AN EXTENSION OF THE NOTION OF CONNECTEDNESS

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by
Kin Chung Wong
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CONTENTS

ABSTRACT

Some results on connectedness can be extended by the concept of almost connectedness. After giving the definitions of a frozen set, $\mathcal{O}(\urcorner, A, ^-)$ and $\bar{\mathcal{O}}(\urcorner, A, ^-)$, we define in succession first frozen spaces and second frozen spaces. An important counterexample is constructed for a frozen set which is not closed. As for the conditions about a frozen set being closed, the following statements hold: Every Frozen set of a first countable or hiding σ-connected space is closed. A countable space is first frozen. Completely regular spaces and σ-closed spaces are second frozen. On the other hand, there are some consequences about the operator $\urcorner$ and continuous functions. An interesting result about the product of almost connected sets is induced by the concept of being second frozen. In the last section, there are extensions of the concepts of connectedness, local connectedness and pathwise connectedness.

Except for new results and definitions, the author has conformed to standard topological terminology. The names of the examples are according to the book: Counterexamples in Topology by Lynn A. Steen and J. Arthur Seeback. Jr., Holt, Rinehart and Winston, Inc. 1970. The notations and conventions come from General Topology by John L. Kelly, D. Van Nostrand Company, Inc., 1955, or from General Topology by Stephen Willard, Addison-Wesley Publishing Company, Inc. 1970.

CHAPTER I

GENERAL PROPERTIES OF ALMOST CONNECTED SPACES

An orchid leaf is sometimes not connected in a pen drawing, though it is connected in the eye of the artist. We do not plan to describe the geometrical view of artists. However, it suggests another concept of connectedness which is less abstract than the above and which is called "almost connectedness", in topology.

This concept was suggested in part by analogies with the *"almost everywhere"* notion of measure theory. However, this is only a superficial comparison, because the two concepts are very different. Technically, the theory of almost connected spaces should rather be considered as a generalization of the concepts of cut points and degree of connectivity in graphs and continua. In addition, it may be related to some notions in lattice theory.

Definition 1 Let X be a topological space and let $A \subset X$. If there exists a countable subset C of X, such that $A \cup C$ is connected, then A is said to be almost connected in X, (or A is called an almost connected subset/subspace of X.)

For example, let $\mathbb{R}$ be the real line, let Q be the set of rational numbers and let P be the set of irrational numbers. Then $P \cup Q = \mathbb{R}$ and P is almost connected but not connected in $\mathbb{R}$, while Q is not almost connected in $\mathbb{R}$. Obviously, almost connectedness is an extension of connectedness, since a connected subset must be almost connected. Many of its properties are similar to the properties of connectedness. For example, the image of an almost connected subset under a continuous function is almost connected, that is, almost connectedness is invariant under continuous mappings. Indeed, both the connectedness and countability of subsets are preserved under a continuous mapping joining two topological spaces. Nevertheless, there are still interesting problems which arise with this concept. We will discuss them in the next chapter. First, we exhibit the following theorems which generalize well-known theorems on connected spaces.

THEOREM 1 Let $\{A_i : i \in I\}$ be a countable family of almost connected subsets of a topological space X. If A_i, A_j are pairwise not separated, then the subset $\bigcup_{i\in I} A_i$ is almost connected in X.

Proof For every $i \in I$, since A_i is almost connected, then there exists a countable subset C_i in X, such that $A_i \cup C_i$ is connected. Now, there are no two elements $i, j \in I$, such that $A_i \cup C_i$ and $A_j \cup C_j$ are separated. Indeed, if there exist $i, j \in I$, such that $A_i \cup C_i$ and $A_j \cup C_j$ are separated, then $A_i \subset A_i \cup C_i$ and $A_j \subset A_j \cup C_j$ imply A_i and A_j are separated, contrary to assumption. Take $C = \bigcup_{i\in I} C_i$; then C is countable in X and $\left(\bigcup_{i\in I} A_i\right) \cup C = \left(\bigcup_{i\in I} A_i\right) \cup \left(\bigcup_{i\in I} C_i\right) = \bigcup_{i\in I}\left[_{i\in I}(A_i \cup C_i)\right] = \bigcup_{i\in I}(A_i \cup C_i)$ is connected. Hence, $\bigcup_{i\in I} A_i$

is almost connected in X .

Corollary The union of a countable family of almost connected subsets with a common point is almost connected.

THEOREM 2 Let X be a topological space and A be an almost connected subset in X . If $A \subset B \subset \overline{A}$, then B is almost connected in X .

Proof Since A is almost connected, there exists a countable subset of X , such that $A \cup C$ is connected. We have

$$A \cup C \subset B \cup C \subset \overline{A} \cup C \subset \overline{A} \cup C = \overline{A \cup C} .$$

Hence, $B \cup C$ is connected and then B is almost connected in X .

Corollary The closure of an almost connected sunset of a topological space is almost connected.

But the converse is not true. For example, consider the set Q of rational numbers of the real line $\mathbb{R}$. It is known that the closure of a connected subset is connected. Theorem 3 indicates that the closure of an almost connected set is connected under some conditions:

THEOREM 3 Let X be a T_3 - space and A be a non-empty subset of X . If there exists a finite subset $C = \{x_0, x_1, \ldots, x_n\}$ of X , such that $Y = A \cup C$ is connected, then $\overline{Y} = \overline{A}$.

Proof It is enough to prove that $\overline{A \cup C} \subset \overline{A}$. First, we prove that $C \subset A$. Suppose on the contrary that $C \not\subset \overline{A}$; then one of the elements of C , say x_0 , is such that $x_0 \in X - \overline{A}$. Since X is T_3 , there exist two disjoint...

open sets U_0, V_0, such that $\overline{A} \subset U_0$ and $x_0 \subset V_0$. On the other hand, since X is T_2, for every $x_i \in C - \{x_0\}$, there exists two disjoint open sets U_i, V_i, such that $x_i \in U_i$ and $x_0 \in V_i$. Let $U = \bigcup_{i=0}^{n} U_i$, $V = \bigcap_{i=0}^{n} V_i$, then U, V are open in X and we obtain:

(i) $Y \subset U \cup V$, since $A \subset U_0$, $x_0 \in V$, $x_i \in U_i \subset U$ for $i = 1, \ldots, n$

(ii) $U \cap N = \emptyset \subset X - Y$; indeed, if $U \cap V \neq \emptyset$, let $x \in U \cap V$; then $x \in U_j$, for some $j \in \{1, \ldots, n\}$ and $x \in V_i$, for every $i = 1, \ldots, n$; thus $U_j \cap V_j \neq 0$.

(iii) $U \cap Y \neq \emptyset$, since $A \neq \emptyset$.

(iv) $V \cap Y \neq \emptyset$, since $x_0 \in V \cap Y$.

It follows that Y is not connected <u>in</u> X. This contradiction shows $C \subset \overline{A}$. Thus, $\overline{C} \subset \overline{A}$ and hence $\overline{A \cup C} = \overline{A} \cup \overline{C} = \overline{A}$.

Corollary Let X be a T_3-space and A be a subset of X. If there exists a finite subset C of X, such that $A \cup C$ is connected, (A is sometime said to be finitely almost connected), then $\overline{A}$ is conenected.

The component of a point a in a topological space X, denoted $\mathrm{Cmp}(a)$, is the maximal connected subset of X containing a. We can extend this definition to the component of a non-empty almost connected subset. First, we consider Theorem 4.

THEOREM 4 Let X be a topological space and A be a non-empty almost connected subset of X. Then there exists a maximal connected subset Z_A of X containing A, and Z_A is the component of a, for every $a \in A$.

Proof Since A is almost connected, then there exists a countable subset C of X, such that $Y = A \cup C$ is connected and then $A \subset Y$. Let $\mathfrak{Z} = \{Z \subset X : A \subset Z$ and Z is connected$\}$. Since Y exists, $\mathfrak{Z}$ is non-empty. Then $Z_A = \bigcup_{Z \in \mathfrak{Z}} Z$ is connected, since $A \neq \emptyset$, and we have $A \subset Z_A$. If B is any connected subset of X containing A, then $B \in \mathfrak{Z}$ and hence $B \subset Z_A$. That is, Z_A is the maximal connected subset of X containing A.

Since for every $a \in A$, $a \in Z_A$ and Z_A is connected, then $Z_A \subset \mathrm{Cmp}(a)$. On the other hand, for every $a \in A$, $A \subset \mathrm{Cmp}(a)$, since $A \subset Z_A$, and so $\mathrm{Cmp}(a) \subset Z_A$. Therefore $Z_A = \mathrm{Cmp}(a)$, for every $a \in A$.

Definition 2 Let X be a topological space and A be a non-empty almost connected subset of X. Then the maximal connected subset of X containing A is called the component of A and denoted by $\mathrm{Cmp}A$.

In a topological space X, since the component of A coincides with the component of an arbitrary point $a \in A$, then the properties of $\mathrm{Cmp}A$ are the same as the properties of $\mathrm{Cmp}(a)$, for every $a \in A$. For example, $\mathrm{Cmp}A$ is closed but not necessarily open in X. Also, the components of two non-empty almost connected subsets either coincide or are disjoint. Besides these, if $\{A_i : i \in I\}$ is any family of non-empty almost connected subsets in a topological space X, and $\bigcup_{i \in I} A_i$ is almost connected in X, then $\text{Cmp} \bigcup_{i\in I} A_i = \text{Cmp}(a_i)$, for every $a_i \in A_i$ and $i \in I$.

$= \text{Cmp}A_i$, for every $i \in I$.

As a corollary, we know that if $\{A_i : i \in I\}$ is a countable family of non-empty almost connected subsets of a topological space X and if A_i, A_j are pairwise not separated, then $\text{Cmp} \bigcup_{i\in I} A_i = \text{Cmp}\ A_i$, for every $i \in I$.

We may also consider the properties of a product of almost connected subsets. Let $\{X_i : i \in I\}$ be any family of topological spaces. If $\prod_{i\in I} A_i$ is almost connected in the product space $\prod_{i\in I} X_i$, then every A_i is almost connected in X_i, for, every projection $p_j : \prod_{i\in I} X_i \to X_j$ is continuous. If $\prod_{i\in I} A_i$ is non-empty and almost connected in $\prod_{i\in I} X_i$, then $A_i = p_i(\prod_{i\in I} A_i)$ is almost connected in X_i, for every $i \in I$. If $\prod_{i\in I} A_i \neq \emptyset$, then $A_i = \emptyset$, for some $i \in I$.

Conversely, we can ask the question from the opposite direction. First, we note that an almost connected subset A of a topological space X is defined as a subset which is contained in a connected subset Y of X, such that $Y - A$ is countable. Then we consider the special case:

<u>*THEOREM 5*</u> Let $\{X_i : i = 1, \ldots, n\}$ be a finite family of topological spaces and let A_i be a countable, almost connected subset in X_i, for every $i \in \{1, \ldots, n\}$, then the subset $A = \prod_{i\in I}^{n} A_i$ is almost connected in $X = \prod_{i=1}^{n} X_i$.

Proof For every $i \in \{1, \ldots, n\}$, since A_i is almost connected in X_i, then there exists a countable subset C_i of X_i, such that $A_i \cup C_i$ is connected. Take $C = \prod_{i=1} (A_i \cup C_i)$, then C is countable and connected in X. Thus $A \subset C$ and $C - A$ is countable. These imply that A is almost connected in X.

Corollary If A_i is an almost connected subset in a countable topological space X_i, for every $i \in \{1, \ldots, n\}$, then the set $\prod_{i=1}^{n} A_i$ is almost connected in $\prod_{i=1}^{n} X_i$.

