Motivic decompositions and Hecke-type algebras

Alexander Neshitov

Thesis submitted to the Faculty of Graduate and Postdoctoral Studies in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics [1]

Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

[1]The Ph.D. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

Let G be a split semisimple algebraic group over a field k. Our main objects of interest are twisted forms of projective homogeneous G-varieties. These varieties have been important objects of research in algebraic geometry since the 1960's.

The theory of Chow motives and their decompositions is a powerful tool for studying twisted forms of projective homogeneous varieties. Motivic decompositions were discussed in the works of Rost, Karpenko, Merkurjev, Chernousov, Calmes, Petrov, Semenov, Zainoulline, Gille and other researchers. The main goal of the present thesis is to connect motivic decompositions of twisted homogeneous varieties to decompositions of certain modules over Hecke-type algebras that allow purely combinatorial description. We work in a slightly more general situation than Chow motives, namely we consider the category of $\mathtt{h}$-motives for an oriented cohomology theory $\mathtt{h}$.

For a group G there is the notion of a versal torsor such that any G-torsor over an infinite field can be obtained as a specialization of a versal torsor. We restrict our attention to the case of twisted homogeneous spaces of the form E/P where P is a special parabolic subgroup of G. The main result of this thesis states that there is a one-to-one correspondence between $\mathtt{h}$-motivic decompositions of the variety E/P and direct sum decompositions of modules $\mathbf{D}_{F,P}^{gr*}$ over the graded formal affine Demazure algebra $\mathbf{D}_F^{gr}$. This algebra was defined by Hoffnung, Malagón-López, Savage and Zainoulline combinatorially in terms of the character lattice, the Weyl group and the formal group law of the cohomology theory $\mathtt{h}$.

In the classical case $\mathtt{h} = \mathrm{CH}$ the graded formal affine Demazure algebra $\mathbf{D}_F^{gr}$ coincides with the nil Hecke ring, introduced by Kostant and Kumar in 1986. So the Chow motivic decompositions of versal homogeneous spaces correspond to decompositions of certain modules over the nil Hecke ring.

As an application, we give a purely combinatorial proof of the indecomposability of the Chow motive of generic Severi-Brauer varieties and the versal twisted form of $\mathbf{HSpin}_8/P_1$.

Acknowledgement

I would like to thank my advisor Kirill Zainoulline for his guidance. Also I would like to thank Baptiste Calmes, Vladimir Chernousov and Nikita Semenov for useful discussions. I acknowledge financial support from Ontario Trillium Foundation, the University of Ottawa and the Fields Institute.

Contents

Chapter 1

Introduction

The theory of Chow motives is an important tool of study of algebraic varieties. Motivic decompositions of Pfister quadric played an essential role in the proof of the Milnor conjecture by Voevodsky, and the motivic decompositions of norm varieties were used to prove the Bloch-Kato conjecture by Rost, Suslin and Voevodsky. Another application to the theory of quadratic forms can be found in the works of Vishik, Karpenko and Merkurjev. Let G be a semisimple algebraic group over a base field k. Our primary objects of interest are projective homogeneous G-varieties. Motivic decompositions of such varieties were intensively investigated in the last two decades. The case of split varieties was established by Köck [26], who showed that in this case the motive decomposes as a sum of Tate motives. The results of Chernousov-Gille-Merkurjev [8] and Brosnan [3] give decompositions of motives of isotropic homogeneous varieties into direct sums of motives of smaller anisotropic varieties. Rost [40] established the motivic decomposition of a Pfister quadric as a sum of twisted copies of an indecomposable motive R called the Rost motive. The case of Severi-Brauer varieties was studied by Karpenko in [24]. In [39] Petrov, Semenov and Zainoulline provided the motivic decomposition of generically split projective varieties as a direct sum of twists of an indecomposable motive.

In the present thesis we consider the case of a versal inner form of a projective homogeneous variety, i.e. a variety of the form E/P where E is a versal (i.e. a generic) torsor of a split semisimple group G and P is a special parabolic subgroup. Note that the groups G and P are uniquely determined by combinatorial data: the root system of G, the character lattice T^* of its split maximal torus T, and the subset subset of simple roots of G defining P.

The main aim of the present thesis is to describe the motivic decompositions of E/P in terms of these combinatorial data. We work in a bit more general situation than the theory of Chow motives. Namely we consider an oriented cohomology theory $\mathtt{h}$ in the sense of Levine-Morel [32] and the theory of $\mathtt{h}$-motives. In the case $\mathtt{h} = \mathrm{CH}$ it coincides with the classical category of Chow motives. The main result of this thesis (Theorem 6.4.10) establishes a $1-1$ correspondence between the $\mathtt{h}$-motivic decompositions of E/P and decompositions of the graded module $\mathbf{D}_{F,P}^{gr*}$ over the graded formal affine Demazure algebra $\mathbf{D}_F^{gr}$ introduced by Hoffnung, Malagon-Lopez, Savage and Zainoulline in [22]. In the case $\mathtt{h} = \mathrm{CH}$ the algebra $\mathbf{D}_F^{gr}$ coincides with the nil-Hecke ring which was introduced by Kostant and Kumar in [28] and is an important object in the geometric representation theory.

As an application, we give a combinatorial proof of indecomposability of the modules $\mathbf{D}_{F,P}^{gr*}$ over $\mathbf{D}_F^{gr}$ for $\mathtt{h} = \mathrm{CH}$ in the following cases: $G = PGL_{p^r}, P = P_1$ and $G = \mathbf{HSpin}_8, P = P_1$. In view of the Theorem 6.4.10 this impies the indecomposabilty of generic Severi-Brauer varieties, previously known due to Karpenko ([24]) and indecomposability of the 6-dimensional involution variety twisted by a generic $\mathbf{HSpin}_8$-torsor.

1.1 Notation and conventions

- Let k denote a base field,
- Let $\mathbf{Sch}_k$ denote the category of separated finite type schemes over k and $\mathbf{Sm}_k$ denote its full subcategory of smooth schemes.
- Let G be a split semisimple algebraic group over k. Fix a split maximal torus T and the corresponding Weyl group W.
- Let $\mathtt{h}_*$ be a graded oriented Borel-Moore homology theory on $\mathbf{Sch}_k$.
- We assume that
  - either the field k is arbitrary and $\mathtt{h}_* = \mathrm{CH}_*$
  - or k has characteristic zero and $\mathtt{h}_*$ is any oriented Borel-Moore homology theory that is generically constant, satisfies the localization property, the coefficient ring $\mathtt{h}_*(k)$ is an integral domain and $|W| \cdot 1 \neq 0$ in $\mathtt{h}^*(k)$ where $|W|$ is the number of elements of the Weyl group.

Let E be a versal torsor of the group G and P be a special parabolic subgroup of G. Restricting the Borel-Moore homology theory $\mathtt{h}_*$ to the category of smooth schemes one gets the oriented cohomology theory $\mathtt{h}^*$. One can associate to $\mathtt{h}^*$ a graded equivariant cohomology theory $\mathtt{h}_G^*$ using the strategy of [19]. In the case $\mathtt{h}^* = \mathrm{CH}^*$ this construction gives the equivariant Chow groups of Edidin-Graham [13]. Replacing CH^* by $\mathtt{h}^*$ in the construction of the category of Chow motives we obtain the category of $\mathtt{h}^*$-motives. Our main object of study is the $\mathtt{h}$-motive of E/P and its decompositions.

Let $d = \dim E/P$. To find a motivic decomposition of E/P is equivalent to find a decomposition $1 = p_1 + \ldots + p_n$ where p_i are orthogonal idempotents in the ring with convolution product $(\mathtt{h}_d(E/P \times E/P), *)$ which is the endomorphism ring of the $\mathtt{h}$-motive of E/P. We approximate the ring $\mathtt{h}_d(E/P \times E/P)$ by a more accessible ring that has the same idempotent decompositions. More precisely, we construct a surjective ring homomorphism (Theorem 6.2.4)

$$\mathtt{h}_d^G(G/P \times G/P) \to \mathtt{h}_d(E/P \times E/P)$$

that lifts decompositions and isomorphisms strictly. Then we use the cellular structure on G/P to get a ring isomorphism

$$\mathtt{h}_d^G(G/P \times G/P) \cong End^0_{\mathbf{D}_F^{gr}}(\mathbf{D}_{F,P}^{gr*})$$

where $\mathbf{D}_F^{gr}$ is the graded formal affine Demazure algebra which is the graded version of the formal affine Demazure algebra introduced in [22] and $\mathbf{D}_{F,P}^{gr*}$ is a certain cyclic $\mathbf{D}_F^{gr}$-module. Note that $\mathbf{D}_F^{gr}$ and $\mathbf{D}_{F,P}^{gr*}$ can be defined in a purely combinatorial way in terms of the root lattice T^*, Weyl group W, set of simple roots S_P defining the parabolic subgroup P and formal group law F of the cohomology theory $\mathtt{h}$.

As a consequence, we deduce the main theorem 6.4.10, that states that there is a surjective homomorphism of rings that lifts decompositions and isomorphisms strictly:

$$End^0_{\mathbf{D}_F^{gr}}(\mathbf{D}_{F,P}^{gr*}) \to End(M_{\mathtt{h}}(E/P))$$

where $M_{\mathtt{h}}(E/P)$ denotes the $\mathtt{h}$-motive of E/P. This implies that there is one-to-one correspondence between motivic decompositions of E/P and direct sum decompositions of the $\mathbf{D}_F^{gr}$-module $\mathbf{D}_{F,P}^{gr*}$.

1.2 Outline of the thesis

In Chapters $2, 3, 4, 5$ we give necessary definitions and state some auxiliary results that will be used in the proof of the main theorem. The results of Chapters 6 and 7 are original to the author. The Appendix A contains a proof of a geometric lemma concering the support of the intersection product for oriented cohomology theories. The case of Chow groups was established in the paper [47]. We use the similar strategy as in [47] to prove the statement in the general case.

In Chapter 2 we review the basic concepts of the theory of algebraic groups, root systems, classification of split semisimple algebraic groups, the notion of torsor, quotients of varieties by group actions and the construction of a versal torsor. In Section 2.7 we give a definition and prove some basic properties of special parabolic subgroups. In Chapter 3 we recall the definition of oriented Borel-Moore homology theories and oriented cohomology theories. In Section 3.4 we discuss a construction of graded oriented Borel-Moore homology theory. Chapter 4 is devoted to construction of the category of $\mathtt{h}_*$-motives associated to oriented Borel-Moore homology theory $\mathtt{h}_*$. In Section 4.2 we construct the category of G-equivariant motives of smooth projective varieties with G-action. In Section 4.4 we discuss the equivariant motivic decomposition of relatively cellular spaces and the equivariant version of Künneth isomorphism. In Section 4.5 we review the cellular structures on split homogeneous varieties and their product that will be used in the proof of the main result. In Chapter 5 we recall the definition of formal affine Demazure algebra $\mathbf{D}_F$ and give a definition of graded formal affine Demazure algebra $\mathbf{D}_F^{gr}$ in Section 5.2. In Section 5.4 we define the $\mathbf{D}_F^{gr}$-modules $\mathbf{D}_{F,P}^{gr*}$ associated to a parabolic subgroup P. In Section 5.5 we construct an isomorphism between the combinatorial module $\mathbf{D}_{F,P}^{gr*}$ and graded equivariant cohomology $\mathtt{h}_T^*(G/P)$. Chapter 6 is devoted to the proof of the main result. In Section 6.1 we consider the convolution algebra associated to a smooth projective morphism and prove the Rost nilpotence theorem for morphisms of such algebras. In Section 6.2 we relate the endomorphism ring of a versal homogeneous variety motive to the endomorphism ring of G-equivariant motive of the split homogeneous variety. In Section 6.3 we construct an isomorphism between the endomorphism ring of a G-equivariant motive of G/B and the graded formal affine Demazure algebra $\mathbf{D}_F^{gr}$. In Section 6.4 we give a proof of the main result for the special parabolic subgroup P and the corresponding versal homogeneous space E/P.

In Chapter 7 we show some applications of the main result in the case $\mathtt{h}^* = \mathrm{CH}^*$. In Section 7.1 we consider the case $G = \mathbf{PGL}_{n+1}$ and $G/P = \mathbb{P}^n$. We check that $\mathbf{D}_P^{gr*}$ the module over the nil Hecke ring is indecomposable. In view of the main result this gives a purely combinatorial proof of indecomposability of Chow motives of versal Severi-Brauer varieties. Section 7.2 considers the case $G = \mathbf{HSpin}_8$ and the parabolic subgroup P_1 which corresponds to the set of all simple roots except the first one. Appendix A contains a proof of a geometric lemma needed for the proof of Rost nilpotence in 6.1.

Chapter 2

Linear algebraic groups, parabolic subgroups and homogeneous spaces

In this chapter we briefly recall the definitions of algebraic groups, torsors and homogeneous spaces. The main references are [2],[43], [20], [25], [48].

2.1 Linear algebraic groups

Definition 2.1.1. *A linear algebraic group over k is an affine variety G with a rational point $1 \in G(k)$ and two regular maps:*

- *Multiplication $m\colon G \times G \to G, (x, y) \mapsto xy$*
- *Inverse map $i\colon G \to G, x \mapsto x^{-1}$*

that satisfy the usual group properties: $x(yz) = (xy)z$ (associativity), $x \cdot 1 = 1 \cdot x = x$ and $xx^{-1} = x^{-1}x = 1$.

For every k-algebra R, multiplication and inverse maps endow the set of R-points $G(R)$ with a group structure. Thus every linear algebraic group defines the functor from the category of k-algebras to the category of groups given by

$$k - \mathbf{Alg} \to \mathbf{Groups}, R \mapsto G(R)$$

called the functor of points. A linear algebraic group is uniquely defined by its functor of points. For any $S \in \mathbf{Sm}_k$ we will denote by $G(S)$ the group of maps $S \to G$ in $\mathbf{Sm}_k$.

Definition 2.1.2. *A homomorphism $f\colon G_1 \to G_2$ of linear algebraic groups is a regular map f that commutes with multiplication maps.*

If G_1 and G_2 are linear algebraic groups then the product $G_1 \times G_2$ has a structure of a linear algebraic group.

Example 2.1.3. The simplest examples are

- Multiplicative group $\mathbb{G}_m$ such that $\mathbb{G}_m(R) = R^\times$ and $m(x, y) = xy$
- Additive group $\mathbb{G}_a$ such that $\mathbb{G}_a(R) = R$ and $m(x, y) = x + y$
- General linear group GL_n such that $GL_n(R) = \{A \in Mat_n(R) \mid \det(A) \in R^\times\}$
- Special linear group SL_n such that $SL_n(R) = \{A \in Mat_n(R) \mid \det(A) = 1\}$

Definition 2.1.4. *A linear algebraic group G is called a split k-torus of rank n if it is isomorphic over k to the group $(\mathbb{G}_m)^n$*

Definition 2.1.5. *A linear algebraic group G is called a torus if over the algebraic closure $\overline{k}$ the group $G \times_k \overline{k}$ is a split $\overline{k}$-torus.*

Definition 2.1.6. *We will call H a closed subgroup of G if H is a closed subvariety of G and H is closed under the multiplication and the inverse map.*

Lemma 2.1.7. *[2, Proposition 1.10] Every linear algebraic group G is a closed subgroup of a general linear group $\mathbf{GL}_n$ for some n.*

Definition 2.1.8. *A linear algebraic group G is called:*

- *Connected, if G is irreducible as a variety.*
- *Simple, if G is connected, nontrivial and $G_{\overline{k}}$ has no nontrivial connected normal subgroups.*
- *Semisimple if G is connected, nontrivial and $G_{\overline{k}}$ has no nontrivial connected solvable normal subgroups.*

Let G be a linear algebraic group and $T \subseteq G$ is a torus in G. We will T a maximal torus if it is not contained in a larger torus inside G. Theorem 18.2 of [2] implies that for a connected group G there is a maximal torus $T \subseteq G$ such that $T_{\overline{k}}$ is a maximal torus in $G_{\overline{k}}$.

Definition 2.1.9. *Linear algebraic group G is called split if it has a maximal torus which is split over k.*

Remark 2.1.10. If G is a split semisimple algebraic group, then all split maximal tori are conjugate.

2.2 Root systems

In this section we recall the combinatorial description of semisimple split groups. Consider an $\mathbb{R}$-vector space V with a Euclidean scalar product $(-,-)$.

Definition 2.2.1. *For any $\alpha \in V$ let s_α denote the orthogonal reflection with respect to α :*

$$s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha$$

Thus $s_\alpha(\alpha) = -\alpha$ and s_α acts trivially on the hyperplane orthogonal to α.

Definition 2.2.2. *A finite set $\Phi \subseteq V$ of nonzero vectors is called a root system if*

- *Φ spans V*
- *For any $\alpha \in \Phi$, s_α leaves Φ invariant*
- *For any $\alpha, \beta \in \Phi$, $\beta - s_\alpha(\beta)$ is an integer multiple of α..*

Definition 2.2.3. *A root system Φ is called reduced if for every $\alpha \in \Phi$ the vectors α and $-\alpha$ are the only multiples of α in Φ.*

Further we will consider only reduced root systems.

Definition 2.2.4. *The Weyl group $W(\Phi)$ of a root system Φ is the group generated by reflections s_α in $Aut(V)$:*

$$W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$$

Definition 2.2.5. *A subset Π of Φ is called a base of a root system if Π is a linearly independent set and for any $\beta \in \Phi$ there is a decomposition*

$$\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha$$

such that $m_\alpha \in \mathbb{Z}$ and either $m_\alpha \geqslant 0$ for all $\alpha \in \Pi$ or $m_\alpha \leqslant 0$ for all $\alpha \in \Pi$.

Lemma 2.2.6. *[41, V. Theorem 1] For any root system Φ there is a base $\Pi \subseteq \Phi$.*

Definition 2.2.7. *For $\alpha, \beta \in \Phi$ define*

$$n(\beta,\alpha) = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$$

By definition of a root system, $n(\beta,\alpha)$ is an integer.

By [41, V,§7], the angle between two nonproportional roots α, β can take value in the set $\{\pi/2, \pi/3, 2\pi/3, \pi/4, 3\pi/4, \pi/6, 5\pi/6\}$.

Fix a base Π of Φ.

Definition 2.2.8. *The Dynkin diagram $D(\Phi)$ is a graph which set of vertices equals to Π. Depending on the angle ϕ between $\alpha, \beta \in \Pi$, it has the following number of edges between α and β*

- 1, *if $\phi = 2\pi/3$*
- 2, *if $\phi = 3\pi/4$*
- 3, *if $\phi = 5\pi/6$*
- 0, *otherwise*

If $(\alpha, \alpha) \neq (\beta, \beta)$, then the edges between α and β are oriented towards α in the case $(\alpha, \alpha) < (\beta, \beta)$, and towards β in the case $(\beta, \beta) < (\alpha, \alpha)$.

The diagram $D(\Phi)$ does not depend on the choice of Π and completely determines Φ by [41, V,§15]

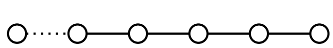
Lemma 2.2.9. *[41, V,§10, Remark 3] The Weyl group $W(\Phi)$ has the following presentation in terms of generators and relations:*

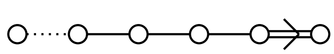
$$W(\Phi) = \langle s_\alpha \mid, s_\alpha^2 = 1, (s_\alpha s_\beta)^{m_{\alpha,\beta}} = 1 \rangle$$

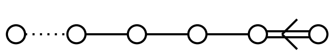
where $\alpha, \beta \in \Pi$ and $m_{\alpha,\beta} = 2, 3, 4, 6$ if the angle between α and β is $\pi/2, 2\pi/3, 3\pi/4, 5\pi/6$ respectively.

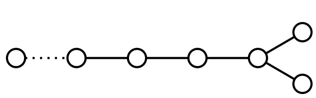
Definition 2.2.10. *A root system Φ is called irreducible if it cannot be partitioned into two proper orthogonal subsets.*

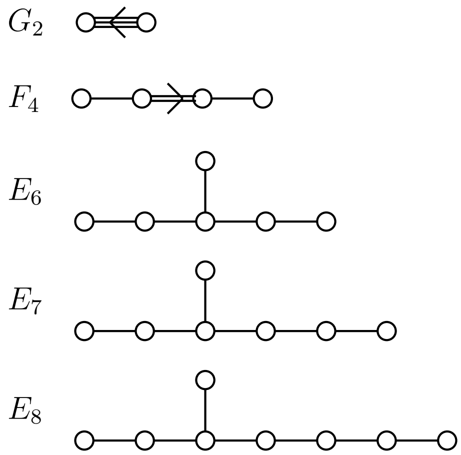
Every root system is an orthogonal union of irreducible root systems, and the Dynkin diagram of any irreducible root system belongs to the following list ([41, V,§15]):

A_n  n vertices, $n \geqslant 1$

B_n  n vertices, $n \geqslant 2$

C_n  n vertices, $n \geqslant 3$

D_n  n vertices, $n \geqslant 4$



2.3 Classification of semisimple split groups

In this section we recall the classification of semisimple split algebraic groups in terms of root systems and character lattices. The main reference for this section is [25, §25]. Let $\Phi \subseteq V$ be a root system. For every $\alpha \in \Phi$ consider the element $\alpha^\vee \in V^*$ defined by

$$\alpha^\vee(v) = 2\frac{(v, \alpha)}{(\alpha, \alpha)}$$

Lemma 2.3.1. *[41, V,Proposition 2] The set $\{\alpha^\vee \mid \alpha \in \Phi\}$ forms a root system $\Phi^\vee$ in V^* called the dual root system.*

Definition 2.3.2. *Define the root lattice $\Lambda_r \subseteq V$ as the $\mathbb{Z}$-span of Φ.*

Definition 2.3.3. *Define the weight lattice $\Lambda_w \subseteq V$ as the lattice dual to the root lattice of $\Phi^\vee$*

$$\Lambda_w = \{v \in V \mid \forall \alpha \in \Phi, \alpha^\vee(v) \in \mathbb{Z}\}$$

Then $\Lambda_r \subseteq \Lambda_w$ and the quotient is a finite abelian group ([20, §0.6]).

2.3.1 Adjoint representation

Definition 2.3.4. *Denote by $L(G)$ the tangent space T_1G of G at the point $1 \in G(k)$. It has a natural structure of a Lie algebra and is called the Lie algebra of G.*

There is a representation $Ad\colon G \to GL(L(G))$ called the adjoint representation ([2, 3.13]) such that for a field extension F/k for any $g \in G(F)$ its image $Ad(g)$ is the differential of the inner automorphism

$$Inn(g)\colon G_F \to G_F, x \mapsto gxg^{-1}$$

2.3.2 Root system of a split semisimple group.

Let G be a split semisimple algebraic group, and $T \subseteq G$ be a split maximal torus. Denote by T^* the character group of T:

$$T^* = Hom(T, \mathbb{G}_m)$$

Since T is a split torus, the group T^* is a free abelian group of the same rank as T.

Consider the restriction of the adjoint representation to T. For any character $\alpha \in T^*$ denote by $L(G)_\alpha$ the eigenspace corresponding to α :

$$L(G)_\alpha = \{v \in L(G) \mid Ad(t)v = \alpha(t)v \text{ for all } t \in T\}$$

Definition 2.3.5. *Define the set of roots to be the characters of T whose eigenspaces are non-trivial*

$$\Phi(G) = \{\alpha \in T^* \mid \alpha \neq 0, L(G)_\alpha \neq 0\}.$$

Definition 2.3.6. *The Weyl group of T in G, $W = W(G,T)$, is defined as the quotient of the normalizer of T in G by T:*

$$W = N_G(T)/T$$

Then W is a constant finite group scheme [2, §11.19].

Remark 2.3.7. Note that $N_G(T)$ acts by conjugation on T. Since T is abelian, this action descends to an action of W on T. Therefore W acts naturally on the lattice T^*.

Consider the space $V = T^* \otimes_{\mathbb{Z}} \mathbb{R}$. Take any scalar product $(-,-)'$ on V. Then one can define a W-invariant scalar product $(-,-)$ on V by the formula

$$(x, y) = \sum_{w \in W} (w(x), w(y))'$$

Lemma 2.3.8. *[25, Theorem 25.1, Proposition 25.2] For a semisimple split group G and a split maximal torus $T \subseteq G$, the set $\Phi(G)$ is a root system in the Euclidean space $V = T^* \otimes_{\mathbb{Z}} \mathbb{R}$ with the scalar product $(-,-)$, and the Weyl group W of G coincides with the Weyl group of the root system: $W = W(\Phi(G))$, and there are inclusions:*

$$\Lambda_r \subseteq T^* \subseteq \Lambda_w$$

Consider a pair (Φ, Λ), where Φ is a root system in V, and Λ is a lattice such that $\Lambda_r \subseteq \Lambda \subseteq \Lambda_w$. We will say that two such pairs are isomorphic: $(\Phi, \Lambda) \cong (\Phi', \Lambda')$ if there is a linear isomorphism $f : V \to V'$ such that $f(\Phi) = \Phi'$ and $f(\Lambda) = \Lambda'$.

Theorem 2.3.9. *(Classification of semisimple split groups)[25, Theorem 25.3, 25.5] For any root system pair* (Φ, Λ) *such that* $\Lambda_r \subseteq \Lambda \subseteq \Lambda_w$*, there is a semisimple split algebraic group* G *such that* $(\Phi(G), T^*) \cong (\Phi, \Lambda)$*. Two semisimple split groups* G_1, G_2 *are isomorphic if and only if* $(\Phi(G_1), T_1^*) \cong (\Phi(G_2), T_2^*)$*.*

Remark 2.3.10. Irreducible root systems correspond to split simple groups via the identification above ([25, Proposition 25.8]).

Definition 2.3.11. *We will call a semisimple split group* G *simply-connected if* $T^* = \Lambda_w$*, and adjoint if* $T^* = \Lambda_r$*.*

Definition 2.3.12. *We call a surjective morphism of linear algebraic groups* $G_1 \to G_2$ *an isogeny if the kernel of* $G_1(F) \to G_2(F)$ *is finite for every field extension* F/k*. Two groups* G_1, G_2 *are called isogeneous if there is a linear algebraic group* H *and isogenies* $H \to G_1$ *and* $H \to G_2$*.*

The classification theorem implies that G_1 and G_2 are isogeneous iff the root systems $\Phi(G_1)$ and $\Phi(G_2)$ are isomorphic. The classification of irreducible root systems implies that isogeny classes of simple split groups are in $1-1$ correspondence with Dynkin diagrams of types $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$.

2.4 Group actions and torsors

In this section we recall the definition and basic properties of torsors. The main reference is [34].

Definition 2.4.1. *For* $X \in \mathbf{Sch}_k$*, a right* G*-action on* X *is a regular map* $m\colon X \times G \to X$ *such that for every* $S \in \mathbf{Sm}_k$ *the map* m *induces a right* $G(S)$*-group action on the set* $X(S)$*. For* $x \in X, g \in G$ *we will write* xg *or* $x \cdot g$ *instead of* $m(x, g)$*.*

Further we will consider only right actions.

