SC

THE INSERTION LOSS SYNTHESIS OF COMMUNICATION NETWORKS USING NON-IDEAL ELEMENTS

by

Gabor C. Temes

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Department of Electrical Engineering
Faculty of Pure and Applied Science
University of Ottawa

1961

Abstract

The purpose of the research work described in this dissertation was to explore available methods for the synthesis of networks from non-ideal elements and to find a new theory upon which a practically tractable synthesis procedure can be founded for circuits with arbitrarily distributed losses.

Since the theory providing the basis for this novel design method turned out to be closely related to the insertion loss synthesis technique, a detailed explanation of the latter was required. It has been provided in Chapter I. Apart from some new derivations involving the relation between the form of the characteristic function and the circuit configuration, and the approximating synthesis procedure for filter-combinations, the fundamental theory described in the first chapter is available in the literature.

The construction of realizable transfer functions for some practical requirements is treated in Chapter II. There the proof of the non-realizability of a quasielliptic response, some realizability-considerations involving elliptic filters, the Section on harmonic suppression filter-sets and the step-by-step design method for Chebyshev pass-band filters are original; the rest of Chapter II recapitulates existing theory.

Chapter III provides a summary of miscellaneous analysis and synthesis methods which take into account the effects of

non-ideal elements. The improved temperature-compensation procedure is new; the other results have been previously published. The derivations contained in Para. III.2.c. have been worked out by the writer, at least one of them, however, probably duplicates Darlington's original (unpublished) calculations.

The new synthesis method and its supporting theorems are described in Chapter IV. First, a large variety of formulae are obtained for the derivatives of various network parameters. These formulae are utilized in the derivation of the main theory; they are also directly applicable to circuit analysis. The formulae can be deduced from the determinant forms of the driving-point and transfer immittances.

The effects of losses on the transfer characteristics are then studied indirectly by examining the changes in various parameters of the transfer function (natural frequencies, attenuation poles, proportionality factor) due to the added dissipation. Using series expansions, from the variation of these parameters the increment of the transfer function itself could be found to a first-order approximation. These results were used to find expressions, in which contributions of individual lossy elements to the distortion of the ideal characteristics were displayed. Even more important, they led to a relatively simple synthesis procedure allowing precorrection (often called predistortion) for the detrimental effects of individual losses.

Since the calculations giving rise to the final formulae involve series expansions and terms containing powers of loss factors greater than unity are omitted, the accuracy decreases with increasing losses. Even for quality-factors in the order of 10, however, examples proved the results to be sufficiently exact for any practical purposes.

Using some special properties of the lossy ladder-networks found in the course of this investigation the general expressions were simplified for this important configuration. For the important case of semi-homogeneous loss distribution (all coils have identical quality-factors, and so do all capacitors, but the quality-factors of the two groups differ), topological considerations were applied to achieve extremely compact formulae.

Procedures to improve the accuracy for extreme cases (large losses, sharply changing response) have been developed; however these are seldom needed. Some alternative procedures applicable in special cases have also been derived.

Finally, the theory was tested with excellent results by applying it to a large number of design problems, a representative cross-section of which has been included in Chapter IV to illustrate the simplicity and usefulness of the method.

LIST OF THE MOST IMPORTANT SYMBOLS

ENGLISH - LETTER SYMBOLS

A	Cascade parameter	p. 16*
	Even part of a Hurwitz polynomial	p. 38
A_i	Insertion loss	p. 21
A_o	Operating loss	p. 19
A_r	Reflection loss	p. 20
A_p	Maximum attenuation variation in the passband	p. 74
A_s	Minimum variation between passband and stopband attenuation	p. 75
B	Cascade parameter	p. 16
	Susceptance function	p. 70
	Phase function	p. 21
pB	Odd part of a Hurwitz polynomial	p. 38
C	Capacitance	
	Cascade parameter	p. 16
D	Cascade parameter	p. 16
	Configuration index	p. 170
d	Dissipation constant	p. 120
E	Voltage	p. 12
E_{20}	Voltage across the load when it is connected directly to the generator	p. 21
e	Suffix used to denote the even part of a function	p. 26
f	Arbitrary even or odd polynomial (operating parameters)	p. 33
G	Conductance	
g	Hurwitz polynomial (operating parameters)	p. 33
h_{ij}	Residue of X_{ij} at a pole	p. 25
h	Arbitrary polynomial (operating parameters)	p. 33

* Page on which the symbol first occurs

I	Current	p. 12
K	Complete elliptic integral of the first kind	p. 80
k	Selectivity parameter	p. 75
k_1	Discrimination parameter	p. 75
M	Voltage transfer ratio	p. 19
N	Current transfer ratio	p. 19
	Network	
n	Number of elements in a network	p. 89
	Degree of a function	p. 77
	Order of a circuit	p. 46
o	Suffix used to denote the odd part of a function	p. 26
P	Even polynomial (operating parameters)	p. 38
P_g	Maximum power available from a generator	p. 19
P_r	Reflected power	p. 20
p	Complex frequency variable	p. 24
P_2	Power delivered to the load	p. 19
P_{20}	Power delivered to the load connected directly to the generator	p. 21
Q	Quality factor of a network element	
q	Modular constant	p. 82
R	Resistance value	
R_o	Impedance normalization factor	
S	Operating factor	p. 19
X	Reactance function	
Y	Admittance function	
X_{ij}, x_{ij}	Immittance function (Z_{ij} or Y_{ij}, z_{ij} or y_{ij})	
Y_D	Driving point admittance function	

Y_{ii} Short circuit driving point admittance function

Y_{ij} $(i \neq j)$ Short circuit transfer admittance function

Z Impedance function

Z_A, Z_B - Impedances of a lattice p. 18

Z_D Driving point impedance function

Z_{ii} Open circuit driving point impedance function

Z_{ij} $(i \neq j)$ Open circuit driving point transfer impedance function

Z_I Image impedance p. 17

GREEK LETTER SYMBOLS

Γ_o	Operating transmission constant	p. 19
Γ_i	Insertion transmission constant	p. 21
Δ	Circuit determinant	p. 13
θ	Image propogation constant	p. 17
Λ	Insertion voltage ratio	p. 21
ρ	Reflection factor	p. 20
Φ	Insertion characteristic function	p. 27
φ	Operating characteristic function	p. 25
ω	Frequency variable	p. 25
ω_p	Normalized passband limit	p. 88
ω_s	Normalized stopband limit	p. 88
Ω_∞	Frequency of infinite attenuation (normalized)	p. 97

INTRODUCTION [1,2,12,13,22,29,33,41]

The title of this work implies a logical division of the subject into two main themes, insertion loss network synthesis and the design of circuits from lossy elements. In fact, the two topics are closely related; it was Darlington's historical work on the insertion loss synthesis of networks[1], that first provided a comprehensive theory for the observation of losses in circuit design, based on his synthesis method for lossless networks.

The present thesis will be divided into four Chapters. The first is devoted to the insertion loss synthesis and its German counterpart, the operating loss design. One reason for the rather complete and logical treatment of these theories was to achieve a self-contained analysis. A second, much more important argument was the scarcity of publications on the subject. Darlington's original work[1] was famous for its compactness as well as for its originality; it provides the results but not much of the derivations or reasoning behind them. While German authors (mainly Cauer and Piloty) gave the details of their work,[2,29-33] largely paralleling Darlington's, the use of different symbols and concepts made the interpretation awkward. In addition, certain parts of Darlington's work, among them the treatment of lossy networks, were not fully explored. The result is a situation, in which prestigeous periodicals (Proc. IRE, IRE Transactions on Circuit Theory, IEE Transaction etc.,) publish articles[12,22,41] by eminent authors like Grossman, Desoer and Zdunek interpreting parts or even single equations of Darlington's paper. Under these circumstances it seemed desirable to give a concise analysis of the method. Some of the derivations of this Chapter as well as those of Chapters II and III have never been published; many results are also new. They are, however embedded in the general theory in order to give a unified treatment of the subject.

Chapter II deals with selected problems of the approximation theory, mainly items relevant to filter design. The treatment is again brief, but logically complete.

Chapter III describes, with certain extensions, mainly already existing methods for the analysis and compensation of the effects of non-ideal element behaviour. It contains a detailed description of Darlington's predistortion theory[1] and gives the derivation (indeed, two different derivations) of his celebrated Eqs. (77) - (81) for the first time.

Finally, Chapter IV demonstrates a new comprehensive theory for the general treatment of lossy networks. Darlington's analysis has been restricted to special loss-distribution and the only attempt to solve the general problem - Desoer's perturbation theory[2] - is rather difficult to apply in most cases and not too well explored.

The theory presented in this Chapter is applicable in all cases;

special attention has, however, been given to certain important specific configurations and loss-distrubitions. For these, the formulae becomes surprisingly simple.

In the course of the calculations, some general network theorems - believed to be new - are stated and proven.

Finally, to illustrate the practical value of this synthesis procedure, a number of design examples has been worked out, and some demonstrated at the end of the Chapter. The accuracy obtained in these instances is much higher than any practical requirement would demand.

While the emphasis in the present text is on the treatment of losses, the principles and many of the results are equally useful in dealing with stray capacitances, element and quality-factor tolerances, etc.

It is the writer's pleasure to express his gratitude for encouragement and fruitful discussions to Professors G.S. Glinski and G.J. van der Maas of the University of Ottawa, and to acknowledge the constructive criticism provided by Drs. A.J. Grossman and G. Szentirmai of the Bell Telephone Laboratories. Also, he is deeply indebted to the Northern Electric Co. Ltd. for sponsoring the research and providing stimulating atmosphere.

TABLE OF CONTENTS

I. INSERTION LOSS SYNTHESIS

The synthesis of communication networks on the insertion loss (or operating loss) basis is performed in the following steps:

1. From the specifications (usually given in a graphical form), a suitable mathematical expression is derived, which approximates the requirements in an acceptable manner and which is at the same time appropriate for the realization procedure of steps 2, and 3,
2. From the algebraic expression obtained in step 1, the immittance-parameters of the network are deduced,
3. The configuration and element values of the network can be derived from the immittance parameters.

This chapter will describe the most important procedures for carrying out steps 2, and 3. From these, the necessary realizability conditions for the algebraic form of the specifications can be derived.

I.1. DEFINITION OF FOUR-POLE PARAMETERS [9]

The networks discussed in this thesis are reactive four-poles (two-terminal-pair circuits), fed by a generator with resistive internal impedance and terminated by a load resistor (Fig. 1). The only resistive elements within the four-pole are the loss resistors associated with the parasitic dissipation of the reactances. All elements of the terminated network will be considered passive, linear, lumped, bilateral and time-invariant.

The following section will define the quantities and parameters which will be found useful in the treatment of these circuits.

a) Driving-point and transfer immittances

For the circuits described above, repeated applications of Kirchhoff's laws lead to two alternative systems of equations:

$$
\begin{aligned}
z_{11} I_1 + z_{12} I_2 + \cdots + z_{1n} I_n &= E_1 \\
z_{21} I_1 + z_{22} I_2 + \cdots + z_{2n} I_n &= E_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \qquad (1) \\
z_{n1} I_1 + z_{n2} I_2 + \cdots + z_{nn} I_n &= E_n .
\end{aligned}
$$

(mesh-equations) and dually

$$
\begin{aligned}
y_{11} E_1 + y_{12} E_2 + \cdots + y_{1n} E_n &= I_1 \\
y_{21} E_1 + y_{22} E_2 + \cdots + y_{2n} E_n &= I_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \qquad (2) \\
y_{n1} E_1 + y_{n2} E_2 + \cdots + y_{nn} E_n &= I_n
\end{aligned}
$$

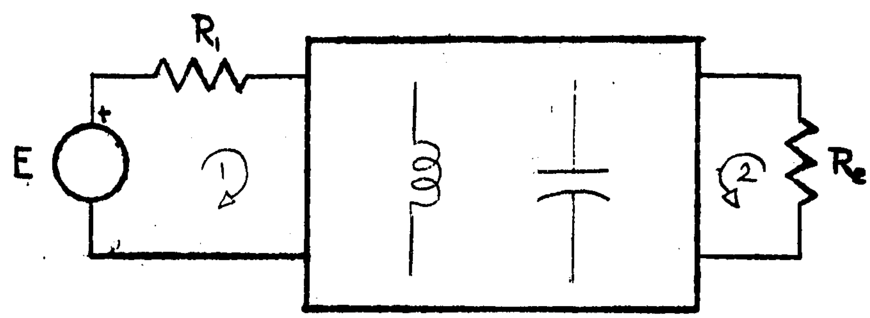


FIGURE 1.

(nodal equations). In the following x_{ij} will denote either z_{ij} or y_{ij}. x_{ij} will be called self-immittance $(i=j)$ or mutual immittance $(i \neq j)$ in Eq. (1) (Eq. (2)) E_i (I_i) are the source voltages (currents), I_j (E_j) the resulting mesh-currents (nodal voltages).

Also

$$x_{ij} = x_{ji} . \tag{3}$$

The ratio of the source voltage E_i (source current I_i) to the current I_j (voltage E_j) will be called driving point $(i=j)$ or transfer $(i \neq j)$ impedance (admittance). Z_{Di} (Y_{Di}) will denote a driving point, Z_{Tij} (Y_{Tij}) a transfer impedance (admittance). X_{Di} or X_{Tij}, the driving point or transfer immittance, might mean either Z or Y. Solving Eq. (1) or (2), we get

$$X_{Di} = \frac{\Delta}{\Delta_{ii}} \tag{4}$$

and

$$X_{Tij} = \frac{\Delta}{\Delta_{ij}} \tag{5}$$

where Δ is the determinant of Eq. (1) or (2) and Δ_{ij} is its minor obtained by omitting the i-th row and the j-th column.

When there will be no danger of confusion with the immittance-parameters (see below), the simpler X_{ii} (driving-point immittance) or X_{ij} (transfer immittance) symbolism will also be used.

b) Impedance - parameters

If in Eq. (1), all $E_i = 0$, $i \neq 1$ and

$$z_{22} = R_2 + z'_{22}$$

(Fig. 1), the output voltage is

$$V_2 = E_1 \frac{R_2}{Z_{T_{12}}} = \frac{E_1 R_2 \Delta_{12}}{\Delta^\circ + z_{22} \Delta_{22}} . \tag{6}$$

Here

$$\Delta^\circ = \Delta \Big|_{z_{22}=0} . \tag{7}$$

If now $R_2 \to \infty$, we get

$$V_2 = E_1 \frac{\Delta_{12}}{\Delta_{22}} .$$

Since further, for $R_2 \to \infty$

$$I_1 = \frac{E_1}{Z_{D_1}} = E_1 \frac{\Delta_{11}}{\Delta} = E_1 \frac{\Delta_{1122}}{\Delta_{22}} ,$$

(where Δ_{1122} is found by deleting the first and second rows, and columns of Δ),

$$\frac{V_2}{I_1}\Big|_{R_2 \to \infty} = Z_{12} = \frac{\Delta_{12}}{\Delta_{1122}} \tag{8}$$

$$\frac{E_1}{I_1}\Big|_{R_2 \to \infty} = Z_{11} = \frac{\Delta_{22}}{\Delta_{1122}} \tag{9}$$

and similarly

$$\left.\frac{V_1}{I_2}\right|_{R_1 \to \infty} = Z_{21} = \frac{\Delta_{21}}{\Delta_{1122}} = Z_{12} \tag{10}$$

and

$$\left.\frac{E_2}{I_2}\right|_{R_1 \to \infty} = Z_{22} = \frac{\Delta_{11}}{\Delta_{1122}} . \tag{11}$$

By superposition now

$$Z_{11} I_1 + Z_{12} I_2 = E_1$$

$$Z_{12} I_1 + Z_{22} I_2 = E_2 . \tag{12}$$

The Z_{ik} are the impedance-parameters of the four-pole.

c) Admittance - parameters

Starting out from Eq. (2) a procedure exactly dual to the one just performed leads to the admittance- parameters Y_{ik} of the four-pole. We get

$$\left.\frac{I_1}{E_1}\right|_{R_2 \to 0} = Y_{11} = \frac{\Delta_{22}}{\Delta_{1122}} \tag{13}$$

$$\left.\frac{I_2}{E_1}\right|_{R_2 \to 0} = Y_{12} = \frac{\Delta_{12}}{\Delta_{1122}} = Y_{21} \tag{14}$$

and

$$\left.\frac{I_2}{E_2}\right|_{R_1 \to 0} = Y_{22} = \frac{\Delta_{11}}{\Delta_{1122}} \tag{15}$$

where the Δ-s now refer to Eq. (2).

Also

$$\begin{aligned} Y_{11} E_1 + Y_{12} E_2 &= I_1 \\ Y_{12} E_1 + Y_{22} E_2 &= I_2 . \end{aligned} \tag{16}$$

d) Cascade parameters

The cascade parameters are defined by the following equations:

$$A = \left.\frac{E_1}{V_2}\right|_{R_2 \to \infty} \tag{17}$$

$$B = \left.\frac{E_1}{-I_2}\right|_{R_2 \to 0} \tag{18}$$

$$C = \left.\frac{I_1}{V_2}\right|_{R_2 \to \infty} \tag{19}$$

$$D = \left.\frac{I_1}{-I_2}\right|_{R_2 \to 0} . \tag{20}$$

Comparison with Eqs. (8) - (16) gives

$$A = \frac{Z_{11}}{Z_{12}} = -\frac{Y_{22}}{Y_{12}} \tag{21}$$

$$B = \frac{[Z]}{Z_{12}} = -\frac{1}{Y_{12}} \tag{22}$$

$$C = \frac{1}{Z_{12}} = -\frac{[Y]}{Y_{12}} \tag{23}$$

$$D = \frac{Z_{22}}{Z_{12}} = -\frac{Y_{11}}{Y_{12}} \tag{24}$$

where $[X] = X_{11}X_{22} - X_{12}^2$,

and $AD - BC = 1$. (25)

e) <u>Image parameters</u>

The image impedances are defined by an impedance-matching condition between the terminations and the driving-point impedances of the four-pole (Fig. 2):

$$Z_{i_1} = Z_{D_1}, \qquad Z_{i_2} = Z_{D_2}. \tag{26}$$

Using Eqs. (8) - (25), we get

$$Z_{i_1}^2 = \frac{AB}{CD} = \frac{Z_{11}}{Y_{11}}, \qquad Z_{i_2}^2 = \frac{BD}{AC} = \frac{Z_{22}}{Y_{22}}. \tag{27}$$

The image propagation function Θ is given by the power transfer of the image-matched network

$$e^{2\Theta} = \frac{E_1 I_1}{-E_2 I_2} = \left(\sqrt{AD} + \sqrt{BC}\right)^2 = \frac{\sqrt{Y_{11}Z_{11}} + 1}{\sqrt{Y_{11}Z_{11}} - 1}. \tag{28}$$

From Eqs. (26) - (28),

$$\cosh\Theta = \sqrt{AD} = \frac{\sqrt{Z_{11}Z_{22}}}{Z_{12}} \tag{29}$$

and

$$\sinh\Theta = \sqrt{BC} = \frac{\sqrt{[Z]}}{Z_{12}} \tag{30}$$

are obtained.

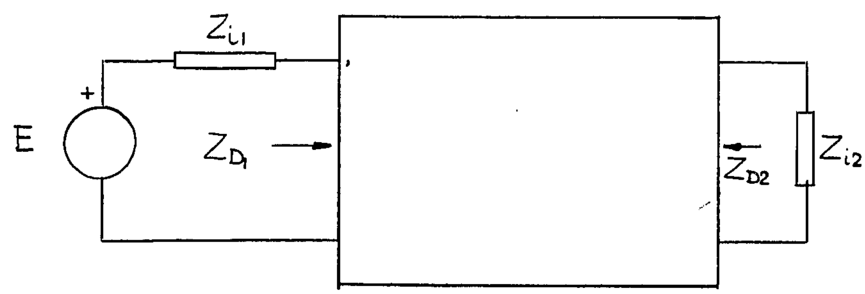


FIGURE 2.

f) Lattice impedances

A symmetrical four-pole is a circuit on which an interchange of input and output terminals could not be detected electrically. This means

$$Z_{11} = Z_{22}, \qquad A = D,$$
$$Z_{i_1} = Z_{i_2}, \qquad Y_{11} = Y_{22}. \tag{31}$$

A symmetrical network can be characterized by two, rather than three parameters. These may be chosen as the impedance - values of its equivalent lattice - circuit (Fig. 3). From Eqs. (8) - (11), it is found that

$$Z_A = Z_{11} - Z_{12}$$
$$Z_B = Z_{11} + Z_{12}. \tag{32}$$

Somewhat analogous to the symmetrical networks are the anti-metric or inverse impedance four-poles. Here

$$\frac{Z_{11}}{Y_{22}} = \frac{Z_{22}}{Y_{11}} = Z_{i_1} Z_{i_2} = R_o^2 = \text{const.} \tag{33}$$

From Eqs. (21) and (22),

$$[Z] = \frac{Z_{11}}{Y_{22}} = R_o^2. \tag{33/a}$$

Consequently, for a symmetrical network (cf Eqs. (29) and (30))

$$\cosh\Theta = \frac{Z_{11}}{Z_{12}}. \tag{29/a}$$

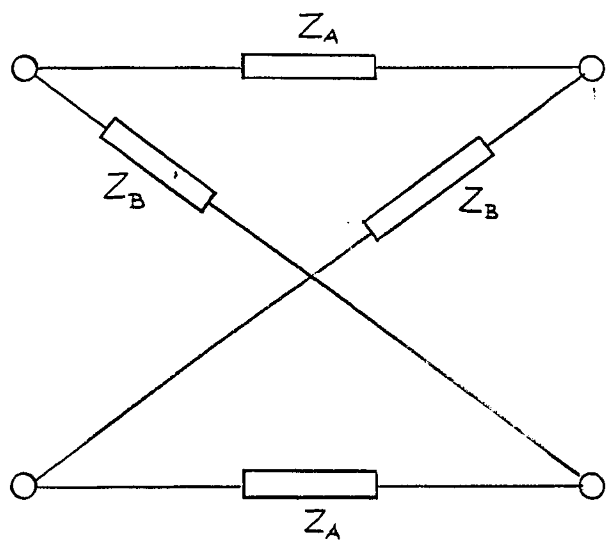


FIGURE 3.

and for an antimetric network

$$\sinh \Theta = \frac{R_0}{Z_{12}} . \qquad (30/a)$$

g) Operating variables

The voltage and current transfer ratios of a terminated four-pole are (Fig. 4)

$$N = \frac{E_1}{E_2} \qquad (34)$$

and

$$M = \frac{I_1}{-I_2} . \qquad (35)$$

The operating factor S is defined by

$$S^2 = \frac{P_{gen}}{-E_2 I_2} \qquad (36)$$

where

$$P_{gen} = \frac{E^2}{4R_1} \qquad (37)$$

is the maximum power obtainable from the generator.

The operating transmission constant Γ_0 is closely related to S

$$\Gamma_0 = \log S = \frac{1}{2} \log \frac{P_{gen}}{-E_2 I_2} = A_0 + jB_0 \qquad (38)$$

A_0 is the operating loss, B_0 the operating phase.

Obviously, for real frequencies

$$A_0 = \frac{1}{2} \log \frac{P_{gen}}{P_2} \geq 0 \qquad (39)$$

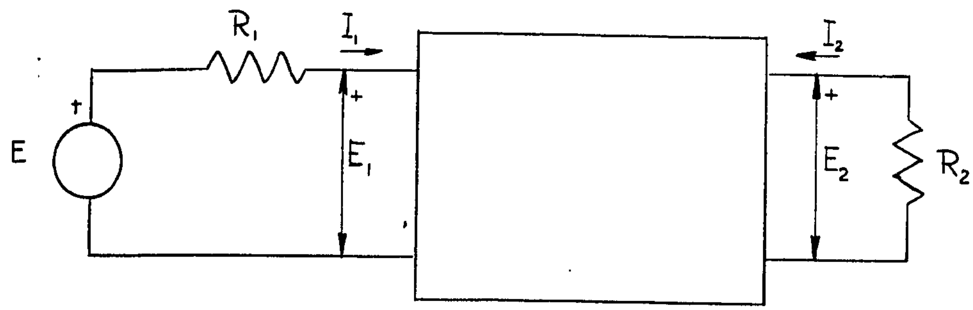


FIGURE 4.

It is customary to extend the concept of reflection loss and coefficient from transmission lines to lumped-element circuits. If the effect of impedance - mismatch between terminations and driving-point impedances is represented by an additional generator counteracting the (matched) source (Fig. 5), it is easy to obtain

$$-E_r = \frac{R_1 - Z_{D_1}}{R_1 + Z_D} E = \varrho_1 E$$

and

$$\varrho_1 = \frac{R_1 - Z_{D_1}}{R_1 + Z_{D_1}} \qquad (40)$$

$$\varrho_2 = \frac{R_2 - Z_{D_2}}{R_2 + Z_{D_2}} ,$$

the reflection factors.

Since

$$|\varrho|^2 = \left|\frac{E_r}{E}\right|^2 = \frac{P_r}{P_{gen}}$$

where P_r is the reflected power, for a loss-free four-pole

$$|\varrho_1|^2 = |\varrho_2|^2 \qquad (41)$$

and

$$\frac{P_2}{P_{gen}} + \frac{P_r}{P_{gen}} = 1, \qquad (42)$$

or

$$|S|^{-2} + |\varrho|^2 = 1. \qquad (42/a)$$

The real part of $\lg \frac{1}{\varrho}$ is usually called the reflection loss

$$A_r = -\lg|\varrho|.$$

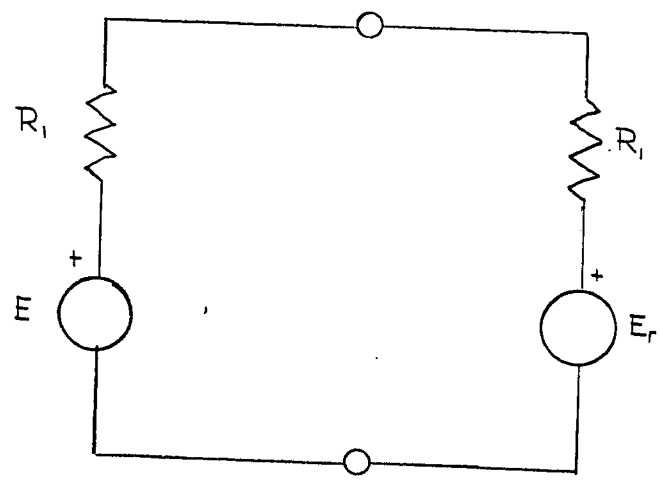


FIGURE 5.

h) Insertion quantities.

An alternative way of specifying the transmission properties of the terminated network is through the insertion quantities.

The insertion voltage ratio Λ is defined by

$$\Lambda = \frac{E_{20}}{E_2} \tag{43}$$

where

$$E_{20} = E\,\frac{R_2}{R_1 + R_2} \tag{44}$$

is the voltage across the load resistor if it is connected directly to the generator. The insertion power ratio, the insertion loss A_i and phase B_i are given by

$$\begin{aligned} \Gamma_i &= A_i + jB_i = \log \frac{\frac{|E_{20}|^2}{R_2}}{-E_2 I_2} = \log \frac{P_{20}}{E_2 I_2} \\ A_i &= \log \frac{P_{20}}{P_2} = \log \left| \frac{E_{20}}{E_2} \right|^2 \end{aligned} \tag{45}$$

Comparison with Eqs. (36) and (39) gives

$$\Lambda = \frac{2\sqrt{R_1 R_2}}{R_1 + R_2}\, S = \sqrt{\eta_G}\, S \tag{46}$$

and

$$A_i = A_0 + \log \eta_G \tag{47}$$

where

$$\eta_G = \frac{4}{\left(\sqrt{\frac{R_1}{R_2}} + \sqrt{\frac{R_2}{R_1}}\right)^2} = \frac{P_{20}}{P_{gen}} \leq 1. \tag{48}$$

is the generator efficiency.

From Eqs. (39) and (47), for real frequencies

$$A_i \geq \lg \eta_G \tag{49}$$

or

$$|\Lambda|^2 \geq \eta_G. \tag{50}$$

I.2. RELATIONS BETWEEN THE IMMITTANCE PARAMETERS AND THE OPERATING AND INSERTION QUANTITIES[9,41]

Since most communication networks are realized in the practically adventageous ladder-configuration, which can best be synthesised from the immittance - parameters (Z_{ij}, Y_{ij}) and since on the other hand the specification of these networks is normally given in terms of the operating or insertion quantities, it is most important to establish relations between these variables. This will be done in this section.

a) Connection between the operating variables and the impedance parameters.[41]

From the definition of the driving-point impedance and Eqs. (12) of Sec. I.1., it follows that

$$Z_{D_1} = \frac{E_1}{I_1} = Z_{11} - \frac{Z_{12}^2}{Z_{22}+R_2} = \frac{[Z] + Z_{11} R_2}{Z_{22} + R_2} . \qquad (1)$$

Also from Eq. (12)

$$-I_2 = \frac{E Z_{12}}{(R_1 + Z_{D1})(R_2 + Z_{22})} = \frac{E Z_{12}}{(R_1 + Z_{11})(R_2 + Z_{22}) - Z_{12}^2} \qquad (2)$$

results.

From Eqs. (1) and (2)

$$M = \frac{E}{(Z_{D_1} + R_1)(-I_2)} = \frac{Z_{22} + R_2}{Z_{12}} , \qquad (3)$$

$$N = \frac{I_1 Z_{D1}}{-I_2 R_2} = \frac{[Z] + Z_{11} R_2}{Z_{12} R_2} = -\frac{Y_{22} + R_2^{-1}}{Y_{12}} \qquad (4)$$

and

$$S = \left(\frac{E^2}{4R_1} \times \frac{1}{I_2^2 \, R_2} \right)^{1/2} = \frac{(R_1 + Z_{11})(R_2 + Z_{22}) - Z_{12}^2}{2 \sqrt{R_1 R_2} \; Z_{12}} \tag{5}$$

could be calculated.

The operating loss can be expressed as

$$A_o = \lg |S| = \lg \left| \frac{(R_1 + Z_{11})(R_2 + Z_{22}) - Z_{12}^2}{2 \sqrt{R_1 R_2} \; Z_{12}} \right| \tag{6}$$

and the reflection factor as

$$\varrho_1 = \frac{R_1 - Z_{D1}}{R_1 + Z_{D1}} = \frac{(R_1 - Z_{11})(R_2 + Z_{22}) + Z_{12}^2}{(R_1 + Z_{11})(R_2 + Z_{22}) - Z_{12}^2} \, . \tag{7}$$

b) Relations between the insertion and impedance parameters.[4]

Since the insertion and operating variables are connected through Eqs. (46) - (48) of Sec. I.l., we get

$$\Lambda = \frac{E_{20}}{E_2} = \sqrt{\eta_G} \; S = \frac{(R_1 + Z_{11})(R_2 + Z_{22}) - Z_{12}^2}{(R_1 + R_2) Z_{12}} \tag{8}$$

and

$$A_i = A_o + \log \eta_G = \log \left| \frac{(R_1 + Z_{11})(R_2 + Z_{22}) - Z_{12}^2}{(R_1 + R_2) Z_{12}} \right| . \tag{9}$$

c) The calculation of the immittance parameters from a prescribed current and/or voltage ratio.[4]

As Eqs. (8) - (16) of Sec. I.l., show, all immittance parameters can be expressed as

$$X_{ij} = \frac{\Delta_{ji}}{\Delta_{11\,22}} \, . \tag{10}$$

Here the determinant - elements are

$$x_{ij} = a_{ij}p + b_{ij}p^{-1} \tag{11}$$

since the (unterminated) network is loss-less.

If follows, that all

$$X_{ij} = \frac{f(p^2)}{p} \tag{12}$$

are odd functions of the complex frequency p,

$$X_{ij}(-p) = -X_{ij}(p). \tag{13}$$

This property will be utilized repeatedly in the derivations that follow.

For a prescribed M, using Eq. (13).[2]

$$M(p) = \frac{Z_{22} + R_2}{Z_{12}}$$

$$M(-p) = \frac{-Z_{22} + R_2}{-Z_{12}}$$

so that

$$Z_{12} = \frac{2R_2}{M(p) - M(-p)} = \frac{R_2}{M_o}, \tag{14}$$

where M_o is the odd part of $M(p)$. Also

$$Z_{22} = \frac{Z_{12}}{2}\left[M(p) + M(-p)\right] = R_2 \frac{M_e}{M_o} \tag{15}$$

where M_e is the even part of $M(p)$.

A dual derivation gives for a prescribed N

$$Y_{12} = -\frac{R_2^{-1}}{N_o} \tag{16}$$

and

$$Y_{22} = R_2^{-1} \frac{N_e}{N_o}. \tag{17}$$

M and N may be prescribed simultaneously. In that case, Eqs. (14) - (17) of this section and Eqs. (21) - (25) of Section I.1. can be used to determine the immittance - parameter matrices.

$$\|Z\| = \frac{R_2}{M_o}\begin{pmatrix} N_e & 1 \\ 1 & M_e \end{pmatrix} \tag{18}$$

$$\|Y\| = \frac{G_2}{N_o}\begin{pmatrix} M_e & -1 \\ -1 & N_e \end{pmatrix} \tag{19}$$

Here
$$G_2 = \frac{1}{R_2}. \tag{20}$$

The driving-point impedance

$$Z_{D_1} = \frac{E_1}{I_1} = \frac{E_1}{E_2} \times \frac{(-I_2 R_2)}{I_1} = \frac{N}{M} R_2 \tag{21}$$

can also be calculated.

d) <u>Calculation from a prescribed operating factor and characteristic function.</u>

At this point it is convenient to define the operating characteristic function

$$\varphi = \rho_1 S \tag{22}$$

From Eqs. (42) and (42/a) of Sec. I.1.

$$|\varphi|^2 = \frac{P_r}{P_2} \tag{23}$$

and

$$|S|^2 = 1 + |\varphi|^2 \tag{24}$$

on the $j\omega$-axis . If now an S and φ pair satisfying Eq. (24) is given, the immittance - parameters can be obtained.

From Eqs. (22), (5) and (7)

$$\varphi = \frac{(R_1 - Z_{11})(R_2 + Z_{22}) + Z_{12}^2}{2\sqrt{R_1 R_2}\, Z_{12}} \tag{25}$$

and by Eq. (12)

$$S_e = \frac{R_2 Z_{11} + R_1 Z_{22}}{2\sqrt{R_1 R_2}\, Z_{12}} \tag{26}$$

$$S_o = \frac{R_1 R_2 + [Z]}{2\sqrt{R_1 R_2}\, Z_{12}}$$

also

$$\varphi_e = \frac{-R_2 Z_{11} + R_1 Z_{22}}{2\sqrt{R_1 R_2}\, Z_{12}}$$

$$\varphi_o = \frac{R_1 R_2 - [Z]}{2\sqrt{R_1 R_2}\, Z_{12}} \quad . \tag{27}$$

Using Eq. (13) a derivation similar to that of Para. I.2.c. gives

$$Z_{11} = R_1 \frac{S_e - \varphi_e}{S_o + \varphi_o}$$

$$Z_{12} = \frac{\sqrt{R_1 R_2}}{S_o + \varphi_o} \tag{28}$$

$$Z_{22} = R_2 \frac{S_e + \varphi_e}{S_o + \varphi_o}$$

and an exactly dual procedure

$$Y_{11} = G_1 \frac{S_e + \varphi_e}{S_o - \varphi_o}$$

$$Y_{12} = \frac{\sqrt{G_1 G_2}}{S_o - \varphi_o} \tag{29}$$

$$Y_{22} = G_2 \frac{S_e - \varphi_e}{S_o - \varphi_o} ,$$

where

$$G_i = R_i^{-1} \quad . \tag{30}$$

e) Calculation from prescribed insertion voltage ratio and characteristic function.[1,41]

If the insertion characteristic function Φ is defined by the equation

$$\Phi = \sqrt{\eta_G}\,\varphi \tag{31}$$

we get

$$|\Phi|^2 = \eta_G |\varphi|^2 = \frac{P_{20}}{P_{gen}} \frac{P_r}{P_2} \tag{32}$$

and from Eqs. (8), (24) and (31') $\quad |\Lambda|^2 = \eta_G + |\Phi|^2 \quad$ (33)

is obtained. Any prescribed Λ, Φ pair must satisfy Eq. (33). A multiplicity of acceptable Φ functions is possible for a given Λ.

The Z and Y parameters can be found as in Para. I.2.d.

$$\begin{aligned}
Z_{11} &= R_1 \frac{\Lambda_e - \Phi_e}{\Lambda_o + \Phi_o} \\
Z_{12} &= \frac{2R_1R_2}{R_1 + R_2} \frac{1}{\Lambda_o + \Phi_o} \\
Z_{22} &= R_2 \frac{\Lambda_e + \Phi_e}{\Lambda_o + \Phi_o}
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
Y_{11} &= G_1 \frac{\Lambda_e + \Phi_e}{\Lambda_o - \Phi_o} \\
Y_{12} &= \frac{2G_1G_2}{G_1 + G_2} \frac{1}{\Lambda_o - \Phi_o} \\
Y_{22} &= G_2 \frac{\Lambda_e - \Phi_e}{\Lambda_o - \Phi_o}
\end{aligned} \tag{35}$$

I.3. REALIZABILITY CONDITIONS [9,25,41]

The physical specifications of the networks discussed in the present work include the passivity of all elements and the purely reactive character of the four-pole components. From these requirements a number of conditions can be set against the parameters describing the circuit. As elsewhere in this thesis, the physical meaning behind the conditions - rather than their mathematical derivation-- will be emphasized. The conditions given will not be mathematically independent, some will be derivable from others. Nevertheless, they will all have important roles in later analysis.

a) Immittance Parameters [2,3,9]

It is convenient to treat the driving point (X_{ii}) and transfer (X_{12}) immittances separately. For X_{ii} the following qualifications can be made:

1. They must be odd rational functions of p, with real coefficients, the degrees of numerator and denominator must differ by 1. (Cf Eqs. (8) - (16) of Sec. I.1. and Eq. (12) of Sec. I.2.)
2. All zeros and poles must be located (interlaced) on the $j\omega$ - axis. They must be simple and the residues must be positive real.

This condition expresses the physical fact, that the networks obtained by open - or short-circuiting the four-pole terminals are passive and loss-less, so that the only possible transient is a un-damped sine-wave. Using Eqs. (8) - (16) of Sec. I.1., the above conclusions are easily derived.

For X_{12} one obtains

3. It must be an odd rational function of p with real coefficients.

4. The denominator must have simple roots on the $j\omega$ - axis.

5. The degree of the numerator cannot be more than one degree higher than that of the denominator.

The difference in conditions, in particular the more lenient requirements against Z_{12} can be traced back to the fact, that its numerator is derived from Δ_{21}, which (unlike Δ_{11}, Δ_{22} or Δ_{1122}), is not the circuit - determinant of any physical circuit and hence transient - considerations do not apply to it.

Two more conditions link all X_{ij} parameters together.

6. At any pole of any X_{ij}, the residues (h_{ij}) must satisfy the condition.

$$\begin{aligned} h_{11} &\geq 0 \\ h_{22} &\geq 0 \\ h_{11}h_{22} - h_{12}^2 &\geq 0 \end{aligned} \tag{1}$$

This condition can be derived e.g. for Z-parameters by considering the (pure reactive) driving-point impedance of the circuit of Fig. 6. From Eqs. (12) of Sec. I.1., this turns out to be

$$Z = \frac{E}{I} = n_2^2\left(x^2 Z_{11} + 2x Z_{12} + Z_{22}\right) \tag{2}$$

where

$$x = \frac{n_1}{n_2}$$

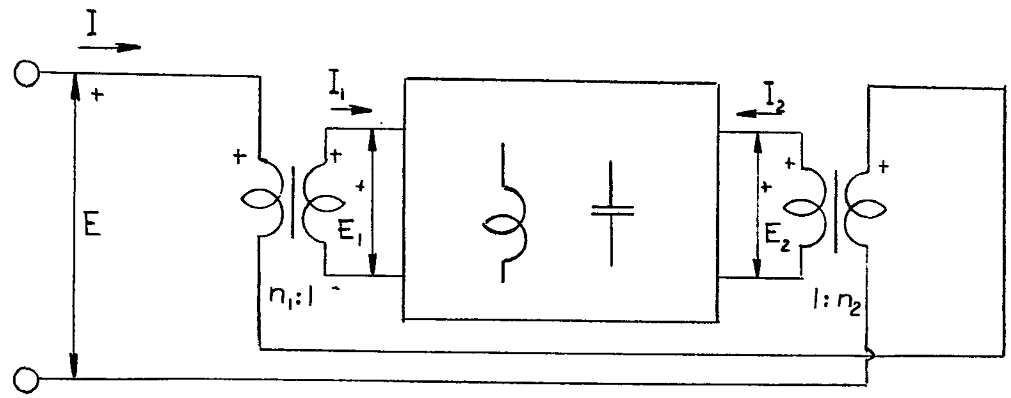


FIGURE 6.

At any ($j\omega$ - axis) pole of Z, its residue must be positive for any x, so that the discriminant

$$4\left(h_{12}^2 - h_{11}h_{22}\right) \leq 0$$

and Eq. (1) follows.

7. From Eqs. (21) - (25) of Sec. I.1 it is easy to derive

$$[Z] = \frac{Z_{11}}{Y_{22}} = \frac{Z_{22}}{Y_{11}} = -\frac{Z_{12}}{Y_{12}}. \tag{3}$$

b) <u>Realizability conditions on cascade parameters</u> [31]

1. $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$ (cf. Eq. (25) of Sec. I.1.)

2. Since (Eqs. (21) - (25) of Sec. I.1.)

$$\begin{aligned} Z_{11}^{-1} &= \frac{C}{A} \\ Z_{22}^{-1} &= \frac{C}{D} \\ Y_{11}^{-1} &= \frac{B}{D} \\ Y_{22}^{-1} &= \frac{B}{A} \end{aligned} \tag{4}$$

all four ratios should satisfy Conditions 1, and 2, of Para. I. 3.a.

3. A and D must be even, B and C odd rational functions, with real coefficients. This follows directly from Eqs. (21) - (25) of Sec. I.1. and Eq. (12) of Sec. I.2.

c) Restraints on voltage and current ratios

Since the voltage and current ratios N and M are dual quantities (see e.g. the equations of Para. I.2.c.), it is sufficient to treat one of them, e.g. M.

The following conditions hold for M:

1. It is a real rational function of p

$$M = \frac{g_m}{f_m} . \tag{5}$$

2. f_m is an even or odd polynomial
3. g_m is a (strictly) Hurwitz polynomial, i.e. all its roots have negative real parts
4. The order of g_m is equal to or higher than the order of f_m .

The first 3 conditions follow from Eq. (3) of Sec. I.2. Defining polynomials p_{ij} and q_{ij} by

$$Z_{ij} = \frac{p_{ij}}{q_{ij}} ,$$

we get

$$M = \frac{p_{22} + R_2 q_{22}}{q_{22} p_{12}/q_{12}} .$$

Now since $p_{22} + R_2 q_{22}$ is the numerator of a physical impedance $(Z_{22} + R_2)$, it is proportional to a physical circuit determinant, hence Condition 3, follows. By Eqs. (1), q_{22} contains all zeros of q_{12} , so $\frac{q_{22}\, p_{12}}{q_{12}}$ is a polynomial (f_m) and according to Condition 1, in Para. I.3.a., it is even or odd.

Finally, since everywhere on the $j\omega$- axis (including $\omega \to \infty$)

$$|M| = \left|\frac{I_1}{I_2}\right| > 0$$

Condition 4, is immediate.

By a dual derivation, identical rules hold for the voltage ratio.

If M and N are prescribed simultaneously, two more restraints apply:

5. The equation

$$\frac{1}{2}\left[M(-p)N(p)+M(p)N(-p)\right]=1 \tag{6}$$

holds for all real frequencies.

6. The roots and poles of $\frac{N}{M}$ must lie in the left half of the complex frequency plane.

Condition 5, can be derived from the fact, that for a loss-less four-pole, the power entering the circuit must be dissipated in the load. Therefore

$$Re\left(E_1\bar{I}_1\right) = Re\left(E_2\bar{I}_2\right) \tag{6/a}$$

and using Eq. (21) of Sec. I.2., Condition 5, follows.

Condition 6, can be seen by considering Eq. (21) of Sec. I.2. and realising that the roots and poles of $\frac{N}{M}$ are natural frequencies of the four-pole terminated in R_2 on the secondary side and left open or short-circuited on the primary terminals.

d) <u>The realisability of operating parameters</u>[2,36,25]

From Eqs. (3) - (5) of Sec. I.2.

$$S=\frac{1}{2}\left[\sqrt{\frac{R_1}{R_2}}\,M+\sqrt{\frac{R_2}{R_1}}\,N\right]=\frac{1}{2\sqrt{R_1R_2}}\left[R_1M+R_2N\right] \tag{7}$$

and from Eqs. (7) and (21) of Sec. I.2.

$$\rho_1=\frac{R_1M-R_2N}{R_1M+R_2N} \tag{8}$$

also

$$\varphi=\rho_1S=\frac{R_1M-R_2N}{2\sqrt{R_1R_2}} \tag{9}$$

Evidently we may define polynomials f, g and h by *

$$S = \frac{g}{f}$$
$$\varphi = \frac{h}{f} \qquad (10)$$
$$\rho_1 = \frac{h}{g}.$$

From these equations and the realizability conditions on M and N, the following restraints follow on S

1. f is even or odd (it is the product of f_m and f_n).
2. $|S|^2 = \left|\frac{P_{gen}}{P_2}\right| \geq 1$ for $p = j\omega$. Therefore, the order of g must be at least equal to the order of f.
3. Since $S = 0$ represents the natural frequencies of the terminated four-pole, g must have strict Hurwitz character (cf. Condition 2.).

For ρ_1:

1. On the $j\omega$-axis
$$|\rho_1|^2 = \frac{|\varphi|^2}{1 + |\varphi|^2} \leq 1$$
so h must have at most the order of g.
2. g must be strictly Hurwitz (see above).

For φ:

1. f is even or odd (see above).

f, g and h are connected through Eq. (24) of Sec. I.2.:

$$|g|^2 = |f|^2 + |h|^2 \qquad (11)$$

for

$p = j\omega$.

e) Realizability conditions on insertion parameters [1,4]

By Eqs. (8) and (31) of Sec. I.2., the realizability-

* h must not be confused with h_{ij} the immittance-residues

conditions on Λ and Φ are identical to those demanded for S and φ, except that the lowest permissible value of $|\Lambda|^2$ on the $j\omega$-axis is given by

$$|\Lambda|^2 \geqslant \eta_G - \frac{4R_1R_2}{(R_1+R_2)^2} \qquad (12)$$

for

$p = j\omega$.

From the Conditions of Para. I.3.d. and this Paragraph, it is obvious, that of the polynomials contained in S and φ (or Λ and Φ) g (or the numerator of Λ) is of the highest degree. This degree n is usually called the order of the network. Its parity plays a very important role in the synthesis procedure, as e.g. Para. I.4.e. will show.

I.4. THE DERIVATION OF RELIZABLE IMMITTANCE PARAMETERS FOR UNTERMINATED, SINGLE-AND DOUBLE-TERMINATED FOUR-POLES[36,41]

The relations derived in Section I.2. and the necessary realizability conditions stated in Section I.3. can now be used to deduce the immittance-parameters of circuits satisfying a wide assortment of termination and attenuation specifications. In the following Section this derivation will be performed for networks in the order of increasing complexity. Throughout this Section it will be assumed that the prescribed parameters satisfy the conditions of Sec. I.3.

a) Unloaded four pole[2]

The source is a pure voltage - or current generator, the load side is open - or short-circuited.

The four possible cases can be handled in two groups. If the generator and load impedances are equal (both zero or infinite), the specified transfer function can be X_{12}, the transfer immittance parameter.

One possible way to find the other immittance parameters X_{ii} is to break X_{12} into partial fractions and then attach X_{ii} satisfying the residue-condition of Para. I.3.a.

If the relation involving all 3 residues is satisfied with equal sign at all poles, Eq. (1) of Sec. I.2. shows that the driving-point immittance remains finite on the whole $j\omega$-axis.[2] This may be a very important requirement (e.g. for filter-combinations, see Sec. I.6.).

Another simple method is to find a lattice equivalent by using Eq. (32) of Sec. I.1. to derive

$$Z_{12} = \frac{1}{2}\left(Z_B - Z_A\right) \tag{1}$$

Assigning the partial fractions with positive residue to $\frac{Z_B}{2}$ and the ones with negative residue to $\frac{-Z_A}{2}$, a realizable pair of lattice impedances will be obtained. Then

$$Z_{11} = Z_{22} = \frac{Z_A + Z_B}{2} \tag{2}$$

will furnish the missing parameters.

If the (extreme) generator-impedance and the load are different, the specified transfer functions are $N(R_2 \rightarrow \infty)$ or $M(R_2 \rightarrow 0)$. From Eqs. (3) and (4) of Sec. I.2., these have the common form of $\pm \frac{X_{22}}{X_{12}}$.

If the numerator and denominator of the given transfer function are divided by a proper polynomial satisfying Conditions 1 - 5 of Para. I.3.a., X_{12} and X_{22} can be separated. If necessary, X_{11} can be added using the residue-condition.

b) Single-loaded four-pole [2]

The source contains a resistor and the load is extreme (0 or ∞) or the source is a voltage- (current-) generator and the load is resistive.

Again, it is sufficient to investigate two basic possibilities. If the loads are resistive and the generators extreme, the specified transfer functions are M or N and Eqs. (14) - (17) of Sec. I.2. give X_{12} and X_{22}.

Since Condition 2, of Para. I.3.c. applies, there are two possibilities:

α) f_l even (l is m or n, L is M or N, P is R or G)

$$L_e = \frac{g_{le}}{f_l}$$

so

$$L_o = \frac{g_{lo}}{f_l}$$

$$X_{12} = \pm P_2 \frac{f_l}{g_{lo}} \tag{3}$$

$$X_{22} = P_2 \frac{g_{le}}{g_{lo}}$$

β) f_l odd

$$L_e = \frac{g_{lo}}{f_l}$$

so

$$L_o = \frac{g_{le}}{f_l}$$

$$X_{12} = \pm P_2 \frac{f_l}{g_{le}}$$

$$X_{22} = P_2 \frac{g_{lo}}{g_{le}} \tag{4}$$

A proper X_{11} can be found, as shown in Para. I.4.a.

If the generator is resistive, and the load extreme, the use of Thevenin's and Norton's theorems and the reciprocity-theorem will always reduce the problem to the one just treated.

For a Thevenin-type generator and open-circuited output, the specified quantity may be the insertion voltage ratio

$$\Lambda = \frac{E_{20}}{E_2} = \frac{E}{E_2} .$$

Switching to Norton-equivalent generator and using reciprocity (Fig. 7), we find, that the circuit is equivalent to one realizing a prescribed current ratio between a current generator feeding the secondary terminals and a resistor connected to the primary side.

Thus, Eqs. (14) - (15) of Sec. I.2. and Eqs. (3) - (4) of this Section apply, if Λ is substituted for M and the suffixes 1 and 2 are interchanged.

Since the insertion voltage ratio is normally written[1] as

$$\Lambda = \frac{A + pB}{P} \tag{5}$$

where A, B and P are even polynomials and the numerator is a Hurwitz-polynomial (cf. Para. I.3.e.), the results are

$$\begin{aligned} Z_{12} &= R_1 \frac{P}{pB} \\ Z_{11} &= R_1 \frac{A}{pB} . \end{aligned} \tag{6}$$

In the dual situation (Norton-type generator, short circuited

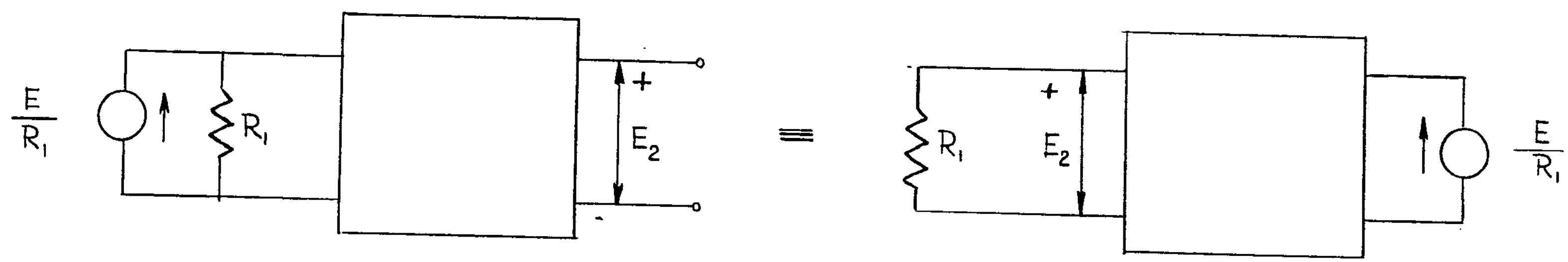


FIGURE 7.

output), an insertion current ratio Π (equal to the insertion voltage ratio, when both exist) may be defined and specified. Then an exactly dual procedure leads to

$$\Pi = \frac{I_{20}}{I_2} = \frac{C + pD}{R} \tag{7}$$

and

$$Y_{12} = G_1 \frac{R}{pD} \tag{8}$$
$$Y_{11} = G_1 \frac{C}{pD} .$$

In certain cases the quantity specified for a single-loaded network is not M or N, but rather $|M|^2$ or $|N|^2$. Then, according to Para. I.3.c., the denominator must be the square of an even or odd polynomial f_m (f_n). If this is not the case, the numerator and demominator must be multiplied by identical factors, to achieve the necessary condition. The square-root of the new denominator (with either sign) will be chosen as f_m (f_n).

Since, also by Para. I.3.c., the numerator of M or N must be a Hurwitz-polynomial, the (enlarged) numerator of $|M|^2$ ($|N|^2$) must be split into a Hurwitz and a non-Hurwitz part. This can be performed[2] by factorizing the enlarged numerator and assigning the factors corresponding to left half-plane roots to g_m (p) (or g_n (p)), the factors corresponding to right half-plane roots to g_m (-p) or g_n (-p). Since on the $j\omega$ - axis

$$g_m(p)\, g_m(-p) = \left| g_m(p) \right|^2 \tag{9}$$

and

$$f_m(p)\,f_m(-p) = \pm\left[f_m(p)\right]^2 \qquad (10)$$

(upper sign for even, lower sign for odd f_m), the ratio of g_m and f_m will satisfy all realizability requirements as well as the specifications.

c, Double-terminated four-pole [1,2,30,36,41]

For a double-terminated network (fed by a generator with resistive internal impedance and terminated by a resistive load) both M and N are finite and they can be specified simultaneously, if Eq. (6) of Sec. I.3. is satisfied. Through Eqs. (7) - (9) of Sec. I.3., then, all operating parameters will be prescribed, so that either Eqs. (18) - (20) or Eqs. (28) - (30), all of Sec. I.2., can be used to derive the immittance - parameters.

If the operating factor S is prescribed, Eq. (24) of Sec. I.2., rewritten as

$$S(p)\,S(-p) = 1 + \varphi(p)\,\varphi(-p) \tag{11}$$

or, using Eq. (10) of this Section and condition 1, of Para I.3.d.,

$$g(p)\,g(-p) = \pm f^2 + h(p)\,h(-p) \tag{12}$$

can be used to find h(p). At this point it is necessary to emphasize again, that Eqs. (11) - (12) are only equivalent to Eq. (24) of Sec. I.2. for $p = j\omega$, which, however, is our range of physical interest.

The distribution of the root-factors between h(p) and h(-p) is not inherently restricted. Certain conditions have to be satisfied, however, if the configuration and/or the value of the terminating resistors is prescribed. Some preferences also exist from the point of view of inherent computational inaccuracies. Because of its importance, this question will be treated in some detail in Para. I.4.e.

After an appropriate h has been found, Eqs. (28) - (30) of Sec. 1.2. can be used to derive the immittance - parameters. For even f:

$$\|Z\| = \frac{1}{g_o + h_o} \begin{pmatrix} R_1(g_e - h_e) & \sqrt{R_1R_2}\, f \\ \sqrt{R_1R_2}\, f & R_2(g_e + h_e) \end{pmatrix} \tag{13}$$

and

$$\|Y\| = \frac{1}{g_o - h_o} \begin{pmatrix} G_1(g_e + h_e) & \sqrt{G_1G_2}\, f \\ \sqrt{G_1G_2}\, f & G_2(g_e - h_e) \end{pmatrix}. \tag{14}$$

For odd f:

$$\|Z\| = \frac{1}{g_e + h_e} \begin{pmatrix} R_1(g_o - h_o) & \sqrt{R_1R_2}\, f \\ \sqrt{R_1R_2}\, f & R_2(g_o + h_o) \end{pmatrix} \tag{15}$$

and

$$\|Y\| = \frac{1}{g_e - h_e} \begin{pmatrix} G_1(g_o + h_o) & \sqrt{G_1G_2}\, f \\ \sqrt{G_1G_2}\, f & G_2(g_o - h_o) \end{pmatrix}. \tag{16}$$

If the insertion voltage ratio Λ is specified, the insertion characteristic function Φ should be calculated from Eq. (33) of Sec. I.2. Using Eq. (5) of this Section and writing

$$\Phi = \frac{A' + pB'}{P} \tag{17}$$

we get

$$A^2 - p^2B^2 = \eta_G P^2 + A'^2 - pB'^2 \qquad (18)$$

from which A' + pB' can be calculated (see the remarks made above on the calculation of h).

The immittance matrices obtained afterwards are identical to those of Eqs. (13) and (14) of this section, if the

$$\begin{aligned} g &= A + pB \\ h &= A' + pB' \\ f &= \sqrt{\eta_G}\, P \end{aligned} \qquad (19)$$

identifications are made.

If not S or Λ, but $\frac{P_{gen}}{P_2}$ or $\frac{P_{2o}}{P_2}$ i.e. $|S|^2$ or $|\Lambda|^2$ is specified, the procedure described in the previous paragraph can be used derive S or Λ.

Often the specifications are not given in an analytic form, but rather in the form of a graph, showing P_2 (ω). In that case it is normally adventageous[36] to derive a tolerance - scheme for $|\varphi|$ using

$$|\varphi| = \left[\frac{P_{gen}}{P_2} - 1\right]^{1/2} \qquad (20)$$

and to try to find a realisable φ satisfying the requirements.

