}$. Then $I_{v'} = (i_2, ..., i_l)$ is a reduced sequence for $v' = s_\beta v$. Moreover, we have

(1) $w \le v'$ implies $w \le v$ and $s_\beta w \le v$ by Lemma 2.2.18;

(2) $\{\beta\} \cup s_\beta\big(v'(\Phi_-) \cap \Phi_+\big) = v(\Phi_-) \cap \Phi_+$ by Corollary 2.2.11;

(3) $w^{-1} \le v$ if and only if $w \le v^{-1}$ by Lemma 2.2.16.

We have

$$\begin{aligned} X_{I_v} = X_{s_\beta} X_{I'_v} &= x_\beta^{-1}(1-\delta_{s_\beta}) \sum_{w \le v'} a_{v',w}\delta_w \\ &= \sum_{w\le v'} x_\beta^{-1} a_{v',w}\delta_w - \sum_{w \le v'} x_\beta^{-1} s_\beta(a_{v',w})\delta_{s_\beta w} \\ &= -x_\beta^{-1} s_\beta \left(a_{v',v'}\delta_v + \sum_{w\le v} a_{w,v}\delta_w \right) \text{ by (1).} \end{aligned}$$

Part (a) holds since $a_{v,w} = 0$ for $w > v$. Also, $a_{v,v} = -x_\beta^{-1} s_\beta(a_{v',v'}) = (-1)^l \prod_{\alpha \in v(\Phi_-)\cap\Phi_+} x_\alpha^{-1}$ by induction hypothesis and (2). By multiplication in Q_W^F we have $q\delta_w = \delta_w w^{-1}(q)$ and we get (c) from (3) and identifying $a'_{v,w} = a_{v,w^{-1}}$.

For the second part of the proof, observe that $(a_{v,w})_{(v,w)\in W^2}$ is a triangular matrix, after fixing a proper ordering on the elements of W, with invertible coefficients on the diagonal. The decomposition of δ_v follows from the fact that the inverse of a triangular matrix is also a triangular matrix with inverse coefficients on the diagonal. The rest follows from the fact that $\delta_{s_\alpha} = 1 - x_\alpha X_\alpha$. ∎

Remark 6.1.7. Note that the coefficient $a_{v,v}$ does not depend on the choice of a reduced expression for v.

We get a similar result to Lemma 6.1.6 for the push-pull elements, as in [CZZ2, Lemma 3.3].

Lemma 6.1.8. *Let I_v be a reduced sequence for $v =\in W$. Then $Y_{I_v} = \sum_{w\le v} a^Y_{v,w}\delta_w$ for some $a^Y_{v,w} \in Q^F$. Moreover, we have $\delta_{I_v} = \sum_{w \le v} b^Y_{v,w} Y_w$ for some $b^Y_{v,w} \in R[\![\Lambda]\!]_F$.*

Proof: This follows by the same arguments as in the proof of Lemma 6.1.6. ∎

Example 6.1.9. For the root system $A_1 \times A_1$, we have $W = \{\mathrm{id}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1}s_{\alpha_2}\}$. Then the matrix of coefficients for the decomposition of X_v, for $v \in W$, in terms of

δ_w, for $w \leq v$, is

$$(a_{v,w})_{v,w\in W} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{x_{\alpha_1}} & \frac{-1}{x_{\alpha_1}} & 0 & 0 \\ \frac{1}{x_{\alpha_2}} & 0 & \frac{-1}{x_{\alpha_2}} & 0 \\ \frac{1}{x_{\alpha_1}x_{\alpha_2}} & \frac{-1}{x_{\alpha_1}x_{\alpha_2}} & \frac{-1}{x_{\alpha_1}x_{\alpha_2}} & \frac{1}{x_{\alpha_1}x_{\alpha_2}} \end{pmatrix}.$$

Since $s_{\alpha_1}s_{\alpha_2} = s_{\alpha_2}s_{\alpha_1}$ and the last row of the matrix is symmetric in α_1 and α_2, then $X_{\alpha_1}X_{\alpha_2} = X_{\alpha_2}X_{\alpha_1}$.

Similarly, the matrix of coefficients for Y_v, for $v \in W$, is

$$(a^Y_{v,w})_{v,w\in W} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{x_{-\alpha_1}} & \frac{1}{x_{\alpha_1}} & 0 & 0 \\ \frac{1}{x_{-\alpha_2}} & 0 & \frac{1}{x_{\alpha_2}} & 0 \\ \frac{1}{x_{-\alpha_1}x_{-\alpha_2}} & \frac{1}{x_{\alpha_1}x_{-\alpha_2}} & \frac{1}{x_{-\alpha_1}x_{\alpha_2}} & \frac{1}{x_{\alpha_1}x_{\alpha_2}} \end{pmatrix}$$

and $Y_{\alpha_1}Y_{\alpha_2} = Y_{\alpha_2}Y_{\alpha_1}$.

As in [CZZ1, Cor. 5.6] and [CZZ2, Cor. 3.4], by Lemma 6.1.6 and Lemma 6.1.8, we obtain the following corollary.

Corollary 6.1.10. *The elements $(X_{I_w})_{w\in W}$ (resp. $(Y_{I_w})_{w\in W}$) form a basis of Q_W^F as a left Q-module.*

6.2 Formal (affine) Demazure algebra

We now have all the tools necessary to define our main objects.

Definition 6.2.1. (cf. [HMSZ, Def. 6.3]) We define the *formal Demazure algebra* D_F to be the R-subalgebra of Q_W^F generated by the formal Demazure elements X_α for all $\alpha \in \Phi^{re}$. The *formal affine Demazure algebra* $\mathbf{D}_F$ is the R-subalgebra of Q_W^F generated by the elements of the formal group algebra $R[\![\Lambda]\!]_F$ and by D_F. In fact, by [CZZ1, Lemma 5.8], $\mathbf{D}_F$ is generated by X_{α_i} for all simple roots α_i, $i = 1, ..., \mathrm{rk}(\Phi)$ together with the elements of $R[\![\Lambda]\!]_F$.

Remark 6.2.2. By definition, D_F is also generated by the formal push-pull elements Y_{α_i} for all simple roots α_i, $i = 1, ..., \mathrm{rk}(\Phi)$.

Note that the action of the operators Δ_α and C_α on $R[\![\Lambda]\!]_F$ extends to an action on Q_W^F. Hence, we get the following generalization of [HMSZ, Lemma 6.5].

Lemma 6.2.3. *For all* $\psi \in Q^F$ *and* $\alpha \in \Phi$, *we have*

$$\psi X_\alpha = X_\alpha s_\alpha(\psi) + \Delta_\alpha(\psi) \quad \textit{and} \quad \psi Y_\alpha = Y_\alpha s_\alpha(\psi) + \Delta_{-\alpha}(\psi).$$

Proof: We have

$$\begin{aligned}
\psi X_\alpha &= \psi\left(\frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\right) = \frac{\psi}{x_\alpha} - \frac{\psi}{x_\alpha}\delta_{s_\alpha} = \frac{\psi}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha} s_\alpha(\psi) \\
&= \frac{\psi}{x_\alpha} - \frac{s_\alpha(\psi)}{x_\alpha} + \frac{s_\alpha(\psi)}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha} s_\alpha(\psi) \\
&= \frac{\psi - s_\alpha(\psi)}{x_\alpha} + \left(\frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}\right) s_\alpha(\psi) = X_\alpha s_\alpha(\psi) + \Delta_\alpha(\psi)
\end{aligned}$$

and

$$\begin{aligned}
\psi Y_\alpha &= \psi\left(\frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_{s_\alpha}\right) = \frac{\psi}{x_{-\alpha}} + \frac{\psi}{x_\alpha}\delta_{s_\alpha} = \frac{\psi}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_{s_\alpha} s_\alpha(\psi) \\
&= \frac{\psi}{x_{-\alpha}} - \frac{s_\alpha(\psi)}{x_{-\alpha}} + \frac{s_\alpha(\psi)}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_{s_\alpha} s_\alpha(\psi) \\
&= \frac{\psi - s_{-\alpha}(\psi)}{x_{-\alpha}} + \left(\frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_{s_\alpha}\right) s_\alpha(\psi) = Y_\alpha s_\alpha(\psi) + \Delta_{-\alpha}(\psi).
\end{aligned}$$

∎

Lemma 6.2.4. [HMSZ, Lemma 6.7] *For all* $\lambda, \mu \in \Lambda \setminus \{0\}$ *with* $\lambda + \mu \neq 0$,

$$\kappa_{\lambda,\mu} := \frac{1}{x_{\lambda+\mu}}\left(\frac{1}{x_\mu} - \frac{1}{x_{-\lambda}}\right) - \frac{1}{x_\lambda x_\mu} \in R[\![\Lambda]\!]_F.$$

For two simple roots α_i and α_j, we will denote $\kappa_{a\alpha_i+v\alpha_j,c\alpha_i+d\alpha_j}$ as $\kappa_{ai+bj,ci+dj}$.

The lemma below is a generalization of [BE, p. 809].

Lemma 6.2.5. *For* $\lambda, \mu \in \Lambda \setminus \{0\}$ *with* $\lambda + \mu \neq 0$, $\kappa_{\lambda,\mu} = 0$ *if and only if the formal group law is of the form* $F(u, v) = u + v - \mu_1 uv$ *for some* $\mu_1 \in R$.

Proof: We follow the proof of [BE]. By definition, we have

$$\kappa_{\lambda,\mu} = \frac{x_\lambda(x_{-\lambda} - x_\mu) - x_{\lambda+\mu}x_{-\lambda}}{x_\lambda x_{-\lambda} x_\mu x_{\lambda+\mu}}.$$

Therefore, $\kappa_{\lambda,\mu} = 0$ if and only if $x_\lambda x_\mu = x_\lambda x_{-\lambda} - x_{\lambda+\mu}x_{-\lambda}$. By (2.1.1) we can write $x_{\lambda+\mu}$ as $x_\alpha + x_\mu + \sum_{m,n\geq 1} a_{mn}x_\lambda^m x_\mu^n$ for some $a_{mn} \in R$. We can also write $x_{-\lambda}$ as $\sum_{k\geq 1} b_k x_\lambda^k$. Then, we have

$$x_\lambda x_\mu = \left(-x_\mu - \sum_{m,n\geq 1} a_{mn}x_\lambda^m x_\mu^n\right)\left(\sum_{k\geq 1} b_k x_\lambda^k\right).$$

The left-hand side has no powers of x_μ greater than 1, so neither does the right-hand side. Since $x_{-\lambda}$ is not a zero divisor, $a_{mn} = 0$ for $n > 1$. By commutativity of the formal group law, $a_{mn} = 0$ for $m > 1$. Thus, $F(x_\lambda, x_\mu) = x_\lambda + x_\mu + a_{11}x_\lambda x_\mu$. ∎

Examples 6.2.6. By Example 6.2.5, $\kappa_{i,j} = 0$ for the additive and multiplicative formal group law. We can also compute κ_{ij} for the Lorentz and hyperbolic formal group law.

(i) For the Lorentz formal group law F_L, we have

$$x_{i+j} = \frac{x_i + x_j}{1 + \mu x_i x_j} \text{ and by Corollary 2.1.7, } x_{-i} = -x_i$$

so

$$\begin{aligned}\kappa_{i,j} &= \frac{1}{x_{i+j}}\left(\frac{1}{x_j} - \frac{1}{x_{-i}}\right) - \frac{1}{x_i x_j} = \frac{1}{x_{i+j}}\left(\frac{1}{x_j} + \frac{1}{x_i}\right) - \frac{1}{x_i x_j} \\ &= \frac{1+\mu x_i x_j}{x_i + x_j}\left(\frac{x_i + x_j}{x_i x_j}\right) - \frac{1}{x_i x_j} = \frac{1}{x_i x_j} + \mu - \frac{1}{x_i x_j} = \mu.\end{aligned}$$

(ii) For the hyperbolic formal group law F_{μ_1,μ_2}, we have

$$x_{i+j} = \frac{x_i + x_j - \mu_1 x_i x_j}{1 + \mu_2 x_i x_j} \text{ and } x_i +_F x_{-i} = \frac{x_i + x_{-i} - \mu_1 x_i x_{-i}}{1 + \mu_2 x_i x_{-i}} = 0.$$

Then,

$$x_i + x_{-i} - \mu_1 x_i x_{-i} = 0. \tag{6.2.1}$$

Using (6.2.1), we get

$$\begin{aligned}\kappa_{i,j} &= \frac{1}{x_{i+j}}\left(\frac{1}{x_j} - \frac{1}{x_{-i}}\right) - \frac{1}{x_i x_j} = \frac{1+\mu_2 x_i x_j}{x_i + x_j - \mu_1 x_i x_j}\left(\frac{x_{-i} - x_j}{x_{-i}x_j}\right) - \frac{1}{x_i x_j} \\ &= \frac{x_{-1} - x_j + \mu_2 x_i x_{-i} x_j - \mu_2 x_i x_j^2}{(x_i + x_j - \mu_1 x_i x_j)x_{-i}x_j} - \frac{x_i + x_j - \mu_1 x_i x_j}{(x_i + x_j - \mu_1 x_i x_j)x_i x_j}\end{aligned}$$

$$= \frac{x_i x_{-i} - x_i x_j + \mu_2 x_i^2 x_{-i} x_j - \mu_2 x_i^2 x_j^2 - x_i x_{-i} - x_{-i} x_j + \mu_1 x_i x_{-i} x_j}{(x_i + x_j - \mu_1 x_i x_j) x_i x_{-i} x_j}$$
$$= \frac{\mu_2 x_i^2 x_{-i} x_j - \mu_2 x_i^2 x_j^2 - x_j(x_i + x_{-i} - \mu_1 x_i x_{-i})}{x_i^2 x_{-i} x_j + x_i x_j^2 (x_{-i} - \mu_1 x_i x_{-i})}$$
$$= \frac{\mu_2 x_i^2 x_{-i} x_j - \mu_2 x_i^2 x_j^2}{x_i^2 x_{-i} x_j - x_i^2 x_j^2} = \mu_2.$$

The next proposition is a generalization of [HMSZ, Prop. 6.8].

