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Relational Models of the Lambda Calculus

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RELATIONAL MODELS OF THE LAMBDA CALCULUS

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Abstract

In [7], Ehrhard et al. present a model of the untyped lambda calculus built from an object without enough points in a cartesian closed category **MRel**. This thesis presents the background needed to construct and understand this model. In particular we describe what it means for models to have enough points and exhibit connections between **MRel** with various categorical models of lambda calculus in the literature. In particular, we are able to relate the graph model to **MRel**. We also describe connections with various kinds of Kleisli categories arising from comonads and their associated theory.

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Contents

1 Introduction

The untyped lambda calculus is a fundamental model of computing that was first introduced by A. Church in the 1930's [8]. Many areas of mathematics consider functions as graphs. That is to say a function $f : X \longrightarrow Y$ is defined by the set $\{(x, y) \in X \times Y \mid f(x) = y\}$ and two functions are said to be equal if they have equal graphs, i.e. they have the same range of arguments and agree on all outputs. This view of functions is called *extensional.* In the lambda calculus, we can have an *intensional* view of functions. We consider functions as rules or formulas. A function is no longer defined by its graph; rather it is defined by how the output of the function is calculated, by its rule. Using this view, we can still say that two functions are extensionally equal if, as before, they range over the same arguments and agree on all outputs. But now we have another notion of equality. We say that two functions are intensionally equal if they are calculated in the same way, in other words, they are equal if they share the same rule.

This notion of thinking of functions as rules is used in many fields. Consider for example computer science: take two programs that are extensionally equal, i.e. yield the same output on the same input. It is of course of interest for a computer scientist to choose the program which is faster and/or requires less space. These differences are intensional differences. Notice that a program can have other programs, or even itself, as input. The same is true in the lambda calculus. The lambda calculus is a type-free structure and therefore functions can be applied to functions (even to themselves). The intuition is to consider a universe $\mathcal{U}$ of functions of a very general kind: functions in $\mathcal{U}$ can apply to any other function in $\mathcal{U}$, so such expressions as $f(g)$ or even $f(f)$ make sense. On the one hand, as we have mentioned, this notion of function has turned out to be obviously useful in computer science, since if we think of programs as functions, this kind of function application allows programs to call themselves. On the other hand, it obviously has set-theoretical problems for mathematicians: how do we make sense of $f(f)$?

Thus lambda calculus studies functions and how they apply to other functions. The two basic underlying ideas are *application* and *abstraction* which are both primitive in the sense that they are built into the definition of the lambda calculus. On the one hand, a function f applied to an argument a is written fa. Abstraction, on the other hand, is the process of starting with any property $\varphi(x)$ and forming the function $x \mapsto \varphi(x)$. In lambda calculus, the latter function is denoted $\lambda x.\varphi(x)$. Thus, in ordinary mathematics, the expression $\lambda x.2x$ denotes the doubling function (on some implicit domain of interpretation) and $\lambda x.x$ denotes the identity function (again on some implicit domain). The fundamental law combining the two is the β-rule:

$$(\lambda x.\varphi(x))a = \varphi(a)$$

This says that to evaluate the function $\lambda x.\varphi(x)$ applied to argument a, do the following: substitute a for all the x's in the expression φ. The key fact about lambda calculus is that (i) there could be millions of such x's and (ii) the a is itself a lambda term. Thus in general we may get non-terminating computations. Consider the term $\omega = \lambda x.xx$. Now consider $\omega\omega$. According to the β-rule,

$$\begin{aligned} \omega\omega &= (\lambda x.xx)\omega \\ &= \omega\omega \\ &= (\lambda x.xx)\omega \\ &= \ldots \end{aligned}$$

As you can see, the term $\omega\omega$ is non-terminating.

The main problem we are interested in investigating is semantical models of this theory: a class of functions and a notion of application which models all the axioms of lambda calculus. A model of the lambda calculus will need to be able to interpret functions and arguments on the same level. For this, we need an object U such that $U^U = U$ or, a slightly relaxed condition, that $U^U \cong U$. This leads us back to the aforementioned set-theoretical problems. As we will see in section 2.2.2, in the world of sets and functions, this isomorphism is not possible unless U is a singleton because of cardinality problems. Obviously, U a singleton set does not lead to any interesting models of the lambda calculus. More generally, we seek a cartesian closed category (i.e. a category with function spaces) with an object $U \neq \mathbf{1}$ such that $U^U \cong U$. It was not until 1969 in [14], that the first interesting mathematical models were developed by Dana Scott. The idea here was to consider a cartesian closed category of lattices with an appropriate topology on the function spaces. Then take the set of continuous functions from U to itself, written $[U \Rightarrow U]$, instead of the set of all functions U^U. This solves the cardinality problem mentioned earlier.

In this thesis, we study a recent new model for untyped lambda calculus defined in a category called **MRel**, due to Ehrhard et al.[7]. This model arises from linear logic. Linear logic was invented by J-Y Girard in 1987 ([9]) as a kind of fine-grained analysis of the traditional laws of logic. If we use a Gentzen sequent-style approach, linear logic modifies the rules of logic by restricting the use of contraction and weakening in the structural rules. This entails a division of the rules of logic into three "levels": multiplicative, additive, and exponential. It turns out that just as lambda calculus is intimately related to cartesian closed categories ([11]) also linear logic has a remarkable connection with monoidal category theory ([6]).

The multiplicative level corresponds categorically (roughly) to $*$-autonomous categories (symmetric monoidal closed categories with a notion of negation or duality $A^\perp = A \multimap \perp$ induced from a dualizing object $\perp$). The multiplicative connectives are $\otimes, ⅋$, and $\multimap$ [1]. The additive level corresponds to adding products (and, by duality, coproducts) to the multiplicative level. Finally, the most subtle level is the exponential level, which correspond to having an endofunctor ! which marks the formulas where the traditional laws of contraction and weakening take place (for example, contraction is the law $!A \rightarrow !A \otimes !A$). The algebraic structure that ! satisfies is intricate and has taken a long time to clarify (It started with the paper of Seely [17] and culminated in more recent work [5, 13].

An important (degenerate) model of linear logic is **Rel** (the category of sets with maps being binary relations) in which the binary connectives (on the objects of **Rel**, i.e. on Sets) are trivially identified with the cartesian product and negation does

[1] ⅋ is the de Morgan dual of $\otimes$ and $\multimap$ (the linear function space) is definable as $A \multimap B = A^\perp ⅋ B$

nothing: $A \otimes B = A \parr B = A \multimap B = A \times B$ and $A^\perp = A$. Note however that $(-)^\perp$ on morphisms (relations) is nontrivial: $R^\perp = R^{op}$, the converse relation. It can be shown that categorical products (denoted &) are given by disjoint union $A \& B = (\{1\} \times A) \cup (\{2\} \times B)$. In this model, it turns out that ! has been effectively modelled by the finite multiset functor, so $!A = \mathcal{M}_f(A)$, the set of all finite multisets over the set A and $! = \mathcal{M}_f$ is a comonad on **Rel**. From the general theory of models of linear logic, the Kleisli category of ! is cartesian closed. This Kleisli category, which is what we have been referring to as **MRel**, is the subject of this thesis. It is both connected with an interesting model of linear logic (**Rel**) as well as with models of lambda calculus (via cartesian closedness). Many of the features we discuss below are motivated from this background.

As we have mentioned, a model of the untyped lambda calculus corresponds to an object U in some ambient cartesian closed category such that $U^U \cong U$. In fact, we are able to model the untyped lambda calculus with a retraction $U^U \lhd U$, but this model will no longer model the η-rule. The η-rule identifies two functions if and only if they agree on all inputs:

$$\lambda x.\varphi x = \varphi$$

where x is not a free variable in φ (see section 2.2.1 for a discussion of free variables). The η-rule essentially identifies a function with its graph, so imposing η is another way of imposing extensional equality between functions. In many models the stronger condition that the terminal object **1** is a generator or that the object U has enough points is true. These models are also known as lambda models.

The work that will follow will present the background needed to further understand the lambda calculus and its models and the category theory needed to understand the construction of **MRel** (the coKleisli category of the finite multiset functor on **Set** lifted to **Rel**). We then try to relate known models of the untyped lambda calculus to this category **MRel**. In the first chapter, we introduce the notation we will be using for finite multisets, give a brief summary of a well known model of the untyped lambda calculus, the graph or $\mathcal{P}\omega$-model due to Dana Scott, and finally, we define some basic category theoretic notions such as functors, monads and Kleisli categories. The examples of these notions will culminate in the construction of the aforementioned category **MRel**. The next chapter will investigate several elements of the structure of **MRel**. First, we show that **MRel** is a cartesian closed category and describe the construction of an extensional reflexive object in **MRel** given in [7]. As we have already mentioned, these two facts combined tell us that we are able to model the untyped lambda calculus in **MRel**. We will then show that the terminal object of **MRel** is not a generator and describe the global sections functor for this category. This structure will be the basis of what is needed for the third chapter, in which we succeed in describing the graph model in terms of objects and maps in **MRel**. We also get a result showing the relationship between retractions in **MRel** and retractions in the category of algebraic lattices **AlgLat**. The last section of this chapter discusses further work that could make our description of the graph model somewhat more natural.

In summary, this thesis is an exposition, expansion and clarification of the paper of Ehrhard et al. [7]. This thesis accomplishes the following:

1. We give the appropriate categorical and lambda calculus background to understand the paper [7] and the work presented in this thesis.

2. We give a detailed treatment of finite multisets.

3. We present a well known model of the lambda calculus: the graph or $\mathcal{P}\omega$ model.

4. Some of our new results include:
   - Study of the connections of the general Ehrhard construction with the graph model (section 4.1, section 4.2, Proposition 4.2.1).
   - Study of the role of distributive laws of monads in the above (section 4.4).
   - Results showing the relationship between reflexive objects in **MRel** and reflexive objects in the category of algebraic lattices (Proposition 4.3.1).
   - Results showing the relationship between the reflexive object given in [7] and C-monoids (Proposition 3.5.1, Proposition 3.5.7).

2 The Basics

2.1 Introduction to multisets

A multiset is a set with multiplicity, that is to say, an element of a multiset may appear more then once. More formally,

Definition 2.1.1. A *multiset* α over a set S is a pair (A, m), where $A \subseteq S$ is called the *support* of α and m, a function $m : A \longrightarrow \mathbb{N}_+$, is called the multiplicity function. The support A is the set of all the distinct elements of α and for $a \in A$, $m(a) =$ multiplicity of a. In other words, $m(a)$ is the number of times a appears in α. We say that two multisets, say $(A, m), (B, n)$ over a set S, are equal when both $A = B$ and $m = n$.

Remark 2.1.2. Note that an ordinary set A is a multiset where every element has multiplicity 1. In other words, the multiplicity function of an ordinary set $m : A \longrightarrow \mathbb{N}_+$ is constantly 1.

Definition 2.1.3. We define the *union* of two multisets α_1,α_2 over a set S, written $\uplus$ as follows: we have α_1 with $A \xrightarrow{m} \mathbb{N}_+$ and α_2 with $B \xrightarrow{n} \mathbb{N}_+$, then $\alpha_1 \uplus \alpha_2$ is defined by the pair $(A \cup B, l)$ where for $x \in A \cup B$

$$l(x) = \begin{cases} m(x) & \textit{if } x \in A \backslash B \\ n(x) & \textit{if } x \in B \backslash A \\ m(x) + n(x) & \textit{if } x \in A \cap B \end{cases}$$

We say that a multiset is *finite* if its support is a finite set. Suppose we have a finite multiset of finite multisets $\mathcal{A}$ over a set S, then $\mathcal{A} = (B, m)$, where $B = \{\alpha_1, \dots, \alpha_k\}$ for multisets $\alpha_1, \dots, \alpha_k$ over S. Then

$$\biguplus \mathcal{A} = (\dots((\alpha_1 \uplus \alpha_2) \uplus \alpha_3) \cdots \uplus \alpha_k)$$

We also write this as $\biguplus_{i=1}^{k} \alpha_i$.

If S is a set, we denote $\mathcal{M}_f(S)$ as the set of all finite multisets over S.

$$\mathcal{M}_f(S) = \{(A, m) \mid A \subseteq_f S, A \xrightarrow{m} \mathbb{N}_+\}$$

Notice that the empty multiset is the pair $(\emptyset, m_\emptyset)$, where $\emptyset \xrightarrow{m_\emptyset} \mathbb{N}_+$ is the unique empty function. Therefore, $\mathcal{M}_f(\emptyset) = \{*\}$, a singleton, namely $\{(\emptyset, m_\emptyset)\}$.

For $\alpha \in \mathcal{M}_f(A)$, A a set, we can represent α as a finite set of pairs, say

$$\alpha = \{(a_1, m(a_1)), \ldots, (a_k, m(a_k))\}$$

The first element of the pair denotes the element of A that is in α and the second element of the pair is the multiplicity of that element. This is in many cases more information then needed; therefore *for simplicity, we will usually denote α as a list* $[a_1, \ldots, a_k]$, $a_i \in A$, *with possibility of repetition.* We write $[\,]$ for the empty multiset.

Definition 2.1.4 (see [7]). An $\mathbb{N}$-indexed sequence of multisets $\{\alpha_i\}$ is *quasi-finite* if $\alpha_i = [\,]$ for all but finitely many indices i.

Definition 2.1.5 (see [7]). Let A be a set then $\mathcal{M}_f(A)^{(\omega)}$ is the set of all quasi-finite $\mathbb{N}$-indexed sequences of multisets over A.

Notation 2.1.6. As a convention, throughout this paper, $\alpha, \beta, \ldots$ will denote multisets. In other words, they are elements of $\mathcal{M}_f(S)$ for some set S. Usually, we will have $\alpha \in \mathcal{M}_f(A)$, $\beta \in \mathcal{M}_f(B)$, but this is not always possible. $\mathcal{A}, \mathcal{B}, \ldots$ will denote elements of $\mathcal{M}_f(\mathcal{M}_f(A)), \mathcal{M}_f(\mathcal{M}_f(B)), \ldots$ for some sets $A, B, \ldots$. In other words, if $\mathcal{A} \in \mathcal{M}_f(\mathcal{M}_f(A))$, $\mathcal{A} = [\alpha_1, \ldots, \alpha_k]$ where each $\alpha_i \in \mathcal{M}_f(A)$. $X, Y, \ldots$ will denote elements of $\mathcal{M}_f(\mathcal{M}_f(\mathcal{M}_f(A)))$, for some set A. Here $X = [\mathcal{A}_1, \ldots, \mathcal{A}_k]$, where each $\mathcal{A}_i \in \mathcal{M}_f(\mathcal{M}_f(A))$.

2.2 Introduction to lambda calculus

The lambda calculus is based on two fundamental operations: function application and abstraction. Abstraction denotes the operation $x \mapsto \varphi(x)$ for some expression φ. In the lambda calculus, we write this as $\lambda x.\varphi(x)$. Suppose $\varphi(x)$ is the polynomial $2x$:

$\varphi(x) = 2x$. We can apply the expression $\lambda x.\varphi(x)$ to another expression, say a:

$$\begin{aligned}(\lambda x.\varphi(x))a &= (\lambda x.2x)a \\ &= 2a \\ &= \varphi(a)\end{aligned}$$

where $\varphi(a)$ is the expression $\varphi(x)$ with all the occurrences of x replaced with a. This can also be written as $\varphi(x)[x := a]$, read $\varphi(x)$ with x replaced by a. This reduction principle of expressions in the lambda calculus is known as the β-rule.

This section will introduce some basic theory concerning lambda calculus. The first part will define the theory of λ or λ-calculus and mention some basic notions such as α-conversion and substitution. The second part will describe what a model of the λ-calculus is from a category theoretic point of view and the last part will give a brief presentation of a well known example of such a model.

2.2.1 The λ-calculus

Here we provide a brief introduction to the syntax of the λ-calculus and follow the presentation given in [2], primarily section 2.1.

Definition 2.2.1. *Lambda terms* are words built from the alphabet consisting of variables $x, y, z, \ldots$, parentheses $(,)$ and an abstractor written λ.

The set of lambda terms Λ is defined inductively as follows: let $\mathcal{V}$ denote the set of variables then

- $\mathcal{V} \subseteq \Lambda$
- if $x \in \mathcal{V}$ and $M \in \Lambda$, then $(\lambda x.M) \in \Lambda$
- if $M, N \in \Lambda$, then $(MN) \in \Lambda$

where MN denotes M applied to N.

Remark 2.2.2. In applied λ-calculi, we may have additional constants and/or term forming operations. For example, in [11], the untyped λ-calculus has projections and surjective pairing operations, $\pi_i(M)$ and $\langle -, - \rangle$, satisfying product equations. These types of λ-calculi are referred to as extended λ-calculi. Here, we will write Δ_A for the set of lambda terms with one constant c_a symbol for each element of a of the set A.

Notation 2.2.3. Notice that when it is not necessary, parentheses in lambda terms will be omitted. Also, for terms $M_1, \dots M_k$, the term $(\cdots((M_1 M_2)M_3)\cdots M_k)$ is abbreviated to $M_1 M_2 \dots M_k$ and a term of the form $(\lambda x_1.(\lambda x_2.(\cdots(\lambda x_n.M)\cdots)))$ is abbreviated to $\lambda x_1 \dots x_n.M$.

Notice that lambda abstraction, $\lambda x.-$, is a variable binding operation similar to $\forall x.\varphi(x)$ or $\int f(x)dx$. Hence, as in logic, we must use the usual conventions of free and bound variables.

Definition 2.2.4. A variable x is *free* in a lambda term M if x is not in the scope of a λx. Otherwise, x is *bound.* The set of free variables in M, written $FV(M)$ is defined as follows:

- $FV(x) = \{x\}$
- $FV(\lambda x.M) = FV(M)\backslash\{x\}$
- $FV(MN) = FV(M) \cup FV(N)$

The set of bound variables in M, written $BV(M)$ is defined as follows:

- $BV(x) = \emptyset$
- $BV(\lambda x.M) = BV(M) \cup \{x\}$
- $BV(MN) = BV(M) \cup BV(N)$

A term with no free variables is *closed.*

Definition 2.2.5. Informally, *α-conversion* or a *change in bound variables* in a lambda term M is the replacement of a subterm of M of the form $\lambda x.N$ by $\lambda y.(N[x := y])$, where y does not occur in N. We therefore syntactically identify terms that are equal up to a change of bound variable.

Notation 2.2.6. We let $M[x := N]$ denote the term obtained by substituting all the occurrences of x in M by N, where α-conversion is performed if necessary to prevent free variables from being bound mistakenly. Consider the expression $(\lambda x.(\lambda y.yx))y$, where the last occurrence of variable y is free. Without performing α-conversion, this reduces to $\lambda y.yy$, the free variable y has become bound by the λy abstraction.

To prevent this, we first convert $(\lambda x.(\lambda y.yx))y$ to $(\lambda x.(\lambda z.zx))y$, then this expression now reduces to $\lambda z.zy$ and no free variables have been inadvertently captured.

Definition 2.2.7. The λ-calculus is a formal theory with formulas $M = N$, for $M, N \in \Lambda$ given as follows:

1. Equality Axioms:

$$M = M \qquad \frac{M = N}{N = M} \qquad \frac{M = N, N = L}{M = L}$$

2. Congruence Axioms:

$$\frac{M = N}{Mx = Nx} \qquad \frac{M = N}{yM = yN} \qquad \frac{M = N}{\lambda x.M = \lambda x.N}\ \xi\text{-rule}$$

3. Lambda Calculus Axiom

$$(\lambda x.M)N = M[x := N] \qquad \beta\text{-rule}$$

Remark 2.2.8. If $M = N$ is a derivable equation in the λ-calculus, many people, such as in [2], will write this as $\vdash M = N$, as in logic. Here we use the usual convention in algebra and write $M = N$ for a derivable equation when this is clear.

In ordinary mathematics, we look at functions extensionally. That is to say, we define functions by their graphs:

$$f : A \longrightarrow B = \{(x, y) \in A \times B \mid f(x) = y\}$$

This is captured in lambda calculus in one of two ways.

Definition 2.2.9.

- *Extensionality rule*

  $$\text{if } Mx = Nx \text{ then } M = N$$

  provided that $x \notin FV(M) \cup FV(N)$.

- The *η-rule* or *η-conversion*

$$\lambda x.Mx = M$$

provided $x \notin FV(M)$. The $\lambda\eta$-calculus is the λ-calculus extended with η.

Proposition 2.2.10 (see [2]). *The λ-calculus extended with the extensionality rule and the $\lambda\eta$-calculus are equivalent.*

2.2.2 Models of the λ-calculus

The difficulty with modelling the untyped λ-calculus is that since such expressions as fg or even ff are allowed, we must be able to interpret them in the model: we need to be able to have functions and arguments live at the same level. In other words, we need a set U where the exponential object U^U is equal to U. In **Set**, even if we relax equality of sets to a bijection $U^U \cong U$, this is not possible unless $|U| = 1$: suppose $|U| > 1$ and suppose that there exists a set theoretic bijection $f : U^U \longrightarrow U$. The fact that this function must be total and injective means that $|U| \geq |U^U|$; but we know that for a set U such that $|U| > 1$, $|U| < |U^U|$, this is a contradiction. Therefore the cartesian closed category **Set** cannot model the untyped λ-calculus. Starting with [14], we know that it is possible to find a cartesian closed category $\mathcal{C}$ with a non-trivial object U such that $U^U \cong U$ via $\psi : U^U \longrightarrow U$ and $\varphi : U \longrightarrow U^U$ where $\psi = \varphi^{-1}$. With such an object, we can then define application for $M, N \in U$ by $MN = ev(\varphi(M), N)$, where ev is the evaluation map in the ambient cartesian closed category $\mathcal{C}$. The laws of the cartesian closedness of $\mathcal{C}$ assure us that we have both the β and η-rules, although to model the λ-calculus without eta, it suffices to have an object U such that $U^U \lhd U$, i.e. that U^U is a retract of U (see definition 2.2.12 below). In this case we model the β-rule without the η-rule.

In many concrete cases, we have that U^U is not the set of all possible maps between U and itself, but instead, the set of all continuous maps between U and itself where U is given an appropriate topology. We write this $[U \Rightarrow U]$ or $[U, U]$. For example, in the category of algebraic lattices **AlgLat** (see the next section 2.2.3), $[U \Rightarrow U]$ is the set of all Scott continuous maps from U to itself, that is maps that preserve joins of directed sets. This restriction to continuous maps avoids the set theoretic problems which arose when working over **Set**.

The question to be answered now is: What is a general model of the untyped λ-calculus? There are three styles of models:

- lambda algebras,
- lambda models and
- categorical models in a cartesian closed category.

The categorical models are easiest to define.

Definition 2.2.11. A category $\mathcal{C}$ is said to be *cartesian closed* if it admits finite products such that for every object B of $\mathcal{C}$, the functor $- \times A : \mathcal{C} \longrightarrow \mathcal{C}$ has a specified right adjoint denoted $(-)^A : \mathcal{C} \longrightarrow \mathcal{C}$. This means there is a natural isomorphism, for all objects A,B and C of $\mathcal{C}$

$$\mathcal{C}(C \times A, B) \cong \mathcal{C}(C, B^A).$$

Equivalently (See [2]), a category $\mathcal{C}$ cartesian closed if

- it has a terminal object $\mathbf{1}$ such that for every object $A \in \mathcal{C}$ there exists a unique map $!_A : A \longrightarrow \mathbf{1}$,
- any two objects A and B of $\mathcal{C}$ have a product $A \times B$ in $\mathcal{C}$ with distinguished projection arrows $\pi_i : A_1 \times A_2 \longrightarrow A_i$ $(i = 1, 2)$, such that for all $f_i : C \longrightarrow A_i$ $(i = 1, 2)$, there is a unique map $\langle f_1, f_2 \rangle : C \longrightarrow A_1 \times A_2$ such that $\pi_i \circ \langle f_1, f_2 \rangle = f_i$ $(i = 1, 2)$,
- any two objects A and B of $\mathcal{C}$ have an exponential B^A in $\mathcal{C}$ with a distinguished evaluation arrows $ev_{A,B} : B^A \times A \longrightarrow B$ such that for all $f : C \times A \longrightarrow B$ there is a unique map $f^* : C \longrightarrow B^A$ such that $f = ev_{A,B} \circ (f^* \times Id_A)$.

Definition 2.2.12. Let $\mathcal{C}$ be a cartesian closed category with terminal object $\mathbf{1}$ and U be an object in $\mathcal{C}$.

(i) U is said to be *reflexive* if U^U is a retract of U, written $U^U \lhd U$. That is to say, there are maps $\psi : U^U \longrightarrow U, \varphi : U \longrightarrow U^U \in \mathcal{C}$ such that $\varphi \circ \psi = Id_{U^U}$. We say that U is an *extensional* reflexive object if we also have that $\psi \circ \varphi = Id_U$, i.e. $U^U \cong U$.

