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OPTIMAL CONTROL OF INFINITE DIMENSIONAL STOCHASTIC SYSTEMS

by

ZHU, Qingxin

A Ph.D. Thesis

submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements for
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University of Ottawa
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To my parents ——

Muyong Zhu and Huiyun Wang

as a little gift for your 80th birthday!

ABSTRACT

In this thesis we study Hamilton-Jacobi-Bellman equation arising from stochastic optimal control problem. More precisely, we study the following second order parabolic partial differential equation

$$(P)\quad \begin{cases} \phi_t(t,x) = & \frac{1}{2}Tr(S\phi_{xx}(t,x)) + (Bx + f(x), \phi_x(t,x)) \\ & + F(t,x,\phi(t,x),\phi_x(t,x)) \\ \phi(0,x) = & \phi_0(x). \end{cases}$$

Where ϕ_0, F are given functions, B is the infinitesimal generator of a strongly continuous semigroup, and S is a positive, self-adjoint nuclear operator in a Banach space X (Chapter 3) or an identity operator in $\mathcal{L}(X^*, X)$ (Chapter 4).

The main contributions of this thesis include:

(1) A direct method suggested recently by Da Prato [26, 28] has been further developed. This method, which is different from ordinary perturbation theorem in semigoup theory, is more constructive, and can be applied to general non-linear parabolic partial differential equations.

(2) Many authors have studied semilinear HJB equations (P), with non-linear term of the form

$$F(x,\phi_x) = \frac{1}{2}|\phi_x|^2 + g(x).$$

Da Prato [31] has also studied a HJB equation with nonlinear term $F(S\phi_x)$. In this thesis we will study the more general non-linear term $F(t, x, \phi, \phi_x)$, and without coercivity or convexity assumptions.

(3) Most of the results about HJB equations obtained before were in R^n, or in a separable Hilbert space. In this thesis we study HJB equations in a separable reflexive Banach space, hence our results are more general.

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Contents

CHAPTER 1

Introduction

1.1 Orientation

Calculus of variation has a long history, starting almost at the same time as the invention of calculus. With the calculus of variations one seeks a curve making a certain function J of the curve a minimum (or a maximum). An easy example is the length of the curve, where the minimum is obtained by the straight line segment joining the given end points.

The original motivation for the calculus of variation came from classical mechanics, optics, and geometry. Since 1950 many new applications have been found, some of them were to problems in the aerospace sciences, industrial process control, economics, biology, and management. To deal with these new applications the theory of calculus of variations has been extended, certain variables called controls were introduced into the system, and a more general theory, which now is called the theory of optimal control, was developed.

Stochastic control theory is a relatively young branch of mathematics. The beginning of its intensive development falls in the late 1950s and early 1960s. During that period an extensive literature appeared on optimal stochastic control using the quadratic performance criterion (see references in Wonham [70]). At the same time, Girsanov [45] and Howard [49] made the first steps in constructing a general theory, based on Bellman's technique of dynamic programming, developed by him somewhat earlier [14].

Two types of engineering problems engendered two different parts of stochastic control theory. Problems of the first type are associated with multistep decision making in discrete time, and are treated in the theory of discrete stochastic dynamic programming. For more on this theory, we note in addition to the work of Howard and Bellman, mentioned above, the books by Derman [37], Mine and Osaki [62], and Dynkin and Yushkevich [39].

Another class of engineering problems which encouraged the development of the theory of stochastic control involves the time continuous control of a dynamic system in the presence of random noise. The case where the system is described by a differential equation and the noise is modeled as a time continuous random process is the core of the optimal control theory of diffusion processes. Our problem falls into this category.

An optimal control problem includes the following information:

(1) A **control** u belonging to some set $\mathcal{U}_{ad}$ (the set of admissible controls, usually a compact subset of a Polish space) which is at our disposition.

(2) For a given control u, the state $x(u)$ of the system which is to be controlled is given by the solution of an equation (**state equation**)

$$\mathcal{P}(x, u) = 0,$$

where $\mathcal{P}$ is a known operator that relates the state and control. Under different situations, the state equation may be given by a difference equation, an ordinary differential equation (lumped system), a partial differential equation (distributed system), an integro-differential equation, or a functional equation, either deterministic or stochastic.

(3) The **observation** $y = f(x, u)$, which is an index of available information about the state given the input history. In general it is not possible to determine the state x when given observation y and input u, such problems are known as partially observed control problems.

(4) The **cost function** $J(u)$, (also called **economic function** or **performance index**), defined in terms of a numerical function $z \to \Phi(z) \geq 0$ on the space of observations, is given by

$$J(u) = \Phi(y) = \Phi(f(x, u)).$$

The optimal control problem is to find a control $u^0 \in \mathcal{U}_{ad}$ such that

$$\inf \{J(u) : u \in \mathcal{U}_{ad}\} = J(u^0).$$

One of the most fascinating and challenging problems in optimal control theory today is the proof of existence of optimal controls, especially feedback controls, i.e., controls dependent on the state of the system. For systems governed by stochastic differential equations or integral equations the existence problem is far more complex (Ahmed [4]). In this thesis we will mainly concern with the question of existence of optimal feedback controls, by means of dynamic programming principle and Hamilton-Jacobi-Bellman equations (HJB equations in short).

1.2 Dynamic Programming Principle and HJB Equations

1.2.1 Stochastic Optimal Feedback Control System

Assume X, U are real Banach spaces. Let $(\Omega, \mathcal{F}, \mathcal{F}_t\uparrow, P)$ be a filtered probability space. That is, $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathcal{F}_t$ is a set of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_r$ if $s < r$. Suppose $\{W_t, t \geq 0\}$ is an $\mathcal{F}_t$ adapted X-valued Brownian motion. We are interested in the optimal control problem of stochastic evolution equations driven by W_t.

Consider the following Controlled Stochastic Evolution Equation (CSEE)

$$(CSEE) \quad \begin{cases} dy(t) & = (By(t) + f^u(t, y(t)))dt + dW_t \\ y(0) & = y_0 \end{cases}$$

where $f^u(t, y(t)) \equiv f(t, y(t), u(t, y(t))), B$ is a linear operator in X.

Definition 1.2.1. (Fleming and Rishel [42, p.156])

A feedback control u is said to be **admissible** if u is a Borel measurable function from $[0, T] \times X \to U$, satisfying the following conditions:

(a) For each $y_0 \in X$, there exists a solution y of (CSEE), unique in probability law.

(b) For each $k > 0$

$$E \int_0^T |u(t, y(t))|^k dt < \infty.$$

We denote by $\mathcal{U}_{ad}$ the class of all admissible feedback controls.

Remark 1.2.2. There are several ways to define admissible control. Some authors define it as a family of probability measures satisfying measurability condition (see Gihman and Skorohod [43, p.79]). Some author defines it a U-valued stochastic process with measurable sample paths satisfying the non-anticipativity condition: for $t \geq s \geq r, W(t) - W(s)$ is independent of $u(z), z \leq r$. (Borkar [16, p.5-6])

Definition 1.2.3. **The performance index (cost function)** of a feedback control system is given by

$$J(u) = E\{\int_0^T \ell(s, y(s), u(s, y(s)))ds + \phi_0(y(T))\}.$$

Where $y(t)$ is a solution of (CSEE), and ℓ, ϕ_0 are known functions.

The optimal feedback control problem considered here is to find a feedback control law u^0 which minimizes $J(u)$. That is,

$$J(u^0) = \inf\{J(u) : u \in \mathcal{U}_{ad}\}.$$

1.2.2 Bellman's Dynamic Programming Principle

There are two different approaches for the optimal control problems described in section 1.2.1. One is probabilistic approach (see Ahmed [5], for example). This approach is based on selection theorems and the so-called "compactness-continuity" argument —— every continuous (more generally, lower semi-continuous) function on a compact set attains its minimum. The distinguishing feature of probabilistic method is that it works with the actual sample paths of the process, so that one does not lose sight of "the stochastic process within". The other one is analytic approach. In this approach the optimal control problem of a stochastic system reduces to an equivalent problem of a deterministic distributed parameter system, governed by a partial differential equation. Compared to the stochastic method, this approach works with average quantities. It is more general and powerful, especially when one deals with analytic properties of the system, such as regularity properties of solutions.

In this thesis we use the analytic approach. The stochastic optimal control problem can be reduced to an equivalent problem of a deterministic system governed by a partial differential equation, called **Hamilton-Jacobi-Bellman equation, or HJB equation** in short. This reduction is done by means of dynamic programming principle, due to R. Bellman [14] (see also Ahmed [3]).

Lemma 1.2.4. (Dynamic Programming Principle)

For $0 \leq t < T, x \in X$ *and* $u \in \mathcal{U}_{ad}$, *let*

$$\ell^u(\theta, y_{t,x}(\theta)) = \ell(\theta, y_{t,x}(\theta), u(\theta, y_{t,x}(\theta))),$$

$$\phi^u(t,x) = E\{\int_t^T \ell^u(\theta, y_{t,x}(\theta))d\theta + \phi_0(y_{t,x}(T))\},$$

where $y_{t,x}$ *is the solution of (CSEE) with initial condition* $y(t) = x$. *Let* u *be an admissible control restricted to the interval* $[t,s]$, *and* $\phi(t,x) = \inf\{\phi^u(t,x) : u \in \mathcal{U}_{ad}\}$, *then for all* $t < s < T$

$$\phi(t,x) = \inf_{u \in \mathcal{U}_{ad}} E\{\int_t^s \ell^u(\theta, y_{t,x}(\theta))d\theta + \phi(s, y_{t,x}(s))\}. \tag{1.1}$$

The dynamic programming principle says that the minimum cost of the control problem equals the minimum of the cost at a given stage plus the minimum cost to go thereafter. In other words, whatever may have been the control policy over the initial period of time $[t,s]$, the subsequent policy must be such that it minimizes the cost of operation for the remaining period $[s,T]$.

The function $\phi(t,x)$ is called the **value function**. Loosely speaking, it gives the minimum cost of future operation of the system starting from (t,x). Using the dynamic programming principle and Itô's formula we give here a formal derivation of HJB equations. For mathematically rigorous proof see for example Krylov [53], Fleming and Rishel [42], Bensoussan [15], Borkar[16].

Let $G = [0,T] \times X$, denote

$$C^{1,2}(G) = \{\phi(t,x) : \phi_t, \phi_x, \phi_{xx} \text{ exists and continuous}\}.$$

Theorem 1.2.5. (Hamilton-Jacobi-Bellman Equation)

Suppose the value function (1.1) belongs to $C^{1,2}(G)$, *then it satisfies the so called Hamilton-Jacobi-Bellman equation*

$$(HJB) \quad \begin{cases} \phi_t(t,x) + \inf_{u \in \mathcal{U}_{ad}} \{A^u(t)\phi(t,x) + \ell^u(t,x)\} = 0 \\ \phi(T,x) = \phi_0(x). \end{cases}$$

where

$$A^u(t)\phi(t,x) = \frac{1}{2} Tr(S\phi_{xx}) + (Bx + f^u(t,x), \phi_x)_{X,X^*}, \tag{1.2}$$

$Tr(\cdot)$ *denotes the trace operator and* S *is the covariance operator of* W_t.

For definition and properties of trace operator, see Reed and Simon [65].

Proof: First we consider finite dimensional case (Ahmed [3, p.402]). By Itô's formula (Ahmed [3, p.338]) and (CSEE), we have

$$\begin{aligned} d\phi(t,y(t)) &= \phi_t(t,y)dt + (\phi_x(t,y), dy) + \frac{1}{2}\langle \phi_{xx}(t,y)dy, dy\rangle \\ &= \phi_t(t,y)dt + (\phi_x(t,y), a^u(t,y))dt + \frac{1}{2}Tr(S\phi_{xx})dt + (\phi_x, dW) \\ &= L^u\phi dt + (\phi_x, dW), \end{aligned}$$

where $L^u\phi = \phi_t + A^u(t)\phi$, $a^u(t) = By(t) + f^u(t, y(t))$. Integrating and taking expectation, we have

$$E\phi(s, y(s)) = \phi(t,x) + E\int_t^s (L^u\phi)(\theta, y(\theta))d\theta.$$

It follows from (1.1) that for all $t < s < T$, we have

$$\inf_{u \in \mathcal{U}_{ad}} E\{\int_t^s \ell(\theta, y, u)d\theta + \int_t^s (L^u\phi)(\theta, y)d\theta\} = 0.$$

Dividing by $s-t$ and letting $s \to t^+$, since $y(t)=x$ we have (formally) that

$$\inf_{u\in\mathcal{U}_{ad}} E\{L^u(t)\phi(t,x)+\ell^u(t,x)\}=0.$$

This gives the HJB equation. The terminal condition follows from the definition of $\phi(t,x)$ (Lemma 1.2.4).

In the infinite dimensional case, we may choose an increasing family of projectors and reduce the problem to a sequence of finite dimensional problems. Then taking the limit we obtain the HJB equation for infinite dimensional systems.

Let P_n denote the projection map from $X \mapsto X_n$, the n dimension linear span of $\{e_i,(1\le i\le n)\}$. That is

$$P_n x=\sum_{i=1}^{n}(x,e_i^*)e_i \equiv x^n, \quad \forall x\in X,$$

where $\{e_i,e_i^*\}$ is a regular biorthogonal system in (X,X^*) (see p.33), and $\{e_i\}\subset\mathcal{D}(B)$. Similarly we define the projection map Q_n for control space U.

Suppose $\phi^n(t,x^n)$ is the value function of the finite dimensional optimal control problem

$$\begin{cases} dy^n(s)=(B_n y^n(s)+f_n^{u_n}(s,y^n(s)))ds+dW_n(s)\\ y^n(t)=x^n\\ J_n(u_n)=E\{\int_t^T \ell^{u_n}(\theta,y^n(\theta))d\theta+\phi_0^n(y^n(T))\}.\end{cases}$$

Where $B_n=P_nBP_n, f_n=P_nfP_n, u_n=Q_nu, W_n=P_nW$, and $\phi_0^n=P_n\phi_0$.

Hence $\phi^n(t,x^n)$ satisfies the HJB equation

$$\begin{cases} \phi_t^n(t,x^n) + \inf_{u_n\in\mathcal{U}_{ad}}\{A_n^{u_n}(t)\phi_n(t,x^n) + \ell^{u_n}(t,x^n)\} = 0 \\ \phi^n(T,x^n) = \phi_0^n(x^n). \end{cases}$$

Assume $\{e_i\} \subset \mathcal{D}(B)$ (in general some more conditions are added to justify the passing to limits, see Da Prato [31, p.68], Barbŭ and Da Prato [11, p.542], Cannarsa and Da Prato [18, p.488-489], for example). Letting $n \to \infty$, and formally define $A^u(t)$ as the infinite dimensional operator given by (1.2), we have $A_n^{u_n}(t) \to A^u(t)$, $\phi^n(t,x^n) \to \phi(t,x)$, $\phi_t^n(t,x^n) \to \phi_t(t,x)$, $\ell^{u_n}(t,x^n) \to \ell^u(t,x)$, $\phi_0^n(x^n) \to \phi_0(x)$. Hence $\phi(t,x)$ satisfies (HJB) equation. (Ahmed [4, p.112]) ∎

The power of HJB equation is enormous. As an example we present a sufficient condition for the existence of optimal control.

Theorem 1.2.6. (Verification Theorem in Finite Dimensions)

If $\phi^* \in C^{1,2}(G)$ is a solution of the HJB equation and the minimum in HJB equation is attained at some control $u^0 \in \mathcal{U}_{ad}$, that is

$$A^{u^0}\phi + \ell(t,x,u^0) = \inf_{u\in\mathcal{U}_{ad}}\{A^u\phi + \ell(t,x,u)\}, \tag{1.3}$$

then the solution of HJB equation coincides with the value function, i.e.,

$$\phi^* = \inf_{u\in\mathcal{U}_{ad}} \phi^u.$$

Thus u^0 is the optimal control we seek. (Ahmed [3, p.403], Fleming and Rishel [42, p.159]).

The verification theorem reduces the optimal control problem to two other problems. The first is to solve the nonlinear second order partial differential equation (HJB equation), with the terminal condition given. The second is to find u^0 by performing the minimization indicated in (1.3). In many applications the second problem is easy, but in general the question of existence of a solution of HJB equation poses a major problem, and it remains an outstanding problem in general Banach spaces. (Ahmed [4, p.113-114])

1.2.3 Other Applications

In the studies of stability, optimal control, and filtering problems, quantities of practical interest are often given by mathematical expectation of certain functionals of the solution process. These quantities can be computed by solving a class of initial or initial-boundary value problems involving second order partial differential equations of the same type as HJB equations. In other words, stochastic problems are converted into equivalent deterministic problems which can be numerically solved to determine the desired quantities, under the assumption of existence of solutions.

Here are some examples.

Example 1.2.7. (Functional of terminal state)

Consider the system

$$\begin{cases} d\xi(s) & = a(s,\xi)dt + \sigma(s,\xi)dW_s \quad (t \leq s \lesssim T) \\ \xi(t) & = x \end{cases}$$

Suppose the coefficients $a : I \times R^n \to R^n, \sigma : I \times R^n \to R(n \times m)$ satisfy the following condition:

(1) $T\|a(t,x)\|^2_{R^n} + 4\|\sigma(t,x)\|^2_{R(n\times m)} \leq K(1 + \|x\|^2)$,

(2) $T\|a(t,x) - a(t,y)\|^2_{R^n} + 4\|\sigma(t,x) - \sigma(t,y)\|^2_{R(n\times m)} \leq K\|x - y\|^2$,

for some constant K. Then for every $f \in C^2$ with $D^\alpha f, (|\alpha| \leq 2)$, having at most polynomial growth, the function ϕ given by

$$\phi(t,x) = E\{f(\xi(T))|\xi(t) = x\} \equiv E\{f(\xi_{t,x}(T))\}$$

satisfies **Kolmogorov Equation**

$$\begin{cases} \phi_t + A(t)\phi = 0 \\ \phi(T,x) = \phi_0(x), \end{cases}$$

where $A(t)\phi = \frac{1}{2}Tr(\sigma S \sigma^* \phi_{xx}) + (a(t,x), \phi_x)$ is the generator of the diffusion process ξ.

For more examples of Kolmogorov equations, see Ahmed [3, p.370-373].

Example 1.2.8. (Fokker-Plank Equation) (Da Prato [28])

Consider the following stochastic system:

$$\begin{cases} d\xi & = B\xi dt + f(\xi)dt + \sigma dW_t \\ \xi(0) & = x \end{cases}$$

where W_t is an X-valued Brownian motion in a probability space $(\Omega, \mathcal{F}, P)$, B is the infinitesimal generator of a C_0-semigroup in X, and $f \in C_b(X)$. Setting

$\phi(t,x) = E\phi_0(\xi_{T-t,x}(T))$, then $\phi(t,x)$ is a solution of

$$\begin{cases} \phi_t = \frac{1}{2}Tr(\sigma S\sigma^*\phi_{xx}) + (Bx + f(x), \phi_x) \\ \phi(0,x) = \phi_0(x), \quad t \in [0,T], \ x \in \mathcal{D}(B). \end{cases}$$

1.3 Some Historical Notes

In this section we give a brief review of some general methods used to study HJB equations, and some well-known results. For references see P.L. Lions [58], Barbu and Da Prato [10], Da Prato [26-32], Havârneanu [47], Cannarsa and Da Prato [18, 19].

First consider the following deterministic optimal control problem Minimize

$$J(u) = \int_0^T \ell(s, y(s), u(s, y(s)))ds + \phi_0(y(T))\},$$

with state equation

$$\frac{dy}{dt} = a^u(t,y).$$

By Dynamic Programming Principle it is related to the following problem

$$\begin{cases} \phi_t + H(t,x,D\phi(t,x)) = 0 \\ \phi(0,x) = \phi_0(x), \quad x \in O, \quad t \in [0,T], \ T > 0. \end{cases} \tag{1.4}$$

Where $D\phi \equiv \phi_x, H(t,x,p) = \inf\{(a^u(t,x),p) + \ell^u(t,x) : u \in \mathcal{U}_{ad}\}$ is the so called **Hamiltonian.** O is an open domain in R^n.

It is well-known that, in general, classical solutions for equation (1.4) do not exist. Because of this difficulty, many authors often seek the mild solution of the equation, that is the solution of an integral equation. M.G. Crandall and P.L. Lions [23] introduced the **viscosity solutions** which allows merely continuous functions to be the solutions of fully non-linear equations. We will define these generalized solutions shortly.