Proposition 6.2.7. *Suppose $i, j \in I$ and let m_{ij} be the order of $s_i s_j$ in W. Then*

$$\underbrace{X_j X_i X_j \cdots}_{m_{ij} \text{ terms}} - \underbrace{X_i X_j X_i \cdots}_{m_{ij} \text{ terms}} = \sum_{w \in W, 1 \le l(w) \le m_{ij}-2} X_w \eta_w \tag{6.2.2}$$

for some $\eta_w \in Q^F$. In paticular, we have the following:

(a) If $\langle \alpha_i, \alpha_j^\vee \rangle = 0$, so that $m_{ij} = 2$, then $X_i X_j = X_j X_i$.

(b) If $\langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_j, \alpha_i^\vee \rangle = -1$, so that $m_{ij} = 3$ (A_2), then $X_j X_i X_j - X_i X_j X_i = X_i \kappa_{i,j} - X_j \kappa_{j,i}$.

(c) If $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ and $\langle \alpha_j, \alpha_i^\vee \rangle = -2$, so that $m_{ij} = 4$ (B_2), then $X_{jiji} - X_{ijij} = X_{ij}(\kappa_{i+2j,-j} + \kappa_{j,i}) - X_{ji}(\kappa_{i+j,j} + \kappa_{i,j}) + X_j(\Delta_i(\kappa_{i+j,j} + \kappa_{i,j})) - X_i(\Delta_j(\kappa_{i+2j,-j} + \kappa_{j,i}))$.

(d) If $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ and $\langle \alpha_j, \alpha_i^\vee \rangle = -3$, so that $m_{ij} = 6$ (G_2), then $X_{jijiji} - X_{ijijij} = X_{ijij}(\kappa_{j,i} + \kappa_{2i+3j,-i-2j} + \kappa_{-i-3j,i+2j} + \kappa_{i+2j,-j}) - X_{jiji}(\kappa_{i,j} + \kappa_{-2i-3j,i+2j} + \kappa_{-i-2j,i+3j} + \kappa_{i+j,j}) + X_{jij}(\Delta_i(\kappa_{i,j} + \kappa_{-2i-3j,i+2j} + \kappa_{-i-2j,i+3j} + \kappa_{i+j,j})) - X_{iji}(\Delta_j(\kappa_{j,i} + \kappa_{2i+3j,-i-2j} + \kappa_{-i-3j,i+2j} + \kappa_{i+2j,-j})) + X_{ij}\xi_{ij} - X_{ji}\xi_{ji} + X_j(\Delta_i(\xi_{ji})) - X_i(\Delta_j(\xi_{ij}))$

where $\xi_{ij} = \frac{1}{x_i x_{i+j} x_{i+2j} x_{2i+3j}} + \frac{1}{x_i x_j x_{i+2j} x_{-2i-3j}} + \frac{1}{x_i x_j x_{2i+3j} x_{-i-j}} - \frac{1}{x_i x_{i+j} x_{i+2j} x_{-i-3j}} - \frac{1}{x_i x_{i+j} x_{i+3j} x_{-j}} + \frac{1}{x_{i+j} x_{i+3j} x_{-j} x_{-2i-3j}} + \frac{1}{x_{i+3j} x_{2i+3j} x_{-j} x_{-i-2j}} + \frac{1}{x_{i+j} x_{i+2j} x_{-i-3j} x_{-2i-3j}} - \frac{1}{x_i x_j x_{i+2j} x_{i+3j}}$

and $\xi_{ji} = \frac{1}{x_i x_j x_{2i+3j} x_{-i-2j}} + \frac{1}{x_i x_j x_{i+2j} x_{-i-3j}} + \frac{1}{x_j x_{i+2j} x_{i+3j} x_{2i+3j}} - \frac{1}{x_i x_j x_{i+j} x_{2i+3j}} + \frac{1}{x_{i+j} x_{i+2j} x_{-i} x_{-2i-3j}} + \frac{1}{x_{i+3j} x_{2i+3j} x_{-i-j} x_{-i-2j}} + \frac{1}{x_{i+j} x_{i+3j} x_{-i} x_{-i-2j}} - \frac{1}{x_j x_{i+3j} x_{2i+3j} x_{-i-j}} - \frac{1}{x_j x_{i+j} x_{i+3j} x_{-i}}$.

Proof: For $\alpha \in \Phi^{re}$, let $\chi_\alpha = \frac{1}{x_\alpha}$ and $\chi'_\alpha = -\frac{1}{x_\alpha}$. We denote $\chi_{ai+bj} = \chi_{a\alpha_i + b\alpha_j}$ and $\chi'_{ai+bj} = \chi'_{a\alpha_i + b\alpha_j}$. Let

$$(n, k) = \begin{cases} (j, i) \text{ if } m_{ij} \text{ is even,} \\ (i, j) \text{ if } m_{ij} \text{ is odd.} \end{cases}$$

Then since $X_i = \frac{1}{x_i} - \frac{1}{x_i}\delta_{s_i} = \chi_i + \chi_i'\delta_{s_i}$, we have

$$\underbrace{X_iX_j\cdots X_nX_k}_{m_{ij}\text{ terms}} = (\chi_j + \chi_j'\delta_{s_j})(\chi_i + \chi_i'\delta_{s_i})\cdots(\chi_n + \chi_n'\delta_{s_n})(\chi_k + \chi_k'\delta_{s_k}). \tag{6.2.3}$$

Since $\delta_w, w \in W$ form a basis of Q_W^F as a right Q^F-module, the expression above can be written as a sum of right Q^F-multiples of δ_w. Recall that

$$q\delta_w = \delta_w w^{-1}(q)$$

for any $q \in Q^F$. Therefore, the leading term of (6.2.3) with respect to the length of w is

$$\delta_{\underbrace{s_js_i\cdots s_ns_k}_{m_{ij}\text{ terms}}}(\chi'_{-\underbrace{s_ks_n\cdots s_js_i(\alpha_j)}_{m_{ij}-1\text{ terms}}}\cdots\chi'_{-s_k(\alpha_n)}\chi'_{-\alpha_k}).$$

Similarly, the leading term for $\underbrace{X_jX_i\cdots X_kX_n}_{m_{ij}\text{ terms}}$ is

$$\delta_{\underbrace{s_is_j\cdots s_ks_n}_{m_{ij}\text{ terms}}}(\chi'_{-\underbrace{s_ns_k\cdots s_is_j(\alpha_i)}_{m_{ij}-1\text{ terms}}}\cdots\chi'_{-s_n(\alpha_k)}\chi'_{-\alpha_n}).$$

By Lemma 2.2.9, $\alpha_k, s_k(\alpha_n), ..., s_ks_n\cdots s_js_i(\alpha_j)$ (resp. $\alpha_n, s_n(\alpha_k), ..., s_ns_k\cdots s_is_j(\alpha_i)$) are precisely the positive roots mapped to negative roots by $s_js_i\cdots s_ns_k$ (resp. $s_is_j\cdots s_ks_n$). Since $s_js_i\cdots s_ns_k = s_is_j\cdots s_ks_n$, the sets of positive roots sent to negative roots coincide. Therefore, the highest order terms in the expression $X_jX_iX_j\cdots - X_iX_jX_i$ cancel.

We can now consider the terms of order $m_{ij}-1$. Let

$$B = \underbrace{\chi_i'\delta_{s_i}\chi_j'\delta_{s_j}\cdots\chi_n'\delta_{s_n}\chi_k'\delta_{s_k}}_{m_{ij}-1\text{ pairs}} \text{ and } B' = \underbrace{\chi_j'\delta_{s_j}\chi_i'\delta_{s_i}\cdots\chi_k'\delta_{s_k}\chi_n'\delta_{s_n}}_{m_{ij}-1\text{ pairs}}.$$

Then the terms of order $m_{ij}-1$ in $X_jX_iX_j\cdots$ and $X_iX_jX_i\cdots$ are $\chi_jB + B'\chi_k$ and $\chi_iB' + B\chi_n$ respectively. We claim that $\chi_jB = B\chi_n$ and $\chi_iB' = B'\chi_k$. We have

$$\begin{aligned}\chi B &= \chi_j\chi_i'\delta_{s_i}\chi_j'\delta_{s_j}\cdots\chi_n'\delta_{s_n}\chi_k'\delta_{s_k}\\ &= \delta_{s_is_j\cdots s_ns_k}(\chi_{s_ks_n\cdots s_js_i(\alpha_j)}\chi'_{-s_j\cdots s_ns_k(\alpha_i)}\cdots\chi'_{-s_k(\alpha_n)}\chi'_{-\alpha_k})\end{aligned}$$

and

$$\begin{aligned}B\chi_n &= \chi_i'\delta_{s_i}\chi_j'\delta_{s_j}\cdots\chi_n'\delta_{s_n}\chi_k'\delta_{s_k}\chi_n\\ &= \delta_{s_is_j\cdots s_ns_k}(\chi_{-s_j\cdots s_ns_k(\alpha_i)}\cdots\chi'_{-s_k(\alpha_n)}\chi'_{-\alpha_k}\chi_{\alpha_n}).\end{aligned}$$

We need to show that $\underbrace{s_k s_n \cdots s_j s_i}_{m_{ij}-1 \text{ terms}}(\alpha_j) = \alpha_n$.
For $m_{ij} = 4, 6$, we have $(n, k) = (j, i)$ and by Lemma 2.2.13

$$\underbrace{s_j(s_i s_j \cdots s_j s_i)}_{m_{ij} \text{ terms}}(\alpha_j) = -\alpha_j.$$

Therefore,

$$\underbrace{s_k s_n \cdots s_j s_i}_{m_{ij}-1 \text{ terms}}(\alpha_j) = \underbrace{s_i s_j \cdots s_j s_i}_{m_{ij}-1 \text{ terms}}(\alpha_j) = \alpha_j = \alpha_n.$$

For $m_{ij} = 3$, we have $(n, k) = (i, j)$ and by Lemma 2.2.13

$$\underbrace{s_i(s_j s_i \cdots s_j s_i)}_{m_{ij} \text{ terms}}(\alpha_j) = -\alpha_i.$$

Therefore,

$$\underbrace{s_k s_n \cdots s_j s_i}_{m_{ij}-1 \text{ terms}}(\alpha_j) = \underbrace{s_j s_i \cdots s_j s_i}_{m_{ij}-1 \text{ terms}}(\alpha_j) = \alpha_i = \alpha_n.$$

Similarly, we can prove that $\chi_i B' = B'\chi_k$. Thus, the terms of order $m_{ij} - 1$ in $X_j X_i X_j \cdots - X_i X_j X_i \cdots$ cancel.

From the definition of the formal Demazure elements, it is clear that $X_i X_j \cdots X_n X_k$ and $X_j X_i \cdots X_k X_n$ consists of nonzero multiple of $\delta_{s_i s_j \cdots s_n s_k}$ and $\delta_{s_j s_i \cdots s_k s_n}$ respectively, plus lower order terms. Therefore by the computations above, this proves (6.2.2), but with the sum on the right hand side over $w \in W$ with $l(w) \leq m_{ij} - 2$ instead of $1 \leq l(w) \leq m_{ij} - 2$. It remains to show that the terms with $l(w) = 0$, i.e. the constant terms or terms of degree zero, cancel. To do so, observe that there is a natural action of Q_W^F on Q^F given by left multiplication by the elements of Q^F and by action of they Weyl group for the elements of $R[W]$. Since the constant terms lie in Q^F, under this action, for any constant term $q \in Q^F$, $q(1) = q$. However, if we act on $1 \in Q^F$ by any element X_p, $p \in I$, we get

$$X_p(1) = \left(\frac{1}{x_p} - \frac{1}{x_p}\delta_{s_p}\right)(1) = \frac{1}{x_p} - \frac{1}{x_p}s_p(1) = \Delta_p(1) = 0.$$

Hence, the constant terms must be zero and the sum on the right hand side of Proposition 6.2.2 is over $w \in W$ with $1 \leq l(w) \leq m_{ij} - 2$.

To complete the proof it remains to show that the relations (a)-(d) hold.

Under the assumption of (a), we have $s_i(\alpha_j) = \alpha_j$ and recall from Example 6.1.9 that

$$X_iX_j = \frac{1}{x_ix_j} - \frac{1}{x_ix_j}\delta_{s_i} - \frac{1}{x_ix_j}\delta_{s_j} + \frac{1}{x_ix_j}\delta_{s_is_j}.$$

Since $s_is_j = s_js_i$ and the expression above is symmetric in i and j, we have $X_iX_j = X_jX_i$.

