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Transition Theorems and Almost Sure Invariance Principles for Strong Martingales

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A Ph.D. Thesis

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ABSTRACT

In this thesis we establish almost sure invariance principles (ASIP's) for strong martingales indexed by two parameters. The method we use is that developed by Berkes and Philipp (Ann. Prob., 7, 1979). This thesis is organized in four chapters.

In Chapter 1, we give a review of invariance principles. We introduce the origin of the concept of the invariance principle, describe the main methods for proving the ASIP's and state some basic results of the almost sure invariance principle.

In Chapter 2, we prove some "transition theorems' which turn two-parameter strong martingales into one-parameter martingales and can help us to prove the ASIP's for two-parameter strong martingales. We also give several simple applications of the transition theorems, such as maximal inequalities with exponential bounds for two-parameter strong martingales and the Prohorov distance between the law of a two-parameter strong martingale and some appropriate normal law.

In Chapter 3, we prove our main theorem—the almost sure invariance principle for two-parameter strong martingales and show some applications, including the functional law of the iterated logarithm for two-parameter strong martingales.

In Chapter 4—the appendix, we state some known results we want to use and give the proofs of Theorem 2.2.1 in Chapter 2 and Lemma 3.2.10 in Chapter 3.

Contents

Chapter 1

Introduction to Invariance Principles

1.1 On the Concept of Invariance Principle

The term "invariance principle" was coined by Erdös and Kac. Let $x_1, x_2, \cdots$ be independent identically distributed (i.i.d.) random variables (r.v.'s) with $Ex_1 = 0, Ex_1^2 = 1$ and distribution function $F(t)$. In their paper [45] (1946), they wanted to evaluate the limit distributions:

$$G_1(t) = \lim_{n\to\infty} P(n^{-1/2} \max_{1\le k\le n} S_k \le t), \tag{1.1}$$

$$G_2(t) = \lim_{n\to\infty} P(n^{-1/2} \max_{1\le k\le n} |S_k| \le t), \tag{1.2}$$

$$G_3(t) = \lim_{n\to\infty} P(n^{1/2} \sum_{k=1}^{n} S_k^2 \le t), \tag{1.3}$$

and

$$G_4(t) = \lim_{n\to\infty} P(n^{-3/2} \sum_{k=1}^{n} |S_k| \le t), \tag{1.4}$$

where $S_k = \sum_{i=1}^{k} x_i$.

Erdös and Kac found that the limits exist and do not depend on the initial distribution of x_1. They also found that the limits can be easily evaluated for some special distribution F. For example, (1.1) and (1.2) can be immediately evaluated if F is the distribution

$P(x_1 = 1) = P(x_1 = -1) = 1/2$ and for (1.3) and (1.4) it is easier if F is the normal distribution. Hence they solved this problem in two steps. First they proved that the limits exist and do not depend on the initial distribution of x_1; second they use some specific distribution F to evaluate the limits. They called this method of proof a principle and their paper has initiated a new methodology for proving limit laws in probability theory.

In their paper [46] (1947), a theorem is proved, that is:

Theorem. Let $x_1, x_2, \cdots$ be independent r.v.'s with mean 0 and variance 1 such that the CLT is applicable. Let $S_k = \sum_{i=1}^{k} x_i$ and N_n denote the number of S_k's, $1 \leq k \leq n$, which are positive. Then

$$\lim_{n\to\infty} P(\frac{N_n}{n} < \alpha) = \frac{2}{\pi} \arcsin \alpha^{1/2}, \qquad 0 \leq \alpha \leq 1$$

They wrote, "If the theorem can be established for one particular sequence of independent r.v.s $y_1, y_2, \cdots$ satisfying the conditions of the theorem, the conclusion of the theorem holds for all sequences of independent r.v.'s which satisfy the conditions of the theorem." That is what their invariance principle means.

Donsker (1951,[31]) also described the invariance principle. Let $S_1, S_2, \cdots$ be the partial sums of i.i.d.r.v.'s $x_1, x_2, \cdots$ having mean 0 and variance 1. He wrote in [31], "One of impacts of the central limit theorem which states

$$\lim_{n\to\infty} P(S_n < \alpha n^{1/2}) = (2\pi)^{-1/2} \int_{-\infty}^{\alpha} \exp(-t^2/2)dt$$

is that the limiting distribution is independent of the original distribution of the r.v.'s. With regard to limit theorems, we say the 'invariance principle' holds in particular case if a limiting distribution exists and is independent of the distribution of the r.v.'s involved." In the same paper he proved an invariance principle which is the famous Donsker theorem and is now called the functional central limit theorem.

Let $C[0,1]$ denote the space of real-valued continuous functions equipped with the supremum norm. If S_k are the partial sums of i.i.d.r.v.'s having mean 0 and variance 1, then for $0 \le t \le 1$,

$$X_n(t) = \begin{cases} n^{-1/2} S_k & \text{if } t = k/n \\ \text{linear on } [(k-1)/n, k/n] & 1 \le k \le n \end{cases}$$

defines a sequence of r.v.'s with values in $C[0,1]$.

Theorem 1.1.1 *(Donsker,1951,[31]) Let $\{B(t); 0 \le t \le 1\}$ be a standard Brownian motion. Considering $B(t)$ and $X_n(t)$ to be $C[0,1]$-valued r.v.'s we have*

$$X_n \xrightarrow{D} B \quad (n \to \infty) \tag{1.5}$$

i.e. for any bounded continuous function $h : C[0,1] \to R^1$,

$$E(h(X_n)) \longrightarrow E(h(B)), (n \to \infty).$$

It is easy to see that if $h(X) = X(1), X(t) \in C[0,1]$, then for any bounded continuous function $g : R^1 \to R^1$,

$$\begin{aligned} E(g(n^{-1/2}S_n)) &= E\{g(h(X_n))\} \\ &\longrightarrow E\{g(h(B))\} = E\{g(B(1))\}, \end{aligned}$$

which implies the usual central limit theorem.

As pointed out by Philipp (1985,[89]), "At present the term 'invariance principle' generally stands as a synonym for an approximation theorem: a given process, such as a partial sum process, an empirical process, an extremal process, a U-statistic process, etc. is approximated in distribution, in probability, in L^p or almost surely by a canonical process, such as a Brownian motion, a Kiefer process, a special extremal process or in case of a U-statistic by a multiple stochastic integral." So we have the corresponding terms: invariance principle in distribution (IPID), invariance principle in probability (IPIP), L^p-invariance principle (L^p-IP) and almost sure invariance principle (ASIP). The ASIP is also called the strong invariance principle.

Obviously the usual central limit theorem and the Donsker theorem are IPID's. The following theorem states a result on IPIP.

Theorem 1.1.2 *(Major,1976,[68]) Let F be a distribution function satisfying*

$$\int t dF(t) = 0 \quad \textit{and} \quad \int t^2 dF(t) = 1. \tag{1.6}$$

Then there are two sequences $\{x_i; i \geq 1\}$ and $\{y_i; i \geq 1\}$ of i.i.d.r.v.'s defined on some probability space with common laws $\mathcal{L}(x_1) = F$ and $\mathcal{L}(y_1) = N(0,1)$, respectively, such that

$$n^{-1/2} \max_{k \leq n} |S_k - T_k| \xrightarrow{P} 0 \quad (n \to \infty) \tag{1.7}$$

where S_k and T_k are the partial sums of x_i's and y_i's respectively.

Since $\{\max_{k \leq n} S_k^2/n; n \geq 1\}$ and $\{\max_{k \leq n} T_k^2/n; n \geq 1\}$ are uniformly integrable, the convergence in (1.7) is even in L^2 which gives an example of L^2-IP (see [88]).

On the ASIP the first result is due to Strassen. Let $S_n = \sum_{i=1}^n x_i$ be the partial sum of i.i.d.r.v.'s $x_1, x_2, \cdots$ with $Ex_1 = 0$ and $Ex_1^2 = 1$. We have

Theorem 1.1.3 *(Strassen,1964,[96]) Without changing its law we can redefine $\{x_i\}$ on a richer probability space on which there exists a Brownian motion $\{B(t); t \geq 0\}$ such that*

$$S_n - B(n) = o\{(n \log\log n)^{1/2}\} \quad a.s. \tag{1.8}$$

Here and hereafter we take $\log x := \log(\max(e, x))$

If we are given a probability space $(\Omega, \mathcal{F}, P)$ with a sequence $\{x_i; i \geq 1\}$ of i.i.d.r.v.'s, then it is not sure at all that a Brownian motion $B(t)$ can be defined on that probability space such that (1.8) holds true. This is the reason why we define a richer probability space and redefine $\{x_i; i \geq 1\}$ on this new space. We will see from Lemma 1.3.3 that taking the product of the given probability space and a copy of the unit interval with Lebesgue measure makes the product space rich enough.

Obviously (1.5) and (1.7) imply the central limit theorem but (1.8) does not. Since the law of the iterated logarithm (LIL) holds for $B(n)$, using (1.8) we can easily prove the LIL for S_n. But (1.5) and (1.7) do not imply the LIL for S_n. If the error term of (1.8) could be improved, for example, if we could obtain (1.8) with error term $o(n^{1/2})$, then the improved result could imply the CLT. But this not true. Major (1976,[68]) proved that the error term in (1.8) is the best possible. So both have earned their rightful places. So far, many results on IPID, IPIP and L^p-ID have been obtained. Three examples are given in section 1.1. For the literature up to 1985 see [89] (Philipp, 1985), [29] (Dehling, 1985), [51] (Goldie and Morrow, 1985) and [86] (Peligrad, 1985).

1.2 Almost Sure Invariance Principles for One-parameter Random Variables

In this section we are going to present some basic results on ASIP's and several main methods and techniques in establishing ASIP's. Our emphasis is on real-valued independent random variables (r.v.s) and martingales.

1.2.1 Main Methods

We have three main methods in establishing ASIP's: the Skorohod embedding method, the quantile transform method and Berkes and Philipp's method.

(a) Skorohod embedding method

This method is based on the Skorohod embedding theorem which is very useful for the proof of four types of invariance principles for martingales. Unfortunately the applicability of this method is essentially restricted to real-valued r.v.s. and to one-dimensional time parameters.

Description 1.2.1. To explain the method, let us sketch Strassen's proof for Theorem 1.1.3. We need the following famous Skorohod embedding theorem (1965,[94],p.180).

"If a sequence $\{x_i; i \geq 1\}$ of i.i.d.r.v.'s satisfying $Ex_1 = 0$ and $Ex_1^2 = 1$ is defined together with a 1-dimensional Brownian motion B on a probability space such that $\{x_i; i \geq 1\}$ and B are mutually independent, then there is a sequence $\{\tau_i; i \geq 1\}$ of i.i.d. nonnegative r.v.'s defined on the same space such that $E\tau_1 = Ex_1^2$ and such that the process

$$B(\tau_1), B(\tau_1 + \tau_2) - B(\tau_1), \cdots, B(\sum_{i=1}^{n} \tau_i) - B(\sum_{i=1}^{n-1} \tau_i), \cdots$$

and the process $\{x_i; i \geq 1\}$ have the same distribution."

To use this result, let us assume that our original sequence $\{x_i; i \geq 1\}$ and the Brownian motion B are already defined on the same probability space and are mutually independent (this of course can always be done). Put $\tau_0 = 0$ and

$$\tilde{x}_j = B(\sum_{i=0}^{j} \tau_i) - B(\sum_{i=0}^{j-1} \tau_i),$$

for $j \geq 1$, also

$$\tilde{S}_n = \sum_{j=1}^{n} \tilde{x}_j = B(\sum_{i=0}^{n} \tau_i), \quad \tilde{S}_0 = 0.$$

Then the processes $\{\tilde{x}_j; j \geq 1\}$ and $\{x_j; j \geq 1\}$ have the same distribution and therefore $\{\tilde{S}_n\}$ and $\{S_n\}$ have the same distribution.

Now estimating $|\tilde{S}_n - B(n)|$ we can obtain

$$|\tilde{S}_n - B(n)| = o((n \log\log n)^{1/2}), \quad \text{a.s.}$$

(For the details see [96] (Strassen,1964)). So the redefined processes $\{\tilde{S}_n\}$ and $\{B(n)\}$ satisfy our requirement.□

To establish ASIP's for martingales we need the following generalized version of the Skorohod embedding theorem.

Theorem 1.2.1 *(Scott and Huggins,1983,[92]) Let* $\{x_k, \mathcal{F}_k; k \geq 1\}$ *be a square integrable martingale difference sequence. Then (on a possibly enlarged version of the underlying probability space) there exists a Brownian motion* $\{B(t), \mathcal{F}_t^*; t \geq 0\}$ *and a nondecreasing sequence* $\{T_n; n \geq 1\}$ *of stopping times such that* $\mathcal{F}_n \subset \mathcal{F}_{T_n}^*$ *and*

$$B(T_n) = \sum_{j=1}^{n} x_j, \quad a.s., \quad n \geq 1$$

Furthermore there exists an increasing sequence $\{\mathcal{S}_n; n \geq 1\}$ *of* σ*-fields such that* $\tau_n = T_n - T_{n-1}$ *is* $\mathcal{S}_n$*-measurable and*

$$E(\tau_n | \mathcal{S}_{n-1}) = E(x_n^2 | \mathcal{F}_{n-1}) \quad a.s.$$

and for $p > 1$,

$$M_p E(\tau_n^p | \mathcal{S}_{n-1}) \leq E(|x_n|^{2p} | \mathcal{F}_{n-1}) \leq N_p E(\tau_n^p | \mathcal{S}_{n-1}) \quad a.s.$$

for some positive constants M_p *and* N_p *depending only on* p.

Note that for continuous martingales an embedding theorem was given by Knight (1971, [62]), which extended a result of Kunita and Watanabe (1967, [64]).

If some random processes can be suitably approximated by martingales then of course the Skorohod embedding method can be used to prove an invariance principle for these processes. For example, Philipp and Stout (1975,[90]) established several ASIP's for some so-called weakly dependent r.v.s in this way.

(b) The quantile transform method.

This is a totally different method developed by Csörgő and Révesz (1975, [17]), refined later by Komlós, Major and Tusnády (1975, 1976, [63]) and extended recently by Berger (1982, [6]) to cover the case of R^d-valued i.i.d. random vectors. The main advantage of this method is that in the ASIP for sums of i.i.d.r.v.s having finite p-th moment or moment generating function it yields sharp error terms. Unfortunately it is restricted to independent r.v.s and random vectors.

Description 1.2.2. To describe this method let $\{x_i, i \geq 1\}$ be a sequence of i.i.d.r.v.s with continuous distribution fuction F such that $Ex_1 = 0$, $Ex_1^2 = 1$ and $E|x_1|^3 < \infty$. We are going to get (1.8) with error term $O(n^{\frac{1}{2}-\lambda})$ for some $\lambda > 0$. Let $n_k = k^\alpha$ for some suitable integer α. Put

$$t_k = \sum_{j=1}^{k+1} n_j, \qquad X_k = n_k^{-1/2} \sum_{j=t_k+1}^{t_{k+1}} x_j, \qquad F_k(t) = P(X_k < t)$$

$$f_k(t) = \Phi^{-1}(F_k(t)), \qquad R_k = f_k(X_k), \qquad T_{t_l} = \sum_{k-1}^{l} \sqrt{n_k} R_k$$

where $\Phi(t)$ denotes the standard normal distribution function. Obviously

$$P(R_k < t) = P(f_k(X_k) < t) = P(F_k(X_k) < \Phi(t)) = \Phi(t).$$

So $\mathcal{L}(R_k) = N(0,1)$ and therefore $\mathcal{L}(T_{t_l}) = N(0, t_l)$. Since $F_k(t) \to \Phi(t)$ we have $f_k(t) \to t$. It is not hard to show that

$$|R_k - X_k| = |f_k(X_k) - X_k| \leq C_0 n_k^{-\beta}, \quad \text{a.s.}$$

where $\beta > 0$ and $C_0 > 0$ are constants independent of k. If $S_n = \sum_{i=1}^{n} x_i$, since $t_l \cong (\alpha+1)^{-1} l^{\alpha+1}$, then

$$\begin{aligned} |S_{t_l} - T_{t_l}| &= |\sum_{k=1}^{l} \sqrt{n_k} X_k - \sum_{k=1}^{l} \sqrt{n_k} R_k| \le C_0 \sum_{k=1}^{l} n_k^{1/2-\beta} \\ &\le C_0 \sum_{k=1}^{l} k^{\alpha(1/2-\beta)+1} \le C_1 l^{\alpha(1/2-\beta)+1} \le C_2 t_l^{1/2-\lambda} \end{aligned}$$

where $\lambda = (2\beta - 1/2)/(\alpha+1) < 0$ if $\alpha > (2\beta)$, and $C_1, C_2 > 0$ are absolute constants. If the probability space under consideration is rich enough, then we can prove that there exists a Brownian motion $B(t)$ such that

$$T_{t_l} = B(t_l).$$

For $t_l < n \le t_{l+1}$ we have

$$\begin{aligned} |S_n - B(n)| &\le |S_n - S_{t_l}| + |S_{t_l} - T_{t_l}| + |B(t_l) - B(n)| \\ &\le C_2 t_l^{1/2-\lambda} + |S_n - S_{t_l}| + |B(t_l) - B(n)|. \end{aligned}$$

Consider

$$P(\max_{t_l < n \le t_{l+1}} |S_n - S_{t_l}| \ge t_l^{1/2-\lambda}) \quad \text{and} \quad P(\max_{t_l < n \le t_{l+1}} |B(t_l) - B(n)| \ge t_l^{1/2-\lambda})$$

and using the Borel-Cantelli lemma, we can show that

$$|S_n - S_{t_l}| = O(t_l^{1/2-\lambda}) \quad \text{a.s.}$$

$$|B(n) - B(t_l)| = O(t_l^{1/2-\lambda}) \quad \text{a.s.}$$

Therefore

$$|S_n - B(n)| = O(n^{1/2-\lambda}) \quad \text{a.s.} \quad \square$$

Let us point out, in Description 1.2.2 we use $\{t_k\}$ to divide the set of natural numbers into groups (blocks): $\{1, 2, \cdots, t_1\}, \{t_1+1, t_1+2, \cdots, t_2\}, \cdots$, then with respect to the k-th group we define X_k and furthermore we define F_k, f_k, R_k, etc. to serve our purpose. This kind of technique is the well-known **blocking technique**.

(c) Berkes and Philipp's method

This method was developed by Berkes and Philipp (1979,[9]) and is widely used. To describe Berkes and Philipp's method, we need the following lemmas. Lemma 1.2.1 is a result concerning the Prohorov distance. Let (D, d) be a metric space and let F and G be probability measures on D. The Prohorov distance between F and G, denoted by $\pi(F, G)$, is defined as follows (see [3], Page 12):

$$\pi(F, G) = \inf\{\varepsilon \geq 0; \quad F(C) \leq G(C_\varepsilon) + \varepsilon, \ C \text{ closed}\}$$

where $C_\varepsilon = \{t \in D; \quad d(t, C) < \varepsilon\}$.

Lemma 1.2.1 *(Yurinskii,1975,[104]) Let $\{x_i; i \geq 1\}$ be a sequence of real-valued i.i.d.r.v.s with $Ex_1 = 0, Ex_1^2 = 1$ and $E|x_1|^3 < \infty$. If $\pi(F, G)$ denotes the Prohorov distance between probability measures F and G, then*

$$\pi(\mathcal{L}(n^{-1/2}\sum_{i=1}^{n} x_i), N(0,1)) < Cn^{-1/2}$$

where $C > 0$ is a constant independent of n.

The following lemma is Lemma 2.1 in [9] (Berkes and Philipp,1979) and is a special case of the Strassen-Dudley theorem (see Theorem 1 in [34](Dudley, 1968)).

Lemma 1.2.2 *Let (D, d) be a separable metric space and $\mathcal{D}$ the σ-fields of all Borel sets of D. Let F and G be two probability measures defined on $\mathcal{D}$ such that*

$$\pi(F, G) < \alpha.$$

Then there exists a probability measure Q on the Borel sets of $D \times D$ with marginals F and G such that

$$Q\{(u, v); d(u, v) > \alpha\} \leq \alpha.$$

Lemma 1.2.3 *(Berkes and Philipp,1979,[9]) Let (D,d) be a complete separable metric space and let F be a probability measure on the Borel sets of D. Let $(\Omega,\mathcal{F},P)$ be a probability space such that $\mathcal{F}$ is atomless. Then there exists a D-valued r.v. X on $(\Omega,\mathcal{F},P)$ with distribution F.*

Lemma 1.2.4 *(Dudley and Philipp,1983,[35],Lemma 2.13) Let $D_i, i = 1,2,3$ be complete separable metric spaces. Let F be a distribution on $D_1 \times D_2$ and G be a distribution on $D_2 \times D_3$ such that the second marginal of F equals the first marginal of G. Then there exists a probability measure on $D_1 \times D_2 \times D_3$ with marginals F on $D_1 \times D_2$ and G on $D_2 \times D_3$.*

Remark 1.2.1 Since there always exists a probability space $(\Omega,\mathcal{F},P)$ with $\mathcal{F}$ being atomless, by Lemma 1.2.3 and 1.2.4 the conclusion of Lemma 1.2.4 can be replaced by the conclusion that there exist a probability space and r.v.'s $Z_i, i = 1,2,3$ on it such that the joint distribution of Z_1 and Z_2 is F and the joint distribution of Z_2 and Z_3 is G.

Note that Lemma 1.2.4 is essentially Lemma A1 in [9] (Berkes and Philipp, 1979) and is also a special case of a generalized Vorob'ev theorem (Shortt, 1982, [93], Theorem 2.6; Vorob'ev, 1962, [100]).

Now let us describe Berkes and Philipp's method.

Description 1.2.3. Let $\{x_i; i \geq 1\}, n_k, t_k$ and X_k be as in Description 1.2.2. We also want to obtain (1.8) with error term $O(n^{1/2-\lambda})$ for some $\lambda > 0$. By Lemma 1.2.1,

$$\pi(\mathcal{L}(X_k), N(0,1)) < Cn_k^{-1/2}, \quad k \geq 1.$$

By Lemma 1.2.2 and Lemma 1.2.3, there exist r.v.s X_k' and Y_k' such that $\mathcal{L}(X_k') = \mathcal{L}(X_k)$, $\mathcal{L}(Y_k') = N(0,1)$ and

$$P(|X_k' - Y_k'| > Cn_k^{-1/2}) \leq Cn_k^{-1/2}, \quad k \geq 1.$$

Let $\{y_i; i \geq 1\}$be a sequence of i.i.d.r.v.'s such that $\mathcal{L}(y_i) = N(0,1)$. Put

$$Y_k'' = n_k^{-1/2} \sum_{i=t_k+1}^{t_{k+1}} y_i, \qquad k \geq 1. \tag{1.9}$$

We have $\mathcal{L}(Y_1', Y_2', \cdots) = \mathcal{L}(Y_1'', Y_2'', \cdots)$. We will now use Lemma 1.2.4 twice to redefine the above r.v.'s on a single probability space such that the relations above hold.

First let

$$F = \mathcal{L}(\{x_i; i \geq 1\}, \{X_k; k \geq 1\}),$$
$$G = \mathcal{L}(\{X_k'; k \geq 1\}, \{Y_k'; k \geq 1\}),$$

and

$$D_1 = D_2 = D_3 = \times_{i=1}^{\infty} R^1.$$

By Lemma 1.2.4, without changing their laws we can redefine $\{x_i; i \geq 1\}, \{X_k; k \geq 1\}$ and $\{Y_k'\}$ on a single probability space such that

$$X_k = n_k^{-1/2} \sum_{i=t_k+1}^{t_{k+1}} x_i \qquad k \geq 1 \tag{1.10}$$

and

$$p_k = P(|X_k - Y_k'| > C n_k^{-1/2}) \leq C n_k^{-1/2}, \quad k \geq 1.$$

Second let

$$F = \mathcal{L}(\{x_i; i \geq 1\}, \{X_k; k \geq 1\}, \{Y_k'; k \geq 1\}),$$
$$G = \mathcal{L}(\{Y_k''; k \geq 1\}, \{y_i; i \geq 1\}),$$

and

$$D_1 = (\times_{i=1}^{\infty} R^1) \times (\times_{i=1}^{\infty} R^1), \quad D_2 = D_3 = \times_{i=1}^{\infty} R^1.$$

By Lemma 1.2.4 again, we can redefine $\{x_i; i \geq 1\}, \{X_k; k \geq 1\}, \{Y_k''; k \geq 1\}$ and $\{y_i; i \geq 1\}$ on a single probability space (for notational convenience we use Y_k to denote Y_k'') such that the relation (1.10),

$$Y_k = n_k^{-1/2} \sum_{i=t_k+1}^{t_{k+1}} y_i \qquad k \geq 1 \tag{1.11}$$

and

$$P(|X_k - Y_k| > C n_k^{-1/2}) \le C n_k^{-1/2} \qquad k \ge 1 \tag{1.12}$$

hold. Let $S_n = \sum_{i=1}^{n} x_i$ and $T_n = \sum_{i=1}^{n} y_i$. For $t_k < n \le t_{k+1}$, consider

$$p_k' = P(\max_{t_k < n \le t_{k+1}} | \sum_{i=t_k+1}^{n} x_i| > t_k^{1/2-\lambda})$$

and

$$p_k'' = P(\max_{t_k < n \le t_{k+1}} | \sum_{i=t_k+1}^{n} y_i| > t_k^{1/2-\lambda}).$$

Properly choosing $\alpha \ge 3$ (α is given in Description 1.2.2) and $\lambda > 0$ we can easily prove that

$$\sum_{k=1}^{\infty} p_k < \infty, \quad \sum_{k=1}^{\infty} p_k' < \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} p_k'' < \infty.$$

By the Borel-Cantelli lemma we have

$$\begin{aligned} &|X_k - Y_k| \le C n_k^{-1/2} \quad \text{a.s.}, \\ &\max_{t_k < n \le t_{k+1}} | \sum_{i=t_k+1}^{n} x_i| \le t_k^{1/2-\lambda} \quad \text{a.s.}, \\ &\max_{t_k < n \le t_{k+1}} | \sum_{i=t_k+1}^{n} y_i| \le t_k^{1/2-\lambda} \quad \text{a.s.}, \end{aligned}$$

and therefore for $t_k < n \le t_{k+1}$,

$$\begin{aligned} |S_n - T_n| &\le |S_{t_k} - T_{t_k}| + | \sum_{i=t_k+1}^{n} x_i| + | \sum_{i=t_k}^{n} y_i| \\ &\le \sum_{j=1}^{k} n_j^{1/2} |X_i - Y_i| + 2 t_k^{1/2-\lambda} \\ &= O(n^{1/2-\lambda}) \quad \text{a.s.} \quad \Box \end{aligned}$$

Description 1.2.3 gives us an idea what Berkes and Philipp's method looks like. From Description 1.2.3 we see that Berkes and Philipp's method is based on estimates of the Prohorov distance and the Strassen-Dudley theorem. Unfortunately if the r.v.'s $\{x_i; i \ge 1\}$ are not independent, then the procedure in Description 1.2.3 cannot be used to establish

ASIP's because when we use Lemma 1.2.4 or Remark 1.2.1 to redefine $\{x_i; i \geq 1\}$, $\{X_k; k \geq 1\}$, $\{X'_k; k \geq 1\}$ and $\{Y_K; k \geq 1\}$ on a single probability space we need that

$$\mathcal{L}(\{X_k; k \geq 1\}) = \mathcal{L}(\{X'_k; k \geq 1\}). \tag{1.13}$$

But from Description 1.2.3 we are not sure at all that (1.13) holds. The following two theorems are useful in the proof of ASIP's for dependent r.v.'s.

Theorem 1.2.2 *(Berkes and Philipp,1979,[9]) Let $\{X_k; k \geq 1\}$ be a sequence of r.v.'s with values in $R^{d_k}, d_k \geq 1$ and let $\{\mathcal{F}_k; k \geq 1\}$ be a non-decreasing sequence of σ-fields such that X_k is $\mathcal{F}$-measurable. Let $\{G_k; k \geq 1\}$ be a sequence of probability distributions on R^{d_k} with characteristic function $g_k(u), u \in R^{d_k}$, respectively. Suppose that for some nonnegative number λ_k, δ_k and $T_k \geq 10^8 d_k$,*

$$E|E[\exp(i < u, X_k >)|\mathcal{F}_{k-1}] - g_k(u)| \leq \lambda_k$$

for all u with $|u| \leq T_k$ and

$$G_k\{u; |u| > \frac{1}{4}T_k\} \leq \delta_k.$$

Then without changing its distribution we can redefine the sequence $\{X_k; k \geq 1\}$ on a richer probability space together with a sequence $\{Y_k; k \geq 1\}$ of independent r.v.s such that Y_k has distribution G_k and

$$P(|X_k - Y_k| \geq \alpha_k) \leq \alpha_k \quad k \geq 1$$

where $\alpha_1 = 1$ and

$$\alpha_k = 16 d_k T_k^{-1} \log T_k + 4\lambda_k^{1/2} T_k^{d_k} + \delta_k \quad k \geq 2.$$

Theorem 1.2.3 *(Berger (1982), see [89] p.240) Let $\{B_k, m_k; k \geq 1\}$ be a sequence of complete separable metric spaces. Let $\mathcal{B}_k$ denotes the Borel σ-field of B_k, let $\{X_k; k \geq 1\}$ be a sequence of r.v.'s with values in B_k and let $\{\mathcal{F}_k; k \leq 1\}$ be a sequence of* *nondecreasing σ-fields such that X_k is $\mathcal{F}_k$-measurable. Suppose that for some sequence $\{\beta_k; k \geq 1\}$ of nonnegative numbers,*

$$E\{\sup_{A \in \mathcal{B}_k} |P(X_k \in A|\mathcal{F}_{k-1}) - P(X_k \in A)|\} \leq \beta_k$$

for all $k \geq 1$. Denote by F_k the distribution of X_k and let $\{G_k; k \geq 1\}$ be a sequence of distributions on $(B_k, \mathcal{B}_k)$ such that

$$F_k(A) \leq G_k(A^{\rho_k}) + \sigma_k \quad \text{for all} \quad A \in \mathcal{B}_k$$

where ρ_k and σ_k are nonnegative numbers and

$$A^{\epsilon} = \bigcup_{t \in A} \{s; m_k(s,t) \leq \epsilon\}.$$

Then without changing its distribution we can redefine the sequence $\{X_k; k \geq 1\}$ on a richer probability space on which there exists a sequence $\{Y_k; k \geq 1\}$ of independent r.v.s with ditribution G_k such that for all $k \geq 1$,

$$P(M_k(X_k, Y_k) > \rho_k) \leq \beta_k + \sigma_k.$$

Note that Theorem 1.2.3 is the latest result in a series of improvements of another approximation theorem of Berkes and Philipp (1979,[9]) and Dehling and Philipp (1982,[32]). For $\rho_k = \sigma_k = 0$, i.e. $F_k = G_k, k \geq 1$, Theorem 1.2.3 was also discussed by Berbee (1987,[5]). Recently Theorem 1.2.2 and Theorem 1.2.3 were extended by Monrad and Philipp (1991,[79]) to cover the case in which the G_k's are all regular conditional distributions.

It is worthwhile to point out that by a slight modification, Berkes and Philipp's method can be used to prove IPIP's. The first paper is due to Philipp (1980, [88]). For others see, e.g. [22] (Dabrowski, 1987) and [24] (Dabrowski, 1990).

1.2.2 Basic Results

(A) Independent random variables

The first result on the ASIP for sums of i.i.d.r.v.'s is Theorem 1.1.3. We have already mentioned that the rate of convergence in Theorem 1.1.3 is not fast enought to prove the CLT. To achieve this we need to replace the error term of (1.8) by $o(n^{1/2})$ at least. However the following theorem says that this is impossible.

Theorem 1.2.4 *(Major,1976,[68]) Let* $\{a_n\}$ *be any sequence of numbers with* $a_n \uparrow \infty$. *Then there exists a distribution function* $F(t)$ *with* $\int t dF(t) = 0$ *and* $\int t^2 dF(t) = 1$ *which has the following property: for any sequence of i.i.d.r.v.'s* $\{x_i; i \geq 1\}$ *with common distribution function* F *and for any Brownian motion* $B(t)$, *one has*

$$\limsup_{n\to\infty} a_n \frac{|S_n - B(n)|}{(\log\log n)^{1/2}} = \infty \qquad a.s.$$

Note that a weaker result of this type was obtained by Breiman (1967, [12]).

Major also showed that if we approximate S_n by Gaussian processes (not Brownian motion) then the rate of covergence in Theorem 1.1.3 can be improved.

