Differential Forms for T-Algebras in Kähler Categories

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Abstract

A Kähler category, as described in [1], axiomatizes the algebraic geometric theory of Kähler Differentials in an abstract categorical setting. To facilitate this, a Kähler category is equipped with an algebra modality, which endows each object in the image of a specified monad with an associative algebra structure; universal derivations are then required to exist naturally for each of these objects. Moreover, it can be demonstrated that for each T-algebra of said monad there is a natural associative algebra structure.

In this paper I will show that under certain conditions on the Kähler category, the universal derivations for the algebras arising from T-algebras exist and arise via a coequalizer. Furthermore, this result is extended to provide an alternative construction for universal derivations for a more general class of algebras, including all algebras in a Kähler category. A prospective categorical formulation of the theory of noncommutative Kähler differentials is then given, and the above said results are shown to apply in this context. Finally, another class of algebras is constructed via a colimit, and the modules of differential forms for these algebras is computed.

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Contents

Chapter 1

Introduction

In Algebraic Geometry, the theory of *Kähler differentials* commences with the observation that the salient features of differentiation can be captured algebraically, most conspicuously in an analogue to the Leibniz rule. One then recognizes that derivations so constructed are in fact maps from algebras over commutative rings into modules over those algebras. The crux of the theory is that for each such algebra there is a universal derivation into a 'module of differential 1-forms'. The theory is then adapted to the theory of schemes where the 1-forms act as a geometric invariant akin to cotangent bundles for manifolds [7, II.8]. Indeed, the theory of Kähler differentials can be used to describe not only 1-forms but n-forms, which provide an algebraic analogue of de Rham Cohomology [4, 16.9].

Kähler categories abstract the theory of Kähler differentials further by subsuming the theory in the required existence of universal derivations [1]. The advantages of this categorification are an increased generality, which possibly allows the theory to be translated to new domains, and also a unique perspective on established material possibly extending the theory's utility in any given domain. It is the latter advantage that is exploited here.

The notion of a Kähler category was developed in an attempt to account for a deficiency in the structure of differential categories; namely, the lack of universal derivations [1]. Differential categories are additive symmetric monoidal categories equipped with a coalgebra modality (a special type of comonad) and a differential combinator. The differential combinator takes a map in the coKleisli category and in a functorial way produces a map which is thought of as the derivative of the original map. Accordingly, the differential combinator must satisfy a set of axioms, each of which is analogous to a corresponding property of the derivative from differential calculus. So if we think of the coKleisli maps as smooth, differential categories provide an abstraction of the notion of linearly approximating smooth maps [2]. In many examples of the dual notion of a codifferential category, universal derivations are ubiquitous, however the axioms for a codifferential category do not explicitly prescribe any universality. Hence a Kähler category is defined to be an additive symmetric monoidal category with an algebra modality and a universal derivation emanating from each object in the image of the algebra modality. An important aspect of their study then is the search for necessary and sufficient conditions under which a codifferential category is a Kähler category [1].

The key motivation for the results in this paper was the discrepancy between the classical theory of Kähler differentials and Kähler categories that whereas modules of differential forms exist for any algebra over a commutative ring in the former, they exist in the latter a priori only for a certain class of algebras [1]. It was observed by Cockett that there are actually many more algebras in a Kähler category; namely, each T-algebra gives rise to an associative algebra. In this paper I will show that if the tensor product in the Kähler category preserves colimits, the modules of universal differential forms can be constructed via a coequalizer. Furthermore, the proof of the validity of this construction requires only the fact that a T-algebra in this category is a specific type of coequalizer, and so is a corollary of a more general result (Theorem 6.0.1).

Additionally, presented in this paper is a prospective formulation of noncommutative Kähler categories due to Cockett, which categorifies the notion of noncommutative Kähler differentials as seen in [6]. In *noncommutative geometry* a frequent method of investigation is the generalization of algebraic formulations of notions from classical mathematics to noncommutative algebraic structures [9]. It was noted above that the theory of Kähler differentials allows one to formulate de Rham cohomology algebraically, and hence noncommutative Kähler differentials ostensibly offer a means for developing de Rham cohomology for noncommutative spaces. [1] In the present formulation of noncommutative Kähler categories, the theory of noncommutative Kähler differentials is abstracted in such a way that makes noncommutative Kähler categories but a subtle variation of the original (commutative) Kähler cate-

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[1] As it happens this attempt is not quite successful, but noncommutative Kähler differentials play an important role in the theory of cyclic cohomology, which is the noncommutative analogue of de Rham cohomology. See [10] for details.

gories. I will show that the results presented here for Kähler categories hold also in the noncommutative context.

Finally, a class of algebras resulting from the colimits of chains of maps in closed Kähler categories is developed here, and it is shown that under certain conditions each of these algebras has an associated module of universal differential forms.

1.1 New Material

The following results are new: Lemma 6.0.1, Proposition 6.0.1, Theorem 6.0.1, Corollary 6.0.1, Corollary 6.0.2, Theorem 6.0.2, Lemma 7.1.1, Theorem 7.1.1, Lemma 8.0.1, Proposition 8.0.1, Lemma 8.0.2, Lemma 8.0.3, Theorem 8.0.1

Chapter 2

Kähler Differentials

2.1 The Commutative Case

The exposition contained herein follows [7, II.8]. For the duration of this subsection, let R be a commutative unital ring, and A a commutative R-algebra. The goal of the theory of Kähler differentials is to develop an analogue of the differential calculus for commutative algebra, thus developing a potent means for the classification of algebras. First though, a strictly algebraic conception of differentiation must be developed. Hence we make the following definition:

Definition 2.1.1. Let M be a left A-module. An A-*derivation* is an R-linear map $\partial : A \longrightarrow M$ satisfying the Leibniz rule. That is:

$$\partial(ab) = a\partial b + b\partial a \qquad\qquad \forall a, b \in A$$

This encapsulates the notion of differentiation in the satisfaction of the Leibniz rule (or product rule) familiar from differential calculus. Note that this definition does not specify exactly how the map ∂ acts on individual elements of the algebra; for instance, $\partial = 0$ is an A-derivation. The real utility of observing the derivations of an algebra then, lies in the notion of a universal derivation.

Definition 2.1.2. *The module of A-differential forms over R* is an A-module $\Omega_{A/R}$ together with an A-derivation $d_A : A \longrightarrow \Omega_{A/R}$ universal in the sense that for any A-derivation $\partial : A \longrightarrow M$ to an A-module M, there exists a unique A-module map $k : \Omega_{A/R} \longrightarrow M$ such that $\partial = d_A; k$ as in the following diagram:

$$\begin{tikzcd}
A \arrow[r, "d_A"] \arrow[dr, "\partial"'] & \Omega_A \arrow[d, dashed, "k"] \\
 & M
\end{tikzcd}$$

.

For any such algebra a module of differential forms can be formally constructed by taking the free A-module generated by the set $\{d_A a | a \in A\}$ and then dividing out by the submodules generated by expressions connoting the appropriate relations (i.e. the submodule generated by expressions of the form $d_A(ab) - ad_A b - bd_A a$). Our universal derivation then sends each $a \in A$ to $d_A a$. A more useful construction is given by the following proposition:

Proposition 2.1.1. *Consider the multiplication of* A, $m_A : A \otimes A \longrightarrow A$ *and let* $I = ker(m_A)$. *Consider* I/I^2 *and the map* $\partial : A \longrightarrow I/I^2$ *defined by* $\partial a = 1 \otimes a - a \otimes 1$. *Then* I/I^2 *is the module of A-differential forms over R with universal derivation* ∂.

Proof: See [12, pg 192] ∎

Note that here we have an explicit description of how the universal derivation acts on elements of the algebra.

2.2 The Noncommutative Case

Founded and articulated largely by Alain Connes, noncommutative geometry aims to extend the techniques used to examine algebra-geometric correspondences of classical mathematics (e.g. the equivalence between the opposite of the category of commutative unital C* algebras and the category of locally compact Hausdorff spaces) to spaces for which the corresponding algebra is noncommutative. The inspiration for this work traces back to Heisenberg's elucidation of the fact that the observables of a quantum mechanical system form a noncommutative algebra, in contrast to the commutative algebra of observables of a classical system. The typical strategy for dealing with such spaces is to develop algebraic analogues of tools used for the analysis of commutative spaces, and then extending to the context of noncommutative algebra [9]. It was pointed out in the introduction that the theory of Kähler differentials facilitates an algebraic analogue of de Rham cohomology, and therefore one way of approaching the development of de Rham cohomology for noncommutative spaces is by formulating the theory of Kähler differentials for noncommutative algebras. There is a slightly different approach to defining modules of differential forms for algebras which are not necessarily commutative; here we consider certain maps from algebras to bimodules as our primary objects of inquiry. Let R be a commutative unital ring, and A an associative R-algebra.

Definition 2.2.1. Let M be an A-bimodule. A *(noncommutative) A-derivation* is an R-linear map $\partial : A \longrightarrow M$ satisfying the (noncommutative) Leibniz rule. That is:

$$\partial(ab) = a(\partial b) + (\partial a)b \qquad \qquad \forall a, b \in A$$

Note that without considering bimodules it would be impossible to construct an analogue of the Leibniz rule that distinguishes between $\partial(ab)$ and $\partial(ba)$ for non-commuting a and b.

Now, with this alternative definition of a derivation, the module of differential forms is defined exactly as in the commutative case. Its construction in this case is surprisingly much simpler. Define $d_A : A \longrightarrow A \otimes A$ to be the linear map defined by

$$d_A a = 1 \otimes a - a \otimes 1$$

We may then define Ω_A to be the submodule of $A \otimes A$ generated by elements of the form $a(d_A b)$ with left and right module actions:

$$c(a(d_A b)) = ca(d_A b)$$

and

$$(a(d_A b))c = a(d_A(bc)) - ab(d_A c),$$

respectively. Then if $\partial : A \longrightarrow M$ is another A-derivation to an A-bimodule M, we can define a bimodule morphism $k : \Omega_A \longrightarrow M$ by its action on simple tensors: $k(a \otimes b) = a(\partial b)$, where $a \otimes b \in \Omega_A$. Commutativity of

$$\begin{tikzcd}
A \arrow[r, "d_A"] \arrow[dr, "\partial"'] & \Omega_A \arrow[d, dashed, "k"] \\
 & M
\end{tikzcd}$$

and uniqueness of k follows easily. It is interesting to note now that when m_A denotes the multiplication on A, $\Omega_A = ker(m_A)$, whereas in the commutative case we needed to divide out by the square of this kernel in order to construct the A-module of differential forms [6, II.8].

Chapter 3

Monads

The exposition contained herein follows [11, VI]. In regards to any category, C, we may consider the monoidal category of endofunctors - functors of the form $T : C \longrightarrow C$ - under the composition of functors. A monad is then a monoid in this category. Explicitly:

Definition 3.0.1. A *monad* in a category, C, is an endofunctor, $T : C \longrightarrow C$ with natural transformations $\eta : I \longrightarrow T$ and $\mu : T^2 \longrightarrow T$ (called the unit and multiplication of the monad, respectively) s.t.

$$\begin{tikzcd}
T^3 \arrow[r, "\mu_T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

and

$$\begin{tikzcd}
IT \arrow[r, "\eta_T"] \arrow[dr, "\cong"'] & T^2 \arrow[d, "\mu"] & TI \arrow[l, "T\eta"'] \arrow[dl, "\cong"] \\
& T &
\end{tikzcd}$$

commute

Examples of monads arise naturally from adjunctions as exhibited in the following

Lemma 3.0.1. *To each adjunction* $(F, G, \eta, \varepsilon) : X \longrightarrow A$ *corresponds a monad* (T, η, μ) *in* X *with* $T = GF$, $\mu = G\varepsilon F$, *and units coinciding.*

Proof: See [11, VI.1] ∎

The close connection between monads and adjunctions can be expatiated further by examining the category of T-algebras for a monad, (T, η, μ).

Definition 3.0.2. A T-*algebra* (A, ν) for a monad (T, η, μ) in C is an object $A \in Ob(C)$ together with a map $\nu : TA \longrightarrow A$ called a structure map, s.t.

$$\begin{tikzcd}
T^2A \arrow[r, "\mu"] \arrow[d, "T\nu"'] & TA \arrow[d, "\nu"] \\
TA \arrow[r, "\nu"'] & A
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "="'] & TA \arrow[d, "\nu"] \\
& A
\end{tikzcd}$$

commute.