Under the assumptions of (b), we have

$$\begin{aligned}
X_jX_iX_j &= \left(\frac{1}{x_j} - \frac{1}{x_j}\delta_{s_j}\right)\left(\frac{1}{x_i} - \frac{1}{x_i}\delta_{s_i}\right)\left(\frac{1}{x_j} - \frac{1}{x_j}\delta_{s_j}\right) \\
&= \frac{1}{x_i}\frac{1}{x_j}\frac{1}{x_j} - \frac{1}{x_j}\frac{1}{x_i}\frac{1}{x_j}\delta_{s_j} - \frac{1}{x_j}\frac{1}{x_i}\delta_{s_i}\frac{1}{x_j} - \frac{1}{x_j}\delta_{s_j}\frac{1}{x_i}\frac{1}{x_j} + \frac{1}{x_j}\frac{1}{x_i}\delta_{s_i}\frac{1}{x_j}\delta_{s_j} \\
&\quad + \frac{1}{x_j}\delta_{s_j}\frac{1}{x_i}\frac{1}{x_j}\delta_{s_j} + \frac{1}{x_j}\delta_{s_j}\frac{1}{x_i}\delta_{s_i}\frac{1}{x_j} - \frac{1}{x_j}\delta_{s_j}\frac{1}{x_i}\delta_{s_i}\frac{1}{x_j}\delta_{s_j} \\
&= \frac{1}{x_ix_j^2} - \frac{\delta_{s_j}}{s_j(x_j)s_j(x_i)s_j(x_j)} - \frac{\delta_{s_i}}{s_i(x_j)s_i(x_i)x_j} - \frac{\delta_{s_j}}{s_j(x_j)x_ix_j} \\
&\quad + \frac{\delta_{s_i}\delta_{s_j}}{s_js_i(x_j)s_js_i(x_i)s_j(x_j)} + \frac{\delta_{s_j}\delta_{s_j}}{s_j^2(x_j)s_j(x_i)s_j(x_j)} + \frac{\delta_{s_j}\delta_{s_i}}{s_is_j(x_j)s_i(x_i)x_j} \\
&\quad - \frac{\delta_{s_j}\delta_{s_i}\delta_{s_j}}{s_js_is_j(x_j)s_js_i(x_i)s_j(x_j)} \\
&= \frac{1}{x_ix_j^2} - \frac{\delta_{s_j}}{x_{i+j}x_{-j}^2} - \frac{\delta_{s_i}}{x_{i+j}x_{-i}x_j} - \frac{\delta_{s_j}}{x_{-j}x_ix_j} + \frac{\delta_{s_is_j}}{x_ix_{-i-j}x_{-j}} + \frac{1}{x_{i+j}x_{-j}x_j} \\
&\quad + \frac{\delta_{s_js_i}}{x_{-i-j}x_{-i}x_j} - \frac{\delta_{s_js_is_j}}{x_{-i-j}x_{-i}x_{-j}} \\
&= \Delta_j\left(\frac{1}{x_ix_j} + \frac{1}{x_{i+j}x_{-j}}\right) + \Delta_i\left(\frac{1}{x_{i+j}x_j}\right) + \frac{\delta_{s_is_j}}{x_ix_{-i-j}x_{-j}} + \frac{\delta_{s_js_i}}{x_{-i-j}x_{-i}x_j} \\
&\quad - \frac{\delta_{s_js_is_j}}{x_{-i-j}x_{-i}x_{-j}} - \frac{1}{x_ix_{i+j}x_j}.
\end{aligned}$$

Since $m_{ij} = 2$, we have $s_is_js_i = s_js_is_j$ and

$$\begin{aligned}
X_jX_iX_j - X_iX_jX_i &= \Delta_i\left(\frac{1}{x_{i+j}x_j} - \frac{1}{x_{i+j}x_{-i}} = \frac{1}{x_ix_j}\right) \\
&\quad - \Delta_j\left(\frac{1}{x_{i+j}x_i} - \frac{1}{x_{i+j}x_{-j}} = \frac{1}{x_ix_j}\right) \\
&= \Delta_i(\kappa_{i,j}) - \Delta_j(\kappa_{j,i}).
\end{aligned}$$

The other computations are similar and will be left to the reader. A Maple script to compute those relations can be found in Appendix A. ∎

Remark 6.2.8. The Maple script in Appendix A was used to initially find the relations (c) and (d) of Proposition 6.2.7. However, the computations were also later checked by hand.

Remark 6.2.9. In the cases $(a),(b)$ and (c), we have $\eta_w \in R[\![\Lambda]\!]_F$ since $\kappa_{\lambda,\mu} \in R[\![\Lambda]\!]_F$ for all $\lambda,\mu \in \Phi, \lambda+\mu \neq 0$ by [HMSZ, Lemma 6.7]. For case (d), we can view $\mathrm{span}_{\mathbb{Z}}\{\alpha_i,\alpha_j\}$ as a finite subroot system of Φ of rank 2 and we also have $\eta_w \in R[\![\Lambda]\!]_F$ by [CZZ1, Lemma 7.1] and [CZZ1, Example 7.3].

Examples 6.2.10. For $F(u,v) = F_a(u,v)$ or $F(u,v) = F_m(u,v)$, since $\kappa_{i,j} = \frac{x_i(x_{-i}-x_j)-x_{i+j}x_{-i}}{x_i x_{-i} x_j x_{i+j}}$ we have that $\kappa_{i,j} = 0$ hence the braid relations $X_jX_i\cdots = X_iX_j\cdots$ hold. For $F(u,v) = F_{\mu_1,\mu_2}(u,v)$, we get by direct computation that

(i) $\kappa_{i,j} = \mu_2$ (see also [Co, Rem. 3.10 and Ex. 3.12.(3)]), so for $m_{ij} = 3$ we have

$$X_{jij} - X_{iji} = \mu_2(X_i - X_j),$$

and for $m_{ij} = 4$ we have

$$X_{jiji} - X_{ijij} = 2\mu_2(X_{ij} - X_{ji}).$$

(ii) $\xi_{ji} = \xi_{ij} = 3\mu_2^2$ (see computer-aided computations in Appendix C), so for $m_{ij} = 6$ we have

$$X_{jijiji} - X_{ijijij} = 4\mu_2(X_{ijij} - X_{jiji}) + 3\mu_2^2(X_{ij} - X_{ji}).$$

(iii) $\kappa_i = \mu_1$, so $X_i^2 = \mu_1 X_i$ for all $i \in I$.

Proposition 6.2.11. *Suppose $i,j \in I$ and let m_{ij} be the order of s_is_j in W. Then*

$$\underbrace{Y_jY_iY_j\cdots}_{m_{ij}\ terms} - \underbrace{Y_iY_jY_i\cdots}_{m_{ij}\ terms} = \sum_{w\in W, 1\le l(w)\le m_{ij}-2} Y_w\eta_w$$

for some $\eta_w \in Q^F$. In paticular, we have the following:

(a) If $\langle \alpha_i, \alpha_j^\vee\rangle = 0$, so that $m_{ij} = 2$, then $Y_iY_j = Y_jY_i$.

(b) If $\langle \alpha_i, \alpha_j^\vee\rangle = \langle \alpha_j, \alpha_i^\vee\rangle = -1$, so that $m_{ij} = 3$ (A_2), then $Y_jY_iY_j - Y_iY_jY_i = Y_i\kappa_{-i,-j} - Y_j\kappa_{-j,-i}$.

(c) If $\langle \alpha_i, \alpha_j^\vee\rangle = -1$ and $\langle \alpha_j, \alpha_i^\vee\rangle = -2$, so that $m_{ij} = 4$ (B_2), then $Y_{jiji} - Y_{ijij} = Y_{ij}(\kappa_{-i-2j,j} + \kappa_{-j,-i}) - Y_{ji}(\kappa_{-i-j,-j} + \kappa_{-i,-j}) + Y_j(\Delta_{-i}(\kappa_{-i-j,-j} + \kappa_{-i,-j})) - Y_i(\Delta_{-j}(\kappa_{-i-2j,j} + \kappa_{-j,-i}))$.

(d) *If* $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ *and* $\langle \alpha_j, \alpha_i^\vee \rangle = -3$, *so that* $m_{ij} = 6$ (G_2), *then* $Y_{jijiji} - Y_{ijijij} = Y_{ijij}(\kappa_{-j,-i} + \kappa_{-2i-3j,i+2j} + \kappa_{i+3j,-i-2j} + \kappa_{-i-2j,j}) - Y_{jiji}(\kappa_{-i,-j} + \kappa_{2i+3j,-i-2j} + \kappa_{i+2j,-i-3j} + \kappa_{-i-j,-j}) + Y_{jij}(\Delta_{-i}(\kappa_{-i,-j} + \kappa_{2i+3j,-i-2j} + \kappa_{i+2j,-i-3j} + \kappa_{-i-j,-j})) - Y_{iji}(\Delta_{-j}(\kappa_{-j,-i} + \kappa_{-2i-3j,i+2j} + \kappa_{i+3j,-i-2j} + \kappa_{-i-2j,j})) + Y_{ij}\xi'_{ij} - Y_{ji}\xi'_{ji} + Y_j(\Delta_{-i}(\xi'_{ji})) - Y_i(\Delta_{-j}(\xi'_{ij}))$

where $\xi'_{ij} = \frac{1}{x_{-i}x_{-i-j}x_{-i-2j}x_{-2i-3j}} + \frac{1}{x_{-i}x_{-j}x_{-i-2j}x_{2i+3j}} + \frac{1}{x_{-i}x_{-j}x_{-2i-3j}x_{i+j}} - \frac{1}{x_{-i}x_{-i-j}x_{-i-2j}x_{i+3j}} - \frac{1}{x_{-i}x_{-i-j}x_{-i-3j}x_{j}} + \frac{1}{x_{-i-j}x_{-i-3j}x_{j}x_{2i+3j}} + \frac{1}{x_{-i-3j}x_{-2i-3j}x_{j}x_{i+2j}} + \frac{1}{x_{-i-j}x_{-i-2j}x_{i+3j}x_{2i+3j}} - \frac{1}{x_{-i}x_{-j}x_{-i-2j}x_{-i-3j}}$

and $\xi'_{ji} = \frac{1}{x_{-i}x_{-j}x_{-2i-3j}x_{i+2j}} + \frac{1}{x_{-i}x_{-j}x_{-i-2j}x_{i+3j}} + \frac{1}{x_{-j}x_{-i-2j}x_{-i-3j}x_{-2i-3j}} - \frac{1}{x_{-i}x_{-j}x_{-i-j}x_{-2i-3j}} + \frac{1}{x_{-i-j}x_{-i-2j}x_{i}x_{2i+3j}} + \frac{1}{x_{-i-3j}x_{-2i-3j}x_{i+j}x_{i+2j}} + \frac{1}{x_{-i-j}x_{-i-3j}x_{i}x_{i+2j}} - \frac{1}{x_{-j}x_{-i-3j}x_{-2i-3j}x_{i+j}} - \frac{1}{x_{-j}x_{-i-j}x_{-i-3j}x_{i}}$.

Proof: The proof is the same as for Proposition 6.2.7 and the Maple script for the computations of the relations can be found in Appendix B. ∎

Examples 6.2.12. For $F(u,v) = F_a(u,v)$ or $F(u,v) = F_m(u,v)$, since $\kappa_{i,j} = \frac{x_i(x_{-i}-x_j)-x_{i+j}x_{-i}}{x_ix_{-i}x_jx_{i+j}}$ we have that $\kappa_{i,j} = 0$ hence the braid relations $Y_jY_i \cdots = Y_iY_j \cdots$ hold. For $F(u,v) = F_{\mu_1,\mu_2}(u,v)$, we get by direct computation that

(i) $\kappa_{i,j} = \mu_2$, so for $m_{ij} = 3$ we have

$$Y_{jij} - Y_{iji} = \mu_2(Y_i - Y_j),$$

and for $m_{ij} = 4$ we have

$$Y_{jiji} - Y_{ijij} = 2\mu_2(Y_{ij} - Y_{ji}),$$

(ii) $\xi_{ji} = \xi_{ij} = 3\mu_2^2$ (one can check this by computer aided computations similar to those in Appendix C), so for $m_{ij} = 6$ we have

$$Y_{jijiji} - Y_{ijijij} = 4\mu_2(Y_{ijij} - Y_{jiji}) + 3\mu_2^2(Y_{ij} - Y_{ji}).$$

(iii) $\kappa_i = \mu_1$, so $Y_i^2 = \mu_1 Y_i$ for all $i \in I$.

6.3 Structure of the formal (affine) Demazure algebra

The next lemma leads to the main results of this chapter. It is a generalization of [HMSZ, Lemma 6.13].

Lemma 6.3.1. *The elements $(X_{I_w})_{w\in W}$ (resp. $(Y_{I_w})_{w\in W}$) form a basis of D_F as a R-module and a basis of $\mathbf{D}_F$ as a $R[\![\Lambda]\!]_F$-module.*

Proof: We follow the proof in [HMSZ] for the X_{I_w}'s, but the same proof also holds for the Y_{I_w}'s. By Lemma 6.1.5(4) and Proposition 6.2.7 we can write any product of formal Demazure elements as R-linear combination of the X_{I_w}'s. Combined with Lemma 6.2.3 we can write any product of formal Demazure elements and elements of $R[\![\Lambda]\!]_F$ as a $R[\![\Lambda]\!]_F$-linear combination of the X_{I_w}'s. Therefore, the X_{I_w}'s span D_F as a R-module and span $\mathbf{D}_F$ as $R[\![\Lambda]\!]_F$-module.

By Corollary 6.1.10, the X_{I_w}'s are linearly independent over Q^F, hence also linearly independent over R and $R[\![\Lambda]\!]_F$. ∎

The theorem below for the formal Demazure elements is a generalization of [HMSZ, Thm 6.14].

Theorem 6.3.2. *Given a formal group law F, the formal affine Demazure algebra $\mathbf{D}_F$ is generated as an R-algebra by $R[\![\Lambda]\!]_F$ and the formal Demazure elements $X_i, i \in I$, and satisfies the following relations:*

- $\psi X_\alpha = X_\alpha s_\alpha(\psi) + \Delta_\alpha(\psi)$ *for all* $\alpha \in \Phi^{re}$ *and* $\psi \in R[\![\Lambda]\!]_F$;
- $X_i^2 = X_i \kappa_i$ *for all* $i \in I$;
- $X_i X_j = X_j X_i$ *for all* $i, j \in I$ *such that* $m_{ij} = 2$;
- *the braid relations of Proposition 6.2.7 for all* $i, j \in I$ *such that* $m_{ij} = 3, 4, 6$;
- *no relations between* X_i *and* X_j *for* $m_{ij} = \infty$.

These relations form a complete set of relations for $\mathbf{D}_F$.

Proof: We follow the proof in [HMSZ]. Let $\widetilde{\mathbf{D}_F}$ be the R-algebra generated by the elements of $R[\![\Lambda]\!]_F$ and the X_i''s subject to the relations given in the theorem. Then we have a surjective ring homomorphism

$$\rho : \widetilde{\mathbf{D}_F} \to \mathbf{D}_F$$

which is the identity on $R[\![\Lambda]\!]_F$ and maps X'_i to X_i for all $i \in I$. To show that ρ is an isomorphism, it remains to show that the map is injective.

For $w \in W$, fix a reduced sequence $I_w = (i_1, ..., i_l)$ and define $X'_{I_w} := X_{i_1} \cdots X'_{i_l}$.