One advantage of solving the approximation problem for φ rather than S is that the realisability conditions on φ are very lenient (cf. Para. I.3.d.) and do not represent a serious restriction on the choice of φ. In addition, it is economical to place the zeros of φ on the $j\omega$-axis (in the pass-bands, to restrict the attenuation), similarly for the poles (to boost the attenuation in the stop-bands). As will be shown in Sec. I.5., the latter restriction is also necessary to ensure the possibility of a ladder realization without mutual inductance. These constraints make φ an easily manageable function.

Finally, in the pass-bands, φ approaches zero, which represents an easier approximation problem, than $|S| \simeq 1$.

Identical remarks apply, of course, to Λ and Φ.

d, Special cases: symmetric and antimetric networks. 1,2,30,36,41

The equally terminated symmetric or antimetric networks are of high importance and could be treated simply. Comparison of Eq. (31) of Sec. I.1, and Eqs. (13) - (16) of this Section show that for symmetric circuits, if f is even (odd), h must be odd (even) and Eq. (12) of this Section simplifies to

$$g(p)\,g(-p) = \pm\left(f+h\right)\left(f-h\right) \tag{21}$$

(upper sign for even f, lower for odd f).

If we set

$$f+h = g_\alpha(p)\,g_\beta(-p) \tag{22}$$

where $g_\alpha(p)$ and $g_\beta(p)$ are Hurwitz, use Eqs. (21) and (13) - (16) of this Section and utilize the opposite parity of f and h, we get:

For f even:

$$Z_A = R\,\frac{g_{\beta o}}{g_{\beta e}} \qquad (23)$$

$$Z_B = R\,\frac{g_{\alpha e}}{g_{\alpha o}}\ .$$

For f odd:

$$Z_A = R\,\frac{g_{\alpha e}}{g_{\alpha o}} \qquad (24)$$

$$Z_B = R\,\frac{g_{\beta e}}{g_{\beta o}}\ ,$$

where

$$R = R_1 = R_2\ .$$

All remarks and results (Eq. (23)) apply to insertion loss quantities equally well.

For antimetric networks, Eqs. (33) of Sec. I.1, and (13)-(16) of this Section show, that both h and f must be even. Therefore,

$$g(p)\,g(-p) = f^2 + h^2 \qquad (25)$$

and the immittance - matrices are obtained by setting h_o equal to zero in Eqs. (13) and (14) of this Section.

e, The choice of characteristic function for prescribed configuration and terminations.[1,41]

As shown by Eqs. (24) and (33) of Sec. I.2. and by Eq. (11) of this Section, the attenuation response will not be affected by any change in φ (or Φ) that leaves

$$\varphi(p)\,\varphi(-p) = \varphi_e^2 - \varphi_o^2 \qquad (26)$$
$$(p = j\omega)$$

unchanged. Since, as mentioned in Para. I,4,c, the zeros of f (p) are normally on the $j\omega$ - axis and since the zeros of all polynomials must be on the real axis or occur in conjugate complex pairs (to keep the coefficients real), these "permissible" changes can be classified as follows:

1. A change in the sign of φ (Φ)
2. A change in the sign of the real root(s) of h ($A' + pB'$)
3. The change of the sign of any complex conjugate root-pair(s) of h.

The effect of these changes on the configuration and terminations will now be analyzed for the important special cases, where the operating factor or the insertion voltage ratio has one real root (the order of the circuit, n, is odd) or no real root (n is even). Also, [illegible] will arbitrarily be chosen even. The procedure to be followed in the general situation will then be obvious.

From Eqs. (1), (7), (24), and (28) of Sec. I.2.,

$$Z_{D_1} = R_1 \frac{g - h}{g + h} = R_1 \frac{A - A' + p(B - B')}{A + A' + p(B + B')} \qquad (27)$$

so

$$Z_{D2} = R_2 \frac{g_e + h_e + g_o - h_o}{g_e - h_e + g_o + h_o} = R_2 \frac{A + A' - p(B - B')}{A - A' + p(B + B')},$$

$$\rho_1 = \frac{R_1 - Z_{D_1}}{R_1 + Z_{D_1}} = \frac{h}{g} = \frac{A' + pB'}{A + pB} \qquad (28)$$

$$\rho_2 = \frac{R_2 - Z_{D2}}{R_2 + Z_{D2}} = \frac{-h_e + h_o}{g} = \frac{-A' + pB'}{A + pB}.$$

At extreme frequencies (0 or ∞) located in the stop-band, the driving-point impedances degenerate into a short- or open-circuit. At extreme frequencies in the pass-band the reflection factors can only have the values $\pm\rho_o$, where

$$\rho_o = \frac{R_1 - R_2}{R_1 + R_2}, \qquad (29)$$

if there is no mutual coupling.

Thus, depending on the configuration and terminations, the reflection factors will have the extreme-frequency values ± 1 in the stop-band and $\pm \rho_o$ in the pass-band. The attenuation will be the same for any sign-combination.

Now for an odd order n and one real root,

$$g = A + pB = (p + \alpha_o)\prod_{(i)}\left[p^2 + 2p\alpha_i + |p_i|^2\right] \qquad (30)$$

where all

$$\alpha_i = -\operatorname{Re}(p_i) > 0 \qquad \left(i = 0, 1, 2, \dots \tfrac{n}{2} - 1\right) \qquad (31)$$

also

$$h = A' + pB' = \pm(p + \alpha_o')\prod_{(j)}\left[p^2 + 2p\alpha_j' + |p_j'|^2\right] \qquad (32)$$

where again

$$\alpha_j = -\operatorname{Re}(p_j), \qquad \left(j = 0, 1, 2, \dots \tfrac{n}{2} - 1\right) \qquad (33)$$

but

$$\alpha_j \lesseqgtr 0 .$$

It is expedient to write

$$h = A' + pB' = (xp + y|\alpha_o'|)\prod_{(j)}\left[p^2 + 2p\alpha_j' + |p_j'|^2\right] \qquad (32/a)$$

with

$$x, y = \pm 1 ,$$

then

$$\left[\rho_1\right]_{p=0} = -\left[\rho_2\right]_{p=0} = y\left|\frac{\alpha_o'\prod_{(j)} p_j'^2}{\alpha_o \prod_{(i)} p_i^2}\right| \qquad (34)$$

and

$$\left[\rho_1\right]_{p\to\infty} = \left[\rho_2\right]_{p\to\infty} = x. \qquad (35)$$

It has been tacitly assumed here, that no degeneracy is present, i.e.

$$A,\ A' \neq 0.$$

Under these circumstances the following formulae will sum up the situation:

$$\underline{x = +1}$$

$$\begin{aligned} &[\varrho_1]_{p\to\infty} = [\varrho_2]_{p\to\infty} = 1 \\ &[Z_{D_1}]_{p\to\infty} = [Z_{D_2}]_{p\to\infty} = 0 \end{aligned} \tag{36}$$

$$\underline{x = -1}$$

$$\begin{aligned} &[\varrho_1]_{p\to\infty} = [\varrho_2] = -1 \\ &[Z_{D_1}]_{p\to\infty} \to [Z_{D_2}]_{p\to\infty} \to \infty \end{aligned} \tag{37}$$

$$\underline{y = +1}$$

$$[\varrho_1]_{p=0} = [-\varrho_2]_{p=0} > 0 \tag{38}$$

$$\underline{y = -1}$$

$$[\varrho_1]_{p=0} = [-\varrho_2]_{p=0} < 0 \ . \tag{39}$$

E.g. for a low-pass filter

$$\underline{y = +1}$$

$$\begin{aligned} &[\varrho_1]_{p=0} = \varrho_0 > 0 \\ &R_1 > R_2 \end{aligned} \tag{40}$$

$$\underline{y = -1}$$

$$\begin{aligned} &\varrho_0 < 0 \\ &R_1 < R_2 \ . \end{aligned} \tag{41}$$

For a desired configuration and for prescribed terminations evidently one has to choose a proper combination of x and y.

For an even n,

$$A' + pB' = z \prod_{(j)} \left[p^2 + 2p\alpha_j' + |p_j'|^2 \right] \tag{42}$$

where

$$z = \pm 1.$$

If

$$\underline{z = +1}$$

$$\left[\rho_1\right]_{p=0} = \left[-\rho_2\right]_{p=0} = z \left| \frac{\prod p_j'^2}{\prod_{(i)} p_i^2} \right| > 0 \tag{43}$$

and

$$\left[\rho_1\right]_{p\to\infty} = \left[\rho_2\right]_{p\to\infty} = 1$$

$$\left[Z_{D_1}\right]_{p\to\infty} = \left[Z_{D_2}\right]_{p\to\infty} = 0 . \tag{44}$$

Similarly, if

$$\underline{z = -1}$$

$$\left[\rho_1\right]_{p=0} = \left[-\rho_2\right]_{p=0} < 0$$

and

$$\left[Z_{D_1}\right]_{p\to\infty} \to \left[Z_{D_2}\right]_{p\to\infty} \to \infty . \tag{45}$$

This is a less flexible case, configuration and terminations being tied together.

Now some special situations can be treated. If

$$A' = 0$$

with (46)

$$\alpha_0' = 0,$$

from Eq. (34) of this Section

$$R_1 = \left[Z_{D_1}\right]_{p=0}$$
$$R_2 = \left[Z_{D_2}\right]_{p=0} . \qquad (47)$$

Eq. (35) of this Section remains valid.

If on the other hand

$$B' = 0, \qquad (48)$$

Eqs. (28) and (43) show, that Eq. (47) will not hold, unless

$$\left[A'\right]_{p=0} = \left|\prod_{(j)} p_j^{\prime 2}\right| = 0, \qquad (49)$$

i.e. unless

$$A' = p^3 A_1', \qquad (50)$$

where A_1' is an even polynomial.

It may be noted here, that a positive sign for

$$g = A + pB$$

has been tacitly assumed throughout this paragraph. If, however, either of the extreme frequencies is in a pass-band, the network configuration will specify the sign of Λ or S at that frequency. E.g. for a low-pass ladder network.

$$\Lambda(0) = \frac{A(0)}{P(0)} > 0 \qquad (51)$$

must hold.

A possible argument in the choice of signs is to make the coefficients in X_{ii}, the immittance parameter used in the ladder development of the circuit (see Sec. I.5.) as large as possible to improve the accuracy of the realization. E.g. for even f and Z_{22}, choosing all roots of h on the left half-plane and using the positive sign of g and h will give the best result.

An example illustrating the usefulness of the analysis given in this paragraph will be provided in connection with the mid-series and mid-shunt low-pass filters in Para. I.5.d. At the same time, a simple method will be presented, that leads to the proper sign-combination in special cases.

I. 5. THE REALIZATION OF FOUR-POLES FROM GIVEN IMMITTANCE-PARAMETERS [5,6,13,23,31,33,41]

Section I.4. described methods, of obtaining the immittance parameters of the four-pole from the attenuation-requirements.
In the present Section, the next logical step, the synthesis of the network proper from these immittance-parameters, will be explored. The realization of circuits in the practically most important ladder-configuration will be given strong preference and a few remarks will be made on the lattice synthesis technique. Some realizability criteria of ladder networks without mutual coupling will be briefly treated.

Throughout this Section, the realization from prescribed impedance-parameters will be analyzed explicitly. Since the synthesis from the admittance-matrix follows exactly dual lines, this does not represent any restriction of generality. Similarly, normally only the development of the primary immittance will be described. Obviously, a similar treatment of X_{22} is tacitly implied.

a, Realization from prescribed X_{11} and X_{12} [3,5,33]

From Eqs. (8) and (9) of Sec. I.1., it is evident, that barring degenerate cases, the poles of X_{12} and X_{11} are identical. Exception to this rule occurs, when X_{11} possesses poles not contained in X_{12} — so-called private poles. (The inverse situation is prevented by Eq. (1) of Sec. I.3.). When no private pole is present, any realization of X_{11} will automatically provide the required poles of X_{12}. The problem is to create a network, that also exhibits the proper zeros of X_{12}. The constant multiplier of X_{12}, representing a frequency-independent loss, is usually unimportant.

If the practically preferred ladder-configuration is chosen, the zeros of X_{12} can only be created through the resonances of shunt arms or through the anti-resonances of series arms. At these frequencies, then, both the attenuation and the open-circuit immittance of the ladder-portion which includes the resonant (or antiresonant) arm must have a pole.

The purpose is therefore to shift one by one the critical frequencies of X_{11} to coincide with the zeros of X_{12}. The creation of proper ladder-arms will then simultaneously realize both sets of critical frequencies. The design of the residual network preceding the ladder-arm is considerably facilitated by the isolation of this network from the rest of the four-pole by the ladder-arm at its critical frequency.

The procedure to be followed is now clear. The arbitrary order of the zeros of X_{12} is established, then the value of X_{11} at the first zero-frequency is found. This value represents the driving-point impedance of the first residual network-fragment. From the sign and value of this impedance the fragment can be designed. Care must be taken, however, at this point, since the creation of this ladder-fragment represents an at least partial removal of a term from the Laurent series expansion of X_{11}. Therefore, it must represent a pole actually present in X_{11} and the residue removed must be smaller than the original value, so that Eq. (1) of Sec. I.3. remains satisfied. After that the new X_{11}' must have a zero and therefore $(X_{11}')^{-1}$ a pole at X_{12}'s zero-frequency and both can be realized through a ladder-arm having this as its critical frequency. The procedure can now be repeated etc.

At this point it is important to notice, that the partial removal of a pole X_{11} through a ladder-arm does not create a transmission zero. This is illustrated in Fig. 8, where Z_k, a series impedance, represents a partial pole-removal from Z_{11}. Therefore, Z_k, Z_1' and in general Z_2' will all have single poles at the anti-resonance frequency of Z_k and

$$E_2 = E_g \frac{Z_2'}{Z_1' + Z_k + Z_g} \neq 0. \qquad (1)$$

If on the other hand Z_k corresponds to a complete pole-removal from Z_1, Z_1' and therefore Z_2' cannot have that pole anymore, consequently at that pole frequency the circuit will have a transmission zero.

The procedure can also be viewed as an alternation of partial and complete pole-removals from X_{11}. Any pole-removal leaves the other poles intact, but shifts the zeros towards the pole with the diminished residue.

This effect is used to bring the poles of X_{11} in coincidence of the zeros of X_{12}.

It is also evident now, how to handle the private poles of X_{11}. The terms corresponding to these poles in the partial-fraction

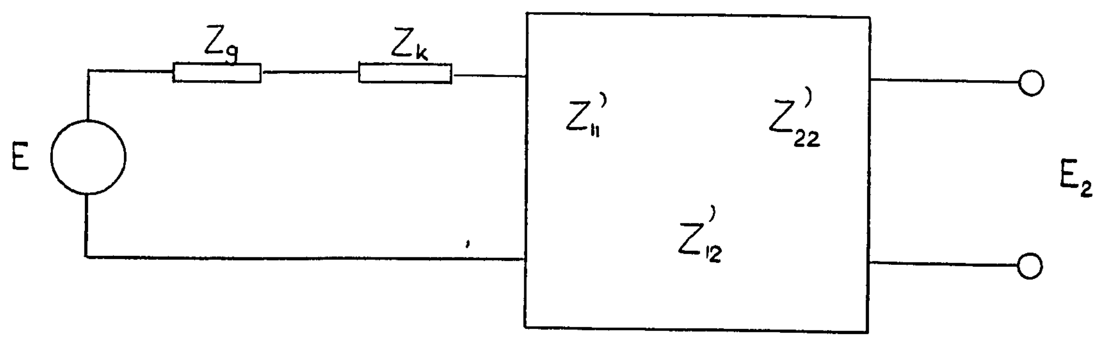


FIGURE 8.

expansion of X_{11} are lumped together into an immittance X_p. The development of the ladder should now begin with the series (parallel) removal of Z_p (Y_p) and thereafter continued as described above.

The constant multiplier of X_{12} is not controlled by the procedure. It can be found from the comparison of the required extreme frequency (0 or ∞) behaviour of X_{12} with that shown by the realized circuit. If necessary, the cascade-connection of an (ideal) transformer will give the specified value.

It may be mentioned, that an equivalent lattice can be found by multiplying X_{12} with a sufficiently small constant, so that at all poles

$$h_{11} \geqslant |h_{12}|. \tag{2}$$

Then it is permissible to select

$$X_{22} = X_{11}$$

(cf. Eq. (1) of Sec. I.3.) and through Eq. (32) of Sec. I.1. the lattice impedances can be calculated.

b. The ladder-expansion of the driving-point immittance.[1,2]

From the complete X_{ij} - matrix or from the operating parameters directly (see e.g. Eq. (27) of Sec. I.4.) the driving point immittances can be obtained. Since at the transmission zero frequencies X_D is pure imaginary or has extreme values, as can be seen from

$$P_2 - |I_2|^2 R_2 - |I_1|^2 \operatorname{Re}(Z_{D1}) - 0 \tag{3}$$

the ladder expansion described in Para. I.5.a. can be performed without any change.

c, Realization of the complete immittance-matrix through the ladder-development of X_{11}. [6,31,36]

It has been shown in Para. I.5.a., that by observing certain rules, it is possible to obtain a ladder-development of X_{11} that simultaneously realizes (except for a constant multiplier) the prescribed X_{12}. Now the still more ambitious problem of realizing the whole immittance-matrix through the development of X_{11} will be investigated.

Suppose first, that X_{11} has been developed into a ladder also incorporating the required X_{12} (this may necessitate a cascaded transformer, as mentioned in Para I.5.a.). Let $X_{22\,min}$ be the most compact expression of X_{22} (containing no private poles) that satisfies the third relation of Eqs. (1) of Sec. I.3. with the equal sign at all poles. Then

$$\begin{aligned} X_{22} &= X_{22\,min} + x \\ X_{22}' &= X_{22\,min} + x' \end{aligned} \tag{4}$$

where X_{22}' is the secondary immittance realized by the ladder-development already performed. x and x' are the "superfluous" parts of X_{22} and X_{22}', respectively. By keeping the ladder-development as simple as possible, x, x' and

$$x - x' = X_{22} - X_{22}' \tag{5}$$

will all be realizable reactance-functions.

If the variables of the network realizing X_{22}' and $X_{22\,min}$

are distinguished by a prime or the suffix "min", respectively, the secondary driving-point immittances of the three networks (with the primary sides terminated in unit resistance) are connected by

$$X_{D2} = X_{D2\,min} + x$$
$$X'_{D2} = X_{D2\,min} + x' . \tag{6}$$

Using Eq. (1) of Sec. I.3., it can be seen that $[X]_{min}$ has only simple poles, consequently (cf. Eq. (1) of Sec. I. 2.) $X_{D2_{min}}$ can have no imaginary pole at all. Since on the other hand, x and x' can have only $j\omega$- axis poles, these can be identified easily from X_{D_2} and X'_{D2}. After $x - x'$ has been found, cascading it with the ladder-network already developed will result in the required circuit.

However, considerable effort is saved, if this cascading is unnecessary, i.e.

$$x - x' = 0 . \tag{7}$$

Since $x - x'$ must be a realizable reactance,

$$x = 0 \tag{8}$$

will guarantee this. Eq. (8) be satisfied, if

$$X_{D_2}^{-1} = \left[\frac{[X] + X_{22}}{1 + X_{11}} \right]^{-1} \neq 0 \tag{9}$$

for

$$p = j\omega .$$

It can be seen, that Eq. (9) is equivalent to the condition that of the immittance parameters only X_{11} is allowed to possess private poles. Then no cascading of additional elements is necessary.

By considering e.g. impedance-parameters and realizing, that a private pole is represented by an additional series branch at the termination, the necessity of this condition is easily visualized.

It is also useful to find conditions, under which Eq. (7) could not be satisfied. If

$$x' \neq 0 \tag{10}$$

$$x \neq 0 \tag{11}$$

and in general, Eq. (7) does not hold.

Eq. (10) means that X'_{D2} has $j\omega$ - axis pole, which can only happen at those critical frequencies of some branch-reactances, which are transmission zeros.

Let $j\omega_i$ be a pole of X'_{D2}. Let us suppose that this attenuation pole has been created by a number of pole-removals from Z_{11}, the last of which was performed through the creation of a series impedance Z_i (Fig. 9). Z_i does not necessarily represent a complete pole-removal of $j\omega_i$; it may be that this pole has been later shifted before final removal.

Now if

a, Z_i was obtained through a full pole-removal, Z_{11A} and thus Z_{12A} does not have a pole at $j\omega_i$. Also,

$$Z'_{22}(j\omega_i) - Z_{22A}(j\omega_i) \tag{12}$$

and since a private pole of Z_{22A} at $j\omega_i$ would create a private pole of Z'_{22} which in turn would cause the condition described by Eq. (10) anyway, we find that Z_{22A} and consequently Z'_{D2} remains finite at $j\omega_i$.

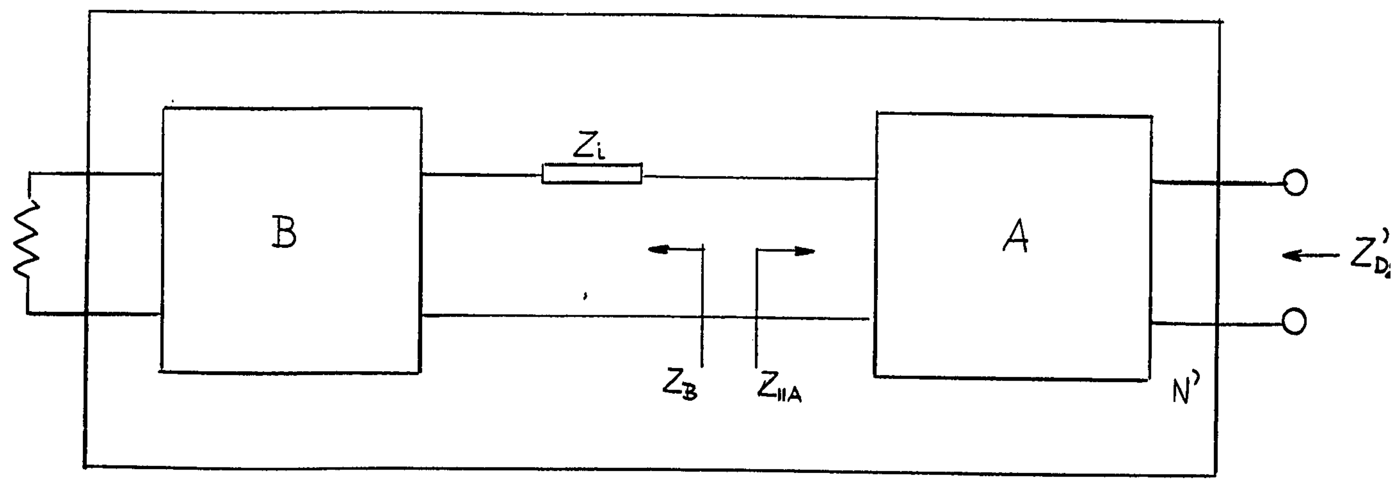


FIGURE 9.

b, If Z_i originated from a partial pole-removal, Z_{11A}, Z_{12A}, Z_{22A} and so Z_{D2}' may all have poles at $j\omega_i$. Generalizing this result, if the cascading of $x-x'$ is to be avoided, partial pole-removals should only be carried out with poles, that can later fully be removed and that are certainly not shifted by other manipulations. This limits the choice to poles at extreme frequencies ($\omega = 0$ or ∞).

An alternative view point is to consider Eqs. (13)-(16) of Sec. I.4. If X_{ii} is a usable immittance, it must have the order of the circuit (the order of g). The analysis of Para I.4.e or the observation of the required network configuration will show, which immittance will be useful. As the equations show, at least one immittance will always be suitable (the cancellation of the highest terms cannot occur simultaneously in $g_e \pm h_e$, $g_o \pm h_o$).

It is also possible to observe, that the development of both X_{11} and X_{22} or Y_{ii} and Z_{ii} is bound to supply the complete circuit and also some numerical check.

In the rare cases, when the transmission zeros are not given, they can be obtained as the zeros of Z_{12} and the private poles of Z_{ii}, as Eq. (5) of Sec. I.2. shows.

Another realization technique, which is, however, only of theoretical interest, is Cauer's partial fractioning method. Essentially, it consists of the following steps:

1, All X_{ij} are expanded into partial fractions.

2, All residues of X_{11} are split up into 2 parts:

parts satisfying Eq. (1) of Sec. I.3. with the equal sign (h'_{11}) and remainders (h''_{11}).

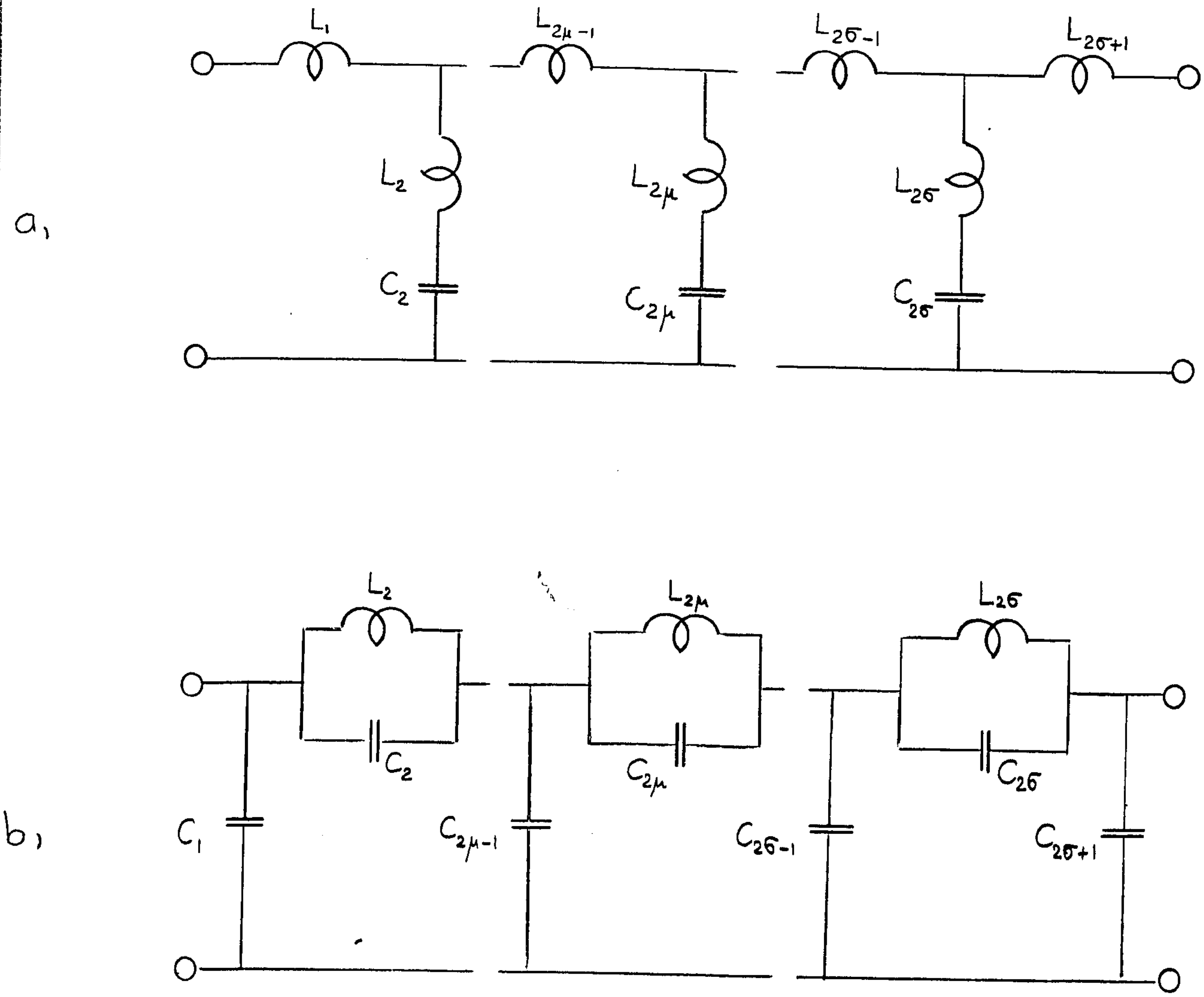
3. Since the addition of X_{ij} parameters corresponds to series (or parallel) connection of the component four-poles, the "canonical" realization of the network is finally built from the series (or parallel) connection of elementary four-poles. The elements are simple circuits realizing the sets of residues $(h'_{11}, h_{12}, h_{22})$ and $(h''_{11}, 0, 0)$, respectively. The number of the component circuits therefore is in general twice the number of the poles.

Since the four poles realizing $(h'_{11},\ h_{12},\ h_{22})$ contain ideal transformers[2], this synthesis procedure is not used in communication circuits.

d. The m-derived low-pass configurations.[1,41]

An important class of ladder-circuits is known for historical reasons as m-derived low pass networks (they have the same configuration, but normally different element values of the m-derived filters of the image-parameter theory). The general diagrams of these circuits are shown on Fig. 10. Fig. 10.a. shows the so-called mid-series low-pass circuit, Fig. 10.b. the mid-shunt configuration.

A common property of these networks is the low-pass character of their response, i.e. the attenuation is low at the near zero frequency, high elsewhere. Since this precludes an attenuation



a)
L_1
$L_{2\mu-1}$
$L_{2\sigma-1}$
$L_{2\sigma+1}$
L_2
$L_{2\mu}$
$L_{2\sigma}$
C_2
$C_{2\mu}$
$C_{2\sigma}$
b)
L_2
$L_{2\mu}$
$L_{2\sigma}$
C_2
$C_{2\mu}$
$C_{2\sigma}$
C_1
$C_{2\mu-1}$
$C_{2\sigma-1}$
$C_{2\sigma+1}$

FIGURE 10.

pole at ω-0 , f must be even. The degree of g and h (n) is, on the other hand, odd for both circuits shown on Fig. 10.

h is of the form described by Eq. (32/a) of Sec. I.4. Since by Eq. (28) of Sec. I.4. for the mid-series circuit $[\varrho_1]_{p\to\infty}$ and $[\varrho_2]_{p\to\infty}$ equal -1, while for the mid-shunt configuration they equal +1, we get (using the symbols of Para. I.4.e.):

$$\begin{aligned} x &= +1 \rightarrow \text{mid-shunt configuration} \\ x &= -1 \rightarrow \text{mid-series configuration} \end{aligned} \tag{12}$$

It is instructive to notice that by Eqs. (13) and (14) of I.4., $x=+1$ means that the order of the Z -matrix will be equal to n, that of the Y-matrix being n-1. For $x=-1$ the reverse situation exists. This corresponds to the fact-easily verified from Fig. (10) - that the mid-shunt (mid-series) configuration is completely characterized by its open-circuit (short-circuit) parameters.

Furthermore, since

$$[Z_{Di}]_{p=0} = R_j \qquad i, j = 1, 2 \quad i \neq j \tag{13}$$

$$y = +1 \rightarrow R_1 > R_2$$

$$y = -1 \rightarrow R_1 < R_2 \tag{14}$$

If R_1/R_2 is specified, but the configuration is indifferent, it is possible to make h originally Hurwitz and then change the sign of h_e (h_o) for $R_2>R_1$ $(R_2<R_1)$. This will result in a $[1, -1]$ $([-1, 1])$ value-pair for $[x, y]$ i.e. it gives the desired termination-condition and a mid-shunt (mid-series) configuration.

If

$$A' = h_e = 0, \tag{15}$$

by Eqs. (32) and (34) of Sec. I.4.,

$$\left[\rho_1\right]_{p=0} = \alpha_0' = 0 \tag{16}$$

and

$$R_1 = R_2 . \tag{17}$$

This is the case of the symmetrical, equally terminated circuit.

If the last two coils of the mid-series circuit have zero inductance or the last two capacitors of the mid-shunt network have zero capacitance, the circuits will have a double attenuation pole at infinite frequency. Such circuits can therefore correspond to

$$pB' = h_o = 0 \tag{18}$$

and if Eq. (50) of Para. I.4.e. is satisfied, the network will be an equally terminated antimetric four-pole.

Finally, it is easy to see, that an odd-order mid-series ladder can be developed by the process of a partial pole removal from the infinite-frequency pole of Y_1^{-1} followed by a complete removal from the reciprocated remainder, repeated as many times as necessary. A dual procedure will supply the odd-order mid-shunt ladder.

An alternative procedure has been developed by Darlington especially for the design of these filters. This is, however, described in detail in Cauer's book and thus it is not necessary to present it here.

e, The realizability of ladder networks without mutual coupling [18,23,38]

The transmission zeros of a ladder network containing no mutual coupling between its elements are necessarily identical to some critical frequencies of its branch-immittances. Therefore, they can only be located on the left half of the complex frequency plane, or, for lossless branches, on the $j\omega$ - axis itself. This is then a necessary (but not sufficient) condition for the ladder-realizability of a transmission function.

No general realizability theory has been developed yet for ladder networks. However, for the important configurations described in the previous paragraph, an adequate analysis exists. It will be briefly presented for the mid-series network only.

The following are necessary and sufficient restraints for the realization of this circuit:

1, Z_{D1} has a critical frequency at $p \to \infty$

2, $\left[Z_{D1}\right]_{p=0} = R_2$ (19)

3, If

$$Z_{D1} = R_1 \frac{m_1 + n_1}{m_2 + n_2} \tag{20}$$

where m_1, m_2 are even, n_1, n_2 odd polynomials,

then

$$m_1 m_2 - n_1 n_2 = k \prod_{(i)} \left(p^2 + \omega_i^2\right)^2 \tag{21}$$

where the ω_i are the finite transmission zeros of the circuit.

4, If the transmission zeros are distributed

$$0 < \omega_1 \leq \omega_2 \leq \ldots \quad \omega_i \leq \ldots \omega_n, \tag{22}$$

then n_1 has i non-zero roots not larger than ω_i on the positive $j\omega$-axis for any i $(1 \leq i \leq n)$.

The sufficiency and necessity of these conditions can be proved by a procedure analog to the complete induction in mathematics; they can be shown to be sufficient and necessary for the elements of a mid-series circuit (series L, shunt resonant circuit or shunt C) and it can be demonstrated, that cascading another elementary four-pole does not change their validity.

However, the necessity of these conditions can also be documented in a more straightforward manner. Conditions 1, and 2, are self-evident.

Condition 3, will be immediate by using Eqs. (27) and (12) of Sec. I.4. to find

$$m_1 m_2 - n_1 n_2 = (g_e - h_e)(g_e + h_e) - (g_o - h_o)(g_o + h_o) = f^2 \tag{23}$$

with f even.

Finally, for an odd n, the circuit is developed from

$$Y_{11}^{-1} = R_1 \frac{g_o - h_o}{g_e + h_e} = R_1 \frac{n_1}{m_2} \tag{24}$$

The zeros of this expression (i.e. the zeros of n_1) must be shifted by partial removals from its pole at infinity to coincide with the finite transmission zeros. But all partial removals shift the zeros of Y_{11}^{-1} up; consequently, Condition 4, must hold for their initial position.

Analog derivations can now easily be found for even n, for mid-shunt configuration, for similar high-pass circuits and, finally, for band-pass filters created by the cascading of a series capacitor to a midseries network.

I. 6 FILTER COMBINATIONS [9, 26, 29, 32, 25]

The separation of a continuous frequency range into separate bands directed to different receivers, or vice versa, the combination of several non-overlapping frequency-bands into a single group without interference, is a frequently arising communication problem. This Section will briefly describe the insertion loss methods for the synthesis of such combining or splitting filter-combinations.

a. The general theory of filter combinations. [26, 29, 32]

Two basic configurations will be treated. In the first (second), all component networks are paralleled (series-connected) on the primary side and individually terminated. These circuits are called front- paralled and front-series combinations. Their treatment can be carried out in exactly dual terms. The analysis will here be restricted to the front-series configuration.

Throughout the Section it will be assumed, that the resultant driving-point impedance of the combination has (exactly or approximately) a finite, real constant value on the whole $j\omega$-axis. Since the residues and real parts of an impedance cannot be negative on the imaginary axis, no individual driving-point impedance will be allowed to have on the $j\omega$-axis a pole or a real part value higher than the required resultant.

If these conditions are satisfied, then it is adequate to make the real part of the driving-point impedance constant. The imaginary part will be automatically zero. Since by Eq. (6/a) of Sec. I.3.

$$|I_1|^2 \, \mathrm{Re}\left(Z_{D_1}\right) = |I_2|^2 R_2 , \tag{1}$$

for the combination

$$\mathrm{Re}\left(Z_D\right) = R_0 - \sum_{(i)} \frac{R_i}{|M_i|^2} \tag{2}$$

where the R_i and M_i are the secondary terminations and current ratios of the component networks. If Eq. (2) is exactly satisfied, all networks are fed by a constant input current:

$$I = \frac{E}{R + R_0} \tag{3}$$

where R is the generator impedance, and the operating factors are given by

$$S_i^2 = \frac{(R + R_0)^2}{4 R_0 R_i} M_i^2 . \tag{4}$$

If

$$R = R_0 = R_i \tag{5}$$

as usually is the case,

$$S_i^2 = M_i^2 . \tag{4/a}$$

Hence, characteristic functions can be defined for each network, with

$$|M_i|^2 = 1 + |\varphi_i|^2 . \tag{6}$$

By Eqs. (2) and (5)

$$\sum_{(i)} \frac{1}{|M_i|^2} = \sum_{(i)} \frac{1}{1 + |\varphi_i|^2} = 1 . \tag{7}$$

b. <u>The exact design of filter-combinations</u> [26, 29, 32]

Two important cases will be treated: The two filter (e.g. low-pass/high-pass) combination and the three filter (e.g. low-pass/band-pass/high-pass) combination.

For the former, from Eq. (7)

$$|\varphi_1|^2 \, |\varphi_2|^2 = 1 \tag{8}$$

and

$$|\varphi_1|^2 = |\varphi_2|^{-2} = \left|\frac{M_1}{M_2}\right|^2 = \left|\frac{S_1}{S_2}\right|^2 = \frac{P_2}{P_1}. \tag{9}$$

Then it is possible to choose

$$\varphi_1 = \frac{h}{f} = \varphi_2^{-1} \tag{10}$$

$$M_1 = \frac{g_1}{f} \qquad M_2 = \frac{g_2}{h} \tag{11}$$

and since

$$|g_1|^2 = |g_2|^2 = |h|^2 + |f^2|, \tag{12}$$

it is convenient to select

$$g_1 = g_2 = g. \tag{13}$$

Now by Eqs. (3) - (4) of Sec. I.4., the impedance-parameters Z_{12} and Z_{22} can be calculated for both networks. If a ladder-realization is desired, the left-most elements must be shunt (to avoid a $j\omega$-axis pole in Z_{Di}) and an expansion of the Z_{22}-s will suffice.

A dual analysis supplies Y_{12} and Y_{22} of front-parallel connected four-poles. Since now the ladder networks start with a series arm, these suffice for the synthesis.

For a 3-filter combination, Eq. (7) gives

$$\begin{aligned} |M_1|^2 &= 1 + |\varphi_1|^2 \\ |M_2|^2 &= 1 + |\varphi_2|^2 \\ |M_3|^2 &= 1 + \frac{2 - |\varphi_1\varphi_2|^2}{|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_1\varphi_2|^2} \end{aligned} \tag{14}$$

The procedure outlined at the end of Para. I.4.b. can be used to derive M_1 M_2 and M_3. The immittance parameters and the circuit are then found as before.

c, The approximating design of filter-combinations

The synthesis method demonstrated in Para. I.6.b. has two important shortcomings. First, the reflection loss of the conducting filter must equal the operating loss of the other filters (see Eqs. (4/a) and (7)). This is because the generator is perfectly matched along the whole $j\omega$-axis (Eq. (2)) and any power rejected by the cut-off filters must be absorbed by the conducting network. As a consequence, the pass-band requirements become absurdly tight. (E.g. for a filter-pair a modest 40 dbs stop-band attenuation of one filter allows about 0.000434 db pass-band ripple for the other).

A second draw-back is that for the same reason the cross-over point between adjacent pass-bands must be at 3 db. Various methods are available to correct this situation, when it could not be tolerated. Some are: the use of band-pass filter to absorb power in any region, where all the other filters have to cut off, the cascading of additional constant-resistance groups having slightly different cut-off frequencies or simply the cascading of conventional filters with the element networks of the filter combination. In this last method care must be exercised to avoid natural frequencies of the cascades in the vicinity of the stop-bands.

Returning to the first difficulty, a method of overcoming it will be described in this Paragraph. It will be treated for front-paralleled networks and it rests upon the following assumptions:

1, The input conductance of a cut-off filter is negligible. Since by taking the dual of Eq. (1)

$$G_D = \mathrm{Re}\,(Y_D) = \frac{G_2}{|N|^2}\,, \tag{15}$$

this supposition will hold even for moderate stop-band attenuations.

2, If the input conductance G of the combination is kept approximately constant and none of the susceptances have $j\omega$ - axis poles, then the driving-point susceptance B_D of the combination is negligible compared to the driving-point conductance. This assumption can be made on the basis of the equations linking the real and imaginary parts of minimum-phase network functions.[9] E.g.

$$\int_0^\infty \left[G_D(\omega) - G_D(\infty)\right]^2 d\omega = \int_0^\infty \left[B_D(\omega)\right]^2 d\omega \tag{16}$$

$$\int_0^\infty \left[B_D(\omega)\right] d\left(\log \omega\right) = \frac{\pi}{2}\left[G_D(\infty) - G_D(0)\right]$$

etc.

3, The pass-band attenuation (A_p) as well as the pass-band voltage transfer loss ($A_{np} = 20 \lg |N|_p$) are small.

Under these assumptions the following results ensue[35] for the circuit of Fig. 11. (in which filter "A" represents the only component network passing energy):

$$A_A = 10 \lg \left| \frac{I_o^2 / 4G}{G_{2A}\, E_{2A}^2} \right| = 10 \lg \left| \frac{I_o^2 / 4G}{G_{DA}\, E_1^2} \right| \tag{17}$$

$$A_A = 10 \lg \frac{\left(G + G_{DA} + G_{DB} + \dots\right)^2 + \left(B_{DA} + B_{DB} + \dots\right)^2}{4 G G_{DA}} \tag{18}$$

$$A_A \cong 10 \lg \frac{\left(G + G_{DA}\right)^2}{4 G G_{DA}}\,. \tag{19}$$

Also

$$A_{nA} = 20 \lg |N|_A = 10 \lg \frac{G_{2A}}{G_{DA}}\,. \tag{20}$$

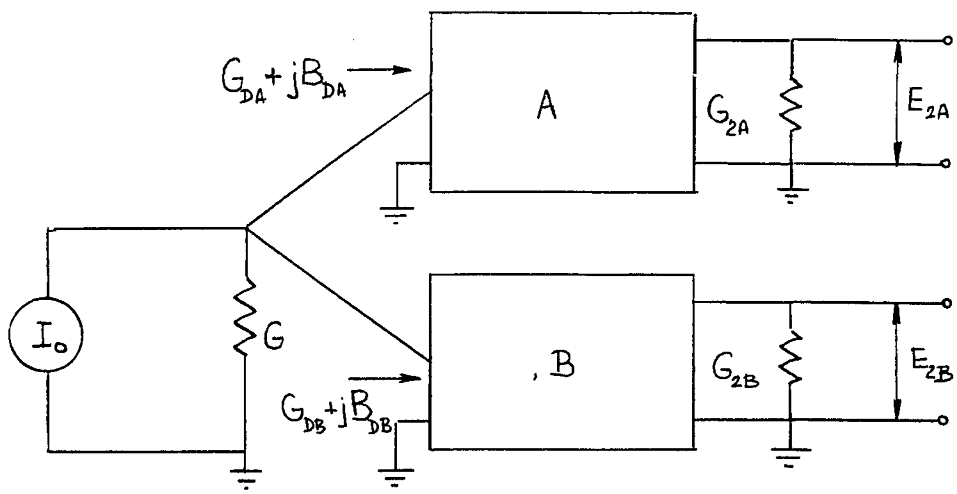


FIGURE II.

If again

$$G = G_o = G_{2A} = G_{2B} = \ldots \cong G_{DA} + G_{DB} + \ldots \qquad (21)$$

$$\frac{G_{2A}}{G_{DA}} = \frac{G}{G_{DA}} = 10^{\frac{A_{nA}}{10}} . \qquad (22)$$

Substituting into Eq. (19) and using series expansion

$$A_A \cong \frac{A_{nA}^2}{17{\cdot}4}\left[1 - \frac{A_{nA}^2}{451}\right] \qquad (23)$$

or, for a specified pass-band ripple A_p, the maximum voltage transfer loss is

$$A_{np} \leq 4{\cdot}175\sqrt{A_p} . \qquad (24)$$

For a filter in the stop-band

$$A_B \cong 10\lg\frac{(G + G_{DA})^2}{4GG_{DB}} = 10\lg\left[\frac{(G + G_{DA})^2}{4GG_{DA}} \times \frac{G_{DA}}{G} \times \frac{G}{G_{DB}}\right] \qquad (25)$$

or for a prescribed stop-band attenuation A_s

$$A_{ns} \geq A_s + A_{np} . \qquad (26)$$

Eqs. (24) and (26) give the allowed extreme values of $|N|^2$ for the component networks. In addition, care must be taken to ensure the constancy of G_D along the whole $j\omega$-axis. According to the assumptions, this is satisfied in the pass and stop-bands. If a linear, equi-slope behaviour of the G_{Di}-s is assumed in the transition bands, then it is sufficient to specify:

$$G_{DL} + G_{DM} \geq G_{Dp} = G \times 10^{-\frac{A_{np}}{10}} \qquad (27)$$

at every cross-over point. Using Eq. (22) and the dual of Eqs. (6) and (2)

$$G_{DL} = G_{DM} = \frac{G}{1 + |\varphi_L|^2} = \frac{G}{1 + |\varphi_M|^2} = \frac{G}{2}\,10^{-\frac{A_{np}}{10}} \qquad (28)$$

so that at cross-over

$$|\varphi_L|^2 = |\varphi_M|^2 = 2 \times 10^{\frac{A_{np}}{10}} - 1 \qquad (29)$$

and from Eqs. (18), (20), and (23).

$$A_L = A_M \cong A_p + 3\,db \qquad (30)$$

i.e. this method results approximately in the same low cross-over point loss as the exact procedure. The measures mentioned at the beginning of this paragraph can be applied to overcome this problem, if necessary.

Several networks have, been designed by this method with very good results. Figs. 12 and 13 demonstrate the circuits and responses of a 2- and a 3-filter combination, respectively.

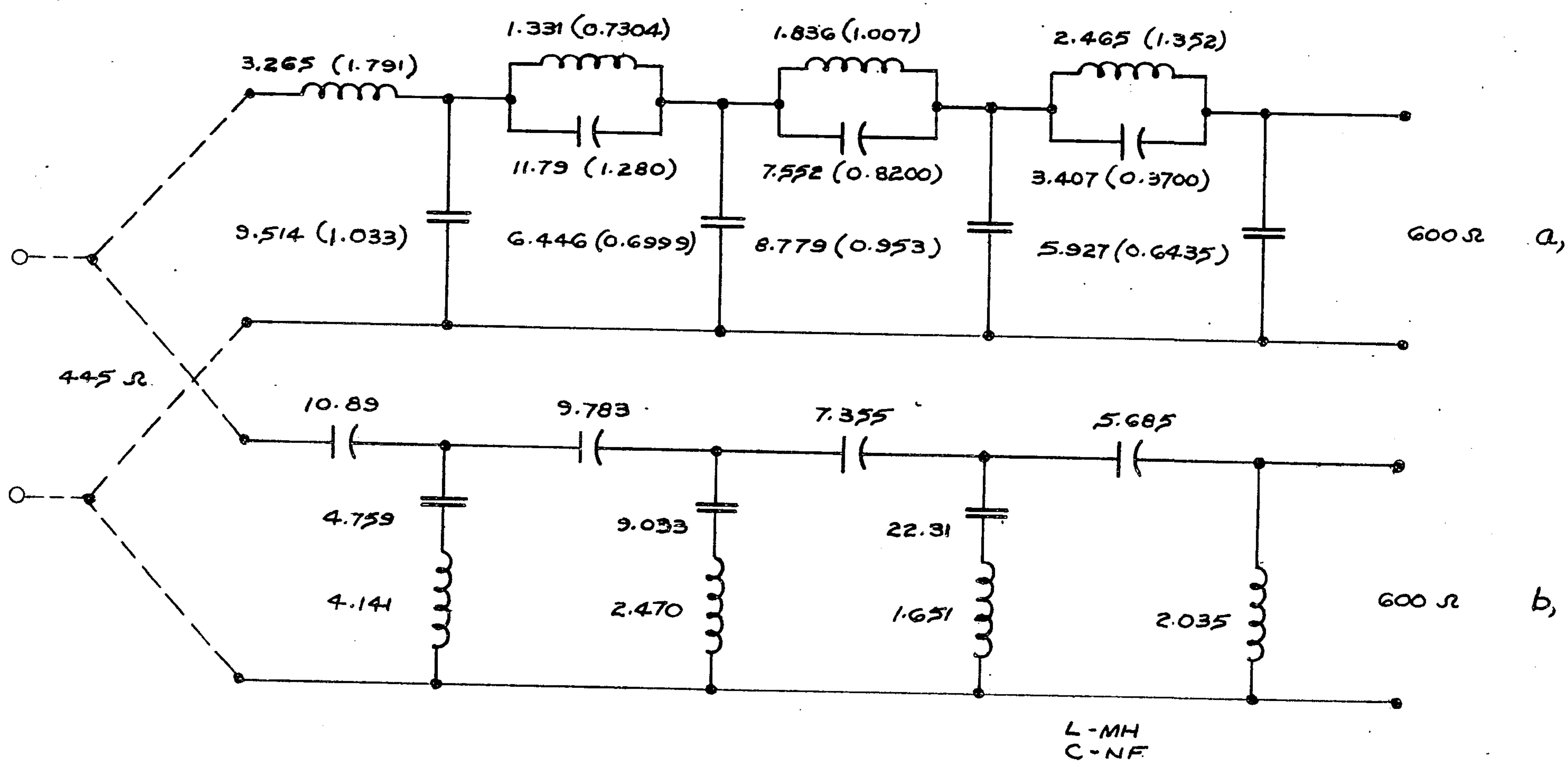

3.265 (1.791)
1.331 (0.7304)
11.79 (1.280)
1.836 (1.007)
7.552 (0.8200)
2.465 (1.352)
3.407 (0.3700)
9.514 (1.033)
6.446 (0.6999)
8.779 (0.953)
5.927 (0.6435)
600 Ω
a,
445 Ω
10.89
9.783
7.355
5.685
4.759
4.141
9.033
2.470
22.31
1.651
2.035
600 Ω
b,
L-MH
C-NF

FIG. 12a.

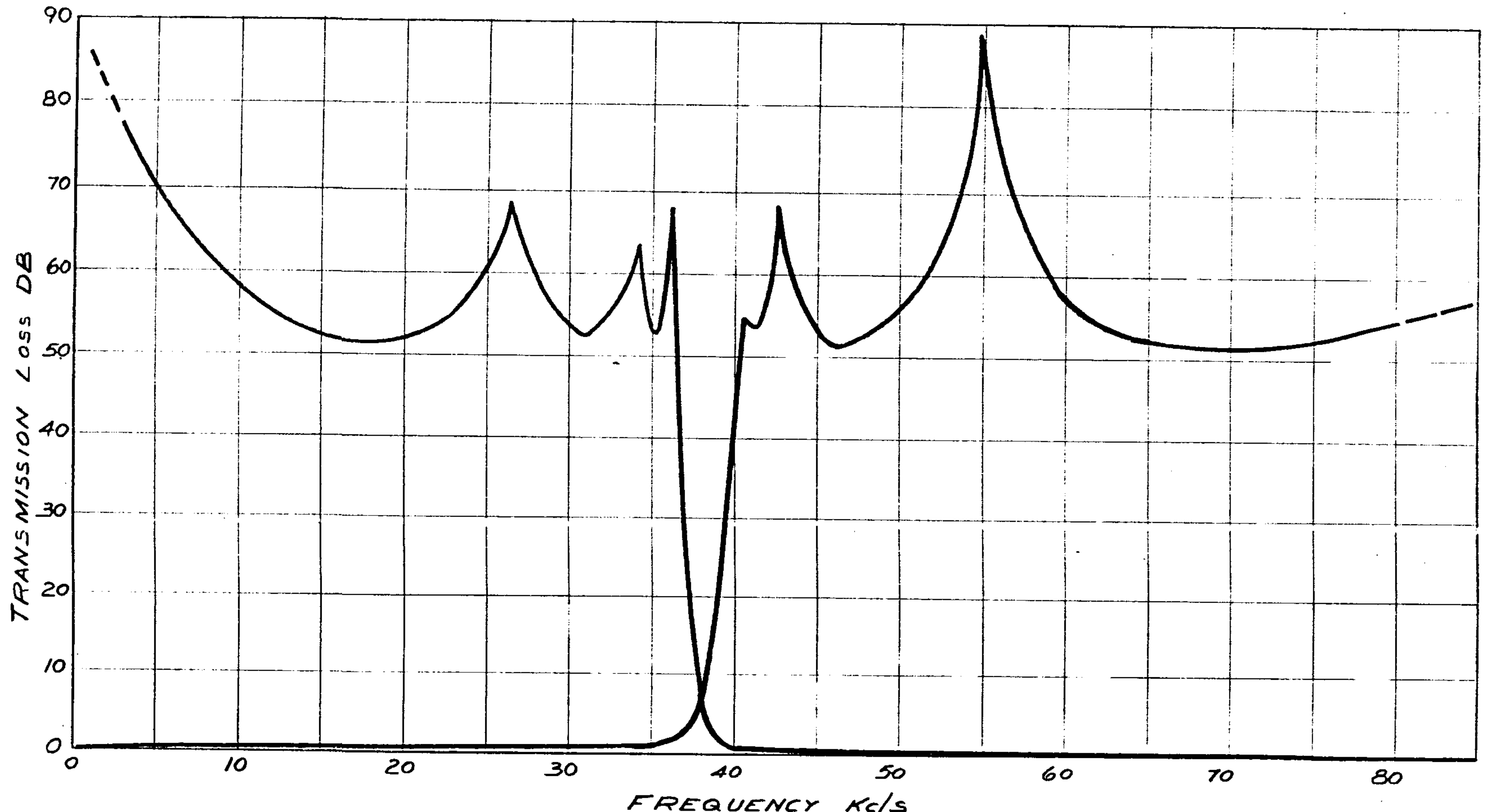
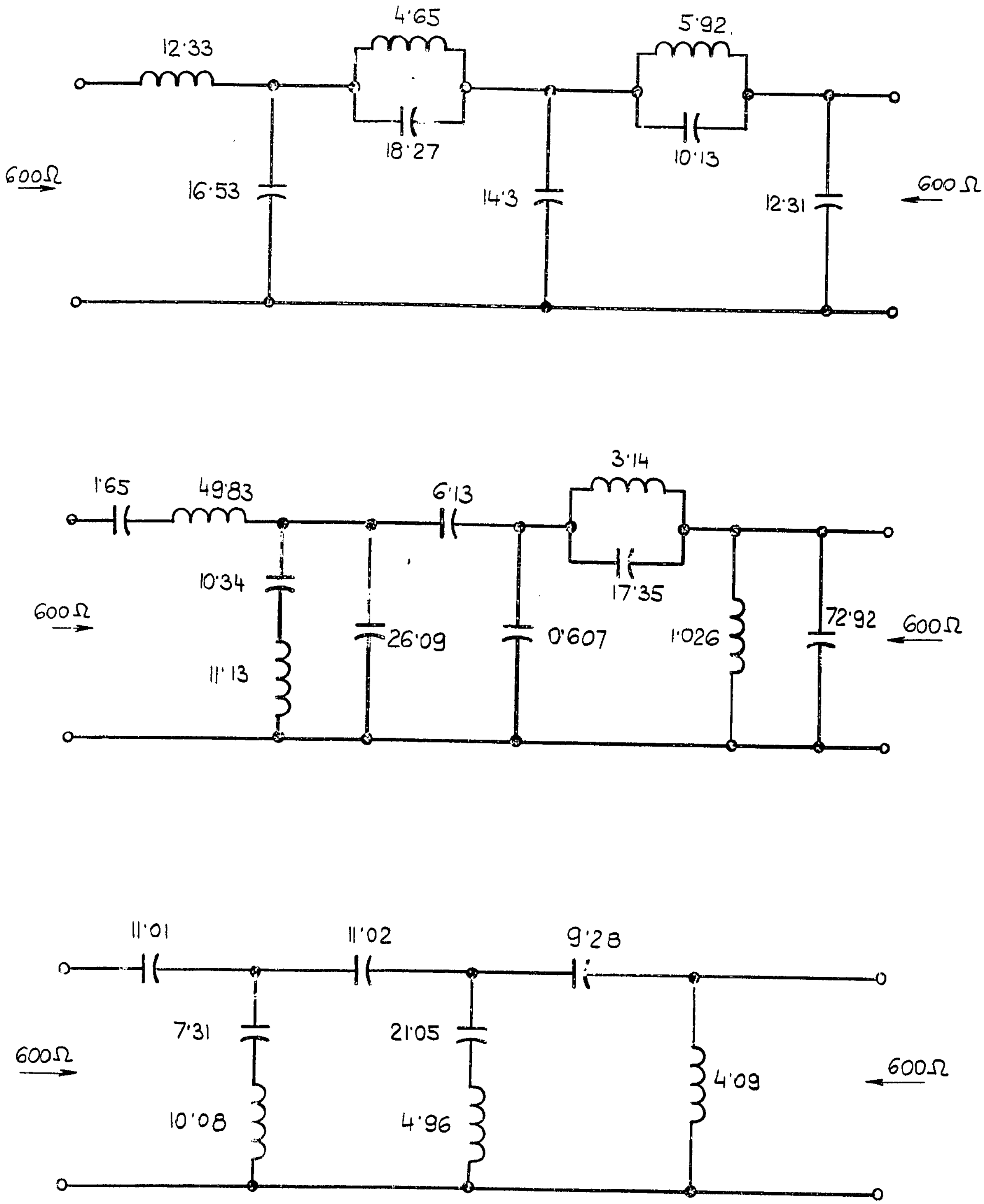

TRANSMISSION LOSS DB
90
80
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
70
80
FREQUENCY Kc/s

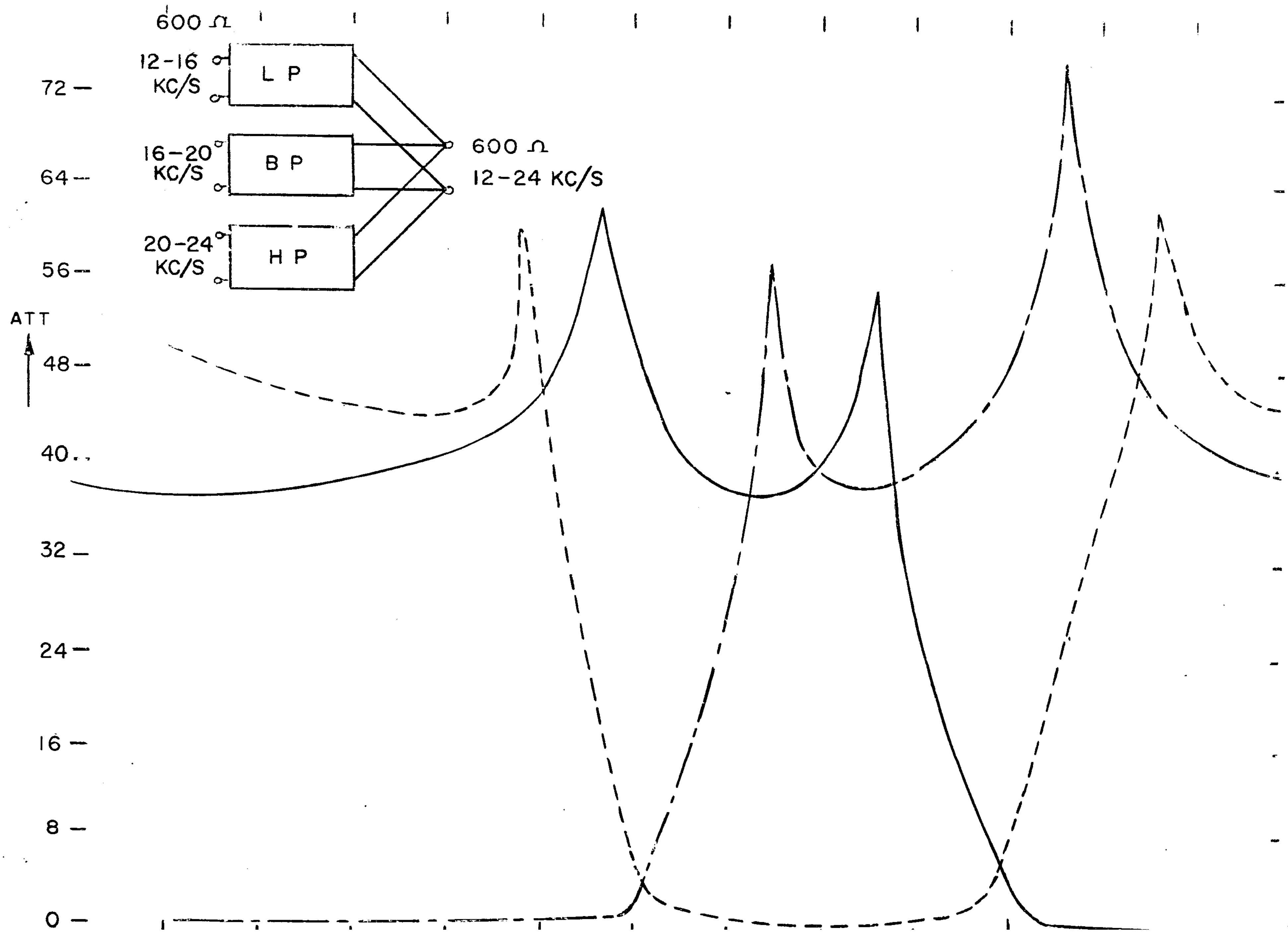
FIGURE 12 b



L - mH
C - nF

FIGURE 13. a.