The classical approach to the equation (1.4) is by **characteristic method.** We explain this method in the following special case. Consider

$$\begin{cases} \phi_t + H(D\phi(t,x)) = 0 \\ \phi(0,x) = \phi_0(x). \end{cases} \tag{1.5}$$

To motivate the idea of the characteristic method, we assume first that

$$H(D\phi) = aD\phi, \quad a \in R.$$

It is clear that in this case the solution is given by

$$\phi(t,x) = \phi_0(x - ta),$$

hence

$$\phi_0(x) = \phi(t, x + ta).$$

Note that

$$D\phi_0(x) = D\phi(t, x + ta), \quad a = H'(D\phi_0(x)).$$

Therefore we consider

$$\phi(t, x + tH'(D\phi_0(x))) = \phi(0,x) + t\Psi(x) = \phi_0(x) + t\Psi(x).$$

Let $u(t,x) = x + tH'(D\phi_0(x))$, we want to find a function $\Psi(x)$ such that $\phi(t, u(t,x))$ satifies

$$\phi_t + H(D\phi) = 0, \quad D\phi_0(x) = D\phi(t, u(t,x)).$$

This can be done as follows. Suppose we have found such a function $\Psi(x)$, then by differentiating the identity $\phi(t,u(t,x)) = \phi_0(x) + t\Psi(x)$ we have

$$\begin{aligned} \Psi(x) &= \frac{\partial}{\partial t}\phi(t,u(t,x)) + \frac{\partial u(t,x)}{\partial t} D\phi(t,u(t,x)) \\ &= -H(D\phi(t,u(t,x))) + H'(D\phi_0(x))D\phi(t,u(t,x)) \\ &= -H(D\phi_0(x)) + H'(D\phi_0(x))D\phi_0(x). \end{aligned}$$

Therefore a solution of problem (1.5) is given by

$$\begin{cases} \phi(t,u(t,x)) = \phi_0(x) + t\{H'(D\phi_0(x))D\phi_0(x) - H(D\phi_0(x))\} \\ u(t,x) = x + H'(D\phi_0(x)). \end{cases}$$

For more results about the method of characteristics, see P.L. Lions [58] and the references therein.

Another widely used method for the first order HJB equations is called **vanishing viscosity** method. The main idea is follows. For any $\epsilon > 0$, consider the approximate problem

$$\begin{cases} \phi^\epsilon_t - \epsilon\Delta\phi^\epsilon + H(t,x,D\phi^\epsilon) = 0 \\ \phi^\epsilon(0,x) = \phi_0(x). \end{cases} \tag{1.6}$$

This is called the parabolic regularization. Here the original first order nonlinear equation is converted to a second order semi-linear equation, and this kind of equations has been well studied by many authors. (See for example, Ladyzenskaya and Ural'steva [56], Gilberg and Trudinger [44], P.L.Lions [58]). After solving the approximate problem, the second step is to pass to the limit as ϵ goes to 0, and establish the existence of solutions for equation (1.4). Obviously this procedure requires *a priori* estimates of solutions of approximate equation, it often causes new difficulties.

The viscosity solution for HJB equations is a relatively new concept. The definition of viscosity solutions for the first order HJB equations in finite dimensions was introduced by M.G. Crandall and P.L. Lions [23]. Viscosity solutions for the second order equations were studied later in P.L. Lions [59], Crandall, Ishii and Lions [22].

The main point of this theory is that it allows merely continuous functions to be the solutions of HJB equations in the viscosity sense. It is worth noting that, in the viscosity sense, the solution of HJB equation coincides with the value function of optimal control problem. Besides, if differentiable, a viscosity solution is the same as classical solution. (see Crandall and Lions, [21, 23]).

To motivate the idea of viscosity solutions for the second order HJB equations, let us consider the following equation

$$F(x, u(x), Du(x), D^2u(x)) = 0, \quad x \in R^N \tag{1.7}$$

where $F : R^N \times R \times R^N \times \mathcal{M}(N) \mapsto R$, and $\mathcal{M}(N)$ is the set of symmetric $N \times N$ matrices. Assume F satisfies the following fundamental monotonicity conditions

(F-1) $F(x,r,p,X) \leq F(x,s,p,Y)$ whenever $r \leq s$ and $X \geq Y$.

Now suppose that $u \in C^2(R^n)$ is a classical solution of (1.7) and ϕ is an arbitrary function from $C^2(R^n)$. If $u - \phi$ attains its local maximum and local minimum at points x^* and x_0 respectively, by a well known result in calculus we have

$$Du(x^*) = D\phi(x^*), \quad D^2u(x^*) \leq D^2\phi(x^*)$$

and

$$Du(x_0) = D\phi(x_0), \quad D^2u(x_0) \geq D^2\phi(x_0).$$

Hence by monotonicity assumption (F-1)

$$F(x^*, u(x^*), D\phi(x^*), D^2\phi(x^*)) \leq F(x^*, u(x^*), Du(x^*), D^2u(x^*)) = 0$$

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq F(x_0, u(x_0), Du(x_0), D^2u(x_0)) = 0$$

Note that the left-hand sides of inequalities do not depend on the derivatives of $u(x)$. This leads to the definition of a kind of generalized solution $u(x)$ which satisfies

$$F(x^*, u(x^*), D\phi(x^*), D^2\phi(x^*)) \leq 0 \quad (sub - solution)$$

whenever $\phi \in C^2$ and x^* is a local maximum of $u - \phi$, and

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0 \quad (super - solution)$$

whenever $\phi \in C^2$ and x_0 is a local minimum of $u - \phi$.

If u is both a sub-solution and super-solution, it is defined as the generalized solution (viscosity solution).

In the definition of viscosity solutions, the space of test functions ($\phi \in C^2$) will generally be replaced by sub and super differentials of $u(x^*)$ and $u(x_0)$. For more details, see Crandall and Lions [21, 23], Lions [58, 59].

The theory of viscosity solutions is powerful in the study of HJB equations and other general partial differential equations, as it provides a way to prove the general existence theorems. The uniqueness is proved by comparison theorem, that is, every subsolution is less than or equal to super solution (see Lions [59], for example). On the other hand, this theory also has some limitations, such as the fundamental monotonicity assumption. We will not consider viscosity solution in this thesis.

Semigroup approach is also a powerful tool for the study of HJB equations in abstract spaces. (Ahmed [1], Aizawa [8, 9], Burch [17], Cannarsa and Da Prato [18, 19], Da Prato [26-32], Havârneanu [47]). The main idea of semigroup approach is quite simple. Rewrite the HJB equation as

$$\phi_t = \mathcal{A}\phi + \mathcal{B}(\phi), \quad \phi(0) = \phi_0 \tag{1.8}$$

where $\mathcal{A}$ is a linear operator and $\mathcal{B}$ is a nonlinear operator defined on some function space Y. If $\mathcal{A}$ generates a semigroup $T(t)$ in Y, then we may consider the so called **mild solutions** of the equation (1.8), defined as the solution of the following integral equation

$$\phi(t) = T(t)\phi_0 + \int_0^t T(t-s)\mathcal{B}(\phi(s))ds$$

and invoke Fixed Point Principles to get the existence of mild solutions.

Since we will follow this approach in this thesis, it is suitable to review some main results in this direction.

A kind of optimal control problems that received the most attention is

$$(State\ \ equation)\quad \begin{cases} dy(t) = (Ay(t) + f(y(t)) + u)dt \mp \sqrt{\epsilon}dW(t) \\ y(0) = x. \end{cases}$$

The cost function is

$$J_\epsilon(x,u) = E\{\int_0^T (g(y(s)) + \frac{1}{2}|u(s)|^2)ds + \phi_0(y(T))\}.$$

And the corresponding HJB equation is given by

$$\begin{cases} \phi_t = \frac{\epsilon}{2}Tr(S\phi_{xx}) + \langle Ax + f(x), \phi_x\rangle - \frac{1}{2}|\phi_x|^2 + g(x) \\ \phi(0,x) = \phi_0(x). \end{cases} \tag{1.9}$$

Da Prato [32] defined **S-differentiable** function space $C^1_S(H)$, where H is a separable Hilbert space. He considered equation (1.9) with modified terms $\langle f(x), S\phi_x\rangle - \frac{1}{2}|S\phi_x|^2$. Using the fact that $\phi \mapsto \frac{\epsilon}{2}Tr(S\phi_{xx}) + \langle Ax, \phi_x\rangle$ generates a semigroup

$$T(t)\phi = E\phi(e^{tA}x + \int_0^t e^{(t-s)A}dW(s)),$$

he proved the existence and uniqueness of solution in $C^1_S(H)$, under the following assumptions:

(H1) S is a self-adjoint, positive nuclear operator in H, given by

$$Sx = \sum_{i=1}^{\infty} \lambda_i (x, e_i) e_i, \quad (\lambda_i > 0),$$

where $\{e_i\}$ is a complete orthonormal system in H.

(H2) A is a self-adjoint negative operator in H such that

$$Ae_i = -\mu_i e_i, \quad \mu_i \geq 0, \; (i = 1, 2, \cdots).$$

(H3) $\phi_0 \in C_S^1(H)$, $f \in C_b(H, H)$, $g \in C_b(H)$.

Havârneanu [47] studied equation (1.9) with $A = 0$. By using the properties of Gauss measure in abstract spaces, he constructed a semigroup with infinitesimal generator $\phi \mapsto Tr(S\phi_{xx})$. Then he used Banach Fixed Point Theorem to prove the existence of the mild solutions.

Recently, Cannarsa and Da Prato [18] considered optimal control problems for stochastic systems driven by a cylindrical Brownian motion. This problem is related to the HJB equation (1.9) with $S = I$, the identity operator in $\mathcal{L}(H)$. They proved the existence and uniqueness of the mild solution under the assumptions (H2) and

(H4) $\sum_{i=1}^{\infty} \mu_i^{2\sigma - 1} < \infty$, $(0 < \sigma < \frac{1}{2})$

(H5) $\phi_0 \in C_b(H)$, $f \in C_b(H, H)$, $g \in C_b(H)$.

Da Prato [31] considered HJB equation with general non-linear term

$$\begin{cases} \phi_t = \frac{1}{2}Tr(S\phi_{xx}) + (Ax, \phi_x) - F(S\phi_x) \\ \phi(0,x) = \phi_0(x). \end{cases} \tag{1.10}$$

Where $F \in Lip(H,H)$ is a Lipschitz continuous function. His strategy is solving the linear equation first, then considering the non-linear term as a perturbation to the linear system, and appealing to the generation theorems of strongly continuous semigroups (C_0-semigroups).

Later in a series of papers, Da Prato [26-30] developed a new technique to construct C_0-semigroup with infinitesimal generator

$$\mathcal{A} : \phi \mapsto \frac{1}{2}Tr(S\phi_{xx}).$$

He calls it **heat semigroup.** (His main result is cited in Chapter 3, Lemma 3.2.1.) Then he added the first order term by using the perturbation theorem. In Da Prato [28] he tried to avoid the perturbation theorem and considered the first order term also as an infinite product of generators of semigroups, but he did not explore this idea thoroughly, and he did not use it for non-linear equations. Besides, in his work, $\mathcal{A}$ is defined only on the space $\overline{C_b^2}(H)$, given by

$$\overline{C_b^2}(H) = \overline{C_b^\infty(H)}^{C_b^2(H)}, \quad C_b^\infty(H) = \bigcap_{k=0}^{\infty} C_b^k(H).$$

These conditions may be too restrictive and consequently the spaces $\overline{C_b^2}(H)$ may be too small. This may limit its applications, for instance, to non-linear HJB equations. In this thesis we will try to overcome this difficulty by introducing a much larger subspace of $C_b(X)$, and further develop Da Prato's method to solve HJB equations with general non-linear term, in an abstract

Banach space.

1.4 Main Results and Contributions

In this thesis we study the HJB equation arising from stochastic optimal control problem of (CSEE). More precisely, we study the following second order parabolic partial differential equation

$$(P)\quad \begin{cases} \phi_t(t,x) = & \frac{1}{2}Tr(S\phi_{xx}(t,x)) + (Bx + f(x), \phi_x(t,x)) \\ & + F(t,x,\phi(t,x),\phi_x(t,x)) \\ \phi(0,x) = & \phi_0(x). \end{cases}$$

Where ϕ_0, F are given functions, B is the infinitesimal generator of a strongly continuous semigroup, and S is a positive nuclear operator in X (Chapter 3) or an identity operator in $\mathcal{L}(X^*, X)$ (Chapter 4).

We give an outline of the contents of this thesis.

Chapter 1 is devoted to the motivations and orientations of our research. In **Chapter 2** some notations and basic theorems in Banach spaces are reviewed. In **Chapter 3** we study HJB equation related to optimal control problems of stochastic systems. In **Chapter 4** we study a more general case when the system is driven by a cylindrical Brownian motion. In **Chapter 5** we give some generalizations and suggestions for future research.

The main contributions of this thesis include:

(1) A direct method suggested recently by Da Prato [26, 28] has been further developed. This method, which is different from ordinary perturbation theorem in semigoup theory, is more constructive, and can be applied to general non-linear parabolic partial differential equations.

(2) Many authors have studied semilinear HJB equations (P), with non-linear term of the form

$$F(x, \phi_x) = \frac{1}{2}|\phi_x|^2 + g(x).$$

Da Prato [31] has also studied a HJB equation with nonlinear term $F(S\phi_x)$. In this thesis we will study the more general non-linear term $F(t, x, \phi, \phi_x)$, and without coercivity or convexity assumptions.

(3) Most of the results about HJB equations obtained before were in R^n, or in a separable Hilbert space. In this thesis we study HJB equations in a separable reflexive Banach space, hence our results are more general.

CHAPTER 2

Preliminaries

2.1 Introduction

Differential equations form a major tool in the study of pure and applied sciences. Depending on the problem, these equations may take the forms of ordinary differential equations, partial differential equations, functional differential equations, and sometimes a combination of interacting systems of ordinary and partial differential equations. In general, under broad assumptions, many of these equations can be reformulated as ordinary differential equations on abstract spaces, for example, Banach spaces.

Semigroup theory provides a unified and powerful tool for the study of differential equations in abstract spaces. In recent years, among many other applications, semigroup theory has been widely used in the study of control and stability of systems governed by differential equations on Banach spaces. It also plays a central role in our study of HJB equations in this thesis.

This chapter is devoted to the mathematical prerequisites that we shall need later. In section 2.2 we review some basic concepts and results from semigroup theory of linear operators. In section 2.3 we present some results on bases in Banach spaces and their duals.

2.2 Semigroup Theory of Linear Operators in Banach Spaces

Let X be a Banach space and $\{T(t), t \geq 0\}$ a family of bounded linear operators in X, i.e., $T(t) \in \mathcal{L}(X),\ t \geq 0$.

Definition 2.2.1. $\{T(t), t \geq 0\}$ is a **semigroup** of bounded linear operators if

(1) $T(0) = I$ (the identity operator on X),

(2) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ (the semigroup property).

$\{T(t), t \geq 0\}$ is said to be a **strongly continuous** semigroup or C_0-semigroup of bounded linear operators on X if

(3) $\lim_{t\downarrow 0} \|T(t)x - x\| = 0$, for every $x \in X$. Moreover, if $\|T(t)\| \leq 1$ for $t \geq 0$, $T(t)$ is called a **C_0-semigroup of contractions.**

Definition 2.2.2 Let $A_t = \frac{1}{t}(T(t) - I)$, an operator A defined by

(i) $D(A) = \{x \in X : \lim_{t\downarrow 0} A_t x \quad \text{exists} \}$,

(ii) $Ax = \lim_{t\downarrow 0} A_t x$, for $x \in D(A)$,

is called **the infinitesimal generator** of the semigroup $\{T(t), t \geq 0\}$ on X.

The limits are understood in the sense of strong topology in X.

The most important properties of a C_0-semigroup are given in the following theorem.

Theorem 2.2.3. *Let $T(t), t \geq 0$ be a C_0-semigroup and let A be its infinitesimal generator. Then*

(a) For $x \in X$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x.$$

(b) For $x \in X$, $\int_0^t T(s)x ds \in D(A)$ and

$$A(\int_0^t T(s)x ds) = T(t)x - x.$$

(c) For $x \in D(A)$, $T(t)x \in D(A)$ and

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax.$$

(d) For $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax d\tau = \int_s^t AT(\tau)x d\tau.$$

Now we give a characterization of the infinitesimal generators of C_0-semigroups of contractions. It is called the generation theorem which forms the corner stone of semigroup theory.

Theorem 2.2.4 (Hille-Yosida). *A linear (possibly unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contractions $\{T(t),\ t \geq 0\}$ if and only if*

(1) A is closed and $\overline{D(A)} = X$,

(2) The resolvent set $\rho(A)$ of A contains the interval $(0, +\infty)$ and

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda} \quad for \quad every \quad \lambda > 0,$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$ is called the resolvent of A corresponding to $\lambda \in \rho(A)$.

The proof of these properties can be found, for example, in Ahmed [1, Chapter 1 and 2], Pazy [64, Chapter 1]. In the following we give an example of a C_0-semigroup of contractions.

Example 2.2.5. Let H be a Hilbert space and $\{e_k\}$ be a complete orthonormal system in H. Denote by $C_b(H)$ the class of bounded uniformly continuous functions $H \to R$, with the supremum norm $\|\phi(x)\|_0 = \sup\{|\phi(x)| : x \in H\}$. Let $\mathcal{X}$ denote the Banach space $C_b(H)$. Define

$$(T(t)\phi)(x) = \frac{1}{\sqrt{2\pi t}} \int_R e^{-\frac{\xi^2}{2t}} \phi(x - \xi e_k) d\xi, \quad \phi \in \mathcal{X}.$$

Since

$$\|T(t)\phi\|_0 \leq \|\phi\|_0 \equiv \sup_{x \in H} |\phi(x)|,$$

one can easily verify that $T(t) \in \mathcal{L}(\mathcal{X})$ and defines a semigroup of contractions. We show it is strongly continuous. Letting $t\eta^2 = \xi^2$ and substituting in the above expression we have

$$(T(t)\phi)(x) = \frac{1}{\sqrt{2\pi}} \int_R e^{-\frac{\eta^2}{2}} \phi(x - \sqrt{t}\eta e_k) d\eta, \quad \phi \in \mathcal{X}.$$

Hence

$$\|T(t)\phi - \phi\|_0 \le \frac{1}{\sqrt{2\pi}} \int_R e^{-\frac{\eta^2}{2}} \|\phi(x - \sqrt{t}\eta e_k) - \phi(x)\|_0 d\eta.$$

Since $\|\phi(x - \sqrt{t}\eta e_k) - \phi(x)\|_0 \to 0$ as $t \to 0$, it follows from the dominated convergence theorem that $\|T(t)\phi - \phi\|_0 \to 0$.

Now we compute the infinitesimal generator of $T(t)$. Let $\phi \in C_b^2(H) \subset \mathcal{X}$, then

$$\begin{aligned} \frac{T(t)\phi - \phi}{t} &= \frac{1}{\sqrt{2\pi} t} \int_R e^{-\frac{\eta^2}{2}} (\phi(x - \sqrt{t}\eta e_k) - \phi(x)) d\eta \\ &= \frac{1}{\sqrt{2\pi} t} \int_R e^{-\frac{\eta^2}{2}} (D_k\phi(x)\sqrt{t}\eta + \frac{1}{2} D_k^2 \phi(x) t\eta^2 + o(t)) d\eta, \end{aligned}$$

where $D_k\phi = \langle \phi_x, e_k \rangle$, $D_k^2\phi = \langle \phi_{xx} e_k, e_k \rangle$ are the first and second order Gâteaux derivatives of ϕ in the direction e_k. By straightforward computations one can verify that

$$\frac{1}{\sqrt{2\pi}} \int_R \eta e^{-\frac{\eta^2}{2}} d\eta = 0, \quad \frac{1}{\sqrt{2\pi}} \int_R \eta^2 e^{-\frac{\eta^2}{2}} d\eta = 1.$$

Thus we have

$$\lim_{t \downarrow 0} \frac{T(t)\phi - \phi}{t} = \frac{1}{2} D_k^2 \phi.$$

That is $A = \frac{1}{2} D_k^2$.

Definition 2.2.6. Let $\delta = \{z : \xi_1 < arg\ z < \xi_2, \xi_1 < 0 < \xi_2\}$ and for

$z \in \Delta$, let $T(z)$ be a bounded linear operator. The family $\{T(z) : z \in \Delta\}$ is an analytic semigroup in Δ if

(1) $z \to T(z)$ is analytic in Δ,

(2) $T(0) = I$ and $T(z)x \to x$ as $z \to 0, z \in \Delta$ for every $x \in X$,

(3) $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.

Clearly, the restriction of an analytic semigroup to the real axis is a C_0-semigroup. On the other hand, we are interested in the possibility of extending a given C_0-semigroup to an analytic semigroup in some sector Δ around the non-negative real axis. Those conditions are given in the following theorem (Ahmed [1, p.82]).

Theorem 2.2.7. *Let $T(t)$ be a uniformly bounded C_0-semigroup. Let A be the infinitesimal generator of $T(t)$ and assume $0 \in \rho(A)$. The following statements are equivalent:*

(a) $T(t)$ can be extended to an analytic semigroup in a sector $\Delta = \{z : |arg\ z| < \delta\}$ and $\|T(z)\|$ is uniformly bounded in every closed subsector $\overline{\Delta}_{\delta'} \subset \Delta_\delta$.