What can we say about the product of almost connected subsets which are not necessarily connected or countable? To answer this question is not so simple, for the available topological tools are not sufficient. We require additional concepts which will be introduced in the next chapter and which will then be used to make an investigation (See theorem 22).

CHAPTER 2

FROZEN SETS, FIRST FROZEN SPACES AND SECOND FROZEN SPACES

A. Some Definitions

Since we have the concept of an almost connected subset, we may think about a special kind of subset which is deduced from that subset in any topological space. An almost connected but not connected subset might be considered as the union of a family of connected subsets, each of which sits suitably close to some of the others in the same family, although not sufficiently close to be almost connected.

Consider the phenomenon of freezing meat. If two pieces are put close enough together and frozen they will form a single block. The result of (physical) freezing depends on the conditions, such as time, temperature, ... and so on, which will not be considered in our topological discussion. However, the notion of freezing may still be useful to describe a concept of topological freezing. Any subset of a topological space may be considered as naturally frozen, without any condition. Two nonempty connected subsets might also be frozen to become a connected subset, a frozen set, if they are sufficiently near each other. In the above sentence, being sufficiently near means being almost connected.

Definition 3 Let X be a topological space and let A be a subset of X. Let $\mathcal{C}_A = \{C \subset X : C$ is countable and $A \cup C$ is connected$\}$. Then the subset $\bigcup_{C \in \mathcal{C}_A} C$ is called the frozen set of A and denoted by $\urcorner A$ (read "fro A") and $\$A = \urcorner A - A$ is called the frosticky set of A. Because, $\$A$ can be regarded as the frost covering the set A with stickiness.

Clearly, $x \in \urcorner A$ iff there exists a countable subset C of X, such that $x \in C$ and $A \cup C$ is connected. In particular if A is almost connected in a non-empty topological space X, there always exists a non-empty countable subset C of X, such that $A \cup C$ is connected. If $\mathcal{C}_A = \{\emptyset\}$, for every almost connected subset A, then $A = \emptyset$. In fact, $X = \emptyset$. Indeed, suppose $A \neq \emptyset$, then

(i) A is connected $\Longrightarrow \{a\} \subset \mathcal{C}_A$, for every $a \in A \Longrightarrow$ $\mathcal{C}_A \neq \{\emptyset\}$.

(ii) A is not connected $\Longrightarrow A \cup \emptyset$ is not connected $\Longrightarrow$ $\mathcal{C}_A \neq \{\emptyset\}$.

Since every singleton $\{x\}$ of X is almost connected, then $\mathcal{C}_{\{x\}} \neq \emptyset$. This contradiction shows X is empty. We also have that A is almost connected iff $\mathcal{C}_A$ is non-empty. If $X = \emptyset$, of course, $\urcorner\emptyset = \emptyset = X$. If $X \neq \emptyset$, then for every $x \in X$, $\emptyset \cup \{x\} = \{x\}$ is connected and $\urcorner\emptyset = \bigcup_{x \in X} x = X$. Hence, $\urcorner\emptyset$ is always closed and $\$\emptyset = X$. If X is connected, then $\urcorner X = X$, and if X is not connected, $\urcorner X = \emptyset$. In either case, $\urcorner X$ is closed. In the latter case,

$$\rceil(\overbrace{\rceil \ldots \rceil}^{n \text{ terms}})\emptyset = \overbrace{\rceil \ldots \rceil}^{n \text{ terms}} X = \begin{cases} \emptyset, & \text{if } n \text{ is odd,} \\ X, & \text{if } n \text{ is even.} \end{cases}$$

When A is not almost connected then $\rceil A = \emptyset$ and hence $\rceil A$ is closed and $\$A = \emptyset - A = \emptyset$. But the converse is not true. For example, if $X = \emptyset$, then $\rceil X = \emptyset$, but X is connected. However, if $X \neq \emptyset$, the converse becomes true. In fact, suppose that A is almost connected, then $\mathcal{C}_A$ is non-empty.

(a). If $A = \emptyset$, then $\rceil A = X \neq \emptyset$.

(b) If $A \neq \emptyset$, then let $a \in A$ and arbitrarily take $C \in \mathcal{C}_A$. Thus, $C_0 = C \cup \{a\} \in \mathcal{C}_A$ and $\emptyset \neq C_0 \subset \rceil A$, so $\rceil A \neq \emptyset$. Hence, if $X \neq \emptyset$, then, $\rceil A \neq \emptyset$ iff A is an almost connected subset of X.

An almost connected subset and a non-almost connected subset sometimes have opposite properties, as in the following lemma:

<u>LEMMA</u> Let X be a topological space.

(i) If A is almost connected, then $\overline{A} \in \rceil A$ and hence $A \in \rceil A$.

(ii) If A is not almost connected, then $\rceil A \subset \overline{A}$.

Proof

(i) If A is almost connected, take $C_0 \in \mathcal{C}_A$. Then, for every $x \in \overline{A}$, we have $A \cup C_0 \subset A \cup C_0 \cup \{x\} \subset \overline{A \cup C_0}$. Let $C = C_0 \cup \{x\}$; then $x \in C$ and $A \cup C$ is connected. Hence $x \in \rceil A$ and $\overline{A} \subset \rceil A$

(ii) If A is not almost connected, then $\rceil A = \emptyset \subset \overline{A}$.

If D is dense in X, then $\urcorner D = \overline{D} = X$ whenever D is almost connected. Otherwise, $\urcorner D = \emptyset$. Hence $\urcorner D$ is always closed.

THEOREM 6 Let X be a topological space and let $A \subset X$. If X is connected or $A \neq \emptyset$, then $\urcorner A$ is connected and $\overline{\urcorner A} \subset \urcorner\urcorner A$. In addition, if A is non-empty and almost connected, then $\urcorner\urcorner A \subset \mathrm{Cmp} A$.

Proof

(i) If A is not almost connected, then $\urcorner A = \emptyset$ is connected. Otherwise, by the previous lemma $A \subset \urcorner A$.

(a) If $A \neq \emptyset$, then $\urcorner A = (\bigcup_{C\in\mathcal{C}_A} C) \cup A = \bigcup_{C\in\mathcal{C}_A}(C \cup A)$ is still connected, since each $C \cup A$ is connected and $A \neq \emptyset$.

(b) If X is connected and $A=\emptyset$, then $\urcorner A = \urcorner\emptyset = X$ is connected.

(ii) Now, if $\urcorner A$ is connected, then for every $a \in \overline{\urcorner A}$, $\urcorner A \subset \urcorner A \cup \{a\} \subset \overline{\urcorner A}$, so $\urcorner A \cup \{a\}$ is connected. Thus $\{a\} \in \mathcal{C}_{\urcorner A}$ and hence $a \in \bigcup_{C\in\mathcal{C}_{\urcorner A}} C = \urcorner\urcorner A$.

(iii) If A is non-empty and almost connected, then $\mathrm{Cmp} A$ exists. Thus by the previous lemma and (ii), we get $A \subset \urcorner A \subset \overline{\urcorner A} \subset \urcorner\urcorner A$. But, since $\urcorner A$ is non-empty, $\urcorner\urcorner A$ is a connected subset of X containing A. Hence, $\urcorner\urcorner A \subset \mathrm{Cmp} A$.

Corollary If A is non-empty and almost connected, then $\urcorner A$ is non-empty and connected (and hence non-empty and almost connected).

Definition 4 Let A be a subset of a topological space X. The family $\mathcal{O}(A,\daleth,{}^{-})$ consisting of all subsets of X generated by $\daleth A$ and $\daleth\overline{A}$ under the operators $\daleth$ and ${}^{-}$ is called a closure organization of the frozen set $\daleth A$. That is,

$$\mathcal{O}(A,\daleth,{}^{-}) = \{\daleth A,\ \overline{\daleth A},\ \daleth\overline{A},\ \overline{\daleth\overline{A}},\ \daleth\daleth A,\ \overline{\daleth\daleth A},\ \daleth\overline{\daleth A},\ \overline{\daleth\overline{\daleth A}},\ \daleth\daleth\overline{A},\ \overline{\daleth\daleth\overline{A}},\ \daleth\overline{\daleth\overline{A}},$$
$$\overline{\daleth\overline{\daleth\overline{A}}},\ \daleth\daleth\daleth A,\ \overline{\daleth\daleth\daleth A},\ \ldots\ldots\}.$$

The proofs of the following theorem and corollaries are quite easy. The corresponding results are useful for observing the relations among the members of a closure organization of a frozen set which can be constructed as a lattice $<\mathcal{O}, \cap, \cup>$, where $\mathcal{O} = \mathcal{O}(A,\daleth,{}^{-})$.

THEOREM 7 If A is non-empty and B is almost connected in a topological space X, then $A \subset B \Longrightarrow \daleth A \subset \daleth B$.

Proof If A is not almost connected, then $\daleth A = \emptyset \subset \daleth B$. Otherwise, A is almost connected and since so also is B, then $\mathscr{C}_A$, $\mathscr{C}_B$ are non-empty. Arbitrarily take $C' \in \mathscr{C}_B$. Then, for every $C \in \mathscr{C}_A$, $A \cup C$ and $B \cup C'$ are connected.

Let $C_0 = C \cup C'$; then C_0 is countable and since $A \subset B$, we have $B \cup C_0 = (A \cup B) \cup C_0 = (A \cup C) \cup (B \cup C')$. Since $A \neq \emptyset$, $(A \cup C) \cap (B \cup C') \neq \emptyset$, and $B \cup C_0$ is connected, i.e. $C_0 \in \mathscr{C}_B$. Hence, for every $C \in \mathscr{C}_A$, there exists $C_0 \in \mathscr{C}_B$, such that $C \subset C_0$. Therefore $\daleth A = \bigcup_{C_A \in \mathscr{C}_A} C_A \subset \bigcup_{C_B \in \mathscr{C}_B} C_B = \daleth B$.

In Theorem 7, the conditions "A is non-empty" and "B is almost connected" are necessary. For example: let $\mathbb{R}$ be the real line and let $A, B \subset \mathbb{R}$.

(i) If $A = \emptyset$, $B = \{0\}$, then $\exists A = \mathbb{R} \neq \{0\} = \exists B$.

(ii) If $A = \{0\}$, $B = \{0,1\}$, then $\exists A = \{0\} \ \emptyset = \exists B$.

Corollary 1 $\exists A \subset \exists \overline{A}$, for every subset A in a topological space X .

Proof (i) If $A = \emptyset$, then $\overline{A} = \emptyset$. We have $\exists A = X = \exists \overline{A}$.

(ii) If $\overline{A}$ is not almost connected, then A is not almost connected. Thus, $\exists A = \emptyset = \exists \overline{A}$.

(iii) If $A \neq \emptyset$ and $\overline{A}$ is almost connected, then $A \subset \overline{A} \Longrightarrow \exists A \subset \exists \overline{A}$.

We can say that if $\exists \overline{A}$ is closed, then $\overline{\exists A} \in \exists \overline{A}$

Corollary 2 Let I be any index set. If A_i , for every $i \in I$, and $\exists(\bigcup_{i \in I} A_i)$ are non-empty, then $\bigcup_{i \in I} \exists A_i \subset \exists(\bigcup_{i \in I} A_i)$. Otherwise, $\exists(\bigcup_{i \in I} A_i) \subset \bigcup_{i \in I} \exists A_i$.

Proof $\exists(\bigcup_{i \in I} A_i) \neq \emptyset$ means $\bigcup_{i \in I} A_i$ is almost connected. For $i \in I$, $A_i \subset \bigcup_{i \in I} A_i \Longrightarrow \exists A_i \subset \exists(\bigcup_{i \in I} A_i)$. Thus $\bigcup_{i \in I} \exists A_i \subset \exists(\bigcup_{i \in I} A_i)$. Otherwise, $A_i = \emptyset$, for some $i \in I$ or $\exists \bigcup_{i \in I} A_i = \emptyset$, and $\exists(\bigcup_{i \in I} A_i) \subset {}_{i \in I} \exists A_i$ is trivial.