FIGURE 13. 6.


600 Ω
12-16 KC/S
L P
16-20 KC/S
B P
20-24 KC/S
H P
600 Ω
12-24 KC/S
ATT
72
64
56
48
40
32
24
16
8
0

II. APPROXIMATION THEORY

As the results of Chapter I. show, a network can always be synthesized for given requirements, provided these requirements satisfy certain restrictions, the realizability conditions (cf. Sec. I.3. and Para. I.5.e). Since, however, the specifications are normally given in a non-mathematical (verbal or graphical) form, it is necessary to translate these into proper algebraic expressions, satisfying the requirements and the realizability conditions simultaneously. Some ways of doing this for the most common types of specifications will be demonstrated in this Chapter.

II. I. THE APPROXIMATION OF THE IDEAL LOW-PASS RESPONSE [20,21,22,24,38]

A large percentage of the approximation problems is equivalent to (or can be simplified to) the simulation of the ideal low-pass response, i.e. of a response, that has zero attenuation below a specified frequency and infinite attenuation above it. Since this ideal response cannot be satisfied with a finite network, a similar requirement postulating the maximum allowed attenuation (the pass-band "ripple") below a specified frequency (the pass-band limit) and the minimum tolerable attenuation (stop-band loss) above a second prescribed frequency (the stop-band limit) will instead be demanded.

The use of the well-known frequency-transformations allows the results derived in this Section to be utilized in the design of high-pass, frequency-symmetrical band-pass, band-stop and multi-band filters.

a, Butterworth-approximation

The simplest characteristic function that leads to a simulation of the ideal low-pass behavior is a monotonic power function with the largest possible number of derivatives zero at the origin. This is

$$\varphi - p^n . \tag{1}$$

From Eq. (24) of Sec. I.2.

$$|S|^2 = 1 + p^n(-p)^n = 1 + \omega^{2n}. \qquad (2)$$

The n transmission zeros will all be at infinite frequency and the natural frequencies of the terminated network can be found from

$$|S|^2 = 0 \qquad (3)$$

to be the left half-plane values satisfying

$$p_k = e^{j\pi \frac{n+1+2k}{2n}}, \qquad k = 0, \pm 1, \pm 2, \dots$$

a total of n. They lie on the unit circle drawn around the origin.

Since all transmission zeros are located at infinite frequency, a ladder-realization built from series coils and shunt capacitors (n components altogether) is possible.

Since by Eqs. (38) and (39) of Sec. I. 1, for the specified attenuation extremes

$$\begin{aligned} A_p^{(db)} &= 10 \log\left[1 + \omega_p^{2n}\right] \\ A_s^{(db)} &= 10 \log\left[1 + \omega_s^{2n}\right], \end{aligned} \qquad (4)$$

one obtains for the necessary order

$$n \geq \frac{\log k_1}{\log k}. \qquad (5)$$

Here k_1 is the discrimination parameter

$$k_1 = \left[\frac{10^{A_p/10} - 1}{10^{A_s/10} - 1}\right]^{1/2} \ll 1 \qquad (6)$$

and k the selectivity parameter

$$k = \frac{\omega_p}{\omega_s} < 1.$$

Evidently, in all the above equations, ω represents a normalized quantity. The unit frequency is the 3-db (cut-off) point.

b. Chebyshev - approximation in the pass-band or in the stop-band[3,21,4]

If we choose

$$|\varphi| = \epsilon \cos n\alpha \tag{7}$$

with

$$\omega = \cos\alpha, \tag{8}$$

this choice will lead to a polynomial $\varphi(p)$, as can be seen from the trigonometric relations giving the cosine of multiple angles. Also, for

$$-1 \leq \omega \leq 1, \tag{9}$$

α can be real and

$$0 \leq |\varphi|^2 \leq \epsilon^2 \tag{10}$$

i.e. the attenuation oscillates (exhibits a Chebyshev-behavior) between zero and

$$A_p^{(db)} = 10 \lg\left(1 + \epsilon^2\right). \tag{11}$$

For

$$\omega > 1, \tag{12}$$

using

$$\cos n\alpha = \frac{e^{jn\alpha} + e^{-jn\alpha}}{2} \tag{13}$$

and

$$e^{j\alpha} = \cos\alpha + j\sin\alpha = \omega + \sqrt{\omega^2 - 1} \tag{14}$$

we get

$$|\varphi| = \epsilon \cos n\alpha = \epsilon\,\frac{\left[\omega + \sqrt{\omega^2-1}\right]^n + \left[\omega + \sqrt{\omega^2-1}\right]^{-n}}{2}, \tag{15}$$

and

$$|\varphi| \to \epsilon\, 2^{n-1}\, \omega^{n} \tag{16}$$

(i.e. exhibits a Butterworth stop-band behavior) for

$$\omega \gg 1. \tag{17}$$

Thus again all (n) transmission zeros are located at $\omega \to \infty$ and the series-coil-shunt-capacitor configuration (the so-called constant-k circuit) is possible.

The natural frequencies can again be found from Eq. (3); now

$$1 + \epsilon^{2} \cos^{2} n\alpha = 0. \tag{18}$$

Using

$$\alpha = \alpha_1 + j\alpha_2, \tag{19}$$

$$\cos(\alpha + j\beta) = \cos\alpha \cosh\beta - j \sin\alpha \sinh\beta, \tag{20}$$

and

$$p_k = \sigma_k + j\omega_k, \tag{21}$$

$$\frac{\sigma_k^2}{\sinh^2 \alpha_2} + \frac{\omega_k^2}{\cosh^2 \alpha_2} = 1 \tag{22}$$

obtains, that is, the natural frequencies lie on a central ellipse, with semiaxes sinh α_2 and cosh α_2.

From Eqs. (7) - (14), the necessary degree is obtained

$$n \geq \frac{\cosh^{-1} k_1}{\cosh^{-1} k}. \tag{23}$$

Again, ω represents a normalized frequency. The unit here is the pass-band limit, as Eqs. (9) and (10) show.

As the comparison of Eqs. (2) and (16) shows, for identical pass-band requirements ($\epsilon = 1$) the Chebyshev pass-band filter has an advantage of $6 \times (n-1)$ db in stop-band over the Butterworth network.

It is evident that the two-stage transformation

$$|\varphi| \longrightarrow \frac{1}{|\varphi|} \tag{24}$$

and

$$\omega \longrightarrow \frac{1}{\omega} \tag{25}$$

results in a response, that is monotonically increasing in the pass-band and oscillates between A_s and infinity in the stop-band, i.e. that has a flat pass-band, Chebyshev stop-band behaviour. The circuit exhibiting this behaviour must be of the 'm-derived' configuration, to be able to possess finite transmission zeros.

c, Chebyshev pass-band and stop-band approximation (elliptic approximation) [1,2,20,22,24]

In Para. II. I.b. it has been demonstrated, that the change from flat to Chebyshev pass-band response improved the stop-band behaviour. This is because the Chebyshev-type behaviour utilizes the permitted attenuation range more fully. It is to be expected, that a Chebyshev response in both bands will result in still more efficient networks.

With the analysis initially being restricted to odd φ (i.e. symmetrical, equally terminated networks), the characteristic function will be chosen

$$|\varphi|^2 = \left(10^{A_p/10} - 1\right) F^2(\omega) \tag{26}$$

where

$$F(\omega) = F_0 \omega \frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) \dots}{(\omega^2 - \omega_a^2)(\omega^2 - \omega_b^2) \dots} \quad . \tag{27}$$

Fig. 14 compares the frequency-diagrams of $|S|^2$, F^2 and two derived functions, $1-F^2$ and $1-k_1^2F^2$ (see Eq. (6) for the definition of k_1). It is evident from the diagrams, that they and $\frac{dF}{d\omega} = \frac{1}{2F}\frac{d(F^2)}{d\omega}$ have the properties listed in the Table below:

	double zeros	simple zeros	double poles	simple poles
$1-F^2$	$\omega_I, \omega_{II}, \dots$	$\sqrt{k}$	$\omega_a, \omega_b, \dots$	∞
$1-k_1^2F^2$	$\omega_\alpha, \omega_\beta, \dots$	$\frac{1}{\sqrt{k}}$	$\omega_a, \omega_b, \dots$	∞
$\frac{dF}{d\omega}$	—	$\omega_I, \omega_{II}, \dots$ $\omega_\alpha, \omega_\beta, \dots$	$\omega_a, \omega_b, \dots$	∞

Since all three quantities are rational functions of ω, by considering the zeros and poles, the equality

$$\frac{dF/d\omega}{\sqrt{(1-F^2)(1-k_1^2F^2)}} = \frac{\pm M_0}{\sqrt{(1-\frac{\omega^2}{k})(1-k\omega^2)}} \tag{28}$$

results. Integrating and substituting

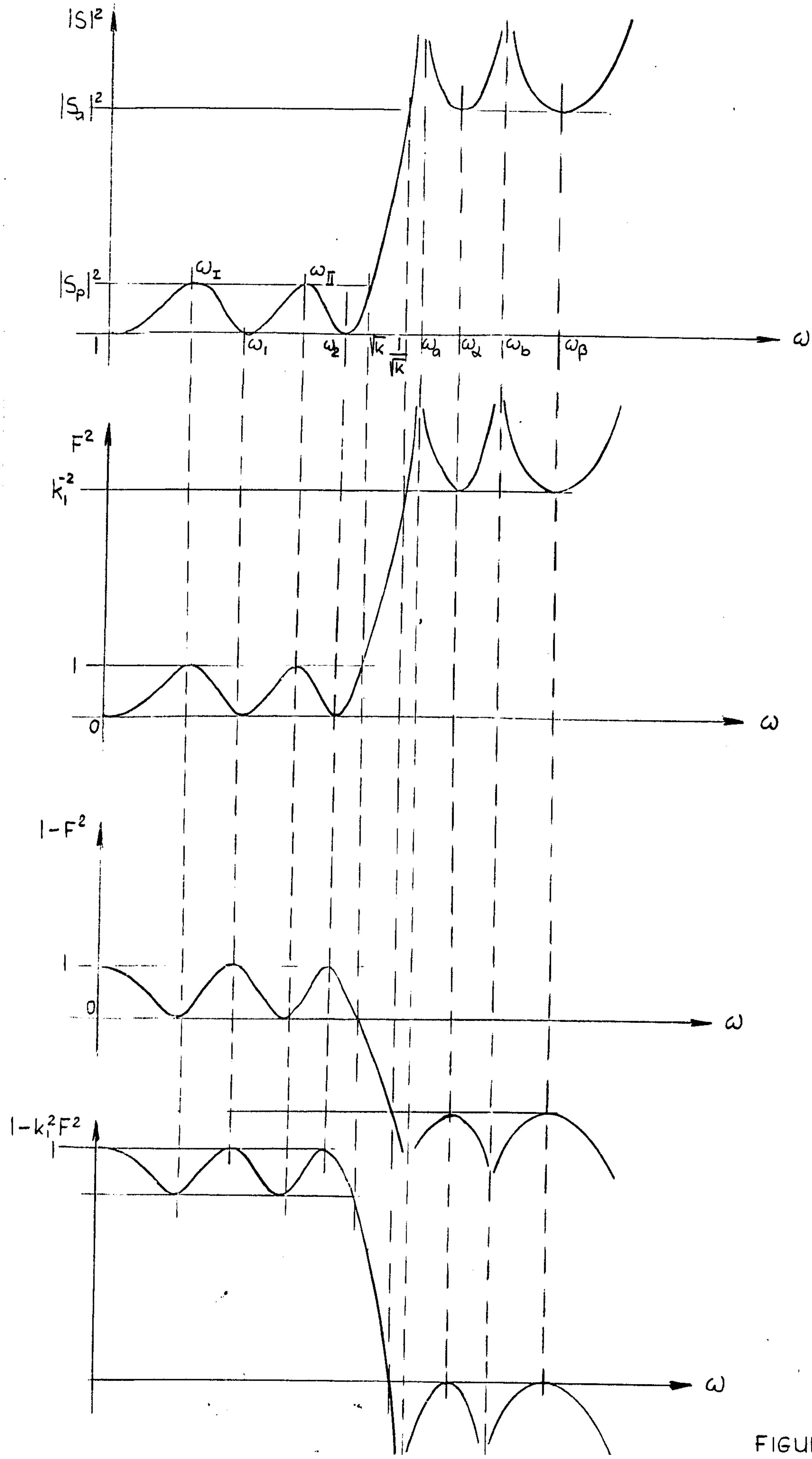
$$\begin{aligned} F &= \sin \Phi_1 \\ \omega &= \sqrt{k} \sin \Phi , \end{aligned} \tag{29}$$

this is transformed to

$$\int_0^{\Phi_1} \frac{d\Theta_1}{\sqrt{1-k_1^2 \sin^2\Theta_1}} = \pm M_0\sqrt{k} \int_0^{\Phi} \frac{d\Theta}{\sqrt{1-k^2\sin^2\Theta}} \quad . \tag{30}$$

Eq. (30) has the parametric solution

$$\begin{aligned} \omega &= \sqrt{k}\ \mathrm{sn}\,(z, k) \\ F &= \mathrm{sn}\,(\pm M_0\sqrt{k}\ z, k_1) \end{aligned} \tag{31}$$



$|S|^2$
$|S_a|^2$
$|S_p|^2$
ω_I
ω_{II}
1
ω_1
ω_2
$\sqrt{k}$
$\frac{1}{\sqrt{k}}$
ω_a
ω_α
ω_b
ω_β
ω
F^2
k_1^{-2}
1
0
ω
$1-F^2$
1
0
ω
$1-k_1^2F^2$
1
ω

FIGURE 14.

where

$$z = \int_0^{\Phi} \frac{d\Theta}{\sqrt{1 - k^2 \sin^2\Theta}} \tag{32}$$

and

$$\operatorname{sn}(z, k) = \sin \Phi \tag{33}$$

is the Jacobian elliptic sine function.

Since the real period of $\operatorname{sn}(z,k)$ is $4K$ (where

$$K = \int_0^{\pi/2} \frac{d\Theta}{\sqrt{1 - k^2 \sin^2\Theta}}$$

is the complete elliptic integral of the first kind) the pass-band $0 \leq \omega \leq \sqrt{k}$ will, correspond to $0 \leq z \leq K$. F has to complete $2m+1$ quarter-periods of oscillation in this same range (Fig. 14) where m is the number of finite attenuation zeros (poles) in the pass-(stop-)band. Therefore

$$M_0 \sqrt{k}\, K = (2m+1) K_1 , \tag{34}$$

and

$$F = \operatorname{sn}\left((2m+1) \frac{K_1}{K} z, \ k_1 \right). \tag{35}$$

Hence, the frequencies of zero attenuation are

$$\omega_i = \sqrt{k} \ \operatorname{sn}\left(\frac{2iK}{2m+1}, \ k \right) \tag{36}$$

In the stop-band, $\frac{1}{\sqrt{k}} \leq \omega \leq \infty$, corresponding to

$$\begin{aligned} z &= u + jK' \\ K &\geq u \geq 0. \end{aligned} \tag{37}$$

Here K' is the complementary complete elliptic integral.

Using

$$\operatorname{sn}(u + jK', k) = \frac{1}{k \ \operatorname{sn}(u, k)} \tag{38}$$

the stop-band is characterized by Eqs. (35) and (37) and

$$\omega = \frac{1}{\sqrt{k}\ \mathrm{sn}\,(u,k)} \quad . \tag{39}$$

Since

$$z = j K' \tag{40}$$

corresponds to infinite ω, where F has a pole, $\mathrm{sn}\left(j(2m+1)\frac{K_1 K'}{K}, k_1\right)$ should be infinite. This will be the case, if

$$K_1' = (2m+1)\ \frac{K_1 K'}{K} \tag{41}$$

as Eq. (38) shows. Then, also by Eq. (38)

$$F = \frac{1}{k_1\,\mathrm{sn}\left(\frac{(2m+1)K_1}{K}\ u, k_1\right)} \tag{42}$$

and the attenuation poles are given by the reciprocals of the ω_i given by Eq. (36), as a comparison of Eqs. (31), (35), (39) and (42) shows.

Finally, in the transition band, $\sqrt{k} < \omega \leq \frac{1}{\sqrt{k}}$ transforms into

$$\begin{aligned} z &= K + jv \\ 0 &\leq v \leq K'. \end{aligned} \tag{43}$$

The unit frequency is transformed into $K + j\,\frac{K'}{2}$,

whence

$$F(1) = F_0 \prod_{i=1}^{n} \left(-\omega_i^2\right) = \frac{1}{\sqrt{k_1}} \tag{44}$$

and F_0 can be calculated.

Now all constants (F_0, ω_1, ω_2, ... ω_a, ω_b, ...) of Eq. (27) are known.

To calculate the order ($n-2m+1$) of F, Eq. (41) will be rewritten:

$$q_1 = q^n \tag{41/a}$$

where

$$q = e^{-\pi \frac{K'}{K}} \tag{42}$$

and

$$q_1 = e^{-\pi \frac{K_1'}{K_1}} \tag{43}$$

are the modular constants.[22] For small k_1 (which is practically always the case)

$$q_1 \cong \frac{k_1^2}{16} = \frac{1}{16} \frac{10^{\frac{A_p}{10}} - 1}{10^{\frac{A_s}{10}} - 1} = q^n \tag{44}$$

thus

$$n \geq \frac{2 \lg k_1 - 1{.}2}{\lg q} \tag{45}$$

and, neglecting -1 in the denominator of k_1^2

$$n \geq \frac{\lg\left(10^{\frac{A_p}{10}} - 1\right) - \frac{A_s}{10} - 1{.}2}{\lg q} . \tag{45/a}$$

Because of its great importance, nomograms and diagrams have been prepared to demonstrate Eq. (45/a) and its various terms. These are shown on Fig. 15.

Since q is a unique function of k, n is thereby given as a function A_p, A_s and k. Also, tables containing $\sqrt{k}\ \mathrm{sn}\left(\frac{i}{n} K, k\right)$ and F_0 are available.[20]

The natural frequencies of the terminated network can be calculated from Eq. (3) or from an approximation procedure described in the literature.[22]

The same analysis can be performed for even φ ($n = 2m$) antimetric networks).[1,2,36] Starting out again from Eq. (26) but with

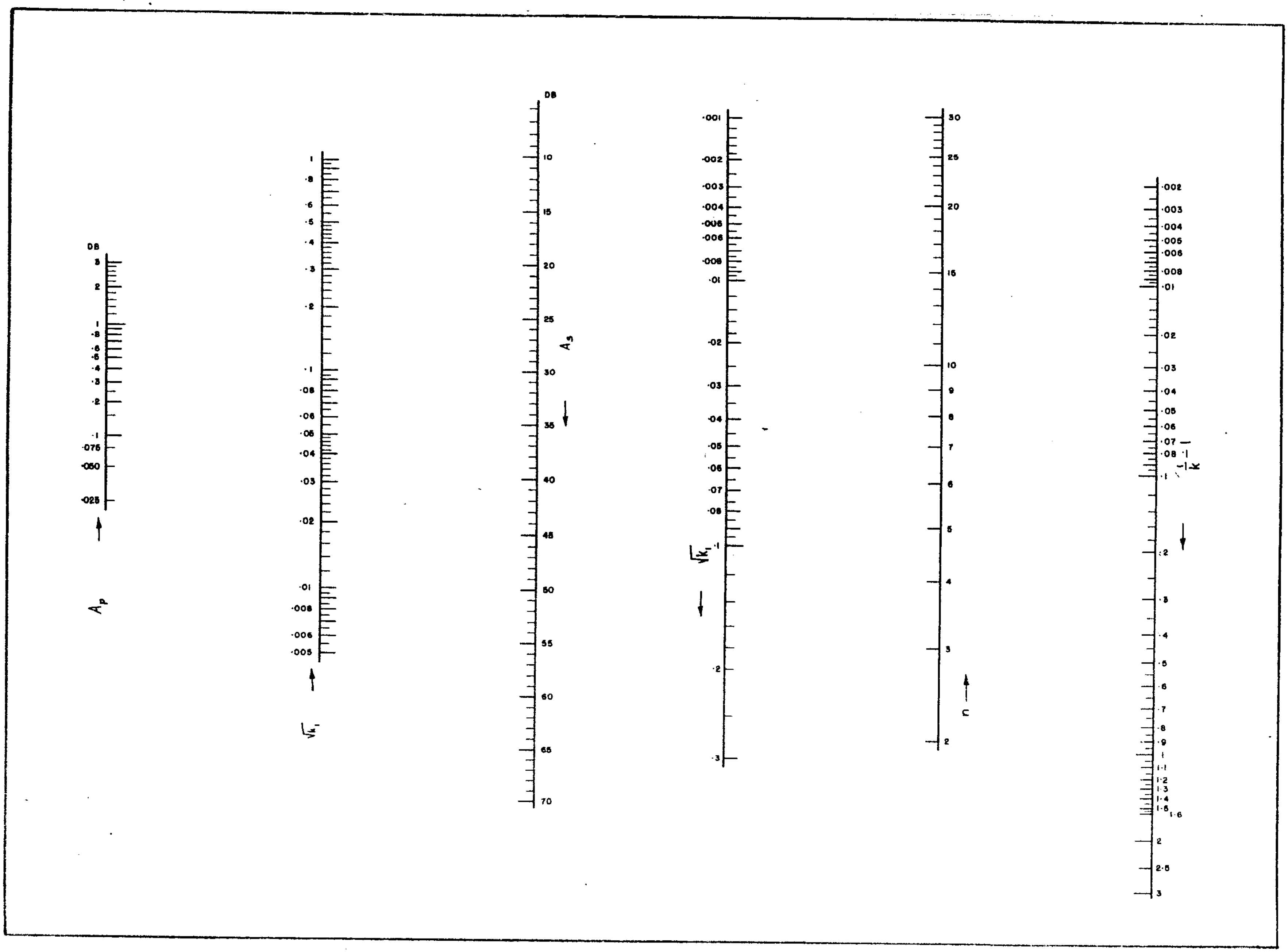


FIGURE 15. a.

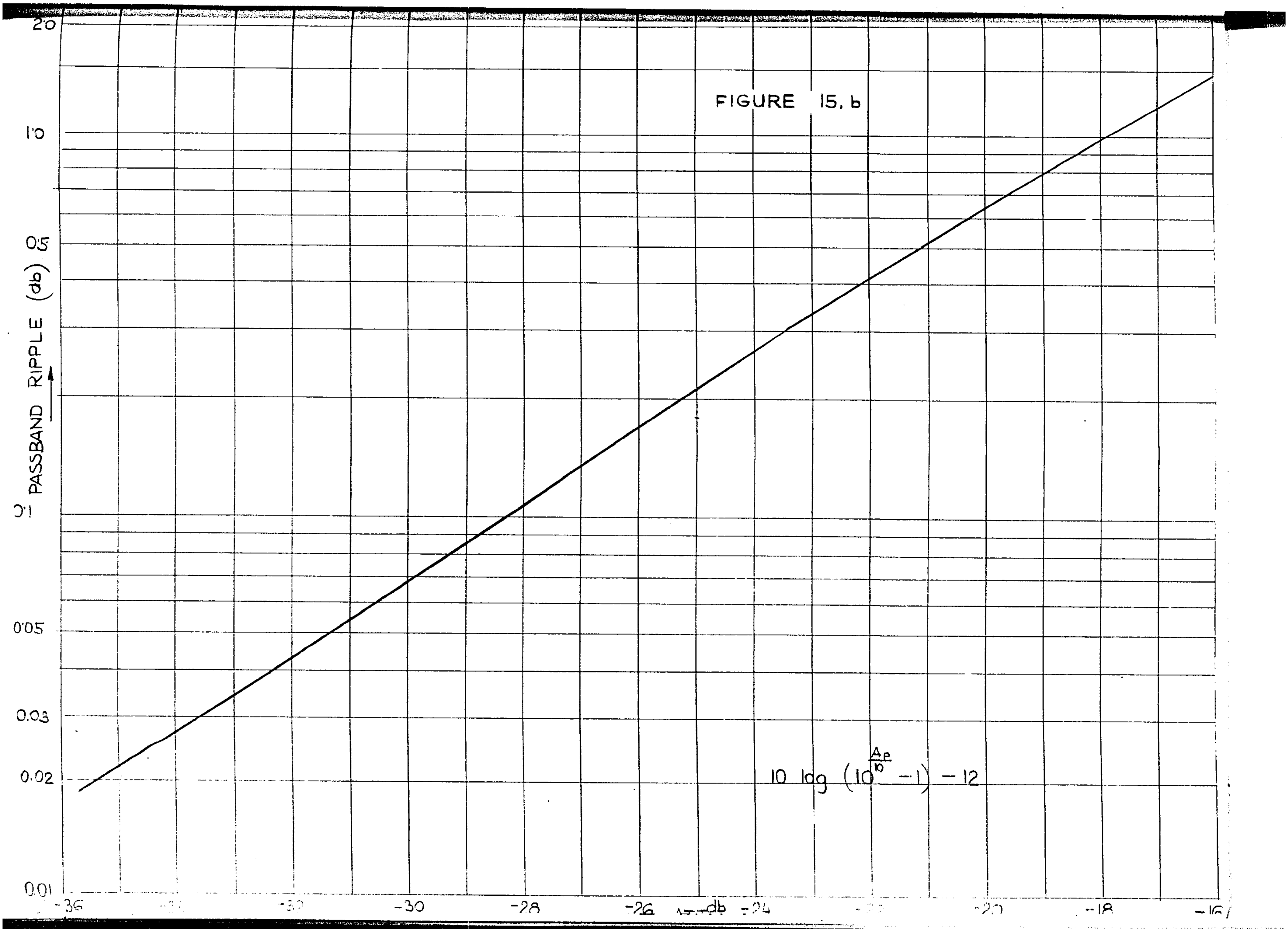
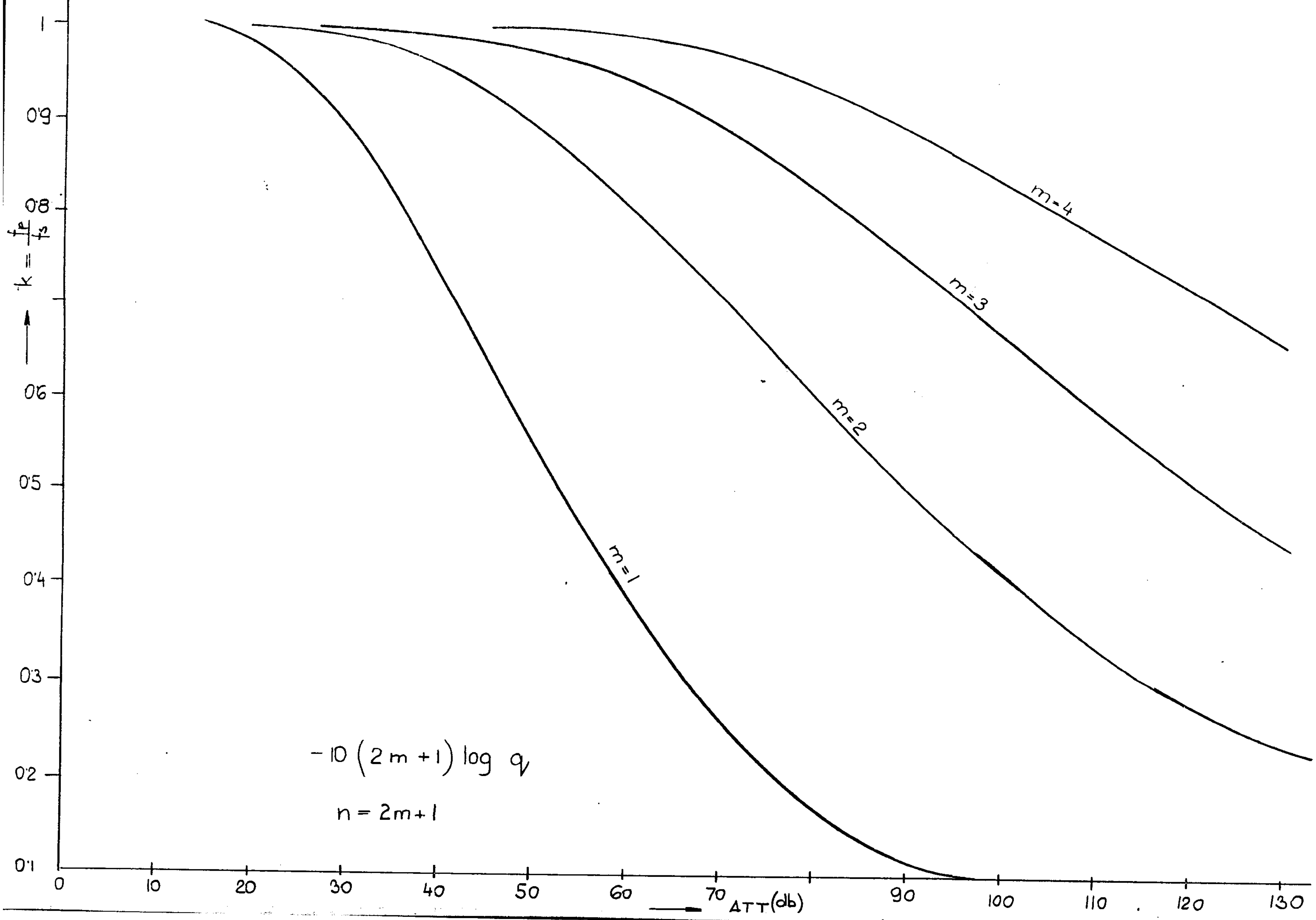


FIGURE 15, b
PASSBAND RIPPLE (db)
2.0
1.0
0.5
0.1
0.05
0.03
0.02
0.01
-36
-30
-28
-26
db
-24
-20
-18
-16
10 log (10^(Ae/10) - 1) - 12

FIGURE 15. c.



1
0.9
0.8
$k = \frac{f_p}{f_s}$
0.6
0.5
0.4
0.3
0.2
0.1
0
10
20
30
40
50
60
70
90
100
110
120
130
ATT (db)
m = 1
m = 2
m = 3
m = 4
$-10\,(2m+1)\log q$
$n = 2m+1$

$$F(\omega) = F_o \frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)\cdots}{(\omega^2 - \omega_a^2)(\omega^2 - \omega_b^2)\cdots} \qquad (46)$$

and carrying out exactly similar calculations, the attennuation zeros (ω_i) and poles ($\omega_i^{)}$) are obtained

$$\omega_i = \frac{1}{\omega_i^{)}} = \sqrt{k}\ \mathrm{sn}\left(\frac{2i-1}{n} K, k\right) \quad . \qquad (47)$$

Eqs. (44) - (45/a) will still hold.

However, as Eqs. (26) and (46) show, now

and
$$[\varphi]_{p=0} \neq 0 \qquad a)$$
$$[\varphi]_{p\to\infty} \not\to \infty \quad . \qquad b) \qquad (48)$$

Inequality (48a) rules out equal terminations, while the second inequality makes the use of mutual inductance necessary.

Bilinear frequency transformations can be used on ω^2 to shift the lowest finite zero to the frequency-origin and/or to shift the highest finite pole to infinite frequency. It is easy to see, that both measures decrease the selectivity slightly.

Since all elliptic networks possess finite attenuation poles, the "m-derived" configuration is necessary for the realized networks.

d, Realizability of elliptic ladder-networks. [23, 38]

The equations and derivations of Para I.5.e. can be used to establish explicit realizability criteria for an elliptic low-pass ladder network without mutual coupling. The results of the calculations are presented on Fig. 16.

The realizability condition can be expressed as

$$\Theta \leq \Theta_m$$

where Θ and Θ_m are the solutions of

$$sn^2\left[\Theta K_1', \ k_1'\right] = 10^{-\frac{A_p}{10}}$$

and

$$sc^2\left[(1-\Theta_m)K', k'\right] = sn\left(\frac{n-1}{n}K, k\right) sn\left(\frac{n-3}{n}K, k\right),$$

respectively. Here

$$sc(u,k) = \frac{sn(u,k)}{cn(u,k)}.$$

Fig. 16.a. illustrates the relation between Θ, k_1, and A_p. If the discrimination is very strong, i.e. if $k_1 << 1$, the approximation

$$\Theta \cong \frac{\log \coth \frac{A_p}{2}}{2 \log \frac{4}{k_1}}$$

can be used.

The equation connecting Θ_m, k and n is plotted on Fig. 16.b. for odd n-s (symmetrical networks). Again, for very sharp filters ($k \cong 1$), an approximation

$$\Theta_m \cong \frac{1}{2} + \frac{x}{\pi}\left[x^{1/n} + x^{-1/n} + x^{3/n} + x^{-3/n}\right]$$

can be used, where

$$x = \frac{1-k}{8k}.$$

In extreme cases

$$\Theta_m \approx \frac{1}{2}.$$

No realizability conditions restrict the design of the one-section filter ($\Theta_m = 1$) or the polynomial networks, where all attenuation poles are at infinity.

For flat pass-band, Chebyshev stop-band filters ($A_p \to 0$, $k \to 0$) the restriction

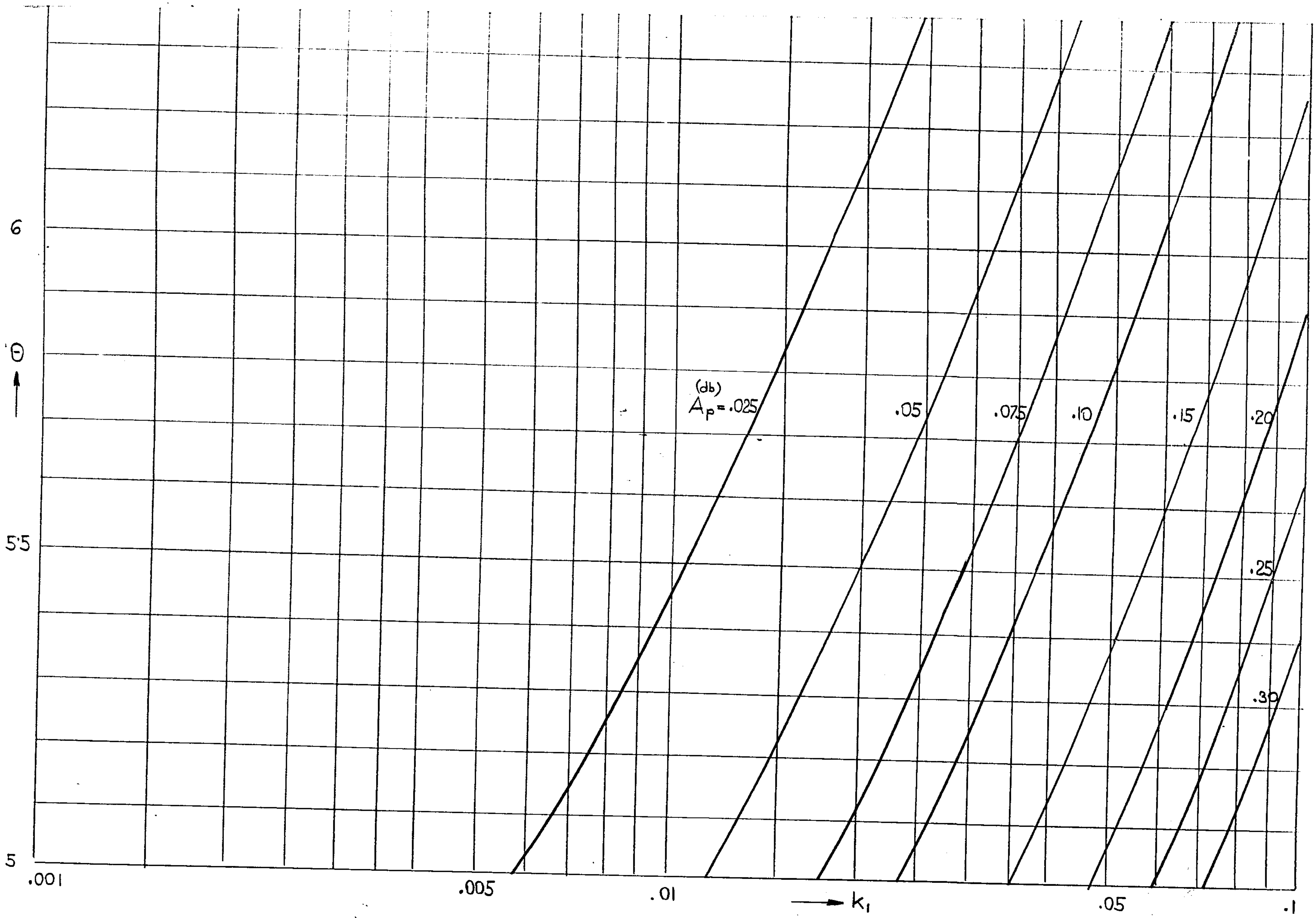
$$A_s^{(db)} \geq 20 \log \frac{1}{2}\left[\left(D+\sqrt{1+D^2}\right)^n + \left(D+\sqrt{1+D^2}\right)^{-n}\right]$$

results, with

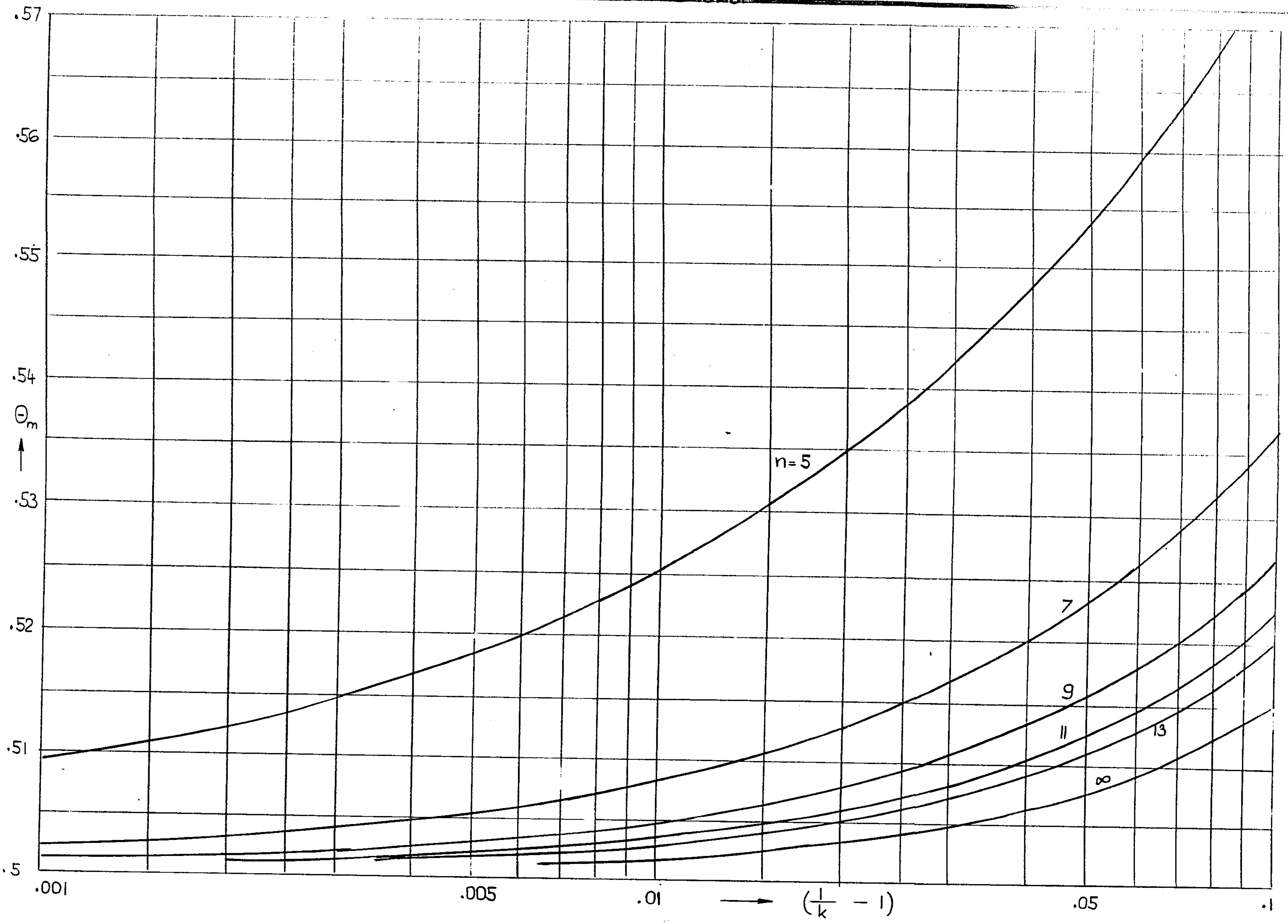
$$D^2 = \sin\frac{(n-1)\pi}{2n} \sin\frac{(n-3)\pi}{2n}.$$

If n is large, this reduces to

$$A_s^{(db)} \geq 7{\cdot}656\,n - 6{\cdot}011 .$$



6
Θ
5·5
5
.001
.005
.01
k₁
.05
·1
(db)
A_p = .025
.05
.075
.10
.15
.20
.25
.30


.57
.56
.55
.54
Θ_m
.53
.52
.51
.5
n=5
7
9
11
13
∞
.001
.005
.01
$(\frac{1}{k} - 1)$
.05
.1

e. A non-realizable low-pass function

The loss functions analyzed in this Chapter have loss minima with the value of zero and loss maxima that are infinite. Neither of these features is possible for a ladder network containing lossy elements. It is of interest therefore to derive a loss-function with finite minima and maxima and a Chebyshev response in both bands. It will be shown, that the only type of these functions that can be handled mathematically is not realizable with finite networks.

The acceptable attenuation response is shown on Fig. 17. Mathematically, for a dissipative four-pole

$$|S|^2 = \frac{P_{gen}}{P_{gen} - P_{diss}}\left[1 + |\varphi|^2\right] = 10^{\frac{A_p}{10}}\left[1 + \left(10^{\frac{A_p}{10}} - 1\right)F^2\right] \quad (49)$$

Carrying out an analysis similar to that yielding Eq. (28),

$$\frac{\frac{dF}{d\omega}}{\sqrt{(1-F^2)(F_a^{2}-F^2)\prod_{(i)}(F_{si}^2-F^2)}} = \pm \frac{M_0}{\sqrt{\left(1 - \frac{\omega^2}{k}\right)\left(1 - k\omega^2\right)}} \quad (50)$$

results. Here F_a' and F_{si} are the F-values corresponding to A_a and A_{si}, respectively.

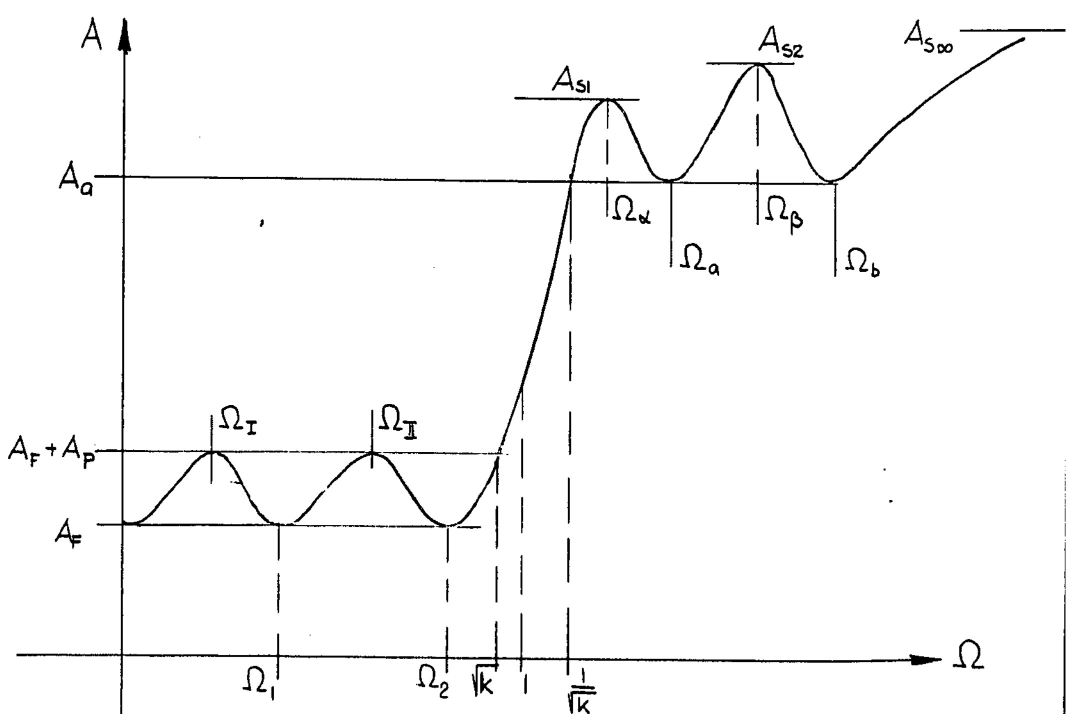
$$F_a^2 = \frac{10^{\frac{A_a - A_p}{10}} - 1}{10^{\frac{A_p}{10}} - 1} \cong k_1^{-2} \quad (51)$$

and

$$F_{si}^2 = \frac{10^{\frac{A_{si} - A_p}{10}} - 1}{10^{\frac{A_p}{10}} - 1}. \quad (52)$$

Eq. (50) leads to a hyperelliptic integral. It is only solvable, if

$$A_{s1} = A_{s2} = \ldots = A_s. \quad (53)$$



A
A_{S1}
A_{S2}
$A_{S\infty}$
A_a
Ω_α
Ω_a
Ω_β
Ω_b
Ω_I
Ω_{II}
$A_F + A_P$
A_F
Ω
Ω_1
Ω_2
$\sqrt{k}$
1
$\frac{1}{\sqrt{k}}$

FIGURE 17.

Then $$\int_0^F \frac{dF}{\sqrt{(1-F^2)(F_a^2-F^2)(F_s^2-F^2)}} = \frac{1}{2}\int_0^x \frac{dx}{\sqrt{x(1-x)(F_a^2-x)(F_s^2-x)}} =$$
$$= C_1\ u_1(\varphi, k_0) + C_2 \qquad (54)$$

with

$$x = F^2 = \frac{a \sin^2\varphi + b}{c \sin^2\varphi + d} \qquad (55)$$

and

$$u_1(\varphi, k_0) = \int_0^\varphi \frac{d\varphi}{\sqrt{1 - k_0^2 \sin^2\varphi}} \ . \qquad (56)$$

In the pass-and stop-band ($0 \leq F^2 \leq 1$ and $F_a^2 \leq F^2 \leq F_s^2$)

$$k_0^2 - \frac{F_s^2 - F_a^2}{F_a^2(F_s^2-1)} \qquad (57)$$

and

$$C_1 = \frac{2}{\sqrt{F_a^2\,(F_s^2-1)}} \ . \qquad (58)$$

Also, in the pass-band

$$\begin{aligned} a &- F_s^2 \\ b &- 0 \\ c &- 1 \\ d &- F_s^2 - 1 . \end{aligned} \qquad (59)$$

Now Eq. (50) will be rewritten (in the pass-band)

$$\omega - \sqrt{k}\ \mathrm{sn}(u, k)$$
$$F - \frac{F_s \sin\varphi}{\sqrt{F_s^2 - \cos^2\varphi}} = \frac{F_s\ \mathrm{sn}\,u_1}{\sqrt{F_s^2 - \mathrm{cn}^2 u_1}} \qquad (60)$$
$$C_1 u_1(\varphi, k_0) + C_2 - \pm M_0 \sqrt{k}\ u \ .$$

$F(\omega)$, as given by Eqs. (60) is not a rational expression, therefore it does not represent a realizable network function unless $\cos^2\varphi - \mathrm{cn}\,u_1$ is neglected. Then it reduces to the solution of Para. II.1.c.

II. 2. OPTIMAL SELECTIVITY OF HARMONIC SUPPRESSING FILTER SETS

An important test facility required in any telecommunication laboratory is a set of test filters. The outputs of most commercial oscillators have a harmonic content in the order of 1%. This is not suitable for many operations and must be improved by additional filtering. Measurement of harmonic distortion and intermodulation calls for separation of fundamental and higher order products and test filters may be used for this purpose.

The filter set may consist of low and high pass filters which could be used together to provide bandpass operation if required.

The purpose of this Section is to find the optimal selectivity for a set of low-pass (or high-pass) filters so that the number of components be kept to a minimum. The calculations involve all three types of approximations and the results emphasize the savings in components using elliptic filters when phase is not a prime parameter.

a. Frequency range and selectivity

The usual requirements for the test set are the following:

A. It should make possible the separation of signals and harmonic contents for any signal frequency in the range of interest.

B. It should provide a fairly high stop-band attenuation.

C. It should have a low passband ripple.

In the following discussion a test set consisting of low-pass filters only will be assumed. The same results apply for high-pass filters.

The low-pass filter can pass signals with frequencies below ω_p and stop signals with frequencies over ω_s. It will pass a signal of frequency ω and stop its harmonics, if

$\omega < \omega_p$ and $2\omega > \omega_s$

giving a useful frequency range

$$\frac{\omega_s}{2} < \omega < \omega_p . \tag{1}$$

If the frequency range to be covered by the filter set is ω_{low} to ω_{high} the filter having the lowest cut-off frequency in the set must have

$$\frac{\omega_s^{(1)}}{2} \leq \omega_{low} \tag{2}$$

Using (2) as an equation, the frequency range covered by this filter will be

$$\omega_{low} < \omega < \omega_{low}\, 2k \tag{3}$$

where $k = \dfrac{\omega_p^{(1)}}{\omega_s^{(1)}}$ is the selectivity parameter of this filter. If all members of the set are derived from the same normalized low-pass filter (which is almost exclusively the case), the selectivity parameter will be the same for all filters.

The useful frequency range of the second lowest filter is then going to be $\omega_{low}(2k)^1 < \omega < \omega_{low}(2k)^2$ and generally for the i-th filter

$$\omega_{low}(2k)^{i-1} < \omega < \omega_{low}(2k)^i . \tag{4}$$

If the number of filters in the set is F, the requirement against the F-th is $\omega_{low}\ (2k)^F \geq \omega_{high}$ giving

$$F \geq \frac{\lg\left(\frac{\omega_{high}}{\omega_{low}}\right)}{\lg(2k)} \tag{5}$$

The useful range is $0.5 < k < 1$. For low k values the number of filters required will be high ($F \to \infty$ for $k \to 0.5$), decreasing with increasing selectivity toward $\lg\left(\frac{\omega_{high}}{\omega_{low}}\right) \Big/ \lg 2$. On the other hand, the number of circuit elements per filter, n, increases rapidly with increasing k and $n \to \infty$ for $k \to 1$. Thus the total number of circuit elements in the filter set, $N = F \cdot n$, tends to infinity as k approaches either end of its range. Obviously, there is an optimum value of k, satisfying requirements A, B, and C, and at the same time requiring a minimum number of elements to be used in the set. To find this optimum value, another equation is needed, linking k to n and the attenuation and ripple specifications (A_s and A_p.). This is supplied by the design equation of the filter-type used in the set.

b. Optimum selectivity of a filter set consisting of Butterworth filters.

The design equation for Butterworth filter is Eq. (5) of Para. II.1.a.

$$n \geq \frac{\lg k_1}{\lg k} \tag{6}$$

where k_1 is the discrimation parameter given by the equation

$$k_1^2 = \frac{10^{0.1 A_p} - 1}{10^{0.1 A_s} - 1}.$$

It is useful to define the discrimation constant

$$G = -\lg k_1 \simeq 0.05 A_s - 0.5 \lg(0.23 A_p) \tag{7}$$

A_S and A_p must be expressed in decibels for substituting into Eq (7).

Using (5) and (6)

$$N = Fn \geq \frac{G \lg\left(\frac{\omega_{high}}{\omega_{low}}\right)}{(-\lg k)\ (\lg 2k)} \quad . \qquad (8)$$

The optimal value of k, leading to the minimum of N, can now be calculated

$$k_{opt} = \frac{\sqrt{2}}{2} \simeq 0.7071 \qquad (9)$$

giving

$$N_{opt} = \left(\frac{2}{\lg 2}\right)^2 G \lg\left(\frac{\omega_{high}}{\omega_{low}}\right) = 44.1\, G \lg\left(\frac{\omega_{high}}{\omega_{low}}\right) \quad . \qquad (10)$$

The value of the other set parameters

$$F_{opt} = \frac{2}{\lg 2} \lg\left(\frac{\omega_{high}}{\omega_{low}}\right) = 6.64 \lg\left(\frac{\omega_{high}}{\omega_{low}}\right) \quad ,$$

$$n_{opt} = \frac{2}{\lg 2} G \cong 6.64\, G \quad . \qquad (11)$$

Naturally, all parameters must be rounded up to the nearest integer.

<u>c. Optimum selectivity of a set of Chebyshev filters</u>

The design equation is

$$n \geqslant \frac{\cosh^{-1} k_1'}{\cosh^{-1} k'} \quad . \qquad (13)$$

To a very good approximation:

$$\cosh^{-1} k_1' \simeq \ln 2 k_1'^{-1} = 0.693 + 2.303 G$$

so

$$N = Fn = \frac{\lg\left(\frac{\omega_{high}}{\omega_{low}}\right)(0.693 + 2.303\, G)}{\lg(2k)\ \cosh^{-1}(1/k)} \qquad (14)$$

The optimum value of k : $k_{opt} \cong 0.7846$.

The other parameters

$$N_{opt} = \lg\left(\frac{\omega_{high}}{\omega_{low}}\right) \times (4.883 + 16.23\, G) \qquad (15)$$

$$F_{opt} = 5{\cdot}11 \lg\left(\frac{\omega_{high}}{\omega_{low}}\right) \qquad (16)$$

$$n_{opt} = 0{\cdot}956 + 3{\cdot}176\, G \ . \qquad (17)$$

d, Optimum selectivity of a set of elliptic filters

The design formula of elliptic filters is

$$m \geqslant \frac{\lg 4 + G}{-\lg q} - \frac{1}{2} \ . \qquad (18)$$

Here $q(k)$ is the modular constant or nome of modulus k and m is the number of filter-sections per filter. The number of elements per filter

$$n = 3m + 1 \geqslant 3 \ \frac{\lg 4 + G}{-\lg q} - \frac{1}{2} \ . \qquad (19)$$

So the total number of circuit elements in the filter set

$$N = Fn = \lg\left(\frac{\omega_{high}}{\omega_{low}}\right) \times \frac{3\ \dfrac{\lg 4 + G}{-\lg q} - \dfrac{1}{2}}{\lg 2k} \ . \qquad (20)$$

For all practical test filters, the second term in the numerator is much smaller than the first, so it is sufficient to find the minimum of

$$\frac{1}{(-\lg q)(\lg 2k)} \ .$$

The minimum will occur for

$$k_{opt} \cong 0{\cdot}8746 \qquad (21)$$

giving

$$N_{opt} = (5{\cdot}034 + 11{\cdot}78 G) \times \lg\left(\frac{\omega_{high}}{\omega_{low}}\right) \qquad (22)$$

$$F_{opt} = 4{\cdot}115 \lg\left(\frac{\omega_{high}}{\omega_{low}}\right) \qquad (23)$$

$$n_{opt} = 1{\cdot}223 + 2{\cdot}863\, G \ . \qquad (24)$$

Naturally, because of Eq. (19), n must be rounded up to the nearest integer having the form

$$n'_{opt} = 3m_{opt} + 1 \qquad\qquad m_{opt} = 1, 2, 3, \ldots \qquad (25)$$

where m_{opt} is the optimal number of filter sections per filter.