(b) There exists a constant C such that for every $\sigma > 0$, $\tau \neq 0$

$$\|R(\sigma + i\tau, A)\| \leq \frac{C}{|\tau|}.$$

(c) There exists $0 < \delta < \frac{\pi}{2}$ and $M > 0$ such that

$$\rho(A) \subset \Sigma = \{\lambda : |arg\ \lambda| < \frac{\pi}{2} + \delta\} \bigcup \{0\},$$

and

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}, \quad for\ \lambda \in \Sigma,\ \lambda \neq 0.$$

(d) $T(t)$ is differentiable (that is, $t \to T(t)x$ is differentiable for every $t > 0$, $x \in X$) and there is a constant C such that

$$\|AT(t)\| \leq \frac{C}{t}, \quad for\ \ t > 0.$$

Let A be a densely defined closed linear operator with $\mathcal{D}(A)$ and $\mathcal{R}(A)$ in X. Suppose the resolvent of A satisfies

$$\rho(A) \supset \Sigma = \{\lambda : 0 < \beta < |arg\ \lambda| \leq \pi\} \bigcup V_0, \quad (\beta < \frac{\pi}{2})$$

where V_0 is a neighbourhood of zero, and

$$\|R(\lambda, A)\| \leq \frac{M}{1 + |\lambda|}, \quad for\ \ \lambda \in \Sigma.$$

For any $\alpha > 0$ we can define fractional powers of A by

$$A^{-\alpha} = \frac{1}{2\pi i} \int_C z^{-\alpha}(A - zI)^{-1} dz$$

and

$$A^{\alpha} = (A^{-\alpha})^{-1}, \quad (A^0 = I)$$

where C is any smooth curve contained in Σ and running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, $\beta < \theta < \pi$.

The following theorem gives some simple properties of A^{α}:

Theorem 2.2.8. *The operator A^{α}, $0 \leq \alpha \leq 1$, satisfies* *(a)* A^α *is a closed operator with* $D(A^\alpha) = R(A^{-\alpha})$.

(b) $\alpha \geq \beta > 0$ *implies* $D(A^\alpha) \subseteq D(A^\beta)$.

(c) $\overline{D(A^\alpha)} = X$ *for every* $\alpha \geq 0$.

(d) $A^{\alpha+\beta}\xi = A^\alpha A^\beta \xi$ *for* $\alpha, \beta \in R$, *and* $\xi \in D(A^\gamma)$ *where* $\gamma = max\{\alpha, \beta, \alpha+\beta\}$. (Ahmed [1, p.95-96])

We have reviewed some properties of one parameter semigroups. We also need some results about n-parameter semigroups of linear bounded operators on Banach spaces (Hille and Phillips [48, p.334-335]).

Definition 2.2.9. *Let* $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ *and denote the unit vectors by* $u_1, \cdots, u_n$ *where* $u_j = (\delta_{jk})$. *A family of linear bounded operators*

$$T(\xi),\ \xi \in R_n^+ \equiv \{(\xi_1, \xi_2, \cdots, \xi_n) : \xi_i \geq 0, i = 1, 2, \cdots, n\} \backslash \{(0, 0, \cdots, 0)\}$$

is an n-**parameter semigroup** *of linear bounded operators on* X *if*

(1) For $\xi, \eta \in R_n^+$, $x \in X$,

$$T(\xi + \eta)x = T(\xi)T(\eta)x.$$

(2) $T(\xi)x$ *is a strongly measurable function of* ξ *in* R_n^+ *for each* $x \in X$.

The relation between one-parameter semigroup and n-parameter semigroup is given by the following

Theorem 2.2.10. *Suppose that* $\|T(\eta u_k)\| \leq M$ *for* $0 < \eta < 1$, $(k = 1, 2, \cdots, n)$, *and the range of* $T(\eta u_k)$ *is dense in* X *for* $\eta > 0$. *Then*

(a) $T(\xi)$ *is strongly continuous in* R_n^+ *and* $T(\xi) \to I$ *strongly as* $\xi \to 0$;

(b) $T(\xi)$ *is the direct product of* n *continuous one-parameter semigroups* $T_k(\xi_k) = T(\xi u_k)$, *i.e.,*

$$T(\xi) = \prod_{k=1}^{n} T_k(\xi_k);$$

(c) $T_j(\xi_j)$ *commutes with* $T_k(\xi_k)$ *and* $\lim_{\eta \to 0} T_k(\eta)x = x$;

(d) The infinitesimal generator A_0 *of* $T(\xi)$, *defined by*

$$A_0 x = \lim_{\eta \to 0} \frac{1}{\eta}(T(\eta\xi) - I)x, \ x \in X,$$

has the form $A_0 = \sum_1^n \xi_k A_k$, $\xi_k \geq 0$, *where* A_k *is infinitesimal generator of* $T_k(\xi_k)$.

For the proof of theorem 2.2.10 see Hille and Phillips [48, p.335].

To conclude this section, we state Trotter-Kato approximation theorem that will be used in Chapter 3 and 4.

Theorem 2.2.11 (Trotter-Kato). *Let* $\{A_n\}$ *be the infinitesimal generators of* C_0*-semigroups* $\{T_n(t)\}$ *such that* $\|T_n(t)\| \leq Me^{\omega t}$, $\forall n$. *Assume*

(1) As $n \to \infty$, $A_n x \to Ax$ *for every* $x \in D$ *where* D *is a dense subset of* X.

(2) There exists a λ_0 *with Re* $\lambda_0 > \omega$ *for which* $(\lambda_0 I - A)D$ *is dense in* X.

Then the closure $\overline{A}$ of A is the infinitesimal generator of a C_0-semigroup $T(t)$, such that

$$\lim_{n\to\infty} T_n(t)x = T(t)x, \quad for \;\; all \;\; t \geq 0, \; x \in X,$$

and the above limit is uniform in t on bounded intervals (Ahmed [1, p.139-140]).

2.3 Some Notations and Results in Banach Spaces

Let X be an infinite dimensional Banach space with a basis $\{e_i\}, i = 1, 2, \cdots$. Without loss of generality we may assume that $\|e_i\| = 1, \; i = 1, 2, \cdots$. (Otherwise replace e_i by $e_i/\|e_i\|$). By Hahn-Banach theorem, there exists a sequence $\{e_i^*\} \subset X^*$ such that $(e_i^*, e_j) = \delta_{ij}, \; (i, j \geq 1)$. By definition the set $\{e_i, e_i^*\}$ constitutes a regular biorthogonal system.

Let $C_b(X)$ denote the set of all bounded and uniformly continuous functions $f : X \to R$. Let $C_b^k(X)$ denote the class of all bounded and uniformly continuous functions on X whose Fréchet differentials up to order k exist and belong to $C_b(X)$. For $\phi \in C_b^k(X)$ and $0 \leq m \leq k$, let $D^m\phi(x; h_1, h_2, \cdots, h_m)$ denote the m-th Fréchet differential of ϕ at x in the direction $(h_1, h_2, \cdots, h_m) \in X^m \equiv X \times X \times \cdots \times X$. For fixed $x \in X, D^m\phi(x; \cdot, \cdots, \cdot)$ is an m-linear form mapping X^m to R. Define

$$\|\phi\|_0 = \sup\{|\phi(x)| : x \in X\},$$

$$\|D^m\phi\|_m \equiv \sup_{x\in X}\{\sup\{|D^m\phi(x;h_1,h_2,\cdots,h_m)| : \|h_i\|_X = 1, i = 1,2,\cdots,m\}\}.$$

For $\phi \in C_b^k(X)$, define

$$\|\phi\|_{C_b^k(X)} = \sum_{m=0}^{k} \|D^m\phi\|_m.$$

It is well known that $C_b(X)$ and $C_b^k(X)$ are Banach spaces with the norm defined above [63].

We shall also use the symbol ϕ_x or $D\phi$ for $D^1\phi$. For each $k \in N$, define $D_k : C_b(X) \to C_b(X)$ by

$$D_k\phi(\cdot) \equiv \lim_{h\to 0}\frac{1}{h}(\phi(\cdot + he_k) - \phi(\cdot)) = \phi_x(\cdot, e_k) = (\phi_x(\cdot), e_k)_{X^*,X}.$$

The last equality holds whenever ϕ is Fréchet differentiable. The domain of D_k is

$\mathcal{D}(D_k) \equiv \{\phi \in C_b(X) : D_k\phi = (\phi_x, e_k)$ exists, uniformly continuous and bounded$\}$.

It can be easily verified that D_k is the infinitesimal generator of a (commutative) C_0-semigroup $R_k(t)$ given by

$$(R_k(t)\phi)(x) \equiv \phi(x + te_k).$$

(In fact $\{R_k(t), t \in R\}$ is a C_0-group.) By Example 2.2.5 (it can easily be extended to Banach space), $A_k \equiv \frac{1}{2}D_k^2$ generates a C_0-semigroup of contractions $P_k(t)$ given by

$$(P_k(t)\phi)(x) \equiv e^{A_kt}\phi(x) = \frac{1}{\sqrt{2\pi t}}\int_R e^{-\xi^2/2t}\phi(x - \xi e_k)d\xi.$$

Now we give an important property of the pair (X, X^*). By definition we say a basis $\{e_i\}$ of X is **shrinking** if and only if $\lim_{n\to\infty} |f|_n = 0$ for each $f \in X^*$, where $|f|_n$ is the norm of f restricted to the linear span of $\{e_i, i > n\}$. We have the following

Lemma 2.3.1. *Suppose X is a reflexive Banach space and $\{e_i, e_i^*\}$ is a regular biorthogonal system, then $\{e_i^*\}$ is also a basis for X^*.*

Proof: It is easy to see that $\{e_i^*\}$ is a w^*-basis of X^*, that is, for every $f \in X^*$, we have

$$f(x) = \sum_{i=1}^{\infty} f(e_i)(e_i^*, x), \quad \forall x \in X.$$

Hence the conclusion follows from the following arguments (Diestel [38,p.52-53]):

(1) For a reflexive Banach space X, every basis $\{e_i\}$ is shrinking;

(2) If $\{e_i^*\}$ is a w^*-basis of X^*, then $\{e_i^*\}$ is a basis of X^* if and only if $\{e_i\}$ is shrinking.

This completes the proof of the lemma. ∎

CHAPTER 3

Optimal Control for Stochastic Systems Driven by a Brownian Motion

3.1 Introduction

From Chapter 1 we know that optimal control problems for stochastic systems give rise to Hamilton-Jacobi-Bellman equations. In the case of finite dimensional stochastic systems, these equations are partial differential equations in R^n or a suitable subset of R^n, and have been extensively studied by many authors (see Lions [58] and references therein). In the case of infinite dimensional stochastic systems, we have similar Hamilton-Jacobi-Bellman equations which are partial differential equations on Banach spaces or suitable subsets of them, as given below:

$$(P)\quad \begin{cases} \phi_t(t,x) &= \frac{1}{2}Tr(S\phi_{xx}(t,x)) + (Bx, \phi_x(t,x)) \\ & \quad + F(t,x,\phi(t,x),\phi_x(t,x)) \\ \phi(0,x) &= \phi_0(x),\ x \in \mathcal{D}(B),\ (t,x) \in I \times \mathcal{D}(B), \end{cases}$$

where $I = [0,T], T > 0$. $\mathcal{D}(B) \equiv$ domain of B.

In problem (P) we assume

(i) B is a linear possibly unbounded operator in X;

(ii) $\phi_0 : X \mapsto R$; $F : I \times X \times R \times X^* \mapsto R$, are given functions;

(iii) $S \in \mathcal{L}_n^+(X^*, X)$, the space of positive nuclear operators from X^* to X, having the standard representation:

$$Sx^* = \sum_{i=1}^{\infty} \lambda_i (x^*, e_i) e_i, \ \ \forall x^* \in X^*, \ (e_i \in X, \ \lambda_i > 0, \ i = 1, 2, \cdots, \ \sum_{i=1}^{\infty} \lambda_i < \infty).$$

As we stated in chapter 1, problem (P) may arise from the following stochastic control problem

$$\begin{aligned} d\xi &= B\xi dt + f(\xi, u)dt + dW_t, \\ J(u) &= E\{\int_0^\tau l(\xi, u)dt + \phi_0(\xi(\tau))\} \to min. \end{aligned}$$

where W_t is an X-valued Brownian motion in a probability space (Ω, Σ, μ) with covariance $cov(W_t) = tS$, and

$$F(t, x, \psi) = \inf\{l(x, u) + (f(x, u), \psi), u \in U\}, \ \psi \in X^*,$$

with U being a closed subset of a Polish space.

The question of existence of solutions of equation (P) and their regularity properties have been studied by Barbu and Da Prato [10, 11], Da Prato [26-32], Cannarsa and Da Prato [18, 19], Havârneanu [47], Lions [59], and others, all in the case when X is a separable Hilbert space. When X is a Banach space fewer efforts have been made before (Ahmed [4]). In this chapter we study problem (P) in a reflexive Banach space. Our results may be regarded as an extension of their work in two aspects: first, from Hilbert space to Banach space; second, from semilinear systems with special nonlinear terms like $F(x, \phi_x) = \frac{1}{2}|\phi_x|^2 + g(x)$ to more general nonlinear system.

In section 3.2 we give some results about the semigroups generated by the sum of a sequence of generators of C_0-semigroups, which will play a central role in our constructions of basic semigroups for problem (P). In section 3.3 we consider the linear case of problem (P), and in section 3.4 we present some results for the nonlinear case.

3.2 Semigroups Generated by Sums of Infinitesimal Generators of C_0-semigroups

In this section we present some general results about the semigroups generated by the sum of a sequence of generators of C_0-semigroups. We will need the following lemma (Da Prato [28]):

Lemma 3.2.1. *Let* $G(0,1,Y)$ *denote the class of infinitesimal generators of* C_0*-semigroups of contractions in a Banach space* Y*. Suppose* $\{C_n\} \in G(0,1,Y)$ *and that the semigroups they generate are mutually commutative. Let*

$$\mathcal{D}(C_0) = \{\phi \in \bigcap_{i=1}^{\infty} \mathcal{D}(C_i) : \sum_{i=1}^{\infty} \|C_i\phi\|_0 < \infty\}, \ C_0\phi = \sum_{i=1}^{\infty} C_i\phi, \ \phi \in \mathcal{D}(C_0).$$

If $\mathcal{D}(C_0)$ is dense in Y, then for any $\phi \in Y$, the limit

$$\lim_{n\to\infty} \prod_{i=1}^{n} e^{tC_i}\phi \equiv T(t)\phi$$

exists uniformly in t on any compact subinterval of $[0,\infty)$; the infinitesimal generator C of $T(t)$ is $\overline{C_0}$, the closure of C_0, and $C \in G(0,1,Y)$.

A modification of the Lemma 3.2.1 is the following

Lemma 3.2.2. *Let C_n satisfy the assumptions of lemma 3.2.1. Define*

$$\mathcal{D}(C_0) = \{\phi \in \bigcap_{i=1}^{\infty} \mathcal{D}(C_i) : \sup_n \|\sum_{i=1}^{n} C_i\phi\|_0 < \infty\},$$

$$C_0\phi = \sum_{i=1}^{\infty} C_i\phi, \quad \phi \in \mathcal{D}(C_0).$$

Suppose $\mathcal{D}(C_0)$ is dense in Y and there exists λ_0 with $Re(\lambda_0) > 0$ for which $(\lambda_0 I - C_0)\mathcal{D}(C_0)$ is dense in Y. Then for any $\phi \in Y$, the limit

$$T(t)\phi \equiv \lim_{n\to\infty} \prod_{i=1}^{n} e^{tC_i}\phi, \ t \geq 0,$$

exists uniformly in t on any compact subinterval of $[0,\infty)$. The infinitesimal generator C of $T(t)$ is $\overline{C_0}$, and $C \in G(0,1,Y)$.

Proof: By assumptions for any $i, j (i \neq j)$, the semigroups generated by C_i and C_j commute, hence $T(t_1, t_2, \cdots, t_n) = \prod_{i=1}^{n} e^{t_i C_i}$ defines an n-parameter semigroup. By Theorem 2.2.9, $\prod_{i=1}^{n} e^{t_i C_i}$ is generated by $\sum_{i=1}^{n} C_i$. Taking $t_1 = t_2 = \cdots = t_n = t$ we see that $\prod_{i=1}^{n} e^{tC_i} = e^{F_n t}$, where $F_n = \sum_{i=1}^{n} C_i$. Now since $F_n\phi \to C_0\phi$ for any $\phi \in \mathcal{D}(C_0)$, the conclusions follow by Trotter-Kato approximation Theorem (Theorem 2.2.11). ∎

3.3 The linear case

As the first step we consider the problem

$$(P1) \quad \begin{cases} \phi_t(t,x) &= \frac{1}{2}Tr(S\phi_{xx}(t,x)) \\ \phi(0,x) &= \phi_0(x), \quad \phi_0 \in C_b(X). \end{cases}$$

Define operator $\mathcal{A}_0$ by

$$\mathcal{A}_0\phi \equiv \frac{1}{2}Tr(S\phi_{xx}), \quad \phi \in \mathcal{D}(\mathcal{A}_0) \equiv C_b^2(X).$$

We remark that $C_b^2(X)$ is not dense in $C_b(X)$ (see Nemirovskii and Semenov [62]), hence recently Da Prato [25, 26] have introduced the following spaces:

$$\overline{C}_b^k(H) \equiv \overline{C_b^\infty(H)}^{C_b^k(H)}, \; C_b^\infty(H) = \bigcap_{k=0}^{\infty} C_b^k(H), \; C_b^0(H) \equiv C_b(H),$$

where H is a Hilbert space, and it is easy to see that $\overline{C}_b^k(H), \; k = 1, 2, \cdots,$ are dense in $\overline{C}_b(H) \equiv \overline{C_b^\infty(H)}^{C_b(H)}$.

We introduce a subspace of $C_b(X)$ as follows. Define $Z(X) \equiv \overline{C_b^2(X)}^{C_b(X)}$. As a closed subspace of $C_b(X), Z(X)$ is also a Banach space and $\mathcal{A}_0$ is densely defined in $Z(X)$.

Now we start to construct a C_0-semigroup for problem (P1).

Theorem 3.3.1. *Suppose X is a reflexive Banach space, then $\mathcal{A}_0$ is closable in $Z(X)$, and its closure $\mathcal{A}$ generates a C_0-semigroup of contractions*

$T(t)$, given by

$$T(t)\phi = \lim_{n\to\infty} \prod_{i=1}^{n} e^{\lambda_i A_i t}\phi, \ \phi \in Z(X). \tag{3.1}$$

where $A_i = \frac{1}{2}D_i^2$ (p.34). Thus the solution of the problem (P1) can be written as $\phi(t,\cdot) = T(t)\phi_0(\cdot)$.

Proof. By Definition

$$\begin{aligned} Tr(S\phi_{xx}(\cdot)) &= \sum_{k=1}^{\infty}(S\phi_{xx}(\cdot)e_k, e_k^*)_{X,X^*} \\ &= \sum_{k=1}^{\infty}\lambda_k(\phi_{xx}(\cdot)e_k, e_k) \\ &= \sum_{k=1}^{\infty}\lambda_k D_k^2\phi(\cdot). \end{aligned}$$

Let $Y \equiv Z(X)$. Since $\mathcal{D}(\mathcal{A}_0)$ is dense in $Z(X)$, by lemma 3.2.1 we have the conclusions. ∎

The regularity properties of $T(t)$ is obtained in the following:

Theorem 3.3.2. *Suppose $t > 0$ and $\phi \in Z(X)$, then $T(t)\phi \in \mathcal{D}(D_k)$, $(k = 1, 2, \cdots)$, and*

$$\|D_k T(t)\phi\|_0 \le \|D_k e^{\lambda_k A_k t}\phi\|_0 \le \frac{C}{\sqrt{\lambda_k t}}\|\phi\|_0. \tag{3.2}$$

Proof: For each $k \in N$, define a semigroup $T_k(t)$ by

$$T_k(t)\phi = \prod_{i\neq k} e^{\lambda_i A_i t}\phi, \ \phi \in C_b(X). \tag{3.3}$$

Clearly $T(t) = T_k(t)e^{\lambda_k A_k t}$. Note that

$$\begin{aligned}
D_k & \left(\int_R e^{-\xi^2/(2t\lambda_k)}\phi(x-\xi e_k)d\xi\right) \\
= & \lim_{h\to 0}\int_R e^{-\xi^2/(2t\lambda_k)}\frac{1}{h}(\phi(x-(\xi-h)e_k)-\phi(x-\xi e_k))d\xi \\
= & \lim_{h\to 0}\frac{1}{h}\{\int_R e^{-\xi^2/(2t\lambda_k)}\phi(x-(\xi-h)e_k)d\xi - \int_R e^{-\xi^2/(2t\lambda_k)}\phi(x-\xi e_k)d\xi\} \\
= & \lim_{h\to 0}\frac{1}{h}\{\int_R e^{-(h+\xi)^2/(2t\lambda_k)}\phi(x-\xi e_k)d\xi - \int_R e^{-\xi^2/(2t\lambda_k)}\phi(x-\xi e_k)d\xi\} \\
= & \lim_{h\to 0}\frac{1}{h}\{\int_R (e^{-(h+\xi)^2/(2t\lambda_k)} - e^{-\xi^2/(2t\lambda_k)})\phi(x-\xi e_k)d\xi\} \\
= & \int_R e^{-\xi^2/(2t\lambda_k)}(-\frac{\xi}{\lambda_k t})\phi(x-\xi e_k)d\xi.
\end{aligned}$$

Hence for $t > 0$,we have

$$\begin{aligned}
\|D_k T(t)\phi\|_0 &= \|T_k(t)D_k e^{\lambda_k A_k t}\phi\|_0 \\
&\le \|D_k e^{\lambda_k A_k t}\phi\|_0 \\
&= \|D_k(\frac{1}{\sqrt{2\pi t\lambda_k}}\int_R e^{-\xi^2/(2t\lambda_k)}\phi(x-\xi e_k)d\xi)\|_0 \\
&\le \frac{1}{\sqrt{2\pi t\lambda_k}}\|\phi\|_0\int_R e^{-\xi^2/(2t\lambda_k)}\frac{|\xi|}{\lambda_k t}d\xi \\
&= \frac{2}{\sqrt{2\pi t\lambda_k}}\|\phi\|_0.
\end{aligned}$$

This gives the inequality (3.2). ∎

Now we consider the following problem:

$$(P2)\qquad \begin{cases} \phi_t(t,x) = \frac{1}{2}Tr(S\phi_{xx}(t,x)) + f(t,x) \\ \phi(0,x) = \phi_0(x). \end{cases}$$

The mild solution of (P2) is given by

$$\phi(t,\cdot) = T(t)\phi_0(\cdot) + \int_0^t T(t-s)f(s,\cdot)ds.$$

We study the regularity properties of the mild solution. First, we generalize the concept of **derivatives in the direction S**, which was introduced by Da Prato [29, 31]. Where S satisfies property (iii) on page 37.