Definition 2.4.2. *Suppose* $X, Y \in \mathbf{Sch}_k$ *are schemes with* G*-actions. A morphism* $f\colon X \to Y$ *is called* G*-equivariant if it respects the* G*-action on* X *and* Y*, i.e. the following diagram commutes*

$$\begin{array}{ccc} X \times G & \xrightarrow{id_G \times f} & Y \times G \\ \downarrow{\scriptstyle m_X} & & \downarrow{\scriptstyle m_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

where m_X *defines the* G*-action on* X *and* m_Y *defines the* G*-action on* Y*.*

Definition 2.4.3. *Denote by $G-\mathbf{Sch}_k$ the category consisting of schemes $X \in \mathbf{Sch}_k$ with G-action and equivariant morphisms. Denote by $G-\mathbf{Sm}_k$ its full subcategory consisting of smooth schemes over k.*

Definition 2.4.4. *Let $p\colon Y \to X$ be a morphism in $G-\mathbf{Sch}_k$ such that p is faithfully flat and the action of G on X is trivial. $Y \to X$ is called a G-torsor if the map*

$$Y \times G \to Y \times_X Y, (y, g) \mapsto (y, yg)$$

is an isomorphism.

Example 2.4.5. Let A be a central simple algebra of degree n over k. Then the functor

$$R \mapsto Iso(A \otimes_k R, Mat_n(R))$$

which assigns to every k-algebra R the set of isomorphisms of algebras between $A \otimes_k R$ and the matrix algebra $Mat_n(R)$ is represented by a scheme which has a natural structure of a $\mathbf{PGL}_n$-torsor over k.

Definition 2.4.6. *For G-torsors $Y_1 \to X_1$ and $Y_2 \to X_2$, a morphism of torsors is a commutative diagram*

$$\begin{array}{ccc} Y_1 & \xrightarrow{F} & Y_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

where F is G-equivariant.

Definition 2.4.7. *A torsor $Y \to X$ is called trivial if it is isomorphic to the torsor $X \times G \to X$.*

If $Y \to X$ is a G-torsor and $f\colon X' \to X$ is a morphism, then the pullback $Y \times_X X' \to X'$ is a G-torsor which we will denote by f^*Y.

Remark 2.4.8. (a) A torsor $p\colon Y \to X$ is trivial if and only if there is a section $s\colon X \to Y$ of p.

(b) If $Y, Y' \to X$ are two G-torsors, and $f\colon Y \to Y'$ is a morphism, then f is an isomorphism.

(c) Any morphism of torsors $(F, f)\colon (Y \to X) \to (Y' \to X')$ establishes an isomorphism $Y \to f^*Y'$.

Proof: A section $s\colon X \to Y$ gives rise to a map of torsors $a\colon X \times G \to Y, (x, g) \mapsto s(x)g$. The base change of a via $Y \times_X Y$ gives the map $Y \times G \to Y \times_X Y$, which is an isomorphism. Thus a is an isomorphism by faithfully flat descent. If $f\colon Y \to Y'$ is a morphism of torsors over the same base X, then the torsor $Y \times_X Y' \to Y$ has a section, hence is trivial by (a). Thus the base change of $f\colon Y \to Y'$ along $Y \times_X Y' \to Y'$ is an isomorphism $G \times Y \to Y \times_X Y'$ thus f is an isomorphism by the faithfully flat descent. Any morphism of torsors $(F, f)\colon (Y \to X) \to (Y' \to X')$ gives a morphism of torsors $Y \to f^*Y'$, which is an isomorphism by (b). ∎

Definition 2.4.9. *Let τ be a Grothendieck topology. We will say that a torsor $Y \to X$ is τ-locally trivial if there is a τ-covering $f\colon X' \to X$ such that f^*Y is trivial.*

Lemma 2.4.10. *[34, Proposition I.3.26,III.4.2] Every G-torsor $Y \to X$ is locally trivial in the étale topology.*

Definition 2.4.11. *Denote by $H^1(X, G)$ the set of isomorphism classes of G-torsors $Y \to X$. For a Grothendieck topology τ denote by $H^1_\tau(X, G)$ a subset of $H^1(X, G)$ consisting of G-torsors that are locally trivial for τ.*

We will consider $H^1(X, G)$ as a pointed set with distinguished point given by the class of the trivial torsor.

Proposition 2.4.12. *[34, Proposition III.4.5, Corollary III.4.7]. Suppose that $1 \to G_1 \to G_2 \to G_3 \to 1$ is a short exact sequence of algebraic groups over k, and $X \in \mathbf{Sm}_k$. Then there is an exact sequence of pointed sets*

$$1 \to G_1(X) \to G_2(X) \to G_3(X) \to H^1(X, G_1) \to H^1(X, G_2) \to H^1(X, G_3)$$

2.4.1 Free action and quotients

Definition 2.4.13. *An action of an abstract group A on a set B is called free if the stabilizer of any point in B is trivial.*

Definition 2.4.14. *[35] Suppose that G is an algebraic group, and $X \in G - \mathbf{Sch}_k$. A G-action on X is called free if the map*

$$X \times G \to X \times X, (g, x) \mapsto (xg, x)$$

is a closed embedding.

Note that if G acts on X freely, then for any $U \in \mathbf{Sm}_k$ the action of $G(U)$ on the set $X(U)$ is free.

Definition 2.4.15. *For a G-action on $X \in \mathbf{Sch}_k$ define the quotient X/G as the étale sheaf associated to the presheaf*

$$U \mapsto X(U)/G(U), U \in \mathbf{Sm}_k$$

We will say that the quotient X/G exists in the category $\mathbf{Sm}_k$ ($\mathbf{Sch}_k$) if there is $Y \in \mathbf{Sm}_k$ ($\mathbf{Sch}_k$) with an isomorphism of étale sheaves:

$$X/G \cong Hom_{\mathbf{Sch}_k}(-, Y).$$

Lemma 2.4.16. *Suppose that $X \in G - \mathbf{Sm}_k$, the G-action on X is free and the quotient X/G exists in $\mathbf{Sm}_k$. Then $X \to X/G$ is a G-torsor.*

Proof: Denote by $Y \in \mathbf{Sm}_k$ the scheme representing X/G. Consider the regular map $X \times G \to X \times_Y X$. For a local strictly Henselian scheme W the map $X(W) \times G(W) \to X(W) \times_{Y(W)} X(W)$ is an isomorphism since $Y(W) = X(W)/G(W)$ and the action of the abstract group $G(W)$ on the set $X(W)$ is free. Then the map of sheaves $X \times G \to X \times_Y X$ is an isomorphism, hence $X \to Y$ is a G-torsor. ∎

Remark 2.4.17. In the case when G is a linear algebraic group and H is a closed subgroup, the quotient G/H exists in $\mathbf{Sch}_k$ by [48, p.121-122].

2.4.2 Versal torsor

In this section we recall the notion of a versal torsor. The main reference is [17].

Definition 2.4.18. *Suppose there is a G-representation V and an open subset $W \subseteq V$ such that G acts on W freely and the quotient W/G exists in $\mathbf{Sm}_k$. Consider the generic fiber E of $W \to W/G$:*

$$\begin{array}{ccc} E & \longrightarrow & W \\ \downarrow & & \downarrow \\ \operatorname{Spec} k(W/G) & \longrightarrow & W/G \end{array}$$

Let $K = k(W/G)$. We will call $E \to K$ a versal torsor.

Note that our definition depends on a choice of a G-representation V and an open subset W.

Lemma 2.4.19. *[17, I.5.3] Versal torsors exist and any versal torsor has the following classifying property: for any infinite field L/k, $Y \in H^1(L, G)$, and any open subset $U \subseteq W/G$ there is an L-point $x\colon \operatorname{Spec} L \to U$ such that x^*U is isomorphic to Y as a G-torsor. Thus any torsor over an infinite field can be obtained as a specialization of a given versal torsor.*

2.5 Borel subgroups and Bruhat decomposition

Definition 2.5.1. *If H_1, H_2 are closed subgroups of G, then there is a closed subgroup $(H_1, H_2) \subseteq G$ ([2, 2.3]) called the commutator subgroup such that the $\overline{k}$-points $(H_1, H_2)(\overline{k})$ are generated by commutator elements $xyx^{-1}y^{-1}, x \in H_1(\overline{k}), y \in H_2(\overline{k})$*

Definition 2.5.2. *An algebraic group H is called solvable if its derived series $D^0(H) = H, D^1(H) = (D^0(H), D^0(H)), D^2(H) = (D^1(H), D^1(H)), \dots$ vanishes, i.e. there is n such that $D^n(G) = 1$.*

Definition 2.5.3. *A Borel subgroup of G is a maximal closed connected solvable subgroup of G.*

If G is a split semisimple algebraic group then all Borel subgroups are $G(k)$-conjugate. For every split maximal torus T there is a Borel subgroup B such that $T \subseteq B \subseteq G$ ([2, §14,21]). If B is a Borel subgroup of G then G/B is a smooth projective variety by [2, 11.18].

Proposition 2.5.4. *(Bruhat decomposition, [2, §14,21]) If G is a split semisimple algebraic group, T its split maximal torus, B a Borel subgroup, and $W = N_G(T)/T$ is the Weyl group, then the group G is a disjoint union of double-B cosets and G/B is a disjoint union of B-orbits of points of $W \subseteq G/B$:*

$$G = \coprod_{w \in W} BwB$$

$$G/B = \coprod_{w \in W} BwB/B$$

Remark 2.5.5. Suppose that G is a split semisimple algebraic group, T is its split maximal torus, and B is a Borel subgroup containing T. Then B is a semi-direct product of its unipotent radical U and T. Since U is isomorphic to an affine space by [2, 15.13], we get that the quotient B/T is isomorphic to an affine space.

2.6 Parabolic subgroups and split homogeneous varieties

Definition 2.6.1. *A closed subgroup $P \subseteq G$ is called parabolic if the quotient variety G/P is projective. Equivalently, a subgroup P is parabolic if and only if P contains a Borel subgroup B ([2, §11]).*

Fix a split maximal torus, Borel subgroup $T \subseteq B \subseteq G$ and the corresponding root system. Fix a set of simple roots Π and a subset $S \subseteq \Pi$. Let W_S denote the subgroup of W generated by simple reflections $s_\alpha, \alpha \in S$. Then there is a parabolic subgroup P_S called standard parabolic subgroup such that $P_S = BW_SB$ and any parabolic subgroup is conjugate to P_S for some S ([20, §29]). If $P = P_S$ is a standard parabolic we will use the notation W_P for the group W_S.

Given a set of simple roots Π, for every w, a reduced word for w is a minimal length expression of w as a product of simple reflections $s_\alpha, \alpha \in \Pi$. The length $l(w)$ is defined as the length of a reduced word. There is a partial order on W called the Bruhat order in which $v \leqslant w$ if and only if for a reduced word $w = s_{\alpha_1} \dots s_{\alpha_k}$ there is a sequence $1 \leqslant i_1 < i_2 \dots < i_m \leqslant k$ such that $v = s_{\alpha_{i_1}} \dots s_{\alpha_{i_m}}$ is a reduced word for v. Any class in W/W_P has a unique representative of minimal length in W. Denote the set of minimal length representatives by W^P.

2.7 Special parabolic subgroups

Definition 2.7.1. *Following [17, II, §3] we will call a linear algebraic group G special if $H^1(L, G)$ is trivial for any field extension L/k.*

Sometimes a special group is defined as a group whose torsors are Zariski-locally trivial. We show that these notions coincide for a linear algebraic group G.

Lemma 2.7.2. *Let G be a linear algebraic group. If G is special then any G-torsor is Zariski - locally trivial.*

Proof: Fix an embedding $G \to \mathbf{GL}_m$ for some m. By Hilbert 90 theorem every torsor in $H^1(S, \mathbf{GL}_m)$ is Zariski-locally trivial. Since G is special, the G-torsor $\mathbf{GL}_m \to \mathbf{GL}_m/G$ is trivial over the generic point of $\mathbf{GL}_m/G$ hence over some open subset of $\mathbf{GL}_m/G$ by [18, Corollary 8.8.2.5]. Then $\mathbf{GL}_m \to \mathbf{GL}_m/G$ is Zariski-locally trivial since $\mathbf{GL}_m$ acts transitively on $\mathbf{GL}_m/G$. Then for any local scheme S the map $\mathbf{GL}_m(S) \to \mathbf{GL}_m/G(S)$ is surjective, so in the exact sequence

$$\mathbf{GL}_m(S) \to \mathbf{GL}_m/G(S) \to H^1(S, G) \to H^1(S, \mathbf{GL}_m)$$

the rightmost arrow has trivial kernel, then $H^1(S, G)$ is trivial. ∎

Example 2.7.3. The groups $\mathbf{GL}_n$, and $\mathbb{G}_a$ are special by [17, II.3.1] and [42, X.1.1]

As a consequence, a Borel subgroup B of any semisimple split linear algebraic group G is special.

Lemma 2.7.4. *Let G be a semisimple split group over k, and $P_1 \subseteq P_2$ be a standard parabolic subgroups. Then for any field extension L/k the map $P_2(L) \to P_2/P_1(L)$ is surjective.*

Proof: By the Bruhat decomposition we have $P_2/P_1 = \coprod_{\overline{w} \in W_{P_2}/W_{P_1}} C(\overline{w})$, where $C(\overline{w}) = B\overline{w}B/B$ is the B orbit of $\overline{w}$. If $w \in W^P$, then $C(w) \to C(\overline{w})$ is an isomorphism, where $C(w)$ is the corresponding B-orbit in P_2/B ([2, 21.29]). Then for any field extension L/k the map $P_2/B(L) \to P_2/P_1(L)$ is surjective, and the map $P_2(L) \to P_2/B(L)$ is surjective since B is special. ∎

Lemma 2.7.5. *Let $P_1 \subseteq P_2$ be two parabolic subgroups of a semisimple split algebraic group G. If P_2 is special, then P_1 is special.*

Proof: For every field extension L/k consider the cohomological exact sequence of pointed sets

$$P_2(L) \to P_2/P_1(L) \to H^1(L, P_1) \to H^1(L, P_2).$$

By Lemma 2.7.4 the leftmost arrow is surjective, hence the rightmost arrow has a trivial kernel. Since $H^1(L, P_2)$ is trivial, then $H^1(L, P_1)$ is trivial, thus P_1 is special. ∎

Chapter 3

Oriented cohomology theories

In this chapter we recall the definition and main properties of oriented cohomology theories. Such theories were studied in particular by Panin and Smirnov ([37],[36]), Levine and Morel ([31],[32]). We will use [32] as a reference.

3.1 Oriented Borel-Moore homology

Let $\mathbf{Sch}'_k$ denote the category consisting of schemes in $\mathbf{Sch}_k$ and projective morphisms. Let $\mathbf{Ab}_*$ denote the category of graded Abelian groups.

Definition 3.1.1. *[32, Definition 5.1.3] An oriented Borel-Moore homology theory $\mathtt{h}_*$ on $\mathbf{Sch}_k$ is given by*

- *an additive functor $\mathtt{h}_*\colon \mathbf{Sch}'_k \to \mathbf{Ab}_*$*
- *for each locally complete intersection morphism $f\colon X \to Y$ in $\mathbf{Sch}_k$ ([15, B.7]) of relative dimension d a homomorphism $f^*\colon \mathtt{h}_*(Y) \to \mathtt{h}_{*+d}(X)$*
- *bilinear external product $\mathtt{h}_*(X) \otimes \mathtt{h}_*(Y) \to \mathtt{h}_*(X \times Y)$ that is associative, commutative and has a unit element $1 \in \mathtt{h}_*(k)$*

The additive functor assigns to each projective morphism $f\colon X \to Y$ a map $f_*\colon \mathtt{h}_*(X) \to \mathtt{h}_*(Y)$ called the push-forward map. These data satisfy axioms $(BM1) - BM(3)$, (PB), (EH), (CD) of [32, Definition 5.1.3]

The axiom $(BM1)$ states that $(f \circ g)^* = g^* \circ f^*$ if f, g is a pair of composable locally complete intersection morphisms. The axiom $(BM2)$ states that if $f\colon X \to Z$ and $g\colon Y \to Z$ are transverse morphisms, f is projective, and g is a locally complete intersection morphism giving the transverse square

$$\begin{array}{ccc} X\times_Z Y & \xrightarrow{g'} & X \\ \downarrow{f'} & & \downarrow{f} \\ Y & \xrightarrow{g} & Z \end{array}$$

then $g^*f_* = f'_*g'^*$. The axiom (EH) implies that for a locally trivial map $f\colon X \to Y$, which fibers are isomorphic to affine spaces $\mathbb{A}^r$, the pullback map $f^*\colon \mathtt{h}_*(Y) \to \mathtt{h}_{*+r}(X)$ is an isomorphism.

Example 3.1.2. The Chow groups give an example of an oriented Borel-Moore homology $X \mapsto \mathrm{CH}_*(X)$, $X \in \mathbf{Sch}_k$ by [32, Example. 5.1.5].

Example 3.1.3. For any $X \in \mathbf{Sch}_k$ consider the theory of coherent sheaves $K'_0(X)$. Consider a graded abelian group $K'_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$ where elements of $K_0(X)$ have degree 0 and $\deg \beta = -1$. Then the functor $X \mapsto K'_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$ defines an oriented Borel-Moore homology by [32, Example. 5.1.6].

Levine and Morel constructed in [32] the algebraic cobordism theory $X \mapsto \Omega_*(X)$ on $\mathbf{Sch}_k$. When the base field k admits resolution of singularitites, this theory is universal ([32, Theorem 7.1.3]) in the following sence:

For any oriented Borel-Moore homology theory $\mathtt{h}_*$ there is a unique homomorphism

$$\Theta\colon \Omega_*(-) \to \mathtt{h}_*(-)$$

of Borel-Moore theories, that is, for any $X \in \mathbf{Sch}_k$ there is a homomorphism $\Theta_X\colon \Omega_*(X) \to \mathtt{h}_*(X)$. This homomorphism is compatible with external product and commutes with push-forwards and pull-back maps:

- $f_* \circ \Theta_X = \Theta_Y \circ f_*$ for any projective morphism $f\colon X \to Y$
- $g^* \circ \Theta_Y = \Theta_Y \circ g^*$ for any locally complete intersection $g\colon X \to Y$

Definition 3.1.4. *[32, Definition 4.4.6] An oriented Borel-Moore homology theory $\mathtt{h}_*$ has the localization property if for any closed embedding $i\colon Z \to X$ with $j\colon U \to X$ its open complement the sequence*

$$\mathtt{h}_*(Z) \xrightarrow{i_*} \mathtt{h}_*(X) \xrightarrow{j^*} \mathtt{h}_*(U) \to 0$$

is exact.

Definition 3.1.5. *For a finitely generated field extension F/k denote by $\mathtt{h}_*(F/k)$ the colimit of $\mathtt{h}_*(X)$ where X runs over the category of models for F ([32, §4.4.1]). In particular, if $F = k(X)$, then $\mathtt{h}_*(k(X)/k) = colim_{U\subseteq X}\mathtt{h}_*(U)$, where U are open subsets of X.*

Definition 3.1.6. *[32, Definition 4.4.1] An oriented Borel-Moore homology theory* $\mathtt{h}_*$ *is generically constant if for every finitely generated separable field extension* F/k *the induced homomorphism* $\mathtt{h}_*(k) \to \mathtt{h}_*(F/k)$ *is an isomorphism.*

Example 3.1.7. Note that $h_* = \mathrm{CH}_*, K_*^0$ or Ω_* give examples of such theories ([32, §4.4.1]).

3.1.1 Degree formula

We call $W \in \mathbf{Sch}_k$ a locally complete intersection scheme if the structure morphism $p\colon W \to \operatorname{Spec} k$ is a locally complete intersection morphism ([15, B.7]). For such a scheme there is a notion of fundamental class $[1_W] \in \mathtt{h}_{\dim(W)}(W)$ given by $p^*(1)$.

Definition 3.1.8. *Let* W *be locally complete intersection scheme and* $f\colon W \to X$ *be a projective morphism in* $\mathbf{Sch}_k$*. Following [32], we will denote by* $[W \to X]$ *the element* $f_*([1_W]) \in \mathtt{h}_*(X)$*.*

For $X \in \mathbf{Sch}_k$ and a closed integral subscheme $Z \subseteq X$ consider a resolution of singularities $f\colon \tilde{Z} \to Z \to X$.

Proposition 3.1.9. *(Generalized degree formula,[32, Theorem 4.4.7]) Suppose* $\mathtt{h}_*$ *is a generically constant oriented Borel-Moore homology that has the localization property, and* $X \in \mathbf{Sch}_k$ *is an irreducible scheme. Then the* $\mathtt{h}_*(k)$*-module* $\mathtt{h}_*(X)$ *is generated by elements of the form* $[\tilde{Z} \to X]$*, where* Z *are proper closed integral subschemes of* X*, and* $\tilde{Z}$ *are given desingularizations of* Z*.*

3.2 Oriented cohomology theory

Let $\mathbf{Ring}^*$ denote the category of graded rings. A functor $F\colon \mathbf{Sm}_k^{op} \to \mathbf{Ring}^*$ is called additive if $F(X \coprod Y) \cong F(X) \times F(Y)$ and $F(\emptyset) = 0$.

Definition 3.2.1. *[32, Definition 1.1.2] An oriented cohomology theory* $\mathtt{h}^*$ *on* $\mathbf{Sm}_k$ *is given by*

- *an additive functor* $\mathtt{h}^*\colon \mathbf{Sm}_k^{op} \to \mathbf{Ring}^*$*. For any* $f\colon X \to Y \in \mathbf{Sm}_k$ *we will denote the corresponding ring homomorphism as* $f^*\colon \mathtt{h}^*(Y) \to \mathtt{h}^*(X)$ *and call it the pull-back homomorphism.*

- *For any projective morphism* $f\colon X \to Y$ *of relative codimension* d*, a homomorphism of graded* $\mathtt{h}^*(X)$*-modules*

  $$f_*\colon \mathtt{h}^*(Y) \to \mathtt{h}^{*+d}(X)$$

  that is called the push-forward map.

These data satisfy the axioms $(A1), (A2), (PB), (EH)$ *of [32, Definition 1.1.2]. The fact that* f_* *is a* $\mathtt{h}^*(X)$ *implies that* $f_*(f^*(x)y) = xf_*(y)$ *which is called the projection formula.*

Often we will denote the product in the ring $\mathtt{h}^*(X)$ by $ab = a \cap b$. This is motivated by the case $\mathtt{h}^* = \mathrm{CH}^*$ where multiplication is given by the intersection product in Chow groups.

Consider an oriented Borel-Moore homology theory $\mathtt{h}_*$ on $\mathbf{Sch}_k$ and its restriction to $\mathbf{Sm}_k$. For any $X \in \mathbf{Sm}_k$ introduce the cohomological grading $\mathtt{h}^*(X) = \mathtt{h}_{\dim(X)-*}(X)$. For any $X \in \mathbf{Sm}_k$ the diagonal embedding $\Delta \colon X \to X \times X$ is an l.c.i morphism, thus one can introduce the ring structure on $\mathtt{h}^*(X)$ via the composition

$$\mathtt{h}^*(X) \otimes \mathtt{h}^*(X) \to \mathtt{h}^*(X \times X) \xrightarrow{\Delta^*} \mathtt{h}^*(X)$$

Then $X \mapsto \mathtt{h}^*(X)$ defines an oriented cohomology theory on $\mathbf{Sm}_k$ ([32, Proposition 5.2.1]).

3.2.1 Formal group laws

In this section we recall basic facts about formal group laws. The main reference is [32, p.4]. Let R be a commutative ring. A formal group law over R is a power series $F \in R[[u, v]]$([32, p. 4]) such that

- $F(u, 0) = F(0, u) = u$
- $F(u, v) = F(v, u)$
- $F(u, F(v, w)) = F(F(u, v), w)$

Often we will use the notation $u +_F v$ for $F(u, v)$. In the rest of the paper we will use this notion when $R = R^*$ is a graded ring and F is homogenous, i.e. in the decomposition $F(u, v) = \sum_{(i,j)} a_{i,j} u^i v^j$ the elements $a_{i,j}$ are homogeneous of degree $1 - i - j$.

Example 3.2.2. The basic examples of formal group laws include the additive formal group law $F_a(u, v) = u + v$ over the ring $R = \mathbb{Z}$ and multiplicative formal group law $F_m(u, v) = u + v - \beta uv$ over the graded ring $\mathbb{Z}[\beta, \beta^{-1}]$ where $\deg(\beta) = -1$.

There is a graded ring $\mathbb{L}$ called Lazard ring and a formal group law F_U over the Lazard ring that is universal: for any formal group law F over R there is a unique ring homomorphism $g \colon \mathbb{L} \to R$ such that $F(u, v) = g(F_U(u, v))$ in $R[[u, v]]$. The ring $\mathbb{L}$ is constructed as $\mathbb{Z}[x_{i,j} \mid i, j \geqslant 0]/I$, where the ideal I is generated by the relations imposed on the coefficients by the properties $F(0, u) = F(u, 0) = 0, F(u, v) = F(v, u), F(F(u, v, w)) = F(u, F(v, w))$ of the power series $F(u, v) =$

$\sum x_{i,j}u^iv^j$. For any formal group law $F = \sum a_{i,j}u^iv^j$ over R the homomorphism $\mathbb{L}^* \to R$ is given by $x_{i,j} + I \mapsto a_{i,j}$. The ring $\mathbb{L}$ is graded with $\deg(a_{i,j}) = 1 - i - j$. The Lazard theorem verifies that the ring $\mathbb{L}^*$ is isomorphic to the polynomial ring in countably many variables $\mathbb{Z}[t_1, t_2, \ldots]$ with $\deg(t_i) = -i$.

3.2.2 Chern classes

For $X \in \mathbf{Sch}_k$ and a line bundle $L \to X$ with the zero section $z\colon X \to L$ there is the Chern class operator

$$\tilde{c}_1(L)\colon \mathtt{h}_*(X) \to \mathtt{h}_{*-1}(X)$$

given by $\tilde{c}_1(L)(x) = z^*(z_*(x))$.

In the case $X \in \mathbf{Sm}_k$ the Chern class operator $\tilde{c}_1(L)$ is given by the multiplication with the Chern class element $c_1(L) = z^*(z_*([1_X])) \in \mathtt{h}^1(X)$:

$$\tilde{c}_1(L)(x) = x \cdot c_1(L)$$

The operators $\tilde{c}_1(L)$ are nilpotent by [32, Lemma 4.1.3], and for two line bundles L, M the operators $\tilde{c}_1(L)$ and $\tilde{c}_1(M)$ commute. There is a formal group law $F_{\mathtt{h}}$ on the ring $\mathtt{h}^*(k)$ such that for every $X \in \mathbf{Sch}_k$ and line bundles L, M on X

$$\tilde{c}_1(L \otimes M) = F_{\mathtt{h}}(\tilde{c}_1(L), \tilde{c}_1(M)).$$

Remark 3.2.3. The formal group law F over $\mathtt{h}^*(k)$ gives rise to a homomorphism $\mathbb{L}^* \to \mathtt{h}^*(k)$. By the construction of the algebraic cobordism theory, one has $\Omega^*(k) \cong \mathbb{L}^*$ and the latter homomorphism coincides with the natural map $\Omega^*(k) \to \mathtt{h}^*(k)$ arising from the universal property of the algebraic cobordism Ω^*. Then for any $X \in \mathbf{Sch}_k$ the homomorphism $\Omega_*(X) \to \mathtt{h}_*(X)$ descends to the homomorphism

$$\Omega_*(X) \otimes_{\mathbb{L}^*} \mathtt{h}^*(k) \to \mathtt{h}_*(X)$$

which is surjective by the degree formula (Proposition 3.1.9).

3.3 Auxiliary results

In this section we establish some standard facts about oriented cohomology theories that will be used later in the thesis.