(ii) (see [1]) Suppose U is a reflexive object via the retraction pair $\psi : U^U \longrightarrow U$ and $\varphi : U \longrightarrow U^U$. Let M be a lambda term with $FV(M) \subseteq \Delta = \{x_1, \dots x_n\}$. Define then:

$$[\![M]\!]_\Delta \in \mathcal{C}(U^n, U) \text{ where } U^n = (\dots(\mathbf{1} \times U) \times \dots) \times U \text{ with } n \text{ copies of } U$$

as follows:

$$\begin{array}{lcl} [\![x_i]\!]_\Delta & = & \pi_2 \circ \pi_1^{n-i} \\ [\![MN]\!]_\Delta & = & ev \circ \langle \varphi \circ [\![M]\!]_\Delta, [\![N]\!]_\Delta \rangle \\ [\![\lambda x.M]\!]_\Delta & = & \psi \circ ([\![M]\!]_{\Delta \cup \{x\}})^* \end{array}$$

where $\pi_1^j = \pi_1 \circ \dots \circ \pi_1, j$ times, ev is evaluation in the cartesian closed category and $(-)^*$ is the curry operation in the cartesian closed category.

With this, we have that a cartesian closed category with a reflexive object is a model of the λ-calculus. If, instead, we have an extensional reflexive object U, $U^U \cong U$, then this is a model of the $\lambda\eta$-calculus.

Proposition 2.2.13 (Soundness, see [2]). *Let $\mathcal{C}$ be a cartesian closed category with reflexive object U and let $M, N \in \Lambda$ such that $FV(MN) \subseteq \Delta$. Then*

$$\lambda \vdash M = N \quad \Longrightarrow \quad [\![M]\!]_\Delta = [\![N]\!]_\Delta$$

and similarly, if U is an extensional reflexive object, $U^U \cong U$, then

$$\lambda\eta \vdash M = N \quad \Longrightarrow \quad [\![M]\!]_\Delta = [\![N]\!]_\Delta$$

We will now briefly define λ-algebras and λ-models which are two other styles of modelling the untyped λ-calculus. Lambda algebras and lambda models are often defined in terms of *combinatory logic*, an area closely related to lambda calculus [2]. The following presentation follows [18].

Definition 2.2.14. *Combinatory logic terms*, CL-terms, are words built from the alphabet consisting of variables $x, y, z, \dots$ ranging over a countable set $\mathcal{V}$, parentheses $(,)$ and constants K, S.

The set of CL-terms $\mathfrak{C}$ is defined inductively as follows:

- if $x \in \mathcal{V}$, then $x \in \mathfrak{C}$

- $K \in \mathfrak{C}, S \in \mathfrak{C}$
- if $P, Q \in \mathfrak{C}$, then $(MN) \in \mathfrak{C}$

where PQ denotes P applied to Q. Here, for a term P, we let $FV(P)$ be the set of all variables in P and we say a term P is *closed* if $FV(P) = \emptyset$. For a set A, $\mathfrak{C}_A$ is the set of CL-terms with one constant symbol c_a for each element a of A.

Notation 2.2.15. Notice that when it is not necessary, parentheses in CL-terms will be omitted.

Definition 2.2.16. *Combinatory logic*, CL, is an equational theory with formulas $P =_{CL} Q$ for $P, Q \in \mathfrak{C}$ given as follows:.

1. Equality Axioms:

$$P =_{CL} P \qquad \frac{P =_{CL} Q}{Q =_{CL} P} \qquad \frac{P =_{CL} Q, Q =_{CL} R}{P =_{CL} R}$$

2. K and S axioms:

$$KPQ =_{CL} P \qquad SPQR =_{CL} PR(QR)$$

3. Congruence and substitution axiom

$$\frac{P =_{CL} Q, R =_{CL} T}{PR =_{CL} QT} \qquad \frac{P =_{CL} Q}{P[x := R] =_{CL} Q[x := R]}$$

Definition 2.2.17 (see [18]). An *applicative structure* $(\mathbf{A}, \bullet)$ is a set $\mathbf{A}$ together with a binary operation $\bullet : \mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}$. We write ab for $a \bullet b$ and $a_1 \cdots a_k$ for $(\cdots(a_1 \bullet a_2) \ldots) \bullet a_k)$. A *combinatory algebra* $(\mathbf{A}, \bullet, k, s)$ is an applicative structure with elements $k, s \in \mathbf{A}$ such that $kxy = x$ and $sxyz = xz(yz)$ for every $x, y, z \in \mathbf{A}$.

It is clear from their definitions that a combinatory algebra is a model of combinatory logic.

Definition 2.2.18. Let $\mathbf{A}$ be a combinatory algebra and $\mathfrak{C}_{\mathbf{A}}$ be the set of combinatory terms with one constant symbol c_a for each element a of $\mathbf{A}$.The *(local)* *interpretation* of CL-terms, with respect to a valuation of variables $\rho : \mathcal{V} \longrightarrow \mathbf{A}$, is defined inductively as follows:

$$\begin{aligned} [\![x]\!]_\rho &= \rho(x) \\ [\![c_a]\!]_\rho &= a \\ [\![K]\!]_\rho &= k \\ [\![S]\!]_\rho &= s \\ [\![PQ]\!]_\rho &= [\![P]\!]_\rho \bullet [\![Q]\!]_\rho \end{aligned}$$

For terms $P, Q \in \mathfrak{C}_\mathbf{A}$, we say the equation $P = Q$ holds (locally) in $\mathbf{A}$, written $\mathbf{A} \models P = Q$, if for all valuations ρ in $\mathbf{A}$, $[\![P]\!]_\rho = [\![Q]\!]_\rho$.

Proposition 2.2.19 (see [18]). *Let $\mathcal{T}$ be a set of equations between combinatory terms. For constant-free combinatory terms P and Q*

$$\mathcal{T} \vdash P = Q \textit{ iff } \mathbf{A} \models P = Q \textit{ for all combinatory algebras } \mathbf{A} \textit{ such that } \mathbf{A} \models \mathcal{T}$$

There is a close connection of combinatory logic with lambda calculus, in that we can simulate a λ-operator in CL as follows.

Definition 2.2.20. (See [2]) Define the *derived lambda abstractor* of combinatory logic, λ^*, as follows: for a combinatory term R and a variable x, define $\lambda^* x.R$ inductively:

$$\begin{aligned} \lambda^* x.x &= SKK & \\ \lambda^* x.P &= KP & \text{if } x \notin FV(P) \\ \lambda^* x.PQ &= S(\lambda^* x.P)(\lambda^* x.Q) & \end{aligned}$$

We now use this derived lambda abstractor to define translations between $\mathfrak{C}_\mathbf{A}$ and $\Delta_\mathbf{A}$. Define $cl : \Delta_\mathbf{A} \longrightarrow \mathfrak{C}_\mathbf{A}$ and $\lambda : \mathfrak{C}_\mathbf{A} \longrightarrow \Delta_\mathbf{A}$ as follows:

$$\begin{aligned} x_{cl} &= x & x_\lambda &= x \\ (c_a)_{cl} &= c_a & (c_a)_\lambda &= c_a \\ (MN)_{cl} &= M_{cl} N_{cl} & (PQ)_\lambda &= P_\lambda Q_\lambda \\ (\lambda x.M)_{cl} &= \lambda^* x.M_{cl} & K_\lambda &= \lambda xy.x \\ & & S_\lambda &= \lambda xyz.xz(yz) \end{aligned}$$

Definition 2.2.21. A combinatory algebra $\mathbf{A}$ is called a *lambda algebra* if for all combinatory terms $P, Q \in \mathfrak{C}_{\mathbf{A}}$

$$\text{if } P_\lambda = Q_\lambda \text{ then } \mathbf{A} \models P = Q$$

Proposition 2.2.22. *The class of lambda algebras can be defined, relative to the class of combinatory algebras, by a set of closed, constant-free equations. In particular, lambda algebras form an algebraic variety.*

We also have the following, due to Curry:

Proposition 2.2.23 (See [2]). *The class of lambda algebras is axiomatized by the equations of combinatory logic and the following five closed equations:*

1. $k = s(s(ks)(s(kk)k))(k(skk))$
2. $s = s(s(ks)(s(k(s(ks)))(s(k(s(kk)))s)))(k(k(skk)))$
3. $s(kk) = s(s(ks)(s(kk)(s(ks)k)))(kk)$
4. $s(ks)(s(kk)) = s(kk)(s(s(ks)(s(kk)(skk)))(k(skk)))$
5. $s(k(s(ks)))(s(ks)(s(ks))) = s(s(ks)(s(kk)(s(ks)(s(k(s(ks)))s))))(ks)$

We have that CL together with these axioms, which are referred to as A_β, is equivalent to λ-calculus [2]. Each of these 5 items represent a rule that holds in the lambda calculus, but not necessarily on CL-terms and the derived lambda abstractor.

We are able to interpret the λ-calculus in a combinatory algebra in the following way: let $\mathbf{A}$ be a combinatory algebra, $M, N \in \Delta_{\mathbf{A}}$ be lambda terms and $\rho : \mathcal{V} \longrightarrow \mathbf{A}$ be a valuation. We define

$$[\![M]\!]_\rho = [\![M_{cl}]\!]_\rho$$

$$\mathbf{A} \models M = N \text{ iff } \mathbf{A} \models M_{cl} = N_{cl}$$

Since lambda algebras are special kinds of combinatory algebras, we are also able to interpret the λ-calculus in lambda algebras. The equations of the λ-calculus hold under this interpretation, but there is still a problem. The ξ-rule does not hold under the local interpretation defined in definition 2.2.18 (For an example of the failure of the ξ-rule, see [18]). This leads several people (e.g. Barendregt, Scott) to define *lambda models.*

Definition 2.2.24. Let $\mathbf{A}$ be a combinatory algebra. We say that $\mathbf{A}$ is *weakly extensional* if for all combinatory terms $P, Q \in \mathfrak{C}_{\mathbf{A}}$ we have

$$\text{if } \mathbf{A} \models P = Q, \text{ then } \mathbf{A} \models \lambda^* x.P = \lambda^* x.Q$$

In other words, $\mathbf{A}$ satisfies the first order formula $P = Q \implies \lambda^* x.P = \lambda^* x.Q$. A *lambda model* is a weakly extensional lambda algebra.

From Proposition 2.2.22, we know that lambda algebras are algebraic in the sense that lambda algebras can be defined relative to combinatory algebras via a set, in this case finite, of equations between terms. Lambda models, on the other hand, are not algebraic: weak extensionality, as defined in 2.2.24, is a first order formula.

As previously mentioned, the ξ-rule fails in lambda algebras. This problem stems from the way free variables are interpreted in the local interpretation previously defined and can be avoided by taking a more categorically inspired viewpoint. In [18], an alternative interpretation, which is called the *absolute interpretation* is suggested. This is related to Lambek's categorical method of indeterminates [11]. Under this interpretation, an equation in the λ-calculus, say $M = N$ with $FV(MN) \subseteq \{x_1, \ldots, x_n\}$, is said to be satisfied if it holds in $\mathbf{A}[x_1, \ldots, x_n]$, the lambda algebra obtained by freely adjoining indeterminates $x_1, \ldots, x_n$ to the lambda algebra $\mathbf{A}$. This interpretation satisfies the ξ-rule, unlike the local interpretation presented above. In fact, with this interpretation, we have the following characterization of lambda models:

Proposition 2.2.25 (see [18]). *Lambda models are lambda algebras in which an equation holds under the local interpretation (with respect to some valuation of the variables) if and only if it holds under the absolute interpretation.*

The connection between cartesian closed categories and lambda algebras uses the notion of points and local well-pointedness(see Section 3.2 for a formal discussion). Let $\mathcal{C}$ be a cartesian closed category with terminal object $\mathbf{1}$. A point x of an object $U \in \mathcal{C}$ is a map $x : \mathbf{1} \longrightarrow U$ and an object U is locally well-pointed, or has enough points, if for every $f, g : U \longrightarrow U$, $f \neq g$ implies that there exists a point $x : \mathbf{1} \longrightarrow U$

such that $f \circ x \neq g \circ x$. Suppose now that we have a reflexive object U via the retraction pair $\psi : U^U \longrightarrow U$ and $\varphi : U \longrightarrow U^U$, so $\varphi \circ \psi = Id_{U^U}$. We have already shown how to interpret an untyped lambda term M with free variables in $\{x_1, \dots x_n\}$ as a map from U^n to U. Define $\mathbf{A}$ to be the set of all maps in $\mathcal{C}$ from $\mathbf{1}$ to U, in other words $\mathbf{A} = \mathcal{C}(\mathbf{1}, U)$, the set of all points of U. Let $\bullet : \mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}$ be the map that sends $a, b \in \mathbf{A}$, to the map $\varphi_* \circ \langle a, b \rangle$, where $\varphi_* = ev \circ (\varphi \times Id_U)$. Then we have the following

Proposition 2.2.26 (see [18]).

1. $\mathbf{A} = (\mathbf{A}, \bullet)$ *is a lambda algebra.*

2. $\mathbf{A}$ *is a lambda model iff* U *is locally well-pointed.*

3. $\mathbf{A}[x] \cong \mathcal{C}(\mathbf{1}, U^U) \cong \mathcal{C}(U, U)$

The second statement of this proposition tells us that in order to model the λ-calculus via lambda models defined using the local interpretation, we need to be working in a category in which either $\mathbf{1}$ is a generator, which is another way of saying that the category has enough points, or at least a category with a reflexive object that has enough points (this is a weaker/local notion of enough points). If we stick to lambda algebras, using the absolute interpretation, we need not restrict ourselves to categories with enough points. Instead, it suffices to looks at cartesian closed category's with reflexive objects as mentioned earlier. In fact, such a model is given in [7], this model is also presented in section 3.4.

The third statement leads us to a connection with another notion: C-monoids (see [11] or section 3.5 for a formal definition). In [11], it is shown that if we have a locally small cartesian closed category $\mathcal{C}$ with a reflexive object U, $U^U \lhd U$, such that $U \times U \cong U$, then $\mathcal{C}(U, U)$ is a weak C-monoid. If we have that U is an extensional reflexive object, $U^U \cong U$, and $U \times U \cong U$ then $\mathcal{C}(U, U)$ is a C-monoid. In fact, in [11] it is shown that the category of C-monoids, i.e. the category whose objects are C-monoids and arrows are structure preserving maps, is isomorphic to the category of extended λ-calculi, i.e. the category whose objects are extended λ-calculi and arrows are *translations*(see [11] for a definition). This isomorphism illustrates the strong connection between C-monoids and the λ-calculus. In [18], there is a similar statement for lambda algebras: The category of lambda theories is equivalent to the category of lambda algebras. Hence, modulo surjective pairing in C-monoids, lambda algebras, lambda theories and C-monoids are equivalent notions.

Remark 2.2.27. From the second item in Proposition 2.2.26, we have that a reflexive object U in a ccc $\mathcal{C}$ can be made into a lambda model if and only if U has enough points in $\mathcal{C}$. The main observation of Ehrhard et al. in [7] is that choosing a different base set, i.e. something other then $\mathcal{C}(\mathbf{1}, U)$, we can get a lambda model from an object without enough points. Indeed, we will present the construction of an ambient cartesian closed category **MRel** which, in a strong sense, is not well-pointed but in which such lambda models do exist.

2.2.3 $\mathcal{P}\omega$ Model

In the untyped lambda calculus, we can form terms like MM, for a term M. This means that a model of the untyped lambda calculus consists of a structure in which objects can be interpreted as functions and also as arguments. The following is a brief overview of one such model: the Graph Model or $\mathcal{P}\omega$ Model and follows the presentation given in [2].

Let us first recall a few definitions.

Definition 2.2.28. A *poset* $(P, \leq)$ or simply P, is a set P together with a binary relation $\leq$ on P such that:

- $\leq$ is reflexive: for all $a \in P$, $a \leq a$;
- $\leq$ is antisymmetric: if $a \leq b$ and $b \leq a$, then $a = b$;
- $\leq$ is transitive: if $a \leq b$ and $b \leq c$, then $a \leq c$.

Definition 2.2.29. A *lattice* L is a poset $(L, \leq)$ that has binary joins, also known as supremum or least upper bound, written $x \vee y$, and meets, also known as infimum or greatest lower bound, written $x \wedge y$.

Definition 2.2.30. A *complete lattice* L is a poset $(L, \leq)$ such that every subset $A \subseteq L$ has both a join, written $\bigvee A$, and a meet, written $\bigwedge A$.

Definition 2.2.31. An element x of a poset $(P, \leq)$ is *compact* if for every non-empty directed subset $D \subseteq P$, if $\bigvee D$ is defined and $x \leq \bigvee D$, then $x \leq d$ for some $d \in D$.

Definition 2.2.32. An *algebraic lattice* is a complete lattice L such that for every $x \in L$, $x = \bigvee\{y \in L \mid y \leq x, \text{ y compact}\}$.

Example 2.2.33. Let A be a set. Then $\mathcal{P}(A)$ is an algebraic lattice. Indeed, $\mathcal{P}(A)$ is a poset ordered by inclusion, $\subseteq$: for $X, Y, Z \in \mathcal{P}(A)$, we have that $X \subseteq X$; if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$; if $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$. For $X, Y \in \mathcal{P}(A)$, $X \vee Y = X \cup Y \in \mathcal{P}(A)$ and $X \wedge Y = X \cap Y \in \mathcal{P}(A)$. For $\mathcal{F} \subseteq \mathcal{P}(A)$, we define $\bigvee \mathcal{F} = \bigcup \mathcal{F} \in \mathcal{P}(A)$ and $\bigwedge \mathcal{F} = \bigcap \mathcal{F} \in \mathcal{P}(A)$, so $\mathcal{P}(A)$ is a complete lattice. The compact elements of $\mathcal{P}(A)$ are the finite subsets: Let $X = \{a_1, \ldots, a_k\}$ be a finite subset of A and let $\mathcal{F} \subseteq \mathcal{P}(A)$ be non-empty and directed. Suppose $X \subseteq \bigcup \mathcal{F}$, then, for $i = 1, \ldots, k$, there is a $D_i \in \mathcal{F}$ such that $a_i \in D_i$. By directedness of $\mathcal{F}$, there is a $D_N \in \mathcal{F}$ such that $D_i \subseteq D_N$, $i = 1, \ldots, k$. Therefore $a_i \in D_N$ for $i = 1, \ldots, k$ and $X \subseteq D_N$ as required. Now to show that $\mathcal{P}(A)$ is an algebraic lattice, it remains to show that for $X \in \mathcal{P}(A)$, $X = \bigcup\{Y \mid Y \subseteq X, \text{ Y finite}\}$. It is clear that $\bigcup\{Y \mid Y \subseteq X, \text{ Y finite}\} \subseteq X$. To see that $X \subseteq \bigcup\{Y \mid Y \subseteq X, \text{ Y finite}\}$, notice that for $x \in X$, $x \in \{x\}$ and $\{x\}$ is a finite subset of X, therefore $x \in \bigcup\{Y \mid Y \subseteq X, \text{ Y finite}\}$.

Definition 2.2.34. The category of algebraic lattices **AlgLat** is the category whose objects are algebraic lattices and maps are set theoretic functions that preserve joins of directed sets.

We now show that **AlgLat** is a cartesian closed category with an object U such that $U^U \lhd U$. Let us first observe that **AlgLat** is a cartesian closed category:

Proposition 2.2.35 (see [2]). **AlgLat** *is a cartesian closed category with the following structure: The terminal object* **1** *is the singleton algebraic lattice. Let* $D = (D, \leq)$ *and* $D' = (D', \leq')$ *be two algebraic lattices. The categorical product of* D *and* D' *will be set theoretical product:*

$$D \times D' = \{(d, d') \mid d \in D, d' \in D'\}$$

together with the usual set-theoretic projections π_1 *and* π_2 *and the usual pairing map* $\langle -, - \rangle$. *Let* $(d, d'), (e, e') \in D \times D'$, *then* $(d, d') \leq^{D \times D'} (e, e')$ *if* $d \leq e$ *and* $d' \leq' e'$. *The exponential object of* D *and* D' *is* $[D, D']$, *the set of morphisms from* D *to* D'

in **AlgLat**. *Let* $f, g \in [D, D']$, *then* $f \leq^{[D,D']} g$ *if for every* $d \in D$, $f(d) \leq' g(d)$. *Evaluation* $ev : [D, D'] \times D \longrightarrow D'$ *is defined as follows: given* $f \in [D, D']$ *and* $d \in D$, *then* $ev(f, d) = f(d)$.

Now that we know that **AlgLat** is a cartesian closed category, we know from section 2.2.2 that we need a reflexive object in **AlgLat** to model the λ-calculus. In this case, the reflexive object will be $\mathcal{P}(\mathbb{N})$. This model is known as the *graph* or $\mathcal{P}\omega$ *model.*

The graph model makes use of two encodings of the natural numbers. Given two natural numbers n and m, we define $\#(n, m) = \frac{1}{2}(n+m)(n+m+1)+m$ and for $n \in \mathbb{N}$, define $e_n = \{k_0, \ldots, k_{m-1}\}$, with $k_0 < k_1 < \cdots < k_{m-1}$, where $n = \sum_{i<m} 2^{k_i}$. The first encoding yields a bijection between $\mathbb{N}^2$ and $\mathbb{N}$ and the second, a bijection between $\mathbb{N}$ and the set of finite subsets of $\mathbb{N}$.[2]

With these encodings, we now have all the background information needed to define the graph model as an object in **AlgLat**. Let $\mathcal{P}\omega = \mathcal{P}(\mathbb{N})$, together with the functions $graph : [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}] \longrightarrow \mathcal{P}\mathbb{N}$ and $fun : \mathcal{P}\mathbb{N} \longrightarrow [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]$ be defined as follows: For $F \in [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]$,

$$graph(F) = \{\#(n, m) \mid m \in F(e_n)\}$$

and for $A \in \mathcal{P}\mathbb{N}$, we define $fun(A) \in [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]$ by

$$fun(A)(B) = \{m \mid \exists e_n \subseteq B, \#(n, m) \in A\}$$

Both *graph* and *fun* preserve the join of directed sets. This is also true for $fun(A) : \mathcal{P}\mathbb{N} \longrightarrow \mathcal{P}\mathbb{N}$, where $A \subseteq N$. Therefore, $graph, fun, fun(A)$ are morphisms in **AlgLat**. It is also shown in [2], that for $f \in [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]$, $fun(graph(f)) = f$. Hence, $[\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]$ is a retract of $\mathcal{P}\mathbb{N}$. From this, we get that $\mathcal{P}\omega = (\mathcal{P}(\mathbb{N}), \cdot)$ is a λ-model, where application, $\cdot : \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N} \longrightarrow \mathcal{P}\mathbb{N}$, is defined as follows: For $A, B \subseteq \mathbb{N}$

$$A \cdot B = fun(A)(B) = \{m \in \mathbb{N} \mid \exists e_n \subseteq B.\#(n, m) \in A\}$$

Notice that for $A \in \mathcal{P}\mathbb{N}$, $A \subseteq graph(fun(A))$: here we do not necessarily get equality. This is why the $\mathcal{P}\omega$ model is not extensional: $\lambda x.ax \neq a$.

2.3 Some basic category theory

2.3.1 Monads and comonads

Let us first recall several functors that will be used in the construction of **MRel**.

Example 2.3.1 (Power set functor). Let $\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}$ be the power set functor. If A is a set, $\mathcal{P}(A) = \{X | X \subseteq A\}$. If $A \xrightarrow{f} B$ is an arrow in **Set** , then $\mathcal{P}(A) \xrightarrow{\mathcal{P}(f)} \mathcal{P}(B)$ and $\mathcal{P}(f)(X) = f[X]$, where $X \subseteq A$ and $f[X] = \{f(x) | x \in X\}$.

To show that $\mathcal{P}$ is a functor we must show that for A a set, $\mathcal{P}(id_A) = id_{\mathcal{P}(A)}$. Indeed $\mathcal{P}(id_A) : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ sends $X \subseteq A$ to $\mathcal{P}(id_A)(X) = id_A[X] = \{id_A(x) | x \in X\} = X = id_{\mathcal{P}(A)}(X)$. It remains to show that for $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, $\mathcal{P}(g \circ f) = \mathcal{P}(g) \circ \mathcal{P}(f)$. Let $X \subseteq A$, then

$$\begin{aligned} \mathcal{P}(g \circ f)(X) &= (g \circ f)[X] \\ &= g[f[X]] \\ &= g[\mathcal{P}(f)(X)] \\ &= \mathcal{P}(g)(\mathcal{P}(f)(X)) \\ &= (\mathcal{P}(g) \circ \mathcal{P}(f))(X) \end{aligned}$$

as required. Therefore, the power set functor $\mathcal{P}$ is indeed a functor.

Notation 2.3.2. If A is a finite subset of B then we write $A \subseteq_f B$.

Example 2.3.3 (Finite power set functor). Similarly, we can define the finite power set functor $\mathcal{P}_f : \mathbf{Set} \longrightarrow \mathbf{Set}$. For a set A, $\mathcal{P}_f(A) = \{X | X \subseteq_f A\}$ and for $A \xrightarrow{g} B$ an arrow in **Set**, $\mathcal{P}_f(A) \xrightarrow{\mathcal{P}_f(g)} \mathcal{P}_f(B)$. For $X \subseteq_f A$, say $X = \{a_1, \ldots, a_k\}$, $\mathcal{P}_f(g)(X) = \{g(a_1), \ldots, g(a_k)\}$. To show this is a functor, we follow the proof as in Example 2.3.1 and use the fact that the finite union of finite sets is again finite.