Theorem 1.2.5 *(Major,1979,[71]) Let a distribution function* $F(t)$ *be given with* $\int t dF(t) = 0$ *and* $\int t^2 dF(t) = 1$. *Define*

$$\sigma_k^2 = \int_{-\sqrt{2^n}}^{\sqrt{2^n}} t^2 dF(t) - \left(\int_{-\sqrt{2^n}}^{\sqrt{2^n}} t dF(t)\right)^2 \quad \text{if} \quad 2^n \leq k < 2^{n+1}.$$

Then a sequence of i.i.d.r.v.'s $\{x_i; i \geq 1\}$ *having the distribution* F *and a sequence of independent normal r.v.s* $\{y_i; i \geq 1\}$ *with* $Ey_k = 0$ *and* $Ey_k^2 = \sigma_k^2$ *can be constructed such that*

$$|S_n - T_n| = o(n^{1/2}) \qquad a.s.$$

where $S_n = \sum_{i=1}^n x_i$ *and* $T_n = \sum_{i=1}^n y_i$.

If the r.v.'s have higher finite moments then the error term of (1.8) may be better. Using the Skorohod embedding method Jain, Jogdeo and Stout proved the following:

Theorem 1.2.6 *(Jain, Jogdeo and Stout, 1975, [59]) Let* $\{x_i; i \geq 1\}$ *be i.i.d.r.v.s with* $Ex_1 = 0$ *and* $Ex_1^2 = 1$*. Assume that for* $\alpha \geq 0$,

$$E(x_1^2(\log\log x_1^2)^\alpha) < \infty.$$

Then upon redefining $\{x_i; i \geq 1\}$ *on a new probability space, if necessary, there exists a Brownian motion* $\{B(t); t \geq 0\}$ *such that*

$$|S_n - B(n)| = o(n^{1/2}(\log\log n)^{(1-\alpha)/2}) \quad a.s.$$

Obviously if $\alpha = 0$, then this theorem is Theorem 1.1.3. If the r.v.'s have a moment of order $r > 2$ or a moment generating function, we have the following two theorems which were proved by applying the quantile transform method.

Theorem 1.2.7 *(Komlós, Major, Tusnady, 1975, 1976, [63] and Major 1976, [69]) Let* $F(t)$ *be a distribution function with* $\int t dF(t) = 0$, $\int t^2 dF(t) = 1$ *and* $\int |t|^p dF(t) < \infty \quad (p > 2)$*. Then a sequence of i.i.d.r.v.'s* $\{x_k; k \geq 1\}$ *having distribution* F *and a Brownian motion* $\{B(t); t \geq 0\}$ *can be constructed such that*

$$|S_n - B(n)| \;=\; o(n^{1/p}) \quad a.s. \tag{1.14}$$

Theorem 1.2.8 *(Komlós, Major, Tusnády, 1975, 1976, [63]) Let* $F(t)$ *be a distribution function with* $\int e^{st} dF(t) < \infty$ *for* s *in some neighborhood of* $s = 0$*. Then the conclusion of Theorem 1.3.2.1 holds with*

$$|S_n - B(n)| \;=\; O(\log n) \quad a.s. \tag{1.15}$$

Note that for $2 < r \leq 3$, (1.14) was obtained by Major (1976, [69]); for $r = 4$, Strassen (1965, [98]) obtained (1.14) with error term $O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4})$; for $r = 8$ with $\int t^3 dF(t) = 0$ Csörgö and Révész (1975,[17]) obtained (1.14) with error term $o(n^{1/6+\epsilon})(\forall\epsilon > 0)$ and for $r > 3$, (1.14) was proved by Komlós, Major and Tusnády (1975, 1976, [63]).

Can we improve the error term in (1.14) and (1.15)? This question was answered by Breiman, and Csörgö and Révész.

Theorem 1.2.9 *(Breiman, 1967, [12]) let $p \geq 2$. Then*

$$\limsup_{n\to\infty} \frac{S_n - B(n)}{n^{1/p}} = \infty \quad a.s.$$

for whatever $B(t)$, provided $\int |t|^p dF(t)$ does not exist, where S_n be the partial sum of i.i.d.r.v.'s $\{x_k; k \geq 1\}$ having distribution function F.

Theorem 1.2.10 *(Csörgö and Révész, 1981, [18]) Let $\{x_k; k \geq 1\}$ be i.i.d.r.v.'s with $Ex_1 = 0$ and $Ex_1^2 = 1$. Denote their distribution function by $F(t)$ and denote the standard normal distribution function by $\Phi(t)$. Let $\{B(t); t \geq 0\}$ be a Brownian motion such that*

$$S_n - B(n) = o(\log n) \quad a.s.$$

Then $F(t) \equiv \Phi(t)$.

Theorem 1.2.9 and Theorem 1.2.10 show that the rate of convergence in (1.14) is the best possible and the rate in (1.15) cannot be improved if $F(t) \neq \Phi(t)$. For independent r.v.'s which are not necessarily identically distributed, a result on the ASIP was given by Major.

Theorem 1.2.11 *(Major, 1977, [70]) Let $\{x_k; k \geq 1\}$ be independent r.v.'s with $Ex_i = 0$ and $Ex_i^2 = \sigma_i^2 < \infty$. Let $S_n = \sum_{i=1}^n x_i$ and $B_n = \sum_{i=1}^n \sigma_i^2$. Suppose that $B_n \to \infty$ and* *that there exists a numerical sequence* $\{M_n\}$ *such that*

$$M_n = o\left(\sqrt{\frac{b_n}{\log\log B_n}}\ \right)$$

and

$$P(|x_n| \le M_n) = 1.$$

If the probability space is rich enough, one can construct a standard Brownian motion $\{B(t); t \ge 0\}$ *such that*

$$|B(n) - S_n| = o((n \log\log n)^{1/2}) \quad a.s..$$

For R^d-valued r.v.'s, a version of Theorem 1.1.3 was proved by Philipp (1979, [87]) using Berkes and Philipp's method. In [6] (Berger, 1982) the quantile transform method and the result in Theorem 1.2.7 were extended to the case of R^d-valued i.i.d.r.v.'s. The multidimensional version of Theorem 1.2.7 was also proved by Einmahl (1984, [40];1987, [42]). In [42] he proved the theorem by using the Stassen-Dudley theorem in connection with an estimate in the multidimensional CLT. For Banach space valued r.v.'s, necessary and sufficient conditions for a version of theorem 1.1.3 to hold were given by Philipp (1979,[87]). In [35] (1983), Dudley and Philipp established ASIP's as well as IPIP's for sums of independent not necessarily measurable r.v.'s with values in a not necessarily separable Banach space. An ASIP for triangular array of i.i.d Banach space valued r.v.'s was proved by Dabrowski, Dehling and Philipp (1984, [26]) which extends an IPIP given by de Acosta (1982, [1]). For r.v.'s in the domain of normal attraction to a stable law and in the domain of attraction to a Gaussian law, some results on the ASIP were given by Stout (1979, [95]), Mijnher (1980, [76]; 1983 [77]), Fisher (1984, [47]), Berkes, Dabrowski, Dehling and Philipp (1986, [7]), and Einmahl (1988, [43]; 1989, [44]).

(B) Martingales

The first result is due to Strassen. Using the Skorohod embedding method, he proved

Theorem 1.2.12 *(Strassen, 1965, [98]) Let* $S_n = \sum_{i=1}^n x_i$*,* $S_0 = 0$ *and* $\mathcal{F}_n = \sigma(x_1, x_2, \cdots, x_n)$ *be the* σ*-field generated by* $x_1, x_2, \cdots, x_n$ *(*$\mathcal{F}_0$ *the trivial* σ*-field) such that* $\{S_n, \mathcal{F}_n, n \geq 0\}$ *forms a martingale with finite second moments. Assume that*

$$V_n = \sum_{i \leq n} E(x_i^2 | \mathcal{F}_{i-1}) \to \infty \quad a.s.$$

as $n \to \infty$ *and that*

$$\sum_{n \geq 1} f(V_n)^{-1} E(x_n^2 I(x_n^2 > f(V_n)) | \mathcal{F}_{n-1}) < \infty \quad a.s.,$$

where f *is a positive nondecreasing function on* R_+*, which increases more slowly than* t*. Then, if the underlying probability space is rich enough, there exists a Brownian motion* $\{B(t); t \geq 0\}$ *such that as* $t \to \infty$*,*

$$S(t) = B(t) + o((tf(t))^{1/4} \log t) \quad a.s.,$$

where $S(t)$ *is obtained by linearly interpolating* S_n *at* V_n*, i.e.*

$$S(t) = S_n + \frac{t - V_n}{V_{n+1} - V_n} \qquad if \quad V_n \leq t < V_{n+1}.$$

Using the same method the following theorem was proved.

Theorem 1.2.13 *(Jain, Jogdeo and Stout, 1975, [59]) For fixed* $\alpha \geq 0$ *let*

$$f_\alpha(t) = t(\log \log t)^{-\alpha}.$$

Suppose that the following conditions hold a.s.:

$$V_n \to \infty \qquad (n \to \infty)$$

and for $\delta > 0$*:*

$$\lim_{n \to \infty} (f_\alpha(V_n))^{-1} \sum_{k=1}^n E\{x_k^2 I(x_k^2 \geq \delta f_\alpha(V_k)) | \mathcal{F}_{k-1}\} = 0,$$

$$\sum_{k=1}^\infty (f_\alpha(V_k))^{-1/2} E\{|x_k| I(x_k^2 \geq \delta f_\alpha(V - k)) | \mathcal{F}_{k-1}\} < \infty,$$

$$\sum_{k=1}^\infty (f_\alpha(V_k))^{-2} E\{x_k^4 I(x_k^2 \leq \delta f_\alpha(V_k)) | \mathcal{F}_{k-1}\} < \infty.$$

Then redefining $\{S_n; n \geq 1\}$, if necessary, on a new probability space, there exists a Brownian motion $\{B(t); t \geq 0\}$ such that

$$|S(t) - B(t)| = o(t^{1/2}(\log\log t)^{(1-\alpha)/2}) \quad a.s.,$$

where $S(t) = S_n$ if $V_n \leq t < V_{n+1}$.

Also using the Skorohod embedding method, Philipp and Stout established an ASIP for martingales under different conditions.

Theorem 1.2.14 *(Philipp and Stout, 1986, [91]) Let f be a nonincreasing differentiable function such that for all $t \geq t_0$,*

$$(\log t)^{-1} \leq f(t) \leq 10^{-3},$$

$f(t)t(\log\log t)^{-1/2}$ is increasing and $g(t) = \log t/f(t) \uparrow \infty$ with $tg'(t)$ bounded. Let $\{x_n, \mathcal{F}_n\}$ be a sequence of martingale differences with finite second moments such that with probability one,

$$V_n \to \infty \qquad (n \to \infty)$$

and

$$|x_n| \leq f(\sqrt{V_n}\,)\sqrt{V_n}\,(\log\log\sqrt{V_n}\,)^{-1/2}.$$

Let $S(t)$ be as in Theorem 1.2.13. Then we can redefine $\{x_n, \mathcal{F}_n, n \geq 1\}$ on a possibly richer probability space on which there exist a standard Brownian motion $\{B(t); t \geq 0\}$ and a r.v. t_0 such that for all $t \geq t_0$,

$$|S(t) - B(t)| \leq 10^3(f(t)t\log\log t)^{1/2} \quad a.s.$$

Using Berkes and Philipp's method, Morrow and Philipp (1982, [83]) established an ASIP for Hilbert space valued martingales. This ASIP was extended later by Monrad and Philipp (1991, [79]). The following theorem is a special case of Theorem 1 in [83].

Theorem 1.2.15 *(Morrow and Philipp, 1982, [83]) Let* $x_i, S_n, \mathcal{F}_n$ *and* V_n *be as in Theorem 1.2.12. Let* $f \neq \infty$ *be a nondecreasing function tending to* ∞ *as* $t \uparrow \infty$ *such that* $f(t)\log^{\alpha} t/t$ *is nonincreasing in* t *for some* $\alpha > 50$. *Suppose that* $V_n \to \infty$ *a.s. and*

$$\sum_{n\geq 1} E\{x_n^2 I(x_n^2 > f(V_n))/f(V_n)\} < \infty.$$

Then without changing its law we can redefine $\{x_n; n \geq 1\}$ *on a richer probability space on which there exists a standard Brownian motion* $\{B(t); t \geq 0\}$ *such that*

$$|\sum_{n\geq 1} x_n I(V_n \leq t) - B(t)| = O(t^{1/2}(f(t)/t)^{1/50}) \quad a.s.$$

For reverse martingales, the first work was done by Scott and Huggins who established an embedding theorem for reverse martingales, then proved an ASIP for them.

Let $\{S_n, \mathcal{F}_n, n \geq 1\}$ be a reverse martingale. We know that $S_\infty = \lim_{n\to\infty} S_n$ exists almost surely. If we put $S'_n = S_n - S_\infty$ then $\{S'_n, \mathcal{F}_n; n \geq 1\}$ is still a reverse martingale and $S'_n = 0$ a.s.

Theorem 1.2.16 *(Scott and Huggins, 1983, [92]) Let* $\{S_n, \mathcal{F}_n; n \geq 1\}$ *be a reverse martingale with* $ES_1^2 < \infty$ *and suppose without loss of generality that* $S_\infty = 0$. *Then (on a possibly enlarged probability space) there exists a Brownian motion* $\{B(t); \mathcal{F}_t^*, t \geq 0\}$ *and a nonincreasing sequence of stopping times* $\{T_n; n \geq 1\}$ *such that* $\mathcal{F}_n \subset \mathcal{F}_{T_n}$, $B(T_n) = S_n$ *a.s. Furthermore there exists a decreasing family of* σ*-fields* $\mathcal{G}_n$ *such that* T_n *is* $\mathcal{G}_n$*-measurable and the following results hold*

$$E(T_n - T_{n+1}|\mathcal{F}_{n+1}) = E((S_n - S_{n+1})^2|\mathcal{F}_{n+1}) \quad a.s.$$

and for $1 < p < \infty$ *there exists positive constants* M_p *and* N_p *depending only on* p *such that with probability one,*

$$M_p E\{(T_n - T_{n+1})^{p/2}|\mathcal{G}_{n+1}\} \leq E\{|S_n - S_{n+1}|^p|\mathcal{F}_{n+1}\} \leq N_p E\{(T - n - T_{n+1})^{p/2}|\mathcal{G}_{n+1}\}.$$

Let $\{W_n; n \geq 1\}$ be a decreasing sequence of positive r.v.s with $W_1^2 < e^{-1}$. Let $\{S_n, \mathcal{F}_n; n \geq 1\}$ be a reverse martingale with $ES_1^2 < \infty$ and $S_\infty = 0$. Define

$$S(t) = \begin{cases} S_{p+1} + x_p(t - W_{p+1}^2)(W_p^2 - W_{p+1}^2)^{-1} & \text{if } t \leq W_1^2 \\ 0 & \text{if } t > W_1^2 \end{cases}$$

where

$$x_n = S_n - S_{n+1} \quad \text{and} \quad p = p(t) = \max\{j; W_j^2 \geq t\}.$$

Scott and Huggins proved a general theorem which implies ASIP's for reverse martingales.

Theorem 1.2.17 *(Scott and Huggins, 1983, [92]) Suppose that $\{S_n; n \geq 1\}$ is a random sequence (not necessarily a reverse martingale) such that $S_n = B_{T_n}$ for some sequence of non-increasing nonnegative r.v.s $\{T_n; n \geq 1\}$. If with probability one $T_n < e$, $T_n \to 0$ and $T_n^{-1} W_n \to 1$, then as $t \to 0$,*

$$|S(t) - B(t)| = o\{(t \log\log t^{-1})^{1/2}\} \quad a.s.$$

Using Theorem 1.2.16, Huggins proved the following result which complements the corresponding martingale result (Theorem 1.2.13) of Jain, Jogdeo and Stout.

Theorem 1.2.18 *(Huggins, 1985, [55]) Let $\{S_n, \mathcal{F}_n; n \geq 1\}$be a reverse martingale with $ES_1^2 < \infty$ and $S_\infty = 0$ a.s. For fixed $\alpha \geq 0$ let $f_\alpha(t) = t(\log\log t^{-1})^{-\alpha}$, $e^{-1} \geq t \geq 0$. Suppose that*

$$W_n = \sum_{k=n}^{\infty} E(x_k^2 | \mathcal{F}_{k+1}) \to 0 \quad a.s. \quad (n \to \infty)$$

and for all $\delta > 0$ with probability one,

$$\lim_{n\to\infty} f_\alpha(W_n)^{-1} \sum_{k=n}^{\infty} E\{x_k^2 I(x_k^2 \geq \delta f_\alpha(W_n)) | \mathcal{F}_{k+1}\} = 0,$$

$$\sum_{k=1}^{\infty} f_\alpha(W_k)^{-1/2} E\{|x_k| I(x_k^2 \geq \delta f_\alpha(W_k)) | \mathcal{F}_{k+1}\} < \infty,$$

and

$$\sum_{k=1}^{\infty} f_\alpha(W_k)^{-2} E\{x_k^4 I(x_k^2 < \delta f_\alpha(W_k)) | \mathcal{F}_{k+1}\} < \infty.$$

Define

$$\begin{aligned} S(t) &= S_{n+1} \qquad \textit{if } W_{n+1} \le t < W_n \\ &= S_1 \qquad \textit{if } t \ge W_1 \end{aligned}$$

Then by extending the probability space if necessary, there exists a Brownian motion $\{B(t), \mathcal{F}_t^*; t \ge 0\}$ *such that as* $t \to 0$,

$$|S(t) - B(t)| = o\{t^{1/2}(\log\log t^{-1})^{(1-\alpha)/2}\} \quad a.s.$$

For a special kind of martingale, the so-called stationary ergodic martingale, some ASIP's were given by Basu (1973,[4]), and Jain, Jogdeo and Stout (1975, [59]). For sums of r.v.'s satisfying some dependence assumptions, several theorems on the ASIP, which apply to martingales, were proved by Eberlein (1986, [38]). In [10] (1991), Besdzied proved several results concerning strong approximation of semimartingales by processes with independent increments.

(C) Other results

There is a large literature on the ASIP's for sums of mixing r.v.s, empirical processes, lacunary sequences, quantile processes and U-statistics etc. For the results up to 1985 see [89] (Philipp, 1985) and [29]. For the results after 1985, see e.g. [30] (Dehling, Denker and Philipp, 1987), [73] (Mason, 1988) and [39] (Eberlein, 1989).

1.3 Invariance Principles for Multi-parameter Random Variables

(I) IPID's for multi-parameter random variables

In recent years some papers on the invariance principle for multi-parameter r.v.'s have appeared. An IPID for i.i.d. two-parameter r.v.'s was proved by Tudor ([99], 1979). Let us introduce IPID's for multi-parameter martingales.

Let $Z_+ = \{0, 1, 2, \cdots\}$ and $Z_+^d = \{(i_1, i_2, \cdots, i_d); i_k \in Z_+, k = 1, 2, \cdots, d\}$. For $i = (i_1, i_2, \cdots, i_d), j = (j_1, j_2, \cdots, j_d) \in Z_+^d$ we write $i < j$ (or $i \leq j$) if and only if $i_k < j_k$ (or $i_k \leq j_k$), $k = 1, 2, \cdots, d$).

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_i; i \in Z_+^d\}$ be an array of sub-σ-fields such that $\mathcal{F}_i \subset \mathcal{F}_j$ if $i \leq j$. We call such an array of sub-σ-fields a filtration.

Definition 1.3.1 *Let $\{x_i; i \in Z_+^d\}$ be an array of r.v.'s. Then $\{x_i, \mathcal{F}_i; i \in Z_+^d\}$ is called a (multiparameter) martingale difference array if*

(i) x_i is $\mathcal{F}_i$-measurable

(ii) $E|x_i| < \infty$

(iii) $E(x_j|\mathcal{F}_i) = 0$ if $i < j$,

For $n \in Z_+^d$ if we put $S_n = \sum_{i \leq n} x_i$, then $\{S_n, \mathcal{F}_i\}$ is called a (multiparameter) martingale.

Now let us state the definition of two-parameter strong martingales. Let $\{\mathcal{F}_{ij}; i, j \in Z_+\}$ be a filtration. Put

$$\mathcal{F}_i^1 = \bigvee_{j \in Z_+} \mathcal{F}_{ij}, \quad \mathcal{F}_j^2 = \bigvee_{i \in Z_+} \mathcal{F}_{ij}, \quad \mathcal{F}_{ij}^* = \mathcal{F}_i^1 \bigvee \mathcal{F}_j^2.$$

Definition 1.3.2 *Let $\{x_{ij}; i, j \in Z_+\}$ be an array of r.v.'s. Then $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ is called a strong martingale difference array (SMDA) if*

1) $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ *is a martingale difference array,*

2) $E(x_{kl}|\mathcal{F}^*_{ij}) = 0$ *if* $(i,j) < (k,l)$.

If we set $S_{mn} = \sum_{(i,j)\leq(m,n)} x_{ij}$, then $\{S_{mn}, \mathcal{F}_{mn}; m, n \in Z_+\}$ is a strong martingale, i.e. it is a martingale and

$$E(S_{mn} - S_{m-1,n} - S_{m,n-1} + S_{m-1,n-1}|\mathcal{F}^*_{m-1,n-1}) = 0.$$

Later on, we will use the (F4) condition. A filtration $\{\mathcal{F}_{ij}; i, j \geq 0\}$ is said to satisfy the (F4) condition if $\mathcal{F}^1_i$ and $\mathcal{F}^2_j$ are conditionally independent given $\mathcal{F}_{ij}$.

We have

Theorem 1.3.1 *(Morkvėnas,1984,[80]) For each* $n \in N$ *let* $\{x^n_{kl}, \mathcal{F}^n_{kl}; 1 \leq k \leq k_n, 1 \leq l \leq l_n\}$ *be a SMDA such that the filtration satisfies the (F4) condition,*

$$\max_{k,l} |x^n_{kl}| \xrightarrow{P} 0, \qquad \sup_n E(\max_{k,l}(x^n_{kl})^2) < \infty,$$

and

$$\sum_{k=1}^{k_n}\sum_{l=1}^{l_n}(x^n_{kl})^2 \xrightarrow{P} 1.$$

Then

$$S^n_{k_n,l_n} = \sum_{k=1}^{k_n}\sum_{l=1}^{l_n} x^n_{kl} \xrightarrow{D} N(0,1).$$

To present the functional CLT we need to recall the definition of the space $D[0,1]^2$ (see [14]).

For $t = (t_1, t_2) \in [0,1]^2$, let

$$\begin{aligned} Q_{\geq,\geq}(t) &= \{(s_1,s_2) \in [0,1]^2; s_1 \geq t_1, s_2 \geq t_2\} \\ Q_{\geq,<}(t) &= \{(s_1,s_2) \in [0,1]^2; s_1 \geq t_1, s_2 < t_2\} \end{aligned}$$

$$\begin{aligned} Q_{<,<}(t) &= \{(s_1,s_2) \in [0,1]^2; s_1 < t_1, s_2 < t_2\} \\ Q_{<,\geq}(t) &= \{(s_1,s_2) \in [0,1]^2; s_1 < t_1, s_2 \geq t_2\}. \end{aligned}$$

Let $x : [0,1]^2 \to R^1$ be a function. We write $x \in D[0,1]^2$ if and only if the following limits exist:

$$\lim_{\substack{s \in Q_{\geq,<}(t) \\ s \to t}} x(s), \qquad \lim_{\substack{s \in Q_{<,<}(t) \\ s \to t}} x(s), \qquad \lim_{\substack{s \in Q_{<,\geq}(t) \\ s \to t}} x(s)$$

and

$$\lim_{\substack{s \in Q_{\geq,\geq}(t) \\ s \to t}} x(s) = x(t).$$

Let Λ be the group of all transformations $\lambda : [0,1]^2 \to [0,1]^2$ of the form $\lambda = (\lambda_1(t_1), \lambda_2(t_2))$, where for $p = 1,2$, $\lambda_p : [0,1] \to [0,1]$ is continuous strictly increasing function with $\lambda_p(0) = 0$ and $\lambda_p(1) = 1$. The "Skorohod" distance between x and y in $D[0,1]^2$ is defined by

$$d(x,y) = \inf\{\min(\|x - y(\lambda)\|, \|\lambda\|) : \lambda \in \Lambda\}$$

where $\|x - y(\lambda)\| = \sup\{|x(t) - y(\lambda(t))|; t \in [0,1]^2\}$ and $\|\lambda\| = \sup\{|\lambda(t) - t|; t \in [0,1]^2\}$. Then $(D[0,1]^2, d)$ is a metric space.

Let $\{k_n(s); 0 \leq s \leq 1\}$ and $\{l_n(t); 0 \leq t \leq 1\}$ be sequences of integer valued positive nondecreasing right continuous functions with $k_n(0) = l_n(0) = 0$. For $0 \leq s \leq 1$ and $0 \leq t \leq 1$, set

$$X^n = X^n(s,t) = \sum_{k=1}^{k_n(s)} \sum_{l=1}^{l_n(t)} x_{kl}^n.$$

Let $W = W(s,t)$ be a two-parameter Brownian motion (see [18], page 58 for the definition). We have

Theorem 1.3.2 *(Morkvènas,1984,[80]) Let* $\{x_{kl}^n, \mathcal{F}_{kl}^n; k = 1,2,\cdots,k_n(1), l = 1,2,\cdots,l_n(1)\}$ *be SMDA's such that the filtrations satisfy the (F4) condition and*

$$\sum_{k=1}^{k_n(s)} \sum_{l=1}^{l_n(t)} (x_{kl}^n)^2 \xrightarrow{L^1} st. \tag{1.16}$$

Then $X^n \xrightarrow{D} W$ *in* $D[0,1]^2$, *i.e. for any bounded continuous functional* $f : D[0,1]^2 \to R^1$,

$$Ef(X^n) \to Ef(W) \qquad (n \to \infty).$$

The following theorem is due to Ivanoff.

Theorem 1.3.3 *(Ivanoff,1983,[57]) Let* $\{x_{kl}^n, \mathcal{F}_{kl}^n; 1 \leq k \leq k_n(1), 1 \leq l \leq l_n(1)\}$ *be SMDA's such that*

$$\sup_n E(\max_{k,l}(x_{kl}^n)^2) < \infty \tag{1.17}$$

$$\sum_{k=1}^{k_n(s)} \sum_{l=1}^{l_n(t)} (x_{kl}^n)^2 \xrightarrow{P} st, \quad (n \to \infty) \tag{1.18}$$

Then $X^n \xrightarrow{D} W$ *in* $D[0,1]^2$.

Clearly, the conditions in Theorem 1.3.3 are weaker than those in Theorem 1.3.2 because unlike Theorem 1.3.2, Theorem 1.3.3 does not need the (F4) condition and (1.16) implies (1.17) and (1.18).

Note that under different conditions, Leonenko and Mišura (1982, [65]) also obtained theorems on the IPID for multi-parameter strong martingales.

(II) ASIP's for Multi-parameter random variables

Let $N = \{1, 2, \cdots\}$ and $i = (i_1, i_2, \cdots, i_d), j = (i_1, j_2, \cdots, j_d) \in N^d$. We write $i \leq j$ iff $i_k \leq j_k, k = 1, 2, \cdots, d$ and write $i < j$ iff $i_k < j_k, k = 1, 2, \cdots, d$.

The first result on the ASIP for multi-parameter r.v.'s is due to Major. Using the quantile transform method he proved

Theorem 1.3.4 *(Major, 1976,[68]) Given a distribution* $F(t)$, $\int t dF(t) = 0$, $\int t^2 dF(t) = 1$ *and a monotone sequence* $n^{(1)} < n^{(2)} < \cdots$, $n^{(i)} \in N^d$, *there exist two sets of i.i.d.r.v.'s* x_n *and* y_n, $n \in N^d$ *with distribution* $F(t)$ *and* $\Phi(t)$ *respectively such that*

$$\sup_{n \leq n^{(k)}} |\sum_{i \leq n} (x_i - y_i)| = o((|n^{(k)}| \log\log |n^{(k)}|)^{1/2}) \qquad a.s.$$

where $|n| = \prod_{i=1}^{d} n_i$ *if* $n = (n_1, n_2, \cdots, n_d)$.

Using Berkes and Philipp's method Morrow (1981,[81]) proved some ASIP's for rectangular sums of multiparameter i.i.d.r.v.'s with values in a separable Banach space. These results were extended or improved later by Li and Wu (1989, [66]). The following theorem is a special case of Theorem 4 in [81].

Theorem 1.3.5 *(Morrow, 1981, [81]) Let* $d \geq 1$ *and* $\{x_n; n \in N^d\}$ *be a sequence of real-valued i.i.d.r.v.'s. Then*

$$Ex_1 = 0 \quad \text{and} \quad \begin{cases} \sigma^2 = Ex_1^2 < \infty & \text{for } d = 1 \\ E(x_1^2 \log^{d-1} |x_1|) < \infty & \text{for } d \geq 2 \end{cases}$$

if and only if there is a d-parameter Brownian motion (explained below) $\{B(t); t \in [0, \infty)^d\}$ *such that* $\mathcal{L}(B(t)) = N(0, \sigma^2 t)$ *and*

$$|S_n - B(n)| = o\{(|n| \log\log |n|)^{1/2}\},$$

where $S_n = \sum_{k \leq n} x_k$.

Note that the definition of d-parameter Brownian motion is similar to that of two-parameter Brownian motion (see [18], page 58). Obviously Theorem 1.3.5 is more general than Theorem 1.3.4 because the conclusion of Theorem 1.3.4 depends on the monotone sequence $n^{(1)} < n^{(2)} < \cdots$.

For the multiple sums of multi-parameter i.i.d.r.v.'s in the domain of attraction of stable laws, some ASIP's were established by Zinchenko (1988, [105]) and for the sums of

R^d-valued multiparameter r.v.'s satisfying a strong mixing condition, several theorems on the ASIP were proved by Berkes and Morrow (1981, [8]).

So far we see that no ASIP is established for multi-parameter martingales. In this thesis we will give an ASIP for two-parameter strong martingales. In Chapter 2, we will prove some "transition theorems" which turn two-parameter strong martingale into one-parameter martingales and will help us to prove the ASIP for the two-parameter strong martingales, and we will also show some applications of the transition theorems. In Chapter 3, the ASIP for two-parameter strong martingales will be stated and proved, and some corollaries of the ASIP will be verified which include a functional LIL for two-parameter strong martingales. In Chapter 4, we will state some known results we want to use and prove two results in this thesis.

Chapter 2

Transition Theorems and Applications

Naturally we can solve some problems of mutiparameter martingales by an approach similar to that by which the problems of one-parameter martingales are solved (see e.g. [57],[60],[61], [75] and [80]). Many authors have used a "transition" idea to prove theorems for strong martingales, that is, the results of one-parameter martingales are applied to strong martingales by turning strong martingales into one-parameter martingales. The transitions may be done by using stopping domain techniques (see e.g. [48] and [101]), or more easily by properly ordering the index set and properly defining the filtration (see e.g. [57]). In this chapter, we will establish a general version of the second type of "transition theorem" and show some applications. We will see that although the transition theorems are useful in the study of strong martingales, not all one-dimensional results can be extended in this way.

2.1 Transition Theorems

Let $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ be a SMDA (see Definition 1.3.2). As usual we assume that $x_{ij} = 0$ if $i = 0$ or $j = 0$. Recall that if $S_{kl} = \sum_{(i,j)\leq(k,l)} x_{ij}$ then $\{S_{kl}, \mathcal{F}_{kl}; k, l \in Z_+\}$ is a strong martingale.