The results for the three filter groups are summarized in the Table.

Filter Type	Butterworth	Chebyshev	Elliptic
Selectivity	0'7071	0'7846	0'8746
Total number of components N	$44'1 \times G \times \lg\left(\frac{\omega_{high}}{\omega_{low}}\right)$	$(4'883 + 16'23 \times G) \times \lg\left(\frac{\omega_{high}}{\omega_{low}}\right)$	$(5'034 + 11'78 \times G) \times \lg\left(\frac{\omega_{high}}{\omega_{low}}\right)$
Number of filters in the set F	$6'64 \lg\left(\frac{\omega_{high}}{\omega_{low}}\right)$	$5'11 \lg\left(\frac{\omega_{high}}{\omega_{low}}\right)$	$4'115 \lg\left(\frac{\omega_{high}}{\omega_{low}}\right)$
Number of components per filter n	$6'64 \times G$	$0'956 + 3'176 \times G$	$1'223 + 2'863 \times G$
Number of sections per filter m	—	—	$0'0743 + 0'9543 \times G$

Table: Optimal values of filter set parameters for a set of low- or high-pass filters.

II. 3. THE APPROXIMATION OF CHEBYSHEV PASS-BAND ARBITRARY STOP-BAND (OR CONVERSE) BEHAVIOUR [7,15,16,34,36,38]

An important class of approximation problems involves the design of networks exhibiting nearly uniform low loss (Chebyshev approximation) in a pass-band and a stop-band attenuation that exceeds consistently a frequency dependent minimum. Alternatively, the specified maximum loss may vary across the pass-band and the stop-band minimum be constant. The design of such filters involves the combination of mathematical and graphical procedures. These will be described in this Section for the most significant cases.

It will be adequate to treat the first situation (Chebyshev pass-band, arbitrary stop-band). The alternative response can then be obtained from it through the steps discussed in Para. II. 1.b. Also, it will initially be assumed, that the suitable location of the attenuation poles is given; Para. II. 3.e. will demonstrate a method for the derivation of the pole-frequencies.

a. The reference-filter method

Chebyshev pass-band response will be exhibited by a network, if its characteristic function is chosen to be

$$\varphi = \sqrt{10^{\frac{A_p}{10}} - 1} \quad \frac{e^{\theta} \pm e^{-\theta}}{2} \qquad (1)$$

with $\theta(p)$ obeying the following restrictions:

1, $e^{\theta} \pm e^{-\theta}$ is a rational function.

2. Θ is imaginary in the pass-band.

3. Θ is of the form $a+jn\pi$ in the stop-band, where 'a' is real and has the (supposedly known) attenuation poles.

Evidently, by Para. I.3.d. the first restriction is a sufficient and necessary realizability condition of φ. The second condition assures $0 \leq |\varphi|^2 \leq 10^{\frac{A_p}{10}}-1$, i.e. Chebyshev-response with the prescribed ripple in the pass-band. Finally, the last restraint provides the necessary attenuation poles.

Observing now Eqs. (29) and (29/a) of Sec. I.1., it is easy to see, that choosing the positive sign in Eq. (1) and selecting as $\Theta(p)$ the image propagation function of a (hypothetical) symmetric image parameter filter satisfies the first restriction. If this so-called reference filter has the specified pass-band and its image propagation zeros are at the postulated pole-frequencies, all conditions are met. The (real) filter to be obtained has then an even φ and is thus antimetric.

Alternatively, the choice of the negative sign in Eq. (1) and an antimetric reference filter will lead (by Eqs. (30) and (30/a) of Sec. I.1.) to a symmetric filter satisfying all necessary conditions. No other choice of reference filter will result in a rational expression for φ; hence only symmetric and antimetric filters can be synthesized by this procedure.

It is important to notice, that the image propagation function of the reference filter must contain all attenuation poles, hence its operating loss will contain the poles coinciding with the reflection peaks (the zeros and poles of the image impedances) in duplicate. Therefore, its (hypothetical) circuit-diagram is normally somewhat more complicated than that of the real network.

b, The design of general parameter low-pass filters[1,2,16]

In the following synthesis of Chebyshev pass-band, arbitrary stop-band (alternatively called general parameter) low-pass networks will be discussed. The analysis will be restricted to antimetric circuits; the design of symmetric networks follows substantially similar lines.

As shown in Para. II.3.a., the characteristic function of such filter is given by

$$\varphi = \sqrt{10^{\frac{A_p}{10}} - 1}\ \cosh\Theta = \sqrt{10^{\frac{A_p}{10}} - 1}\ \frac{Z_{11I}}{Z_{12I}}, \qquad (2)$$

where Θ is the image propagation function and the Z_{ijI} are the impedance parameters of a symmetric image parameter filter. Since the sections of the reference-filter are matched,

$$\Theta = \sum_{(i)} \Theta_i \qquad (3)$$

where the Θ_i are exhibited by the individual sections. If one attenuation pole is associated with each (symmetric) section, one has from Eqs. (29/a) and (32) of Sec. I.1.

$$\Theta_i = \cosh^{-1}\left(\frac{Z_{Ai}+Z_{Bi}}{Z_{Bi}-Z_{Ai}}\right) = \cosh^{-1}\left(\frac{q_i^2+1}{q_i^2-1}\right) \qquad (4)$$

and

$$e^{\Theta_i} = \frac{q_i+1}{q_i-1} \qquad (5)$$

with

$$q_i = \sqrt{\frac{Z_{Bi}}{Z_{Ai}}} \quad . \qquad (6)$$

The attenuation poles $\Omega_{\infty i}$ therefore must coincide with the one-points of the q_i (frequencies, where $q_i = 1$).

From Eq. (3), for the cascade of σ symmetrical reference-filter sections

$$e^{\Theta} = \frac{q+1}{q-1} = \prod_{i=1}^{\sigma} \frac{q_i+1}{q_i-1} \, . \qquad (7)$$

If the q_i are chosen as

$$q_i = \frac{m_i p}{\sqrt{p^2+1}} \qquad (8)$$

with

$$m_i^2 = 1 - \Omega_{\infty i}^{-2} , \qquad (9)$$

the following observations can be made:

1. The one-points will be correctly located at $\Omega_{\infty i}$.

2. The second and third restrictions of Para. II.3.a. will be satisfied for $\omega_p = 1$.

It is also easy to see that Θ of Eq. (7) corresponds to a symmetrical reference-filter.

However,

$$\lim_{p\to\infty} \Theta \not\to \infty ,$$

therefore, the real filter would have finite attenuation at infinite frequencies, requiring mutual coupling.

This objection can be removed by making

$$q_6 = \frac{p}{\sqrt{p^2+1}} \tag{10}$$

corresponding to

$$\Omega_{\infty 6} \to \infty \,. \tag{11}$$

Using Eqs. (2) - (5),

$$\varphi = \sqrt{10^{\frac{A_p}{10}}-1}\;\frac{1}{2}\left(\frac{q+1}{q-1}+\frac{q-1}{q+1}\right) = \frac{1}{2}\sqrt{10^{\frac{A_p}{10}}-1}\left(\prod_{i=1}^{6}\frac{q_i+1}{q_i-1}+\prod_{i=1}^{6}\frac{q_i-1}{q_i+1}\right). \tag{12}$$

From Eq. (8), after some straightforward calculations.

$$\varphi = \frac{1}{2}\sqrt{10^{\frac{A_p}{10}}-1}\;\frac{\prod_{i=1}^{6}(m_i p + w)^2 - \prod_{i=1}^{6}(m_i p - w)^2}{2\prod_{i=1}^{5}\left(1+\frac{p^2}{\Omega_{\infty i}^2}\right)} \tag{13}$$

results. Here

$$w^2 = p^2+1, \tag{14}$$

but the φ of Eq. (13) is nevertheless rational, since all odd powers of w (and p) cancel in it.

It is important to notice, that for the limiting process of Eq. (11), φ (and thus the attenuation) will have a double pole at infinity.

For a symmetric filter, an analog derivation results in a similar φ, but the first term in the numerator will be multiplied by (p + w), the second by (p - w). This is due to the matched half-section necessarily used in the (antimetric) reference filter.

c. Step-by-step design of symmetrical Chebyshev pass-band low and high-pass filters 38.

For the special case of symmetry, the theory outlined in Para I.4.d. (Eqs. (23) - (24)) can be used to derive the lattice-impedances. If an m-derived configuration is used, explicit relations may be deducted between the coefficients of Z_A, and Z_B, and the ladder element values. By this method the following step-by-step design procedure is obtained for 1, 2, 3, and 4 section filters (low-pass or high-pass):

A. 1 pole (1 section) general parameter filters

1. Given: f_p pass band limit frequency in c/s

f_∞ pole frequency in c/s

$R_{gen} = R_{load}$ terminating impedances in ohms

A_p pass band ripple in db-s

2. $$\Omega_\infty = \frac{f_\infty}{f_p} \qquad \text{for low pass}$$

$$\Omega_\infty = \frac{f_p}{f_\infty} \qquad \text{for high pass}$$

3. $$m = \frac{\sqrt{\Omega_\infty^2 - 1}}{\Omega_\infty}$$

4. $$Z = \frac{1}{\Omega_\infty^2}$$

5. $$b_0 = 1 + 2m$$

$$b_2 = (1+m)^2$$

6. $$H = \sqrt{10^{0.1 A_p}} :$$

7. $$p^3 + \frac{Z}{b_2 H} p^2 + \frac{b_o}{b_2} p + \frac{1}{b_2 H} = 0$$

The solution of the equation gives a negative real root ($-r_1$) and a pair of conjugate complex roots with positive real parts ($r_2 \pm j\, i_2$).

Here $r_1,\ r_2,\ i_2 > 0$.

8. $$a_1 = a_3 = \frac{1}{r_1}$$

$$a_2 = \frac{4 r_2}{r_2^2 + i_2^2}$$

9. $$C_o = \frac{1}{2\pi f_p R_{gen}}$$

$$L_o = R_{gen}^2 C_o$$

10. a. For low pass, mid-series network (Fig. 18a)

$$L_1 = L_3 = a_1 L_o$$

$$L_2 = \frac{Z}{a_2} L_o$$

$$C_2 = a_2 C_o$$

b. Low pass, mid shunt network (Fig. 18b)

$$C_1 = C_3 = a_1 C_o$$

$$C_2 = \frac{Z}{a_2} C_o$$

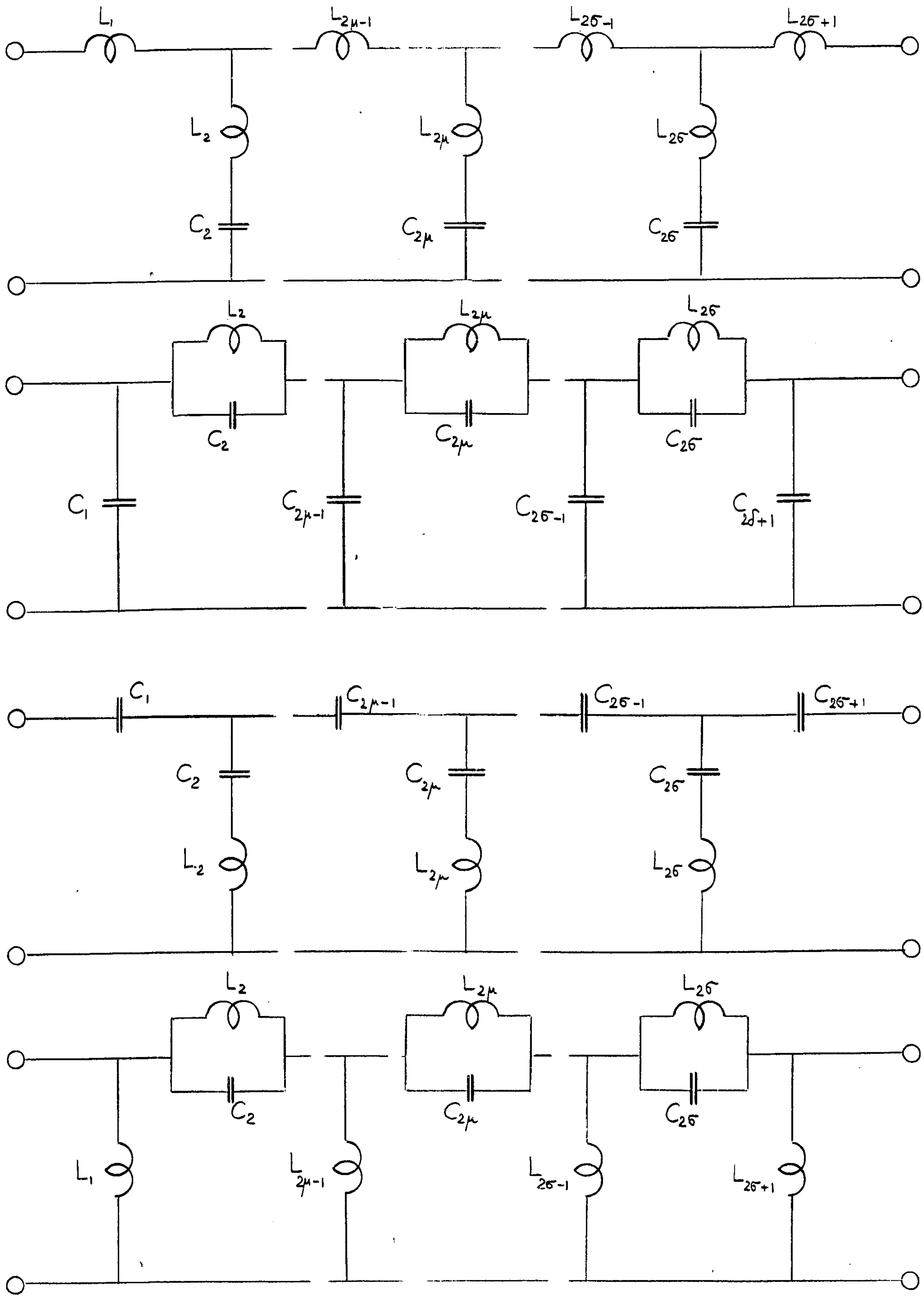
$$L_2 = a_2 L_o$$

c. High pass, mid-series (Fig. 18c)

$$C_1 = C_3 = \frac{C_o}{a_1}$$

$$C_2 = \frac{a_2}{Z} C_o$$

$$L_2 = \frac{L_o}{a_2}$$



L_1
$L_{2\mu-1}$
$L_{2\sigma-1}$
$L_{2\sigma+1}$
L_2
$L_{2\mu}$
$L_{2\sigma}$
C_2
$C_{2\mu}$
$C_{2\sigma}$
L_2
$L_{2\mu}$
$L_{2\sigma}$
C_2
$C_{2\mu}$
$C_{2\sigma}$
C_1
$C_{2\mu-1}$
$C_{2\sigma-1}$
$C_{2\sigma+1}$
C_1
$C_{2\mu-1}$
$C_{2\sigma-1}$
$C_{2\sigma+1}$
C_2
$C_{2\mu}$
$C_{2\sigma}$
L_2
$L_{2\mu}$
$L_{2\sigma}$
L_2
$L_{2\mu}$
$L_{2\sigma}$
C_2
$C_{2\mu}$
$C_{2\sigma}$
L_1
$L_{2\mu-1}$
$L_{2\sigma-1}$
$L_{2\sigma+1}$

FIGURE 18.

d. High pass, mid-shunt (Fig. 18d) $L_1 = L_3 = \frac{L_o}{a_1}$

$$L_2 = \frac{a_2}{Z} L_o$$

$$C_2 = \frac{C_o}{a_2}$$

B. 2 pole (2 section) filters

1. Input data:

Pass band limit, f_p

Pass band ripple, A_p

Pole frequencies, $f_{\infty 1}$

$f_{\infty 2}$

Termination: $R_{gen} = R_{load}$

2.

$$\Omega_{\infty i} = \frac{f_{\infty i}}{f_p} \qquad \text{for low-pass}$$

$$\Omega_{\infty i} = \frac{f_p}{f_{\infty i}} \qquad \text{for high-pass}$$

3. $$m_i = \frac{\sqrt{\Omega_{\infty i}^2 - 1}}{\Omega_{\infty i}}$$

4. $$z_i = \frac{1}{\Omega_{\infty i}^2}$$

5. $$\delta_1 = m_1 + m_2$$

$$\delta_2 = m_1 m_2$$

6. $$H = \sqrt{10^{0.1 A_p} - 1}$$

7. $$b_0 = 1 + 2\delta_1$$

$$b_2 = b_0 + b_4 - \delta_2^2$$

$$b_4 = (1 + \delta_1 + \delta_2)^2$$

8. $$p^5 + \frac{z_1 z_2}{b_4 H} p^4 + \frac{b_2}{b_4} p^3 + \frac{z_1 + z_2}{b_4 H} p^2 + \frac{b_0}{b_4} p + \frac{1}{b_4 H} = 0$$

The solution of this equation gives a negative real root ($-r_1$) and two pairs of conjugate complex roots, one pair with negative real parts ($-r_2 \pm j i_2$) and one pair with positive real parts ($r_4 \pm j i_4$). Here

$$r_1,\ r_2,\ r_4,\ i_2,\ i_4 > 0 .$$

9.
$$\gamma_1 = \frac{1}{r_1 \left(r_2^2 + i_2^2 \right)}$$
$$\gamma_2 = \gamma_1 \left(2 r_1 r_2 + r_2^2 + i_2^2 \right)$$
$$\gamma_3 = \frac{1}{2 r_4}$$
$$\gamma_4 = \gamma_3 \left(r_4^2 + i_4^2 \right)$$
$$\beta_1 = \gamma_1 \left(r_1 + 2 r_2 \right)$$

10.
$$a_1 = \gamma_3 - \gamma_4 z_1$$
$$a_2 = 2 \ \frac{\beta_1 - z_1}{a_3}$$
$$a_3 = \gamma_4 \left(2\beta_1 - z_1 - z_2 \right)$$
$$a_4 = 2 \ \frac{\beta_1 - z_2}{a_3}$$
$$a_5 = \gamma_3 - \gamma_4 z_2$$

11.
$$C_o = \frac{1}{2 \pi f_p R_{gen}}$$
$$L_o = R_{gen}^2 C_o$$

12. a. Low-pass, mid-series filter

$$L_1 = a_1 L_o$$
$$L_2 = \frac{z_1}{a_2} L_o \qquad C_2 = a_2 C_o$$
$$L_3 = a_3 L_o$$
$$L_4 = \frac{z_2}{a_4} L_o \qquad C_4 = a_4 C_o$$
$$L_5 = a_5 L_o$$

b. For low-pass, mid-shunt filter

$$C_1 = a_1 C_0$$

$$C_2 = \frac{Z_1}{a_2} C_0 \qquad L_2 = a_2 L_0$$

$$C_3 = a_3 C_0$$

$$C_4 = \frac{Z_3}{a_4} C_0 \qquad L_4 = a_4 L_0$$

$$C_5 = a_5 C_0$$

c. For high-pass, mid-series filter

$$C_1 = \frac{C_0}{a_1}$$

$$C_2 = \frac{a_2}{Z_1} C_0 \qquad L_2 = \frac{L_0}{a_2}$$

$$C_3 = \frac{C_0}{a_3}$$

$$C_4 = \frac{a_4}{Z_2} C_0 \qquad L_4 = \frac{L_0}{a_4}$$

$$C_5 = \frac{C_0}{a_5}$$

d. for high pass, mid-shunt filter

$$L_1 = \frac{L_0}{a_1}$$

$$L_2 = \frac{a_2}{Z_1} L_0 \qquad C_2 = \frac{C_0}{a_2}$$

$$L_3 = \frac{L_0}{a_3}$$

$$L_4 = \frac{a_4}{Z_2} L_0 \qquad C_4 = \frac{C_0}{a_4}$$

$$L_5 = \frac{L_0}{a_5}$$

C. 3 pole (3 section) filters

1. Input data:

f_p pass-band limit frequency

A_p pass-band ripple

$f_{\infty 1}$, $f_{\infty 2}$, $f_{\infty 3}$ pole frequencies

$R_{gen} = R_{load}$ terminations

2.

$$\Omega_{\infty i} = \frac{f_{\infty}}{f_p} \qquad i = 1,\ 2,\ 3$$ for low pass

$$\Omega_{\infty i} = \frac{f_p}{f_{\infty i}}$$ for high pass

3. $$m_i = \frac{\sqrt{\Omega^2_{\infty i} - 1}}{\Omega_{\infty i}}$$

4. $$Z = \frac{1}{\Omega^2_{\infty i}}$$

5. $$\sigma_1 = m_1 + m_2 + m_3$$

$$\sigma_2 = m_1 m_2 + m_1 m_3 + m_2 m_3$$

$$\sigma_3 = m_1 m_2 m_3$$

6. $$b_0 = 1 + 2\sigma_1$$

$$b_2 = 3b_0 + 2(\sigma_1\sigma_2 + \sigma_2 + \sigma_3) + \sigma_1^{\,2}$$

$$b_4 = b_2 + b_6 - b_0 - \sigma_3^{\,2}$$

$$b_6 = (1 + \sigma_1 + \sigma_2 + \sigma_3)^2$$

7. $$H = \sqrt{10^{0.1 A_{p\,dB}} - 1}$$

8. $$p^7 + \frac{Z_1 Z_2 Z_3}{b_6 H} p^6 + \frac{b_4}{b_6} p^5 + \frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}{b_6 H} p^4 + \frac{b_2}{b_6} p^3 + \frac{Z_1 + Z_2 + Z_3}{b_6 H} p^2 + \frac{b_0}{b_6} p + \frac{1}{b_6 H} = 0$$

The roots with negative real parts:

$$-r_1, \; -r_2 \pm j i_2 .$$

The roots with positive real parts

$$r_3 \pm j i_3, \quad r_4 \pm j i_4 ,$$

where again all

$$r_k, \; i_k > 0.$$

9. $$\rho_k^2 = r_k^2 + i_k^2 \qquad k = 1, 2, 3, 4$$

10. $$\gamma_1 = \left(r_1 \rho_2^2 \right)^{-1} \qquad \beta_1 = \left(r_1 + 2 r_2 \right) \gamma_1$$

$$\gamma_2 = \left(2 r_1 r_2 + \rho_2^2 \right) \gamma_1$$

$$\gamma_3 = \left[2 \left(r_3 \rho_4^2 + r_4 \rho_3^2 \right) \right]^{-1} \qquad \beta_2 = 2 \left(r_3 + r_4 \right) \gamma_3$$

$$\gamma_0 = \rho_3^2 \rho_4^2 \gamma_3$$

11. $$T_1 = \left(\beta_1 - z_1 \right)\left(\beta_2 - z_3 \right) + \left(\beta_2 - z_1 \right)\left(\beta_1 - z_3 \right)$$

$$T_2 = 2 \left(\beta_1 - z_1 \right)\left(\beta_1 - z_3 \right)\left(\beta_2 - z_1 \right) + \left(z_1 - z_2 \right) T_1$$

$$T_3 = 2 \left(\beta_1 - z_1 \right)\left(\beta_1 - z_3 \right)\left(\beta_2 - z_3 \right) + \left(z_3 - z_2 \right) T_1$$

$$T_4 = \frac{\gamma_2}{\beta_1 - \beta_2}$$

12. $$a_1 = \frac{\gamma_1 - \gamma_2 z_1}{\beta_1 - z_1} \qquad a_2 = \frac{2 \left(\beta_1 - z_1 \right)\left(\beta_2 - z_1 \right)}{T_2 T_4}$$

$$a_3 = \frac{T_2 T_4}{T_1}\left(\beta_1 - z_1 \right) \qquad a_4 = \frac{2 \left(\beta_1 - z_2 \right)\left(\beta_2 - z_2 \right) T_1^2}{\left(\beta_1 - \beta_2 \right) T_2 T_3 T_4}$$

$$a_5 = \frac{T_3 T_4}{T_1}\left(\beta_1 - z_1 \right) \qquad a_6 = \frac{2 \left(\beta_1 - z_3 \right)\left(\beta_2 - z_3 \right)}{T_3 T_4}$$

$$a_7 = \frac{\gamma_1 - \gamma_2 z_3}{\beta_1 - z_3}$$

13.

$$C_o = \frac{1}{2\pi f_p R_{gen}}$$

$$L_o = R_{gen}^2 C_o$$

14. a. Low pass mid-series

$$L_{2\mu\pm1} = a_{2\mu\pm1} L_o \qquad \mu = 1, 2, 3$$

$$L_{2\mu} = \frac{z_\mu}{a_{2\mu}} L_o$$

$$C_{2\mu} = a_{2\mu} C_o$$

b. Low pass mid-shunt

$$C_{2\mu\pm1} = a_{2\mu\pm1} C_o \qquad \mu = 1, 2, 3$$

$$C_{2\mu} = \frac{z_\mu}{a_{2\mu}} C_o$$

$$L_{2\mu} = a_{2\mu} L_o$$

c. High pass, mid-series

$$C_{2\mu\pm1} = \frac{C_o}{a_{2\mu\pm1}} \qquad \mu = 1, 2, 3$$

$$C_{2\mu} = \frac{a_{2\mu}}{z_\mu} C_o$$

$$L_{2\mu} = \frac{L_o}{a_{2\mu}}$$

d. High pass, mid-shunt

$$L_{2\mu\pm1} = \frac{L_o}{a_{2\mu\pm1}} \qquad \mu = 1, 2, 3$$

$$L_{2\mu} = \frac{a_{2\mu}}{z_\mu} L_o$$

$$C_{2\mu} = \frac{C_o}{a_{2\mu}}$$

D. 4 pole (4 section) filters

1. Input data

$$f_p,\ f_{\infty 1},\ f_{\infty 2},\ f_{\infty 3},\ f_{\infty 4}$$

$$R_{gen} = R_{load},\qquad A_p$$

2. $$\Omega_{\infty i} = \frac{f_{\infty i}}{f_P} \qquad \text{for low pass}$$

$$\Omega_{\infty i} = \frac{f_P}{f_{\infty i}} \qquad \text{for high pass}$$

3. $$Z_i = \frac{1}{\Omega_{\infty i}^2}$$

4. $$m_i = \sqrt{1 - z_i}$$

5. $$\delta_1 = m_1 + m_2 + m_3 + m_4$$

$$\delta_2 = m_1m_2 + m_1m_3 + m_1m_4 + m_2m_3 + m_2m_4 + m_3m_4$$

$$\delta_3 = m_1m_2m_3 + m_1m_2m_4 + m_1m_3m_4 + m_2m_3m_4$$

$$\delta_4 = m_1m_2m_3m_4$$

6. $$b_0 = 1 + 2\delta_1$$

$$b_2 = 4b_0 + 2(\delta_1\delta_2 + \delta_2 + \delta_3) + \delta_1^2$$

$$b_4 = 3b_2 - 6b_0 + \delta_2^2 + 2\left[\delta_3(\delta_1 + \delta_2) + \delta_4(1 + \delta_1)\right]$$

$$b_6 = b_0 + b_4 + b_8 - b_2 - \delta_4^2$$

$$b_8 = (1 + \delta_1 + \delta_2 + \delta_3 + \delta_4)^2$$

7. $$H = \sqrt{10^{0.1 A_p(db)} - 1}$$

8. $$\mathcal{E}_1 = Z_1 + Z_2 + Z_3 + Z_4 \qquad \mathcal{E}_2 = Z_1Z_2 + Z_1Z_3 + Z_1Z_4 + Z_2Z_3 + Z_2Z_4 + Z_3Z_4$$

$$\mathcal{E}_3 = Z_1Z_2Z_3 + Z_1Z_2Z_4 + Z_1Z_3Z_4 + Z_2Z_3Z_4 \qquad \mathcal{E}_4 = Z_1Z_2Z_3Z_4$$

9. $$p^9 + \frac{\mathcal{E}_4}{b_8H}p^8 + \frac{b_6}{b_8}p^7 + \frac{\mathcal{E}_3}{b_8H}p^6 + \frac{b_4}{b_8}p^5 +$$

$$+ \frac{\mathcal{E}_2}{b_8H}p^4 + \frac{b_2}{b_8}p^3 + \frac{\mathcal{E}_1}{b_8H}p^2 + \frac{b_0}{b_8}p + \frac{1}{b_8H} = 0$$

Roots with negative real parts:

$$-r_1 , \quad -r_2 \pm j i_2 , \quad -r_3 \pm j i_3 \quad .$$

Roots with positive real parts:

$$r_4 \pm j i_4 , \quad r_5 \pm j i_5 \quad .$$

10. $$p_k^2 = r_k^2 + i_k^2$$

11. $$\gamma_1 = \left(2 r_1 p_2^2 p_3^2\right)^{-1}$$

$$\gamma_2 = \gamma_1 \left[p_2^2 (r_1 + 2 r_3) + p_3^2 (r_1 + 2 r_2) + 4 r_1 r_2 r_3 \right]$$

$$\gamma_3 = 2 \gamma_1 \left(r_1 + 2 r_2 + 2 r_3 \right)$$

$$\gamma_4 = \left[2 \left(r_4 p_5^2 + r_5 p_4^2 \right) \right]^{-1}$$

$$\gamma_5 = \gamma_4 \left(p_4^2 + p_5^2 + 4 r_4 r_5 \right)$$

$$\gamma_6 = \gamma_4 \, p_4^2 \, p_5^2$$

$$\beta_1 = 2 \gamma_4 \left(r_4 + r_5 \right)$$

$$\beta_2 = \gamma_2 + \sqrt{\gamma_2^2 - \gamma_3^2}$$

$$\beta_3 = 2 \gamma_2 - \beta_2$$

12. $$d_{ij} = \beta_i - z_j \qquad i, j = 1, 2, 3, 4$$

13. $$T_1 = d_{13} d_{14} + (\beta_2 - \beta_1)(\beta_1 - \beta_3)$$

$$T_2 = d_{11} d_{12} + (\beta_2 - \beta_1)(\beta_1 - \beta_3)$$

$$T_3 = d_{11} d_{24} d_{34} + d_{14} d_{21} d_{31}$$

$$T_4 = d_{12} d_{23} d_{33} + d_{13} d_{22} d_{32}$$

$$T_5 = d_{13} T_3 + (z_4 - z_1) d_{11} T_1$$

$$T_6 = d_{12} T_3 + (z_1 - z_4) d_{14} T_2$$

$$T_7 = 2\, d_{11} d_{21} d_{31} T_1 + (z_1 - z_2)\, T_5$$

$$T_8 = 2\, d_{14} d_{24} d_{34} T_2 + (z_4 - z_3)\, T_6$$

$$T_9 = T_3 T_4 + (z_4 - z_1)(z_3 - z_2)\, T_1 T_2$$

$$T_{10} = \frac{2(\beta_2 - \beta_1)(\beta_1 - \beta_3)}{\gamma_6}$$

10.

$$a_1 = \frac{\gamma_4 - \gamma_5 z_1 + \gamma_6 z_1^2}{d_{11}}$$

$$a_2 = \frac{d_{11} d_{21} d_{31} T_{10}}{T_7}$$

$$a_3 = \frac{2 T_1 T_7}{T_5 T_{10}}$$

$$a_4 = \frac{2 d_{12} d_{22} d_{32} T_5^2}{\gamma_6 T_7 T_9}$$

$$a_5 = \frac{\gamma_6 T_3 T_9}{T_5 T_6}$$

$$a_6 = \frac{2 d_{13} d_{23} d_{33} T_6^2}{\gamma_6 T_8 T_9}$$

$$a_7 = \frac{2 T_2 T_8}{T_6 T_{10}}$$

$$a_8 = \frac{d_{14} d_{24} d_{34} T_{10}}{T_8}$$

$$a_9 = \frac{\gamma_4 - \gamma_5 z_4 + \gamma_6 z_4^2}{d_{14}}$$

11. a. Low pass mid-series

$$L_{2\mu\pm 1} = a_{2\mu\pm 1} L_0 \qquad \mu = 1, 2, 3, 4$$

$$L_{2\mu} = \frac{z_\mu}{a_{2\mu}} L_0$$

$$C_{2\mu} = a_{2\mu} C_0$$

b. Low pass mid-shunt

$$C_{2\mu\pm1} = a_{2\mu\pm1} C_o \qquad \mu = 1, 2, 3, 4$$
$$C_{2\mu} = \frac{Z_\mu}{a_{2\mu}} C_o$$
$$L_{2\mu} = a_{2\mu} L_o$$

c. High pass, mid-series

$$C_{2\mu\pm1} = \frac{C_o}{a_{2\mu\pm1}} \qquad \mu = 1, 2, 3, 4$$
$$C_{2\mu} = \frac{a_{2\mu}}{Z_\mu} C_o$$
$$L_{2\mu} = \frac{L_o}{a_{2\mu}}$$

d. High pass, mid-shunt

$$L_{2\mu\pm1} = \frac{L_o}{a_{2\mu\pm1}} \qquad \mu = 1, 2, 3, 4$$
$$L_{2\mu} = \frac{a_{2\mu}}{Z_\mu} L_o$$
$$C_{2\mu} = \frac{C_o}{a_{2\mu}}$$

By using frequency transformation, the results given above can of course be directly applied to the design of band-pass and band-stop filters with frequency-symmetrical attenuation characteristics.

If the poles fed into the procedure have been calculated using Para. II.1.c. or selected from special tables (observing the different normalizations) an elliptic response is obtained for the filter.

3. Frequency-asymmetrical band-pass filters

Eq. (5) shows, that a properly selected q_n must be imaginary in the pass-bands, real in the stop-bands and it must have the value 1 at attenuation poles. For frequency-asymmetrical band-pass filters (not derivable by a reactance-transformation from a low-pass network) a suitable form of the q -function is therefore [15,16,34]

$$q_i = m_i \left[\frac{p^2 + \omega_p^2}{p^2 + \omega_{-p}^2} \right]^{\frac{1}{2}}, \tag{15}$$

with

$$m_i^2 = \frac{-\Omega_{\infty i}^2 + \omega_{-p}^2}{-\Omega_{\infty i}^2 + \omega_p^2} . \tag{16}$$

Here ω_{-p} and ω_p are the lower and upper pass-band limits.

This choice of q_i leads to a proper characteristic function through the steps described in Para. II.3.b. The network itself can then be obtained by conventional technique.

It should be mentioned here, that identical results and formulae can also be obtained for general parameter networks using potential analog rather than reference filter technique. [7,36] The latter, however, provides somewhat more insight into the physical background of the process.

e. A graphical method of obtaining the pole-frequencies. [34,36]

Hitherto it has been assumed, that the pole-frequencies are prescribed or known. In some cases this happens (e.g. for filters suppressing specific frequencies, elliptic filters), often however a graphical specification or frequency-dependent requirements are to be met with the establishment of a suitable number of poles at proper places. The situation is alleviated by the close connection between the operating loss of the real filter and the image progation function of the reference network. By Eq. (1), with proper approximations valid in the stop-band

$$A_0(p) \cong 20\,\mathrm{Re}\left\{\lg \varphi\right\} \cong A_i + 10\lg\left[10^{\frac{A_p}{10}} - 1\right] - 6 \,. \qquad (17)$$

Here A_0 is the operating loss in the stop-band, $A_i = \mathrm{Re}(\Theta)$ is the image attenuation of the reference filter and A_p the pass-band ripple, all in decibels.

Using Eq. (17) and the curve of Fig. 19, the problem simplifies to the establishment of poles for the reference-filter. A graphical process using special templates is available for carrying out that task.

For the use of a single template, it is necessary, that the attenuation-contributions of the individual sections of the reference-filter (corresponding to different $\Omega_{\infty i}$-s) be plotted by identical (though shifted) curves on a distorted frequency-scale.

Since in general

$$q_i = f(\Omega_{\infty i})\,F(\omega)$$

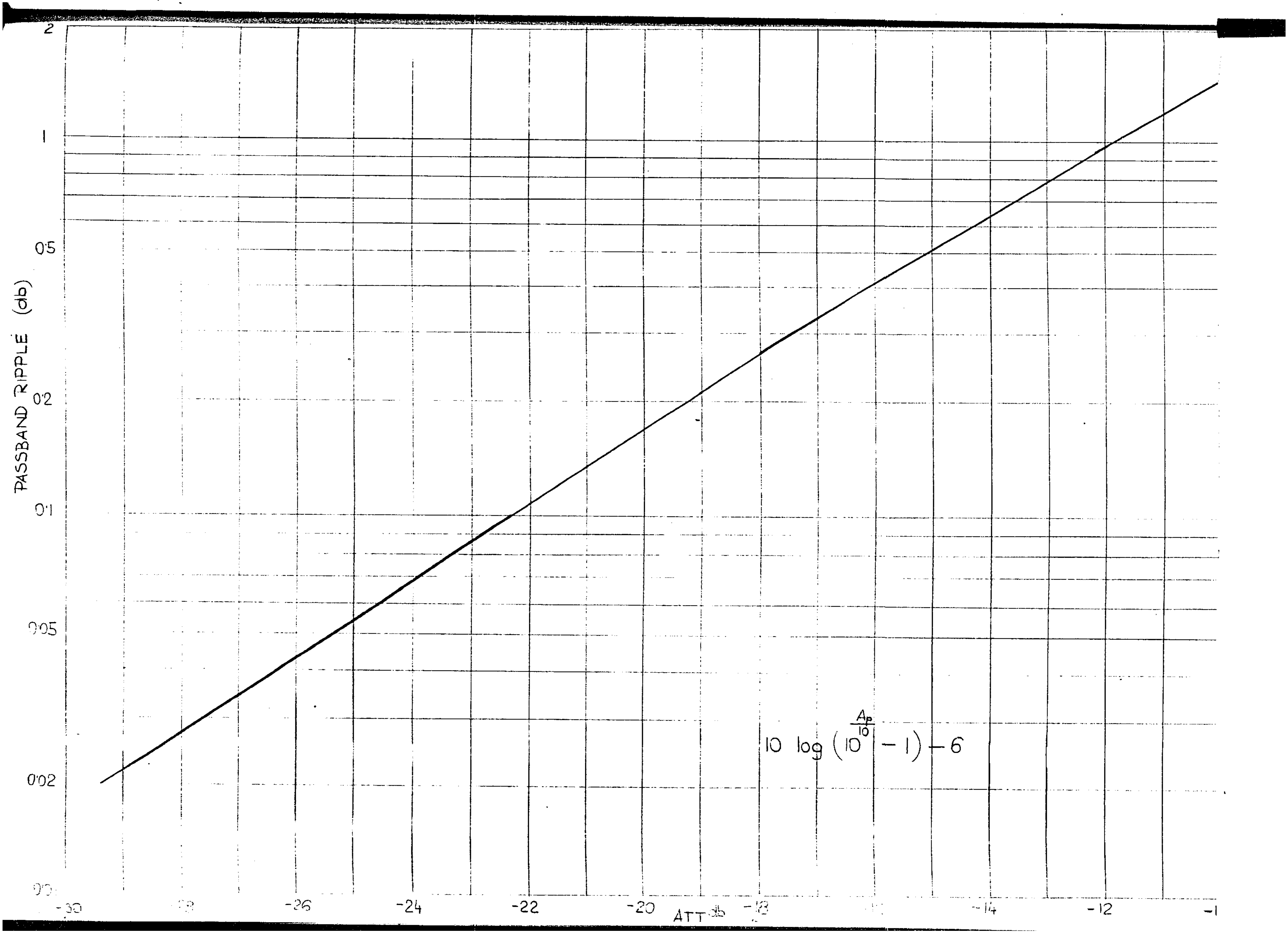
and

$$q_i(\Omega_{\infty i}) = f(\Omega_{\infty i})\,F(\Omega_{\infty i}) = 1$$

$$q_i = \frac{F(\omega)}{F(\Omega_{\infty i})} \,. \qquad (18)$$

Also, by Eq. (5)

$$A_i = \ln\left|\frac{q_i + 1}{q_i - 1}\right| \qquad (19)$$


PASSBAND RIPPLE (db)
2
1
0·5
0·2
0·1
0·05
0·02
-30
-26
-24
-22
-20
ATT db
-18
-14
-12
-1
10 log (10^(Ap/10) - 1) - 6

(A_i in Nepers). If now γ and $\gamma_{\infty i}$ are defined by

$$F(\omega) = e^{\gamma}$$
$$F(\Omega_{\infty i}) = e^{\gamma_{\infty i}}, \tag{20}$$

we get

$$A_i = \ln \frac{e^{|\gamma - \gamma_{\infty i}|} + 1}{e^{|\gamma - \gamma_{\infty i}|} - 1} \tag{21}$$

as the loss-contribution of the section associated with $\Omega_{\infty i}$.
As required, A_i is only dependent on $\gamma - \gamma_{\infty i}$.

When an antimetric reference filter is utilized, a loss of $\frac{A_i}{2}$ will also have to be associated with the half-section.

A_i and $\frac{A_i}{2}$ describe the contour of the templates. They are plotted on Fig. 20.

Now for a low-pass filter, by Eq. (8)

$$\gamma = \ln F(\omega) = \ln \frac{\omega}{\sqrt{\omega^2 - 1}} \tag{22}$$

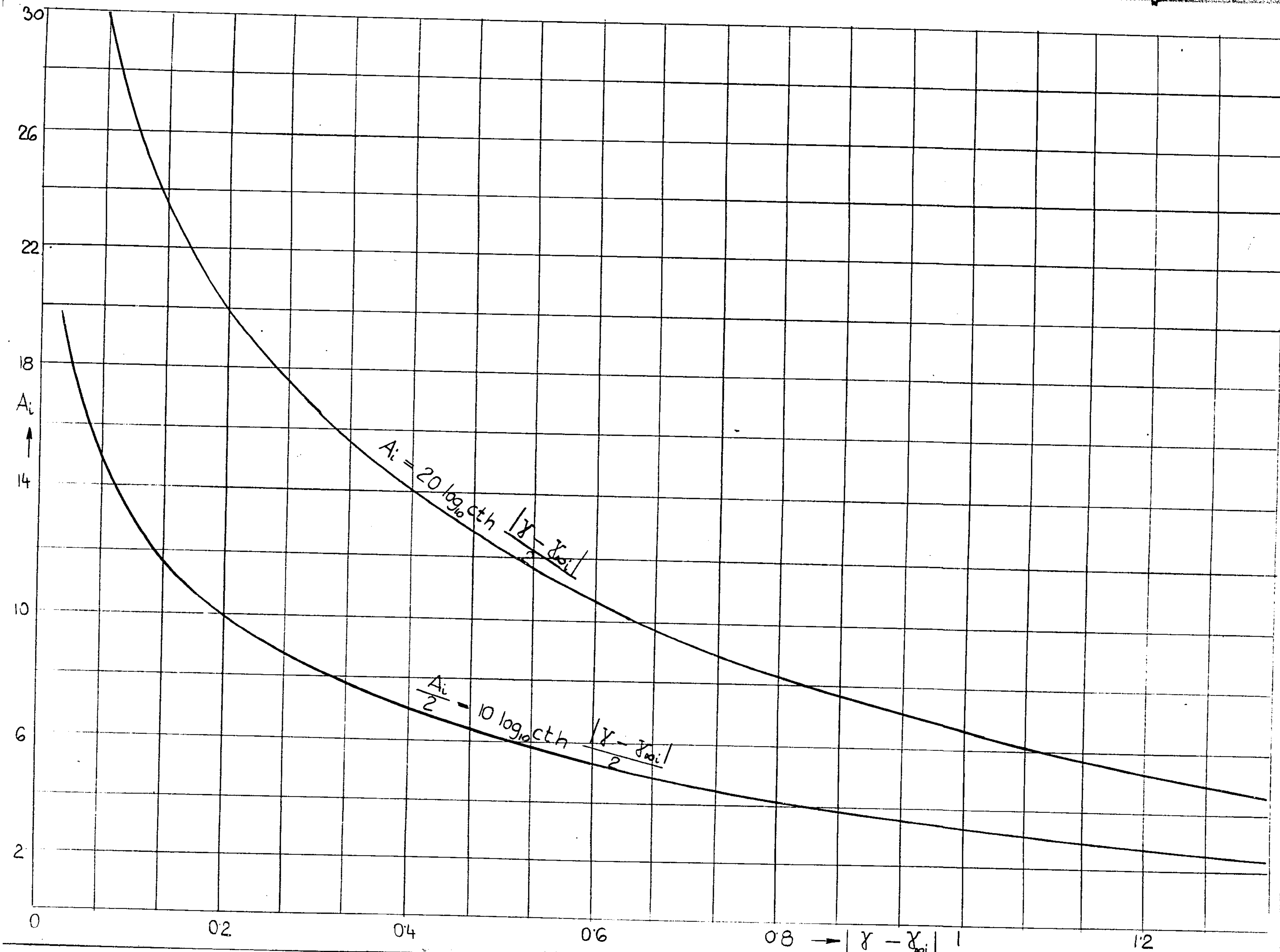
gives the suitable frequency-transformation. This can be plotted permanently on the abscissa-axis of the graph-paper used.

For the asymmetrical band-pass filter Eq. (15) is the basis of the frequency-transformation, which will depend now, however, on the ω_{-p}/ω_p ratio. It can be built up from a constant transformation of the form

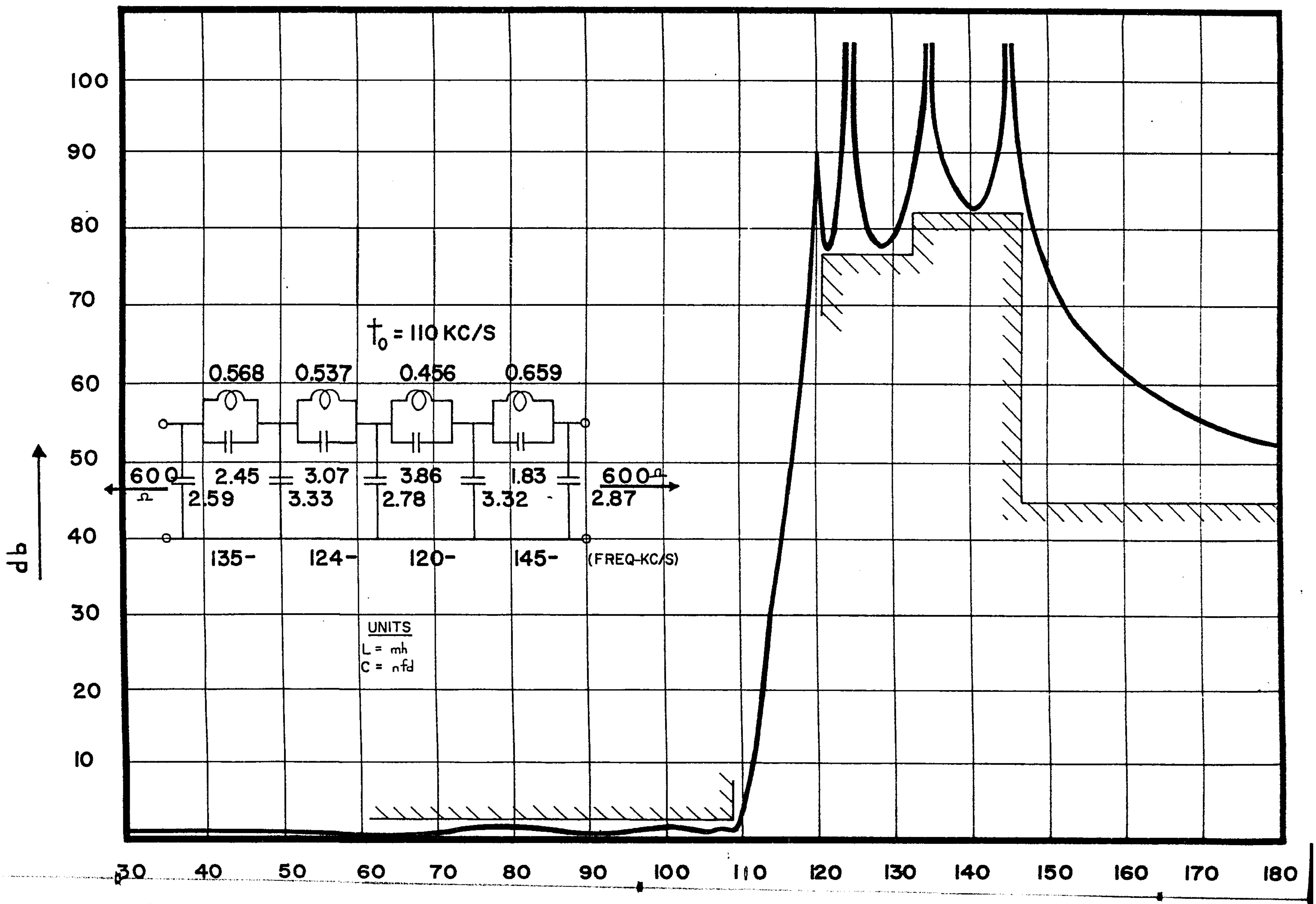
$$\Omega = - \coth \gamma$$

and a simpler relation between ω and Ω, which needs to be calculated for every filter specification.

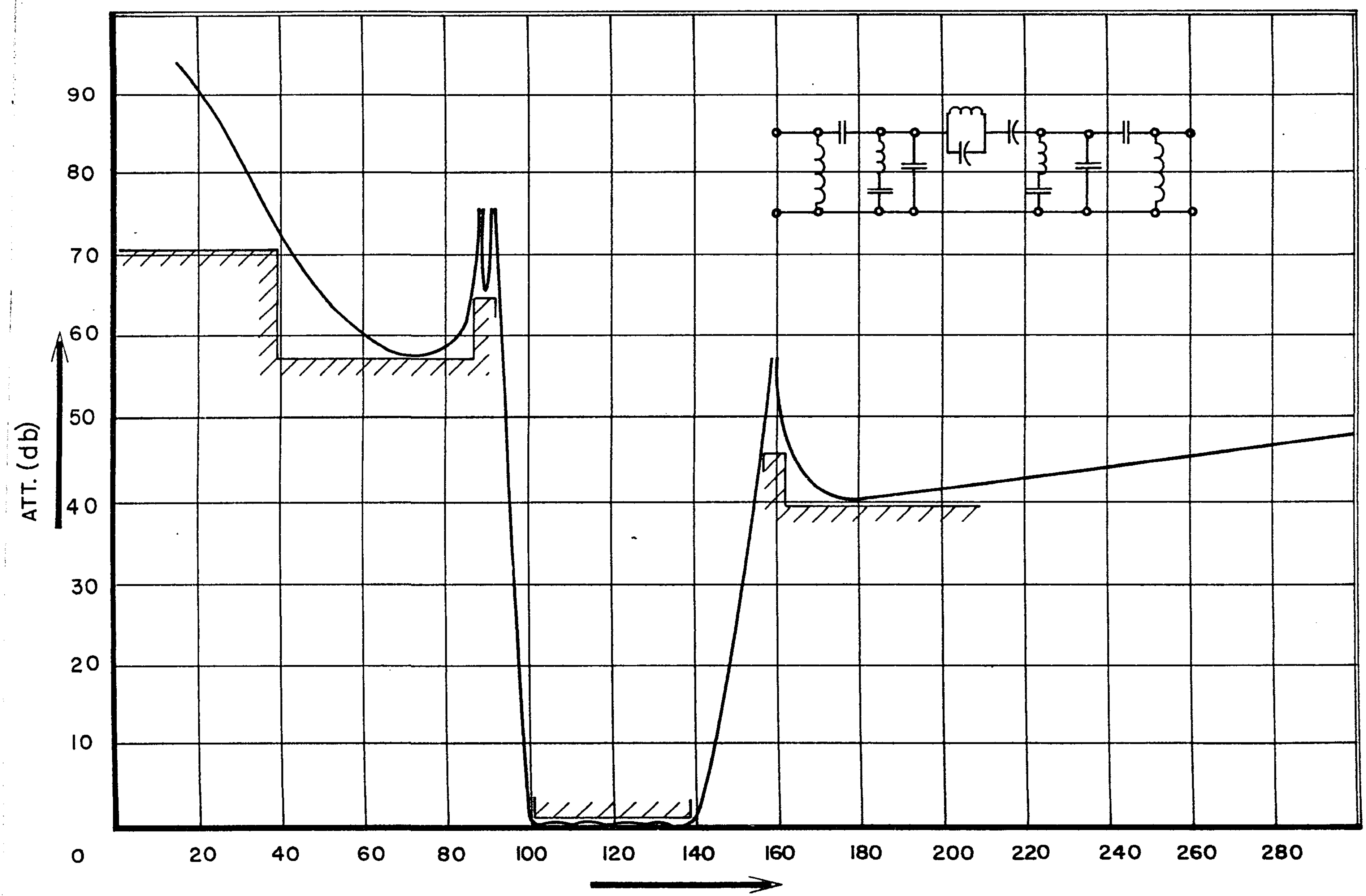
FIGURE 20.


30
26
22
18
A_i
14
10
6
2
0
0·2
0·4
0·6
0·8
$\rightarrow |\gamma - \gamma_{oi}|$
1
1·2
$A_i = 20 \log_{10} cth \frac{|\gamma - \gamma_{oi}|}{2}$
$\frac{A_i}{2} = 10 \log_{10} cth \frac{|\gamma - \gamma_{oi}|}{2}$

Figs. (21) - (22) show examples of low-pass and asymmetrical band-pass filters with general parameters designed using the methods described in this Section.



db
100
90
80
70
60
50
40
30
20
10
f_0 = 110 KC/S
0.568
0.537
0.456
0.659
600 Ω
2.45
2.59
3.07
3.33
3.86
2.78
1.83
3.32
600 Ω
2.87
135-
124-
120-
145-
(FREQ-KC/S)
UNITS
L = mh
C = nfd
30
40
50
60
70
80
90
100
110
120
130
140
150
160
170
180



ATT. (db)
90
80
70
60
50
40
30
20
10
0
20
40
60
80
100
120
140
160
180
200
220
240
260
280
FREQ. (kc/s)

FIGURE 22

III. INSERTION LOSS DESIGN OBSERVING PARASITIC EFFECTS

In the derivations of Chapters I. and II. the ideal behavior of the components was tacitly assumed. Contrary to this supposition, parasitic phenomena due to the stray capacitances, inductances and element losses, temperature effects, aging, tolerances etc. have often detrimental effects on the response. This Chapter will provide - with certain additions - a summary of existing methods for dealing with this situation. Obviously, the problem is twofold: it is necessary to estimate and if required, to correct for the parasitic effects. With one exception, the existing procedures can only be used to analyse some specially distributed stray phenomena.

After that, in Chapter IV, a comprehensive new theory is presented for the estimation and precorrection of parasitic effects on the response. It has special computational advantages, provides valuable insight into the physical back-ground, and imposes few restrictions on the distribution of the disturbing elements.

III. 1. NON-IDEAL CONSIDERATIONS IN COMMUNICATION NETWORKS[4,9,10,36,37]

This Section will present some simple statements and methods for dealing with parasitic effects without using special mathematical techniques. The estimation of the effects of non-ideal element characteristics on the behaviour will also be discussed for some specific situations.

a. Temperature effects

It will be assumed that the temperature-dependence of all elements can be described by

$$X_T = X_{T_0}\left[1 + \delta_X\left(T - T_0\right)\right] \tag{1}$$

where δ_X is the temperature coefficient of element X. Let now all capacitors have coefficient δ_C, that is under the designer's control and the terminations have δ_R, also freely chosen by the designer, while the common temperature-coefficient δ_L of the coils is prescribed. Then the changes in element values

$$\frac{\Delta L_i}{L_i} = \delta_L \Delta T$$
$$\frac{\Delta C_j}{C_j} = \delta_C \Delta T \tag{2}$$
$$\frac{\Delta R_k}{R_k} = \delta_R \Delta T$$

can be interpreted as the results of a change in the denormalizing frequency and impedance-units.

Since

$$L_i = \frac{l_i R_0}{\omega_0}$$

$$C_j = \frac{c_j}{\omega_o R_o} \tag{3}$$

and

$$R_k = r_k R_o ,$$

the units are changed according to

$$\omega_o^2 = \frac{l_i c_j}{L_i C_j} \longrightarrow \frac{l_i c_j}{L_i C_j (1+\delta_L \Delta T)(1+\delta_C \Delta T)} \tag{4}$$

and

$$\frac{c_j}{l_i}\,\frac{L_i}{C_j} = \left(\frac{R_k}{r_k}\right)^2 \longrightarrow \frac{c_j}{l_i}\,\frac{L_i}{C_j}\,\frac{1+\delta_L \Delta T}{1+\delta_C \Delta T} = \frac{R_k^2 (1+\delta_R \Delta T)^2}{r_k^2} \tag{5}$$

It can be seen, that the response will remain undisturbed, if the frequency-unit does not change, i.e. if

$$(1+\delta_L \Delta T)(1+\delta_C \Delta T) = 1 \tag{6}$$

and if the terminations are re-normalized with the new impedance-unit:

$$\frac{1+\delta_L \Delta T}{1+\delta_C \Delta T} = (1+\delta_R \Delta T)^2 , \tag{7}$$

Neglecting higher-order terms of $\delta \Delta T$,

$$\delta_C = -\delta_L \tag{8}$$

and

$$\delta_R = \delta_L \tag{9}$$

are obtained. Choosing δ_C and δ_R this way, an approximate compensation is achieved in a wide temperature range. It is interesting to see, that this implies

$$\sum_{(i)} L_i \frac{\partial F}{\partial L_i} - \sum_{(j)} C_j \frac{\partial F}{\partial C_j} + \sum_{(k)} R_k \frac{\partial F}{\partial R_k} = 0 \qquad (10)$$

for any transmission ratio of like quantities ($F = \frac{X_1}{X_2}$).