A morphism of T-algebras $f : (A, \nu) \longrightarrow (B, \omega)$ is then a morphism $f : A \longrightarrow B$ in C which makes

$$\begin{tikzcd}
TA \arrow[r, "\nu"] \arrow[d, "Tf"'] & A \arrow[d, "f"] \\
TB \arrow[r, "\omega"'] & B
\end{tikzcd}$$

commute. We may then define the category of T-algebras in the obvious way. We obtain the following result:

Theorem 3.0.1. *Let T be a monad on the category C and let C^T be the corresponding category of T-algebras. Then there exists an adjunction (which we will call the T-algebra adjunction) $(F^T, G^T, \eta^T, \epsilon^T) : C \longrightarrow C^T$ giving rise to the monad T. Furthermore, if T is borne of the adjunction $(F, G, \eta, \epsilon) : C \longrightarrow A$ then there is a unique functor $K : A \longrightarrow C^T$ such that $K; G^T = G$ and $F; K = F^T$.*

Proof: See [11, VI.2-3]. ∎

Precisely dual to the notion of a monad is a comonad. Given a monad T we can define what is called the Kleisli category, and so using duality, given a comonad S in a category C we may define the coKleisli category.

Definition 3.0.3. Given a comonad (S, ϵ, δ) in a category C we may define the *coKleisli category* K of S to have the same objects as C, and morphisms coinciding with those in C of the form $f : SA \longrightarrow B$. Given another morphism $g : SB \longrightarrow C$ composition in K is defined by:

$$f; g = \delta_A; Sf; g$$

It is an easy exercise to confirm that this properly defines a category, and furthermore that the dual notion of a Kleisli category is also a properly defined category. Moreover,

Theorem 3.0.2. *If C_T is the Kleisli category determined by a monad T on a category C, there is a canonical adjunction $(F_T, G_T, \eta_T, \epsilon_T) : C \longrightarrow C_T$ giving $(T, \eta, \mu) = (G_T F_T, \eta_T, G_T \epsilon_T F_T)$. Furthermore, if $(F, G, \eta', \epsilon') : C \longrightarrow A$ is another adjunction giving rise to T, then there is a unique functor $L : C_T \longrightarrow A$ such that $L; G = G_T$ and $F_T; L = F$.*

Proof: See [11, VI.5] ∎

The relationship between monads and adjunctions may now be summarized as follows:

Theorem 3.0.3. *Given a monad T in a category C, we may consider the category of adjunctions $(F, G, \eta, \epsilon) : C \longrightarrow A$ giving rise to this monad. This category has the*

Kleisli construction outlined in Theorem 3.2 as its initial object and the T-algebra adjunction as its terminal object.

3.1 Algebra Modalities

We now define a specific type of monad called an algebra modality. Dual to this is the notion of a coalgebra modality, which of course is a specific type of comonad. In what follows, the term 'additive category' refers to a category whose Hom-sets are commutative monoids whose operation distributes over composition. [1]

Definition 3.1.1. In an additive, symmetric monoidal category, C, a monad, (T, η, μ) is an *algebra modality* if for each object $A \in C$, there exist natural transformations, $m_A : TA \otimes TA \longrightarrow TA$ and $e_A : I \longrightarrow A$, where I is the unit of the tensor, s.t.

$$\begin{tikzcd}
TA \otimes TA \otimes TA \arrow[r, "Id_{TA} \otimes m_A"] \arrow[d, "m_A \otimes Id_{TA}"'] & TA \otimes TA \arrow[d, "m_A"] \\
TA \otimes TA \arrow[r, "m_A"'] & TA
\end{tikzcd}$$

and

$$\begin{tikzcd}
I \otimes TA \arrow[r, "e_A \otimes Id_{TA}"] \arrow[dr, "\cong"'] & TA \otimes TA \arrow[d, "m_A"] & TA \otimes I \arrow[l, "Id_{TA} \otimes e_A"'] \arrow[dl, "\cong"] \\
 & TA &
\end{tikzcd}$$

commute, giving TA an associative algebra structure. Also,

$$\begin{tikzcd}
T^2A \otimes T^2A \arrow[r, "\mu_A \otimes \mu_A"] \arrow[d, "m_{TA}"'] & TA \otimes TA \arrow[d, "m_A"] \\
T^2A \arrow[r, "\mu_A"'] & TA
\end{tikzcd}$$

and

$$\begin{tikzcd}
I \arrow[r, "e_{TA}"] \arrow[dr, "e_A"'] & T^2A \arrow[d, "\mu_A"] \\
 & TA
\end{tikzcd}$$

---

[1] Most sources call these categories preadditive

commute, so that μ_A is an associative algebra morphism. Furthermore, for an arbitrary morphism $f : A \longrightarrow B$ in C, $Tf : TA \longrightarrow TB$ is an associative algebra morphism [2].

A *coalgebra modality* is a comonad S equipped with a counit usually denoted by e and a comultiplication, usually denoted by Δ. It may furthermore be the case that a coalgebra modality has an algebra structure; this indeed occurs in some important examples of differential categories. Hence we make the following definition:

Definition 3.1.2. A *bialgebra modality* is a comonad $(!, \delta, \epsilon)$ such that each $!A$ is naturally a bialgebra $(!A, \nabla, i, \Delta, e)$ with δ a homomorphism of coalgebra structures, and ϵ satisfies the following equations: $i; \epsilon = 0$ and $\nabla; \epsilon = \epsilon \otimes e + e \otimes \epsilon$ [2]

If a category has an algebra modality, then in addition to TA having an associative algebra structure for all $A \in Ob(C)$ it turns out that each T-algebra gives rise to an associative algebra. Precisely:

Lemma 3.1.1 (Cockett, private communication)**.** *Let C be a category with an algebra modality, T. Then if (A, ν) is a T-algebra, there exist maps $m_\nu : A \otimes A \longrightarrow A$ and $e_\nu : I \longrightarrow A$ in C making (A, m_ν, e_ν) into an associative algebra such that ν is a morphism of associative algebras.*

In order to prove this, recall the following lemma due to Peter Johnstone (see [8, 0.17])

Lemma 3.1.2. *Let*

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\alpha_2"] & X_2 \arrow[r, "f_3"] \arrow[d, shift right, "\beta_1"'] \arrow[d, shift left, "\beta_2"] & X_3 \arrow[d, shift right, "\gamma_1"'] \arrow[d, shift left, "\gamma_2"] \\
Y_1 \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, "\alpha_3"'] & Y_2 \arrow[r, "g_3"] \arrow[d, "\beta_3"'] & Y_3 \arrow[d, "\gamma_3"'] \\
Z_1 \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Z_2 \arrow[r, "h_3"] & Z_3
\end{tikzcd}$$

be a diagram in any category satisfying the ”obvious” commutativity conditions (i.e. $f_j;\beta_i = \alpha_i;g_j$ *for* $i = 1,2, j = 1,2,$ *etc.), in which the rows and columns are coequalizers and there exist maps* $j : X_2 \longrightarrow X_1$ *and* $k : Y_1 \longrightarrow X_1$ *such that* $j;f_1 = j;f_2 = id_{X_2}$ *and* $k;\alpha_1 = k;\alpha_2 = id_{Y_1}$*. Then the diagonal*

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "f_1;\beta_!"] \arrow[r, shift right, "f_2;\beta_2"'] & Y_2 \arrow[r, "g_3;\gamma_3"] & Z_3
\end{tikzcd}$$

is a coequalizer.

Proof: (**Proof of Lemma 3.1.1**)

First note that

$$\begin{tikzcd}
T^2A \arrow[r, shift left, "T\nu"] \arrow[r, shift right, "\mu"'] & TA \arrow[r, "\nu"] & A
\end{tikzcd}$$

is not only a coequalizer, but also a split coequalizer, which makes it necessarily an absolute coequalizer. This means that the rows and columns of

$$\begin{tikzcd}
T^2A\otimes T^2A \arrow[r, shift left, "T\nu\otimes id"] \arrow[r, shift right, "\mu\otimes id"'] \arrow[d, shift right, "id\otimes T\nu"'] \arrow[d, shift left, "id\otimes\mu"] & TA\otimes T^2A \arrow[r, "\nu\otimes id"] \arrow[d, shift right, "id\otimes T\nu"'] \arrow[d, shift left, "id\otimes\mu"] & A\otimes T^2A \arrow[d, shift right, "id\otimes T\nu"'] \arrow[d, shift left, "id\otimes\mu"] \\
T^2A\otimes TA \arrow[r, shift left, "T\nu\otimes id"] \arrow[r, shift right, "\mu\otimes id"'] \arrow[d, "id\otimes\nu"'] & TA\otimes TA \arrow[r, "\nu\otimes id"] \arrow[d, "id\otimes\nu"'] & A\otimes TA \arrow[d, "id\otimes\nu"'] \\
T^2A\otimes A \arrow[r, shift left, "T\nu\otimes id"] \arrow[r, shift right, "\mu\otimes id"'] & TA\otimes A \arrow[r, "\nu\otimes id"] & A\otimes A
\end{tikzcd}$$

are all coequalizers. Since the coequalizers are reflexive it follows from the above lemma that

$$\begin{tikzcd}
T^2A\otimes T^2A \arrow[r, shift left, "T\nu\otimes T\nu"] \arrow[r, shift right, "\mu\otimes\mu"'] & TA\otimes TA \arrow[r, "\nu\otimes\nu"] & A\otimes A
\end{tikzcd}$$

is also a split coequalizer. Hence it is an absolute coequalizer and so similarly we can deduce that

$$\begin{tikzcd}
T^2A\otimes T^2A\otimes T^2A \arrow[r, shift left, "T\nu\otimes T\nu\otimes T\nu"] \arrow[r, shift right, "\mu\otimes\mu\otimes\mu"'] & TA\otimes TA\otimes TA \arrow[r, "\nu\otimes\nu\otimes\nu"] & A\otimes A\otimes A
\end{tikzcd}$$

is also a coequalizer. Hence we may define the map $m_\nu : A\otimes A \longrightarrow A$ as the unique map making the right hand square of

$$\begin{tikzcd}
T^2A\otimes T^2A \arrow[r, shift left, "T\nu\otimes T\nu"] \arrow[r, shift right, "\mu\otimes\mu"'] \arrow[d, "m_{T^2A}"'] & TA\otimes TA \arrow[r, "\nu\otimes\nu"] \arrow[d, "m_{TA}"] & A\otimes A \arrow[d, dashed, "m_\nu"] \\
T^2A \arrow[r, shift left, "T\nu"] \arrow[r, shift right, "\mu"'] & TA \arrow[r, "\nu"] & A
\end{tikzcd}$$

commute (note that the top and bottom left-hand squares commute since $T\nu$ and μ are algebra maps). Furthermore, since

$$id\otimes m_{TA}; m_{TA}; \nu = id\otimes m_{TA}; \nu\otimes\nu; m_\nu = \nu\otimes\nu\otimes\nu; id\otimes m_\nu; m_\nu$$

and

$$m_{TA} \otimes id; m_{TA}; \nu = m_{TA} \otimes id; \nu \otimes \nu; m_\nu = \nu \otimes \nu \otimes \nu; m_\nu \otimes id; m_\nu$$

the uniqueness of k in

$$\begin{tikzcd}
T^2A \otimes T^2A \otimes T^2A \arrow[r, shift left, "T\nu \otimes T\nu \otimes T\nu"] \arrow[r, shift right, "\mu \otimes \mu \otimes \mu"'] \arrow[d, "id \otimes m_{T^2A}; m_{T^2A}"] & TA \otimes TA \otimes TA \arrow[r, "\nu \otimes \nu \otimes \nu"] \arrow[d, "id \otimes m_{TA}; m_{TA} = m_{TA} \otimes id; m_{TA}"] & A \otimes A \otimes A \arrow[d, dashed, "k"] \\
T^2A \arrow[r, shift left, "T\nu"] \arrow[r, shift right, "\mu"'] & TA \arrow[r, "\nu"] & A
\end{tikzcd}$$

implies that $id \otimes m_\nu; m_\nu = m_\nu \otimes id; m_\nu$. Finally, we may define $e_\nu := e_{TA}; \nu$. Then if $l : I \otimes TA \longrightarrow TA$ and $l' : TA \longrightarrow I \otimes TA$ are isomorphisms

$$l'; id_I \otimes \nu; e_\nu \otimes id_A; m_\nu = l'; e_{TA} \otimes id_{TA}; \nu \otimes \nu; m_\nu = l'; e_{TA} \otimes id_{TA}; m_{TA}; \nu = l'; l; \nu = \nu$$

Since ν is split epi, it follows that e_ν so defined is a left unit for m_ν. It can be similarly shown that e_ν is a right unit for m_ν and so it follows that (A, m_ν, e_ν) is an associative algebra making ν into an algebra morphism. ∎

3.2 Examples

1. In $Vect_K$ we may consider the functor which takes a vector space V to its freely generated symmetric associative algebra $\bigoplus_{n=0}^{\infty} Sym^n V$ with $Sym^0 V = I$, the base field, and $Sym^n V$ is the n-fold tensor product of V with itself modulo the subspace generated by all $v_1 \otimes v_2 \otimes ... \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes ... \otimes v_{\sigma(n)}$; the algebra structure may then be defined with concatenation as the multiplication [5, B.2]. Then F is left adjoint to a forgetful functor G, which forgets the algebra structure and so we may define a monad $T = GF$. We have then, for instance, that $T(< x, y >)$ is the vector space of polynomials in variables x and y. It is not hard to check that this monad is an algebra modality with multiplication the usual multiplication of polynomials and unit the inclusion of the field. It should also be noted that $Vect_K^{op}$ has a canonical coalgebra structure on the comonad resulting from the reversal of the maps of the monad in $Vect_K$.