Lemma 3.3.1. *Suppose $X \in \mathbf{Sch}_k$. Then there is a number $n(X)$ such that for any vector bundle $Y \to X$ of rank r and open subscheme $U \subseteq Y$ such that* $\operatorname{codim} Y \setminus U > n(X)$ *the pullback homomorphism $p^*\colon \mathtt{h}_*(X) \to \mathtt{h}_{*+r}(U)$ is an isomorphism.*

Proof: Note that the [19, Proposition 15] establishes the result for $\mathtt{h}_* = \Omega_*$. The same arguments prove the statement for general $\mathtt{h}_*$: by the Jouanalou's device [23] there is an affine torsor $\tilde{X} \to X$ with affine $\tilde{X}$. Denote by $\tilde{Y}$ the pullback of Y. Then $\mathtt{h}_*(X) \to \mathtt{h}_*(\tilde{X})$ and $\mathtt{h}_*(Y) \to \mathtt{h}_*(\tilde{Y})$ are isomorphisms. Thus it sufficient to prove the statement for affine X. In this case there is a surjection from a trivial vector bundle $\mathbb{A}^N \times X \to Y$. For the trivial vector bundle there is a section $s\colon X \to U$ when $\operatorname{codim} Y \setminus U > \dim X$ thus p^* is an isomorphism. ∎

Lemma 3.3.2. *Suppose that a morphism $f\colon X \to Y$ in $\mathbf{Sm}_k$ factors as $f\colon X \xrightarrow{z} L \xrightarrow{j} Y$ where $p\colon L \to X$ is a vector bundle, $z\colon X \to L$ is a zero section and j is an open embedding.*

Then for every projective map $a\colon Y' \to Y$ and $X' = X \times_Y Y'$ the following diagram of pull-back and push-forward maps commutes (we omit the grading):

$$\begin{array}{ccc} \mathtt{h}(X') & \xrightarrow{a'_*} & \mathtt{h}(X) \\ \uparrow{\scriptstyle f'^*} & & \uparrow{\scriptstyle f^*} \\ \mathtt{h}(Y') & \xrightarrow{a_*} & \mathtt{h}(Y). \end{array}$$

Proof: Observe that the map $f'\colon X' \to Y'$ factors as $X' \xrightarrow{z'} L \times_Y Y' \xrightarrow{j'} Y'$, where z' is the zero section of the vector bundle $p'\colon L' = L \times_Y Y' \to X'$, and j' is an open embedding. Let b denote the canonical map $L' \to L$. Since j and j' are flat, we have $j^* a_* = b_* j'^*$. Note that by the homotopy invariance property $z^* = (p^*)^{-1}$ and $z'^* = (p'^*)^{-1}$. Since p and p' are flat, $p^* a'_* = b_* p'^*$. Then $z^* b_* = a'_* z'^*$ and

$$f^* a_* = z^* j^* a_* = z^* b_* j'^* = a'_* z'^* j'^* = a'_* f'^*.$$

∎

3.4 Equivariant oriented cohomology theory

According to [19], for an algebraic group G and an oriented Borel-Moore homology theory $\mathtt{h}_*$ one can associate its G-equivariant version defined on schemes with G-action. The construction of [19] is carried out for algebraic cobordism Ω_* but it can be applied to any oriented Borel-Moore homology with localization property by [19, §5]. In this section we recall the construction. Let G be a linear algebraic group.

Definition 3.4.1. *[19, Definition 10] A good system of representations (V_i, U_i) consists of a sequence of G-representations V_i, open subschemes $U_i \subseteq V_i$ such that*

- *G acts freely on U_i and the quotient U_i/G exists in the category* $\mathbf{Sch}_k$
- *for any i there is a representation W_i such that $V_{i+1} = V_i \oplus W_i$*
- *there are inclusions $U_i \subseteq U_i \oplus W_i \subseteq U_{i+1}$*
- $\lim_{i\to\infty} \dim V_i = \infty$
- $\operatorname{codim}(V_i \setminus U_i) > \operatorname{codim}(V_j \setminus U_j)$ *for $i > j$*

Such systems exist by [45, Remark 1.4]. For $X \in G - \mathbf{Sch}_k$ the quotient $(X \times U_i)/G$ exists in $\mathbf{Sch}_k$ by [19, Lemma 9]. We will denote $(X \times U_i)/G$ by $X \times^G U_i$.

Remark 3.4.2. If (V_i, U_i) is a good system of representations, then for any G-variety X the connecting maps $X \times^G U_i \to X \times^G U_{i+1}$ factor as in 3.3.2, i.e., we have $X \times^G U_i \to X \times^G (U_i \oplus W_i) \to X \times^G U_{i+1}$.

Definition 3.4.3. *For a linear algebraic group G and $X \in G - \mathbf{Sch}_k$ define*

$$\mathtt{h}^G_n(X) = \lim_{i\to\infty} \mathtt{h}_{n+\dim U_i - \dim G}(X \times^G U_i),$$

and the graded oriented Borel-Moore homology theory

$$\mathtt{h}^G_*(X) = \bigoplus_{n\in\mathbb{Z}} \mathtt{h}^G_n(X).$$

Remark 3.4.4. The definition of $\mathtt{h}^G_*(X)$ does not depend on a choice of a good system of representations (V_i, U_i)[19, Theorem 16,§5]. There is also an ungraded version of oriented Borel-Moore homology theory $\mathtt{h}_G(X) = \lim \mathtt{h}_*(X \times^G U_i)$[19, Definintion 12]. Being different, these theories share the list of formal properties [19, Remark 13].

When $X \in G - \mathbf{Sm}_k$, we will use the cohomological grading $\mathtt{h}^n_G(X) = \mathtt{h}^G_{\dim X - n}(X)$ and call $\mathtt{h}^*_G(-)$ the graded equivariant cohomology theory.

Remark 3.4.5. In the case when $\mathtt{h} = \mathrm{CH}$ the graded equivariant groups $\mathtt{h}^G_*$ coincide with the equivariant Chow groups CH^G_* of Edidin-Graham [13].

3.4.1 Pull-back, push-forward maps and action restriction

Suppose that $f\colon X \to Y$ is a G-equivariant locally complete intersection morphism of relative dimension d. For any i it gives rise to a locally complete intersection morphism $f \times^G id\colon X \times^G U_i \to Y \times^G U_i$. The limit of induced pull-back maps gives rise to a pull-back for equivariant theory $f^*\colon \mathtt{h}^G_*(Y) \to \mathtt{h}^G_{*+d}(X)$[19, §5]. The same procedure establishes a push-forward map $f_*\colon \mathtt{h}^G_*(X) \to \mathtt{h}^G_*(Y)$ for any G-equivariant projective morphism $f\colon X \to Y$.

If $H \subseteq G$ is a subgroup of G, then any good system of G-representations (U_i, V_i) is also a good system of H-representations ([19, 4.6.1]). Thus for any i there is a natural projection $X \times^H U_i \to X \times^G U_i$. The pullback of this projection gives rise to the forgetful map

$$\mathtt{h}_*^G(X) \to \mathtt{h}_*^H(Y).$$

Lemma 3.4.6. *([19, Theorem 26]) For any $X \in G - Sch_k$ there is an isomorphism $\mathtt{h}_*^H(X) \cong \mathtt{h}_*^G(X \times G/H)$ and the forgetful map $\mathtt{h}_*^G(X) \to \mathtt{h}_*^H(X)$ coincides with the pullback $\mathtt{h}_*^G(X) \to \mathtt{h}_*^G(X \times G/H)$.*

Proof: The statement follows from the isomorphism of schemes $(X \times G/H) \times^G U_i \cong X \times^H U_i$ that fits into the commutative diagram

$$\begin{array}{ccc} (X \times G/H) \times^G U_i & \longrightarrow & X \times^H U_i \\ \uparrow & \nearrow & \\ X \times^G U_i & & \end{array}$$

with vertical arrows given by natural projections. ∎

Remark 3.4.7. By 2.5.5 for any $X \in G - Sch_k$ and any U_i in a good system of representations of B, the projection $X \times^T U_i \to X \times^B U_i$ is an affine fibration, then by homotopy invariance property the natural homomorphism $\mathtt{h}_*^B(X) \to \mathtt{h}_*^T(X)$ is an isomoprhism.

3.5 Weyl group action on T-equivariant cohomology

Suppose G is a split semisimple algebraic group, T its split maximal torus and $N = N_G(T)$ its normalizer. Let $W = N/T$ be the Weyl group and $X \in N - \mathbf{Sch}_k$. Fix a good system of N-representations (U_i, V_i). For any $w = nT \in W$ there is an automorphism

$$m_w \colon X \times^T U_i \to X \times^T U_i, (x, u) \cdot T \mapsto (xn, un) \cdot T$$

These maps commute with inclusions $X \times^T U_i \to X \times^T U_{i+1}$. Thus the pull-back maps of m_{nT} induce an automorphism map $\mathtt{h}_*^T(X) \to \mathtt{h}_*^T(X)$, and we get a W-action on $\mathtt{h}_*^T(X)$.

Lemma 3.5.1. *Suppose $\lambda \in T^*$. Let V_λ be the corresponding 1-dimensional representation of T. Then for any $X \in T - \mathbf{Sch}_k$ the quotient $L_\lambda = (V_\lambda \times X) \times^T U_i$ is a vector bundle over $X \times^T U_i$. Consider $w \in W$ and the map $m_w \colon X \times^T U_i \to X \times^T U_i$. Then the vector bundle $m_w^*(L_\lambda)$ is isomorphic to the vector bundle $L_{w(\lambda)}$ where $w(\lambda)$ denotes the usual W-action on the character group T^*.*

Proof: Let $w = nT, n \in N(k)$. Note that $V_{w(\lambda)}$ has the same underlying vector space as V_λ. Consider a morphism

$$f_w \colon L_{w(\lambda)} = (V_{w(\lambda)} \times X) \times^T U_i \to (V_\lambda \times X) \times^T U_i = L_\lambda, (v, x, u) \cdot T \mapsto (v, xn, un) \cdot T$$

This morphism is well defined since $(v \cdot \lambda(n^{-1}tn), xtn, utn) \cdot T = (v, xn, un) \cdot T$. This morphism fits into the commutative diagram:

$$\begin{array}{ccc} L_{w(\lambda)} & \xrightarrow{f_w} & L_\lambda \\ \downarrow & & \downarrow \\ X \times^T U_i & \xrightarrow{m_w} & X \times^T U_i \end{array} .$$

Thus f_w induces a homomorphism of line bundles $L_{w(\lambda)} \to m_w^*(L_\lambda)$. It is injective since f_w is. Thus it is an isomorphism. ∎

Chapter 4

Motives associated to oriented cohomology theories

The category of motives associated to a cohomology theory was considered in [33]. For the classical construction of Chow motives we refer to [14, Ch. XII]. In this chapter we recall the construction of the category of $\mathtt{h}$-motives associated to an oriented cohomology theory $\mathtt{h}$ and establish some necessary auxiliary results.

4.1 Motives over a smooth base scheme

Let $S \in \mathbf{Sm}_k$ be a smooth irreducible base scheme. Denote by $\mathbf{SmProj}_S$ the full subcategory of $\mathbf{Sm}_S$ consisting of pairs (X, f) where $f\colon X \to S$ is a smooth projective map. For any $X, Y \in \mathbf{SmProj}_S$ denote by

$$Corr_m(X, Y) = \mathtt{h}_{\dim X+m}(X \times_S Y)$$

Consider the category $Corr_{\mathtt{h}}(S)$ in which objects are pairs (X, i), $X \in \mathbf{SmProj}_S, i \in \mathbb{Z}$. For $X, Y \in \mathbf{SmProj}_S$ let $X_1, \ldots X_n$ be irreducible components of X. Then the morphism set is defined as

$$Hom_{Corr_{\mathtt{h}}}((X, i), (Y, j)) = \bigoplus_{l=1}^{n} Corr_{i-j}(X_l, Y).$$

Composition in the category $Corr_{\mathtt{h}}(S)$ is given by the convolution product defined as follows. Suppose $X_1, X_2, X_3 \in \mathbf{SmProj}_S$ are irreducible. Let $p_{ij}\colon X_1 \times_S X_2 \times_S X_3 \to X_i \times_S X_j$ denote the corresponding projections for $1 \leqslant i < j \leqslant 3$. Let α be a morphism from (X_1, i_1) to (X_2, i_2), and β be a morphism from (X_2, i_2) to (X_3, i_3), so $\alpha \in \mathtt{h}_{\dim(X_1/S)+i_1-i_2}(X_1 \times X_2), \beta \in \mathtt{h}_{\dim(X_2/S)+i_2-i_3}(X_2 \times X_3)$. Then the composition $\gamma = \beta \circ \alpha$ is given by the element

$$\gamma = p_{13*}(p_{12}^*(\alpha) \cap p_{23}^*(\beta)) \in \mathtt{h}_{\dim(X_1/S)+i_1-i_3}(X_1 \times_S X_3).$$

Note that in the category $Corr_{\mathtt{h}}(S)$ the object $(X \sqcup Y, i)$ is a direct sum of (X,i) and (Y,i). Consider the additive completion of $Corr^+_{\mathtt{h}}(S)$ of the category $Corr_{\mathtt{h}}(S)$ so the objects of $Corr^+_{\mathtt{h}}(S)$ are formal direct sums $\oplus(X_i, n_i)$ where $X_i \in \mathbf{SmProj}_S$ are irreducible and the homomorphism between direct sums $\oplus_{i=1}^m (X_i, n_i)$ and $\oplus_{i=1}^{m'}(X'_i, n'_i)$ are given by $m \times m'$ matrices with entries in $Corr_{n_i - n'_j}(X_i, X'_j)$.

Define the category of $\mathtt{h}$-motives $\mathcal{M}_{\mathtt{h}}(S)$ as the idempotent completion of the category $Corr^+_{\mathtt{h}}(S)$. The objects of $\mathcal{M}_{\mathtt{h}}(S)$ will be the pairs (A,p) where $A \in Corr^+_{\mathtt{h}}(S)$ and $p \in End(A)$ is an idempotent $p \circ p = p$. The homomorphisms are given by

$$Hom_{\mathcal{M}_{\mathtt{h}}(S)}((A,p),(B,q)) = q \circ Hom_{Corr^+_{\mathtt{h}}(S)}(A,B) \circ p$$

So the category $\mathcal{M}_{\mathtt{h}}(S)$ is additive and has image and kernel for any idempotent. In the case $S = \operatorname{Spec} k$, $\mathtt{h}_* = \mathrm{CH}_*$ the category obtained is the classical category of Chow motives [14, Ch. XII].

Definition 4.1.1. *For $X, Y \in \mathbf{SmProj}_S$ consider an isomorphism $t\colon X \times_S Y \to Y \times_S X$ that flips the coordinates. For any $\alpha \in \mathtt{h}_d(X \times_S Y)$ define its trasnpose $\alpha^t \in \mathtt{h}_d(Y \times_S X)$ as the image $t_*(\alpha) \in \mathtt{h}_d(Y \times_S X)$.*

Definition 4.1.2. *For any $X \in \mathbf{SmProj}_S$ denote by $M_{\mathtt{h}}(X)(i) \in \mathcal{M}_{\mathtt{h}}(S)$ the image of the object $(X,i) \in Corr_{\mathtt{h}}(S)$.*

Remark 4.1.3. There is a functor $\mathbf{SmProj}_S \to \mathcal{M}_{\mathtt{h}}(S)$ which sends X to $M(X)$ and any morphism $f\colon X \to Y$ to $\Gamma_{f*}([1_X]) \in \mathtt{h}_{\dim(X/S)}(X \times_S Y) = Hom_{\mathcal{M}_{\mathtt{h}}(k)}(M(X), M(Y))$ where $\Gamma_f\colon X \to X \times_S Y$ is the graph morphism. We will write $[\Gamma_f]$ instead of $\Gamma_{f*}([1_X])$.

For any $X \in \mathbf{SmProj}_S$ by definition one has

$$\mathtt{h}_{\dim S + i}(X) = Hom_{\mathcal{M}_{\mathtt{h}}(S)}(S(i), M(X))$$

$$\mathtt{h}^i(X) = Hom_{\mathcal{M}_{\mathtt{h}}(S)}(M(X), S(i))$$

The pull-back map $f^*\mathtt{h}^i(Y) \to \mathtt{h}^i(X)$ and the push-forward map $f_*\colon \mathtt{h}_i(X) \to \mathtt{h}_i(Y)$ then can be realized as the pre-composition and composition with morphism Γ_f respectively.

The natural transformation $\Omega_*(-) \to \mathrm{CH}_*(-)$ gives rise to a functor $F\colon Corr_\Omega(S) \to Corr_{\mathrm{CH}}(S)$ and as a consequence, the functor $\mathcal{M}_\Omega(S) \to \mathcal{M}_{\mathrm{CH}}(S)$. We will check analogues of the properties of [46, §2].

Lemma 4.1.4. *The functor $F\colon Corr_\Omega(S) \to Corr_{\mathrm{CH}}(S)$ has the following properties*

(1) F is surjective on isomorphism class of objects

(2) F is surjective on homomorphisms

(3) *the kernel of* $F_X\colon End_{Corr_\Omega(S)}(X) \to End_{Corr_{\mathrm{CH}}(S)}(F(X))$ *consists of nilpotents for any* $X \in Corr_\Omega(S)$.

Proof: We repeat the arguments of [46, §2] The properties (1), (2) are obvious. For any $(X, i) \in Corr_\Omega(S)$ we have to check that $\Omega_{\dim X}(X \times_S X) \to \mathrm{CH}_{\dim X}(X \times_S X)$ has nilpotent kernel. By [32, Remark 4.5.6] the kernel coincides with $\Omega_{\geqslant 1}(k) \cdot \Omega_*(X \times_S X)$. So for every y in this kernel $y^d \in \Omega_{\dim(X/S)}(X \times_S X) \cap (\Omega_{\geqslant d}(k)\Omega_*(X \times_S X))$ which is zero for $d > \dim(X \times_S X)$ since $\Omega_{<0}(X \times_S X) = 0$. So the kernel of $\mathrm{End}_{Corr^+_\Omega(S)}(X, i) \to \mathrm{End}_{Corr^+_{CH}(S)}(X, i)$ consists of nilpotents. ∎

Corollary 4.1.5. *If* $f\colon M_1 \to M_2$ *is a morphism in* $\mathcal{M}_\Omega(S)$ *such that its image is an isomorphism in* $\mathcal{M}_{\mathrm{CH}}(S)$, *then* f *is an isomorphism.*

Proof: This follows from 4.1.4 and [46, Lemma 2.1,Proposition 2.5]. ∎

4.2 Equivariant motives

Let G be a linear algebraic group. Consider the category $G - \mathbf{SmProj}_k$ of smooth projective k-varieties with G-action and G-equivariant maps. We repeat the construction of the category $\mathcal{M}_{\mathtt{h}}(k)$ in context of G-equivariant varieties and construct the category of G-equivariant motives $\mathcal{M}_{G,\mathtt{h}}(k)$.

For $X, Y \in G - \mathbf{SmProj}_k$ define

$$G - Corr_m(X, Y) = \mathtt{h}^G_{\dim X + m}(X \times Y).$$

Consider the category $G - Corr_{\mathtt{h}}(k)$ in which objects are pairs (X, i), $X \in G - \mathbf{SmProj}_k$, $i \in \mathbb{Z}$. For $X, Y \in G - \mathbf{SmProj}_S$ let $X_1, \dots X_n$ be irreducible components of X. Then the morphism set is defined as

$$Hom_{G-Corr_{\mathtt{h}}(k)}((X, i), (Y, j)) = \bigoplus_{l=1}^{n} G - Corr_{i-j}(X_l, Y).$$

The composition in the category $G - Corr_{\mathtt{h}}(k)$ is given by the convolution product defined using equivariant pull-back and push-forward maps as follows. Suppose $X_1, X_2, X_3 \in G - \mathbf{SmProj}_k$ are irreducible. Let $p_{ij}\colon X_1 \times_S X_2 \times_S X_3 \to X_i \times_S X_j$ denote the corresponding projections for $1 \leqslant i < j \leqslant 3$. Let α is a morphism from (X_1, i_1) to (X_2, i_2) and β is a morphism from (X_2, i_2) to (X_3, i_3), so $\alpha \in$

$\mathtt{h}^G_{\dim(X_1/S)+i_1-i_2}(X_1\times X_2), \beta \in \mathtt{h}^G_{\dim(X_2/S)+i_2-i_3}(X_2\times X_3)$ then the composition $\gamma = \beta\circ\alpha$ is given by the element

$$\gamma = p_{13*}(p_{12}^*(\alpha) \cap p_{23}^*(\beta)) \in \mathtt{h}^G_{\dim(X_1/S)+i_1-i_3}(X_1 \times_S X_3).$$

Consider the additive completion $G - Corr^+_{\mathtt{h}}(k)$ of the category $G - Corr_{\mathtt{h}}(k)$. So the objects of $G - Corr^+_{\mathtt{h}}(k)$ are formal direct sums $\oplus(X_i, n_i)$, where $X_i \in G - \mathbf{SmProj}_k$ are irreducible and the homomorphism between direct sums $\oplus_{i=1}^m(X_i, n_i)$ and $\oplus_{i=1}^{m'}(X'_i, n'_i)$ are given by $m \times m'$ matrices with entries in $G - Corr_{n_i-n'_j}(X_i, X'_j)$.

Define the category of G-equivariant $\mathtt{h}$-motives $\mathcal{M}_{G,\mathtt{h}}(k)$ as the idempotent completion of the category $G - Corr^+_{\mathtt{h}}(k)$. The objects of $\mathcal{M}_{G,\mathtt{h}}(k)$ are the pairs (A, p) where $A \in G - Corr^+_{\mathtt{h}}(k)$ and $p \in End(A)$ is an idempotent, i.e. $p \circ p = p$. The homomorphisms are given by

$$Hom_{\mathcal{M}_{G,\mathtt{h}}(k)}((A,p),(B,q)) = q \circ Hom_{Corr^+_{\mathtt{h}}(S)}(A,B) \circ p.$$

So the category $\mathcal{M}_{G,\mathtt{h}}(k)$ is additive and has image and kernel for any idempotent.

Definition 4.2.1. *For any $X \in \mathbf{SmProj}_S$ denote by $M_{G,\mathtt{h}}(X)(i) \in \mathcal{M}_{G,\mathtt{h}}(k)$ the image of the object $(X, i) \in G - Corr_{\mathtt{h}}(k)$.*

Remark 4.2.2. There is a functor $G - \mathbf{SmProj}_k \to \mathcal{M}_{G,\mathtt{h}}(k)$ which sends X to $M_{G,\mathtt{h}}(X)$, and any morphism $f\colon X \to Y$ to $\Gamma_{f*}([1_X]) \in \mathtt{h}^G_{\dim(X/S)}(X \times_S Y)$ where $\Gamma_f\colon X \to X \times_G Y$ is the graph morphism. We will write $[\Gamma_f]$ instead of $\Gamma_{f*}([1_X])$.

For any $X \in G - \mathbf{SmProj}_k$ by definition one has

$$\mathtt{h}^G_{\dim S+i}(X) = Hom_{\mathcal{M}_{G,\mathtt{h}}(k)}(M_G(pt)(i), M_G(X)),$$

$$\mathtt{h}^i_G(X) = Hom_{\mathcal{M}_{G,\mathtt{h}}(k)}(M_G(X), M_G(pt)(i)).$$

The pull-back map $f^*\colon \mathtt{h}^i_G(Y) \to \mathtt{h}^i_G(X)$ and push-forward map $f_*\colon \mathtt{h}_i^G(X) \to \mathtt{h}_i^G(Y)$ then can be realized by the pre-composition and composition with the morphism Γ_f.

4.3 Lifting of idempotents

Suppose that A^*, B^* are associative unital graded rings, and $f\colon A^* \to B^*$ is a graded homomorphism. Two idempotents $p, q \in B^0$ are called orthogonal if $pq = qp = 0$. We will say that there exists an isomorphism of degree d between p and q if there is $a \in pB^dq$ and $b \in qB^{-d}p$, such that $ab = p$ and $ba = q$.

Definition 4.3.1. *Following [39, §2], we will say that f lifts decompositions and isomorphisms strictly if*

- *for every decomposition* $1 = q_1 + \ldots + q_n$, *where* q_i *are pairwise orthogonal idempotents in* B^0, *there is a decomposition* $1 = p_1 + \ldots + p_n$ *where* p_i *are pairwise orthogonal idempotents in* A^0 *and* $f(p_i) = q_i, i = 1 \ldots n$.
- *for any idempotents* $p_1, p_2 \in A^0$ *and elements* $a \in f(p_1)B^d f(p_2), b \in f(p_2)B^{-d}f(p_1)$ *such that* $ab = f(p_1)$ *and* $ba = f(p_2)$, *there are* $a' \in p_1A^dp_2$ *and* $b \in p_2A^{-d}p_1$ *such that* $f(a') = a, f(b') = b, a'b' = p_1, b'a' = p_2$.

Lemma 4.3.2. *[39, Proposition 2.6] Suppose that* $f\colon A^* \to B^*$ *is a surjective homomorphism of associative unital graded rings and the kernel of* f *restricted to* A^0 *consists of nilpotents. Then* f *lifts decompositions and isomorphisms strictly.*

Lemma 4.3.3. *Consider a commutative square of abelian groups*

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{h} & B' \end{array}$$

Suppose that f *and* f' *are surjective and the induced map* $\ker(f') \to \ker(f')$ *is surjective. Then the homomorphism* $A \to B \times_{B'} A'$ *is surjective.*

Proof: Consider $(b, a') \in B \times_{B'} A'$. Then there is $a_1 \in A$ such that $f(a_1) = b$. Then $a' - g(a_1) \in \ker(f')$ so there is $a_2 \in \ker(f)$ such that $g(a_2) = a' - g(a_1)$. Then $a = a_1 + a_2$ is a preimage of (b, a') in A. ∎

Lemma 4.3.4. *Consider two directed sequences of graded rings* $A^*_{i+1} \to A^*_i$ *and* $B^*_{i+1} \to B^*_i, i \in \mathbb{N}_0$, *and a homomorphism of these systems* $f_i\colon A^*_i \to B^*_i$. *Suppose that*

(1) f_i *is surjective and* $\ker(f_i)$ *consists of nilpotents for every* i

(2) *the map* $\ker(f_{i+1}) \to \ker(f_i)$ *is surjective*

Consider $A^n = \varprojlim_i A^n_i, B^n = \varprojlim_i B^n_i$ *and* $A^* = \oplus_{n\in\mathbb{Z}} A^n$ *and* $B^* = \oplus_{n\in\mathbb{Z}} B^n$. *Then the limit* $f\colon A^* \to B^*$ *is surjective and lifts decompositions and isomorphisms strictly.*

Proof: Lemma 4.3.3 implies that f is surjective. Let us check that it lifts decompositions and isomorphisms strictly. Suppose that $1 = q_1 + \ldots + q_n$ is a decomposition in B^0. Each q_i is given by a sequence $(q_{i,j}) \in B^0_j$ such that $q_{i,j} \mapsto q_{i,j-1}$ under the homomorphism $B_j \to B_{j-1}$. By Lemma 4.3.2 there is a decomposition $1 = p_{1,0} + \ldots + p_{n,0}$ in A^*_0 such that $f_0(p_{i,0}) = q_{i,0}$. We construct a sequence of decompositions $(p_{i,j}) \in A^0_j$ by induction. For every i consider a homomorphism $A^*_j \to A^*_{j-1} \times B^*_j$. There is decomposition $1 = (1,1) = (p_{1,j-1}, q_{1,j}) + \ldots + (p_{n,j-1}, q_{n,j})$. This decomposition lies in the image of A_j^* by Lemma 4.3.3, thus by Lemma 4.3.2 there is a decomposition $1 = p_{1,j} + \ldots + p_{n,j}$ that is a lift of decomposition $q_{i,j}$ and is mapped to $p_{i,j-1}$ via the homomorphism $A_j^* \to A_{j-1}^*$. Then the sequence $(p_{i,j})_{j\in\mathbb{N}_0}$ defines a decomposition $1 = p_1 + \ldots + p_n \in A^*$ that lifts $1 = q_1 + \ldots + q_n$. The same reasoning shows that the map $A^* \to B^*$ lifts isomorphisms strictly. ∎

4.4 Cellular spaces and equivariant Künneth isomorphism

Foe $X, Y \in \mathbf{Sch}_S$ one can consider the K-homology group $A(X \times_S Y, K_*)$ as correspondences. In the case when $X, Z \in \mathbf{Sch}_S$ and $Y \in \mathbf{SmProj}_S$ there is a pairing for K-homology groups [14, §62]

$$A(Y \times_S Z, K_*) \times A(X \times_S Y, K_*) \to A(X \times_S Z, K_*)$$

playing the role of the composition of correspondences.