In a practical application, this compensation technique gave an accuracy of $\pm$ .001 db in an extremely wide temperature-range.

b, High-frequency considerations

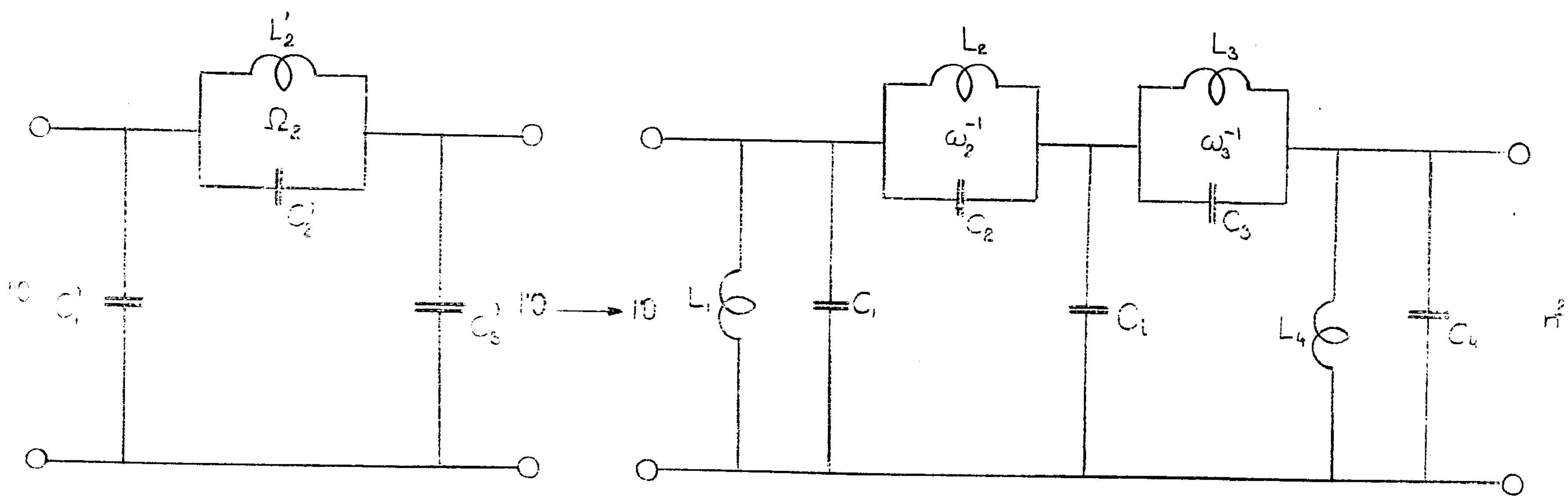
Several special precautions must be made for filters intended for very high frequency (5-200 Mc) operation. Most of these are needed because the element-values tend to be very low at these frequencies and thus the effect on stray elements will be emphasized. Apart from mechanical construction, material, shielding, tuning, etc. problems, which are beyond the scope of the present analysis, the following synthesis considerations apply:

1, All elements must have a parallel capacitor absorbing the stray capacitances.

2, All nodes must be capacitively loaded to ground, for the same reason.

3, No element value must be comparable with the value of any stray component disturbing it.

Conditions 1, and 2, are satisfied e.g. for the symmetrical mid-shunt low pass configuration, but not for the frequency-



FIGURE, 23.

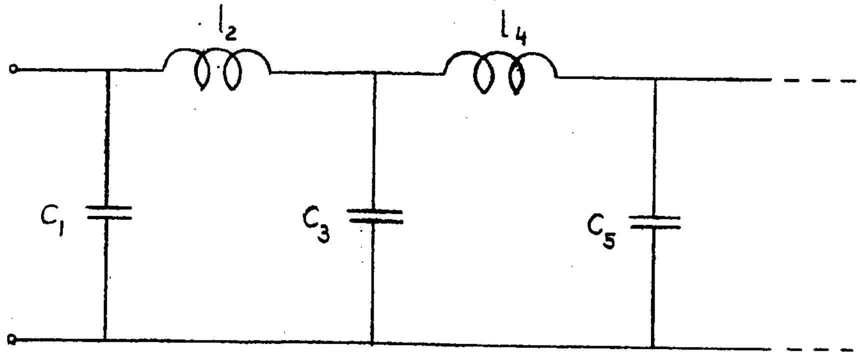


Figure 24 a   Normalized Low Pass Network

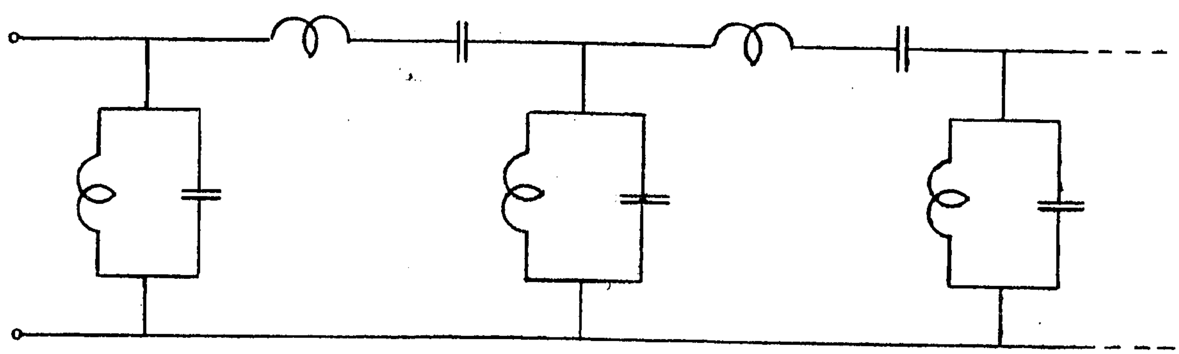


Figure 24 b   Normalized Band Pass Network Transformed From Low Pass Networks of Figure 24 a

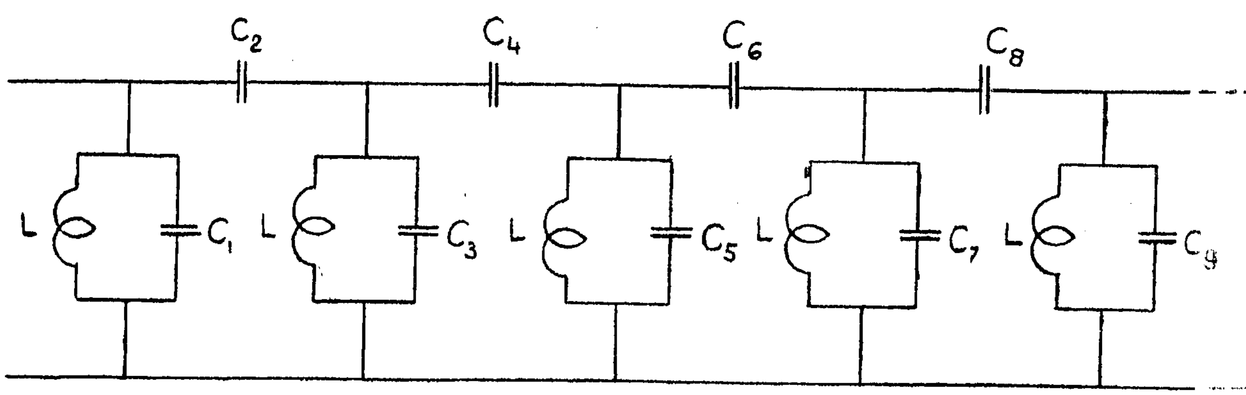
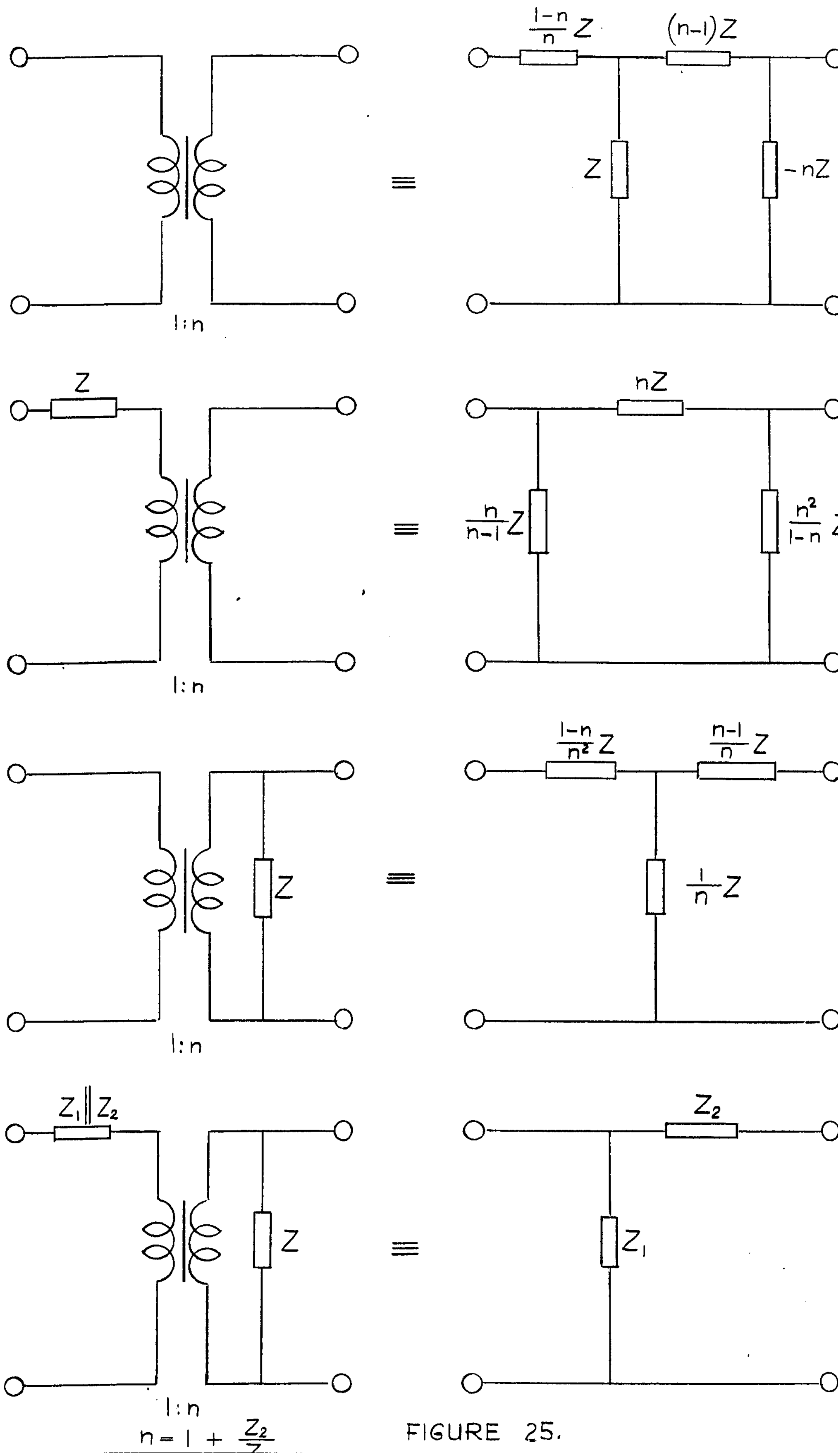


Figure 24 c   Normalized Band Pass Network Equivalent to Figure 24 b



$\frac{1-n}{n}Z$
$(n-1)Z$
Z
$-nZ$
$\equiv$
1:n
Z
nZ
$\frac{n}{n-1}Z$
$\frac{n^2}{1-n}Z$
$\equiv$
1:n
$\frac{1-n}{n^2}Z$
$\frac{n-1}{n}Z$
Z
$\frac{1}{n}Z$
$\equiv$
1:n
$Z_1 \| Z_2$
Z
Z_2
Z_1
$\equiv$
1:n
$n = 1 + \frac{Z_2}{Z}$

FIGURE 25.

symmetrical band-pass network, that could be derived from it using reactance transformation.

Several exact and approximate network equivalences are available for obtaining a network satisfying these conditions. Fig. 23 shows a low-pass to band-pass frequency-transformation[36] of a mid-shunt circuit to a proper band-pass circuit meeting Conditions 1, and 2,.

Fig. 24 shows an approximating procedure[4] for the transformation of constant -k type networks into narrow-band band-pass filters.

Finally, Fig. 25 shows a series of network identities[2] useful for the fulfillment of Conditions 2, and 3, (the simple delta-star transformation is also frequently required).

The equivalences of Figs. 23 - 25 can be checked e.g. by comparing the immittance parameters of the circuits.

Figs. 26 - 27 show the circuit and the (measured) characteristics of a 65 Mc filter-pair designed by these techniques.

The estimation of dissipation effects on the response of networks.[2,10]

The simplest treatment of dissipation effects is possible, if it is supposed that the dissipation constants

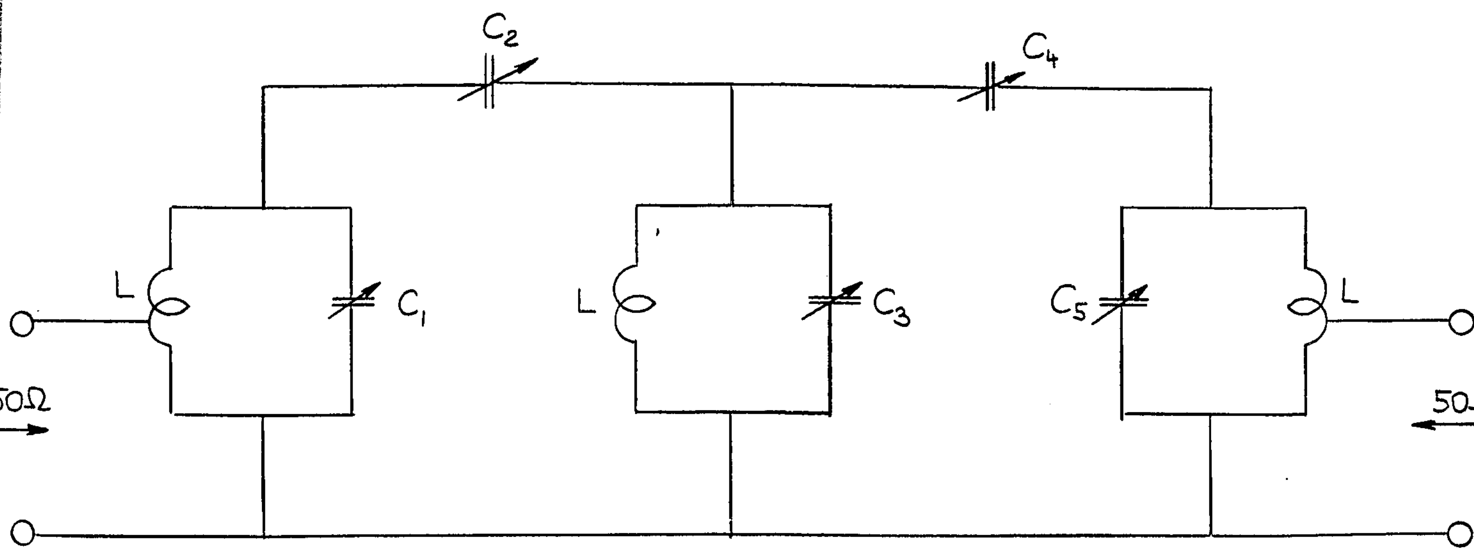


FIGURE 26.

FIGURE 27. a.

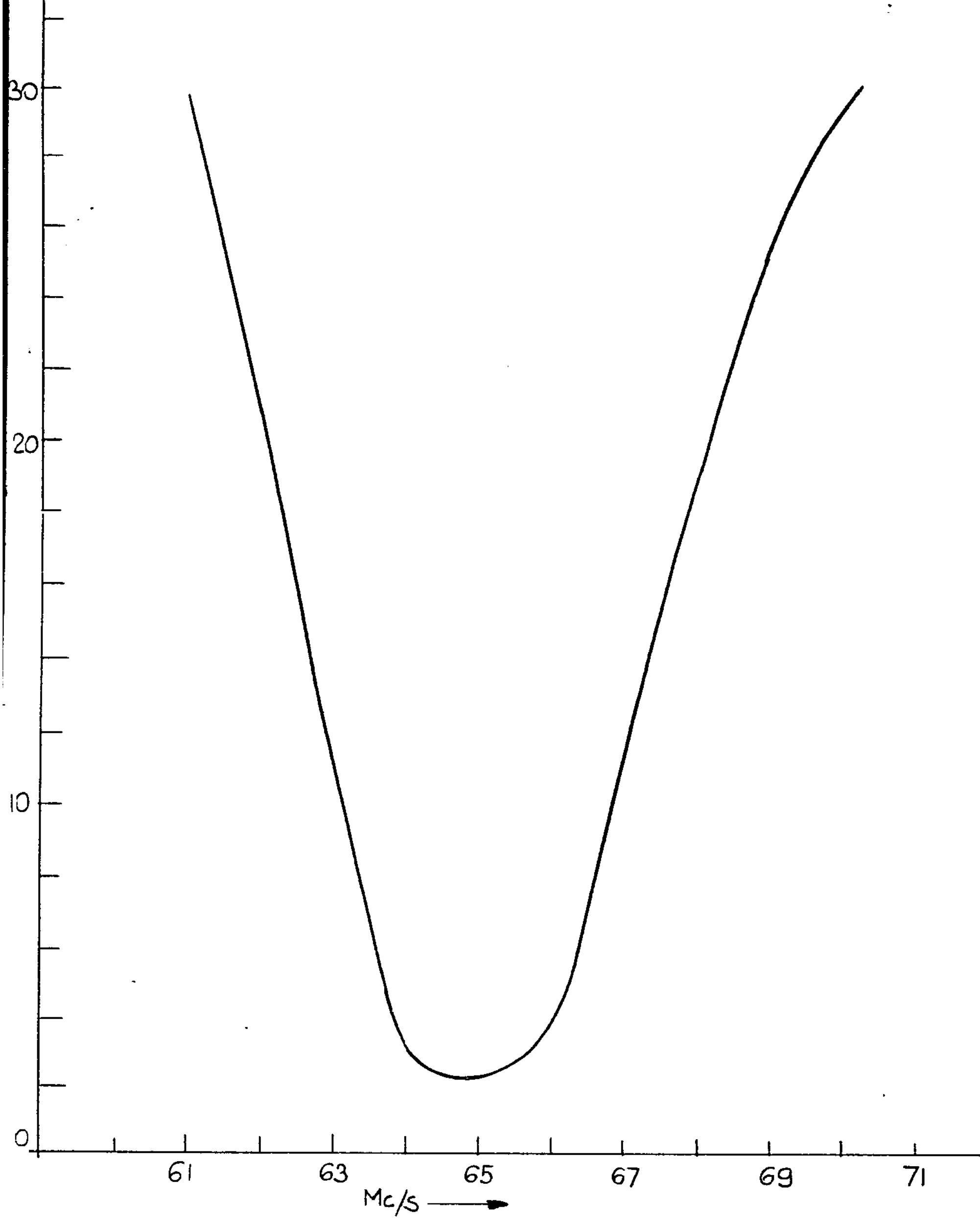
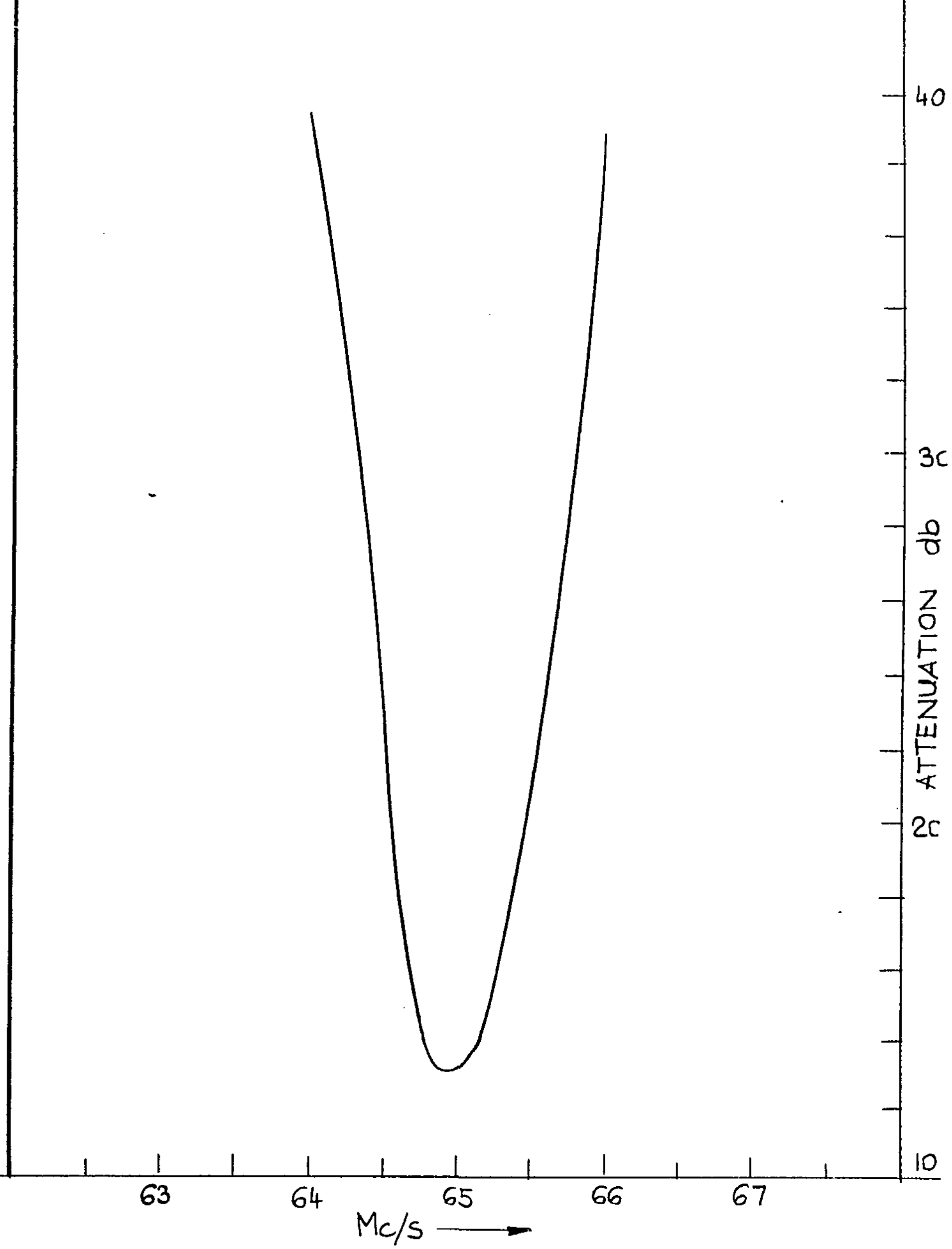


FIGURE 27. b.



$$d_L = \frac{R_L}{L} \tag{11}$$

or

$$d_c = \frac{G_c}{C} \tag{12}$$

are the same for each coil and capacitor and that also

$$d_L = d_c = d . \tag{13}$$

In that case the effect can be described by

$$\begin{aligned} pL &\rightarrow pL + R_L - (p+d)L \\ pC &\rightarrow pC + G_c - (p+d)C \end{aligned} \tag{14}$$

i.e.

$$p \rightarrow p+d . \tag{15}$$

This change can be interpreted as a shift of the $j\omega$ -axis to the right or the shift of all extreme frequencies to the left by d. Thus evaluating the response along the $d+j\omega$ line will give the lossy behaviour. More explicitly, if the (complex) transmission is

$$F(j\omega) = A(\omega) + jB(\omega), \tag{16}$$

$$F(d+j\omega) \simeq A + jB + d\left[\frac{\partial A}{\partial(j\omega)} + j\frac{\partial B}{\partial(j\omega)}\right] \tag{17}$$

results, using the Cauchy-Riemann equations. Thus

$$\begin{aligned} A(\omega) &\rightarrow A(\omega) + d\frac{\partial B(\omega)}{\partial\omega} \\ B(\omega) &\rightarrow B(\omega) - d\frac{\partial A(\omega)}{\partial\omega} \end{aligned} \tag{18}$$

is the approximate effect of this "homogeneous" dissipation.

It is important to notice, that pure real or imaginary quantities do not change B (ω) and thus do not contribute (in first approximation) to the change in A. E.g. in the voltage ratio.

$$\Lambda = \frac{A + pB}{P} \tag{19}$$

P is even and thus real for $p = j\omega$. Consequently the change in $|\Lambda|$ can be analyzed by replacing p by $(p + d)$ in the numerator only. Obviously, this does not hold in the neighborhood of any $j\omega$-axis pole, where higher-order terms of Eq. (17) are no longer negligible.

A more practical assumption is to make all d_L-s equal among each other and similarly for the d_c-s, but to take

$$d_c \neq d_L . \tag{20}$$

Now the general circuit-determinant element of the four-pole (cf. Eq. (11) of Sec. I.2.) will change according to [1,2,12]

$$Z_{ij} = \frac{p^2 L_{ij} C_{ij} + 1}{p C_{ij}} \longrightarrow \frac{(p + d_L)(p + d_c) L_{ij} C_{ij} + 1}{(p + d_c) C_{ij}} \tag{21}$$

and thus e.g. the impedance-parameters (Eq. (12) of Sec. I.2.)

$$Z_{ij} = \frac{f[p^2]}{p} \longrightarrow \frac{f[(p + d_L)(p + d_c)]}{p + d_c} \tag{22}$$

or

$$Z_{ij}(p) \longrightarrow \Delta \times \frac{f[p_d^2]}{p_d} = \Delta \times Z_{ij}(p_d), \tag{23}$$

where

$$p_d = [(p + d_L)(p + d_c)]^{1/2} \simeq p + \frac{d_L + d_c}{2} = p + d_o \tag{24}$$

and

$$\Delta = \left[\frac{p + d_L}{p + d_c}\right]^{1/2} \cong 1 + \frac{d_L - d_c}{2} = 1 + \delta \tag{25}$$

It is important to see, how much error arises, if

$$\Delta \cong 1 \tag{26}$$

is used, i.e. if the actual loss-distribution is approximated with a homogeneous dissipation characterized by d_o. The error could be measured by the change in attenuation caused by it.

Normalizing each Z_{ij} to $(R_i R_j)^{1/2}$, Eq. (6) of Sec. I.2. gives

$$\Delta A \cong \log \left| \Delta^{-1} \right| \left| 1 + \delta \frac{Z_{11} + Z_{22} + 2[Z]}{1 + Z_{11} + Z_{22} + [Z]} \right|. \tag{27}$$

where all $Z_{ij} = Z_{ij}(p_d)$.

Two special cases are of interest. For symmetrical networks, using Eq. (32) of Sec. I.1., after some calculation.

$$|\Delta A| \leq \frac{2\delta}{\omega} = 2\left(Q_L^{-1} - Q_c^{-1}\right) \tag{28}$$

is obtained.

Similarly, for antimetric circuits, Eq. (33/a) of Sec. I.1. leads to

$$|\Delta A| \leq \frac{1}{2}\left(\frac{\delta}{\omega}\right)^2. \tag{29}$$

In certain cases it is possible to predict the accuracy of this approximation even without resorting to Eqs. (28) - (29). From the mesh-equations (1) of Sec. I.1. it can be shown, that the driving-point admittance at any points can be written as

$$Y_D = 4\left[F + j\omega\left(V - T\right)\right] \tag{30}$$

where F, V and T are the average loss, the electrical and the magnetic energy for unit generator voltage, respectively.[3,9] If Y_D is pure susceptive

$$Y_D = jB_D = j4\omega\left(V - T\right) \tag{31}$$

and

$$X_D = -B_D^{-1} \tag{32}$$

is the driving-point reactance,

$$\frac{dX_D}{d\omega} = \frac{X_D}{\omega}\,\frac{T+V}{T-V} \geqslant \frac{|X_D|}{\omega} \tag{33}$$

results.

Obviously the total <u>incidental</u> dissipation can be written[9] as

$$2\left(d_L T + d_c V\right) = 2\left(T+V\right)d_0 + 2\left(T-V\right)\delta \tag{34}$$

Here the first term corresponds to the homogeneous approximation, the second to its error. According to Eqs. (30) - (33), the error can be neglected, if Y_D is purely conductive, or if it is a sharply frequency-variant susceptance. Since it is easy to show that for a large variety of circuits (all-pass, symmetric, antimetric networks) the total electromagnetic energy is $\frac{R_{gen}}{2}\,\frac{dB}{d\omega}$,for networks exhibiting large delays the first term will be large and the approximation accurate.

In any case, the accuracy of the approximation may deteriorate for very low frequencies and large losses, as

Eqs. (28) - (29) and later Example c, will show.

For networks derived by frequency-transformation, it is possible to calculate the hypothetical loss to be associated with the (low-pass) prototype. If

$$P = f(p) \tag{35}$$

gives the frequency-variable of the prototype, the lossy response can be found by substituting

$$P_d \cong f(p) + f'(p)\, d_o \cong P + f'(p_i)\, d_o \tag{36}$$

where p_i is a properly chosen frequency. E.g. for low-pass to band-pass transformation.

$$P = \frac{p^2 + \omega_p \omega_{-p}}{p\sqrt{(\omega_p - \omega_{-p})(\omega_a - \omega_{-a})}} \tag{37}$$

and choosing

$$p_i = j\sqrt{\omega_p \omega_{-p}} \tag{38}$$

$$P_d = P + D = P + \frac{2\, d_o}{\sqrt{(\omega_p - \omega_{-p})(\omega_a - \omega_{-a})}} \tag{39}$$

results. Especially for narrow relative bandwidth, this will be a good approximation. Now e.g. Eq. (18) can be used to derive the lossy low-pass response and then, from Eq. (37) the final result.

For sharp networks, the largest part of the distortion will be caused at the pass-band limit(s) by the shift of the complex zero(s) closest to the cut-off point. This effect can be

calculated from the ratio of the original and changed root-factors

$$\Delta A^{(N)} = \frac{1}{2} \ln \frac{P_2}{P_{2d}} = \ln \left| \frac{p_c + \alpha_0 - (-\alpha_i + j\omega_i)}{p_c - (-\alpha_i + j\omega_i)} \right| \qquad (40)$$

(Fig. 28.) and since the cut-off frequency

$$p_c \cong j\omega_i, \qquad (41)$$

the distortion is approximately

$$\Delta A \cong 8.686 \frac{\alpha_0}{\alpha_i} \qquad (42)$$

in decibels.

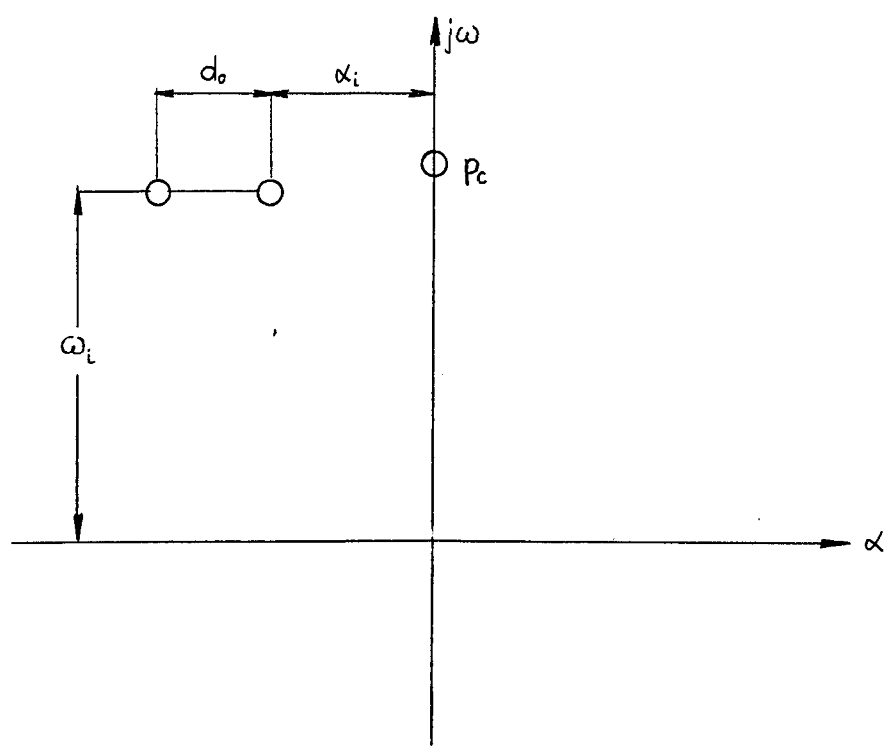


FIGURE 28.

III. 2. METHODS FOR THE COMPENSATION OF PARASITIC LOSS-EFFECTS 8, 9, 12, 13, 14, 19, 25, 27,

Para. III.1.c. gave methods for the estimation of loss-effects on communication networks for some special distribution of the losses, viz. for homogeneous loss (all elements equally lossy) and semi-homogeneous dissipation (all elements of the same type equally lossy). Now some techniques will be presented which seek to prevent these effects. Some of these methods can be incorporated in the original design procedure, others can only correct the already synthesized network.

a. Predistortion for homogeneous dissipation.

It was demonstrated in Para. III.1.c., that the effect of homogeneously distributed loss can be represented by the $p \rightarrow p+d$ frequency-transformation. It is rather obvious, that the replacement of p by p - d in the original transmission function acts as an anticipation of this effect. If homogeneous loss is added to a network designed from the modified - "predistorted" - function, it will tend to restore the desired response.

Special care must be exercised in the course of this procedure[1,25] to avoid the violation of the realizability conditions of Sec.I.3., especially the conditions of Para. I.3.d. or I.3.e.

Various cases will now be treated in some detail.[25]

If all attenuation poles are imaginary, the replacement

$$p \rightarrow p - d \tag{1}$$

will destroy the even or odd character of the denominator(f or P). However, by making the

$$S(p) = \frac{g(p)}{f(p)} \longrightarrow S_{pr}'(p) = k \frac{g(p-d)\,f(p+d)}{f(p-d)\,f(p+d)} \tag{2}$$

transition, this is prevented. Also, since f (p + d) is Hurwitz, if the condition

$$d < |\alpha_i| \tag{3}$$

(where α_i is the real part of the natural frequency closest to the imaginary axis) is satisfied, the new numerator will be Hurwitz. The factor

$$k > 1 \tag{4}$$

is normally necessary for satisfying

$$|S_{pr}(j\omega)| \geqslant 1 \tag{5}$$

For the predistorted insertion voltage ratio

$$|\Lambda_{pr}(j\omega)|^2 \geqslant \eta_{G_{pr}} = \frac{4R_1R_{2pr}}{(R_1 + R_{2pr})^2} \tag{6}$$

specifies the new termination ratio. (See also Para. I.3.e.)

If some attenuation poles are complex and

$$d < \mathrm{Re}(p_j) \tag{7}$$

for some pole p_j , Eq. (2) does not give a Hurwitz numerator. However, by splitting $f(p) = f_H(p)\, f_{NH}(p)$ into Hurwitz and non-Hurwitz factors and making

$$S_{pr} = k\,\frac{g(p-d)\; f_H(p+d)\; f_{NH}(-p-d)}{f_H(p+d)\; f_H(-p+d)\, f_{NH}(p-d)\; f_{NH}(-p-d)}\,, \tag{8}$$

the numerator has only left half-plane roots, the denominator has proper parity (even) and, $|S_{pr}|$ remains unchanged.

Associating common factors with the numerator and the denominator of S (cf. Eqs. (2) and (8)) increases the order of the circuit. Fortunately, as expounded in Para. III.1.c., the frequency-shift in the denominator has only minor effect on the response; consequently it may be omitted. As the derivations of Para. I.5.c. show, this may be necessary anyway, if a ladder realization is required.

For a large amount of loss, the stop-band attenuation near the pass-band(s) may fall intolerably. In that case, a compromise is possible by predistorting exactly the pole-factors corresponding to the nearest poles, and leaving intact the others. If

$$f(p) = f_1(p)\; f_2(p) \tag{9}$$

where $f_1(p)$ contains the critical pole-factors mentioned

$$S_{pr} = k\,\frac{g(p-d)\; f_1(p+d)}{f_1(p+d)\; f_1(p-d)\; f_2(p)} \tag{10}$$

will have all the required properties.

According to the analysis presented in Para. III.1.c., the application of this procedure to networks obtained by reactance transformation[2] is straightforward. A predistortion of the prototype network by

$$P \rightarrow P - f'(p_i)\, d_o \tag{11}$$

followed by the reactance-transformation will provide the final predistorted circuit.

For networks with sharp cut-off, Eqs. (41)- (42) of Sec. III.1. can be used to find the proper terminations satisfying Eq. (6). Some very useful diagrams summing up the calculations are shown on Fig. 29.

b. Precorrection of single-loaded circuits with semi-homogeneous dissipation[27]

For a single-terminated network the immittance-parameters can be obtained from Eqs. (3) - (8) of Sec. I.4. E.g. for $R_2 \rightarrow \infty$ using insertion loss parameters:

$$Z_{11} = R_1 \frac{A}{pB} \tag{12}$$
$$Z_{12} = R_1 \frac{P}{pB} \; .$$

As Eqs. (22) - (25) of Sec. III. 1. show, the effect of a semi-homogeneous loss can be described by

FIGURE 29. a.

THE APPROXIMATE EFFECT OF HOMOGENEOUS DISSIPATION AT THE PASSBAND LIMIT.

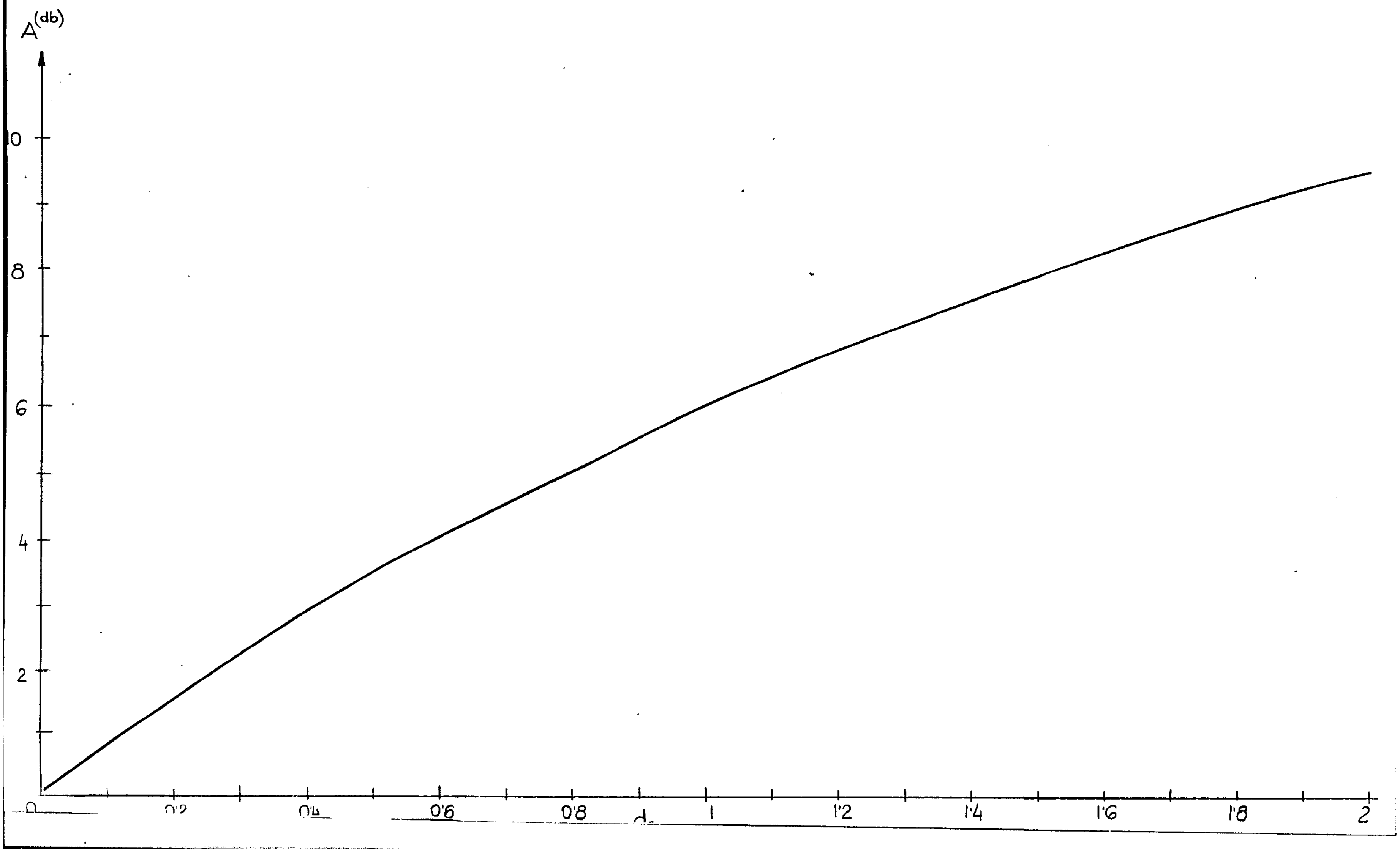
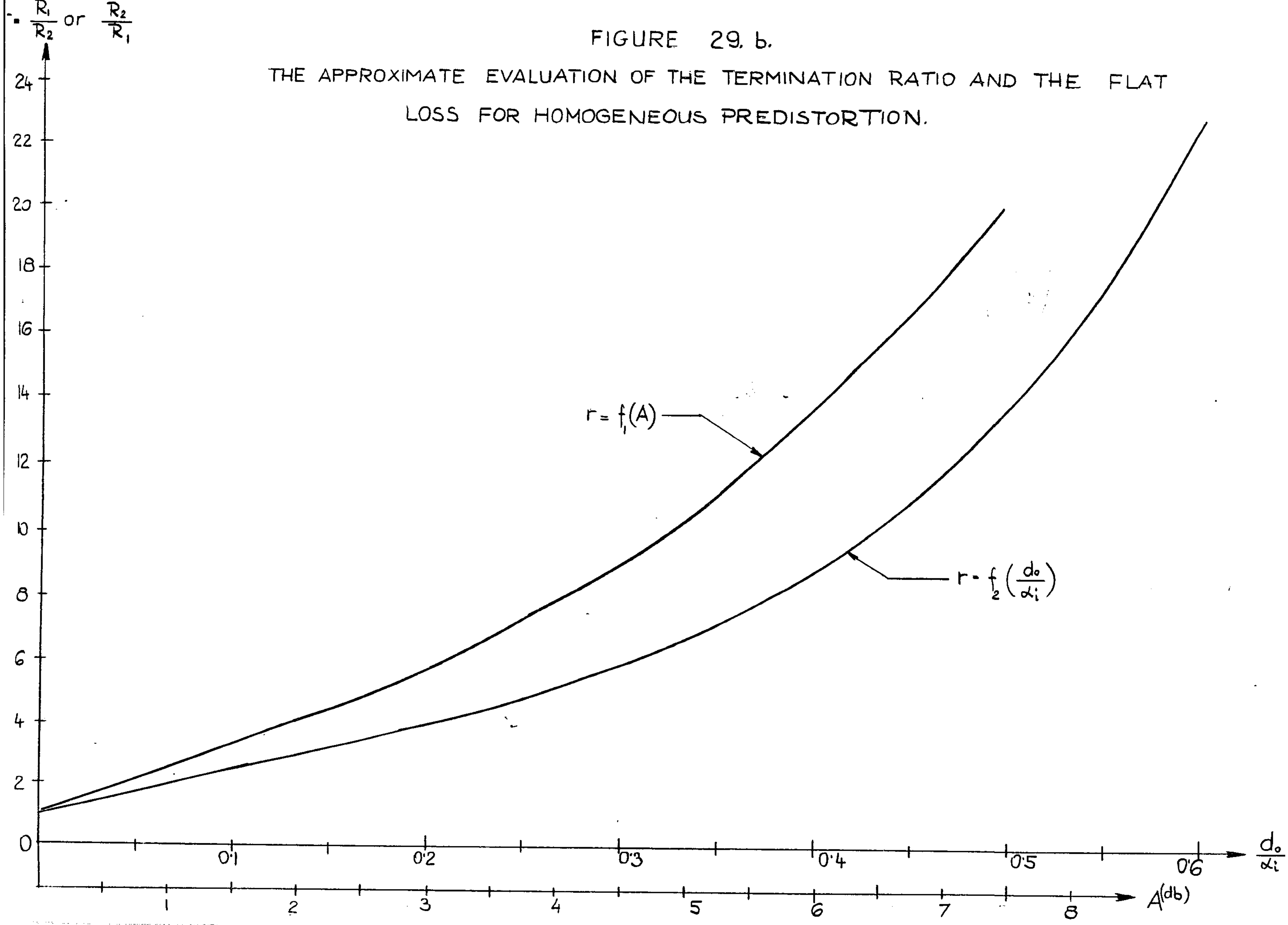


FIGURE 29. b.

THE APPROXIMATE EVALUATION OF THE TERMINATION RATIO AND THE FLAT LOSS FOR HOMOGENEOUS PREDISTORTION.



$$Z_{ij} = \frac{f[p^2]}{p} \longrightarrow Z_{ijd} = \frac{f\left[(p+d_o)^2 - \delta^2\right]}{p+d_o-\delta} \qquad (13)$$

or

$$Z_{ijd} = \frac{f\left[p_d^2 - \delta^2\right]}{p_d - \delta} \,. \qquad (14)$$

Accordingly, the proper predistortion procedure for the Z_{ij}-s can be performed in two steps. First

$$p \rightarrow p_{-d} = p - d_o \qquad (15)$$

is substituted, i.e. a homogeneous predistortion performed on the Z_{ij}-s. Next, the replacement

$$p_{-d}^2 \rightarrow p_R^2 + \delta^2 \qquad (16)$$

in f (p - d) and the substitution

$$p_{-d} - \delta \rightarrow p_R \qquad (17)$$

of the linear divisor factor are carried out. The expressions thus obtained must have the form

$$Z_{ij} = \frac{f_R\,(p_R^2)}{p_R} \qquad (18)$$

to be realizable by the final predistorted reactance network.

The following symbols will be used for the voltage ratios obtained by the individual steps.

For the original network (N) with dissipation factors d_L and d_c:

$$\Lambda = \frac{A + pB}{P} \,. \qquad (19)$$

The (hypothetical) network (N_δ) obtained by homogeneous predistortion, possessing dissipation factors $\pm\,\delta$, has

$$\Lambda_\delta = \frac{A_\delta + pB_\delta}{P_\delta} . \qquad (20)$$

Final predistorted reactance network (N_R):

$$\Lambda_R = \frac{A_R + pB_R}{P_R} \; . \tag{21}$$

Obviously, as before, with a good approximation

$$\begin{aligned} A_\delta(p) + pB_\delta(p) &= A(p-d_0) + (p-d_0)B(p-d_0) \\ P_\delta(p) &= P(p) \; . \end{aligned} \tag{22}$$

Also, by Eq. (14)

$$\begin{aligned} Z_{11R} &= R_1 \frac{A_R(p^2)}{pB_R(p^2)} \longleftrightarrow Z_{1\delta} = R_1 \frac{A_R(p^2-\delta^2)}{(p-\delta)B_R(p^2-\delta^2)} \\ Z_{12R} &= R_1 \frac{P_R(p^2)}{pB_R(p^2)} \longleftrightarrow Z_{12\delta} = R_1 \frac{P_R(p^2-\delta^2)}{(p-\delta)B_R(p^2-\delta^2)} \; . \end{aligned} \tag{23}$$

Using Eq. (8) of Para. I.2.b. with $R_2 \to \infty$:

$$\Lambda_\delta = \frac{A(p-d_0)+(p-d_0)B(p-d_0)}{P(p)} = \frac{R_1 + Z_{11\delta}}{Z_{12\delta}} \tag{24}$$

or by Eqs. (22) and (23)

$$\Lambda_\delta = \frac{A_\delta + pB_\delta}{P} = \frac{A_R(p^2-\delta^2)+(p-\delta)B_R(p^2-\delta^2)}{P_R(p^2-\delta^2)} \tag{25}$$

from which the predistorted polynomials A_R , B_R, P_R , and the impedance-parameters could easily be found. Clearly it is possible to write

$$\begin{aligned} A_R(p^2-\delta^2) - \delta B_R(p^2-\delta^2) &= A_\delta(p^2) \\ B_R(p^2-\delta^2) &= B_\delta(p^2) \end{aligned} \tag{26}$$

so that $P_R(p^2-\delta^2) = P(p^2)$,

$$\begin{aligned} A_R(p^2) &= A_\delta(p^2+\delta^2) + \delta B_\delta(p^2+\delta^2) \\ B_R(p^2) &= B_\delta(p^2+\delta^2) \\ P_R(p^2) &= P(p^2+\delta^2) \; . \end{aligned} \tag{27}$$

Eqs. (27), (22) and (23) constitute the solution of the problem.

It is worth mentioning, that for the special case of constant-k low-pass networks, explicit formulae can be found connecting the coefficients of A + pB to $A_R + \rho B_R$ $\left(P = P_R = 1\right)$.

The predistortion procedure just described evidently applies to the precorrection of filter-combinations equally well (cf. Sec. I.6.).

c. The predistortion of double-terminated networks with equally lossy coils and equally lossy capacitors.

The physical argument of the previous Paragraph is equally valid for the double-terminated case. The same mathematical derivation, however, does not prove to be useful. A completely analogous process using Eqs. (13) and (19) of Sec. I.2.b. results in

$$
\begin{aligned}
A_R(p^2-\delta^2) - \delta B_R'(p^2-\delta^2) &= A_\delta \\
B_R(p^2-\delta^2) &= B_\delta \\
P_R(p^2-\delta^2) &= P_\delta
\end{aligned}
\qquad (28)
$$

with

$$
\begin{aligned}
&A_\delta(p^2-\delta^2) - \left[p^2-\delta^2\right] B_R(p^2-\delta^2) - \eta_G P_R(p^2-\delta^2) = \\
&= A_R'(p^2-\delta^2) - \left[p^2-\delta^2\right] B_R'^2 ,
\end{aligned}
\qquad (29)
$$

a system of 4 equations with 5 unknowns (A_R , B_R , P_R , A'_R , B'_R). It is necessary therefore to use a different approach.[1,12,25] To make the equations somewhat simpler, operating parameters will be used.

By Eq. (12) of Para. I.2.c. and Eq. (14) **it is advisable to write:**

$$Z_{ijR} = \sqrt{R_i R_j} \; \frac{f_{ijR}(p^2)}{p} \tag{30}$$

$$Z_{ij\delta} = \sqrt{R_i R_j} \; \frac{f_{ijR}(p^2 - \delta^2)}{p - \delta} = \sqrt{R_i R_j} \; \frac{f_{ij}(p^2)}{p - \delta}$$

and thus from Eq. (5) of Sec. I.2., with an even f

$$S_\delta - \frac{g_\delta}{f_\delta} = \frac{g_{\delta e} + g_{\delta o}}{f_{\delta e}} - \frac{1}{2}\left[\frac{f_{11} + f_{22}}{f_{12}} + (p+\delta)\frac{[f] + (p-\delta)^2}{(p^2-\delta^2)\, f_{12}} \right] \tag{31}$$

where

$$[f] - f_{11} f_{22} - f_{12}^2 \, . \tag{32}$$

Equating even and odd parts and simplifying results in

$$\frac{f_{11} + f_{22}}{f_{12}} + \delta \frac{[f]}{f_{12}^*(p^2-\delta^2)} - \frac{\delta}{f_{12}} = 2\frac{g_{\delta e}}{f_{\delta e}} \tag{33}$$

and

$$p\frac{[f]}{f_{12}^*(p^2-\delta^2)} + \frac{p}{f_{12}} = 2\frac{g_{\delta o}}{f_{\delta e}} \, . \tag{34}$$

These expressions contain the linear combinations of $\frac{f_{ii}}{f_{12}}$, $\frac{1}{f_{12}}$ and $\frac{[f]}{f_{12}^*(p^2-\delta^2)}$. It is rather obvious, that the symbolism defining h'_o and h_e by

$$p\frac{[f]}{f_{12}^*(p^2-\delta^2)} - \frac{p}{f_{12}} = 2\frac{h'_o}{f_{\delta e}} \tag{35}$$

$$\frac{f_{11} - f_{22}}{f_{12}} = 2\frac{h_e}{f_{\delta e}} \tag{36}$$

is expedient. At this point, however, there are still only 5 equations, Eqs. (32) - (36), at our disposal, while the number of unknowns is 6 (f_{ij}, $[f]$, h_e, h'_o). The rational character of all functions involved can be used to overcome this difficulty. From Eqs. (34) - (35)

$$f_{12} = \frac{p f_{\delta_e}}{g_{\delta_o} - h'_o} \tag{37}$$

and

$$[f] = \frac{p^2 - \delta^2}{p} f_{12} \frac{g_{\delta_o} + h'_o}{f_{\delta_e}} = (p^2 - \delta^2)\frac{g_{\delta_o} + h'_o}{g_{\delta_o} - h'_o}. \tag{38}$$

From Eqs. (33) and (36), after some calculation

$$f_{11} = \frac{p\,(g_{\delta_e} + h_e) - \delta h'_o}{g_{\delta_o} - h'_o} \tag{39}$$

and

$$f_{22} = \frac{p\,(g_{\delta_e} - h_e) - \delta h'_o}{g_{\delta_o} - h'_o} \tag{40}$$

follows.

Substituting Eqs. (37) - (40) into Eq. (32) gives

$$\frac{p^2 - \delta^2}{p^2}\left(g^2_{\delta_o} - h'^2_o\right) = \left(g_{\delta_e} - \frac{\delta}{p} h'_o\right)^2 - h_e^2 - f^2_{\delta_e} \tag{41}$$

and after bringing into a more convenient form

$$\left(1 - \frac{\delta^2}{p^2}\right)\left(g^2_{\delta_e} - g^2_{\delta_o}\right) = f^2_{\delta_e} + \left[h_e^2 - \left(g_{\delta_e}\frac{\delta}{p} - h'_o\right)^2\right]. \tag{42}$$

Observing that g_{δ_e}, f_{δ_e} and h_e are even, while g_{δ_o} and h_o are odd (see Eqs. (35) - (36)), for

$$p = j\omega \tag{43}$$

$$\left|\left(1 - \frac{\delta}{p}\right) g_\delta\right|^2 = f_\delta^2 + |h|^2 \tag{44}$$

with

$$h = h_e + h_o = h_e + \left(g_{\delta_e}\frac{\delta}{p} - h'_o\right). \tag{45}$$

Eq. (44) is completely analogous to characteristic equation of the lossless insertion-loss design process (cf. Eq. (12) of Sec. I.4.). It can be solved, although not uniquely, for h. After that, the even part and odd part of h supplies h_e

... and Eqs. (37) - (40) give f_{ij}. Eq. (30) can then be used ... find $Z_{ij\delta}$ and Z_{ijR}. We get

$$Z_{11\delta} = \frac{R_1}{p-\delta} \; \frac{(p^2-\delta^2)\, g_{\delta e} + p^2 h_e + p h_o}{p g_{\delta o} - \delta g_{\delta e} + p h_o}$$

$$Z_{22\delta} = \frac{R_2}{p-\delta} \; \frac{(p^2-\delta^2)\, g_{\delta e} - p^2 h_e + p h_o}{p g_{\delta o} - \delta g_{\delta e} + p h_o} \qquad (46)$$

$$Z_{12\delta} = \frac{\sqrt{R_1 R_2}}{p-\delta} \; \frac{p^2 f_{\delta e}}{p g_{\delta o} - \delta g_{\delta e} + p h_o}\,.$$

Finally the Z_{ijR} are obtained by substituting the $p - \delta$ divisor for p and changing p^2 into $p^2+\delta^2$ in all (even) polynomials.

Eq. (44) suggests, that simpler mathematical procedure may be ...ble, if S_δ is written as

$$S_\delta = \frac{g_\delta}{f_\delta} = \frac{\left(1-\frac{\delta}{p}\right)^{-1} g(p)}{f_e(p)} \qquad (47)$$

This assumption will be proved by the following derivation.

..., as before, h is defined by

$$|g|^2 = f_e^2 + |h|^2, \qquad (48)$$

... procedure very similar to the one just performed, gives

$$\frac{(f_{11}+f_{22})(p-\delta) + [f] + (p-\delta)^2}{2\,(p-\delta)\, f_{12}} = \frac{p}{p-\delta} \; \frac{g_e + g_o}{f_e}\,. \qquad (49)$$

... separating even and odd parts

$$\frac{g_o}{f_e} = \frac{[f] - \delta(f_{11}+f_{22}) + p^2 + \delta^2}{2p f_{12}} = \frac{p^2 + (f_{11}-\delta)(f_{22}-\delta) - f_{12}^2}{2 p f_{12}} \qquad (50)$$

$$\frac{g_e}{f_e} = \frac{f_{11}+f_{22}-2\delta}{2 f_{12}} = \frac{(f_1-\delta)+(f_{22}-\delta)}{2 f_{12}} \qquad (51)$$

...ined.

From Eq. (48), on the $j\omega$-axis

$$\frac{h_e^2 - h_o^2}{f_e^2} = \frac{g_e^2 - g_o^2}{f_e^2} - 1 \tag{52}$$

holds. Substituting from Eqs. (50) - (51),

$$\frac{h_e^2 - h_o^2}{f_e^2} = \frac{p^2\left[(f_{11}-\delta)+(f_{22}-\delta)\right]^2 - \left[p^2+(f_{11}-\delta)(f_{22}-\delta)-f_{12}^2\right]^2 - 4p^2 f_{12}^2}{4p^2 f_{12}^2} \tag{53}$$

results. Carrying out the operations, the right side of the equation turns out to be

$$\frac{\left\{p^2\left[(f_{22}-\delta)-(f_{11}-\delta)\right]\right\}^2 - \left\{p\left[p^2-(f_{11}-\delta)(f_{22}-\delta)+f_{12}^2\right]\right\}^2}{\left(2p^2 f_{12}\right)^2} . \tag{54}$$

Hence the identifications

$$\frac{h_e}{f_e} = \frac{(f_{22}-\delta)-(f_{11}-\delta)}{2\, f_{12}} \tag{55}$$

and

$$\frac{h_o}{f_e} = \frac{p^2-(f_{11}-\delta)(f_{22}-\delta)+f_{12}^2}{2p f_{12}} \tag{56}$$

may be made.

From Eqs. (50) and (56)

$$f_{12} = p\,\frac{f_e}{g_o + h_o}\,, \tag{57}$$

From Eqs. (51) and (55)

$$f_{11} = \delta + f_{12}\,\frac{g_e - h_e}{f_e} = \delta + p\,\frac{g_e - h_e}{g_o + h_o} \tag{58}$$

$$f_{22} = \delta + p\,\frac{g_e + h_e}{g_o + h_o} \tag{59}$$

follows.

Therefore

$$\begin{aligned} Z_{11\delta} &= \frac{R_1}{p-\delta}\;\frac{\delta(g_o+h_o) + p(g_e - h_e)}{g_o+h_o} \\ Z_{22\delta} &= \frac{R_2}{p-\delta}\;\frac{\delta(g_o+h_o) + p(g_e + h_e)}{g_o + h_o} \\ Z_{12\delta} &= \frac{\sqrt{R_1R_2}}{p-\delta}\;\frac{p f_e}{g_o + h_o} . \end{aligned} \tag{60}$$

Again, by the simple steps described following Eq. (46), the Z_{ij_R} can be obtained.

It has been tacitly assumed, that S_δ contains a constant factor to assure realizability condition 2, of Para. 1.3.d. for N_R . Practically, if this condition is satisfied for S_δ with a small safety margin, S_R will also be realizable.