Example 2.3.4 (Finite multiset functor on **Set**). Let $\mathcal{M}_f(-) : \mathbf{Set} \longrightarrow \mathbf{Set}$ be the finite multiset functor over **Set** defined as follows: If A is a set, $\mathcal{M}_f(A) = \{\alpha | \alpha \text{ is a finite multiset over } A\}$. If $A \xrightarrow{f} B$ is an arrow in **Set** , then $\mathcal{M}_f(A) \xrightarrow{\mathcal{M}_f(f)} \mathcal{M}_f(B)$,where if $\alpha \in \mathcal{M}_f(A)$, say $\alpha = [a_1, \ldots, a_k]$, then $\mathcal{M}_f(f)(\alpha) = \mathcal{M}_f(f)([a_i, \ldots, a_k]) = [f(a_1), \ldots, f(a_k)]$.

To show that $\mathcal{M}_f(-)$ is a functor we must show that for A a set, $\mathcal{M}_f(id_A) = id_{\mathcal{M}_f(A)}$. Indeed $\mathcal{M}_f(id_A) : \mathcal{M}_f(A) \longrightarrow \mathcal{M}_f(A)$ sends $\alpha = [a_1, \ldots, a_k] \in \mathcal{M}_f(A)$ to $\mathcal{M}_f(id_A)(\alpha) = [id_A(a_1), \ldots, id_A(a_k)] = [a_1, \ldots, a_k] = \alpha$. It remains to show that for

$A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, $\mathcal{M}_f(g \circ f) = \mathcal{M}_f(g) \circ \mathcal{M}_f(f)$. Let $\alpha = [a_1, \ldots, a_k] \in \mathcal{M}_f(A)$; then

$$\begin{aligned}
\mathcal{M}_f(g \circ f)(\alpha) &= \mathcal{M}_f(g \circ f)[a_1, \ldots, a_k] \\
&= [(g \circ f)a_1, \ldots, (g \circ f)a_k] \\
&= [(g(f(a_1)), \ldots, (g(f(a_k))] \\
&= \mathcal{M}_f(g)[f(a_1), \ldots, f(a_k)] \\
&= (\mathcal{M}_f(g) \circ \mathcal{M}_f(f))([a_1, \ldots, a_k] \\
&= (\mathcal{M}_f(g) \circ \mathcal{M}_f(f))(\alpha)
\end{aligned}$$

as required. Therefore, the multiset functor $\mathcal{M}_f$ over **Set** is indeed a functor.

Definition 2.3.5. A *natural transformation* between two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ is a family of arrows $\{\theta_C : FC \longrightarrow GC \mid C \in |\mathcal{C}|\}$ such that for every $f : C \longrightarrow D$, the following diagram commutes:

$$\begin{array}{ccc}
FA & \xrightarrow{\theta_A} & GA \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\
FB & \xrightarrow{\theta_B} & GB
\end{array}$$

Example 2.3.6. Let Id : **Set** $\longrightarrow$ **Set** be the identity functor on **Set** and $\mathcal{P}$: **Set** $\longrightarrow$ **Set** be the power set functor from above. Then $\{-\} : Id \longrightarrow \mathcal{P}$, defined by the family $\{\{-\}_A : A \longrightarrow \mathcal{P}(A) \mid \text{A a set}\}$, where for $a \in A$, $\{-\}_A(a) = \{a\}$, is a natural transformation. We must verify that for every $f : A \longrightarrow B \in$ **Set**, the following diagram commutes:

$$\begin{array}{ccc}
A & \xrightarrow{\{-\}_A} & \mathcal{P}(A) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}(f)} \\
B & \xrightarrow{\{-\}_B} & \mathcal{P}(B)
\end{array}$$

Indeed,

$$\begin{aligned}
(\mathcal{P}(f) \circ \{-\}_A)(x) &= \mathcal{P}(f)\{x\} \\
&= f[\{x\}] \\
&= \{f(x)\} \\
&= \{-\}_B(f(x)) \\
&= (\{-\}_B \circ f)(x)
\end{aligned}$$

So we have that $\mathcal{P}(f) \circ \{-\}_A = \{-\}_B \circ f$ as required. Therefore $\{-\}$ is a natural transformation.

Definition 2.3.7. A *monad* on a category $\mathcal{C}$ is a triple (T, η, μ) , where $T : \mathcal{C} \longrightarrow \mathcal{C}$ is a functor, $\eta : Id \longrightarrow T$ and $\mu : T^2 \longrightarrow T$ are two natural transformations called unit and multiplication, such that the following diagrams commute:

$$\begin{array}{ccc} T & \xrightarrow{T\eta} & T^2 \\ {\scriptstyle \eta T}\downarrow & \overset{id}{\searrow} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

$$\begin{array}{ccc} T^3 & \xrightarrow{\mu T} & T^2 \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

where, for $A \in |\mathcal{C}|$, $T\eta(A) = T(\eta_A)$, $(\eta T)A = \eta_{T(A)}$, $T\mu(A) = T(\mu_A)$ and $(\mu T)A = \mu_{T(A)}$. Or equivalently, we require that $\mu \circ \mu T = \mu \circ T\mu$ and $\mu \circ T\eta = \mathrm{id}_{\mathcal{C}} = \mu \circ \eta T$.

Example 2.3.8 (Power set monad on **Set**). Let $\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}$ be the power set functor from above and $\mathcal{P}^2 = \mathcal{P} \circ \mathcal{P}$. Let $\mu : \mathcal{P}^2 \longrightarrow \mathcal{P}$ be defined by the family $\{\mu_A : \mathcal{P}^2(A) \longrightarrow \mathcal{P}(A) \mid \text{A a set}\}$, where for $\mathcal{F} \subseteq \mathcal{P}(A)$, $\mu_A(\mathcal{F}) = \bigcup \mathcal{F}$. Then the triple $(\mathcal{P}, \eta, \mu)$ is a monad on **Set**, where $\eta = \{-\}$ from above.

First, we must verify that both η and μ are natural transformations. We know that η is a natural transformation, since $\eta = \{-\}$ and $\{-\}$ is a natural transformation. To show that μ is a natural transformation, we must show that for every $f : A \longrightarrow B \in \mathbf{Set}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}^2(A) & \xrightarrow{\mu_A} & \mathcal{P}(A) \\ {\scriptstyle \mathcal{P}^2(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{P}(f)} \\ \mathcal{P}^2(B) & \xrightarrow{\mu_B} & \mathcal{P}(B) \end{array}$$

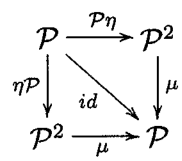
Let $\mathcal{F} \subseteq \mathcal{P}(A)$, then

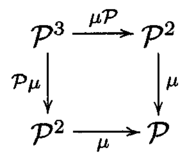
$$\begin{aligned} (\mathcal{P}(f) \circ \mu_A)(\mathcal{F}) &= \mathcal{P}(f)(\bigcup \mathcal{F}) \\ &= f[\bigcup \mathcal{F}] \\ &= \bigcup\{f(a) \mid a \in X \ \wedge \ X \in \mathcal{F}\} \end{aligned}$$

and

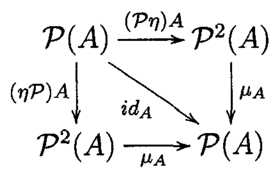
$$\begin{aligned}
(\mu_B \circ \mathcal{P}^2(f))\mathcal{F} &= \mu_B((\mathcal{P}^2(f))(\mathcal{F})) \\
&= \mu_B(\{\mathcal{P}(f)(X) \mid X \in \mathcal{F}\}) \\
&= \bigcup\{f[X] | \; X \in \mathcal{F}\} \\
&= \bigcup\{f(a) | \; a \in X \;\wedge\; X \in \mathcal{F}\}
\end{aligned}$$

Now we must show that the following diagrams commute





Let A be a set. Then, for the first diagram, we must show that the following commutes:



Let $X \subseteq A$, then we have

$$\begin{aligned}
(\mu_A \circ (\mathcal{P}\eta)A)X &= (\mu_A \circ (\mathcal{P}\eta_A))X \\
&= (\mu_A(\eta_A[X])) \\
&= \mu_A(\{\eta_A(x) \mid x \in X\}) \\
&= \mu_A(\{\{x\} \mid x \in X\}) \\
&= \bigcup\{\{x\} \mid x \in X\} \\
&= X \\
&= id_A(X) \\
(\mu_A \circ (\eta\mathcal{P})A)X &= (\mu_A \circ (\eta_{\mathcal{P}(A)}))X \\
&= \mu_A(\{X\}) \\
&= X \\
&= id_A(X)
\end{aligned}$$

Therefore $\mu_A \circ (\mathcal{P}\eta)A = \mu_A \circ (\eta\mathcal{P})A = id_A$ as required. For the second diagram, we must show that the following commutes:

$$\begin{array}{ccc} \mathcal{P}^3(A) & \xrightarrow{(\mu\mathcal{P})A} & \mathcal{P}^2(A) \\ {\scriptstyle (\mathcal{P}\mu)A}\downarrow & & \downarrow{\scriptstyle \mu_A} \\ \mathcal{P}^2(A) & \xrightarrow[\mu_A]{} & \mathcal{P}(A) \end{array}$$

Let $\mathcal{F} \in \mathcal{P}^3(A)$. Then

$$\begin{aligned} (\mu_A \circ (\mathcal{P}\mu)A)\mathcal{F} &= (\mu_A \circ \mathcal{P}(\mu_A))\mathcal{F} \\ &= \mu_A(\mathcal{P}(\mu_A)(\mathcal{F})) \\ &= \mu_A(\{\mu_A X \mid X \in \mathcal{F}\}) \\ &= \bigcup\{\bigcup X \mid X \in \mathcal{F}\} \end{aligned}$$

and

$$\begin{aligned} z \in (\mu_A \circ (\mathcal{P}\mu)A)\mathcal{F} \quad &\text{iff} \quad z \in \bigcup\{\bigcup X \mid X \in \mathcal{F}\} \\ &\text{iff} \quad \exists X \in \mathcal{F} \exists \mathcal{B}(\mathcal{B} = \cup X \ \wedge \ z \in \mathcal{B}) \\ &\text{iff} \quad \exists X(X \in \mathcal{F} \wedge \ z \in \cup X) \\ &\text{iff} \quad \exists X(X \in \mathcal{F} \wedge \exists B.(B \in X \ \wedge \ z \in B)) \qquad (*) \end{aligned}$$

Now, in the other direction,

$$\begin{aligned} (\mu_A \circ (\mu\mathcal{P})A)\mathcal{F} &= (\mu_A \circ \mu_{\mathcal{P}(A)})\mathcal{F} \\ &= \mu_A(\mu_{\mathcal{P}(A)})(\mathcal{F})) \\ &= \mu_A(\bigcup \mathcal{F}) \\ &= \bigcup\bigcup \mathcal{F} \end{aligned}$$

and

$$\begin{aligned} z \in (\mu_A \circ (\mu\mathcal{P})A)\mathcal{F} \quad &\text{iff} \quad \exists B.B \in \bigcup \mathcal{F} \ \wedge \ z \in B \\ &\text{iff} \quad \exists B(B \in \cup\mathcal{F} \ \wedge \ z \in B) \end{aligned}$$

Translating this, we get that the above says

$$z \in (\mu_A \circ (\mu\mathcal{P})A)\mathcal{F} \quad \text{iff} \quad \exists B(\exists X(X \in \mathcal{F} \wedge B \in X) \ \wedge \ z \in B) \qquad (**)$$

Clearly the two expressions $(*)$ and $(**)$ are equal. Therefore $\mu_A \circ (\mathcal{P}\mu)A = \mu_A \circ (\mu\mathcal{P})A$ as required and $(\mathcal{P}, \eta, \mu)$ is indeed a monad.

Example 2.3.9 (Finite power set monad on **Set**). Again, we can define the finite power set monad $(\mathcal{P}_f, \eta, \mu)$. Where η and μ are the same as in Example 2.3.8. To show this is a monad, we follow the proof as in Example 2.3.8 and use the fact that the finite union of finite sets is again finite.

Example 2.3.10 (Finite multiset monad on **Set**). Let $\mathcal{M}_f(-) : \mathbf{Set} \longrightarrow \mathbf{Set}$ be the multiset functor over *Set* from above and $\mathcal{M}_f^2 = \mathcal{M}_f \circ \mathcal{M}_f$. Let $\eta : Id \longrightarrow \mathcal{M}_f$ be defined by the family $\{\eta_A : A \longrightarrow \mathcal{M}_f(A) \mid \text{A a set}\}$, where for $a \in A$, $\eta_A(a) = [a]$. Let $\mu : \mathcal{M}_f^2 \longrightarrow \mathcal{M}_f$ be defined by the family $\{\mu_A : \mathcal{M}_f^2(A) \longrightarrow \mathcal{M}_f(A) \mid \text{A a set}\}$, where for $\mathcal{A} \in \mathcal{M}_f(\mathcal{M}_f(A))$, $\mu_A(\mathcal{A}) = \biguplus \mathcal{A}$. Then the triple $(\mathcal{M}_f, \eta, \mu)$ is a monad on **Set**.

First, we must verify that both η and μ are natural transformations. To show that η is a natural transformation we must verify that for every $f : A \longrightarrow B \in \mathbf{Set}$, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathcal{M}_f(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_f(f)} \\ B & \xrightarrow{\eta_B} & \mathcal{M}_f(B) \end{array}$$

Indeed,

$$\begin{aligned} (\mathcal{M}_f(f) \circ \eta_A)(x) &= \mathcal{M}_f(f)([x]) \\ &= [f(x)] \\ &= \eta_B(f(x)) \\ &= (\eta_B \circ f)(x) \end{aligned}$$

So we have that $\mathcal{M}_f(f) \circ \eta_A = \eta_B \circ f$ as required. Therefore η is a natural transformation.

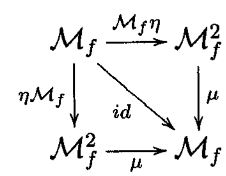
To show that μ is a natural transformation, we must show that for every $f : A \longrightarrow B \in \mathbf{Set}$, the following diagram commutes:

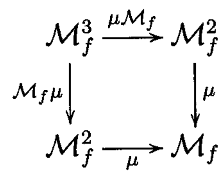
$$\begin{array}{ccc} \mathcal{M}_f^2(A) & \xrightarrow{\mu_A} & \mathcal{M}_f(A) \\ {\scriptstyle \mathcal{M}_f^2(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_f(f)} \\ \mathcal{M}_f^2(B) & \xrightarrow{\mu_B} & \mathcal{M}_f(B) \end{array}$$

Let $\mathcal{A} \in \mathcal{M}_f^2(A)$, say $\mathcal{A} = [\alpha_1, \ldots, \alpha_k]$ and for $i = 1, \ldots k$, $\alpha_i = [a_i^1, \ldots, a_i^{l_i}]$ then

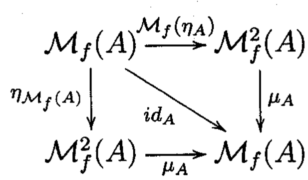
$$\begin{aligned}
(\mathcal{M}_f(f) \circ \mu_A)(\mathcal{A}) &= \mathcal{M}_f(f)(\mu(\mathcal{A})) \\
&= \mathcal{M}_f(f)(\alpha_1 \uplus \cdots \uplus \alpha_k) \\
&= \mathcal{M}_f(f)([a_1^1, \ldots, a_1^{l_1}, \ldots, a_k^1, \ldots, a_k^{l_k}]) \\
&= [f(a_1^1), \ldots, f(a_1^{l_1}), \ldots, f(a_k^1), \ldots, f(a_k^{l_k})]) \\
&= \mu_B([[f(a_1^1), \ldots, f(a_1^{l_1})], \ldots, [f(a_k^1), \ldots, f(a_k^{l_k})]]) \\
&= \mu_B([\mathcal{M}_f(f)(\alpha_1), \ldots, \mathcal{M}_f(f)(\alpha_k)]) \\
&= (\mu_B \circ \mathcal{M}_f^2(f))(\mathcal{A})
\end{aligned}$$

So we have that $\mathcal{M}_f(f) \circ \mu_A = \mu_B \circ \mathcal{M}_f^2(f)$ as required. Therefore μ is a natural transformation. To show that we have a monad we must show that the following diagrams commute





Let A be a set. Then, for the first diagram, we must show that the following commutes:



Let $\alpha \in \mathcal{M}_f(A)$ with $\alpha = [a_1, \ldots, a_k]$, then we have

$$\begin{aligned}
(\mu_A \circ \mathcal{M}_f(\eta_A))\alpha &= \mu_A(\mathcal{M}_f(\eta_A)[a_1, \ldots, a_k]) \\
&= \mu_A([\eta_A(a_1), \ldots, \eta_A(a_k)]) \\
&= \mu_A([[a_1], \ldots, [a_k]]) \\
&= [a_1] \uplus \cdots \uplus [a_k] \\
&= [a_1, \ldots, a_k] \\
&= \alpha \\
&= id_A(\alpha) \\
\mu_A \circ \eta_{\mathcal{M}_f(A)})\alpha &= \mu_A(\eta_{\mathcal{M}_f(A)}(\alpha)) \\
&= (\mu_A([\alpha]) \\
&= \alpha \\
&= id_A(\alpha)
\end{aligned}$$

Therefore $\mu_A \circ \mathcal{M}_f(\eta_A) = \mu_A \circ \mathcal{M}_f(\eta_A)) = id_A$ as required. For the second diagram, we must show that the following commutes:

$$\begin{array}{ccc}
\mathcal{M}_f^3(A) & \xrightarrow{\mu_{\mathcal{M}_f A}} & \mathcal{M}_f^2(A) \\
{\scriptstyle \mathcal{M}_f(\mu_A)}\downarrow & & \downarrow{\scriptstyle \mu_A} \\
\mathcal{M}_f^2(A) & \xrightarrow[\mu_A]{} & \mathcal{M}_f(A)
\end{array}$$

Let $X \in \mathcal{M}_f^3(A)$, where $X = [\mathcal{A}_1, \ldots, \mathcal{A}_k]$ and for $i = 1, \ldots k$, $\mathcal{A}_i = [\alpha_i^1, \ldots, \alpha_i^{l_i}]$. Then

$$\begin{aligned}
(\mu_A \circ \mathcal{M}_f(\mu_A))X &= \mu_A(\mathcal{M}_f(\mu_A)([\mathcal{A}_1, \ldots, \mathcal{A}_k])) \\
&= \mu_A([\alpha_1^1 \uplus \cdots \uplus \alpha_1^{l_1}, \ldots, \alpha_k^1 \uplus \cdots \uplus \alpha_k^{l_k}]) \\
&= \alpha_1^1 \uplus \cdots \uplus \alpha_1^{l_1} \uplus \cdots \uplus \alpha_k^1 \uplus \cdots \uplus \alpha_k^{l_k}
\end{aligned}$$

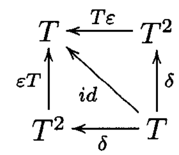
and

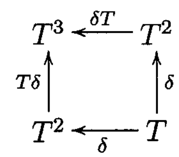
$$\begin{aligned}
(\mu_A \circ \mu_{\mathcal{M}_f A})X &= \mu_A(\mu_{\mathcal{M}_f A}([\mathcal{A}_1, \ldots, \mathcal{A}_k])) \\
&= \mu_A(\mathcal{A}_1 \uplus \cdots \uplus \mathcal{A}_k) \\
&= \alpha_1^1 \uplus \cdots \uplus \alpha_1^{l_1} \uplus \cdots \uplus \alpha_k^1 \uplus \cdots \uplus \alpha_k^{l_k}
\end{aligned}$$

Therefore $\mu_A \circ \mathcal{M}_f(\mu_A) = \mu_A \circ \mu_{\mathcal{M}_f A}$ as required and $(\mathcal{M}_f, \eta, \mu)$ is indeed a monad.

We now dualize the notion of a monad by moving from $\mathcal{C}$ to $\mathcal{C}^{op}$.

Definition 2.3.11. A *comonad* on a category $\mathcal{C}$ is a triple (T, ε, δ) , where $T : \mathcal{C} \longrightarrow \mathcal{C}$ is a functor, $\varepsilon : T \longrightarrow Id$ and $\delta : T \longrightarrow T^2$ are two natural transformations called counit and comultiplication, such that the following diagrams commute:





where, for $A \in |\mathcal{C}|$, $T\varepsilon(A) = T(\varepsilon_A)$, $(\varepsilon T)A = \varepsilon_{T(A)}$, $T\delta(A) = T(\delta_A)$ and $(\delta T)A = \delta_{T(A)}$. Or equivalently, we require that $\delta T \circ \delta = T\delta \circ \delta$ and $T\varepsilon \circ \delta = \mathrm{id}_{\mathcal{C}} = \varepsilon T \circ \delta$.

2.3.2 Kleisli categories

Definition 2.3.12. Given a monad (T, η, μ) on a category $\mathcal{C}$, the *Kleisli category* $Kl(T)$ is the category whose objects are the objects of $\mathcal{C}$ and an arrow $f : A \longrightarrow B \in Kl(T)$ is an arrow $f : A \longrightarrow TB \in \mathcal{C}$. The identity $id : A \longrightarrow A$ is $\eta_A : A \longrightarrow TA$ and the composition of $f : A \longrightarrow TB$ and $g : B \longrightarrow TC$, $g \bullet f : A \longrightarrow TC$ is defined as follows: $g \bullet f = \mu_C \circ T(g) \circ f$.

We must check that the Kleisli category does indeed form a category:

Proof. First, we must show that, for $f : A \longrightarrow B \in Kl(T)$, $f \bullet id_A = f = id_B \bullet f$. We have $f : A \longrightarrow B \in Kl(T)$, therefore $f : A \longrightarrow TB \in \mathcal{C}$. We also have that $id_A : A \longrightarrow A \in Kl(T)$, which is $\eta_A : A \longrightarrow A \in \mathcal{C}$ and similarly, that $id_B : B \longrightarrow B \in Kl(T)$, which is $\eta_B : B \longrightarrow B \in \mathcal{C}$. From the definition of composition in $Kl(T)$, we have:

$$\begin{aligned} f \bullet id_A &= \mu_B \circ T(f) \circ \eta_A \\ &= \mu_B \circ \eta_{TB} \circ f && \text{since } \eta \text{ is a n.t.} \\ &= (\mu_B \circ (\eta T)_B) \circ f \\ &= id_B \circ f && \text{by properties of monad } (T, \mu, \eta) \\ &= f \end{aligned}$$

$$\begin{aligned} id_B \bullet f &= \mu_B \circ T(\eta_B) \circ f \\ &= id_B \circ f \qquad\qquad \text{by properties of monad } (T, \mu, \eta) \\ &= f \end{aligned}$$

So we have $f \bullet id_A = f = id_B \bullet f$ as required.

Second, we must show that, for $f \in Hom_{\mathcal{C}_T}(A, B)$, $g \in Hom_{\mathcal{C}_T}(B, A)$ and $h \in Hom_{\mathcal{C}_T}(C, D)$, $(h \bullet g) \bullet f = h \bullet (g \bullet f)$:

$$\begin{aligned} (h \bullet g) \bullet f &= \mu_D \circ T(h \bullet g) \circ f \\ &= \mu_D \circ T[\mu_D \circ T(h) \circ g] \circ f \\ &= \mu_D \circ T\mu_D \circ T^2(h) \circ T(g) \circ f \\ h \bullet (g \bullet f) &= \mu_D \circ T(h) \circ g \bullet f \\ &= \mu_D \circ T(h) \circ \mu_C \circ T(g) \circ f \end{aligned}$$

Therefore we must show that $T(\mu_D) \circ T^2(h) = T(h) \circ \mu_C$, or equivalently, we must show the following diagram commutes in $Kl(T)$:

$$\begin{array}{ccc} T^2C & \xrightarrow{\mu_C} & TC \\ {\scriptstyle T^2h}\downarrow & & \downarrow{\scriptstyle Th} \\ T^3D & \xrightarrow{T\mu_D} & T^2D \end{array}$$

This is equivalent to showing the following diagram commutes in $\mathcal{C}$:

$$\begin{array}{ccc} T^2C & \xrightarrow{\mu_C} & TC \\ {\scriptstyle T^2h}\downarrow & & \downarrow{\scriptstyle Th} \\ T^2D & \xrightarrow{\mu_D} & TD \end{array}$$

But this diagram does commute since μ is a natural transformation. Therefore, $(h \bullet g) \bullet f = h \bullet (g \bullet f)$ as required. □

Proposition 2.3.13. *[See [11]] The Kleisli category $Kl(T)$ of a monad (T, η, μ) on a category $\mathcal{C}$ gives rise to an adjunction $F : \mathcal{C} \longrightarrow Kl(T)$ and $U : Kl(T) \longrightarrow \mathcal{C}$ such that $UF = T$.*

Example 2.3.14 (Kleisli category of power set monad on **Set**)**.** We have shown that $(\mathcal{P}, \eta, \mu)$ is a monad on **Set**. $Kl(\mathcal{P})$ is the category whose objects are the objects of

Set. The objects of **Set** are sets, therefore the objects of $Kl(\mathcal{P})$ are sets. For the arrows, notice

$$\begin{aligned} f : A \longrightarrow B \in Kl(\mathcal{P}) &\text{ iff } f : A \longrightarrow \mathcal{P}(B) \in \mathbf{Set} \\ &\text{ iff } A \longrightarrow \mathbf{2}^B \\ &\text{ iff } A \times B \longrightarrow \mathbf{2} \\ &\text{ iff } f \subseteq A \times B \end{aligned}$$

Therefore, the arrows of $Kl(\mathcal{P})$ are subsets of $A \times B$, and can be considered as relations from A to B. Let R_f be the relation associated to $f : A \longrightarrow B \in Kl(\mathcal{P})$.