If $f\colon X \to Y$ in $\mathbf{Sch}_S$ is a flat morphism, $\beta \in A(Y \times_S Z, K_*)$ there is a notion of composition $\beta \circ f \in A(X \times_S Z, K_*)$

Such that for $Z \in \mathbf{SmProj}_S, Y, T \in \mathbf{Sch}_S$ and flat morphism $f\colon X \to Y$ then for any $\alpha \in A(Y \times_S Z, K_*), \beta \in A(Z \times_S T, K_*)$ one has $(\beta \circ \alpha) \circ f = \beta \circ (\alpha \circ f)$.[14, Proposition 62.8]

Definition 4.4.1. *[14, §66] A morphism $f\colon X \to Y$ in $\mathbf{Sch}_S$ is called an affine fibration of rank d if f is flat and for every point $y \in Y$ the fiber $X_y = X \times_Y y$ is isomorphic to the affine space $\mathbb{A}^d_{k(y)}$.*

Definition 4.4.2. *[14, §66] A scheme $X \in \mathbf{Sch}_S$ is called relatively cellular if there is a filtration by closed subschemes*

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_2 \subseteq \ldots \subseteq X_n = X$$

and affine bundles $p_i\colon U_i = X_i \setminus X_{i-1} \to Y_i$ of rank d_i in $\mathbf{Sm}_S$ for $0 \leqslant i \leqslant n$. The schemes $Y_i \in \mathbf{Sm}_S$ are called bases of cellular filtration.

Lemma 4.4.3. *[14, Theorem 66.2] Let $X \in \mathbf{SmProj}_S$ and X is relatively cellular (definition 4.4.2) with bases $Y_i \in \mathbf{SmProj}_S$. Let $\alpha_i \in CH_{\dim X_i}(X_i \times_S Y_i)$ be any preimage of $[\Gamma_{p_i}]$ under the pullback map $\mathrm{CH}_{\dim X_i}(X_i \times_S Y_i) \to \mathrm{CH}_{\dim X_i}(U_i \times_S Y_i)$ and $a_i = f_{i*}((\alpha_i)^t) \in \mathrm{CH}(Y_i \times_S X)$ where $f_i\colon X_i \to X$ is a closed embedding. Then*

$$\sum a_i\colon \oplus_{i=1}^n M_{\mathrm{CH}}(Y_i)(d_i) \to M_{\mathrm{CH}}(X)$$

is an isomorphism in $\mathcal{M}_{\mathrm{CH}}(S)$.

Proof: Let $g_i\colon U_i \to X_i$ be the inclusion. For any $x \in A(Y_i \times_S Z, K_*)$ we have $(x \circ \alpha_i) \circ g_i = x \circ (\alpha_i \circ g_i)$ by [14, 62.8]. Thus the composition

$$A(Y_i \times_S Z, K_*) \xrightarrow{\alpha_i^*} A(X_i \times_S Z, K_*) \xrightarrow{g_i^*} A(U_i \times_S Z, K_*)$$

equals to the isomorphism p_i^*, hence g_i^* is a split surjection. Then the connecting homomorphism $A_{k+1}(U_i \times_S Z, K_{-k}) \to \mathrm{CH}_k(X_{i-1} \times_S Z)$ is zero and there is a short exact sequence

$$0 \to \mathrm{CH}_k(X_{i-1} \times_S Z) \to \mathrm{CH}_k(X_i \times_S Z) \to \mathrm{CH}_k(U_i \times_S Z) \to 0$$

which is a split surjection for any i, thus the map

$$\sum a_i\colon\ \oplus_{i=1}^n \mathrm{CH}(Y_i \times_S Z) \to \mathrm{CH}(X \times_S Z)$$

is an isomorphism, so by Yoneda lemma the map

$$\sum a_i\colon\ \oplus_{i=1}^n M_{\mathrm{CH}}(Y_i)(d_i) \to M_{\mathrm{CH}}(X)$$

is an isomorphism in $\mathcal{M}_{\mathrm{CH}}(S)$. ∎

Lemma 4.4.4. *Assume that $X \in \mathbf{SmProj}_S$ and there is a cellular structure*

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_2 \subseteq \ldots \subseteq X_n = X$$

$p_i\colon U_i \to Y_i$ and $Y_i \in \mathbf{SmProj}_S$. Suppose for any i there is $M_i \in \mathbf{SmProj}_S$ and an S-map $f_i\colon M_i \to X_i \times_S Y_i$ such that the fiber of f_i over $U_i \times_S Y_i$ is isomorphic to the graph morphism $U_i \to U_i \times_S Y_i$. Denote by α_i the image of $(f_{i*}([1_{M_i}]))^t$ in $\mathtt{h}_{\dim X_i}(Y_i \times_S X)$. Then

$$\sum_{i=1}^n \alpha_i\colon\ \sum_{i=0}^n M_{\mathtt{h}}(Y_i)(d_i) \to M_{\mathtt{h}}(X)$$

is an isomorphism in $\mathcal{M}_{\mathtt{h}}(S)$.

Proof: Consider the case $\mathtt{h}_* = \Omega_*$ and the functor $\mathcal{M}_\Omega(k) \to \mathcal{M}_{\mathrm{CH}}(S)$. Then the image of $\sum_{i=1}^n \alpha_i$ in $Hom_{\mathcal{M}_{\mathrm{CH}}(S)}(\oplus M_{\mathrm{CH}}(Y_i)(d_i), M_{\mathrm{CH}}(X))$ is an isomorphism by Lemma 4.4.3. Then $\sum_{i=1}^n \alpha_i$ is an isomorphism by 4.1.5. Then its specialization to any theory $\mathtt{h}_*$ is an isomoprhism. ∎

Corollary 4.4.5. *Consider a flat map $p\colon X \to Y$ in $\mathbf{SmProj}_S$. Assume that there is a cellular structure*

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_2 \subseteq \ldots \subseteq X_n = X$$

such that $p\colon U_i = X_i \setminus X_i \to Y$ is an affine fibration of rank d_i. Assume there is $\tilde{X}_i \in \mathbf{SmProj}_S$ and an S-map $f_i\colon \tilde{X}_i \to X_i$ such that f_i is an isomorphism over U_i. Denote by $[\tilde{X}_i]$ the elements $[\tilde{X}_i \to X]$ in $\mathtt{h}_{\dim X_i}(X)$. Then

(1) $[\tilde{X}_i]$ form a basis of $\mathtt{h}^*(Y)$-module $\mathtt{h}^*(X)$

(2) In the case $Y = S$ the pairing $\mathtt{h}(X) \times \mathtt{h}(X) \to \mathtt{h}(S)$ given by $(a,b) = p_*(ab)$ is perfect.

Proof: Applying Lemma 4.4.4 to the case $Y_i = Y$ we get an isomorphism of motives $\oplus_{i=1}^n M_{\mathtt{h}}(Y)(d_i) \to M_{\mathtt{h}}(X)$ given by $\alpha_i = [\tilde{X}_i \to Y \times X]$. Then $\mathtt{h}_*(X)$ has $\mathtt{h}^*(Y)$ -basis given by $[\tilde{X}_i \to X]$. In the case $Y = S$ the inverse isomorphism of motives $F\colon M_{\mathtt{h}}(X) \to \oplus_{i=1}^n S(d_i)$ is given by elements $a_i \in \mathtt{h}^{d_i}(X)$. Then the basis a_i is dual to the basis $[\tilde{X}_i] \in \mathtt{h}_{\dim S + d_i}(X)$ with respect to the pairing $(a,b) = p_*(ab)$. ∎

Corollary 4.4.6. *(Künneth isomorphism) Under the hypothesis of 4.4.5(2) the map $f\colon \mathtt{h}(X \times_S X) \to \operatorname{End}_{\mathtt{h}(S)} \mathtt{h}(X)$ given by $a \mapsto f_a$, $f_a(x) = p_{2*}(p_1^*(x) \cdot a)$ is an isomorphism of $\mathtt{h}(S)$-modules.*

Proof: The pairing $(\cdot,\cdot)$ gives an isomorphism $\mathtt{h}(X) \to Hom_{\mathtt{h}(S)}(\mathtt{h}(X), \mathtt{h}(S))$ and, hence, an isomorphism $\operatorname{End}_{\mathtt{h}(S)} \mathtt{h}(X) \xrightarrow{\simeq} \mathtt{h}(X) \otimes_{\mathtt{h}(S)} \mathtt{h}(X)$. Consider the composition

$$\rho\colon \mathtt{h}(X \times_S X) \xrightarrow{f} \operatorname{End}_{\mathtt{h}(S)} \mathtt{h}(X) \xrightarrow{\simeq} \mathtt{h}(X) \otimes_{\mathtt{h}(S)} \mathtt{h}(X)$$

and a map $\pi\colon \mathtt{h}(X) \otimes \mathtt{h}(X) \to \mathtt{h}(X \times_S X)$ given by $\pi(a \otimes b) = p_1^*(a) \cdot p_2^*(b)$.

By definition, we have $f_{p_1^*(a)p_2^*(b)}(x) = p_{2*}(p_1^*(x)p_1^*(a)p_2^*(b)) = (x,a)b$. Hence, $\rho(\pi(a \otimes b)) = a \otimes b$ and the map ρ is surjective. Note that Corollary 4.4.5 applied to $X \times_S X \to X$ implies that $\mathtt{h}(X \times_S X)$ is a free $\mathtt{h}(X)$-module of rank $(n+1)$. Then it is a free $\mathtt{h}(S)$-module of rank $(n+1)^2$. Thus, ρ is a surjective homomorphism between free modules of the same rank, hence, it is an isomorphism. ∎

We now provide the equivariant analogues of Lemma 4.4.4 and Corollaries 4.4.5 and 4.4.6.

Definition 4.4.7. *For $X, Y \in G - \mathbf{Sch}_k$ we call $f\colon X \to Y$ an equivariant affine fibration if f is equivariant and f is an affine fibration.*

Definition 4.4.8. *[14, §66] A scheme $X \in G-\mathbf{Sch}_k$ is called relatively G-equivariant cellular if there is a filtration by closed subschemes*

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_2 \subseteq \ldots \subseteq X_n = X$$

and G-equivariant affine fibrations $p_i \colon U_i = X_i \setminus X_{i-1} \to Y_i$ of rank d_i for $0 \leqslant i \leqslant n$. The schemes $Y_i \in G - \mathbf{Sm}_k$ are called bases of cellular filtration.

Lemma 4.4.9. *Suppose $X \in G-\mathbf{SmProj}_k$ and there is a sequence of G-equivariant closed subschemes*

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X$$

with G-equivariant affine fibrations $X_i \setminus X_{i-1} = U_i \to Y_i$ with $Y_i \in G - \mathbf{SmProj}_k$. Suppose for any i there is $M_i \in G-\mathbf{SmProj}_k$ and G-equivariant map $f_i \colon M_i \to X_i \times Y_i$ such that $f_i \colon f_i^{-1}(U_i) \to U_i$ is isomorphic to the graph map $\Gamma_{p_i} \colon U_i \to U_i \times Y_i$. Let α_i denote the image of $(f_{i*}([1_{M_i}]))^t$ in $\mathtt{h}^G_{\dim X_i}(Y_i \times X)$. Then the homomorphism

$$\sum \alpha_i \colon \ \oplus_{i=1}^n M_{G,\mathtt{h}}(Y_i)(d_i) \to M_{G,\mathtt{h}}(X)$$

is an isomorphism in $\mathcal{M}_{G,\mathtt{h}}(k)$

Proof: By Yoneda lemma it is sufficient to check that for any $Z \in G - \mathbf{SmProj}_k$ the induced map

$$\sum \alpha_{i*} \colon \ \oplus_{i=1}^n \mathtt{h}^G_{m-d_i}(Z \times Y_i) \to \mathtt{h}^G_m(Z \times X) \qquad (*)$$

is an isomoprhism for any $m \in \mathbb{Z}$. Note that for any U_j in the good system of representations of G we have $(Z \times Y_i) \times^G U_j = (Z \times^G U_j) \times_{U_j/G} (Y_i \times^G U_j)$, thus the arrow in question a limit of the maps

$$\mathtt{h}_{m+r-d_i}((Z \times^G U_j) \times_{U_j/G} (Y_i \times^G U_j)) \to \mathtt{h}^G_{m+r}((Z \times^G U_j) \times_{U_j/G} (X \times^G U_j)), \quad (**)$$

where $r = \dim U_j - \dim G$.

For any U_j in the system of good representations (V_j, U_j) the schemes $X_i \times^G U_j$ give cellular filtration with bases $Y_i \times^G U_j$ and schemes $M_i \times^G U_j$ satisfy the conditions of Lemma 4.4.4 over the base scheme $S = U_j/G$. Thus by Lemma 4.4.4 the homomorphism $(**)$ is an isomorphism thus $(*)$ is an isomorphism. ∎

Corollary 4.4.10. *Consider a flat map $p \colon X \to Y$ in $G - \mathbf{SmProj}_k$ and a G-equivariant cellular fibration*

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n = X$$

such that $p \colon U_i = X_i \setminus X_{i-1} \to Y$ is a G-equivariant affine fibration of rank d. Assume there are $\tilde{X}_i \in G - \mathbf{SmProj}_k$ and equivariant maps $f_i \colon \tilde{X}_i \to X_i$ such that $f^{-1}(U_i) \to U_i$ is an isomorphism. Denote by $[\tilde{X}_i]$ the elements $[\tilde{X}_i \to X] \in \mathtt{h}^G_{\dim X_i}(X)$. Then

(1) $[\tilde{X}_i]$ *form a basis of* $\mathtt{h}^*_G(Y)$*-module* $\mathtt{h}^*_G(X)$

(2) *In the case* $Y = \operatorname{Spec} k$ *the pairing* $\mathtt{h}^*_G(X) \times \mathtt{h}^*_G(X) \to \mathtt{h}^*_G(k)$ *given by* $(a, b) = p_*(ab)$ *is perfect*

Proof: By Lemma 4.4.9 there is an isomorphism of motives $\oplus_{i=1}^n M_{G,\mathtt{h}}(Y)(d_i) \to M_{G,\mathtt{h}}(X)$ given by $\alpha_i = [\tilde{X}_i \to Y \times X]$. Then $\mathtt{h}^G_*(X)$ has $\mathtt{h}^*_G(Y)$ -basis given by $[\tilde{X}_i \to X] = [\tilde{X}_i]$. In the case $Y = k$ the inverse isomorphism of motives $F\colon M_{G,\mathtt{h}}(X) \to \oplus_{i=1}^n M_{G,\mathtt{h}}(k)(d_i)$ is given by elements $a_i \in \mathtt{h}^{d_i}_G(X)$. Then the basis a_i is dual to the basis $[\tilde{X}_i] \in \mathtt{h}^G_{d_i}(X)$ with respect to the pairing $(a, b) = p_*(ab)$. ∎

Corollary 4.4.11. *(Equivariant Künneth isomorphism). In the hypothesis of 4.4.10(2) the map* $f\colon \mathtt{h}^G_*(X \times X) \to \operatorname{End}_{\mathtt{h}^*_G(k)}(\mathtt{h}^*_G(X))$ *given by* $a \mapsto f_a$, $f_a(x) = p_{2*}(p_1^*(x) \cdot a)$ *is an isomorphism of* $\mathtt{h}^*_G(k)$*-modules.*

Proof: The pairing $(\cdot,\cdot)$ gives an isomorphism $\mathtt{h}^G_*(X) \to Hom_{\mathtt{h}^*_G(k)}(\mathtt{h}^*_G(X), \mathtt{h}^*_G(k))$ and, hence, an isomorphism $\operatorname{End}_{\mathtt{h}^*_G(k)}(\mathtt{h}^*_G(X)) \xrightarrow{\simeq} \mathtt{h}^*_G(X) \otimes_{\mathtt{h}^*_G(k)} \mathtt{h}^*_G(X)$. Consider the composition

$$\rho\colon \mathtt{h}^*_G(X \times X) \xrightarrow{f} \operatorname{End}_{\mathtt{h}^*_G(k)} \mathtt{h}^*_G(X) \xrightarrow{\simeq} \mathtt{h}^*_G(X) \otimes_{\mathtt{h}^*_G(k)} \mathtt{h}^*_G(X)$$

and the map $\pi\colon \mathtt{h}^*_G(X) \otimes \mathtt{h}^*_G(X) \to \mathtt{h}^*_G(X \times X)$ given by $\pi(a \otimes b) = p_1^*(a) \cdot p_2^*(b)$.

By definition, we have $f_{p_1^*(a)p_2^*(b)}(x) = p_{2*}(p_1^*(x)p_1^*(a)p_2^*(b)) = (x, a)b$. Hence, $\rho(\pi(a \otimes b)) = a \otimes b$ and the map ρ is surjective. Note that by 4.4.10 applied to $X \times X \to X$, $\mathtt{h}^*_G(X \times X)$ is a free $\mathtt{h}^*_G(X)$-module of rank $(n+1)$, hence, it is a free $\mathtt{h}^*_G(k)$-module of rank $(n+1)^2$. Thus, ρ is a surjective homomorphism between free modules of the same rank. Hence, it is an isomorphism. ∎

4.5 Cellular structure on projective homogeneous varieties

In this section we give a list of equivariant cellular structures on projective homogeneous varieties and their products that will be used. G is a semisimple split algebraic group, T its split maximal torus, W the corresponding Weyl group and $T \subseteq B$ a Borel subgroup and $B \subseteq P$ a standard parabolic subgroup. Denote by $\leqslant$ the Bruhat order on W.

B-equivariant cellular structure on G/P

The Bruhat decomposition implies that B orbits on G/B are indexed by the elements of W. For $w \in W$ let $C(w) = BwB/B$ be the corresponding B-orbit. Then $C(w)$ is isomorphic to the affine space $\mathbb{A}^{l(w)}$([2, 14.12]) Denote by X_w the closure of $C(w)$ in G/B and call it a Schubert cell. Then $X_w = \coprod_{v \leqslant w} C(v)$ and for the filtration X_i given by $X_i = \cup_{l(w) \leqslant i} X_w$ we have a B-equivariant filtration X_i with

$$X_i \setminus X_{i-1} = \coprod_{w | l(w)=i} C(w)$$

Then we can subdivide this filtration by considering any linear order $1 <' w_1 <' \ldots <' w_n$ on W that extends the Bruhat order and take $X'_w = \cup_{v \leqslant' w} X_v$. Then $X'_{w_i} \setminus X'_{w_{i-1}} = C(w_i) \cong \mathbb{A}^{l(w_i)}$ thus

$$\emptyset \subseteq X'_1 \subseteq X'_{w_1} \subseteq \ldots \subseteq X'_{w_n} = X$$

is a B-equivariant cellular filtration (in the sense of Definition 4.4.8) whose bases coincide with $\operatorname{Spec} k$.

For any $w \in W$ let $w = s_1 \ldots s_n$ be a reduced decomposition. Denote $I_w = (s_1, \ldots, s_n)$ the sequence of simple reflections. Then the Bott-Samelson variety X_{I_w} ([12],[7, §7]) given by

$$X_{I_w} = P_{\alpha_1} \times^B P_{\alpha_2} \times^B \ldots^B P_{\alpha_n}/B$$

is smooth and projective and provides B-equivariant maps $X_{I_w} \to X_w$ that is an isomorphism over $C(w)$.

Note that B-orbits on G/P are given by the cosets $\overline{w} \in W/W_P$ and have the form $C(\overline{w}) = B\overline{w}P/P$. Denote by $X_{\overline{w}}$ the closure of $C(\overline{w})$. If $w \in W^P$ is a minimal length representative of the coset $\overline{w} \in W/W_P$ then the projection $C(w) \to C(\overline{w})$ is an isomorphism, then X_{I_w} gives a resolution of singularities of $X_{\overline{w}}$, thus applying Lemma 4.4.9 and Corollaries 4.4.10 and 4.4.11 we get that

$$M_{B,\mathtt{h}}(G/P) \cong \bigoplus_{w \in W^P} M_{B,\mathtt{h}}(pt)(l(w)),$$

the elements $[X_{I_w}]$, $w \in W^P$ give a basis of $\mathtt{h}^*_B(G/P)$ over $\mathtt{h}^*_B(k)$ and $\mathtt{h}^*_B(G/P \times G/P) \cong End_{\mathtt{h}^*_B(k)}(\mathtt{h}^*_B(G/P))$.

G-equivariant cellular structure on $G/B \times G/B$

The G-orbits of the diagonal G-action on $G/B \times G/B$ are given by $O_w = (B, wB) \cdot G$ for $w \in W$. The projection $O(w) \to G/B$ is an affine fibration of rank $l(w)$. For $w \in W$

let X_w denote the closure of $O(w)$. Let $w = s_1 \dots s_n$ be a reduced decomposition. Then the variety

$$X_{I_w} = G/B \times_{G/P_{i_1}} \times \dots \times_{G/P_{i_n}} G/B$$

is smooth projective and the fiber of $X_{I_w} \to X_w$ over $O(w)$ is an isomorphism. Then $X_i = \cup_{l(w)\leqslant i} X_w$ forms a filtration on $G/B \times G/B$ with $X_i \setminus X_{i-1} = \coprod_{l(w)=i} O(w)$. Subdividing this filtration we get a G-equivariant filtration X'_{w_i} with $X'_{w_i} \setminus X'_{w_{i-1}} = O_{w_i}$, thus by Corollary 4.4.10 $[X_{I_w}]$ form a basis of $\mathtt{h}_G(G/B \times G/B)$ over $\mathtt{h}(G/B)$.

G-equivariant cellular structure on $G/P \times G/P'$

Suppose that P and P' are two standard parabolic subgroups. We recall the cellular structure on $G/P \times G/P'$ constructed by Chernousov and Merkurjev in [9]. By [9, Lemma 2.1] the assignment $w \mapsto (P, wP') \cdot G$ gives a bijection between $W_P \backslash W / W'_P$ and the set of G-orbits on $G/P \times G/P'$. For a class $D \in W_P \backslash W / W'_P$ let O_D denote the corresponding orbit. For any class $D \in W_P \backslash W / W'_P$ there is a standard parabolic subgroup P_D and a G-equivariant affine fibration $O_D \to G/P_D$ of rank $l(D)$ where $l(D)$ is the length of the minimal coset representative of D ([9, Proposition 4.1]). By [9, Proposition 5.1] there is a G-equivariant filtration V_i on $G/P \times G/P'$ such that $V_i \setminus V_{i-1} = O_{D_i}$. In case when k has characteristic zero we can take any G-equivariant desingularisation of V_i by [27, 3.9.1], thus by 4.4.9 we get that

$$M_{G,\mathtt{h}}(G/P \times G/P') \cong \bigoplus_{D \in W_P \backslash W / W'_P} M_{G,\mathtt{h}}(G/P_D)(l(D))$$

Chapter 5

Formal affine Demazure algebras

In this chapter we recall the theory of formal affine Demazure algebras developed by Calmes, Hoffnung, Malagon-Lopez, Petrov, Savage, Zainoulline and Zhong in [22], [5],[6],[7],[4]. We will follow the exposition of [22].

5.1 Graded formal group algebra

Let $\mathtt{h}$ be an oriented cohomology theory with the coefficient ring R and the formal group law F. We recall the notion of formal group algebra studied in [4] [22],[5].

For an abelian group Λ let $R[x_\Lambda]$ denote the polynomial ring with variables x_λ indexed by the elements $\lambda \in \Lambda$ and $\varepsilon\colon R[x_\Lambda] \to R$ be the augmentation map that maps x_λ to 0 for any $\lambda \in \Lambda$. Define $R[[x_\Lambda]]$ to be the $\ker(\varepsilon)$-adic completion of $R[x_\Lambda]$.

Let J_F denote the closure of the ideal generated by elements $x_{\lambda+\mu} - (x_\lambda +_F x_\mu)$ for all $\lambda, \mu \in \Lambda$.

Definition 5.1.1. *Denote the formal group algebra of* Λ *and formal group law* (R, F) *as the quotient*

$$R[[\Lambda]]_F = R[[x_\Lambda]]/J_F.$$

Proposition 5.1.2. *[7, Theorem 3.3] In case* $\Lambda = T^*$ *the formal group algebra* $R[[T^*]]_F$ *coincides with the ungraded oriented cohomology of a point*

$$R[[T^*]]_F \cong \mathtt{h}_T(\operatorname{Spec} k)$$

Following the notation of [6] and [7] we will denote $R[[T^*]]_F$ by S. We will need a similar description for the graded equivariant theory $\mathtt{h}^*_T$.

Let $R[[T^*]]$ be the power series in variables x_λ indexed by elements of $\lambda \in T^*$. Declare the grading of each x_λ to be 1. Together with grading on R this allows to define a grading of any monomial using the formula

$$\deg r x_{\lambda_1} \dots x_{\lambda_m} = \deg r + m.$$

Definition 5.1.3. *For $i \in \mathbb{Z}$ define $R[[T^*]]^i$ to be the subgroup consisting of power series f such that every monomial of f has degree i. Let $R[[T^*]]^i_F$ denote the image of the natural map $R[[T^*]]^i \to R[[T^*]]_F$.*

Definition 5.1.4. *Define the graded formal group algebra S^{gr} to be the graded ring*

$$S^{gr} = \bigoplus_{i \in \mathbb{Z}} R[[T^*]]^i_F.$$

Note that choice of a basis $\lambda_1, \dots, \lambda_n$ in T^* provides an isomorphism ([4, Corollary 2.12]):

$$R[[T^*]]_F \cong R[[x_1, \dots, x_n]], x_{\lambda_i} \mapsto x_i.$$

Definition 5.1.5. *We will call a power series in $R[[T^*]]_F \cong R[[x_1, \dots, x_n]]$ homogeneous if all the monomials on the power series have the same degree. Note that this property does not depend on the choice of a basis of T^* since for any $\lambda, \mu \in T^*$ $x_\lambda +_F x_\mu$ is a homogeneous power series of degree 1.*

Remark 5.1.6. The graded formal group algebra S^{gr} is a subring of the formal group algebra S consisting of finite sums of homogeneous power series.