Definition 3.3.3. An element $\phi \in C_b(X)$ is said to be differentiable in the direction S if and only if for any $x \in X$ and $y \in X^*$, $\lim_{h\to 0^+} \frac{1}{h}(\phi(x+hSy) - \phi(x)) \equiv \mathcal{L}_\phi(x,y)$ exists.

Note that if $\phi \in C_b^1(X)$, then $\mathcal{L}_\phi(x,y) = (\phi_x(x), Sy) = (S^*\phi_x(x), y)$ for all $y \in X^*$, where S^* is the dual of the operator S. Now we define an operator D_S as follows

Definition 3.3.4. $C_S^1(X) = \{\phi \in C_b(X) : \mathcal{L}_\phi(x,y) = (D_S\phi(x), y)$ for some $D_S\phi(x) \in X^{**}$, and $\|D_S\phi\|_0 \equiv \sup_{x\in X}\{\|D_S\phi(x)\|_{X^{**}} < \infty\}$.

$C_S^1(X)$ is a Banach space with the norm $\|\phi\|_{C_S^1} \equiv \|\phi\|_0 + \|D_S\phi\|_0$.

In the following we assume that X is a reflexive Banach space. It is easy to see that $C_b^1(X) \subset C_S^1(X) \subset \mathcal{D}(D_k)$ for all $k \in N$. On the other hand we have

Theorem 3.3.5. *Suppose* $\phi \in \bigcap_{i=1}^\infty \mathcal{D}(D_i)$ *and* $\sum_{i=1}^\infty \lambda_i \|D_i\phi\|_0 < \infty$. *Then*

$\phi \in C_S^1(X)$ and

$$D_S\phi(\cdot) = \sum_{i=1}^{\infty} \lambda_i(\phi_x(\cdot), e_i)e_i = \sum_{i=1}^{\infty} \lambda_i D_i\phi(\cdot)e_i.$$

Proof. Suppose $\phi \in \bigcap_{i=1}^{\infty} \mathcal{D}(D_i)$. For any finite linear combination $\sum_{i=1}^{n} x_i e_i$, we have

$$\lim_{h\to 0^+} \frac{1}{h}(\phi(x + h\sum_{i=1}^{n} x_i e_i) - \phi(x)) = \sum_{i=1}^{n} x_i(\phi_x(x), e_i) = \sum_{i=1}^{n} x_i(D_i\phi)(x).$$

For any $y \in X^*$, it follows from Lemma 2.3.1 that there exists a sequence of scalars $\{a_i\}$ with $\sup_i\{|a_i|\} \equiv a' < \infty$, such that $y = \sum_{i=1}^{\infty} a_i e_i^* \in X^*$. Define $y_n = \sum_{i=1}^{n} a_i e_i^*$, then

$$\begin{aligned} \mathcal{L}_\phi(x, y_n) &= \lim_{h\to 0^+} \frac{1}{h}\{\phi(x + hSy_n) - \phi(x)\} \\ &= \lim_{h\to 0^+} \frac{1}{h}\{\phi(x + h\sum_{i=1}^{n} \lambda_i a_i e_i) - \phi(x)\} \\ &= \sum_{i=1}^{n} \lambda_i a_i (D_i\phi)(x). \end{aligned}$$

Let $g_n(x) \equiv \mathcal{L}_\phi(x, y_n)$. Since $\sup\{|g_n(x)| : x \in X\} \leq a' \sum_{i=1}^{n} \lambda_i \|D_i\phi\|_0$, it follows from the assumption of theorem that $\{g_n\}$ is a Cauchy sequence in $C_b(X)$. Hence there exists a $g \in C_b(X)$, such that

$$g(x) \equiv \lim_n g_n(x) = \lim_n \mathcal{L}_\phi(x, y_n) = \mathcal{L}_\phi(x, y) = \sum_{i=1}^{\infty} a_i\lambda_i(D_i\phi)(x).$$

This implies that ϕ is Gâteaux differentiable in the direction Sy, and the Gâteaux differential is continuous, hence coincides with the Fréchet differential (see Hille and Philips [48, p.109-113]) and $y \mapsto \mathcal{L}_\phi(x, y)$ is linear. It follows that

$$\mathcal{L}_\phi(x, y) = \sum_{i=1}^{\infty} a_i\lambda_i(D_i\phi)(x) = (\sum_{i=1}^{\infty} \lambda_i(D_i\phi)(x)e_i, y) = (D_S\phi(x), y)$$

for some $D_S\phi(x) \in X^{**} = X$. Thus we have

$$D_S\phi(x) = \sum_{i=1}^{\infty} \lambda_i (D_i\phi)(x) e_i.$$

This completes the proof of the theorem. ∎

Theorem 3.3.6. *Suppose* $f \in C([0,T], Z(X))$ *and* $\sum_{i=1}^{\infty} \sqrt{\lambda_i} < \infty$, *then*

$$v(t) = \int_0^t T(t-s)f(s)ds \in C([0,T], C_S^1(X))$$

and there exists a constant $C > 0$ *such that* $\|D_S v(t)\|_0 \leq C\sqrt{t}\|f\|_{C([0,T],Z(X))}$. *Where* $\|f\|_{C([0,T],Z(X))} = \sup\{\|f\|_0 : t \in [0,T]\}$.

Proof. Since $T(t) = T_k(t)e^{\lambda_k A_k t}$, from the proof of Theorem 3.3.2 we have

$$\begin{aligned} \|D_k T(t-s)f(s)\|_0 &\leq \|T_k(t-s)\| \, \|D_k e^{\lambda_k A_k (t-s)} f(s)\|_0 \\ &\leq \|D_k e^{\lambda_k A_k (t-s)} f(s)\|_0 \\ &\leq \frac{C}{\sqrt{\lambda_k (t-s)}} \|f\|_{C([0,T],Z(X))}. \end{aligned}$$

Hence

$$\|D_k v(t)\|_0 \leq 2C\sqrt{\frac{t}{\lambda_k}} \|f\|_{C([0,T],Z(X))}, \quad k \in N.$$

Thus $v(t) \in \bigcap_{i=1}^{\infty} \mathcal{D}(D_i)$, and

$$\sum_{i=1}^{\infty} \lambda_i \|D_i v(t)\|_0 \leq 2C\sqrt{t}\|f\|_{C([0,T],Z(X))} (\sum_{i=1}^{\infty} \sqrt{\lambda_i}) < \infty.$$

Hence it follows from Theorem 3.3.5 that $v(t) \in C_S^1(X)$ for each $t > 0$, and

$$D_S v(t) = \sum_{i=1}^{\infty} \lambda_i D_i v(t) e_i.$$

Thus

$$\begin{aligned} \|D_S v(t)\|_0 &= \|\sum_{i=1}^{\infty} \lambda_i D_i v(t) e_i\|_0 \\ &\le \sum_{i=1}^{\infty} \lambda_i \|D_i v(t)\|_0 \\ &\le 2C\sqrt{t}\|f\|_{C([0,T],Z(X))}(\sum_{i=1}^{\infty} \sqrt{\lambda_i}). \quad \blacksquare \end{aligned}$$

Corollary 3.3.7. *Suppose that* $\phi_0 \in C_S^1(X), f \in C([0,T], Z(X))$, *and* $\sum_{i=1}^{\infty} \sqrt{\lambda_i} < \infty$, *then the mild solution of (P2) belongs to* $C([0,T], C_S^1(X))$.

Now we add a first order term to Problem (P2). Consider the following problem

$$(P3) \quad \begin{cases} \phi_t(t,x) = \frac{1}{2} Tr(S\phi_{xx}(t,x)) + (Bx, \phi_x(t,x)) + f(t,x) \\ \phi(0,x) = \phi_0(x), \quad (t,x) \in I \times \mathcal{D}(B), \end{cases}$$

with the assumptions that B generates a C_0-semigroup in X and satisfies

$$Be_i = -\mu_i e_i, \quad (\mu_i > 0, \ i = 1, 2, \cdots).$$

Theorem 3.3.8. *For each* $i \in N$,

(i) $\{S_i(t), t \ge 0\}$, *given by*

$$(S_i(t)\phi)(x) \equiv \phi(x - (1 - e^{-\mu_i t})(x, e_i^*)e_i), \quad \phi \in C_b(X), \tag{3.4}$$

is a C_0*-semigroup of contractions in* $C_b(X)$, *with infinitesimal generator given by* $B_i\phi(x) = -\mu_i(x, e_i^*)D_i\phi(x)$;

(ii) $\{\Gamma_i(t), t \geq 0\}$, *given by*

$$\Gamma_i(t) \equiv e^{\lambda_i \alpha_i(t) A_i} e^{tB_i}, \ \alpha_i(t) = \frac{e^{2\mu_i t} - 1}{2\mu_i}, \ t > 0, \tag{3.5}$$

is a C_0*-semigroup of contractions in* $C_b(X)$*, and its infinitesimal generator is* $G_i \equiv \overline{\lambda_i A_i + B_i}$.

Proof. Since for $t > s > 0$,

$$\begin{aligned}
& S_i(t)S_i(s)\phi(x) \\
= \ & S_i(t)\phi(x - (1 - e^{-\mu_i s})(x, e_i^*)e_i) \\
= \ & \phi(x - (1 - e^{-\mu_i s})(x, e_i^*)e_i - (1 - e^{-\mu_i t})(x - (1 - e^{-\mu_i s})(x, e_i^*)e_i, e_i^*)e_i) \\
= \ & \phi(x - (1 - e^{-\mu_i s})(x, e_i^*)e_i - (1 - e^{-\mu_i t})((x, e_i^*)e_i - (1 - e^{-\mu_i s})(x, e_i^*)e_i) \\
= \ & \phi(x - ((1 - e^{-\mu_i s}) + (1 - e^{-\mu_i t}) - (1 - e^{-\mu_i s})(1 - e^{-\mu_i t}))(x, e_i^*)e_i) \\
= \ & \phi(x - (1 - e^{-\mu_i (s+t)})(x, e_i^*)e_i) \\
= \ & S_i(t+s)\phi(x)
\end{aligned}$$

It is clear that $S_i(t)$ defines a C_0-semigroup of contractions in $C_b(X)$. Its infinitesimal generator B_i is given by

$$\begin{aligned}
(B_i\phi)(x) &= \lim_{h\to 0} \frac{\phi(x - (1 - e^{-\mu_i h})(x, e_i^*)e_i) - \phi(x)}{h} \\
&= \lim_{h\to 0} \frac{\phi(x + (e^{-\mu_i h} - 1)(x, e_i^*)e_i) - \phi(x)}{(e^{-\mu_i h} - 1)(x, e_i^*)} \left(\frac{e^{-\mu_i h} - 1}{h}\right)(x, e_i^*) \\
&= -\mu_i(x, e_i^*)(\phi_x(x), e_i) = -\mu_i(x, e_i^*)D_i\phi(x), \ \phi \in \mathcal{D}(D_i).
\end{aligned}$$

Thus (i) has been established. Next we prove (ii). We claim that

$$e^{tB_i}e^{sA_i} = e^{se^{2\mu_i t}A_i}e^{tB_i}, \quad 0 \le t, s < \infty. \tag{3.6}$$

In fact

$$\begin{aligned}
&(e^{tB_i}e^{sA_i}\phi)(x) \\
= \;& e^{tB_i}\Big(\frac{1}{\sqrt{2\pi s}}\int_R e^{-\xi^2/2s}\phi(x-\xi e_i)d\xi\Big) \\
= \;& \frac{1}{\sqrt{2\pi s}}\int_R e^{-\xi^2/2s}\phi(x-(1-e^{-\mu_i t})(x,e_i^*)e_i-\xi e_i)d\xi \\
= \;& \frac{1}{\sqrt{2\pi s}}\int_R e^{-\xi^2/2s}\phi(x-(1-e^{-\mu_i t})(x-e^{\mu_i t}\xi e_i,e_i^*)e_i-e^{\mu_i t}\xi e_i)d\xi \\
& (let \;\; \eta \equiv e^{\mu_i t}\xi) \\
= \;& \frac{1}{\sqrt{2\pi s}}\int_R e^{-\eta^2 e^{-2\mu_i t}/2s}\phi(x-(x-\eta e_i,e_i^*)(1-e^{-\mu_i t})e_i-\eta e_i)e^{-\mu_i t}d\eta \\
= \;& \frac{1}{\sqrt{2\pi s e^{2\mu_i t}}}\int_R e^{-\eta^2/(2se^{2\mu_i t})}\phi(x-\eta e_i-(x-\eta e_i,e_i^*)(1-e^{-\mu_i t})e_i)d\eta \\
= \;& e^{se^{2\mu_i t}A_i}\phi(x-(x,e_i^*)(1-e^{-\mu_i t})e_i) \\
= \;& (e^{se^{2\mu_i t}A_i}e^{tB_i}\phi)(x).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\Gamma_i(t)\Gamma_i(s) &= e^{\lambda_i\alpha_i(t)A_i}e^{tB_i}e^{\lambda_i\alpha_i(s)A_i}e^{sB_i} \\
&= e^{\lambda_i\alpha_i(t)A_i}e^{\lambda_i\alpha_i(s)e^{2\mu_i t}A_i}e^{tB_i}e^{sB_i} \\
&= e^{\lambda_i\frac{e^{2\mu_i t}-1+e^{2\mu_i s}e^{2\mu_i t}-e^{2\mu_i t}}{2\mu_i}A_i}e^{(t+s)B_i} \\
&= e^{\lambda_i\frac{e^{2\mu_i(t+s)}-1}{2\mu_i}A_i}e^{(t+s)B_i} \\
&= \Gamma_i(t+s).
\end{aligned}$$

From this one can verify easily that $\{\Gamma_i(t), t \ge 0\}$ is a C_0-semigroup of contractions on $C_b(X)$. We compute the infinitesimal generator. For

$\phi \in C_b^2(X)$ we have

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{\Gamma_i(t)\phi-\phi}{t} &= \lim_{t\downarrow 0} \frac{e^{\lambda_i \frac{e^{2\mu_i t}-1}{2\mu_i} A_i} e^{tB_i}\phi-\phi}{t} \\
&= \lim_{t\downarrow 0} \{\frac{e^{\lambda_i \frac{e^{2\mu_i t}-1}{2\mu_i} A_i} e^{tB_i}\phi - e^{tB_i}\phi}{t} + \frac{e^{tB_i}\phi-\phi}{t}\} \\
&= \lim_{t\downarrow 0} \{e^{tB_i}(\frac{e^{\lambda_i \frac{e^{2\mu_i t}-1}{2\mu_i} e^{-2\mu_i t} A_i}\phi-\phi}{t})\} + B_i\phi \\
&= \lim_{t\downarrow 0}\ e^{tB_i}\{\frac{e^{\lambda_i \frac{1-e^{-2\mu_i t}}{2\mu_i} A_i}\phi-\phi}{1-e^{-2\mu_i t}} \cdot \frac{1-e^{-2\mu_i t}}{t}\} + B_i\phi \\
&= (\frac{\lambda_i}{2\mu_i} A_i(2\mu_i) + B_i)\phi \\
&= (\lambda_i A_i - \mu_i(x, e_i^*)D_i)\phi.
\end{aligned}$$

Hence (ii) holds and $G_i = \overline{\lambda_i A_i + B_i}$. ∎

Now we define a linear operator $\mathcal{G}_0$ by:

$$\mathcal{D}(\mathcal{G}_0) \equiv \{\phi \in C_b^2(X) : \| \sum_{i=1}^{\infty} \lambda_i A_i \phi - \sum_{i=1}^{\infty} \mu_i(x, e_i^*) D_i \phi \|_0 < \infty\},$$

$$\mathcal{G}_0 \phi \equiv \sum_{i=1}^{\infty} G_i \phi, \quad \phi \in \mathcal{D}(\mathcal{G}_0).$$

Since B is closed, it follows that

$$(Bx, \phi_x) = (-\sum_{i=1}^{\infty} (x, e_i^*) \mu_i e_i, \phi_x) = -\sum_{i=1}^{\infty} \mu_i(x, e_i^*) D_i \phi.$$

Hence

$$\mathcal{G}_0 \phi = \frac{1}{2} Tr(S\phi_{xx}) + (Bx, \phi_x), \quad \phi \in \mathcal{D}(\mathcal{G}_0). \tag{3.7}$$

Remark 3.3.9. It is clear that $\| \sum_{i=1}^{\infty} \lambda_i A_i \phi \|_0 < \infty$ for all $\phi \in C_b^2(X)$, hence

$$\mathcal{D}(\mathcal{G}_0) \supset \{\phi \in C_b^2(X) : \| \sum_{i=1}^{\infty} \mu_i (x, e_i^*) D_i \phi \|_0 < \infty \}.$$

For any $j \in N$, define $\phi_j(x) = e^{-x_j^2}$, where $x_j = (e_j^*, x)$. Then $D_i \phi_j(x) = -2x_j e^{-x_j^2} \delta_{ij}$, hence $\phi_j(x) \in \mathcal{D}(\mathcal{G}_0)$, $\forall j \in N$. Therefore $\mathcal{D}(\mathcal{G}_0)$ is non-trivial.