Theorem 2.1.1 *Let* $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ *be a SMDA and let* $\prec$ *be a complete order in* Z_+^2*. Define*

$$\mathcal{G}_{kl} = \bigvee_{(i,j) \preceq (k,l)} \mathcal{F}_{ij}$$

If for all $(i,j), (k,l) \in Z_+^2$ *such that* $(i,j) \prec (k,l)$ *one has either* $i \le k-1$ *or* $j \le l-1$*, then* $\{x_{ij}, \mathcal{G}_{ij}; i, j \in Z_+\}$ *forms a martingale difference sequence (MDS) with respect to the order* $\prec$*. That is*

a) $\mathcal{G}_{ij} \subset \mathcal{G}_{kl}$ *if* $(i,j) \preceq (k,l)$*,*

b) x_{kl} *is* $\mathcal{G}_{kl}$*-measurable for all* $k, l \in Z_+$*,*

c) $E|x_{kl}| < \infty$ *for all* $k, l \in Z_+$*,*

d) $E(x_{kl}|\mathcal{G}_{ij}) = 0$ *if* $(i,j) \prec (k,l)$*.*

Proof. By the assumptions and the definition of $\mathcal{G}_{kl}$, we see that a), b), and c) are obviously true. Since $\mathcal{G}_{ij} \subset \mathcal{F}^*_{k-1,l-1}$ for $(i,j) \prec (k,l)$ we have

$$E(x_{kl}|\mathcal{G}_{ij}) = E\{E(x_{kl}|\mathcal{F}^*_{k-1,l-1})|\mathcal{G}_{ij}\} = 0. \quad \square$$

Let $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ be a SMDA. As in [57], we define "$\dot{<}$" as follows: for $(i,j), (k,l) \in Z_+^2$,

$$\begin{aligned} (i,j) \dot{<} (k,l) \quad &\text{iff} \quad i+j < k+l \\ &\text{or} \quad i+j = k+l \quad \text{and} \quad i < k. \end{aligned}$$

Clearly, $\dot{<}$ is a complete order on Z_+^2 (see Figure 2.1.1, page 37). Putting

$$\dot{S}_{mn} = \sum_{(i,j) \dot{\le} (m,n)} x_{ij}$$
$$\dot{\mathcal{F}}_{mn} = \bigvee_{(i,j) \dot{\le} (m,n)} \mathcal{F}_{ij}$$

we have

Corollary 2.1.1 *With respect to the order $\dot{<}$, $\{\dot{S}_{mn}, \dot{\mathcal{F}}_{mn}; m, n \in Z_+^2\}$ is a martingale.*

Proof. By Theorem 2.1, we only need to show that $i \leq k-1$, or $j \leq l-1$ for $(i,j)\dot{<}(k,l)$. Suppose $(i,j)\dot{<}(k,l)$ and $k, l \geq 1$, then $i+j < k+l$ or $i+j = k+l$, and $i \leq k$. It is easy to see that if $i < k$ then $i \leq k-1$, if $j < l$ then $j \leq l-1$. □

Definition 2.1.1 *Let $\Theta = \{(p_n, q_n); n \in Z_+\} \subset Z_+^2$. Then Θ is called a ruling sequence if $p_0 = q_0 = 0$ and $(p_0, q_0) < (p_1, q_1) < (p_2, q_2) < \cdots$. Let $(k,l) \in Z_+^2$. We say that the ruling sequence goes through (k,l) if there exists n such that $(p_n, q_n) = (k,l)$.*

Let $Q = \{(p_n, q_n); n \in Z_+\}$ be a ruling sequence. For $n \geq 1$, set

$$\begin{aligned} L_Q(n) &= \{(i,j) \in Z_+^2; p_{n-1} < i \leq p_n, 0 \leq j \leq q_n\} \\ &\quad \bigcup \{(i,j) \in Z_+^2; 0 \leq i \leq p_n, q_{n-1} < j \leq q_n\}. \end{aligned}$$

Definition 2.1.2 *For $(i,j), (k,l) \in Z_+^2$, we write*

$$(i,j)\ddot{<}(k,l)$$

if $(i,j) \in L_Q(n)$ and $(k,l) \in L_Q(n+t)$ (for some $t \geq 1$) or if $(i,j), (k,l) \in L_Q(n)$ and $(i,j)\dot{<}(k,l)$.

We can see that $\ddot{<}$ is a complete order on Z_+^2 (see Figure 2.1.2, page 37). Now put

$$\bar{S}_{mn} = \sum_{(i,j)\ddot{\leq}(m,n)} x_{ij}, \qquad \bar{\mathcal{F}}_{mn} = \bigvee_{(i,j)\ddot{\leq}(m,n)} \mathcal{F}_{ij}.$$

We have

Corollary 2.1.2 *$\{\bar{S}_{mn}, \bar{\mathcal{F}}_{mn}; m, n \in Z_+\}$ is a martingale with respect to the order $\ddot{<}$.*

Proof. Suppose that $(i,j)\ddot{<}(m,n)$ and $m, n \geq 1$. If $(i,j) \in L_Q(n')$ and $(m,n) \in L_Q(n'+t), (t \geq 1)$, by the definition of $L_Q(n')$, we obtain

$$\mathcal{F}_{ij} \subset \mathcal{F}_{p_{n'}, q_{n'}} \subset \mathcal{F}^*_{m-1,n-1}.$$

If $(i,j),(m,n) \in L_Q(n')$, then $(i,j)\dot{<}(m,n)$. As in the proof of Corollary 2.1.1, we have

$$i \le k-1, \text{ or } j \le l-1 \Longrightarrow \mathcal{F}_{ij} \subset \mathcal{F}^*_{m-1,n-1}.$$

So for $(k,l)\ddot{<}(m,n)$,

$$\bar{\mathcal{F}}_{kl} = \bigvee_{(i,j)\ddot{\le}(k,l)} \mathcal{F}_{i,j} \subset \mathcal{F}^*_{m-1,n-1}.$$

By Theorem 2.1.1, we reach the conclusion. □

Let $Q = \{(p_n, q_n); n \in Z_+\}$ be a ruling sequence. Define

$$R_{st} = \{(i,j) \in Z_+^2; p_{s-1} < i \le p_s, q_{t-1} < j \le q_t\}.$$

Definition 2.1.3 *For $(i,j),(k,l) \in Z_+^2$, we write*

$$(i,j)\tilde{<}(k,l)$$

iff $(i,j) \in R_{st}, (k,l) \in R_{uv}$ and $(s,t)\dot{<}(u,v)$, or $(i,j),(k,l) \in R_{st}$ and $(i,j)\dot{<}(k,l)$.

Obviously $\tilde{<}$ is a complete order in Z_+^2 (see Figure 2.1.3, page 38).

Corollary 2.1.3 *Let*

$$\tilde{\mathcal{F}}_{kl} = \bigvee_{(i,j)\tilde{\le}(k,l)} \mathcal{F}_{i,j}. \tag{2.1}$$

Then $\{x_{ij}, \tilde{\mathcal{F}}_{ij}; i,j \in Z_+\}$ is a MDS with respect to the order $\tilde{<}$.

Proof. Suppose that $(i,j)\tilde{<}(k,l)$. Let us show that $i \le k-1$, or $j \le l-1$.

a) If $(i,j),(k,l) \in R_{st}$, by the definition of $\tilde{<}$ we have $(i,j)\dot{<}(k,l)$. By the proof of Corollary 2.1.1, $i \le k-1$, or $j \le l-1$.

b) If $(i,j) \in R_{st}$, $(k,l) \in R_{uv}$ and $(s,t)\dot{<}(u,v)$, then $s \le u-1$ or $t \le v-1$. Therefore if $s \le u-1$, then $i \le p_s \le p_{u-1} \le k-1$ and if $t \le v-1$, then $j \le p_t \le p_{v-1} \le l-1$. □

Definition 2.1.4 *We write*

$$(i,j) \overset{(1)}{<} (k,l), \qquad (i,j),(k,l) \in Z_+^2$$

if $i < k$*, or if* $i = k$ *and* $j < l$*, and we write*

$$(i,j) \overset{(2)}{<} (k,l) \qquad (i,j),(k,l) \in Z_+^2$$

if $j < l$*, or if* $j = l$ *and* $i < k$*.*

It is easy to see that $\overset{(1)}{<}$ and $\overset{(2)}{<}$ are also complete orders on Z_+^2 (see Figure 2.1.4, page 38-39). Set

$$\mathcal{F}_{ij}^{(1)} = \mathcal{F}_{ij} \bigvee \mathcal{F}_{i-1}^{1}, \quad \mathcal{F}_{ij}^{(2)} = \mathcal{F}_{ij} \bigvee \mathcal{F}_{j-1}^{2}, \qquad i,j \geq 1.$$

If $i = 0$ or $j = 0$, $\mathcal{F}_{ij}$ is set to be the trivial σ-field. We have

Corollary 2.1.4 *Let* $n \in N$*. Then* $\{x_{ij}, \mathcal{F}_{ij}^{(1)}; (i,j) \in Z_+^2, j \leq n\}$ *and* $\{x_{ij}, \mathcal{F}_{ij}^{(2)}; (i,j) \in Z_+^2, i \leq n\}$ *are sequences of martingale differences with respect to the orders* $\overset{(1)}{<}$ *and* $\overset{(2)}{<}$ *respectively.*

Proof. For $(i,j) \overset{(1)}{<} (k,l)$, we see from Definition 2.1.4 that $i \leq k-1$ or $i = k$ and $j \leq l-1$. Similarly if $(i,j) \overset{(2)}{<} (k,l)$ we have that $i \leq k-1$ or $j \leq l-1$. By Theorem 2.1.1 we complete the proof. □

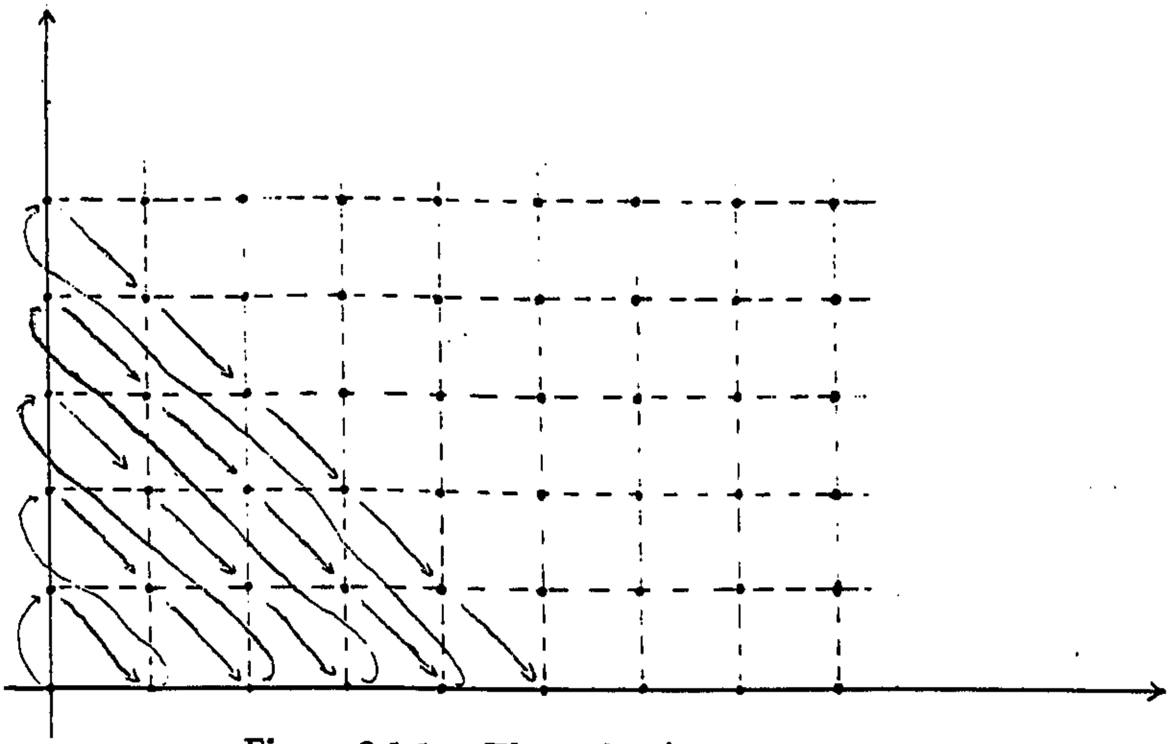

Figure 2.1.1 The order $\dot{<}$

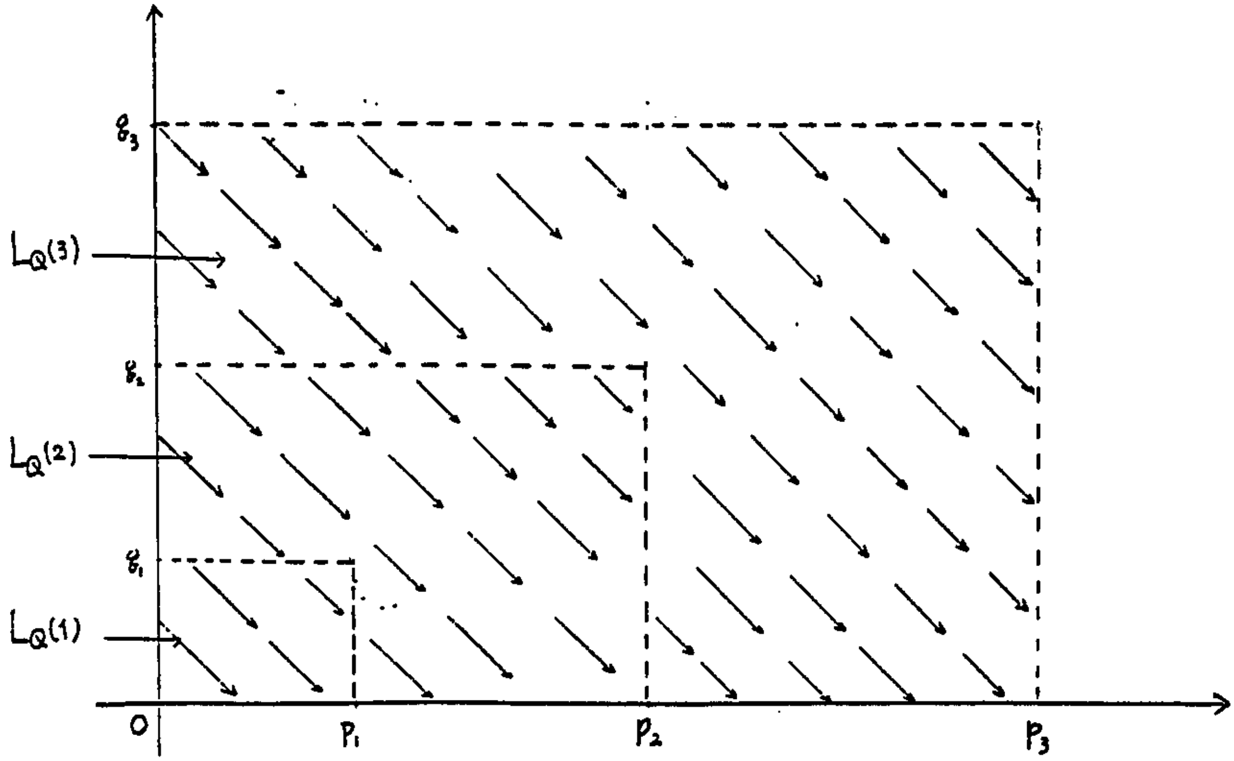

Figure 2.1.2 The order $\ddot{<}$

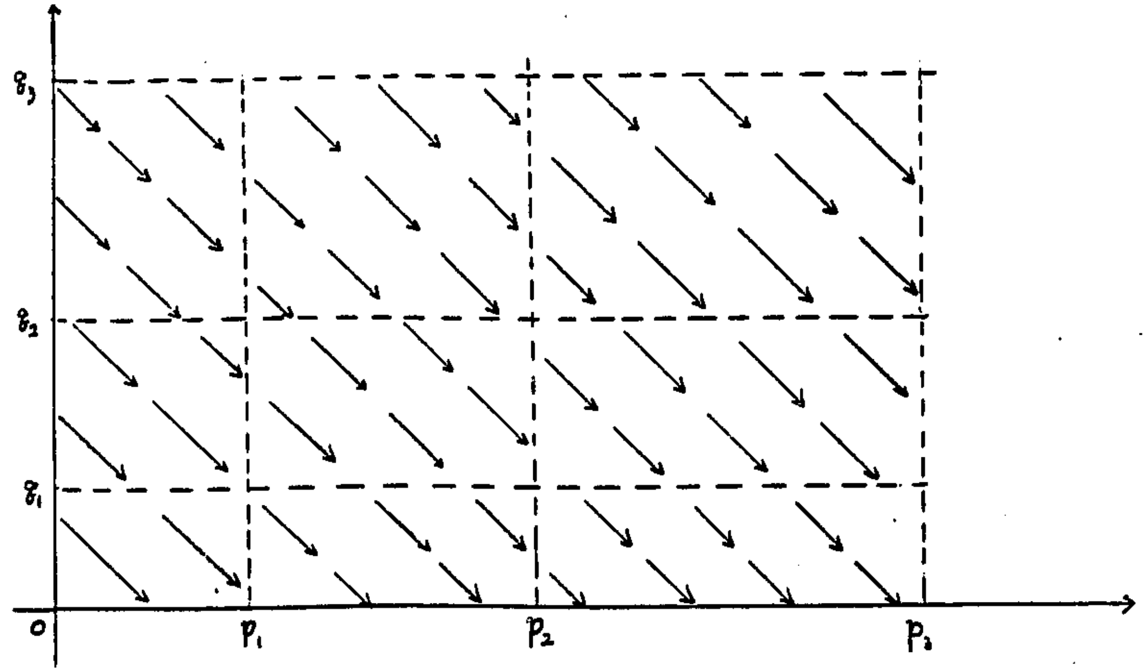

Figure 2.1.3 The order $\tilde{<}$

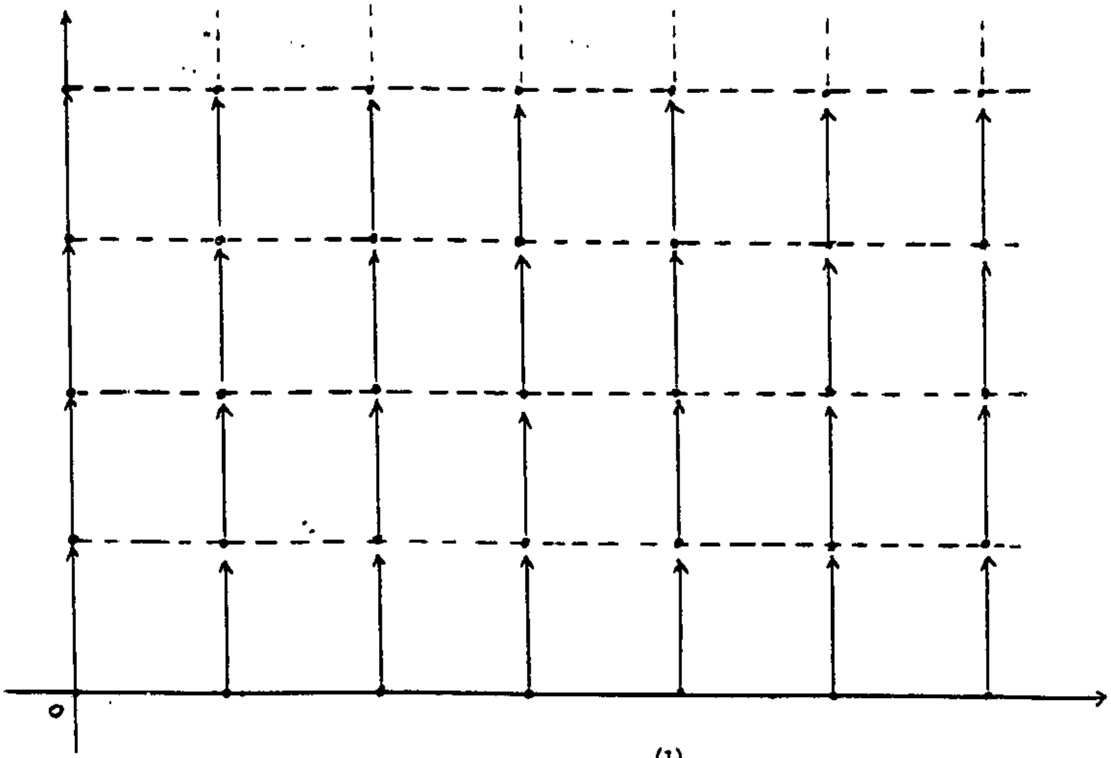

Figure 2.1.4 The order $\overset{(1)}{<}$

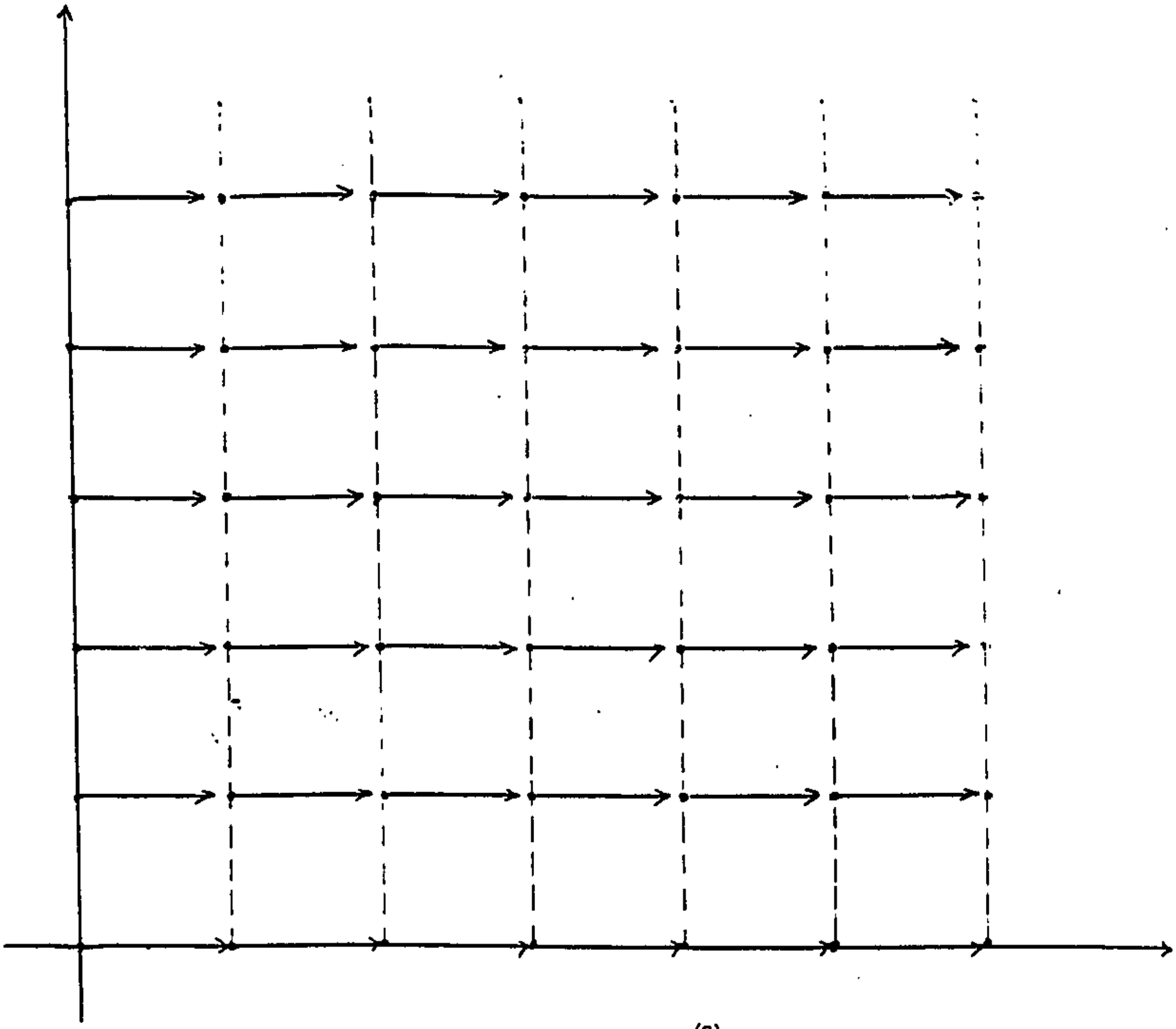

Figure 2.1.5 The order $\overset{(2)}{<}$

2.2 Applications

In this section we will use the transition theorems and the results from one-parameter martingale theory (which will be stated in Chapter 4 and will be labeled by "Theorem A1", "Theorem A2", etc.) to prove some corresponding results for two-parameter strong martingales. For convenience, here and hereafter, we use the notation "$i, j \to \infty$" to denote "$i \to \infty$ and $j \to \infty$" and in this section, $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ stands for a SMDA.

Lemma 2.2.1 *Suppose that* $\{b_{ij}; i, j \in Z_+^2\}$ *is an array of real numbers. If for all ruling sequences* $\{(p_n, q_n)\}$, $b_{p_n, q_n} \to 0$ *as* $n \to \infty$ *then* $b_{ij} \to 0$ *as* $i, j \to \infty$

Proof. If $b_{ij} \not\to 0$ as $i, j \to \infty$, then there exists $\varepsilon_0 > 0$ such that for all $k > 0$, there exists $i_k, j_k \geq k$ satisfying $(i_{k-1}, j_{k-1}) < (i_k, j_k), (k = 1, 2, \cdots)$ such that $|b_{i_k, j_k}| \geq \varepsilon_0$, which contradicts the assumptions. □

Lemma 2.2.2 *If for* $p > 1$ *and* $i, j \in Z_+$, $E|x_{ij}|^p < \infty$, *then there exist two constants* $C_1, C_2 > 0$ *depending only on* p *such that*

$$C_1 E(\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2)^{p/2} \leq E|S_{mn}|^p \leq E(\max_{(k,l) \leq (m,n)} |S_{kl}|)^p \leq C_2 E(\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2)^{p/2}$$

where $S_{kl} = \sum_{i=1}^{k} \sum_{j=1}^{l} x_{ij}$.

Proof. This lemma is a combination of Theorem 1 in [75] and Theorem 1 in [60]. □

By using the transition theorems in section 2.1, we can easily prove

$$C_1 E(\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2)^{p/2} \leq E|S_{mn}|^p \leq C_2 E(\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2)^{p/2}. \tag{2.2}$$

In fact, let $Q = \{(p_k, q_k); k \in Z_+\}$ be a ruling sequence which goes through (m, n) and let $\ddot{<}$ be defined with respect to Q, then

$$S_{mn} = \bar{S}_{mn} = \sum_{(i,j) \ddot{\leq} (m,n)} x_{ij} = \sum_{(i,j) \leq (m,n)} x_{ij}$$

$$\sum_{(i,j)\le(m,n)} x_{ij}^2 = \sum_{(i,j)\ddot{\le}(m,n)} x_{ij}^2$$

By Corollary 2.1.2 and Theorem A3, we get (2.2).

Unfortunately we cannot prove (2.2) with $E|S_{mn}|^p$ being replaced by $E(\max_{(k,l)\le(m,n)}|S_{kl}|)^p$ just by using the transition theorems and the corresponding result for one-parameter martingales. In general, $\max_{(k,l)\le(m,n)}|S_{kl}|$, taken with respect to the partial order $\le$, cannot be expressed as $\max_{(k,l)\preceq(m,n)}|S_{kl}|$, taken with respect to some complete order $\prec$ with respect to which $\{x_{ij},\mathcal{G}_{ij}; i,j\in Z_+\}$ is a MDS, where $\mathcal{G}_{ij}$ is defined in Theorem 2.1.1.

The following examples illustrate this. Consider the complete order $\dot{<}$. Let $(\Omega,\mathcal{F},P)$ is the probability space and $A\subset\Omega$ with $P(A)>0$. Suppose that on A,

$$S_{kl}=\begin{cases} k+l & \text{if } k=1 \text{ or } l=1\\ 0 & \text{otherwise}\end{cases}$$

If $m,n>1$, then on A

$$\max_{(k,l)\le(m,n)}|S_{kl}|=\max\{m+1,n+1\},$$

but

$$\max_{(k,l)\dot{\le}(m,n)}|S_{kl}|=m+n+1>\max\{m+1,n+1\}.$$

The situation is similar if we replace $\dot{<}$ by complete order $\ddot{<}$, $\tilde{<}$, $\overset{(1)}{<}$ or $\overset{(2)}{<}$. What we need to do is to properly choose $\{S_{kl}\}$.

Lemma 2.2.3 *(Theorem 3.3 in [101]). For $\lambda>0$, we have*

$$\lambda P(\max_{(k,l)\le(m,n)}|S_{kl}|\ge\lambda)\le 13E(|S_{mn}|).$$

Now let us use the transition theorems to show some propositions.

Proposition 2.2.1 *Let $1 \leq p < 2$. Suppose that $\{|x_{ij}|^p; i, j \in Z_+\}$ is uniformly integrable, then as $m, n \to \infty$,*

$$(mn)^{-1} E|S_{mn}|^p \to 0 \tag{2.3}$$

$$(mn)^{-1} \max_{(k,l)\leq(m,n)} |S_{kl}|^p \xrightarrow{P} 0. \tag{2.4}$$

If $1 < p < 2$, then

$$(mn)^{-1} E(\max_{(k,l)\leq(m,n)} |S_{kl}|^p) \to 0. \tag{2.5}$$

Proof. Let $Q = \{(p_n, q_n)\}$ be a ruling sequence and $\ddot{<}$ the complete order in Z_+^2 with respect to Q. Then by Corollary 2.1.2, $\{\bar{S}_{kl}, \bar{\mathcal{F}}_{kl}; k, l \in Z_+\}$ is a martingale. By Theorem A5, as $n \to \infty$,

$$(p_n q_n)^{-1} E|S_{p_n, q_n}|^p \to 0.$$

Therefore by Lemma 2.2.1, we obtain (2.3). Using Lemma 2.2.2, for $1 < p < 2$, there exists some constant $C > 0$ such that

$$(mn)^{-1} E(\max_{(k,l)\leq(m,n)} |S_{kl}|^p) \leq C(mn)^{-1} E|S_{mn}|^p$$

which together with (2.3) implies (2.5). Finally we prove (2.4). If $1 < p < 2$, then by (2.5) for each $\varepsilon > 0$,

$$\begin{aligned} &P((mn)^{-1} \max_{(k,l)\leq(m,n)} |S_{kl}|^p \geq \varepsilon) \\ &\quad \leq (mn\varepsilon)^{-1} E(\max_{(k,l)\leq(m,n)} |S_{kl}|^p) \to 0. \end{aligned}$$

If $p = 1$, then by Lemma 2.2.3 and (2.3), (2.4) is also true. □

Proposition 2.2.2 *Let η^2 be an almost surely finite random variable and $\{h_{ij}; i, j \in Z_+\}$ an array of positive numbers. Suppose that*

$$h_{mn}^{-1} \max_{(i,j)\leq(m,n)} |x_{ij}| \xrightarrow{P} 0, \qquad (m, n \to \infty)$$

$$h_{mn}^{-2} \sum_{(i,j)\leq(m,n)} x_{ij}^2 \xrightarrow{P} \eta^2, \quad (m, n \to \infty)$$

$$h_{mn}^{-2} E(\max_{(i,j)\leq(m,n)} x_{ij}^2) \leq M < \infty, \quad (\forall m, n \in Z_+).$$

Then

$$h_{mn}^{-1} \sum_{(i,j)\le(m,n)} x_{ij} \xrightarrow{D} Z, \tag{2.6}$$

where Z *has characteristic function* $E\exp(-\eta^2 t^2/2)$.