2. We may equip the category of *Sets and relations* with a tensor via the cartesian product and give its Hom-sets monoidal structure via the union operation. A monad (T, μ, η) may be constructed for this category, where T is the multi-set functor, μ takes the union of a given set of multi-sets, and η takes an element to the singleton set containing that element. One may prescribe an algebra structure for this monad by defining the multiplication map $m_{TA} : TA \otimes TA \longrightarrow TA$ to take the union of the sets in each component. Mapping to the empty set is clearly the unit for this algebra structure, and it is clear that T is an algebra modality.

3. In a symmetric monoidal category C with finite products and a comonad $(!, \delta, \epsilon)$ it is sometimes the case that there is an isomorphism $!(A \times A) \cong !A \otimes !A$ for each $A \in Ob(C)$ and furthermore that these isomorphisms are compatible with the monoidal structure on C (e.g. in models for linear logic). Then applying the functor ! to the diagonal map $\Delta : A \longrightarrow A \times A$ and composing with the above isomorphism gives a map $!A \longrightarrow !(A \times A) \longrightarrow !A \otimes !A$. One can check that this defines a comultiplication map for a coalgebra modality on C, where the counits are the maps to the terminal object. Similarly, if C has biproducts and an analogous isomorphism, it can be shown that ! is furthermore a bialgebra modality.

Chapter 4

Differential Categories

The exposition contained herein follows [2]. A differential category is an additive symmetric monoidal category with a coalgebra modality and a differential combinator, which will be defined shortly. Differential categories provide a categorical definition of differential structure; the maps in the coKleisli category are to be thought of as smooth while those of the base category can be thought of as linear and the differential combinator gives a transformation from the former to the latter. A proper differential structure in any category should in some sense give the best local linear approximations of any given smooth map. To this end the axioms which define a differential combinator precisely model the most important properties deduced directly from the definition of the derivative from differential calculus. Explicitly:

Definition 4.0.1. Let C be an additive symmetric monoidal category with coalgebra modality S. Then a *left differential combinator* $D_{AB} : C(SA, B) \longrightarrow C(A \otimes SA, B)$ produces for each coKleisli map $f : SA \longrightarrow B$ a (left) derivative $D_{AB}[f] : A \otimes SA \longrightarrow B$ in such a way that D_{AB} is natural in A and B, $D[0] = 0$, and $D[f+g] = D[f] + D[g]$. Furthermore D must satisfy the following axioms:
[D.1] Constant maps:

$$D[e_A] = 0$$

[D.2] Product rule:

$$D[\Delta(f \otimes g)] = (1 \otimes \Delta); {a_\otimes}^{-1}; (D[f] \otimes g) + (1 \otimes \Delta); a_\otimes; (c_\otimes \otimes 1); {a_\otimes}^{-1}; (f \otimes D[g])$$

where $f : SA \longrightarrow B, g : SA \longrightarrow C$ and $a_\otimes$, $c_\otimes$ are the associativity and commutativity isomorphisms

[D.3] Linear maps:

$$D[\epsilon_A; f] = (1 \otimes e_A); {u^R}_\otimes; f$$

where $f : A \longrightarrow B$ and $u_\otimes$ is the unit isomorphism

[D.4] The chain rule:

$$D[\delta; Sf; g] = (1 \otimes \Delta); {a_\otimes}^{-1}; (D[f] \otimes (\delta; Sf)); D[g]$$

Given a coKleisli map $f : SA \longrightarrow B$, naturality of D_{AB} and the fact that $id_{SA}; f = S(id_A); f$ gives $D[id_{SA}]; f = id_A \otimes id_{SA}; D[f]$, and hence any such $D[f]$ may be described in this way. Because of this privileged position of $D[id_{SA}]$, we will give it the special denotation d_A. We observe then that a commutative diagram in a differential category such as:

$$\begin{tikzcd}
SA \arrow[r, "id_{SA}"] \arrow[d, "S(id_A)"'] & SA \arrow[d, "f"] \\
SA \arrow[r, "f"'] & B
\end{tikzcd}$$

can be transformed via differential combination to the diagram:

$$\begin{tikzcd}
A \otimes SA \arrow[r, "d_A"] \arrow[d, "id_{A \otimes SA}"'] & SA \arrow[d, "f"] \\
A \otimes SA \arrow[r, "D(f)"'] & B
\end{tikzcd}$$

and hence $D[f] = d_A; f$ so that these d_A's generate the differential structure of the category. We may therefore encapsulate the properties of a differential combinator in those of a deriving transformation:

Definition 4.0.2. In an additive category with a coalgebra modality a natural transformation $d_A : A \otimes SA \longrightarrow SA$ is a *(left) deriving transformation* in case it satisfies the following conditions:

[d.1] Constant maps: $d_A; e_A = 0$

[d.2] Copying $d_A; \Delta = (id_A \otimes \Delta); a_{\otimes}^{-1}; (d_A \otimes id_A) + (id_A \otimes \Delta); a_{\otimes}^{-1}; (c_{\otimes} \otimes id_S A); a_{\otimes}; (id_{SA} \otimes d_A)$

[d.3] Linearity: $d_A; \epsilon_A = (id_A \otimes e); u_{\otimes}^R$

[d.4] Chaining: $d_A; \delta = (id_A \otimes \Delta); a_{\otimes}^{-1}; (d_A \otimes \delta); d_{SA}$

To summarize, we have the following:

Theorem 4.0.1. [2, 2.6] *The following are equivalent:*

i)An additive symmetric monoidal category with a deriving transformation for its coalgebra modality;

ii)A differential category

There is another formulation of differential categories that although less general still exhibits appropriate structure for some important examples. We require a preliminary definition:

Definition 4.0.3. A *storage category* is a symmetric monoidal category possessing products and a *storage modality* . A storage modality is a comonad $(!, \delta, \epsilon)$ which is symmetric monoidal and has each object of the form $!A$ naturally a commutative comonoid $(!A, \Delta, e)$ such that the comonoid structure is a morphism for the coalgebras of the comonad.

We then have

Definition 4.0.4. A *differential storage category* is an additive storage category with a deriving transformation d, such that the following is satisfied:

[∇-rule]: $(d_A \otimes id_{!A}); \nabla = (id_{!A} \otimes \nabla); d_A$

This additionally specified structure allows one to formulate specific types of differential categories using the following:

Definition 4.0.5. A categorical model of the *differential calculus* is an additive category with biproducts with a bialgebra modality consisting of a comonad $(!, \delta, \epsilon)$ such that each object $!X$ has a natural bialgebra structure $(!X, \nabla_X, i_X, \Delta_X, e_X)$, and a natural map $\eta_X : X \longrightarrow !X$ satisfying the following:
[dC.1]: $\eta; e = 0$
[dC.2]: $\eta; \Delta = \eta \otimes i + i \otimes \eta$
[dC.3]: $\eta; \epsilon = id_X$
[dC.4]: $(\eta \otimes id_{!X}); \nabla; \delta = (\eta \otimes \Delta); ((\nabla; \eta) \otimes \delta); \nabla$

The notions of differential storage categories and categorical models of the differential calculus are related in the following way:

Theorem 4.0.2. [2, 4.13] *Models of the differential calculus on additive storage categories correspond precisely to differential storage categories, that is, to deriving transformations on these categories satisfying the* ∇*-rule.*

4.1 Examples

1. As mentioned above, we may invoke a coalgebra modality on the opposite of the category of vector spaces over a fixed field K, denoted $Vect_K^{op}$. Let $\{x_i\}_{i=1}^n$ be a basis for the vector space V and consider the traditional differential structure on polynomials given by $d(f) = \sum_{i=1}^n x_i \otimes \partial(f)/x_i$. If we denote the opposite of this map (which has a natural realization in the category $Vect_K^{op}$) by d_V and consider the maps opposite to those comprising the coalgebra modality (namely, the maps of the algebra modality on $Vect_K$), it can be shown that d_V is a deriving transformation by proving that the opposites of the axioms $[d.1] - [d.4]$ hold for d. Hence $Vect_K^{op}$ is a differential category.[2]

2. Similarly, we may produce a coalgebra modality on the category of *Sets and relations* by taking the opposites of the maps described in section 3.2. For example, if we denote the comonad by S then the comultiplication map $\Delta : SA \longrightarrow SA \otimes SA$ maps to all pairs of sets whose union produces the original set in SA. A deriving transformation $d_A : A \otimes SA \longrightarrow SA$ in this category then, is the map defined by $(a, U) \mapsto \{a\} \cup U$ where $a \in A$, $U \subseteq A$.[2]

3. It is shown in [3] that the category of *convenient vector spaces* is a differential category.

Chapter 5

Kähler Categories

In any additive symmetric monoidal category we may define what it means for a map to be a derivation:

Definition 5.0.1. Let (A, m_A, e_A) be an associative algebra and $(M, \bullet)$ be a left A-module in an additive symmetric monoidal category C. Then a map $\partial : A \longrightarrow M$ is an A-*derivation* if $e_A; \partial = 0$ and

$$m_A; \partial = (id_A \otimes \partial + c; id_A \otimes \partial); \bullet \qquad \text{(Leibniz)}$$

It is not difficult to see that a derivation postcomposed with a module map will also be a derivation, and so a universal A-derivation in this context is simply an A-derivation through which all other A-derivations factor uniquely via a module map. We may now formally state the definition of a Kähler category:

Definition 5.0.2. A *(commutative) Kähler category* K is an additive symmetric monoidal category with an algebra modality T s.t. $\forall A \in Ob(K)$ there is a TA-module of differential forms Ω_{TA} together with a universal TA-derivation $d_{TA} : TA \longrightarrow \Omega_{TA}$. That is, for any TA-derivation $\partial : TA \longrightarrow M$ there is a unique TA-module map $m : \Omega_{TA} \longrightarrow M$ making the following diagram commute:

$$\begin{tikzcd}
TA \arrow[r, "d_{TA}"] \arrow[dr, "\partial"'] & \Omega_{TA} \arrow[d, dashed, "m"] \\
& M
\end{tikzcd}$$

A feature of this definition that suggests a need for investigation is the fact that although we have associative algebras arising from any T-algebra in the category (by Lemma 3.1.1), universal derivations are defined to exist only for objects in the image of the monad. Determining the precise criteria a Kähler category must fulfill in order for these T-algebra associative algebras to have universal derivations was the initial motivation for this paper.

However, this was not pertinent in the initial conception of Kähler categories, in which Kähler categories served as a framework for the comparison between differential categories and the theory of Kähler differentials. Since Kähler differentials provide a means for abstract differential structure for algebras rather than coalgebras, this comparison is manifested in the question: what conditions are necessary and sufficient for a *codifferential* category to be Kähler?

Now, for each object in the image of the monad of a codifferential category we have a canonical derivation, the deriving transformation, $d_A : TA \longrightarrow TA \otimes A$, and so we can ask whether or not this happens to be a universal derivation for each such TA. From [1] it appears as though the answer to this question is intimately connected to the relation between the given algebra modality for the codifferential category and the monad induced by the free associative algebra functor:

$$FA = I + A + A \otimes A + A \otimes A \otimes A + ...$$

which itself has an associative (but not necessarily commutative) algebra modality. Since FA is a coproduct, a map $\alpha_{TA} : FTA \longrightarrow TA$ can be constructed through maps $\alpha_n : TA^{\otimes n} \longrightarrow TA$ for all n. For $n = 0$ and $n = 1$ these are e_A and the identity respectively, and for all other n we may take the n-fold multiplication induced by the algebra modality of T. We may then construct a map $\varphi : F \longrightarrow T$ by setting $\varphi_A := F\eta_A; \alpha_A$. By [1, 4.5] φ is a morphism of monads, and the following definition is made:

Definition 5.0.3. The monad T satisfies *property* K if the natural transformation $\varphi : F \longrightarrow T$ is a component-wise epimorphism.