Proof: Choose a basis of T^* and identify $R[[T^*]]_F \cong R[[x_1, \dots, x_n]]$. This isomorphism identifies $R[[T^*_F]]^i$ with a subgroup of homogeneous power series of degree i. Thus $R[[T^*]]^i_F$ are pairwise disjoint, so S^{gr} injects into $R[[T^*]]_F = S$ and is a graded ring and S^i equals to the subgroup of homogeneous power series of degree i. ∎

Proposition 5.1.7. *There is an isomorphism of graded rings with W-action*

$$S^{gr} \cong \mathtt{h}^*_T(k), x_\lambda \mapsto c^T_1(L_\lambda)$$

Proof: Choose an isomorphism $T \cong \mathbb{G}^n_m$. This gives a basis $\lambda_1, \dots, \lambda_n$ of T^*. Choose a sequence $U_i = (\mathbb{A}^i \setminus 0)^n$ as a system of good representations. Then $U_i/T = (\mathbb{P}^{i-1})^n$ and $\mathtt{h}((\mathbb{P}^{i-1})^n) = R[x_1, \dots x_n]/(x^i_1, \dots, x^i_n)$ where $x_j = c_1(L(\lambda_j))$. Then for any $m \in \mathbb{Z}$ $\mathtt{h}^m(U_i/T) = (R[x_1, \dots x_n]/(x^i_1, \dots, x^i_n))^m$ and $\lim_{i \mapsto \infty} \mathtt{h}^m(U_i/T)$ coincides with the group $R[[x_1, \dots, x_n]]^m$ of homogeneous power series of degree m. Thus $\mathtt{h}^*_T(k)$ coincides with S^{gr} inside $R[[T^*]]_F = R[[x_1, \dots, x_n]]$. Note that $w \cdot c_1(L_\lambda) = c_1(L_{w(\lambda)})$ by Lemma 3.5.1, thus the map $S^{gr} \to \mathtt{h}^*_T(k)$ is W-equivariant on generators. Hence it is W-equivariant. ∎

Example 5.1.8. In the case when $R = \mathbb{Z}$ and $F = F_a$ is the additive formal group law, the graded formal group algebra S^{gr} coincides with the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ where $\lambda_1, \dots, \lambda_n$ is a basis of T^*.

5.2 Graded formal affine Demazure algebra

In this section we recall the definition of the formal affine Demazure algebra $\mathbf{D}_F$ introduced in [22] and develop its graded analogue $\mathbf{D}_F^{gr}$.

Let G be a semisimple split algebraic group. Fix a maximal split torus T and the root system Σ. Denote by Λ_r the root lattice, by Λ_w the weight lattice, and by $\Lambda = T^*$ the intermediate character lattice. The corresponding Weyl group $W = N_G(T)/T$ acts on $T^* = \Lambda$. This action gives rise to a W-action on $R[[\Lambda]]_F$ given by the formula

$$w(x_\lambda) = x_{w(\lambda)} \text{ for any } w \in W, \lambda \in \Lambda.$$

As in the previous section, consider the formal group algebra S and the graded formal group algebra $S^{gr} \subseteq S$. Recall that the elements of S^{gr} are finite sums of homogeneous power series. Since R is an integral domain, then S and S^{gr} are integral domains by [4, 2.13].

Definition 5.2.1. *For any root $\alpha \in \Sigma$ and any element $x \in R[[\Lambda]]_F$ the difference $x - s_\alpha(x)$ is uniquely divisible by x_α by [4, Cor. 3.4]. Then the formal Demazure operator Δ_α^F is defined by*

$$\Delta_\alpha^F(x) = \frac{x - s_\alpha(x)}{x_\alpha} \textit{ for any } x \in R[[\Lambda]]_F.$$

Note that if $x \in S^{gr}$ then $\Delta_\alpha^F(x) \in S^{gr}$.

In the case when F is additive or multiplicative, the formal Demazure operators coincide with classical ones defined in [11]. The formal Demazure operators are R-linear.

Remark 5.2.2. The operators Δ_α^F have degree -1 on S^{gr} and for any $u, v \in S$

$$\Delta_\alpha^F(uv) = \Delta_\alpha^F(u)v + s_\alpha(u)\Delta_\alpha^F(v) \text{ in } S$$

Remark 5.2.3. As operators on S the Demazure operators have the following commuting relation with the multiplication operators. For any $x \in S$ we will write x for the multiplication by x operator on S. Then

$$\Delta_\alpha^F x = s_\alpha(x)\Delta_\alpha^F + \Delta_\alpha^F(x) \text{ in } End_R(S).$$

Let $Q = S[\frac{1}{x_\alpha} \mid \alpha \in \Sigma]$ (respectively $Q^{gr} = S^{gr}[\frac{1}{x_\alpha} \mid \alpha \in \Sigma]$) be the localization of S (respectively S^{gr}) in the variables corresponding to the roots. Then the W-action on S naturally descends to a W-action on Q and Q^{gr}

Let $R[W]$ be the group ring of the Weyl group W. Denote the basis elements by $\delta_w, w \in W$.

Definition 5.2.4. *([22, Definition 6.1]) The twisted formal group algebra Q_W (respectively Q_W^{gr}) is defined as*

$$Q_W = Q \otimes_R R[W], (Q_W^{gr} = Q^{gr} \otimes_R R[W]) \text{ as an } R\text{-module}$$

with multiplication qiven by

$$q\delta_w \cdot q'\delta_{w'} = qw(q')\delta_{ww'}.$$

Definition 5.2.5. *For each root $\alpha \in \Sigma$ define the corresponding formal Demazure element as*

$$X_\alpha = \frac{1}{x_\alpha}(1 - \delta_{s_\alpha}).$$

Definition 5.2.6. *([22, Definition 6.3]) The formal affine Demazure algebra $\mathbf{D}_F$ (resp. graded formal affine Demazure algebra $\mathbf{D}_F^{gr}$) is the subalgebra of Q_W (resp. Q_W^{gr}) generated by S (S^{gr}) and formal Demazure elements $X_\alpha, \alpha \in \Sigma$.*

For any $w \in W$ fix a reduced decomposition $w = s_{i_1}s_{i_2}\dots s_{i_k}$ as a product of simple reflections. Denote the sequence $i_1, \dots i_k$ by I_w. Define $X_{I_w} = X_{i_1}X_{i_2}\dots X_{i_k}$. Note that in general X_{I_w} does depend on the choice of the reduced decomposition.

5.3 Generators and relations of the formal affine Demazure algebra

Definition 5.3.1. *For a root $\alpha \in \Sigma$ denote by*

$$\kappa_\alpha^F = \frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}$$

This element lies in S according to [22, Definition 4.2]. For $\alpha = \alpha_i$ we will write κ_i for κ_{α_i}.

Remark 5.3.2. It is convenient to introduce another set of generators ([7, Definition 5.2])

$$Y_\alpha = \kappa_\alpha - X_\alpha$$

called push-pull elements.

For any $\alpha, \beta \in \Lambda$ the element

$$\kappa_{\alpha,\beta} = \frac{1}{x_{\alpha+\beta}x_\beta} - \frac{1}{x_{\alpha+\beta}x_{-\alpha}} - \frac{1}{x_\alpha x_\beta}$$

lies in S by [22, Lemma 6.7]. For linear combinations of simple roots $\alpha = n_1\alpha_i + m_1\alpha_j$ and $\beta = n_2\alpha_i + m_2\alpha_j$ $\kappa_{n_1 i + m_1 j, n_2 i + m_2 j}$ for $\kappa_{\alpha,\beta}$. Note that since a formal group law is a homogeneous power series, the elements κ_α and $\kappa_{\alpha,\beta}$ lie in S^{gr}.

Remark 5.3.3. Choose a set of simple roots $\alpha_1, \dots, \alpha_n$. Let us denote X_{α_i} by X_i. Let $s_i \in W$ be the simple reflections corresponding to the simple roots α_i Then the Weyl group has the following presentation in terms of generators and relations:

$$W = \langle s_i \mid s_i^2 = 1, (s_i s_j)^{m_{i,j}} = 1 \rangle$$

where
$m_{i,j} = 2$ if $\langle \alpha_i^\vee, \alpha_j \rangle = 0$,
$m_{i,j} = 3$ if $\langle \alpha_i^\vee, \alpha_j \rangle = \langle \alpha_j^\vee, \alpha_i \rangle - 1$,
$m_{i,j} = 4$ if $\langle \alpha_i^\vee, \alpha_j \rangle = -1, \langle \alpha_j^\vee, \alpha_i \rangle = -2$,
$m_{i,j} = 6$ if $\langle \alpha_i^\vee, \alpha_j \rangle = -1, \langle \alpha_j^\vee, \alpha_i \rangle = -3$.

There is a presentation of the formal affine Demazure algebra $\mathbf{D}_F$ in terms of generators and relations.

Proposition 5.3.4. *([5, Theorem 7.9],[22, Proposition 6.8]) The formal affine Demazure algebra is generated as R-algebra by S and elements X_i, $i = 1 \dots, n$ modulo the following relations:*

- $X_i \phi = s_i(\phi) X_i + \Delta_i(\phi)$ *for* $\phi \in S$,
- $X_i^2 = \kappa_i X_i$,
- *if* $m_{i,j} = 2$ *then* $X_{ij} = X_{ji}$,
- *if* $m_{i,j} = 3$ *then* $X_{jij} - X_{iji} = X_i \kappa_{i,j} - X_j \kappa_{j,i}$,
- *if* $m_{i,j} = 4$ *then* $X_{jiji} - X_{ijij} = X_{ij}(\kappa_{i+2j,-j} + \kappa_{j,i}) - X_{ji}(\kappa_{i+j,j} + \kappa_{i,j}) + X_j(\Delta_i(\kappa_{i+j,j} + \kappa_{i,j})) - X_i(\Delta_j(\kappa_{i+2j,-j} + \kappa_{j,i}))$,
- *if* $m_{i,j} = 6$ *then* $X_{jijiji} - X_{ijijij} = X_{ijij}(\kappa_{j,i} + \kappa_{2i+3j,-i-2j} + \kappa_{-i-3j,i+2j} + \kappa_{i+2j,-j}) - X_{jiji}(\kappa_{i,j} + \kappa_{-2i-3j,i+2j} + \kappa_{-i-2j,i+3j} + \kappa_{i+j,j}) + X_{jij}(\Delta_i(\kappa_{i,j} + \kappa_{-2i-3j,i+2j} + \kappa_{-i-2j,i+3j} + \kappa_{i+j,j})) - X_{iji}(\Delta_j(\kappa_{j,i} + \kappa_{2i+3j,-i-2j} + \kappa_{-i-3j,i+2j} + \kappa_{i+2j,-j})) + X_{ij}\xi_{ij} - X_{ji}\xi_{ji} + X_j(\Delta_i(\xi_{ji})) - X_i(\Delta_j(\xi_{ij}))$,
where $\xi_{ij} = \frac{1}{x_i x_{i+j} x_{i+2j} x_{2i+3j}} + \frac{1}{x_i x_j x_{i+2j} x_{-2i-3j}} + \frac{1}{x_i x_j x_{2i+3j} x_{-i-j}} - \frac{1}{x_i x_{i+j} x_{i+2j} x_{-i-3j}} - \frac{1}{x_i x_{i+j} x_{i+3j} x_{-j}} + \frac{1}{x_{i+j} x_{i+3j} x_{-j} x_{-2i-3j}} + \frac{1}{x_{i+3j} x_{2i+3j} x_{-j} x_{-i-2j}} + \frac{1}{x_{i+j} x_{i+2j} x_{-i-3j} x_{-2i-3j}} - \frac{1}{x_i x_j x_{i+2j} x_{i+3j}}$
and $\xi_{ji} = \frac{1}{x_i x_j x_{2i+3j} x_{-i-2j}} + \frac{1}{x_i x_j x_{i+2j} x_{-i-3j}} + \frac{1}{x_j x_{i+2j} x_{i+3j} x_{2i+3j}} - \frac{1}{x_i x_j x_{i+j} x_{2i+3j}} + \frac{1}{x_{i+j} x_{i+2j} x_{-i} x_{-2i-3j}} + \frac{1}{x_{i+3j} x_{2i+3j} x_{-i-j} x_{-i-2j}} + \frac{1}{x_{i+j} x_{i+3j} x_{-i} x_{-i-2j}} - \frac{1}{x_j x_{i+3j} x_{2i+3j} x_{-i-j}} - \frac{1}{x_j x_{i+j} x_{i+3j} x_{-i}}$.

Proposition 5.3.5. *The graded formal affine Demazure algebra $\mathbf{D}_F^{gr}$ is a free left S^{gr}-module with the basis $\{X_{I_w}\}_{w \in W}$.*

Proof: Note that $\mathbf{D}_F^{gr}$ is a subalgebra of $\mathbf{D}_F$ and the elements of X_{I_w} form a basis of $\mathbf{D}_F$ over S by [5, proposition 7.7]. Then X_{I_w} are linearly independent over S^{gr} in $\mathbf{D}_F^{gr}$. Since the elements $\kappa_i, \kappa_{i,j}$ and $\xi_{i,j}$ lie in S^{gr}, we get that X_{I_w} generate $\mathbf{D}_F^{gr}$ a S^{gr}-module. ∎

Proposition 5.3.6. *The graded formal affine Demazure algebra is generated as R-algebra by S^{gr} and X_i module the same list of relations as in Proposition 5.3.4*

Proof: The same set of relations holds in $\mathbf{D}_F^{gr}$, so there is a homomorphism from the algebra A generated by S^{gr}, X_i modulo the relations to $\mathbf{D}_F^{gr}$. Note that A is a free S^{gr}-module with the basis X_{I_w}, thus the homomorphism $A \to \mathbf{D}_F^{gr}$ sends basis to basis, hence is an isomorphism. ∎

In the case when F is additive FGL or multiplicative FGL the graded formal affine Demazure algebra $\mathbf{D}_F^{gr}$ becomes a classical object: nil Hecke ring introduce by Kostant and Kumar ([28]) or the 0-Hecke ring respectively:

Proposition 5.3.7. *([22, Proposition 7.1])*

In the case $R = \mathbb{Z}$ and F is the additive formal group law $S^{gr} = \mathbb{Z}[x_{\lambda_1}, \dots x_{\lambda_n}]$ where λ_i form a basis of $\Lambda = T^*$ and

$\mathbf{D}_F^{gr}$ is generated as $\mathbb{Z}$ - algebra by S^{gr} and elements X_i, $i = 1, \dots, n$ modulo the relations:

$$X_i f = s_i(f) X_i + \Delta_i(f) \text{ for any } f \in S$$
$$X_i^2 = 0$$
$$(X_i X_j)^{m_{i,j}/2} = (X_j X_i)^{m_{i,j}/2} \text{ if } m_{i,j} \text{ is even}$$
$$X_i X_j X_i = X_j X_i X_j \text{ if } m_{i,j} = 3.$$

Thus it is isomorphic to the nil Hecke ring ([28, Definition 4.12]).

In the case $R = \mathbb{Z}$ and F is the multiplicative formal group law specialized at $\beta = 1$, then $S^{gr} = \mathbb{Z}[T^*]$ is the group ring of T^* and $\mathbf{D}_F^{gr}$ is generated as $\mathbb{Z}$-algebra by S and X_i, $i = 1, \dots, n$ modulo the usual braid relations:

$$X_i f = s_i(f) X_i + \Delta_i(f) \text{ for any } f \in S$$
$$X_i^2 = X_i$$
$$(X_i X_j)^{m_{i,j}/2} = (X_j X_i)^{m_{i,j}/2} \text{ if } m_{i,j} \text{ is even}$$
$$X_i X_j X_i = X_j X_i X_j \text{ if } m_{i,j} = 3.$$

Remark 5.3.8. It follows that in both additive and multiplicative cases the elements X_{I_w} do not depend on the choice of the reduced decompositions I_w of w.

5.4 Modules corresponding to parabolic subgroups

Let $P \subseteq G$ be a standard parabolic subgroup and $W_P \subseteq W$ the corresponding subgroup of the Weyl group. The main result of [7] establishes an isomorphism $\mathtt{h}_T(G/P) \cong \mathbf{D}_P^*$ between the ungraded T-equivariant theory of the homogeneous space G/P and a certain projective $\mathbf{D}_F$-module $\mathbf{D}_P^*$, defined in terms of W_P. In this section we adapt this result for the graded case to get an isomorphism $\mathtt{h}_T^*(G/P) \cong \mathbf{D}_P^{gr*}$ between the graded T-equivariant cohomology of G/P and some projective $\mathbf{D}_F^{gr}$-module $\mathbf{D}_{F,P}^{gr*}$.

Definition 5.4.1. *Let $\mathbf{D}_F^*$ ($\mathbf{D}_F^{gr*}$) denote the S-dual of $\mathbf{D}_F$ (S^{gr}-dual of $\mathbf{D}_F^{gr}$)*

$$\mathbf{D}_F^* = Hom_S(\mathbf{D}_F, S), \mathbf{D}_F^{gr*} = Hom_{S^{gr}}(\mathbf{D}_F^{gr}, S^{gr}),$$

where $\mathbf{D}_F$ ($\mathbf{D}_F^{gr}$) is considered as a left S-module (S^{gr}-module).

Analogously,

$$Q_W^* = Hom_Q(Q_W, Q), Q_W^{gr*} = Hom_{Q^{gr}}(Q_W^{gr}.Q^{gr})$$

Then Q_W^* (Q_W^{gr*}) is a free left Q-module (Q^{gr}-module) with the dual basis f_v defined by the rule $f_v(\delta_w) = \delta_{w,v}^{Kr}$, where $\delta_{w,v}^{Kr}$ equals to 1 if $w = v$ and 0 otherwise.

Remark 5.4.2. Note that $\mathbf{D}_F = \mathbf{D}_F^{gr} \otimes_{S^{gr}} S$. Thus there is a map $\mathbf{D}_F^{gr*} \to \mathbf{D}_F^*$ which is injective, since it sends the S^{gr}-basis of $\mathbf{D}_F^{gr}$ to S-basis of $\mathbf{D}_F^*$.

The natural inclusion $\mathbf{D}_F \to Q_W$ induces an isomorphism $\mathbf{D}_F \otimes_S Q \cong Q_W$. Therefore any S-linear map $F \cong \mathbf{D}_F \to S$ gives rise to a Q-linear map $F_Q \colon Q_W \cong \mathbf{D}_F \otimes_S Q \to Q$. The assignment $F \mapsto F_Q$ defines a homomorphism $D_F^* \to Q_W^*$ which is injective by [6, Theorem 10.7]. Then the corresponding homomorphism $D_F^{gr*} \to Q_W^{gr*}$ is injective. Following [30] we introduce the following:

Definition 5.4.3. *Define the $\odot$-action of Q_W on Q_W^* by the rule*

$$q\delta_w \odot pf_v = qw(p)f_{wv}$$

Lemma 5.4.4. *The $\odot$ action endows Q_W^* with the structure of a left Q_W-module*

Proof: Note that $q_1\delta_{w_1} \odot (q\delta_w \odot pf_v) = q_1w_1(q)w_1w(p)f_{w_1wv} = (q_1\delta_{w_1}q\delta_w) \odot pf_v$. ∎

Note that $\odot$ descends to an action of Q_W^{gr} on Q_W^{gr*}.

Proposition 5.4.5. *The $\odot$-action of $\mathbf{D}_F$ ($\mathbf{D}_F^{gr}$) on Q_W^* restricts to the action on $\mathbf{D}_F^*$ ($\mathbf{D}_F^{gr*}$.)*

Proof: The statement about $\mathbf{D}_F$ is proved in [30, Theorem 2.5]. The statement for $\mathbf{D}_F^{gr}$ follows since $\mathbf{D}_F^{gr*} = \mathbf{D}_F^* \cap Q_W^{gr*}$ inside Q_W^*. ∎

Definition 5.4.6. *Define S_{W/W_P} (resp. S^{gr}_{W/W_P}) as a free S (resp. S^{gr})-module with basis $\delta_{\overline{w}}, \overline{w} \in W/W_P$. Define $Q_{W/W_P} = S_{W/W_P} \otimes_S Q$ and $Q^{gr}_{W/W_P} = S^{gr}_{W/W_P} \otimes_{S^{gr}} Q^{gr}$.*

Note that Q_{W/W_P} (resp. Q^{gr}_{W/W_P}) has a natural structure of the left Q_W (resp. Q_W^{gr})-module given by the formula

$$q\delta_w \cdot q_1\delta_{\overline{w_1}} = qw(q_1)\delta_{\overline{ww_1}},$$

and there is a canonical Q_W (resp. Q_W^{gr})-module epimorphism $Q_W \to Q_{W/W_P}$ (resp. $Q_W^{gr} \to Q^{gr}_{W/W_P}$) given by $\delta_w \mapsto \delta_{\overline{w}}$.

Definition 5.4.7. *[6, §11] Define $\mathbf{D}_{F,P}$ (resp. $\mathbf{D}^{gr}_{F,P}$) as the image of the composition*

$$\mathbf{D}_F \to Q_W \to Q_{W/W_P} (\textit{resp. } \mathbf{D}_F^{gr} \to Q_W^{gr} \to Q^{gr}_{W/W_P}).$$

The canonical inclusion $S_W \to \mathbf{D}_F$ (resp. $S_W^{gr} \to \mathbf{D}_F^{gr}$) induces an inclusion $S_{W/W_P} \to \mathbf{D}_{F,P}$ (resp. $S^{gr}_{W/W_P} \to \mathbf{D}^{gr}_{F,P}$) and the dual map $\mathbf{D}^*_{F,P} \to S^*_{W/W_P}$ (resp. $\mathbf{D}^{gr*}_{F,P} \to S^{gr*}_{W/W_P}$).

The same reasoning as in Remark 5.4.2 shows that there is an inclusion $\mathbf{D}^{gr*}_{F,P} \subseteq \mathbf{D}^*_{F,P}$.

Lemma 5.4.8.

$$\mathbf{D}_F^{gr*} = \mathbf{D}_F^* \cap S_W^{gr*} \textit{ in } S_W^*$$
$$\mathbf{D}^{gr*}_{F,P} = \mathbf{D}^*_{F,P} \cap S^{gr*}_{W/W_P} \textit{ in } S^*_{W/W_P}$$

Proof: The lemma follows from the fact that Q^{gr} is faithfully flat S^{gr}-module and the equalities hold after taking tensor product with Q^{gr}: $DF^{gr*} \otimes_{S^{gr}} Q^{gr} = Q_W^{gr*}$ and $\mathbf{D}^{gr*}_{F,P} \otimes_{S^{gr}} Q^{gr} = Q^{gr*}_{W/W_P}$. ∎

Lemma 5.4.9. *The dual homomorphism $\mathbf{D}^*_{F,P} \to S^*_{W/W_P}$ (resp. $\mathbf{D}^{gr*}_{F,P} \to S^{gr*}_{W/W_P}$) is injective.*

Proof: The ungraded statement is given by [6, Lemma 11.5]. The graded homomorphism $\mathbf{D}^{gr*}_{F,P} \to S^{gr*}_{W/W_P}$ is injective since it is given as a restriction of the ungraded homomorphism $\mathbf{D}^*_{F,P} \to S^*_{W/W_P}$ to $\mathbf{D}^{gr*}_{F,P} \subseteq \mathbf{D}^*_{F,P}$. ∎

5.5 Demazure algebra and cohomology

In this section we connect the modules $\mathbf{D}_{F,P}^{gr*}$ of the previous section to the cohomology of homogeneous spaces. Note that T-fixed points on G/P correspond to the right cosets W/W_P. So there is an embedding

$$W/W_P \to G/P,$$

where W/W_P is considered as a disjoint union of rational points indexed by the set W/W_P. There is a natural action of W on W/W_P, thus a $N_G(T)$-action on W/W_P, and the embedding $W/W_P \to G/P$ is $N_G(T)$-equivariant. Thus there is a W-action on $\mathtt{h}_T^*(W/W_P)$ defined in Section 3.5, and the pullback of embedding gives a W-equivariant map of S^{gr}-modules

$$\mathtt{h}_T^*(G/P) \to \mathtt{h}_T^*(W/W_P).$$

We will call this map restriction to the fixed point locus. Since W/W_P is a disjoint union of rational points indexed by W/W_P, there is a natural identification

$$\mathtt{h}_T^*(W/W_P) \cong S_{W/W_P}^{gr*},$$

which sends class of a point indexed by $\overline{w} \in W/W_P$ to the basis element $f_{\overline{w}}$.

Lemma 5.5.1. *The isomorphism*

$$\mathtt{h}_T^*(W/W_P) \cong S_{W/W_P}^{gr*}$$

is W-equivariant, where the W-action on $\mathtt{h}_T^*(W/W_P)$ is constructed in Section 3.5 and the W-action on S_{W/W_P}^{gr*} is given by the restriction of $\odot$-action (Definition 5.4.3) of S_W^{gr} on S_{W/W_P}^{gr*}

Proof: Take $q \in S^{gr} = \mathtt{h}_T^*(k)$ and take the element $q[pt_{\overline{w}}]$, where $[pt_{\overline{w}}]$ denotes the class of the point indexed by $\overline{w} \in W/W_P$. Then for any $v \in W$ we have $v \cdot (q[pt_{\overline{w}}]) = (v \cdot q)[pt_{v \cdot \overline{w}}] = v(q)[pt_{\overline{vw}}]$. Thus the isomorphism sends the element $v \cdot q[pt_{\overline{w}}]$ in $\mathtt{h}_T^*(W/W_P)$ to $\delta_v \odot q f_{\overline{w}}$ in S_{W/W_P}^{gr*}. ∎

According to Section 4.5, the cells of G/P are given by the Schubert varieties $X_w^P = \overline{BwP/P}$ and the classes of Bott-Samelson varieties $[X_{I_w}^P]$ form a basis of $\mathtt{h}_T^*(G/P)$ over S^{gr} and $\mathtt{h}_T(G/P)$ over S.