The relations between the X'_w's allow us to write any element of $\widetilde{\mathbf{D}_F}$ as

$$\sum_{w \in W} X'_w a_w, \quad a_w \in R[\![\Lambda]\!]_F.$$

Suppose such an element of the form above is in the kernel of ρ. Then

$$0 = \rho(\sum_{w \in W} X'_w a_w) = \sum_{w \in W} X_w a_w.$$

However, since by Lemma 6.3.1 the X_w's form a basis of $\mathbf{D}_F$ as a $R[\![\Lambda]\!]_F$-module, this implies that $a_w = 0$ for all $w \in W$. Therefore, ρ is injective. ∎

Remark 6.3.3. Note that without the relations $\psi X_\alpha = X_\alpha s_\alpha(\psi) + \Delta_\alpha(\psi)$, we have a complete set of relations for the formal Demazure algebra D_F.

Since the Y_{I_w}'s also form a basis of the formal affine Demazure algebra as a $R[\![\Lambda]\!]_F$-module, we have the following result.

Theorem 6.3.4. *Given a formal group law F, the formal affine Demazure algebra $\mathbf{D}_F$ is generated as an R-algebra by $R[\![\Lambda]\!]_F$ and the formal push-pull elements $Y_i, i \in I$, and satisfies the following relations:*

- $\psi Y_\alpha = Y_\alpha s_\alpha(\psi) + \Delta_{-\alpha}(\psi)$ *for all $i \in I$ and $\psi \in R[\![\Lambda]\!]_F$;*
- $Y_i^2 = Y_i \kappa_i$ *for all $i \in I$;*
- $Y_i Y_j = Y_j Y_i$ *for all $i, j \in I$ such that $m_{ij} = 2$;*
- *the braid relations of Proposition 6.2.11 for all $i, j \in I$ such that $m_{ij} = 3, 4, 6$;*
- *no relations between Y_i and Y_j for $m_{ij} = \infty$.*

These relations form a complete set of relations for $\mathbf{D}_F$.

Proof: The proof is the same as for Theorem 6.3.2. ∎

Chapter 7

Hecke Algebras

7.1 Definition of a Hecke algebra

The purpose of the present section is to generalize [CZZ2, Prop. 9.2] to a Kac-Moody root system of arbitrary type, see Theorem 7.2.1. In Corollaries 7.3.1, 7.3.2 and 7.3.4, we obtain different algebras isomorphic to the affine Demazure algebra depending on our choice of root system and formal Demazure lattice. Some of the results of this chapter appeared in [L], a paper written as part of the research leading to this thesis.

Definition 7.1.1. (cf. [CMHL, p. 72]) Let R be a (commutative unital) ring containing $\mathbb{Z}[t, t^{-1}]$. In Lusztig's notation, the *Hecke algebra* H associated with the Coxeter group W is the associative R-algebra with 1 generated by elements $T_i := T_{s_i}$, $i \in I$, and satisfying

(i) the quadratic relations $(T_i + t)(T_i - t^{-1}) = T_i^2 + (t - t^{-1})T_i - 1 = 0$ for all $i \in I$, and

(ii) the braid relations $T_iT_jT_i \cdots = T_jT_iT_j \cdots$ (m_{ij} factors on both sides of the equation) for any $i \neq j$ in I with $s_{\alpha_i}s_{\alpha_j}$ of order m_{ij} in W. If $m_{ij} = \infty$, there are no relations between T_i and T_j.

7.2 Isomorphism between the formal Demazure algebra and a Hecke algebra

Here is our main theorem.

Theorem 7.2.1. [L, Thm 5.2] *Let R be a (commutative unital) ring containing $\mathbb{Z}[t, t^{-1}]$ and let $u \in R$. Set $\mu_1 = u(t + t^{-1})$ and $\mu_2 = -u^2$. Then if F is the* *hyperbolic formal group law* $F_{\mu_1,\mu_2}(u,v) = \frac{u+v-\mu_1 uv}{1+\mu_2 uv}$*, the assignment* $X_i \mapsto u(T_i + t)$ *defines a morphism of* R*-algebras from the formal Demazure algebra* D_F *to the Hecke algebra* H *over* R*. If* $u \in R^\times$*, this morphism is an isomorphism.*

Proof: Let A be the free associative unital algebra on generators $X_i, i \in I$. There exists a unique (well-defined) algebra homomorphism $\psi : A \to H$ such that $\psi(X_i) = u(T_i + t)$. We check that ψ annihilates the ideal generated by the relations of 6.3.2 defining D_F. We have

$$\begin{aligned}\psi(X_i^2 - \mu_1 X_i) &= \psi(X_i)^2 - \mu_1\psi(X_i)\\ &= u^2T_i^2 + 2u^2tT_i + u^2t^2 - \mu_1 uT_i - \mu_1 ut\\ &= u^2((t^{-1}-t)T_i + 1) + (2u^2t - u^2(t+t^{-1}))T_i\\ &\quad + u^2t^2 - u^2t(t+t^{-1}) \text{ (by 7.1.1)}\\ &= (u^2t^{-1} - u^2t + 2u^2t - u^2t - u^2t^{-1})T_i + u^2 + u^2t^2 - u^2t^2 - u^2 = 0\end{aligned}$$

We can check that $\psi(X_iX_j - X_jX_i) = 0$ since $T_iT_j = T_jT_i$ by 7.1.1. We also have

$$\begin{aligned}\psi(X_{jij} - X_{iji} + \mu_2X_j - \mu_2X_i) &\overset{(7.1.1)}{=} u^3t(T_j^2 - T_i^2) + u^2t^2(T_j - T_i)\\ &\quad + \mu_2u(T_j + t) - \mu_2u(T_i + t)\\ &= u^3t((t^{-1}-t)T_j + 1 - (t^{-1}-t)T_i - 1)\\ &\quad + u^2t^2(T_j - T_i) + \mu_2u(T_j - T_i) + \mu_2ut - \mu_2ut\\ &= (u^3t(t^{-1}-t) + u^2t^2 + \mu_2u)(T_j - T_i)\\ &= (u^3 - u^3t^2 + u^3t^2 - u^3)(T_j - T_i) = 0.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\psi(X_{jiji} - X_{ijij} + 2\mu_2(X_{ji} - X_{ij})) =\\ &\quad u^4(T_jT_iT_jT_i - T_iT_jT_iT_j) + 2u^2(u^2t(t^{-1}-t) + u^2t^2 - u^2)(T_jT_i - T_iT_j) = 0\end{aligned}$$

and

$$\begin{aligned}\psi(X_{jijiji} - X_{ijijij} - 4\mu_2(X_{ijij} - X_{jiji}) - 3\mu_2^2(X_{ij} - X_{ji})) &= u^6(T_jT_iT_jT_iT_jT_i\\ &- T_iT_jT_iT_jT_iT_j) + 4u^4(u^2t(t^{-1}-t) + u^2t^2 - u^2)(T_jT_iT_jT_i - T_iT_jT_iT_j)\\ &+ (2(t^{-1}-t)u^6t + 2u^6t^2 + 3(t^{-1}-t)^2u^6t^2 + 6(t^{-1}-t)u^6t^3 + 2u^6t^4\\ &- 8u^6t(t^{-1}-t) - 9u^6t^2 + 3u^6)(T_jT_i - T_iT_j)\\ &= u^6(2 - 2t^2 + 2t^2 + 3 - 6t^2 + 3t^4 + 6t^2 - 6t^4 + 3t^4 - 8 + 8t^2\\ &- t^2 + 3)(T_jT_i - T_iT_j) = 0.\end{aligned}$$

Since ψ annihilates the relations defining D_F, it descends to a unital algebra homomorphism $\bar{\psi}\colon D_F \to H$ mapping $X_i \in D_F$ onto $\psi(X_i) \in H$.

Moreover, if u is invertible, $\bar{\psi}$ is surjective since the image of $\bar{\psi}$ contains the generators of H, i.e. $\bar{\psi}(u^{-1}X_i - t) = T_i \in \mathrm{im}(\bar{\psi})$. Also, we can find a surjective homomorphism in the other direction. Let B be the free associative unital algebra on generators $T_i, i \in I$. There exists a unique (well-defined) algebra homomorphism $\phi\colon B \to D_F$ such that $\phi(T_i) = u^{-1}X_i - t$. We check that ϕ annihilates the ideal generated by the relations (i) and (ii) of 7.1.1 defining H. We have

$$\begin{aligned}
\phi(T_i)^2 = (u^{-1}X_i - t)^2 &= u^{-2}X_i^2 - 2u^{-1}tX_i + t^2 \\
&= u^{-1}(t+t^{-1})X_i - 2u^{-1}tX_i + t^2 \text{ (by 6.2.10)} \\
&= u^{-1}t^{-1}X_i - u^{-1}tX_i - 1 + t^2 + 1 \\
&= (t^{-1} - t)(u^{-1}X_i - t) + 1 \\
&= (t^{-1} - t)\phi(T_i) + 1.
\end{aligned}$$

Therefore, $(\phi(T_i) + t)(\phi(T_i) - t^{-1}) = \phi(T_i)^2 + (t - t^{-1})\phi(T_i) - 1 = 0$. We can also check that $\phi(T_iT_j - T_jT_i) = 0$ since $X_iX_j = X_jX_i$ by 6.3.2. We also have

$$\begin{aligned}
\phi(T_{iji} - T_{jij}) &= u^{-3}(X_{iji} - X_{jij}) - u^{-2}t(X_i^2 - X_j^2) + u^{-1}t^2(X_i - X_j) \\
&= u^{-3}\mu_2(X_j - X_i) - u^{-2}t\mu_1(X_i - X_j) + u^{-1}t^2(X_i - X_j) \text{ (by 6.2.10)} \\
&= (-u^{-3}\mu_2 - u^{-2}t\mu_1 + u^{-1}t^2)(X_i - X_j) \\
&= (-u^{-3}u^2 - u^{-2}tu(t+t^{-1}) + u^{-1}t^2)(X_i - X_j) \\
&= (-u^{-1} - u_{-1}t^2 - u^{-1} + u^{-1}t^2)(X_i - X_j) = 0.
\end{aligned}$$

We also have

$$\begin{aligned}
\phi(T_{jiji} - T_{ijij}) &= u^{-4}(X_{jiji} - X_{ijij}) + (-2u^{-2}t(t+t^{-1}) + 2u^{-2}t^2)(X_{ji} - X_{ij}) \\
&= -2u^{-2}(X_{ij} - X_{ji}) + 2(-2u^{-2}t(t+t^{-1}) + 2u^{-2}t^2)(X_{ji} - X_{ij}) \\
&\quad + 2u^{-2}t^2(X_{ji}) \\
&= 2(u^{-2} - u^{-2}t(t+t^{-1}) + u^{-2}t^2)(X_{ji} - X_{ij}) = 0
\end{aligned}$$

and

$$\begin{aligned}
\phi(T_{jijiji} - T_{ijijij}) &= u^{-6}(X_{jijiji} - X_{ijijij}) - 4u^{-4}t(t+t^{-1})(X_{jiji} - X_{ijij}) \\
&\quad + u^{-4}t^2(4X_{jiji} - 4X_{ijij} + 3X_{ji} - 3X_{ij}) \\
&\quad - 6u^{-2}t^3(t+t^{-1})(X_{ji} - X_{ij}) + 3u^{-2}t^4(X_{ji} - X_{ij}) \\
&= 4(u^{-4} - u^{-4}t(t+t^{-1}) + u^{-4}t^2)(X_{jiji} - X_{ijij}) \\
&\quad + 3(-u^{-2} + u^{-2}t^2(t+t^{-1})^2 - 2u^{-2}t^3(t+t^{-1}) + u^{-2}t^4)\cdot \\
&\quad \cdot (X_{ji} - X_{ij}) = 0.
\end{aligned}$$

Similarly, we get $\phi(T_{jiji} - T_{ijij}) = 0$ and $\phi(T_{jijiji} - T_{ijijij}) = 0$. Therefore, ϕ descends to a unital algebra homomorphism $\bar{\phi}\colon H \to D_F$, which is surjective and such that $\bar{\psi}\circ\bar{\phi} = \mathrm{id}_H$ and $\bar{\phi}\circ\bar{\psi} = \mathrm{id}_{D_F}$, hence, the isomorphism between D_F and H. ∎

Remark 7.2.2. The isomorphism of 7.2.1 was used in [LZ2] to tackle the problem of defining Schubert classes, indepedently of the choice of reduced expression of the Weyl group, in the case of singular elliptic cohomology, which is the cohomology theory corresponding to the hyperbolic formal group law. By applying the isomorphism to the Khazhdan-Lusztig basis, they obtain elements of the formal Demazure algebra that do not depend on the choice of reduced expression, in the case of the rank 2 root systems, because of the twisted braid relations of 6.2.7.

7.3 Examples of algebras isomorphic to formal affine Demazure algebras

We obtain similar results to Theorem 7.2.1 for formal affine Demazure algebras.

Corollary 7.3.1. [L, Cor. 5.3] *Let R be a (commutative unital) ring containing $\mathbb{Z}[t,t^{-1}]$, let Λ be a formal Demazure lattice, and let Φ be a Kac-Moody root system. Let $\mu_1 = t+t^{-1}$ and $\mu_2 = -1$ and let $F_{\mu_1,\mu_2}(u,v) = \frac{u+v-(t+t^{-1})uv}{1-uv}$ be the hyperbolic formal group law. Let $\mathbf{H}$ be the R-algebra generated by the elements of $R[\![\Lambda]\!]_F$ and by T_i, $i \in I$, subject to the relations (i) and (ii) of 7.1.1 and for all $\gamma \in R[\![\Lambda]\!]_F$ and $i \in I$*

$$\gamma T_i - T_i s_{\alpha_i}(\gamma) = (1 - t x_{\alpha_i})\Delta_{\alpha_i}(\gamma).$$

Then, the formal affine Demazure algebra $\mathbf{D}_F$ is isomorphic to $\mathbf{H}$ by a ring isomorphism preserving $R[\![\Lambda]\!]_F$, in particular as an R-algebra.