The comparison between codifferential categories and Kähler categories culminates in the following theorem:

Theorem 5.0.1. [1, 4.8] *A codifferential category with a monad, T, satisfying property K is a Kähler category.*

Proof: Given an object A, we first require a candidate for the TA-module of universal differential forms; we consider the TA-module $TA \otimes A$ with module action inherited from the associative algebra structure on TA. This is indeed the free TA-module over A: given a TA-module $(M, \bullet_M)$ and any map $h : A \longrightarrow M$ we have a morphism $id_{TA} \otimes h; \bullet_M : TA \otimes A \longrightarrow M$, which satisfies

$$\begin{aligned} id_{TA} \otimes id_{TA} \otimes h; id_{TA} \otimes \bullet_M; \bullet_M &= id_{TA} \otimes id_{TA} \otimes h; m_{TA} \otimes id_M; \bullet_M \\ &= m_{TA} \otimes id_M; id_{TA} \otimes h; \bullet_M \end{aligned}$$

and is therefore a morphism of TA-modules, which furthermore has

$$l; e_{TA} \otimes id_A; id_{TA} \otimes h; \bullet_M = l; id_I \otimes h; e_{TA} \otimes id_M; \bullet_M = l; id_I \otimes h; l' = h$$

where $l : A \cong I \otimes A$ and $l' : I \otimes M \cong M$. $id_{TA} \otimes h; \bullet_M$ is unique in this regard since if we have a module map $k : TA \otimes A \longrightarrow M$ such that $l; e_{TA} \otimes id_A; k = h$, then we have

$$\begin{aligned} k &= id_{TA} \otimes l; id_{TA} \otimes e_{TA} \otimes id_A; m_{TA} \otimes id_A; k \\ &= id_{TA} \otimes l; id_{TA} \otimes e_{TA} \otimes id_A; id_{TA} \otimes k; \bullet_M \end{aligned}$$

$$= id_{TA} \otimes h; \bullet_M$$

Now, if there is a derivation $\partial : TA \longrightarrow M$ to some TA-module, M, then $\eta; \partial : A \longrightarrow M$ factors uniquely through $TA \otimes A$ via some module map $k : TA \otimes A \longrightarrow M$, so that $\eta; \partial = l; e_{TA}; k$. In a codifferential category we have that $l; e_{TA} = \eta; d_A$ so that $\eta; \partial = \eta; d_A; k$. The completion of the proof requires that we cancel the η's to show that k is the unique module map making $\partial = d_A; k$. Invoking property K we may reduce this to the problem of showing that the maps $F\eta; \alpha, d_A; k$ and $F\eta; \alpha; \partial$ are equal. Since the domain of each of these maps is a coproduct, it suffices to show that the maps are equal on each component.

For the I component, both maps are 0 by definition. Equality in the A component is already established, since we know that $\eta; \partial = \eta; d_A; k$. For the $A \otimes A$ component, we must show that $\eta \otimes \eta; m_{TA}; d_A; k = \eta \otimes \eta; m_{TA}; \partial$. Observe that

$$\begin{aligned} \eta \otimes \eta; m_{TA}; d_A; k &= \eta \otimes \eta; (id_{TA} \otimes d_A + c; id_{TA} \otimes d_A); m_{TA} \otimes id_A; k \\ = \eta \otimes l; id_{TA} \otimes e_A \otimes id_A; m_{TA} \otimes id_A; k &+ l \otimes \eta; id_{TA} \otimes e_A \otimes id_A; c; m_{TA} \otimes id_A; k \\ &= \eta \otimes id_A; k + id_A \otimes \eta; c; k \end{aligned}$$

and that

$$\begin{aligned} \eta \otimes \eta; m_{TA}; \partial &= \eta \otimes \eta; id_{TA} \otimes \partial; \bullet_M + \eta \otimes \eta; \partial \otimes id_{TA}; c; \bullet_M \\ &= \eta \otimes id_A; id_{TA} \otimes (\eta; \partial); \bullet_M + id_A \otimes \eta; (\eta; \partial) \otimes id_{TA}; c; \bullet_M \end{aligned}$$

The result then follows from the universal property of k.

The remainder of the proof requires similar calculations; details can be found in [1, 4.2] ∎

Chapter 6

Differential Forms for T-Algebras

The material contained in this section is the result of original work.

As shown above, for each T-algebra in a Kähler category we may construct an associative algebra, which makes the structure map of the T-algebra into a morphism of associative algebras. Since universal derivations are a priori only defined for the algebras arising axiomatically in a Kähler category, it is natural to ask then if universal derivations from these new associative algebras exist and, if so, how they are constructed. In what follows it is proven that if the tensor product preserves coequalizers (for instance if the category is closed), these universal derivations (and in fact, many others) can be constructed via a coequalizer.

Let (A, ν) be a T-algebra in a Kähler category, K. We then have a naturally arising algebra structure (A, m_ν, e_ν), which makes ν into an associative algebra map. If $\partial : A \longrightarrow M$ is an A-derivation to an A-module M then it follows easily that $\nu;\partial$ is a derivation from TA to M (viewed here as a TA-module). In general:

Lemma 6.0.1. *Let $f : X \longrightarrow Y$ be a morphism of associative algebras (X, m_X, e_X) and (Y, m_Y, e_Y), and let $\partial : Y \longrightarrow M$ be a derivation to a Y-module M. Then there exists an X-module structure on M such that $f;\partial : X \longrightarrow M$ is an X-derivation.*

Proof: Let $\bullet : Y \otimes M \longrightarrow M$ be a module action of Y on M. Then:

$$\begin{aligned} id_X \otimes f \otimes id_M; id_X \otimes \bullet; f \otimes id_M; \bullet &= f \otimes f \otimes id_M; id_X \otimes \bullet; \bullet \\ &= f \otimes f \otimes id_M; m_Y \otimes id_M; \bullet = m_X \otimes id_M; f \otimes id_M; \bullet \end{aligned}$$

and so $f \otimes id_M; \bullet : X \otimes M \longrightarrow M$ is an X-module action on M. We then have:

$$\begin{aligned} m_X; f; \partial = f \otimes f; m_Y; \partial = f \otimes f; (id_Y \otimes \partial + c; id_Y \otimes \partial); \bullet \\ = (f \otimes (f;\partial) + c; f \otimes (f;\partial)); \bullet = (id_X \otimes (f;\partial) + c; id_X \otimes (f;\partial)); f \otimes id_M; \bullet \end{aligned}$$

and

$$e_X; f; \partial = f; e_Y; \partial = 0$$

and so $f;\partial$ is an X-derivation. ∎

Hence if there exists a universal A-derivation $d_{A,\nu} : A \longrightarrow \Omega_{A,\nu}$, then there exists a TA-module structure on $\Omega_{A,\nu}$ making $\nu; d_{A,\nu}$ a TA-derivation, and so there exists a unique TA-module map Ω_ν making

$$\begin{tikzcd} \Omega_{TA} \arrow[r, "\Omega_\nu"] & \Omega_{A,\nu} \\ TA \arrow[u, "d_{TA}"] \arrow[r, "\nu"'] & A \arrow[u, "d_{A,\nu}"'] \end{tikzcd}$$

commute, where d_{TA} is the universal derivation from TA to Ω_{TA}. Motivated by the desire to investigate the would-be properties of this hypothetical map, we make the following definition:

Definition 6.0.1. Let $f : A \longrightarrow B$ be an arbitrary map in a Kähler category. (This implies that $Tf : TA \longrightarrow TB$ is a morphism of associative algebras.) Define Ω_{Tf} :

$\Omega_{TA} \longrightarrow \Omega_{TB}$ to be the unique map making

$$\begin{tikzcd}
\Omega_{TA} \arrow[r, "\Omega_{Tf}"] & \Omega_{TB} \\
TA \arrow[u, "d_{TA}"] \arrow[r, "Tf"'] & TB \arrow[u, "d_{TB}"']
\end{tikzcd}$$

commute, which exists by universality of d_{TA} and by Lemma 6.0.1.

Proposition 6.0.1. $\Omega_{T(-)}$ *is a functor from* K *to the category of modules and module maps in* K.

Proof: Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be arbitrary maps. Then the inner and outer squares of

$$\begin{tikzcd}
\Omega_{TA} \arrow[r, "\Omega_{Tf}"'] \arrow[rr, bend left, "\Omega_{T(f;g)}"] & \Omega_{TB} \arrow[r, "\Omega_{Tg}"'] & \Omega_{TC} \\
TA \arrow[u, "d_{TA}"] \arrow[r, "Tf"] \arrow[rr, bend right, "T(f;g)"'] & TB \arrow[u, "d_{TB}"'] \arrow[r, "Tg"] & TC \arrow[u, "d_{TC}"']
\end{tikzcd}$$

commute. But by functoriality $Tf;Tg = T(f;g)$ and hence by uniqueness it follows that $\Omega_{Tf};\Omega_{Tg} = \Omega_{T(f;g)}$. The preservation of identities follows similarly. ∎

Consequently, we have $\Omega_{T\eta};\Omega_{T\nu} = id_{\Omega_{TA}}$. Similarly, since the definition of an algebra modality makes $\mu : T^2A \longrightarrow TA$ into an algebra map, we can define $\Omega_\mu : \Omega_{T^2A} \longrightarrow \Omega_{TA}$ in the exact same way and it can be shown that $\Omega_{T\eta};\Omega_\mu = id_{\Omega_{TA}}$. Presumably then, if such an Ω_ν exists as described above, we should have that $\Omega_\mu;\Omega_\nu = \Omega_{T\nu};\Omega_\nu$. Indeed, we should expect

$$\begin{tikzcd}
\Omega_{T^2A} \arrow[r, shift left, "\Omega_\mu"] \arrow[r, shift right, "\Omega_{T\nu}"'] & \Omega_{TA} \arrow[r, "\Omega_\nu"] & \Omega_{A,\nu}
\end{tikzcd}$$

to be a coequalizer. The preceding remarks motivate the following result:

Theorem 6.0.1. *Let (A, m_A, e_A), (B, m_B, e_B) and (C, m_C, e_C) be associative algebras in an additive symmetric monoidal category K, the tensor product of which preserves colimits (i.e. $X \otimes \_\_$ preserves colimits $\forall X \in Ob(K)$). Furthermore, let A and B have universal derivations $d_A : A \longrightarrow \Omega_A$ and $d_B : B \longrightarrow \Omega_B$, respectively. Then if $f, g : A \longrightarrow B$ and $h : B \longrightarrow C$ are morphisms of associative algebras s.t.*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd}$$

is a coequalizer, and there exists an associative algebra morphisms $r : B \longrightarrow A$ s.t. $r; f = id_B = r; g$, then the module of differential forms for C is given by the coequalizer

$$\begin{tikzcd} \Omega_A \arrow[r, shift left, "\Omega_f"] \arrow[r, shift right, "\Omega_g"'] & \Omega_B \arrow[r, "\Omega_h"] & \Omega_C \end{tikzcd}$$

where Ω_f and Ω_g are the unique A-module maps with $f; d_B = d_A; \Omega_f$ and $g; d_B = d_A; \Omega_g$. Furthermore, the universal derivation $d_C : C \longrightarrow \Omega_C$ is the map uniquely determined by the universal property of the coequalizer in:

$$\begin{tikzcd} \Omega_A \arrow[r, shift left, "\Omega_f"] \arrow[r, shift right, "\Omega_g"'] & \Omega_B \arrow[r, "\Omega_h"] & \Omega_C \\ A \arrow[u, "d_A"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[u, "d_B"] \arrow[r, "h"] & C \arrow[u, dashed] \end{tikzcd}$$

Note that the structure map of a T-algebra satisfies all conditions required of h in this theorem, so the construction of the universal derivation $d_{A,\nu} : A \longrightarrow \Omega_{A,\nu}$ follows as a natural corollary.

Proof: (**Proof of Theorem**)

We will proceed by proving a series of lemmas.