Corollary 3 If $\exists A_i$, for every i in some index set I , and $\bigcap_{i \in I} A_i$ are non-empty, then $\exists(\bigcap_{i \in I} A_i) \subset \bigcap_{i \in I} \exists A_i$. Otherwise, $\bigcap_{i \in I} \exists A_i \subset \exists(\bigcap_{i \in I} A_i)$.

Proof Similar to the above Corollary 2.

From the results of Corollaries 2 and 3, given a countable set I , if $\bigcap_{i \in I} A_i \neq \emptyset$ and A_i are almost connected, for every $i \in I$, then.

$$\urcorner(\bigcap_{i\in I} A_i) \subset \bigcap_{i\in I} \urcorner A_i \subset \bigcup_{i\in I} \urcorner A_i \subset \urcorner(\bigcup_{i\in I} A_i) .$$

Corollary 4 If the following two conditions are satisfied:

(i) $A \neq \emptyset$ and A, B are almost connected,

(ii) X is connected or $B \neq \emptyset$,

then $A \subset B \Longrightarrow \urcorner\urcorner A \subset \urcorner\urcorner B$

Proof There are three steps.

(a) if $A \neq \emptyset$ and B is almost connected then $A \subset B \Rightarrow \urcorner A \subset \urcorner B$.

(b) if $A \neq B$ and A is almost connected then $\urcorner A \neq \emptyset$.

(c) if X is connected or $B \neq \emptyset$ then $\urcorner B$ is connected.

Using (a), (b), (c) and Theorem 7, the result follows immediately.

Corollary 5 If A is non-empty and almost connected, then $\urcorner \bar{A} \subset \urcorner\urcorner A$.

Proof $A \neq \emptyset \Longrightarrow \bar{A} \neq \emptyset$ and $\urcorner A$ is connected. Since A is almost connected, we have $\bar{A} \subset \urcorner A$. Thus $\urcorner \bar{A} \subset \urcorner\urcorner A$.

We now give two important definitions.

Definition 5 Let X be a topological space.

(i) If all members of $\mathcal{O}(A,\urcorner,{}^-)$ are equal, for each non-empty almost connected subset A of X, then X is said to be first frozen.

(ii) If all members in the family $\bar{\mathcal{O}}(A,\urcorner,{}^-) = \mathcal{O}(A,\urcorner,{}^-) \cup \{\bar{A}\}$ are equal, for each non-empty almost connected subset A of X, then X is said to be second frozen.

By definition, if a topological space is second frozen, then it must be first frozen. But the converse is not true. For example, let $X = \{a,b\}$, $\mathcal{T} = \{X, \{b\}, \emptyset\}$. Then the topological space $(X, \mathcal{T})$ is first frozen, but not second frozen, since $A = \{a\} = \overline{A}$, but $\urcorner A = X \neq \overline{A}$.

The following theorem allows one to determine whether a given topological space is first or second frozen.

THEOREM 8 A topological space X is first frozen iff $\urcorner A = \urcorner\urcorner A$, for each nonempty almost connected subset A. X is second frozen iff $\urcorner A = \overline{A}$, for each nonempty almost connected subset A.

Proof

(1) Assume that $\urcorner A = \urcorner\urcorner A$, for each nonempty almost connected subset A in X.

(a) A is nonempty $\Longrightarrow \overline{\urcorner A} \subset \urcorner\urcorner A$ (by Theorem 6) and A is nonempty and almost connected $\Longrightarrow \urcorner A = \overline{\urcorner A} = \urcorner\urcorner A$ (by assumption).

(b) A is nonempty and almost connected $\Longrightarrow \urcorner\overline{A} \subset \urcorner\urcorner A$ (by Theorem 7, Cor. 5). Since $\urcorner A = \urcorner\overline{A}$ is always true (by Theorem 7, Cor. 1), then $\urcorner A = \urcorner\overline{A} = \urcorner\urcorner A$.

(c) Since $\urcorner\overline{A} = \overline{\urcorner A}$ is closed, $\urcorner\overline{A} = \overline{\urcorner\overline{A}}$.

Hence, we have $\urcorner A = \overline{\urcorner A} = \urcorner\urcorner A = \urcorner\overline{A} = \overline{\urcorner\overline{A}}$. Now an element of the family $\mathcal{O}(A, \urcorner, {}^{-})$ employing more than two symbols "$\urcorner$" can be simplified in the following way. The last three terms in such an element must be one of the following: $\urcorner\urcorner A$, $\overline{\urcorner\urcorner A}$, $\urcorner\overline{\urcorner A}$, $\overline{\urcorner\overline{\urcorner A}}$, $\urcorner\urcorner\overline{A}$, $\overline{\urcorner\urcorner\overline{A}}$, $\urcorner\overline{\urcorner\overline{A}}$ and $\overline{\urcorner\overline{\urcorner\overline{A}}}$. Each of these can be simplified to $\urcorner A$ by using the equalities $\urcorner A = \overline{\urcorner A} = \urcorner\overline{A} = \overline{\urcorner\overline{A}} = \urcorner\urcorner A$ Thus, every element of $O(A,\urcorner,^{-})$ can be simplified to $\urcorner A$ in finitely many steps. Hence, X is first frozen. To prove the converse is trivial.

(ii) Assume that $\urcorner A = \overline{A}$, for each nonempty almost connected subset A in X .

(a) $\urcorner A = \overline{A} \Longrightarrow \urcorner A$ is closed $\Longrightarrow \urcorner A = \overline{\urcorner A}$.

(b) A is nonempty and almost connected $\Longrightarrow \urcorner A$ is nonempty and almost connected $\Longrightarrow \overline{\urcorner A} = \urcorner\urcorner A$ (by assumption).

Hence, we get $\urcorner A = \urcorner\urcorner A$, and so X is first frozen. Now we can say that all members in the family $\overline{O}(A,\urcorner,^{-})$ are equal. That is, X is second frozen. To prove the converse is trivial.

In general, if A is a non-empty almost connected subset of a topological space X , then we may consider the family $L_0 = O(A,\urcorner,^{-}) \cup \{\mathrm{Cmp}A\}$ or $L_{\overline{0}} = O(A,\urcorner,^{-}) \cup \{\mathrm{Cmp}A\}$. With the two canonical binary operations $\cap$ and $\cup$ on L_0 or $L_{\overline{0}}$, one obtains an infinite lattice which has the least element $\urcorner A$ or $\overline{A}$ and the greatest element $\mathrm{Cmp}A$. But, if X is a first frozen space or second frozen space, then the lattice $< L_0, \cap, \cup >$ or $< L_{\overline{0}}, \cap, \cup >$ is not very interesting.

B. A Counter example: a frozen set which is not closed.

Though covered with ice, a frozen article still gives a strong impression of having a boundary in it. In a first frozen or second frozen space, it is obvious that every frozen set is closed. One may ask whether a frozen set must contain its boundary in general. In other words, is a frozen set always closed in a topological space?. In addition, one may ask whether there exists a non-empty almost connected subset A such that $\rceil A$ is not equal to $\rceil\rceil A$, that is, which is almost connected but not connected. We have seen an example of a non-Hausdorff space which is first frozen but not second frozen (page 15). Is there a topological space which is not first frozen? Each of the preceding questions is very hard to answer. The following important counter-example will provide an answer to some of these questions.

First, we consider a countable connected Hausdorff space (See Counterexamples in Topology, by L. A. Steen and J. A. Seebach Jr., Holt, Rinehart and Winston, Inc., 1970, page 93).

Let $X = \{(x,y) : y \geq 0,\ x,\ y \in Q\}$, where Q is the set of rational numbers. For every $(x,y) \in X$ and $\varepsilon > 0$, a basic neighborhood $N_\varepsilon((x,y))$ of (x,y) is defined as

$$N_\varepsilon((x,y)) = \{(x,y)\} \cup B_\varepsilon(x + y/\sqrt{3}) \cup B_\varepsilon(x - y/\sqrt{3}),$$

$$\text{where } B_\varepsilon(z) = \{q \in Q : |q - z| < \varepsilon\}.$$

The points (x,y), $(x + y/\sqrt{3},0)$ and $(x - y/\sqrt{3},0)$ are three vertices of an equilateral triangle with each side of length $2y/\sqrt{3}$ whenever $y > 0$. If $y = 0$, then $N_\varepsilon((x,y))$ is just an interval containing the rational numbers with center $(x,0)$ and length 2ε. A simple calculation shows that if $(x_1,y_1) \neq (x_2,y_2)$, then $(x_1 + y_1/\sqrt{3}, 0) \neq (x_2 + y_2/\sqrt{3}, 0)$ and $(x_1 - y_1/\sqrt{3}, 0) \neq (x_2 - y_2/\sqrt{3}, 0)$. Thus, for any two distinct points in X, it is easy to find disjoint basic neighborhoods of these points. Hence, X is a Hausdorff space.

The closure of $N_\varepsilon((x,y))$ is a set of straight lines lying in X, such that every intersection of such a line and the x-axis belongs to the closure of either $B_\varepsilon(x + \sqrt{y}/ 3)$ or $B (x - y/ 3)$ on the x-axis. Thus, the intersection of the closures of two non-empty open sets in X is non-empty. This implies that any two non-empty open sets, whose union is the whole space X, cannot be disjoint. As a consequence, X is a countable connected Haussdorf space.

In the above example we may replace $\sqrt{3}$ by any other irrational number. Thus, by a similar argument, we can get other countable connected Hausdorff spaces.

We now construct a topological space X_Ω. Let Ω be the first uncountable ordinal number, and let X be a countable connected Hausdorff space as above. Choose $x_0, x_1 \in X$, $x_0 \neq x_1$. Consider the cartesian product $X \times [0,\Omega]$, where $[0,\Omega]$ is the closed ordinal space.

Let X_Ω be the quotient of $X \times [0,\Omega]$ under the identification: (x_1,α) identified with $(x_0, \alpha+1)$. The set X_Ω will be topologized as follows:

(i) If $x \neq x_0$ or x_1, then a basic neighborhood of (x,α) is $U \times \{\alpha\}$ where U is a neighborhood of x in X.

(ii) If $x = x_1$, then a basic neighborhood of $(x_1,\alpha) = (x_0, \alpha+1)$ is $(U\times\{\alpha\}) \cup (V \times \{\alpha+1\})$, where U is a neighborhood of x_1 in X and V is a neighborhood of x_0 in X.

(iii) If $x = x_0$, then there are two cases:

(A) if α is not a limit ordinal, then let α' be the immediate predecessor of α, (i.e. $\alpha = \alpha' + 1$). A basic neighborhood of $(x_0, \alpha) = (x_1, \alpha'+1)$ is $(U \times \{\alpha'\}) \cup (V \times \{\alpha\})$, where U is a neighborhood of x_1 in X and V a neighborhood of x_0 in X.

(B) if α is a limit ordinal, then a basic neighborhood of (x_0, α) is $U_{\alpha'} \cup (V \times \{\alpha\})$, where V is a neigh-

bor borhood of x_0 in X and $U_{\alpha'} = \{(x,\beta) : \alpha' < \beta \leq \alpha,$ $x \in X\} = \cup\,\{X \times \{\beta\} : \alpha < \beta \leq \alpha\}$.

It is not difficult to check that X_Ω is a topological space.