Proposition 5.5.2. *The restriction to fixed points locus homomorphism is injective and identifies $\mathtt{h}_T^*(G/P)$ with $\mathbf{D}_{F,P}^{gr*}$ inside S_{W/W_P}^{gr*} :*

$$\begin{array}{ccc} \mathtt{h}_T^*(G/P) & \longrightarrow & \mathtt{h}_T^*(W/W_P) \\ \downarrow & & \downarrow \\ \mathbf{D}_{F,P}^{gr*} & \longrightarrow & S_{W/W_P}^{gr*}. \end{array}$$

Proof: By [7, Theorem 8.11], the statement is true for the ungraded theory: $\mathtt{h}_T(G/P) \to \mathtt{h}_T(W/W_P) = S^*_{W/W_P}$ is injective and its image is $\mathbf{D}^*_{F,P}$. Note that $\mathtt{h}^*_T(G/P)$ is a subgroup of $\mathtt{h}_T(G/P)$ given by linear combinations of $[X^P_{I_w}], w \in W^P$ with coefficients in $S^{gr} \subseteq S$, and the graded restriction map $\mathtt{h}^*_T(G/P) \to \mathtt{h}^*_T(W/W_P)$ is a restriction of corresponding ungraded map. Then the graded restriction map is injective. Note that $\mathbf{D}^*_{F,P}$ in S^*_{W/W_P} is a free S-module and its basis lies in $\mathbf{D}^{gr*}_{F,P} = S^{gr*}_{W/W_P} \cap \mathbf{D}^*_{F,P}$ by Lemma 5.4.8, then the image of $\mathtt{h}^*_T(G/P)$ coincides with $\mathbf{D}^{gr*}_{F,P}$. ∎

Lemma 5.5.3. $\mathbf{D}^{gr*}_F$ *is a free left* $\mathbf{D}^{gr}_F$*-module of rank one.*

Proof: Let $[pt]$ denote the class of a point in $\mathbf{D}^{gr*}_F = \mathtt{h}^*_T(G/B)$. By [30, Theorem 3.4] the elements $Y_{I_w} \odot [pt]$ equal to the Bott-Samelson classes $[X_{I_w}]$ in $\mathtt{h}_T(G/B)$. Then $Y_{I_w} \odot [pt]$ form a S^{gr}-basis of $\mathbf{D}^{gr*}_F$, then $\mathbf{D}^{gr*}_F$ is a free $\mathbf{D}^{gr}_F$-module with basis $[pt]$. ∎

Lemma 5.5.4. $\mathbf{D}^{gr*}_{F,P}$ *has a structure of the* $\mathbf{D}^{gr}_F$*-module and the projection* $\mathbf{D}^{gr*}_F \to \mathbf{D}^{gr*}_{F,P}$ *corresponding to the push-forward map* $\mathtt{h}^*_T(G/B) \to \mathtt{h}^*_T(G/P)$ *is a* $\mathbf{D}^{gr}_F$*-module homomorphism*

Proof: By [30, Lemma 3.3], the $\mathbf{D}_F$-action on $\mathbf{D}^*_F$ descends to $\mathbf{D}_F$-action on $\mathbf{D}^*_{F,P}$. Note that $\mathbf{D}^{gr}_F$ acts on Q^{gr*}_{W/W_P}, thus $\mathbf{D}^{gr}_F$ acts on $\mathbf{D}^{gr*}_{F,P}$ by Lemma 5.4.8. Note that the push-forward map $\mathtt{h}^*_T(G/B) \to \mathtt{h}^*_T(G/P)$ is W-equivariant. Thus the corresponding homomorphism $\mathbf{D}^{gr*}_F \to \mathbf{D}^{gr*}_{F,P}$ is a S^{gr}_W-module homomorphism. Then it is a $\mathbf{D}^{gr}_F$-module homomorphism since the elements x_α are regular in S^{gr}. ∎

Chapter 6

Motives of homogeneous spaces

Throughout this chapter we consider a split semisimple group G, a special parabolic subgroup P, a versal G-torsor $E \to \operatorname{Spec} K$ (2.4.18) and the corresponding twisted homogeneous space E/P. In this chapter we prove the main result (Theorem 6.4.10) of this thesis. The results of this chapter are original to the author.

6.1 The convolution algebra of a smooth projective morphism

Let $X \in Sm_k$. Consider a smooth projective map $f\colon Y \to X$ of relative dimension d.

Definition 6.1.1. *Denote by Y_X^n the n-fold fiber product $Y \times_X Y \times_X \ldots \times_X Y$ (n times). Then $Y_X^\bullet$ will form a simplicial scheme with standard projections as face maps.*

Lemma 6.1.2. *Let $p_{12}, p_{23}, p_{13}\colon Y_X^3 \to Y_X^2$ be standard projections. Then the rule*

$$x * y = p_{13*}(p_{12}^*(x) \cap p_{23}^*(y))$$

defines an associative product on $\mathtt{h}^*(Y_X^2)$ and the product of m elements is given by the formula

$$x_1 * x_2 \ldots * x_m = p_{1,m+1}(p_{1,2}^*(x_1) \cap p_{2,3}^*(x_2) \cap \ldots \cap p_{m,m+1}^*(x_m)).$$

Proof: For $a, b, c \in \mathtt{h}^*(Y_X^2)$ we have that $a * (b * c) = p_{13*}(p_{12}^*(a) \cap p_{23}^* p_{13*}(p_{12}^*(b) \cap p_{23}^*(c))) = p_{13*}(p_{12}^*(a) \cap p_{124*} p_{234}^*(p_{12}^*(b) \cap p_{23}^*(c))) = p_{13*}(p_{124*}(p_{12}^*(a) \cap p_{23}^*(b) \cap p_{34}^*(c))) = p_{14*}(p_{12}^*(a) \cap p_{23}^*(b) \cap p_{34}^*(c)) = (a * b) * c$. The analogous formula for n elements is obtained by induction. ∎

Remark 6.1.3. Note that the convolution product acts on graded components in the following way

$$\mathtt{h}^m(Y_X^2) \otimes \mathtt{h}^n(Y_X^2) \to \mathtt{h}^{m+n-d}(Y_X^2).$$

Then $\mathtt{h}^d(Y_X^2)$ is an algebra with respect to the convolution product.

This construction is functorial with respect to base change:

Lemma 6.1.4. *Let $Y \to X$ be smooth projective, $A \in \mathbf{Sm}_k$ and $f\colon A \to X$. Take $B = A \times_X Y$. Then the pullback of the map $B_A^2 \to Y_X^2$ induces a homomorphism of convolution algebras $\mathtt{h}^*(Y_X^2) \to \mathtt{h}^*(B_A^2)$*

Proof: Note that for every $1 \leqslant i < j \leqslant 3$ there is a Cartesian square

$$\begin{array}{ccc} B_A^3 & \longrightarrow & Y_X^3 \\ \downarrow{\scriptstyle p_{ij}} & & \downarrow{\scriptstyle p_{ij}} \\ B_A^2 & \longrightarrow & Y_X^2 \end{array}$$

The vertical arrows are flat, so the square is transverse. Thus the pushforwards commute with pullbacks and the pullback $\mathtt{h}(Y_X^2) \to \mathtt{h}(B_A^2)$ is a homomorphism of convolution algebras. ∎

Assume that $X' \to X$ is an open inclusion and let Y' be the fiber: $Y' = Y \times_X X'$.

Proposition 6.1.5. *The kernel of the convolution algebra homomorphism $\mathtt{h}^*(Y_X^2) \to \mathtt{h}^*(Y'^2_{X'})$ consists of nilpotents (with respect to the convolution product).*

Proof: Let Z be the closed complement of X' in X and $W = Z \times_X Y$ be the fiber over Z. Then for any n two squares on the diagram

$$\begin{array}{ccccc} W_Z^n & \longrightarrow & Y_X^n & \longleftarrow & Y'^n_{X'} \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & X & \longleftarrow & X' \end{array} \qquad (1)$$

are Cartesian. Then for any projection $p_{ij}\colon Y_X^n \to Y_X^2$ the diagram

$$\begin{array}{ccc} W_Z^n & \longrightarrow & Y_X^n \\ \downarrow & & \downarrow{\scriptstyle p_{ij}} \\ W_Z^2 & \longrightarrow & Y_X^2 \end{array} \qquad (2)$$

is Cartesian. Now if an element x lies in the kernel of $\mathtt{h}(Y_X^2) \to \mathtt{h}(Y'^2_{X'})$, then by the localization sequence x lies in the image of $\mathtt{h}(W_Z^2)$. Take $d = \dim X + 1$. Then the d-fold convolution product of x equals to

$$x^{*d} = p_{1,d*}(p_{12}^*(x) \cap p_{23}^*(x) \cap \ldots \cap p_{d,d+1}^*(x))$$

Where p_{ij} are projections $Y_X^{d+1} \to Y_X^2$. Note that each $p_{ij}^*(x)$ lies in the image of $\mathtt{h}(W_Z^{d+1}) \to \mathtt{h}(Y_X^{d+1})$. Since the left square in (1) is Cartesian, applying Corollary A.0.9 to the map $Y_X^{d+1} \to X$ implies that the product $p_{12}^*(x) \cap p_{23}^*(x) \cap \ldots \cap p_{d,d+1}^*(x)$ is zero, thus d-fold convolution product x^{*d} is zero.

∎

6.2 Endomorphism ring of a versal homogeneous space

The aim of this section is to prove the Theorem 6.2.4. We apply the general setting of Section 6.1 to relate the endomorphism ring of motive $M_{\mathtt{h}}(E/P)$ to the convolution algebra $\mathtt{h}_G^*(G/P \times G/P)$. Let P be a parabolic subgroup of G. Fix a system of good representations (V_i, U_i) of G. For any U_i in the system consider a smooth projective map.

$$U_i/P \to U_i/G$$

Then by Lemma 6.1.2 there is a convolution product on $\mathtt{h}^*((U_i/P)^2_{U_i/G})$. For a versal torsor $E \to \operatorname{Spec} K$ let V denote its ambient G-representation (2.4.18). By definition there is an open subset U of V such that $E \to U$ is a fiber over the generic point $\operatorname{Spec} K \to U/G$. Then $E \to U$ is a limit of open embeddings

$$E = \lim_{W \to U/G} E_W$$

where W runs over all open subsets of U/G and E_W is the fiber over the open subset: $E_Z = U \times_{U/G} (W)$.

There is the following diagram consisting of Cartesian squares:

$$\begin{array}{ccccc}
E \times^P U_i & \longrightarrow & V \times^P U_i & \longrightarrow & U_i/P \\
\downarrow & & \downarrow & & \downarrow \\
E \times^G U_i & \longrightarrow & V \times^G U_i & \longrightarrow & U_i/G
\end{array}$$

Proposition 6.2.1. *Consider the induced map of convolution algebras*

$$f_i\colon \mathtt{h}^*((U_i/P)^2_{(U_i/G)}) \to \mathtt{h}^*((E\times^P U_i)^2_{(E\times^G U_i)})$$

Then f_i is surjective, $\ker(f_i)$ *consists of nilpotents and the inclusion $U_i \to U_{i+1}$ induces a surjective homomorphism* $\ker(f_{i+1}) \to \ker(f_i)$

Proof: Since $(E\times^P U_i)^2_{E\times^G U_i} = \lim_{W\to U_i/G}(E_W \times^P U_i)^2_{E_W\times^G U_i}$, then

$$\mathtt{h}((E\times^P U_i)^2_{(E\times^G U_i)}) = \operatorname{colim}_{W\to U_i/G}\mathtt{h}((E_W\times^P U_i)^2_{(E_W\times^G U_i)}).$$

For every open W in U/G the map of convolution algebras

$$\mathtt{h}((V\times^P U_i)^2_{(V\times^G U_i)}) \to \mathtt{h}((E_W\times^P U_i)^2_{(E_W\times^G U_i)})$$

is surjective and its kernel consists of nilpotents by Proposition 6.1.5. Note that $V\times^G U_i \to U_i/G$ is a vector bundle, therefore the pullback

$$\mathtt{h}((U_i/P)^2_{(U_i/G)}) \to \mathtt{h}((V\times^P U_i)^2_{(V\times^G U_i)})$$

is an isomorphism. Then the homomorphism f_k is a colimit of surjective maps with kernel consisting of nilpotents. Therefore it is surjective with kernel consisting of nilpotents. The kernel of f_i is covered by $\operatorname{colim}_Z \mathtt{h}_*((Z\times^P U_i)^2_{(Z\times^G U_i)})$ where Z runs over closed subsets of V that include $V\setminus U$ and closures of preimages of closed subsets of U/G. For any Z the inclusion $U_i \to U_{i+1}$ induces a surjection $\mathtt{h}_*((Z\times^P U_{i+1})^2_{(Z\times^G U_{i+1})}) \to \mathtt{h}_*((Z\times^P U_i)^2_{(Z\times^G U_i)})$, hence the map $\ker(f_{i+1}) \to \ker(f_i)$ is surjective. ∎

Lemma 6.2.2. *There are isomorphisms of convolution algebras*

$$\mathtt{h}^*((U_i/P)^2_{(U_i/G)}) \cong \mathtt{h}^*((G/P\times G/P)\times^G U_i)$$

where the action of G on $G/P\times G/P$ is given by $(g_1P, g_2P)\cdot g = (g^{-1}g_1P, g^{-1}g_2P)$. and the convolution product on the right hand side is given by three projections $(G/P^3)\times^G U_i \to (G/P^2)\times^G U_i$.

Proof: There is a commutative diagram

$$\begin{array}{ccc} G/P\times^G U_i & \longrightarrow & U_i/P \\ \downarrow & & \downarrow \\ U_i/G & = & U_i/G \end{array}$$

where the upper arrow is an isomorphism given by $(gP, u)\cdot G \mapsto ugP$. Then by functoriality Lemma 6.1.4 there is an isomorphism of convolution algebras

$$\mathtt{h}((U_i/P)^2_{(U_i/G)}) = \mathtt{h}((G/P\times^G U)^2_{(U_i/G)}).$$

The isomorphism $(G/P\times^G U_i)^n_{(U_i/G)} \cong (G/P)^n \times^G U_i$ respects the projections, therefore there is an isomorphism of convolution algebras $\mathtt{h}((G/P\times^G U)^2_{(U_i/G)}) \cong \mathtt{h}((G/P\times G/P)\times^G U_i)$. ∎

By functoriality 6.1.4 the inclusion $U_i \to U_{i+1}$ induces a convolution algebra homomorphism

$$\mathtt{h}(E\times^P U_{i+1})^2_{(E\times^G U_{i+1})} \to \mathtt{h}(E\times^P U_i)^2_{(E\times^G U_i)}$$

Lemma 6.2.3. *There is an isomorphism of convolution algebras*

$$\varprojlim_{i\to\infty} \mathtt{h}^*((E\times^P U_i)^2_{(E\times^G U_i)}) \cong \mathtt{h}^*(E/P\times E/P)$$

Proof: Note that $E\times^G U_i$ is an open subscheme in the affine space $E\times^G V_i$ and $E\times^P U_i$ is on open subset of the vector bundle $E\times^P V_i$ over E/P and the codimension of its complement is equal to the codimension of the complement of U_i in V_i. Then by Lemma 3.3.1 for sufficiently large i there projection induces an isomorphism

$$\mathtt{h}((E/P)^n) \to \mathtt{h}((E\times^P U_i)^n_{(E\times^G U_i)}), n=2,3.$$

This isomorphism respects pullbacks and pushforwards of projection, thus for $n=2$ it induces an isomorphism of convolution algebras $\mathtt{h}(E/P\times E/P) \to \mathtt{h}((E\times^P U_i)^2_{(E\times^G U_i)})$ for sufficiently large i. ∎

Theorem 6.2.4. *Let $d = \dim G/P$ There is a surjective homomorphism of graded convolution algebras*

$$\mathtt{h}^*_G(G/P\times G/P) \to \mathtt{h}^*(E/P\times E/P)$$

that lifts idempotents and isomorphisms strictly.

Proof: By Proposition 6.2.1 and Lemma 6.2.2 for any number i there is a surjective convolution algebra homomorphism with kernel consisting of nilpotents and $\ker(f_{i+1}) \to \ker(f_i)$ is surjective where

$$f_i\colon \mathtt{h}^*((G/P\times G/P)\times^G U_i) \to \mathtt{h}^*((E\times^P U_i)^2_{(E\times^G U_i)}).$$

Then the limit homomorphism f

$$f\colon \mathtt{h}^*_G(G/P\times G/P)\to \varprojlim_i \mathtt{h}^*((E\times^P U_i)^2_{(E\times^G U_i)})$$

is surjective and lifts decompositions and isomorphisms strictly by Lemma 4.3.4. Now the statement follows from Lemma 6.2.3. ∎

6.3 Relation to the formal affine Demazure algebra.

Let $X=G/B$. Recall that $S^{gr}=\mathtt{h}^*_T(k)=\mathtt{h}^*_B(k)$ (5.1.7,3.4.7). Since the cellular structure on X given by the Bruhat decomposition is B-equivariant (4.5), Corollary 4.4.11 implies existence of the Künneth isomorphism $\mathtt{h}^*_B(X\times X)\simeq End_{S^{gr}}(\mathtt{h}^*_B(X))$ between the respective convolution algebra and the ring of S^{gr}-linear endomorphisms.

Theorem 6.3.1. *The composition of ring maps*

$$\mathtt{h}^*_G(X\times X)\to \mathtt{h}^*_B(X\times X)\xrightarrow{\simeq} End_{S^{gr}}(\mathtt{h}^*_B(X)) \qquad (6.3.1)$$

is injective and its image is the subalgebra in $End_{S^{gr}}(\mathtt{h}^*_B(X))$ generated by multiplication by elements of $\mathtt{h}^*_G(X)=\mathtt{h}^*_B(k)=S^{gr}$ and the push-pull operators

$$p_i^*p_{i*}\colon \mathtt{h}^*_B(X)\to \mathtt{h}^*_B(G/P_i)\to \mathtt{h}^*_B(X),$$

where α_i is a simple root, P_i is the corresponding minimal parabolic subgroup and $p_i\colon X\to G/P_i$ is the corresponding projection.

Before proving Theorem 6.3.1 we need several lemmas. For any $w\in W$ let $\mathcal{O}_w$ denote the G-orbit $(B,wB)\cdot G$ in $X\times X$. Let X_w be the closure of $\mathcal{O}_w$. For the i-th simple reflection s_i we denote X_{s_i} simply by X_i.

Lemma 6.3.2. *We have $X_i=X\times_{G/P_i}X$ and, in particular, X_i is smooth.*

Proof: We have $(g_1B,g_2B)\in X\times_{G/P_i}X$ iff $g_1P_i=g_2P_i$, so $g_2=g_1h$ for some $h\in P_i$. Since $P_i=B\cup Bs_iB$, it means that either $g_2B=g_1B$ or $g_2B=g_1Bs_iB$, so $(g_1B,g_2B)\in\mathcal{O}_i\cup\Delta_X=X_i$. ∎

For an element $w\in W$ consider its reduced decomposition $w=s_{i_1}\dots s_{i_k}$. Let $I_w=(i_1,\dots,i_k)$. Define by X_{I_w} the variety

$$X_{I_w}=X\times_{G/P_{i_1}}X\times_{G/P_{i_2}}\dots\times_{G/P_{i_k}}X.$$

Then X_{I_w} is a smooth projective variety with diagonal G-action. Note that $X_{I_w} = X_{i_1} \times_X X_{i_2} \times_X \ldots \times_X X_{i_k}$. Consider the projection on the first and the last factor $p_{1,k+1}\colon X_{I_w} \to X \times X$. Then the fiber of $p_{1,k+1}$ over $\mathcal{O}_w$ is isomorphic to $\mathcal{O}_{s_{i_1}} \times_X \times \ldots \times_X \mathcal{O}_{s_{i_k}} = \mathcal{O}_w$. Let $[X_{I_w}]$ denote the image $p_{1,k+1*}(1) \in \mathtt{h}_B(X \times X)$.

Lemma 6.3.3. *The image of* $[X_{I_w}]$ *under the Künneth isomorphism* $\mathtt{h}^*_B(X \times X) \to End_{S^{gr}}(\mathtt{h}^*_B(X))$ *is the composition of push-pull operators* $p^*_{i_k} p_{i_k *} \circ \ldots \circ p^*_{i_1} p_{i_1 *}$.

Proof: By definition of the Künneth isomorphism, the image of $[X_{I_w}]$ is the S^{gr}-linear operator

$$\mathtt{h}^*_B(X) \overset{pr_1^*}{\to} \mathtt{h}^*_B(X \times X) \overset{\cdot [X_{I_w}]}{\longrightarrow} \mathtt{h}^*_B(X \times X) \overset{pr_{2*}}{\to} \mathtt{h}^*_B(X).$$

It equals to the composition

$$\mathtt{h}^*_B(X) \overset{p_1^*}{\to} \mathtt{h}^*_B(X_{I_w}) \overset{p_{k+1*}}{\to} \mathtt{h}^*_B(X).$$

Consider the diagram:

$$\begin{array}{ccccccccc}
\mathtt{h}^*_B(X) & \xrightarrow{pr_1^*} & \mathtt{h}^*_B(X_{i_1}) & \longrightarrow & \mathtt{h}^*_B(X_{(i_1,i_2)}) & \longrightarrow & \cdots & \longrightarrow & \mathtt{h}^*_B(X_{I_w}) \\
\downarrow{\scriptstyle p_{i_1*}} & & \downarrow{\scriptstyle pr_{2*}} & & \downarrow & & & & \downarrow \\
\mathtt{h}^*_B(G/P_{i_1}) & \xrightarrow{p_{i_1}^*} & \mathtt{h}^*_B(X) & \longrightarrow & \mathtt{h}^*_B(X_{i_2}) & & & & \cdots \\
 & & \downarrow{\scriptstyle p_{i_2*}} & & \downarrow & & & & \downarrow \\
 & & \mathtt{h}^*_B(G/P_{i_2}) & \xrightarrow{p_{i_2}^*} & \mathtt{h}^*_B(X) & & & & \cdots \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & \cdots & \longrightarrow & \cdots & \longrightarrow & \mathtt{h}^*_B(X)
\end{array}$$

Each square in this diagram commutes, so we get that $p_{k+1*} \circ p_1^*$ equals to the composition $p^*_{i_k} p_{i_k *} \circ \ldots \circ p^*_{i_1} p_{i_1 *}$. ∎

Lemma 6.3.4. *For any* $w \in W$ *fix some reduced decomposition* I_w *of* w*. Then the classes* $[X_{I_w}]_G$*,* $w \in W$ *form an* S^{gr}*-basis of* $\mathtt{h}^*_G(X \times X)$*.*

Proof: The closures X_w form a relative cellular G-equivariant filtration on $X \times X$ over the second X and X_{I_w} satisfy the conditions of 4.4.10. Then the statement follows from Lemma 4.4.10. ∎

Proof: (**Proof of theorem 6.3.1**)
The same reasoning as in Lemma 6.3.4 shows that the classes $[X_{I_w}]_B$ form an $\mathtt{h}^*_B(X)$-basis of $\mathtt{h}^*_B(X\times X)$. Moreover, the first map in (6.3.1) maps $[X_{I_w}]_G$ to $[X_{I_w}]_B$. Therefore, to check the injectivity it is enough to verify that the map $S^{gr} = \mathtt{h}^*_G(X) \to \mathtt{h}^*_B(X)$ is injective. The latter morphism coincides with the G-equivariant pullback $\mathtt{h}^*_G(X) \to \mathtt{h}^*_G(X\times X)$ by Lemma 3.4.6 which is injective by 4.4.10. Then the statement follows from Lemma 6.3.3. ∎

Theorem 6.3.5. *The convolution algebra $\mathtt{h}^*_G(X\times X)$ is isomorphic as R-algebra to the graded formal affine Demazure algebra $\mathbf{D}^{gr}_F$.*

Proof: By Theorem 6.3.1 the ring $\mathtt{h}^*_G(X\times X)$ is isomorphic to the subalgebra of $End_{S^{gr}}(\mathtt{h}^*_B(X))$ generated by S^{gr} and push-pull operators $p_i^* p_{i*}$. Since the map $B \to B/T$ is an affine fibration, the natural map $\mathtt{h}^*_B(X) \to \mathtt{h}^*_T(X)$ is an isomorphism. Hence we may identify S^{gr} with $\mathtt{h}^*_T(k)$ and $End_{S^{gr}}(\mathtt{h}^*_B(X))$ with $End_{S^{gr}}(\mathtt{h}^*_T(X))$. Observe that these identifications preserve push-pull operators. The inclusion $W \to X$ gives an embedding $\mathtt{h}_T(X) \to \mathtt{h}_T(W) = S^*_W \subseteq Q^*_W$. By [7, Corollary 8.7] there is the following commutative diagram:

$$\begin{array}{ccccc} \mathtt{h}_T(X) & \longrightarrow & S^*_W & \hookrightarrow & Q^*_W \\ \downarrow{\scriptstyle p_i^* p_{i*}} & & & & \downarrow{\scriptstyle A_i} \\ \mathtt{h}_T(X) & \longrightarrow & S^*_W & \hookrightarrow & Q^*_W \end{array} \qquad (6.3.2)$$

where the operator A_i is given by $A_i(f)(x) = f(x\cdot Y_i)$ for $x \in Q_W$, $f \in Q^*_W$. Note that A_i descends to an operator on Q^{gr*}_W, and the push-pull operator $p_i^* p_{i*}$ descends to an operator on $\mathtt{h}^*_T(X)$. Then the analogous commutative square exists for $\mathtt{h}^*_T(X)$ and Q^{gr*}_W.

Since Q^{gr}_W is a finite free Q^{gr}-module, the natural map $i\colon Q^{gr}_W \to End_{Q^{gr}}(Q^{gr*}_W)$ given by $i(x)(f)(y) = f(yx)$ is an inclusion. Note that every A_i lies in the image of Q^{gr}_W. The base extension form S^{gr} to Q^{gr} gives an inclusion $End_{S^{gr}}(\mathtt{h}^*_T(X)) \to End_{Q^{gr}}(Q^*_W)$ that maps push-pull operators $p_i^* p_{i*}$ to A_i. Then the subalgebra of $End_{S^{gr}}(\mathtt{h}^*_T(X))$ generated by S^{gr} and push-pull operators $p_i^* p_{i*}$ is isomorphic to a subalgebra of Q^{gr}_W generated by S^{gr} and Y_i, which is $\mathbf{D}^{gr}_F$. ∎

6.4 Equivariant endomorphism ring for a special parabolic subgroup

In this section we study the convolution ring $\mathtt{h}^*_G(G/P \times G/P)$ when P is a special parabolic subgroup of G.

Lemma 6.4.1. $\mathtt{h}^*(P/B)^{W_P} \cong \mathtt{h}^*(k)$.

Proof: Denote $\mathtt{h}^*(k)$ by R. First let us check that $\Omega(P/B)^{W_P} \cong \Omega(k)$. Here we consider $\Omega(k)$ as a submodule $\Omega(k) \cdot [P/B]$ in $\Omega(P/B)$. By [19, Lemma 32] we have $\Omega(P/B)^{W_P}_{\mathbb{Q}} \cong \Omega(k)_{\mathbb{Q}}$. For any $x \in \Omega(P/B)^{W_P}$ there is $n \in \mathbb{Z}$ such that $nx = p^*(y)$ for some $y \in \Omega(k)$. Since P/B is cellular, $\Omega(P/B)$ has a $\Omega(k)$-basis, including $[P/B]$. Then $\Omega(P/B)/\Omega(k)$ is a free $\Omega(k)$-module, and nx vanishes in this factor. Since the factor is free over $\Omega(k) = \mathbb{L}$, it has no torsion, therefore $x \in \Omega(k)$.