Proof. Let $Q = \{(p_n, q_n)\}$ be a ruling sequence and $\ddot{<}$ be defined with respect to Q. For $n = 1, 2, \cdots$, set

$$\begin{aligned} x_{n,(i,j)} &= h_{p_n,q_n}^{-1} x_{ij} \\ S_{n,(k,l)} &= \sum_{(i,j)\ddot{\le}(k,l)} x_{n,(i,j)} \\ \mathcal{F}_{n,(k,l)} &= \bar{\mathcal{F}}_{kl}. \end{aligned}$$

Then by Corollary 2.1.2, $\{S_{n,(k,l)}, \mathcal{F}_{n,(k,l)}; (k,l) \le (p_n, q_n)\}$ is a martingale and by the assumptions

$$\begin{aligned} &\max_{(i,j)\le(p_n,q_n)} |x_{n,(i,j)}| = h_{p_n,q_n}^{-1} \max_{(i,j)\le(p_n,q_n)} |x_{ij}| \xrightarrow{P} 0, \\ &\sum_{(i,j)\le(p_n,q_n)} x_{n,(i,j)}^2 = h_{p_n,q_n}^{-2} \sum_{(i,j)\le(p_n,q_n)} x_{ij}^2 \xrightarrow{P} \eta^2, \\ &E(\max_{(i,j)\le(p_n,q_n)} x_{n,(i,j)}^2) = h_{p_n,q_n}^{-2} E(\max_{(i,j)\le(p_n,q_n)} x_{ij}^2) \le M < \infty, \quad (\forall n \in Q). \end{aligned}$$

Clearly $\mathcal{F}_{n,(k,l)} = \mathcal{F}_{n+1,(k,l)}$. By Theorem A6, we have, as $n \to \infty$

$$h_{p_n,q_n}^{-1} \sum_{(i,j)\le(p_n,q_n)} x_{ij} \xrightarrow{D} Z;$$

that is , for each real number t,

$$P(h_{p_n,q_n}^{-1} \sum_{(i,j)\le(p_n,q_n)} x_{ij} \le t) \to P(Z \le t).$$

By Lemma 2.2.1, we get (2.6). □

Proposition 2.2.3 *Let* $\{h_{kl}\}$, η^2 *and* Z *be as in Proposition 2.2.2. Suppose that as* $m, n \to \infty$,

$$\begin{aligned} &h_{mn}^{-2} \sum_{(i,j)\le(m,n)} E\{x_{ij}^2 I(|x_{ij}| > \varepsilon h_{mn}) | \mathcal{F}_{i,j-1}^{(1)}\} \xrightarrow{P} 0, \quad (\textit{for each } \varepsilon > 0) \\ &h_{mn}^{-2} \sum_{(i,j)\le(m,n)} E(x_{ij}^2 | \mathcal{F}_{i,j-1}^{(1)}) \xrightarrow{P} \eta^2 \end{aligned}$$

where I_A denotes the indicator of the event A. Then

$$h_{mn}^{-1} \sum_{(i,j)\leq(m,n)} x_{ij} \xrightarrow{D} Z \quad (m,n \to \infty).$$

Proof. Let $Q = \{(p_n, q_n)\}$ be a ruling sequence. By the assumptions, we have, as $n \to \infty$,

$$h_{p_n,q_n}^{-2} \sum_{(i,j)\leq(p_n,q_n)} E\{x_{ij}^2 I(|x_{ij}| > \varepsilon h_{p_n,q_n})|\mathcal{F}_{i,j-1}^{(1)}\} \xrightarrow{P} 0, \quad (\varepsilon > 0)$$

$$h_{p_n,q_n}^{-2} \sum_{(i,j)\leq(p_n,q_n)} E(x_{ij}^2|\mathcal{F}_{i,j-1}^{(1)}) \xrightarrow{P} \eta^2.$$

Since by Corollary 2.1.4, $\{x_{ij}, \mathcal{F}_{ij}^{(1)}; (i,j) \leq (p_n, q_n)\}$ is a set of martingale differences with respect to $\overset{(1)}{<}$, by Theorem A7, as $n \to \infty$,

$$h_{p_n,q_n} \sum_{(i,j)\leq(p_n,q_n)} x_{ij} \xrightarrow{D} Z.$$

As in the proof of Proposition 2.2.2, using Lemma 2.2.1, we finish the proof. □

Similarly we have

Proposition 2.2.4 *Let $\{h_{kl}\}$, η^2 and Z be as in Proposition 2.2.2. Suppose that as $m, n \to \infty$,*

$$h_{mn}^{-2} \sum_{(i,j)\leq(m,n)} E\{x_{ij}^2 I(|x_{ij}| > \varepsilon h_{mn})|\mathcal{F}_{i-1,j}^{(2)}\} \xrightarrow{P} 0 \quad (\textit{for each } \varepsilon > 0),$$

$$h_{mn}^{-2} \sum_{(i,j)\leq(m,n)} E(x_{ij}^2|\mathcal{F}_{i-1,j}^{(2)}) \xrightarrow{P} \eta^2.$$

Then

$$h_{mn}^{-1} \sum_{(i,j)\leq(m,n)} x_{ij} \xrightarrow{D} Z \quad (m,n \to \infty).$$

Proposition 2.2.5 *Let η^2 and Z be as in Proposition 2.2.2. Suppose that $\sum_{i,j=1}^{\infty} Ex_{ij}^2 < \infty$. Write $s_{mn}^2 = \sum_{(i,j)\not\leq(m,n)} Ex_{ij}^2$. If*

$$s_{mn}^{-1} \sup_{(i,j)\not\leq(m,n)} |x_{ij}| \xrightarrow{P} 0 \quad (m,n \to \infty),$$

$$s_{mn}^{-2} \sum_{(i,j)\not\leq(m,n)} x_{ij}^2 \xrightarrow{P} \eta^2 \quad (m,n \to \infty),$$

$$E(s_{mn}^{-2} \sup_{(i,j)\not\leq(m,n)} x_{ij}^2) \leq M < \infty \quad (\forall m,n \in Z_+),$$

then

$$s_{mn}^{-1} \sum_{(i,j)\nleq(m,n)} x_{ij} \xrightarrow{D} Z.$$

If $P(\eta^2 > 0) = 1$, *then*

$$\Big(\sum_{(i,j)\nleq(m,n)} x_{ij}^2 \Big)^{-1/2} \sum_{(i,j)\nleq(m,n)} x_{ij} \xrightarrow{D} N(0,1).$$

Proof. Let $Q = \{(p_n, q_n)\}$ be a ruling sequence and $\ddot{<}$ be defined with respect to Q. For $n = 1, 2, \cdots$, put

$$\begin{aligned} x_{n,(i,j)} &= s_{p_n,q_n}^{-1} x_{ij} \\ S_{n,(i,j)} &= \sum_{(p_n,q_n)\ddot{<}(k,l)\ddot{\leq}(i,j)} x_{n,(k,l)} \\ \mathcal{F}_{n,(i,j)} &= \bar{\mathcal{F}}_{ij} \end{aligned}$$

If $(i,j)\ddot{\leq}(p_n,q_n)$ or $(i,j) \in Z^2 - Z_+^2$ (Z=$\{$all integers$\}$), then we define $S_{n,(i,j)} = 0$ and if $(i,j) \in Z^2 - Z_+^2$, we set $\mathcal{F}_{n,(i,j)} = \mathcal{F}_{0,0}$. Suppose that a complete order in the set $Z^2 - Z_+^2$ is defined. If we assign (i,j) to be "less than" $(0,0)$ for $(i,j) \in Z^2 - Z_+^2$, then we have defined a complete order in Z^2. We still denote it by $\ddot{<}$. So $\{S_{n,(i,j)}, \mathcal{F}_{n,(i,j)}; (i,j) \in Z^2\}$ is a martingale with respect to $\ddot{<}$ with differences

$$y_{n,(i,j)} = \begin{cases} 0 & \text{if } (i,j)\ddot{\leq}(p_n,q_n) \\ & \\ x_{n,(i,j)} & \text{if } (i,j)\ddot{>}(p_n,q_n). \end{cases}$$

By the assumptions, we have

$$\begin{aligned} &\sup_{n,i,j} E(S_{n,(i,j)}^2) \leq \sup_{n,i,j} \sum_{(i,j)\nleq(p_n,q_n)} s_{p_n,q_n}^{-2} E x_{ij}^2 = 1, \\ &\sup_{i,j} |y_{n,(i,j)}| = s_{p_n,q_n}^{-1} \sup_{(i,j)\nleq(p_n,q_n)} |x_{ij}| \xrightarrow{P} 0, \\ &\sum_{i,j} y_{n,(i,j)}^2 = s_{p_n,q_n}^{-2} \sum_{(i,j)\nleq(p_n,q_n)} x_{ij}^2 \xrightarrow{P} \eta^2, \\ &E\Big(\sum_{i,j} y_{n,(i,j)}^2\Big) = E\Big(s_{p_n,q_n}^{-2} \sum_{(i,j)\nleq(p_n,q_n)} x_{ij}^2\Big) \leq M < \infty, \quad (\forall n \in Q). \end{aligned}$$

Clearly

$$\begin{aligned} S_{n,\infty} &= \sum_{i,j} y_{n,(i,j)} = s_{p_n,q_n}^{-1} \sum_{(i,j)\nleq(p_n,q_n)} x_{ij}, \\ U_{n,\infty}^2 &= \sum_{i,j} y_{n,(i,j)}^2 = s_{p_n,q_n}^{-2} \sum_{(i,j)\nleq(p_n,q_n)} x_{ij}^2, \\ \mathcal{F}_{n,(i,j)} &= \mathcal{F}_{n+1,(i,j)}. \end{aligned}$$

By Theorem A8, we get

$$s_{p_n,q_n}^{-1} \sum_{(i,j)\nleq(p_n,q_n)} x_{ij} \xrightarrow{D} Z,$$

and if $P(\eta^2 > 0) = 1$, then

$$\Big(\sum_{(i,j)\nleq(p_n,q_n)} x_{ij}^2\Big)^{-1/2} \sum_{(i,j)\nleq(p_n,q_n)} x_{ij} \xrightarrow{D} N(0,1).$$

Finally using Lemma 2.2.1, we reach the conclusion. □

Proposition 2.2.6 *Let $\delta > 0$. If*

$$\begin{aligned} S_{mn} &= \sum_{(i,j)\le(m,n)} x_{ij}, \qquad L_{mn}(\delta) = \sum_{(i,j)\le(m,n)} E|x_{ij}|^{2+2\delta}, \\ N_{mn}^{(1)}(\delta) &= E\Big|\sum_{(i,j)\le(m,n)} E(x_{ij}^2|\mathcal{F}_{i,j-1}^{(1)}) - 1\Big|^{1+\delta}, \\ N_{mn}^{(2)}(\delta) &= E\Big|\sum_{(i,j)\le(m,n)} E(x_{ij}^2|\mathcal{F}_{i-1,j}^{(2)}) - 1\Big|^{1+\delta}, \end{aligned}$$

then there exists a constant C_δ depending only on δ such that

$$\sup_{-\infty<t<\infty} |P(S_{mn} \le t) - \Phi(t)| \le C_\delta \{L_{mn}(\delta) + N_{mn}^{(i)}(\delta)\}^{1/(3+2\delta)}$$

where $i = 1$ or 2 and $\Phi(t)$ denotes the standard normal distribution function.

Proof. By Corollary 2.1.4, $\{x_{ij}, \mathcal{F}_{ij}^{(1)}; (i,j) \le (m,n)\}$ is a set of martingale differences with respect to the order $\overset{(1)}{<}$ and $\{x_{ij}, \mathcal{F}_{ij}^{(2)}; (i,j) \le (m,n)\}$ is also a set of martingale differences with respect to the order $\overset{(2)}{<}$. So Theorem A9 together with the above facts implies this proposition. □

Now let us use the transition theorems to prove a maximal inequality with exponential bounds for SMDA's. Maximal inequalities are very important in probability theory. Some maximal inequalities concerning multiparameter martingales may be found , for example, in [60] and [19]. However maximal inequalities with exponential bounds are not available and we need them to prove ASIP's for strong martingales. To establish the maximal inequality with exponential bound, instead of the bound in Lemma 2.2.2 or Lemma 2.2.3, we need to show the following lemmas.

Lemma 2.2.4 *Let* $\{x_i, \mathcal{F}_i; i = 0, 1, \cdots,\}$ *be a set of martingale differences with* $|x_i| \leq M$ *a.s. and* $E(x_i^2|\mathcal{F}_{i-1}) \leq b$ *a.s. If* $S_k = \sum_{i=1}^k x_i$ *and* $0 \leq t \leq 1/M$, *then*

$$E(e^{tS_n}) \leq \exp(nbt^2).$$

Proof. Since

$$e^{tx_i} = 1 + tx_i + \frac{t^2}{2!}x_i^2 + \frac{t^3}{3!}x_i^3 + \cdots, \qquad (i = 1, 2, \cdots, n)$$

by martingale properties and the assumptions, we have

$$\begin{aligned} E(e^{tx_i}|\mathcal{F}_{i-1}) &= 1 + \frac{t^2}{2!}E(x_i^2|\mathcal{F}_{i-1}) + \frac{t^3}{3!}E(x_i^3|\mathcal{F}_{i-1}) + \cdots \\ &\leq 1 + \frac{t^2}{2!}b(1 + \frac{tM}{3} + \frac{(tM)^2}{4\cdot 3} + \frac{(tM)^3}{5\cdot 4\cdot 3} + \cdots) \\ &\leq 1 + \frac{t^2}{2}b(1 + \frac{1}{3} + \frac{1}{4\cdot 3} + \cdots) \\ &\leq 1 + bt^2 \leq e^{bt^2} \end{aligned}$$

and therefore

$$\begin{aligned} E(e^{tS_n}) &= E\{e^{tS_{n-1}}E(e^{tx_n}|\mathcal{F}_{n-1})\} \\ &\leq e^{bt^2}E(e^{tS_{n-1}}) \\ &\leq \cdots \leq \exp(nbt^2). \quad \square \end{aligned}$$

Lemma 2.2.5 *Let* $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ *be a SMDA with* $|x_{ij}| \leq M$ *a.s. Then for* $K > 0$,

$$P(\max_{k\leq m, l\leq n} S_{kl} \geq K) \leq 3E(e^{tS_{mn}})e^{-tK} \tag{2.7}$$

where $S_{mn} = \sum_{i=1}^m \sum_{j=1}^n x_{ij}$ *and* $t > 0$.

Proof. Obviously

$$\begin{aligned} P(\max_{k\leq m, l\leq n} S_{kl} \geq K) &= P(\exp(\max_{k\leq m, l\leq n} tS_{kl}) \geq e^{tK}) \\ &= P(\max_{k\leq m, l\leq n} e^{tS_{kl}} \geq e^{tK}) \\ &= P(\max_{k\leq m} M_k \geq e^{tK}), \end{aligned} \tag{2.8}$$

where

$$M_k = \max_{l\leq n} e^{tS_{kl}}.$$

Since

$$tS_{kl} = E(tS_{k+1,l}|\mathcal{F}_k^1),$$

$\{e^{tS_{kl}}, \mathcal{F}_k^1; k \geq 0\}$ is a submartingale. Thus

$$e^{tS_{kl}} \leq E(e^{tS_{k+1,l}}|\mathcal{F}_k^1) \leq E(\max_{l\leq n} e^{tS_{k+1,l}}|\mathcal{F}_k^1)$$

and we get

$$M_k \leq E(M_{k+1}|\mathcal{F}_k^1).$$

Hence $\{M_k, \mathcal{F}_k^1\}$ is a submartingale. By (2.8) and Theorem A1,

$$\begin{aligned} P(\max_{k\leq m, l\leq n} e^{tS_{kl}} \geq e^{tK}) &= P(\max_{k\leq m} M_k \geq e^{tK}) \\ &\leq e^{-tK} E(M_m) \\ &= e^{-tK} E(\max_{l\leq n} e^{tS_{ml}}). \end{aligned} \tag{2.9}$$

It is easy to see that $\{e^{tS_{ml}}, \mathcal{F}_l^2; l \geq 0\}$ is a submartingale. By Theorem A2, for $k > 1$,

$$\begin{aligned} E(\max_{l\leq n} e^{tS_{ml}}) &= E(\max_{l\leq n}(e^{tS_{ml}/k})^k) \\ &\leq (\frac{k}{k-1})^k E(e^{tS_{mn}}). \end{aligned}$$

Letting $k \to \infty$ and combining (2.8) and (2.9), we get (2.7). □

Proposition 2.2.7 *Let $\{x_{ij}, \mathcal{F}_{ij}\}$ be a SMDA satisfying $E(x_{ij}^2|\mathcal{F}_{i,j-1}^{(1)}) \leq b$ for $i \leq m$ and $j \leq n$ and $|x_{ij}| \leq M$ a.s. Then*

$$P(\max_{k\leq m, l\leq n} |S_{kl}| \geq K) \leq \begin{cases} 6\exp(-\frac{K^2}{4mnb}) & \text{if } K \leq 2mnb/M \\ 6\exp(-\frac{KM-mnb}{M^2}) & \text{if } K > 2mnb/M. \end{cases}$$

Proof. By Corollary 2.1.4, Lemma 2.2.4 and Lemma 2.2.5,

$$P(\max_{k\le m, l\le n} S_{kl} \ge K) \le 3\exp(-tK + mnbt^2).$$

Put

$$f(t) = -tK + mnbt^2.$$

Then $f'(t) = 0$ implies $t = K/(2mnb)$. If $K \le 2mnb/M$, then $f(\frac{K}{2mnb}) = -K^2/(2mnb)$ minimizes $f(t)$; if $K > 2mnb/M$, then $f(1/M) = -(KM - mnb)/M^2$ minimizes $f(t)$. So we have

$$P(\max_{k\le m, l\le n} S_{kl} \ge K) \le \begin{cases} 3\exp(-\frac{k^2}{4mnb}) & \text{if } K \le 2mnb/M \\ 3\exp(-\frac{KM-mnb}{M^2}) & \text{if } K > 2mnb/M \end{cases}$$

To complete the proof, observe that

$$[\max_{k\le m, l\le n} |S_{kl}| \ge K] \subset [\max_{k\le m, l\le n} S_{kl} \ge K] \bigcup [\max_{k\le m, l\le n} (-S_{kl}) \ge K]$$

and that $\{-x_{ij}, \mathcal{F}_{ij}\}$ is also a SMDA. □

Using the transition theorems we can also obtain an estimate of the Prohorov distance between the law of a two-parameter strong martingale and some appropriate normal law. This result is not used in the sequel, but is of independent interest. To prove this result we need the following theorem.

Let $\{x_i, \mathcal{F}_i; i \in Z_+\}$ be a MDS. Put

$$s_n^2 = \sum_{i=1}^{n} Ex_i^2, \qquad v_i^2 = E(x_i^2|\mathcal{F}_{i-1})$$

and denote by $\mathcal{L}(x)$ the distribution or law of the random variable x, by $\pi(F, G)$ the Prohorov distance between the probability measures F and G, and by $N(a, \sigma^2)$ the normal law with mean a and variance σ^2. We have

Theorem 2.2.1 *Let $\{x_i, \mathcal{F}_i; i \in Z_+\}$ be a MDS with $E|x_i|^3 < \infty$ and set*

$$V_n = E|s_n^2 - \sum_{i=1}^{n} v_i^2|^{3/2} + \sum_{i=1}^{n} E|x_i|^3.$$

Then

$$\pi(\mathcal{L}(\sum_{i=1}^{n} x_i), N(0, s_n^2)) \le 21 V_n^{1/6} \tag{2.10}$$

and if $V_n < 1/2^5$, *then*

$$\begin{aligned}&\pi(\mathcal{L}(\sum_{i=1}^{n} x_i), N(0, s_n^2))\\ &\qquad \le 6V_n^{2/9}(1 + \frac{\sqrt{2}}{3}|\log V_n|^{1/2}) + 7V_n^{1/4}(1 + \frac{1}{2}|\log V_n|^{1/2}).\end{aligned} \tag{2.11}$$

Since the proof of this theorem is long and independent of the transition theorems, we will do it in Chapter 4. Let $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ be a SMDA with $E|x_{ij}|^3 < \infty$. Set

$$s_{mn}^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} Ex_{ij}^2, \qquad v_{ij}^2 = E(x_{ij}^2 | \mathcal{F}_{i,j-1}^{(1)})$$

$$V_{mn} = E|s_{mn}^2 - \sum_{i=1}^{m}\sum_{j=1}^{n} v_{ij}^2\ |^{3/2} + \sum_{i=1}^{m}\sum_{j=1}^{n} E|x_{ij}|^3.$$

We have a two-parameter version of Theorem 2.2.1.

Proposition 2.2.8 *Let* $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ *be a SMDA with* $E|x_{ij}|^3 < \infty$. *Then*

$$\pi(\mathcal{L}(\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}), N(0, s_{mn}^2)) \le 21 V_{mn}^{1/6}$$

and if $V_{mn} < 1/2^5$, *then*

$$\begin{aligned}&\pi(\mathcal{L}(\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}), N(0, s_{mn}^2))\\ &\qquad \le 6V_{mn}^{2/9}(1 + \frac{\sqrt{2}}{3}|\log V_{mn}|^{1/2}) + 7V_{mn}^{1/4}(1 + \frac{1}{2}|\log V_{mn}|^{1/2}).\end{aligned}$$

Proof. By Corollary 2.1.4, $\{x_{ij}, \mathcal{F}_{ij}^{(1)}; i, j \in Z_+\}$ is a MDS with respect to the order $\overset{(1)}{<}$. Using Theorem 2.2.1, we get our conclusion. □

As the final corollary of the transition theorems we give a "one-dimensional" ASIP for two-parameter strong martingales. Let Q be a ruling sequence and $\ddot{<}$ the complete order with respect to Q. We have

Proposition 2.2.9 *Let* $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ *be a real-valued SMDA with* $\mathcal{F}_{kl} = \sigma(x_{ij}; (i,j) \leq (k,l))$ *and let* f *be a real-valued nondecreasing function tending to* ∞ *along the positive real axis such that* $f(t)\log^{\alpha} t/t$ *is nonincreasing for* $\alpha > 50$. *Suppose that as* $m, n \to \infty$,

$$W_{mn} = \sum_{(k,l) \ddot{\leq} (m,n)} E(x_{kl}^2 | \bar{\mathcal{F}}_{kl}^{-}) \to \infty,$$

where

$$\bar{\mathcal{F}}_{kl}^{-} = \bigcup_{(k,l) \ddot{<} (m,n)} \mathcal{F}_{kl}.$$

If

$$\sum_{m,n} E\{x_{mn}^2 I(x_{mn}^2 > f(W_{mn}))/f(W_{mn})\} < \infty,$$

then without changing its law we can redefine $\{x_{ij}; i, j \in Z_+\}$ *on a richer probability space on which there exists a standard Brownian motion* $\{B(t); t \geq 0\}$ *such that*

$$|\sum_{m,n} x_{mn} I(W_{mn} \leq t) - B(t)| = O(t^{1/2}(f(t)/t)^{1/50}) \quad a.s.$$

Proof. It follows from Corollary 2.1.2 and Theorem 1.2.15. □

If $\{x_{ij}; i, j \geq 1\}$ is a sequence of i.i.d.r.v's with $E|x_{ij}|^3 < \infty$ and $f(t) = t^{3/4}$, then it is easy to check that the conditions in Proposition 2.2.9 are satisfied and therefore the conclusion holds with

$$|\sum_{m,n} x_{mn} I(W_{mn} \leq t) - B(t)| = O(t^{1/2 - 1/200}) \quad \text{a.s.}$$

where

$$W_{mn} = \sum_{(k,l) \ddot{\leq} (m,n)} E x_{kl}^2.$$

Remark 2.2.1 If we replace the order $\ddot{<}$ by $\dot{<}$ or $\tilde{<}$, then Proposition 2.2.9 still holds.

Remark 2.2.2 Although we can use transition theorems together with ruling sequences to prove IPID's for strong martingales such as the proofs of Proposition 2.2.2, 2.2.3, 2.2.4 and

2.2.5, we cannot prove ASIP's for strong martingales in the same way.

The basic reason for Remark 2.2.2 is the difference between convergence in distribution and convergence almost surely. Let $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ be a SMDA defined on probability space $(\Omega, \mathcal{F}, P)$. Suppose that for each ruling sequence Q_u, $u \in U$, there exists a null set $A_u \subset \Omega$ such that

$$(\bar{s}_{mn})^{-1} \sum_{(i,j) \ddot{\leq} (m,n)} x_{ij} \to N(0,1) \text{ everywhere on } \Omega - A_u, \tag{2.12}$$

where

$$\bar{s}_{mn}^2 = \sum_{(i,j) \ddot{\leq} (m,n)} E x_{ij}^2.$$

So on $\Omega - \bigcup_{u \in U} A_u$, (2.12) holds for all ruling sequences. Since U is an uncountable set (there are uncountably many ruling sequences), we are not sure at all if $\bigcup_{u \in U} A_u$ is a null set or not. Therefore we are not sure if the following conclusion is correct or not:

$$s_{mn}^{-1} \sum_{(i,j) \leq (m,n)} x_{ij} \to N(0,1) \quad \text{a.s.}$$

Thus, we cannot prove ASIP's for strong martingales directly just by using the corresponding ASIP's for one-parameter martingales, transition theorems and ruling sequences.

Chapter 3

Almost Sure Invariance Principles

In this chapter we want to establish ASIP's for strong martingale difference arrays. As stated in Chapter 1 we have three main methods to prove ASIP's. Since SMDA's are not independent, the quantile transform cannot be used here. Then what about the Skorohod embedding method ? Unlike one-parameter martingales, not every multi-parameter strong martingale can be embedded in a Brownian sheet. Two counterexamples may be found in [20] and [58]. In [84], a theorem is proved that using stopping sets (or domains) to change the "time", a special class of two-parameter strong martingales may become two-parameter Brownian motion. Unfortunately the boundaries of the stopping sets are very irregular. It is difficult to control these boundaries if we use this time-change theorem to prove ASIP's for two-parameter strong martingales. So we choose to use Berkes and Philipp's method. We will use Theorem 1.3.2 to prove the ASIP's. Therefore we need to get the quantities such as λ, δ etc., and to estimate the error terms as in Description 1.3.3. Section 3.1 will present our main theorem—an ASIP for two-parameter strong martingales and some applications which include the functional law of iterated logarithm for strong martingales. Section 3.2 will prove some lemmas for the quantities and the error terms, and Section 3.3 will prove the main theorem.

3.1 The Main Theorem and its Applications

Let $\{x_{ij}, \mathcal{F}_{ij}; i,j \in Z_+\}$ be a SMDA with $E|x_{ij}|^{2+\delta} < \infty$ for some δ $(0 < \delta \leq 1)$. Let $t_0 = 0$ and $t_k = [\exp(k^\rho)]$, where $\rho = \delta/2(2+\delta)$. Then $Q_* = \{(t_k, t_k); k \geq 0\}$ is a ruling sequence. In this chapter, $\tilde{<}$ always denotes the complete order defined in Definition 2.1.3 with respect to Q_*. Put

$$\tilde{\mathcal{F}}_{kl}^{-} = \bigvee_{(i,j)\tilde{<}(k,l)} \mathcal{F}_{ij}.$$

Then by Corollary 2.1.3, $\{x_{ij}, \tilde{\mathcal{F}}_{ij}; i,j \in Z_+\}$ is a MDS under the complete order $\tilde{<}$. Let

$$s_{mn}^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} Ex_{ij}^2, \qquad q_{ij}^2 = E(x_{ij}^2|\tilde{\mathcal{F}}_{ij}^{-})$$

and define

$$R_{kl} = \{(i,j) \in Z_+^2; t_{k-1} < i \leq t_k, t_{l-1} < j \leq t_l\}$$

$$h_{kl} = \text{the number of the elements in } R_{kl}$$

$$X_{kl} = h_{kl}^{-1/2} \sum_{(i,j)\in R_{kl}} x_{ij} \tag{3.1}$$

$$\sigma_{kl}^2 = h_{kl}^{-1} \sum_{(i,j)\in R_{kl}} Ex_{ij}^2$$

$$V_{k,l,\delta} = E|\sum_{(i,j)\in R_{kl}} (Ex_{ij}^2 - q_{ij}^2)|^{(2+\delta)/2}. \tag{3.2}$$

We have the following main theorem.

Theorem 3.1.1 *Let* $\{x_{ij}, \mathcal{F}_{ij}; i,j \in Z_+\}$ *be a SMDA, where* $\mathcal{F}_{kl} = \sigma(x_{ij}; i \leq k, j \leq l)$. *If there exist positive constants* b, C *and* D *such that*

$$E(x_{ij}^2|\mathcal{F}_{i,j-1}^{(1)}) \leq b \text{ a.s} \qquad E(x_{ij}^2|\mathcal{F}_{i-1,j}^{(2)}) \leq b \text{ a.s}$$

and

$$\sup_{i,j} E|x_{ij}|^{2+\delta} \leq C \qquad \sup_{k,l}(V_{k,l,\delta}/h_{kl}) \leq D,$$

then without changing its distribution we can redefine $\{x_{ij}\}$ *on a richer probability space on which there exists an array of independent random variables* $\{y_{ij}\}$ *with* $\mathcal{L}(y_{ij}) =$

$N(0, Ex_{ij}^2)$ such that

$$\sup_{(k,l)\le(m,n)} |\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}-y_{ij})| = O\left\{(mn)^{1/2}\left(\frac{\sqrt{\log\log m}}{(\log n)^{2\delta}}+\frac{\sqrt{\log\log n}}{(\log m)^{2\delta}}+((\log m)(\log n))^{-1/4}\right)\right\} \quad a.s. \tag{3.3}$$

where $\log t = \ln(\max(e,t))$.

If the filtration $\{\mathcal{F}_{ij}; i,j \in Z_+\}$ satisfies the (F4) condition, that is , $\mathcal{F}_i^1$ and $\mathcal{F}_j^2$ are conditionally independent given $\mathcal{F}_{ij}$, then we obtain

Corollary 3.1.1 *Let $\{x_{ij}, \mathcal{F}_{ij}; i,j \in Z_+\}$ be a SMDA with $E|x_{ij}|^{2+\delta} < \infty$ $(0 < \delta \le 1)$, where $\mathcal{F}_{kl} = \sigma(x_{ij}; i \le k, j \le l)$. If (F4) is satisfied, and there exist positive constants b, C and D such that*

$$E(x_{ij}^2|\mathcal{F}_{i-1,j-1}^*) \le b \ a.s, \qquad \sup_{i,j} E|x_{ij}|^{2+\delta} \le C$$

and for all k and l,

$$h_{kl}^{-1}E|\sum_{(i,j)\in R_{kl}}(Ex_{ij}^2 - E(x_{ij}^2|\mathcal{F}_{i-1,j-1}^*))|^{(2+\delta)/2} \le D,$$

where $\mathcal{F}_{ij}^*$ is as in Definition 1.3.2, then without changing its distribution we can redefine $\{x_{ij}\}$ on a richer probability space on which there exists an array of independent random variables $\{y_{ij}\}$ with $\mathcal{L}(y_{ij}) = N(0, Ex_{ij}^2)$ such that (3.3) holds.

If $\{x_{ij}\}$ are independent, then Theorem 3.1.1 becomes

Corollary 3.1.2 *Let $\{x_{ij}; i,j \in Z_+\}$ be an array of independent r.v.'s such that $Ex_{ij} = 0$ and $\sup_{i,j} E|x_{ij}|^{2+\delta} < \infty$ for $0 < \delta \le 1$. Then without changing its distribution we can redefine $\{x_{ij}\}$ on a richer probability space on which there exists an array of independent random variables $\{y_{ij}\}$ with $\mathcal{L}(y_{ij}) = N(0, Ex_{ij}^2)$ such that (3.3) holds.*

Note that Corollary 3.1.2 is different from Theorem 1 in [82]. For example, the error bound in (3.3) is a function of m and n, but the error bound in Theorem 1 of [82] is a function of n. However if we take $m = n$ in (3.3), then the conclusion of Corollary 3.1.2 is a special case of Theorem 1 of [82].

Clearly the r.v.'s $\{x_{ij}; i, j \in Z_+\}$ in Corollary 3.1.1 forms a simple SMDA. Now let us consider a SMDA with non-trivial dependence structure.

Lemma 3.1.1 *Let* $\{\gamma_{ij}; i, j \in Z_+\}$ *be any array of r.v.'s. with* $E|\gamma_{ij}| < \infty$ *and let* $\{Z_{ij}; i, j \in Z_+\}$ *be an array of independent r.v.'s with* $EZ_{ij} = 0$ *and independent of* $\{\gamma_{ij}; i, j \in Z_+\}$. *Define*

$$x_{ij} = \gamma_{ij} Z_{ij}$$

$$\mathcal{F}_{ij} = \sigma(\gamma_{kl}; (k,l) \le (i,j)) \bigvee \sigma(Z_{kl}; (k,l) \le (i,j))$$

Then $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ *is a SMDA.*

Proof. Trivial. □

Corollary 3.1.3 *Let* $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ *be the SMDA defined in Lemma 3.1.1. Suppose that*

(a) $|\gamma_{ij}| < M_1$ *a.s. for some* $M_1 > 0$ *and all* $i, j \in Z_+$

(b) $E|\gamma_{ij}^2 - E\gamma_{ij}^2|^{(2+\delta)/2} < M_2 h_{kl}^{-\delta/2}$ *if* $(i,j) \in R_{kl}$.

(c) $\sup_{i,j} E|Z_{ij}|^{2+\delta} = M_3 < \infty$

where $0 < \delta \le 1$. *Then without changing its distribution we can redefine* $\{x_{ij}\}$ *on a richer probability space on which there exists an array of independent random variables* $\{y_{ij}\}$ *with* $\mathcal{L}(y_{ij}) = N(0, Ex_{ij}^2)$ *such that (3.3) is true.*

Proof. By the assumptions,

$$E(x_{ij}^2 | \mathcal{F}_{i,j-1}^{(1)}) \quad = \quad E(\gamma_{ij}^2 Z_{ij}^2 | \mathcal{F}_{i,j-1}^{(1)})$$

$$\leq M_1^2 E(Z_{ij}^2) \leq M_1^2 M_3^{2/(2+\delta)} \tag{3.4}$$

$$E(x_{ij}^2 | \mathcal{F}_{i-1,j}^{(2)}) \leq M_1^2 M_3^{2/(2+\delta)} \tag{3.5}$$

and

$$\begin{aligned}
& h_{kl}^{-1} E| \sum_{(i,j)\in R_{kl}} (Ex_{ij}^2 - q_{ij}^2)|^{(2+\delta)/2} \\
& \leq h_{kl}^{\delta/2-1} \sum_{(i,j)\in R_{kl}} E|Ex_{ij}^2 - q_{ij}^2|^{(2+\delta)/2} \\
& \leq h_{kl}^{\delta/2-1} \sum_{(i,j)\in R_{kl}} E|Z_{ij}|^{2+\delta} E|\gamma_{ij}^2 - E\gamma_{ij}^2|^{(2+\delta)/2} \\
& \leq M_2 M_3.
\end{aligned}$$

Hence the conditions of Theorem 3.1.1 are satisfied and the proof is complete. □

Now let us give an example for the SMDA in Corollary 3.1.3 which satisfies the conditions (a), (b) and (c). Let γ_{ij}, $i, j \in Z_+$, be uniform in $[1, 1+e^{-i-j}]$ and z_{ij}, $i, j \in Z_+$ be i.i.d.r.v.'s with $\mathcal{L}(z_{ij}) = N(0,1)$. Then the conditions (a) and (c) in Corollary 3.1.3 are automatically satisfied. Since

$$E(\gamma_{ij}^2) = \int_1^{1+e^{-i-j}} \frac{1}{e^{-i-j}} x^2 dx = (1+qe^{-i-j})^2$$

where $0 < q < 1$, we have, with probability 1,

$$\begin{aligned}
|\gamma_{ij}^2 - E\gamma_{ij}^2| &\leq |1-(1+qe^{-i-j})^2| + |(1+e^{-i-j})^2 - (1+qe^{-i-j})^2| \\
&\leq 8e^{-i-j}.
\end{aligned}$$

It is easy to see that there exists $M_2 > 0$ such that for all $(i,j) \in R_{kl}$,

$$E|\gamma_{ij}^2 - E\gamma_{ij}^2|^{3/2} \leq 8^{3/2} e^{-3(i+j)/2} \leq M_2 h_{kl}^{-1/2} \quad (\delta = 1),$$

i.e. (b) is also satisfied. Actually we can replace $[1, 1+e^{-i-j}]$ by $[1-a_{ij}, 1+b_{ij}]$ with $a_{ij}, b_{ij} > 0$. If $a_{ij} + b_{ij} \to 0$ fast enough, then the conditions (a), (b) and (c) in Corollary 3.1.3 will be satisfied.