<u>THEOREM 9</u> In the topological space X_Ω, let $0 = (x_0, 0) \in X \times \{0\}$, then we obtain

(i) $\urcorner\{0\} = X_\Omega - \Omega$ is not closed in X_Ω.

(ii) $\urcorner\{0\} \neq \urcorner\urcorner\{0\}$,

(iii) there exists an almost connected subset A, such that $\overline{A}$ is not connected.

Proof

(i) Suppose that C is a countable subset of X_Ω, such that $\{0\} \cup C$ is connected. Then for each $c \in C - \{\Omega\}$, there exists $\alpha_c \in [0,\Omega]$, such that $c \in X \times \{\alpha_c\}$. Let $\alpha^* = \sup\{\alpha_c : c \in C - \{\Omega\}\}$. Since Ω is the first uncountable ordinal and α_c is countable, for every $c \in C - \{\Omega\}$, α^* is countable so $\alpha^* \in [0,\Omega]$. We have $\{0\} \cup C \subset X_\Omega - \{\alpha^* + 1\}$, by the definition of α^* and $\alpha^* + 1 \neq \Omega$. Consider $O_1 = \{(x,\alpha) : \alpha < \alpha^* + 1\}$, $O_2 = \{(x,\alpha) : \alpha > \alpha^* + 1\}$, two disjoint open sets in X_Ω, i.e. $X_\Omega - \{\alpha^* + 1\} = O_1 \cup O_2$. Thus, we can say that $\Omega \notin C$. Otherwise, $(\{0\} \cup C) \cap O_1 \supset \{0\} \neq \emptyset$ and $(\{0\} \cup C) \cap O_2 \supset \{\Omega\} \neq \emptyset$ imply $\{0\} \cup C$ is not connected. Since C is an arbitraty countable subset of X_Ω, such that $\{0\} \cup C$ is connected, then $\Omega \notin \daleth\{0\}$.

On the other hand, let $(x,\alpha) \in X_\Omega$, for any arbitrary $\alpha < \Omega$ i.e. α is a countable ordinal. Then $X_\alpha = \{(x,\beta) : x \in X, \beta \leq \alpha\} = \bigcup_{\beta \leq \alpha} (X \times \{\beta\})$ is still countable. We are going to prove that it is connected by transfinite induction on the limit ordinals smaller than or equal to α. Since X_0 is connected, assume that X_β is connected for all $\beta < \gamma \in [0,\Omega]$. Consider $X_\gamma = \{(x,\beta) : \beta < \gamma\} \cup (X \times \{\gamma\}) = (\bigcup_{\beta<\gamma} X_\beta) \cup (X \times \{\gamma\})$. Thus (x_0,γ) belongs to the connected subset $\bigcup_{\beta<\gamma} X_\beta$ and $X \times \{\gamma\}$ is connected subset $\overline{\bigcup_{\beta<\gamma} X_\beta}$ and $X \times \{\gamma\}$ is connected imply that X_γ is connected. We can assert that X_α is connected by transfinite induction. It follows that $(x,\alpha) \in \daleth\{0\}$, for every $x \in X$, $\alpha < \Omega$.

Therefore, $X_\Omega - \{\Omega\} = X \times [0,\Omega] \subset \urcorner\{0\}$. But we have seen that $\Omega \notin \urcorner\{0\}$, so $\urcorner\{0\} = X_\Omega - \{\Omega\}$. Since Ω is an accumulation point of $X_\Omega - \{\Omega\}$, then $X_\Omega - \{\Omega\}$ is not closed. Hence, $\urcorner\{0\}$ is not closed.

(ii) Since $X_\Omega - \{\Omega\}$ is connected, then $X_\Omega = \overline{X_\Omega - \{\Omega\}}$ is still connected. Thus, we obtain $\urcorner\urcorner\{0\} = (X_\Omega - \{\Omega\}) \cup \{\Omega\} = X_\Omega \neq \urcorner\{0\}$.

(iii) Let $1 = (x_0,1)$ and $A = \{0,1\} \subset X_\Omega$, then A is obviously almost connected. Since X and $[0,\Omega]$ are Hausdorff, then so is X . It follows that A is closed, i.e. $A = \overline{A}$, but $\overline{A}$ is not connected.

C. <u>The conditions for a frozen set to be closed</u>.

Now, we ask what conditions on a topological space are sufficient to make each frozen set to be closed. Furthermore, under what conditions, is a topological space first frozen or second frozen? We express our discussion in the following theorems.

<u>*THEOREM 10*</u> Let X be a topological space. If for every subset E of X and for every $p \in \overline{E}$, there exists a sequence in E which converges to p, then every frozen set is closed in X.

<u>*Proof*</u> Take an arbitrary subset A in X.

(i) If A is empty, then $\urcorner A = X$ is closed.

(ii) If A is not almost connected, then $\urcorner A = \emptyset$ is closed.

(iii) If A is non-empty and almost connected, then $\rceil A \neq \emptyset$. Thus for each $x \in \overline{\rceil A}$, there is a sequence $\langle x_n \rangle$ in $\rceil A$, such that $x_n \to x$. Since $x_n \in \rceil A$, for each $n = 1, 2, \ldots$, there exists a countable set C_n such that $x_n \in C_n$ and $A \cup C_n$ is connected.

Let $C_0 = \bigcup_{n=1}^{\infty} C_n$, then $A \cup C_0$ is connected, since $A \neq \emptyset$ and $A \cup C_n$ is connected for each n. Moreover, $x_n \to x$, so that $x \in \overline{\{x_n\}} \subset \overline{C_0} \subset \overline{A \cup C_0}$. Let $C = C_0 \cup \{x\}$, then C is countable. We have $A \cup C_0 \subset A \cup C = (A \cup C_0) \cup \{x\} \subset \overline{A \cup C_0}$. Thus $x \in C$ and $A \cup C$ is connected and hence $x \in \rceil A$. It follows that $\rceil A = \overline{\rceil A}$ is closed.

Corollary If X is first countable, then every frozen set of a subset of X is closed.

The real line $\mathbb{R}$ is first countable and so in $\mathbb{R}$ every frozen set is closed.

THEOREM 11 If X is a completely regular topological space then X is second frozen. (i.e. $\rceil A = \overline{A}$, for every non-empty almost connected subset A of X.)

Proof Let A be a non-empty almost connected subset in X. It is enough to show that $\rceil A \subset \overline{A}$. Suppose on the contrary that there exists $x \in \rceil A$ such that $x \notin \overline{A}$. Since X is completely regular, then there exists a Urysohn function f, such that $f(\overline{A}) = \{0\}$, $f(x) = 1$. Since $x \in \rceil A$, there exists a countable subset C of X, such that $x \in C$ and $A \cup C$ is connected. Then $f(A \cup C)$ is connected, since f is continuous. $A \neq \emptyset$ and $f(A) \subset f(\overline{A}) = \{0\}$ implies $f(A) = \{0\}$. Since C is countable, $f(A \cup C) = f(A) \cup f(C)$, containing the points 0 and 1, is a countable subset of $[0,1]$ and hence is not connected. This contradiction shows $\urcorner A \subset \overline{A}$.

If we consider any disconnected subspace X of the real line, then $\overline{\emptyset} = \emptyset \neq X = \urcorner\emptyset$. On the other hand, for every non-almost connected subset A, then $A \neq \emptyset = \urcorner A$. These indicate that the property of being second frozen only guarantees $\urcorner A = \overline{A}$, for every non-empty almost connected subset.

From the above theorem, we gain a lot of results about being a second frozen space. If X is a topological space and is one of the following: T_4; locally compact and Hausdorff; regular and normal; regular and Lindelöf; zero dimensional; uniformizable; ... etc. , then X is completely regular and hence second frozen.

We already know that the image of an almost connected subset under a continuous function is almost connected. Between a uniform space and a completely uniform space, we have an analogous but more interesting result.

__THEOREM 12__ Let X be a uniform space and Y a complete uniform space. If A is an almost connected subset of X and the mapping $f : A \to Y$ is uniformly continuous, then $f(A)$ is almost connected in Y. (Note: the domain of f is A).

Proof Since X is a uniform space, Y a complete uniform space and $f : A \to Y$ is uniformly continuous, then there exists an extension $\overline{f} : \overline{A} \to Y$, such that $\overline{f}$ is uniformly continuous and hence is continuous.

Since A is almost connected in X, then there exists a countable subset C of X, such that $A \cup C$ is connected. Then $c \in \mathcal{C}_A$ and $C \subset \urcorner A = \overline{A}$, since X is completely regular. Thus we have $\overline{f}(C) \subset \overline{f}(\overline{A})$. Hence $\overline{f}(A \cup C) = \overline{f}(A) \cup f(C) \subset \overline{f}(\overline{A})$. $\overline{f}$ is continuous, and so $\overline{f}(A \cup C)$ is connected in $\overline{f}(\overline{A}) \subset Y$. That is, $\overline{f}(A) \cup \overline{f}(C)$ is connected in Y. And since $\overline{f}(C)$ is countable, $\overline{f}(A)$ is almost connected in Y. $\overline{f}(A) = f(A)$ implies that $f(A)$ is almost connected in Y.

Corollary Let X be a metric space and Y a complete metric space. If A is an almost connected subset of X and the mapping $f : A \to Y$ is uniformly continuous, then $f(A)$ is almost connected in Y.

Let us talk a little about first frozen spaces. There is a rather interesting sufficient condition to be a first frozen space, namely that the space has a countable number of elements.

THEOREM 13 Let X be a topological space and A a non-empty almost connected subset of X. If the frosticky set $\$A$ of A is countable, then $\urcorner A = \urcorner\urcorner A$.

Proof Since A is non-empty and almost connected, then $X \neq \emptyset$ and by Theorem 6, we have $A \subset \urcorner A \subset \urcorner\urcorner A$. i.e. $\urcorner\urcorner A \neq \emptyset$.

For every $x \in \urcorner\urcorner A$, there exists a countable subset C_0 of X, such that $x \in C_0$ and $C_0 \cup \urcorner A$ is connected. Take $C = C_0 \cup \$A$; then C is countable, $x \in C$ and $C \cup A = C_0 \cup \$A \cup A = C_0 \cup \urcorner A$ is connected, since $A \subset \urcorner A$ and $\$A \cup A = \urcorner A$. Therefore $x \in \urcorner A$ and $\urcorner\urcorner A \subset \urcorner A$. The result is immediate.

Corollary. If X is a countable space, then X is first frozen.

Proof. If $X = \emptyset$, then it is trivial.

If $X \neq \emptyset$, then every frosticky set is countable.

We have seen an example of a space which is first frozen but not second frozen. In that space we can find a non-empty almost connected subset A, such that $\urcorner A \neq \overline{A}$. It indicates that the property of being countable can only guarantee $\urcorner A = \urcorner\urcorner A$. A definition of a hiding σ-connected space will be given for looking after spaces which are very different from the real line, spaces which are discrete; indiscrete; countable; totally disconnected; of finite complement; ... and so on. It will be seen that every frozen set in such a space is closed. However, the countable complement topological space (if its underlying set is uncountable) and the real line will not have this property, and the proof of the latter will be obvious.

Definition 6 A space X is said to be hiding σ-connected, if for any non-empty subset $A \subset X$ and any point $x \in X$, the following condition is satisfied: If $A \cup \{x\}$ is connected, then there exists a nonempty countable subset C of A, such that $C \cup \{x\}$ is connected.

To test whether X is hiding σ-connected it is sufficient to observe whether an arbitrary uncountable subset A of X satisfies the condition of the definition. Indeed, if A is non-empty and countable and $\{x\} \cup A$ is connected, for some $x \in X$, then we may take $C = A$.

THEOREM 14 Every frozen set in a hiding σ-connected topological space X is closed.