Lemma 6.0.2.

$$\begin{tikzcd} A \otimes \Omega_A \arrow[r, shift left, "f \otimes \Omega_f"] \arrow[r, shift right, "g \otimes \Omega_g"'] & B \otimes \Omega_B \arrow[r, "h \otimes \Omega_h"] & C \otimes \Omega_C \end{tikzcd}$$

is a coequalizer

Proof: First note that since r is an associative algebra morphism that splits both f and g, there exists an $\Omega_r : \Omega_B \longrightarrow \Omega_A$, which can easily be shown to split both Ω_f and Ω_g. Using the fact that the tensor preserves colimits we may build the following diagram

$$\begin{tikzcd}
A \otimes \Omega_A \arrow[r, shift left, "id\otimes\Omega_f"] \arrow[r, shift right, "id\otimes\Omega_g"'] \arrow[d, shift right, "g\otimes id"'] \arrow[d, shift left, "f\otimes id"] & A \otimes \Omega_B \arrow[r, "id\otimes\Omega_h"] \arrow[d, shift right, "g\otimes id"'] \arrow[d, shift left, "f\otimes id"] & A \otimes \Omega_C \arrow[d, shift right, "g\otimes id"'] \arrow[d, shift left, "f\otimes id"] \\
B \otimes \Omega_A \arrow[r, shift left, "id\otimes\Omega_f"] \arrow[r, shift right, "id\otimes\Omega_g"'] \arrow[d, "h\otimes id"'] & B \otimes \Omega_B \arrow[r, "id\otimes\Omega_h"] \arrow[d, "h\otimes id"'] & B \otimes \Omega_C \arrow[d, "h\otimes id"'] \\
C \otimes \Omega_A \arrow[r, shift left, "id\otimes\Omega_f"] \arrow[r, shift right, "id\otimes\Omega_g"'] & C \otimes \Omega_B \arrow[r, "id\otimes\Omega_h"] & C \otimes \Omega_C
\end{tikzcd}$$

where the rows and columns are coequalizers, all "obviously" commutative squares commute, and the two pairs of parallel arrows in the top left-hand corner each have common sections. Lemma 3.1.2 states that under these conditions, the result follows. ∎

Observe that in the definition of Ω_g in

$$\begin{tikzcd}
\Omega_A \arrow[r, "\Omega_g"] & \Omega_B \\
A \arrow[u, "d_A"] \arrow[r, "g"'] & B \arrow[u, "d_B"']
\end{tikzcd}$$

$g; d_B$ is a derivation from A to Ω_B where the latter is an A-module with action $g \otimes id_{\Omega_B}; \bullet_B$. So Ω_g is an A-module morphism respecting this particular module action and so we have that

$$\begin{tikzcd}
A \otimes \Omega_A \arrow[r, "g\otimes\Omega_g"] \arrow[d, "\bullet_A"'] & B \otimes \Omega_B \arrow[d, "\bullet_B"] \\
\Omega_A \arrow[r, "\Omega_g"'] & \Omega_B
\end{tikzcd}$$

commutes. In exactly the same way, we have that

$$\begin{tikzcd}
A \otimes \Omega_A \arrow[r, "f\otimes\Omega_f"] \arrow[d, "\bullet_A"'] & B \otimes \Omega_B \arrow[d, "\bullet_B"] \\
\Omega_A \arrow[r, "\Omega_f"'] & \Omega_B
\end{tikzcd}$$

commutes, and so we may define a map from $C \otimes \Omega_C$ to Ω_C as in the following diagram

$$\begin{tikzcd}
A \otimes \Omega_A \arrow[r, shift left, "f\otimes\Omega_f"] \arrow[r, shift right, "g\otimes\Omega_g"'] \arrow[d, "\bullet_A"'] & B \otimes \Omega_B \arrow[r, "h\otimes\Omega_h"] \arrow[d, "\bullet_B"] & C \otimes \Omega_C \arrow[d, dashed, "\bullet_C"] \\
\Omega_A \arrow[r, shift left, "\Omega_f"] \arrow[r, shift right, "\Omega_g"'] & \Omega_B \arrow[r, "\Omega_h"] & \Omega_C
\end{tikzcd}$$

using the universal property of the coequalizer from the preceding lemma.

Lemma 6.0.3. $\bullet_C$ *as defined above is a* C*-module action*

Proof: First note that

$$h \otimes h \otimes \Omega_h; m_C \otimes id_{\Omega_C}; \bullet_C = m_B \otimes id_{\Omega_B}; h \otimes \Omega_h; \bullet_C$$

$$= m_B \otimes id_{\Omega_B}; \bullet_B; \Omega_h = id_B \otimes \bullet_B; \bullet_B; \Omega_h$$

$$= id_B \otimes \bullet_B; h \otimes \Omega_h; \bullet_C = h \otimes h \otimes \Omega_h; id_C \otimes \bullet_C; \bullet_C$$

Now, exactly as in the previous lemma, we may deduce that

$$\begin{tikzcd}
A \otimes A \otimes \Omega_A \arrow[r, shift left, "f\otimes f\otimes\Omega_f"] \arrow[r, shift right, "g\otimes g\otimes\Omega_g"'] & B \otimes B \otimes \Omega_B \arrow[r, "h\otimes h\otimes\Omega_h"] & C \otimes C \otimes \Omega_C
\end{tikzcd}$$

is a coequalizer from the following diagram:

$$\begin{tikzcd}
A \otimes A \otimes \Omega_A \arrow[r, shift left, "id\otimes f\otimes\Omega_f"] \arrow[r, shift right, "id\otimes g\otimes\Omega_g"'] \arrow[d, shift right, "g\otimes id\otimes id"'] \arrow[d, shift left, "f\otimes id\otimes id"] & A \otimes B \otimes \Omega_B \arrow[r, "id\otimes h\otimes\Omega_h"] \arrow[d, shift right, "g\otimes id\otimes id"'] \arrow[d, shift left, "f\otimes id\otimes id"] & A \otimes C \otimes \Omega_C \arrow[d, shift right, "g\otimes id\otimes id"'] \arrow[d, shift left, "f\otimes id\otimes id"] \\
B \otimes A \otimes \Omega_A \arrow[r, shift left, "id\otimes f\otimes\Omega_f"] \arrow[r, shift right, "id\otimes g\otimes\Omega_g"'] \arrow[d, "h\otimes id\otimes id"'] & B \otimes B \otimes \Omega_B \arrow[r, "id\otimes h\otimes\Omega_h"] \arrow[d, "h\otimes id\otimes id"'] & B \otimes C \otimes \Omega_C \arrow[d, "h\otimes id\otimes id"'] \\
C \otimes A \otimes \Omega_A \arrow[r, shift left, "id\otimes f\otimes\Omega_f"] \arrow[r, shift right, "id\otimes g\otimes\Omega_g"'] & C \otimes B \otimes \Omega_B \arrow[r, "id\otimes h\otimes\Omega_h"] & C \otimes C \otimes \Omega_C
\end{tikzcd}$$

and hence we can construct a map k from $C \otimes C \otimes \Omega_C$ to Ω_C as in the following diagram

$$\begin{tikzcd}
A \otimes A \otimes \Omega_A \arrow[r, shift left, "f\otimes f\otimes\Omega_f"] \arrow[r, shift right, "g\otimes g\otimes\Omega_g"'] \arrow[d, "id\otimes\bullet_A;\bullet_A"] & B \otimes B \otimes \Omega_B \arrow[r, "h\otimes h\otimes\Omega_h"] \arrow[d, "id\otimes\bullet_B;\bullet_B=m_B\otimes id;\bullet_B"] & C \otimes C \otimes \Omega_C \arrow[d, dashed, "k"] \\
\Omega_A \arrow[r, shift left, "\Omega_f"] \arrow[r, shift right, "\Omega_g"'] & \Omega_B \arrow[r, "\Omega_h"] & \Omega_C
\end{tikzcd}$$

which is unique by the universal property of the coequalizer. But both $m_C \otimes id_{\Omega_C}; \bullet_C$ and $id_C \otimes \bullet_C; \bullet_C$ satisfy this, and hence must be equal. Finally, if we let l and l'

denote our isomorphisms between Ω_C and $I \otimes \Omega_C$ then we have

$$\Omega_h; l; e_C \otimes id_{\Omega_C}; \bullet_C = l; e_B \otimes id_{\Omega_B}; h \otimes \Omega_h; \bullet_C = l; e_B \otimes id_{\Omega_B}; \bullet_B; \Omega_h \ = \Omega_h$$

and since Ω_h is epi, the proof is complete. ∎

So now we have that Ω_C is a C-module.

Lemma 6.0.4. *d_C is a derivation.*

Proof: First,

$$e_C; d_C = e_B; h; d_C = e_B; d_B; \Omega_h \ = 0$$

and now,

$$\begin{aligned} m_C; d_C &= h \otimes h; m_C; d_C = s \otimes s; m_B; h; d_C = s \otimes s; m_B; d_B; \Omega_h \\ &= (id_B \otimes d_B + c; id_B \otimes d_B); \bullet_B; \Omega_h && \text{(Leibniz)} \\ &= (id_B \otimes d_B + c; id_B \otimes d_B); h \otimes \Omega_h; \bullet_C \\ &= (h \otimes (d_B; \Omega_h) + c; h \otimes (d_B; \Omega_h)); \bullet_C \\ &= (h \otimes (h; d_C) + c; h \otimes (h; d_C)); \bullet_C \\ &= h \otimes h; (id_A \otimes d_C + c; id_C \otimes d_C); \bullet_C \end{aligned}$$

We may use Johnstone's lemma once again to prove that $h \otimes h$ is a coequalizing map, and hence epi, and so d_C satisfies the Leibniz rule, and the proof is complete. ∎

We may now complete the proof of the theorem with the following lemma:

Lemma 6.0.5. *d_C is the universal derivation for C.*

Proof: Let ∂ be a derivation from C to a C-module M. Then $h; \partial$ is a derivation from B to M (which is here viewed as a B-module). So by the universal property of d_B there exists a unique B-module map, $k : \Omega_B \longrightarrow M$ s.t. $d_B; k = h; \partial$. Also,

$$d_A; \Omega_f; k = f; d_B; k = f; h; \partial = g; h; \partial = g; d_B; k = d_A; \Omega_g; k$$

and by the universal property of d_A, $\Omega_f; k = \Omega_g; k$. So then by the universal property of the coequalizer there is a unique $q : \Omega_C \longrightarrow M$ s.t. $\Omega_h; q = k$. Hence $h; \partial = d_B; k = d_B; \Omega_h; q = h; d_C; q$ so that $\partial = d_C; q$, since h is epi.

Now, let $\bullet_M$ denote the action of C on M. Then

$$id_C \otimes \Omega_h; id_C \otimes q; \bullet_M = id_C \otimes k; \bullet_M = \bullet_C; k = \bullet_C; \Omega_h; q = id_C \otimes \Omega_h; \bullet_M; q$$

and since Ω_h is a coequalizer and colimits are preserved by tensor, $id_C \otimes \Omega_h$ is a coequalizer, and so $id_C \otimes q; \bullet_M = \bullet_M; q$, which means that q is a C-module morphism. Finally, suppose there is another C-module morphism, $j : \Omega_C \longrightarrow M$ satisfying $d_C; j = \partial$. Then

$$\begin{aligned} h; d_C; j &= h; \partial \\ d_B; \Omega_h; j &= d_B; k \\ \Omega_h; j &= k = \Omega_h; q \qquad \text{(universal property of } d_B\text{)} \end{aligned}$$

and so uniqueness follows from Ω_h being epi and the theorem is proven. ∎

∎

We can now easily confirm our intended result:

Corollary 6.0.1. *If (A, m_ν, e_ν) is an associative algebra arising from a T-algebra (A, ν) in a Kähler category K, the tensor product of which preserves colimits (i.e. $X \otimes \_\_$ preserves colimits $\forall X \in Ob(K)$), then the module of differential forms is given by the coequalizer*

$$\begin{tikzcd} \Omega_{T^2A} \arrow[r, shift left, "\Omega_{\mu_A}"] \arrow[r, shift right, "\Omega_{T\nu}"'] & \Omega_{TA} \arrow[r, "\Omega_\nu"] & \Omega_{A,\nu} \end{tikzcd}$$

and the universal derivation $d_\nu : A \longrightarrow \Omega_{A,\nu}$ is the map uniquely determined by the universal property of the coequalizer in:

$$\begin{tikzcd}
\Omega_{T^2A} \arrow[r, shift left, "\Omega_{T\nu}"] \arrow[r, shift right, "\Omega_{T\mu}"'] & \Omega_{TA} \arrow[r, "\Omega_{\nu}"] & \Omega_{A,\nu} \\
T^2A \arrow[u, "d_{T^2A}"] \arrow[r, shift left, "T\nu"] \arrow[r, shift right, "\mu"'] & TA \arrow[u, "d_{TA}"'] \arrow[r, "\nu"] & A \arrow[u, dashed]
\end{tikzcd}$$