Using Theorem 3.1.1 we can easily see that the following corollary, a central limit theorem, is valid.

Corollary 3.1.4 *Let the conditions in Theorem 3.1.1 be satisfied and* $s_{mn}^2/(mn) \to \gamma^2 > 0$. *For* $\alpha, \beta > 1$ *let*

$$\Delta_{\alpha,\beta} = \{(i,j) \in N^2; (\alpha^{-1}i)^{1/\beta} \le j \le \alpha i^{\beta}\}.$$

Then for $m, n \in \Delta_{\alpha,\beta}$ *and as* $m, n \to \infty$, *we have*

$$\frac{\sum_{i=1}^m \sum_{j=1}^n x_{ij}}{\sqrt{mn}} \xrightarrow{D} N(0, \gamma^2).$$

Note that if m goes to infinity too much faster than n or n goes to infinity too much faster than m, then Theorem 3.1.1 cannot guarantee that the conclusion of Corollary 3.1.4 is true. For example, if $m = \exp(\exp(n))$, then the conclusion of Theorem 3.1.1 becomes

$$\sup_{(k,l)\le(m,n)} |\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij} - y_{ij})| = O\left\{(mn)^{1/2}\frac{\sqrt{\log\log m}}{(\log n)^{2\delta}}\right\} \quad \text{a.s., as } m, n \to \infty.$$

Therefore even if $\sum_{i=1}^m \sum_{j=1}^n y_{ij}/\sqrt{mn} \xrightarrow{D} N(0, \gamma^2)$, we are not sure whether the conclusion Corollary 3.1.4 holds because $\sqrt{\log\log m}/(\log n)^{2\delta} \to \infty$ as $m, n \to \infty$.

We can also use Theorem 3.1.1 to prove the functional law of the iterated logarithm (FLIL) for two-parameter strong martingales. Let $\{x_{ij}; i, j \in Z_+\}$ be a SMDA with finite variance. Put

$$\sigma_{ij}^2 = Ex_{ij}^2, \quad s_{mn}^2 = \sum_{i=1}^m\sum_{j=1}^n \sigma_{ij}^2 \text{ and } S_{mn} = \sum_{i=1}^m\sum_{j=1}^n x_{ij}.$$

We define

$$S(s,t) = S_{mn} \quad \text{if } m \le s < m+1, \quad n \le t < n+1$$

and

$$H_{s,t}(u,v) = S(us, vt)/(4s_{st}^2 \log\log s_{st}^2)^{1/2}$$

where $(u,v) \in [0,1]^2$ and $s_{st}^2 = E\{(S(s,t))^2\}$. Then for each fixed (s,t), $H_{s,t}(u,v) \in D[0,1]^2$ (see Section 1.3 for the definition of $D[0,1]^2$). For $x, y \in D[0,1]^2$, if we define

$$d'(x,y) = \sup_{(u,v)\in[0,1]^2} |x(u,v) - y(u,v)| = \|x - y\|,$$

then $(D[0,1]^2, d')$ is also a metric space. Let $L([0,1]^2)$ be the family of real-valued integrable functions defined on $[0,1]^2$ and K a subset of $D[0,1]^2$ such that for $x \in K$,

$$x(s,t) = \int_0^s \int_0^t y(u,v)dudv, \quad 0 \le s,t \le 1$$

where $y \in L([0,1]^2)$ and $\int_{[0,1]^2} y^2(u,v)dudv \le 1$. To prove the FLIL for SMDAs, we will first establish the FLIL for the case of independent normal entries, then extend it to suitable SMDAs. We need the following proposition which is a direct corollary of Theorem 4 in [102].

Proposition 3.1.1 *Suppose that*

(1) $\lim_{m,n\to\infty} s_{mn}^2/(mn) = 1$.

(2) $\lim_{m,n\to\infty}(\frac{\log\log(mn)}{mn})^{1/2} \max_{(k,l)\le(m,n)} \|x_{kl}\|_\infty = 0$.

(3) $\{x_{ij}; i,j \in Z_+\}$ *are independent.*

Then

$$P(\lim_{s,t\to\infty} \sup_{(s',t')\ge(s,t)} d'(H_{s',t'}, K) = 0)$$
$$= P(\bigcap_{x\in K}\{\lim_{s,t\to\infty} \inf_{(s',t')\ge(s,t)} d'(H_{s',t'}, x) = 0\}) = 1.$$

Let $\{y_{ij}; i.j \in Z_+\}$ be independent normal r.v.'s with mean 0 and variance σ_{ij}^2. Set $T_{mn} = \sum_{i=1}^m \sum_{j=1}^n y_{ij}$,

$$T(s,t) = T_{kl} \text{ if } k \le s < k+1 \text{ and } l \le t < l+1$$

and

$$J_{s,t}(u,v) = T(us,vt)/(4s_{st}^2 \log\log s_{st}^2)^{1/2},$$

where $s_{st}^2 = E[(T(s,t))^2]$. We have

Lemma 3.1.2 *If* $\sup_{i,j}\sigma_{ij}^{2+\delta} < \infty$ *and (1) in Proposition 3.1.1 is satisfied, then*

$$\begin{aligned} &P(\lim_{s,t\to\infty} \sup_{(s',t')\geq(s,t)} d'(J_{s',t'}, K) = 0) \\ &= P(\bigcap_{x\in K}\{\lim_{s,t\to\infty}\inf_{(s',t')\geq(s,t)} d'(J_{s',t'}, x) = 0\}) = 1. \end{aligned}$$

Proof. In this proof we will use the following fact:

$$\sup_{i,j} E|y_{ij}|^{2+\delta} < \infty \quad \text{for} \quad \delta > 0$$

which follows from

$$E|y_{ij}|^{2+\delta} = E|N(0,\sigma_{ij}^2)|^{2+\delta} = \sigma_{ij}^{2+\delta}E|N(0,1)|^{2+\delta}.$$

Let

$$\begin{aligned} &y'_{ij} = y_{ij}I(|y_{ij}| \leq (ij)^{\frac{3+\delta}{3(2+\delta)}}), \qquad T'_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} y'_{ij}, \\ &T'(s,t) = T'_{kl} \ \text{ if } k \leq s < k+1 \text{ and } l \leq t < l+1 \end{aligned}$$

and

$$J'_{s,t}(u,v) = T'(us,vt)/(4s'^2_{st}\log\log s'^2_{st})$$

where $s'^2_{st} = E[(T'(s,t))^2]$. Since

$$Ey'^2_{ij} = \sigma_{ij}^2 - E\{y_{ij}^2 I(|y_{ij}| > (ij)^{\frac{3+\delta}{3(2+\delta)}})\}$$

and

$$\begin{aligned} &\sum_{i=1}^{m}\sum_{j=1}^{n} E\{y_{ij}^2 I(|y_{ij}| > (ij)^{\frac{3+\delta}{3(2+\delta)}})\} \\ &\quad \leq \sum_{i=1}^{m}\sum_{j=1}^{n}(E|y_{ij}|^{2+\delta})^{2/(2+\delta)}\{EI(|y_{ij}| > (ij)^{\frac{3+\delta}{3(2+\delta)}})\}^{\delta/(2+\delta)} \\ &\quad \leq \sum_{i=1}^{m}\sum_{j=1}^{n}(E|y_{ij}|^{2+\delta})^{2/(2+\delta)}\{(ij)^{-(3+\delta)/3}E|y_{ij}|^{2+\delta}\}^{\delta/(2+\delta)} \\ &\quad \leq \sum_{i=1}^{m}\sum_{j=1}^{n} E|y_{ij}|^{2+\delta}(ij)^{-\frac{\delta(3+\delta)}{3(2+\delta)}} \\ &\quad \leq C(mn)^{1-\frac{\delta(3+\delta)}{3(2+\delta)}}, \end{aligned}$$

we obtain

$$s_{mn}^{'2}/(mn) = s_{mn}^2/(mn) + o(1) \quad (m, n \to \infty).$$

It is easy to see that we also have

$$s_{st}^{'2}/(st) = s_{st}^2/(st) + o(1) \quad (s, t \to \infty). \tag{3.6}$$

Clearly

$$\begin{aligned} &\lim_{m,n\to\infty} \left(\frac{\log\log(mn)}{mn}\right)^{1/2} \sup_{(i,j)\leq(m,n)} \|y_{ij}'\|_\infty \\ &\leq \lim_{m,n\to\infty} \left(\frac{\log\log(mn)}{mn}\right)^{1/2} (mn)^{\frac{3+\delta}{6+3\delta}} = 0. \end{aligned} \tag{3.7}$$

Combining (3.6), (3.7) and Proposition 3.1.1, we get

$$\begin{aligned} &P(\lim_{s,t\to\infty} \sup_{(s',t')\geq(s,t)} d'(J_{s',t'}', K) = 0) \\ &= P(\bigcap_{x\in K} \{\lim_{s,t\to\infty} \inf_{(s',t')\geq(s,t)} d'(J_{s',t'}', x) = 0\}) = 1. \end{aligned} \tag{3.8}$$

Let

$$a_{st} = (4s_{st}^2 \log\log s_{st}^2)^{1/2}, \quad a_{st}' = (4s_{st}^{'2} \log\log s_{st}^{'2})^{1/2}$$

and

$$J_{s,t}''(u, v) = T'(us, vt)/a_{st}.$$

We are going to prove

$$\begin{aligned} &P(\lim_{s,t\to\infty} \sup_{(s',t')\geq(s,t)} d'(J_{s',t'}'', K) = 0) \\ &= 1 = P(\bigcap_{x\in K} \{\lim_{s,t\to\infty} \inf_{(s',t')\geq(s,t)} d'(J_{s',t'}'', x) = 0\}). \end{aligned} \tag{3.9}$$

In view of (3.8), let Ω_1 be the set such that $P(\Omega_1) = 1$ and on Ω_1,

$$\lim_{s,t\to\infty} \sup_{(s',t')\geq(s,t)} d'(J_{s',t'}', K) = 0 \tag{3.10}$$

and for all $x \in K$,

$$\lim_{s,t\to\infty} \inf_{(s',t')\geq(s,t)} d'(J_{s,t}', x) = 0. \tag{3.11}$$

Let us consider a fixed $\omega \in \Omega_1$. Since for $x \in K$,

$$\begin{aligned} d'(J''_{s,t}, x) &\le d'(J'_{s,t} \cdot \frac{a'_{st}}{a_{st}}, x \cdot \frac{a'_{st}}{a_{st}}) + d'(x \cdot \frac{a'_{st}}{a_{st}}, x) \\ &\le \frac{a'_{st}}{a_{st}} \cdot d'(J'_{s,t}, x) + \|x\|_\infty |\frac{a'_{st}}{a_{st}} - 1| \end{aligned}$$

and for any $\varepsilon > 0$, there exists $x_0 \in K$ such that

$$\frac{a'_{st}}{a_{st}} \cdot d'(J'_{s,t}, x_0) < \inf_{x \in K} \{\frac{a'_{st}}{a_{st}} \cdot d'(J'_{s,t}, x)\} + \varepsilon$$

we have

$$\begin{aligned} d'(J''_{s,t}, K) &\le \inf_{x \in K} \{\frac{a'_{st}}{a_{st}} \cdot d'(J'_{s,t}, x) + \|x\|_\infty |\frac{a'_{st}}{a_{st}} - 1|\} \\ &\le \frac{a'_{st}}{a_{st}} \cdot d'(J'_{s,t}, x_0) + \|x_0\|_\infty |\frac{a'_{st}}{a_{st}} - 1| \\ &< \inf_{x \in K} \{\frac{a'_{st}}{a_{st}} \cdot d'(J'_{s,t}, x)\} + \varepsilon + \|x_0\|_\infty |\frac{a'_{st}}{a_{st}} - 1| \\ &= \frac{a'_{st}}{a_{st}} \cdot d'(J'_{s,t}, K) + \varepsilon + \|x_0\|_\infty |\frac{a'_{st}}{a_{st}} - 1| \end{aligned}$$

which combines (3.10) and (3.6) to give

$$\lim_{s,t \to \infty} \sup_{(s',t') \ge (s,t)} d'(J''_{s',t'}, K) = 0$$

Since $\omega \in \Omega_1$ is arbitary, we get the first equality of (3.9). On the other hand, for $x \in K$ and the fixed $\omega \in \Omega_1$, by (3.11),

$$\begin{aligned} &\lim_{s,t \to \infty} \inf_{(s',t') \ge (s,t)} d'(J''_{s,t}, x) \\ &\quad \le \lim_{s,t \to \infty} \inf_{(s',t') \ge (s,t)} \{\frac{a'_{st}}{a_{st}} \cdot d'(J'_{s,t}, x) + \|x\|_\infty |\frac{a'_{st}}{a_{st}} - 1|\} = 0. \end{aligned}$$

By (3.6) and (3.8) again, the second equality in (3.9) is also true.

Finally let us finish the proof. By (3.9) we only need to show that

$$d'(J_{s,t}, J''_{s,t}) \to 0 \quad \text{a.s.} \quad (s, t \to \infty). \tag{3.12}$$

Let

$$y''_{ij} = y_{ij} I(|y_{ij}| > (ij)^{\frac{3+\delta}{3(2+\delta)}}).$$

We have

$$\begin{aligned} d'(J_{s,t}, J''_{s,t}) &= \sup_{(u,v)\in[0,1]^2} \left|\frac{T(us,vt)}{a_{st}} - \frac{T'(s,t)}{a_{st}}\right| \\ &\le a_{st}^{-1} \sum_{i=1}^{[s]} \sum_{j=1}^{[t]} |y''_{ij}|. \end{aligned} \tag{3.13}$$

Since

$$\begin{aligned} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(y''_{ij} \neq 0) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(|y_{ij}| > (ij)^{\frac{3+\delta}{3(2+\delta)}}) \\ &\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E|y_{ij}|^{2+\delta}(ij)^{-1-\delta/3} \\ &\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{-1-\delta/3} < \infty, \end{aligned}$$

by the Borel-Cantelli lemma we obtain

$$P(y''_{ij} \neq 0 \ \text{ i.o.}) = 0,$$

which together with (3.13) and the condition (1) in Proposition 3.1.1 implies (3.12). Therefore we complete our proof. □

Now it is time to prove the following corollary.

Corollary 3.1.5 *Let $\{x_{ij}, \mathcal{F}_{ij}; i,j \in Z_+\}$ be a SMDA satisfying the conditions of Theorem 3.1.1. Suppose that as $m, n \to \infty$,*

$$s_{mn}^2/(mn) \longrightarrow 1. \tag{3.14}$$

Then the conclusion in Proposition 3.1.1 holds true with the independent r.v.'s being replaced by this SMDA.

Proof. Let $\{y_{ij}; i,j \in Z_+\}$ be independent normal r.v.s with $\mathcal{L}(y_{ij}) = N(0, Ex_{ij}^2)$. By the assumptions and Lemma 3.1.2, we only need to show that

$$d'(H_{s,t}, J_{s,t}) \longrightarrow 0 \quad \text{a.s.} \quad (s,t \to \infty).$$

In fact, by Theorem 3.1.1 and the assumptions, with probability one,

$$\begin{aligned} d'(H_{s,t}, J_{s,t}) &= \sup_{(u,v)\in[0,1]^2} \{\frac{|S(us,vt) - T(us,vt)|}{a_{st}}\} \\ &= \frac{\sup_{(u,v)\in[0,1]^2} o\{\sqrt{[us][vt]\log\log([us][vt])}\ \}}{a_{st}} \\ &= \frac{o\{\sqrt{[s][t]\log\log([s][t])}\}}{a_{st}} = o(1). \end{aligned}$$

So the corollary is true. □

Note that the condition (3.14) in Corollary 3.1.5 is important. To explain this, let us use an example given by Li, Bhaskara and Wang (1992, [67]).

Let $\{x_{ij}; i,j \in Z_+\}$ be independent r.v.s with

$$P(x_{1,j} = 1) = P(x_{1,j} = -1) = P(x_{i,1} = 1) = P(x_{i,1} = -1) = 1/2$$

and $x_{ij} = 0$ a.s. for $i \neq 1$ or $j \neq 1$. In this case,

$$s_{mn}^2/(mn) \longrightarrow 0,$$

i.e. the condition (3.14) fails, and

$$\begin{aligned} &\limsup_{m,n\to\infty} \frac{S_{mn}}{\sqrt{4s_{mn}^2 \log\log s_{mn}^2}} \\ &\quad = \limsup_{(m+n)\to\infty} \frac{|\sum_{i=1}^m x_{i,1} + \sum_{j=2}^n x_{1,j}|}{\sqrt{4(m+n-1)\log\log(m+n-1)}} = 1/\sqrt{2} \neq 1 \end{aligned}$$

which means that the conclusion in Corollary 3.1.5 also fails for this "SMDA".

3.2 Lemmas for the Proof of the Main Theorem

We will use Theorem 1.2.2 to prove Theorem 3.1.1. To do this we need some lemmas.

Lemma 3.2.1 *(Example 34.3 in [11]) Let X and Y be random vectors of dimensions j and k respectively. Let $\mathcal{G}$ be a σ-field. Suppose that X is $\mathcal{G}$-measurable and that Y is independent of $\mathcal{G}$. Let $\phi(s,t)$ be a bounded Borel function on $R^j \times R^k$. If $f_\phi(u) = E(\phi(u,Y))$, then*

$$E(\phi(X,Y)|\mathcal{G}) \;=\; f_\phi(X).$$

Lemma 3.2.2 *Let $\{x_j, \mathcal{F}_j; i = 0, 1, \cdots, n\}$ be a MDS such that $E|x_j|^{2+\delta} < \infty \quad (0 < \delta \leq 1)$. Put*

$$s_n^2 = \sum_{j=1}^{n} Ex_j^2, \qquad v_j^2 = E(x_j^2|\mathcal{F}_{j-1}).$$

If $\sum_{j=1}^n v_j^2 = s_n^2$, then

$$E|E(e^{it\sum_{j=1}^n x_j}|\mathcal{F}_0) - e^{-t^2 s_n^2/2}| \leq 3|t|^{2+\delta} \sum_{k=1}^{n} E|x_k|^{2+\delta}.$$

Proof. Let $z_k, k = 1, 2, \cdots, n$ be $N(0,1)$ mutually independent and independent of $\mathcal{F}_n$. By the assumptions and Lemma 3.2.1,

$$\begin{aligned}
E(e^{it\sum_{j=1}^n v_j z_j}|\mathcal{F}_0) &= E\{E(e^{it\sum_{j=1}^n v_j z_j}|\mathcal{F}_n)|\mathcal{F}_0\} \\
&= E(e^{-(t^2/2)\sum_{j=1}^n v_j^2}|\mathcal{F}_0) \\
&= E(e^{-t^2 s_n^2/2}|\mathcal{F}_0) \\
&= e^{-t^2 s_n^2}.
\end{aligned}$$

Hence

$$E(e^{it\sum_{j=1}^n x_j}|\mathcal{F}_0) - e^{-t^2 s_n^2/2} \;=\; E(e^{it\sum_{j=1}^n x_j} - e^{it\sum_{j=1}^n v_j z_j}|\mathcal{F}_0). \tag{3.15}$$

Let

$$w_k = \sum_{j=1}^{k-1} x_j + \sum_{j=k+1}^{n} v_j z_j$$

and we use the convention $\sum_{j=1}^{0} x_j = \sum_{j=n+1}^{n} v_j z_j = 0$. Then

$$e^{it\sum_{j=1}^{n} x_j} - e^{it\sum_{j=1}^{n} v_j z_j} = \sum_{k=1}^{n}(e^{it(w_k+x_k)} - e^{it(w_k+v_k z_k)}). \tag{3.16}$$

Since by Lemma 3.2.1,

$$\begin{aligned} E(e^{it(w_k+x_k)}|\mathcal{F}_n) &= e^{it\sum_{j=1}^{k} x_j} E(e^{it\sum_{j=k+1}^{n} v_j z_j}|\mathcal{F}_n) \\ &= \exp\{it\sum_{j=1}^{k} x_j - \frac{t^2}{2}\sum_{j=k+1}^{n} v_j^2\} \\ &= \exp\{it\sum_{j=1}^{k} x_j - \frac{t^2}{2}(s_n^2 - \sum_{j=1}^{k} v_j^2)\}, \end{aligned}$$

$$\begin{aligned} &E(e^{it(w_k+v_k z_k)}|\mathcal{F}_n \bigvee \sigma(z_k)) \\ &= \exp(it\sum_{j=1}^{k-1} x_j + itv_k z_k)E\{e^{it\sum_{j=k+1}^{n} v_j z_j}|\mathcal{F}_n \bigvee \sigma(z_k)\} \\ &= \exp(it\sum_{j=1}^{k-1} x_j + itv_k z_k)E(e^{it\sum_{j=k+1}^{n} v_j z_j}|\mathcal{F}_n) \\ &= \exp\{it\sum_{j=1}^{k-1} x_j + itv_k z_k - \frac{t^2}{2}(s_n^2 - \sum_{j=1}^{k} v_j^2)\}. \end{aligned}$$

Hence, if $w_k' = it\sum_{j=1}^{k-1} x_j - (t^2/2)(s_n^2 - \sum_{j=1}^{k} v_j^2)$, then w_k' is $\mathcal{F}_{k-1}$-measurable.

$$\begin{aligned} &E(e^{it(w_k+x_k)} - e^{it(w_k+v_k z_k)}|\mathcal{F}_0) \\ &= E(e^{w_k'+itx_k} - e^{w_k'+itv_k z_k}|\mathcal{F}_0) \\ &= E[E(e^{w_k'}(e^{itx_k} - e^{itv_k z_k})|\mathcal{F}_{k-1})|\mathcal{F}_0] \\ &= E[e^{w_k'}(E(e^{itx_k}|\mathcal{F}_{k-1}) - E(e^{itv_k z_k}|\mathcal{F}_{k-1}))|\mathcal{F}_0]. \end{aligned} \tag{3.17}$$

Since

$$e^{it} = 1 + it - \frac{t^2}{2} + R(t)$$

with $|R(t)| \le |t|^{2+\delta}$, we have

$$\begin{aligned} E(e^{itx_k}|\mathcal{F}_{k-1}) &= 1 - \frac{t^2}{2}E(x_k^2|\mathcal{F}_{k-1}) + E(R(tx_k)|\mathcal{F}_{k-1}) \\ &= 1 - \frac{t^2}{2}v_k^2 + E(R(tx_k)|\mathcal{F}_{k-1}), \quad \text{and} \\ E(e^{itv_k z_k}|\mathcal{F}_{k-1}) &= 1 - \frac{t^2}{2}v_k^2 + E(R(tv_k z_k)|\mathcal{F}_{k-1}). \end{aligned} \tag{3.18}$$

Combining (3.15), (3.16), (3.17), and (3.18) we obtain

$$\begin{aligned}
&E|E(e^{it\sum_{j=1}^n x_j}|\mathcal{F}_0) - e^{t^2 s_n^2/2}| \\
&\quad \le \sum_{k=1}^n E|E(e^{it(w_k+x_k)} - e^{it(w_k+v_k z_k)}|\mathcal{F}_0)| \\
&\quad \le \sum_{k=1}^n E|E(e^{itx_k}|\mathcal{F}_{k-1}) - E(e^{itv_k z_k}|\mathcal{F}_{k-1})| \\
&\quad \le \sum_{k=1}^n (E|R(tx_k)| + E|R(tv_k z_k)|) \\
&\quad \le \sum_{k=1}^n (|t|^{2+\delta}E|x_k|^{2+\delta} + |t|^{2+\delta}E|v_k|^{2+\delta}E|z_k|^{2+\delta}) \\
&\quad \le |t|^{2+\delta}\sum_{k=1}^n (E|x_k|^{2+\delta} + E|E(x_k|\mathcal{F}_{k-1})|^{(2+\delta)/2}E|z_k|^{2+\delta}) \\
&\quad \le |t|^{2+\delta}\sum_{k=1}^n (1 + E|z|^{2+\delta})E|x_k|^{2+\delta} \\
&\quad \le 3|t|^{2+\delta}\sum_{k=1}^n E|x_k|^{2+\delta}. \quad \square
\end{aligned}$$

Usually $\sum_{j=1}^n v_j^2 = s_n^2$ is not true; we need something more. Define

$$\tau = \max\{k \le n; \sum_{j=1}^k v_j^2 \le s_n^2\}.$$

Obviously τ is a stopping time with respect to $\{\mathcal{F}_j\}$. Let I_A denote the indicator of the set A. Put

$$x_j' = x_j I_{(j\le\tau)}, \quad x_j'' = x_j I_{(j>\tau)}, \quad j = 0, 1, 2, \cdots, n$$

and

$$x_{n+1}' = x_{n+1}'' = (s_n^2 - \sum_{i=1}^{\tau} v_i^2)^{1/2} Z$$

where Z is $N(0,1)$ and independent of $\mathcal{F}_n \bigvee \sigma(z_i, N_i; i = 0, 1, 2, \cdots, n)$. Set $\mathcal{F}_{n+1} = \mathcal{F}_n \bigvee \sigma(Z)$. The following three lemmas establish a version of Lemma 3.2.2 when $\sum_{j=1}^n v_j^2 = s_n^2$ is not true—Corollary 3.2.1.

Lemma 3.2.3 *We have that* $\{x_j', \mathcal{F}_j; j = 0, 1, 2, \cdots, n+1\}$ *and* $\{x_j'', \mathcal{F}_j; j = 0, 1, 2, \cdots, n+1\}$ *are two MDS's with*

$$v_j'^2 = E(x_j'^2|\mathcal{F}_{j-1}) = v_j^2 I_{(j\le\tau)}$$

$$v_j'^2 = E(x_j''^2|\mathcal{F}_{j-1}) = v_j^2 I_{(j>\tau)}$$

and

$$v_{n+1}'^2 = v_{n+1}''^2 = E(x_{n+1}'^2|\mathcal{F}_n) = s_n^2 - \sum_{j=1}^{\tau} v_j^2.$$

Proof. This lemma follows from the facts that $I_{(i\le\tau)}$ and $I_{(i>\tau)}$ are $\mathcal{F}_{i-1}$-measurable and $(s_n^2 - \sum_{i=1}^{\tau} v_i^2)^{1/2}$ is $\mathcal{F}_n$-measurable, as well as the properties of Z. □

Lemma 3.2.4 *If* $E|x_j|^{2+\delta} < \infty \ \ (0 < \delta \le 1)$, *then*

$$E|E(e^{it\sum_{j=1}^{n+1} x_j'}|\mathcal{F}_0) - e^{t^2 s_n^2/2}| \le 9|t|^{2+\delta} V_{n\delta}$$

where $V_{n,\delta} = \sum_{j=1}^{n} E|x_j|^{2+\delta} + E|s_n^2 - \sum_{i=1}^{n} v_i^2|^{(2+\delta)/2}$.