Proof: We proceed in the same way as in 7.2.1 with $u = 1$. We have a unital algebra homomorphism ψ defined as the identity on $R[\![\Lambda]\!]_F$ and mapping $X_i \mapsto T_i + t$. To show that ψ annihilates the ideal generated by the relations 6.3.2 defining $\mathbf{D}_F$ it remains to show that

$$\psi(\gamma X_i - X_i s_{\alpha_i}(\gamma) - \Delta_{\alpha_i}(\gamma)) = 0$$

for all $i \in I$ and all $\gamma \in R[\![\Lambda]\!]_F$. We have

$$\begin{aligned}\psi(\gamma X_i - X_i s_{\alpha_i}(\gamma) - \Delta_{\alpha_i}(\gamma)) &= \gamma(T_i + t) - (T_i + t)s_{\alpha_i}(\gamma) - \Delta_{\alpha_i}(\gamma)\\ &= \gamma T_i - T_i s_{\alpha_i}(\gamma) + t(\gamma - s_{\alpha_i}(\gamma) - \Delta_{\alpha_i}(\gamma)\\ &= \gamma T_i - T_i s_{\alpha_i}(\gamma) - (1 - t x_{\alpha_i})\Delta_{\alpha_i}(\gamma) = 0.\end{aligned}$$

Therefore, we get a unital algebra homomorphism $\bar{\psi}\colon \mathbf{D}_F \to \mathbf{H}$ and we can show it is an isomorphism as in the proof of 7.2.1. ∎

Recall that for a formal Demazure lattice Λ of rank n, by 3.1.10 we have

$$R[\![\Lambda]\!]_F \cong R[\![x_1, x_2, ..., x_n]\!].$$

Therefore, we can replace the elements of $R[\![\Lambda]\!]_F$ in $\mathbf{H}$ by elements of $R[\![x_1, x_2, ..., x_n]\!]$ and still have an isomorphism between $\mathbf{D}_F$ and $\mathbf{H}$. We will use this idea in the following corollaries for some specific Kac-Moody root systems and formal Demazure lattices.

Corollary 7.3.2. *Let*

$$\Phi = A_1^{(1)}, \quad \Lambda = \mathbb{Z}\alpha_0 + \mathbb{Z}\tfrac{1}{2}\delta, \text{ and } R = \mathbb{Z}[t, t^{-1}, (t+t^{-1})^{-1}].$$

Let $\mu_1 = t + t^{-1}$ and $\mu_2 = -1$ and let $F_{\mu_1,\mu_2}(u,v) = \frac{u+v-(t+t^{-1})uv}{1-uv}$ be the hyperbolic formal group law. Let $\mathbf{H}$ be the R-algebra generated by T_i, $i \in I = \{0,1\}$, subject to the relations (i) and (ii) of 7.1.1 and by elements of $R[\![c_0, c_1]\!]$ such that for $i, j \in I$

$$c_i c_j = c_j c_i, \quad c_1 T_j - T_j c_1 = 0,$$

$$c_0 T_0 - T_0 \tfrac{c_0}{(t+t^{-1})c_0 - 1} = (1 - tc_0)(1 - \tfrac{1}{(t+t^{-1})c_0 - 1}),$$

$$c_0 T_1 - T_1 \tfrac{p}{q} = -t(c_0 - \tfrac{p}{q}) + \tfrac{r}{s}$$

where

$$\begin{aligned} p = {} & (-t^5 - t^3 - t)c_0c_1^4 + (4t^4 + 4t^2)c_0c_1^3 - 6t^3c_0c_1^2 + t^3c_0 \\ & + (t^6 + t^4 + t^2 + 1)c_1^4 + (-4t^5 - 4t^3 - 4t)c_1^3 + (6t^4 + 6t^2)c_1^2 - 4t^3c_1, \\ q = {} & t((-t^3 - t)c_0c_1^4 + 4t^2c_0c_1^3 - 4t^2c_0c_1 + (t^3 + t)c_0 + (t^4 + t^2 + 1)c_1^4 \\ & + (-4t^3 - 4t)c_1^3 + 6t^2c_1^2 - t^2), \\ r = {} & (2c_0c_1t - (t^2 + 1)c_0 - tc_1^2 + t)((-t^3 - t)c_0c_1^2 + 4t^2c_0c_1 + (-t^3 - t)c_0 \\ & + (t^4 + 1)c_1^2 + (-2t^3 - 2t)c_1 + 2t^2), \end{aligned}$$

and

$$\begin{aligned} s = {} & t((-t^3 - t)c_0c_1^4 + 4t^2c_0c_1^3 - 4t^2c_0c_1 + (t^3 + t)c_0 + (t^4 + t^2 + 1)c_1^4 \\ & + (-4t^3 - 4t)c_1^3 + 6t^2c_1^2 - t^2). \end{aligned}$$

Then, the formal affine Demazure algebra $\mathbf{D}_F$ is isomorphic to $\mathbf{H}$ as R-algebra.

Proof: We proceed in the same way as in 7.2.1 with $u = \mu_1/(t + t^{-1}) = 1$. We have a unital algebra homomorphism ψ mapping

$$X_i \mapsto T_i + t, \quad x_0 \mapsto c_0, \quad x_{\frac{1}{2}\delta} \mapsto c_1.$$

We have $qX_\alpha = X_\alpha s_\alpha(q) + \Delta_\alpha(q)$ for all $\alpha \in \Phi^{re}$ and $q \in Q_W^F$. Since, $\langle \delta, \alpha_i^\vee \rangle = 0$ for $i \in I$, we have $\Delta_i(x_{\frac{1}{2}\delta}) = 0$ and $x_{\frac{1}{2}\delta} X_j = X_j x_{\frac{1}{2}\delta}$ for $j \in I$. We also have

$$\psi(x_{\frac{1}{2}\delta} X_j - X_j x_{\frac{1}{2}\delta}) = c_1(T_j + t) - (T_j + t)c_1 = 0$$

since $c_1 T_j - T_j c_1 = 0$ for all $j \in I$. We can also look at the relation $x_{\alpha_0} X_i = X_i s_i(x_{\alpha_0}) + \Delta_i(x_{\alpha_0})$ for $i \in I$. If $i = 0$, we have $s_0(x_{\alpha_0}) = x_{s_0(\alpha_0)} = x_{-\alpha_0} = \frac{x_{\alpha_0}}{(t+t^{-1})x_{\alpha_0}-1}$ and $\Delta_0(x_{\alpha_0}) = 1 - \frac{1}{(t+t^{-1})x_{\alpha_0}-1}$. Then we get that

$$\begin{aligned}
&\psi\left(x_{\alpha_0} X_0 - X_0 s_0(x_{\alpha_0}) - \Delta_0(x_{\alpha_0})\right) = \\
&= \psi\left(x_{\alpha_0} X_0 - X_0 \tfrac{x_{\alpha_0}}{(t+t^{-1})x_{\alpha_0}-1} - (1 - \tfrac{1}{(t+t^{-1})x_{\alpha_0}-1})\right) \\
&= c_0 T_0 - T_0 \tfrac{c_0}{(t+t^{-1})c_0-1} - (1 - tc_0)(1 - \tfrac{1}{(t+t^{-1})c_0-1}) = 0.
\end{aligned}$$

The last relation $x_{\alpha_0} X_1 = X_1 s_1(x_{\alpha_0}) + \Delta_1(x_{\alpha_0})$ requires more computation since we have to write x_{α_1} in terms of x_{α_0} and $x_{\frac{1}{2}\delta}$. We get that

$$x_{\alpha_1} = x_{-\alpha_0+2(\frac{1}{2}\delta)} = x_{-\alpha_0} +_F x_{\frac{1}{2}\delta} +_F x_{\frac{1}{2}\delta} = \frac{t^2 x_{\frac{1}{2}\delta} - t x_{\alpha_0} x_{\frac{1}{2}\delta}^2 - 2t x_{\frac{1}{2}\delta} + x_{\frac{1}{2}\delta}^2}{t^2 x_{\alpha_0} - 2t x_{\alpha_0} x_{\frac{1}{2}\delta} + t x_{\frac{1}{2}\delta}^2 - t + x_{\alpha_0}}.$$

Then we have

$$\psi(x_{\alpha_1}) = \frac{t^2 c_1 - t c_0 c_1^2 - 2t c_1 + c_1^2}{t^2 c_0 - 2t c_0 c_1 + t c_1^2 - t + c_0}.$$

Moreover, we need to compute $s_{\alpha_1}(x_{\alpha_0})$ by writing it in terms of x_{α_0} and $x_{\frac{1}{2}\delta}$. We get

$$s_{\alpha_1}(x_{\alpha_0}) = x_{\alpha_0+2\alpha_1} = x_{-\alpha_0+4(\frac{1}{2}\delta)} = x_{-\alpha_0} +_F x_{\frac{1}{2}\delta} +_F x_{\frac{1}{2}\delta} +_F x_{\frac{1}{2}\delta} +_F x_{\frac{1}{2}\delta} = \frac{a}{b}$$

where

$$\begin{aligned}
a = {} & (-t^5 - t^3 - t)x_{\alpha_0} x_{\frac{1}{2}\delta}^4 + (4t^4 + 4t^2) x_{\alpha_0} x_{\frac{1}{2}\delta}^3 - 6t^3 x_{\alpha_0} x_{\frac{1}{2}\delta}^2 + t^3 x_{\alpha_0} \\
& + (t^6 + t^4 + t^2 + 1) x_{\frac{1}{2}\delta}^4 + (-4t^5 - 4t^3 - 4t) x_{\frac{1}{2}\delta}^3 + (6t^4 + 6t^2) x_{\frac{1}{2}\delta}^2 - 4t^3 x_{\frac{1}{2}\delta}
\end{aligned}$$

and

$$\begin{aligned}
b = {} & t((-t^3 - t) x_{\alpha_0} x_{\frac{1}{2}\delta}^4 + 4t^2 x_{\alpha_0} x_{\frac{1}{2}\delta}^3 - 4t^2 x_{\alpha_0} x_{\frac{1}{2}\delta} + (t^3 + t) x_{\alpha_0} + (t^4 + t^2 + 1) x_{\frac{1}{2}\delta}^4 \\
& + (-4t^3 - 4t) x_{\frac{1}{2}\delta}^3 + 6t^2 x_{\frac{1}{2}\delta}^2 - t^2).
\end{aligned}$$

By combining the computations for $s_{\alpha_1}(x_{\alpha_0})$ and x_{α_1} and after doing some simplifications, we get

$$\Delta_1(x_{\alpha_0}) = \frac{1 - s_{\alpha_1}(x_{\alpha_0})}{x_{\alpha_1}} = \frac{c}{d}$$

where

$$c = (2x_{\alpha_0}x_{\frac{1}{2}\delta}t - (t^2+1)x_{\alpha_0} - tx^2_{\frac{1}{2}\delta} + t)((-t^3-t)x_{\alpha_0}x^2_{\frac{1}{2}\delta} + 4t^2x_{\alpha_0}x_{\frac{1}{2}\delta} + (-t^3-t)x_{\alpha_0} + (t^4+1)x^2_{\frac{1}{2}\delta} + (-2t^3-2t)x_{\frac{1}{2}\delta} + 2t^2)$$

and

$$d = t((-t^3-t)x_{\alpha_0}x^4_{\frac{1}{2}\delta} + 4t^2x_{\alpha_0}x^3_{\frac{1}{2}\delta} - 4t^2x_{\alpha_0}x_{\frac{1}{2}\delta} + (t^3+t)x_{\alpha_0} + (t^4+t^2+1)x^4_{\frac{1}{2}\delta} + (-4t^3-4t)x^3_{\frac{1}{2}\delta} + 6t^2x^2_{\frac{1}{2}\delta} - t^2).$$

Now we can check that ψ annihilates the relation $x_{\alpha_0}X_1 - X_1s_1(x_{\alpha_0}) - \Delta_1(x_{\alpha_0})$. Note that $\psi(a) = p, \psi(b) = q, \psi(c) = r$ and $\psi(d) = s$. Then we have

$$\begin{aligned}\psi(x_{\alpha_0}X_1 - X_1s_1(x_{\alpha_0}) - \Delta_1(x_{\alpha_0})) &= \psi(x_{\alpha_0}X_1 - X_1s_1(x_{\alpha_0}) - \tfrac{c}{d}) \\ &= c_0(T_1+t) - (T_1+t)\tfrac{p}{q} - \tfrac{r}{s} \\ &= c_0T_1 - T_1\tfrac{p}{q} + t(c_0 - \tfrac{p}{q}) - \tfrac{r}{s} = 0.\end{aligned}$$

By 7.2.1, we get a unital algebra homomorphism $\bar{\psi}\colon \mathbf{D}_F \to \mathbf{H}$ and we can show it is an isomorphism as in the proof of 7.2.1. ∎

Remark 7.3.3. All the fractions appearing in the corollary are power series since they come from formal sums and formal inverses. We write them as fractions in order to simplify the notation.

We can also have a similar result for a Kac-Moody root system of indefinite type and the formal Demazure lattice being the weight lattice.