Furthermore, we are now equipped with a number of alternative descriptions for our canonical modules of differential forms. For one, each object of the form TA gives rise to the T-algebra (TA, μ_A), and since μ_A is an associative algebra homomorphism for TA's canonical algebra structure, the coequalizer

$$\begin{tikzcd}
\Omega_{T^3A} \arrow[r, shift left, "\Omega_{T\mu_A}"] \arrow[r, shift right, "\Omega_{\mu_{TA}}"'] & \Omega_{T^2A} \arrow[r, dashed, "\Omega_{\mu_A}"] & \Omega_{TA,\mu_A}
\end{tikzcd}$$

gives $\Omega_{TA,\mu_A} \cong \Omega_{TA}$ by universality of the module of differential forms. Clearly, we may replace 'A' by 'T^nA' in the preceding. Similarly,

Corollary 6.0.2. *Let (A, ν) be a T-algebra in a Kähler category K whose tensor product preserves colimits (i.e. $X \otimes \_\_$ preserves colimits $\forall X \in Ob(K)$). Then the colimit*

$$\begin{tikzcd}
\Omega_{T^{n+2}A} \arrow[r, shift left, "\Omega_{T^{n+1}\mu_A}"] \arrow[r, shift right, "\Omega_{T^{n+2}\nu}"'] & \Omega_{T^{n+1}A} \arrow[r, dashed, "\Omega_{T^n\nu}"] & \Omega_{T^nA,T\nu}
\end{tikzcd}$$

describes $\Omega_{T^nA,T^n\nu} \cong \Omega_{T^nA}$

Proof: Since

$$\begin{tikzcd}
T^2A \arrow[r, shift left, "\mu"] \arrow[r, shift right, "T\nu"'] & TA \arrow[r, "\nu"] & A
\end{tikzcd}$$

is an absolute coequalizer, it follows that

$$\begin{tikzcd}
T^{n+2}A \arrow[r, shift left, "T^n\mu"] \arrow[r, shift right, "T^{n+1}\nu"'] & T^{n+1}A \arrow[r, "T^n\nu"] & T^nA
\end{tikzcd}$$

is a coequalizer. Furthermore, $T^{n+1}\eta$ is a section for both $T^{n+1}\nu$ and $T^n\mu$, so the result follows immediately from Theorem 6.0.1 and by universality of the module of differential forms. ∎

From the above corollaries it is clear that in an additive symmetric monoidal category with an algebra modality T, if for some object A and some positive integer n there exist modules of differential forms for both T^nA and $T^{n+1}A$, there must exist a module of differential forms for $T^{n-1}A$ and therefore one for $T^{n-2}A$, etc. So then if it can be shown that for each object A in the category there exists a positive integer N_A such that modules of differential forms exist for T^nA for all $n \geq N_A$ the category must be Kähler. We explore a possible approach to exploiting this fact in Chapter 8.

One can recognize from the proof of Theorem 6.0.1 that the requirement that the splitting map r be an algebra map is only necessary in order to obtain a canonical splitting map Ω_r. Hence we may reformulate the theorem in the following manner:

Theorem 6.0.2. *Let* (A, m_A, e_A)*,* (B, m_B, e_B) *and* (C, m_C, e_C) *be associative algebras in an additive symmetric monoidal category* K*, the tensor product of which preserves colimits (i.e.* $X \otimes \_\_$ *preserves colimits* $\forall X \in Ob(K)$*). Furthermore, let* A *and* B *have universal derivations* $d_A : A \longrightarrow \Omega_A$ *and* $d_B : B \longrightarrow \Omega_B$*, respectively. Then if* $f, g : A \longrightarrow B$ *and* $h : B \longrightarrow C$ *are morphisms of associative algebras s.t.*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd}$$

is a coequalizer, and there exist morphisms $r : B \longrightarrow A$ *s.t.* $r; f = id_B = r; g$ *and* $q : \Omega_B \longrightarrow \Omega_A$ *s.t.* $q; \Omega_f = id_{\Omega_B} = q; \Omega_g$ *(where* Ω_f *and* Ω_g *are the unique* A*-module maps with* $f; d_B = d_A; \Omega_f$ *and* $g; d_B = d_A; \Omega_g$*) then the module of differential forms for* C *is given by the coequalizer*

$$\begin{tikzcd} \Omega_A \arrow[r, shift left, "\Omega_f"] \arrow[r, shift right, "\Omega_g"'] & \Omega_B \arrow[r, "\Omega_h"] & \Omega_C \end{tikzcd}$$

Furthermore, the universal derivation $d_C : C \longrightarrow \Omega_C$ *is the map uniquely determined by the universal property of the coequalizer in:*

$$\begin{tikzcd}
\Omega_A \arrow[r, shift left, "\Omega_f"] \arrow[r, shift right, "\Omega_g"'] & \Omega_B \arrow[r, "\Omega_h"] & \Omega_C \\
A \arrow[u, "d_A"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[u, "d_B"] \arrow[r, "h"] & C \arrow[u, dashed]
\end{tikzcd}$$

Chapter 7

Noncommutative Kähler Categories

Using the abstraction of the classical theory of Kähler differentials to Kähler categories as a guide we may begin the process of generalizing the notion of a Kähler category to the noncommutative setting. A key difference that needs to be accounted for in modelling this situation is that our algebras no longer need be commutative, which means that derivations will have to be sent to bimodules. That is:

Definition 7.0.1. Let (A, m_A, e_A) be an associative algebra in an additive monoidal category C. Then a *(noncommutative) A-derivation* $\partial : A \longrightarrow M$ to an A-bimodule (M, a_L, a_R) is a morphism satisfying $e_A; \partial = 0$ and

$$m_A; \partial = (\partial \otimes id_A); a_R + (id_A \otimes \partial); a_L \qquad \text{(NC Leibniz)}$$

With this notion of derivation, the definition of a universal derivation and of a module of differential forms (here a bimodule) is exactly as in the commutative case. Finally,

Definition 7.0.2. A *(noncommutative) Kähler category K* is an additive monoidal category with an algebra modality T s.t. $\forall A \in Ob(K)$ there is a TA-bimodule of differential forms Ω_{TA} together with a universal TA-derivation $d_{TA} : TA \longrightarrow \Omega_{TA}$.

7.1 Differential Forms for Noncommutative Kähler Categories

The material contained in this section is the result of original work.

The above results about commutative Kähler categories carry over to the case of a noncommutative Kähler category. The key difference is that here we are dealing with bimodules of differential forms and with them an altered Leibniz rule as described in the preceding section. To begin, we require the following lemma:

Lemma 7.1.1. *Let* (A, m_A, e_A) *be an associative algebra in a noncommutative Kähler category,* K*, and let* $\partial : A \longrightarrow M$ *be a derivation to an* A*-bimodule* (M, a_L, a_R)*. Then given an associative algebra* (B, m_B, e_B) *and an associative algebra morphism* $f : B \longrightarrow A$ *there exists a* B*-bimodule structure on* M *making* $f;\partial$ *a* B*-derivation.*

Proof: We first prove that $f \otimes id_M; a_L$ is a left B-module action on M. The proof that $id_M \otimes f; a_R$ is a right B-module action on M is similar:

$$\begin{aligned} id_B \otimes f \otimes id_M; id \otimes a_L; f \otimes id_M; a_L &= f \otimes f \otimes id_M; id_A \otimes a_L; a_L \\ &= f \otimes f \otimes id_M; m_A \otimes id_M; a_L = m_B \otimes id_M; f \otimes id_M; a_L \end{aligned}$$

That these maps give M a B-bimodule structure follows from

$$\begin{aligned} id_B \otimes id_M \otimes f; id_B \otimes a_R; f \otimes id_M; a_L &= f \otimes id_M \otimes f; id_A \otimes a_R; a_L \\ &= f \otimes id_M \otimes f; a_L \otimes id_A; a_R = f \otimes id_M \otimes id_B; a_L \otimes id_B; id_M \otimes f; a_R \end{aligned}$$

We then have that $f;\partial$ is a B-derivation since

$$\begin{aligned} m_B; f; \partial = f \otimes f; m_A; \partial &= f \otimes f; ((id_A \otimes \partial); a_R + (\partial \otimes id_A); a_L) \\ &= (id_B \otimes (f;\partial)); f \otimes id_M; a_R + ((f;\partial) \otimes id_B); id_M \otimes f; a_L \end{aligned}$$

∎

Hence we have the following theorem:

Theorem 7.1.1. *Let* (A, m_A, e_A), (B, m_B, e_B) *and* (C, m_C, e_C) *be associative algebras in a noncommutative Kähler category* K*, the tensor product of which preserves colimits (i.e.* $X \otimes \_\_$ *and* $\_\_ \otimes X$ *preserve colimits* $\forall X \in Ob(K)$*). Furthermore, let* A *and* B *have universal derivations* $d_A : A \longrightarrow (\Omega_A, a_L, a_R)$ *and* $d_B : B \longrightarrow (\Omega_B, b_L, b_R)$*, respectively. Then if* $f, g : A \longrightarrow B$ *and* $h : B \longrightarrow C$ *are morphisms of associative algebras s.t.*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd}$$

is a coequalizer, and there exists an associative algebra morphisms $r : B \longrightarrow A$ *s.t.* $r; f = id_B = r; g$*, the module of differential forms for* C *is given by the coequalizer*

$$\begin{tikzcd} \Omega_A \arrow[r, shift left, "\Omega_f"] \arrow[r, shift right, "\Omega_g"'] & \Omega_B \arrow[r, "\Omega_h"] & \Omega_C \end{tikzcd}$$

where Ω_f *and* Ω_g *are the unique* A*-module maps with* $f; d_B = d_A; \Omega_f$ *and* $g; d_B = d_A; \Omega_g$*. Furthermore, the universal derivation* $d_C : C \longrightarrow \Omega_C$ *is the map uniquely determined by the universal property of the coequalizer in:*

$$\begin{tikzcd} \Omega_A \arrow[r, shift left, "\Omega_f"] \arrow[r, shift right, "\Omega_g"'] & \Omega_B \arrow[r, "\Omega_h"] & \Omega_C \\ A \arrow[u, "d_A"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[u, "d_B"] \arrow[r, "h"] & C \arrow[u, dashed] \end{tikzcd}$$

Proof: We may define our left C-module action on Ω_C exactly as in the commutative case, and defining our right C-module action can be done in the obvious analogous way. What remains is then to confirm that our left and right module actions interact properly. To this end, from the fact that

$$\begin{tikzcd} \Omega_A \arrow[r, shift left, "\Omega_f"] \arrow[r, shift right, "\Omega_g"'] & \Omega_B \arrow[r, "\Omega_h"] & \Omega_C \end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes \Omega_A \arrow[r, shift left, "f\otimes\Omega_f"] \arrow[r, shift right, "g\otimes\Omega_g"'] & B \otimes \Omega_B \arrow[r, "h\otimes\Omega_h"] & C \otimes \Omega_C
\end{tikzcd}$$

are coequalizers, we construct the following diagram:

$$\begin{tikzcd}
A \otimes \Omega_A \otimes A \arrow[r, shift left, "f\otimes\Omega_f\otimes id"] \arrow[r, shift right, "g\otimes\Omega_g\otimes id"'] \arrow[d, shift right, "id\otimes id\otimes g"'] \arrow[d, shift left, "id\otimes id\otimes f"] & B \otimes \Omega_B \otimes A \arrow[r, "h\otimes\Omega_h\otimes id"] \arrow[d, shift right, "id\otimes id\otimes g"'] \arrow[d, shift left, "id\otimes id\otimes f"] & C \otimes \Omega_C \otimes A \arrow[d, shift right, "id\otimes id\otimes h"'] \arrow[d, shift left, "id\otimes id\otimes f"] \\
A \otimes \Omega_A \otimes B \arrow[r, shift left, "f\otimes\Omega_f\otimes id"] \arrow[r, shift right, "g\otimes\Omega_g\otimes id"'] \arrow[d, "id\otimes id\otimes h"'] & B \otimes \Omega_B \otimes B \arrow[r, "h\otimes\Omega_h\otimes id"] \arrow[d, "id\otimes id\otimes h"'] & C \otimes \Omega_C \otimes B \arrow[d, "id\otimes id\otimes h"'] \\
A \otimes \Omega_A \otimes C \arrow[r, shift left, "f\otimes\Omega_f\otimes id"] \arrow[r, shift right, "g\otimes\Omega_g\otimes id"'] & B \otimes \Omega_B \otimes C \arrow[r, "h\otimes\Omega_h\otimes id"] & C \otimes \Omega_C \otimes C
\end{tikzcd}$$

and it can easily be confirmed that it satisfies the conditions of Lemma 3.1.2, so that

$$\begin{tikzcd}
A \otimes \Omega_A \otimes A \arrow[r, shift left, "f\otimes\Omega_f\otimes f"] \arrow[r, shift right, "g\otimes\Omega_g\otimes g"'] & B \otimes \Omega_B \otimes B \arrow[r, "h\otimes\Omega_h\otimes h"] & C \otimes \Omega_C \otimes C
\end{tikzcd}$$

is a coequalizer. Hence the uniqueness of k in the following diagram confirms that we have proper bimodule actions:

$$\begin{tikzcd}
A \otimes \Omega_A \otimes A \arrow[r, shift left, "f\otimes\Omega_f\otimes f"] \arrow[r, shift right, "g\otimes\Omega_g\otimes g"'] \arrow[d, "id\otimes a_R;a_L"] & B \otimes \Omega_B \otimes B \arrow[r, "h\otimes\Omega_h\otimes h"] \arrow[d, "id\otimes b_R;b_L=b_L\otimes id;b_R"] & C \otimes \Omega_C \otimes C \arrow[d, dashed, "k"] \\
\Omega_A \arrow[r, shift left, "\Omega_f"] \arrow[r, shift right, "\Omega_g"'] & \Omega_B \arrow[r, "\Omega_h"] & \Omega_C
\end{tikzcd}$$