Now we prove that $\mathcal{G}_0$ generates a C_0-semigroup in space

$$E(X) = \overline{\mathcal{D}(\mathcal{G}_0)}^{C_S^1}. \tag{3.8}$$

It is a closed subspace of $C_S^1(X)$, thus also a Banach space and $\mathcal{G}_0$ is densely defined in $E(X)$. By straightforward computations we see that $\Gamma_i \Gamma_j = \Gamma_j \Gamma_i$, $(i \neq j)$. Indeed,

$$\begin{aligned}
e^{sA_i} e^{sB_j} \phi(x) &= e^{sA_i} (\phi(x - (1 - e^{-\mu_j s})(x, e_j^*) e_j) \\
&= \frac{1}{\sqrt{2\pi s}} \int_R e^{-\eta^2/2s} \phi(x - (x - \eta e_i, e_j^*)(1 - e^{-\mu_j t}) e_j - \eta e_i) d\eta \\
&= \frac{1}{\sqrt{2\pi s}} \int_R e^{-\eta^2/2s} \phi(x - (x, e_j^*)(1 - e^{-\mu_j t}) e_j - \eta e_i) d\eta \\
&= e^{sB_j} (\frac{1}{\sqrt{2\pi s}} \int_R e^{-\eta^2/2s} \phi(x - \eta e_i) d\eta) \\
&= e^{sB_j} e^{sA_i} \phi(x).
\end{aligned}$$

Therefore

$$\Gamma_i \Gamma_j = e^{\lambda_i \frac{e^{2\mu_i t} - 1}{2\mu_i} A_i} e^{tB_i} e^{\lambda_j \frac{e^{2\mu_j t} - 1}{2\mu_j} A_j} e^{tB_j} = e^{\lambda_j \frac{e^{2\mu_j t} - 1}{2\mu_j} A_j} e^{tB_j} e^{\lambda_i \frac{e^{2\mu_i t} - 1}{2\mu_i} A_i} e^{tB_i} = \Gamma_j \Gamma_i.$$

Now we want to show that $(\lambda_0 I - \mathcal{G}_0) \mathcal{D}(\mathcal{G}_0)$ is dense in $E(X)$ for some $\lambda_0 > 0$. It suffices to show that $\lambda_0 \in \rho(\mathcal{G}_0)$, the resolvent set of $\mathcal{G}_0$. This can be done in two steps. Write $\mathcal{G}_0 = \mathcal{A}_0 + \mathcal{B}_0$, where $\mathcal{A}_0\phi = \frac{1}{2}Tr(S\phi_{xx})$, and

$$\mathcal{B}_0\phi(\cdot) = (B\cdot, \phi_x), \quad \mathcal{D}(\mathcal{B}_0) = \{\phi \in C_b(X) : (B\cdot, \phi_x) \in C_b(X)\}.$$

First we show that $\rho(\mathcal{B}_0)$ contains $(0, +\infty)$. Define a bounded linear operator $T_B(t)$, $(t > 0)$, in $E(X)$ by

$$T_B(t)\phi(x) = \phi(e^{tB}x), \quad x \in E(X).$$

It is clear that $T_B(t+s)\phi(x) = \phi(e^{tB}e^{sB}x) = T_B(t)T_B(s)\phi(x)$. Since

$$\begin{aligned}\lim_{h\downarrow 0}\frac{T_B(h)\phi(x) - \phi(x)}{h} &= \lim_{h\downarrow 0}\frac{\phi(e^{hB}x) - \phi(x)}{h}\\ &= (\phi_x, Bx)\\ &= \mathcal{B}_0\phi(x), \quad \phi \in \mathcal{D}(\mathcal{B}_0)\end{aligned}$$

and $\mathcal{D}(\mathcal{B}_0) \supset \mathcal{D}(\mathcal{G}_0)$, it follows that $\mathcal{B}_0$ is densely defined in $E(X)$ and therefore $T_B(t)$ is a C_0-semigroup of contractions on $E(X)$. Hence by Hille-Yosida Theorem (Theorem 2.2.4) we have $\rho(\mathcal{B}_0) \supset (0, +\infty)$, and

$$\|(\lambda I - \mathcal{B}_0)^{-1}\| < \frac{1}{\lambda}, \quad \forall\, \lambda > 0,$$

$$((\lambda I - \mathcal{B}_0)^{-1}\phi)(x) = \int_0^\infty e^{-\lambda t}\phi(e^{tB}x)dt. \tag{3.9}$$

Next we prove that $\mathcal{G}_0 = \mathcal{A}_0 + \mathcal{B}_0$ is closable and that the resolvent set of its closure $\overline{\mathcal{G}_0}$, $\rho(\overline{\mathcal{G}_0})$, contains $(0, +\infty)$. We follow the proof of Da Prato [30, p.290]. By a result of Da prato and Grisvard [33], it is sufficient to find another Banach space U continuously imbedded in $\mathcal{D}(\mathcal{A}_0^2)$ such that the restrictions of $\mathcal{A}_0$ and $\mathcal{B}_0$ on U, denoted by $\mathcal{A}_0|_U$ and $\mathcal{B}_0|_U$ respectively, satisfy the following estimates

$$\|(\lambda I - \mathcal{A}_0|_U)^{-1}\|_{\mathcal{L}(U)} \leq \frac{1}{\lambda}, \quad \|(\lambda I - \mathcal{B}_0|_U)^{-1}\|_{\mathcal{L}(U)} \leq \frac{1}{\lambda}.$$

Let $U = C_b^4(X)$, then $U \subset \mathcal{D}(\mathcal{A}_0^2)$ continuously embedded and we have the estimate

$$\|(\lambda I - \mathcal{A}_0|_U)^{-1}\|_{\mathcal{L}(U)} \leq \frac{1}{\lambda}, \quad \lambda > 0$$

by Hille-Yosida Theorem (Theorem 2.2.4). Moreover, since

$$\frac{d^4(\lambda I - \mathcal{B}_0|_U)^{-1}\phi(x)}{d^4x} = \int_0^\infty e^{-\lambda t}\phi^{(4)}(e^{tB}x)(e^{tB}h_1, e^{tB}h_2, e^{tB}h_3, e^{tB}h_4))dt,$$

recalling that $\|e^{tB}\| \leq 1$, we get

$$\|(\lambda I - \mathcal{B}_0|_U)^{-1}\|_{\mathcal{L}(U)} < \frac{1}{\lambda}, \quad \forall\, \lambda > 0.$$

Thus we have that $(\lambda_0 I - \mathcal{G}_0)\mathcal{D}(\mathcal{G}_0)$ is dense in $E(X)$ for $\lambda_0 > 0$. It follows by Lemma 3.2.1 that the closure $\mathcal{G} = \overline{\mathcal{G}_0}$ is an infinitesimal generator of a C_0-semigroup $R(t)$ in $E(X)$. ∎

We give some properties of the semigroup $R(t)$.

Theorem 3.3.10. *Let $\{R(t), t \geq 0\}$ be the C_0-semigroup of contractions generated by the linear operator $\mathcal{G}$. Then for any bounded linear operator $L \in \mathcal{L}(C_b(X))$, we have*

$$R(t)(L\phi) = L(R(t)\phi), \quad \forall \phi \in E(X).$$

Proof: Note that

$$R(t)\phi = \lim_{n\to\infty} \prod_{i=1}^{n} \Gamma_i(t)\phi, \quad \phi \in E(X).$$

it suffices to show that $\Gamma_i(t)$ commutes with L for every $i \in N$. Denote

$$\Gamma_i(t) = e^{\lambda_i \alpha_i(t) A_i} e^{tB_i} \equiv P_i(\alpha_i(t))S_i(t), \quad (\alpha_i(t) = (e^{2\mu_i t} - 1)/2\mu_i, \ t > 0)$$

we need only to show that $P_i L = LP_i$ and $S_i L = LS_i$ for all $i \in N$.

By Definition

$$L(S_i(t)\phi)(x) = L(\phi(x - (1 - e^{-\mu_i t})(x, e_i^*)e_i)) = S_i(t)(L\phi)(x),$$

$$(P_k(s)\phi)(x) \equiv e^{A_k s}\phi(x) = \frac{1}{\sqrt{2\pi s}} \int_R e^{-\xi^2/2s}\phi(x - \xi e_k)d\xi.$$

Hence

$$\begin{aligned}
(P_k(s)L\phi)(x) &= \frac{1}{\sqrt{2\pi s}} \int_R e^{-\xi^2/2s} L\phi(x - \xi e_k)d\xi \\
&= L(\frac{1}{\sqrt{2\pi s}} \int_R e^{-\xi^2/2s}\phi(x - \xi e_k))d\xi \\
&= LP_i(s)\phi.
\end{aligned}$$

Since L is bounded, we have

$$\begin{aligned}
R(t)L(\phi) &= \lim_{n\to\infty} \prod_{i=1}^{n} e^{\lambda_i A_i t} L(\phi) \\
&= \lim_{n\to\infty} L(\prod_{i=1}^{n} e^{\lambda_i A_i t}\phi) \\
&= L(\lim_{n\to\infty} \prod_{i=1}^{n} e^{\lambda_i A_i t}\phi) \\
&= L(R(t)\phi). \ \blacksquare
\end{aligned}$$

3.4 The Non-linear Case

Now we consider the following Cauchy problem

$$(P4)\quad \begin{cases} \phi_t(t,x) &= \frac{1}{2}Tr(S\phi_{xx}(t,x)) + (Bx, \phi_x(t,x)) \\ & + \ F(t,x,\phi(t,x), D_S\phi(t,x)) \\ \phi(0,x) &= \phi_0(x), \quad x \in \mathcal{D}(B), \quad t \in [0,T] \end{cases}$$

To simplify the notations we may also write the problem in the following abstract form

$$(P4)\quad \begin{cases} \phi_t &= \mathcal{G}_0\phi + G(\phi) \\ \phi(0) &= \phi_0, \end{cases}$$

where $\phi \in C(I, C_b(X))$, $G(\phi)(t) \equiv F(t,\cdot,\phi(t,x), D_S\phi(t,x))$, and $\mathcal{G}_0$ is the operator defined by (3.7), (p.49).

The problem (P4) is said to have a **mild solution** ϕ if ϕ satisfies the following integral equation

$$\phi(t) = R(t)\phi_0 + \int_0^t R(t-s)G(\phi)(s)ds, \quad t \in [0,T].$$

where $R(t)$ is the C_0 semigroup generated by $\mathcal{G}$.

We remark that Da Prato [28-30] has studied some special cases of this problem in Hilbert spaces (see section 1.3). Following we present a direct method for more general case. We will solve the problem (P4) in Banach space $\Sigma_S^1(X)$, as defined below:

Definition 3.4.1. Define

$\Sigma(X) = \{\phi \in Z(X) : \lim_n \prod_{i=1}^n \Gamma_i(t)\phi$ exists uniformly in t on every compact subinterval of $[0,\infty)$, and belongs to $Z(X)\}$.
($\Gamma_i(t) \equiv e^{\lambda_i \alpha_i(t) A_i} e^{tB_i}$, see (3.5), p.47).
Let

$$\Sigma_S^1(X) = \Sigma(X) \bigcap C_S^1(X).$$

Define the norm in $\Sigma_S^1(X)$ by

$$\|\phi\|_{\Sigma_S^1} = \|\phi\|_0 + \|D_S\phi\|_0.$$

It is clear that $E(X) \subset \Sigma_S^1(X) \subset C_S^1(X)$. We prove that $\Sigma_S^1(X)$ is a Banach space.

Theorem 3.4.2. *$\Sigma(X)$ is a closed subspace of $Z(X)$, hence $\Sigma_S^1(X)$ (with the graph norm defined above) is a Banach space. Moreover, if $\phi \in \Sigma(X)$ and $\sum_{i=1}^\infty \mu_i^{-\sigma} < \infty$ for some $\sigma > 0$, then $R(t)\phi \in \Sigma_S^1(X)$.*

Proof. Take any sequence $\{\phi_n\} \subset \Sigma(X)$ such that $\phi_n \to \phi$ $(n \to \infty)$ in $Z(X)$. Since $\|R(t)\phi_n - R(t)\phi_m\|_0 \leq \|\phi_n - \phi_m\|_0$, $\{R(t)\phi_n\}$ is a Cauchy sequence in $Z(X)$. Thus there exists $\psi \in Z(X)$ such that $R(t)\phi_n \to \psi$ in $Z(X)$. Now let $R_n(t) = \prod_{i=1}^n e^{\lambda_i \alpha_i(t) A_i} e^{tB_i}$, then for any $m \in N$ we have

$$\begin{aligned} & \|R_n(t)\phi - \psi\|_0 \\ \leq\ & \|R_n(t)\phi - R_n(t)\phi_m\|_0 + \|R_n(t)\phi_m - R(t)\phi_m\|_0 + \|R(t)\phi_m - \psi\|_0 \\ \leq\ & \|\phi - \phi_m\|_0 + \|R_n(t)\phi_m - R(t)\phi_m\|_0 + \|R(t)\phi_m - \psi\|_0. \end{aligned}$$

Hence for any $\epsilon > 0$, we can choose N so large that makes $\|R_n(t)\phi - \psi\|_0 < \epsilon$ for all $m, n \geq N$. Hence $R_n(t)\phi \to \psi$ $(n \to \infty)$ and $R(t)\phi = \psi$. Further it is easy to check that $t \to R(t)\phi$ converges uniformly on every bounded interval $[0,T]$, $T > 0$. Hence $\phi \in \Sigma(X)$ and $\Sigma(X)$ is a closed subspace of $Z(X)$.

We show that $\Sigma^1_S(X)$ is a closed subspace of $C^1_S(X)$. Indeed, suppose $\{\phi_n\} \subset \Sigma^1_S(X)$ and $\phi_n \to \phi$ in $\Sigma^1_S(X)$ ($C^1_S(X)$), that is

$$\|\phi_n - \phi\|_0 + \|D_S\phi_n - D_S\phi\|_0 \to 0, \quad (n \to \infty). \tag{3.10}$$

Hence $\phi \in C^1_S(X)$. Since (3.10) also implies that $\|\phi_n - \phi\|_0 \to 0$, $\{\phi_n\}$ converges to ϕ in $Z(X)$, and ϕ must belong to $\Sigma(X)$ as $\Sigma(X)$ is closed in $Z(X)$. Therefore $\phi \in \Sigma^1_S(X)$ and $\Sigma^1_S(X)$ is a closed subspace of $C^1_S(X)$, and hence a Banach Space.

Now suppose $\phi \in \Sigma(X)$. Since $R(t)(R(t)\phi) = R(2t)\phi \in Z(X), R(t)\phi \in \Sigma(X)$. In the following theorem we will prove that $R(t)\phi \in C^1_S(X)$ for all $\phi \in \Sigma(X)$, hence $R(t)\phi \in \Sigma^1_S(X)$. ∎

Theorem 3.4.3. *Let* $\phi \in \Sigma(X)$. *Suppose that* $\sum_{i=1}^{\infty} \mu_i^{-\sigma} < \infty$ *for some* $\sigma > 0$, *then* $R(t)\phi \in C^1_S(X)$, *and*

$$\|D_S R(t)\phi\|_0 \le \frac{C}{\sqrt{t}}\|\phi\|_0. \tag{3.11}$$

Proof. Let

$$R_k(t)\phi \equiv \prod_{i \neq k} \Gamma_i(t)\phi = \prod_{i \neq k} e^{\lambda_i \alpha_i(t) A_i} e^{tB_i}\phi.$$

Since $R(t)\phi = R_k(t)e^{\lambda_k\alpha_k(t)A_k}e^{tB_k}\phi$, we have

$$\begin{aligned}\|D_kR(t)\phi\|_0 &= \|R_k(t)D_k(e^{\lambda_k\alpha_k(t)A_k}e^{B_kt}\phi)\|_0\\ &\leq \|D_k(e^{\lambda_k\alpha_k(t)A_k}e^{B_kt}\phi)\|_0.\end{aligned}$$

Now by Theorem 3.3.2 we have

$$\|D_ke^{\lambda_ktA_k}\phi\|_0 \leq \frac{C}{\sqrt{\lambda_kt}}\|\phi\|_0,$$

hence

$$\|D_kR(t)\phi\|_0 \leq \frac{C}{\sqrt{\lambda_k\alpha_k(t)}}\|e^{B_kt}\phi\|_0 \leq \frac{C}{\sqrt{\lambda_k\alpha_k(t)}}\|\phi\|_0. \tag{3.12}$$

It is easy to verify that

$$\alpha_k(t) = \frac{e^{2\mu_kt}-1}{2\mu_k} \geq te^{\mu_kt}$$

therefore

$$\|D_kR(t)\phi\|_0 \leq \frac{C}{\sqrt{\lambda_kt}}e^{-\frac{1}{2}\mu_kt}\|\phi\|_0.$$

We show that if $t > 0, \sigma > 0$, $\mu_i > 0$ $(\forall i)$, and $\sum_{i=1}^{\infty}\mu_i^{-\sigma} < \infty$, then $\sum_{i=1}^{\infty}e^{-\mu_it} < \infty$. Since

$$\sum_{i=1}^{\infty}\mu_i^{-\sigma} < \infty, \quad \lim_{i\to\infty}\frac{e^{-\mu_it}}{\mu_i^{-\sigma}} = 0,$$

the conclusion follows from the Comparison Criterion in calculus.

Hence $R(t)\phi \in \bigcap_{i=1}^{\infty}\mathcal{D}(D_i)$ and by Cauchy-Schwartz inequality we have

$$\begin{aligned}\sum_{i=1}^{\infty}\lambda_i\|D_iR(t)\phi\| &\leq \frac{C}{\sqrt{t}}(\sum_{i=1}^{\infty}\sqrt{\lambda_i}e^{-\frac{1}{2}\mu_it})\|\phi\|_0\\ &\leq \frac{C}{\sqrt{t}}(\sum_{i=1}^{\infty}\lambda_i)^{1/2}(\sum_{i=1}^{\infty}e^{-\mu_it})^{1/2}\|\phi\|_0\\ &< \infty.\end{aligned}$$

Hence by Theorem 3.3.5 $R(t)\phi \in C_S^1(X)$ and

$$\|D_S R(t)\phi\|_0 \leq \frac{C}{\sqrt{t}}\|\phi\|_0.$$

This completes the proof of the theorem. ∎

Now we are ready to prove the main result of this section.

Theorem 3.4.4. *Let $\phi_0 \in E(X)$ and $\sum_{i=1}^{\infty} \mu_i^{-\sigma} < \infty$ for some $\sigma > 0$. Suppose that the operator G satisfies the following assumptions:*

(A1). $G : \Sigma_S^1(X) \mapsto \Sigma(X)$;

(A2). For each $r > 0$, there exists a constant K_r such that

$$\begin{aligned} \|G(\xi) - G(\eta)\|_0 &\leq K_r\|\xi - \eta\|_{\Sigma_S^1}; && (3.13) \\ \|G(\xi)\|_0 &\leq K_r(1 + \|\xi\|_{\Sigma_S^1}), && (3.14) \end{aligned}$$

for all $\xi, \eta \in B_r \equiv \{\phi \in Z(X) : \|\phi - \phi_0\|_{\Sigma_S^1} \leq r\}$.

Then there exists $\tau_r \in (0, T]$ such that the problem (P4) has a unique mild solution $\phi \in C(I_r, \Sigma_S^1(X))$, where $I_r \equiv [0, \tau_r]$.

Proof: Let $\phi \in C(I_r, \Sigma_S^1(X))$ such that $\phi(t) \in \Sigma_S^1(X)) \bigcap B_r$ for all $t \in I_r$. Define the operator Γ by

$$\Gamma\phi = R(t)\phi_0 + \int_0^t R(t-s)G(\phi)ds, \quad \phi_0 \in E(X), \quad t \in [0, T],$$

where $E(X)$ is defined by (3.8), p.50. Since $R(t)$ is a C_0-semigroup on $E(X)$, there exists a $\sigma \in (0, T]$ such that

$$\|R(t)\phi_0 - \phi_0\|_{E(X)} \leq \frac{r}{2}, \quad t \in [0, \sigma].$$

By assumption (A2) we have

$$\begin{aligned}\|\int_0^t R(t-s)G(\phi)(s)ds\|_{\Sigma_S^1} &\le \int_0^t K_r(1+\sup_{s\le t}\|\phi(s)\|_{\Sigma_S^1})ds \\ &\le \int_0^t K_r(1+r+\|\phi_0\|)ds.\end{aligned}$$

Hence there exists $\delta \in [0,T]$ such that

$$\|\int_0^t R(t-s)G(\phi)(s)ds\|_{\Sigma_S^1} \le \frac{r}{2}, \quad t \in [0,\delta].$$

Letting $\tau_r = min(\sigma,\delta)$, we have

$$\|\Gamma\phi(t)-\phi_0\|_{\Sigma_S^1} \le \|\Gamma\phi(t)-R(t)\phi_0\|_{\Sigma_S^1} + \|R(t)\phi_0-\phi_0\|_{E(X)} \le r, \quad t \in [0,\tau_r].$$

Hence $\Gamma B_r \subset B_r$. By Theorem 3.4.3 we have $R(t)\Sigma(X) \subset \Sigma_S^1(X)$; hence $\Gamma(\Sigma_S^1(X)) \subset \Sigma_S^1(X)$.

Define $Z_r \equiv \{\phi \in C(I_r,\Sigma_S^1(X)) : \phi(0)=\phi_0,\ \phi(t) \in B_r,\ \forall t \in I_r\}$. Then $\Gamma : Z_r \mapsto Z_r$. We prove that Γ is a contraction on Z_r. For any $\phi,\psi \in Z_r$, By Theorem 3.4.3 and assumption (A2) we have

$$\begin{aligned}&\|\Gamma\phi-\Gamma\psi\|_{C(I_r,\Sigma_S^1(X))} \\ = \ &\sup_{t\in I_r}\|\Gamma\phi(t)-\Gamma\psi(t)\|_{\Sigma_S^1} \\ = \ &\sup_{t\in I_r}\|\int_0^t R(t-s)(G(\phi)(s)-G(\psi)(s))ds\|_{\Sigma_S^1} \\ \le \ &\sup_{t\in I_r}\int_0^t\{\|R(t-s)(G(\phi)-G(\psi))\|_0+\|D_S R(t-s)(G(\phi)-G(\psi))\|_0\}ds \\ \le \ &\sup_{t\in I_r}\int_0^t\{\|(G(\phi)-G(\psi))\|_0+\frac{C}{\sqrt{t-s}}\|(G(\phi)-G(\psi))\|_0\}ds \\ \le \ &\sup_{t\in I_r}\int_0^t K_r\{\|\phi-\psi\|_{\Sigma_S^1}+\frac{C}{\sqrt{t-s}}\|\phi-\psi\|_{\Sigma_S^1}\}ds \\ \le \ &\sup_{t\in I_r}\{K_r(C\sqrt{\tau_r}+\tau_r)\|\phi-\psi\|_{\Sigma_S^1})\} \\ \le \ &M\sqrt{\tau_r}\|\phi-\psi\|_{C(I_r,\Sigma_S^1(X))}.\end{aligned}$$

Therefore Γ is a contraction for τ_r small enough. It follows from Banach fixed point principle that Γ has a unique fixed point in Z_r. This completes the proof. ∎

Remark 3.4.5. If G satisfies the uniform growth condition

$$\|G(\xi)\|_0 \leq K(1 + \|\xi\|_{\Sigma_S^1}), \quad (\forall \xi \in \Sigma_S^1)$$

for some fixed constant K, then the solution of (P4) can be continued indefinitely.