Proof. Clearly

$$E|x_j'|^{2+\delta} \le E|x_j|^{2+\delta} < \infty, \quad j = 0, 1, 2, \cdots, n \tag{3.19}$$

and by the definition of x_{n+1}' (see page 67),

$$\begin{aligned} E|x_{n+1}'|^{2+\delta} &= E|(s_n^2 - \sum_{i=1}^{\tau} v_i^2)^{1/2} Z|^{2+\delta} \\ &= E|s_n^2 - \sum_{i=1}^{\tau} v_i^2|^{(2+\delta)/2} E|Z|^{2+\delta} \\ &\le 2(E|s_n^2 - \sum_{i=1}^{n} v_i^2|^{(2+\delta)/2} + E|v_{\tau+1}^2|^{(2+\delta)/2}). \end{aligned} \tag{3.20}$$

Since

$$E|v_{\tau+1}^2|^{(2+\delta)/2} \le \sum_{i=1}^{n} E|x_i|^{2+\delta},$$

we have

$$E|x_{n+1}'|^{2+\delta} \le 2(E|s_n^2 - \sum_{i=1}^{n} v_i^2|^{(2+\delta)/2} + \sum_{i=1}^{n} E|x_i|^{2+\delta}). \tag{3.21}$$

By Lemma 3.2.3,

$$\sum_{j=1}^{n+1} v_j'^2 = \sum_{j=1}^{n} v_j^2 I_{(j\le\tau)} + s_n^2 - \sum_{j=1}^{\tau} v_j^2 = s_n^2.$$

Using Lemma 3.2.3, Lemma 3.2.2, (3.19) and (3.21), we obtain

$$\begin{aligned}
&E|E(e^{it\sum_{j=1}^{n+1} x_j'}|\mathcal{F}_0) - e^{t^2 s_n^2/2}| \\
&\leq 3|t|^{2+\delta}\sum_{j=1}^{n+1} E|x_j'|^{2+\delta} \\
&\leq 3|t|^{2+\delta}\{\sum_{j=1}^{n} E|x_j|^{2+\delta} + 2(E|s_n^2 - \sum_{i=1}^{n} v_i^2|^{(2+\delta)/2} + \sum_{i=1}^{n} E|x_i|^{2+\delta})\} \\
&\leq 9|t|^{2+\delta}(\sum_{j=1}^{n} E|x_j|^{2+\delta} + E|s_n^2 - \sum_{i=1}^{n} v_i^2|^{(2+\delta)/2}). \quad \square
\end{aligned}$$

Lemma 3.2.5 *If* $E|x_i|^{2+\delta} < \infty$ *for* $0 < \delta \leq 1$*, then*

$$E|E(e^{it\sum_{j=1}^{n} x_j}|\mathcal{F}_0) - E(e^{it\sum_{j=1}^{n+1} x_j'}|\mathcal{F}_0)| \leq \frac{3}{2}|t|^2 V_{n,\delta}^{2/(2+\delta)}.$$

Proof.

$$\begin{aligned}
&E|E(e^{it\sum_{j=1}^{n} x_j}|\mathcal{F}_0) - E(e^{it\sum_{j=1}^{n+1} x_j'}|\mathcal{F}_0)| \\
&\quad \leq E|E(e^{it\sum_{j=1}^{n} x_j}|\mathcal{F}_0) - E(e^{it\sum_{j=1}^{n} x_j'}|\mathcal{F}_0)| \\
&\qquad + E|E(e^{it\sum_{j=1}^{n} x_j'}|\mathcal{F}_0) - E(e^{it\sum_{j=1}^{n+1} x_j'}|\mathcal{F}_0)| \\
&\quad = J_1 + J_2.
\end{aligned} \tag{3.22}$$

If $R_1(t) = e^{it} - (1 + it)$ then $|R_1(t)| \leq |t|^2/2$. Since $\sum_{j=1}^{n} x_j' = \sum_{j=1}^{\tau} x_j$ and $\sum_{j=1}^{n} x_j = \sum_{j=1}^{\tau} x_j + \sum_{j=1+\tau}^{n} x_j$,

$$J_1 = E|E\{e^{it\sum_{j=1}^{\tau} x_j}(it\sum_{j=1+\tau}^{n} x_j + R_1(t\sum_{j=1+\tau}^{n} x_j))|\mathcal{F}_0\}|. \tag{3.23}$$

Clearly

$$\begin{aligned}
&E|E(e^{it\sum_{j=1}^{\tau} x_j} R_1(t\sum_{j=1+\tau}^{n} x_j)|\mathcal{F}_0)| \\
&\leq E|R_1(t\sum_{j=1+\tau}^{n} x_j)| \\
&\leq \frac{|t|^2}{2} E(\sum_{j=1+\tau}^{n} x_j)^2.
\end{aligned}$$

By Lemma 3.2.3,

$$\begin{aligned} E(\sum_{j=1+\tau}^{n} x_j)^2 &= E(\sum_{j=1}^{n} x_j I_{(j>\tau)})^2 = E(\sum_{j=1}^{n} x_j'')^2 \\ &= \sum_{j=1}^{n} E(x_j''^2) = E(\sum_{j=1}^{n} E(x_j''^2|\mathcal{F}_{j-1})) \\ &= E(\sum_{j=1}^{n} v_j''^2) = E(\sum_{j=1+\tau}^{n} v_j^2) \\ &\leq E|s_n^2 - \sum_{j=1}^{n} v_j^2| + E|s_n^2 - \sum_{j=1}^{\tau} v_j^2|. \end{aligned}$$

Since

$$E|s_n^2 - \sum_{j=1}^{n} v_j^2| \leq (E|s_n^2 - \sum_{j=1}^{n} v_j^2|^{(2+\delta)/2})^{2/(2+\delta)} \leq V_{n,\delta}^{2/(2+\delta)} \tag{3.24}$$

and

$$\begin{aligned} E|s_n^2 - \sum_{j=1}^{\tau} v_j^2| &\leq (E|s_n^2 - \sum_{j=1}^{\tau} v_j^2|^{(2+\delta)/2})^{2/(2+\delta)} \\ &\leq (E|s_n^2 - \sum_{j=1}^{n} v_j^2|^{(2+\delta)/2} + E|v_{1+\tau}|^{(2+\delta)/2})^{2/(2+\tau)} \\ &\leq V_{n,\delta}^{2/(2+\delta)}, \end{aligned} \tag{3.25}$$

we have

$$E|E(e^{it\sum_{j=1}^{\tau} x_j} R_1(t \sum_{j=1+\tau}^{n} x_j)|\mathcal{F}_0)| \leq |t|^2 V_{n,\delta}^{2/(2+\delta)}. \tag{3.26}$$

Since

$$\begin{aligned} &E(e^{it\sum_{j=1}^{\tau} x_j}(it \sum_{j=1+\tau}^{n} x_j)|\mathcal{F}_0) \\ &= E(\sum_{k=1}^{n} e^{it\sum_{j=1}^{k} x_j}(it \sum_{j=k+1}^{n} x_j) I_{(\tau=k)}|\mathcal{F}_0) \\ &= \sum_{k=1}^{n} E\{E(e^{it\sum_{j=1}^{k} x_j}(it \sum_{j=k+1}^{n} x_j) I_{(\tau=k)}|\mathcal{F}_0\} \\ &= \sum_{k=1}^{n} E\{e^{it\sum_{j=1}^{k} x_j} I_{(\tau=k)} it E(\sum_{j=k+1}^{n} x_j|\mathcal{F}_0)\} \\ &= 0, \end{aligned} \tag{3.27}$$

by (3.23), (3.25), (3.26) and (3.27),

$$J_1 \le |t|^2 V_{n,\delta}^{2/(2+\delta)}. \tag{3.28}$$

Since

$$e^{it\sum_{j=1}^{n+1} x_j'} = e^{it\sum_{j=1}^{n} x_j'}(1 + itx_{n+1}' + R_1(tX_{n+1}')),$$

we have

$$J_2 = E|E(e^{it\sum_{j=1}^{n} x_j'}(itx_{n+1}' + R_1(tx_{n+1}'))|\mathcal{F}_0)|. \tag{3.29}$$

By the definition of x_{n+1}' (see page 67),

$$\begin{aligned}
&E(e^{it\sum_{j=1}^{n} x_j'} itx_{n+1}'|\mathcal{F}_0)\\
&= E[E(e^{it\sum_{j=1}^{n} x_j'} it(s_n^2 - \sum_{j=1}^{\tau} v_j^2)^{1/2} Z|\mathcal{F}_n)|\mathcal{F}_0]\\
&= E[e^{it\sum_{j=1}^{n} x_j'} it(s_n^2 - \sum_{j=1}^{\tau} v_j^2)^{1/2} E(Z|\mathcal{F}_n)|\mathcal{F}_0]\\
&= 0.
\end{aligned}$$

Hence by (3.24),

$$\begin{aligned}
J_2 &\le E|R_1(tx_{n+1}')| \le \frac{|t|^2}{2} E x_{n+1}'^2\\
&\le \frac{|t|^2}{2} E|s_n^2 - \sum_{j=1}^{\tau} v_j^2| E Z^2\\
&\le \frac{|t|^2}{2} V_{n,\delta}^{2/(2+\delta)}
\end{aligned}$$

which together with (3.22) and (3.28) gives the conclusion of Lemma 3.2.5. □

Corollary 3.2.1 *Let $\{x_j, \mathcal{F}_j; j = 0, 1, 2, \cdots, n\}$ be a MDS with $E|x_j|^{2+\delta} < \infty$ for $0 < \delta \le 1$. Then*

$$E|E(e^{it\sum_{j=1}^{n} x_j}|\mathcal{F}_0) - e^{-t^2 s_n^2/2}| \le \frac{3}{2}|t|^2 V_{n,\delta}^{2/(2+\delta)} + 9|t|^{2+\delta} V_{n,\delta}$$

where

$$V_{n,\delta} = \sum_{j=1}^{n} E|x_j|^{2+\delta} + E|s_n^2 - \sum_{j=1}^{n} v_j^2|^{(2+\delta)/2}.$$

Proof. It follows from Lemma 3.2.4 and Lemma 3.2.5. □

Replacing t by $t/\sqrt{n}$ in Corollary 3.2.1, we get

$$\begin{aligned} &E|E(e^{i\frac{t}{\sqrt{n}}\sum_{j=1}^{n}x_j}|\mathcal{F}_0) - e^{-\frac{t^2}{2}\frac{s_n^2}{n}}| \\ &\quad\le \frac{3}{2}\frac{|t|^2}{n}V_{n,\delta}^{2/(2+\delta)} + 9\frac{|t|^{2+\delta}}{n^{(2+\delta)/2}}V_{n,\delta}. \end{aligned} \tag{3.30}$$

Therefore we have

Corollary 3.2.2 *Let the conditions in Corollary 3.2.1 be satisfied and* $\sup_{n\ge 1}(V_{n.\delta}/n) = D < \infty$. *Then*

$$E|E(e^{i\frac{t}{\sqrt{n}}\sum_{j=1}^{n}x_j}|\mathcal{F}_0) - e^{-\frac{t^2}{2}\frac{s_n^2}{n}}| \le D_1(|t|^2 + |t|^{2+\delta})n^{-\delta/(2+\delta)} \tag{3.31}$$

where $D_1 = \max(\frac{3}{2}D^{2/(2+\delta)}, 9D)$.

Proof. By the assumptions we have

$$\begin{aligned} &\frac{3}{2}\frac{|t|^2}{n}V_{n,\delta}^{2/(2+\delta)} + 9\frac{|t|^{2+\delta}}{n^{(2+\delta)/2}}V_{n,\delta} \\ &\quad\le \frac{3}{2}\frac{|t|^2}{n^{\delta/(2+\delta)}}D^{2/(2+\delta)} + 9\frac{|t|^{2+\delta}}{n^{\delta/2}}D \\ &\quad\le D_1(|t|^2 + |t|^{2+\delta})n^{-\delta/(2+\delta)}. \end{aligned}$$

By (3.30), we get (3.31). □

In order to prove ASIP's for strong martingales, we need some maximal inequalities.

Let $\{\mathcal{F}_{ij}; i, j \in Z_+\}$ be a filtration and $\{\mathcal{F}_{ij}^{(1)}\}$ and $\{\mathcal{F}_{ij}^{(2)}\}$ be as in Corollary 2.1.4. Let $\{x_{ij}, \mathcal{F}_{ij}^{(1)}\}$ and $\{y_{ij}, \mathcal{F}_{ij}^{(2)}\}$ be two MDS's with respect to the orders $\overset{(1)}{<}$ and $\overset{(2)}{<}$ respectively. We have

Lemma 3.2.6 *Suppose that* $|x_{ij}|$, $|y_{ij}| \le M$ *a.s. and*

$$E(x_{ij}|\mathcal{F}_{i,j-1}^{(1)}), \quad E(y_{ij}|\mathcal{F}_{i-1,j}^{(2)}) \le b \quad a.s..$$

If for $K > 0$,

$$p_1 = P(\max_{k\leq m} |S_{kn}| \geq K), \qquad p_2 = P(\max_{l\leq n} |T_{ml}| \geq K),$$

then

$$p_1 \quad or \quad p_2 \leq \begin{cases} 2\exp(-\frac{K^2}{4mnb}) & \text{if } K \leq mnb/M \\ 2\exp(-\frac{KM-mnb}{4M^2}) & \text{if } K > mnb/M \end{cases}$$

where

$$S_{kl} = \sum_{i=1}^{k}\sum_{j=1}^{l} x_{ij}, \qquad T_{kl} = \sum_{i=1}^{k}\sum_{j=1}^{l} y_{ij}.$$

Proof. Since

$$\begin{aligned} E(S_{kn}|\mathcal{F}^1_{k-1}) &= S_{k-1,n} + E(\sum_{j=1}^{n} x_{kj}|\mathcal{F}^1_{k-1}) \\ &= S_{k-1,n} + \sum_{j=1}^{n} E\{E(x_{kj}|\mathcal{F}^{(1)}_{k,j-1})|\mathcal{F}^1_{k-1}\} \\ &= S_{k-1,n} \end{aligned}$$

and $S_{k,n}$ is $\mathcal{F}^1_k$-measurable, we have that $\{S_{k,n}, \mathcal{F}^1_k\}$ is a martingale for fixed n. So $\{e^{tS_{k,n}}, \mathcal{F}^1_k\}$ is a submartingale. Therefore

$$\begin{aligned} P(\max_{k\leq m} S_{k,n} \geq K) &= P(\max_{k\leq m} e^{tS_{k,n}} \geq e^{tK}) \\ &\leq e^{-tK} E(e^{tS_{mn}}). \end{aligned}$$

By Lemma 2.2.4 and an argument similar to that of the proof of Proposition 2.2.7, we conclude that Lemma 3.2.6 is valid for p_1 . Similarly it is also true for p_2. □

Lemma 3.2.7 *Let $\{x_{ij}, \mathcal{F}^{(1)}_{ij}\}$ and $\{y_{ij}, \mathcal{F}^{(2)}_{ij}\}$ be MDS' with respect to the orders $\overset{(1)}{<}$ and $\overset{(2)}{<}$ respectively. If $E(x^2_{ij}) < \infty$ and $E(y^2_{ij}) < \infty$, then for $K > 0$,*

$$P(\max_{k\leq m} |S_{kl}| \geq K) \leq \frac{C}{K^2}\sum_{i=1}^{m}\sum_{j=1}^{n} Ex^2_{ij}$$

$$P(\max_{l\leq n} |T_{kl}| \geq K) \leq \frac{C}{K^2}\sum_{i=1}^{m}\sum_{j=1}^{n} Ey^2_{ij}$$

where $C > 0$ is an absolute constant.

Proof. Simple. □

Lemma 3.2.8 *(Lemma 2.13 in [35]) Let A, B and C be complete and separable metric spaces. Suppose F is a distribution on $A \times B$ and G is a distribution on $B \times C$ such that F and G have the same marginal on B. Then there exists a distribution on $A \times B \times C$ with marginals F on $A \times B$ and G on $B \times C$.*

Lemma 3.2.9 *(Lemma 2.3 in [9]) Let A be a complete separable metric space and let F be a probability measure on the Borel sets of A. Let $(\Omega, \mathcal{F}, P)$ be a probability space such that $\mathcal{F}$ is atomless. Then there exists a random variable X on $(\Omega, \mathcal{F}, P)$ with values in A and distribution F.*

Remark 3.2.1 Since there always exists a probability space $(\Omega, \mathcal{F}, P)$ with $\mathcal{F}$ being atomless, by Lemma 3.1 and Lemma 3.2, the conclusion of Lemma 3.1 can be replaced by the conclusion that there exist a probability space and random variables X, Y and Z on it such that the joint distribution of X and Y is F and the joint distribution of Y and Z is G.

Lemma 3.2.10 *Let $R_i = R$ be the real line with the usual distance, and let $\mathcal{B}_i = \mathcal{B}$ be the σ-field of Borel sets of R. If for $u = (u_1, u_2, \cdots), v = (v_1, v_2, \cdots) \in \times_{i=1}^{\infty} R_i$, we define*

$$d(u,v) = \sum_{k=1}^{\infty} 2^{-k} \frac{|u_k - v_k|}{1 + |u_k - v_k|},$$

then $(\times_{i=1}^{\infty} R_i, d)$ is a complete and separable metric space. If $\mathcal{B}_{\infty}$ is the σ-field of Borel sets of $(\times_{i=1}^{\infty} R_i, d)$, then

$$\mathcal{B}_{\infty} = \times_{i=1}^{\infty} \mathcal{B}_i. \tag{3.32}$$

Finally if we set $D_k = \times_{i=1}^{\infty} R_i, k = 1, 2, \cdots, n$ and for $t = (t_1, t_2, \cdots, t_n), t' = (t'_1, t'_2, \cdots, t'_n) \in \times_{k=1}^{n} D_k$, we define

$$\bar{d}(t, t') = \max_{1 \le k \le n} d(t_k, t'_k),$$

then $(\times_{k=1}^{n} D_k, \bar{d})$ *is a complete and separable metric space. If* $\mathcal{D}$ *denotes the Borel* σ*-field of* $(\times_{k=1}^{n} D_k, \bar{d})$*, then*

$$\mathcal{D} = \underbrace{\mathcal{B}_\infty \times \mathcal{B}_\infty \times \cdots \times \mathcal{B}_\infty}_{n}.$$

We will prove this lemma in Chapter 4.

Remark 3.2.2 By Lemma 3.2.10, if $X = (x_1, x_2, \cdots)$ and $Y = (y_1, y_2, \cdots)$ are $\times_{i=1}^{\infty} R_i$-valued r.v.s such that they have the same distribution, then for each n, $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ are R^n-valued r.v.s and they have the same distribution on R^n. Conversely, if for each n, $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ are R^n-valued r.v.s and they have the same distribution on R^n, then X and Y are $\times_{i=1}^{\infty} R_i$-valued r.v.s such that they have the same distribution on $\times_{i=1}^{\infty} R_i$.

3.3 Proof of the Main Theorem

In this section, we will use the notations in Section 3.1 and Theorem 1.2.2 will be applied to prove our main theorem—Theorem 3.1.1.

Proof of Theorem 3.1.1. To apply Theorem 1.2.2 to prove our theorem, we need to estimate some constants such as λ_{kl}, δ_{kl} and α_{kl} which are similar to λ_k, δ_k and α_k in Theorem 1.2.2, to define and redefine random variables such as $\{x_{ij}\}$ and $\{y_{ij}\}$, and to estimate the error between the summation of $\{x_{ij}\}$ and the summation of $\{y_{ij}\}$. We divided the proof into three steps: estimation of some constants, defining $\{y_{ij}\}$ and redefining $\{x_{ij}\}$ and estimating the error term.

<u>Step 1. Estimation of some constants.</u>

Obviously $\{\tilde{\mathcal{F}}_{t_k,t_l}\}$ is a sequence of σ-fields which is nondecreasing with respect to the order

$\tilde{<}$ and X_{kl} is $\{\tilde{\mathcal{F}}_{t_k,t_l}\}$-measurable, where $\tilde{<}$ is defined in Definition 2.1.3, $\tilde{\mathcal{F}}_{t_k,t_l}$ is defined as in Corollary 2.1.3 and X_{kl} is as in Theorem 3.1.1. If G_{kl} is the distribution of $N(0, \sigma_{kl}^2)$, then G_{kl} has characteristic function

$$g_{kl}(u) = e^{-u^2\sigma_{kl}^2/2}.$$

If $(k', l') = \max\{(i,j); (i,j)\dot{<}(k,l)\}$, then $\{\tilde{\mathcal{F}}_{t_{k'},t_{l'}}\}$ is the "$\mathcal{F}_{k-1}$" in Theorem 1.2.2. By the assumptions and Corollary 3.2.2

$$E|E(e^{itX_{kl}}|\tilde{\mathcal{F}}_{t_{k'},t_{l'}}) - g_{kl}(t)| \leq D_1(|t|^2 + |t|^{2+\delta})h_{kl}^{-\delta/(2+\delta)}, \tag{3.33}$$

where h_{kl} is as in Theorem 3.1.1. By a simple estimation we get

$$h_{kl}^{-\delta/(2+\delta)} \leq C_0 \exp(-\rho(k^\rho + l^\rho)), \tag{3.34}$$

where C_0 is an absolute constant and $\rho = \delta/2(2+\delta)$. From now on, C_i's always denote positive absolute constants. Let $d = \delta/4(2+\delta)^2$ and define

$$T_{kl} = \max\{10^8, \exp(d(k^\rho + l^\rho))\}. \tag{3.35}$$

Then for $|t| \leq T_{kl}$, by (3.33) and (3.34), we have

$$E|E(e^{itX_{kl}}|\tilde{\mathcal{F}}_{t_{k'},t_{l'}}) - g_{kl}(t)| \leq \lambda_{kl} \tag{3.36}$$

with $\lambda_{kl} = C_1 \exp(-\rho(k^\rho + l^\rho)/2)$, and for k, l large enough,

$$\begin{aligned}
G_{kl}\{u; |u| > T_{kl}/4\} &= P(|N(0, \sigma_{kl}^2)| > T_{kl}/4) \\
&\leq P(|N(0, \sigma_{kl}^2)| > \frac{1}{4}\exp[d(k^\rho + l^\rho)]) \\
&\leq C_2 \exp(-d(k^\rho + l^\rho))\sigma_{kl}^2,
\end{aligned}$$

where

$$\begin{aligned}
\sigma_{kl}^2 &= h_{kl}^{-1} \sum_{(i,j)\in R_{kl}} Ex_{ij}^2 \\
&\leq h_{kl}^{-1} \sum_{(i,j)\in R_{kl}} (E|x_{ij}|^{2+\delta})^{2/(2+\delta)} \\
&\leq \sup_{i,j}(E|x_i|^{2+\delta})^{2/(2+\delta)} \\
&\leq C^{2/(2+\delta)}.
\end{aligned}$$

Hence, we have

$$G_{kl}\{u; |u| > T_{kl}/4\} \leq \delta_{kl} \tag{3.37}$$

with $\delta_{kl} = C_3 \exp(-d(k^\rho + l^\rho))$.

<u>Step 2. Defining $\{y_{ij}\}$ and redefining $\{x_{ij}\}$.</u>

By Theorem 1.2.2, we can redefine $\{X_{kl}\}$ (X_{kl} is defined as in Theorem 3.1.1) on a richer probability space together with a sequence $\{Y'_{kl}; k, l \geq 1\}$ of independent random variables such that $Y'_{kl} \overset{D}{=} N(0, \sigma^2_{kl})$ and if $\{X'_{kl}\}$ is the redefined version of $\{X_{kl}\}$, then

$$P(|X'_{kl} - Y'_{kl}| > \alpha_{kl}) \leq \alpha_{kl} \tag{3.38}$$

with

$$\begin{aligned} \alpha_{kl} &= 16T_{kl}^{-1}\log T_{kl} + 4\lambda_{kl}^{1/2}T_{kl} + \delta_{kl} \\ &\leq 16\exp(-d(k^\rho + l^\rho))d(k^\rho + l^\rho) \\ &\quad +4C_1\exp(-d(k^\rho + l^\rho)(1 + \delta/2) + d(k^\rho + l^\rho)) \\ &\quad +C_3\exp(-d(k^\rho + l^\rho)) \\ &\leq C_4\exp(-d\delta(k^\rho + l^\rho)/2) \\ &= C_4\exp(-d_1(k^\rho + l^\rho)) \end{aligned} \tag{3.39}$$

where $d_1 = \delta^2/8(2+\delta)^2$. Obviously

$$\sum_{k,l} \alpha_{kl} \leq C_5 \sum_{k,l} \exp(-d_1(k^\rho + l^\rho)) < \infty. \tag{3.40}$$

Since for any m and n,

$$\mathcal{L}(X_{11}, X_{12}, X_{21}, X_{13}, X_{22}, X_{31}, \cdots, X_{mn}) = \mathcal{L}(X'_{11}, X'_{12}, X'_{21}, X'_{13}, X'_{22}, X'_{31}, \cdots, X'_{mn}),$$

by Remark 3.2.2, $(X_{11}, X_{12}, X_{21}, X_{13}, X_{22}, X_{31}, \cdots)$ and $(X'_{11}, X'_{12}, X'_{21}, X'_{13}, X'_{22}, X'_{31}, \cdots)$ are $\times_{i=1}^\infty R_i$-valued r.v.s and they have the same distribution on $\times_{i=1}^\infty R_i$. Furthermore,

$(x_{11}, x_{12}, x_{21}, x_{13}, x_{22}, x_{31}, \cdots)$ and $(Y'_{11}, Y'_{12}, Y'_{21}, Y'_{13}, Y'_{22}, Y'_{31}, \cdots)$ are $\times_{i=1}^{\infty} R_i$-valued r.v.s. Let $A = B = C = (\times_{i=1}^{\infty} R_i, d)$ and

$$F = \mathcal{L}(\{x_{ij}; i, j \in Z_+\}, \{X_{kl}; k, l \geq 1\})$$

$$G = \mathcal{L}(\{X'_{kl}; k, l \geq 1\}, \{Y'_{kl}; k, l \geq 1\}).$$

Then by Remark 3.2.1, we have a probability space on which there exist $(\times_{i=1}^{\infty} R_i, d)$-valued r.v.s X, Y and Z such that

$$\mathcal{L}(X, Y) = F, \qquad \mathcal{L}(Y, Z) = G. \tag{3.41}$$

If

$$X = \{x''_{ij}; i, j \in Z_+\}, \qquad Y = \{X''_{kl}; i, j \in Z_+\}$$
$$Z = \{Y'_{kl}; k, l \geq 1\}$$

then

$$\begin{aligned} \mathcal{L}(\{x''_{ij}; i, j \in Z_+\}) &= \mathcal{L}(\{x_{ij}; i, j \in Z_+\}) \\ \mathcal{L}(\{X''_{kl}; i, j \in Z_+\}) &= \mathcal{L}(\{X_{kl}; k, l \geq 1\}) \\ &= \mathcal{L}(\{X'_{kl}; k, l \geq 1\}) \end{aligned}$$

and

$$\mathcal{L}(\{Y''_{kl}; k, l \geq 1\}) = \mathcal{L}(\{Y'_{kl}; k, l \geq 1\}).$$

Therefore by (3.41), the definition of X_{kl} and (3.38),

$$X''_{kl} = h_{kl}^{-1/2} \sum_{(i,j) \in R_{kl}} x''_{ij} \tag{3.42}$$

$$P(|X''_{kl} - Y''_{kl}| > \alpha_{kl}) \leq \alpha_{kl}. \tag{3.43}$$

Let $\{y_{ij}; i, j \in Z_+\}$ be an array of independent r.v.'s with $\mathcal{L}(y_{ij}) = N(0, Ex_{ij}^2)$. Define

$$Y_{kl} = h_{kl}^{-1/2} \sum_{(i,j) \in R_{kl}} y_{ij}. \tag{3.44}$$

Evidently, $(y_{11}, y_{12}, y_{21}, y_{13}, y_{22}, y_{31}, \cdots)$ and $(Y_{11}, Y_{12}, Y_{21}, Y_{13}, Y_{22}, Y_{31}, \cdots)$ are $(\times_{i=1}^{\infty} R_i, d)$-valued r.v.'s such that

$$\mathcal{L}(Y_{11}, Y_{12}, Y_{21}, Y_{13}, Y_{22}, Y_{31}, \cdots) = \mathcal{L}(Y''_{11}, Y''_{12}, Y''_{21}, Y''_{13}, Y''_{22}, Y''_{31}, \cdots).$$

Let $A = (\times_{i=1}^{\infty} R_i) \times (\times_{i=1}^{\infty} R_i)$, $B = C = \times_{i=1}^{\infty} R_i$ and

$$F = \mathcal{L}(\{x''_{ij}; i, j \in Z_+\}, \{X''_{kl}; k, l \geq 1\}, \{Y''_{k}; k, l \geq 1\})$$

$$G = \mathcal{L}(\{Y_{kl}; k, l \geq 1\}, \{y_{ij}; i, j \in Z_+\}).$$

Then by Remark 3.2.1 again, without changing their laws, we can redefine $\{x''_{ij}\}, \{X''_{kl}\}, \{Y''_{kl}\}$ and $\{y_{ij}\}$ on a new probability space (for notational convenience, we denote them by $\{x_{ij}\}$, $\{X_{kl}\}$, $\{Y_{kl}\}$ and $\{y_{ij}\}$ respectively) such that (3.1) and (3.44) hold and

$$P(|X_{kl} - Y_{kl}| > \alpha_{kl}) \leq \alpha_{kl}. \tag{3.45}$$

Step 3. Estimating the error term.

Finally let us estimate the error term:

$$e_{mn} := \sup_{(k,l) \leq (m,n)} | \sum_{(i,j) \leq (k,l)} (x_{ij} - y_{ij})|.$$

We will do this by using truncation, Lemma 3.2.6, Lemma 3.2.7 and the Borel-Cantelli lemma. Let

$$\xi_{ij} = x_{ij} I(|x_{ij}| \leq (ij)^{1/(2+\delta)}), \qquad \eta_{ij} = x_{ij} I(|x_{ij}| > (ij)^{1/(2+\delta)})$$

$$\xi'_{ij} = \xi_{ij} - E(\xi_{ij} | \mathcal{F}^{(1)}_{i,j-1}), \qquad \eta'_{ij} = \eta_{ij} - E(\eta_{ij} | \mathcal{F}^{(1)}_{i,j-1})$$

$$\xi''_{ij} = \xi_{ij} - E(\xi_{ij} | \mathcal{F}^{(2)}_{i-1,j}), \qquad \eta''_{ij} = \eta_{ij} - E(\eta_{ij} | \mathcal{F}^{(2)}_{i-1,j}).$$

Then $\{\xi'_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$, $\{\eta'_{ij}, \mathcal{F}^{(1)}_{ij}; i, j \in Z_+\}$ are MDS's with respect to the order $\overset{(1)}{<}$ and $\{\xi''_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$, $\{\eta''_{ij}, \mathcal{F}^{(1)}_{ij}; i, j \in Z_+\}$ are MDS's with respect to the order $\overset{(2)}{<}$. Since $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ is a SMDA, we have

$$x_{ij} = \xi_{ij} + \eta_{ij} = \xi'_{ij} + \eta'_{ij} = \xi''_{ij} + \eta''_{ij}.$$

By the assumption of Theorem 3.1.1,

$$E(\xi_{ij}'^2|\mathcal{F}^{(1)}_{i,j-1}) \le E(\xi_{ij}^2|\mathcal{F}^{(1)}_{i,j-1}) \le E(x_{ij}^2|\mathcal{F}^{(1)}_{i,j-1}) \le b \text{ a.s.}$$

Using Lemma 3.2.6 and Lemma 3.2.7, for $t_{m_*} < m \le t_{m_*+1}, t_{n_*} < n \le t_{n_*+1}$ and defining

$$K_1 := \frac{(t_{m_*}t_{n_*}\log\log t_{n_*})^{1/2}}{(\log t_{m_*})^{2\delta}}$$

we have

$$\begin{aligned}
P(&\max_{t_{m_*}<m\le t_{m_*+1}} |\sum_{t_{m_*}<i\le m}\sum_{j\le t_{n_*}} x_{ij}| \ge 2K_1) \\
&\le P(\max_{t_{m_*}<m\le t_{m_*+1}} |\sum_{t_{m_*}<i\le m}\sum_{j\le t_{n_*}} \xi_{ij}'| \ge K_1) \\
&\quad + P(\max_{t_{m_*}<m\le t_{m_*+1}} |\sum_{t_{m_*}<i\le m}\sum_{j\le t_{n_*}} \eta_{ij}'| \ge K_1) \\
&\le 2\exp(-\frac{K_1^2}{4t_{n_*}(t_{m_*+1}-t_{m_*})b}) + K_1^{-2}\sum_{i=t_{m_*}+1}^{t_{m_*+1}}\sum_{j=1}^{t_{n_*+1}} E(\eta_{ij}^2).
\end{aligned} \tag{3.46}$$

Since

$$t_{m_*+1} - t_{m_*} \le C_6 t_{m_*+1}(\log t_{m_*})^{1-1/\rho}$$

we get

$$\begin{aligned}
\frac{K_1^2}{4t_{n_*}(t_{m_*+1}-t_{m_*})b} &= \frac{t_{m_*}t_{n_*}\log\log t_{n_*}}{4t_{n_*}(t_{m_*+1}-t_{m_*})b(\log t_{m_*})^{4\delta}} \\
&\ge C_7(\log t_{m_*})(\log\log t_{n_*}) \\
&\ge C_8 m_*^{\rho}\log n_*.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E(\eta_{ij}'^2) &\le E(\eta_{ij}^2) = E(x_{ij}^2 I(|x_{ij}| > (ij)^{1/(2+\delta)})) \\
&\le (E|x_{ij}|^{2+\delta})^{2/(2+\delta)}(P(|x_{ij}| > (ij)^{1/(2+\delta)}))^{\delta/(2+\delta)} \\
&= (E|x_{ij}|^{2+\delta})^{2/(2+\delta)}(E|x_{ij}|^{2+\delta}/(ij))^{\delta/(2+\delta)} \\
&= E|x_{ij}|^{2+\delta}/(ij)^{\delta/(2+\delta)} \\
&\le \sup_{i,j} E|x_{ij}|^{2+\delta}/(ij)^{\delta/(2+\delta)} \\
&\le C(ij)^{-\delta/(2+\delta)}
\end{aligned}$$

and

$$\begin{aligned}\sum_{i=t_{m_*}+1}^{t_{m_*+1}} \sum_{j=1}^{t_{n_*+1}} (ij)^{-\delta/(2+\delta)} &\leq (\sum_{i=1}^{t_{m_*+1}} i^{-\delta/(2+\delta)})(\sum_{j=1}^{t_{n_*+1}} j^{-\delta/(2+\delta)}) \\ &\leq C_9(t_{m_*}t_{n_*})^{2/(2+\delta)}.\end{aligned}$$

Hence the right side of (3.46) is less than

$$\begin{aligned}&2n_*^{-C_8 m_*^\rho} + C_9 C(t_{m_*}t_{n_*})^{2/(2+\delta)}/(\frac{t_{m_*}t_{n_*}\log\log t_{n_*}}{(\log t_{m_*})^{4\delta}}) \\ &\quad\leq 2n_*^{-C_8 m_*^\rho} + C_{10}(t_{m_*}t_{n_*})^{-\delta/(2+\delta)}(\log t_{m_*})^{4\delta}/\log\log t_{n_*} \\ &\quad\leq 2n_*^{-C_8 m_*^\rho} + C_{11}(t_{m_*}t_{n_*})^{-\delta/4},\end{aligned}$$

and so

$$\begin{aligned}&P(\max_{t_{m_*}<m\leq t_{m_*+1}} |\sum_{t_{m_*}<i\leq m}\sum_{j\leq t_{n_*}} x_{ij}| \geq 2K_1) \\ &\quad\leq C_{12}(n_*^{-C_8 m_*^\rho} + \exp(-\delta(m_*^\rho + n_*^\rho)/4)).\end{aligned} \tag{3.47}$$

Similarly we can prove that if

$$K_2 := \frac{(t_{m_*}t_{n_*}\log\log t_{m_*})^{1/2}}{(\log t_{n_*})^{2\delta}}.$$

then

$$\begin{aligned}&P(\max_{t_{n_*}<n\leq t_{n_*+1}} |\sum_{i\leq t_{m_*}}\sum_{t_{n_*}\leq j\leq n} x_{ij}| \geq 2K_2) \\ &\quad\leq C_{12}(m_*^{-C_8 n_*^\rho} + \exp(-\delta(m_*^\rho + n_*^\rho)).\end{aligned} \tag{3.48}$$

Furthermore, if

$$K_3 := (t_{m_*}t_{n_*})^{1/2}(m_* n_*)^{-\rho/4},$$

then by Lemma 2.2.2, we have

$$\begin{aligned}&P(\max_{\substack{t_{m_*}<m\leq t_{m_*+1} \\ t_{n_*}<n\leq t_{n_*+1}}} |\sum_{i=t_{m_*}+1}^{m}\sum_{j=t_{n_*}+1}^{n} x_{ij}| \geq K_3) \\ &\quad\leq C_{13}E(\sum_{i=t_{m_*}+1}^{t_{m_*+1}}\sum_{t_{n_*}+1}^{t_{n_*+1}} x_{ij}^2)^{(2+\delta)/2}/(t_{m_*}t_{n_*})^{(2+\delta)/2}(m_* n_*)^{-\delta/8} \\ &\quad\leq C_{14}h_{m_*+1,n_*+1}^{(2+\delta)/2}/(t_{m_*}t_{n_*})^{(2+\delta)/2}(m_* n_*)^{-\delta/8} \\ &\quad\leq C_{15}(m_* n_*)^{-1-\delta/8}.\end{aligned} \tag{3.49}$$

Note that if we replace x_{ij} by y_{ij} in (3.47), (3.48) and (3.49), the inequalities remain true. Therefore,

$$\begin{aligned} &P(\max_{t_{m_*}<m\le t_{m_*+1}} |\sum_{t_{m_*}<i\le m}\sum_{j\le t_{n_*}}(x_{ij}-y_{ij})|\ge 4K_1) \\ &\le 2C_{12}(n_*^{-C_8 m_*^{\rho}}+\exp(-\delta(m_*^{\rho}+n_*^{\rho})/4)) \end{aligned} \tag{3.50}$$

$$\begin{aligned} &P(\max_{t_{n_*}<n\le t_{n_*+1}} |\sum_{i\le t_{m_*}}\sum_{t_{n_*}\le j\le n}(x_{ij}-y_{ij})|\ge 4K_2) \\ &\le 2C_{12}(m_*^{-C_8 n_*^{\rho}}+\exp(-\delta(m_*^{\rho}+n_*^{\rho})/4)) \end{aligned} \tag{3.51}$$

$$\begin{aligned} &P(\max_{\substack{t_{m_*}<m\le t_{m_*+1}\\ t_{n_*}<n\le t_{n_*+1}}} |\sum_{i=t_{m_*}+1}^{m}\sum_{j=t_{n_*}+1}^{n}(x_{ij}-y_{ij})|\ge 2K_3) \\ &\le 2C_{15}(m_*n_*)^{-1-\delta/8}. \end{aligned} \tag{3.52}$$

Let

$$\begin{aligned} A_{kl} &= \{\omega; |X_{kl}-Y_{kl}|>\alpha_{kl}\} \\ B_{m_*n_*} &= \{\omega; \max_{t_{m_*}<m\le t_{m_*+1}} |\sum_{t_{m_*}<i\le m}\sum_{j\le t_{n_*}}(x_{ij}-y_{ij})|\ge 4K_1\} \\ C_{m_*n_*} &= \{\omega; \max_{t_{n_*}<n\le t_{n_*+1}} |\sum_{i\le t_{n_*}}\sum_{t_{n_*}\le j\le n}(x_{ij}-y_{ij})|\ge 4K_2\} \\ D_{m_*,n_*} &= \{\omega; \max_{\substack{t_{m_*}<m\le t_{m_*+1}\\ t_{n_*}<n\le t_{n_*+1}}} |\sum_{i=t_{m_*}+1}^{m}\sum_{j=t_{n_*}+1}^{n}(x_{ij}-y_{ij})|\ge 2K_3\}. \end{aligned}$$

Since

$$\sum_{m_*,n_*}(m_*n_*)^{-1-\delta/8}<\infty$$

and there exists $n_0>0$ such that

$$\sum_{m_*,n_*>n_0}(n_*^{-C_8 m_*^{\rho}}+\exp(-\delta(m_*^{\rho}+n_*^{\rho})/4))<\infty,$$

$$\sum_{m_*,n_*>n_0}(m_*^{-C_8 n_*^{\rho}}+\exp(-\delta(m_*^{\rho}+n_*^{\rho})/4))<\infty$$

and (3.38), (3.40) hold, by the Borel-Cantelli Lemma, we have

$$P(A\bigcap B\bigcap C\bigcap D)=1$$

where

$$
\begin{aligned}
A &= \bigcup_{k,l} \bigcap_{(i,j)\ddot{>}(k,l)} A_{ij}^c \\
B &= \bigcup_{k,l} \bigcap_{(i,j)\ddot{>}(k,l)} B_{ij}^c \\
C &= \bigcup_{k,l} \bigcap_{(i,j)\ddot{>}(k,l)} C_{ij}^c \\
D &= \bigcup_{k,l} \bigcap_{(i,j)\ddot{>}(k,l)} D_{ij}^c
\end{aligned}
$$

and $\ddot{<}$ is defined with respect to $Q = \{(p_n, q_n) = (n, n)\}$.