- Identity: The identity arrow is $\eta_A : A \longrightarrow \mathcal{P}(A)$ and

$$\begin{aligned} (a, b) \in R_{id_A} &\text{ iff } (a, b) \in R_{\eta_A} \\ &\text{ iff } b \in \eta_A(a) \\ &\text{ iff } b \in \{a\} \\ &\text{ iff } b = a \end{aligned}$$

  Therefore $R_{id_A} = \{(a, a) \mid a \in A\}$.

- Composition: Let $f : A \longrightarrow \mathcal{P}(B)$ and $g : B \longrightarrow \mathcal{P}(C)$ be two arrows in $Kl(\mathcal{P})$ then $g \bullet f : A \longrightarrow \mathcal{P}(C)$ is defined to be $\mu_B \circ \mathcal{P}(g) \circ f$. Let $a \in A$, then $(\mu_B \circ \mathcal{P}(g) \circ f)(a) = \mu_B(g[f(a)]) = \bigcup\{g(b) \mid b \in f(a)\}$. Therefore

$$\begin{aligned} (a, c) \in R_{g \bullet f} &\text{ iff } c \in (g \bullet f)(a) \\ &\text{ iff } c \in \bigcup\{g(b) \mid b \in f(a)\} \\ &\text{ iff } \exists b \text{ s.t. } b \in f(a), c \in g(b) \\ &\text{ iff } \exists b \text{ s.t. } (a, b) \in R_f, (b, c) \in R_g \\ &\text{ iff } (a, c) \in R_g \circ R_f \\ &\quad \text{(i.e. Composition in } Kl(\mathcal{P}) \text{ is composition in } \mathbf{Rel}) \end{aligned}$$

From this and previous remarks, it is easy to see that $Kl(\mathcal{P}) \cong$ **Rel**. We can apply Proposition 2.3.13, to get an adjunction pair $F : \mathbf{Set} \longrightarrow \mathbf{Rel}$ and $U : \mathbf{Rel} \longrightarrow \mathbf{Set}$. Given a morphism $f : A \longrightarrow B \in \mathbf{Rel}$, let $f^* : A \longrightarrow \mathcal{P}(B)$ be the associated map in **Set** and given a map $f : A \longrightarrow \mathcal{P}(B) \in \mathbf{Set}$, let $f^\dagger : A \longrightarrow B \in \mathbf{Rel}$ be the associated

Klesli map. Let A be a set then $U(A) = \mathcal{P}(A)$ and $F(A) = A$. Let $R : A \longrightarrow B$ be a map in **Rel**, then we define $U(R) : \mathcal{P}(A) \longrightarrow \mathcal{P}(B) \in$ **Set** to be $(\mu_B^{\mathcal{P}} \circ \mathcal{P}(R^*))$. For a set $X \subseteq A$,

$$
\begin{aligned}
U(R)(X) &= (\mu_B^{\mathcal{P}} \circ \mathcal{P}(R^*))(X) \\
&= \mu_B(\{R^*(x) \mid x \in X\}) \\
&= \mu_B(\{\{y \mid (x,y) \in R\} \mid x \in X\}) \\
&= \bigcup\{\{y \mid (x,y) \in R\} \mid x \in X\} \\
&= \{y \mid \exists x \in X.(x,y) \in R\}
\end{aligned}
$$

Now suppose that we have a function $f : A \longrightarrow B$ in **Set**, define $F(f)$ to be $(\eta_B^{\mathcal{P}} \circ f)^\dagger$. Let a be an element of A, then

$$
\begin{aligned}
(\eta_B^{\mathcal{P}} \circ f)^\dagger(A) &= \eta_B(f(a)) \\
&= \{f(a)\}
\end{aligned}
$$

Therefore $F(f) = (\eta_B^{\mathcal{P}} \circ f)^\dagger = \{(a, f(a)) \mid a \in A\}$. Now, from Proposition 2.3.13, $UF = \mathcal{P}$. This can easily be checked: for a set A, $UF(A) = U(A) = \mathcal{P}(A)$. Let $f : A \longrightarrow B$ be a map in **Set** and $X \subseteq A$. Then

$$
\begin{aligned}
(UF(f))(X) &= U(F(f))(X) \\
&= \{y \mid \exists x \in X.(x,y) \in F(f)\} \\
&= \{f(x) \mid x \in X\} \\
&= \mathcal{P}(f)(X)
\end{aligned}
$$

Remark 2.3.15. Throughout this thesis we will need some properties of the category of sets and relations **Rel**. We list here several of these properties without going into details. We have just shown that we can think of **Rel** as the Kleisli category of the power set monad. **Rel** is a degenerate model of linear logic in the sense that many of the connectives collapse to being the same in **Rel**. For example, multiplicative disjunction and conjunction $\otimes$ are both modelled by cartesian product $\times$. Linear implication $\multimap$ is also modelled by cartesian product. Additive disjunction and conjunction are modelled by disjoint union. For A an object of **Rel**, A a set, negation in **Rel** does nothing: $A^\perp = A$. On a morphism R, R a relation, $R^\perp = R^{op}$, the converse relation. The terminal object of **Rel** is the emptyset $\emptyset$. The categorical product of

Rel is also disjoint union, for two sets A and B, we write this as $A\&B$. **Rel** is a $*$-autonomous (i.e. symmetric monoidal closed) category.

We now dualize the notion of Kleisli category by moving from $\mathcal{C}$ to $\mathcal{C}^{op}$.

Definition 2.3.16. Given a comonad (T, ε, δ) on a category $\mathcal{C}$, the *coKleisli category* $coKl(T)$ is the category whose objects are the objects of $\mathcal{C}$ and an arrow $f : A \longrightarrow B \in coKl(T)$ is an arrow $f : TA \longrightarrow B \in \mathcal{C}$. The identity $id : A \longrightarrow A$ is $\varepsilon_A : A \longrightarrow TA$ and the composition of $f : TA \longrightarrow B$ and $g : TB \longrightarrow C$, $g \bullet f : TA \longrightarrow C$ is defined as follows: $g \bullet f = g \circ T(f) \circ \delta_C$.

The proof that this is indeed a category follows the proof that a Kleisli category is indeed a category, by duality

2.3.3 Distributive laws on monads

Definition 2.3.17. Let (T, η^T, μ^T) and (S, η^S, μ^S) be two monads on a category $\mathcal{C}$. A *distributive law of* S *over* T is a natural transformation $l : TS \longrightarrow ST$ such that the following four diagrams commute:

$$\begin{array}{ccccc} TSS & \xrightarrow{lS} & STS & \xrightarrow{Sl} & SST \\ {\scriptstyle T\mu^S}\downarrow & & & & \downarrow{\scriptstyle \mu^S T} \\ TS & & \xrightarrow[l]{} & & ST \end{array}$$

$$\begin{array}{ccccc} TTS & \xrightarrow{Tl} & TST & \xrightarrow{lT} & STT \\ {\scriptstyle \mu^T S}\downarrow & & & & \downarrow{\scriptstyle S\mu^T} \\ TS & & \xrightarrow[l]{} & & ST \end{array}$$

$$\begin{array}{ccc} & T & \\ {\scriptstyle T\eta^S}\swarrow & & \searrow{\scriptstyle \eta^S T} \\ TS & \xrightarrow[l]{} & ST \end{array} \qquad \begin{array}{ccc} & S & \\ {\scriptstyle \eta^T S}\swarrow & & \searrow{\scriptstyle S\eta^T} \\ TS & \xrightarrow[l]{} & ST \end{array}$$

Proposition 2.3.18. *Two monads (T, η^T, μ^T) and (S, η^S, μ^S) on a category $\mathcal{C}$ together with a distributive law $l : TS \longrightarrow ST$ of S over T induce a monad structure on T on the Kleisli category of S $Kl(S)$.*

Proof. Define the induced monad on $Kl(S)$ by the triple $(\hat{T}, \eta^{\hat{T}}, \mu^{\hat{T}})$ as follows. Given a morphism $f : A \longrightarrow B \in Kl(S)$, let $f^* : A \longrightarrow SB$ be the associated map in $\mathcal{C}$

and given a map $f : A \longrightarrow SB \in \mathcal{C}$, let $f^\dagger : A \longrightarrow B \in Kl(S)$ be the associated Klesli map. Let A be an object of $Kl(S)$, therefore $A \in \mathcal{C}$, then $\hat{T}(A) = T(A)$. Let $f : A \longrightarrow B$ be a morphism in $Kl(S)$, then $f^* : A \longrightarrow SB$ is a morphism in $\mathcal{C}$ and we can therefore apply T to it to get the morphism $T(f^*) : T(A) \longrightarrow TS(B) \in \mathcal{C}$. We can now compose $T(f^*)$ with our distributive law l at B to get the morphism $l_B \circ T(f^*) : T(A) \longrightarrow ST(B) \in \mathcal{C}$. We can now move back to the Kleisli category by performing the $\dagger$ operation and get a morphism $(l_B \circ T(f^*))^\dagger : T(A) \longrightarrow T(B) \in Kl(S)$. We define $\hat{T}(f)$ to be $(l_B \circ T(f^*))^\dagger$. To get a monad structure, define $\eta^{\hat{T}}$ to be $(\eta^S_{TA} \circ \eta^T_A)^\dagger$ and $\mu^{\hat{T}}$ to be $(l_B \circ T(\eta^S_A) \circ \mu^T_A)^\dagger$. We omit the proof that $\hat{T}$ is a functor and that $(\hat{T}, \eta^{\hat{T}}, \mu^{\hat{T}})$ is a monad and prove it for the special case where T is the finite multiset monad and S is the power set monad via the remark 2.3.21 and examples of the next section example 2.3.22 and example 2.3.24. □

The following remark summarizes a folklore result discussed in [12] p.616. (See also [4] p.262)

Remark 2.3.19 (Equivalence of liftings and distributive laws)**.** Given two monads (T, η^T, μ^T) and (S, η^S, μ^S) on a category $\mathcal{C}$, the monad T admits a lifting to the Kleisli category of S, $Kl(S)$, if and only if there is a distributive law $l : TS \longrightarrow ST$ of S over T.

Example 2.3.20 (Distributive law of power set monad over the multiset monad on **Set**)**.** Let $(\mathcal{P}, \eta^{\mathcal{P}}, \mu^{\mathcal{P}})$ be the power set monad on **Set** and $(\mathcal{M}_f(-), \eta^{\mathcal{M}_f}, \mu^{\mathcal{M}_f})$ be the multiset monad on **Set**. Define a natural transformation $l : \mathcal{M}_f \circ \mathcal{P} \longrightarrow \mathcal{P} \circ \mathcal{M}_f$ by the family

$$\{l_X : \mathcal{M}_f\mathcal{P}(X) \longrightarrow \mathcal{P}\mathcal{M}_f(X) \mid X \text{ a set}\}$$

where $l_X([A_1, \ldots, A_n]) = \{[a_1, \ldots, a_n] \mid a_i \in A_i\}$, $A_i \subseteq X$[1]. Then l is a distributive law of the power set monad over the multiset monad on **Set**. To show this, we must check that l is indeed a natural transformation and that all four diagrams from the definition commute. We will start by showing l is a natural transformation. To do

[1]Notice that the A_i are not necessarily distinct subsets of X

this we must show that for any $f : X \longrightarrow Y \in \mathbf{Set}$ the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{M}_f\mathcal{P}(X) & \xrightarrow{l_X} & \mathcal{P}\mathcal{M}_f(X) \\
{\scriptstyle \mathcal{M}_f\mathcal{P}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{P}\mathcal{M}_f(f)} \\
\mathcal{M}_f\mathcal{P}(Y) & \xrightarrow{l_Y} & \mathcal{P}\mathcal{M}_f(Y)
\end{array}$$

Let $[A_1, \ldots, A_k] \in \mathcal{M}_f\mathcal{P}(X)$ then we have

$$\begin{aligned}
(\mathcal{P}\mathcal{M}_f(f) \circ l_X)[A_1, \ldots, A_n] &= \mathcal{P}\mathcal{M}_f(f)(\{[a_1, \ldots, a_n] \mid a_i \in A_i\}) \\
&= \{\mathcal{M}_f(f)([a_1, \ldots, a_n]) \mid a_i \in A_i\}) \\
&= \{[f(a_1), \ldots, f(a_n)]) \mid a_i \in A_i\}) \\
&= \{[b_1, \ldots, b_n]) \mid b_i \in f[A_i]\}) \\
&= l_Y([f[A_1], \ldots, f[A_n]]) \\
&= l_Y([\mathcal{P}(f)(A_1), \ldots, \mathcal{P}(f)(A_n)]) \\
&= (l_Y \circ \mathcal{M}_f\mathcal{P}(f))[A_1, \ldots, A_n]
\end{aligned}$$

as required. Now, according to the definition, it remains to show that the following four diagrams commute:

$$\begin{array}{ccccc}
\mathcal{M}_f\mathcal{P}\mathcal{P}(X) & \xrightarrow{l_{\mathcal{P}(X)}} & \mathcal{P}\mathcal{M}_f\mathcal{P}(X) & \xrightarrow{\mathcal{P}(l_X)} & \mathcal{P}\mathcal{P}\mathcal{M}_f(X) \\
{\scriptstyle \mathcal{M}_f(\mu^{\mathcal{P}}_X)}\downarrow & & & & \downarrow{\scriptstyle \mu^{\mathcal{P}}_{\mathcal{M}_f(X)}} \\
\mathcal{M}_f\mathcal{P}(X) & & \xrightarrow{\qquad l_X \qquad} & & \mathcal{P}\mathcal{M}_f(X)
\end{array} \qquad (1)$$

$$\begin{array}{ccccc}
\mathcal{M}_f\mathcal{M}_f\mathcal{P}(X) & \xrightarrow{\mathcal{M}_f(l_X)} & \mathcal{M}_f\mathcal{P}\mathcal{M}_f(X) & \xrightarrow{l_{\mathcal{M}_f(X)}} & \mathcal{P}\mathcal{M}_f\mathcal{M}_f(X) \\
{\scriptstyle \mu^{\mathcal{M}_f}_{\mathcal{P}(X)}}\downarrow & & & & \downarrow{\scriptstyle \mathcal{P}(\mu^{\mathcal{M}_f}_X)} \\
\mathcal{M}_f\mathcal{P}(X) & & \xrightarrow{\qquad l_X \qquad} & & \mathcal{P}\mathcal{M}_f(X)
\end{array}$$

$$\begin{array}{ccc}
 & \mathcal{M}_f(X) & \\
{\scriptstyle \mathcal{M}_f(\eta^{\mathcal{P}}_X)}\swarrow & & \searrow{\scriptstyle \eta^{\mathcal{P}}_{\mathcal{M}_f(X)}} \\
\mathcal{M}_f\mathcal{P}(X) & \xrightarrow{\qquad l_X \qquad} & \mathcal{P}\mathcal{M}_f(X)
\end{array}$$

$$\begin{array}{ccc}
 & \mathcal{P}(X) & \\
{\scriptstyle \eta^{\mathcal{M}_f}_{\mathcal{P}(X)}}\swarrow & & \searrow{\scriptstyle \mathcal{P}(\eta^{\mathcal{M}_f}_X)} \\
\mathcal{M}_f\mathcal{P}(X) & \xrightarrow{\qquad l_X \qquad} & \mathcal{P}\mathcal{M}_f(X)
\end{array}$$

where X is a set.

To show that the first diagram commutes, we let $[\mathcal{F}_1, \ldots, \mathcal{F}_n] \in \mathcal{M}_f\mathcal{P}\mathcal{P}(X)$, $\mathcal{F}_i \subseteq \mathcal{P}(X)$, then

$$\begin{aligned}
(\mu^{\mathcal{P}}_{\mathcal{M}_f(X)} \circ \mathcal{P}(l_X) \circ l_{\mathcal{P}(X)})[\mathcal{F}_1, \ldots, \mathcal{F}_n] &= (\mu^{\mathcal{P}}_{\mathcal{M}_f(X)} \circ \mathcal{P}(l_X))\{[A_1, \ldots, A_n] \mid A_i \in \mathcal{F}_i\} \\
&= \mu^{\mathcal{P}}_{\mathcal{M}_f(X)}\{l_X([A_1, \ldots, A_n]) \mid A_i \in \mathcal{F}_i, i = 1 \ldots, n\} \\
&= \mu^{\mathcal{P}}_{\mathcal{M}_f(X)}\{\{[a_1, \ldots, a_n] \mid a_i \in A_i\} \mid A_i \in \mathcal{F}_i\} \\
&= \textstyle\bigcup_{A_i \in \mathcal{F}_i}\{[a_1, \ldots, a_n] \mid a_i \in A_i \text{ and } A_i \in \mathcal{F}_i\} \\
& \quad RHS
\end{aligned}$$

and

$$\begin{aligned}
(l_X \circ \mathcal{M}_f(\mu^{\mathcal{P}}_X))[\mathcal{F}_1, \ldots, \mathcal{F}_n] &= l_X([\mu^{\mathcal{P}}_X(\mathcal{F}_1), \ldots, \mu^{\mathcal{P}}_X(\mathcal{F}_n)]) \\
&= l_X([\textstyle\bigcup_{A_{i_1} \in \mathcal{F}_1}(A_{i_1}), \ldots, \bigcup_{A_{i_n} \in \mathcal{F}_n}(A_{i_n})]) \\
&= \{[a_i, \ldots, a_n] \mid a_1 \in \textstyle\bigcup_{A_{i_1} \in \mathcal{F}_1}(A_{i_1}), \ldots, a_n \in \bigcup_{A_{n_1} \in \mathcal{F}_n}(A_{i_n})\} \\
& \quad LHS
\end{aligned}$$

Notice

$$\begin{aligned}
[a_1, \ldots, a_n] \in LHS \quad & \text{iff} \quad a_1 \in \textstyle\bigcup_{A_{i_1} \in \mathcal{F}_1}(A_{i_1}), \ldots, a_n \in \bigcup_{A_{n_1} \in \mathcal{F}_n}(A_{i_n}) \\
& \text{iff} \quad \exists A_{i_1} \in \mathcal{F}_1 \text{ s.t. } a_1 \in A_{i_1}, \ldots, \exists A_{i_n} \in \mathcal{F}_n \text{ s.t. } a_n \in A_{i_n} \\
& \text{iff} \quad a_1 \in A_{i_1} \text{ and } A_{i_1} \in \mathcal{F}_1, \ldots, a_n \in A_{i_n} \text{ and } A_{i_n} \in \mathcal{F}_n \\
& \text{iff} \quad [a_1, \ldots, a_n] \in RHS
\end{aligned}$$

Therefore the first diagram commutes.

For the second diagram, let $[\alpha_1, \ldots, \alpha_n] \in \mathcal{M}_f\mathcal{M}_f\mathcal{P}(X)$, $\alpha_i = [A^i_1, \ldots, A^i_{k_i}]$, then

$$\begin{aligned}
(\mathcal{P}(\mu^{\mathcal{M}_f}_X) \circ l_{\mathcal{M}_f(X)} \circ \mathcal{M}_f(l_X))[\alpha_1, \ldots, \alpha_n] &= (\mathcal{P}(\mu^{\mathcal{M}_f}_X) \circ l_{\mathcal{M}_f(X)})[l_{\mathcal{M}_f(X)}(\alpha_1), \ldots, l_{\mathcal{M}_f(X)}(\alpha_n)] \\
&= (\mathcal{P}(\mu^{\mathcal{M}_f}_X) \circ l_{\mathcal{M}_f(X)})[B_1, \ldots, B_n] \\
& \quad \text{where } B_i = \{[a^i_1, \ldots, a^i_{k_i}] \mid a^i_j \in A^i_j\} \\
&= \mathcal{P}(\mu^{\mathcal{M}_f}_X)\{[b_1, \ldots, b_n] \mid b_i \in B_i\} \\
&= \{(\mu^{\mathcal{M}_f}_X)([b_1, \ldots, b_n]) \mid b_i \in B_i\} \\
&= \{b_1 \uplus \cdots \uplus b_n \mid b_i \in B_i\}
\end{aligned}$$

In the other direction, we get

$$\begin{aligned}(l_X \circ \mu^{\mathcal{M}_f}_{\mathcal{P}(X)})[\alpha_1, \ldots, \alpha_n] &= l_X(\alpha_1 \uplus \cdots \uplus \alpha_n) \\ &= l_X([A^1_1, \ldots, A^1_{k_1}, \ldots, A^n_1, \ldots, A^n_{k_n}]) \\ &= \{[a^1_1, \ldots, a^1_{k_1}, \ldots, a^n_1, \ldots, a^n_{k_n}] \mid a^j_i \in A^j_i\} \\ &= \{b_1 \uplus \cdots \uplus b_n \mid b_i \in B_i\}\end{aligned}$$

Therefore the second diagram commutes.

For the third diagram, let $\alpha = [a_1, \ldots, a_n] \in X$ then

$$\begin{aligned}(l_X \circ \mathcal{M}_f(\eta^{\mathcal{P}}_X))(\alpha) &= l_X([\eta^{\mathcal{P}}_X(a_1), \ldots, \eta^{\mathcal{P}}_X(a_n)]) \\ &= l_X([\{a_1\}, \ldots, \{a_n\}]) \\ &= \{[b_1, \ldots, b_n] \mid b_i \in \{a_i\}, i = 1, \ldots, n\} \\ &= \{[a_1, \ldots, a_n]\} \\ &= \{\alpha\} \\ &= \eta^{\mathcal{P}}_{\mathcal{M}_f(X)}(\alpha)\end{aligned}$$

Therefore the third diagram commutes.

For the fourth diagram, let $A \subseteq X$ then

$$\begin{aligned}(l_X \circ \mathcal{M}_f(\eta^{\mathcal{M}_f}_{\mathcal{P}(X)}))(A) &= l_X([A]) \\ &= \{[a] \mid a \in A\} \\ &= \{\eta^{\mathcal{M}_f}_X(a) \mid a \in A\} \\ &= \eta^{\mathcal{M}_f}_X[A] \\ &= \mathcal{P}(\eta^{\mathcal{M}_f}_X)(A)\end{aligned}$$

Therefore the fourth diagram commutes. We have shown that l is a natural transformation and that all four diagrams commute, therefore l is indeed a distributive law of the power set monad over the multiset monad on **Set**.