Remark 3.4.6. The assumption $\sum_i \mu_i^{-\sigma} < \infty$ also implies that $(-B)^{-\sigma}$ exists and is a bounded operator. It can be verified that B is the infinitesimal generator of an analytic semigroup in X, given by

$$e^{tB}x = \sum_{i=1}^{\infty} e^{-t\mu_i}(x, e_i^*)e_i, \ x \in X.$$

Thus the above requirements are naturally fulfilled (Theorem 2.2.8).

CHAPTER 4

Optimal Control for Stochastic Systems Driven by a Cylindrical Brownian Motion

4.1 Introduction

In this chapter we study the optimal control problem for stochastic systems driven by a cylindrical Brownian motion, which is a generalization of white noise in infinite dimensions. In other words, we study the stochastic system perturbed by a random noise which can be assumed to be a white noise. A direct method which was introduced by Da Prato [29] is further developed for this problem, some results in Cannarsa and Da Prato [18] have been extended.

In section 4.2 we formulate our problem and give some properties of cylindrical Brownian motion. In section 4.3 we study the linear case of Hamilton-Jacobi-Bellman equation. As in Chapter 3, we construct some C_0-semigroups corresponding to linear operators associated with the HJB equation. In section 4.4 we study the analytic properties of these semigroups. Finally, in section 4.5 we will present some results about semilinear HJB equation, which can be regarded as an extension of the results in Chapter 3, section 3.4.

4.2 Cylindrical Brownian Motions and Control System

In order to generalize "white noise" in infinite dimensional Banach spaces we follow the definition of Cannarsa and Da Prato [18].

Let $\{\mu_k\}$ be a sequence of positive real numbers. Then there exists a unique self-adjoint operator B in X such that $Be_k = -\mu_k e_k$. It is clear that B is the infinitesimal generator of an analytic semigroup in X, given by (Curtain and Pritchard [25, p.11]):

$$e^{tB}x = \sum_{k=1}^{\infty} e^{-t\mu_k}(x, e_k^*)e_k.$$

Consider now a complete probability space $(\Omega, \mathcal{F}, P)$, and a sequence $\{w_k\}$ of standard one-dimensional Brownian motions, mutually independent. Denote by $W^n(t)$ the Brownian motion given by

$$W^n(t) = \sum_{k=1}^{n} w_k(t)e_k.$$

Consider the stochastic convolution

$$W_B^n(t) = \int_0^t e^{(t-s)B}dW^n(s).$$

By some computations we have

$$E(W_B^n(t))^2 = \sum_{k=1}^{n} E\{\int_0^t e^{-(t-s)\mu_k}dw_k(s)\}^2 = \sum_{k=1}^{n}\int_0^t e^{-2\mu_k(t-s)}ds = \sum_{k=1}^{n}\beta_k(t),$$

where

$$\beta_k(t) = \frac{1-e^{-2\mu_k t}}{2\mu_k}.$$

It is clear that

$$\lim_{n\to\infty} E(W_B^n(t))^2 = \sum_{k=1}^{\infty} \beta_k(t) < \infty, \quad if \ \sum_{k=1}^{\infty} \frac{1}{\mu_k} < \infty,$$

hence $W_B^n(t)$ converges in $L^2(\Omega, \mathcal{F}, P)$:

$$W_B^n(t) \to W_B(t) \equiv \sum_{k=1}^{\infty} e_k \int_0^t e^{-(t-s)\mu_k} dw_k(s) \equiv \int_0^t e^{(t-s)B} dW(s), \qquad (4.1)$$

where $W(t) = \sum_{k=1}^{\infty} w_k(t) e_k$ is called a **cylindrical Brownian motion.**

Now we formulate our problem. In this chapter we study the following stochastic optimal control problem:

Let $(\Omega, \mathcal{F}, \mathcal{F}_t \uparrow, P)$ be a filtered probability space, and suppose $\{W_t, t \geq 0\}$ is an X-valued cylindrical Brownian motion. Let y^u denote the solution of the controlled stochastic evolution equation (CSEE)

$$(CSEE) \quad \begin{cases} dy^u &= (By^u + f(y^u, u))dt + \sqrt{\epsilon} dW_t \\ y^u(0) &= x_0 \end{cases}$$

corresponding to the control $u \in \mathcal{U}_{ad}$ (the class of admissible controls). The optimal control problem here is to find a minimal point $u^0 \in \mathcal{U}_{ad}$ such that

$$\begin{aligned} J(u^0) &= E\{\int_0^T \ell(y^0, u^0)dt + \phi_0(y^0(T))\} \\ &\leq E\{\int_0^T \ell(y^u, u)dt + \phi_0(y^u(T))\} \equiv J(u) \end{aligned}$$

for all $u \in \mathcal{U}_{ad}$, where y^0 is the solution of (CSEE) corresponding to the control u^0.

Define

$$\psi^u(t,x) = E\{\int_t^T \ell(y^u(s),u(s)) + \phi_0(y^u(T))\},\ 0 \le t \le T,\ u \in \mathcal{U}_{ad},$$

where $y^u(s),\ t \le s \le T$ is the solution of the (CSEE) starting from the state x at time t. By using the Bellman's Principle of Optimality (also called Dynamic Programming Principle, see Lemma 1.2.3) one can formally verify that $\psi(t,x) \equiv \inf\{\psi^u(t,x) : u \in \mathcal{U}_{ad}\}$ satisfies the following Hamilton-Jacobi-Bellman equation

$$(HJB)\quad \begin{cases} \psi_t(t,x) & + \frac{\epsilon}{2}Tr(\psi_{xx}) + (Bx,\psi_x) + F(x,\psi_x) = 0 \\ \psi(T,x) & = \ \phi_0(x), \quad x \in \mathcal{D}(B), \quad t \in [0,T] \end{cases}$$

where $F(x,\psi_x) = \inf\{l(x,u) + (f(x,u),\psi_x) : u \in \mathcal{U}_{ad}\}$. By reversing the flow of time $t \to T-t$, and defining $\phi(t,x) = \psi(T-t,x)$, the equation (HJB) can be rewritten as an initial value problem:

$$(E)\quad \begin{cases} \phi_t(t,x) & = \frac{\epsilon}{2}Tr(\phi_{xx}) + (Bx,\phi_x) + F(x,\phi_x) \\ \phi(0,x) & = \phi_0(x), \quad x \in \mathcal{D}(B), \quad t \in [0,T] \end{cases}$$

We use semigroup approach to prove the existence and uniqueness of a mild solution for the system (E), and study the regularity properties of the solution. These properties are important in the study of problem (E) in the general nonlinear case.

We remark that recently Cannarsa and Da Prato[18] have studied a special case of system (E) using probabilistic approach. Our results in this chapter are more general and could be regarded as an extension of their work in two aspects: First, from Hilbert space to Banach space; second, from $F(x,\phi_x) = \frac{1}{2}|\phi_x|^2 + g(x)$ to the general nonlinear term $F(x,\phi_x)$.

4.3 The Linear Case

Without loss of generality we take $\epsilon = 1$. First we consider the equation

$$(E1) \quad \begin{cases} \phi_t(t,x) &= \frac{1}{2}Tr(\phi_{xx})(t,x) \\ \phi(0,x) &= \phi_0(x). \end{cases}$$

Define

$$\mathcal{D}(\mathcal{A}_0) \equiv \{\phi \in C_b(X) : Tr(\phi_{xx}) \in C_b(X)\},$$
$$\mathcal{A}_0\phi \equiv \frac{1}{2}Tr(\phi_{xx}), \quad \phi \in \mathcal{D}(\mathcal{A}_0).$$

It is easy to see that $\overline{\mathcal{A}}_0$ generates a C_0 semigroup $\hat{T}(t)$ in $Y = \overline{\mathcal{D}(\mathcal{A}_0)}^{C_b(X)}$, namely

$$(\hat{T}(t)\phi)(x) = E\{\phi(x + W_t)\}. \tag{4.2}$$

In fact, for each $\phi \in \mathcal{D}(\mathcal{A}_0)$,

$$\begin{aligned} \hat{T}(t)\phi(x) - \phi(x) &= E\{\phi(x + W_t) - \phi(x)\} \\ &= E\{(\phi_x, W_t) + \frac{1}{2}(\phi_{xx}W_t, W_t) + o(t)\} \\ &= \frac{1}{2}E(\phi_{xx}W_t, W_t) + o(t) \\ &= \frac{t}{2}Tr(\phi_{xx}) + o(t). \end{aligned}$$

Hence we have

$$\lim_{t\downarrow 0} \frac{\hat{T}(t)\phi - \phi}{t} = \frac{1}{2}Tr(\phi_{xx})$$

and

$$\lim_{t\downarrow 0} \|\hat{T}(t)\phi - \phi\|_0 = 0.$$

Thus the assertion follows.

By Lemma 3.2.2 we can deduce a non-probabilistic representation of $\hat{T}(t)$.

Theorem 4.3.1 *Suppose X is a reflexive Banach space. Then $\mathcal{A} = \overline{\mathcal{A}_0}$ generates a C_0-semigroup of contractions $\hat{T}(t)$ in Y given by:*

$$\hat{T}(t)\phi = \lim_{n\to\infty} \prod_{i=1}^{n} e^{A_i t}\phi, \quad \phi \in Y. \tag{4.3}$$

Thus the solution of the problem (E1) is given by $\phi(t,\cdot) = \hat{T}(t)\phi_0(\cdot)$, $\phi_0 \in Y$.

Proof. By definition

$$\frac{1}{2}Tr(\phi_{xx}(\cdot)) = \frac{1}{2}\sum_{k=1}^{\infty}(\phi_{xx}(\cdot)e_k, e_k)_{X^*,X} = \frac{1}{2}\sum_{k=1}^{\infty} D_k^2\phi(\cdot) = \sum_{i=1}^{\infty} A_k\phi(\cdot).$$

Since $\mathcal{D}(\mathcal{A}_0)$ is dense in Y and $\rho(\mathcal{A}_0) \supset (0,\infty)$, the conclusion follows from Lemma 3.2.2. ∎

The smoothing properties of $\hat{T}(t)$ is given in the following:

Theorem 4.3.2. *Suppose $t > 0$ and $\phi \in Y$, then $\hat{T}(t)\phi \in \mathcal{D}(D_k)$, $(k = 1, 2, \cdots)$, and*

$$\|D_k\hat{T}(t)\phi\|_0 \leq \|D_k e^{A_k t}\phi\|_0 \leq \frac{C}{\sqrt{t}}\|\phi\|_0. \tag{4.4}$$

Proof. Similar to the proof of Theorem 3.3.2. ∎

Now we consider the following initial value problem:

$$(E2)\quad \begin{cases} \phi_t(t,x) &= \frac{1}{2}Tr(\phi_{xx}(t,x)) + (Bx, \phi_x) \\ \phi(0,x) &= \phi_0(x). \end{cases}$$

where B is a closed operator in X such that

$$Be_i = -\mu_i e_i, \quad \mu_i > 0, \quad \mu_i \to \infty \ \ as \ \ i \to \infty.$$

It is clear that B is the infinitesimal generator of an analytic semigroup e^{tB} in X, given by

$$e^{tB}x = \sum_{i=1}^{\infty} e^{-t\mu_i}(x, e_i^*)e_i, \quad x \in X.$$

Define the operator $\mathcal{B}_0$ by

$$\mathcal{D}(\mathcal{B}_0) = \{\phi \in C_b^1(X) : \|(Bx, \phi_x(x))\|_0 < \infty\},$$

$$(\mathcal{B}_0\phi)(x) = (Bx, \phi_x(x))_{X,X^*}, \quad \phi \in \mathcal{D}(\mathcal{B}_0).$$

Since $(Bx, \phi_x(x)) = -\sum_{i=1}^{\infty} \mu_i(x, e_i^*)D_i\phi$, we have the following theorem

Theorem 4.3.3.

(1) For each $i \in N$,

$$(S_i(t)\phi)(x) \equiv \phi(x - (1 - e^{-\mu_i t})(x, e_i^*)e_i), \quad \phi \in C_b(X), \tag{4.5}$$

defines a C_0-semigroup of contractions in $C_b(X)$ with infinitesimal generator given by

$$B_i\phi(x) = -\mu_i(x, e_i^*)D_i\phi(x), \quad \phi \in C_b^1(X).$$

(2) The resolvent set $\rho(\mathcal{B}_0) \supset (0, \infty)$, and, for any $\phi \in C_b(X)$, we have

$$(\lambda - \mathcal{B}_0)^{-1}\phi(x) = \int_0^\infty e^{-\lambda t}\phi(e^{tB}x)dt, \ \lambda \in \rho(\mathcal{B}_0).$$

(3) The closed operator $\mathcal{B} \equiv \overline{\mathcal{B}}_0$ generates a C_0-semigroup of contractions $S(t)$ given by

$$S(t)\phi(x) = \lim_{n\to\infty}\prod_{i=1}^{n} S_i(t)\phi(x), \quad \phi \in \overline{\mathcal{D}(\mathcal{B}_0)}. \tag{4.6}$$

Proof. Part (1) is proved in Theorem 3.3.8. For part (2), let $\lambda > 0$ and $\phi \in C_b(X)$, and define

$$(F(\lambda)\phi)(x) = \int_0^\infty e^{-\lambda t}\phi(e^{tB}x)dt.$$

It is clear that $F(\lambda)$ defines a bounded linear operator in $C_b(X)$. Furthermore, for $h > 0$

$$\begin{aligned} &\frac{(F(\lambda)\phi)(e^{hB}x) - F(\lambda)\phi(x)}{h} \\ = \ &\frac{1}{h}\{\int_0^\infty e^{-\lambda t}\phi(e^{(t+h)B}x)dt - \int_0^\infty e^{-\lambda t}\phi(e^{tB}x)dt\} \\ = \ &\frac{1}{h}\{(e^{\lambda h} - 1)\int_h^\infty e^{-\lambda t}\phi(e^{tB}x)dt - \int_0^h e^{-\lambda t}\phi(e^{tB}x)dt\}. \end{aligned}$$

Letting $h \to 0$, we have

$$\mathcal{B}_0F(\lambda)\phi(x) = \lambda F(\lambda)\phi(x) - \phi(x).$$

This means that for any $\phi \in C_b(X)$ and $\lambda > 0, F(\lambda)\phi \in \mathcal{D}(\mathcal{B}_0)$ and $(\lambda I - \mathcal{B}_0)F(\lambda) = I$. On the other hand, if $\phi \in \mathcal{D}(\mathcal{B}_0)$, it is easy to see that

$$F(\lambda)(\lambda I - \mathcal{B}_0)\phi = (\lambda I - \mathcal{B}_0)F(\lambda)\phi = \phi.$$

Hence $(\lambda I - \mathcal{B}_0)^{-1} = F(\lambda)$. Finally part (3) holds by Lemma 3.2.2. This completes the proof of the theorem. ∎

Define $D(X) = \overline{\mathcal{D}(\mathcal{A}_0) \bigcap \mathcal{D}(\mathcal{B}_0)}^{C_b^1(X)}$. Being a closed subspace of $C_b^1(X)$, it is also a Banach space and both $\mathcal{A}_0$ and $\mathcal{B}_0$ are densely defined in $D(X)$.

Theorem 4.3.4. *Let $\hat{T}(t)$ and $S(t)$ be the C_0-semigroups of contractions generated by $\mathcal{A}$ and $\mathcal{B}$ resp., and*

$$\beta_i(t) = \frac{1 - e^{-2\mu_i t}}{2\mu_i}$$

and

$$V(t)\phi = \prod_{i=1}^{\infty} e^{\beta_i(t)A_i}\phi,$$

then $Q(t) \equiv S(t)V(t)$ defines a C_0-semigroup of contractions in $D(X)$, with the infinitesimal generator $\overline{\mathcal{A}_0 + \mathcal{B}_0}$.

Proof. It is easy to see that $V(t)\phi$ is well defined for each $\phi \in \mathcal{D}(\mathcal{A}_0)$. Indeed, set

$$V_n(t) = \prod_{i=1}^{n} e^{\beta_i(t)A_i},$$

since $V_n(t)$ is a C_0-semigroup of contractions , we have (Ahmed [1, Theorem 1.3.3, p.8])

$$V_{n+1}(t)\phi - V_n(t)\phi = V_n(t) \int_0^{\beta_{n+1}(t)} e^{A_{n+1}s} A_{n+1}\phi ds.$$

Hence

$$\|V_{n+1}(t)\phi - V_n(t)\phi\|_0 \leq \beta_{n+1}(t)\|A_{n+1}\phi\|_0.$$

Since $\phi \in \mathcal{D}(\mathcal{A}_0)$ implies that $\|\sum_k A_k\phi\|_0 < \infty$, there exists a finite positive number M such that $\|A_k\phi\|_0 < M$, $\forall k$. Hence

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|V_{n+1}(t)\phi - V_n(t)\phi\|_0 \to 0, \quad T < \infty.$$

Thus $\{V_n(\cdot)\phi\}$ is Cauchy and hence converges uniformly in $t \in [0,T]$ for all $\phi \in \mathcal{D}(\mathcal{A}_0)$. Now recall that

$$\hat{T}(t)\phi = \lim_{n\to\infty} \prod_{i=1}^{n} e^{tA_i}\phi, \quad (e^{tA_i}\phi)(x) = \frac{1}{\sqrt{2\pi t}} \int_R e^{-\xi^2/2t}\phi(x - \xi e_i)d\xi, \tag{4.7}$$

$$S(t)\phi = \lim_{n\to\infty} \prod_{i=1}^{n} e^{tB_i}\phi, \quad (e^{tB_i}\phi)(x) = \phi(x - (1 - e^{-\mu_i t})(x, e_i^*)e_i). \tag{4.8}$$

The following identity can be verified by straightforward computations. For any $\phi \in C_b(X)$ and $0 \leq s, t < \infty$,

$$e^{tB_i}e^{sB_j}\phi = e^{sB_j}e^{tB_i}\phi; \tag{4.9}$$

$$e^{tA_i}e^{sA_j}\phi = e^{sA_j}e^{tA_i}\phi; \tag{4.10}$$

$$e^{tB_i}e^{sA_j}\phi = e^{sA_j}e^{tB_i}\phi, \quad (i \neq j); \tag{4.11}$$

$$e^{tB_i}e^{sA_i}\phi = e^{se^{2\mu_i t}A_i}e^{tB_i}\phi. \tag{4.12}$$

In fact, equality (4.12) has been proved in Chapter 3, see (3.6). Hence we have

$$\begin{aligned} Q(t_1 + t_2)\phi &= S(t_1 + t_2)V(t_1 + t_2)\phi \\ &= \lim_{n\to\infty} \prod_{i=1}^{n} e^{(t_1+t_2)B_i}\Big(\lim_{m\to\infty} \prod_{i=1}^{m} e^{\beta_i(t_1+t_2)A_i}\Big)\phi. \end{aligned}$$

Recalling the proof of Lemma 3.2.2 we see that if e^{tC_1} and e^{tC_2} are C_0-semigroups which commute with each other, then $e^{tC_1}e^{tC_2} = e^{t(C_1+C_2)}$ is also a C_0-semigroup. Note that

$$\begin{aligned}\beta_i(t_1+t_2) &= (1-e^{-2\mu_i(t_1+t_2)})/2\mu_i \\ &= ((1-e^{-2\mu_i t_2}) + (1-e^{-2\mu_i t_1})e^{-2\mu_i t_2})/2\mu_i \\ &= \beta_i(t_2) + \beta_i(t_1)e^{-2\mu_i t_2}.\end{aligned}$$

Let

$$V_n(t) \equiv \prod_{i=1}^{n} e^{\beta_i(t)A_i}, \quad S_n(t) \equiv \prod_{i=1}^{n} e^{tB_i}, \quad \psi = V(t)\phi.$$

Since e^{tB_i} is a contraction for any $i \geq 1$, it follows that

$$\begin{aligned}&\|S_n(t)V_n(t)\phi - S(t)V(t)\phi\|_0 \\ \leq\ &\|S_n(t)V_n(t)\phi - S_n(t)V(t)\phi\|_0 + \|S_n(t)V(t)\phi - S(t)V(t)\phi\|_0 \\ \leq\ &\|V_n(t)\phi - V(t)\phi\|_0 + \|S_n(t)\psi - S(t)\psi\|_0.\end{aligned}$$