Very careful considerations must be given to the exact algebraic form of S and h. An analysis similar to that of Para. 1.4.e. should clarify the connection between the necessary form of these variables and the desired configuration and terminations. The derivations can be based upon the reflection coefficients of N_δ , obtained from Eqs. (60), (48) and Eq. (1) of Sec. 1.2.

$$\rho_{1\delta} = \frac{h(p)}{\left(1-\frac{\delta}{p}\right)^{-1} g(p)} - \frac{\delta}{p} \tag{61}$$

$$\rho_{2\delta} = \frac{-h(-p)}{\left(1-\frac{\delta}{p}\right)^{-1} g(p)} - \frac{\delta}{p} .$$

d. The perturbation method[13]

Throughout this Chapter special loss-distributions have been asssumed. A more general method will now be described for the correction of network responses for arbitrary Q-s.

Let the transfer function of a network containing ideal elements be

$$T = T\,(p,\ p_i,\ z_j\ ,\ 0) \tag{62}$$

where p_i and z_j are the zeros and poles, respectively, and the 0 symbolizes the absence of dissipation. The transfer function of the same circuit with a small amount of loss will be

$$T_d = T_d\left(p,\ p_i+\Delta p_i,\ z_j+\Delta z_j\ ,\ r_k\right) \tag{63}$$

since the critical frequencies have been shifted slightly by the dissipation. By definition

$$T_d\left(z_l+\Delta z_l\ ,\ p_i+\Delta p_i\ ,\ z_j+\Delta z_j\ ,\ r_k\right) = 0 \tag{64}$$

or, using

$$T_d \cong T + \sum_{(k)} \frac{\partial T}{\partial r_k}\, r_k\ , \tag{65}$$

$$T_{d(p=z_l)} + \left(\frac{\partial T}{\partial p} + \sum_{(k)} \frac{\partial^2 T}{\partial p\, \partial r_k}\, r_k\right)_{p=z_l} \Delta z_l \cong 0\ . \tag{66}$$

Observing, that r_k and Δz_l are both small,

$$\Delta z_l \cong -\left(\frac{T_d}{\frac{\partial T}{\partial p}}\right)_{p=z_l} . \tag{67}$$

A similar analysis involving $1/T$ and p_i rather than T and z_l obviously gives

$$\Delta p_i \cong -\left(\frac{\frac{1}{T_d}}{\frac{\partial}{\partial p}\,\frac{1}{T}}\right)_{p=p_i} . \tag{68}$$

Now if the transmission function T is precorrected to exhibit $p-\Delta p_i$ and $z_l-\Delta z_l$ as its critical frequencies, evidently a first order correction has been achieved.

Unfortunately, this method is extremely laborious, since it involves the computation of T_d, $\frac{\partial T}{\partial p}$, also the factorization and reassembling of T. Also, it does not simplify for degenerate loss-distributions. (Semi-homogeneous dissipation, lossless capacitors or a single lossy element).

e, Special Techniques

Numerous design procedures have been developed to compensate for dissipation effects in special situations. Many of these are only applicable to circuits designed on the image-parameter basis and are thus beyond the scope of this thesis. A few others will be briefly demonstrated in this Paragraph.

For the case of a narrow-band band-pass filter that consists of series-connected resonant and shunt-connected antiresonant tank-circuits, Butterworth or Chebyshev response can be achieved even with lossy elements.[14] If all circuits are tuned to ω_0, the mid-band frequency, and the circuit is fed from a current-source, the lossy transmission response can be written as a polynomial in

$$P - j\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right). \tag{69}$$

Comparing this with the required Butterworth or Chebyshev behaviour and equating the like coefficients, the element and Q-values can be established.

A simple method is available[8] for achieving infinite attenuation of the pole-frequencies of a lossy network, even for poles located very close to the pass-band. It is illustrated for a symmetrical mid-shunt low-pass filter section (Fig. 30.a.).

By selecting the component values in the lossy network of Fig. 30.b. properly, after a delta-star transformation of L_1-2R-L_1 the equivalent network of Fig. 30.c. will have the same series and shunt inductive reactance (at the attenuation pole-frequency) as the ideal network of Fig. 30.a. In addition, the total complex shunt impedance can have a resonance at this frequency. These conditions provide the required equations for L_1, L_2, R and C_2.

The dual procedure is illustrated on Fig. 30.d. Here star-delta transformation may be utilized.

Unfortunately, since the practical application of the procedure requires the extremely accurate tuning of several elements simultaneously, it has very limited use in communication networks.

An important special case is represented by the lattice networks. For these, a large number of network equivalances (Fig. 31) exist,[3,9] which can be applied to transform resistances into the internal branches of the network. These resistors can then be used to absorb the element losses.

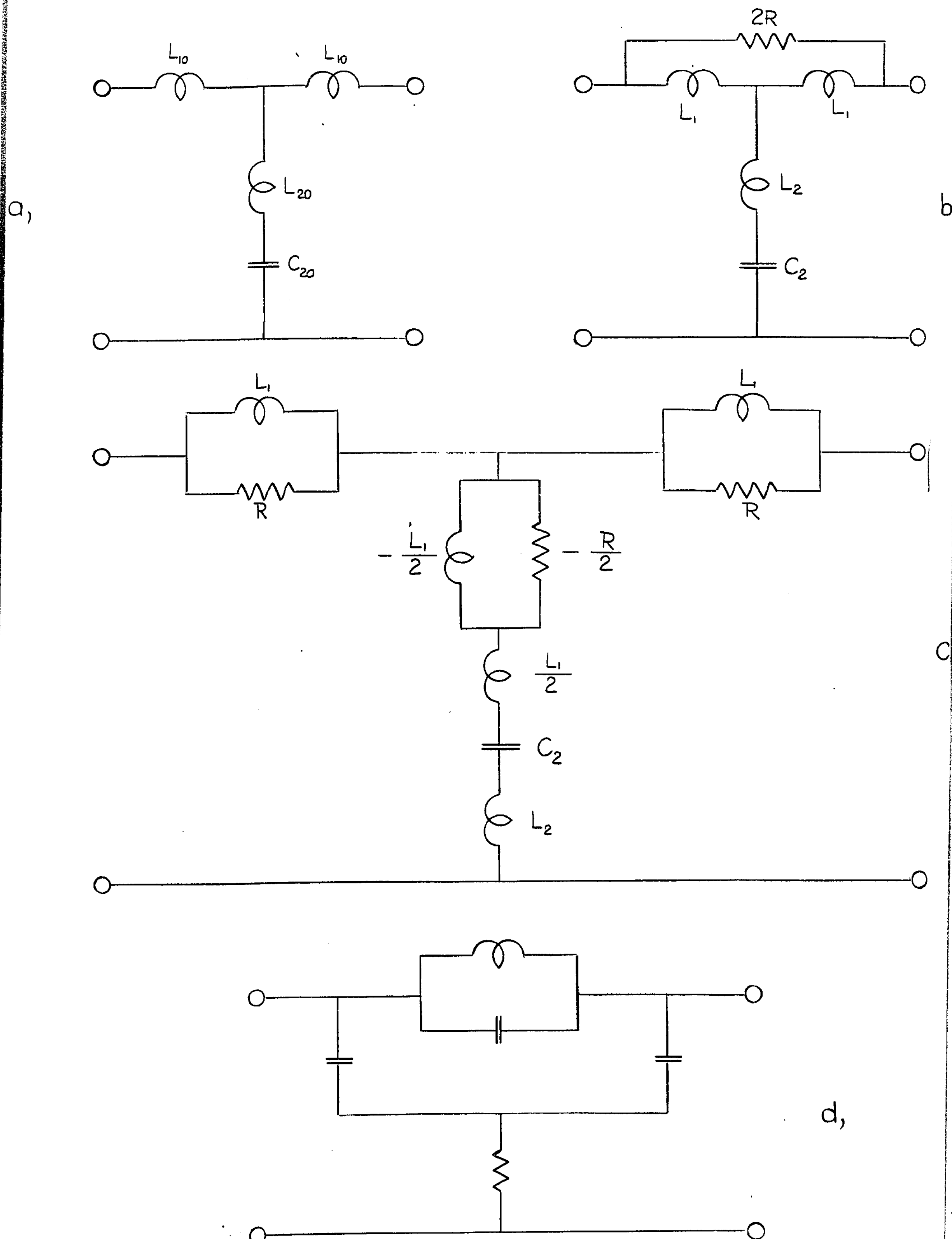
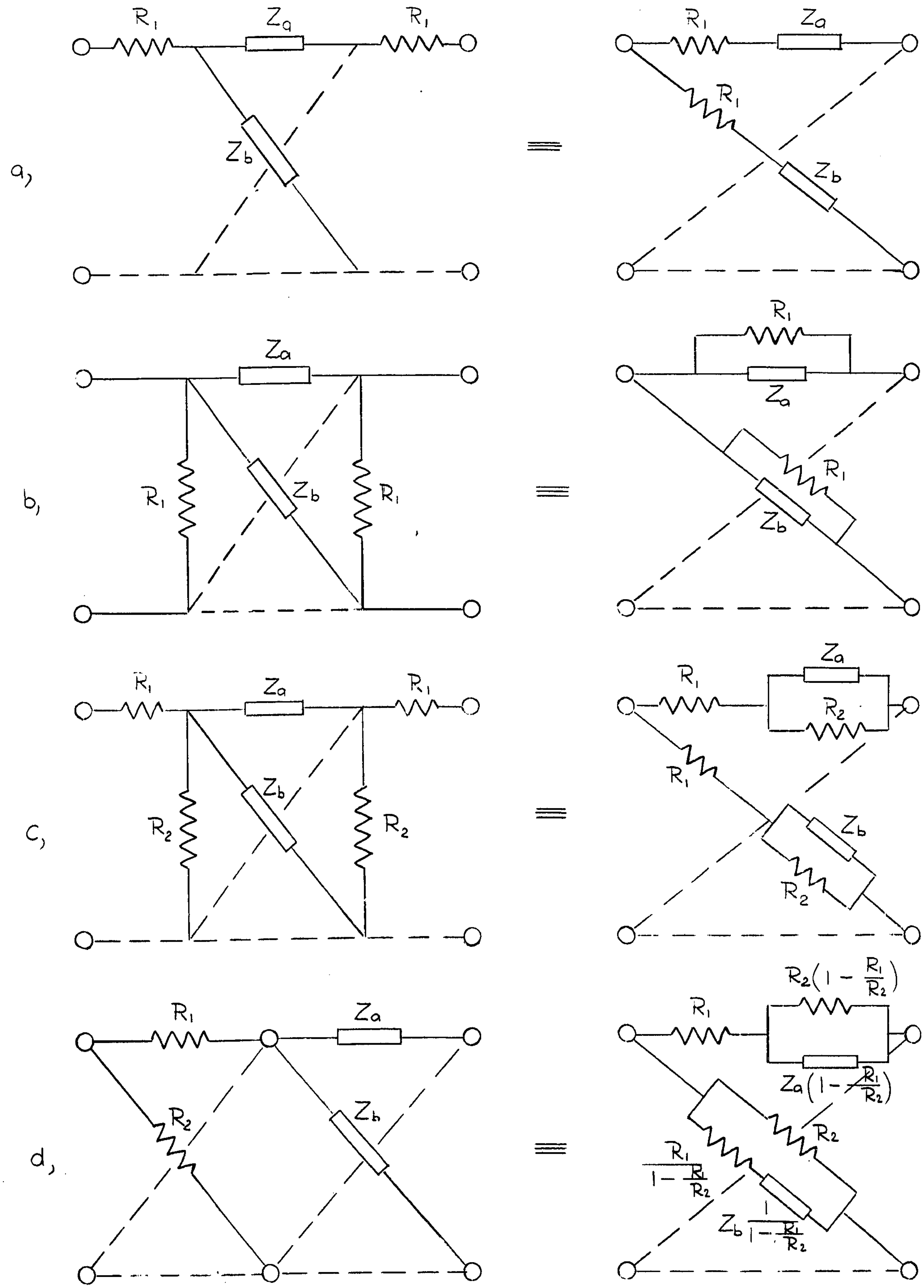
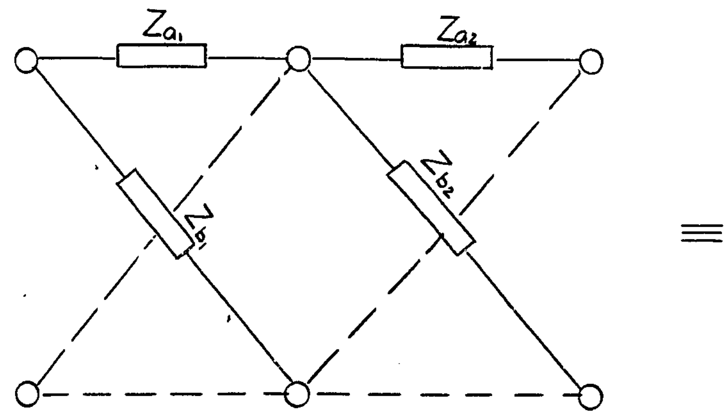


FIGURE 30.



$R_1 R_2 = Z_a Z_b = R_o^2$

FIGURE 31.



$$Z_{a_1} Z_{b_1} = Z_{a_2} Z_{b_2}$$

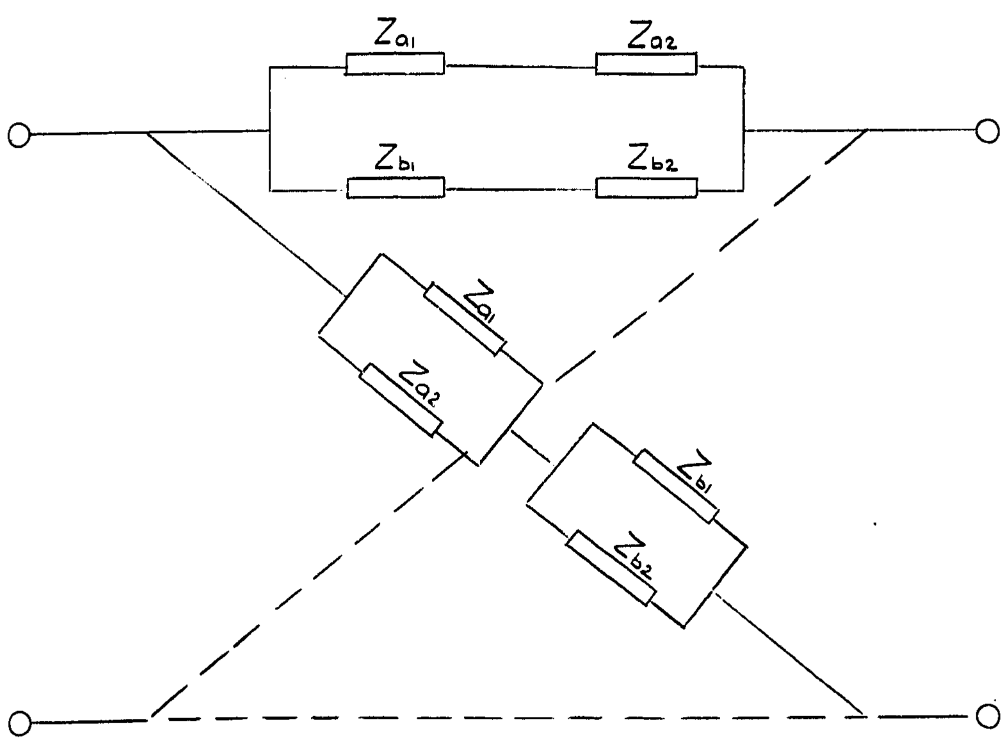


FIGURE 31. e.

IV A COMPREHENSIVE THEORY FOR THE ESTIMATION AND PRECORRECTION OF THE EFFECTS OF LOSSES IN COMMUNICATION NETWORKS.

In this Chapter a new theory is presented for the estimation and precompensation of element losses in resistance-terminated LC-circuits. It is based on some fundamental network properties. A number of new general theorems for the special cases of ladders, insertion loss-designed networks etc. are also proven.

The design method built upon these theoretical considerations is described and demonstrated through several numerical examples. The process is valid for all commonly occuring loss distributions. The procedure uses quantities that are available from the loss less design and supplies:

a, the displacement of transmission poles and zeros due to dissipation,

b, the individual effects of the different elements in the resulting distortion,

c, explicit expressions for the predistorted transmission function,

d, the effects of Q-tolerances.

The method is not restricted to dissipation effects. It could equally well be used for other parasitic phenomena, e.g. stray capacitances, element tolerances, etc.

The formulae are specially simple and useful for the important ladder-configuration and/or for the case of equal loss-factors in the like elements. In the latter situation, it offers an attractive alternative to Darlington's procedure (Para.III.2.c.).

Examples are given to show the relative simplicity and the extreme accuracy of the method in practical network synthesis.

IV. 1. THE SENSITIVITY OF DRIVING-POINT AND TRANSFER PARAMETERS [9, 11, 40]

A number of simple relations can be derived between the driving-point and transfer quantities defined in Section I.1. and their derivaties with respect to an arbitrary circuit immittance. These relations will prove very useful in the analysis of and compensation for the effects of non-ideal element behavior (element variations, tolerances, dissipation etc.) on communication networks. A few direct applications are also stated in this Section.

a. Sensitivity of a parameter [3]

The sensitivity of a parameter P to the variations of element x is defined as:

$$S_P^x = \frac{\partial \ln P}{\partial \ln x} = \frac{x}{P}\,\frac{\partial P}{\partial x}\,. \tag{1}$$

Since P and x are normally known, the determination of S_P^x only requires the finding of $\frac{\partial P}{\partial x}$.

If P is a driving-point or transfer impedance (Z_{lm}) and x a bilateral impedance in the j-th mesh (z), Eqs. (4) and (5) of Section I.1. give

$$Z_{lm} = \frac{E_l}{I_m} = \frac{\Delta}{\Delta_{lm}} = \frac{\Delta^{\circ} + z\,\Delta_{jj}}{\Delta^{\circ}_{lm} + z\,\Delta_{lmjj}}, \tag{2}$$

where Δ is the circuit-determinant and $\Delta^{\circ} = \Delta\big|_{z=0}$.

b. The sensitivity of driving-point impedances

Differentiating Eq. (2) and using Eq. (1), and

$$\Delta\Delta_{abcd} = \Delta_{ab}\,\Delta_{cd} - \Delta_{ad}\,\Delta_{cb}, \tag{3}$$

the following relations are obtained for a driving-point impedance ($l = m = 1$):

$$\frac{\partial Z_{11}}{\partial z} = \frac{\Delta_{jj}}{\Delta_{11}} - \frac{\Delta_{11jj}\,\Delta}{\Delta_{11}^2} = \frac{Z_{11}^{(oc)} - Z_{11}}{Z_{jj}^{(oc)}}\,, \tag{4}$$

$$\frac{\partial Z_{11}}{\partial z} = \frac{Z_{11}}{Z_{jj}} - \frac{Z_{11}}{Z_{jj}^{(oc)}}\,, \tag{5}$$

$$\frac{\partial Z_{11}}{\partial z} = \frac{Z_{11}}{Z_{jj}}\left[1 - \frac{Z_{11}}{Z_{11}^{(oc)}}\right]. \tag{6}$$

$Z_{11}^{(oc)}$ and $Z_{jj}^{(oc)}$ are illustrated in Fig. 32.* It is interesting to note, that

$$\frac{Z_{11}^{(oc)}}{Z_{jj}^{(oc)}} = \frac{\Delta_{jj}}{\Delta_{11jj}}\cdot\frac{\Delta_{11jj}}{\Delta_{11}} = \frac{\Delta}{\Delta_{11}}\;\frac{\Delta_{jj}}{\Delta} = \frac{Z_{11}}{Z_{jj}}\,.$$

Also

$$\frac{\partial Z_{11}}{\partial z} = \left(\frac{Z_{11}}{Z_{1j}}\right)^2 = \left(\frac{I_j}{I_1}\right)^2 \tag{7}$$

and similarly

$$\frac{\partial Z_{11}}{\partial y} = -z^2\,\frac{\partial Z_{1}}{\partial z} = -\left(\frac{E_j}{I_1}\right)^2 \tag{8}$$

$$\frac{\partial (Z_{11})^{-1}}{\partial z} = -Z_{11}^{-2}\,\frac{\partial Z_{11}}{\partial z} = -Z_{1j}^{-2} = -\left(\frac{I_j}{E_1}\right)^2 \tag{9}$$

$$\frac{\partial (Z_{1})^{-1}}{\partial y} = \left(\frac{z}{Z_{11}}\right)^2\frac{\partial Z_{11}}{\partial z} = \left(\frac{E_j}{E_1}\right)^2. \tag{10}$$

Eqs. (7) - (10) include the familiar Cohn-Vratsanos theorem.[11,40]

c. The sensitivity of transfer impedances

For transfer impedance ($l = 1$, $m = 2$)

$$\frac{\partial Z_{12}}{\partial z} = \frac{\Delta_{jj}}{\Delta_{12}} - \frac{\Delta_{12jj}\,\Delta}{\Delta_{12}^2} = \frac{Z_{12}}{Z_{jj}}\left[1 - \frac{Z_{12}}{Z_{12}^{(oc)}}\right] \tag{11}$$

* Z_{jj} and Y_{jj} - also illustrated - are sometimes called "plier-type" impedance and "soldering-iron type" admittance, respectively.[3]

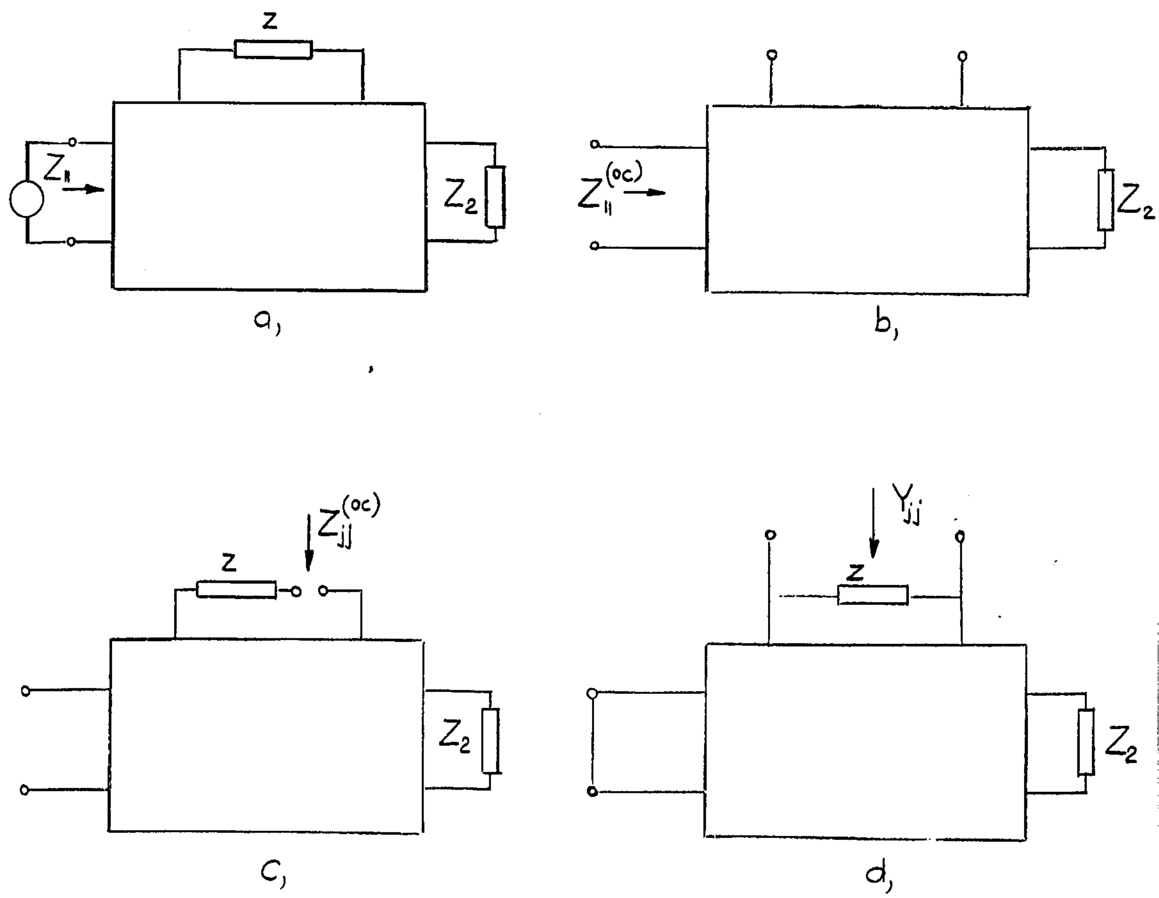


FIGURE 32.

$$\frac{\partial Z_{12}}{\partial y} = \frac{Z_{12}}{Y_{jj}}\left[1 - \frac{Z_{12}}{Z_{12}^{(sc)}}\right] \tag{12}$$

$$\frac{\partial(Z_{12}^{-1})}{\partial z} = -\left(Z_{1j}\,Z_{2j}\right)^{-1} \tag{13}$$

$$\frac{\partial Z_{12}}{\partial z} = \frac{Z_{12}^{2}}{Z_{1j}\,Z_{2j}}\,. \tag{14}$$

Y_{jj} is illustrated on Fig. 32, $Z_{12}^{(sc)}\left(Z_{12}^{(oc)}\right)$ is the value of Z_{12} for z = o (∞). If $Z_{12}^{(oc)}$ or $Z_{12}^{(sc)}$ is zero, as it is normally the case for the configurations of Fig. 33 (which include the practically important ladder networks), Eqs. (11) and (12) simplify to

$$\frac{\partial Z_{12}}{\partial z} = \frac{Z_{12}}{Z_{jj}} \tag{11/a}$$

and

$$\frac{\partial Z_{12}}{\partial y} = \frac{Z_{12}}{Y_{jj}}\,. \tag{12/a}$$

Thus for these cases

$$S_{Z_{12}}^{z} = \frac{z}{Z_{jj}} = \frac{z}{z + Z_{jj}'} \tag{15}$$

and

$$S_{Z_{12}}^{y} = \frac{y}{Y_{jj}} = \frac{y}{y + Y_{jj}'} \tag{16}$$

respectively (Fig. 33).

d. Applications

The main application of the formulae derived above will be shown in the next Section. However, they can also be used in analyzing the effects of element tolerances and parasitic phenomena on the insertion loss or reflection factor of passive

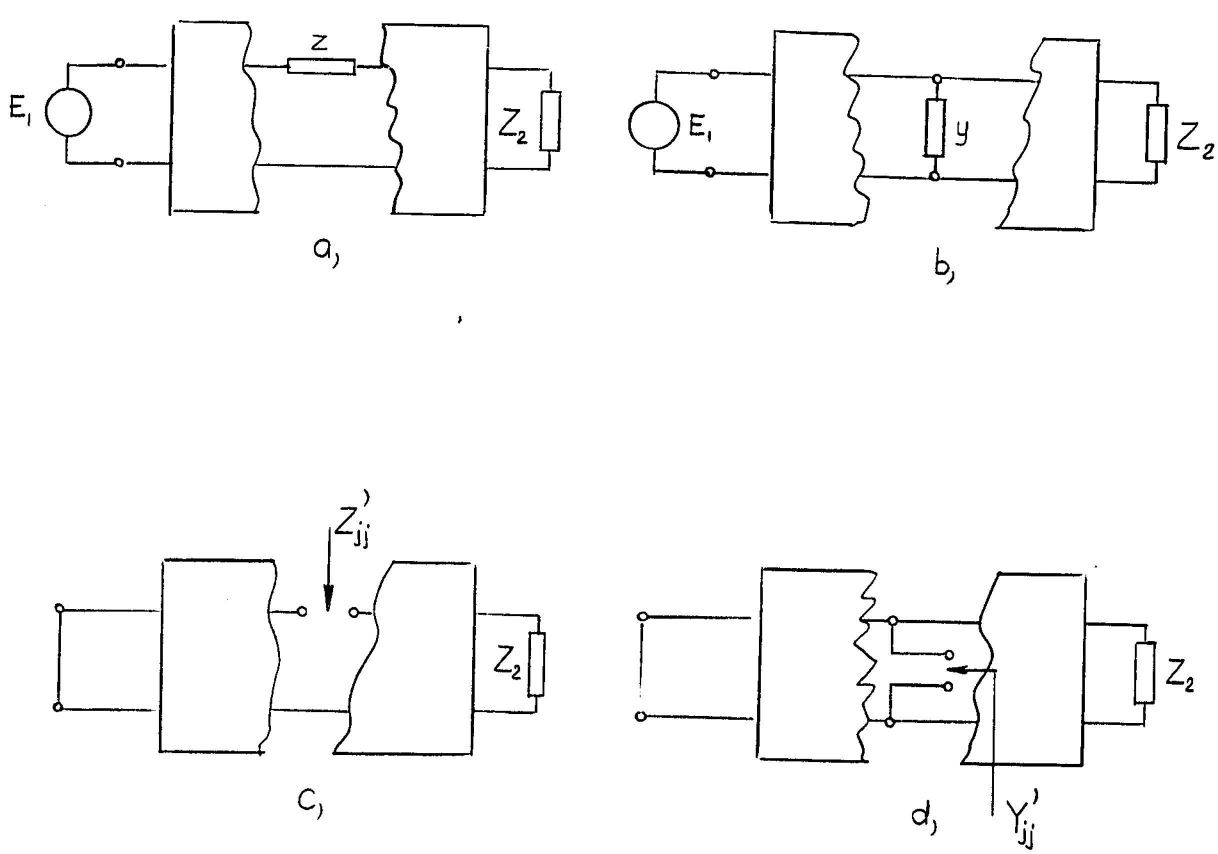


FIGURE 33.

circuits.

The insertion constant of a double-loaded network (Fig. 34) is:

$$\Gamma_i = A_i + jB_i = \frac{1}{2}\log\frac{E^2\left(\frac{R_2}{R_1+R_2}\right)^2}{E^2 Z_{12}^{-2} R_2} = \log\frac{Z_{12}}{R_1+R_2}, \quad (17)$$

where Z_{12} is the transfer impedance of the whole network. Now if z, a "series" element (Fig. 33, a,) changes to $z+dz$, and if the loss and phase are measured in nepers and radians, we get

$$d\Gamma_i = \ln\left[1+\frac{dZ_{12}}{Z_{12}}\right] \simeq \frac{dZ_{12}}{Z_{12}}. \quad (18)$$

Using Eq. (11, a,)

$$d\Gamma_i \simeq \frac{\partial Z_{12}}{\partial z}\frac{dz}{Z_{12}} = \frac{dz}{Z_{jj}}. \quad (19)$$

Similarly, for a "shunt" element y

$$d\Gamma_i \simeq \frac{\partial Z_{12}}{\partial y}\frac{dy}{Z_{12}} = \frac{dy}{Y_{jj}}. \quad (20)$$

One interesting conclusion that could be drawn from Eqs. (19) and (20), refers to networks operating in their pass-band. Since in this case Z_{jj} and Y_{jj} are approximately real, the effect of dissipation, resulting in real dz-s and dy-s, changes, in first approximation, only the loss and not the phase for any distribution of dissipation (with certain restrictions, this has been shown by Bode). Parasitic capacitances in the "shunt" branches, on the other hand, will mainly affect the phase-shift.

At frequencies, where z (y) becomes an open (short) circuit (attenuation poles), Eqs. (19) and (20) no longer hold. Instead,

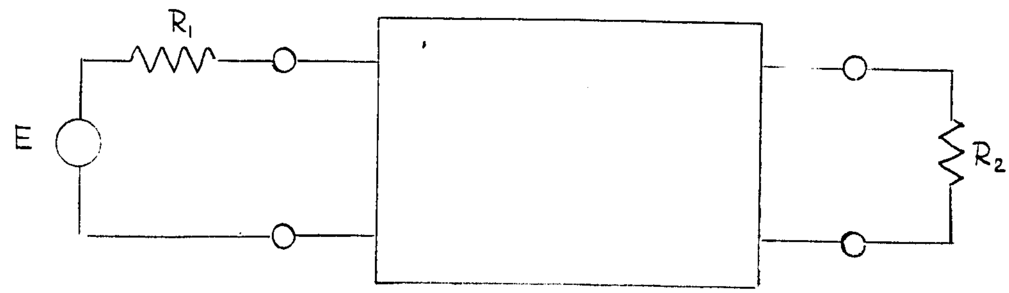


FIGURE 34.

for "series" element $z = y^{-1}$

$$Z_{12} \cong \frac{\Delta_{jj}}{\Delta^{\circ}_{12}} \frac{1}{dy} - \frac{Z_{12}^{(sc)}}{Z'_{jj}\, dy} \tag{21}$$

$$\Gamma_i \cong \ln \frac{Z_{12}^{(sc)}}{(R_1+R_2)\, Z'_{jj}\, dy} . \tag{22}$$

Similarly, for "shunt" element

$$Z_{12} \cong \frac{\Delta^{\circ}}{\Delta_{12jj}} \frac{1}{dz} = \frac{Z_{12}^{(oc)}}{Y'_{jj}\, dz} \tag{23}$$

(Fig. 33,b,) and

$$\Gamma_i \cong \ln \frac{Z_{12}^{(oc)}}{(R_1+R_2)\, Y'_{jj}\, dz} . \tag{24}$$

For the reflection factor

$$\rho_1 = \frac{R_1 - (Z_{11} - R_1)}{R_1 + (Z_{11} - R_1)} = \frac{2R_1}{Z_{11}} - 1 \tag{25}$$

and, using Eq. (7)

$$\frac{\partial \rho_1}{\partial z} = \frac{\partial \rho_1}{\partial Z_{11}} \frac{\partial Z_{11}}{\partial z} = - \frac{2R_1}{Z_{ij}^2} . \tag{26}$$

Equations (15), (16) and a different form of (22) can also be proved using continuants for ladder networks.

If the circuit has to satisfy stiff reflection loss requirements,

$$Z_{D_1} \cong R_1 \tag{27}$$

throughout the pass-band. Then the distortion (in Nepers) caused by the loss in impedance is approximately

$$\Delta A \cong \frac{\partial A}{\partial Z_{D_1}} \Delta Z_{D_1} + \frac{\partial A}{\partial P_d} P_d \simeq \frac{1}{2} \ln \frac{P_2}{P_2 - P_d} \tag{28}$$

where the dissipated power is

$$P_d = | I_j + \Delta I_j |^2 r_j . \tag{29}$$

The first term of Eq. (28) is negligible, since

$$\left(\frac{\partial P_2}{\partial Z_{D_1}}\right)_{Z_{D_1}=R_1} = 0 \ . \tag{30}$$

Using Eqs. (11.a.) and (29),

$$P_d \simeq \frac{E^2 r_j}{|Z_{1j}|^2} \left| 1 - 2\frac{r_j}{Z_{jj}} \right| \tag{31}$$

and

$$\Delta A \simeq \frac{r_j}{2R_j} \left| \frac{Z_{12}}{Z_{1j}} \left(1 - \frac{r_j}{Z_{jj}}\right) \right|^2 \simeq \frac{r_j}{2R_j} \left| \frac{Z_{12}}{Z_{1j}} \right|^2 . \tag{32}$$

The new load current

$$I_2 = E\left(\frac{1}{Z_{12}} - \frac{r_j}{Z_{1j} Z_{2j}}\right) \tag{33}$$

can also be obtained.

For that special case, also

$$Z_{jj} \simeq \frac{|Z_{2j}|^2}{2R_2} . \tag{34}$$

<u>e. Second derivates</u>

The second derivates of Z_{11} , Z_{12} can be calculated similarly:

$$\frac{\partial^2 Z_{11}}{\partial z^2} = -2\,\frac{\Delta_{11jj}\,\Delta_{1j}^2}{\Delta_{11}^3} = -\frac{2}{Z_{jj}^{(\infty)}}\left(\frac{Z_{11}}{Z_{1j}}\right)^2 = -\frac{2}{Z_{jj}^{(\infty)}}\,\frac{\partial Z_{11}}{\partial z} \tag{35}$$

$$\frac{\partial^2 (Z_{11}^{-1})}{\partial z^2} = 2\,\frac{\Delta_{1j}^2\,\Delta_{jj}}{\Delta^3} = \frac{2}{Z_{1j}^2\,Z_{jj}} = -\frac{2}{Z_{jj}}\,\frac{\partial (Z_{11}^{-1})}{\partial z} \tag{36}$$

$$\frac{\partial^2 Z_{12}}{\partial z^2} = -2\,\frac{\Delta_{12jj}\,\Delta_{1j}\,\Delta_{2j}}{\Delta_{12}^3} = -2\,\frac{Z_{12}}{Z_{jj}\,Z_{12}^{(\infty)}}\,\frac{\partial Z_{12}}{\partial z} \tag{37}$$

$$\frac{\partial^2 (Z_{12}^{-1})}{\partial z^2} = 2\,\frac{\Delta_{jj}\,\Delta_{1j}\,\Delta_{2j}}{\Delta^3} = \frac{2}{Z_{1j}\,Z_{2j}\,Z_{jj}} = -\frac{2}{Z_{jj}}\,\frac{\partial (Z_{12}^{-1})}{\partial z} . \tag{38}$$

An important application of these equations is the network containing one element, the impedance of which varies as an approximately exponential function of a parameter τ . E.g., the element may be a thermistor and τ the temperature, or the element a varistor and τ the value of the DC control current.

It is normally desirable to operate such networks at the inflexion point of the (Z_{11}, τ) or $\left(\frac{E_2}{E}, \tau\right)$ curve, where the response is linear and most sensitive. Now, if

$$z = z_0 e^{\kappa\tau}, \tag{39}$$

$$\frac{\partial^2 Z_{11}}{\partial \tau^2} = \frac{\partial Z}{\partial \tau}\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial \tau}\;\frac{\partial Z_{11}}{\partial z}\right) = \kappa^2 z\;\frac{\partial Z_{11}}{\partial z}\;\frac{Z_{jj}^{(oc)} - 2z}{Z_{jj}^{(oc)}}, \tag{40}$$

for

$$z = Z_{jj}^{(oc)} - z = Z_{jj}^{(oc)}, \tag{41}$$

(Fig. 35,a,) the required inflexion is obtained, since normally

$$\frac{\partial^3 Z_{11}}{\partial \tau^3} = -\frac{\kappa^3 Z_{jj}^{(oc)}}{4}\;\frac{\partial Z_{11}}{\partial z} \neq 0. \tag{42}$$

Similarly,

$$\frac{\partial^2}{\partial \tau^2}\left(\frac{E_2}{E_1}\right) = Z_2\,\frac{\partial^2 (Z_{12}^{-1})}{\partial \tau^2} = Z_2 \kappa^2 z\;\frac{\partial (Z_{12}^{-1})}{\partial z}\;\frac{Z_{jj} - 2z}{Z_{jj}} \tag{43}$$

gives the inflexion for

$$z = Z_{jj} - z = Z_{jj}' \tag{44}$$

(Fig. 35,b), since normally

$$\frac{\partial^3}{\partial \tau^3}\left(\frac{E_2}{E_1}\right) = Z_2\;\frac{\kappa^3 Z_{jj}}{4}\;\frac{\partial (Z_{12}^{-1})}{\partial z} \neq 0. \tag{45}$$

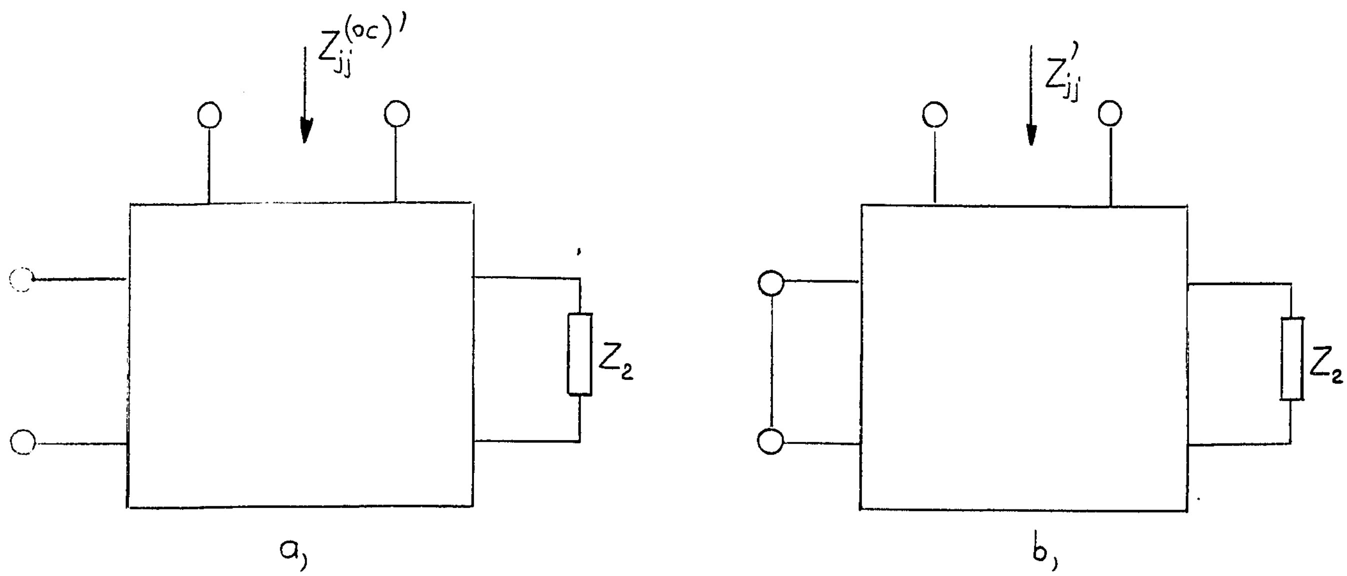


FIGURE 35

IV. 2. THE ESTIMATION AND PRECORRECTION OF THE EFFECTS OF ARBITRARILY DISTRIBUTED PARASITIC ELEMENTS

A general theory for the analysis and synthesis of terminated lossy LC-networks will be presented in this Section. The practical application to specific cases will be treated later (in Sec. IV.3.).

As usual, it will be assumed, that

a, the losses can be represented by a frequency-independent series resistance (parallel conductance) for every coil (capacitor),

b, the dissipation is small, i.e. terms containing Q_i^{-n}, $(n \geqslant 2)$ are normally negligible compared to terms of the order Q_i^{-1}.

No other restriction will be postulated on the distribution etc. of losses.

a, The effect of losses on the transmission function

Let the transmission function be represented by Z_{12}, the transfer impedance (Para. I.1.a.) of the double-terminated network. Since

$$Z_{12} = \frac{\Delta}{\Delta_{12}} = \frac{E_{gen}}{I_2} = (R_1 + R_2)\Lambda - 2\sqrt{R_1 R_2}\; S, \qquad (1)$$

this does not represent any restriction.

The results of Sec. IV.1. can then be applied directly.

If

$$Z_{12} = C \, \frac{\prod\limits_{(i)} (p - p_i)}{\prod\limits_{(j)} (p - p_j)} \qquad (2)$$

(cf., Eq. (1) and Para. I.3.d.), the change due to dissipation will not effect C. This is because C is specified by the behaviour for $p \to \infty$, which will remain unaltered under condition a, given in the introduction of this Section.

The explicit formula for the change in Z_{12} can be found from Eqs. (11) - (12) of Sec. IV.1.

$$\Delta Z_{12} = \sum_{(k)} \frac{\partial Z_{12}}{\partial Z_k} r_k + \sum_{(l)} \frac{\partial Z_{12}}{\partial y_l} g_l \tag{3}$$

$$\Delta Z_{12} = Z_{12} \left\{ \sum_{(k)} \frac{r_k}{Z_{kk}'} \left[1 - \frac{Z_{12}}{Z_{12}^{(0,k)}} \right] + \sum_{(l)} \frac{g_l}{Y_{ll}} \left[1 - \frac{Z_{12}}{Z_{12}^{(s,l)}} \right] \right\}, \tag{4}$$

where

$$Z_{12}^{(0,k)} = \lim_{Z_k \to \infty} Z_{12} \tag{5}$$

and

$$Z_{12}^{(s,l)} = \lim_{Z_l \to 0} Z_{12} . \tag{6}$$

While Eq. (4) could be used as a basis for further analysis, it will be more useful to derive an alternative form of ΔZ_{12}. The development leading to this equivalent form will also provide some important intermediate results.

b. An alternative derivation of the perturbed response

It is obvious from Eq. (2) of Sec. IV.1., that at a natural frequency of the terminated network any "plier-type" impedance Z_{kk} is zero[3] (Fig. 36.):

$$Z_{kk}(p_i) = 0 . \tag{7}$$

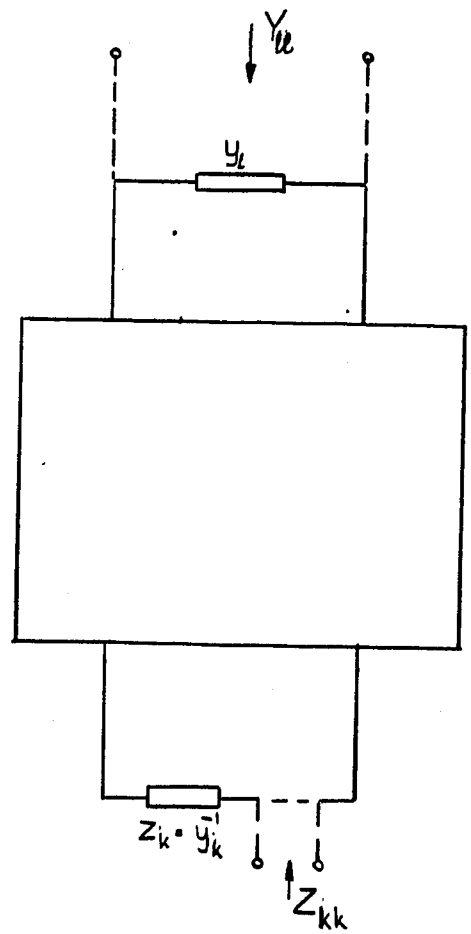


FIGURE 36.

If z_k is changed due to the losses by

$$\Delta z_k = r_k = \text{const.}\,, \tag{8}$$

p_i will be shifted by Δp_{ik} and from

$$Z_{kk}(p_i + \Delta p_{ik}) + r_k = 0 \tag{9}$$

neglecting higher-order terms

$$\Delta p_{ik} \cong - \frac{\Delta z_k}{\left[\frac{\partial}{\partial p}\left(z_{kk} + \Delta z_k\right)\right]_{p_i}} = - \left(\frac{p - p_i}{Z_{kk}}\right)_{p_i} r_k \tag{10}$$

follows. From Eq. (10)

$$\Delta p_{ik} \cong - h_{ik}\, r_k\,, \tag{11}$$

where h_{ik} is the residue of Z_{kk}^{-1} at p_i.

Eq. (10) could also be transformed into

$$\Delta p_{ik} \cong - \left(\frac{p - p_i}{z_k}\;\frac{Z_{11}^{(o,k)}}{Z_{11}}\right) \Delta r_k\,. \tag{10/a}$$

Since the property expressed by Eq. (7) is equally true for any "soldering-iron" type admittance Y_{ll}, the dual equation:

$$\Delta p_{il} = - k_{il}\, g_l \tag{12}$$

is also valid. Here g_l is the (constant) increase in y_l and k_{il} is the residue of Y_{ll}^{-1} at p_i.

For several lossy elements the resultant displacement is

$$\Delta p_i \cong - \sum_{(k)} h_{ik} r_k - \sum_{(l)} k_{il}\, g_l\,. \tag{13}$$

At a (finite) attenuation pole (cf. Eq. (2) of Sec. IV.1)

$$\Delta_{12}(p_j) = \left(\Delta_{12}^{o} + z_k\, \Delta_{12\,kk}\right)_{p_j} = 0 \tag{14}$$

and a derivation similar to that yielding Eq. (11)

gives
$$\Delta p_{jk} \cong - l_{jk}\, r_k\,, \tag{15}$$

where l_{jk} is the residue of

$$\frac{\Delta_{12\,kk}}{\Delta_{12}} - \frac{Z_{12}}{Z_{12}^{(0,k)}\,Z_{kk}} \tag{16}$$

at p_j.

By duality, for

$$y_l \rightarrow y_l + g_l\,, \tag{17}$$

$$\Delta p_{jl} \simeq -\, m_{il}\, g_l \tag{18}$$

results, where m_{il} is the residue of $\frac{Y_{12}}{Y_{12}^{(s,l)}\,Y_{ll}}$ and

$$Y_{12}^{(s,l)} = \lim_{y_l \to \infty} Y_{12}\,. \tag{19}$$

Finally, for the general case, the displacement of the attenuation pole is given by

$$\Delta p_j \simeq -\sum_{(k)} l_{jk}\, r_k - \sum_{(l)} m_{jl}\, g_l\,. \tag{20}$$

A straightforward way of calculating the losses is to use Eqs. (13) and (20) and the invariancy of the constant multiplier in Z_{12} to derive the lossy response. Similarly, a precompensation could be achieved by synthesizing the circuit to exhibit $p_i - \Delta p_i$ and $p_j - \Delta p_j$ — calculated from Eqs. (13) and (20) — as its critical frequencies. This approach would be analog to the one demonstrated in Para. III.2.d. It is, however, very much preferable to pursue the following derivation

$$\Delta(\ln Z_{12}) \simeq \frac{\Delta Z_{12}}{Z_{12}} = \ln\left\{\prod_{(i)} \frac{p - (p_i + \Delta p_i)}{p - p_i} \prod_{(j)} \frac{p - p_j}{p - (p_j + \Delta p_j)}\right\} \tag{21}$$

and

$$\frac{\Delta Z_{12}}{Z_{12}} \cong \sum_{(i)} \ln\left(1 - \frac{\Delta p_i}{p-p_i}\right) - \sum_{(j)} \ln\left(1 - \frac{\Delta p_j}{p-p_j}\right). \quad (22)$$

Using Eqs. (13) and (20)

$$\Delta Z_{12} \cong -Z_{12}\left\{\sum_{(i)}(p-p_i)^{-1}\left[\sum_{(k)} h_{ik} r_k + \sum_{(l)} k_{il} g_l\right] - \sum_{(j)}(p-p_j)^{-1}\left[\sum_{(k)} l_{jk} r_k + \sum_{(l)} m_{jl} g_l\right]\right\}. \quad (23)$$

By the definition of h_{ik}, k_{il}, l_{jk} and m_{jl}

$$\begin{aligned} \sum_{(i)} \frac{h_{ik}}{p-p_i} &= \frac{1}{Z_{kk}} \\ \sum_{(i)} \frac{k_{il}}{p-p_i} &= \frac{1}{Y_{ll}} \\ \sum_{(j)} \frac{l_{jk}}{p-p_j} &= \frac{Z_{12}}{Z_{12}^{(o,k)} Z_{kk}} \\ \text{and} \quad \sum_{(j)} \frac{m_{jl}}{p-p_j} &= \frac{Y_{12}}{Y_{12}^{(s,l)} Y_{ll}}. \end{aligned} \quad (24)$$

Thus, after inverting the order of summations in Eq. (23)

$$\Delta Z_{12} = Z_{12}\left\{\sum_{(k)} \frac{r_k}{Z_{kk}}\left(1 - \frac{Z_{12}}{Z_{12}^{(o,k)}}\right) + \sum_{(l)} \frac{g_l}{Y_{ll}}\left(1 - \frac{Y_{12}}{Y_{12}^{(s,l)}}\right)\right\} \quad (25)$$

is obtained. Eq. (25) is equivalent to Eq. (4), since

$$\frac{Z_{12}}{Z_{12}^{(s,l)}} = \frac{R_1 R_2 Y_{12}}{R_1 R_2 Y_{12}^{(s,l)}} = \frac{Y_{12}}{Y_{12}^{(s,l)}}. \quad (26)$$

Now since the effect of dissipation is to change Z_{12} into

$$Z_{12d} \cong Z_{12} + \Delta Z_{12}, \quad (27)$$

a first-order precompensation will be achieved by choosing

$$Z_{12\,pred} = Z_{12} - \Delta Z_{12}. \quad (28)$$

The validity of Eq. (28) can also be seen from

$$\Delta Z_{12}(-\Delta p_i, -\Delta p_j) = -\Delta Z_{12}(\Delta p_i, \Delta p_j). \qquad (29)$$

c, <u>The influence of individual element-losses on the attenuation</u>

To calculate the distortion caused by the element losses (r_k-s and g_l-s), one can write by Eq. (17) of Sec. IV.1.

$$\Delta A \cong \mathrm{Re}\left(\frac{\partial \Gamma_i}{\partial Z_{12}} \Delta Z_{12}\right) = \mathrm{Re}\left(\frac{\Delta Z_{12}}{Z_{12}}\right) \qquad (30)$$

and by Eq. (25)

$$\Delta A^{(db)} = 8686\ \mathrm{Re}\left\{\sum_{(k)} \frac{r_k}{Z_{kk}}\left(1 - \frac{Z_{12}}{Z_{12}^{(0,k)}}\right) + \sum_{(l)} \frac{g_l}{Y_{ll}}\left(1 - \frac{Y_{12}}{Y_{12}^{(s,l)}}\right)\right\}. \qquad (31)$$

It is easy to see, that Eq. (31) gives the distortion for all $j\omega$-values, including $j\omega$-axis attenuation poles, where also

$$Z_{12d} \cong -\frac{Z_{12}^2}{\Delta Z_{12}} = \frac{1}{\sum\limits_{(k)} \frac{r_k}{Z_{kk} Z_{12}^{(0,k)}} + \sum\limits_{(l)} \frac{g_l}{Y_{ll} Y_{12}^{(s,l)}}}. \qquad (32)$$

Obviously,

$$Z_{12}^{(0,k)} \neq 0$$
$$Y_{12}^{(s,l)} \neq 0 \qquad (33)$$

for $\quad p = j\omega$.

One important aspect of Eq. (31) is that it yields the individual contributions of various lossy elements to the overall effect of the dissipation. It may be used to establish the necessary quality factors of the components and also the tolerances on the quality factors for networks designed using a pre-corrected synthesis procedure. The application of crystal, mechanical etc. elements may be decided through similar analysis.

d. The predistortion of the transmission function

Eqs. (25) and (28) demonstrated the fundamental principle of the precorrection procedure to be followed in the general case. It can be seen, that $Z_{12\,pred}$ as given by this equation is of higher order and is thus represented by more complicated network than Z_{12}. This is always the situation, when the attenuation poles as well as the zeros are pre-shifted[1,25] (cf. Eq. (8) of Sec. III.2.). It will be shown in Sec. IV.3., that for the most important situations drastic simplifications occur and no increase in the order of Z_{12} is necessary.

No consideration has yet been given to the realizability of $Z_{12\,pred}$. From Eq. (1), obviously

$$\left| Z_{12\,pred} \right|_{p=j\omega} \geqslant 2\sqrt{R_1 R_2}\,. \tag{34}$$

If the pass-band of the network includes any extreme frequencies ($p_e = 0$ or ∞), for a hypothetical symmetrically loaded filter

$$\frac{Z_{12}(p_e)}{2R_1} = \Lambda(p_e) = S(p_e) - 1. \tag{35}$$

A constant factor assuring Eq. (35) must be associated with the predistorted Z_{12}, Λ or S before the realizability criteria of Sec. I.3. are applied to establish R_{2pred}. E.g. for a low-pass filter this procedure leads to

$$\frac{1}{2}\left(\sqrt{\frac{R_1}{R_{2\,pred}}} + \sqrt{\frac{R_{2pred}}{R_1}}\right) = \frac{(S - \Delta S)_{p=0}}{|S - \Delta S|_{min,\,j\omega}} \tag{36}$$

from which $R_{2\,pred}$ could be obtained.

Normally $|S|_{min,\,j\omega}$ is known to be at ω_{min}. Then the location of $|S - \Delta S|_{min,\,j\omega}$)

$$\omega_{min,\,pred} = \omega_{min} + \Delta\omega_{min} \tag{37}$$

can be found from

$$\left(\frac{\partial |S|}{\partial \omega}\right)_{\omega_{min}} = \left(\frac{\partial |S - \Delta S|}{\partial \omega}\right)_{\omega_{min,\,pred}} = 0 \tag{38}$$

or

$$\left(\frac{\partial \ln |S|}{\partial \omega}\right)_{\omega_{min}} = \left(\frac{\partial \ln |S - \Delta S|}{\partial \omega}\right)_{\omega_{min,\,pred}} = 0. \tag{39}$$

A number of equivalent expressions can be found. E.g.

$$\omega_{min\,pred} \cong \omega_{min} - \left(\frac{\dfrac{\partial^2 |S|}{\partial\omega\,\partial S}}{\dfrac{\partial^2 |S|}{\partial \omega^2}}\,\Delta|S|\right)_{\omega_{min}} \tag{40}$$

$$\omega_{min\,pred} = \omega_{min} - \left[\frac{(\omega-\omega_{min})^2}{|\varphi|}\,\frac{\partial}{\partial\omega}\,\mathrm{Re}\,\frac{\Delta S}{S}\right]_{\omega_{min}} \tag{41}$$

$$\omega_{min\,pred} \cong \omega_{min} - \left[\frac{(\omega-\omega_{min})^2}{|S|-1}\,\frac{\partial}{\partial\omega}\left(\frac{\mathrm{Re}\,S\,\mathrm{Re}\,\Delta S + \mathrm{Im}\,S\,\mathrm{Im}\,\Delta S}{|S|}\right)\right]_{\omega_{min}}. \tag{42}$$

Using similar techniques, approximate expressions could also be found for $|S - \Delta S|_{min,\,j\omega}$. Practically, however, it is simpler and more accurate to find the location and value of $|S-\Delta S|_{min,\,j\omega}$ in a purely numerical way.

The synthesis procedure starts then with the realization of a lossless network from Z_{12}. Proper Q - s must be chosen and ΔZ_{12} calculated. From $Z_{12} - \Delta Z_{12}$ a realizable termination ratio and a new, topologically unchanged network should be realized. The Q - s or preferably the loss-resistances and conductances of the original network can be retained.

e. Accuracy and higher-order effects

The accuracy of the zero and pole-displacement formulae can be improved through an iterative procedure. E.g. Eq. (10) can be iterated using

$$\Delta p_{ik}^{(n)} = - r_k \left(\frac{p - p_i}{Z_{kk}} \right)_{p_i + \Delta p_{ik}^{(n-1)}} . \qquad (43)$$

Similar equations hold for Δp_{il}, Δp_{jk}, and Δp_{jl}.

The predistortion procedure described in Para. IV. 2.b. and d. presupposes the approximate identity of the original loss-less network and the predistorted loss-less circuit. For a large amount of loss this assumption will be inaccurate. However, by iterating the whole procedure, good results can be obtained. The iteration can be carried out by using Z_{12}, Z_{kk}, Y_{ll}, ... of the once predistorted network to derive a higher-order approximation of $\Delta Z_{12}^{(2)}$ for obtaining a second predistorted circuit.