Remark 2.3.21 (Lifting $\mathcal{M}_f(-)$ on **Set** to **Rel**)**.** This distributive law l of the power set monad over the multiset monad on **Set** allows us to lift the multiset monad $\mathcal{M}_f(-)$ on **Set** to a multiset monad on **Rel** which we will call, for now, $\hat{\mathcal{M}}_f(-)$. Since the objects of **Rel** are the same as the objects of **Set**, for any object A of **Rel**, i.e. for any set A, then $\hat{\mathcal{M}}_f(A) = \mathcal{M}_f(A) = \{\alpha | \alpha \text{ is a finite multiset over } A\}$. Now suppose we have a relation $R : X \longrightarrow Y \in$ **Rel**. Recall that $Kl(\mathcal{P}) =$ **Rel**, where $\mathcal{P}$ is the power set monad on **Set**. Therefore our relation $R : X \longrightarrow Y \in \mathbf{Rel}$ is equivalent to some arrow, say $R^* : X \longrightarrow \mathcal{P}(Y) \in \mathbf{Set}$. Now that we have an arrow in **Set**, we can make use of our multiset monad on **Set**: apply $\mathcal{M}_f$ to R^* to get an arrow $\mathcal{M}_f(R^*) : \mathcal{M}_f(X) \longrightarrow \mathcal{M}_f\mathcal{P}(Y) \in \mathbf{Set}$. Here is where we use the distributive law of $\mathcal{P}$ over $\mathcal{M}_f(-)$: we compose $\mathcal{M}_f(R^*)$ with l_Y to get $l_Y \circ \mathcal{M}_f(R^*) : \mathcal{M}_f(X) \longrightarrow \mathcal{P}\mathcal{M}_f(Y) \in \mathbf{Set}$. Again, we use the fact that $Kl(\mathcal{P}) = \mathbf{Rel}$, and notice that $l_Y \circ \mathcal{M}_f(R^*) : \mathcal{M}_f(X) \longrightarrow \mathcal{P}\mathcal{M}_f(Y) \in \mathbf{Set}$ is equivalent to an arrow $(l_Y \circ \mathcal{M}_f(R^*))^\dagger : \mathcal{M}_f(X) \longrightarrow \mathcal{M}_f(Y) \in \mathbf{Rel}$. Therefore, for $\alpha = [x_1, \ldots, x_k] \in \mathcal{M}_f(X)$

$$\begin{aligned}(l_Y \circ \mathcal{M}_f(R^*))(\alpha) &= l_Y(\mathcal{M}_f(R^*)(\alpha)) \\ &= l_Y([R^*(x_1), \ldots, R^*(x_k)]) \\ &= l_Y([\{y \mid (x_1, y) \in R\}, \ldots, \{y \mid (x_k, y) \in R\}]) \\ &= \{[y_1, \ldots, y_k] \mid y_i \in \{y \mid (x_i, y) \in R\}\} \\ &= \{[y_1, \ldots, y_k] \mid (x_i, y_i) \in R\}\end{aligned}$$

Define $\hat{\mathcal{M}}_f(R) : \mathcal{M}_f(X) \longrightarrow \mathcal{M}_f(Y)$ to be $(l_Y \circ \mathcal{M}_f(R^*))^\dagger$. From the above calculation we get the following definition:

$$\begin{aligned}(\alpha, \beta) \in \hat{\mathcal{M}}_f(R) \quad &\text{iff} \quad \text{whenever } \alpha = [x_1, \ldots, x_k] \in \mathcal{M}_f(X), \beta = [y_1, \ldots, y_k] \in \mathcal{M}_f(Y), \\ &\qquad \text{we have } (x_i, y_i) \in R \\ &\text{iff} \quad \exists c \subseteq_f R \text{ s.t. } \pi_1[c] = \alpha, \pi_2[c] = \beta\end{aligned}$$

viewing R as a subset of $X \times Y$, $\mathcal{M}_f(R)$ as a subset of $\mathcal{M}_f(X) \times \mathcal{M}_f(Y)$ and $\pi_i[c]$ as the multiset direct image for $i = 1, 2$. Now, from Propostion 2.3.18, for a set A, $\hat{\mathcal{M}}_f(A) = \mathcal{M}_f(A)$. If $A \xrightarrow{R} B$ is an arrow in **Rel** , then $\mathcal{M}_f(A) \xrightarrow{\hat{\mathcal{M}}_f(R)} \mathcal{M}_f(B)$ and $(\alpha, \beta) \in \mathcal{M}_f(R)$ iff $\exists c \subseteq_f R$ such that $\pi_1[c] = \alpha$ and $\pi_2[c] = \beta$. For the monad structure, Proposition 2.3.18 tells us that $(\hat{\mathcal{M}}_f, \eta^{\hat{\mathcal{M}}_f}, \mu^{\hat{\mathcal{M}}_f})$ is a monad on $Kl(\mathcal{P})$, which we can think of as **Rel** by example 2.3.14. For a set A, $\eta_A^{\hat{\mathcal{M}}_f}$ is given by $(\eta^{\mathcal{P}}_{\mathcal{M}_f(A)} \circ \eta_A^{\mathcal{M}_f})^\dagger$. For a an element of A,

$$\begin{aligned}(\eta^{\mathcal{P}}_{\mathcal{M}_f(A)} \circ \eta_A^{\mathcal{M}_f})(a) &= \eta^{\mathcal{P}}_{\mathcal{M}_f(A)}([a]) \\ &= \{[a]\}\end{aligned}$$

Therefore,

$$\begin{aligned}\eta_A^{\hat{\mathcal{M}}_f} &= (\eta^{\mathcal{P}}_{\mathcal{M}_f(A)} \circ \eta_A^{\mathcal{M}_f})^{\dagger} \\ &= \{(a, [a]) \mid a \in A\}\end{aligned}$$

Lastly, $\mu_A^{\hat{\mathcal{M}}_f}$ is given by $(l_A \circ \mathcal{M}_f(\eta_A^{\mathcal{P}}) \circ \mu_A^{\mathcal{M}_f})^{\dagger}$. Let $\mathcal{A}$ be an element of $\mathcal{M}_f^2(A)$, say $\mathcal{A} = [\alpha_1, \ldots, \alpha_k]$ with $\alpha_i = [a_1^i, \ldots, a_{l_i}^i]$ for $i = 1, \ldots k$. Then

$$\begin{aligned}(l_A \circ \mathcal{M}_f(\eta_A^{\mathcal{P}}) \circ \mu_A^{\mathcal{M}_f})(\mathcal{A}) &= (l_A \circ \mathcal{M}_f(\eta_A^{\mathcal{P}}))(\alpha_1 \uplus \cdots \uplus \cdots \alpha_k) \\ &= (l_A \circ \mathcal{M}_f(\eta_A^{\mathcal{P}}))([a_1^1, \ldots, a_{l_1}^1, \ldots, a_1^k, \ldots, a_{l_k}^k]) \\ &= l_A([\{a_1^1\}, \ldots, \{a_{l_1}^1\}, \ldots, \{a_1^k\}, \ldots, \{a_{l_k}^k\}]) \\ &= \{[a_1^1, \ldots, a_{l_1}^1, \ldots, a_1^k, \ldots, a_{l_k}^k]\}\end{aligned}$$

Therefore,

$$\begin{aligned}\mu_A^{\hat{\mathcal{M}}_f} &= (l_A \circ \mathcal{M}_f(\eta_A^{\mathcal{P}}) \circ \mu_A^{\mathcal{M}_f})^{\dagger} \\ &= \{([\alpha_1, \ldots, \alpha_k], \alpha_1 \uplus \cdots \uplus \alpha_k) \mid \alpha_i \in \mathcal{M}_f(A), i = 1, \ldots, k\}\end{aligned}$$

Example 2.3.22 and example 2.3.24 verify that $\hat{\mathcal{M}}_f$ is indeed an endofunctor on **Rel** and that $(\hat{\mathcal{M}}_f, \eta^{\hat{\mathcal{M}}_f}, \mu^{\hat{\mathcal{M}}_f})$ is a monad on **Rel**.

2.3.4 Properties of the lifted $\mathcal{M}_f$: Rel $\longrightarrow$ Rel

From this point on, we will write $\mathcal{M}_f(-)$ for the lifted multiset comonad on **Rel**, unless specified otherwise.

Example 2.3.22 (Finite multiset functor on **Rel**). Let $\mathcal{M}_f(-)$: **Rel** $\longrightarrow$ **Rel** be the finite multiset functor defined above. If A is a set, $\mathcal{M}_f(A) = \{\alpha | \alpha \text{ is a finite multiset over } A\}$. If $A \xrightarrow{R} B$ is an arrow in **Rel** , then $\mathcal{M}_f(A) \xrightarrow{\mathcal{M}_f(R)} \mathcal{M}_f(B)$ and $(x, y) \in \mathcal{M}_f(R)$ iff $\exists c \subseteq_f R$ such that $\pi_1[c] = x$ and $\pi_2[c] = y$.

To show that $\mathcal{M}_f(-)$ is a functor we must show that for A a set, $\mathcal{M}_f(id_A) = id_{\mathcal{M}_f(A)}$. Indeed $\mathcal{M}_f(id_A) : \mathcal{M}_f(A) \longrightarrow \mathcal{M}_f(A)$ and for $X \subseteq A$, $(x, y) \in \mathcal{M}_f(id_A)$ iff $\exists c \subseteq_f id_A$ such that $\pi_1[c] = x$ and $\pi_2[c] = y$. But, since $c \subseteq_f id_A$, $\pi_1[c] = \pi_2[c]$; therefore $x = y$ and we have $(x, y) \in \mathcal{M}_f(id_A)$ iff $x = y$ iff $(x, y) \in id_{\mathcal{M}_f(A)}$. It remains to show that for $A \xrightarrow{R} B$ and $B \xrightarrow{R'} C$, $\mathcal{M}_f(R' \circ R) = \mathcal{M}_f(R') \circ \mathcal{M}_f(R)$.

Notice

$$\begin{aligned}(x,y)\in\mathcal{M}_f(R'\circ R) \quad &\text{iff} \quad \exists c\subseteq_f R'\circ R \text{ s.t. } \pi_1[c]=x \text{ and } \pi_2[c]=y\\ &\text{iff} \quad \exists(a_1,c_1),\ldots,(a_n,c_n)\in R'\circ R \text{ and } x=[a_1,\ldots,a_n], y=[c_1,\ldots,c_n]\\ &\text{iff} \quad \exists b_1,\ldots,b_n\in B \text{ s.t. } (a_i,b_i)\in R, (b_i,c_i)\in R',\\ &\qquad x=[a_1,\ldots,a_n], y=[c_1,\ldots,c_n]\\ &\text{iff} \quad (x,y)\in\mathcal{M}_f(R')\circ\mathcal{M}_f(R)\end{aligned}$$

as required. Therefore, the finite multiset functor $\mathcal{M}_f(-)$ is indeed a functor on **Rel**.

Example 2.3.23 (Finite multiset monad on **Rel**). As discussed in 2.3.21, $\eta^{\mathcal{M}_f}$: $Id \longrightarrow \mathcal{M}_f(-)$ is defined by the family

$$\{\eta_A^{\mathcal{M}_f} : A \longrightarrow \mathcal{M}_f(A) \mid \text{A a set}\}\ ,\ \text{where } \eta_A^{\mathcal{M}_f} = \{(a,[a]) \mid a\in A)\}$$

and $\mu^{\mathcal{M}_f} : \mathcal{M}_f^2(-) \longrightarrow \mathcal{M}_f(-)$ is defined by the family

$$\{\mu_A^{\mathcal{M}_f} : \mathcal{M}_f^2(A) \longrightarrow \mathcal{M}_f(A) \mid \text{A a set}\}, \text{ where } \delta_A^{\mathcal{M}_f} = \{([\alpha_1,\ldots,\alpha_k],\alpha_1\uplus\ldots\uplus\alpha_k) \mid \alpha_i\in\mathcal{M}_f(A)\}$$

We claim that $(\mathcal{M}_f,\eta,\mu)$ is a monad. Since **Rel** is self-dual, $\mathcal{M}_f$ can also be defined as a comonad. The proof that $(\mathcal{M}_f,\eta,\mu)$ is a monad is dual to the proof that $\mathcal{M}_f$ is a comonad and hence, we omit proving this claim and instead show that $\mathcal{M}_f$ is a comonad in the next example (example 2.3.24).

Example 2.3.24 (Finite multiset comonad on **Rel**). Let $\mathcal{M}_f : \mathbf{Rel} \longrightarrow \mathbf{Rel}$ be the finite multiset functor from above. Recall that we can define $\varepsilon : \mathcal{M}_f(-) \longrightarrow Id$ by the family

$$\{\varepsilon_A : \mathcal{M}_f(A) \longrightarrow A \mid \text{A a set}\}\ ,\ \text{where } \varepsilon_A = \{([a],a) \mid a\in A)\}$$

and $\delta : \mathcal{M}_f(-) \longrightarrow \mathcal{M}_f(\mathcal{M}_f(-))$ by the family

$$\{\delta_A : \mathcal{M}_f(A) \longrightarrow \mathcal{M}_f(\mathcal{M}_f(A)) \mid \text{A a set}\}, \text{ where } \delta_A = \{(\alpha_1\uplus\ldots\uplus\alpha_k,[\alpha_1,\ldots,\alpha_k] \mid \alpha_i\in\mathcal{M}_f(A)\}$$

such that $(\mathcal{M}_f(-),\varepsilon,\delta)$ is a comonad on **Rel**.

To check that we have a comonad we must verify that both ε and δ are natural transformations. To show that ε is a natural transformation, we must show that for every $R : A \longrightarrow B \in \mathbf{Rel}$, viewed as a subset of $A \times B$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_f(A) & \xrightarrow{\varepsilon_A} & A \\ {\scriptstyle \mathcal{M}_f(R)}\downarrow & & \downarrow{\scriptstyle R} \\ \mathcal{M}_f(B) & \xrightarrow{\varepsilon_B} & B \end{array}$$

Notice

$$\begin{aligned} (\alpha, b) \in R \circ \varepsilon_A \quad & \text{iff} \quad \exists a \in A \text{ s.t. } (\alpha, a) \in \varepsilon_A, (a, b) \in R \\ & \text{iff} \quad \exists a \in A \text{ s.t. } \alpha = [a], (a, b) \in R \end{aligned}$$

and

$$(\alpha, b) \in \varepsilon_B \circ \mathcal{M}_f(R) \quad \text{iff} \quad \exists \beta \in \mathcal{M}_f(B) \text{ s.t. } (\alpha, \beta) \in \mathcal{M}_f(R), (\beta, b) \in \varepsilon_B$$

but $(\beta, b) \in \varepsilon_B$ means $\beta = [b]$. Therefore

$$\begin{aligned} (\alpha, b) \in \varepsilon_B \circ \mathcal{M}_f(R) \quad & \text{iff} \quad (\alpha, [b]) \in \mathcal{M}_f(R) \\ & \text{iff} \quad \exists c \subseteq_f R \text{ s.t. } \pi_1[c] = \alpha \text{ and } \pi_2[c] = [b] \end{aligned}$$

Therefore α has one element say a, which implies $c = \{(a, b)\}$ with $(a, b) \in R$. So we have

$$\begin{aligned} (\alpha, b) \in \varepsilon_B \circ \mathcal{M}_f(R) \quad & \text{iff} \quad \exists a \in A \text{ s.t. } \alpha = [a], (a, b) \in R \\ & \text{iff} \quad (\alpha, b) \in \varepsilon_B \circ \mathcal{M}_f(R) \end{aligned}$$

Therefore $R \circ \varepsilon_A = \varepsilon_B \circ \mathcal{M}_f(R)$ as required and we have that ε is a natural transformation.

To show that δ is a natural transformation, we must show that for every $R : A \longrightarrow B \in \mathbf{Rel}$, viewed as a subset of $A \times B$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_f(A) & \xrightarrow{\delta_A} & \mathcal{M}_f(\mathcal{M}_f(A)) \\ {\scriptstyle \mathcal{M}_f(R)}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_f(\mathcal{M}_f(R))} \\ \mathcal{M}_f(B) & \xrightarrow{\delta_B} & \mathcal{M}_f(\mathcal{M}_f(A)) \end{array}$$

Notice

$$(\alpha, \mathcal{B}) \in \mathcal{M}_f(\mathcal{M}_f(R)) \circ \delta_A \quad \text{iff} \quad \exists \mathcal{A} \in \mathcal{M}_f(\mathcal{M}_f(A)) \text{ s.t. } (\alpha, \mathcal{A}) \in \delta_A, (\mathcal{A}, \mathcal{B}) \in \mathcal{M}_f(\mathcal{M}_f(R))$$

but

$$(\alpha, \mathcal{A}) \in \delta_A \quad \text{iff} \quad \text{when } \mathcal{A} = [\alpha_1, \ldots, \alpha_k], \alpha_i \in \mathcal{M}_f(A), \text{ then } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k$$

$$([\alpha_1, \ldots, \alpha_k], \mathcal{B}) \in \mathcal{M}_f(\mathcal{M}_f(R)) \quad \text{iff} \quad \exists c \subseteq_f \mathcal{M}_f(R) \text{ s.t. } \pi_1[c] = [\alpha_1, \ldots, \alpha_k], \pi_2[c] = \mathcal{B}$$

So $\mathcal{B} = [\beta_1, \ldots, \beta_k], \beta_i \in \mathcal{M}_f(B)$, where $(\alpha_i, \beta_i) \in \mathcal{M}_f(R)$. This happens if and only if $\exists d_i \subseteq_f R$ s.t. $\pi_1[d_i] = \alpha_i$, $\pi_2[d_i] = \beta_i$. This means that each α_i and β_i have the same number of elements. Now, supposing that $\alpha_i = [a^i_1, \ldots, a^i_{l_i}], \beta_i = [b^i_1, \ldots, b^i_{l_i}]$, then $(\alpha_i, \beta_i) \in \mathcal{M}_f(R)$ if and only if $(a^i_1, b^i_1), \ldots, (a^i_{l_i}, b^i_{l_i}) \in R$ for every i. Summarizing, we have

$$\begin{aligned}(\alpha, \mathcal{B}) \in \mathcal{M}_f(\mathcal{M}_f(R)) \circ \delta_A \quad \text{iff} \quad & \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k, \mathcal{B} = [\beta_1, \ldots, \beta_k], \alpha_i = [a^i_1, \ldots, a^i_{l_i}], \\ & \beta_i = [b^i_1, \ldots, b^i_{l_i}], (a^i_1, b^i_1), \ldots, (a^i_{l_i}, b^i_{l_i}) \in R, i = 1 \ldots k\end{aligned}$$

Now for the other direction,

$$(\alpha, \mathcal{B}) \in \delta_B \circ \mathcal{M}_f(R) \quad \text{iff} \quad \exists \beta \in \mathcal{M}_f(B) \text{ s.t. } (\alpha, \beta) \in \mathcal{M}_f(R), (\beta, \mathcal{A}) \in \delta_B$$

but

$$(\beta, \mathcal{B}) \in \delta_B \quad \text{iff} \quad \text{when } \mathcal{B} = [\beta_1, \ldots, \beta_k], \beta_i \in \mathcal{M}_f(B), \text{ then } \beta = \beta_1 \uplus \cdots \uplus \beta_k$$

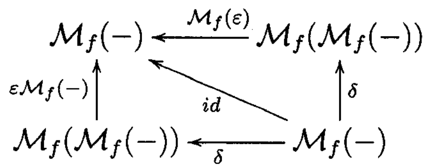
$$(\alpha, \beta_1 \uplus \cdots \uplus \beta_k) \in \mathcal{M}_f(R) \quad \text{iff} \quad \exists c \subseteq_f R \text{ s.t. } \pi_1[c] = \alpha, \pi_2[c] = \beta_1 \uplus \cdots \uplus \beta_k$$

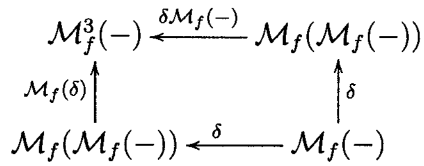
Supposing that $\beta_i = [b^i_1, \ldots, b^i_{l_i}]$, then $\alpha = [a^1_1, \ldots, a^1_{l_1}, \ldots, a^k_1, \ldots, a^k_{l_k}]$ with $(a^i_1, b^i_1), \ldots, (a^i_{l_i}, b^i_{l_i}) \in R$ for every i. Notice that we can write $\alpha = \alpha_1 \uplus \cdots \uplus \alpha_k$, where $\alpha_i = [a^i_1, \ldots, a^i_{l_i}]$. In summary,

$$\begin{aligned}(\alpha, \mathcal{B}) \in \delta_B \circ \mathcal{M}_f(R) \quad \text{iff} \quad & \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k, \mathcal{B} = [\beta_1, \ldots, \beta_k], \alpha_i = [a^i_1, \ldots, a^i_{l_i}], \\ & \beta_i = [b^i_1, \ldots, b^i_{l_i}], (a^i_1, b^i_1), \ldots, (a^i_{l_i}, b^i_{l_i}) \in R, i = 1 \ldots k \\ \text{iff} \quad & (\alpha, \mathcal{B}) \in \mathcal{M}_f(\mathcal{M}_f(R)) \circ \delta_A\end{aligned}$$

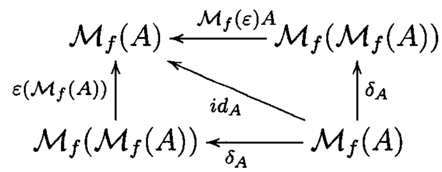
So we have that $\mathcal{M}_f(\mathcal{M}_f(R)) \circ \delta_A = \delta_B \circ \mathcal{M}_f(R)$ as required and that δ is indeed a natural transformation.

To complete the proof that $(\mathcal{M}_f(-), \varepsilon, \delta)$ is a comonad, we must show that the following two diagrams commute





Let A be a set. Then, for the first diagram, we must show that the following commutes:



$$
\begin{aligned}
(\alpha, \beta) \in \mathcal{M}_f(\varepsilon)A \circ \delta_A \quad &\text{iff} \quad (\alpha, \beta) \in \mathcal{M}_f(\varepsilon_A) \circ \delta_A \\
&\text{iff} \quad \exists \mathcal{A} \in \mathcal{M}_f(\mathcal{M}_f(A)) \text{ s.t. } (\alpha, \mathcal{A}) \in \delta_A, (\mathcal{A}, \beta) \in \mathcal{M}_f(\varepsilon_A)
\end{aligned}
$$

but

$$
\begin{aligned}
(\alpha, \mathcal{A}) \in \delta_A \quad &\text{iff} \quad \text{when } \mathcal{A} = [\alpha_1, \ldots, \alpha_k], \alpha_i \in \mathcal{M}_f(A), \text{ then } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k \\
(\mathcal{A}, \beta) \in \mathcal{M}_f(\varepsilon_A) \quad &\text{iff} \quad \exists c \subseteq_f \varepsilon_A \text{ s.t. } \pi_1[c] = [\alpha_1, \ldots, \alpha_k], \pi_2[c] = \beta \\
&\text{iff} \quad k = 1, \text{ and when } \alpha_1 = [a_1, \ldots, a_j] \text{ then } c = \{([a_1], a_1), \ldots, ([a_j], a_j)\} \\
&\qquad \text{and } \beta = \pi_1[c] = [a_1, \ldots, a_j] = \alpha_1
\end{aligned}
$$

So, we have

$$
\begin{aligned}
(\alpha, \beta) \in \mathcal{M}_f(\varepsilon_A) \circ \delta_A \quad &\text{iff} \quad \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k = \alpha_1 \text{ since } k = 1 \text{ and } \beta = \alpha_1, \alpha_1 \in \mathcal{M}_f(A) \\
&\text{iff} \quad \alpha = \beta \\
&\text{iff} \quad (\alpha, \beta) \in Id_{\mathcal{M}_f(A)}
\end{aligned}
$$

Now

$$
\begin{aligned}
(\alpha, \beta) \in \varepsilon(\mathcal{M}_f(A)) \circ \delta_A \quad &\text{iff} \quad (\alpha, \beta) \in \varepsilon_{\mathcal{M}_f(A)} \circ \delta_A \\
&\text{iff} \quad \exists \mathcal{A} \in \mathcal{M}_f(\mathcal{M}_f(A)) \text{ s.t. } (\alpha, \mathcal{A}) \in \delta_A, (\mathcal{A}, \beta) \in \varepsilon_{\mathcal{M}_f(A)}
\end{aligned}
$$

but

$$\begin{array}{rll}(\alpha, \mathcal{A}) \in \delta_A & \text{iff} & \text{when } \mathcal{A} = [\alpha_1, \ldots, \alpha_k], \alpha_i \in \mathcal{M}_f(A), \text{ then } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k \\ (\mathcal{A}, \beta) \in \varepsilon_{\mathcal{M}_f(A)} & \text{iff} & k = 1, \mathcal{A} = [\alpha_1], \beta = \alpha_1\end{array}$$

since

$$\varepsilon_{\mathcal{M}_f(A)} = \{([m], m) \mid m \in \mathcal{M}_f(A)\}$$

So, we have

$$\begin{array}{rll}(\alpha, \beta) \in \mathcal{M}_f(\varepsilon_A) \circ \delta_A & \text{iff} & \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k = \alpha_1 \text{ since } k = 1 \text{ and } \beta = \alpha_1, \alpha_1 \in \mathcal{M}_f(A) \\ & \text{iff} & \alpha = \beta \\ & \text{iff} & (\alpha, \beta) \in Id_{\mathcal{M}_f(A)}\end{array}$$

Therefore $\mathcal{M}_f(\varepsilon)A \circ \delta_A = Id_{\mathcal{M}_f(A)} = \varepsilon(\mathcal{M}_f(A)) \circ \delta_A$ as required. For the second diagram, we must show that the following commutes:

$$\begin{array}{ccc}\mathcal{M}_f^3(A) & \xleftarrow{\delta(\mathcal{M}_f(A))} & \mathcal{M}_f(\mathcal{M}_f(A)) \\ {\scriptstyle \mathcal{M}_f(\delta)(A)} \uparrow & & \uparrow {\scriptstyle \delta_A} \\ \mathcal{M}_f(\mathcal{M}_f(A)) & \xleftarrow{\delta_A} & \mathcal{M}_f(A)\end{array}$$

$$\begin{array}{rll}(\alpha, X) \in \delta(\mathcal{M}_f(A)) \circ \delta_A & \text{iff} & (\alpha, X) \in \delta_{\mathcal{M}_f(A)} \circ \delta_A \\ & \text{iff} & \exists \mathcal{A} \in \mathcal{M}_f(\mathcal{M}_f(A)) \text{ s.t. } (\alpha, \mathcal{A}) \in \delta_A, (\mathcal{A}, X) \in \delta_{\mathcal{M}_f(A)}\end{array}$$

but

$$(\alpha, \mathcal{A}) \in \delta_A \quad \text{iff} \quad \text{when } \mathcal{A} = [\alpha_1, \ldots, \alpha_k], \alpha_i \in \mathcal{M}_f(A), \text{ then } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k$$

$$(\mathcal{A}, X) \in \delta_{\mathcal{M}_f(A)} \quad \text{iff} \quad \text{when } X = [\mathcal{A}_1, \ldots, \mathcal{A}_j], \text{ then } \mathcal{A} = \mathcal{A}_1 \uplus \cdots \uplus \mathcal{A}_j$$

Suppose $\mathcal{A}_l = [\alpha_1^l, \ldots, \alpha_{i_l}^l], l = 1, \ldots, j$, then $\mathcal{A} = \mathcal{A}_1 \uplus \cdots \uplus \mathcal{A}_j = [\alpha_1^1, \ldots, \alpha_{i_1}^1, \ldots \alpha_1^j, \ldots, \alpha_{i_j}^j]$. Therefore $\alpha = \alpha_1^1 \uplus \cdots \uplus \alpha_{i_j}^j$ Summarizing, we have

$$\begin{array}{rll}(\alpha, X) \in \delta(\mathcal{M}_f(A)) \circ \delta_A & \text{iff} & \text{when } X = [\mathcal{A}_1, \ldots, \mathcal{A}_j], \mathcal{A}_l = [\alpha_1^l, \ldots, \alpha_{i_l}^l], \\ & & \text{then } \alpha = \alpha_1^1 \uplus \cdots \uplus \alpha_{i_j}^j\end{array}$$

Now for the other direction,

$$\begin{array}{rll}(\alpha, X) \in \mathcal{M}_f(\delta)(A) \circ \delta_A & \text{iff} & (\alpha, X) \in \mathcal{M}_f(\delta_A) \circ \delta_A \\ & \text{iff} & \exists \mathcal{A} \in \mathcal{M}_f(\mathcal{M}_f(A)) \text{ s.t. } (\alpha, \mathcal{A}) \in \delta_A, (\mathcal{A}, X) \in \mathcal{M}_f(\delta_A)\end{array}$$

but

$$(\alpha, \mathcal{A}) \in \delta_A \quad \text{iff} \quad \text{when } \mathcal{A} = [\alpha_1, \dots, \alpha_k], \alpha_i \in \mathcal{M}_f(A), \text{ then } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k$$

$$(\mathcal{A}, X) \in \mathcal{M}_f(\delta_A) \quad \text{iff} \quad \exists c \subseteq_f \delta_A \text{ s.t. } \pi_1[c] = \mathcal{A} = [\alpha_1, \dots, \alpha_k], \pi_2[c] = X$$

Supposing that $X = [\mathcal{A}_1, \dots, \mathcal{A}_j]$ and $\mathcal{A}_l = [\alpha_1^l, \dots, \alpha_{i_l}^l], l = 1, \dots, j$, then $c = \{(\biguplus_{n=1}^{l_1} \alpha_1^n, \mathcal{A}_1), \dots, (\biguplus_{n=1}^{l_j} \alpha_j^n, \mathcal{A}_j)\}$ to ensure that $\pi_2[c] = X$ and $c \subseteq_f \delta_A$. This means, in particular, that $k = j$ and that $\alpha_1 = \biguplus_{n=1}^{i_1} \alpha_n^1, \dots, \alpha_k = \biguplus_{n=1}^{i_k} \alpha_n^k$. Then since, from above, we have that $\alpha = \alpha_1 \uplus \cdots \uplus \alpha_k$, therefore

$$\begin{aligned} \alpha &= \alpha_1 \uplus \cdots \uplus \alpha_k \\ &= (\biguplus_{n=1}^{i_1} \alpha_n^1) \uplus \cdots \uplus (\biguplus_{n=1}^{i_k} \alpha_n^k) \\ &= \alpha_1^1 \uplus \cdots \uplus \alpha_{i_j}^j \end{aligned}$$

Summarizing, we have

$$\begin{aligned} (\alpha, X) \in \mathcal{M}_f(\delta_A) \circ \delta_A \quad &\text{iff} \quad \text{when } X = [\mathcal{A}_1, \dots, \mathcal{A}_j], \mathcal{A}_l = [\alpha_1^l, \dots, \alpha_{i_l}^l], \\ &\qquad \text{then } \alpha = \alpha_1^1 \uplus \cdots \uplus \alpha_{i_j}^j \\ &\text{iff} \quad (\alpha, X) \in \delta_{\mathcal{M}_f(A)} \circ \delta_A \end{aligned}$$

So $\mathcal{M}_f(\delta_A) \circ \delta_A = \delta_{\mathcal{M}_f(A)} \circ \delta_A$ and the diagram commutes. $(\mathcal{M}_f(-), \varepsilon, \delta)$ satisfies all the necessary requirements and is indeed a comonad on **Rel**.