To complete our proof, we need only show that d_C satisfies the non-commutative Leibniz rule:

$$h \otimes h; m_C; d_C = m_B; h; d_C = m_B; d_B; \Omega_h$$

$$
\begin{aligned}
&= ((d_B \otimes id_B); b_R + (id_B \otimes d_B); b_L); \Omega_h && \text{(NC Leibniz)}\\
&= ((d_B \otimes id_B); \Omega_h \otimes h; c_R + (id_B \otimes d_B); h \otimes \Omega_h; c_L)\\
&= (((d_B; \Omega_h) \otimes h); c_R + (h \otimes (d_B; \Omega_h)); c_L)\\
&= (((h; d_C) \otimes h); c_R + (h \otimes (h; d_C)); c_L)\\
&= h \otimes h; (d_C \otimes id_C); c_R + (id_C \otimes d_C); c_L
\end{aligned}
$$

and once again $h \otimes h$ is epi. Universality then follows exactly as before, and we are done. ∎

Chapter 8

The Limiting Case

The material contained in this section is the result of original work.

One consequence of the preceding theorems is that to prove that a monoidal category with an algebra modality T is a Kähler category requires that one only prove that for each object A, $\exists N_A$ such that $\forall n \geq N_A$ there exists a module of differential forms for $T^n A$. That is, if we know that there is a module of differential forms associated to $T^n A$ $\forall n \geq N_A$, we may use the above theorems to derive those for $k < n$ as well as those for all T-algebras. Although this does not appear to offer any sort of immediate computational advantage, it suggests that we observe what happens as $n \longrightarrow \infty$. To this end, let T be a monad in a closed monoidal category C, and consider the following:

Definition 8.0.1. Denote by $colim\,\eta$ the colimit of the following chain of maps

$$\begin{tikzcd} TA \arrow[r, "T\eta"] & T^2A \arrow[r] & \ldots \arrow[r] & T^nA \arrow[r, "T^n\eta"] & T^{n+1}A \arrow[r, "T^{n+1}\eta"] & T^{n+2}A \arrow[r] & \ldots \end{tikzcd}$$

with associated maps denoted $j_n : T^n A \longrightarrow colim\,\eta$

Before we proceed, we require the following lemma:

Lemma 8.0.1. *Given the following chains in C*

$$\begin{tikzcd} A_1 \arrow[r, "a_1"] & A_2 \arrow[r] & \ldots \arrow[r] & A_n \arrow[r, "a_n"] & A_{n+1} \arrow[r, "a_{n+1}"] & A_{n+2} \arrow[r] & \ldots \end{tikzcd}$$

$$\begin{tikzcd}
B_1 \arrow[r, "b_1"] & B_2 \arrow[r] & \dots \arrow[r] & B_n \arrow[r, "b_n"] & B_{n+1} \arrow[r, "b_{n+1}"] & B_{n+2} \arrow[r] & \dots
\end{tikzcd}$$

with respective colimits in C, A with maps α_n and B with maps β_n, the colimit of the chain

$$\begin{tikzcd}
\dots \arrow[r] & A_n \otimes B_n \arrow[r, "a_n \otimes b_n"] & A_{n+1} \otimes B_{n+1} \arrow[r, "a_{n+1} \otimes b_{n+1}"] & A_{n+2} \otimes B_{n+2} \arrow[r] & \dots
\end{tikzcd}$$

is $A \otimes B$ with maps $\alpha_n \otimes \beta_n$

Note that this result is not trivial: although the category being closed prescribes that colimits are preserved when tensoring with a fixed object, it does not necessarily follow that taking two colimits in parallel will result in the tensor product of the two colimits.

Proof: Let there be a cocone over

$$\begin{tikzcd}
\dots \arrow[r] & A_n \otimes B_n \arrow[r, "a_n \otimes b_n"] & A_{n+1} \otimes B_{n+1} \arrow[r, "a_{n+1} \otimes b_{n+1}"] & A_{n+2} \otimes B_{n+2} \arrow[r] & \dots
\end{tikzcd}$$

with base H, and maps $h_n : A_n \otimes B_n \longrightarrow H$. Then for all n we can form a cocone with base H via the following diagram:

$$\begin{tikzcd}
 & & A_n \otimes B_n \arrow[r, "a_n \otimes b_n"] & A_{n+1} \otimes B_{n+1} \arrow[r] & \dots \arrow[r] & H \\
A_n \otimes B_1 \arrow[r] & \dots \arrow[r] & A_n \otimes B_n \arrow[r, "id \otimes b_n"] \arrow[u, "id \otimes id"] & A_n \otimes B_{n+1} \arrow[r] \arrow[u, "a_n \otimes id"'] & \dots &
\end{tikzcd}$$

Note that since C is closed the bottom row has colimit $A_n \otimes B$ with maps $id_{A_n} \otimes \beta_m$, and hence $\forall n\ \exists! k_n : A_n \otimes B \longrightarrow H$ factoring this cocone.

Now, let $m \leq n$. Then

$$\begin{aligned}
id_{A_n} \otimes \beta_m; k_n = id_{A_n} \otimes (b_m; \dots; b_{n-1}; \beta_n); k_n &= id_{A_n} \otimes (b_m; \dots; b_{n-1}); a_n \otimes b_n; h_{n+1} \\
&= a_n \otimes id_{B_m}; id_{A_{n+1}} \otimes (b_m; \dots; b_{n-1}; b_n); h_{n+1} \\
&= a_n \otimes id_{B_m}; id_{A_{n+1}} \otimes \beta_m; k_{n+1} = id_{A_n} \otimes \beta_m; a_n \otimes id_B; k_{n+1}
\end{aligned}$$

Similarly, if $m > n$,

$$id_{A_n} \otimes \beta_m; k_n = (a_n; a_{n+1}; ...; a_{m-1}) \otimes id_{B_m}; h_m = a_n \otimes id_{B_m}; (a_{n+1}; ...; a_{m-1}) \otimes id_{B_m}; h_m$$
$$= a_n \otimes id_{B_m}; id_{A_{n+1}} \otimes \beta_m; k_{n+1} = id_{A_n} \otimes \beta_m; a_n \otimes id_B; k_{n+1}$$

Hence, by universality, $k_n = a_n \otimes id_B; k_{n+1}$. Using this fact we may build the following diagram:

$$\begin{tikzcd}
H \arrow[r] & ... \arrow[r] & H \arrow[r, "id"] & H \arrow[r] & ... \\
A_1 \otimes B \arrow[u, "k_1"] \arrow[r] & ... \arrow[r] & A_n \otimes B \arrow[u, "k_n"] \arrow[r, "a_n \otimes id"] & A_{n+1} \otimes B \arrow[u, "k_{n+1}"] \arrow[r] & ... \\
A_1 \otimes B_1 \arrow[u, "id \otimes \beta_1"] \arrow[r] & ... \arrow[r] & A_n \otimes B_n \arrow[u, "id \otimes \beta_n"] \arrow[r, "a_n \otimes b_n"] & A_{n+1} \otimes B_{n+1} \arrow[u, "id \otimes \beta_{n+1}"] \arrow[r] & ...
\end{tikzcd}$$

Commutativity of the lower squares follows by definition of the maps β_n. Now, note that the top row has colimit H with all maps the identity, and the second row has the colimit $A \otimes B$ (since C is closed) with maps $\alpha_n \otimes id$. A cocone may be formed over the third row with base $A \otimes B$ and maps $\alpha_n \otimes \beta_n$. By the universal property of the colimit in the second row, there is a unique map $\omega : A \otimes B \longrightarrow H$ s.t. $k_n = \alpha_n \otimes id_B; \omega$ for all n. By construction, $\forall n \; id \otimes \beta_n; k_n = h_n$, and so $\alpha_n \otimes \beta_n; id_A \otimes id_B; \omega = \alpha_n \otimes \beta_n; \omega = id \otimes \beta_n; k_n = h_n$. Hence the cocone over

$$\begin{tikzcd}
... \arrow[r] & A_n \otimes B_n \arrow[r, "a_n \otimes b_n"] & A_{n+1} \otimes B_{n+1}
\end{tikzcd}$$
$$\begin{tikzcd}
{} \arrow[r, "a_{n+1} \otimes b_{n+1}"] & A_{n+2} \otimes B_{n+2} \arrow[r] & ...
\end{tikzcd}$$

with base H factors through $A \otimes B$ via the map ω.

To show uniqueness, suppose there is another map ω' factoring this cocone. Then $\forall n \; \alpha_n \otimes \beta_n; \omega' = h_n$ so that $id_{A_n} \otimes \beta_n; \alpha_n \otimes id_B; \omega' = id_{A_n} \otimes \beta_n; k_n$ for all n. We may form a cocone over

$$\begin{tikzcd}
A_n \otimes B_1 \arrow[r] & ... \arrow[r] & A_n \otimes B_n \arrow[r, "id \otimes b_n"] & A_n \otimes B_{n+1} \arrow[r] & ...
\end{tikzcd}$$

with base H via the maps $id_{A_n} \otimes \beta_m; k_n$. Clearly this cocone factors through $A_n \otimes B$ via the map k_n. We will now show that it also factors via the map $\alpha_n \otimes id_B; \omega'$:
Let $m > n$

$$\begin{aligned} id_{A_n} \otimes \beta_m; \alpha_n \otimes id_B; \omega' &= (a_n; ...; a_{m-1}) \otimes id_{B_m}; \alpha_m \otimes \beta_m; \omega' = (a_n; ...; a_{m-1}) \otimes id_{B_m}; h_m \\ &= (a_n; ...; a_{m-1}) \otimes id_{B_m}; id_{A_m} \otimes \beta_m; k_m \\ &= id_{A_n} \otimes \beta_m; (a_n; ...; a_{m-1}) \otimes id_B; k_m \\ &= id_{A_n} \otimes \beta_m; (a_n; ...; a_{m-2}) \otimes id_B; k_{m-1} = ... = id_{A_n} \otimes \beta_m; k_n \end{aligned}$$

Now, let $m \leq n$

$$\begin{aligned} id_{A_n} \otimes \beta_m; \alpha_n \otimes id_B; \omega' &= id_{A_n} \otimes (b_m; ...; b_{n-1}); \alpha_n \otimes \beta_n; \omega' \\ &= id_{A_n} \otimes (b_m; ...; b_{n-1}); h_n = id_{A_n} \otimes (b_m; ...; b_{n-1}); id_{A_n} \otimes \beta_n; k_n \\ &= id_{A_n} \otimes \beta_m; k_n \end{aligned}$$

and so by uniqueness, $k_n = \alpha_n \otimes id_B; \omega'$ for all n. But then the cocone formed over

$$\begin{tikzcd} A_1 \otimes B \arrow[r] & ... \arrow[r] & A_n \otimes B \arrow[r, "a_n \otimes id"] & A_{n+1} \otimes B \arrow[r] & ... \end{tikzcd}$$

with base H via the maps k_n factors through $A \otimes B$ via both ω and ω', and so by uniqueness $\omega = \omega'$. ∎

It is clear from the associativity of the tensoral structure that this result extends to provide the colimit of any number of parallel chains with given colimits. Now, we will assume that *colim* η exists in what follows. We then have the following:

Proposition 8.0.1. *If the monad T is an algebra modality, then there exists an algebra structure on colim η making j_n into an associative algebra map for all n.*

Proof: By the above lemma

$$\begin{tikzcd} ... \arrow[r] & T^nA \otimes T^nA \arrow[r, "T^n\eta \otimes T^n\eta"] & T^{n+1}A \otimes T^{n+1}A \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "T^{n+1}\eta \otimes T^{n+1}\eta"] & T^{n+2}A \otimes T^{n+2}A \arrow[r] & ... \end{tikzcd}$$

has colimit $\mathit{colim}\,\eta \otimes \mathit{colim}\,\eta$. We may form a cocone over this chain with base $\mathit{colim}\,\eta$ via the maps $m_{T^nA}; j_n$, since $T^n\eta \otimes T^n\eta; m_{T^{n+1}A}; j_{n+1} = m_{T^nA}; T^n\eta; j_{n+1} = m_{T^nA}; j_n$. Hence there is a unique map, call it $m_\eta : \mathit{colim}\,\eta \otimes \mathit{colim}\,\eta \longrightarrow \mathit{colim}\,\eta$ factoring this cocone.