This implies that

$$Q(t)\phi = S(t)V(t)\phi = \lim_{n\to\infty} S_n(t)V_n(t)\phi$$

uniformly in $t \in [0,T]$. Thus

$$\begin{aligned}Q(t_1+t_2)\phi &= \lim_{n\to\infty}\{\prod_{i=1}^{n} e^{(t_1+t_2)B_i}\prod_{i=1}^{n} e^{\beta_i(t_1+t_2)A_i}\phi\} \\ &= \lim_{n\to\infty}\prod_{i=1}^{n} e^{t_1B_i}e^{t_2B_i}(\prod_{i=1}^{n} e^{\beta_i(t_1)e^{-2\mu_i t_2}A_i}e^{\beta_i(t_2)A_i})\phi \\ &= \lim_{n\to\infty}\prod_{i=1}^{n} e^{t_1B_i}\prod_{i=1}^{n} e^{t_2B_i}\prod_{i=1}^{n} e^{\beta_i(t_1)e^{-2\mu_i t_2}A_i}\prod_{i=1}^{n} e^{\beta_i(t_2)A_i}\phi\end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty}\prod_{i=1}^{n} e^{t_1 B_i}\prod_{i=1}^{n} e^{\beta_i(t_1)A_i}\prod_{i=1}^{n} e^{t_2 B_i}\prod_{i=1}^{n} e^{\beta_i(t_2)A_i}\phi \\
&= S(t_1)V(t_1)S(t_2)V(t_2)\phi \\
&= Q(t_1)Q(t_2)\phi
\end{aligned}$$

Now for $\phi \in \mathcal{D}(\mathcal{A}_0)\bigcap\mathcal{D}(\mathcal{B}_0)$, we have

$$\begin{aligned}
\lim_{h\downarrow 0}\frac{Q(h)\phi-\phi}{h} &= \lim_{h\downarrow 0}\lim_{n\to\infty}(\prod_{i=1}^{n} e^{hB_i}e^{\beta_i(h)A_i}\phi-\phi)/h \\
&= \lim_{n\to\infty}\lim_{h\downarrow 0}(\prod_{i=1}^{n} e^{hB_i}e^{\beta_i(h)A_i}\phi-\phi)/h \\
&= \lim_{n\to\infty}\lim_{h\downarrow 0}\frac{S_n(h)V_n(h)\phi-\phi}{h} \\
&= \lim_{n\to\infty}\lim_{h\downarrow 0}\{(S_n(h)\frac{V_n(h)\phi-\phi}{h}+\frac{S_n(h)\phi-\phi}{h}\}.
\end{aligned}$$

It is easy to verify that

$$\lim_{h\downarrow 0}\{S_n(h)(V_n(h)\phi-\phi)/h\} = (\sum_{i=1}^{n} A_i)\phi \equiv \mathcal{A}_n\phi.$$

Indeed, let $\Gamma_n(h)\phi \equiv (V_n(h)\phi-\phi)/h$, then

$$\begin{aligned}
&\|S_n(h)\Gamma_n(h)\phi-\mathcal{A}_n\phi\|_0 \\
\le\ &\|S_n(h)\Gamma_n(h)\phi-S_n(h)\mathcal{A}_n\phi\|_0+\|S_n(h)\mathcal{A}_n\phi-\mathcal{A}_n\phi\|_0 \\
\le\ &\|\Gamma_n(h)\phi-\mathcal{A}_n\phi\|_0+\|S_n(h)\mathcal{A}_n\phi-\mathcal{A}_n\phi\|_0 \to 0,\ (h\downarrow 0).
\end{aligned}$$

Thus we have

$$\lim_{h\downarrow 0}(Q(h)\phi-\phi)/h = \lim_{n\to\infty}\sum_{i=1}^{n}(A_i+B_i)\phi = (\mathcal{A}_0+\mathcal{B}_0)\phi.$$

Finally we prove that $Q(t)$ is a C_0-semigroup. Since

$$\lim_{h\downarrow 0}\|(Q(h)\phi-\phi\|_0 = \lim_{n\to\infty}\lim_{h\downarrow 0}\|S_n(h)V_n(h)\phi-\phi\|_0,$$

we can change the order of limits because

$$Q(h)\phi = \lim_{n\to\infty} S_n(h)V_n(h)\phi$$

uniformly in h (Lemma 3.2.2). Now since $V_n(h)S_n(h)$ is the product of C_0-semigroups, the C_0 property of $Q(t)$ can be seen accordingly. This completes the proof of the theorem. ∎

4.4 Some Properties of Semigoup $Q(t)$

In the previous section we have constructed a C_0-semigroup $Q(t)$ in the Banach space $D(X)$. In the following we prove that $\hat{R}(t)$ is not an analytic semigroup. Recall that a C_0-semigroup $S(t)$, with the infinitesimal generator C, is an analytic semigroup (more precisely, can be extended to an analytic semigroup in a sector containing positive real half line) if and only if $S(t)$ is differentiable and there exists a constant M such that

$$\|\frac{d}{dt}S(t)\| = \|CS(t)\| \leq \frac{M}{t}, \quad \forall t > 0. \tag{4.13}$$

Theorem 4.4.1. *$Q(t)$ is not an analytic semigoup in $D(X)$.*

Proof. We prove that there does not exist a constant M such that

$$\|\frac{d}{dt}Q(t)\| \leq \frac{M}{t}, \quad t > 0.$$

Consider

$$Q_n(t) = S_n(t)V_n(t) = \prod_{i=1}^{n} e^{tB_i} \prod_{i=1}^{n} e^{\beta_i(t)A_i} = \prod_{i=1}^{n} e^{\beta_i(t)A_i} e^{tB_i},$$

where

$$\beta_i(t) = \frac{1 - e^{-2\mu_i t}}{2\mu_i}.$$

Since

$$Q(t)\phi = \lim_{n\to\infty} Q_n(t)\phi, \quad \forall \phi \in D(X)$$

uniformly in $t \in [0, T]$, it is sufficient to show that

$$t\|\frac{d}{dt}Q_n(t)\| \to \infty \quad as \quad n \to \infty, \quad (\forall\, t > 0).$$

From the proof of Theorem 4.3.4 we can see that $Q_n(t)$ is a C_0-semigroup of contractions, with the infinitesimal generator

$$\sum_{i=1}^{n}(A_i + B_i) = \sum_{i=1}^{n}(A_i - \mu_i x_i D_i), \quad x_i = (x, e_i^*).$$

Denote

$$\frac{df}{dt} = \dot{f}.$$

Hence by properties of C_0-semigroups (see Ahmed [1, p.7 Theorem 1.3.3(iii)]) we have

$$\dot{Q}_n(t) = Q_n(t)\sum_{i=1}^{n}(A_i + B_i).$$

Define

$$\phi_j(x) = e^{-x_j^2}, \quad x_j = (x, e_j^*).$$

It is clear that $\phi_j \in C_b^2(X)$ and

$$D_i\phi_j(x) = -2x_j\phi_j(x)\delta_{ij},$$

$$A_i\phi_j(x) = \frac{1}{2}D_i^2\phi_j(x) = (2x_j^2 - 1)\phi_j(x)\delta_{ij},$$

where δ_{ij} is the Kronecker delta. Hence

$$\begin{aligned}
\dot{Q}_n(t)\phi_j(x) &= Q_n(t)\sum_{i=1}^{n}(A_i - \mu_i x_i D_i)\phi_j(x) \\
&= Q_n(t)\{(2x_j^2 - 1) - \mu_j x_j(-2x_j)\}\phi_j(x) \\
&= Q_n(t)\{2(1 + \mu_j)x_j^2 - 1\}\phi_j(x) \\
&= 2(1 + \mu_j)Q_n(t)(x_j^2\phi_j(x)) - Q_n(t)\phi_j(x).
\end{aligned}$$

To compute $Q_n(t)\phi_j(x)$, note that if $i \neq j$, then

$$\begin{aligned}
e^{tB_i}e^{\beta_i(t)A_i}\phi_j(x) &= e^{tB_i}\frac{1}{\sqrt{2\pi\beta_i(t)}}\int_R e^{-\xi^2/(2\beta_i(t))}\phi_j(x - \xi e_i)d\xi \\
&= \frac{1}{\sqrt{2\pi\beta_i(t)}}\int_R e^{-\xi^2/(2\beta_i(t))}\phi_j(x - \xi e_i - (1 - e^{-\mu_i t})(x, e_i^*)e_i)d\xi \\
&= \frac{1}{\sqrt{2\pi\beta_i(t)}}\int_R e^{-\xi^2/(2\beta_i(t))}e^{-(x-(1-e^{-\mu_i t})x_i e_i - \xi e_i, e_j^*)^2}d\xi \\
&= e^{-(x,e_j^*)^2}\frac{1}{\sqrt{2\pi\beta_i(t)}}\int_R e^{-\xi^2/(2\beta_i(t))}d\xi \\
&= \phi_j(x).
\end{aligned}$$

Hence for $n > j$

$$\begin{aligned}
&Q_n(t)\phi_j(x) = e^{tB_j}e^{\beta_j(t)A_j}\phi_j(x) \\
&= \frac{1}{\sqrt{2\pi\beta_j(t)}}\int_R e^{-\xi^2/(2\beta_j(t))}\phi_j(x - \xi e_j - (1 - e^{-\mu_j t})x_j e_j)d\xi \\
&= \frac{1}{\sqrt{2\pi\beta_{.}(t)}}\int_R e^{-\xi^2/(2\beta_j(t))}e^{-(x-\xi e_j-(1-e^{-\mu_j t})x_j e_j, e_j^*)^2}d\xi \\
&= \frac{1}{\sqrt{2\pi\beta_j(t)}}\int_R e^{-\xi^2/(2\beta_j(t))}e^{-(x_j-\xi-(1-e^{-\mu_j t})x_j)^2}d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi\beta_j(t)}} \int_R e^{-\xi^2/(2\beta_j(t))} e^{-(e^{-\mu_j t}x_j-\xi)^2} d\xi \\
&\quad (denote\ \ a^2 = \frac{1}{2\beta_j(t)}, \quad b = e^{-\mu_j t}x_j) \\
&= \frac{a}{\sqrt{\pi}} \int_R e^{-(a^2\xi^2+(b-\xi))^2} d\xi \\
&= \frac{a}{\sqrt{\pi}} \int_R e^{-((a^2+1)\xi^2-2b\xi+b^2)} d\xi \\
&= \frac{a}{\sqrt{\pi}} \int_R e^{-(a^2+1)(\xi^2-\frac{2b}{a^2+1}\xi+\frac{b^2}{a^2+1})} d\xi \\
&= \frac{a}{\sqrt{\pi}} \int_R e^{-(a^2+1)((\xi-\frac{b}{a^2+1})^2+\frac{a^2b^2}{(a^2+1)^2})} d\xi \\
&= \frac{a}{\sqrt{\pi}} e^{-\frac{a^2b^2}{(a^2+1)}} \int_R e^{-(a^2+1)(\xi-\frac{b}{a^2+1})^2} d\xi \\
&= \frac{a}{\sqrt{a^2+1}} e^{-\frac{a^2b^2}{a^2+1}}.
\end{aligned}
$$

The last equality is from the identity

$$\int_0^\infty e^{-a\eta^2} d\eta = \frac{\sqrt{\pi}}{2\sqrt{a}}. \tag{4.14}$$

Since $\mu_j \to \infty$ as $j \to \infty$, and

$$a^2 = \frac{1}{2\beta_j(t)} = \frac{\mu_j t}{1-e^{-\mu_j t}} \approx \mu_j \gg 1, \quad t > 0, \tag{4.15}$$

it is clear that $a^2 \to \infty$ as $\mu_j \to \infty$ for any $t > 0$. Thus $a^2 + 1 \approx a^2$ for sufficient large j. This implies that

$$Q_n(t)\phi_j(x) \approx e^{-b^2} = e^{-x_j^2 e^{-2\mu_j t}}, \quad t > 0, \quad j > N \tag{4.16}$$

for some integer N.

Similarly we can prove that

$$e^{tB_i}e^{\beta_i(t)A_i}x_j^2\phi_j(x) = x_j^2\phi_j(x), \quad (j \neq i).$$

Hence

$$
\begin{aligned}
&Q_n(t)x_j^2\phi_j(x) = e^{tB_j}e^{\beta_j(t)A_j}x_j^2\phi_j(x)\\
= &\frac{1}{\sqrt{2\pi\beta_j(t)}}\int_R e^{-\xi^2/(2\beta_j(t))}\phi_j(x-\xi e_j-(1-e^{-\mu_j t})x_je_j)(x-\xi e_j-(1-e^{-\mu_j t})x_je_j,e_j^*)^2d\xi\\
= &\frac{1}{\sqrt{2\pi\beta_i(t)}}\int_R e^{-\xi^2/(2\beta_j(t))}e^{-(e^{-\mu_j t}x_j-\xi)^2}(e^{-\mu_j t}x_j-\xi)^2d\xi\\
&\qquad\qquad (denote\ \ a^2=\frac{1}{2\beta_j(t)},\ \ b=e^{-\mu_j t}x_j)\\
= &\frac{a}{\sqrt{\pi}}\int_R e^{-(a^2\xi^2+(b-\xi))^2}(b-\xi)^2d\xi\\
= &\frac{a}{\sqrt{\pi}}e^{-\frac{a^2b^2}{(a^2+1)}}\int_R e^{-(a^2+1)(\xi-\frac{b}{a^2+1})^2}(\xi-\frac{b}{a^2+1}-\frac{a^2b}{a^2+1})^2d\xi\\
= &\frac{a}{\sqrt{\pi}}e^{-\frac{a^2b^2}{(a^2+1)}}\int_R e^{-(a^2+1)(\xi-\frac{b}{a^2+1})^2}\{(\xi-\frac{b}{a^2+1})^2+(\frac{a^2b}{a^2+1})^2-2(\frac{a^2b}{a^2+1})(\xi-\frac{b}{a^2+1})\}d\xi.
\end{aligned}
$$

Recall formula (4.14) and the following identities

$$\int_R \eta e^{-a\eta^2}d\eta=0, \tag{4.17}$$

$$\int_R \eta^2e^{-a\eta^2}d\eta=\frac{\sqrt{\pi}}{2a\sqrt{a}},\quad (a>0). \tag{4.18}$$

We have

$$
\begin{aligned}
Q_n(t)x_j^2\phi_j(x) &= \frac{a}{\sqrt{\pi}}e^{-\frac{a^2b^2}{a^2+1}}\{\frac{\sqrt{\pi}}{2\sqrt{(a^2+1)^3}}+(\frac{a^2b}{a^2+1})^2\frac{\sqrt{\pi}}{2\sqrt{a^2+1}}\}\\
&= \frac{a}{\sqrt{\pi}}e^{-\frac{a^2b^2}{(a^2+1)}}\{\frac{1}{2(a^2+1)}+(\frac{a^2b}{a^2+1})^2\}\\
&\approx e^{-b^2}(-\frac{1}{2(a^2+1)}+b^2).
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\dot{Q}_n(t)\phi_j(x) &= \{2(1+\mu_j)(-\frac{1}{2(a^2+1)}+b^2)-1\}e^{-b^2}\\
&= \{\frac{1+\mu_j}{a^2}-1+2(1+\mu_j)b^2\}e^{-b^2}.
\end{aligned}
$$

From (4.15) we have $a^2 \approx \mu_j$. It follows that

$$\begin{aligned}\dot{Q}_n(t)\phi_j(x) &= (\frac{1}{\mu_j} + 2(1+\mu_j)b^2)e^{-b^2} \\ &= (\frac{1}{\mu_j} + 2(1+\mu_j)e^{-2\mu_j t}x_j^2)e^{-x_j^2 e^{-2\mu_j t}}.\end{aligned}$$

Hence

$$\begin{aligned}\|\dot{Q}_n(t)\phi_j(x)\|_0 &= \sup_{x\in X}|(\frac{1}{\mu_j} + 2(1+\mu_j)e^{-2\mu_j t}x_j^2)e^{-x_j^2 e^{-2\mu_j t}}| \\ &= \sup_{b\in R}|(\frac{1}{\mu_j} + 2(1+\mu_j)b^2)e^{-b^2}| \\ &= \sup_{r\in R^+}|(\frac{1}{\mu_j} + 2(1+\mu_j)r)e^{-r}|.\end{aligned}$$

Let

$$f(r) = (\frac{1}{\mu_j} + 2(1+\mu_j)r)e^{-r},$$

it is easy to see that $f(r)$, $r \geq 0$ attains its maximum at

$$r_0 = 1 - \frac{1}{2\mu_j(1+\mu_j)},$$

hence

$$\begin{aligned}\|\dot{Q}_n(t)\phi_j(x)\|_0 &= (\frac{1}{\mu_j} + 2(1+\mu_j)r_0)e^{-r_0} \\ &= 2(1+\mu_j)e^{-r_0}.\end{aligned}$$

Thus we have

$$\|\dot{Q}_n(t)\phi_j(x)\|_0 \to \infty, \quad as \ \ j \to \infty.$$

Since $\|\phi_j(x)\|_0 = 1$, $\forall j$, it follows that

$$\|\dot{Q}_n(t)\| = \sup_{\|\phi\|=1} \|\dot{Q}_n(t)\phi(x)\|_0 \geq \sup_j \|\dot{Q}_n(t)\phi_j(x)\|_0 = \infty.$$

Therefore there does not exist a constant M such that

$$\|\frac{d}{dt}Q(t)\| \leq \frac{M}{t}, \quad t > 0.$$

Hence $Q(t)$ is not an analytic semigroup.

4.5 The Non-linear Case

First we consider a special case. Assume that H is a separable Hilbert space, consider the system

$$(E3) \quad \begin{cases} \phi_t(t,x) &= \frac{1}{2}Tr(\phi_{xx}(t,x)) + (Bx, \phi_x(t,x)) + \frac{1}{2}|\phi_x(t,x)|^2 + g(x) \\ \phi(0,x) &= \phi_0(x), \quad x \in \mathcal{D}(B), \quad t \in [0,T] \end{cases}$$

where $\phi_0 \in D(H)$ and $g \in C_b(H)$.

Equation (E3) is associated with the following control problem: Minimize

$$J(u) = E\{\int_0^T [g(y(s)) + \frac{1}{2}|u(s)|^2]\}ds$$

over all controls $u \in M_W^2(0,T;H)$, the space of H-valued stochastic process which are square integrable and adapted to $\sigma\{W_s, s \leq t\} \equiv \mathcal{F}_t^W$, subject to the state equation

$$\begin{aligned} dy(t) &= By(t)dt + dW_t, \\ y(0) &= x. \end{aligned}$$

where W_t is a H-valued cylindrical Brownian motion in a probability space $(\Omega, \mathcal{F}, P)$, with covariance operator $cov\ W_t = tI$.

We may change (E3) into a linear system by using the well-known logarithmic transformation (Fleming [41]). Setting

$$\psi = e^{-\phi}$$

we have

$$\psi_t = -\psi\phi_t, \quad \psi_x = -\psi\phi_x, \quad \psi_{xx} = -\psi(\phi_{xx} - \phi_x \circ \phi_x),$$

where $(x \circ y)h = x\langle y, h\rangle$ for any $x, y, h \in H$. Substituting into (E3) we have

$$\begin{aligned} \psi_t &= \frac{1}{2}Tr(\psi_{xx}) + \langle Bx, \psi_x\rangle - g\psi \\ \psi(0) &= e^{-\phi_0}. \end{aligned}$$

Since $g \in C_b(H)$, the term $-g\psi$ represents a bounded perturbation, this system can be solved by means of perturbation theory (Ahmed [1]). But since $Tr(\phi_x \circ \phi_x) \neq |\phi_x|^2$ in general, it is clear that this method fails in Banach spaces.

Below we consider the more general case. We consider the following Cauchy problem

$$(E4) \quad \begin{cases} \phi_t(t,x) = \frac{1}{2}Tr(\phi_{xx}) + (Bx, \phi_x) + F(x, \phi_x) \\ \phi(0,x) = \phi_0(x), \quad x \in \mathcal{D}(B), \quad t \in [0,T] \end{cases}$$

We say that the problem (E4) has a mild solution $\phi(t,x)$ if ϕ is the solution of the following integral equation

$$\phi(t,x) = Q(t)\phi_0(x) + \int_0^t Q(t-s)F(x,\phi_x(s,x))ds, \quad t \in [0,T]$$

where $Q(t)$ is the C_0 semigroup defined in Theorem 4.6.

We remark that in 1985 Da Prato [31] has studied a similar problem in separable Hilbert spaces with ϕ_x here replaced by $S\phi_x$, the so called derivative in the direction S (see Definition 3.2.3 and the comments thereafter), by using the Crandall-Ligget generation theorem. In this section we present a direct method for more general case. We will solve the problem (E4) in the Banach space $\mathbf{E}_b^1(X)$, defined as follows:

Definition 4.5.1. Let $\mathbf{E}(X) = \{\phi \in Y : \lim_{n\to\infty} \prod_{i=1}^n e^{\beta_i(t)A_i} e^{tB_i}\phi$ exists uniformly in t on every compact subinterval of $[0,\infty)$, and belongs to $Y\}$. (Y is defined in section 4.3, p.66)
Define $\mathbf{E}_b^1(X) = \mathbf{E}(X) \bigcap C_b^1(X)$, with the norm $\|\phi\|_{\mathbf{E}_b^1(X)} = \|\phi\|_0 + \|D\phi\|_0$.