Example 2.3.25 (coKleisli category of finite multiset comonad on **Rel**). We have shown that $(\mathcal{M}_f(-), \varepsilon, \delta)$ is a comonad on **Rel**. $coKl(\mathcal{M}_f(-))$ is the category whose objects are the objects of **Rel**. The objects of **Rel** are sets, therefore the objects of $coKl(\mathcal{M}_f(-))$ are sets. For the arrows, notice

$$\begin{aligned} f : A \longrightarrow B \in coKl(\mathcal{M}_f(-)) \quad &\text{iff} \quad f : \mathcal{M}_f(A) \longrightarrow B \in \mathbf{Rel} \\ &\text{iff} \quad f \subseteq \mathcal{M}_f(A) \times B \end{aligned}$$

Therefore, the arrows of $coKl(\mathcal{M}_f(-))$ are subsets of $\mathcal{M}_f(A) \times B$, or equivalently, relations from $\mathcal{M}_f(A)$ to B.

- Identity: The identity arrow is $\varepsilon_A : \mathcal{M}_f(A) \longrightarrow A$, $Id_A = \varepsilon_A = \{([a], a) \mid a \in A)\}$

- Composition: Let $f : C \longrightarrow D$ and $g : D \longrightarrow E$ be two arrows in $coKl(\mathcal{M}_f(-))$, $f \subseteq \mathcal{M}_f(C) \times D$ and $g \subseteq \mathcal{M}_f(D) \times E$, then $g \bullet f : C \longrightarrow E$ is defined to be $g \circ \mathcal{M}_f(f) \circ \delta_C$.

First notice that $g \circ \mathcal{M}_f(f) \subseteq \mathcal{M}_f(\mathcal{M}_f(C)) \times E$ and

$$(\mathcal{A}, e) \in g \circ \mathcal{M}_f(f) \quad \text{iff} \quad \exists \beta \in \mathcal{M}_f(D) \text{ s.t. } (\mathcal{A}, \beta) \in \mathcal{M}_f(f), (\beta, e) \in g$$

but

$$\begin{aligned}(\mathcal{A}, \beta) \in \mathcal{M}_f(f) \quad &\text{iff} \quad \exists c \subseteq_f f \text{ s.t. } \pi_1[c] = \mathcal{A}, \pi_2[c] = \beta \\ &\text{iff} \quad \exists (\alpha_1, d_1), \ldots, (\alpha_k, d_k) \in f \text{ s.t. } [\alpha_1, \ldots, \alpha_k] = \mathcal{A}, \\ &\quad\quad [d_1, \ldots, d_k] = \beta, \alpha_i \in \mathcal{M}_f(C)\end{aligned}$$

Therefore

$$(\mathcal{A}, e) \in g \circ \mathcal{M}_f(f) \quad \text{iff} \quad \exists (\alpha_1, d_1), \ldots, (\alpha_k, d_k) \in f \text{ s.t. } \mathcal{A} = [\alpha_1, \ldots, \alpha_k], ([d_1, \ldots, d_k], e) \in g$$

Now, notice that $g \bullet f = g \circ \mathcal{M}_f(f) \circ \delta_C \subseteq \mathcal{M}_f(C) \times E$ and

$$(\alpha, e) \in g \circ \mathcal{M}_f(f) \circ \delta_C \quad \text{iff} \quad \exists \mathcal{A} \in \mathcal{M}_f(\mathcal{M}_f(C)) \text{ s.t. } (\alpha, \mathcal{A}) \in \delta_C, (\mathcal{A}, e) \in g \circ \mathcal{M}_f(f)$$

but

$$(\alpha, \mathcal{A}) \in \delta_C \quad \text{iff} \quad \text{when } \mathcal{A} = [\alpha_1, \ldots, \alpha_k] \text{ then } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k$$

$$(\mathcal{A}, e) \in g \circ \mathcal{M}_f(f) \quad \text{iff} \quad \exists (\alpha_1, d_1), \ldots, (\alpha_k, d_k) \in f \text{ s.t. } \mathcal{A} = [\alpha_1, \ldots, \alpha_k], ([d_1, \ldots, d_k], e) \in g$$

Putting this all together we get

$$\begin{aligned}(\alpha, e) \in g \circ \mathcal{M}_f(f) \circ \delta_C \quad \text{iff} \quad &\exists (\alpha_1, d_1), \ldots, (\alpha_k, d_k) \in f \text{ s.t. } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k], \\ &([d_1, \ldots, d_k], e) \in g\end{aligned}$$

$$\boxed{g \bullet f = \{(\alpha, e) \mid \exists (\alpha_1, d_1), \ldots, (\alpha_k, d_k) \in f \text{ s.t. } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k, ([d_1, \ldots, d_k], e) \in g\}}$$

We call the coKleisli category of the lifted $\mathcal{M}_f$ on **Rel** the finite multiset relation category or **MRel**. We can apply Proposition 2.3.13, to get an adjunction pair $F : \textbf{Rel} \longrightarrow \textbf{MRel}$ and $U : \textbf{MRel} \longrightarrow \textbf{Rel}$. Given a morphism $f : A \longrightarrow B \in \textbf{MRel}$, let $f^* : \mathcal{M}_f(A) \longrightarrow B$ be the associated map in **Rel** and given a map

$f : \mathcal{M}_f(A) \longrightarrow B \in$ **Rel**, let $f^\dagger : A \longrightarrow B \in$ **MRel** be the associated Klesli map. Let A be a set then $U(A) = \mathcal{M}_f(A)$ and $F(A) = A$. Let $f : A \longrightarrow B$ be a map in **MRel**, then we define $U(f) : \mathcal{M}_f(A) \longrightarrow \mathcal{M}_f(B) \in$ **Rel** to be $(\mathcal{M}_f(f^*) \circ \delta_A^{\mathcal{M}_f})$. Then

$$\begin{array}{rll} (\alpha,\beta) \in U(f) & \text{iff} & (\alpha,\beta) \in (\mathcal{M}_f(f^*) \circ \delta_A^{\mathcal{M}_f}) \\ & \text{iff} & \exists \mathcal{A} \in \mathcal{M}_f^2(A).(\alpha,\mathcal{A}) \in \delta_A^{\mathcal{M}_f} \ \wedge \ (\mathcal{A},\beta) \in \mathcal{M}_f(f^*) \\ & \text{iff} & \exists \alpha_1,\ldots,\alpha_k \in \mathcal{M}_f(A).\alpha = \alpha_1 \uplus \cdots \uplus \alpha_k \ \wedge \ ([\alpha_1,\ldots,\alpha_k],\beta) \in \mathcal{M}_f(f^*) \\ & \text{iff} & \exists \alpha_1,\ldots,\alpha_k \in \mathcal{M}_f(A).\alpha = \alpha_1 \uplus \cdots \uplus \alpha_k \ \wedge \ (\alpha_1,b_1)\ldots,(\alpha_k,b_k) \in f^*, \\ & & \qquad \text{where } \beta = [b_1,\ldots,b_k] \end{array}$$

Therefore

$$U(f) = \{(\alpha_1 \uplus \cdots \uplus \alpha_k, [b_1,\ldots,b_k]) \mid (\alpha_i,b_i) \in f^*, i = 1,\ldots,k\}$$

Now suppose that we have a relation $R : A \longrightarrow B$ in **Rel**, define $F(R)$ to be $(R \circ \varepsilon_A^{\mathcal{M}_f})^\dagger$. Then

$$\begin{array}{rll} (\alpha,b) \in R \circ \varepsilon_A^{\mathcal{M}_f} & \text{iff} & \exists a \in A.(\alpha,a) \in \varepsilon_A \ \wedge \ (a,b) \in R \\ & \text{iff} & \alpha = [a] \ \wedge \ (a,b) \in R \end{array}$$

Therefore

$$(R \circ \varepsilon_A^{\mathcal{M}_f}) = \{([a],b) \mid (a,b) \in R\}$$

and $F(R) = (R \circ \varepsilon_A^{\mathcal{M}_f})^\dagger$. Now, from Proposition 2.3.13, $UF = \mathcal{M}_f$. This can easily be checked: for a set A, $UF(A) = U(A) = \mathcal{M}_f(A)$. Let $f : A \longrightarrow B$ be a relation. Then

$$\begin{array}{rcl} UF(R) & = & \{(\alpha_1 \uplus \cdots \uplus \alpha_k, [b_1,\ldots,b_k]) \mid (\alpha_i,b_i) \in F(R)^*, i = 1,\ldots,k\} \\ & = & \{(\alpha_1 \uplus \cdots \uplus \alpha_k, [b_1,\ldots,b_k]) \mid (\alpha_i,b_i) \in ((R \circ \varepsilon_A^{\mathcal{M}_f})^\dagger)^*, i = 1,\ldots,k\} \\ & = & \{(\alpha_1 \uplus \cdots \uplus \alpha_k, [b_1,\ldots,b_k]) \mid (\alpha_i,b_i) \in (R \circ \varepsilon_A^{\mathcal{M}_f}), i = 1,\ldots,k\} \\ & = & \{([a_1,\ldots,a_k],[b_1,\ldots,b_k]) \mid (a_i,b_i) \in R, i = 1,\ldots,k\} \\ & = & \mathcal{M}_f(R) \end{array}$$

3 Investigating the structure of MRel

3.1 MRel is a cartesian closed category

In the previous section, we introduced the category **MRel**. Recall that **MRel** is the category whose objects are sets. Let A and B be two sets, then an arrow from A to B in **MRel** is a relation from $\mathcal{M}_f(A)$ to B. The identity on A, A a set, is

$$Id_A = \{([a], a) \mid a \in A)\}$$

Given two arrows $f \in \mathbf{MRel}(A, B)$, $g \in \mathbf{MRel}(B, C)$, then

$$g \circ f = \{(\alpha, c) \mid \exists(\alpha_1, b_1), \ldots, (\alpha_k, b_k) \in f \text{ s.t. } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k, ([b_1, \ldots, b_k], c) \in g\}$$

In fact, **MRel** is a cartesian closed category.

Proposition 3.1.1. **MRel** *is a cartesian closed category*

There are several ways of showing that **MRel** is a cartesian closed category. One of which is to use the following fact due to Seely:

Proposition 3.1.2 (see [17]). *Let $(!, \epsilon, \delta)$ be a comonad on a $*$-autonomous category $\mathcal{C}$ such that for all objects $A, B \in \mathcal{C}$ the following isomorphism holds:*

$$!(A \times B) \cong !A \otimes !B.$$

Then the coKleisli category of the comonad $(!, \epsilon, \delta)$ is a cartesian closed category, in which $A \Rightarrow B = !A \multimap B$

Proof. The terminal object of $coKl(!)$ is the terminal object of $\mathcal{C}$. The product $A \times B$

of $coKl(!)$ is the same as in $\mathcal{C}$ and $A \Rightarrow B$ is $!A \multimap B$. Then

$$\begin{aligned} coKl(!)(A \times B, C) &\cong \mathcal{C}(!(A \times B), C) \\ &\cong \mathcal{C}(!A \otimes !B, C) \\ &\cong \mathcal{C}(!A, !B \multimap C) \\ &\cong \mathcal{C}(!A, B \Rightarrow C) \\ &\cong coKl(!)(A, B \Rightarrow C) \end{aligned}$$

□

In our case we have the comonad $(\mathcal{M}_f, \varepsilon, \delta)$ over **Rel**. From section 2.3.2, we know that **Rel** is a $*$-autonomous category. Notice that in **Rel** both $\otimes$ and $\multimap$ are set theoretical product $\times$ and that categorical product is &. We have also already seen that

$$\mathcal{M}_f(A\&B) \cong \mathcal{M}_f(A) \times \mathcal{M}_f(B)$$

Combining this we have that $coKleisli(\hat{\mathcal{M}}_f) = \mathbf{MRel}$ is a cartesian closed category from Proposition 3.1.2 where the exponential $A \Rightarrow B$ in **MRel** is $\mathcal{M}_f(A) \times B$.

We may also show that **MRel** is a cartesian closed category directly. Here we take the structure presented in [7] and show that this structure satisfies the equations of a cartesian closed category.

- Terminal object $\mathbf{1}$: In **MRel**, $\mathbf{1} = \emptyset$ and for a set A, the unique element of $\mathbf{MRel}(A, \emptyset)$ is the empty relation. The empty relation is indeed in $\mathbf{MRel}(A, \emptyset)$ since $\emptyset \subseteq \mathcal{M}_f(A) \times \emptyset$. Also notice

$$\begin{aligned} f \in \mathbf{MRel}(A, \emptyset) \quad &\text{iff} \quad f \subseteq \mathcal{M}_f(A) \times \emptyset \\ &\text{iff} \quad f \subseteq \emptyset \\ &\text{iff} \quad f = \emptyset \end{aligned}$$

  This shows that the empty relation is the unique element of $\mathbf{MRel}(A, \emptyset)$.

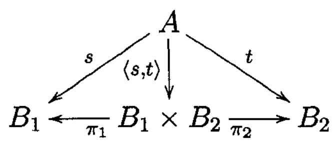
- Categorical product: Let B_1 and B_2 be sets and define the product $B_1\&B_2$ as the disjoint union

$$B_1\&B_2 = (\{1\} \times B_1) \cup (\{2\} \times B_2)$$

together with

$$\pi_i = \{([(i, b)], b) \mid b \in B_i\} \in \mathbf{MRel}(B_1 \times B_2, B_i), \text{ for } i = 1, 2$$

To show that this is indeed a categorical product in **MRel** we must check that for any set A and arrows $s \in \mathbf{MRel}(A, B_1)$, $t \in \mathbf{MRel}(A, B_2)$ there is a unique map $\langle s, t\rangle \in \mathbf{MRel}(A, B_1 \times B_2)$ making the following diagram commute:



Define

$$\langle s, t\rangle = \{(\alpha, (1, b)) \mid (\alpha, b) \in s\} \cup \{(\alpha, (2, b)) \mid (\alpha, b) \in t\}$$

We have that $\langle s, t\rangle \subseteq \mathcal{M}_f(A) \times B_1 \& B_2$, therefore $\langle s, t\rangle \in \mathbf{MRel}(A, B_1 \& B_2)$. Now we must check that $\pi_1 \circ \langle s, t\rangle = s$ and $\pi_2 \circ \langle s, t\rangle = t$:

$$\pi_1 \circ \langle s, t\rangle = \{(\alpha, b) \mid \exists (\alpha_1, b_1), \ldots, (\alpha_k, b_k) \in \langle s, t\rangle \text{ s.t. } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k, ([b_1, \ldots, b_k], b) \in \pi_1\}$$

But $([b_1, \ldots, b_k], b) \in \pi_1$ means that $b \in B_1$, $k = 1$ and $b_1 = (1, b)$. Therefore

$$\begin{array}{lll} (\alpha, b) \in \pi_1 \circ \langle s, t\rangle & \text{iff} & \exists \alpha_1 \in \mathcal{M}_f(A) \text{ s.t. } (\alpha, (1, b)) \in \langle s, t\rangle, \alpha = \alpha_1 \\ & \text{iff} & \exists \alpha_1 \in \mathcal{M}_f(A) \text{ s.t. } (\alpha, (1, b)) \in \{(\alpha, (1, b)) \mid (\alpha, b) \in s\}, \alpha = \alpha_1 \\ & \text{iff} & \exists \alpha_1 \in \mathcal{M}_f(A) \text{ s.t. } (\alpha, b) \in s, \alpha = \alpha_1 \end{array}$$

So we get that

$$\begin{array}{rcl} \pi_1 \circ \langle s, t\rangle & = & \{(\alpha, b) \mid \exists \alpha_1 \in \mathcal{M}_f(A) \text{ s.t. } (\alpha, b) \in s, \alpha = \alpha_1\} \\ & = & \{(\alpha, b) \mid (\alpha, b) \in s\} \\ & = & s \end{array}$$

Similarly, we have that $\pi_2 \circ \langle s, t\rangle = t$. To finish, we must show that $\langle s, t\rangle$ is the unique map in $\mathbf{MRel}(A, B_1 \times B_2)$ making the diagram commute. Again, let A be any set and $s \in \mathbf{MRel}(A, B_1)$, $t \in \mathbf{MRel}(A, B_2)$. Now suppose that for some $f \in \mathbf{MRel}(A, B_1 \& B_2)$, $\pi_1 \circ f = s$ and $\pi_2 \circ f = t$. We must show that $f = \langle s, t\rangle$:

$$\pi_1 \circ f = \{(\alpha, b) \mid \exists (\alpha_1, b_1), \ldots, (\alpha_k, b_k) \in f \text{ s.t. } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k, ([b_1, \ldots, b_k], b) \in \pi_1\}$$

Again, $([b_1, \ldots, b_k], b) \in \pi_1$ means that $b \in B_1$, $k = 1$ and $b_1 = (1, b)$. Hence, the above set becomes

$$\pi_1 \circ f = \{(\alpha, b) \mid (\alpha, (1, b)) \in f\}$$

And since $\pi_1 \circ f = s$, we get that

$$s = \{(\alpha, b) \mid (\alpha, (1, b)) \in f\}$$

Similarly,

$$t = \{(\alpha, b) \mid (\alpha, (2, b)) \in f\}$$

Summarizing, we get

$$\begin{aligned} (\alpha, (i, b)) \in f \quad & \text{iff} \quad i = 1 \text{ and } (\alpha, b)) \in \pi_1 \circ f \\ & \qquad \text{or } i = 2 \text{ and } (\alpha, b) \in \pi_2 \circ f \\ & \text{iff} \quad (\alpha, b)) \in s \text{ or } (\alpha, b) \in t \end{aligned}$$

Therefore

$$\begin{aligned} f &= \{(\alpha, (1, b)) \mid (\alpha, b) \in s\} \cup \{(\alpha, (2, b)) \mid (\alpha, b) \in t\} \\ &= \langle s, t \rangle \end{aligned}$$

as required.

Remark 3.1.3. Let A and B be sets. Then $\mathcal{M}_f(A\&B)$ and $\mathcal{M}_f(A) \times \mathcal{M}_f(B)$ are isomorphic sets. An element of $\mathcal{M}_f(A\&B)$ is a multiset with elements that are either marked with a 1 or a 2, those marked with a 1 are paired with an element of A and those marked with a 2 are paired with an element from B. Consider $([a_1, \ldots, a_k], [b_1, \ldots, b_{k'}]) \in \mathcal{M}_f(A) \times \mathcal{M}_f(B)$. To get an element of $\mathcal{M}_f(A\&B)$ simply mark all the a's with a 1 and all the b's with a 2 like so: $[(1, a_1), \ldots, (1, a_k), (2, b_1), \ldots, (2, b_{k'})] \in \mathcal{M}_f(A\&B)$. Similarly, with an element $[(1, a_1), \ldots, (1, a_k), (2, b_1), \ldots, (2, b_{k'})] \in \mathcal{M}_f(A\&B)$, unmark the a's and b's, and group them together as: $([a_1, \ldots, a_k], [b_1, \ldots, b_{k'}]) \in \mathcal{M}_f(A) \times \mathcal{M}_f(B)$. It is clear that these are inverse to each other. For simplicity, we will sometimes consider this isomorphism as equality.