Also, since a large part of the change in element-values is due to the modification of R_2 , a proper amount of flat loss may be built into the original network. This preamble will make the alteration of the load unnecessary and it will also decrease the necessary changes in the reactive element values.

IV.3. APPLICATION OF THE GENERAL THEORY TO SPECIAL CONFIGURATIONS AND LOSS-DISTRIBUTIONS

The general principles described in Sec. IV.2. will now be applied to some specific configurations and loss-distributions. It will be shown that for the most important cases the rather involved general equations (cf. Eq. (25) of Sec. IV.2.) reduce to straightforward, easily applicable formulae.

All statements given in this Section are true in first-order approximation only, unless explicitly stated otherwise.

a. Ladder networks and other special configurations.

The following important theorem will now be proven:

For any terminated LC ladder network the shift of the attenuation poles due to losses has only second-order effect on the attenuation. The immediate vicinity of any ($j\omega$-axis) attenuation pole is of course excluded from the range of validity.

Only a very specialized form of this theorem, that referring to networks with homogeneous loss, seems to be known[12] (cf. Para. III.1.c.).

To prove [illegible] the statement, first it will be shown, that if Z_n is a composite branch-reactance built from reactive immittances Z_1 and Z_2, the change of Z_n due to losses is pure [illegible] the [illegible].

Let the change

$$\Delta Z_n \cong \sum_{(k)} \frac{\partial Z_n}{\partial z_k} r_k + \sum_{(l)} \frac{\partial Z_n}{\partial y_l} g_l \tag{1}$$

is real, since all $\frac{\partial Z_n}{\partial z_k}$ and $\frac{\partial Z_n}{\partial y_l}$ are real.

Since

$$\Delta Y_m \cong - \frac{\Delta Z_m}{Z_m^2}, \tag{2}$$

the assertion is equally true for composite branch-susceptances.

Next it will be demonstrated, that for a reactive ladder the displacement of any attenuation pole caused by the losses is real. In a ladder an attenuation pole can only be due to

$$\dot{Z}_n(p_j) = 0 \tag{3}$$

or

$$Y_m(p_j) = 0 \; . \tag{4}$$

If the dissipation in its elements changes the branch-immittance, the pole-shift Δp_j can be expressed. For a shunt branch from

$$Z_n(p_j) \cong Z_n(p_j) + \left(\frac{\partial Z_n}{\partial p}\right)_{p_j} \Delta p_j + \Delta Z_n(p_j) \cong 0 \tag{5}$$

or

$$\Delta p_j \cong - \left(\frac{\Delta Z_n}{\frac{\partial Z_n}{\partial p}}\right)_{p_j} = - \left(\frac{j(\omega - \omega_j)\Delta Z_n}{Z_n(j\omega)}\right)_{\omega = \omega_j} \tag{6}$$

which is real according to what has been proven about the real character of ΔZ_n . A dual proof holds for the pole-displacement caused by losses in a series branch.

If for instance the pole was produced by a series-connected antiresonant (shunt-connected resonant) circuit and the coil has a quality-factor Q,

$$\Delta p_j \cong - \frac{\omega}{2Q}, \tag{7}$$

as can readily be obtained.

The main theorem can now be demonstrated from

$$\Delta_p A = \Delta_p \left[\log |Z_{12}(j\omega)| \right] = \mathrm{Re} \left\{ \frac{\Delta_p Z_{12}}{Z_{12}} \right\} \tag{8}$$

where Δ_p indicates a change due to pole-shifts only. Using Eq. (22) of Sec. IV.2.

$$\Delta_p A \simeq \mathrm{Re} \left\{ \sum_{(j)} \frac{\Delta p_j}{j(\omega - \omega_j)} \right\} = 0 \tag{9}$$

since all Δp_j are real, as shown above.

As a consequence of Eq. (9), in the case of ladder networks the terms corresponding to the effect of pole-displacements could be deleted from Eqs. (25) and (31) of Sec. IV.2. These equations simplify then to

$$\Delta Z_{12} \cong Z_{12} \left\{ \sum_{(k)} \frac{r_k}{Z_{kk}} + \sum_{(l)} \frac{g_l}{Y_{ll}} \right\} \tag{10}$$

and

$$\Delta A^{(db)} \cong 8{\cdot}686\ \mathrm{Re} \left\{ \sum_{(k)} \frac{r_k}{Z_{kk}} + \sum \frac{g_l}{Y_{ll}} \right\}. \tag{11}$$

Even these simple expressions are cut to half in the usual case of quasi-ideal capacitors.

The most important feature of Eq. (10) is that in contrast to the general formula, the new

$$Z_{12\,pred} - Z_{12} \left\{ 1 - \left[\sum_{(k)} \frac{r_k}{Z_{kk}} + \sum_{(l)} \frac{g_l}{Y_{ll}} \right] \right\} \tag{12}$$

preserves the original $j\omega$-axis location of the poles and usually also the order and structure of Z_{12}. To show this, Eq. (2) of Sec. IV.1. can be used to derive

$$\frac{1}{Y_{ll}} = \left[y_l + \frac{1}{Z_{ll} - z_l}\right]^{-1} = z_l \frac{\Delta_l^{\circ}}{\Delta}$$

where

$$\Delta_l^{\circ} = (\Delta)_{z_l = 0}$$

and all Δ-s refer to the mesh (z_{ij}) equations.

Then

$$Z_{12\,pred} = \frac{\Delta}{\Delta_{12}}\left\{1 - \sum_{(k)} \frac{\Delta_{kk} r_k}{\Delta} - \sum_{(l)} \frac{\Delta_l^{\circ} z_l g_l}{\Delta}\right\}$$

or

$$Z_{12\,pred} = \frac{\Delta - \sum_{(k)} \Delta_{kk} r_k - \sum_{(l)} \Delta_l^{\circ} z_l g_l}{\Delta_{12}}. \qquad (14)$$

Here

$$\Delta = \frac{P_{2n}(p)}{p^n}$$

$$\Delta_{kk} = \frac{R_{2n-2}(p)}{p^{n-1}}$$

and normally

$$\Delta_l^{\circ} = \frac{S_{2n}(p)}{p^n},$$

where P, R, and S are polynomials of the order shown by the suffixes and

$$z_l = \frac{1}{pC_L}.$$

Hence $Z_{12\,pred}$ retains the order and structure of the original Z_{12}, if any one of the following conditions is satisfied:

1, $d_c = 0$ (the usual case)

2. For special configurations (e.g. if the circuit contains only one capacitor) the denominator of Δ_l° may have lower degree, than n.

3. It will be shown in Para. IV.3.d. that circuits with semi-homogeneous loss distribution also yield proper predistorted transfer functions.

Under any of these circumstances, if the Hurwitz character of the numerator is not affected by the correction terms, i.e. for sufficiently low losses, $Z_{n\,pred}$ can always be realised with a ladder-network that possesses the same configuration as the original loss-less circuit.

It may be observed, that Eqs.(9) - (11) are not restricted to ladder networks. If for all lossy coils (L_k) $Z_{k}^{(0,k)} \rightarrow \infty$ and for all lossy capacitors (C_l) $Y_{ll}^{(s,l)} \rightarrow \infty$, Eqs. (9) - (11) hold. This has already been stated in connection with Eqs. (11/a) and (12/a) of Sec. IV.1.

Some simplification will also occur, if all $Z_{12}^{(s,k)}$ and $Y_{12}^{(0,l)}$ are equal to zero (this can only happen for special configurations and terminations). Then

$$\frac{Z_{12}}{Z_{12}^{(0,k)}\, Z_{kk}} = z_k^{-1}\left(1-\frac{\Delta_{12}^{\circ}}{\Delta_{12}}\right) = \frac{1}{z_k} \tag{15}$$

and

$$\frac{Y_{12}}{Y_{12}^{(s,l)}\; Y_{ll}} = \frac{1}{y_l} \qquad (16)$$

can be substituted into Eqs. (25) and (31) of Sec. IV.2.

Finally, for very low quality-factors and sharp transition regions, it may be important to estimate and (partially) compensate for the effect of the pole-displacements, at least for the shift of the pole located next to the pass-band.

By Eq. (22) of Sec. IV.2. and Eq. (8)

$$\Delta_p A \simeq \mathrm{Re}\left\{\frac{\Delta p_j}{p-p_j} - \frac{1}{2}\left(\frac{\Delta p_j}{p-p_j}\right)^2\right\} - \frac{1}{2}\left(\frac{\Delta p_j}{\omega-\omega_j}\right)^2 \qquad (17)$$

and using Eqs. (6) and (1), for a shunt Z_n

$$\Delta_p A \simeq \frac{1}{2}\left[\frac{\sum\limits_{(k)} \frac{\partial Z_n}{\partial z_k} r_k + \sum\limits_{(l)} \frac{\partial Z_n}{\partial y_l} g_l}{\frac{\partial Z_n}{\partial p}}\right]^2_{p_j} (\omega-\omega_j)^{-2} \qquad (17/a)$$

results (in nepers). For a series Y_m a dual expression can be found.

If e.g. the attenuation poles were realized by simple tuned circuits - either series-connected anti-resonant or shunt-connected resonant networks - substitution into Eq.(17/a) gives the distortion at pass-band limit as

$$\Delta_p A^{(db)} \simeq -\frac{1{\cdot}086}{\sum\limits_{(j)}\left[Q_j(\Omega_j-1)^2\right]} \qquad (17/b)$$

where Q_j is the quality factor of the coil in a pole-producing tank circuit, measured at the pass-band limit (ω_p) and Ω_j is the corresponding pole frequency, normalized to ω_p.

Two comments should be made about Eq. (17/b). First, as predicted, it describes an effect of second order in $1/Q$. Also, its applicability is quite wide, since the branch-reactances of the ladder network can always be represented by their Foster-equivalents. This results in the special situation to which Eq. (17/b) applies.

For very sharp filters, i.e. when some $\Omega_j \cong 1$, it may be necessary to compensate for the shift of the pole p_j nearest to ω_p. If the ladder configuration is to be preserved, this must be done by reshifting the natural frequency p_i nearest to $j\omega_p$. $A(\omega_p)$ will not be effected by Δp_j if

$$\frac{j\omega_p - (p_i + \Delta p_i)}{j\omega_p - (p_j + \Delta p_j)} = \frac{j\omega_p - p_i}{j\omega_p - p_j},$$

or the compensating zero-displacement chosen

$$\Delta p_i = \frac{\omega_p - \frac{p_i}{j}}{2Q_j(\Omega_j - 1)} \qquad (18)$$

where Eq. (7) has been utilized.

b. A generalized network-theorem

The generalized predistortion theory described in Sec. IV.2. and its simplified version derived in Para. IV. 3.a. gives particularly useful results for semi-homogeneous loss-distribution. This application requires the generalization of the following useful theorem stated by Foster.[17]

For a connected network of lumped elements, the relation

$$\sum_{j=1}^{n} \frac{y_j}{Y_{jj}} = V - 1 \tag{19}$$

holds. In Eq. (19), the y_j-s are the branch admittances, the Y_{jj} are the soldering-iron type admittances measured across y_j and V is the number of nodes in the network. The summation is extended over all branches.

The formula could be converted into a more general and - for the present analysis - more useful form. If for some branches z_i, the branch impedance and Z_{ii}, the plier-type impedance for z_i rather then y_i and Y_{jj} are given, from Fig. 36

$$\frac{y_i}{Y_{ii}} = \frac{z_i^{-1}}{z_i^{-1} + (Z_{ii} - z_i)^{-1}} = 1 - \frac{z_i}{Z_{ii}}. \tag{20}$$

Substituting into Eq. (19),

$$\sum_{j=1}^{k} \frac{y_i}{Y_{jj}} - \sum_{i=k+1}^{n} \frac{z_i}{Z_{ii}} = V + k - n - 1 \tag{21}$$

where k and n - k are the numbers of branches treated on an admittance or impedance-basis, respectively.

Let branch i contain two series-connected elements z_{i_1} and z_{i_2}. Then

$$\frac{y_i}{Y_{ii}} = 1 - \left(\frac{z_{i_1}}{Z_{ii}} + \frac{z_{i_2}}{Z_{ii}}\right). \tag{22}$$

Depending on whether the two elements are handled on a (z_{i_1}, z_{i_2}), (z_{i_1}, y_{i_2}) or (y_{i_1}, y_{i_2}) basis, the following equivalent expressions are obtained in addition to Eq. (22):

$$\frac{y_i}{Y_{ii}} = \frac{y_{i2}}{Y_{ii_2}} - \frac{z_{i1}}{Z_{ii}} \tag{23}$$

and

$$\frac{y_i}{Y_{ii}} = \frac{y_{i1}}{Y_{ii_1}} + \frac{y_{i2}}{Y_{ii_2}} - 1 \,, \tag{24}$$

where

$$Y_{ii_j} = y_{ij} + \left(Z_{ii} - z_{ij}\right)^{-1} \tag{25}$$

is the soldering-iron type admittance measured across the terminals of the single impedance z_{ij} .

Using Eqs. (21) - (25), the generalized expression is found to be

$$\sum_{j=1}^{l} \frac{y_j}{Y_{jj}} - \sum_{i=1}^{m} \frac{z_i}{Z_{ii}} = V - q - \frac{t-u}{2} - 1 \tag{26}$$

where q and s are the numbers of single - element branches, treated on the impedance or admittance basis, respectively. t and u are the corresponding element numbers in two-element branches. Obviously

$$\begin{aligned} l &= s + u \,, \\ m &= q + t \,. \end{aligned} \tag{27}$$

The summation here extends to all <u>elements</u> of the circuit. In Eq. (26) Y_{jj} and Z_{ii} are referring to the soldering-iron and plier-type immittances adjunct to the individual <u>elements</u>.

<u>c. The pre-correction of dissipation effects for semi-homogeneous loss distribution.</u>

If the distribution of losses is semi-homogeneous, i.e. if all coils have the same quality-factor and the same is true for the capacitors, but $Q_L \neq Q_c$, a two-step procedure can be applied for the deduction of the predistorted transmission function. The derivation of this process will now be carried out in terms of insertion loss parameters; using Eq. (1) of Sec. IV.2, the transition to other transfer variables is straightforward.

Suppose that a homogeneous dissipation with loss-factor

$$d_0 = \frac{d_L + d_c}{2} = \frac{\omega}{2}\left(\frac{1}{Q_L} + \frac{1}{Q_c}\right) \tag{28}$$

is removed from the prescribed loss-less response. As in Para. III. 2.c., the new response will correspond to a network N_δ built from lossy coils and negatively lossy capacitors[1,12] with loss-factors:

$$d_{L\delta} = -d_{C\delta} = \delta \tag{29}$$

where

$$\delta = \frac{1}{2}\left(d_L - d_c\right). \tag{30}$$

The predistortion of Λ_δ, the insertion voltage ratio of N_δ can be executed using Eq. (12):

$$\Lambda_R = \left(\frac{V_{20}}{V_2}\right)_R = \Lambda_\delta\left\{1 - \left[\sum_{(i)} \frac{r_{\delta i}}{Z_{\delta ii}} + \sum_{(j)} \frac{g_{\delta j}}{Y_{\delta jj}}\right]\right\}$$

or, substituting

$$\begin{aligned} r_{\delta i} &= \delta L_{\delta i} = \frac{\delta}{p} z_{\delta i} \\ g_{\delta j} &= -\delta C_j = -\frac{\delta}{p} y_{\delta j}, \end{aligned} \tag{32}$$

the result is

$$\Lambda_R = \Lambda_\delta\left\{1 + \frac{\delta}{p}\left[\sum_{(j)} \frac{y_{\delta j}}{Y_{\delta jj}} - \sum_{(i)} \frac{z_{\delta i}}{Z_{\delta ii}}\right]\right\}. \tag{33}$$

Here the first summation is extended to all capacitors and the second includes all coils and the terminations.

The use of Eq. (26) of the previous Paragraph now leads to

$$\Lambda_R = \Lambda_\delta \left[1 + \delta \; \frac{D + 1 - T_\delta(p)}{p} \right], \tag{34}$$

where

$$T_\delta(p) = \frac{Z_{D1\delta}}{R_{1\delta} + Z_{D1\delta}} + \frac{Z_{D2\delta}}{R_{2\delta} + Z_{D2\delta}} \tag{35}$$

and $Z_{D_1\delta}, Z_{D_2\delta}$ are the driving-point impedances of the terminated four-pole N_δ. Also

$$D = V - m - 2, \tag{36}$$

where m is the number of branches comprising single coils diminished by the number of capacitors in series with R_1 or R_2.

Since both the numerator of Λ_δ and the denominator of $T_\delta(p)$ exhibit the natural frequencies of N_δ as roots, simplifications occur and - as it was pointed out in connection with Eq. (12), - for sufficiently low losses the ladder-realization of N_R corresponding to Λ_R is usually possible, if the original Λ was ladder-realizable. This statement implies the use of the approximate method (Para. III.2.a.) in the execution of the homogeneous predistortion.

d. The application of the method to networks synthesized on the insertion loss basis.

Eq. (34) can be rewritten by using the insertion loss polynomials A, pB, A' and pB'. From Eq. (27) of Sec. I.4., the driving-point impedances of N_δ can be expressed as

$$Z_{D_1\delta} = R_{1\delta}\frac{A_\delta - A'_\delta + p\left(B_\delta - B'_\delta\right)}{A_\delta + A'_\delta + p\left(B_\delta + B'_\delta\right)} \tag{37}$$

and

$$Z_{D_2\delta} = R_{2\delta}\frac{A_\delta + A'_\delta + p\left(B_\delta - B'_\delta\right)}{A_\delta - A'_\delta + p\left(B_\delta + B'_\delta\right)}\ . \tag{38}$$

Substituting into Eqs. (35) and (34)

$$1 - T_\delta(p) = \frac{p\,B'_\delta}{A_\delta + pB_\delta} \tag{39}$$

and

$$\Lambda_R = \Lambda_\delta\left[1 + \frac{\delta}{p}\left(D + \frac{pB'_\delta}{A_\delta + pB_\delta}\right]\right.$$

are obtained. Thus, the predistortion can be described simply by

$$\Lambda_\delta = \frac{A_\delta + pB_\delta}{P_\delta} \longrightarrow \Lambda_R = \frac{A_\delta + pB_\delta + \delta\left[\frac{D}{p}\left(A_\delta + pB_\delta\right) + B'_\delta\right]}{P_\delta} \tag{40}$$

For a wide variety of networks (mid-series and mid-shunt m-derived low-pass filters, constant -k configuration low-pass circuits, or any circuit passing energy at zero freqnency) D is zero and Eq. (40) is reduced to

$$\Lambda_R = \frac{A_\delta + pB_\delta + \delta B'_\delta}{P_\delta}\ . \tag{41}$$

The change in attenuation due to the opposite dissipations $\pm\delta$ in the L-s and C-s can be calculated from Eqs. (11), (26) and (39):

$$\Delta A^{(db)} \approx 8{\cdot}686\,\mathrm{Re}\left\{\delta\,\frac{T_\delta - D - 1}{p}\right\} = -8{\cdot}686\,\mathrm{Re}\left\{\frac{\delta}{p}D + \frac{\delta B'_\delta}{A_\delta + pB_\delta}\right\} \tag{42}$$

evaluated for $p = d_o + j\omega$.

Choosing arbitrarily $D = 0$ introduces an error of

$$\Delta' A^{(db)} \cong 8{\cdot}686\,\frac{\delta d_o D}{\omega^2 + d_o^2} \cong 2{\cdot}172\,D\left(Q_L^{-2} - Q_C^{-2}\right) \tag{43}$$

with the Q-s measured at the lowest pass-band limit.

An important aspect of Eqs. (42) - (43) is that they demonstrate the indifference of the D-value employed. Thus the simpler Eq. (41) rather than Eq. (40) can be used in all cases.

e. General comments on the precorrection procedure

Several precautions and remarks can be made in connection with the analysis and synthesis processes demonstrated in Sections IV.2.

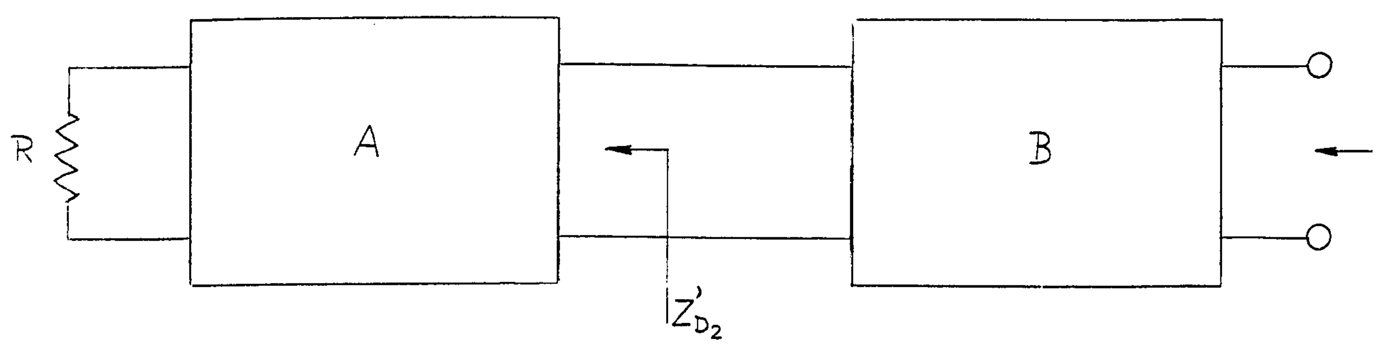
The general procedure (for arbitrary loss-distribution) will be greatly alleviated, if the Z_{kk} and Y_{ll} of the loss-less network are made available at the outset. This will be the case e.g. if the circuit has been developed simultaneously from both the primary and secondary driving-point immittances. Even for one-sided realization, the calculations can be simplified using a network theorem demonstrated on Fig. 37. The computation effort is cut to half for structurally symmetrical circuits.

If the special procedure of Para. IV.3.d. is used, special care must be exercised to establish the sign of B_δ' correctly for the configuration .

It has been shown in Para. I.4.e. how to utilize the expressions for the reflection coefficients to arrive at a proper characteristic function.[41] If one of the terminations is extreme, the formulae for the open and short-circuit immittances (Eqs. (34) - (35) of Sec. I.2.) can also be used to find such signs for A' and pB' that will prevent all immittances from becoming identically zero or infinite.

The use of the precorrected synthesis procedure outlined in

IF



THEN

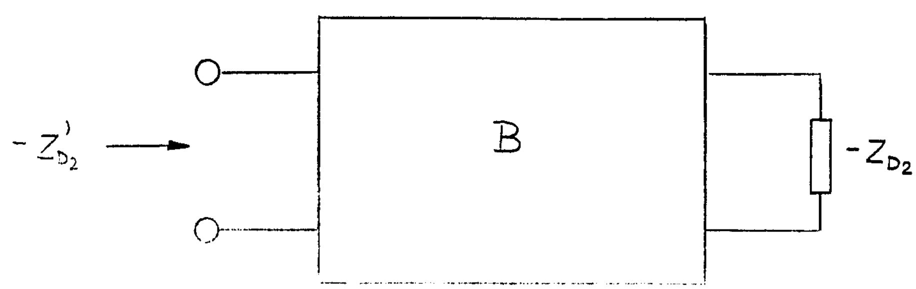


FIGURE 37.

Secs. IV.2.3. normally involves the solution of two polynomial equations, the first one to find $A'_\delta + pB'_\delta$, the second to find $A'_R + pB'_R$. The latter may be omitted, if the method of undetermined coefficients is used in conjuction with the equations

$$A'^2_\delta - p^2 B'^2_\delta - A^2_\delta - p^2 B^2_\delta - \eta_G P^2_\delta \tag{44}$$

and

$$A'^2_R - p^2 B'^2_R = (A'_\delta + \Delta A'_\delta)^2 - p^2(B'_\delta + \Delta B'_\delta)^2 = (A_\delta + \Delta A_\delta)^2 - p^2(B_\delta + \Delta B_\delta)^2 - \eta_G P^2_\delta \tag{45}$$

For single-terminated network usually one polynomial equation needs to be solved.

Several interesting observations can be made about Eqs. (40)-(41). Their common form is identical to that of Λ_δ with an added correction factor (proportional to δ) in the numerator. Without the correction the procedure would be equivalent to a homogeneous frequency-shift by $\frac{d_L + d_C}{2}$, a method often applied in network synthesis.[9] The correction represents a very significant improvement, as some examples of Section IV. 4. will show. The accuracy is particularly enhanced around the origin, where the homogeneous method becomes very crude, as Eqs. (28) and (29) of Sec. III.1. suggest.

For $\delta \to 0$, Eqs. (40) - (41) predictably degenerate into a homogeneous predistortion.

It is also of some interest that in the case of the important class of networks for which D is zero, the predistortion of the oppositely dissipative network only effects the even part of the numerator.

Because of the relative simplicity of the procedure outlined in Para. IV.3.d., if the spread of the Q_{Li} -s is not very large and the same is true for the Q_{cj} -s , it is normally acceptable to use

$$Q_L = \frac{\sum_{i=1}^{n} Q_{Li}}{n}$$
$$Q_c = \frac{\sum_{j=1}^{m} Q_{cj}}{m} \qquad (46)$$

and apply the method of Para. IV.3.d.

Eq. (11) can be used to establish the distortion introduced by this approximation.

Finally, it should be emphasized, that the general principles of Chapter IV are equally applicable for the estimation and correction of any parasitic effect (stray capacitances and inductances, element value tolerances, etc.)

f. Alternative methods for obtaining the predistorted network

Using the fundamental philosophy of the predistortion procedure developed above, numerous basically similar synthesis processes could to be constructed. Some will be briefly treated in this paragraph.

The derivation performed to obtain Eq. (25) of Sec. IV.2. could be duplicated for the frequency-displacement formulae [13], Eqs. (67) and (68) of Para. III.2.d. Observing from these equations, that the shift of any zero of $T(p)$ is the negative residue of $\frac{T_d}{T}$ at that root and that any pole-shift is the negative residue of $\frac{T}{T_d}$ at the pole, the steps expressed by

Eqs. (21)- (25) of Sec. IV.2. lead to

$$\Delta T(p) \cong T_d(p) - \frac{[T(p)]^2}{T_d(p)} . \tag{47}$$

T is a transmission quantity $\left(\frac{X_2}{Y_1}\right)$. For an attenuation quantity $\frac{Y_1}{X_2}$ (e.g. Λ, S, Z_{12} etc.)

$$\Delta U = \Delta\left(\frac{1}{T}\right) \cong - U^2 \Delta T \tag{48}$$

or

$$\Delta U \cong U_d(p) - \frac{[U(p)]^2}{U_d(p)} . \tag{49}$$

From the derivation leading to Eq. (47) - (49) it is easy to see, that the first term on the right side of Eq. (49) represents the effect of the natural frequency displacements, while the second term corresponds to the pole-shifts. Therefore, for ladder-networks

$$U_{pred} = U - \Delta U \cong 2U - U_d . \tag{50}$$

To avoid a change in the configuration, if

$$\begin{aligned} U &= \frac{E}{P} \\ U_d &= \frac{E_d}{P_d} \end{aligned} \tag{51}$$

the approximation

$$U_{pred} \cong \frac{2E - E_d}{P} \tag{52}$$

may be used.

The simplest way to find E_d, the polynomial proportional to the circuit determinant of the dissipative network, is to observe, that it is contained in the numerator of

$$Z_{1d} = R_1 + Z_{D1d} . \tag{53}$$

It is also possible to equate the circuit determinants of the predistorted lossy and the original loss-less circuit (including a proportionality factor). If in addition the attenuation poles of the loss-less predistorted network are chosen to agree with those of the original circuit, neglecting high-order terms of small quantities a linear system of equations results for the ΔL_i and ΔC_j, the changes in element values. A proper value of $R_{2\,pred}$ must be selected in advance.

An interesting extension of the basic idea used in Para. III. 1.a. for the temperature-compensation of networks can also be applied for loss-effect correction. If semi-homogeneous loss-distribution is supposed and, as in Para. IV.3.c., a homogeneous predistortion by d_o is performed, the usual oppositely dissipative network exhibiting loss-factors $\pm\delta$ results.

This consists of immittances

$$\begin{aligned} pL_{i\delta} + R_{Li\delta} &= pL_{i\delta}\left(1+\frac{\delta}{p}\right); \\ pC_{j\delta} + G_{Cj\delta} &= pC_{j\delta}\left(1-\frac{\delta}{p}\right); \end{aligned} \tag{54}$$

An analysis similar to that leading Eq.(9) of Sec. III.1. shows that a change of the terminations

$$R_{k\delta} \to R_{k\delta}\left(1+\frac{\delta}{p}\right) = R_{k\delta} + \frac{1}{C_{k\delta}\,p} \tag{55}$$

would serve to compensate the effect of opposite dissipations. Obviously, the power series that are truncated in the derivation of Eq. (55) do not converge for $p \approx 0$. Hence this process is not accurate for low frequencies.

Eq. (55) suggests the cascading a capacitor $C_{k\delta}$ with each termination of the δ-network. When the dissipation corresponding to d_L and d_c is associated with the elements of the corrected δ-network the usual way,

$$pC_{k\delta} \rightarrow pC_{k\delta} + G_{k\delta} \tag{56}$$

so that the final network will consist of the homogenously predistorted reactive elements and a pair of compensating impedances cascaded with the terminations. The compensating impedance in series with R_k will be according to its derivation:

$$\frac{1}{pC_{k\delta} + G_{k\delta}} = \frac{\delta R_k}{p + d_c} \tag{57}$$

a parallel RC combination.

It is unfortunate that this method breaks down exactly where it is most needed, at the vicinity of the frequency-origin (cf. Eqs. (28) and (29) of Sec. III.I.).

While it is beyond the scope of this work, it is interesting to notice, that by an argument paralleling the one advanced in Para. IV.2.b., a predistortion formula can also be developed for the synthesis of an LCR - impedance from lossy reactive elements. From Eqs. (5) and (8) of Sec. IV. 1.,

$$Z_{11pred} = Z_{11} - \Delta Z_{11} \cong Z_{11}\left[1 - \sum_{(k)} r_k \left(\frac{1}{Z_{kk}} - \frac{1}{Z_{kk}^{(0,1)}}\right) + \sum_{(l)} \frac{g_l}{Y_l^2}\left(\frac{1}{Z_{ll}} - \frac{1}{Z_{ll}^{(0,1)}}\right)\right] \tag{58}$$

where $Z_{kk}^{(0,1)}$ and $Z_{ll}^{(0,1)}$ are measured with open circuited, Z_{kk} and Z_{ll} with short-circuited input terminals.

IV. 4 THE PRECORRECTED SYNTHESIS OF COMMUNICATION NETWORKS

NUMERICAL EXAMPLES [9,13]

A variety of numerical examples will be given to illustrate the efficiency and accuracy of the new precorrection method. All calculations, checking and response-plotting have been carried out on a digital computer to avoid any numerical or experimental errors. An attempt was made to recalculate some problems already worked out by previous authors to obtain a comparative measure of simplicity and accuracy. In all examples shown - and in several others not described here - the computations were relatively straightforward and the precision far surpassed any practical requirements.

a, A single-loaded Chebyshev pass-band filter

To test the accuracy of Eqs. (10) - (20) of Para. IV.2.b., the example used in an article by Desoer[13] was recalculated. This example describes the predistortion of a fifth-order Chebyshev-passband low-pass filter, driven by a current source and terminated in a 1 Ω resistor. For a passband $0 \leq \omega \leq 1$, the network could be realized in the form shown on Fig. 38. If the coils are assigned a Q of 40 (following Desoer)[13], Eq. (10) of Sec. IV.2. can be used to obtain first-order approximations of the natural frequency displacements.

The results are shown in the Table.

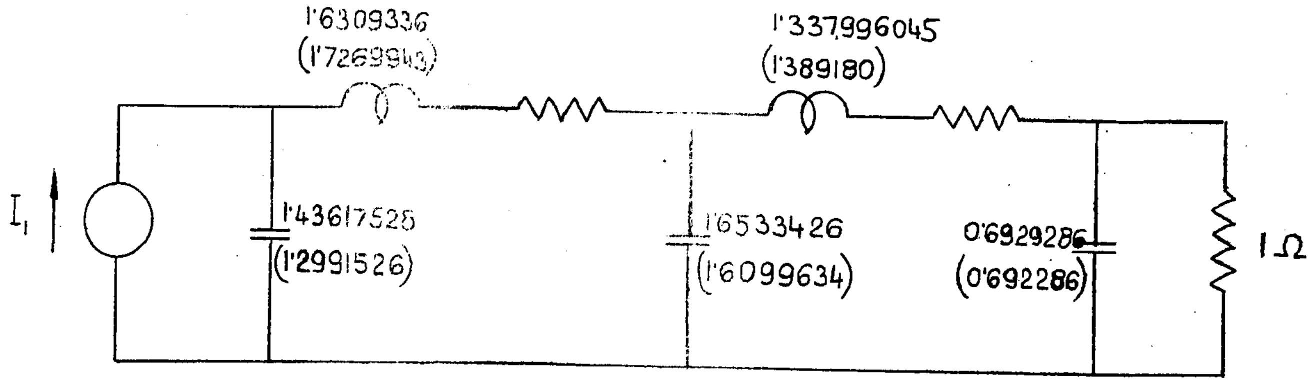


Figure 28.

Chebyshev-passband low-pass filter. Predistorted element values in brackets.

Natural frequencies (p_i)	-0.46141	-.37329$\pm$j0.64734	-0.14258$\pm$j1.04791
Exact displacement (Δp_i)	+0.02664	-0.2461$\pm$j0.012195	-0.01370$\pm$j0.00259
Desoer's approximant for Δp_i	+0.02708	-0.02462$\pm$j0.01107	-0.01392$\pm$j0.00184
First approximant (Eq. (10))	+0.02734	-0.02495$\pm$j0.01165	+0.01392$\pm$j0.00184
Error (Desoer)	-0.00044	+0.00001$\pm$j0.00112	+0.00021$\pm$j0.00075
Error (Eq. (10))	-0.00070	+0.00034$\pm$j).00054	+0.00002$\pm$j0.00013

Table:

Natural frequency displacements

Evidently the results are comparable to Desoer's. The resulting pass-band response is shown in Fig. 39. The maximum deviation from the ideal Chebyshev-response is 0.03 db, negligible in most practical applications, even with this unrealistically low Q.

While this is larger than the value obtained by using Desoer's equation (5), in this example the second-order errors caused by the error in Δp_i on one hand and by the change in the value of $\frac{\partial Z_{ii}}{\partial p}$ (or $\frac{\partial U}{\partial p}$) when $p_i \rightarrow p_i - \Delta p_i$ on the other hand tends to compensate. Thus, the frequency-response computed from Eq. (10) is much closer to the response obtained using <u>exact</u> Δp_i, than the curve given by Desoer's Eq. (5).[13]

The effect of uncompensated dissipation is indicated on Fig. 40. The continuous curve has been computed exactly, the separate points from Eq. (31) of Sec. IV. 2.

Fig. 41 shows the compensation obtained using Eq. (12) of Sec. IV.3. This method requires much less computation work than the procedure based on Eq. (10) and the results are slightly better (the maximum deviation from ideal is about 0.02 db).

b, <u>The predistortion of a double-loaded elliptic low-pass filter using the general method</u>

A 3 - section low-pass filter using lossy coils (Q = 300) and (practically) loss-less capacitors should satisfy the following specifications.

$$A_p = 0.1 \text{ db}$$
$$A_s = 40 \text{ db}$$
$$\Omega_p = \frac{1}{\Omega_s} = 0.952$$
$$R_1 = R_2 = 1$$

FIGURE 39.

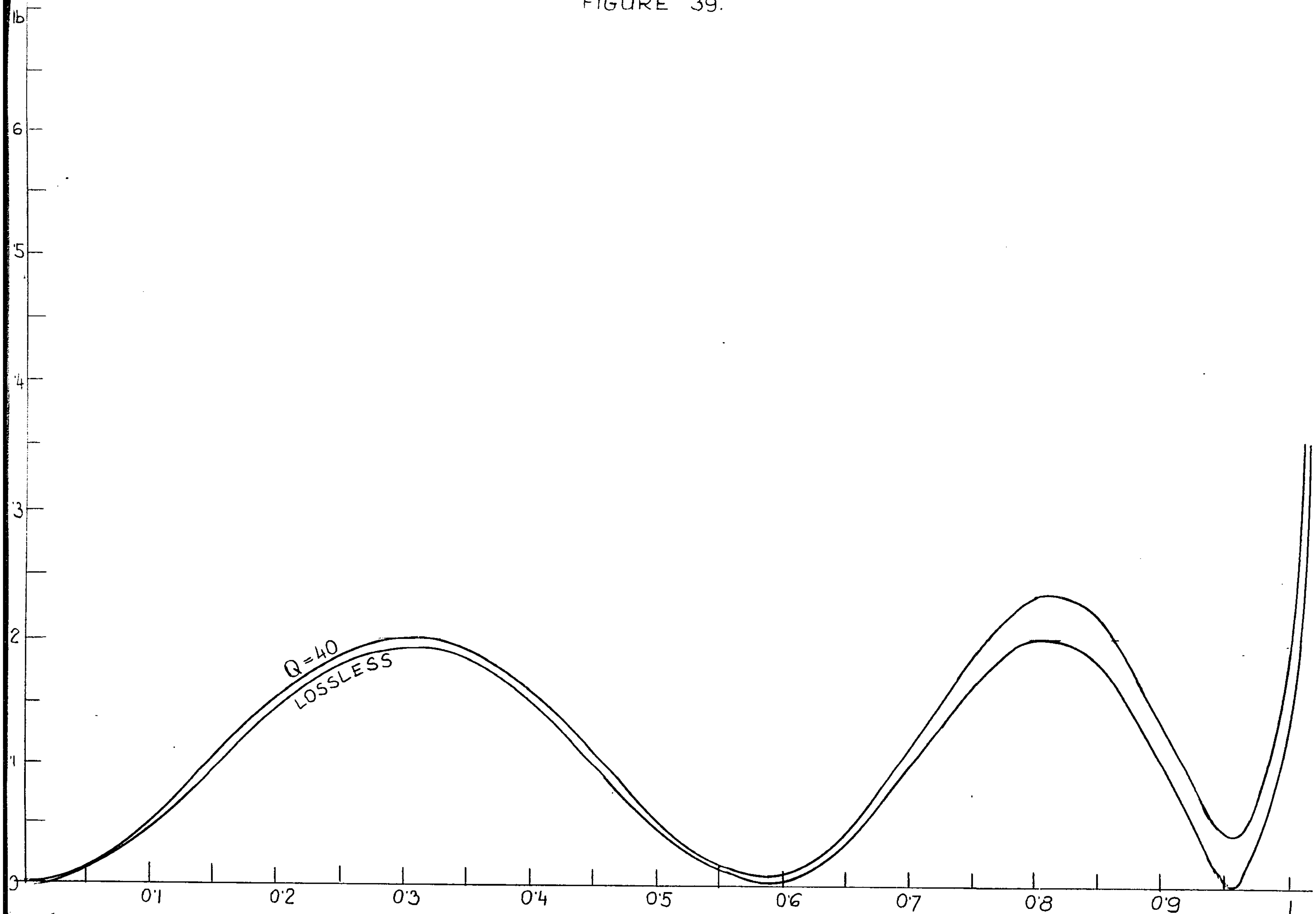


FIGURE 40

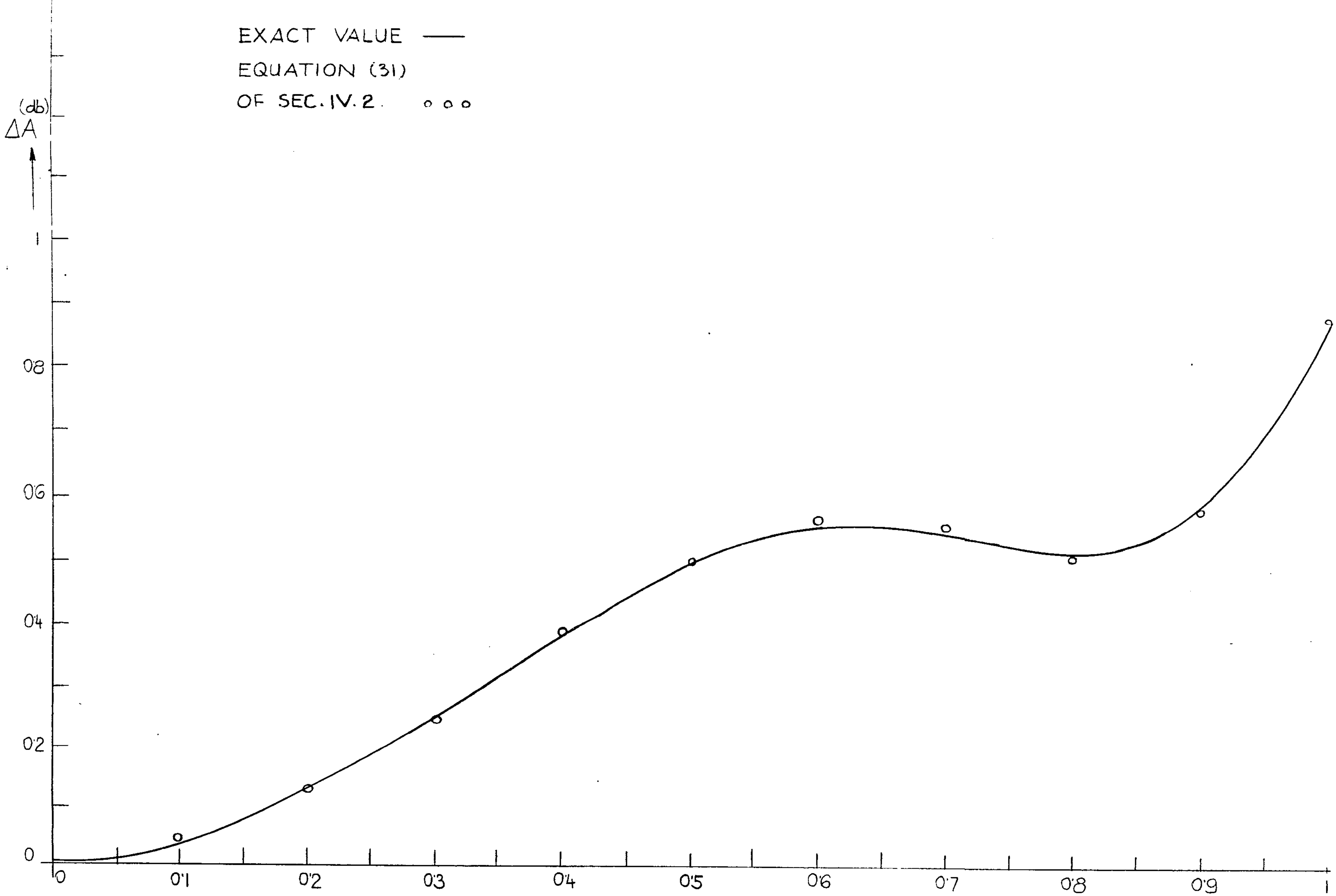
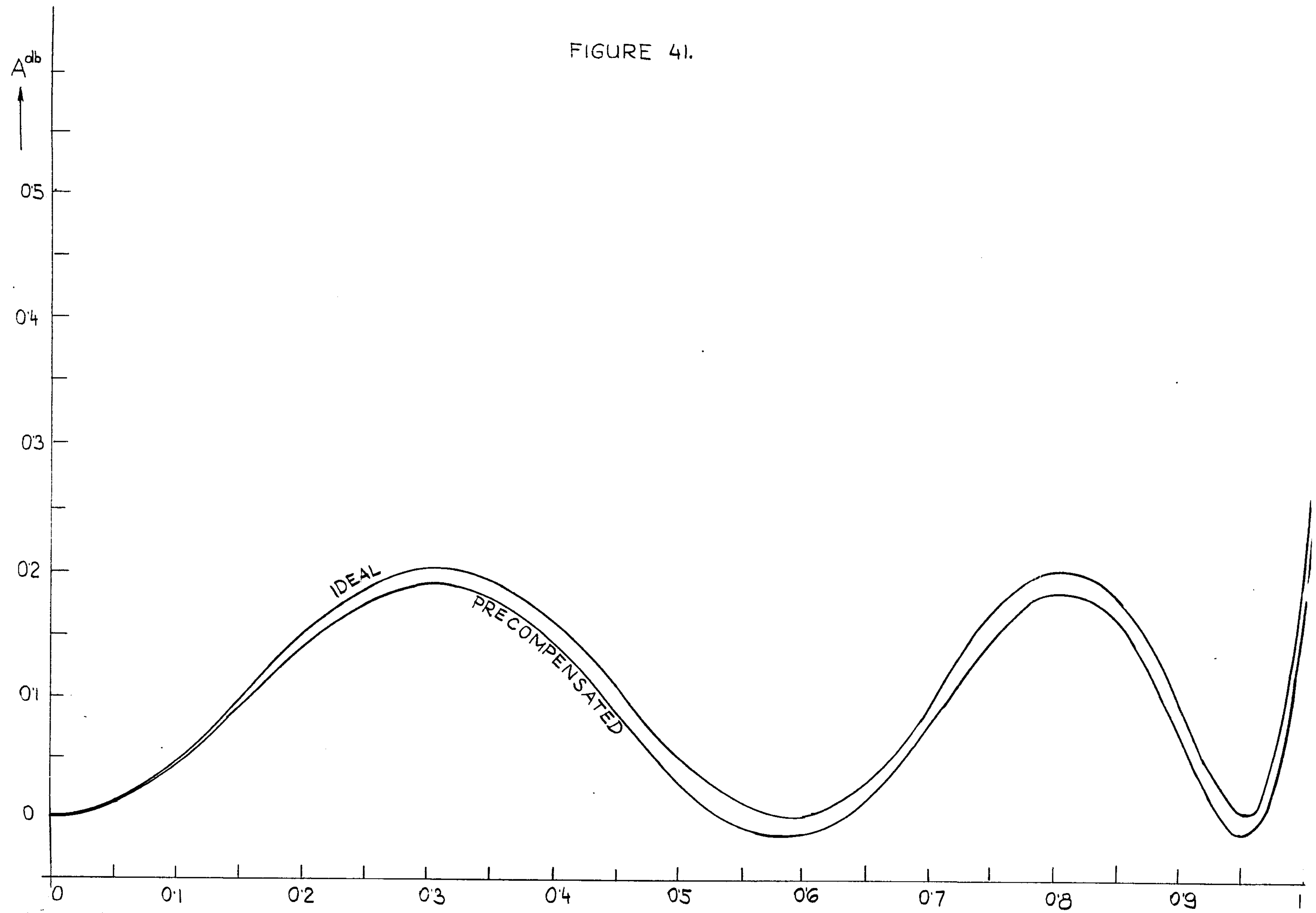

EXACT VALUE
EQUATION (31)
OF SEC. IV. 2
(db)
ΔA
1
0·8
0·6
0·4
0·2
0
0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
1

FIGURE 41.



A^{db}
0.5
0.4
0.3
0.2
0.1
0
IDEAL
PRECOMPENSATED
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

The characteristic function of a symmetric circuit satisfying these requirements is

$$\varphi = \frac{h}{f} =$$

$$= \frac{p^7 + 1.915116384p^5 + 1.1341150241p^3 + 0.197196423351p}{0.05088855 42p^6 + 0.292669982919p^4 + 0.494215372808p^2 + 0.25806022913}$$

By separating the Hurwitz-part of

$$|g|^2 = f^2 - h^2$$

the operating factor

$$S = \frac{g}{f} =$$

$$= \frac{p^7 + 1.57974755008p^6 + 3.16162272252p^5 + 3.1967976697p^4 + 3.0052610299p^3 + 1.87144975298p^2 + 0.865855223020p + 0.258060229130}{0.0508885542p^6 + 0.292669982919p^4 + 0.494215372808p^2 + 0.25806022913}$$

is obtained.

The driving-point impedances

$$Z_{D_1} = Z_{D_2} = \frac{g - h}{g + h} =$$

$$= \frac{0.78987377504p^6 + 0.6235316926p^5 + 1.59839883485p^4 + 0.93557299844p^3 + 0.935724876 49p^2 + 0.334329400289p + 0.129030114565}{p^7 + 0.78987377504p^6 + 2.53836955326p^5 + 1.59839883485p^4 + 2.06988022545p^3 + 0.93572487649p^2 + 0.5315258236 4p + 0.129030114565}$$

can now be calculated. Expanding from both ends, one obtains the element values (Fig. 42) and simultaneously the partial results

$Z_a =$

$$= \frac{0.0254442770829}{0.01580901587252p + 0.0254442771687}$$

$Z_A =$

$$= \frac{\text{numerator of } Z_{D1}}{\begin{array}{l}0.509236325708p^7 + 0.402634667906p^6 + 1.545253804801p^5 + 1.017109448121p^4 + \\ + 1.48830427105p^3 + 0.7279962961p^2 + 0.451356947119p + 0.129030114565\end{array}}$$

$Z_b =$

$$= \frac{\text{numerator of } Z_c}{0.0158090158752p^3 + 0.0254442771528p^2 + 0.0220375672862p + 0.0354689991603}$$

$Z_B =$

$$= \frac{\begin{array}{l}0.276950156915p^4 + 0.217703093617p^3 + 0.370900121303p^2 + 0.172955144592p + \\ + 0.0925619012616\end{array}}{\begin{array}{l}0.409236325708p^5 + 0.402634668375p^4 + 0.835384836041p^3 + 0.455841835904p^2 + \\ + 032378842205p + 0.0925619012861\end{array}}$$

$Z_c =$

$$= \frac{0.0413677633246p^2 + 0.0256285149888p + 0.0354689991563}{0.0839490652604p^3 + 0.0676489904728p^2 + 0.0804613049969p + 0.0354689991603}$$

$Z_C =$

$$= \frac{\text{numerator of } Z_B}{\begin{array}{l}0.0530502561070p^5 + 0.0440390289163p^4 + 0.224446498123p^3 + 0.170954116\overline{56}p^2 + \\ + 0.171322540096p + 0.0925619012861\end{array}}$$

$Z_d =$

$$= \frac{\text{numerator of } Z_e}{\begin{array}{l}0.0839490652604p^5 + 0.0676589906634p^4 + 0.351738545445p^3 + 0.254105670983p^2 + \\ + 0.260006716096\,p + 0.114616311441\end{array}}$$

$Z_D =$

$$= \frac{0.0293352049979p^2 + 0.0121485317068p + 0.0286440730467}{0.053050256107p^3 + 0.0440390294051p^2 + 0.053017227337p + 0.028644073062}$$

$Z_e =$

$$= \frac{\begin{array}{l}0.433204586209p^4 + 0.341430528709p^3 + 0.544704386789p^2 + 0.248370755153p + \\ + 114616311428\end{array}}{\begin{array}{l}0.747019481083p^5 + 0.490258450226p^4 + 1.185472480208p^3 + 0.634266235160p^2 + \\ + 0.435440420397p + 0.114616311441\end{array}}$$

$$Z_E =$$

$$= \frac{\text{numerator of } Z_D}{\begin{array}{l} 0.00814928527325p^3 + 0.025444277199 4p^2 + 0.00917411499053p + \\ 0.028644073062 \end{array}}$$

$$Z_f =$$

$$= \frac{\text{numerator of } Z_{D1}}{\begin{array}{l} 0.747019481083p^7 + 0.590258450214p^6 + 2.02653488793p^5 + 1.29875381547p^4 + \\ + 1.76999435971p^3 + 0.828645963571p^2 + 0.490200099999p + 0.129030114565 \end{array}}$$

$$Z_F = \frac{0.0254442770139}{0.00914928527325p + 0.0254442771756}$$

Now it is easy to calculate the necessary plier-type impedances

$$Z_{22} = pL_2 + \frac{1}{pC_2} \Big\| \left(Z_a + Z_B \right) -$$

$$= \frac{\begin{array}{l} 0.0182528758996p^7 + 0.0288349359756p^6 + 0.0677087072105p^5 + \\ + 0.583507511538p^4 + 0.0549546564948p^3 + 0.0341593501061p^2 \\ + 0.0158043479539p + 0.00471034133461 \end{array}}{\begin{array}{l} 0.0252613650307p^6 + 0.0399065779 27p^5 + 0.0558752116 99p^4 + \\ + 0.052061398299p^3 + 0.0307185554745p^2 + 0.0143783546701p + \\ + 0.00235517067158 \end{array}},$$

$$Z_{44} - pL_4 + \frac{1}{pC_4} \Big\| \left(Z_c + Z_D \right) -$$

$$= \frac{\begin{array}{l} 0.00787394951307p^7 + 0.0124388524631p^6 + 0.0248944576981p^5 + \\ + 0.0251714234559p^4 + 0.0236632735367p^3 + 0.0147357008506p^2 + \\ + 0.00681770030336p + 0.0020319532061 \end{array}}{\begin{array}{l} 0.00545130822185p^6 + 0.0086116908154p^5 + 0.0140106748966p^4 + \\ + 0.0131439955012p^3 + 0.0089121399 4437p^2 + 0.00462054322848p + \\ + 0.00101597660338 \end{array}},$$

$$Z_{66} - pL_6 + \frac{1}{pC_6} \,\Big\|\, (Z_e + Z_F) =$$

$$= \frac{\begin{array}{l} 0.0226019273022p^7 + 0.0357053391704p^6 + 0.0714587662783p^5 + \\ + 0.0722537876418p^4 + 0.0679246899328p^3 + 0.0422983703944p^2 + \\ + 0.0195699963336p + 0.0058326583754 \end{array}}{\begin{array}{l} 0.0532913171301p^6 + 0.0841868274118p^5 + 0.1153472080024p^4 + \\ + 0.1024004270422p^3 + 0.0580844576454p^2 + 0.024229616056p + \\ + 0.0029163291971 \end{array}} .$$

Using Eq. (12) of Sec. IV.3., one gets

$$S^{(0)}_{pred} = S\Big(1 - \sum_{(j)} \frac{L_j}{Q_j\, Z_{jj}}\Big) =$$

$$= \frac{\begin{array}{l} p^7 + 1.569747550009p^6 + 3.14582524703p^5 + 3.17364267434p^4 + \\ + 2.983940903656p^3 + 1.85831363961p^2 + 0.859617058108p + \\ 0.256945797801 \end{array}}{\begin{array}{l} 0.0608885542p^6 + 0.292669982919p^4 + 0.494215372808p^2 + \\ + 0.25806022913 \end{array}}$$

$\left| S^{(0)}_{pred} \right|$ has a $j\omega$-axis minimum value very slightly greater than 0.949360 at $\omega = 0.94822$. Thus it is possible to use

$$S_{pred} = \frac{S^{(0)}_{pred}}{\left| S^{(0)}_{pred} \right|_{min}} = \frac{S^{(0)}_{pred}}{0.949360}$$

in computing the predistorted network.

Now

$h_{pred} =$

$$= p^7 + 0.807191634505p^6 + 2.24071773226p^5 + 1.41706514939p^4 +$$

$$+ 1.59709794891p^3 + 0.707712692852p^2 + 0.351430267764p + 0.0774598862193$$

is obtained by choosing the Hurwitz-part of $|g_{pred}|^2 - |f_{pred}|^2$.

Finally, the open-circuit driving-point impedances

$$Z_{11} = \frac{Ev\,(g_{pred} + h_{pred})}{Odd\,(g_{pred} + h_{pred})} =$$

$$\frac{2.376939185p^6 + 4.590707823p^4 + 2.566026332p^2 + 0.334405684}{2p^7 + 5.386542979p^5 + 4.5810469 85p^3 + 1.2110573258p}$$

$$Z_{22} = \frac{\text{Even } (g \text{ pred} - h \text{ pred})}{\text{Odd } (g \text{ pred} - h \text{ pred})} =$$

$$= \frac{0.762555916p^6 + 1.756577525p^4 + 1.150600946p^2 + 0.1794859116}{2p^7 + 5.386542979p^5 + 4.581046985p^3 + 1.2110573258p}$$

and the termination-ratio

$$\frac{R_2}{R_1} = Z_{D_1}(o) = \left. \frac{g_{\text{pred}} - h_{\text{pred}}}{g_{\text{pred}} + h_{\text{pred}}} \right|_{p=0} = 1.86308657$$

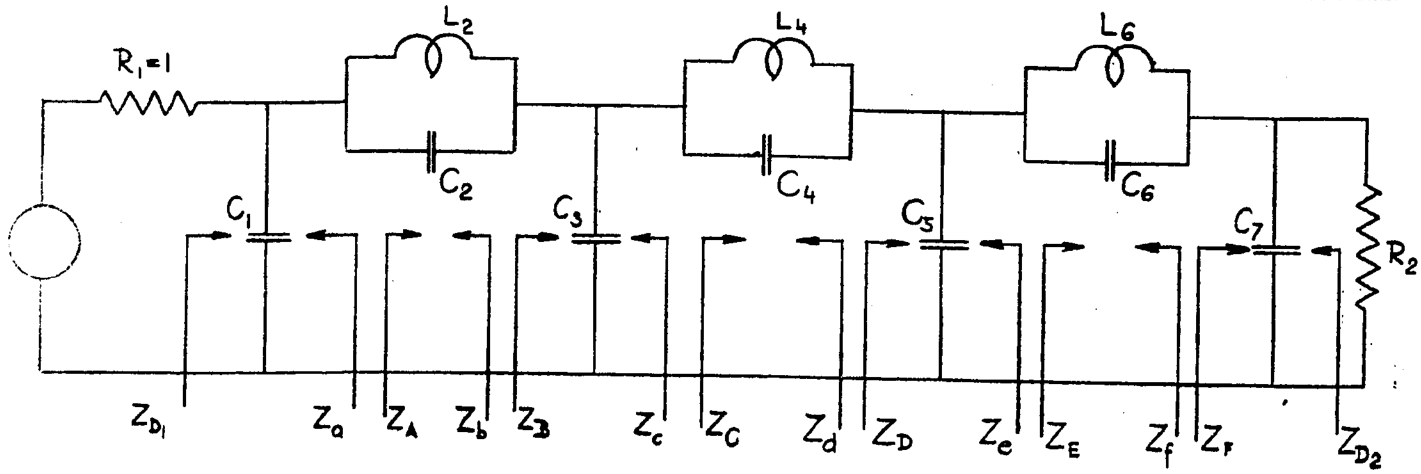
can be calculated.

By ladder-expansion, the circuit of Fig. 43 results. Figs. 44 - 45 show the ideal response, the effects of uncompensated losses, the response of the predistorted network without losses and, finally, the compensated lossy response. Due to second-order effects, the ripple is 0.1087 db instead of the specified 0.1 db.