It is clear that $D(X) \subset \mathbf{E}_b^1(X) \subset C_b^1(X)$. we prove that $\mathbf{E}_b^1(X)$ is a Banach space.

Theorem 4.5.2. $\mathbf{E}(X)$ *is a closed subspace of* Y, *and* $\mathbf{E}_b^1(X)$, *furnished with the norm topology, is a Banach space. Moreover if* $\phi \in \mathbf{E}(X)$ *and* $\sum_{i=1}^\infty \mu_i^{-\sigma} < \infty$ *for some* $\sigma > 0$, *then* $Q(t)\phi \in \mathbf{E}_b^1(X)$.

Proof. Take any sequence $\{\phi_n\} \subset \mathbf{E}(X)$ such that $\phi_n \to \phi \quad (n \to \infty)$ in Y. Since $\|Q(t)\phi_n - Q(t)\phi_m\|_0 \leq \|\phi_n - \phi_m\|_0$, $\{Q(t)\phi_n\}$ is a Cauchy sequence in Y. Thus there exists $\psi \in Y$ such that $Q(t)\phi_n \to \psi$ in Y. Now let

$$Q_n(t) = \prod_{i=1}^{n} e^{\beta_i(t)A_i} e^{tB_i}.$$

Then for any $m \in N$ we have

$$\begin{aligned}
&\|Q_n(t)\phi - \psi\|_0 \\
\leq\ & \|Q_n(t)\phi - Q_n(t)\phi_m\|_0 + \|Q_n(t)\phi_m - Q(t)\phi_m\|_0 + \|Q(t)\phi_m - \psi\|_0 \\
\leq\ & \|\phi - \phi_m\|_0 + \|Q_n(t)\phi_m - Q(t)\phi_m\|_0 + \|Q(t)\phi_m - \psi\|_0
\end{aligned}$$

Hence for any $\epsilon > 0$, we can choose N sufficiently large so that $\|Q_n(t)\phi - \psi\|_0 < \epsilon$ for all $m, n \geq N$. Hence $Q_n(t)\phi \to \psi \quad (n \to \infty)$. This implies that $\phi \in \mathbf{E}(X)$ and $Q(t)\phi = \psi$. Further it is easy to see that $Q_n(t)\phi \to Q(t)\phi$ converges uniformly in $t \in [0, T]$, $T > 0$. Hence $\mathbf{E}(X)$ is a closed subspace of Y.

It is clear that $\mathbf{E}_b^1(X)$ is a closed subspace of $C_b^1(X)$. Indeed, suppose $\{\phi_n\} \subset \mathbf{E}_b^1(X)$ and $\phi_n \to \phi$ in $\mathbf{E}_b^1(X)$ $(C_b^1(X))$, that is

$$\|\phi_n - \phi\|_0 + \|D\phi_n - D\phi\|_1 \to 0, \quad (n \to \infty). \tag{4.19}$$

Hence $\phi \in C_b^1(X)$.

Since (4.19) also implies that $\|\phi_n - \phi\|_0 \to 0$, $\{\phi_n\}$ converges to ϕ in Y, and ϕ must belong to $\mathbf{E}(X)$ as $\mathbf{E}(X)$ is closed in Y. Therefore $\phi \in \mathbf{E}_b^1(X)$ and $\mathbf{E}_b^1(X)$ is a closed subspace of $C_b^1(X)$, and hence a Banach Space.

Now suppose $\phi \in \mathbf{E}(X)$. Since $Q(t)(Q(t)\phi) = Q(2t)\phi \in Y$, $Q(t)\phi \in \mathbf{E}(X)$. In the following theorem we will prove that $Q(t)\phi \in C_b^1(X)$ for all $\phi \in \mathbf{E}(X)$, hence $Q(t)\phi \in \mathbf{E}_b^1(X)$. ∎

Now we prove a regularity property of semigroup $Q(t)$, which seems similar to Theorem 3.4.3 but really different because the parameters $\beta_k(t)$ and $\alpha_k(t)$ are different.

Theorem 4.5.3. *Let* $D\phi \equiv \phi_x$. *Suppose that* $\phi \in \mathbf{E}(X)$ *and* $\sum_{i=1}^{\infty} \mu_i^{-\sigma} < \infty$ *for some* $\sigma > 0$, *then* $Q(t)\phi \in C_b^1(X)$, *and*

$$\|DQ(t)\phi\|_0 \le \frac{C}{\sqrt{t}}\|\phi\|_0. \tag{4.20}$$

Proof. Let

$$Q_k(t)\phi = \prod_{i\neq k} e^{tB_i}e^{\beta_i(t)A_i}\phi.$$

Since $Q(t)\phi = Q_k(t)e^{tB_k}e^{\beta_k(t)A_k}\phi$, we have

$$\begin{aligned} &\|D_kQ(t)\phi\|_0 = \|Q_k(t)D_k(e^{tB_k}e^{\beta_k(t)A_k}\phi)\|_0 \\ \le\ & \|D_k(e^{tB_k}e^{\beta_k(t)A_k}\phi)\|_0 \end{aligned}$$

Since

$$\begin{aligned} & D_ke^{tB_k}\psi(x) \\ = \ & D_k\psi(x-(1-e^{-\mu_k t})(x,e_k^*)e_k) \\ = \ & \lim_{h\to 0^+}(\psi(x+he_k-(1-e^{-\mu_k t})(x+he_k,e_k^*)e_k)-\psi(x-(1-e^{-\mu_k t})(x,e_k^*)e_k))/h \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0^+} (\psi(x-(1-e^{-\mu_k t})(x,e_k^*)e_k + e^{-\mu_k t}he_k) - \psi(x-(1-e^{-\mu_k t})(x,e_k^*)e_k))/h \\
&= \lim_{h\to 0^+} (e^{tB_k}\psi(x+e^{-\mu_k t}he_k) - e^{tB_k}\psi(x))/h \\
&= e^{tB_k}D_k\psi(x)e^{-\mu_k t}.
\end{aligned}
$$

By Theorem 4.3.2

$$\|D_k e^{tA_k}\phi\|_0 \le \frac{C}{\sqrt{t}}\|\phi\|_0,$$

hence

$$\|D_k Q(t)\phi\|_0 \le \frac{Ce^{-\mu_k t}}{\sqrt{\beta_k(t)}}\|\phi\|_0. \tag{4.21}$$

Since

$$\beta_k(t) = \frac{1-e^{-2\mu_k t}}{2\mu_k} \ge te^{-\mu_k t},$$

it follows that

$$\|D_k Q(t)\phi\|_0 \le \frac{C}{\sqrt{t}}e^{-\frac{1}{2}\mu_k t}\|\phi\|_0.$$

We have proved in Theorem 3.4.3 (p.58) that $\sum_{i=1}^{\infty}\mu_i^{-\sigma} < \infty$, $(\sigma > 0)$ implies

$$\sum_{k=1}^{\infty} e^{-\frac{1}{2}\mu_k t} < \infty, \quad t > 0, \ \mu_k > 0 \ \ (\forall k).$$

Hence for $t > 0$ we have $Q(t)\phi \in \bigcap_{i=1}^{\infty}\mathcal{D}(D_i)$ and

$$\sum_{i=1}^{\infty}\|D_i Q(t)\phi\|_0^2 \le \frac{C^2}{t}(\sum_{i=1}^{\infty} e^{-\frac{1}{2}\mu_i t})\|\phi\|_0^2 < \infty.$$

This implies that $Q(t)\phi \in C_b^1(X)$ and

$$(DQ(t)\phi(x), h) = \sum_{i=1}^{\infty}(D_i Q(t)\phi(x)e_i, h); \ \forall h \in X.$$

Hence

$$\|DQ(t)\phi\|_0^2 \le \sum_{i=1}^{\infty}\|D_i Q(t)\phi(x)\|_0^2 \le \frac{C}{t}\|\phi\|_0^2.$$

This completes the proof of the theorem. ∎

Now we prove the main theorem of this section. It generalizes the similar results of Da Prato and others (see [18],[30,31],[47], or section 1.3).

Theorem 4.5.4. *Let* $\phi_0 \in D(X)$ *and* $\sum_{i=1}^{\infty} \mu_i^{-\sigma} < \infty$ *for some* $\sigma > 0$. *Let* G *denote the operator* $G(\phi)(\cdot) \equiv F(\cdot, D\phi)$ *satisfying*

(H1). $G : \mathbf{E}_b^1(X) \mapsto \mathbf{E}(X)$;

(H2). For each $r > 0$, *there exists a constant* K_r *such that*

$$\|G(\xi) - G(\eta)\|_0 \leq K_r \|\xi - \eta\|_{\mathbf{E}_b^1(X)}; \tag{4.22}$$

$$\|G(\xi)\|_0 \leq K_r(1 + \|\xi\|_{\mathbf{E}_b^1(X)}), \tag{4.23}$$

for all $\xi, \eta \in B_r \equiv \{\phi \in \mathbf{E}_b^1(X) : \|\phi - \phi_0\|_{\mathbf{E}_b^1(X)} \leq r\}$.

Then there exists a $\tau_r \in (0, T]$ *such that the problem (E4) has a unique mild solution* $\phi \in C(I_r, \mathbf{E}_b^1(X))$, *where* $I_r \equiv [0, \tau_r]$.

Proof. Let $\phi \in C(I_r, \mathbf{E}_b^1(X))$ such that $\phi(t) \in \mathbf{E}_b^1(X) \bigcap B_r$ for all $t \in I_r$. Define the operator Γ by

$$(\Gamma\phi)(t) = Q(t)\phi_0 + \int_0^t Q(t-s)G(\phi(s))ds, \quad t \in [0, T].$$

Since $Q(t)$ is a C_0-semigroup on $D(X)$, there exists a $\sigma \in (0, T]$ such that

$$\|Q(t)\phi_0 - \phi_0\|_{D(X)} \leq \frac{r}{2}, \quad t \in [0, \sigma].$$

By assumption (H2) and Theorem 4.5.3 we have

$$\| \int_0^t Q(t-s)G(\phi(s))ds\|_{\mathbf{E}_b^1(X)} \leq \int_0^t CK_r(1 + \sup_{s \leq t} \|\phi(s)\|_{\mathbf{E}_b^1(X)})ds$$

$$\leq \int_0^t CK_r(1+r+\|\phi_0\|)ds,$$

where C is a constant. Hence there exists $\delta \in [0,T]$ such that

$$\| \int_0^t Q(t-s)G(\phi(s))ds\|_{\mathbf{E}_b^1(X)} \leq \frac{r}{2}, \quad t \in [0,\delta].$$

Letting $\tau_r = \min(\sigma, \delta)$, we have

$$\begin{aligned}
\|(\Gamma\phi)(t) - \phi_0\|_{\mathbf{E}_b^1(X)} &\leq \|(\Gamma\phi)(t) - Q(t)\phi_0\|_{\mathbf{E}_b^1(X)} + \|Q(t)\phi_0 - \phi_0\|_{D(X)} \\
&\leq r, \quad t \in [0,\tau_r].
\end{aligned}$$

Define

$$Y_r \equiv \{\phi \in C(I_r, \mathbf{E}_b^1(X)) : \phi(0) = \phi_0\} \bigcap C(I_r, B_r).$$

By Theorem 4.5.3 we have $\Gamma : Y_r \mapsto Y_r$, i.e., Y_r is invariant under the mapping Γ.

Now we prove that Γ is a contraction on Y_r. For any $\phi, \psi \in Y_r$, by Theorem 4.5.3 and assumption (H2) we have

$$\begin{aligned}
& \|\Gamma\phi - \Gamma\psi\|_{C(I_r,\mathbf{E}_b^1(X))} \\
= \ & \sup_{t\in I_r} \|\Gamma\phi(t) - \Gamma\psi(t)\|_{\mathbf{E}_b^1(X)} \\
= \ & \sup_{t\in I_r} \| \int_0^t Q(t-s)(G(\phi)(s) - G(\psi)(s))ds\|_{\mathbf{E}_b^1(X)} \\
\leq \ & \sup_{t\in I_r} \int_0^t \{\|Q(t-s)(G(\phi) - G(\psi))\|_0 + \|DQ(t-s)(G(\phi) - G(\psi))\|_0\}ds \\
\leq \ & \sup_{t\in I_r} \int_0^t \{\|(G(\phi) - G(\psi))\|_0 + \frac{C}{\sqrt{t-s}}\|(G(\phi) - G(\psi))\|_0\}ds \\
\leq \ & \sup_{t\in I_r} \int_0^t K_r\{\|\phi - \psi\|_{\mathbf{E}_b^1(X)} + \frac{C}{\sqrt{t-s}}\|\phi - \psi\|_{\mathbf{E}_b^1(X)}\}ds \\
\leq \ & \sup_{t\in I_r} \{K_r(C\sqrt{\tau_r} + \tau_r)\|\phi - \psi\|_{\mathbf{E}_b^1(X)})\} \\
\leq \ & M\sqrt{\tau_r}\|\phi - \psi\|_{C(I_r,\mathbf{E}_b^1(X))}.
\end{aligned}$$

Hence Γ is a contraction for τ_r sufficiently small. Now it follows from Banach fixed point theorem that Γ has a unique fixed point in Y_r. This completes the proof. ∎

Remark 4.5.5 If G satisfies the uniform growth condition

$$\|G(\xi)\|_0 \leq K(1 + \|\xi\|_{\mathbf{E}_b^1}), \quad (\forall \xi \in \mathbf{E}_b^1)$$

for a fixed constant K, then the solution of (E4) can be continued indefinitely.

CHAPTER 5

Comments and Suggestions for Future Research

5.1 Some Comments

In this thesis we have made an attempt to study HJB equations in Banach spaces. It seems that this topic has not been well explored before. Clearly it is a very difficult subject. There are many extra obstacles in formulating and solving HJB equations in a Banach space rather than in a Hilbert space. For instance, the optimal control problem we mentioned in Chapter 1 (page 18-19, and equation 1.9) in a Banach space setting becomes

$$dy = (Ay + f(y))dt + Budt + dW, \quad y(0) = y_0,$$

$$J(u) = E\{\int_0^T (g(y(s)) + \frac{1}{2}\|u\|_U^2)ds + \phi_0(y(T)) \rightarrow min,$$

where $A : \mathcal{D}(A) \mapsto X$; $B : X \mapsto U$ are linear operators. The associated HJB equation has the form [Ahmed 4]:

$$\begin{cases} \phi_t & = \frac{1}{2}Tr(S\phi_{xx}) + (Bx + f(x), \phi_x)_{X,X^*} + \frac{1}{2}\|B^*\phi_x\|_{U^*}^2 + g(x) \\ \phi(0,x) & = \phi_0(x), \end{cases}$$

with optimal control $u_0 = \nu(-B^*\phi_x)$, where ν is the duality map from $U^*\backslash\{0\} \mapsto U$ (usually multivalued map). Obviously its non-linearity is stronger than that of its counterpart in Hilbert spaces.

The work presented in this thesis is only a beginning in attacking this vast and formidable area. It is natural that the results obtained here have many drawbacks. One example is, in our treatment in Chapter 3 and 4, (page 46 and 68), the linear unbounded operator B is assumed to be the infinitesimal generator of an analytic semigroup, it would be very nice if B could only be the infinitesimal generator of a C_0-semigroup. Another problem is that it is hard to characterize the domains of the semigroups $R(t)$ and $Q(t)$, (i.e., the spaces $E(X)$ and $D(X)$), although we know they are not empty. A simple characterization of these spaces would be very useful in applications.

We can also improve our work by presenting some applications whatsoever of the abstract results in this thesis. The first application will be the stochastic optimal control problems mentioned in Sections 1.2, 3.1, and 4.1. In this direction we should deduce a feedback law for each of the optimal control problems, from the mild solution of the corresponding HJB equation.

There are several directions in which we may extend the results in this thesis. One is the optimal control problem for stochastic systems governed by non-linear stochastic differential equations, another is the viscosity solutions for second order non-linear HJB equations in infinite dimensions.

Below we discuss these topics briefly. We mention that the study of computational methods for stochastic optimal control problems is also an interesting subject.

5.2 Non-linear Stochastic Systems

Consider the following non-linear stochastic differential equation

$$d\xi(t) = (B\xi(t) + f(\xi(t)))dt + dW(t) \tag{5.1}$$

where $W(t)$ is a Wiener process in Banach space X, with covariance operator S; B is the infinitesimal generator of a C_0-semigroup; and $f(x)$ is a continuous function.

We are interested in the study of the Kolmogorov (forward) equation relative to the equation (5.1), that is

$$\begin{cases} \phi_t(t,x) &= \frac{1}{2}Tr(S\phi_{xx}(t,x)) + (Bx + f(x), \phi_x(t,x)) + g(x) \\ \phi(0,x) &= \phi_0(x). \end{cases}$$

Even more generally, we will study the following semilinear HJB equation

$$\begin{cases} \phi_t(t,x) &= \frac{1}{2}Tr(S\phi_{xx}(t,x)) + (Bx + f(x), \phi_x(t,x)) + F(x,\phi,\phi_x) \\ \phi(0,x) &= \phi_0(x) \end{cases} \tag{5.2}$$

where S is a self-adjoint positive nuclear operator. $f(x)$, $g(x)$, $\phi_0(x)$ are given functions.

Using the results in Chapter 3, we may consider the mild solution of equation (5.2).

$$\phi(t,x) = (R(t)\phi_0)(x) + \int_0^t R(t-s)\{(f(x),\phi_x) + F(x,\phi(s,x),\phi_x(s,x))\}ds,$$

where $R(t)$ is the C_0-semigroup defined in section 3.3. Under some fairly moderate assumptions on functions $f(x)$ and F, we may be able to prove the existence and uniqueness results of the mild solutions for equation (5.2), in a suitable subspace of $C(X)$. In particular, when $F(x, \phi, \phi_x) = -\frac{1}{2}\|\phi_x\|^2_{X^*} + g(x)$, we can get an extension of the results in Havârneanu [47] by combining the strategies therein and the properties of semigroup $R(t)$.

Another approach to equation (5.2) is to treat the extra term $(f(x), \phi_x)$ as a perturbation to the infinitesimal generator of $R(t)$. By the perturbation theory of semigroups we may prove that

$$\frac{1}{2}Tr(S\phi_{xx}(t,x)) + (Bx + f(x), \phi_x(t,x))$$

generates a semigroup in a subspace of $C_b(X)$. In this direction, a simple example is given in Da Prato [29], with $f(x) = Sg(x)$, $g(x) \in C_b(X)$.

5.3 Viscosity Solutions for Second Order HJB Equations in Infinite Dimensions

As we have mentioned in Chapter 1, section 1.4, the theory of viscosity solutions provides a powerful tool for the study of HJB equations. For the first order HJB equations in a finite dimensional space, this theory has been developed extensively, see Crandall, Ishii and Lions [22,23,24], for example. However, these cases do not fit in the framework of optimal stochastic control problems, where the associated HJB equations are second order partial differential equations (PDEs).

In 1988, P.L.Lions considered the second order PDE of the following form

$$F(D^2u, Du, u, x) = 0, \quad x \in H, \tag{5.3}$$

where H is a separable Hilbert space. F is bounded, uniformly continuous on bounded set, and satisfies the monotonicity condition (or degenerate ellipticity, see section 1.4). He then introduced a notion of viscosity solutions as follows:

Definition 5.2.1. Let

$$\begin{gathered} Q = \{\phi \in C^1(H, R) \mid D\phi \text{ is Lipschitz on bounded sets of } H; \\ \text{for all } k, h \in H, \lim_{t \to 0^+}(1/t)(D\phi(x + tk) - D\phi(x), h) \\ \text{exists and is uniformly continuous on bounded sets of } H\}. \end{gathered}$$

Let $u \in BUC_{loc}(H)$. We say that u is a viscosity subsolution of equation (5.3) if for each $\phi \in Q$ and at each local maximum x^* of $u - \phi$, we have

$$\liminf_{y \to x^*} F(D^2\phi(y), D\phi(x^*), u(x^*), x^*) \leq 0;$$

We say that u is a viscosity supersolution of equation (5.3) if for each $\phi \in Q$ and at each local minimum x_0 of $u - \phi$, we have

$$\liminf_{y \to x_0} F(D^2\phi(y), D\phi(x_0), u(x_0), x_0) \geq 0.$$

We say u is a viscosity solution of equation (5.3) if u is both a viscosity subsolution and supersolution.

Clearly this notion is adapted from the definition of viscosity solutions for the first order equations in finite dimensions. Using this definition, P.L.Lions has proved that the value function of a specific class of stochastic optimal control problems is the viscosity solution of the associated HJB equation, under some severe restrictions. (Lions [59]).

In the finite dimensional case, general existence and uniqueness results for the second order equations have been recently proved, see Jensen et al [52], Lions and Souganidis [60], Ishii and Lions [51]. All these proofs have used in a fundamental way the existence of second order expansions at almost all points for convex or concave functions on R^n, a classical result due to Alexandrov, whose counterpart in infinite dimensions is not clear. This seems to prevent a straightforward adaptation of these arguments to infinite dimensions.

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