Remark 3.1.4. Since **MRel** has products, we can define & as a functor $\mathbf{MRel}^2 \xrightarrow{-\&-} \mathbf{MRel}$ as follows: let $f : A \longrightarrow C$ and $g : B \longrightarrow D$ be two maps on **MRel**, then $f\&g : A\&B \longrightarrow C\&D \in \mathbf{MRel}$ is defined as

$$\begin{aligned} f\&g &= \{([(1,a_1),\ldots,(1,a_k)],(1,c)) \mid ([a_1,\ldots,a_k],c) \in f\} \cup \\ &\quad \{([(2,b_1),\ldots,(2,b_l)],(2,d)) \mid ([b_1,\ldots,b_l],d) \in g\} \end{aligned}$$

From Remark 3.1.3, we have that $\mathcal{M}_f(A\&B) \cong \mathcal{M}_f(A) \times \mathcal{M}_f(B)$. Using this, we can rewrite $f\&g$ as a subset of $(\mathcal{M}_f(A) \times \mathcal{M}_f(B)) \times (C\&D)$ as follows:

$$f\&g = \{((\alpha,[\,]),(1,c)) \mid (\alpha,c) \in f\} \cup \{((\beta,[\,]),(2,d)) \mid (\beta,d) \in g\}$$

- Exponential objects: Let A and B be sets, the exponential object $A \Rightarrow B = \mathcal{M}_f(A) \times B$ and $ev_{A,B} : (A \Rightarrow B)\&A \longrightarrow B$ is defined as follows:

$$ev_{A,B} = \{(([(\alpha,b)],\alpha),b) \mid \alpha \in \mathcal{M}_f(A), b \in B\}$$

To show that this is a genuine exponential object we must show that for any set C and map $g : (C \times A) \longrightarrow B \in \mathbf{MRel}(C \times A, B)$ there is a unique map $\lambda g : C \longrightarrow (A \Rightarrow B)$ such that $ev_{A,B} \circ (\lambda g\&Id_A) = g$. Define

$$\lambda g = \{(\gamma,(\alpha,b)) \mid ((\gamma,\alpha),b) \in g\}$$

Then

$$\lambda g\&Id_A = \{(\gamma,[\,],(1,(\alpha,b))) \mid ((\gamma,\alpha),b) \in g\} \cup \{([\,],[a],(2,a)) \mid a \in A\}$$

and

$$\begin{aligned} ev_{A,B} \circ (\lambda g\&Id_A) &= \{((\gamma,\alpha),b) \mid \exists(\gamma_1,\alpha_1,X_1),\ldots,(\gamma_k,\alpha_k,X_k) \in \lambda g\&Id_A \text{ s.t.} \\ &\quad \gamma = \gamma_1 \uplus \cdots \uplus \gamma_k,\ \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k, \\ &\quad ([X_1,\ldots,X_k],b) \in ev_{A,B}\} \end{aligned}$$

But $([X_1,\ldots,X_k],b) \in ev_{A,B}$ means that $X_i = (1,([a_1,\ldots,a_{k-1}],b))$ such that $((\gamma_i,[a_1,\ldots,a_{k-1}]),b) \in g$ for some i, say $i = 1$ and $X_2 = (2,a_1),\ldots,X_k = (2,a_{k-1})$. With this, we get that $\alpha_1 = \gamma_2 = \cdots = \gamma_k = [\,]$ and that $\alpha_2 = [a_1], \alpha_3 = [a_2],\ldots,\alpha_k = [a_{k-1}]$. Summarizing,we get

$$\begin{aligned} ev_{A,B} \circ (\lambda g\&Id_A) &= \{((\gamma,\alpha),b) \mid ((\gamma,\alpha),b) \in g\} \\ &= g \end{aligned}$$

Hence, the proposed candidate for λg satisfies the required equation. Now, it remains to show that it is unique. Suppose that $h : C \longrightarrow (A \Rightarrow B)$ is a map such that

$$ev_{A,B} \circ (h \& Id_A) = g$$

We must show that $h = \lambda g$. From the above calculations, we have that

$$\begin{aligned} ev_{A,B} \circ (h\&Id_A) &= \{((\gamma,\alpha),b) \mid \exists(\gamma_1,\alpha_1,X_1),\dots,(\gamma_k,\alpha_k,X_k) \in h\&Id_A \text{ s.t.} \\ &\qquad \gamma = \gamma_1 \uplus \cdots \uplus \gamma_k,\ \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k, \\ &\qquad ([X_1,\dots,X_k],b) \in ev_{A,B}\} \end{aligned}$$

Now, $([X_1,\dots,X_k],b) \in ev_{A,B}$ means that for some i, say $i = 1$, $X_1 = (1,([a_1,\dots a_k],b)$ such that $(\gamma_1,(\alpha,b) \in h$ and that $\alpha_1 = \gamma_2 = \cdots = \gamma_k = [\,]$. This also implies that $X_2 = (2,a_1),\dots,X_k = (2,a_{k-1})$ and $\alpha_2 = [a_1], \alpha_3 = [a_2],\dots,\alpha_k = [a_{k-1}]$. Therefore, we can rewrite $ev_{A,B} \circ (h\&Id_A)$ as

$$\begin{aligned} ev_{A,B} \circ (h\&Id_A) &= \{((\gamma,\alpha),b) \mid \exists(\gamma_1,[\,],(1,([a_1,\dots a_{k-1}],b)), \\ &\qquad ([\,],[a_1],(2,a_1)),\dots,([\,],[a_{k-1}],(2,a_{k-1})) \in h\&Id_A \\ &\qquad \text{s.t. } \gamma = \gamma_1 \uplus [\,] \uplus \cdots \uplus [\,],\ \alpha = [\,] \uplus [a_1] \uplus \cdots \uplus [a_k]\} \\ &= \{((\gamma,\alpha),b) \mid (\gamma,(\alpha,b)) \in h\} \end{aligned}$$

But we know that $ev_{A,B} \circ (h\&Id_A) = g$, therefore $g = \{((\gamma,\alpha),b) \mid (\gamma,(\alpha,b)) \in h\}$ and so h must be $\{(\gamma,(\alpha,b)) \mid ((\gamma,\alpha),b) \in g\}$ and we have that $h = \lambda g$ as required.

3.2 Enough points

Definition 3.2.1. Let A be an object in a category $\mathcal{C}$ with terminal object $\mathbf{1}$. The *points* of A in $\mathcal{C}$ are all the morphisms $x \in \mathcal{C}(\mathbf{1}, A)$

Example 3.2.2.

1. Consider the category of sets and set-theoretic functions **Set**. The terminal object $\mathbf{1}$ of **Set** is the singleton set $\{*\}$, therefore a point of a set A is a function from $\{*\}$ to A. For every element a of A there is a map $f_a : \{*\} \longrightarrow A$, where $f_a(*) = a$. Therefore the set of all points of a set A in **Set** is isomorphic to A, $\mathbf{Set}(\mathbf{1}, A) \cong A$.

2. Consider the category $K\text{-}Vect$, the category whose objects are vector spaces over a fixed field K and maps are K-linear transformations. The terminal object of $K\text{-}Vect$ is $\{0\}$ the zero vector space. A point of a vector space V in $K\text{-}Vect$ is a K-linear transformation from $\{0\}$ to V. There is only one such transformation: the transformation that sends 0 to 0. Therefore the set of all points of a vector space V is a singleton, $K\text{-}Vect(\mathbf{1}, V) = \{*\}$.

3. Consider the category of sets and relations **Rel**. The terminal object of **Rel** is the empty set $\emptyset$, therefore a point of a set A is a relation R from $\emptyset$ to A. We can think of R as a subset of $\emptyset \times A$, but $\emptyset \times A = \emptyset$. Therefore R is the empty relation and hence, there is only one point of A. The set of all points of a set A is a singleton, $\mathbf{Rel}(\mathbf{1}, A) = \{*\}$.

4. Let A be an object in **AlgLat**, i.e. A is an algebraic lattice. The points of A in **AlgLat** are all the morphisms $x \in \mathbf{AlgLat}(\mathbf{1}, A)$. The terminal object of **AlgLat** is $\{*\}$ the singleton algebraic lattice, therefore a point of A is an arrow from $\{*\}$ to A, which is a set theoretic map preserving the join of directed subsets of $\{*\}$. For every $a \in A$, there is one such map $f_a : \{*\} \longrightarrow A$, where $f_a(*) = a$ and this map preserves the join of directed subsets of $\{*\}$ since there is only one such subset, $\{*\}$ itself. Therefore, the set of all points of an algebraic lattice $A \in \mathbf{AlgLat}$ is isomorphic to A, $\mathbf{AlgLat}(\mathbf{1}, A) = \{f_a : \{*\} \longrightarrow A \mid a \in A\} \cong A$

Example 3.2.3. Let A be an object in **MRel**, i.e. A a set. The points of A in **MRel** are all the morphisms $x \in \mathbf{MRel}(\mathbf{1}, A)$. The terminal object of **MRel** is $\emptyset$, therefore a point of A is an arrow from $\emptyset$ to A, which is a relation from $\mathcal{M}_f(\emptyset)$ to A. Recall that $\mathcal{M}_f(\emptyset) = \{[\,]\} = \{*\}$, a singleton. These are, up to isomorphim, the subsets of A. $\mathbf{MRel}(\mathbf{1}, A) = \mathcal{P}(A)$

Definition 3.2.4. A category $\mathcal{C}$ with terminal object $\mathbf{1}$, is said to have *enough points* [7], or to be *well-pointed* [18], if for any A, B objects in $\mathcal{C}$ and $f, g \in \mathcal{C}(A, B)$, whenever $f \neq g$ there exists a point $x \in \mathcal{C}(\mathbf{1}, A)$ such that $f \circ x \neq g \circ x$.

An object U of a category $\mathcal{C}$ with terminal object $\mathbf{1}$, is said to have *enough points* [7], or to be *locally well-pointed* [18], if for any $f, g \in \mathcal{C}(U, U)$, whenever $f \neq g$ there exists a point $x \in \mathcal{C}(\mathbf{1}, U)$ such that $f \circ x \neq g \circ x$.

It is clear that if there is an object in a category $\mathcal{C}$ that is not locally well-pointed, then $\mathcal{C}$ is not well-pointed.

Example 3.2.5.

1. The category **Set** has enough points: let A, B be two sets and $g, h \in \mathbf{Set}(A, B)$. Suppose that $g \neq h$, then there is an a on A such that $g(a) \neq h(a)$, but $g(a) = (g \circ f_a)(*)$, where f_a is the point of A that sends $*$ to a. Therefore, $g \circ f_a \neq h \circ f_a$.

2. The category K-$Vect$ does not have enough points: let V, W be two vector spaces and $g, h \in K\text{-}Vect(V, W)$. Suppose that $g \neq h$, then $g(x) \neq h(x)$ for some $x \neq 0$ (since $g(0) = h(0) = 0$, this is true for all linear transformations). The only point of V is the map from $\{0\}$ to V, call it f, that sends 0 to 0 and $g \circ f = g(0) = h(0) = h \circ f$. Therefore, there does not exist a point $x \in K\text{-}Vect(\mathbf{1}, V)$ such that $g \circ x \neq h \circ x$.

3. Using similar reasoning as for **Set**, it is clear that **AlgLat** does have enough points.

Example 3.2.6. **MRel** does not have enough points. For every non-empty object $A \in |\mathbf{MRel}|$, A is not locally well-pointed. In fact, for every object $A \in |\mathbf{MRel}|$, there exists $f, g \in \mathbf{MRel}(A, A)$ such that $f \neq g$, but $f \circ x = g \circ x$ for every $x \in \mathbf{MRel}(\mathbf{1}, A)$. $f, g \in \mathbf{MRel}(A, A)$, therefore $f, g \subseteq \mathcal{M}_f(A) \times A$.

Let $f = \{(\alpha_1, b)\}$ and $g = \{(\alpha_2, b)\}$, where $\alpha_1, \alpha_2 \in \mathcal{M}_f(A)$ and $\alpha_1 \neq \alpha_2$, but $supp(\alpha_1) = supp(\alpha_2)$. Let $x \in \mathbf{MRel}(\mathbf{1}, A)$. Then, as previously discussed, we can think of x as, up to isomorphism, a subset of A. Now, notice

$$f \circ x = \{(\alpha, a) \mid \exists (\alpha_1, a_1), \ldots, (\alpha_k, a_k) \in x \text{ s.t. } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k, ([a_1, \ldots, a_k], a) \in f\}$$

but $(\alpha_i, a_i) \in x$ means $\alpha_i = [\,]$ for $i = 1, \ldots, k$, $\alpha = [\,]$ and $a_i \in A$. Therefore $f \circ x$ becomes

$$f \circ x = \{([\,], a) \mid \exists a_1, \ldots, a_k \in A \text{ s.t. } ([\,], a_i) \in x, ([a_1, \ldots, a_k], a) \in f\}$$

and similarly

$$g \circ x = \{([\,], a) \mid \exists a_1, \ldots, a_k \in A \text{ s.t. } ([\,], a_i) \in x, ([a_1, \ldots, a_k], a) \in g\}$$

If $supp(\alpha_1) \subseteq x$, thinking of x as a subset of A, then $f \circ x = \{([\,], b)\}$. But since $supp(\alpha_1) = supp(\alpha_2)$, we also have that $supp(\alpha_2) \subseteq x$ and then, similarly, $g \circ x = \{([\,], b)\}$. Therefore, $f \circ x = g \circ x$. If $supp(\alpha_1) \not\subseteq x$, then $f \circ x = \emptyset$, but since $supp(\alpha_1) = supp(\alpha_2)$, we also have that $supp(\alpha_2) \not\subseteq x$ and then, similarly, $g \circ x = \emptyset$. Therefore, $f \circ x = g \circ x$. In all cases, $f \circ x = g \circ x$ for every $x \in \mathbf{MRel}(\mathbf{1}, A)$ and we have that A is not locally well-pointed and in turn, **MRel** does not have enough points.

The notion of having enough points relates back to how we view functions. If a category has enough points then two morphisms are equal if they are extensionally equal: morphisms are defined by how they act on points. The lambda calculus allows us to view functions intensionally, therefore models of the lambda calculus should be able to reflect this fact. The full force of adding extensional equality (as an internal first-order axiom) to the intensional β-equality of lambda calculus means that we identify the two notions of equality (intensional and extensional) in the strongest possible way. But categorically, 1 is a generator is not a natural "internal" condition. Hence we would like to move away from this requirement and allow models of the untyped lambda calculus that do not have enough points.

3.3 Global sections functor

Definition 3.3.1. The *global section functor* Γ *of a category* $\mathcal{C}$ *with terminal object* $\mathbf{1}$ is a functor from $\mathcal{C}$ to **Set**. For an object $A \in \mathcal{C}$, $\Gamma(A)$ is the set of all points of A, $\Gamma(A) = \mathcal{C}(\mathbf{1}, A)$. For an arrow $f : A \longrightarrow B \in \mathcal{C}(A, B)$, $\Gamma(f) : \Gamma(A) \longrightarrow \Gamma(B)$, for $a : \mathbf{1} \longrightarrow A \in \Gamma(A)$, then $\Gamma(f)(a)$ is given by $f \circ a$, $\Gamma(f)(a) = f \circ a$.

Remark 3.3.2. If $\mathcal{C}$ is a category with products $\times$ then the global sections functor $\Gamma : \mathcal{C} \longrightarrow$ **Set** preserves them: for two objects A,B of $\mathcal{C}$, $\Gamma(A \times B) \cong \Gamma(A) \times \Gamma(B)$. If $\mathcal{C}$ has exponentials it is not true that Γ preserves exponentials:for two objects A,B of $\mathcal{C}$, $\Gamma(B^A) \not\cong \Gamma(B)^{\Gamma(A)}$. However, there is a canonical comparison map of $\Gamma(B^A)$ and $\Gamma(B)^{\Gamma(A)}$. To construct this map, take the evaluation map $ev : B^A \times A \longrightarrow B$

in $\mathcal{C}$ and apply Γ to it to get the map $\Gamma(ev) : \Gamma(B^A \times A) \longrightarrow \Gamma(B)$ in **Set**. Since Γ preserves products, we get a map $\tilde{\Gamma}(ev) : \Gamma(B^A) \times \Gamma(A) \longrightarrow \Gamma(B)$ and since **Set** is cartesian closed, we can get a map $\widetilde{\Gamma}(ev)^* : \Gamma(B^A) \longrightarrow \Gamma(B)^{\Gamma(A)}$. This map has the property that it is an embedding if and only if the object A has enough points. More generally, it is easy to see that the terminal object $\mathbf{1}$ is a generator in $\mathcal{C}$ if and only if the functor $\Gamma = Hom(1, -) : \mathcal{C} \longrightarrow \mathbf{Set}$ is faithful (i.e. injective on hom-sets).

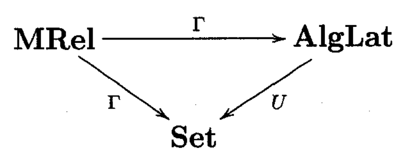
Example 3.3.3. Consider the category **MRel**. Then the global sections functor of **MRel**, $\Gamma : \mathbf{MRel} \longrightarrow \mathbf{Set}$, is defined as follows: let A be an object in **MRel**, so A is an arbitrary set. Then $\Gamma(A) = \mathbf{MRel}(\mathbf{1}, A)$. In section 3.2, we have already shown that, up to isomorphism, $\mathbf{MRel}(\mathbf{1}, A) = \mathcal{P}(A)$. Therefore, we have that $\Gamma(A) = \mathcal{P}(A)$. Now, let $f : A \longrightarrow B \in \mathbf{MRel}(A, B)$; by definition this means that $f \subseteq \mathcal{M}_f(A) \times B$. Then $\Gamma(f) : \Gamma(A) \longrightarrow \Gamma(B) \in \mathbf{Set}$. Let $X \in \Gamma(A) = \mathcal{P}(A)$; we can think of X as a subset of A. Using the definition of composition in **MRel** we have that

$$\begin{aligned} \Gamma(f)(X) &= f \circ X \\ &= \{([\,], b) \mid \exists a_1, \ldots, a_k \in X \text{ s.t. } ([a_1, \ldots, a_k], b) \in f\} \\ &\cong \{b \mid \exists a_1, \ldots, a_k \in X \text{ s.t. } ([a_1, \ldots, a_k], b) \in f\} \end{aligned}$$

Therefore, $\Gamma(f) : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$, where for $X \subseteq A$, $\Gamma(f)(X) = \{b \mid \exists a_1, \ldots, a_k \in X \text{ s.t. } ([a_1, \ldots, a_k], b) \in f\}$

In fact, we can show that this global sections functor Γ for **MRel** actually sends objects and morphisms in **MRel** to objects and morphisms in **AlgLat**.

Proposition 3.3.4. Γ *factors through* **AlgLat***:*



Here, U is the forgetful functor from **AlgLat** *to* **Set***.*

Proof. For any object $A \in \mathbf{MRel}$, $\Gamma(A) = \mathcal{P}(A)$ and we know that the power set of any set is indeed an algebraic lattice. It remains to show that for any $f : A \longrightarrow B \in \mathbf{MRel}(A, B)$, for A and B sets, $\Gamma(f) \in \mathbf{AlgLat}(\mathcal{P}(A), \mathcal{P}(B))$. To do this, we must check that $\Gamma(f)$ preserves directed joins. Let $\mathcal{F} \subseteq \mathcal{P}(A)$ be a directed family of subsets of A, say $\mathcal{F} = \{X_i \mid i \in I\}$. Then we have that

$$\Gamma(f)(\bigcup_{i\in I} X_i) = \{b \mid \exists a_1, \ldots, a_k \in \bigcup_{i\in I} X_i \text{ s.t. } ([a_1, \ldots, a_k], b) \in f\}$$
$$LHS$$

and

$$\bigcup_{i\in I} \Gamma(f)(X_i) = \bigcup_{i\in I} \{b \mid \exists a_1, \ldots, a_k \in X_i \text{ s.t. } ([a_1, \ldots, a_k], b) \in f\}$$
$$RHS$$

Now

$$\begin{array}{lll} b \in RHS & \text{iff} & \exists i \in I \text{ s.t. } b \in \{b \mid \exists a_1, \ldots, a_k \in X_i \text{ s.t. } ([a_1, \ldots, a_k], b) \in f\} \\ & \text{iff} & \exists i \in I. \exists a_1, \ldots, a_k \in X_i \text{ s.t. } ([a_1, \ldots, a_k], b) \in f \\ & \text{iff} & \exists a_1, \ldots, a_k \in \bigcup_{i\in I} X_i \text{ s.t. } ([a_1, \ldots, a_k], b) \in f \qquad (*) \\ & \text{iff} & b \in LHS \end{array}$$

The only "only if" direction of the step marked with a $(*)$ is obvious, but the "if" direction is not as clear and is due to the fact that $\mathcal{F}$ is directed. Suppose that there are $a_1, \ldots, a_k \in \bigcup_{i\in I} X_i$ such that $([a_1, \ldots, a_k], b) \in f$. Then for each a_j, $j = 1, \ldots, k$, there is an $X_j \in \mathcal{F}$ such that $a_j \in X_j$. Now by the directedness of $\mathcal{F}$, there is an $X_N \in \mathcal{F}$ such that $X_j \subseteq X_N$ for $j = 1, \ldots, k$. Therefore each $a_j \in X_N$ and we have that there is an $X \in \mathcal{F}$, namely X_N, such that there are $a_1, \ldots, a_k \in X$ with $([a_1, \ldots, a_k], b) \in f$ as required. This shows that for any $f \in \mathbf{MRel}(A, B)$, $\Gamma(f) \in \mathbf{AlgLat}(A, B)$ and that Γ factors through **AlgLat**. □

3.4 The existence of a reflexive object in MRel

Definition 3.4.1. An object U in a category $\mathcal{C}$ is said to be *reflexive* if it is equipped with morphisms $e : U^U \longrightarrow U$ and $p : U \longrightarrow U^U$ such that $p \circ e = Id_{U^U}$. U is *extensional* if, in addition, $e \circ p = Id_U$.

In [7], such an object is given: it shown that there is a extensional reflexive object D in **MRel** and that this object is a model of λ-calculus. This object D is defined as follows:

$$D = \bigcup_{n\in\mathbb{N}} D_n$$

where

$$\begin{aligned} D_0 &= \emptyset \\ D_{n+1} &= \mathcal{M}_f(D_n)^{(\omega)} \end{aligned}$$

Taking a closer look, notice $D_1 = \mathcal{M}_f(D_0)^{(\omega)} = \mathcal{M}_f(\emptyset)^{(\omega)}$. This is the set of all quasi-finite $\mathbb{N}$-indexed sequences of finite multisets over $\emptyset$. But there is only one finite multiset over $\emptyset$, namely $[\ ]$; therefore there is only one element of $\mathcal{M}_f(\emptyset)^{(\omega)}$, which is $* = ([\ ],[\ ],[\ ],\dots)$, the sequence in which every element is the empty sequence. $D_1 = \{*\}$, a singleton.

Now $D_2 = \mathcal{M}_f(D_1)^{(\omega)} = \mathcal{M}_f(\{*\})^{(\omega)}$. This is the set of all quasi-finite $\mathbb{N}$-indexed sequences of finite multisets over singleton $\{*\}$. An element of D_2, say $\sigma = (\alpha_i)_{i\in\mathbb{N}}$, where $\alpha_i = [a_1,\dots,a_k]$ where $a_j = *$ for $j = 1,\dots,k$ and $k = 0$ for all but finitely many $i \in \mathbb{N}$. In general, if σ is in D, then σ is in $\mathcal{M}_f(D_n)^{(\omega)}$ for some $n \in \mathbb{N}$. We think of σ as a sequence $\sigma = (\sigma_0,(\sigma_1,\sigma_2,\dots))$.

Let $\sigma \in D$, with say $\sigma = (\sigma_0,(\sigma_1,\sigma_2,\dots))$ and $m \in \mathcal{M}_f(D)$. We write $m\cdot\sigma$ for the element $\tau \in D$, where $\tau = (m,(\sigma_0,\sigma_1,\dots))$.

Proposition 3.4.2. *$\mathcal{D} = (D,p,e)$ is an extensional categorical model of λ-calculus where*

$$p = \{([m\cdot\sigma],(m,\sigma)) \mid m \in \mathcal{M}_f(D), \sigma \in D\} \in \mathbf{MRel}(D, D \Rightarrow D)$$

$$e = \{([(m,\sigma)], m\cdot\sigma) \mid m \in \mathcal{M}_f(D), \sigma \in D\} \in \mathbf{MRel}(D \Rightarrow D, D)$$

Proof. To show that $\mathcal{D}$ is a model we must show $p \circ e = Id_{D\Rightarrow D}$ and to show that it is extensional we must show that $e \circ p = Id_D$.

$$p \circ e = \{(n,c) \mid \exists (n_1,b_1)\dots(n_k,b_k) \in e \text{ s.t. } n = n_1 \uplus \cdots \uplus n_k, ([b_1,\dots,b_k],c) \in p\}$$

$(n_i,b_i) \in e$ means $n_i = [(m_i,\sigma_i)], b_i = m_i\cdot\sigma_i, m_i \in \mathcal{M}_f(D), \sigma \in D$

$([b_1,\dots,b_k],c) \in p$ means $b_1 = \cdots = b_k = b = m\cdot\sigma$ where $c = (m,\sigma)$, also, $n_i = [(m,\sigma)]$ for all i since $b_i = m\cdot\sigma$. Therefore

$$p \circ e = \{([(m,\sigma)],(m,\sigma)) \mid m \in \mathcal{M}_f(D), \sigma \in D\} = Id_{D\Rightarrow D}$$

as required. The proof that $e \circ p = Id_D$ is very similar. □

3.5 C-Monoids and MRel

The previous section has shown that there is an extensional reflexive object D is **MRel** $D \cong D \Rightarrow D$. A natural question to ask ourselves is whether or not $D \cong D\&D$. In this section, we show that $D\&D \lhd D$ and that $End(D)$ is a weak C-monoid.

Proposition 3.5.1. *$D\&D$ is a retract of D, i.e. $D\&D \lhd D$.*

Proof. Let

$$\varphi : D \longrightarrow D\&D \text{ and } \psi : D\&D \longrightarrow D$$

So we have that $\varphi \subseteq \mathcal{M}_f(D) \times D\&D$ and $\psi \subseteq \mathcal{M}_f(D\&D) \times D \cong (\mathcal{M}_f(D) \times \mathcal{M}_f(D)) \times D$, where

$$\varphi = \{([\sigma], (1, \sigma)) \mid \sigma \in D\} \cup \{([\sigma], (2, \sigma)) \mid \sigma \in D\}$$

$$\psi = \{(([\sigma], [\,]), \sigma) \mid \sigma \in D\} \cup \{(([\,], [\sigma]), \sigma) \mid \sigma \in D\}$$

$$\varphi \circ \psi = \{(m, b) \mid \exists (m_1, b_1) \dots (m_k, b_k) \in \psi \text{ s.t. } m = \uplus m_i, ([b_1, \dots, b_k], b) \in \varphi\}$$

but $([b_1, \dots, b_k], b) \in \varphi$ means $b = (j, \sigma)$ for some $\sigma \in D$, therefore $k = 1$ and $b_1 = \sigma$. Therefore

$$\varphi \circ \psi = \{(m, (j, \sigma)) \mid \exists (m_1, \sigma_1) \in \psi, m = m_1, ([\sigma], (j, \sigma)) \in \varphi\}$$

$(m, \sigma) \in \psi$ means $m = [(1, \sigma)]$ or $m = [(2, \sigma)]$. So we have that

$$\begin{aligned} \varphi \circ \psi &= \{([(i, \sigma)], (i, \sigma)) \mid (i, \sigma) \in D\&D\} \\ &= id_{D\&D} \end{aligned}$$

as required. □

Note that $\psi \circ \varphi \neq Id_D$, in fact, $\psi \circ \varphi = Id_D \uplus Id_D$, disjoint union, two copies of Id_D.