Proof Suppose on the contrary that there exists $x \in \overline{\urcorner A}$, but $x \notin \urcorner A$. Since $A \neq \emptyset$, $\urcorner A$ is connected and hence $\{x\} \cup \urcorner A$ is connected. Since X is hiding σ-connected, there exists a non-empty countable subset C_1 in $\urcorner A$, such that $C_1 \cup \{x\}$ is connected.

Take an arbitrary point $c \in C_1$, then $c \in \urcorner A$, and there exists a countable subset C_2 of X, such that $c \in C_2$ and $C_2 \cup A$ is connected. Let $C = C_1 \cup C_2 \cup \{x\}$, then C is countable, $x \in C$ and $A \cup C = (A \cup C_2) \cup (C_1 \cup \{x\})$ is connected, since $C_1 \cap C_2 = \emptyset$. Thus, $x \in \urcorner A$. Hence, $A = \overline{\urcorner A}$.

THEOREM 15 If X is a totally disconnected space, then X is hiding σ-connected. The converse is not true.

Proof Let $x \in X$ and A be any non-empty subset of X such that $\{x\} \cup A$ is connected. Then $\{x\} \cup A$ is singleton, since X is totally disconnected. It follows that $A = \{x\}$, since A is non-empty. Take $C = A$; then C is a non-empty countable subset of A, such that $C \cup \{x\}$ is connected.

To see that the converse is not true, let X be an indiscrete topological space consisting of more than one point. Then X is hiding σ-connected, but not totally disconnected.

Corollary Every frozen set in a totally disconnected space is closed.

THEOREM 16 A finite complement topological space X is hiding σ-connected.

Proof . It is sufficient to show the result when X is an uncountable space. First, we prove that every infinite subset Y of X is connected. Suppose on the contrary that Y is not connected, then there exists a closed and open proper subset B of Y. B is proper closed in Y implies that there exists a closed set H of X with $H \neq X$, such that $B = H \cap Y$. B is proper open in Y implies that there exists an open set U of X with $U \neq \emptyset$, such that $B = U \cap Y$.

(i) Since $H \neq X$, H is finite and then B must be finite.

(ii) $Y - B = Y - (U \cap Y) = (Y - U) \cap (Y - Y) = Y - U = Y \cap U^c$. Since $U \neq \emptyset$ and $U^c \neq X$, then U^c is finite, so $Y - B$ is finite.

(i) and (ii) imply Y is finite. This contradiction shows Y is connected.

Now, we are going to prove that X is hiding σ-connected. Let A be any uncountable subset of X such that $A \cup \{x\}$ is connected, for some $x \in X$. Take an arbitrary countable infinite subset C in A. Then, $C \cup \{x\}$ is an infinite subset of X and is connected. Hence, X is hiding σ-connected.

Corollary Every frozen set in a finite complement topological space is closed.

It is not the case that a countable complement topological space is hiding σ-connected. Moreover, such a space does not satisfy the conditions of theorems 10, 11 and 13. However, one can still obtain similar results for such a space.

THEOREM 17 A countable complement topological space X is not hiding σ-connected. (We assume that the underlying set is uncountable, otherwise it is discrete).

Proof Let A be an uncountable subset of X and $x \notin A$. Then $A \cup \{x\}$ is also uncountable and is connected. In fact, we can prove that each uncountable subset of X is connected in the same way as in Theorem 16. On the other hand, every countable subset Y which contains at least two distinct points y_1, y_2 of X is not connected. Indeed, there exist two open sets U, V in X, where $X - U = \{y_1\}$ and $X - V = Y - \{y_1\}$, such that:

(i) $Y \subset U \cup V$. In fact, for every $y \in Y$,

- (A) if $y \neq y_1$, then $y \in U$.
- (B) if $y \neq y_1$, then $y \in Y^c \cup \{y_1\} = (Y \cap \{y_1\}^c)^c = (Y - \{y_1\})^c = V$

(ii) $U \cap V \subset X - Y$. In fact, $X - (U \cap V) = (X - U) \cup (X - V) = \{y_1\} \cup (Y - \{y_1\}) = Y$, since $y_1 \in Y$ and so $U \cap V = X - Y$.

(iii) $y_2 \in U \Longrightarrow Y \cap U \neq \emptyset$. In fact, $y_1 \neq y_2 \Longrightarrow y_2 \in U$.

(iv) $y_1 \in V \Longrightarrow Y \cap V \neq \emptyset$.

Thus, for any non-empty countable subset C of A, $C \cup \{x\}$ contains at least two distinct points in X. Hence, $C \cup \{x\}$ is not connected. This shows that X is not hiding σ-connected.

Definition 7 Let X be a topological space. If each countable subset of X is closed, then X is said to be σ-closed.

We see immediately from this definition that X must be a T_1-space. The converse is obviously not true, since the real line and the finite complement topological space are not σ-closed. Nevertheless, the discrete topological space and the countable complement topological space supply us examples of that definition. It is easily seen that being σ-closed is hereditary. This will be used in the following theorem.

THEOREM 18 If X is a σ-closed topological space, then X is second frozen.

Proof Let A be a non-empty almost connected subset of X and $x \in \urcorner A$. Then there exists a countable subset C of X, such that $x \in C$ and $A \cup C$ is connected. Then $Y = \overline{A} \cup C$ is also connected.

Suppose that $x \notin \overline{A}$ and let $D = Y - \overline{A}$. Then $D \cap \overline{A} = \emptyset$ and $D = (\overline{A} \cup C) - \overline{A} = C - \overline{A}$ is countable, so D is closed in Y. Since $x \in D$ and $A \neq \emptyset$, $D \neq \emptyset$ and $\overline{A} \neq \emptyset$. Thus $Y = \overline{A} \cup D$, that is, Y is a union of two non-empty disjoint closed subsets of Y, and so Y is not connected. This contradiction shows that $\urcorner A = \overline{A}$

D. The relations between the operator $\urcorner$ and continuous functions

Let X, Y be two topological spaces, $f : X \to Y$ a continuous mapping and $A \subset X$. One may ask if $f(\urcorner A) = \urcorner f(A)$. Of course, when A is not non-empty and almost connected in X, the answer is "no". In fact, the equality does not hold, even when X and Y are second fr[illegible]en spaces and A is non-empty and almost connected. However, in general, we have that $f(\urcorner A) \subset \urcorner f(A)$.

THEOREM 19. Let X, Y be two topological spaces and let f be a continuous function from X to Y. Then we have $f(\urcorner A) \subset \urcorner f(A)$, for every subset A of X.

Proof

(i) If $X = \emptyset$ or A is not almost connected, then $A = \emptyset$. We have $f(\urcorner A) = f(\emptyset) = \emptyset \subset \urcorner f(A)$.

(ii) If A is almost connected, then $\urcorner A \neq \emptyset$, since $X \neq \emptyset$. For every $y \in f(\urcorner A)$, there exists $x \in \urcorner A$, such that $y = f(x)$. Thus, there exists a countable subset C of X, such that $x \in C$ and $A \cup C$ is connected in Y. Since $f(A) \cup f(C) = f(A \cup C)$ is connected in Y and $f(C)$ is countable, $f(C) \in \mathscr{C}_{f(A)}$. Hence $y = f(x) \in f(C) \subset \urcorner f(A)$. i.e. $y \in \urcorner f(A)$. We have, $f(\urcorner A) \subset \urcorner f(A)$.

In the above theorem, the continuity of f is necessary. Indeed, if we consider $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

and take $A = [-1,1]$, a non-empty almost connected subset of $\mathbb{R}$, then

$$f(\urcorner A) = \{-1,1\} \neq \emptyset = \urcorner f(A) .$$

On the other hand, $f(\urcorner A)$ need not equal $\urcorner f(A)$, even if f is continuous and A is non-empty and almost connected. For example, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \tan^{-1} x$, for every $x \in \mathbb{R}$. Take $A = \mathbb{R}$, then

$$f(\urcorner A) = f(\urcorner \mathbb{R}) = f(\mathbb{R}) = (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

$$\urcorner(f(A)) = \urcorner f(\mathbb{R}) = \urcorner(-\tfrac{\pi}{2}, \tfrac{\pi}{2}) = [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

Thus, $f(\urcorner A) \neq \urcorner f(A)$.

Corollary If $f : X \to Y$ is continuous and Y is first frozen, then we have $\urcorner f(A) = \urcorner f(\urcorner A)$, for each non-empty almost connected subset A in X.

Proof Since A is non-empty and almost connected, $\urcorner A$ is non-empty and almost connected and $f(\urcorner A)$ is non-empty and almost connected. On the other hand, since $A \neq \emptyset$, $f(A) \neq \emptyset$ and then $\urcorner f(A)$ is connected. By Theorem 7 and Theorem 19, we have

$f(\urcorner A) \subset \urcorner f(A) \Longrightarrow \urcorner f(\urcorner A) \subset \urcorner\urcorner f(A) = \urcorner f(A)$, since Y is first frozen.

On the other hand, $f(A) \neq \emptyset$ and $f(\urcorner A)$ is almost connected.

By Theorem 7 again, we have $\urcorner f(A) \subset \urcorner f(\urcorner A)$.

* Imitating Theorem 19, can we say that $\rceil f^{-1}(B) \subset f^{-1}(\rceil B)$, for every $B \subset Y$? First, consider the example of the function $f : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0) \\ x, & \text{if } x \in [,1] \\ 1, & \text{if } x \in (1, +\infty) \end{cases}$$

Then f is continuous. Take $B = (0,1) \cup \{2\}$, then $f^{-1}(B) = (0,1)$ and hence $\rceil f^{-1}(B) = [0,1]$. On the other hand, $\rceil B = \emptyset$, and hence $f^{-1}(\rceil B) = \emptyset$. Thus $\rceil f^{-1}(B) = [0,1] \neq \emptyset = f^{-1}(\rceil B)$. However, Theorem 20 will answer this question.

<u>*THEOREM 20*</u> Let X, Y be two topological spaces and let f be a surjective and continuous function from X onto Y. Then we have $\rceil f^{-1}(B) \subset f^{-1}(\rceil B)$, for every subset B of Y. In particular, if $f^{-1}(B)$ is almost connected in X, then so is B in Y.

<u>*Proof*</u> If $X = \emptyset$, the result is trivial. We may assume that $X \neq \emptyset$.

(i) If $f^{-1}(B)$ is not almost connected, then $\rceil f^{-1}(B) = \emptyset$ $f^{-1}(\rceil B)$.

(ii) If $f^{-1}(B)$ is almost connected, then $\rceil f^{-1}(B) \neq \emptyset$, since $X \neq \emptyset$.

For every $x \in \rceil f^{-1}(B)$, there exists a countable subset C of X, such that $x \in C$ and $C \cup f^{-1}(B)$ is connected. Since f is surjective, $f(C \cup f^{-1}(B)) = f(C) \cup f(f^{-1}(B)) = f(C) \cup B$. Since f is continuous, $f(C) \cup B$ is connected. Also, B is almost connected and $f(C) \in \mathscr{C}_B$, since $f(C)$ is countable. Thus $f(C) \subset \urcorner B$ and $x \in C \Longrightarrow f(x) \in f(C) \subset \urcorner B \Longrightarrow x \in f^{-1}(\urcorner B)$. Hence, $\urcorner f^{-1}(B) \subset f^{-1}(\urcorner B)$.