Now we fix $\omega \in A \bigcap B \bigcap C \bigcap D$. Then there exists a positive integer $n_\omega > n_0$ such that for $(k,l)\ddot{>}(n_\omega, n_\omega)$ and $(m_*, n_*) > (n_\omega, n_\omega)$,

$$|X_{kl} - Y_{kl}| \le \alpha_{kl} \overset{(3.39)}{\le} C_4 \exp(-d_1(k^\rho + l^\rho)) \tag{3.53}$$

$$\max_{t_{m_*} < m \le t_{m_*+1}} \Big| \sum_{t_{m_*} < i \le m} \sum_{j \le t_{n_*}} (x_{ij} - y_{ij}) \Big| \le 4K_1 \tag{3.54}$$

$$\max_{t_{n_*} < n \le t_{n_*+1}} \Big| \sum_{i \le t_{n_*}} \sum_{t_{n_*} \le j \le n} (x_{ij} - y_{ij}) \Big| \le 4K_2 \tag{3.55}$$

$$\max_{\substack{t_{m_*} < m \le t_{m_*+1} \\ t_{n_*} < n \le t_{n_*+1}}} \Big| \sum_{i=t_{m_*}+1}^{m} \sum_{j=t_{n_*}+1}^{n} (x_{ij} - y_{ij}) \Big| \le 2K_3. \tag{3.56}$$

It is time for us to use the above inequalities (3.53)—(3.56) to estimate the error term. It is easy to see that

$$
\begin{aligned}
& \Big| \sum_{(i,j) \le (m,n)} (x_{ij} - y_{ij}) \Big| \\
\le & \Big| \sum_{(i,j) \le (t_{n_\omega}, t_{n_\omega})} (x_{ij} - y_{ij}) \Big| + \Big| \sum_{(t_{n_\omega}, t_{n_\omega}) \ddot{<} (i,j) \le (t_{m_*}, t_{n_*})} (x_{ij} - y_{ij}) \Big| \\
& + \Big| \sum_{t_{m_*} < i \le t_{m_*}} \sum_{j \le t_{n_*+1}} (x_{ij} - y_{ij}) \Big| + \Big| \sum_{i \le t_{m_*}} \sum_{t_{n_*} \le j \le t_{n_*}} (x_{ij} - y_{ij}) \Big| \\
& + \Big| \sum_{i=t_{m_*}+1}^{m} \sum_{j=t_{n_*}+1}^{n} (x_{ij} - y_{ij}) \Big| \\
= & \, Q_1 + Q_2 + Q_3 + Q_4 + Q_5
\end{aligned}
$$

in which

$$Q_1 = C_\omega,$$

$$
\begin{aligned}
Q_2 &\leq \sum_{(n_\omega,n_\omega)\not<(k,l)\leq(m_*,n_*)} h_{kl}^{-1/2}|X_{kl}-Y_{kl}| \\
&\leq C_4 \sum_{(k,l)\leq(m_*,n_*)} h_{kl}^{-1/2}\exp(-d_1(k^\rho+l^\rho\ )) \\
&= C_4 \sum_{(k,l)\leq(m_*,n_*)} \{(t_k-t_{k-1})(t_l-t_{l-1})\}^{1/2}\exp(-d_1(k^\rho+l^\rho\ )) \\
&\leq C_{16} \sum_{(k,l)\leq(m_*,n_*)} \exp((1/2-d_1)(k^\rho+l^\rho)) \\
&= C_{16}(\sum_{k=1}^{m_*}\exp((1/2-d_1)k^\rho))(\sum_{l=1}^{n_*}\exp((1/2-d_1)l^\rho)) \\
&\leq C_{16}m_*n_*\exp((1/2-d_1)(m_*^\rho+n_*^\rho)) \\
&\leq C_{17}(t_{m_*}t_{n_*})^{(1-d_1)/2} \\
&\leq C_{17}(mn)^{(1-d_1)/2}
\end{aligned}
$$

where $d_1=\delta^2/8(2+\delta)^2$, and

$$
\begin{aligned}
&Q_3+Q_4+Q_5 \\
&\leq 4K_1+4K_2+2K_3 \\
&= C_{18}(t_{m_*}t_{n_*})^{1/2}\left(\frac{(\log\log t_{m_*})^{1/2}}{(\log t_{n_*})^{2\delta}}+\frac{(\log\log t_{n_*})^{1/2}}{(\log t_{m_*})^{2\delta}}+((\log t_{m_*})(\log t_{n_*}))^{-1/4}\right) \\
&\leq C_{19}(mn)^{1/2}\left(\frac{(\log\log m)^{1/2}}{(\log n)^{2\delta}}+\frac{(\log\log n)^{1/2}}{(\log m)^{2\delta}}+((\log m)(\log n))^{-1/4}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|\sum_{(i,j)\leq(m,n)}(x_{ij}-y_{ij})| \\
&\leq C_\omega+C_{20}(mn)^{1/2}\left(\frac{(\log\log m)^{1/2}}{(\log n)^{2\delta}}+\frac{(\log\log n)^{1/2}}{(\log m)^{2\delta}}+((\log m)(\log n))^{-1/4}\right) \\
&= C_\omega+H_{mn}. \qquad (3.57)
\end{aligned}
$$

Finally by (3.57),

$$
\begin{aligned}
e_{mn} &= \max_{(k,l)\leq(m,n)}|\sum_{(i,j)\leq(k,l)}(x_{ij}-y_{ij})| \\
&\leq \max_{(k,l)\leq(t_{n_\omega},t_{n_\omega})}|\sum_{(i,j)\leq(k,l)}(x_{ij}-y_{ij})|
\end{aligned}
$$

$$\begin{aligned} & + \max_{(t_{n_\omega}, t_{n_\omega}) \ddot{<} (k,l) \leq (m,n)} \Big| \sum_{(i,j) \leq (k,l)} (x_{ij} - y_{ij}) \Big| \\ \leq \; & D_\omega + \max_{(t_{n_\omega}, t_{n_\omega}) \ddot{<} (k,l) \leq (m,n)} (C_\omega + H_{kl}) \\ \leq \; & D_\omega + C_\omega + H_{mn} \\ \leq \; & C_{21}(mn)^{1/2} \left(\frac{(\log \log m)^{1/2}}{(\log n)^{2\delta}} + \frac{(\log \log n)^{1/2}}{(\log m)^{2\delta}} + ((\log m)(\log n))^{-1/4} \right). \end{aligned}$$

By the facts that $P(A \bigcap B \bigcap C \bigcap D) = 1$ and that the above inequality holds for each fixed $\omega \in A \bigcap B \bigcap C \bigcap D$, we get (3.3).

The proof of Theorem 3.1.1 is complete. □

3.4 Comments on Applicability

For Theorem 3.1.1, the strongest condition is the following (see page 54):

$$\sup_{k,l} (V_{k,l,\delta} / h_{kl}) \leq D. \tag{3.58}$$

In this section, we wish to discuss further when this condition and the other conditions in Theorem 3.1.1 may be satisfied.

<u>Infinite Array of Exchangeable Random Variables</u>

The r.v.'s $x_1, x_2, \cdots, x_n$ are said to be exchangeable if $(x_1, x_2, \cdots, x_n)$ and $(x_{p(1)}, x_{p(2)}, \cdots, x_{p(n)})$ have the same distribution for any permutation p of $\{1, 2, \cdots, n\}$.

Let I be a set. The r.v.'s $x_\alpha, \alpha \in I$ are said to be exchangeable if for any finite subset I_1 of I, $x_\alpha, \alpha \in I_1$ are exchangeable.

For exchangeable sequences an ASIP has been given by Dabrowski (1989, [23]). Here we consider arrays of exchangeable r.v.'s.

Let $\{x_{ij}; i,j \in Z_+\}$ be an array of exchangeable r.v.'s and let

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(x_{ij}; i \geq n \text{ or } j \geq n)$$

Then by the representation theorem of DeFinetti (1937, [25]), $\{x_{ij}\}$ are conditionally independent given $\mathcal{T}$ and identically distributed. For these exchangeable r.v.'s, using Theorem 3.1.1 we can prove the following ASIP.

Corollary 3.4.1 *Let $\{x_{ij}; i,j \in Z_+\}$ be an array of exchangeable r.v.'s such that*

(a) $Ex_{11}^2 = 1$, $E|x_{11}|^{2+\delta} < \infty \quad (0 < \delta \leq 1)$

(b) $Ex_{11}x_{12} = 0$, $E(x_{11}^2 - 1)(x_{12}^2 - 1) = 0$.

Then, without changing its law, we can redefine $\{x_{ij}\}$ on a richer probability space on which there exists an array of i.i.d.r.v's $\{y_{ij}\}$ with $\mathcal{L}(y_{ij}) = N(0,1)$ such that (3.3) holds.

Proof. Define

$$\mathcal{F}_{kl} = \sigma(x_{ij}; i \leq k, j \leq l) \bigvee \mathcal{T}.$$

Then,

$$\mathcal{F}_{kl}^* = \sigma(x_{ij}; i \leq k) \bigvee \sigma(x_{ij}; j \leq l) \bigvee \mathcal{T}.$$

We claim that $\{x_{ij}, \mathcal{F}_{ij}; i,j \in Z_+\}$ is a SMDA. In fact we can easily prove the following.

(I) x_{ij} is $\mathcal{F}_{ij}$-measurable,

(II) $\mathcal{F}_{ij} \subset \mathcal{F}_{kl}$ if $(i,j) \leq (k,l)$,

(III) $E|x_{ij}| < \infty$,

(IV) $E(x_{kl}|\mathcal{F}_{k-1,l-1}^*) = 0$.

(I), (II) and (III) are clearly true by the definition of $\mathcal{F}_{ij}$ and the assumptions in Corollary 3.4.1. We prove that (IV) is also true. Since

$$0 = E(x_{11}x_{12}) = E\{E(x_{11}x_{12}|\mathcal{T})\} = E\{(E(x_{11}|\mathcal{T}))^2\}$$

we get

$$E(x_{11}|\mathcal{T}) = 0.$$

Hence

$$\begin{aligned} E(x_{kl}|\mathcal{F}^*_{k-1,l-1}) &= E(x_{kl}|\sigma(x_{ij}; i \leq k-1) \bigvee \sigma(x_{ij}; j \leq l-1) \bigvee \mathcal{T}) \\ &= E(x_{kl}|\mathcal{T}) = E(x_{11}|\mathcal{T}) = 0. \end{aligned}$$

Now let us check the other conditions of Theorem 3.1.1. We have

$$\begin{aligned} E(x^2_{kl}|\mathcal{F}^{(1)}_{k,l-1}) &= E\{E(x^2_{kl}|\mathcal{F}^*_{k-1,l-1})|\mathcal{F}^{(1)}_{k,l-1}\} \\ &= E\{E(x^2_{kl}|\mathcal{T})|\mathcal{F}^{(1)}_{k,l-1}\} \\ &= E\{E(x^2_{11}|\mathcal{T})|\mathcal{F}^{(1)}_{k,l-1}\}. \end{aligned}$$

Since

$$\begin{aligned} 0 &= E(x^2_{11}-1)(x^2_{12}-1) = E\{E((x^2_{11}-1)(x^2_{12}-1)|\mathcal{T})\} \\ &= E\{(E(x^2_{11}-1|\mathcal{T}))^2\} \end{aligned}$$

we obtain

$$1 = E(x^2_{11}|\mathcal{T}) = E(x^2_{kl}|\mathcal{F}^{(1)}_{k,l-1}) \quad \text{a.s.}$$

Similarly

$$E(x^2_{kl}|\mathcal{F}^{(2)}_{k-1,l}) = E(x^2_{11}|\mathcal{T}) = 1$$

and

$$q^2_{kl} = E(x^2_{kl}|\tilde{\mathcal{F}}^-_{kl}) = 1.$$

The latter implies that

$$V_{k,l,\delta} = 0.$$

Finally,

$$\begin{aligned} \sup_{i,j} E|x_{11}|^{2+\delta} &= \sup_{i,j} E\{E(|x_{ij}|^{2+\delta}|\mathcal{T}\} \\ &= \sup_{i,j} E\{E(|x_{11}|^{2+\delta}|\mathcal{T})\} \\ &= E|x_{11}|^{2+\delta} < \infty. \end{aligned}$$

All the conditions of Theorem 3.1.1 are satisfied and the proof is complete. □

Using Corollary 3.4.1 we can easily prove the following result.

Corollary 3.4.2 *Let* $\{x_{ij}; i,j \in Z_+\}$ *be an array of exchangeable r.v.'s such that*

$$v^2 = E(x_{11}^2|\mathcal{T}) > 0 \qquad a.s..$$

Suppose that $Ex_{11}x_{12} = 0$ *and* $E|\frac{x_{11}}{v}|^{2+\delta} < \infty$ *for some* δ, $0 < \delta \leq 1$. *Then without changing its law, we can redefine* $\{x_{ij}\}$ *on a richer probability space on which there exists an array of i.i.d.r.v.'s* $\{y_{ij}\}$ *with* $\mathcal{L}(y_{ij}) = N(0,1)$ *such that*

$$\begin{aligned} &\max_{k\leq m, l\leq n} |\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij} - vy_{ij})| \\ &\overset{a.s.}{=} O\left\{(mn)^{1/2}\left(\frac{\sqrt{\log\log m}}{(\log n)^{2\delta}} + \frac{\sqrt{\log\log n}}{(\log m)^{2\delta}} + ((\log m)(\log n))^{-1/4}\right)\right\} \end{aligned} \tag{3.59}$$

Proof. Let

$$x'_{ij} = x_{ij}/v,$$

then it is easy to see that $\{x'_{ij}; i,j \in Z_+\}$ is an array of exchangeable r.v.'s such that

$$\begin{aligned} Ex'^2_{11} &= E[E(x_{11}^2/v^2|\mathcal{T})] \\ &= E[E(x_{11}^2|\mathcal{T})/v^2] = 1 \quad \text{a.s.} \\ E|x'_{11}|^{2+\delta} &= E|x_{11}/v|^{2+\delta} < \infty \\ Ex'_{11}x'_{12} &= E[E(x_{11}x_{12}/v^2|\mathcal{T})] = E[E(x_{11}x_{12}|\mathcal{T})/v^2] \\ &= E\{[E(x_{11}|\mathcal{T})]^2/v^2\} = 0 \end{aligned}$$

and

$$\begin{aligned} E(x'^2_{11} - 1)(x'^2_{12} - 1) &= E[E((x'^2_{11} - 1)(x'^2_{12} - 1)|\mathcal{T})] \\ &= E[E((x_{11}^2 - v^2)(x_{12}^2 - v_2)|\mathcal{T})/v^4] = 0. \end{aligned}$$

Applying Corollary 3.4.1 to $\{x'_{ij}; i,j \in Z_+\}$ we obtain (3.3). Hence (3.59) holds. □

Note that to get a better estimate for the error term in Corollary 3.4.1, we can use the technique in Dabrowski [23].

Sufficient Moment Condition.

The "tough" condition in Theorem 3.1.1 is (3.58). In general, $\{Ex_{ij}^2 - q_{ij}^2; i,j \in Z_+\}$ is not a SMDA. So the results for SMDAs cannot be used to check (3.58). However if for some constant D and all k, l we have

$$EW_{kl}^2/h_{kl}^{1+\gamma} \leq D \tag{3.60}$$

where $\gamma = (2-\delta)/(2+\delta)$ and

$$W_{kl} = \sum_{(i,j)\in R_{kl}} (Ex_{ij} - q_{ij}^2),$$

then

$$\begin{aligned} E|W_{kl}|^{(2+\delta)/2}/h_{kl} &\leq (EW_{kl}^2)^{(2+\delta)/4}/h_{kl} \\ &\leq (Dh_{kl}^{1+\gamma})^{(2+\delta)/4}/h_{kl} \\ &= D^{(2+\delta)/4}h_{kl}/h_{kl} = D^{(2+\delta)/4}. \end{aligned}$$

(3.60) is trivally true if x_{ij}'s are independent and if (3.60) holds for some bounded SMDA, then all the conditions in Theorem 3.1.1 will be satisfied.

A Contact Infection Model

We consider the integer lattice Z_+^2. At each site (i,j) $(i,j \in Z_+)$, we imagine that there is a tree. Some disease can spread among these trees. Suppose that the tree on site (i,j) may get the disease only from the trees on sites $(i-1,j)$ and $(i,j-1)$ (see Cressie page 411 for a similar example). We define

$$z_{ij} = \begin{cases} 1 & \text{if the tree at } (i,j) \text{ is infected,} \\ 0 & \text{if not.} \end{cases}$$

Let

$$\mathcal{F}_{kl} = \sigma(z_{ij}; i \leq k, j \leq l)$$

Clearly

(1) we have (F4) for the filtration $\{\mathcal{F}_{kl}; k, l \in Z_+\}$, that is, $\mathcal{F}_k^1$ and $\mathcal{F}_l^2$ are conditionally independent given $\mathcal{F}_{kl}$.

(2) For any real-valued function $f(x)$ such that $E|f(z_{ij})| < \infty$, we have

$$E(f(z_{ij})|\mathcal{F}^*_{i-1,j-1}) = E(f(z_{ij})|z_{i-1,j}, z_{i,j-1})$$

If we let

$$x_{ij} = z_{ij} - E(z_{ij}|z_{i-1,j}, z_{i,j-1})$$

then the following lemma is true.

Lemma 3.4.1 $\{x_{ij}, \mathcal{F}_{ij}; i, j \in Z_+\}$ is a SMDA.

Proof. It suffices to prove that

$$E(x_{ij}|\mathcal{F}^*_{i-1,j-1}) = 0.$$

In fact,

$$\begin{aligned} E(x_{ij}|\mathcal{F}^*_{i-1,j-1}) &= E(z_{ij}|\mathcal{F}^*_{i-1,j-1}) - E\{E(z_{ij}|\mathcal{F}^*_{i-1,j-1})|z_{i-1,j}, z_{i,j-1}\} \\ &= E(z_{ij}|z_{i-1,j}, z_{i,j-1}) - E\{E(z_{ij}|z_{i-1,j}, z_{i,j-1})|z_{i-1,j}, z_{i,j-1}\} \\ &= 0. \quad \square \end{aligned}$$

Since $|x_{ij}| \leq 1$, if we can prove that (3.60) holds for this SMDA, then all conditions in Theorem 3.1.1 will be satisfied. Put

$$W_{kl} = \sum_{(i,j)\in R_{kl}} (Ex_{ij}^2 - q_{ij}^2) = \sum_{(i,j)\in R_{kl}} (Ex_{ij}^2 - E(x_{ij}^2|z_{i-1,j}, z_{i,j-1})).$$

(1) If (3.60) holds for $\{W_{kl}\}$, of course, (3.58) also holds.

(2) If $Ex_{ij}^2 - q_{ij}^2$, $i, j \in Z_+$ happen to be independent, then

$$EW_{kl}^2/h_{kl} = h_{kl}/h_{kl} = 1.$$

Therefore (3.58) holds.

(3) If we can show that for some $0 < \delta \leq 1$, $D > 0$ and all k, l

$$h_{kl}^{\delta/2} P(|W_{kl}| \geq h_{kl}^{2/(2+\delta)}) \leq D$$

then (3.58) holds. In fact from

$$|Ex_{ij}^2 - E(x_{ij}^2 | z_{i-1,j}, z_{i,j-1})| \leq 1$$

we have

$$\begin{aligned} V_{k,l,\delta} &= \int_{|W_{kl}| \leq h_{kl}^{2/(2+\delta)}} |W_{kl}|^{(2+\delta)/2} dP + \int_{|W_{kl}| > h_{kl}^{2/(2+\delta)}} |W_{kl}|^{(2+\delta)/2} dP \\ &\leq h_{kl} + h_{kl}^{(2+\delta)/2} P(|W_{kl}| \geq h_{kl}^{2/(2+\delta)}). \end{aligned}$$

Hence

$$V_{k,l,\delta}/h_{kl} \leq 1 + D.$$

Chapter 4

Appendix

4.1 Some Results for Martingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space: Ω is a set, $\mathcal{F}$ is a σ-field of subsets of Ω and P is a probability measure on $\mathcal{F}$. Let J be an ordered set having a complete order $\prec$.

Definition A1. Suppose that $\{\mathcal{F}_i; i \in J\}$ is a set of sub-σ-fields of $\mathcal{F}$ and $\{S_i; i \in J\}$ is a set of real-valued random variables on Ω satisfying

a) $\mathcal{F}_i \subset \mathcal{F}_j$ if $i \preceq j$,

b) for all $i \in J$, S_i is $\mathcal{F}_i$-measurable,

c) $E|S_i| < \infty$ for all $\in J$,

d) $E(S_j|\mathcal{F}_i) = S_i$ a.s. for all $i, j \in J$ with $i \prec j$,

then $\{S_i, \mathcal{F}_i; i \in J\}$ is called a martingale. If d) is replaced by d′) $E(S_j|\mathcal{F}_i) \geq S_i$ a.s. for all $i, j \in J$ with $i \prec j$, then $\{S_i, \mathcal{F}_i; i \in J\}$ is called a submartingale.

Definition A2. If $\{S_i, \mathcal{F}_i; i \in J\}$ is a martingale with $J = Z$ or Z_+ and $x_i = S_i - S_{i-1}$, then $\{x_i, \mathcal{F}_i; i \in J\}$ is called a MDS (martingale difference sequence).

Note that if $J = Z_+$ in Definition A2, then we define $S_{-1} = 0$.

Theorem A1 (Doob inequality, Theorem 2.1 in [53]). If $\{S_i, \mathcal{F}_i; 1 \le i \le n\}$ is a submartingale, then for each $\lambda > 0$,

$$\lambda P(\max_{i \le n} S_i > \lambda) \le E[S_n I(\max_{i \le n} S_i > \lambda)].$$

Theorem A2 (Theorem 3.4 in [33]). If $\{S_i, \mathcal{F}_i; 1 \le i \le n\}$ is a submartingale and if $S_i \ge 0$, then for $\alpha > 1$,

$$E(\max_{i \le n} S_i^\alpha) \le (\frac{\alpha}{\alpha - 1})^\alpha E(S_n^\alpha).$$

Theorem A3 (Burkholder's inequality, Theorem 2.10 in [53]). If $\{S_i, \mathcal{F}_i; 1 \le i \le n\}$ is a martingale and $1 < p < \infty$, then there exist constants C_1 and C_2 depending only on p such that

$$C_1 E|S_n|^p \le E|\sum_{i=1}^n x_i^2|^{p/2} \le C_2 E|S_n|^p$$

where $x_i = S_i - S_{i-1}$ if $i > 1$ and $x_1 = S_1$.

Theorem A4 (Theorem 2.5 and 2.6 in [53]). Let $\{S_i, \mathcal{F}_i; i \in Z\}$ be a martingale. We have:

a) There exists an a.s. finite random variable $S_{-\infty}$ such that $\lim_{n \to -\infty} S_n = S_{-\infty}$ a.s.

b) If $\sup_{n \ge 1} |S_n - S_0| < \infty$, then S_n converges a.s. to a random variable S_∞ such that $E|S_\infty| < \infty$.

Theorem A5 (Theorem 2.22 in [53]). Let $\{x_i, \mathcal{F}_i; i \ge 0\}$ be a MDS and $1 \le p < 2$. If $\{|x_i|^p; i \ge 1\}$ is uniformly integrable, then $n^{-1} E|S_n|^p \to 0$ as $n \to \infty$, where $S_n = \sum_{i=1}^n x_i$.

Theorem A6 (Theorem 3.2 in [53]). Let η^2 be an a.s. finite random variable and for each $n \in N$, let $\{x_{n,i}, \mathcal{F}_{n,i}; 1 \le i \le k_n\}$ be a MDS such that $E x_{n,i}^2 < \infty$. Suppose that

$$\max_i |x_{n,i}| \xrightarrow{P} 0,$$

$$\sum_i x_{n,i}^2 \xrightarrow{P} \eta^2,$$

$E(\max_i x_{n,i}^2)$ is bounded in n, and that the σ-fields are nested: $\mathcal{F}_{n,i} \subset \mathcal{F}_{n+1,i}$ for $1 \leq i \leq k_n$. Then

$$S_{n,k_n} = \sum_{i=1}^{k_n} x_{n,i} \xrightarrow{D} Z$$

where the random variable Z has characteristic function $E\exp(-\eta^2 t^2/2)$.

Theorem A7 (Corollary 3.1 in [53]). Let $\{S_{n,i}, \mathcal{F}_{n,i}; 1 \leq i \leq k_n\}$ and η^2 be as in Theorem A6. Suppose that for all $\varepsilon > 0$,

$$\sum_{i=1}^{k_n} E\{x_{n,i}^2 I(|x_{n,i}| > \varepsilon) | \mathcal{F}_{n,i-1}\} \xrightarrow{P} 0$$

and

$$\sum_i E(x_{n,i}^2 | \mathcal{F}_{n,i-1}) \xrightarrow{P} \eta^2.$$

If the σ-fields are nested, then the conclusion of Theorem A6 remains true.

Let $\{S_i, \mathcal{F}_i; i \in Z\}$ be a martingale and $x_i = S_i - S_{i-1}$. If $\sup_i E(S_i^2) < \infty$, then

$$E(\sum_i x_i^2) = \sum_i Ex_i^2 \leq \sup_i E(S_i^2) < \infty.$$

Therefore $\sum_i x_i^2 < \infty$ a.s., and by Theorem A4,

$$S_{-\infty} = \lim_{n \to -\infty} S_n, \qquad S_\infty = \lim_{n \to \infty} S_n$$

are a.s. finite random variables. So we see that the assumptions in the following theorem are reasonable.

Theorem A8 (Theorem 3.6 in [53]). Let η^2 be an a.s. finite random variable and for each $n \in N$, $\{S_{n,i}, \mathcal{F}_{n,i}; i \in Z\}$ be an integrable martingale such that $\sup_{n,i} E(S_{n,i}^2) < \infty$ and $S_{n,-\infty} = \lim_{i \to -\infty} S_{n,i} = 0$ a.s. If $x_{n,i} = S_{n,i} - S_{n,i-1}$,

$$\sup_i |x_{n,i}| \xrightarrow{P} 0,$$

$$\sum_i x_{n,i}^2 \xrightarrow{P} \eta^2,$$

$E(\sup_i x_{n,i}^2)$ is bounded in n, and for all n and i, $\mathcal{F}_{n,i} \subset \mathcal{F}_{n+1,i}$, then $S_{n,\infty} \xrightarrow{D} Z$, where Z has characteristic function $E\exp(\eta^2 t^2/2)$. If $P(\eta^2 > 0) = 1$, then

$$S_{n,\infty}/U_{n,\infty} \xrightarrow{D} N(0,1),$$

where $U^2_{n,\infty} = \sum_{i=-\infty}^{\infty} x^2_{n,i}$.

Theorem A9 (Theorem 1 in [52]). Let $\{x_i, \mathcal{F}_i; i \geq 1\}$ be a MDS with $E(x_i^2) < \infty$. Then for any $\delta > 0$, there exists a constant C_δ depending only on δ such that

$$\sup_{-\infty<t<\infty} |P(S_n \leq t) - \Phi(t)| \leq C_\delta(L_{n,2\delta} + N_{n,2\delta})^{1/(3+2\delta)}$$

where $S_n = \sum_{i=1}^n x_i$, $\Phi(t)$ denotes the standard normal distribution function and

$$L_{n,2\delta} = \sum_{i=1}^n E|x_i|^{2+2\delta}, \qquad N_{n,2\delta} = E(|\sum_{i=1}^n E(x_i^2|\mathcal{F}_{i-1}) - 1|^{1+\delta}).$$

4.2 Proof of Theorem 2.2.1

First let us restate Theorem 2.2.1:

"Let $\{x_i, \mathcal{F}_i; i \in Z_+\}$ be a MDS with $E|x_i|^3 < \infty$ and set

$$V_n = E|s_n^2 - \sum_{i=1}^n v_i^2|^{3/2} + \sum_{i=1}^n E|x_i|^3.$$

Then

$$\pi(\mathcal{L}(\sum_{i=1}^n x_i), N(0, s_n^2)) \leq 21 V_n^{1/6} \tag{4.1}$$

and if $V_n < 1/2^5$, then

$$\begin{aligned} &\pi(\mathcal{L}(\sum_{i=1}^n x_i), N(0, s_n^2)) \\ &\leq 6V_n^{2/9}(1 + \frac{\sqrt{2}}{3}|\log V_n|^{1/2}) + 7V_n^{1/4}(1 + \frac{1}{2}|\log V_n|^{1/2})." \end{aligned} \tag{4.2}$$

To prove Theorem 2.2.1, we need some lemmas. Let $f : R^1 \to R^1$ be a bounded function with first three derivatives continuous and bounded, and set $\|f\| = \sup_t |f(t)|$.