Corollary 7.3.4. *Let Φ be the root system corresponding to the generalized Cartan matrix $A = \left(\begin{smallmatrix} 2 & -3 \\ -3 & 2 \end{smallmatrix}\right)$ and $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\{\omega_1, \omega_2\}$ is a set of fundamental weights. Let $R = \mathbb{Z}[t, t^{-1}, (t+t^{-1})^{-1}]$ and $F_{(t+t^{-1}),-1}(u,v) = \frac{u+v-(t+t^{-1})uv}{1-uv}$ be the hyperbolic formal group law. Let $\mathbf{H}$ be the R-algebra generated by T_i, $i \in I = \{1,2\}$, subject to the relations (i) and (ii) of 7.1.1 and by elements of $R[\![c_1, c_2]\!]$ such that for $i, j \in I, i \neq j$*

$$c_ic_j = c_jc_i, \quad c_iT_j - T_jc_i = 0$$

$$c_iT_i - T_i\tfrac{p_{ij}}{q_{ij}} = -t(c_i - \tfrac{p_{ij}}{q_{ij}}) + \tfrac{r_{ij}}{s_{ij}}$$

where

- $p_{ij} = (-t^3 - t)c_i^3 c_j + (t^4 + t^2 + 1)c_i^3 + 3t^2 c_i^2 c_j + (-3t^3 - 3t)c_i^2 + 3t^2 c_i - t^2 c_j$,
- $q_{ij} = t(-tc_i^3 c_j + (t^2 + 1)c_i^3 - 3tc_i^2 + 3tc_i c_j + (-t^2 - 1)c_j + t)$,
- $r_{ij} = t^6 c_j^3 - 2t^5 c_i c_j^3 + t^4 c_i^2 c_j^3 - 3t^5 c_j^2 + 6t^4 c_i c_j^2 + t^4 c_j^3 - 3t^3 c_i^2 c_j^2 - 2t^3 c_i c_j^3 + t^2 c_i^2 c_j^3 + 3t^4 c_j + t^3 c_i^2 - 6t^3 c_i c_j - 3t^3 c_j^2 + 6t^2 c_i c_j^2 + t^2 c_j^3 - 2tc_i c_j^3 - t^3 + 3t^2 c_j - 3tc_j^2 + c_j^3$,
- $s_{ij} = (-t^2 c_j^3 + tc_i c_j^3 + t^2 c_i - 3tc_i c_j + 3tc_j^2 - c_j^3 - t + c_i)t^2$.

Then, the formal affine Demazure algebra $\mathbf{D}_F$ *is isomorphic to* $\mathbf{H}$ *as* R*-algebra.*

Proof: Same proof as 7.3.2. ∎

Appendix A

Computations of relations for Demazure elements

The following is a Maple script to compute the relations in Proposition 6.2.7. To run the script, one has to load the Coxeter package by John Stembridge. You can dowload the package here:

http://www.math.lsa.umich.edu/ jrs/maple.html#coxeter

In the header of the script you must include:

read(”Coxeter2.4v.txt”); withcoxeter(); withweyl;

Here is the complete script with explanations of every piece of code.

Action of the n-th simple reflection on an expression P for a root system of type $A_1 \times A_1$.

```
reflA1A1 := proc (n, P) if n = 2 then return subs({x[j] = x[-j], x[-j] = x[j]}, P)
else return subs({x[i] = x[-i], x[-i] = x[i]}, P)
end if
end proc:
```

Action of the n-th simple reflection on an expression P for a root system of type A_2.

```
reflA2 := proc (n, P)
if n = 2 then return subs({x[i] = x[j+i], x[j] = x[-j], x[-i] = x[-j-i], x[-j] = x[j], x[-j-i]
= x[-i], x[j+i] = x[i]}, P)
else return subs({x[i] = x[-i], x[j] = x[j+i], x[-i] = x[i], x[-j] = x[-j-i], x[-j-i] = x[-j],
x[j+i] = x[j]}, P)
```

end if
end proc:

Action of the n-th simple reflection on an expression P for a root system of type B_2.

```
reflB2 := proc (n, P)
if n = 2 then return subs({x[i] = x[2*j+i], x[j] = x[-j], x[-i] = x[-2*j-i], x[-j] = x[j],
x[-2*j-i] = x[-i], x[2*j+i] = x[i]}, P)
else return subs({x[i] = x[-i], x[j] = x[j+i], x[-i] = x[i], x[-j] = x[-j-i], x[-j-i] = x[-j],
x[j+i] = x[j]}, P)
end if
end proc:
```

Action of the n-th simple reflection on an expression P for a root system of type G_2.

```
reflG2:=proc(n,P)
if n=2 then return subs({x[j]=x[-j],x[i]=x[3*j+i],x[-j]=x[j],x[-i]=x[-3*j-i],
x[j+i]=x[2*j+i],x[-j-i]=x[-2*j-i],x[3*j+i]=x[i],x[-3*j-i]=x[-i],x[2*j+i]=x[j+i],
x[-2*j-i]=x[-j-i]},P)
else return subs({x[j]=x[j+i],x[i]=x[-i],x[-j]=x[-j-i],x[-i]=x[i],x[j+i]=x[j],x[-j-i]=x[-j],
x[3*j+i]=x[3*j+2*i],x[-3*j-i]=x[-3*j-2*i],x[3*j+2*i]=x[3*j+i],x[-3*j-2*i]=x[-3*j-i]},P)
end if
end proc:
```

Inverts an element of the Weyl group for a root system of type $A_1 \times A_1$.

```
invA1A1 := proc (w, P) local u, p: p := P:
for u to nops(w) do
p := reflA1A1(w[u], p):
od:
return p:
end proc:
```

Multiply two basis elements of the twisted group algebra for a root system of type $A_1 \times A_1$.

```
multA1A1p:=proc(P,Q)
return [reduce([op(P[1]),op(Q[1])],A1*A1),invA1A1(Q[1],P[2])*Q[2]]:
end proc:
```

Multiply a linear combination for a root system of type $A_1 \times A_1$.

```
multA1A1:=proc(P,Q)
local r,i,j: r:=[]:
for i from 1 to nops(P) do
for j from 1 to nops(Q) do
r:=[op(r),multA1A1p(P[i],Q[j])]:
od:
od:
return r:
end proc:
```

Inverts an element of the Weyl group for a root system of type A_2.

```
invA2 := proc (w, P) local u, p: p := P:
for u to nops(w) do
p := reflA2(w[u], p):
od:
return p:
end proc:
```

Multiply two basis elements of the twisted group algebra for a root system of type A_2.

```
multA2p:=proc(P,Q)
return [reduce([op(P[1]),op(Q[1])],A2),invA2(Q[1],P[2])*Q[2]]:
end proc:
```

Multiply a linear combination for a root system of type A_2.

```
multA2:=proc(P,Q)
local r,i,j: r:=[]:
for i from 1 to nops(P) do
for j from 1 to nops(Q) do
r:=[op(r),multA2p(P[i],Q[j])]:
od:
od:
return r:
end proc:
```

Inverts an element of the Weyl group for a root system of type B_2.

```
invB2 := proc (w, P) local u, p: p := P:
```

```
for u to nops(w) do
p := reflB2(w[u], p):
od:
return p:
end proc:
```

Multiply two basis elements of the twisted group algebra for a root system of type B_2.

```
multB2p:=proc(P,Q)
return [reduce([op(P[1]),op(Q[1])],B2),invB2(Q[1],P[2])*Q[2]]:
end proc:
```

Multiply a linear combination for a root system of type B_2.

```
multB2:=proc(P,Q)
local r,i,j: r:=[]:
for i from 1 to nops(P) do
for j from 1 to nops(Q) do
r:=[op(r),multB2p(P[i],Q[j])]:
od:
od:
return r:
end proc:
```

Inverts an element of the Weyl group for a root system of type G_2.

```
invG2:=proc(w,P) local u,p: p:=P:
for u from 1 to nops(w) do
p:=reflG2(w[u],p):
od:
return p:
end proc:
```

Multiply two basis elements of the twisted group algebra for a root system of type G_2.

```
multG2p:=proc(P,Q)
return [reduce([op(P[1]),op(Q[1])],G2),invG2(Q[1],P[2])*Q[2]]:
end proc:
```

Multiply a linear combination for a root system of type G_2.

```
multG2:=proc(P,Q)
local r,i,j: r:=[]:
for i from 1 to nops(P) do
for j from 1 to nops(Q) do
r:=[op(r),multG2p(P[i],Q[j])]:
od:
od:
return r:
end proc:
```

Translates the list-notation into the x[..]-notation.

```
myexpr := proc (P) local k, s:
s := 0:
for k to nops(P) do
if nops(P[k][1])> 0 then s := s+delta[op(subs({1 = i, 2 = j}, P[k][1]))]*P[k][2]
else s := s+P[k][2] end if:
od:
return s:
end proc:
```

Creates the $\kappa_{i,j}$'s.

```
mkappa:=proc(i,j)
return x[i+j]*x[j]-x[i+j]*x[-i]-x[i]*x[j]
end proc:
print('kappaij =',mkappa(i,j)):
```

Creates the formal Demazure elements X_i and X_j.

```
X:=[[[],x[i]],[[1],-x[-i]]]:
Y:=[[[],x[j]],[[2],-x[-j]]]:
Demazelt[i]:=myexpr(X);
Demazelt[j]:=myexpr(Y);
```

Creates X_iX_j and X_jX_i.

```
Demazelt[j, i] := myexpr(multA1A1(Y, X));
Demazelt[i, j] := myexpr(multA1A1(X, Y));
```

Checks that $X_jX_i = X_iX_j$ for $m_{ij} = 2$.

```
Demazelt[j, i]-Demazelt[i, j];
```

Creates $X_iX_jX_i$ and $X_jX_iX_j$ for $m_{i,j} = 3$.

```
Demazelt[j, i, j] := myexpr(multA2(multA2(Y, X), Y));
Demazelt[i, j, i] := myexpr(multA2(multA2(X, Y), X));
```

Computes the difference $X_jX_iX_j - X_iX_jX_i$.

```
DIFF := Demazelt[j, i, j]-Demazelt[i, j, i];
```

Coefficients for X_i and X_j in the difference $X_jX_iX_j - X_iX_jX_i$.

```
C[j]:=simplify(coeff(DIFF,delta[j])/coeff(Demazelt[j],delta[j]));

C[i]:=simplify(coeff(DIFF,delta[i])/coeff(Demazelt[i],delta[i]));
```

Checks that the coefficient for X_j is $-\kappa_{j,i}$.

```
C[j]+mkappa(j,i);
print(`C[j] =`,-kappa[j,i]):
```

Checks that the coefficient for X_i is $\kappa_{i,j}$.

```
C[i]-mkappa(i,j);
print(`C[i] =`,kappa[i,j]):
```

Creates X_jX_i and X_iX_j for $m_{i,j} = 4$.

```
Demazelt[j, i] := myexpr(multB2(Y, X));
Demazelt[i, j] := myexpr(multB2(X, Y));
```

Creates $X_jX_iX_jX_i$ and $X_iX_jX_iX_j$ for $m_{i,j} = 4$.

```
Demazelt[i,j,i,j]:=myexpr(multB2(multB2(multB2(X,Y),X),Y));
Demazelt[j,i,j,i]:=myexpr(multB2(multB2(multB2(Y,X),Y),X));
```

Computes the difference $X_jX_iX_jX_i - X_iX_jX_iX_j$.

DIFF := Demazelt[j, i, j, i]-Demazelt[i, j, i, j]:

Coefficients for X_jX_i and X_iX_j in the difference $X_jX_iX_jX_i - X_iX_jX_iX_j$.

C[j,i]:=simplify(coeff(DIFF,delta[j,i])/coeff(Demazelt[j,i],delta[j,i]));

C[i,j]:=simplify(coeff(DIFF,delta[i,j])/coeff(Demazelt[i,j],delta[i,j]));

Removes the terms of degree 2 in the difference $X_jX_iX_jX_i - X_iX_jX_iX_j$.

DIFF1:=expand(DIFF-Demazelt[j,i]*C[j,i]-Demazelt[i,j]*C[i,j]):

Coefficients for X_j and X_i in the difference $X_jX_iX_jX_i - X_iX_jX_iX_j$.

C[j]:=simplify(coeff(DIFF1,delta[j])/coeff(Demazelt[j],delta[j]));

C[i]:=simplify(coeff(DIFF1,delta[i])/coeff(Demazelt[i],delta[i]));

print("Checking that the presentation is correct: has to be 0 at the end"):
simplify(DIFF-C[i, j]*Demazelt[i, j]-C[j, i]*Demazelt[j, i]-C[i]*Demazelt[i]-C[j]*Demazelt[j]);

Checks that the coefficient for X_jX_i is $-\kappa_{i+j,j} - \kappa_{i,j}$.

C[j,i]+mkappa(i+j,j)+mkappa(i,j);
print('C[j,i] =',-kappa[j+i,j]-kappa[i,j]):

Checks that the coefficient for X_iX_j is $\kappa_{i+2j,-j} + \kappa_{j,i}$.

C[i,j]-mkappa(i+2*j,-j)-mkappa(j,i);
print('C[i,j] =',kappa[2*j+i,-j]+kappa[j,i]):

Action of the formal Demazure operator Δ_{α_n} on P.

DemB2:=proc(n,P)
if n=1 then return expand(x[i]*(P-reflB2(1,P))) else return expand(x[j]*(P-reflB2(2,P)))
end if:
end proc:

Checks that the coefficient for X_i is $-\Delta_{\alpha_j}(\kappa_{i+2j,-j} + \kappa_{j,i})$.

expand(C[i]+DemB2(2,C[i,j]));
print(‘C[i]= - Delta_j (Demazure operator) applied to C[i,j]‘):

Checks that the coefficient for X_j is $-\Delta_{\alpha_i}(-\kappa_{i+j,j} - \kappa_{i,j})$.

expand(C[j]+DemB2(1,C[j,i]));
print(‘C[j]= - Delta_i (Demazure operator) applied to C[j,i]‘):

Creates X_jX_i and X_iX_j for $m_{i,j} = 6$.

Demazelt[j, i] := myexpr(multG2(Y, X));
Demazelt[i, j] := myexpr(multG2(X, Y));

Creates $X_jX_iX_j$ and $X_iX_jX_i$ for $m_{i,j} = 6$.

Demazelt[j,i,j]:=myexpr(multG2(multG2(Y,X),Y));
Demazelt[i,j,i]:=myexpr(multG2(multG2(X,Y),X));

Creates $X_jX_iX_jX_i$ and $X_iX_jX_iX_j$ for $m_{i,j} = 6$.

Demazelt[i,j,i,j]:=myexpr(multG2(multG2(multG2(X,Y),X),Y));
Demazelt[j,i,j,i]:=myexpr(multG2(multG2(multG2(Y,X),Y),X));

Creates $X_jX_iX_jX_iX_jX_i$ and $X_iX_jX_iX_jX_iX_j$ for $m_{i,j} = 6$.

Demazelt[i,j,i,j,i,j]:=myexpr(multG2(multG2(multG2(multG2(multG2(X,Y),X),Y),X),Y));
Demazelt[j,i,j,i,j,i]:=myexpr(multG2(multG2(multG2(multG2(multG2(Y,X),Y),X),Y),X));

Computes the difference $X_jX_iX_jX_iX_jX_i - X_iX_jX_iX_jX_iX_j$.