In the above theorem, the continuity of f is necessary. As a counter-example, consider $f : [0,3] \to [0,1]$, where $[0,3]$ and $[0,1]$ are subspaces of $\mathbb{R}$, and

$$f(x) = \begin{cases} x, & \text{if} \quad x \in [0,1] \\ 0, & \text{if} \quad x \in [1,2] \\ 1, & \text{if} \quad x \in (2,3] . \end{cases}$$

Then f is surjective but not continuous. Take $B = \{1\}$, then $f^{-1}(B) = (2,3]$ and hence $\urcorner f^{-1}(B) = [2,3]$. On the other hand, $\urcorner B = \{1\}$ and hence $f^{-1}(\urcorner B) = (2,3]$. Thus $\urcorner f^{-1}(B) \not\subset f^{-1}(\urcorner B)$.

$\urcorner f^{-1}(B)$ need not equal $f^{-1}(\urcorner B)$, even if f is surjective and continuous and B is non-empty and almost connected. For example, let X be the discrete topological space of all real numbers and $f : X \to \mathbb{R}$ be the identity function. Take $B = [0,1] \subset \mathbb{R}$, then $\urcorner B = [0,1] = B$ and $f^{-1}(B) = B$. Then $\urcorner f^{-1}(B) = \urcorner B = \urcorner[0,1]$ in X and so $\urcorner f^{-1}(B) = \emptyset$, since $[0,1]$ is not almost connected in X. Thus $f^{-1}(\urcorner B) = f^{-1}(B) = [0,1] \not\subset \emptyset = \urcorner f^{-1}(B)$.

Nevertheless, if X and Y are second frozen and $f : X \to Y$ is only continuous, then we also have $\urcorner f^{-1}(B) \subset f^{-1}(\urcorner B)$, for each subset B of Y where both B and $f^{-1}(B)$ are non-empty and almost connected. In fact, we obtain $\urcorner f^{-1}(B) = \overline{f^{-1}(B)} \subset f^{-1}(\bar{B}) = f^{-1}(\urcorner B)$. In this remark, if $f^{-1}(B)$ is almost connected in X, then so also is B in Y, but the converse is clearly not true, even if f is continuous and surjective. For example, consider the function $f : X \to Y$, where $X = [-1,1]$ and $Y = [0,1]$ are subspaces of $\mathbb{R}$, such that $f(x) = |x|$, for all $x \in [-1,1]$. Take $B = \{1\}$; then B is non-empty and almost connected in Y, but $f^{-1}(B) = \{-1,1\}$ is not almost connected in X.

Corollary Let $f : X \to Y$ be surjective and continuous, and Y be first frozen. Then we have, $\daleth B = \daleth f(\daleth f^{-1}(B))$, for each subset B of Y, such that both B and $f^{-1}(B)$ are non-empty and almost connected.

Proof Since $\daleth f^{-1}(B) \subset f^{-1}(\daleth B)$, for every subset B of Y, and f is surjective we have $f(\daleth f^{-1}(B)) \subset f(f^{-1}(\daleth B)) = \daleth B$. Since $f^{-1}(B)$ is non-empty and almost connected, so is $\daleth f^{-1}(B)$. Also, since f is continuous, $f(\daleth f^{-1}(B))$ is non-empty and almost connected. $B \neq \emptyset$ implies that $\daleth B$ is almost connected. We have $f(\daleth f^{-1}(B)) \subset \daleth B \Longrightarrow \daleth f(\daleth f^{-1}(B)) \subset \daleth\daleth B = \daleth B$, since Y is first frozen. On the other hand, the almost connectedness of $f^{-1}(B)$ implies that $f^{-1}(B) \subset \daleth f^{-1}(B)$, so that $B = f(f^{-1}(B)) \subset f(\daleth f^{-1}(B))$. Thus, B is non-empty and $f(\daleth f^{-1}(B))$ is almost connected. Hence the result $\daleth B = \daleth f(\daleth f^{-1}(B))$ follows. Now we can say that $\daleth B = \daleth f(\daleth f^{-1}(B))$.

E. Some results on second frozen spaces; products of almost connected subsets.

The real line and many other topological spaces are second frozen. Some ideal results hold in such spaces. Recall that the closure of an almost connected subset is almost connected.

This conclusion will be shown to be stronger here; namely, the closure of an almost connected subset A is connected in a second frozen space. Indeed, if $A = \emptyset$, the result is trivial; if $A = \emptyset$, then $\overline{A} = \urcorner A$ is connected.

The following result can be easily obtained from properties of closure: In a second frozen space X, let $A_i : i = 1, \ldots, n$ be a finite family of non-empty almost connected subsets of X. If $\bigcup_{i=1}^{n} A_i$ is almost connected, then $\urcorner \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \urcorner A$.

An additional result follows. Let A be a subset of a topological space X, and Y be a subspace of X containing $\urcorner A$. Then the frozen set $\urcorner_Y A$ of A in Y coincides with $\urcorner A$, i.e., $\urcorner_Y A = \urcorner A$. Also, if A is almost connected in X, then A is almost connected in Y. In fact, if A is not almost connected in X, then A is not almost connected in Y. Hence $\urcorner_Y A = \emptyset = \urcorner A$. If A is almost connected in X, then for every $x \in \urcorner A$, there exists a countable subset C in $\urcorner A \subset Y$, such that $x \in C$ and $A \cup C$ is connected in Y, so $x \in \urcorner_Y A$, i.e., $\urcorner A \subset \urcorner_Y A$. To prove $\urcorner_Y A \subset \urcorner A$ is trivial. Now, let A be a subset of a second frozen space X, and Y be a closed almost connected subspace of X. If $A \cap Y \neq \emptyset$, then we have $\urcorner_Y (A \cap Y) = \urcorner (A \cap Y)$. In fact, $\urcorner (A \cap Y) \subset Y$. Furthermore, if A is also almost connected, then $\urcorner (A \cap Y) \subset \urcorner A \cap Y$.

As for the relation between the frozen set of a product of non-empty subsets in a topological space and the product of the frozen sets of those subsets, we have the following theorem to describe it.

THEOREM 21 Let $\{X_i : i \in I\}$ be a family of topological spaces, and A_i be non-empty subsets of X_i, for every $i \in I$. Then $\daleth \prod_{i\in I} A_i \subset \prod_{i\in I} \daleth A_i$

Proof If $\prod_{i\in I} A_i$ is not almost connected in $\prod_{i\in I} X_i$, then $\prod_{i\in I} A_i = \emptyset \subset \prod_{i\in I} \daleth A_i$

If $\prod_{i\in I} A_i$ is almost connected in $\prod_{i\in I} X_i$ and $x \in \daleth \prod_{i\in I} A_i$, then there exists a countable subset C in $\prod_{i\in I} X_i$, such that $x \in C$ and $\prod_{i\in I} A_i \cup C$ is connected, for every $i \in I$, we have,

$x \in C \Longrightarrow x(i) \in p_i(C)$, where $p_i(C)$ is the i-th projection.

Consider that $p_i(\prod_{i\in I} A_i \cup C) = p_i(\prod_{i\in I} A_i) \cup p_i(C) = A_i \cup p_i(C)$. Since p_i is continuous, $A_i \cup p_i(C)$ is connected. C is countable implies that $p_i(C)$ is also countable. Thus $x(i) \in \daleth A_i$, for every $i \in I$, i.e. $x \in \prod_{i\in I} \daleth A_i$.

Corollary Let $\{X_i : i \in I\}$ be a family of second frozen spaces, and A_i be a non-empty subset of X_i, for every $i \in I$; then

(i) if there exists some A_{i_0} which is not almost connected, for some $i_0 \in I$, then $\daleth \prod_{i\in I} A_i = \prod_{i\in I} \daleth A_i = \emptyset$.

(ii) if $\prod_{i\in I} A_i$ is almost connected, then $\daleth \prod_{i\in I} A_i = \prod_{i\in I} \daleth A_i = \overline{\prod_{i\in I} A_i} = \prod_{i\in I} \overline{A_i}$.

Proof

(i) It is trivial.

(ii) Since $\prod_{i\in I} A_i$ is non-empty and almost connected, then A_i is non-empty and almost connected in X_i, for every $i \in I$. Then, by Theorem 23, $\prod_{i\in I} \rceil A_i = \prod_{i\in I} \overline{A_i} = \overline{\prod_{i\in I} A_i} \subset \rceil \prod_{i\in I} A_i \subset \rceil A_i$.

In the above, we have said that if $\prod_{i\in I} A_i$ is non-empty and almost connected in the product space $\prod_{i\in I} X_i$ with the hypothesis of that corollary, then we get the nice result that $\rceil \prod_{i\in I} A_i = \prod_{i\in I} \rceil A_i$. However, the problem arises to determine under what conditions $\prod_{i\in I} A_i$ is almost connected in $\prod_{i\in I} X_i$? This problem does not appear to be difficult, but in fact, it is far from easy. In the previous chapter, we briefly talked about the product of finitely many almost connected subsets. With the concept of being second frozen, we seek further information about this problem.

__THEOREM 22__ Let A_1, A_2 be two almost connected subsets with all components being open in second frozen spaces X_1, X_2 respectively. (That is, A_1, A_2 are two open, locally connected and almost connected subsets of X_1, X_2 respectively). Then $A_1 \times A_2$ is almost connected in $X_1 \times X_2$.

__Proof__ We may assume that A_1, A_2 are non-empty and that at least one of A_1 and A_2 is not connected. Let C_1, C_2 be two non-empty countable subsets in X_1, X_2 respectively, such that $A_1 \cup C_1$ and $A_2 \cup C_2$ are connected. Specify that if A_1 is connected, then we take $C_1 \subset A_1$. Of course, the other $C_j \not\subset A_j$.

We are going to prove that $Y = (A_1 \times A_2) \cup (C_1 \times C_2)$ is connected. Suppose on the contrary that Y is not connected. Then there exist two non-empty disjoint closed subsets M, N of Y, such that $Y = M \cup N$.

Take arbitrary components B_1 in A_1 and B_2 in A_2. Then $B_1 \times B_2 \subset M$ or $B_1 \times B_2 \subset N$. Otherwise,

$M \cap (B_1 \times B_2)$ is non-empty and closed in $B_1 \times B_2$.

$N \cap (B_1 \times B_2)$ is non-empty and closed in $B_1 \times B_2$ and $(M \cap (B_1 \times B_2)) \cup (N \cap (B_1 \times B_2)) = (M \cup N) \cap (B_1 \times B_2) = Y \cap (B_1 \times B_2) = B_1 \times B_2$. Thus, $B_1 \times B_2$ is not connected. This is a contradiction which shows that $B_1 \times B_2$ must be entirely contained in either M or N.

Since X_1, X_2 are second frozen and A_1, A_2 are almost connected in X_1, X_2 respectively, $p_1 \in C_1 \Longrightarrow p_1 \in \exists A_1 = \overline{A}_1$ and $p_2 \in C_2 \Longrightarrow p_2 \in \exists A_2 = \overline{A}_2$. Hence, $p = (p_1, p_2) \in C_1 \times C_2 \Longrightarrow p \in \overline{A}_1 \times \overline{A}_2 = \overline{A_1 \times A_2}$. i.e. $C_1 \times C_2 \subset \overline{A_1 \times A_2}$. Let

$\mathcal{M} = \{B_1 \times B_2 \subset M : B_1$ is a component in A_1 and B_2 is a component in $A_2\}$,

$\mathcal{N} = \{B_3 \times B_4 \subset N : B_3$ is a component in A_1 and B_4 is a component in $A_2\}$.

Then $\mathcal{M}$ and $\mathcal{N}$ are non-empty. Otherwise, we may assume that $N \subset C_1 \times C_2$, thus $A_1 \times A_2 \subset M$. Then we get $N \subset C_1 \times C_2 \subset \overline{A_1 \times A_2} \cap Y \subset \overline{M} \cap Y = M$, since M is closed in Y. But this is impossible.