The comparison of the uncompensated lossy response with the loss-less behaviour (Fig. 44) shows that even a slight dissipation has detrimental effects on the performance of a network designed to meet difficult requirements. This emphasizes the importance of loss compensation in high-quality circuits.

c. The lossy design of a two-section mid-series low-pass filter

As a next example, the two-section low-pass prototype filter used in Darlington's example[1] has been predistorted. This choice made it possible to obtain a numerical check on the early stages of the computations. The filter has to satisfy the following requirements:

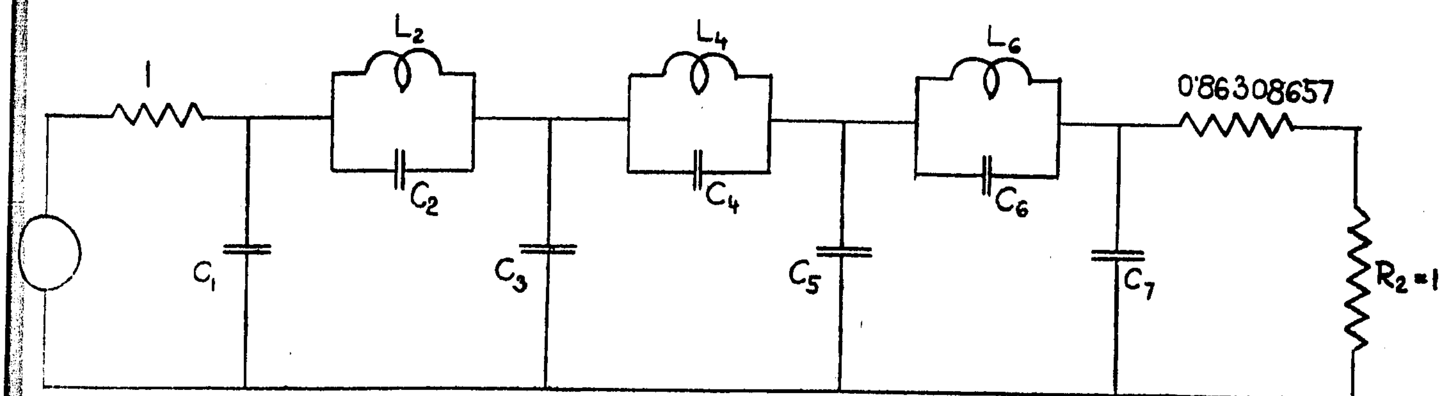


C_1 = 0.62131911
C_2 = 0.99281122
C_3 = 1.64717751
C_4 = 0.214244962
C_5 = 1.53061725
C_6 = 2.094432
C_7 = 0.320279695

L_2 = 0.72256098
L_4 = 1.44441466
L_6 = 0.4241202

$\Omega_{\infty 2}$ = 1.180672392
$\Omega_{\infty 4}$ = 1.79762354168
$\Omega_{\infty 6}$ = 1.06101696354

Fig. 42.
Loss-less ladder network



C_1 = 0.164787488731
C_2 = 1.14648198855
C_3 = 1.47942996085
C_4 = 0.18985569393
C_5 = 1.33022292737
C_6 = 1.54306180371
C_7 = 0.647051074137

L_2 = 0.625711225725
L_4 = 1.62996725788
L_6 = 0.575667917763

$\Omega_{\infty 2}$ = 1.180672392
$\Omega_{\infty 4}$ = 1.79762354168
$\Omega_{\infty 6}$ = 1.06101696354

Fig. 43.
Predistorted lossy network

FIGURE 44.

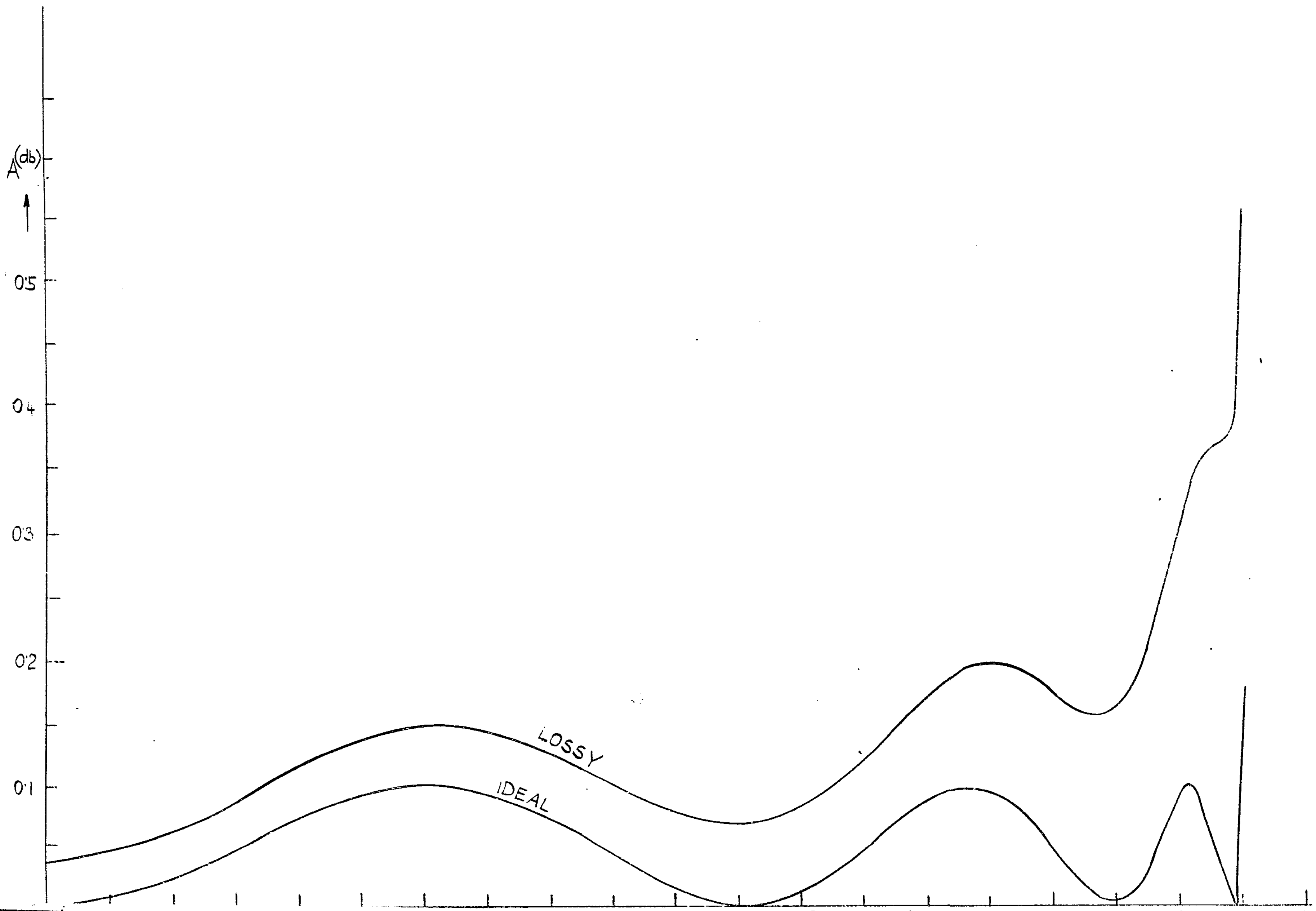
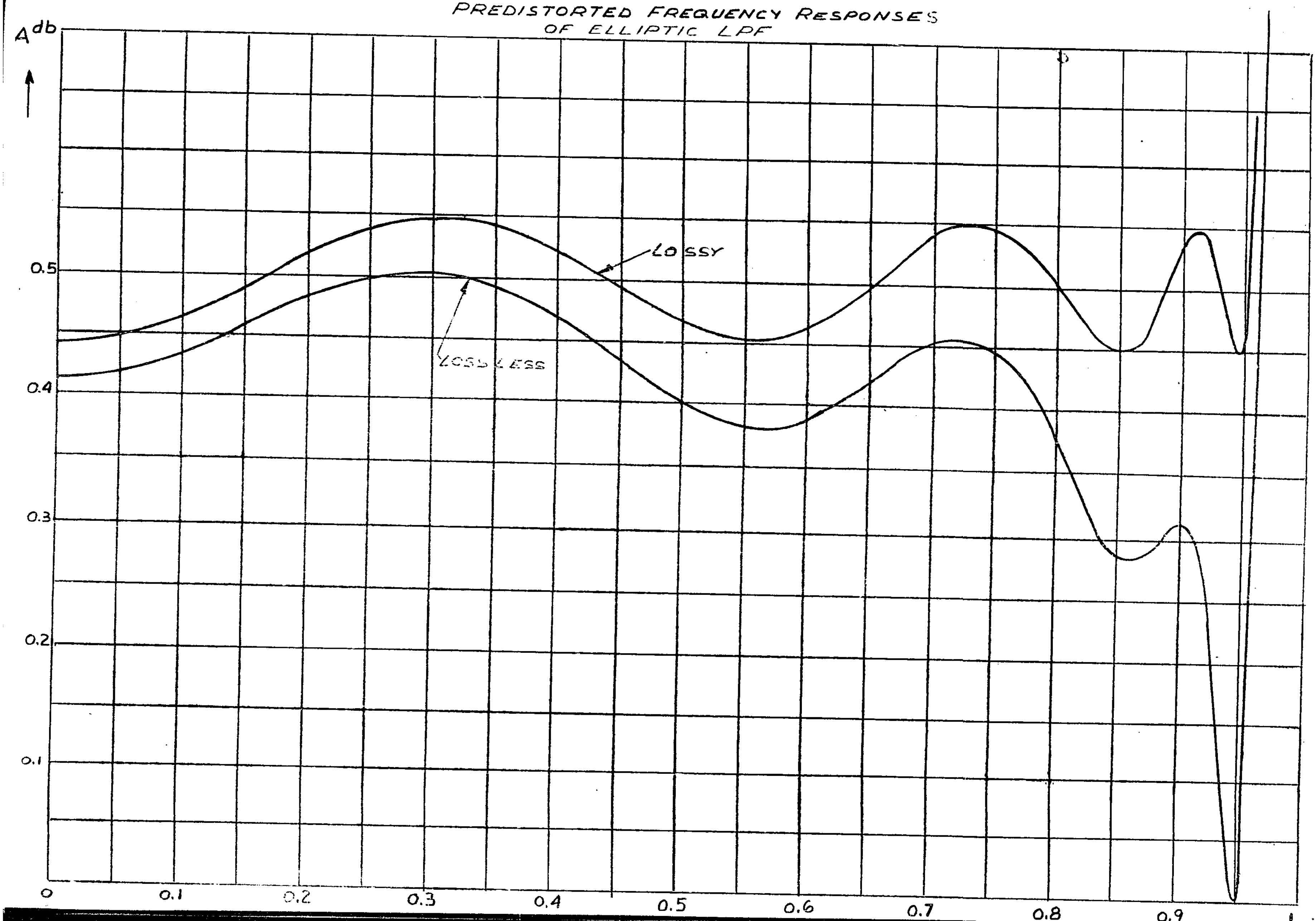


FIG. 45.

PREDISTORTED FREQUENCY RESPONSES
OF ELLIPTIC LPF



$$\Omega_p = \sqrt{0.62} = 0.7874007874$$

$$\Omega_s = \Omega_p^{-1} = 1.27000127$$

$$A_p = 0.3 \text{ db}$$

$$A_a = 52.4 \text{ db.}$$

The average dissipation constant

$$d_o = 0.04263$$

is supposed to originate from the losses in the coils only, so that

$$d_L = 2\, d_o = 2\delta$$

$$d_c = 0 \,.$$

Since for this network (Fig. 46)

$$D = 3 - 1 - 2 = 0$$

Eq. (41) of Sec. IV.3., applies.

Assuming Chebyshev pass and stop-and response, one obtains

$$\frac{V_{20}}{V_2} = \frac{A(p) + pB(p)}{P} =$$

$$= \frac{p^5 + 1.051186000p^4 + 1.37424382992p^3 + 0.82461084482p^2 + 0.401855813603p + 0.0944748555409}{0.013476236924p^4 + 0.07764321517p^2 + 0.0944748555409}$$

$$\left(\frac{V_{20}}{V_2}\right)_\delta = \frac{A(p-d_o) + (p-d_o)B(p-d_o)}{P} = \frac{A_1 + pB_1}{P} =$$

$$= \frac{p^5 + 0.838036000p^4 + 1.21316754 22p^3 + 0.659542873p^2 + 0.338731894 9p + 0.078739008610}{0.0112316185 09p^4 + 0.064710866807p^2 + 0.078739008610}$$

and using Eq. (41)

$$\left(\frac{V_{20}}{V_2}\right)_R = \frac{p^5 + 0.79540600p^4 + 1.21316754 22p^3 + 0.612657438p^2 + 0.338731 8949p + 0.06620625193}{0.00944390052 6p^4 + 0.05441094 6079p^2 + 0.06620625193}$$

(obviously all $\frac{V_{20}}{V_2} = 1$ for $p = 0$).

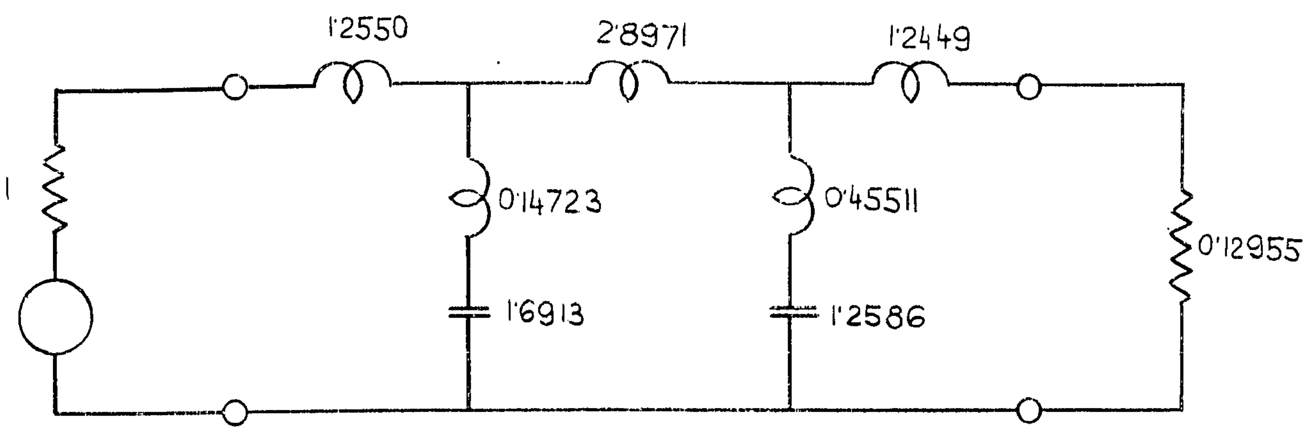
By finding the minimum value of $\left(\frac{V_{20}}{V_2}\right)_R$ on the $j\omega$-axis and using Eq.(36) of Sec. IV. 2., it is established that a termination ratio of 0.129554696 is sufficient to ensure the realizability.

The (conventional) realization of $\left(\frac{V_{20}}{V_2}\right)_R$ gives now the element values of Fig. 46. Adding the losses, the pass-band response shown on Fig. 47. is obtained. While the ripple is slightly less than 0.4 db rather than 0.30 db, considering the high amount of losses (the Q_L is about 9 at Ω_p!), this performance is quite creditable. As a comparison, Figs. 48. and 49. show the response obtained by Bode's method[9] of homogeneous predistortion by d_o (the ripple is 1.1525 db) and the uncorrected effect of losses.

The deviation from the specified ripple is about nine times higher for the Bode-compensated circuit.[9] The uncorrected response is distorted beyond recognition.

As the next exaple will show, the effects of second-order errors on the final response disappear almost completely, when more realistic quality-factors are selected.

d. <u>The precorrected design of a double-loaded elliptic mid-shunt filter using the simplified method.</u>



$$\frac{r_i}{L_i} = 0{\cdot}08526$$

$$\frac{g_j}{C_j} = 0$$

FIGURE 46.

FIGURE 47.

THE LOSSY RESPONSE OF THE NETWORK OF FIG. 46.

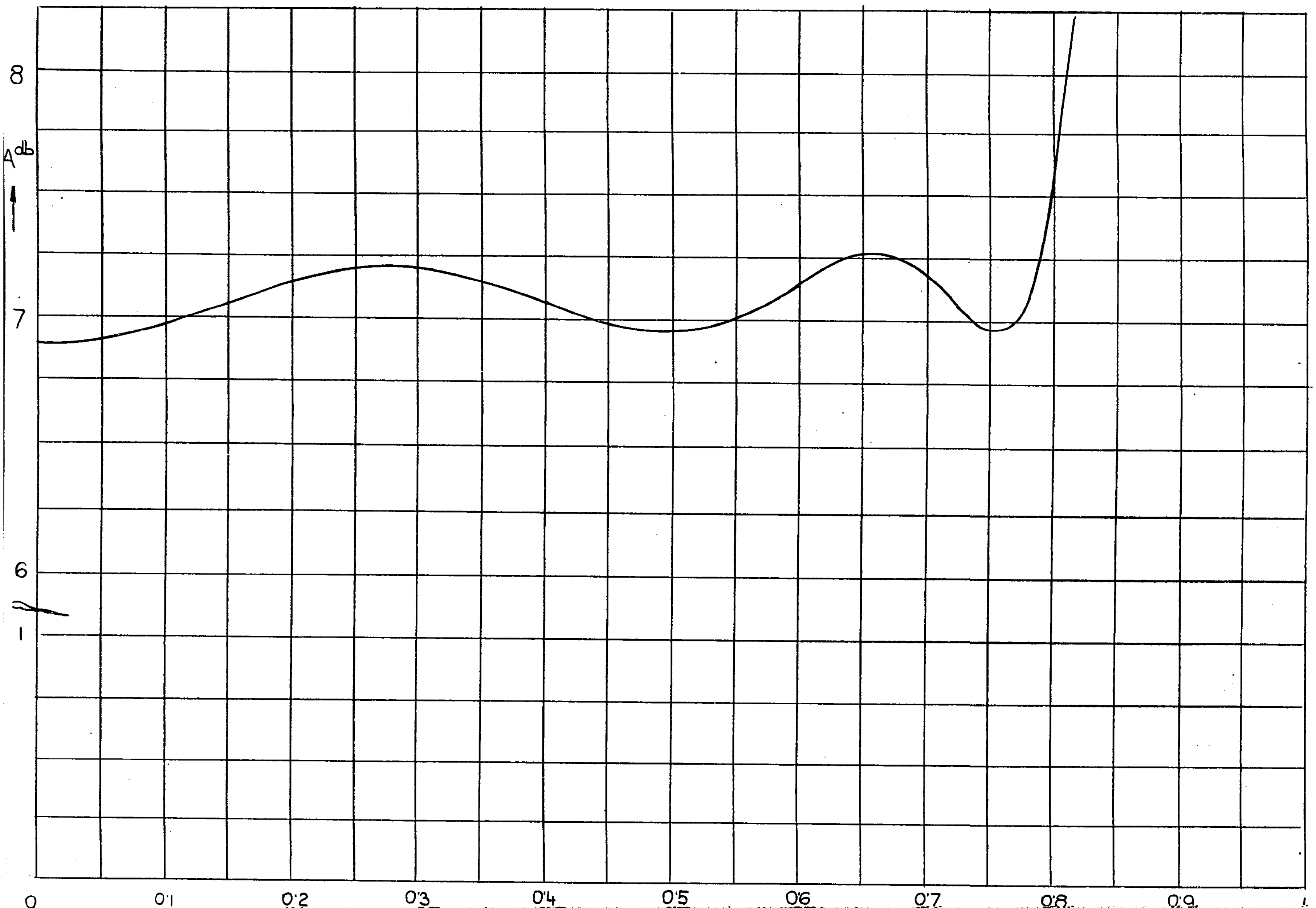


FIGURE 48.

THE LOSSY RESPONSE OF THE NETWORK SYNTHESIZED BY BODE'S METHOD.

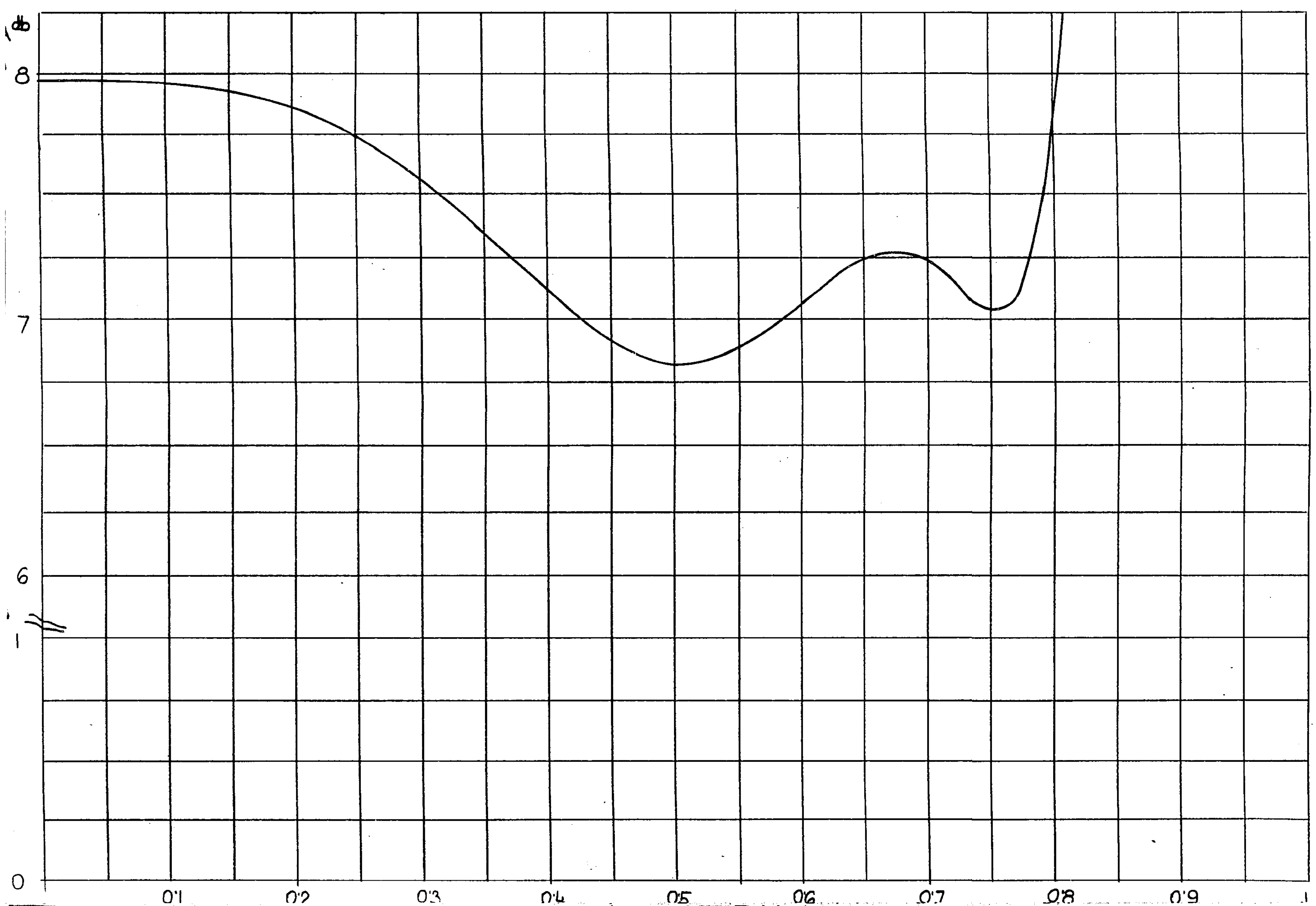
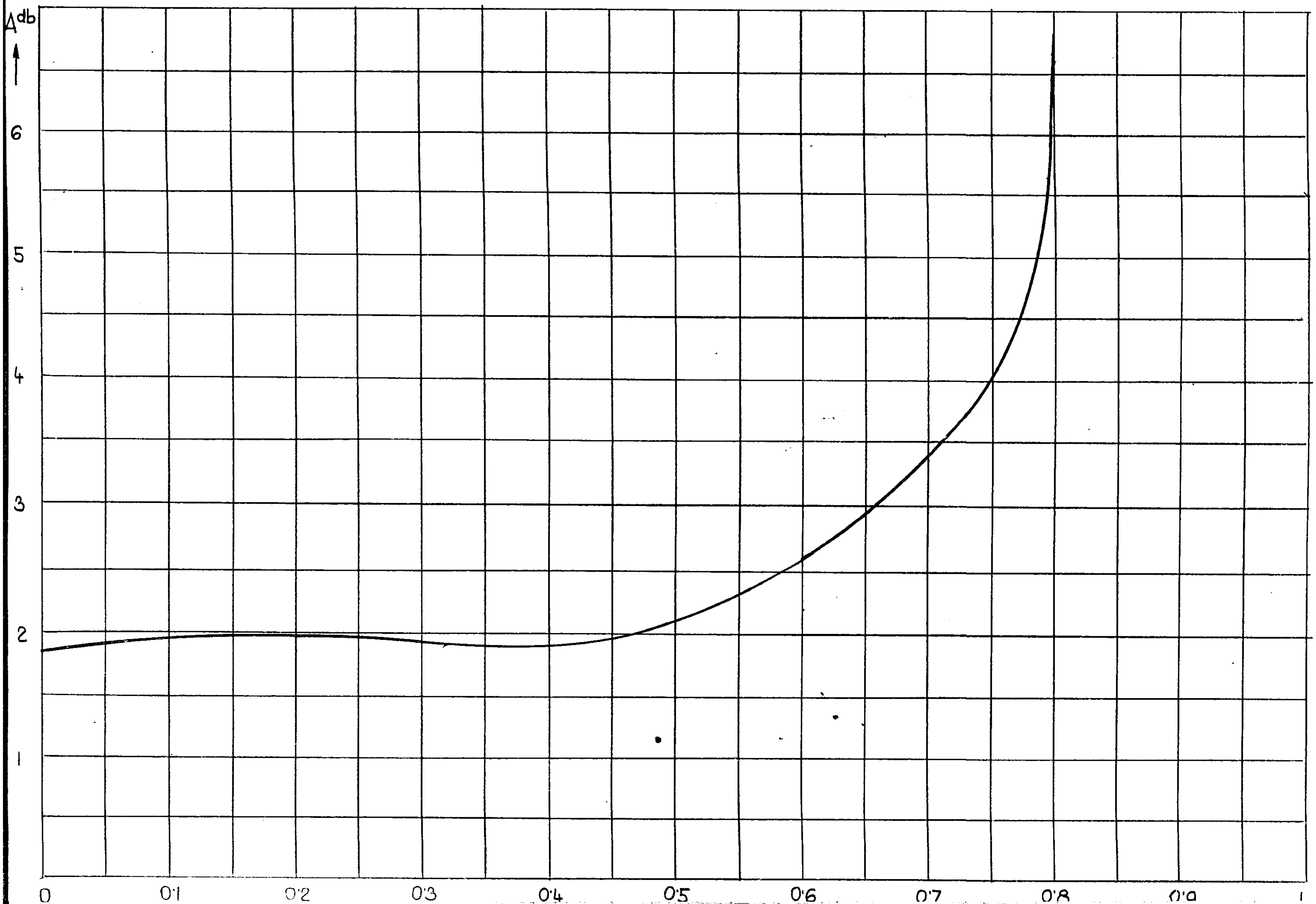


FIGURE 49.

UNCOMPENSETED LOSSY RESPONSE.



The three-section low pass filter with lossy coils ($\frac{L}{r} = 300$) and loss-less capacitors used as Example b) will now be recalculated using the method developed in Para. IV.3.d.

The specifications are the following

$$\Omega_p = \Omega_s^{-1} = 0.952$$

$$A_p = 0.1 \text{ db}$$

$$A_s = 40 \text{ db.}$$

The suitable voltage ratio is

$$\frac{V_{20}}{V_2} = \frac{\begin{array}{l} p^7 + 1.57974755009p^6 + 3.16162272252p^5 + 3.19679766970p^4 + \\ + 3.00526102099p^3 + 1.87144975298p^2 + 0.865855223929p + \\ + 0.258060229130 \end{array}}{\begin{array}{l} 0.06088554200p^6 + 0.292669982919p^4 + \\ 0.494215372808p^2 + 0.25806022913 \end{array}}$$

and exactly as before (D is again zero)

$$\left(\frac{V_{20}}{V_2}\right)_J = \frac{\begin{array}{l} p^7 + 1.568088342p^6 + 3.14588358036p^5 + 3.17051647447p^4 + \\ + 2.98403671338p^3 + 1.85647658165p^2 + 0.85964204 2852p \\ + 0.256622321674 \end{array}}{\begin{array}{l} 0.05060500399p^6 + 0.291039230438p^6 + \\ + 0.49146161265p^2 + 0.256622321674 \end{array}}$$

and

$$\left(\frac{V_{20}}{V_2}\right)_R = \frac{\begin{array}{l} p^7 + 1.56974755008p^6 + 3.14588358036p^5 + \\ + 3.17425746416p^4 + 2.98403671338p^3 + 1.85914347162p^2 \\ + 0.859642042852p + 0.257208376724 \end{array}}{\begin{array}{l} 0.05072057194 7p^6 + 0.29170388426p^4 + \\ \quad + 0.49258397625p^2 + 0.25720837667 \end{array}}$$

can be calculated.

The circuit of Fig. 50 results. The lossy response of this network is shown on Fig. 51. The ripple is now 0.1002 db, i.e. within 0.0002 db (20 μb) of the specified value.

The response given by a network synthesized with Bode's method[9] is shown on Fig. 52. The ripple is 0.1134 db.

e. <u>The precorrection of an elliptic filter for different coil quality factors.</u>

The double-terminated elliptic filter used in Examples b, and d, will be precorrected for the more general case, when

$$Q_{L_1} = Q_{L_2} = 250$$
$$Q_{L_3} = 400.$$

To make the procedure more accurate, the loss-less circuit is designed to possess the anticipated $\frac{R_2}{R_1} = 2$ in advance (cf. Para. IV.2.e.). The lossless voltage ratio of Example d, in conjunction with an η_G of $\frac{8}{9}$ leads then to the element values of the lossless network (in brackets on Fig. 53).

Eq. (12) of Sec. IV.3. gives now the predistorted voltage ratio

$$\left(\frac{V_{20}}{V_2}\right)_{pred} = \frac{\begin{array}{l} p^7 + 1.56924755006p^6 + 3.14503537333p^5 + 3.17338738793p^4 + \\ + 2.98267450900p^3 + 1.8587060l248p^2 + 0.859078325901p + \\ + 0.25718301634 9 \end{array}}{\begin{array}{l} 0.0507155709771p^6 + 0.291675122725p^4 + 0.492535408239p^2 \\ + 0.25718301634 9 \end{array}}$$

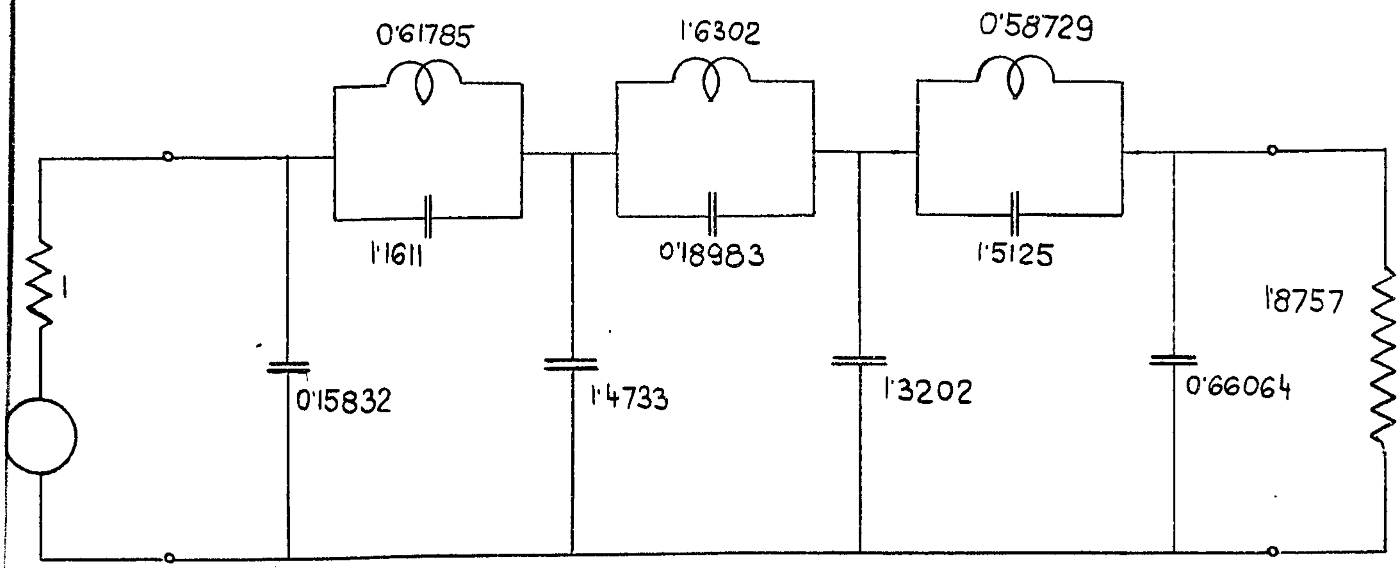


FIGURE 50.

PREDISTORTED MID-SHUNT LOW PASS FILTER.

FIGURE 51.

PREDISTORTED LOSSY RESPONSE.

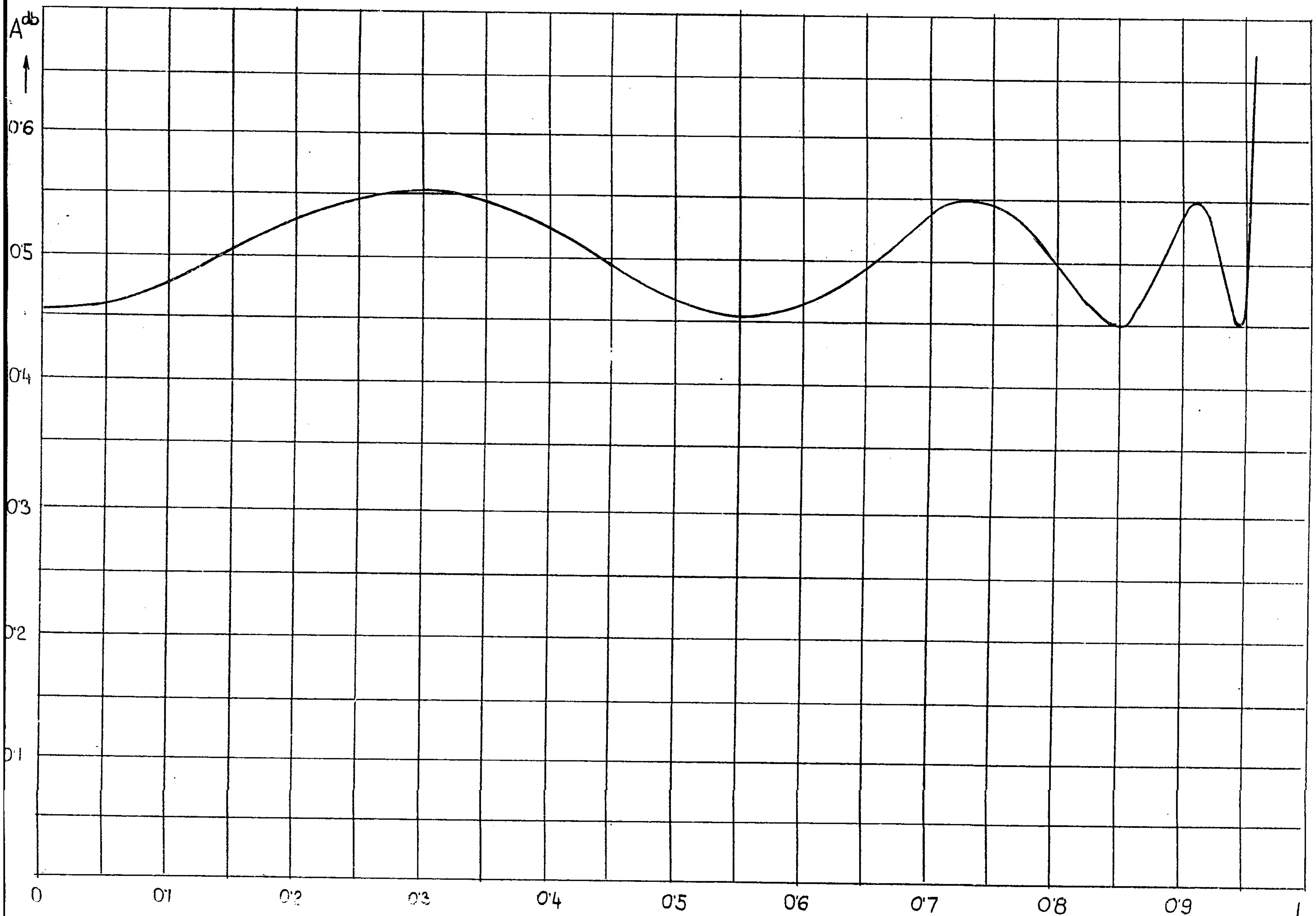
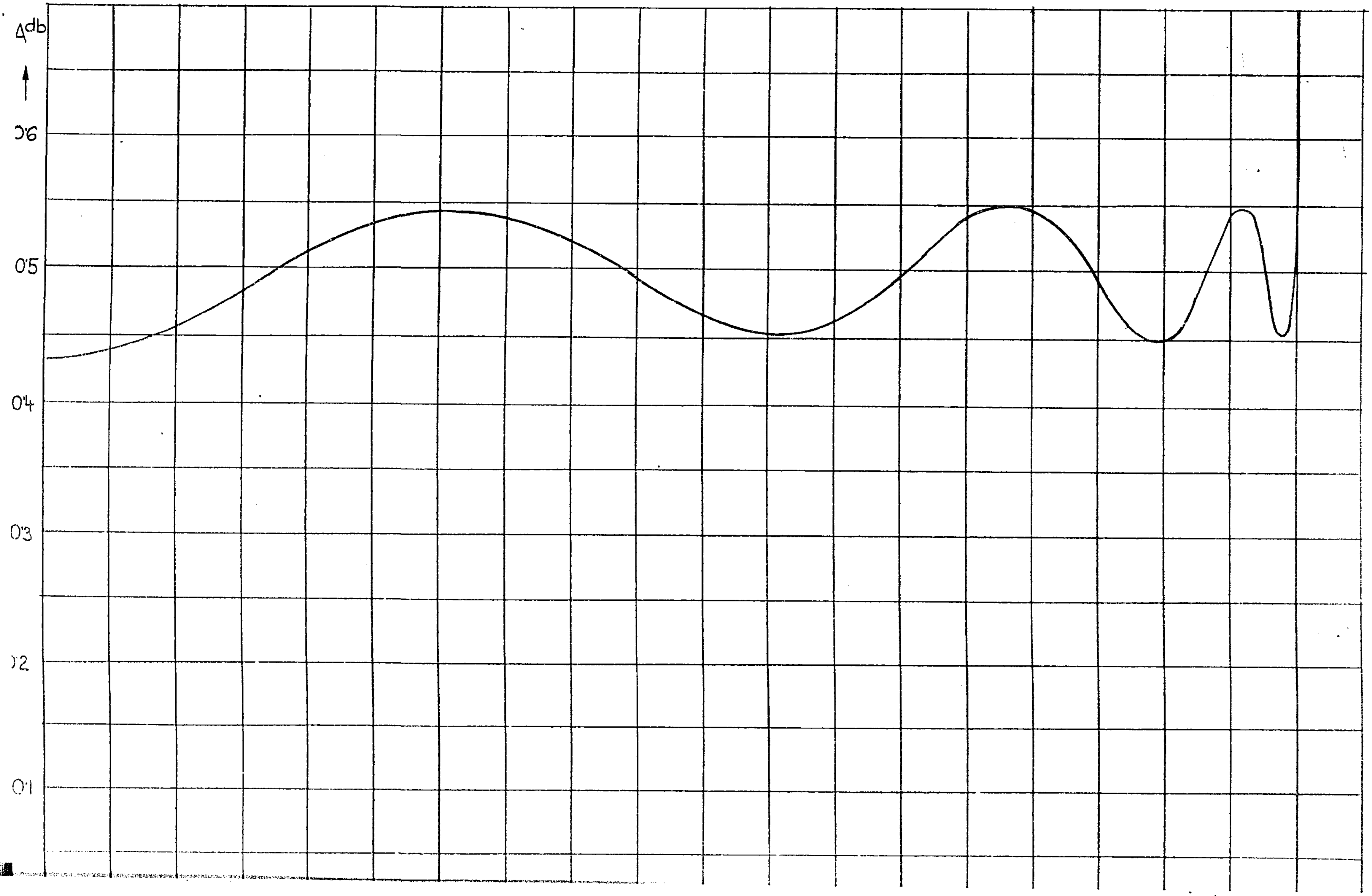


FIGURE 52.

LOSSY RESPONSE OF THE NETWORK SYNTHESIZED BY BODE'S METHODE.



This is realized by the circuit [illegible] element values [illegible] Fig. 53. It is significant to notice the relatively minor changes in the reactance values. This demonstrates the usefulness of the termination-adjustment.

The lossy response of the network exhibits a pass-band ripple of 0.1017 db, or a distortion of 0.0017 db.

f. The predistortion procedure applied to a filter-combination

To demonstrate the use of the new predistortion procedure on filter-groups, the lossy design of a low-pass/high-pass filter combination is carried out in this Example.

Using the (approximate) method of [illegible] I.6.c., the voltage transfer ratios of a filter pair satisfying the specifications,

$$f_p^{(LP)} = 930 \text{ kc}$$

$$f_s^{(LP)} = 980 \text{ kc}$$

$$f_p^{(HP)} = 934 \text{ kc}$$

$$f_s^{(HP)} = 890 \text{ kc}$$

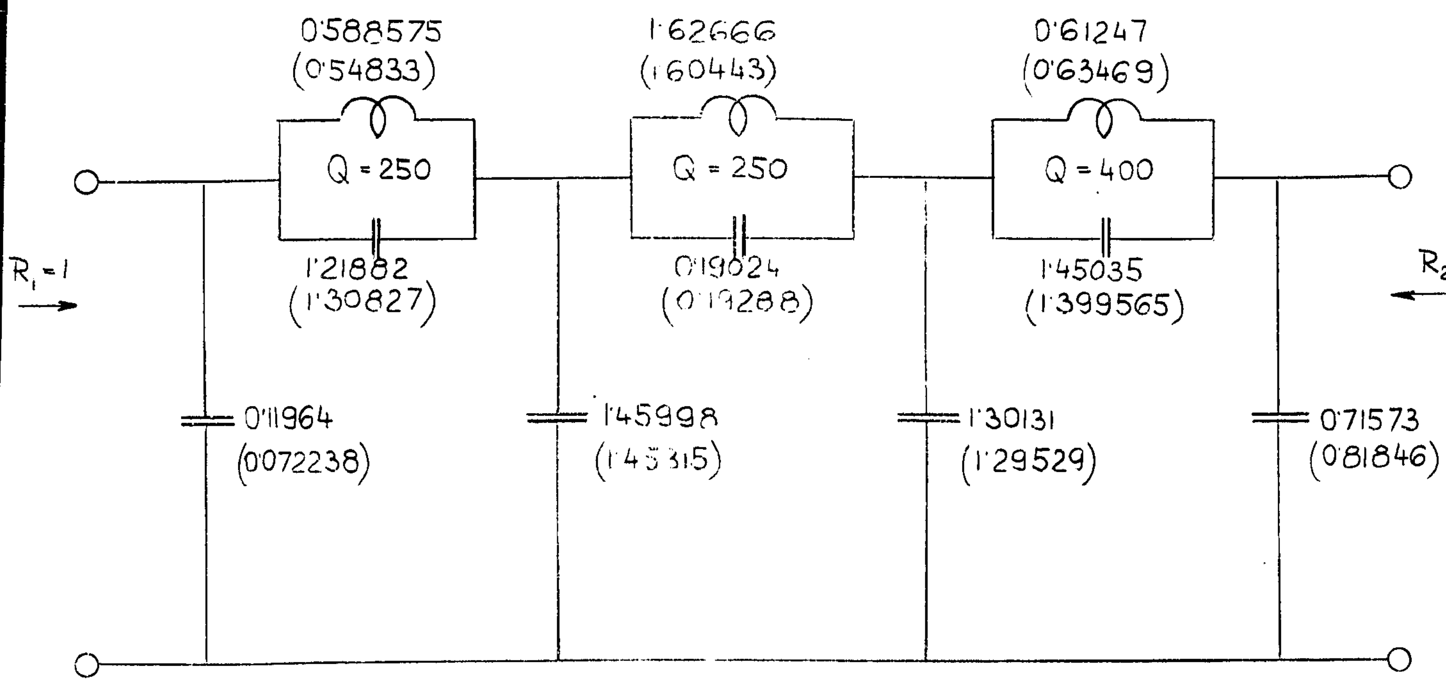
$$f_c = 932 \text{ kc}$$

$$A_p^{(LP)} = A_p^{(HP)} = 0.2[illegible] \text{ db}$$

$$A_s^{(LP)} = A_s^{(HP)} = [illegible] \text{ db}$$

are found to be

[illegible]



PREDISTORTED AND LOSSLESS MID-SHUNT FILTERS.
LOSSLESS ELEMENT VALUES IN BRACKETS.

FIGURE 53

$$= \frac{\begin{aligned}&p^{10} + 0.85519495013 9p^9 + 3.31094244886p^8 + 2.41456363574p^7 + \\ &+ 4.07658027698p^6 + 2.44096375785p^5 + 2.24224562599p^4 + \\ &+ 1.02554514655 9p^3 + 0.500894734185p^2 + 0.144788847466p + \\ &+ 0.024437993101\end{aligned}}{\begin{aligned}&0.0057098147476 2p^8 + 0.0352630690561p^6 + 0.0771146615050p^4 \\ &+ 0.0719763229228p^2 + 0.024437993101\end{aligned}} ,$$

$$N^{(HP)} = \Lambda^{(HP)} = \left(\frac{A + pB}{P}\right)_{HP} =$$

$$= \frac{\begin{aligned}&p^{10} + 5.92474372866p^9 + 20.4965573727p^8 + 41.9613607380p^7 \\ &+ 91.7524437285p^6 + 99.8839663224p^5 + 166.813215485p^4 \\ &+ 98.8036753274p^3 + 135.483400757p^2 + 34.9944819388p \\ &+ 40.9198904291\end{aligned}}{\begin{aligned}&p^{10} + 2.94526324749p^8 + 3.15552349926p^6 + 1.44296092197p^4 + \\ &+ 0.23364499384 3p^2\end{aligned}}$$

Here Eq. (18) of Sec. I.4. leads (with $\eta_G = 0$) to

$$A' = \pm A$$

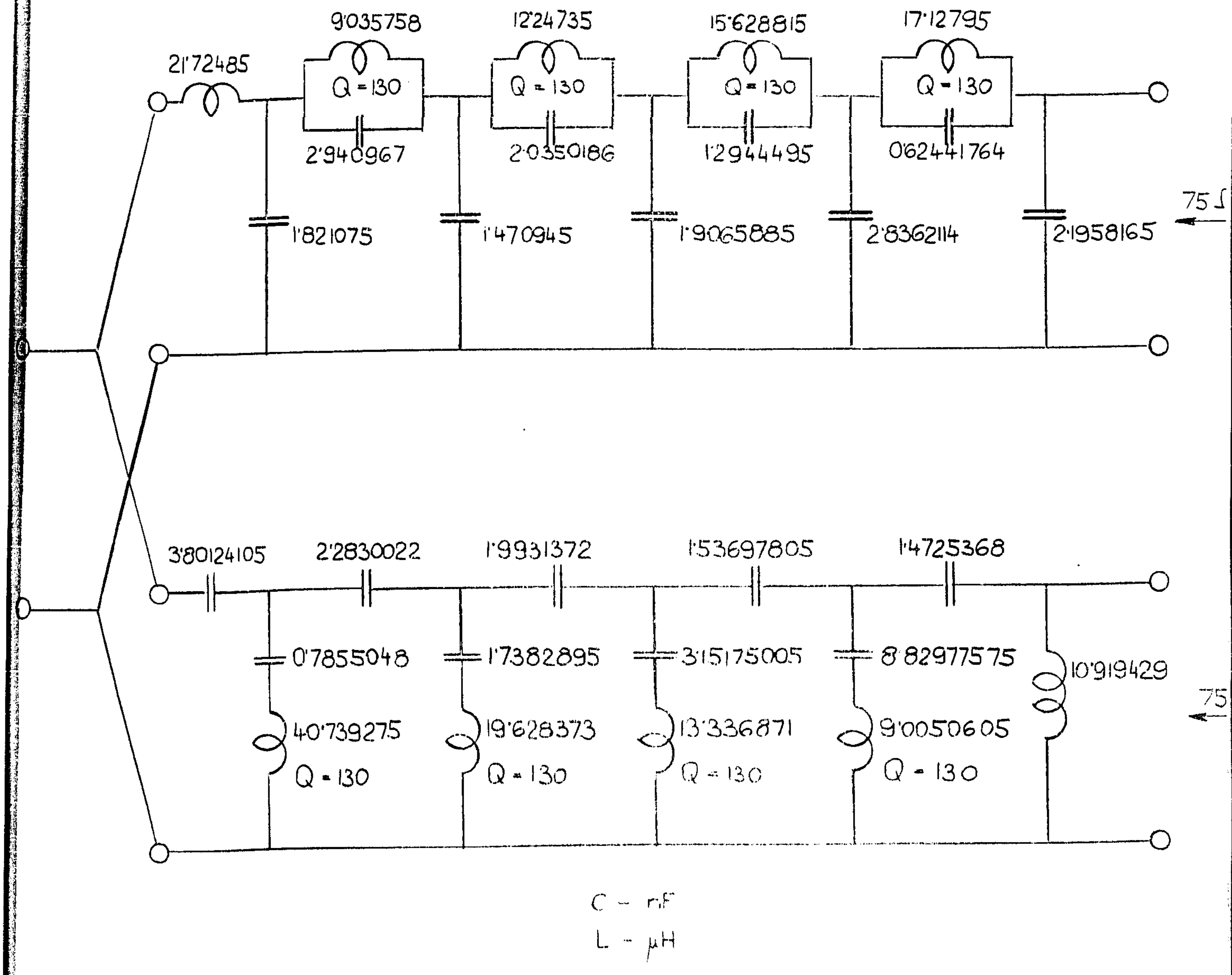
$$B' = \pm B .$$

The proper signs make all immittances of Eqs. (34) - (35) of Sec. I.2. finite, or

$$A' = -A$$

$$B' = -B .$$

For an anticipated Q of 130, using Eq. (41) of Sec. IV.3. for both networks (even though $D^{(HP)} \neq 0$, cf. Para. IV.3.d.), the predistorted voltage ratios and -through the methods of Para. I.4.b. - the predistorted networks can be obtained (Fig. 54.).



21'72485
9'035758
Q = 130
2'940967
12'24735
Q = 130
2'0350186
15'628815
Q = 130
1'2944495
17'12795
Q = 130
0'62441764
1'821075
1'470945
1'9065885
2'8362114
2'1958165
75 Ω
3'80124105
2'2830022
1'9931372
1'53697805
1'4725368
0'7855048
1'7382895
3'15175005
8'82977575
10'919429
40'739275
Q = 130
19'628373
Q = 130
13'336871
Q = 130
9'0050605
Q = 130
75
C - nF
L - μH

FIGURE 54.

The lossy transfer voltage ratios of these networks exhibit a ripple of 2.087 db, (low-pass) and 2.101 db (high-pass), corresponding to actual pass-band ripples of 0.24988 db and 0.25324 db, respectively from Eq. (24) of Para. I. 6. c. Thus the specifications are tolerably satisfied.

The predistortion of the voltage ratios usually upsets the impedance - matching, however, and impedance - equalization way be needed at the branching point.

LITERATURE

General references, used throughout the work:

1. S. DARLINGTON:

   Synthesis of Reactance 4-Poles which Produce Prescribed Insertion Loss Characteristics, Journal of Mathematics and Physics, Vol. 18, pp. 257-353. September, 1939.

2. W. CAUER:

   Synthesis of Linear Communication Networks, McGraw-Hill, 1958.

3. E. A. GUILLEMIN

   Synthesis of Passive Networks, Wiley, 1957.

Special references:

4. F. S. ATIYA:

   Theory der maximal - geebneten und quasi - Tschebyscheffscher Filter, Arch. Elekt. Übertragung, Vol. 7, pp. 441-450; September, 1953.

5. W. BADER:

   Kopplungsfreie Kettenschaltungen, Telegr. Fernspr. Tech., Vol. 31. July, 1942.

6. W. BADER:

   Kettenschaltungen mit vorgeschriebener Kettenmatrix Telegr. Fernspr. Tech., Vol. 32. pp. 119-125, 144-147; June/July, 1943.

7. B. J. BENNETT:

Synthesis of Electric Filters with Arbritrary Phase Characteristics, 1953, IRE Conv. Record, Pt. 5.

8. H. W. BODE:

U. S. Patent, No. 2,002,216.

9. H. W. BODE:

Network Analysis and Feedback Amplifier Design, Van Nostrand, 1945.

10. W. CAUER:

Siebschaltungen, Verein deutscher Ingenieure Verlag, Berlin, 1931.

11. R. M. Cohn:

The Resistance of an Electrical Network, Proc. Amer. Math. Soc. 1950, Vol. 1.

12. C. A. DESOER:

Notes commenting on Darlington's design procedure for networks made of uniformly dissipative coils and uniformly dissipative capacitors. Trans. IRE on C.T. December, 1959.

13. C. A. DESOER:

Network Design by First-Order Predistortion Technique, IRE Trans. on Circuit Theory, September, 1957.

14. M. DISHAL:

The Design of Dissipative Band-Pass Filters Producing Desired Exact Amplitude Frequency Characteristics, Proc. IRE, 1949, Vol. 37, page 1050, September, 1949.

15. V. FETZER:

Explizite Berechnungsformeln für Filterschaltungen mit allgemeinen Parameter, Archiv für Elektrotechnik, 1954, Vol. 8, page 31.

16. V. FETZER:

Die numerische Berechnung von Filterschaltungen mit allgemeinen Parametern nach der modernen Theorie unter besonderer Berücksichtigung der Cauerschen Arbeiten. Arch. Elekt. Übertragung, Vol. 5, pp. 499-508. November, 1951.

17. R. M. FOSTER:

An Extension of a Network Theorem. IRE Trans. on CT. March, 1961.

18. T. FUJISAWA:

Realizability Theorem on Mid-Series or Mid-Shunt Low-Pass Ladders Without Mutual Induction. IRE Trans. On Circuit Theory, Vol. CT-2, pp.320-325, December, 1955.

19. GEFFE - BENNETT:

A Note On Predistortion, Trans. IRE on CT. September, 1957.

20. E. GLOWATSKI:

Sechsstellige Tafel der Cauer-Parameter, Verlag der Bayerischen Akademie der Wissenschaften, Munich, 1955.

21. E. GREEN:

Synthesis of Ladder Networks to Give Butterworth or Chebyshev Response in the Pass-Band, Proc. I.E.E., Monograph No. 88R, January, 1954.

22. A. J. GROSSMAN:

Synthesis of Tchebycheff Parameter Symmetrical Filters, Proc. IRE Vol. 45, April, 1957.

23. J. MEINGUET AND V. BELEVITCH:

On the Realizability of Ladder Filters. IRE Trans. on Circuit Theory, Vol. CT-5, page 253, December, 1958.

24. W. MILORT:

Theory and Design Data for the Synthesis of Tchebycheff Parameter Filters. Philips Telecom. Review Vol. 20 No. 4, May, 1959.

25. N. MING:

Verwirklichung von linearen Vierpolschaltungen... Archiv für Elektrotechnik, Vol. 39, pp. 425-471 and 496-507, 1949.

26. E. L. NORTON:

Constant Resistance Networks with Application To Filter Groups, Bell System Technical Journal, 1937, April, Vol. 16, page 178.

27. H. J. ORCHARD:

Predistortion of Singly - Loaded Reactance Networks Trans IRE on CT, 1960, June.

28. A. PAPOULIS:

Displacement of the Zeros of the Impedance Z(p) due to Incremental Variations in the Network Elements. Proc. IRE. January, 1955. Vol. 42. pp. 79 - 82.

29. H. PILOTY:

Weichenfilter. Telegr. Fernspr. Tech. Vol. 28, pp. 291 - 298, 333 - 344, August/September 1939.

30. H. PILOTY:

Wellenfilter, insbesonder symmetrische und antimetrische mit vorgeschriebenem Betriebsverhalten, Telegr. Fernsp. Tech. Vol. 28, pp. 363 - 375, October, 1939.

31. H. PILOTY:

Kanonische Kettenschaltungen für Reaktanzvierpole mit vorgeschriebenem Betriebseigenschaften, Telegr. Fernspr. Tech. Vol. 29, September/November, 1940.

32. H. PILOTY:

Konjugierte Impedanzen und Weichenfilter, Telegr. Fernspr. Tech. Vol. 31, 10 October, 1942.

33. H. PILOTY:

Reaktanzvierpol mit gegebenem einseitigen Leerlauf - oder Kurzschlusswiderstand. Arch. Elekt. Übertragung, Vol. 1, pp. 59 - 70, July/August, 1947.

34. E. RUMPELT:

Schablonenverfahen für den Entwurf elektrischer Wellenfilter auf der Grundlage der Wellenparameter, Telegr. Fernspr. Tech. Vol. 31, 8 August, 1942.

35. E. RUMPELT:

Über den Entwurf elektrischer Wellenfilter mit vorgeschriebenem Betriebsverhalten, Ph.D. Thesis for Technischen Hochschule, München, 1947.

36. SAAL AND ULBRICH:

On the Design of Filters by Synthesis. Trans. IRE on CT., December, 1958.

37. G. SZENTIRMAI:

The Analysis of the Effects of Tolerance and Dissipation on Networks, to be published.

38. G. SZENTIRMAI AND G. TEMES:

Realizability Nomographs for Elliptic Filters, to be published.

39. G. C. TEMES:

Synthesis of General Parameter Insertion Loss Filters Using a Digital Computer, AIEE Transactions, Pt. 1., Communication and Electronics, May, 1961.

40. J. VRATSANOS

Zur Berechnung der Stromverteilung in einem linearen Netzwerk. Arch. Elekt. Übertragung, February, 1957.

41. J. ZDUNEK:

Network Synthesis on the Insertion Loss Basis, Proc. I.E.E., Vol. 105, Part C, March, 1958.