Note that

$$\Omega(P/B)^{W_P} \otimes_{\mathbb{L}} R[\frac{1}{|W_P|}] \to \mathtt{h}(P/B)^{W_P} \otimes_R R[\frac{1}{|W_P|}]$$

is surjective. Therefore $\mathtt{h}(P/B)^{W_P} \otimes_R R[\frac{1}{|W_P|}] = \mathtt{h}(k) \otimes_R R[\frac{1}{|W_P|}]$. Then for any $x \in \mathtt{h}(P/B)^{W_P}$ we have that $|W_P|^m x \in \mathtt{h}(k)$ for some $m \in \mathbb{N}$. So $|W_P|^m \overline{x} = 0$ in the quotient $\mathtt{h}(P/B)/\times \mathtt{h}(k)$. Then $\overline{x} = 0$ since $\mathtt{h}(P/B)/\times \mathtt{h}(k)$ is a free R-module and $|W_P|$ is a regular element of R. ∎

Lemma 6.4.2. *The map $\mathtt{h}^*_P(k) \to \mathtt{h}^*_B(k)^{W_P}$ is an isomorphism.*

Proof: We use the same arguments as [19, Remark 35]. Fix U_i in the system of good representations. Since P is special, $U_i/B \to U_i/P$ is Zariski locally-trivial with cellular fiber P/B. Then same arguments as in [19, Proposition 7] show that $\mathtt{h}^*(U_i/B)$ is a free $\mathtt{h}^*(U_i/P)$-module. We may choose a basis inductively, so in the limit this gives a basis of $\mathtt{h}^*_B(k)$ over $\mathtt{h}^*_P(k)$. Thus we get an isomorphism $\mathtt{h}^*_B(k) \cong \mathtt{h}^*_P(k) \otimes_R \mathtt{h}^*(P/B)$. After inverting $|W_P|$ we get that

$$\mathtt{h}^*_B(k)^{W_P}[1/|W_P|] \cong \mathtt{h}^*_P(k) \otimes \mathtt{h}^*(P/B)^{W_P}[1/|W_P|] = \mathtt{h}^*_P(k)[1/|W_P|].$$

Then $\mathtt{h}^*_B(k)^{W_P}/\mathtt{h}^*_P(k) \otimes_R R[1/|W_P|] = 0$, so $\mathtt{h}^*_B(k)^{W_P}/\mathtt{h}^*_P(k)$ is a $|W_P|$-torsion submodule of $\mathtt{h}^*_B(k)/\mathtt{h}^*_P(k)$. Since $\mathtt{h}^*_B(k)$ has a $\mathtt{h}^*_P(k)$-basis containing 1, we have that $\mathtt{h}^*_B(k)/\mathtt{h}^*_P(k)$ is a free $\mathtt{h}^*_P(k)$-module therefore it has no $|W_P|$ torsion, since $\mathtt{h}^*_P(k) \subseteq \mathtt{h}^*_B(k) = S^{gr}$ and S^{gr} is a free R-module. Then $\mathtt{h}^*_B(k)^{W_P}/\mathtt{h}^*_P(k) = 0$. ∎

Remark 6.4.3. Consider the W-action on $\mathrm{Maps}(W/W_P, S^{gr})$ given by

$$(w\cdot f)(x) = w\cdot f(w^{-1}x),\quad x\in W/W_p,\ f\in \mathrm{Maps}(W/W_p, S^{gr}).$$

Then the isomorphism $S^{gr*}_{W/W_P} \cong \mathrm{Maps}(W/W_P, S^{gr})$ is W-equivariant. Then by Lemma 5.5.1 the pullback map

$$\mathtt{h}_T(G/P)\to \mathtt{h}_T(W/W_P) = \mathrm{Maps}(W/W_P, \mathtt{h}_T(k)) \text{ is } W\text{-equivariant.}$$

Lemma 6.4.4. *Consider the restriction map $\mathtt{h}^*_G(G/P)\to \mathtt{h}^*_T(G/P)$. It induces an isomorphism*

$$\mathtt{h}^*_G(G/P)\to \mathtt{h}^*_T(G/P)^W.$$

Proof: By Lemma 3.4.6 the composition of restriction homomorphism $\mathtt{h}^*_G(G/P)\to \mathtt{h}^*_T(G/P)$ with the isomorphism $\mathtt{h}^*_T(G/P)\simeq \mathtt{h}^*_G(G/T\times G/P)$ is given by the pullback of the second factor projection

$$\mathtt{h}^*_G(G/P)\to \mathtt{h}^*_G(G/T\times G/P).$$

Then the image is contained in $\mathtt{h}^*_G(G/T\times G/P)^W$. Recall that the T-fixed points of G/P are given by the natural embedding $W/W_P\to G/P$. Here we consider W/W_P as finite constant scheme with trivial T-action. For any U_i in the good system of representations of G one has a commutative diagram of schemes:

$$\begin{array}{ccc} G/P\times^G U_i & \longleftarrow & G/P\times^T U_i \\ \simeq\uparrow & & \uparrow \\ U_i/P & \xleftarrow{\ f\ } & W/W_P\times^T U_i. \end{array}$$

The leftmost arrow is a scheme isomorphism given by $uP\to (P,u)G$, the upper arrow is the projection, and the rightmost arrow arises from the fixed-point embedding $W/W_P\to G/P$. Then the bottom arrow f is given by

$$f\colon (wW_P, u)T\mapsto u\sigma\cdot P \text{ where } \sigma\in N_G(T) \text{ such that } \sigma T W_P = wW_P.$$

This diagram is compatible with the embedding $U = U_i\to U_{i+1}$ in the Borel construction, hence it induces the commutative diagram of equivariant pullbacks:

$$\begin{array}{ccc} \mathtt{h}^*_G(G/P) & \longrightarrow & \mathtt{h}^*_T(G/P) \\ \downarrow\simeq & & \downarrow \\ \mathtt{h}^*_P(k) & \xrightarrow{\ f^*\ } & \mathtt{h}^*_T(W/W_P). \end{array}$$

By lemma 6.4.3 the rightmost map is W-equivariant, so we have a diagram

$$\begin{CD} \mathtt{h}^*_G(G/P) @>>> \mathtt{h}^*_T(G/P)^W \\ @V{\simeq}VV @VVV \\ \mathtt{h}^*_P(k) @>{f^*}>> \mathtt{h}^*_T(W/W_P)^W. \end{CD} \qquad (*)$$

Recall that $\mathtt{h}^*_T(W/W_P) = \mathrm{Maps}(W/W_P, \mathtt{h}^*_T(k))$ and by definition of W-action on this set we have

$$\mathrm{Maps}(W/W_P, \mathtt{h}^*_T(k))^W = \mathrm{Maps}_W(W/W_P, \mathtt{h}^*_T(k)) = (\mathtt{h}^*_T(k))^{W_P}.$$

By the construction of f we see that the map $f^* \colon \mathtt{h}^*_P(k) \to \mathrm{Maps}(W/W_P, \mathtt{h}^*_T(k))$ is given by $x \mapsto f_x$, $f_x(w) = w \cdot \pi^*(x)$ where $\pi^* \colon \mathtt{h}^*_P(k) \to \mathtt{h}^*_T(k)$ is the restriction map. Thus, via the identification

$$f^* \colon \mathtt{h}_P(k) \to Maps(W/W_P, \mathtt{h}^*_T(k))^W = \mathtt{h}^*_T(k)^{W_P}$$

the map f^* is given by the usual restriction map $\mathtt{h}^*_P(k) \to \mathtt{h}^*_T(k)^{W_P}$ which is an isomorphism by Lemma 6.4.2. The fixed-point pullback $\mathtt{h}^*_T(G/P) \to \mathtt{h}^*_T(W/W_P)$ is injective by Proposition 5.5.2. Thus in the diagram $(*)$ the rightmost arrow is injective and the bottom arrow is an isomorphism, then the upper arrow is an isomorphism as well. ∎

We will need the following generalization of the previous lemma for equivariant motives.

Lemma 6.4.5. *Suppose X is a smooth G-variety of dimension d such that $\mathtt{h}^*_G(X) \to \mathtt{h}^*_T(X)^W$ is an isomorphism. Then for any idempotent $\rho \in \mathtt{h}^d_G(X \times X)$ the map*

$$\mathtt{h}^*_G(X, \rho) \to \mathtt{h}^*_T(X, \bar\rho)^W \text{ is an isomorphism,}$$

where $\bar\rho$ is the image of ρ in $\mathtt{h}^d_T(X \times X)$.

Proof: Recall that $\mathtt{h}^*_G(X, \rho) = \rho \cdot \mathtt{h}^*_G(X)$ and we have a direct sum decomposition $\mathtt{h}^*_G(X) = \mathtt{h}^*_G(X, \rho) \oplus \mathtt{h}^*_G(X, id - \rho)$. Let $\bar\rho$ denote the image of ρ in $\mathtt{h}^d_T(X \times X)$. Then the decomposition $\mathtt{h}^*_T(X) = \mathtt{h}^*_T(X, \bar\rho) \oplus \mathtt{h}^*_T(X, id - \bar\rho)$ is W-equivariant. Thus the surjection $\mathtt{h}^*_G(X) \to \mathtt{h}^*_T(X)^W$ equals to the direct sum of its restrictions to $\mathtt{h}^*_G(X, \rho)$ and $\mathtt{h}^*_G(X, id - \rho)$:

$$\mathtt{h}^*_G(X, \rho) \oplus \mathtt{h}^*_G(X, id - \rho) \to (\mathtt{h}^*_T(X, \bar\rho) \oplus \mathtt{h}^*_T(X, id - \bar\rho))^W = \mathtt{h}^*_T(X, \bar\rho)^W \oplus \mathtt{h}^*_T(X, id - \bar\rho)^W.$$

Then each of the maps $\mathtt{h}^*_G(X, \rho) \to \mathtt{h}^*_T(X, \bar\rho)^W$ and $\mathtt{h}^*_G(X, id - \rho) \to \mathtt{h}^*_T(X, id - \bar\rho)^W$ is an isomorphism. ∎

Let $\mathcal{M}_{sp,\mathtt{h}}$ denote the additive subcategory of the category of G-equivariant motives $\mathcal{M}_{G,\mathtt{h}}$, generated by the motives $M_G(G/P)(i)$ where P is a special subgroup of G, $i \in \mathbb{Z}$

Lemma 6.4.6. *For any two special parabolic subgroups P_1, P_2 the G-equivariant motive $M_G(G/P_1 \times G/P_2)$ belongs to $\mathcal{M}_{sp,\mathtt{h}}$.*

Proof: By Section 4.5, the variety $G/P_1 \times G/P_2$ is relatively G-equivariant cellular over varieties of the form G/P_w where $P_w = R_uP_1 \cdot (P_1 \cap wP_2w^{-1})$ Since P_1 is special, $P_w \subseteq P_1$ is special by Lemma 2.7.5. Then by Section 4.5 the motive of $M_G(G/P_1 \times G/P_2)$ is isomorphic to a sum of motives of varieties $M_G(G/P_w)(i_w)$ for some $i_w \in \mathbb{Z}$ and $M_G(G/P_w)(i_w)$ lies in $\mathcal{M}_{sp,\mathtt{h}}$. ∎

Corollary 6.4.7. *For a special subgroup P the homomorphism*

$$\mathtt{h}^*_G(G/P \times G/P) \to \mathtt{h}^*_T(G/P \times G/P)^W$$

is an isomorphism.

Proof: By Lemmas 6.4.4 and 6.4.5 for every motive M in $\mathcal{M}_{sp}$ the map $\mathtt{h}^*_G(M) \to \mathtt{h}^*_T(M)^W$ is an isomorphism and by Lemma 6.4.6 the motive $M(G/P \times G/P)$ belongs to $\mathcal{M}_{sp}$. ∎

Lemma 6.4.8. *The subgroup $\mathtt{h}^*_T(G/P \times G/P)^W$ is identified via the Künneth isomorphism $\mathtt{h}^*_T(G/P \times G/P) \to End_{S^{gr}}(\mathtt{h}^*_T(G/P))$ with $End_{S^{gr}_W}(\mathtt{h}^*_T(G/P))$.*

Proof: An element $a \in \mathtt{h}^*_T(G/P \times G/P)$ let $f_a \in End_{S^{gr}}(\mathtt{h}^*_T(G/P))$ be its image under the Künneth isomorphism. Then for any and $w \in W$ the endomorphism $f_{a^w} \in End_S(\mathtt{h}_T(G/P))$ is given by

$$x \mapsto p_{2*}(p_1^*(x) \cap a^w) = (p_{2*}(p_1^*(x^{w^{-1}}) \cap a))^w$$

Thus f_{a^w} equals to the composition

$$\mathtt{h}^*_T(G/P) \xrightarrow{w^{-1}} \mathtt{h}^*_T(G/P) \xrightarrow{f} \mathtt{h}^*_T(G/P) \xrightarrow{w} \mathtt{h}^*_T(G/P).$$

Then $\mathtt{h}^*_T(G/P \times G/P)^W$ consists of W-equivariant endomorphisms. Thus $\mathtt{h}^*_T(G/P \times G/P)^W = End_{S^{gr}_W}(\mathtt{h}^*_T(G/P))$. ∎

Proposition 6.4.9. *There is a graded algebra isomorphism*

$$End_{S_W^{gr}}(\mathtt{h}_T^*(G/P)) \cong End_{\mathbf{D}_F^{gr}}(\mathbf{D}_{F,P}^{gr*}).$$

Proof: Proposition 5.5.2 identifies $\mathtt{h}_T^*(G/P)$ with $\mathbf{D}_{F,P}^{gr*}$ as S_W^{gr}-module, where S_W^{gr}-action on $\mathbf{D}_{F,P}^{gr*}$ is given as a restriction of the $\odot$-action $\mathbf{D}_F^{gr}$ to $S_W^{gr} \subseteq \mathbf{D}_F^{gr}$. Note that for every element $a \in \mathbf{D}_F^{gr}$ there is $b \in S^{gr}$ such that $ba \in S_W^{gr}$. Then since S^{gr} is integral and $\mathbf{D}_{F,P}^{gr*}$ is a free S^{gr}-module, every S_W^{gr}-linear endomorphism is $\mathbf{D}_F^{gr}$-linear. ∎

This is the last step in establishing the main result of the present thesis:

Theorem 6.4.10. *Suppose that G is a semisimple split algebraic group over k, P is a special parabolic subgroup and E is a versal G-torsor. Then there is a surjective homomorphism that lifts idempotents and isomorphisms strictly between the ring of degree-preserving endomorphisms of the $\mathbf{D}_F^{gr}$-module $\mathbf{D}_{F,P}^{gr*}$ and the endomorphism ring of the $\mathtt{h}$-motive $M_{\mathtt{h}}(E/P)$:*

$$End^0_{\mathbf{D}_F^{gr}}(\mathbf{D}_{F,P}^{gr*}) \to End(M_{\mathtt{h}}(E/P)).$$

As a consequence, there is $1-1$ correspondence between $\mathtt{h}$-motivic decompositions of the versal homogeneous space E/P and direct sum decompositions of the graded projective $\mathbf{D}_F$-module $\mathbf{D}_{F,P}$.

Proof: Let $d = \dim G/P$. By Theorem 6.2.4 there is a surjective algebra homomorphism $\mathtt{h}_G^d(G/P \times G/P) \to \mathtt{h}^d(E/P \times E/P)$ that lifts idempotents and isomorphisms strictly. By Corollary 6.4.7 the graded convolution algebra $\mathtt{h}_G^*(G/P \times G/P)$ is isomorphic to $\mathtt{h}_T^*(G/P \times G/P)^W$ which coincides with $End_{\mathbf{D}_F^{gr}}(\mathbf{D}_{F,P}^{gr*})$ by Lemma 6.4.8 and Proposition 6.4.9. Restricting the latter equality to the degree d component we get $\mathtt{h}^d(G/P \times G/P) = End^0_{\mathbf{D}_F^{gr}}(\mathbf{D}_{F,P}^{gr*})$. ∎

Chapter 7

Application to Chow motives

In this chapter we show some application of the main result for the case of Chow motives. We set the coefficient ring R equal to $\mathbb{Z}$ and take the additive formal group law F.

In this case the algebraic objects $\mathbf{D}_F^{gr}, \mathbf{D}_F^{gr*}, \mathbf{D}_{F,P}^{gr*}$ simplify in many aspects.

- The graded ring S^{gr} becomes isomorphic to ring $\mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ of polynomials in a basis of T^*.

- The formal affine Demazure algebra $\mathbf{D}_F^{gr}$ becomes isomorphic to the the nil Hecke ring $\mathbf{D}$ (5.3.7).

- The classes of Schubert cells $[X_w] \in \mathbf{D} =^* \mathbf{D}_F^{gr*} = \mathrm{CH}_T(G/P)$ do not depend on the choice of reduced decomposition of the words w and $[X_w] = \Delta_w \odot [pt]$ where $\Delta_w \in \mathbf{D}$ are Demazure operators.

Let us denote in $\mathbf{D}_{F,P}^{gr*}$ by $\mathbf{D}_P^*$ in this case.

Lemma 7.0.11. *The kernel of the map $\mathbf{D}^* \to \mathbf{D}_P^*$ is generated as $\mathbf{D}$-module by elements $\Delta_i \odot [pt]$ where s_i is a simple reflection in W_P.*

Proof: Consider the restriction of the projection $\pi\colon G/B \to G/P$ onto a Schubert cell $\pi\colon X_w \to \pi(X_w)$. It is birational if $w \in W^P$ and its fibers have positive dimension if $w \notin W^P$. Then the pushforward map $\pi_*\colon \mathrm{CH}_T(G/B) \to \mathrm{CH}_T(G/P)$ maps the basis elements $[X_w] \in \mathrm{CH}_T(G/B)$ to basis elements $[X_w] \in \mathrm{CH}_T(G/P)$ if $w \in W^P$ or to 0 if $w \notin W^P$. Note that the class of the Schubert cell $[X_W]$ in $\mathbf{D}^* = \mathrm{CH}_T(G/B)$ and in $\mathbf{D}_P^* = \mathrm{CH}_T(G/P)$ is given by $\Delta_w \odot [pt]$. Thus the S-basis of the kernel of the pushforward map $\pi_*\colon \mathbf{D}^* \to \mathbf{D}_P^*$ consists of $\Delta_w \odot [pt]$ where $w \notin W^P$. Note that $w \notin W^P$ if and only if there is a reduced decomposition of the form $w = s_{i_1} \ldots s_{i_k}$ where $s_{i_k} \in W_P$. Then the kernel is contained in the $\mathbf{D}$-submodule generated by $\Delta_i \odot [pt]$ where $s_i \in W_P$. Since every element $\pi_*(\Delta_i \odot [pt]) = 0$ for every $s_i \in W_P$, and π_* is a D-module homomorphism, the kernel coincides with the D-submodule generated by $\Delta_i \odot [pt]$ where $s_i \in W_P$. ∎

Remark 7.0.12. Note that $\mathbf{D}_P^*$ is a cyclic $\mathbf{D}$-module generated by $[pt]$. Then any endomorphism $F \in \mathrm{End}^0_{\mathbf{D}}(\mathbf{D}_P^*)$ is determined by the value $F([pt]) = \sum_{w\in W^P} a_w[X_w]$ where $a_w \in S, \deg(a_w) = l(w)$.

Remark 7.0.13. If $w \in W^P$, s_i is a simple reflection and $l(s_iw) = l(w) - 1$ then $s_iw \in W^P$

Proof: Assume the converse, then there is a reduced expression $s_iw = s_{i_1} \dots s_{i_k}$ with $s_{i_k} \in W_P$ and $k = l(w) - 1$. Then $w = s_is_{i_1} \dots s_{i_k}$ is an expression of the length $k + 1 = l(w)$, so it is reduced and $s_{i_k} \in W_P$, then $w \notin W^P$. ∎

Now we give a criterion for which values $F([pt])$ can take.

Lemma 7.0.14. *There exists an endomorphism $F \in \mathrm{End}^0_{\mathbf{D}}(\mathbf{D}_P^*)$ with $F([pt]) = \sum_{v\in W^P} a_v\Delta_v \odot [pt]$ if and only if for every simple reflection $s_j \in W_P$ the following holds:*

$$\Delta_j(a_v) = \begin{cases} -a_{s_jv} & \text{if } l(s_jv) = l(v) - 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof: Since $\mathbf{D}_P^*$ is a direct summand of $\mathbf{D}^*$, every $\mathbf{D}$-module endomorphism descends from an endomorphism of a free $\mathbf{D}$-module $\mathbf{D}^*$. By Lemma 7.0.11 any $F' \in \mathrm{End}_{\mathbf{D}}(\mathbf{D}^*)$ descends to an endomorphism of $\mathbf{D}_P^*$ if and only if for any $s_j \in W_P$ the element $F'(\Delta_j \odot [pt])$ lies in the kernel of $\mathbf{D}^* \to \mathbf{D}_P^*$ If $F'([pt]) = \sum_{w\in W} a_w\Delta_w \odot [pt]$ then

$$F'(\Delta_j \odot [pt]) = \sum_{w\in W} \Delta_j \odot (a_w\Delta_w \odot [pt]) = \sum_{w\in W} (s_j(a_w)\Delta_j\Delta_w + \Delta_j(a_w)\Delta_w) \odot [pt] =$$

$$\sum_{v\in W} (\delta^{Kr}_{l(s_jv),l(v)-1} s_j(a_{s_jv}) + \Delta_j(a_v))\Delta_v \odot [pt]$$

The latter element is in the kernel of $\mathbf{D}^* \to \mathbf{D}_P^*$ iff for every $v \in W^P$ the element $\delta^{Kr}_{l(s_jv),l(v)-1} s_j(a_{s_jv}) + \Delta_j(a_v) = 0$ in S. Since $s_j\Delta_j = \Delta_j$ in $\mathbf{D}$ we get the desired condition on a_v when $v \in W^P$ ∎

For any $F \in \mathrm{End}_D(\mathbf{D}_P^*)$ write the corresponding matrix for the S-basis $\Delta_v \odot [pt], v \in W^P$:

$$F(\Delta_w \odot [pt]) = \sum_{v\in W^P} a_{v,w}\Delta_v \odot [pt]$$

By Remark 7.0.12 all entries of the matrix $a_{v,w}$ depend on the first column $a_v = a_{v,1}$.

Lemma 7.0.15. *If $s_j \in W$ is a simple reflection such that $l(s_jw) > l(w)$ and $s_jw \in W^P$ then*

$$a_{v,s_jw} = \begin{cases} s_j(a_{s_jv,w}) + \Delta_j(a_{v,w}) & \text{if } l(s_jv) = l(v) - 1 \\ \Delta_j(a_{v,w}) & \text{if } l(s_jv) = l(v) + 1 \end{cases}$$

Proof: If $F(\Delta_w \odot [pt]) = \sum_{v\in W^P} a_{v,w}\Delta_v \odot [pt]$ then

$$F(\Delta_{s_jw} \odot [pt]) = \sum_{v\in W^P} (\Delta_j a_{v,w}\Delta_v) \odot [pt] = \sum_{v\in W^P} (s_j(a_{v,w})\Delta_j\Delta_v + \Delta_j(a_{v,w})\Delta_v) \odot [pt] =$$

$$= \sum_{v\in W^P} (\delta^{Kr}_{l(s_jv),l(v)-1} s_j(a_{s_jv,w}) + \Delta_j(a_{v,w}))\Delta_v \odot [pt].$$

∎

When $F \in \mathrm{End}_D(\mathbf{D}^*_P)$ has degree 0, we have that $\deg(a_{v,w}) = l(v) - l(w)$, thus the matrix corresponding to F is block-diagonal, where the blocks corresponds to the lengths of elements in W^P. Since the augmentation map $Mat_{|W^P|}(S) \to Mat_{|W^P|}(\mathbb{Z})$ lifts isomorphisms and idempotents strictly, we get

Lemma 7.0.16. *There is an idempotent in $\mathrm{End}^0_{\mathbf{D}}(\mathbf{D}^*_P)$ if and only if there is an endomorphism $F \in \mathrm{End}^0_{\mathbf{D}}(\mathbf{D}^*_P)$ such that its diagonal blocks consist of idempotent matrices with $\mathbb{Z}$-entries.*

Proof: Consider the composition $\mathrm{End}^0_{\mathbf{D}}(\mathbf{D}^*_P) \to Mat^0_{|W^P|}(S) \to Mat_{|W^P|}(\mathbb{Z})$ and apply the Lemma 4.3.4. ∎

7.1 Versal Severi-Brauer varieties

Consider $G = \mathbf{PGL}_{n+1}$ and P corresponding to last $n-1$ simple roots, so $G/P = \mathbb{P}^n$. So $W = S_{n+1} = \langle s_1, \dots s_n\rangle$ and $W_P = \langle s_2, \dots, s_n\rangle$ where s_i is the transposition $(i, i+1)$. The set of minimal representatives is $W^P = \{1, s_1, s_2s_1, \dots, s_n \dots s_2s_1\}$.

In this case $S = \mathbb{Z}[\alpha_1, \dots, \alpha_n]$ is the polynomial ring in simple roots and D is the corresponding nil Hecke ring generated by S and Δ_i for $i = 1, \dots, n$ modulo the relations $\Delta_i f = s_i(f)\Delta_i + \Delta_i(f)$, $\Delta_i^2 = 0$ and usual braid relations between Δ_i and Δ_j.

Proposition 7.1.1. *If $n = p^r - 1$ for a prime number p and some r then the graded $\mathbf{D}$-module $\mathbf{D}^*_P$ is indecomposable.*

Proof: Suppose that $F \in \mathrm{End}^0_{\mathbf{D}}(\mathbf{D}^*_P)$ is an idempotent. Note that in W^P there is only one element of each length. Thus the matrix $a_{v,w}$ of F is diagonal. Let us check that all entries on the diagonal of F are simultaneously 0 or 1.

Denote by $c_{i,j} = a_{s_i \dots s_2 s_1, s_j \dots s_2 s_1}$ the (i,j)-entry of the matrix. So $\deg(c_{i,j}) = i - j$. By Lemma 7.0.14 we get that $c_{i,0} = (-1)^{n-i} \Delta_{i+1 \dots n} c_{n,0}$ for $i \geqslant 1$.

Now the recurrent formula of Lemma 7.0.15 allows to find all the matrix entries: $c_{i,i} = s_i(c_{i-1,i-1}) + \Delta_i(c_{i,i-1})$ and $c_{i,j} = \Delta_j(c_{i,j-1})$ for $i > j$. Then

$$c_{i,i} = c_{0,0} + (-1)^{n-1}\Delta_{1,\dots,n}(c_{n,0}) + (-1)^{n-2}\Delta_{213\dots n}(c_{n,0}) + \ldots + (-1)^{n-i}\Delta_{i \dots 1, i+1, \dots n}(c_{n,0})$$

Thus the matrix is fully determined by $c_{0,0} \in \mathbb{Z}$ and a homogeneous degree n polynomial $c_{n,0}$. Note that Lemma 7.0.14 implies the restriction on $c_{n,0}$:

$$\Delta_2(c_{n,0}) = \Delta_3(c_{n,0}) = \ldots = \Delta_{n-1}(c_{n,0}) = 0.$$

Then for any $k \neq i$ we have that $\Delta_k \Delta_{i-1,\dots 1, i+1 \dots n}(c_{n,0}) = 0$

Then $c_{i,i-1}$ lies in the invariant subring $S^{s_1,\dots,s_{i-1},s_{i+1},\dots s_n}$. Since $c_{i,i-1}$ has degree 1 we can write

$$c_{i,i-1} = b_1\alpha_1 + b_2\alpha_2 + \ldots b_n\alpha_n.$$

The condition that $s_k(c_{i,i-1}) = c_{i,i-1}$ for any $k \neq i$ implies that

$$b_2 = 2b_1, b_3 = 3b_1, \ldots b_i = ib_1, b_i = (n+1-i)b_n, \ldots, b_{n-2} = 3b_n, b_{n-1} = 2b_n$$

Then

$$\Delta_i(c_{i,i-1}) = 2b_i - b_{i-1} - b_{i+1} = b_1 + b_n$$

Since $ib_1 = (n+1-i)b_n$ we get that $i(b_1 + b_n) = (n+1)b_n$ is divisible by p^r. For any $i = 1, \dots n$ we have $v_p(i) < r = v_p(n+1)$, then $b_1 + b_n$ is divisible by p. Thus any $\Delta_i(c_{i,i-1})$ is divisible by p, hence any

$$c_{i,i} = c_{0,0} + \sum_{k=1}^{i} \Delta_k(c_{k,k-1}) \equiv c_{0,0} \mod p$$

Then if F is an idempotent each $c_{i,i}$ is either 0 or 1. Then $c_{i,i} = c_{0,0}$. Thus there is no non-trivial idempotent $F \in \mathrm{End}^0_{\mathbf{D}}(\mathbf{D}^*_P)$. ∎

Remark 7.1.2. By the main Theorem 6.4.10 this result gives a combinatorial proof of the theorem of Karpenko about the indecomposability of the versal Severi-Brauer variety.