DIFF:=Demazelt[j,i,j,i,j,i]-Demazelt[i,j,i,j,i,j]:

Coefficients for $X_jX_iX_jX_i$ and $X_iX_jX_iX_j$ in the difference $X_jX_iX_jX_iX_jX_i - X_iX_jX_iX_jX_iX_j$.

C[j,i,j,i]:=simplify(coeff(DIFF,delta[j,i,j,i])/coeff(Demazelt[j,i,j,i],delta[j,i,j,i]));

C[i,j,i,j]:=simplify(coeff(DIFF,delta[i,j,i,j])/coeff(Demazelt[i,j,i,j],delta[i,j,i,j]));

Removes the terms of degree 4 in the difference $X_jX_iX_jX_iX_jX_i - X_iX_jX_iX_jX_iX_j$.

DIFF1:=expand(DIFF-Demazelt[j,i,j,i]*C[j,i,j,i]-Demazelt[i,j,i,j]*C[i,j,i,j]):

Coefficients for $X_jX_iX_j$ and $X_iX_jX_i$ in the difference $X_jX_iX_jX_iX_jX_i - X_iX_jX_iX_jX_iX_j$.

C[j,i,j]:=simplify(coeff(DIFF1,delta[j,i,j])/coeff(Demazelt[j,i,j],delta[j,i,j]));

C[i,j,i]:=simplify(coeff(DIFF1,delta[i,j,i])/coeff(Demazelt[i,j,i],delta[i,j,i]));

Removes the terms of degree 3 in the difference $X_jX_iX_jX_iX_jX_i - X_iX_jX_iX_jX_iX_j$.

DIFF2:=expand(DIFF1-Demazelt[j,i,j]*C[j,i,j]-Demazelt[i,j,i]*C[i,j,i]):

Coefficients for X_jX_i and X_iX_j in the difference $X_jX_iX_jX_iX_jX_i - X_iX_jX_iX_jX_iX_j$.

C[j,i]:=simplify(coeff(DIFF2,delta[j,i])/coeff(Demazelt[j,i],delta[j,i]));

C[i,j]:=simplify(coeff(DIFF2,delta[i,j])/coeff(Demazelt[i,j],delta[i,j]));

Removes the terms of degree 2 in the difference $X_jX_iX_jX_iX_jX_i - X_iX_jX_iX_jX_iX_j$.

DIFF3:=expand(DIFF2-Demazelt[j,i]*C[j,i]-Demazelt[i,j]*C[i,j]):

Coefficients for X_j and X_i in the difference $X_jX_iX_jX_iX_jX_i - X_iX_jX_iX_jX_iX_j$.

C[j]:=simplify(coeff(DIFF3,delta[j])/coeff(Demazelt[j],delta[j]));

C[i]:=simplify(coeff(DIFF3,delta[i])/coeff(Demazelt[i],delta[i]));

print("Checking that the presentation is correct: has to be 0 at the end"):
simplify(DIFF-Demazelt[i,j,i,j]*C[i,j,i,j]-Demazelt[j,i,j,i]*C[j,i,j,i]-Demazelt[i,j,i]*C[i,j,i]-Demazelt[j,i,j]*C[j,i,j]-Demazelt[i,j]*C[i,j]-Demazelt[j,i]*C[j,i]-Demazelt[i]*C[i]-Demazelt[j]*C[j]);

Checks that the coefficients for $X_iX_jX_iX_j$ is $\kappa_{j,i} + \kappa_{2i+3j,-i-2j} + \kappa_{-i-3j,i+2j} + \kappa_{i+2j,-j}$.

C[i,j,i,j]-mkappa(j,i)-mkappa(2*i+3*j,-i-2*j)-mkappa(-i-3*j,i+2*j)-mkappa(i+2*j,-j);
print('C[i,j,i,j] =',kappa[j,i]+kappa[2*i+3*j,-i-2*j]+kappa[-i-3*j,i+2*j]+kappa[i+2*j,-j]):

Checks that the coefficients for $X_jX_iX_jX_i$ is $-\kappa_{i,j} - \kappa_{-2i-3j,i+2j} - \kappa_{-i-2j,i+3j} - \kappa_{i+j,j}$.

```
C[j,i,j,i]+mkappa(i,j)+mkappa(-2*i-3*j,i+2*j)+mkappa(-i-2*j,i+3*j)+mkappa(i+j,j);
print(`C[j,i,j,i] =`,-kappa[i,j]-kappa[-2*i-3*j,i+2*j]-kappa[-i-2*j,i+3*j]-kappa[i+j,j]):
```

Action of the formal Demazure operator Δ_{α_n} on P.

```
Dem:=proc(n,P)
if n=1 then return expand(x[i]*(P-reflG2(1,P))) else return expand(x[j]*(P-reflG2(2,P)))
end if:
end proc:
```

Checks that the coefficients for $X_i X_j X_i$ is $-\Delta_{\alpha_j}(\kappa_{j,i} + \kappa_{2i+3j,-i-2j} + \kappa_{-i-3j,i+2j} + \kappa_{i+2j,-j})$.

```
expand(C[i,j,i]+Dem(2,C[i,j,i,j]));
print(`C[i,j,i]= - Delta_j (Demazure operator) applied to C[i,j,i,j]`):
```

Checks that the coefficients for $X_j X_i X_j$ is $-\Delta_{\alpha_i}(\kappa_{i,j}+\kappa_{-2i-3j,i+2j}+\kappa_{-i-2j,i+3j}+\kappa_{i+j,j})$.

```
expand(C[j,i,j]+Dem(1,C[j,i,j,i]));
print(`C[j,i,j]= - Delta_i (Demazure operator) applied to C[j,i,j,i]`):
```

Shows that the coefficient for $X_i X_j$ is ξ_{ij} and checks that the coefficient for X_i is $-\Delta_{\alpha_j}(\xi_{ij})$.

```
C[i,j];
expand(C[i]+Dem(2,C[i,j]));
print(`C[i]= - Delta_j applied to C[i,j]`):
```

Shows that the coefficient for $X_j X_i$ is ξ_{ji} and checks that the coefficient for X_j is $-\Delta_{\alpha_i}(\xi_{ji})$.

```
C[j,i];
expand(C[j]+Dem(1,C[j,i]));
print(`C[j]= - Delta_i applied to C[j,i]`):
```

Appendix B

Computations of relations for push-pull elements

The following is a Maple script to compute the relations in Proposition 6.2.11. See Appendix A for details about the package you need to load in order to run the script.

Action of the n-th simple reflection on an expression P for a root system of type $A_1 \times A_1$.

```
reflA1A1 := proc (n, P)
if n = 2 then return subs({x[j] = x[-j], x[-j] = x[j]}, P)
else return subs({x[i] = x[-i], x[-i] = x[i]}, P)
end if
end proc:
```

Action of the n-th simple reflection on an expression P for a root system of type A_2.

```
reflA2 := proc (n, P)
if n = 2 then return subs({x[i] = x[j+i], x[j] = x[-j], x[-i] = x[-j-i], x[-j] = x[j], x[-j-i]
= x[-i], x[j+i] = x[i]}, P)
else return subs({x[i] = x[-i], x[j] = x[j+i], x[-i] = x[i], x[-j] = x[-j-i], x[-j-i] = x[-j],
x[j+i] = x[j]}, P)
end if
end proc:
```

Action of the n-th simple reflection on an expression P for a root system of type B_2.

```
reflB2 := proc (n, P)
if n = 2 then return subs({x[i] = x[2*j+i], x[j] = x[-j], x[-i] = x[-2*j-i], x[-j] = x[j],
```

```
x[-2*j-i] = x[-i], x[2*j+i] = x[i]}, P)
else return subs({x[i] = x[-i], x[j] = x[j+i], x[-i] = x[i], x[-j] = x[-j-i], x[-j-i] = x[-j],
x[j+i] = x[j]}, P)
end if
end proc:
```

Action of the n-th simple reflection on an expression P for a root system of type G_2.

```
reflG2:=proc(n,P)
if n=2 then return subs({x[j]=x[-j],x[i]=x[3*j+i],x[-j]=x[j],x[-i]=x[-3*j-i],x[j+i]=x[2*j+i],
x[-j-i]=x[-2*j-i],x[3*j+i]=x[i],x[-3*j-i]=x[-i],x[2*j+i]=x[j+i],x[-2*j-i]=x[-j-i]},P)
else return subs({x[j]=x[j+i],x[i]=x[-i],x[-j]=x[-j-i],x[-i]=x[i],x[j+i]=x[j],x[-j-i]=x[-j],
x[3*j+i]=x[3*j+2*i],x[-3*j-i]=x[-3*j-2*i],x[3*j+2*i]=x[3*j+i],x[-3*j-2*i]=x[-3*j-i]},P)
end if
end proc:
```

Inverts an element of the Weyl group for a root system of type $A_1 \times A_1$.

```
invA1A1 := proc (w, P) local u, p: p := P:
for u to nops(w) do
p := reflA1A1(w[u], p):
od:
return p:
end proc:
```

Multiply two basis elements of the twisted group algebra for a root system of type $A_1 \times A_1$.

```
multA1A1p:=proc(P,Q)
return [reduce([op(P[1]),op(Q[1])],A1*A1),invA1A1(Q[1],P[2])*Q[2]]:
end proc:
```

Multiply a linear combination for a root system of type $A_1 \times A_1$.

```
multA1A1:=proc(P,Q)
local r,i,j: r:=[]:
for i from 1 to nops(P) do
for j from 1 to nops(Q) do
r:=[op(r),multA1A1p(P[i],Q[j])]:
od:
od:
```

```
return r:
end proc:
```

Inverts an element of the Weyl group for a root system of type A_2.

```
invA2 := proc (w, P) local u, p: p := P:
for u to nops(w) do
p := reflA2(w[u], p):
od:
return p:
end proc:
```

Multiply two basis elements of the twisted group algebra for a root system of type A_2.

```
multA2p:=proc(P,Q)
return [reduce([op(P[1]),op(Q[1])],A2),invA2(Q[1],P[2])*Q[2]]:
end proc:
```

Multiply a linear combination for a root system of type A_2.

```
multA2:=proc(P,Q)
local r,i,j: r:=[]:
for i from 1 to nops(P) do
for j from 1 to nops(Q) do
r:=[op(r),multA2p(P[i],Q[j])]:
od:
od:
return r:
end proc:
```

Inverts an element of the Weyl group for a root system of type B_2.

```
invB2 := proc (w, P) local u, p: p := P:
for u to nops(w) do
p := reflB2(w[u], p):
od:
return p:
end proc:
```

Multiply two basis elements of the twisted group algebra for a root system of type B_2.

```
multB2p:=proc(P,Q)
return [reduce([op(P[1]),op(Q[1])],B2),invB2(Q[1],P[2])*Q[2]]:
end proc:
```

Multiply a linear combination for a root system of type B_2.

```
multB2:=proc(P,Q)
local r,i,j: r:=[]:
for i from 1 to nops(P) do
for j from 1 to nops(Q) do
r:=[op(r),multB2p(P[i],Q[j])]:
od:
od:
return r:
end proc:
```

Inverts an element of the Weyl group for a root system of type G_2.

```
invG2:=proc(w,P) local u,p: p:=P:
for u from 1 to nops(w) do
p:=reflG2(w[u],p):
od:
return p:
end proc:
```

Multiply two basis elements of the twisted group algebra for a root system of type G_2.

```
multG2p:=proc(P,Q)
return [reduce([op(P[1]),op(Q[1])],G2),invG2(Q[1],P[2])*Q[2]]:
end proc:
```

Multiply a linear combination for a root system of type G_2.

```
multG2:=proc(P,Q)
local r,i,j: r:=[]:
for i from 1 to nops(P) do
for j from 1 to nops(Q) do
r:=[op(r),multG2p(P[i],Q[j])]:
od:
od:
return r:
```

```
end proc:
```

Translates the list-notation into the x[..]-notation.

```
myexpr := proc (P) local k, s:
s := 0:
for k to nops(P) do
if nops(P[k][1])>0 then s := s+delta[op(subs({1 = i, 2 = j}, P[k][1]))]*P[k][2] else s
:= s+P[k][2] end if:
od:
return s:
end proc:
```

Creates the $\kappa_{i,j}$'s.

```
mkappa:=proc(i,j)
return x[i+j]*x[j]-x[i+j]*x[-i]-x[i]*x[j] end proc:
print(`kappa_ij =`,mkappa(i,j)):
```

Creates the formal Push-pull elements Y_i and Y_j.

```
X:=[[[],x[-i]],[[1], x[-i]]]:
Y:=[[[],x[-j]],[[2], x[-j]]]:
Pushpull[i]:=myexpr(X);
Pushpull[j]:=myexpr(Y);
```

Creates Y_iY_j and Y_jY_i.

```
Pushpull[j, i] := myexpr(multA1A1(Y, X));
Pushpull[i, j] := myexpr(multA1A1(X, Y));
```

Checks that $Y_jY_i = Y_iY_j$ for $m_{ij} = 2$.

```
Pushpull[j, i]-Pushpull[i, j];
```

Creates $Y_iY_jY_i$ and $Y_jY_iY_j$ for $m_{i,j} = 3$.

```
Pushpull[j, i, j] := myexpr(multA2(multA2(Y, X), Y));
Pushpull[i, j, i] := myexpr(multA2(multA2(X, Y), X));
```

Computes the difference $Y_j Y_i Y_j - Y_i Y_j Y_i$.

DIFF := Pushpull[j, i, j]-Pushpull[i, j, i];

Coefficients for Y_i and Y_j in the difference $Y_j Y_i Y_j - Y_i Y_j Y_i$.

C[j]:=simplify(coeff(DIFF,delta[j])/coeff(Pushpull[j],delta[j]));

C[i]:=simplify(coeff(DIFF,delta[i])/coeff(Pushpull[i],delta[i]));

Checks that the coefficient for Y_j is $-\kappa_{-j,-i}$.