Let $M^* = \bigcup_{B_1 \times B_2 \in \mathcal{M}} B_1 \times B_2 \subset X_1 \times X_2$ and

$N^* = \bigcup_{B_2 \times B_4 \in \mathcal{N}} B_3 \times B_4 \subset X_1 \times X_2$, then

$\emptyset \neq M^* \subset M$ and $\emptyset \neq N^* \subset N$. On the other hand, $M^*, N^* \subset A_1 \times A_2$. It will be useful to show that $\overline{A}_1 \times \overline{A}_2 = \overline{M}^* \cup \overline{N}^*$. In fact,

$(a_1,a_2) \in A_1 \times A_2 \Longrightarrow (a_1,a_2) \in B_{a_1} \times B_{a_2}$, for some component B_{a_1} in A_1 and B_{a_2} in A_2

$\Longrightarrow (a_1,a_2) \in M^* \cup N^*$.

Thus, $A_1 \times A_2 \subset M^* \cup N^* \Longrightarrow A_1 \times A_2 = M^* \cup N^*$

$\Longrightarrow \overline{A}_1 \times \overline{A}_2 = \overline{A_1 \times A_2} = \overline{M^* \cup N^*} = \overline{M^*} \cup \overline{N^*}$.

Now, $M^* \subset M \Longrightarrow \overline{M^*} \cap Y \subset \overline{M} \cap Y = M$. Similarly, $\overline{N^*} \cap Y \subset N$. Since $M \cap N \neq \emptyset$, $\overline{M^*} \cap \overline{N^*} \cap Y = \emptyset$.

On the other hand, by assumption, A_1 and A_2 are non-empty, so $\overline{A_1 \times A_2} = \overline{A}_1 \times \overline{A}_2 = A_1 \times A_2$ is connected. Therefore, $\overline{M^*} \cap \overline{N^*} \neq \emptyset$, since $\overline{M^*}$ and $\overline{N^*}$ are non-empty closed subsets in $\overline{A_1 \times A_2} = \overline{M^*} \cup \overline{N^*}$.

We are going to prove that $\overline{M^*} \cap \overline{N^*} \cap Y \neq \emptyset$. Since C_1, C_2 are non-empty, we claim that $(C_1 \times C_2) \cap (\overline{M^*} \cap \overline{N^*}) \neq \emptyset$. Let $Z = (A_1 \cup C_1) \times (A_2 \cup C_2)$, then $Z \subset \urcorner A_1 \times \urcorner A_2 = \overline{A}_1 \times \overline{A}_2 = \overline{M^*} \cup \overline{N^*}$. $\overline{M^*} \cap Z$ is closed in Z and so is $\overline{N^*} \cap Z$. Also, $Z = (\overline{M^*} \cap Z) \cup (\overline{N^*} \cap Z)$. Since Z is connected, then there exists some $q = (q_1,q_2) \in \overline{M^*} \cap \overline{N^*} \cap Z$. But $q \notin A_1 \times A_2$, as otherwise, $q \in Y$ implies $\overline{M^*} \cap \overline{N^*} \cap Y \neq \emptyset$, which completes the proof. Hence,

$$q \in (M^* \cap N^* \cap Z) - (a_1 \times A_2)$$

$$\Longrightarrow q \in Z - (A_1 \times A_2) = [(A_1 \cup C_1) \times (A_2 \cup C_2)] - (A_1 \times A_2)$$

$$\Longrightarrow q_1 \in C_1 - A_1 \text{ and } q_2 \in A_2 \cup C_2, \text{ or } q_1 \in A_1 \cup C_1 \text{ and } q_2 \in C_2 - A_2 .$$

First, we assume that $q_1 \in C_1 - A_1 \subset C_1$ and $q_2 \in A_2 \cup C_2$. If $q_2 \in C_2$, then $q = (q_1,q_2) \in C_1 \times C_2$, i.e., $(C_1 \times C_2) \cap (M^* \cap N^*) \neq \emptyset$, which completes the proof. If $q_2 \notin C_2$, then there exists a component B_2 in A_2, such that $q_2 \in B_2 \subset A_2$. We claim that $\{q_1\} \times B_2 \subset \overline{M^*} \cap \overline{N^*}$.

For every $(q_1,b) \in \{q_1\} \times B_2$ and any basic open set $U_1 \times U_2$ in $X_1 \times X_2$ which contains (q_1,b), then U_1 is open in X_1 and contains q_1. Consider the basic open set $U_1 \times B_2$ which contains $q = (q_1,q_2)$. Since $q \in \overline{M^*} \cap \overline{N^*}$, then there exist $(m_1,m_2) \in M^*$ and $(n_1,n_2) \in N^*$, such that (m_1,m_2), $(n_1,n_2) \in U_1 \times B_2$. Thus, $(m_1,m_2) \in M^* \cap (U_1 \times B_2) \Longrightarrow m_1 \in B_1 \cap U_1 \subset B_1$, for some component B_1 in A_1, and $m_2 \in B_2$. Also $(n_1,n_2) \in N^* \cap (U_1 \times B_2) \Longrightarrow n_1 \in B_3 \cap U_1 \subset B_3$, for some B_3 in A_1, and $n_2 \in B_2$.

$$(m_1,m_2) \in M^* \Longrightarrow (m_1,m_2) \quad B_1 \times B_2 \in \quad .$$
$$(n_1,n_2) \in N^* \Longrightarrow (n_1,n_2) \quad B_3 \times B_2 \in \quad .$$

Hence, $\{m_1\} \times B_2 \subset B_1 \times B_2$, $\{n_1\} \times B_2 \subset B_3 \times B_2$. On the other hand, $(m_1,b) \in U_1 \times U_2$ and $(n_1,b) \in U_1 \times U_2$. Also, $(m_1,b) \in B_1 \times B_2 \in \overline{M^*}$ and $(n_1,b) \in B_3 \times B_2 \in \overline{N^*}$. Thus, there is a net in $\overline{M^*}$ which converges to (q_1,b) and a net in $\overline{N^*}$ which converges to (q_1,b). These imply that $(q_1,b) \in \overline{M^*} \cap \overline{N^*}$, that is, $\{q_1\} \times B_2 \subset \overline{M^*} \cap \overline{N^*}$.

If A_2 is connected, by specification, then $C_2 \subset A_2 = B_3$. Take an arbitrary $c \in C_2$; then $(q_1,c) \in \{q_1\} \times B_2 \subset \overline{M^*} \cap \overline{N^*}$ and so $(C_1 \times C_2) \cap (\overline{M^*} \cap \overline{N^*}) \neq \emptyset$. The proof is complete again.

If A_2 is not connected, we observe that $\overline{B}_2 \cap (A_2 \cup C_2) \neq B_2$. Otherwise, B_2 is a closed and open proper subset in $A_2 \cup C_2$, since $C_2 \not\subset A_2$. It is impossible. Take $b_2 \in [\overline{B}_2 \cap (A_2 \cup C_2)] - B_2$; then $b_2 \notin A_2$. Otherwise, $B_2 \subset B_2 \cup \{b_2\} \subset \overline{B}_2 \Longrightarrow B_2 \cup \{b_2\}$ is connected $\Longrightarrow B_2$ is not a component of A_2. Thus, $b_2 \notin A_2$ and $b_2 \in A_2 \cup C_2 \Longrightarrow b_2 \in C_2$. Therefore, $(q_1, b_2) \in (C_1 \times C_2) \cap (\{q_1\} \times \overline{B}_2)$. Since $\{q_1\} \times B_2 \subset \overline{M^*} \cap \overline{N^*}$, we have $\{q_1\} \times \overline{B}_2 \subset \overline{\{q_1\} \times B_2} \subset \overline{M^*} \cap \overline{N^*}$. Thus, $(C_1 \times C_2) \cap (\overline{M^*} \cap \overline{N^*}) \neq \emptyset$.

Second, if $q_1 \in A_1 \cup C_1$ and $q_2 \in C_2 - A_2$, then we can similarly prove the same result. In this case, we need all components of A_1 to be open in X_1.

Corollary Let $\{X_i : i = 1, \ldots, n\}$ be a finite family of second frozen spaces and let A_i be an almost connected subset of X_i with all components open in X_i, for $i = 1, \ldots, n$. Then the product set $\prod_{i=1}^{n} A_i$ is almost connected in the product space $\prod_{i=1}^{n} X_i$.

Proof By induction.

CHAPTER III

AN EXTENSION OF CONNECTEDNESS

So far we have seen many results associated with the concept of almost connectedness. Let us now consider the relation of this concept with other notions of connectedness. The following statements are not difficult to obtain. An almost connected subset need not be totally disconnected, totally pathwise disconnected, locally connected or locally pathwise connected and vice versa, in each case. In addition, a pathwise connected subset is connected and hence is almost connected (see page 51).

To go from the concept of being almost connected to the concept of being connected, it is necessary to consider an appropriate countable subset along with the almost connected subset. In a similar way we can extend many other concepts in mathematics. In the theory of topology itself, we can introduce concepts such as almost normal, almost compact, ... etc. We do not intend to investigate all such concepts here. However, we will study those which are relative to connectedness.

Definition 8 Let X be a topological space and $A \subset X$. If there exists a countable subset C of X, such that $A \cup C$ is locally connected in X. Then A is said to be almost locally connected in X (or A is called an almost locally connected subset/subspace of X.)

Notions of almost pathwise connected and almost locally pathwise connected can be similarly defined.

A topological space X is almost connected in itself (or almost locally connected, almost pathwise connected, almost locally pathwise connected) iff X is connected (or locally connected, pathwise connected, locally pathwise connected), for the countable set C in such definitions are in X. Accordingly, it is quite clear that the following statements hold:

(i) A locally connected subset is almost locally connected.

(ii) A pathwise connected subset is almost pathwise connected.

(iii) A locally pathwise connected subset is almost locally pathwise connected.

(iv) The irrational subset P of ℝ provides a counter example to show that the converses of (i), (ii) and (iii) are not true. P is not connected, not locally connected and not locally pathwise connected.

(v) An almost locally pathwise connected subset is almost locally connected. The converse is not true, for the countable complement topological space is a counter example.

(vi) An almost locally pathwise connected subset hense also almost locally connected, need not be almost connected and so need not be almost pathwise connected. A counter example is the subset $U = (0,1) \cup (2,3)$ of ℝ.

(vii) An almost pathwise connected subset and then an almost connected subset need not be almost locally connected and then need not be almost locally pathwise connected. A well known counter example, the closed infinite broom can be used here. Let $X = B$ be the subspace of the metric space $(\mathbb{R}^2, d)$, where B is the union of the family of closed line segments joining the point $(0,0)$ to the points $(1,0)$ and $(1, 1/n)$, $n = 1,2,3, \ldots$ Consider $B \subset X$.

(viii) An almost pathwise connected subset is almost connected, but the converse is not true. As a counter example we may consider the topologist's sine curve. Let $X = E = E_1 \cup E_2$ be the subspace of $(\mathbb{R}^2, d)$, where $E_1 = \{(0,1): -1 \leq y \leq 1\}$ and $E_2 = \{(x,y) : 0 < x \leq 1, y = \sin 1/x\}$. Then consider $E \subset X$.

We have defined the idea of being almost locally connected. Interestingly, we can also define the distinct concept of being locally almost connected. In order to avoid confusion, we shall refer to the latter as being La-connected. Moreover, we can introduce the concept of being almost La-connected (i.e. almost locally almost connected.) In fact, we can also have the concepts of being locally almost locally connected, locally La-connected, Locally almost La-connected (or, briefly, $(La)^2$-connected), ..., $(La)^n$-connected etc. However, we do not intend to discuss these.