7.2 The case $G = \mathbf{HSpin}_8$ and $P = P_1$

In this case $P = P_1$ is the maximal parabolic subgroup generated by all roots except the first one. The image of a versal $\mathbf{HSpin}_8$ torsor E in the set of $\mathbf{PGO}_8$-torsor satisfies condition of [38, Theorem 3.3], then E/P is generically split, then P is a special parabolic subgroup. Note that $W = \langle s_1, s_2, s_3, s_4\rangle$, $W_P = \langle s_2, s_3, s_4\rangle$, and the set of minimal coset representatives W^P is given by the Hasse diagram

$$\begin{array}{ccccccccccccc}
1 & \xrightarrow{s_1\cdot} & s_1 & \xrightarrow{s_2\cdot} & s_2s_1 & \xrightarrow{s_4\cdot} & s_4s_2s_1 & & & & & & \\
 & & & & \downarrow{\scriptstyle s_3\cdot} & & \downarrow{\scriptstyle s_3\cdot} & & & & & & \\
 & & & & s_3s_2s_1 & \xrightarrow{s_4\cdot} & s_4s_3s_2s_1 & \xrightarrow{s_2\cdot} & s_2s_4s_3s_2s_1 & \xrightarrow{s_1\cdot} & s_1s_2s_4s_3s_2s_1 & &
\end{array}$$

By the recurrent formulas of Lemma 7.0.15 we obtain:

$$\begin{aligned}
a_{1,1} &= a_\emptyset + \Delta_{12342}(a_{24321}),\\
a_{21,21} &= a_{1,1} - \Delta_{21342}(a_{24321}),\\
a_{321,321} &= a_{21,21} + \Delta_{32142}(a_{24321}) \text{ and } a_{421,421} = a_{21,21} + \Delta_{42132}(a_{24321}),\\
a_{321,421} &= \Delta_{42142}(a_{24321}) \text{ and } a_{421,321} = \Delta_{32132}(a_{24321}),\\
a_{4321,4321} &= a_{421,421} + \Delta_{32142}(a_{24321}) - \Delta_{34212}(a_{24321})\\
&= a_{321,321} + \Delta_{42132}(a_{24321}) - \Delta_{43212}(a_{24321}),\\
a_{24321,24321} &= a_{4321,4321} + (\Delta_{24321} - \Delta_{243}s_2\Delta_{12})(a_{24321}),\\
a_{124321,124321} &= a_{24321,24321} + \Delta_{12432}(s_1(a_{24321}) + \Delta_1(a_{124321})).
\end{aligned}$$

Let $\Delta^d_{i_1,i_2,..}$ denote the image $\Delta_{i_1,i_2,..}(S^d(T^*))$ modulo 2. Recall that in case $G = \mathbf{HSpin}_8$ the lattice T^* is generated by $\alpha_2, \alpha_3, \alpha_4, \omega_4$.

We claim that $\Delta_{12342}(a_{24321}) \equiv 0$ modulo 2. Indeed, let $f = \Delta_{2342}(a_{24321}) \in S^1(T^*)$. Then

$$\Delta_3(f) = \Delta_{32342}(a_{24321}) = \Delta_{23242}(a_{24321}) = \Delta_{2342}(\Delta_4(a_{24321})) = 0.$$

by Lemma 7.0.14. Now for $f = a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 + b\omega_4$ we get $\Delta_1(f) \equiv a_2 \mod 2$ but $\Delta_3(f) \equiv a_2 \mod 2$ as well.

Similarly, $\Delta_{32142}(a_{24321}) \equiv 0$. In this case denote $f = \Delta_{2142}(a_{24321})$. Then

$$\Delta_1(f) = \Delta_{12142}(a_{24321}) = \Delta_{21242}(a_{24321}) = \Delta_{2142}(\Delta_4(a_{24321})) = 0.$$

And $\Delta_3(f) \equiv \Delta_1(f) \equiv 0$.

By the same arguments, $\Delta_{32132}(a_{24321}) \equiv \Delta_{34212} \equiv 0$.

Consider now $\Delta_{21342}(a_{24321})$. Let $g = \Delta_{342}(a_{24321})$. We then have $\Delta_{32}(g) = 0$. Let $g = \sum_{2\le i\le j} c_{ij}\alpha_i\alpha_j + \sum_{2\le i} b_i\omega_4\alpha_i + d\omega_4^2$. Then

$$\Delta_2(g) \equiv c_{22}(\alpha_2 + \alpha_3 + \alpha_4) + \alpha_2(c_{23} + c_{24}) + \omega_4(b_3 + b_4).$$

The fact that $\Delta_{32}(g) \equiv 0$ implies that $c_{22} + c_{23} + c_{24} \equiv 0$. But

$$\Delta_1(g) \equiv c_{22}\alpha_1 + c_{23}\alpha_3 + c_{24}\alpha_4 + b_2\omega_4.$$

So that $\Delta_{21}(g) \equiv (c_{22} + c_{23} + c_{24}) \equiv 0$. Combining we obtain that

$$a_\emptyset \equiv a_{1,1} \equiv a_{21,21} \equiv a_{321,321} \equiv a_{421,321}$$

and

$$a_{421,421} \equiv a_{321,421} \equiv a_{4321,4321} \equiv a_{24321,24321} \equiv a_{124321,124321}.$$

Then the $\mathbf{D}$-module $\mathbf{D}_P^*$ is either indecomposable or splits into two irreducible direct summands with a generating function $1 + t + t^2 + t^3$ (over S) each.

Remark 7.2.1. The main result implies that the motive of the versal twisted form of $\mathbf{HSpin}_8/P_1$ is either indecomposable, or splits as a direct sum of motives $M = N \oplus N(3)$, where N is indecomposable with a generating function $1+t+t^2+t^3$. Using know result on motives of quadratic forms (e.g. that after splitting the algebra, the motive of a $Spin_8$-generic quadratic form splits into 2-fold Rost motives) it follows that the second decomposition is impossible, i.e., M has to be indecomposable.

Appendix A

Support of intersection products

In the remaining part of the appendix we give a proof of Corollary A.0.9.

Lemma A.0.2. *Let $X \in \mathbf{Sch}_k$ and $E \to X$ be a rank d vector bundle with zero section $z\colon X \to E$. Then the following diagram commutes:*

$$\begin{array}{ccc} \Omega_*(\mathbb{P}(E\oplus 1)) & \xrightarrow{\widetilde{c}_d(q^*E\otimes\mathcal{O}(1))\cap -} & \Omega_{*-d}(\mathbb{P}(E\oplus 1)) \\ \downarrow & & \downarrow q_* \\ \Omega_*(E) & \xrightarrow{z^*} & \Omega_{*-d}(X) \end{array}$$

Proof: There is a global section of the sheaf $q^*E \otimes \mathcal{O}(1) = \underline{Hom}(\mathcal{O}(-1), q^*E)$, given by an element $s \in Hom(\mathcal{O}(-1), q^*E)$ given by the composition of the natural embedding and projection

$$\mathcal{O}(-1) \to q^*(E \oplus 1) \to q^*E$$

One can see that the zero set $Z(s)$ consists of those points of $\mathbb{P}(E\oplus 1)$, that corresponds to additional lines 1 in $E_x \oplus 1$, over every point of $x \in X$. So X is the zero subscheme of s with regular embedding $\bar{s}\colon X \to \mathbb{P}(E \oplus 1)$ given by

$$X \stackrel{z,1}{\to} (E \oplus 1) \setminus (Z(E), 0) \to \mathbb{P}(E \oplus 1).$$

By [32, Lemma 6.6.7], the operator $\widetilde{c}_d(q^*E \otimes \mathcal{O}(1)) \cap -$ on $\Omega(\mathbb{P}(E \oplus 1))$ is given by $\bar{s}_*\bar{s}^*$. Then the right-down pass is given by $q_*\bar{s}_*\bar{s}^* = \bar{s}^*$

Note that $\bar{s}^*$ equals to the down-right pass $\Omega_*(\mathbb{P}(E \oplus 1)) \to \Omega_*(E) \stackrel{z^*}{\to} \Omega_{*-d}(X)$. ∎

Lemma A.0.3. *(Splitting) Let $X \in \mathbf{Sch}_k$ and $E \to X$ be a vector bundle of rank d. Then for any point $x \in X$ there is an open subscheme U of X with $x \in U$, and a projective morphism $p\colon X' \to X$ such that*

- *p^*E has a filtration by subbundles with linear subsequent quotients.*
- *There is an open subset U' such that $p\colon U' \to U$ is an isomorphism.*

Proof: Consider the standard splitting procedure $p\colon Fl(E) \to X$ where $Fl(E)$ is the variety of complete flags $(e_0 \subset e_1 \subset \ldots \subset e_d)$ of the vector bundle E. Then p^*E has a filtration by tautological subbundles. Since the flag varieties are projective, the map p is projective, and in some neighborhood U of any point $x \in X$ the bundle E trivializes, so $Fl(E)|_U \cong Fl \times U$, thus there is a section $s\colon U \to p^{-1}(U)$. Taking X' to be the closure of $s(U)$ in $Fl(E)$, we get a map $p\colon X' \to X$ with desired properties. ∎

Lemma A.0.4. *Let $X, Y \in \mathbf{Sch}_k$ and $p\colon X \to Y$ be a projective birational morphism. Then $p_*\colon \Omega_*(X) \to \Omega_*(Y)$ is surjective.*

Proof: First, consider the case when X and Y are smooth. Then for any $\alpha \in \Omega_*(Y)$ we have $p_*(p^*\alpha) = \alpha \cdot p_*(1_X)$ and by the degree formula $p_*(1_X) = 1_Y + a$ where $a \in \mathbb{L} \cdot \Omega^{>0}(Y)$, hence a is nilpotent in the ring $\Omega^*(Y)$, therefore $p_*(1_X)$ is invertible.

Now consider the general case $p\colon X \to Y$ with $X, Y \in \mathbf{Sch}_k$. Take an element $\beta \in \Omega_*(Y)$. Since algebraic cobordism is detected by smooth schemes by Lemma [32, 2.4.15], there is a smooth scheme Y' and projective morphism $q\colon Y' \to Y$ and an element $\beta' \in \Omega_*(Y')$ such that $q_*(\beta') = \beta$. Let $X' = X \times_Y Y'$. Then the morphism $P\colon X' \to Y$ is projective birational. Take X'' to be a resolution of singularities of X':

$$\begin{array}{ccccc} X'' & \xrightarrow{F} & X' & \xrightarrow{P} & Y' \\ & & \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle q} \\ & & X & \xrightarrow{p} & Y \end{array}$$

Then $F\colon X'' \to X$ is projective birational. Thus $X'', Y' \in \mathbf{Sm}_k$ and the map $P \circ F$ is projective birational. Then by the first case there is $\alpha' \in \Omega_*(X'')$ such that $P_*F_* = \beta'$. Then $\beta = q_*\beta' = q_*P_*F_*(\alpha') = p_*Q_*F_*$ where $Q\colon X' \to X$ is the projection. ∎

Definition A.0.5. *Let $X \in \mathbf{Sch}_k$ and Z be a closed subset of X. We say that an element $\alpha \in \Omega_*(X)$ is supported on Z if α lies in the image $\Omega_*(Z) \to \Omega_*(X)$.*

Lemma A.0.6. *Let $X, X' \in \mathbf{Sch}_k$, $f\colon X' \to X$ a smooth morphism of schemes, E is a vector bundle of rank d on X and $E' = f^*E$ and Z_i, $i = 1 \ldots m$ be irreducible closed subsets of X and $Z'_i = f^{-1}(Z_i)$. Then there are closed subsets $\widetilde{Z}_i \to Z_i$ of codimension d such that for any $\alpha \in \Omega_*(X')$ supported on Z'_i the element $\widetilde{c}_d(E') \cap \alpha$ is supported on $f^{-1}(\widetilde{Z}_i)$.*

Proof: First, consider the case when E is a line bundle. Then we have $E \cong E_1 \otimes E_2^\vee$ for some very ample line bundles E_1, E_2. Then by [32, Lemma 2.3.10] we have $\widetilde{c}_1(E) = \widetilde{c}_1(E_1) -_F \widetilde{c}_1(E_2)$. By Lemma [32, 6.6.7], the operator $\widetilde{c}_1(E_1') \cap -$ is given by $s_* s^*$ where $s\colon f^{-1}(D_1) \to X'$ and D_1 in X is the divisor of the very ample line bundle E_1. Thus $\widetilde{c}_1(E_1') \cap \alpha$ is supported on $f^{-1}(Z_i \cap D_1)$. Similarly, $\widetilde{c}_1(E_2') \cap \alpha$ is supported on $f^{-1}(Z_i \cap D_2)$, where D_2 is a divisor of E_2. Since E_1, E_2 are very ample we may choose D_1 and D_2 to intersect each Z_i by codimension 1, thus $\widetilde{c}_1(E_1') -_F \widetilde{c}_1(E_2') \cap (\alpha)$ is supported on some $f^{-1}(\widetilde{Z}_i)$ of codimension 1 for each i.

Consider the general case. For each $i = 1 \dots m$ there is a projective map $p_i\colon Y_i \to X$ given by lemma A.0.3 such that an open subset of Y_i is isomorphic to some open neighborhood of the generic point of Z_i. Let $W_i = \overline{(p_i^{-1}(U \cap Z_i))}$ be the proper transform of Z_i. Then $W_i \to Z_i$ is projective birational. Let $p_i'\colon Y_i' \to X'$ denote the pullback of p_i along f and $W_i' = f^{-1}(W_i)$.Then by lemma A.0.4 for any $\alpha \in \Omega_*(X')$ supported on Z_i' we may found a preimage $\alpha' \in \Omega_*(Y_i')$ supported on W_i'. Now by Whitney formula [32, Definition 1.1.2] we have $\widetilde{c}_d((p_i')^* E') \cap - = \prod_{j=1}^d \widetilde{c}_1(E_j')$ where E_j are linear subsequent quotients of lemma A.0.3. Now, applying inductively the case $d = 1$ for α' we can find the subset $\widetilde{W}_i$ of codimension d in W_i, such that $\widetilde{c}_d((p_i')^* E') \cap \alpha' = (\prod_{j=1}^d \widetilde{c}_1(E_j)) \cap \alpha'$ is supported on $f^{-1}(\widetilde{W}_i)$, then its pushforward is equal to $\widetilde{c}_d(E') \cap \alpha$ and is supported on $f^{-1}(\widetilde{Z}_i) = f^{-1}(p(\widetilde{W}_i))$. ∎

This allows us to proceed using the strategy of [47, Lemma 6.3]:

Lemma A.0.7. *Let $V \to B \leftarrow T$ be closed embeddings with regular f and smooth quasi-projective B. Let $\varepsilon\colon W \to B$ be a smooth morphism. Consider two Cartesian diagrams:*

$$\begin{array}{ccccc} W_V & \xrightarrow{f_W} & W & \xleftarrow{g_W} & W_T \\ \downarrow & & \downarrow{\scriptstyle \varepsilon} & & \downarrow \\ V & \xrightarrow{f} & B & \xleftarrow{g} & T \end{array} \quad \text{and} \quad \begin{array}{ccc} T & \xrightarrow{g} & B \\ {\scriptstyle \tilde{f}}\uparrow & & \uparrow{\scriptstyle f} \\ \tilde{T} & \xrightarrow{\tilde{g}} & V \end{array}$$

Then there exists a closed embedding $h\colon Z \to V$ such that $\operatorname{codim} h \geqslant \operatorname{codim} g$ *and* $\operatorname{im}(f_W^* \circ g_{W*}) \subseteq \operatorname{im}(h_{W*})$ *inside* $\Omega_*(V)$

Proof: Consider the Cartesian square

$$\begin{array}{ccc} W_T & \xrightarrow{g_W} & W \\ {\scriptstyle \widetilde{f}_W}\uparrow & & \uparrow{\scriptstyle f_W} \\ W_{\widetilde{T}} & \xrightarrow{\widetilde{g}_W} & W_V \end{array}$$

By [32, Proposition 6.6.3] $f_W^* \circ g_{W*} = \widetilde{g}_{W*} \circ f_W^!$ where the refined pullback $f_W^!$ is given by the composition [32, 6.6.2]

$$\Omega_*(W_T) \to \Omega_*(C_W) \to \Omega_*(N_W) \to \Omega_{*-d}(W_{\widetilde{T}})$$

where C_W is the normal cone of $\widetilde{f}_W$ and N_W is the the normal bundle pullback $\widetilde{g}_W^*(N_{f_W})$ and d is the codimension of f.

Let N be the pullback of the normal bundle $\widetilde{g}^*(N_f)$ and $q\colon \mathbb{P}(N\oplus 1)\to \widetilde{T}$ be the projection and E denote the vector bundle $q^*N\otimes\mathcal{O}(1)$. Consider the closed subscheme $\mathbb{P}(C\oplus 1)$ inside $\mathbb{P}(N\oplus 1)$ where C is the normal cone of the map $\widetilde{f}\colon \widetilde{T}\to T$. Applying the lemma A.0.6 to the vector bundle E and irreducible components of $\mathbb{P}(C\oplus 1)$ we get a closed subset Z' of codimension at least d in $\mathbb{P}(C\oplus 1)$ such that for every cobordism class x supported on $\mathbb{P}(C_W\oplus 1)$ the class $x\cap \widetilde{c}_d(\varepsilon^*E)$ is supported on $\varepsilon^{-1}(Z')$. Thus in view of lemma A.0.2 one has that the image of the composition

$$\Omega_*(C_W)\to\Omega_*(N_W)\to\Omega_{*-d}(W_{\widetilde{T}})$$

is supported on $\varepsilon^{-1}(Z)$ where $Z=q(Z')$. Then $h\colon Z\to V$ is the desired embedding. ∎

Lemma A.0.8. *Let $\pi\colon Y\to X$ be a smooth morphism and X be a smooth quasiprojective variety. Then for any closed embeddings $i_1\colon Z_1\to X$ and $i_2\colon Z_2\to X$ there exists a closed embedding $i_3\colon Z_3\to X$ with* $\operatorname{codim} Z_3\geqslant\operatorname{codim} Z_1+\operatorname{codim} Z_2$ *and* $\operatorname{im}(i_1')_*\cdot\operatorname{im}(i_2')_*\subseteq im(i_3')_*$ *in $\Omega_*(Y)$ where $i'\colon Y_{Z_l}\to Y, l=1,2,3$ is obtained from the respective Cartesian square.*

Proof: The diagonal embedding $Y\to Y\times Y$ factors as $Y\xrightarrow{\phi} Y\times_X Y\xrightarrow{f_W} Y\times Y$. By lemma A.0.7 applied to $B=X\times X$, $V=X$, $f\colon\Delta_X$, $T=Z_1\times Z_2$ and $W=Y\times Y$ we get a closed embedding $h\colon Z\to X$ such that

$$\operatorname{codim} Z\geqslant\operatorname{codim} Z_1+\operatorname{codim} Z_2 \text{ and } \operatorname{im}(f_W^*\circ(i_1'\times i_2')_*)\subseteq\operatorname{im}(h_W)_*.$$

Consider the Cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Y\times_X Y \\ \uparrow{\scriptstyle h'} & & \uparrow{\scriptstyle h_W} \\ Y_Z & \xrightarrow{\phi_Z} & (Y\times_X Y)_Z \end{array}$$

According to [32, Proposition 6.6.3] we have $\phi^*\circ h_{W*}=h'_*\circ\phi_Z^!$ Thus $\operatorname{im}(i_1')_*\cdot\operatorname{im}(i_2')_*\subseteq \operatorname{im}\Delta_Y^*\circ(i_1'\times i_s')_*\subseteq\operatorname{im}(h_*')$. ∎

Corollary A.0.9. *The statement of lemma A.0.8 holds for the cohomology theory* $\mathtt{h}$

Proof: The natural map $\Omega_*(-)\otimes_{\mathbb{L}}\mathtt{h}(k)\to\mathtt{h}(-)$ is surjective and compatible with push-forwards and intersection product. ∎

Bibliography

[1] F. Anderson and K. Fuller. *Rings and categories of modules*, volume 13 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1974.

[2] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1991.

[3] P. Brosnan. On motivic decompositions arising from the method of Bialynicki-Birula, *Invent. Math.* 161, 91-111, 2005.

[4] B. Calmès, V. Petrov, and K. Zainoulline. Invariants, torsion indices and oriented cohomology of complete flags. *Ann. Sci. Éc. Norm. Supér.* (4) 46 (2013), no. 3, 405–448, 2013.

[5] B. Calmès, K. Zainoulline, and C. Zhong. A coproduct structure on the formal affine Demazure algebra. *Math. Z.* 282, no. 3-4, 1191–1218, 2016.

[6] B. Calmès, K. Zainoulline, and C. Zhong. Push-pull operators on the formal affine Demazure algebra and its dual. *Preprint*, arXiv:1312.0019.

[7] B. Calmès, K. Zainoulline, and C. Zhong. Equivariant oriented cohomology of flag varieties. *Doc. Math.*, Extra vol.: Alexander S. Merkurjev's sixtieth birthday, 113–144, 2015.

[8] V. Chernousov, S. Gille, and A. Merkurjev. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J.* 126, no. 1, 137–159, 2005.

[9] V. Chernousov and A. Merkurjev. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. *Transform. Groups* 11, no. 3, 371–386, 2006.

[10] B. Conrad, O. Gabber, and G. Prasad. *Pseudo-reductive groups.* Cambridge University Press, 2010

[11] M. Demazure. Invariants symétriques entiers des groupes de Weyl et torsion. *Invent. Math.* 21, 287–301, 1973.

[12] M. Demazure. Désingularisation des variétés de Schubert généralisées. Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I. *Ann. Sci. cole Norm. Sup.* (4) 7 , 53–88, 1974.

[13] D. Edidin and W. Graham. Equivariant intersection theory. *Invent. Math.* 131, no. 3, 595–634, 1998.

[14] R. Elman, N. Karpenko, and A. S. Merkurjev. *The algebraic and geometric theory of quadratic forms*, volume 56 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2008.

[15] W. Fulton. *Intersection theory.* Springer-Verlag, Berlin Heidelberg, 1998.

[16] S. Gille and K. Zainoulline. Equivariant pretheories and invariants of torsors. *Transform. Groups* 17, no. 2, 471–498, 2012.

[17] S. Garibaldi, A. Merkujev, and J.-P. Serre. *Cohomological Invariants in Galois Cohomology*, volume 28 of *University Lecture Series.* American Mathematical Society, Providence, RI, 2003.

[18] A. Grothendieck and J. Dieudonné. *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I.* Inst. Hautes tudes Sci. Publ. Math. No. 20 259 pp. 1964.

[19] J. Heller and J. Malagón-López. Equivariant algebraic cobordism. *J. Reine Angew. Math.* 684, 87–112, 2013.

[20] J. E. Humphreys. *Linear algebraic groups.* Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975.

[21] J. E. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, volume 94 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2008.

[22] A. Hoffnung, J. Malagón-López, A. Savage, and K. Zainoulline. Formal Hecke algebras and algebraic oriented cohomology theories. *Selecta Math. (N.S.)* 20, no. 4, 1213–1245, 2014.

[23] J. P. Jouanolou. Une suite exacte de Mayer-Vietoris en K-théorie algébrique. *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, Springer, Berlin. Lecture Notes in Math., Vol. 341, 293–316, 1973.

[24] N. Karpenko. Grothendieck Chow motives of Severi-Brauer varieties, *St. Petersburg Math. J.* 7, 649-661, 1996.

[25] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol. *The book of involutions. With a preface in French by J. Tits.* American Mathematical Society Colloquium Publications, 44. American Mathematical Society, Providence, RI, 1998.

[26] B. Köck. Chow motif and higher Chow theory of G/P, *Manuscripta Math.* 70, 363-372, 1991.

[27] J. Kollár. *Lectures on resolution of singularities.* Annals of Mathematics Studies, 166. Princeton University Press, Princeton, NJ, 2007.

[28] B. Kostant and S. Kumar. The nil Hecke ring and cohomology of G/P for a Kac-Moody group G. *Adv. in Math.* 62, no. 3, 187–237, 1986.

[29] C. Lenart and K. Zainoulline. Towards generalized cohomology Schubert calculus via formal root polynomials. *Preprint*, arXiv:1408.5952.

[30] C. Lenart, K. Zainoulline, and C. Zhong. Parabolic Kazhdan-Lusztig basis, Schubert classes and equivariant elliptic cohomology. *Preprint.*

[31] M. Levine. Oriented cohomology, Borel-Moore homology, and algebraic cobordism. *Michigan Math. J.* 57, Special volume in honor of Melvin Hochster. 523–572, 2008

[32] M. Levine and F. Morel. *Algebraic cobordism.* Springer Monographs in Mathematics. Springer, Berlin, 2007.

[33] Y. Manin. Correspondences, motifs and monoidal transformations *Math. USSR-Sb.* 6, 439–470, 1968.

[34] J.S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*, Princeton University Press, 1980.

[35] D. Mumford, J. Fogarty, F. Kirwan. *Geometric invariant theory.* Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], 34. Springer-Verlag, Berlin, 1994.

[36] I. Panin. Oriented cohomology theories of algebraic varieties. II (After I. Panin and A. Smirnov). *Homology, Homotopy Appl.* 11, no. 1, 349–405, 2009.

[37] I. Panin and A. Smirnov. Push-forwards in oriented cohomology theories of algebraic varieties. *Preprint.*

[38] V. Petrov and N. Semenov. Generically split projective homogeneous varieties. II. *J. of K-theory* 10, issue 1, 1–8, 2012.

[39] V. Petrov, N. Semenov, and K. Zainoulline. J-invariant of linear algebraic groups. *Ann. Sci. c. Norm. Supr.* (4) 41, no. 6, 1023–1053, 2008.

[40] M. Rost. The motive of a Pfister form, *preprint* http://www.mathematik.uni-bielefeld.de/ rost/data/motive.pdf, 1998.

[41] J.-P. Serre. *Complex semisimple lie algebras.* Springer-Verlag New York, 1987.

[42] J.-P. Serre. *Local fields.* Springer-Verlag New York, 1979.

[43] T. Springer. *Linear algebraic groups.* Modern Birkhuser Classics. Birkhuser Boston, Inc., Boston, MA, 2009.

[44] J. Tits. Classification of algebraic semisimple groups. *Algebraic Groups and Discontinuous Subgroups* (Proc. Sympos. Pure Math., Boulder, Colo., 1965) pp. 33–62 Amer. Math. Soc., Providence, R.I., 1966

[45] B. Totaro. The Chow ring of a classifying space. *Algebraic K-theory* (Seattle, WA, 1997), 249–281, Proc. Sympos. Pure Math., 67, Amer. Math. Soc., Providence, RI, 1999.

[46] A. Vishik and N. Yagita. Algebraic cobordisms of a Pfister quadric. *J. Lond. Math. Soc.* (2) 76, no. 3, 586–604, 2007.

[47] A. Vishik and K. Zainoulline. Motivic splitting lemma. *Doc. Math.* 13, 81–96, 2008.

[48] W. Waterhouse. *Introduction to affine group schemes.* Springer-Verlag, New York, 1979.