Lemma 4.2.1 *Let $\{x_i, \mathcal{F}_i; 0 \le i \le n\}$ be a MDS such that $E|x_i|^3 < \infty$ and $\sum_{i=1}^n v_i^2 = s_n^2$. Then*

$$|E\{f(\sum_{i=1}^n x_i) - f(N(0, s_n^2))\}| \;\; \le \;\; \|f^{(3)}\| \sum_{i=1}^n E|x_i|^3. \tag{4.3}$$

Proof. Let $Z_i, 1 \le i \le n$ be independent $N(0,1)$-random variables independent of $\mathcal{F}_n$. By Lemma 3.1.1 and the assumptions,

$$E(f(N(0, s_n^2)) \;\; = \;\; E(f(\sum_{i=1}^n v_i Z_i)). \tag{4.4}$$

Let

$$W_k = \sum_{i=1}^{k-1} x_i + \sum_{i=k+1}^{n} v_i Z_i, \qquad 1 \le k \le n.$$

Here we assign $\sum_{i=1}^0 x_i = \sum_{i=n+1}^n v_i Z_i = 0$. Then

$$f(\sum_{i=1}^n x_i) - f(\sum_{i=1}^n v_i Z_i) \;\; = \;\; \sum_{k=1}^n \{f(W_k + x_k) - f(W_k + v_k Z_k)\}. \tag{4.5}$$

By Taylor's formula, there exist ξ_k and ξ_k' such that

$$\begin{aligned} & f(W_k + x_k) - f(W_k + v_k Z_k) \\ & = f'(W_k)x_k + \frac{1}{2}f''(W_k)x_k^2 + \frac{1}{6}f^{(3)}(\xi_k)x_k^3 \\ & \quad - f'(W_k)v_k Z_k - \frac{1}{2}f''(W_k)v_k^2 Z_k^2 - \frac{1}{6}f^{(3)}(\xi_k')v_k Z_k^3. \end{aligned} \tag{4.6}$$

By the assumptions it is easy to see that

$$\begin{aligned} E(f'(W_k)x_k) &= E\{E(f'(W_k)x_k|\mathcal{F}_{k-1} \bigvee \sigma(Z_{k+1}, \cdots, Z_n))\} \\ &= E\{f'(W_k)E(x_k|\mathcal{F}_{k-1})\} = 0, \\ E(f''(W_k)x_k^2) &= E\{E(f''(W_k)x_k^2|\mathcal{F}_{k-1} \bigvee \sigma(Z_{k+1}, \cdots, Z_n))\} \\ &= E(f''(W_k)v_k^2), \\ E(f'(W_k)v_k Z_k) &= E\{f'(W_k)v_k E(Z_k)\} = 0, \end{aligned}$$

and

$$E(f''(W_k)v_k^2Z_k^2) \;=\; E\{f''(W_k)v_k^2\}E(Z_k^2) = E(f''(W_k)v_k^2).$$

By (4.4) and the above facts,

$$\begin{aligned}&|E\{f(W_k+x_k)-f(W_k+v_kZ_k)\}|\\ &\le \frac{1}{6}\|f^{(3)}\|(E|x_k|^3+E|x_k|^3E|Z_k|^3).\end{aligned} \tag{4.7}$$

Combining (4.4)-(4.7), we get (4.3). □

As we know, $\sum_{i=1}^n v_i^2 = s_n^2$ is not always true. We have to do something further. Let τ, Z and x_i', x_i'' $(i=1,2,\cdots,n+1)$ be defined as in Section 3.2 (see page 67). We have

Lemma 4.2.2 *If* $E|x_i|^3<\infty$, *then*

$$|E\{f(\sum_{i=1}^{n+1}x_i')-f(N(0,s_n^2))\}| \le \frac{3}{2}\|f^{(3)}\|V_n. \tag{4.8}$$

Proof. By Lemma 3.2.3,

$$\sum_{i=1}^{n+1}v_i'^2 = \sum_{i=1}^{n}v_i^2 I_{(i\le\tau)} + s_n^2 - \sum_{i=1}^{\tau}v_i^2 = s_n^2.$$

By Lemma 4.2.1,

$$E\{f(\sum_{i=1}^{n+1}x_i')-f(N(0,s_n^2))\}| \le \frac{1}{2}\|f^{(3)}\|\sum_{i=1}^{n+1}E|x_i'|^3.$$

Since for $i\le n$, we have $E|x_i'|^3 \le E|x_i|^3$ and

$$\begin{aligned}E|x_{n+1}'|^3 &= E|s_n^2-\sum_{i=1}^{\tau}v_i^2|^{3/2}E|Z|^3\\ &\le 2E|s_n^2-\sum_{i=1}^{\tau}v_i^2|^{3/2}\\ &\le 2E\{|s_n^2-\sum_{i=1}^{\tau}v_i^2|^{3/2}I_{(\tau=n)}\}+2E\{|s_n^2-\sum_{i=1}^{\tau}v_i^2|^{3/2}I_{(\tau<n)}\}\\ &\le 2E|s_n^2-\sum_{i=1}^{n}v_i^2|^{3/2}+2E|v_{\tau+1}|^3\\ &\le 2E|s_n^2-\sum_{i=1}^{n}v_i^2|^{3/2}+2\sum_{i=1}^{n}E|x_i|^3.\end{aligned} \tag{4.9}$$

Hence we get (4.8). □

Lemma 4.2.3 *If $E|x_i|^3 < \infty$, then*

$$|E\{f(\sum_{i=1}^{n} x_i) - f(\sum_{i=1}^{n+1} x_i')\}| \le \|f''\| V_n^{2/3}. \tag{4.10}$$

Proof. By Taylor's formula, there exists ξ such that

$$\begin{aligned} f(\sum_{i=1}^{n} x_i) &= f(\sum_{i=1}^{\tau} x_i) + f'(\sum_{i=1}^{\tau} x_i)(\sum_{i=\tau+1}^{n} x_i) \\ &\quad + \frac{1}{2} f''(\xi)(\sum_{i=\tau+1}^{n} x_i)^2. \end{aligned}$$

Since $\sum_{i=1}^{\tau} x_i' = \sum_{i=1}^{\tau} x_i$ and

$$\begin{aligned} &E\{f'(\sum_{i=1}^{\tau} x_i)(\sum_{i=\tau+1}^{n} x_i)\} \\ &= \sum_{k=1}^{n} E\{f'(\sum_{i=1}^{k} x_i) I_{(\tau=k)} E(\sum_{i=k+1}^{n} x_i|\mathcal{F}_k)\} = 0, \end{aligned}$$

we obtain

$$|E\{f(\sum_{i=1}^{n} x_i) - f(\sum_{i=1}^{n} x_i')\}| \le \frac{1}{2}\|f''\| E(\sum_{i=\tau+1}^{n} x_i)^2.$$

By Lemma 3.2.3,

$$\begin{aligned} E(\textstyle\sum_{i=\tau+1}^{n} x_i)^2 &= E(\sum_{i=1}^{n} x_i'')^2 = E X_i''^2 \\ &= E(\sum_{i=1}^{n} v_i''^2) = E(\sum_{i=\tau+1}^{n} v_i^2) \\ &\le E|s_n^2 - \sum_{i=1}^{n} v_i^2| + E|s_n^2 - \sum_{i=1}^{\tau} v_i^2|. \end{aligned}$$

Clearly

$$E|s_n^2 - \sum_{i=1}^{n} v_i^2| \le (E|s_n^2 - \sum_{i=1}^{n} v_i^2|^{3/2})^{2/3},$$

and by (4.9),

$$E|s_n^2 - \sum_{i=1}^{\tau} v_i^2| \le (E|s_n^2 - \sum_{i=1}^{\tau} v_i^2|^{3/2})^{2/3} \le V_n^{2/3}. \tag{4.11}$$

So

$$|E\{f(\sum_{i=1}^{n} x_i) - f(\sum_{i=1}^{n} x_i')\}| \le \|f''\| V_n^{3/2}. \tag{4.12}$$

By Taylor's formula again, there exists ξ' such that

$$f(\sum_{i=1}^{n+1} x_i') = f(\sum_{i=1}^{n} x_i') + f'(\sum_{i=1}^{n} x_i')x_{n+1}' + \frac{1}{2}f''(\xi')x_{n+1}'^2.$$

Since

$$E(f'(\sum_{i=1}^{n} x_i')x_{n+1}') = E\{f'(\sum_{i=1}^{n} x_i')(s_n^2 - \sum_{i=1}^{\tau} v_i^2)^{1/2}E(Z|\mathcal{F}_n)\} = 0,$$

by (4.11) we have

$$\begin{aligned} &|E\{f(\sum_{i=1}^{n} x_i) - f(\sum_{i=1}^{n+1} x_i')\}| \\ &\leq \frac{1}{2}E|s_n^2 - \sum_{i=1}^{\tau} v_i^2|EZ^2 \\ &\leq \frac{1}{2}\|f''\|V_n^{2/3}. \end{aligned}$$

Together with (4.12) this gives (4.10). □

Given a metric space (A, d), the space of bounded Lipschitz functions $BL(A, d)$ is the set of real functions on A such that

$$\begin{aligned} \|f\|_\infty &= \sup_{t \in A} |f(t)| < \infty \\ \|f\|_L &= \sup_{\substack{s,t \in A \\ s \neq t}} \frac{|f(t) - f(s)|}{d(s,t)} < \infty. \end{aligned}$$

We have the following lemma.

Lemma 4.2.4 *Let $B \subset A$. For every $f \in BL(B, d)$, there exists $g \in BL(A, d)$ such that $g(t) = f(t)$ for $t \in B$, $\|f\|_\infty = \|g\|_\infty$ and $\|f\|_L = \|g\|_L$.*

This lemma is Corollary 1.4 in [3]. Sometimes we write $BL(A)$ for $BL(A, d)$ if d is already clear.

Lemma 4.2.5 *If $K \subset R^1$ is a closed subset and $\varepsilon > 0$, then there exists a function $g_\varepsilon(t)$ on R^1 such that $0 \leq g_\varepsilon(t) \leq 1$, $g_\varepsilon(t) = 1$ if $t \in K$, $g_\varepsilon(t) = 0$ if $t \in K_\varepsilon^c$ and for all*

$s, t \in R^1$,

$$|g_\varepsilon(t) - g_\varepsilon(s)| \le \frac{1}{\epsilon}|t - s| \tag{4.13}$$

where

$$K_\varepsilon = \{t; \inf_{s \in K} |s - t| < \varepsilon\}$$

and K_ε^c denotes the complement of K_ε.

Proof. On $K \bigcup K_\varepsilon^c$, we define

$$v(t) = \begin{cases} 1 & \text{if } t \in K \\ 0 & \text{if } t \in K_\varepsilon^c. \end{cases}$$

Then $\|v\|_\infty = 1$ and

$$\|v\|_L = \sup\{\frac{|v(t) - v(s)|}{|t - s|}\} = \frac{1}{\varepsilon}.$$

Hence $v(t) \in BL(K \bigcup K_\varepsilon^c)$. By Lemma 4.2.4, there exists $g_\varepsilon(t) \in BL(R^1)$, extending $v(t)$, such that $\|g_\varepsilon\|_\infty = 1$ and $\|g_\varepsilon\|_L = 1/\varepsilon$. Evidently $g_\varepsilon(t) = v(t) = 1$ if $t \in K$, $g_\varepsilon(t) = v(t) = 0$ if $t \in K_\varepsilon^c$ and for all $s, t \in R^1$, (4.13) holds. Now we only need to show that $0 \le g_\varepsilon(t) \le 1$. It is true for $t \in K \bigcup K_\varepsilon^c$. Suppose that there exists $t_0 \in R^1 - K \bigcup K_\varepsilon^c$ such that $g_\varepsilon(t_0) < 0$. Then we can find $s_0 \in K$ such that $|s_0 - t_0| < \varepsilon$. Therefore

$$\|g_\varepsilon\|_L \ge \frac{|g_\varepsilon(t_0) - g_\varepsilon(s_0)|}{|t_0 - s_0|} > \frac{1 + |g_\varepsilon(s_0)|}{\varepsilon} > \frac{1}{\varepsilon},$$

which contradicts the fact that $\|g_\varepsilon\| = 1/\varepsilon$. □

For $g_\varepsilon(t)$, we define

$$g_{\varepsilon,\rho}(t) = \rho^{-3} \int_0^\rho \int_0^\rho \int_0^\rho g_\varepsilon(t + u + v + w) du\ dv\ dw.$$

We have

Lemma 4.2.6 *$g_{\varepsilon,\rho}(t)$ has the following properties.*

a) $\|g_{\varepsilon,\rho}(\cdot) - g_\varepsilon(\cdot)\|_\infty \le 3\rho/\varepsilon$,

b) $\|g''_{\varepsilon,\rho}\|_\infty \le 2/(\rho\varepsilon)$, *and*

c) $\|g^{(3)}_{\varepsilon,\rho}\|_\infty \le 4/(\rho^2\varepsilon)$.

Proof. Part a) is simple. Part b) and c) follow from (4.13) and the following equations.

$$g''_{\varepsilon,\rho}(t) = \rho^{-3}\int_0^{\rho}(g_\varepsilon(t+2\rho+w) - 2g_\varepsilon(t+\rho+w) + g_\varepsilon(t+w))dw$$

$$g^{(3)}_{\varepsilon,\rho}(t) = \rho^{-3}(g_\varepsilon(t+3\rho) - 3g_\varepsilon(t+2\rho) + 3g_\varepsilon(t+\rho) - g_\varepsilon(t)). \quad \Box$$

Lemma 4.2.7 *(Lemma 1 in [103]). Let* $0 < \varepsilon < 1/2$, $\varepsilon_1 > 0$ *and* $K \subset R^1$. *If* $\phi(t)$ *is a continuous function defined on* R^1 *such that* $0 \le \phi(t) \le 1$, $\phi(t) \ge 1 - \varepsilon$ *if* $t \in K$ *and* $\phi(t) \le \varepsilon$ *if* $t \in R^1 - K_{\varepsilon_1}$, *then for probability measures* F *and* G,

$$F(K) \le G(K_{\varepsilon_1}) + 4\varepsilon + \rho,$$

where $\rho = |\int \phi(t)F\,(dt) - \int \phi(t)G\,(dt)|$.

Lemma 4.2.8 *(Lemma 4 in [103] and Lemma 2.3 in [28]). Let* $K \subset R^1$ *be a closed set. Then for* $0 < \varepsilon < 1/2$ *there exists a three times continuously differentiable function* $\phi_\varepsilon(t)$ *on* R^1 *such that* $0 \le \phi_\varepsilon(t) \le 1$ *with* $\phi_\varepsilon(t) \ge 1 - 2\varepsilon$ *if* $t \in K$ *and* $\phi_\varepsilon(t) \le 2\varepsilon$ *if* $t \in R^1 - K_{\varepsilon_1}$, *where* $\varepsilon_1 = 4\varepsilon(|\log\varepsilon|^{1/2} + 1)$ *and*

$$|\phi'_\varepsilon(t)| \le \varepsilon^{-1}, \quad |\phi''_\varepsilon(t)| \le \varepsilon^{-2}, \quad |\phi^{(3)}_\varepsilon(t)| \le 2\varepsilon^{-3}.$$

Proof of Theorem 2.2.1 Let

$$F = \mathcal{L}(\sum_{i=1}^{n} x_i), \quad G = \mathcal{L}(\sum_{i=1}^{n+1} x'_i), \quad N = N(0, s_n^2),$$

where x'_i is as in Lemma 3.2.3. We have

$$\pi(F,N) \le \pi(F,G) + \pi(G,N). \tag{4.14}$$

For a closed set $K \subset R^1$ and $\varepsilon > 0$, let $g_\varepsilon(t)$ and $g_{\varepsilon,\rho}(t)$ be as in Lemma 4.2.6 and Lemma 4.2.7. Then for probability measures F' and G', by Lemma 4.2.6 and Lemma 4.2.7,

$$F'(K) \;\le\; \int g_\varepsilon(t)F'(dt)$$

$$\begin{aligned}
&\le \int g_\varepsilon(t)G'(dt) + |\int g_\varepsilon(t)F'(dt) - \int g_\varepsilon(t)G'(dt)| \\
&\le G'(K_\varepsilon) + |\int g_{\varepsilon,\rho}(t)F'(dt) - \int g_{\varepsilon,\rho}(t)G'(dt)| \\
&\quad +|\int (g_\varepsilon(t) - g_{\varepsilon,\rho}(t))F'(dt)| + |\int (g_\varepsilon(t) - g_{\varepsilon,\rho}(t))G'(dt)| \\
&\le G'(K_\varepsilon) + |\int g_{\varepsilon,\rho}(t)F'(dt) - \int g_{\varepsilon,\rho}(t)G'(dt)| + \frac{6\rho}{\varepsilon}. \qquad (4.15)
\end{aligned}$$

By Lemma 4.2.2 and Lemma 4.2.6,

$$|E\{g_{\varepsilon,\rho}(\sum_{i=1}^{n+1} x_i') - g_{\varepsilon,\rho}(N(0,s_n^2))\}| \le \frac{6}{\rho^2\varepsilon}V_n.$$

By (4.15),

$$G(K) \le N(K_\varepsilon) + \frac{6\rho}{\epsilon} + \frac{6}{\rho^2\varepsilon}V_n.$$

Let $\rho = \varepsilon^2$ and $\varepsilon = V_n^{1/6}$, we get

$$\pi(G,N) \le 12V_n^{1/6}. \qquad (4.16)$$

On the other hand, using Lemma 4.2.3 and Lemma 4.2.6,

$$|E\{g_{\varepsilon,\rho}(\sum_{i=1}^{n} x_i) - g_{\varepsilon,\rho}(\sum_{i=1}^{n+1} x_i')\}| \le \frac{3}{\rho\varepsilon}V_n^{2/3}.$$

By (4.15) again,

$$F(K) \le G(K_\varepsilon) + \frac{6\rho}{\epsilon} + \frac{3}{\rho^2\varepsilon}V_n^{2/3}.$$

If $\rho = \varepsilon^2$ and $\varepsilon = V_n^{1/6}$, then

$$\pi(F,G) \le 9V_n^{1/6}. \qquad (4.17)$$

Evidently (4.1) follows from (4.14), (4.16) and (4.17).

Now let us prove (4.2). For closed set $K \subset R^1$ and $0 < \varepsilon < 1/2$, let $\phi(t)$ be as in Lemma 4.2.8. Then by Lemma 4.2.2 and Lemma 4.2.8,

$$|E\{\phi_\varepsilon(\sum_{i=1}^{n+1} x_i') - \phi(N(0,s_n^2))\}| \le 3\varepsilon^{-3}V_n.$$

By Lemma 4.2.7,

$$G(K) \leq N(K_{\epsilon_1}) + 4\varepsilon + 3\varepsilon^{-1}V_n,$$

where $\epsilon_1 = 4\epsilon(1 + |\log \varepsilon|^{1/2})$. If $V_n < 1/2^5$, then $v^{1/4} < 1/2$. Letting $\varepsilon = V_n^{1/4}$, we have

$$G(K) \leq N(K_{\epsilon_1}) + 7V_n^{1/4}.$$

Therefore

$$\pi(G, N) \leq 7V_n^{1/4}(1 + \frac{1}{2}|\log V_n|^{1/2}). \tag{4.18}$$

Similarly, using Lemma 4.2.3 and Lemma 4.2.8,

$$|E\{\phi_\varepsilon(\sum_{i=1}^{n} x_i) - \phi_\varepsilon(\sum_{i=1}^{n+1} x_i^{'})\}| \leq \frac{3}{2}\varepsilon^{-2}V_n^{2/3},$$

and by Lemma 4.2.7 again,

$$F(K) \leq G(K_{\epsilon_1}) + 4\varepsilon + \frac{3}{2}\varepsilon^{-1}V_n^{2/3}.$$

If $V_n < 1/2^5$, then $V_n^{2/9} < 1/2$. Letting $\varepsilon = V_n^{2/9}$, we get

$$\pi(F, G) \leq \frac{11}{2}V_n^{2/9}(1 + \frac{\sqrt{2}}{3}|\log V_n|^{1/2}). \tag{4.19}$$

Combining (4.14), (4.18) and (4.19), we get (4.2). □

4.3 Proof of Lemma 3.2.10

Before we do the proof, let us recall Lemma 3.2.10.

"Let $R_i = R$ be the real line with the usual distance, and let $\mathcal{B}_i = \mathcal{B}$ be the σ-field of Borel sets of R. If for $u = (u_1, u_2, \cdots), v = (v_1, v_2, \cdots) \in \times_{i=1}^{\infty} R_i$, we define

$$d(u, v) = \sum_{k=1}^{\infty} 2^{-k} \frac{|u_k - v_k|}{1 + |u_k - v_k|},$$

then $(\times_{i=1}^{\infty} R_i, d)$ is a complete and separable metric space. If $\mathcal{B}_\infty$ is the σ-field of Borel sets of $(\times_{i=1}^{\infty} R_i, d)$, then

$$\mathcal{B}_\infty = \times_{i=1}^{\infty} \mathcal{B}_i. \tag{4.20}$$

Finally if $D_k = \times_{i=1}^{\infty} R_i$, $k = 1, 2, \cdots, n$ and for $t = (t_1, t_2, \cdots, t_n), t' = (t'_1, t'_2, \cdots, t'_n) \in \times_{k=1}^{n}$, we define

$$\bar{d}(t, t') = \max_{1 \le k \le n} d(t_k, t'_k),$$

then $(\times_{k=1}^{n} D_k, \bar{d})$ is a complete and separable metric space. If $\mathcal{D}$ denotes the Borel σ-field of $(\times_{k=1}^{n} D_k, \bar{d})$, then

$$\mathcal{D} = \underbrace{\mathcal{B}_\infty \times \mathcal{B}_\infty \times \cdots \times \mathcal{B}_\infty}_{n}."$$

Proof of Lemma 3.2.10. Let $\mathcal{T}$ be the family of open sets of R and $\mathcal{T}_i = \mathcal{T}$. Let $\mathcal{T}_d$ be the topology of $(\times_{i=1}^{\infty} R_i, d)$. Then we have the following well-known topological result.

$$\mathcal{T}_d = \times_{i=1}^{\infty} \mathcal{T}_i \tag{4.21}$$

By the definition of the product topology, $\times_{i=1}^{\infty} \mathcal{T}_i$ has a basis

$$\mathcal{H} = \{\times_{i=1}^{\infty} W_i; W_i \in \mathcal{F}_i \text{ and } W_i = R \text{ for all but at most finitely many } i\}.$$

Let $\times_{i=1}^{\infty} W_i \in \mathcal{H}$ and $t \in \times_{i=1}^{\infty} W_i$. Suppose that there exists an n such that $W_{n+1} = W_{n+2} = \cdots = R$. If $t = (t_1, t_2, \cdots)$ then there exists $\delta_i > 0$ such that

$$t_i \in I_i = (t_i - \delta_i, t_i + \delta_i) \subset W_i \quad i = 1, 2, \cdots, n.$$

Let

$$p_n = \min\{2^{-i} \frac{\delta_i}{1 + \delta_i}; i = 1, 2, \cdots, n\}$$

and

$$N(t, p_n) = \{t' \in \times_{i=1}^{\infty} R_i; d(t', t) < p_n\}.$$

We prove that $N(t,p_n) \subset \times_{i=1}^{\infty} J_i$, where $J_i = I_i$ for $i = 1,2,\cdots,n$, $J_i = R$ for $i > n$. If $t' \in N(t,p_n)$, then

$$d(t',t) = \sum_{i=1}^{\infty} 2^{-i} \frac{|t'_i - t_i|}{1 + |t'_i - t_i|} < p_n$$

which implies

$$2^{-i} \frac{|t'_i - t_i|}{1 + |t'_i - t_i|} < p_n \le 2^{-i} \frac{\delta_i}{1+\delta_i}.$$

Since $x/(1+x)$ is an increasing function for $x \ge 0$, we see that

$$|t'_i - t_i| < \delta_i \qquad i = 1,2,\cdots,n$$

and hence that $t' \in \times_{i=1}^{\infty} J_i \subset \times_{i=1}^{\infty} W_i$. So we conclude that

$$\times_{i=1}^{\infty} \mathcal{T}_i \subset \mathcal{T}_d. \tag{4.22}$$

On the other hand, let $N(t,\delta)$ be any neighborhood in $\mathcal{T}_d$. Let n be the smallest number such that

$$\sum_{i=n+1}^{\infty} 2^{-i} < \delta/2.$$

Let $\delta_i > 0$ $(i = 1,2,\cdots,n)$ be any sequence of real numbers such that

$$\delta_1 + \delta_2 + \cdots + \delta_n < \delta/2.$$

Let I_i and J_i be as before. We claim that

$$\times_{i=1}^{\infty} J_i \subset N(t,\delta).$$

In fact, for $t' \in \times_{i=1}^{\infty} J_i$, we have

$$\begin{aligned} d(t',t) &= \sum_{i=1}^{\infty} 2^{-i} \frac{|t'_i - t_i|}{1 + |t'_i - t_i|} \\ &\le \sum_{i=1}^{n} 2^{-i} \frac{\delta_i}{1+\delta_i} + \frac{\delta}{2} \\ &< \sum_{i=1}^{n} \delta_i + \frac{\delta}{2} < \delta. \end{aligned}$$

Therefore we get

$$\mathcal{T}_d \subset \times_{i=1}^{\infty} \mathcal{T}_i$$

which together with (4.22) gives (4.21).

Now let us prove that

$$\mathcal{B}_\infty \subset \times_{i=1}^{\infty}\mathcal{B}_i. \tag{4.23}$$

We know that $\mathcal{B}_\infty = \sigma(\mathcal{T}_d) = \sigma(\times_{i=1}^{\infty}\mathcal{T}_d)$, so it is enough to show that

$$\times_{i=1}^{\infty}\mathcal{T}_i \subset \times_{i=1}^{\infty}\mathcal{B}_i. \tag{4.24}$$

Let

$$\mathcal{T}^0 = \{(a,b); a, b \text{ are rational numbers or } \infty\}.$$

If $A \in \times_{i=1}^{\infty}\mathcal{T}_i$, then there exists a set of indices Λ such that

$$A = \bigcup_{\alpha\in\Lambda} A_\alpha, \qquad A_\alpha \in \mathcal{H}.$$

Let

$$\Lambda_i = \{\alpha \in \Lambda; A_\alpha = W_1 \times W_2 \times \cdots \times W_i \times R \times R \times \cdots, W_i \in \mathcal{T}_j\}.$$

Since each W_j is the union of countable sets in $\mathcal{T}^0$, we get for $\alpha \in \Lambda_i$,

$$A_\alpha = \bigcup G_\beta^{(i,\alpha)},$$

where $G_\beta^{(i,\alpha)}$ has the form $I_1 \times I_2 \times \cdots \times I_i \times R \times R \times \cdots$ with $I_i \in \mathcal{T}^0$. Therefore,

$$\bigcup_{\alpha\in\Lambda_i} A_\alpha = \bigcup G_{r_i}^{(i)},$$

where $G_{r_i}^{(i)}$ has the form $I_1 \times I_2 \times \cdots \times I_i \times R \times R \times \cdots$. Finally

$$A = \bigcup_{i=1}^{\infty}\Big(\bigcup_{\alpha\in\Lambda_i} A_\alpha\Big) = \bigcup_{i=1}^{\infty}\Big(\bigcup G_{r_i}^{(i)}\Big).$$

It is easy to see that A is a countable union of sets which have the form $I_1 \times I_2 \times \cdots \times I_i \times R \times R \times \cdots, \quad i = 1, 2, \cdots$. Now we conclude that $A \in \times_{i=1}^{\infty}\mathcal{B}_i$. So (4.24) is true and therefore (4.23) holds.

In order to prove

$$\mathcal{B}_\infty \supset \times_{i=1}^{\infty} \mathcal{B}_i \tag{4.25}$$

we need to prove that

$$\begin{aligned}&\sigma(R \times \cdots \times R \times \mathcal{T}_i \times R \times \cdots)\\ &\quad = R \times \cdots \times R \times \mathcal{B}_i \times R \times \cdots .\end{aligned} \tag{4.26}$$

Obviously

$$R \times \cdots \times R \times \mathcal{T}_i \times R \times \cdots \subset R \times \cdots \times R \times \mathcal{B}_i \times R \times \cdots$$

and the latter is a σ-field. So

$$\begin{aligned}&\sigma(R \times \cdots \times R \times \mathcal{T}_i \times R \times \cdots)\\ &\quad \subset R \times \cdots \times R \times \mathcal{B}_i \times R \times \cdots\end{aligned} \tag{4.27}$$

Let

$$\mathcal{F} = \{B \in \mathcal{B}_i; R \times \cdots \times R \times B \times R \times \cdots \in \sigma(R \times \cdots \times R \times \mathcal{T}_i \times R \times \cdots)\}.$$

Clearly $\mathcal{F} \supset \mathcal{T}_i$ and $\mathcal{F}$ is a σ-field. We have

$$\mathcal{B}_i = \sigma(\mathcal{T}_i) \subset \mathcal{F}$$

which implies

$$\begin{aligned}&R \times \cdots \times R \times \mathcal{B}_i \times R \times \cdots\\ &\quad \subset \sigma(R \times \cdots \times R \times \mathcal{T}_i \times R \times \cdots).\end{aligned} \tag{4.28}$$

By (4.27) and (4.28) we get (4.26). Since

$$\begin{aligned}\mathcal{B}_\infty &= \sigma(\times_{i=1}^{\infty}) \supset \sigma(R \times \cdots \times R \times \mathcal{T}_i \times R \times \cdots)\\ &= R \times \cdots \times R \times \mathcal{B}_i \times R \times \cdots ,\end{aligned}$$

we have

$$\mathcal{B}_\infty \supset \bigcup_{i=1}^{\infty}(R \times \cdots \times R \times \mathcal{B}_i \times R \times \cdots),$$

and hence

$$\mathcal{B}_\infty \supset \sigma(\bigcup_{i=1}^{\infty}(R \times \cdots \times R \times \mathcal{B}_i \times R \times \cdots)) = \times_{i=1}^{\infty}\mathcal{B}_i$$

i.e. (4.25) holds. From (4.23) and (4.25), (4.21) follows.

The conclusion that $(\times_{i=1}^{\infty}R_i, d)$ is a complete and separable metric space is obvious because R is complete and the countable set

$$\Gamma = \{(r_1, r_2, \cdots, r_n, 0, 0, \cdots); r_i\text{'s are rational}, n = 1, 2, \cdots\}$$

is dense in $(\times_{i=1}^{\infty}R_i, d)$.

From the proof of (4.24), we can see that $\mathcal{T}_d$ has a countable basis. With this fact at hand, the rest of the proof of Lemma 3.2.10 is simple. □

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Transition Theorems and Almost Sure Invariance Principles for Strong Martingales

(A thesis summary)

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The study of invariance principles is an important field in probability theory. The almost sure invariance principle (ASIP) or strong invariance principle is a special type of invariance principle. Many results for the ASIP have been established. For example, ASIP's have been proven for the sums of one-parameter random variables, including one-parameter martingales, as well as ASIP's for the rectangular sums of independent identically distributed multi-parameter random variables (see Chapter 1 of the thesis for the details). In the thesis we consider strong martingales indexed by two parameters (or two-parameter strong martinales) and establish ASIP's for the strong martingales. We use the properties of one-parameter martingales and apply the method developed by Berkes and Philipp (Ann. Prob., 7, 1979) to prove ASIP's for two-parameter strong martingales. The bridges

between two-parameter strong martingales and one-parameter martingales are the so-called "transition theorems" which turn two-parameter strong martingales into one-parameter martingales by giving the index set a suitable total order (see Chapter 2 of the thesis for the details).

This thesis is organized in four chapters:

In Chapter 1, we give a review of invariance principles. We introduce the origin of the concept of the invariance principle, describe the main methods for proving the ASIP's and state some basic results of the almost sure invariance principle.

In Chapter 2, we prove the transition theorems and give several applications, such as maximal inequalities with exponential bounds for two-parameter strong martingales and the Prohorov distance between the law of a two-parameter strong martingale and some appropriate normal law.

In Chapter 3, we prove our main theorem—the almost sure invariance principle for two-parameter strong martingales and show some applications. The most important application is the functional law of the iterated logarithm for two-parameter strong martingales.

In Chapter 4—the appendix, we state some known results we want to use and give the proofs of Theorem 2.2.1 in Chapter 2 and Lemma 3.2.10 in Chapter 3.