Definition 9 Let X be a topological space and $x \in X$. The subset N of X is called an almost connected neighborhood of x in X, if N is a neighborhood of x and is almost connected in X.

In a topological space X, let Y be a subspace of X and $B \subset Y$. If B is almost connected in Y, then B is almost connected in X. The subset $B = P \cap (0,1) \subset Y = P$ in $\mathbb{R}$ shows that the converse is not true. Unfortunately, a neighborhood of some point in Y is not necessarily a neighborhood of that point in X, although the converse is true. It follows that there is no law relating an almost connected neighborhood of some point in a topological space and in a subspace. That is,

(i) if N is an almost connected neighborhood of y in Y, then N need not be an almost connected neighborhood of y in X. For example, let $Y = N = [0,1]$ in $\mathbb{R}$; then N is an almost connected neighborhood of 0 in Y, but N is not a neighborhood of 0 in $\mathbb{R}$.

(ii) if M is an almost connected neighborhood of x in X, then M need not be an almost connected neighborhood of x in Y. For example, let $Y = M = (0,2) \cup P$ in $\mathbb{R}$, then M is an almost connected neighborhood of 1 in $\mathbb{R}$. Moreover, M is not almost connected in Y, since $M = Y$ is not connected.

Definition 10 Let X be a topological space and $x \in X$. If each neighborhood of x contains an almost connected neighborhood of x, then X is said to be locally almost connected (or La-connected).

A subset A of X is said to be La-connected in X, if for every $a \in A$, each neighborhood of a in A contains a neighborhood K of a in A and K is almost connected in X.

As in Definition 8, a subset A of X is said to be almost La-connected in X, if there exists a countable subset C of X, such that $A \cup C$ is La-connected in X.

From the definition, we know that a topological space X is almost La-connected in itself iff X is La-connected. Moreover, an La-connected subset in X is almost La-connected in X. We can also ask if the converse is true. For example, $P \subset \mathbb{R}$ is almost La-connected and also La-connected in $\mathbb{R}$.

THEOREM 23 Let X be a topological space and $A \subset X$. A is almost La-connected in X iff A is La-connected in X.

Proof One side of the proof is trivial.

Assume that A is almost La-connected in X, then there exists a countable subset C_0 of X, such that $A \cup C_0$ is La-connected in X. For every $a \in A$ and for every neighborhood N of a in A, there exists a neighborhood M of a in $A \cup C_0$, such that $N = M \cap A$. Since $A \cup C_0$ is La-connected inn X, there exists a neighborhood V of a in $A \cup C_0$, such that

$V \subset M$ and V is almost connected in X. i.e. there exists a countable subset C_1 of X, such that $V \cup C_1'$ is connected.

Let $U = V \cap A$, then U is a neighborhood of a in A, $U \subset V$ and $U = V \cap A \subset M \cap A = N$. Observe that $V \subset U \cup C_0$. In fact, $x \in V \Longrightarrow x \in M \Longrightarrow x \in A \cup C_0 \Longrightarrow x \in (A \cap V) \cup C_0 \Longrightarrow x \in U \cup C_0$.

Let $C_2 = C_1 \cup (V \cap C_0)$; then C_2 is a countable subset of X. Consider

$$\begin{aligned} U \cup C_2 &= U \cup (C_1 \cup (V \cap C_0)) = C_1 \cup (U \cup (V \cap C_0)) \\ &= C_1 \cup ((U \cup V) \cap (U \cup C_0)) = C_1 \cup (V \cap (U \cup C_0)) \\ &= C_1 \cup V . \end{aligned}$$

Hence, $U \cup C_2$ is connected. i.e. U is a neighborhood of a in A which is contained in N and is almost connected in X.

According to the theorem, being La-connected in X and being almost La-connected have the same meaning. We prefer to select the former as representing the latter. Define A to be a locally La-connected subset of a topological space X if for every $a \in A$ and for every neighborhood of a in A there exists a neighborhood V of a in A, such that V is La-connected in X, i.e. V is almost La-connected in X. Then we find that a locally La-connected subset of X is equal to an $(La)^2$-connected subset of X.

<u>THEOREM 24</u> If A is almost locally connected in a topological space X, then A is La-connected in X.

<u>Proof</u> Since A is almost locally connected, then there exists a countable subset C of X, such that $Y = A \cup C$ is locally connected.

For every $x \in A$ and every neighborhood N of x in A, there exists a neighborhood M of a in Y, such that $N = M \cap A$, and then there exists a connected neighborhood L of a in Y, such that $L \subset M$. Let $K = L \cap A$, then K is a neighborhood of a in A and $L = L \cap Y = (L \cap A) \cup (L \cap C) = K \cup (L \cap C)$.

Since C is countable, $L \cap C$ is a countable subset of Y. Also, since L is connected, K is almost connected in Y and hence is almost connected in X. In other words, for every $x \in A$, each neighborhood of x in A contains a neighborhood K of x in A and K is almost connected in X.

Because every open subset in $\mathbb{R}^2$ must contain at least a disc (in fact, it contains infinitely many discs), the closed infinite broom informs us that an almost pathwise connected subset and so an almost connected subset need not be La-connected. Conversely, by Theorem 24, an La-connected subset in X need not be almost connected, almost pathwise connected nor even almost locally pathwise connected. Furthermore, it need not be almost locally connected. Thus, the converse of theorem 24 is not true.

For example, let X be an uncountable discrete topological space and let P be the irrational subset of $\mathbb{R}$. Then $X \times P$ is La-connected in the product space $X \times \mathbb{R}$. In fact, for every point $(x,p) \in X \times P$ and every neighborhood N of (x,p) in $X \times P$, N must contains a basic open set $\{x\} \times U$ of $X \times P$, such that $(x,p) \in \{x\}\ U$. Obviously, $\{x\} \times U$ is almost connected in $X \times \mathbb{R}$, though $X \times P$ is not almost locally connected in X. In fact, for every countable subset C of $X \times \mathbb{R}$, C can at most meet countably many copies of P. Thus there exists $y \in X$, such that $\{y\} \times P$ is still not locally connected. Take an arbitrary point $(y,p) \in \{y\} \times P$; then $\{y\} \times P$ is a neighborhood of (y,p) in $X \times \mathbb{R}$, but it contains no connected neighborhood of (y,p) in $X \times \mathbb{R}$. Hence, $(X \times \mathbb{R}) \cup C$ is not locally connected.

On the other hand, an almost locally pathwise connected subset is almost locally connected and hence is La-connected. After defining a locally almost locally connected subset/subspace A of a topological space X as follows: for every $a \in A$, every neighborhood of a in A contains a neighborhood L of a in A, such that L is almost locally connected in X, we get that A is locally La-connected, since L is La-connected in X.

Similarly, we can successively define an almost locally La-connected subset, a locally ... locally La-connected subset, ... and so on in a topological space. It is, however, clear that an almost almost something-subset is almost that thing in the same topological space.

Since we have several concepts generalizing connectedness, it is desirable to make a list of their relationship with each other. In the list we will use the symbols:

(i) $\longrightarrow$ means "implies".

(ii) - - - A - - ▶ means "need not be (with the counter example A)"

(iii) $\mathbb{R}$: real line

(iv) Q : space of rational numbers $\subset \mathbb{R}$

(v) P : space of irrational numbers $\subset \mathbb{R}$

(vi) U : $(0,1) \cup (2,3) \subset \mathbb{R}$

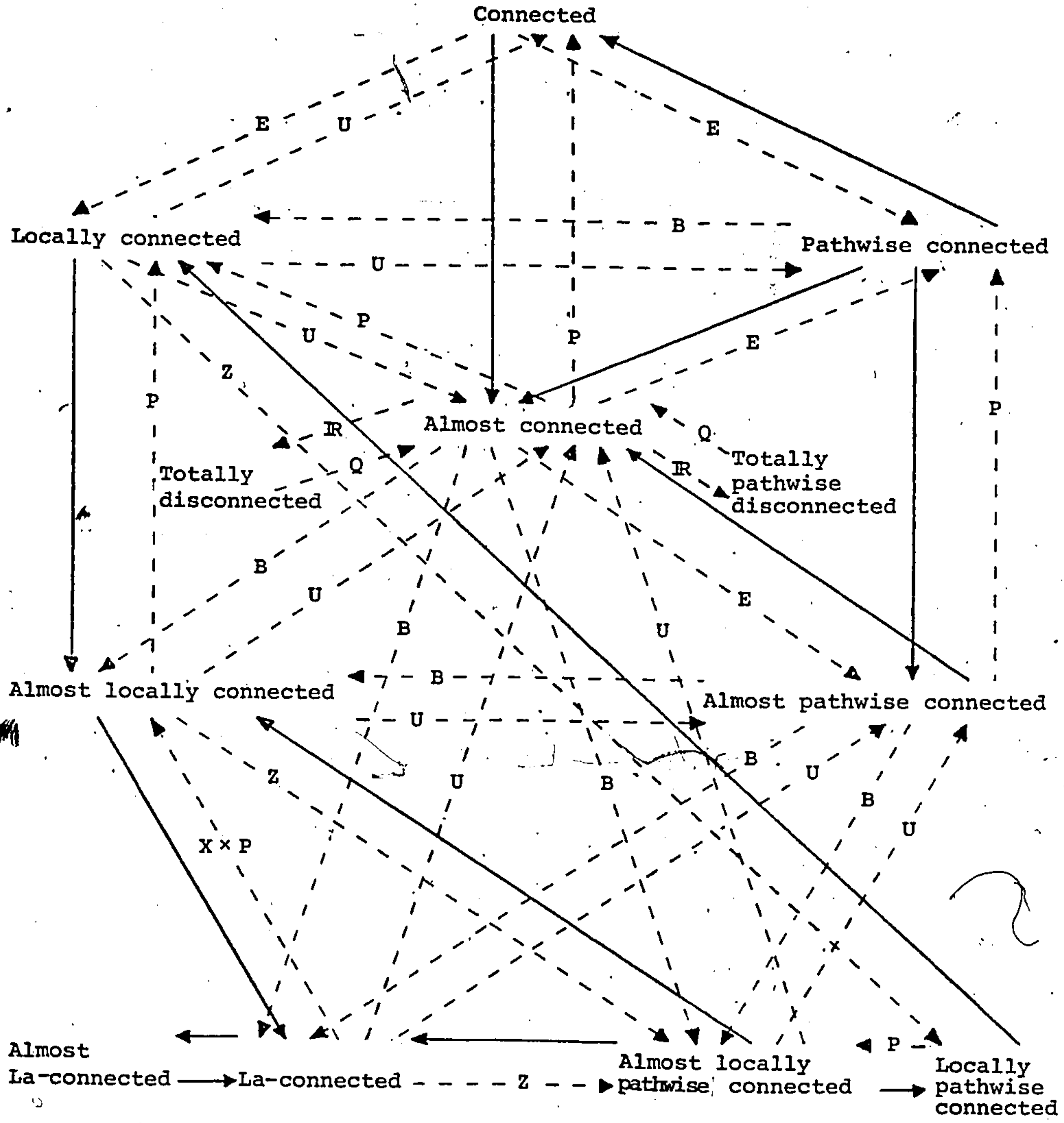
(vii) Z : countable complement topological space

(viii) X : uncountable discrete topological space

(ix) B : closed infinite broom (see page 44)

(x) E : topologist's sine curve (see page 44)

A LIST OF RELATIONS ABOUT THE SYSTEM OF CONNECTEDNESS



IR : real line, Q : space of rational numbers, P : space of irrational numbers, U : (0,1) ∪ (2,3), Z : countable complement space, X : uncountable discrete space, B : closed infinite broom, E : topologist's sine curve.

INDEX

NOTATIONS