Now, $\mathit{colim}\,\eta \otimes \mathit{colim}\,\eta \otimes \mathit{colim}\,\eta$ is the colimit of the obvious chain, and we can form two cocones over this chain with base $\mathit{colim}\,\eta$ via maps $j_n \otimes j_n \otimes j_n; m_\eta \otimes id_{\mathit{colim}\,\eta}; m_\eta$ or via maps $j_n \otimes j_n \otimes j_n; id_{\mathit{colim}\,\eta} \otimes m_\eta; m_\eta$. Observe though, that

$$\begin{aligned} j_n \otimes j_n \otimes j_n; m_\eta \otimes id_{\mathit{colim}\,\eta}; m_\eta &= m_{T^nA} \otimes id_{T^nA}; j_n \otimes j_n; m_\eta \\ = m_{T^nA} \otimes id_{T^nA}; m_{T^nA}; j_n &= id_{T^nA} \otimes m_{T^nA}; m_{T^nA}; j_n \\ &= j_n \otimes j_n \otimes j_n; id_{\mathit{colim}\,\eta} \otimes m_\eta; m_\eta \end{aligned}$$

so that these two cocones are indeed the same, and hence there is a unique map from $\mathit{colim}\,\eta \otimes \mathit{colim}\,\eta \otimes \mathit{colim}\,\eta$ to $\mathit{colim}\,\eta$ factoring both of them. Hence $m_\eta \otimes id_{\mathit{colim}\,\eta}; m_\eta = id_{\mathit{colim}\,\eta} \otimes m_\eta; m_\eta$. Constructing a unit for this algebra is similar. ∎

Note that in case we are working in a symmetric monoidal category, a symmetry map for $\mathit{colim}\,\eta \otimes \mathit{colim}\,\eta$ can be constructed in a manner similar to that of the construction of the algebra multiplication m_η. Since our goal is to prove that given certain conditions this algebra has a module of differential forms, we will require a candidate module. We will proceed by taking C to be a Kähler category. We may then make the following definition:

Definition 8.0.2. *Denote by Ω_η the colimit of the following chain of maps*

$$\begin{tikzcd} \Omega_{TA} \arrow[r, "\Omega_{T\eta}"] & \Omega_{T^2A} \arrow[r] & \ldots \arrow[r] & \Omega_{T^nA} \arrow[r, "\Omega_{T^n\eta}"] & \Omega_{T^{n+1}A} \arrow[r, "\Omega_{T^{n+1}\eta}"] & \Omega_{T^{n+2}A} \arrow[r] & \ldots \end{tikzcd}$$

with associated maps denoted $\Omega_{j_n} : \Omega_{T^nA} \longrightarrow \Omega_\eta$

We now have the following lemmas:

Lemma 8.0.2. *Ω_η has a canonical $\mathit{colim}\,\eta$-module structure*

Proof: By Lemma 8.0.1, $colim\,\eta\otimes\Omega_\eta$ is the colimit of the obvious chain. We may construct a cocone over this chain with base Ω_η and maps $\bullet_n;\Omega_{j_n} : T^nA\otimes\Omega_{T^nA}\longrightarrow\Omega_\eta$, where $\bullet_n$ is the T^nA-module action on Ω_{T^nA}. Showing that these maps form a cocone is routine, and hence we have a unique map, which we denote $\bullet_\eta : colim\,\eta\otimes\Omega_\eta\longrightarrow\Omega_\eta$, factoring the cocone. Since $colim\,\eta\otimes colim\,\eta\otimes\Omega_\eta$ is the colimit of the obvious chain, we may relabel our argument from Proposition 8.1 to prove that $id_{colim\,\eta}\otimes\bullet_\eta;\bullet_\eta = m_\eta;\bullet_\eta$.

▮

Lemma 8.0.3. *There exists a canonical derivation from* $colim\,\eta$ *to* Ω_η

Proof: We may form a cocone over

$$\begin{tikzcd}
TA \arrow[r, "T\eta"] & T^2A \arrow[r] & \ldots \arrow[r] & T^nA \arrow[r, "T^n\eta"] & T^{n+1}A \arrow[r, "T^{n+1}\eta"] & T^{n+2}A \arrow[r] & \ldots
\end{tikzcd}$$

with base Ω_η via the maps $d_{T^nA};\Omega_{j_n}$, since $T^n\eta; d_{T^{n+1}A};\Omega_{j_{n+1}} = d_{T^nA};\Omega_{T^nA};\Omega_{j_{n+1}} = d_{T^nA};\Omega_{j_n}$. Hence there is a unique map, which we denote $d_\eta : colim\,\eta\longrightarrow\Omega_\eta$ factoring this cocone. $\forall n$ we have

$$\begin{aligned}
j_n\otimes j_n; m_\eta; d_\eta = m_{T^nA}; j_n; d_\eta &= m_{T^nA}; d_{T^nA}; \Omega_{j_n}\\
&= (id_{T^nA}\otimes d_{T^nA} + c; id_{T^nA}\otimes d_{T^nA}); \bullet_n; \Omega_{j_n}\\
&= (id_{T^nA}\otimes d_{T^nA} + c; id_{T^nA}\otimes d_{T^nA}); j_n\otimes\Omega_{j_n}; \bullet_\eta\\
&= j_n\otimes j_n; (id_{colim\,\eta}\otimes d_\eta + c; id_{colim\,\eta}\otimes d_\eta); \bullet_\eta
\end{aligned}$$

Since this holds for all n, by the universal property of the colimit (specifically $colim\,\eta\otimes colim\,\eta$) the Leibniz rule holds for d_η and it follows that d_η is a derivation. ▮

Finally, we have the following theorem:

Theorem 8.0.1. *If* C *is a closed Kähler category then there exists a module of differential forms for* $colim\,\eta$

Proof: Let $\partial : colim\,\eta \longrightarrow M$ be a derivation to a $colim\,\eta$-module M. Then $\forall n$, $j_n; \partial$ is a derivation from T^nA to M (thence viewed as a T^nA-module), and hence there exists a unique T^nA-module map $M_n : \Omega_{T^nA} \longrightarrow M$ s.t. $d_{T^nA}; M_n = j_n; \partial$. Since

$$d_{T^nA}; M_n = j_n; \partial = T^n\eta; j_{n+1}; \partial = T^n\eta; d_{T^{n+1}A}; M_{n+1} = d_{T^nA}; \Omega_{T^nA}; M_{n+1}$$

it follows by the universal property of d_{T^nA} that $M_n = \Omega_{T^nA}; M_{n+1}$ and so by the universal property of the colimit Ω_η, there exists a unique map $k : \Omega_\eta \longrightarrow M$ s.t. $M_n = \Omega_{j_n}; k$ for all n. Furthermore, $\forall n$, we have

$$\begin{aligned} j_n \otimes \Omega_{j_n}; \bullet_\eta; k = \bullet_n; \Omega_{j_n}; k = \bullet_n; M_n = id_{T^nA} \otimes M_n; j_n \otimes id_M; \bullet_M \\ = j_n \otimes \Omega_{j_n}; id_{colim\,\eta} \otimes k; \bullet_M \end{aligned}$$

so that $\bullet_\eta; k = id_{colim\,\eta} \otimes k; \bullet_M$ and so k is a module map. It remains to be shown that $\partial = d_\eta; k$, and that k is the unique map with this property. To this end, note that $\forall n$

$$j_n; d_\eta; k = d_{T^nA}; \Omega_{j_n}; k = d_{T^nA}; M_n = j_n; \partial$$

so that by the universal property of the colimit $colim\,\eta$, $d_\eta; k = \partial$. Now, if there is another module map $q : \Omega_\eta \longrightarrow M$ such that $d_\eta; q = \partial$, then $\forall n$

$$d_{T^nA}; M_n = j_n; \partial = j_n; d_\eta; q = d_{T^nA}; \Omega_{j_n}; q$$

so that by the universal property of d_{T^nA}, $M_n = \Omega_{j_n}; q$ (and so $\Omega_{j_n}; q = \Omega_{j_n}; k$) for all n and by the universal property of the colimit Ω_η, $q = k$. ∎

Chapter 9

Concluding Remarks

The initial purpose of the investigation resulting in this paper was simply to fill a noticeable gap in the development of the theory of Kähler categories. As it turns out though, the results contained herein may both aid in the development of the theory of Kähler categories as well as foster a new understanding of the relationship between Kähler categories and differential categories. In regards to the latter, it was stipulated in the final theorem of this paper that a category be a Kähler category in order for $colim\,\eta$ to have a module of differential forms. However, it is not clear that this result will not hold for general codifferential categories. Since the constructed derivation inherits its universal property from a colimit which may exist in any category with an algebra modality, the main difficulty in proving that this result holds for general codifferential categories is in constructing a canonical derivation for $colim\,\eta$. If this can be accomplished, it would be interesting to investigate the extent to which the converse of the above theorem holds; that is, under what conditions is a monoidal category with an algebra modality, in which $colim\,\eta$ has a module of differential forms, a Kähler category?

An obvious motivation for such an investigation is to eliminate, weaken, or at least better understand the current requirement that in order for a codifferential category to be a Kähler category, Property K must hold. But there would also be significant computational advantages to relating the module of differential forms of $colim\, \eta$ to those of $T^n A$ for all n. For instance, in subsequent work we will be investigating the notion of Hochschild homology in the context of a Kähler category and will use the classical proof of the Hochschild-Kostant-Rosenberg (HKR) theorem as a guide to understanding the relationship between Hochschild homology and Kähler differentials in this context. A seemingly unavoidable technique utilized in the proof of the HKR theorem is the calculation of the module of differential forms for the localization of an algebra at a maximal ideal [10, 3.4.4]. If we are able to express the localization of an algebra categorically, then it will remain for us to discover what the modules of differential forms for localized algebras are; if these algebras can be expressed as T-algebras, then Corollary 6.0.1 gives us an immediate answer. If not though, it is clear that for any algebra we may construct $colim\, \eta$ and compute its module of differential forms, and then use the results of the investigation delineated in the above paragraph to aid in the computation of the module of differential forms for the original algebra.

Bibliography

[1] Richard Blute, Robin Cockett, Tim Porter, Robert Seely, *Kähler Categories*, Cahiers de Topologie et Geometrie Differentielle 52, pp. 253-268, 2012.

[2] Richard Blute, Robin Cockett, Robert Seely, *Differential Categories*, Mathematical Structures in Computer Science 16, pp. 1049-1083, 2006

[3] Richard Blute, Thomas Ehrhard, Chirstine Tasson, *A Convenient Differential Category*, 2012.

[4] David Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, 2004.

[5] William Fulton, Joe Harris, *Representation Theory*, Springer-Verlag, 1991.

[6] Jose M. Gracia-Bondia, Hector Figueroa, Joseph C. Varilly, *Elements of Noncommutative Geometry*, Birkhäuser, 2001.

[7] Robin Hartshorne, *Algebraic Geometry*, Springer-Verlag, 1974.

[8] Peter Johnstone, *Topos Theory*, Academic Press, 1977.

[9] Masoud Khalkhali, *Basic Noncommutative Geometry*, European Mathematical Society, 2009.

[10] Jean-Louis Loday, *Cyclic Homology*, Springer-Verlag, 1998

[11] Saunders MacLane, *Categories for the Working Mathematician*, Springer-Verlag, 1971.

[12] Hideyuki Matsumura, *Commuative Ring Theory*, Cambridge University Press, 1986.