C[j]+mkappa(-j,-i);
print('C[j] =',-kappa[-j,-i]):

Checks that the coefficient for Y_i is $\kappa_{-i,-j}$.

C[i]-mkappa(-i,-j);
print('C[i] =',kappa[-i,-j]):

Creates $Y_j Y_i$ and $Y_i Y_j$ for $m_{i,j} = 4$.

Pushpull[j, i] := myexpr(multB2(Y, X));
Pushpull[i, j] := myexpr(multB2(X, Y));

Creates $Y_j Y_i Y_j Y_i$ and $Y_i Y_j Y_i Y_j$ for $m_{i,j} = 4$.

Pushpull[i,j,i,j]:=myexpr(multB2(multB2(multB2(X,Y),X),Y));
Pushpull[j,i,j,i]:=myexpr(multB2(multB2(multB2(Y,X),Y),X));

Computes the difference $Y_j Y_i Y_j Y_i - Y_i Y_j Y_i Y_j$.

DIFF := Pushpull[j, i, j, i]-Pushpull[i, j, i, j]:

Coefficients for $Y_j Y_i$ and $Y_i Y_j$ in the difference $Y_j Y_i Y_j Y_i - Y_i Y_j Y_i Y_j$.

C[j,i]:=simplify(coeff(DIFF,delta[j,i])/coeff(Pushpull[j,i],delta[j,i]));

C[i,j]:=simplify(coeff(DIFF,delta[i,j])/coeff(Pushpull[i,j],delta[i,j]));

Removes the terms of degree 2 in the difference $Y_jY_iY_jY_i - Y_iY_jY_iY_j$.

DIFF1:=expand(DIFF-Pushpull[j,i]*C[j,i]-Pushpull[i,j]*C[i,j]):

Coefficients for Y_j and Y_i in the difference $Y_jY_iY_jY_i - Y_iY_jY_iY_j$.

C[j]:=simplify(coeff(DIFF1,delta[j])/coeff(Pushpull[j],delta[j]));

C[i]:=simplify(coeff(DIFF1,delta[i])/coeff(Pushpull[i],delta[i]));

print("Checking that the presentation is correct: has to be 0 at the end"):
simplify(DIFF-C[i, j]*Pushpull[i, j]-C[j, i]*Pushpull[j, i]-C[i]*Pushpull[i]-C[j]*Pushpull[j]);

Checks that the coefficient for Y_jY_i is $-\kappa_{-i-j,-j} - \kappa_{-i,-j}$.

C[j,i]+mkappa(-i-j,-j)+mkappa(-i,-j);
print('C[j,i] =',-kappa[-j-i,-j]-kappa[-i,-j]):

Checks that the coefficient for Y_iY_j is $\kappa_{-i-2j,j} + \kappa_{-j,-i}$.

C[i,j]-mkappa(-i-2*j,j)-mkappa(-j,-i);
print('C[i,j] =',kappa[-2*j-i,j]+kappa[-j,-i]):

Action of the formal Demazure operator Δ_{α_n} on P.

DemB2:=proc(n,P)
if n=1 then return expand(x[-i]*(P-reflB2(1,P))) else return expand(x[-j]*(P-reflB2(2,P)))
end if:
end proc:

Checks that the coefficient for Y_i is $-\Delta_{-\alpha_j}(\kappa_{-i-2j,j} + \kappa_{-j,-i})$.

expand(C[i]+DemB2(2,C[i,j]));
print('C[i]= - Delta_-j (Demazure operator) applied to C[i,j]'):

Checks that the coefficient for Y_j is $-\Delta_{\alpha_i}(-\kappa_{-i-j,-j} - \kappa_{-i,-j})$.

expand(C[j]+DemB2(1,C[j,i]));
print('C[j]= - Delta_-i (Demazure operator) applied to C[j,i]'):

Creates X_jX_i and X_iX_j for $m_{i,j} = 6$.

Pushpull[j, i] := myexpr(multG2(Y, X));
Pushpull[i, j] := myexpr(multG2(X, Y));

Creates $Y_jY_iY_j$ and $Y_iY_jY_i$ for $m_{i,j} = 6$.

Pushpull[j,i,j]:=myexpr(multG2(multG2(Y,X),Y));
Pushpull[i,j,i]:=myexpr(multG2(multG2(X,Y),X));

Creates $Y_jY_iY_jY_i$ and $Y_iY_jY_iY_j$ for $m_{i,j} = 6$.

Pushpull[i,j,i,j]:=myexpr(multG2(multG2(multG2(X,Y),X),Y));
Pushpull[j,i,j,i]:=myexpr(multG2(multG2(multG2(Y,X),Y),X));

Creates $Y_jY_iY_jY_iY_jY_i$ and $Y_iY_jY_iY_jY_iY_j$ for $m_{i,j} = 6$.

Pushpull[i,j,i,j,i,j]:=myexpr(multG2(multG2(multG2(multG2(multG2(X,Y),X),Y),X),Y));
Pushpull[j,i,j,i,j,i]:=myexpr(multG2(multG2(multG2(multG2(multG2(Y,X),Y),X),Y),X));

Computes the difference $Y_jY_iY_jY_iY_jY_i - Y_iY_jY_iY_jY_iY_j$.

DIFF:=Pushpull[j,i,j,i,j,i]-Pushpull[i,j,i,j,i,j]:

Coefficients for $Y_jY_iY_jY_i$ and $Y_iY_jY_iY_j$ in the difference $Y_jY_iY_jY_iY_jY_i - Y_iY_jY_iY_jY_iY_j$.

C[j,i,j,i]:=simplify(coeff(DIFF,delta[j,i,j,i])/coeff(Pushpull[j,i,j,i],delta[j,i,j,i]));

C[i,j,i,j]:=simplify(coeff(DIFF,delta[i,j,i,j])/coeff(Pushpull[i,j,i,j],delta[i,j,i,j]));

Removes the terms of degree 4 in the difference $Y_jY_iY_jY_iY_jY_i - Y_iY_jY_iY_jY_iY_j$.

DIFF1:=expand(DIFF-Pushpull[j,i,j,i]*C[j,i,j,i]-Pushpull[i,j,i,j]*C[i,j,i,j]):

Coefficients for $Y_jY_iY_j$ and $Y_iY_jY_i$ in the difference $Y_jY_iY_jY_iY_jY_i - Y_iY_jY_iY_jY_iY_j$.

C[j,i,j]:=simplify(coeff(DIFF1,delta[j,i,j])/coeff(Pushpull[j,i,j],delta[j,i,j]));

C[i,j,i]:=simplify(coeff(DIFF1,delta[i,j,i])/coeff(Pushpull[i,j,i],delta[i,j,i]));

Removes the terms of degree 3 in the difference $Y_jY_iY_jY_iY_jY_i - Y_iY_jY_iY_jY_iY_j$.

DIFF2:=expand(DIFF1-Pushpull[j,i,j]*C[j,i,j]-Pushpull[i,j,i]*C[i,j,i]):

Coefficients for Y_jY_i and Y_iY_j in the difference $Y_jY_iY_jY_iY_jY_i - Y_iY_jY_iY_jY_iY_j$.

C[j,i]:=simplify(coeff(DIFF2,delta[j,i])/coeff(Pushpull[j,i],delta[j,i]));

C[i,j]:=simplify(coeff(DIFF2,delta[i,j])/coeff(Pushpull[i,j],delta[i,j]));

Removes the terms of degree 2 in the difference $Y_jY_iY_jY_iY_jY_i - Y_iY_jY_iY_jY_iY_j$.

DIFF3:=expand(DIFF2-Pushpull[j,i]*C[j,i]-Pushpull[i,j]*C[i,j]):

Coefficients for Y_j and Y_i in the difference $Y_jY_iY_jY_iY_jY_i - Y_iY_jY_iY_jY_iY_j$.

C[j]:=simplify(coeff(DIFF3,delta[j])/coeff(Pushpull[j],delta[j]));

C[i]:=simplify(coeff(DIFF3,delta[i])/coeff(Pushpull[i],delta[i]));

print("Checking that the presentation is correct: has to be 0 at the end"):
simplify(DIFF-Pushpull[i,j,i,j]*C[i,j,i,j]-Pushpull[j,i,j,i]*C[j,i,j,i]-Pushpull[i,j,i]*C[i,j,i]-Pushpull[j,i,j]*C[j,i,j]-Pushpull[i,j]*C[i,j]-Pushpull[j,i]*C[j,i]-Pushpull[i]*C[i]-Pushpull[j]*C[j]);

Checks that the coefficients for $Y_iY_jY_iY_j$ is $\kappa_{-j,-i} + \kappa_{-2i-3j,i+2j} + \kappa_{i+3j,-i-2j} + \kappa_{-i-2j,j}$.

C[i,j,i,j]-mkappa(-j,-i)-mkappa(-2*i-3*j,i+2*j)-mkappa(i+3*j,-i-2*j)-mkappa(-i-2*j,j);
print('C[i,j,i,j] =',kappa[-j,-i]+kappa[-2*i-3*j,i+2*j]+kappa[i+3*j,-i-2*j]+kappa[-i-2*j,j]):

Checks that the coefficients for $Y_jY_iY_jY_i$ is $-\kappa_{-i,-j} - \kappa_{2i+3j,-i-2j} - \kappa_{i+2j,-i-3j} - \kappa_{-i-j,-j}$.

C[j,i,j,i]+mkappa(-i,-j)+mkappa(2*i+3*j,-i-2*j)+mkappa(i+2*j,-i-3*j)+mkappa(-i-j,-j);
print('C[j,i,j,i] =',-kappa[-i,-j]-kappa[2*i+3*j,-i-2*j]-kappa[i+2*j,-i-3*j]-kappa[-i-j,-j]):

Action of the formal Demazure operator Δ_{α_n} on P.

Dem:=proc(n,P)
if n=1 then return expand(x[-i]*(P-reflG2(1,P))) else return expand(x[-j]*(P-reflG2(2,P)))
end if:

end proc:

Checks that the coefficients for $Y_iY_jY_i$ is $-\Delta_{-\alpha_j}(\kappa_{-j,-i} + \kappa_{-2i-3j,i+2j} + \kappa_{i+3j,-i-2j} + \kappa_{-i-2j,j})$.

expand(C[i,j,i]+Dem(2,C[i,j,i,j]));
print(‘C[i,j,i]= - Delta_-j (Demazure operator) applied to C[i,j,i,j]‘):

Checks that the coefficients for $Y_jY_iY_j$ is $-\Delta_{-\alpha_i}(\kappa_{-i,-j} + \kappa_{2i+3j,-i-2j} + \kappa_{i+2j,-i-3j} + \kappa_{-i-j,-j})$.

expand(C[j,i,j]+Dem(1,C[j,i,j,i]));
print(‘C[j,i,j]= - Delta_-i (Demazure operator) applied to C[j,i,j,i]‘):

Shows that the coefficient for Y_iY_j is ξ'_{ij} and checks that the coefficient for Y_i is $-\Delta_{-\alpha_j}(\xi'_{ij})$.

C[i,j];
expand(C[i]+Dem(2,C[i,j]));
print(‘C[i]= - Delta_-j applied to C[i,j]‘):

Shows that the coefficient for Y_jY_i is ξ'_{ji} and checks that the coefficient for Y_j is $\Delta_{-\alpha_i}(\xi'_{ji})$.

C[j,i];
expand(C[j]+Dem(1,C[j,i]));
print(‘C[j]= - Delta_-i applied to C[j,i]‘):

Appendix C

Computations for the coefficients ξ_{ij} and ξ_{ji} for the hyperbolic formal group law

The following is a Maple script to compute the value of the coefficients in Example 6.2.10. You do not need to load any package to run this script.

Let $a := -\mu_1$ and $b := \mu_2$.

Procedure for the formal sum for the hyperbolic formal group law F_{μ_1,μ_2}.

```
F := proc (u, v) → (u+v+a*u*v)/(1+b*u*v) end proc;
```

We denote $i := x := x_i$ and $j := y := x_j$.

```
i := x;
j := y;
```

Creates x_{i+j}.

```
i_j := F(x, y);
```

Creates x_{i+2j}.

```
i_j2 := F(F(x, y), y);
```

Creates x_{i+3j}.

i_j3 := F(F(F(x, y), y), y);

Creates x_{2i+3j}.

i2_j3 := F(F(F(F(x, x), y), y), y);

We denote $k := z := x_{-i}$ and $l := w := x_{-j}$.

k := z;
l := w;

Creates x_{-i-j}.

k_l := F(z, w);

Creates x_{-i-2j}.

k_l2 := F(F(z, w), w);

Creates x_{-i-3j}.

k_l3 := F(F(F(z, w), w), w);

Creates x_{-2i-3j}.

k2_l3 := F(F(F(F(z, z), w), w), w);

Equations for the formal inverses, i.e. $F(x_i, x_{-i}) = F(x_j, x_{-j}) = 0$.

eqns := {F(i, k) = 0, F(j, l) = 0};

Creates ξ_{ji} and simplifies it with respect the equations.

```
Xji := 1/(i*j*i2_j3*k_l2)+1/(i*j*i_j2*k_l3)+1/(j*i_j2*i_j3*i2_j3)-1/(i*j*i_j*i2_j3)+
+1/(i_j*i_j2*k*k2_l3)+1/(i_j3*i2_j3*k_l*k_l2)+1/(i_j*i_j3*k*k_l2)-1/(j*i_j3*i2_j3*k_l)-
-1/(j*i_j*i_j3*k);
simplify(Xij, eqns)
```

Creates ξ_{ij} and simplifies it with respect the equations.

Xij := -1/(i*i_j*i_j2*i2_j3)-1/(i*j*i_j2*k2_l3)-1/(i*j*i2_j3*k_l)+1/(i*i_j*i_j2*k_l3)+
+1/(i*i_j*i_j3*l)-1/(i_j*i_j3*l*k2_l3)-1/(i_j3*i2_j3*l*k_l2)-1/(i_j*i_j2*k_l3*k2_l3)+
+1/(i*j*i_j2*i_j3);
simplify(Xij, eqns)

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