Definition 3.5.2. A *monoid* is a non-empty set $\mathcal{M}$ with a binary operation $\cdot : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ and identity element e such that $\cdot$ is associative and $e \cdot a = a \cdot e = a$ for all $a \in G$.

We can view a monoid $\mathcal{M}$ as a category with one object. The elements of $\mathcal{M}$ are the arrows of the category, the identity arrow in the category is $e \in \mathcal{M}$ and composition in the category is multiplication, $\cdot$, in $\mathcal{M}$.

Definition 3.5.3. A *C-monoid* is a monoid $\mathcal{M}$ together with $(\pi_1, \pi_2, \varepsilon, *, \langle\rangle)$ where $\pi_1, \pi_2, \varepsilon \in \mathcal{M}$, $(-)^* : \mathcal{M} \longrightarrow \mathcal{M}$ a unary operation and $\langle -, - \rangle : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ a binary operation satisfying:

C1. $\pi_1 \langle a, b \rangle = a$

C2. $\pi_2 \langle a, b \rangle = b$

C3. $\langle \pi_1 c, \pi_2 c \rangle = c$

C4. $\varepsilon \langle h^* \pi_1, \pi_2 \rangle = h$

C5. $(\varepsilon \langle k \pi_1, \pi_2 \rangle)^* = k$

for every $a, b, h, k \in \mathcal{M}$.

Notice that the definition of a C-monoid is analogous to the axioms of a cartesian closed category except for the existence of a terminal object. We omit the terminal object since if a cartesian closed category has only 1 object, then that object is the terminal object. With this, we would have that $|Hom(1,1)| = 1$: there is only 1 arrow in the category.

Lambek and Scott prove the following consequences of the axioms of C-monoids given in Definition 3.5.3:

Proposition 3.5.4 (see [11]). *In a C-monoid, the following laws hold:*

C3a. $\langle a, b \rangle c = \langle ac, bc \rangle$

C3b. $\langle \pi_1, \pi_2 \rangle = e$

C4a. $\varepsilon \langle h^* a, b \rangle = h \langle a, b \rangle$

C5a. $h^* k = (h \langle k \pi_1, \pi_2 \rangle)^*$

Proposition 3.5.5 (see [11]). *Let C be any locally small cartesian closed category with an object U such that $U^U \cong U$ and $U \times U \cong U$, then $End(U)$ is a C-monoid.*

Definition 3.5.6. A *weak C-Monoid* is a monoid $\mathcal{M}$ together with $(\pi_1, \pi_2, \varepsilon, *, \langle\rangle)$ where $\pi_1, \pi_2, \varepsilon \in \mathcal{M}$, $(-)^* : \mathcal{M} \longrightarrow \mathcal{M}$ a unary operation and $\langle -, - \rangle : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ a binary operation satisfying $C1, C2, C3a, C4a$.

We do not know if D is isomorphic to $D\&D$. We only know that $D\&D$ is a retract of D. Hence, we cannot use Proposition 3.5.5 to show that $End(D)$ is a C-monoid. In fact, with respect to the retraction pair $\varphi : D \longrightarrow D\&D$ and $\psi : D\&D \longrightarrow D$ given in the proof of Proposition 3.5.1, $End(D)$ is a weak C-monoid.

Proposition 3.5.7. *Let D be the reflexive object in* **MRel** *presented in [7]. Then $End(D)$ is a weak C-monoid.*

Proof. First, we must define structure of the weak C-monoid, namely the 5 tuple $(\pi_1, \pi_2, \{\}, \varepsilon, ()^\star)$, as follows:

$$\begin{aligned}
\pi_1 &= \pi_{1_{D,D}} \circ \varphi \\
\pi_2 &= \pi_{2D,D} \circ \varphi \\
\{a, b\} &= \psi \circ \langle a, b\rangle \\
\varepsilon &= ev_{D,D} \circ (p \times Id_D) \circ \varphi \\
h^\star &= e \circ (h \circ \psi)^*
\end{aligned}$$

These are all indeed morphisms from D to D, and are therefore elements of $End(D)$ as required. We will now check which of the C-monoid axioms hold here:

- C1:

$$\begin{aligned}
\pi_1\{a, b\} &= \pi_{1_{D,D}} \circ \varphi \circ \psi \circ \langle a, b\rangle \\
&= \pi_{1_{D,D}} \circ \langle a, b\rangle \\
&= a
\end{aligned}$$

  C1 holds.

- C2:

$$\begin{aligned}
\pi_2\{a, b\} &= \pi_{2_{D,D}} \circ \varphi \circ \psi \circ \langle a, b\rangle \\
&= \pi_{2_{D,D}} \circ \langle a, b\rangle \\
&= b
\end{aligned}$$

  C2 holds.

- C3:

$$\begin{aligned}\{\pi_1 c, \pi_2 c\} &= \psi \circ \langle \pi_1 c, \pi_2 c\rangle \\ &= \psi \circ \langle {\pi_1}_{D,D} \circ \varphi \circ c, {\pi_2}_{D,D} \circ \varphi \circ c\rangle \\ &= \psi \circ \langle \pi_1, \pi_2\rangle \circ \varphi \circ c \\ &= \psi \circ \varphi \circ c \\ &\neq c\end{aligned}$$

C3 does not hold, since $\psi \circ \varphi \neq Id_D$.

- C3a:

$$\begin{aligned}\{a, b\}c &= \psi \circ \langle a, b\rangle \circ c \\ &= \psi \circ \langle ac, bc\rangle \\ &= \{ac, bc\}\end{aligned}$$

C3a holds.

- C3b:

$$\begin{aligned}\{\pi_1, \pi_2\} &= \psi \circ \langle \pi_1, \pi_2\rangle \\ &= \psi \circ \langle {\pi_1}_{D,D} \circ \varphi, {\pi_2}_{D,D} \circ \varphi\rangle \\ &= \psi \circ \langle {\pi_1}_{D,D}, {\pi_2}_{D,D}\rangle \circ \varphi \\ &= \psi \circ \varphi \\ &\neq Id_D\end{aligned}$$

C3b does not hold, since $\psi \circ \varphi \neq Id_D$.

- C4:

$$\begin{aligned}\varepsilon\{h^{\star}\pi_1, \pi_2\} &= ev_{D,D} \circ (p \times Id_D) \circ \varphi \circ \psi \circ \langle e \circ (h \circ \psi)^{*} {\pi_1}_{D,D} \circ \varphi, {\pi_2}_{D,D} \circ \varphi\rangle \\ &= ev_{D,D} \circ (p \times Id_D) \circ \langle e \circ (h \circ \psi)^{*} {\pi_1}_{D,D} \circ \varphi, {\pi_2}_{D,D} \circ \varphi\rangle \\ &= ev_{D,D} \circ (p \times Id_D) \circ \langle e \circ (h \circ \psi)^{*} {\pi_1}_{D,D}, {\pi_2}_{D,D}\rangle \circ \varphi \\ &= ev_{D,D} \circ \langle p \circ e \circ (h \circ \psi)^{*} {\pi_1}_{D,D}, Id_D \circ {\pi_2}_{D,D}\rangle \circ \varphi \\ &= ev_{D,D} \circ \langle (h \circ \psi)^{*} {\pi_1}_{D,D}, {\pi_2}_{D,D}\rangle \circ \varphi \\ &= (h \circ \psi) \circ \varphi \\ &= h \circ \psi \circ \varphi \\ &\neq h\end{aligned}$$

C4 does not hold, since $\psi \circ \varphi \neq Id_D$.

- C4a:

$$\begin{aligned}
\varepsilon\{h^{\star}a, b\} &= ev_{D,D} \circ (p \times Id_D) \circ \varphi \circ \psi \circ \langle e \circ (h \circ \psi)^{*}a, b\rangle \\
&= ev_{D,D} \circ (p \times Id_D) \circ \langle e \circ (h \circ \psi)^{*}a, b\rangle \\
&= ev_{D,D} \circ \langle p \circ e \circ (h \circ \psi)^{*}a, Id_D \circ b\rangle \\
&= ev_{D,D} \circ \langle h \circ \psi)^{*}a, b\rangle \\
&= h \circ \psi \circ \langle a, b\rangle \\
&= h\{a, b\}
\end{aligned}$$

  C4a holds.

- C5:

$$\begin{aligned}
(\varepsilon\{k\pi_1, \pi_2\})^{\star} &= (ev_{D,D} \circ (p \times Id_D) \circ \varphi \circ \psi \circ \langle k\pi_{1_{D,D}} \circ \varphi, \pi_{2_{D,D}} \circ \varphi\rangle)^{\star} \\
&= (ev_{D,D} \circ (p \times Id_D) \circ \langle k\pi_{1_{D,D}} \circ \varphi, \pi_{2_{D,D}} \circ \varphi\rangle)^{\star} \\
&= (ev_{D,D} \circ (p \times Id_D) \circ \langle k\pi_{1_{D,D}}, \pi_{2_{D,D}}\rangle \circ \varphi)^{\star} \\
&= e \circ (ev_{D,D} \circ (p \times Id_D) \circ \langle k\pi_{1_{D,D}}, \pi_{2_{D,D}}\rangle \circ \varphi \circ \psi)^{*} \\
&= e \circ p \circ k \\
&= k
\end{aligned}$$

  C5 holds.

- C5a:

$$\begin{aligned}
h^{*}k &= e \circ (h \circ \psi)^{*}k \\
&= e \circ (ev_{D,D}\langle (h \circ \psi)^{*}k\pi_{1_{D,D}}, \pi_{2_{D,D}}\rangle)^{*} \\
&= e \circ (ev_{D,D}\langle (h \circ \psi)^{*}k\pi_{1_{D,D}}\langle k\pi_{1_{D,D}}, \pi_{2_{D,D}}\rangle, \pi_{2_{D,D}}\langle k\pi_{1_{D,D}}, \pi_{2_{D,D}}\rangle\rangle)^{*} \\
&= e \circ (ev_{D,D}\langle (h \circ \psi)^{*}k\pi_{1_{D,D}}, \pi_{2_{D,D}}\rangle\langle k\pi_{1_{D,D}}, \pi_{2_{D,D}}\rangle)^{*} \\
&= e \circ (h \circ \psi \circ \langle k\pi_{1_{D,D}}, \pi_{2_{D,D}}\rangle)^{*} \\
&= e \circ (h \circ \psi \circ \langle k\pi_1 \circ \psi, \pi_2 \circ \psi\rangle)^{*} \\
&= e \circ (h \circ \psi \circ \langle k\pi_1, \pi_2\rangle \circ \psi)^{*} \\
&= e \circ (h \circ \{k\pi_1, \pi_2\} \circ \psi)^{*} \\
&= (h \circ \{k\pi_1, \pi_2\})^{*}
\end{aligned}$$

  C5a holds.

Hence, we conclude that $End(D)$ is a weak C-monoid. □

Notice that under the structure defined above, $End(D)$ is not a weak C-monoid. From an exercise in [11] p.100 Ex.3(b), this means that the Karoubi envelope (which is the idempotent splitting completion) of $End(D)$ is a cartesian closed category.

4 MRel and the $\mathcal{P}\omega$ model

In section 2.2.3, we gave a brief description of Scott's $\mathcal{P}\omega$ model of the untyped lambda calculus. In this model, we interpret the lambda calculus in the power set of the natural numbers $\mathbb{N}$. The set of natural numbers $\mathbb{N}$ is an object in **MRel** and as we have already seen in section 3.3, the global sections functor Γ applied to $\mathbb{N}$ is isomorphic to $\mathcal{P}(\mathbb{N})$. This chapter explores the possible relationship between well known models of the untyped lambda calculus, in particular the graph model, and **MRel**. The first section defines a notion of application in **MRel** and compares Γ of this function to application in the graph model. The second describes the setting in which we get are eble to get the graph model from **MRel** using Γ. The third section describes how we get reflexive objects in **AlgLat** from reflexive objects in **MRel**.

4.1 Application

Observe that there is a faithful, identity-on-objects embedding **Set** $\hookrightarrow$ **MRel** mapping $f : A \longrightarrow B \in$ **Set** to $\overline{f} : A \longrightarrow B \in$ **MRel**, where $\overline{f} = \{([a], b) \mid f(a) = b\}$. Suppose that we have an object in **MRel**, say a set A, and a set theoretic isomorphism $A \cong A \Rightarrow A$ via $\varphi : (A \Rightarrow A) \longrightarrow A \in$ **Set** and $\psi : A \longrightarrow (A \Rightarrow A) \in$ **Set**, where $A \Rightarrow A$ is $\mathcal{M}_f(A) \times A$ (the exponential in **MRel**). We define an application map $App : A\&A \longrightarrow A$ using the above isomorphism and the evaluation map $ev_{A,A} : (A \Rightarrow A)\&A \longrightarrow A$ by letting $App = ev_{A,A} \circ (\overline{\psi}\&Id_A)$. Then we can apply Γ to this map which will give us a map $\widetilde{\Gamma}(App) : \mathcal{P}(A) \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$. We will then compare this notion of application to application in the $\mathcal{P}\omega$ model.

First, let us examine what App looks like. The map $\overline{\psi}$ is a map from A to $A \Rightarrow A$ in **MRel** and can be regarded as a subset of $\mathcal{M}_f(A) \times (\mathcal{M}_f(A) \times A)$. Therefore, $\overline{\psi}\&Id_A : A\&A \longrightarrow (A \Rightarrow A)\&A$ will be a subset of $\mathcal{M}_f(A\&A) \times (\mathcal{M}_f(A) \times A)\&A$ or equivalently, a subset of $\mathcal{M}_f(A) \times \mathcal{M}_f(A) \times (\mathcal{M}_f(A) \times A)\&A$ defined as follows

$$\overline{\psi}\&Id_A = \{([(1, a_1)], (1, (\beta, a_2))) \mid \psi(a_1) = (\beta, a_2)\} \cup \{([(2, a)], (2, a)) \mid a \in A\}$$

Also, recall

$$\begin{aligned} ev_{A,A} &= \{(([(\alpha,n)],\alpha),n) \mid \alpha \in \mathcal{M}_f(A), n \in A\} \\ &= \{([(1,([a_1,\ldots,a_k],n)),(2,a_1),\ldots,(2,a_k)],n) \mid a_i, n \in A\} \end{aligned}$$

Now, $App : A\&A \longrightarrow A$, which is equal to $ev_{A,A} \circ (\overline{\psi}\&Id_A)$, will be be a subset of of $\mathcal{M}_f(A\&A) \times A$ or equivalently, a subset of $(\mathcal{M}_f(A) \times \mathcal{M}_f(A)) \times A$. From the definition of composition in **MRel**, we have

$$\begin{aligned} App &= ev_{A,A} \circ (\overline{\psi}\&Id_A) \\ &= \{(\alpha,a) \mid \exists(\alpha_1,X_1),\ldots,(\alpha_k,X_k) \in (\overline{\psi}\&Id_A) \text{ s.t. } \alpha = \alpha_1 \uplus \cdots \uplus \alpha_k, \\ &\qquad ([X_1,\ldots,X_k],a) \in ev_{A,A}\} \end{aligned}$$

The fact that $([X_1,\ldots,X_k],a) \in ev_{A,A}$ tells us that for some i, say for $i = 1$,$X_i = X_1 = (1,([a_1,\ldots,a_{k-1}],a))$ and $X_2 = (2,a_1),\ldots,X_k = (2,a_{k-1})$. Since $(\alpha_i,X_i) \in (\overline{\psi}\&Id_A)$, we have that $\alpha_1 = [(1,b)]$ such that $\psi(b) = ([a_1,\ldots,a_{k-1}],a)$ and $\alpha_2 = [(2,a_1)],\ldots,\alpha_k = [(2,a_{k-1})]$. Finally, given that $\alpha = \alpha_1 \uplus \cdots \uplus \alpha_k$, we have that $\alpha = [(1,b),(2,a_1),\ldots,(2,a_{k-1})]$. Summarizing, we have

$$\begin{aligned} App &= \{([(1,b),(2,a_1),\ldots,(2,a_{k-1})],a) \mid \psi(b) = ([a_1,\ldots,a_{k-1}],a)\} \\ &\cong \{(([b],\alpha),a) \mid \psi(b) = ([a_1,\ldots,a_{k-1}],a)\} \end{aligned}$$

The isomorphism above is due to the fact that $\mathcal{M}_f(A\&A) \cong \mathcal{M}_f(A) \times \mathcal{M}_f(A)$. By Proposition 3.3.4 we have that applying Γ to App gives us $\Gamma(App) : \mathcal{P}(A\&A) \longrightarrow \mathcal{P}(A)$ which is a map in **AlgLat**. Let $V \subseteq A\&A$, then we have

$$\Gamma(App)(V) = \{a \in A \mid \exists a_1,\ldots,a_k \in V \text{ s.t. } ([a_1,\ldots,a_k],a) \in App\}$$

Notice that to any $V \subseteq A\&A$, we can associate a pair of of subsets $(X,Y) \in \mathcal{P}(A) \times \mathcal{P}(A)$, all those elements in V that are marked with a 1 are in X and all those marked with a 2 are in Y. This gives a map $\widetilde{\Gamma}(App) : \mathcal{P}(A) \times \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$, where for $(X,Y) \in \mathcal{P}(A) \times \mathcal{P}(A)$

$$\begin{aligned} \widetilde{\Gamma}(App)(X,Y) &= \{a \in A \mid \exists\alpha \in \mathcal{M}_f(X), \beta \in \mathcal{M}_f(Y) \text{ s.t. } ((\alpha,\beta),a) \in App\} \\ &= \{a \in A \mid \exists x \in X, \beta \in \mathcal{M}_f(Y) \text{ s.t. } , \psi(x) = (\beta,a))\} \end{aligned}$$

It is true that $\mathbb{N} \Rightarrow \mathbb{N} = \mathcal{M}_f(\mathbb{N}) \times \mathbb{N} \cong \mathbb{N} \in \mathbf{Set}$. Call this isomorphism $\psi : \mathbb{N} \longrightarrow (\mathbb{N} \Rightarrow \mathbb{N}) \in \mathbf{Set}$ and therefore $\mathbb{N} \Rightarrow \mathbb{N} \cong \mathbb{N} \in \mathbf{MRel}$ via $\overline{\psi} : \mathbb{N} \longrightarrow (\mathbb{N} \Rightarrow \mathbb{N}) \in \mathbf{MRel}$. Letting A be $\mathbb{N}$, we get a notion of application on $\mathcal{P}\mathbb{N}$, $\bullet : \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N} \longrightarrow \mathcal{P}\mathbb{N}$, and for $A, B \subseteq \mathbb{N}$

$$A \bullet B = \bigcup_{a \in A, \beta \in \mathcal{M}_f(B)} \{n \in \mathbb{N} \mid \psi(a) = (\beta, n)\}$$

This definition of application is similar to application in the graph model given in section 2.2.3, the only difference being that the union ranges over finite multisets in our case and finite sets in the graph model.

4.2 Retrieving the graph model from MRel

At the end of section 4.1, we notice that $\mathbb{N} \Rightarrow \mathbb{N} \cong \mathbb{N}$ in **MRel**. Since Γ is a functor it preserves isomorphims. Also, by Proposition 3.3.4, Γ factors through **AlgLat**. These facts combined tell us that $\mathcal{P}(\mathbb{N} \Rightarrow \mathbb{N}) \cong \mathcal{P}(\mathbb{N}) \in \mathbf{AlgLat}$. The idea is to find the correct isomorphism $\psi : \mathbb{N} \Rightarrow \mathbb{N} \longrightarrow \mathbb{N} \in \mathbf{MRel}$, and maps $H, G \in \mathbf{AlgLat}$ so that the following diagram represents the graph model:

$$\mathcal{P}(\mathbb{N}) \underset{\Gamma(\psi)}{\overset{\Gamma(\psi^{-1})}{\rightleftarrows}} \mathcal{P}(\mathbb{N} \Rightarrow \mathbb{N}) \underset{G}{\overset{H}{\rightleftarrows}} [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]$$

In other words, we need an isomorphism $\psi : \mathbb{N} \Rightarrow \mathbb{N} \longrightarrow \mathbb{N} \in \mathbf{MRel}$, and maps $H, G \in \mathbf{AlgLat}$ such that $H \circ \Gamma(\psi^{-1}) = fun$ and $\Gamma(\psi) \circ G = graph$. Since **MRel** is a cartesian closed category, it has an evaluation map. It would be nice if H arose from such an evaluation map in **MRel**: $\mathbb{N}$ is an object in **MRel** therefore $ev_{\mathbb{N},\mathbb{N}} : (\mathbb{N} \Rightarrow \mathbb{N}) \longrightarrow \mathbb{N}$, which we will refer to as ev in this section for simplicity, is an evaluation map in **MRel**. By Proposition 3.3.4, $\Gamma(ev) : \mathcal{P}((\mathbb{N} \Rightarrow \mathbb{N})\&\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$ is a morphism in **AlgLat** which is equivalent to a map $\widetilde{\Gamma}(ev) : \mathcal{P}(\mathbb{N} \Rightarrow \mathbb{N}) \times \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$, since $\mathcal{P}((\mathbb{N} \Rightarrow \mathbb{N})\&\mathbb{N}) \cong \mathcal{P}(\mathbb{N} \Rightarrow \mathbb{N}) \times \mathcal{P}(\mathbb{N})$. Now since **AlgLat** is a cartesian closed category, we can take the curry of this map to get $\widetilde{\Gamma}(ev)^* : \mathcal{P}(\mathbb{N} \Rightarrow \mathbb{N}) \longrightarrow [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]$ and let $H = \widetilde{\Gamma}(ev)^*$. Then we could define define a map $G : [\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{N})] \longrightarrow \mathcal{P}(\mathbb{N} \Rightarrow \mathbb{N})$ such that $\widetilde{\Gamma}(ev)^* \circ G = Id_{[\mathcal{P}(\mathbb{N}),\mathcal{P}(\mathbb{N})]}$. With this and the previously mentioned isomorphism $\Gamma(\psi)$, we will get a retraction $[\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{N})] \triangleleft \mathcal{P}(\mathbb{N})$. The idea is that for the correct choice of isomorphism ψ, this retraction corresponds to the graph model. This, unfortunately, is not possible to do. This method does not work for the exact reason why our notion of application, defined in section 4.1, did not coincide with application in the graph model. The graph model is given in terms of finite sets, not finite multisets. We need a way to move away from $\mathcal{M}_f(\mathbb{N})$ and into $\mathcal{P}_f(\mathbb{N})$. This is done by adding an extra step into our diagram:

$$\mathcal{P}(\mathbb{N}) \underset{\Gamma(\psi)}{\overset{\Gamma(\psi^{-1})}{\rightleftarrows}} \mathcal{P}(\mathbb{N} \Rightarrow \mathbb{N}) \underset{\Gamma(\overline{g})}{\overset{\Gamma(\overline{g^{-1}})}{\rightleftarrows}} \mathcal{P}(\mathcal{P}_f(\mathbb{N}) \times \mathbb{N}) \underset{G}{\overset{H}{\rightleftarrows}} [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]$$

where g is a set theoretic isomorphism from $\mathcal{P}_f(\mathbb{N}) \times \mathbb{N}$ to $\mathbb{N} \Rightarrow \mathbb{N} \in$ **Set** defined as follows: for $(A, n) \in \mathcal{P}_f(\mathbb{N}) \times \mathbb{N}$, $A = \{b_1, \ldots b_k\}$

$$g(A,n) = (\{(n_1, n_1'), \ldots, (n_k, n_k')\}, n), \text{ where } b_i = \#(n_i, n_i'), i = 1, \ldots, k$$

and $g^{-1} : \mathbb{N} \Rightarrow \mathbb{N} \longrightarrow \mathcal{P}_f(\mathbb{N}) \times \mathbb{N} \in$ **Set**, and for $(\alpha, n) \in \mathbb{N} \Rightarrow \mathbb{N}$, $\alpha = \{(n_1, m(n_1)), \ldots, (n_k, m(n_k))\}$

$$g^{-1}(\alpha, n) = (\{\#(n_1, m(n_1)), \ldots, \#(n_k, m(n_k))\}, n)$$

where,for $a, b \in \mathbb{N}$, $\#(a, b)$ is the coding of pairs used in the graph model defined in section 2.2.3. Now, since $\#$ is a bijection, this is shown in [2], it is clear that g and g^{-1} define an isomorphism of sets and so $\overline{g}$ and $\overline{g^{-1}}$ define an isomorphism in **MRel**. We will use this isomorphism and the coding of finite sets and pairs from the graph model to define the correct choice of $\psi : \mathbb{N} \Rightarrow \mathbb{N} \longrightarrow \mathbb{N} \in$ **MRel**.

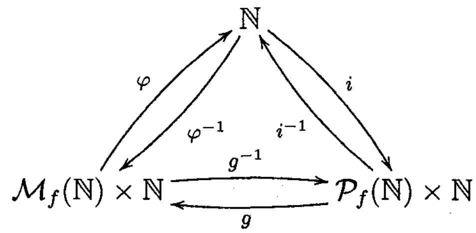
We know that $\mathbb{N} \cong \mathcal{P}_f(\mathbb{N}) \times \mathbb{N} \in$ **Set** via $i : \mathbb{N} \longrightarrow \mathcal{P}_f(\mathbb{N}) \times \mathbb{N}$ and $i^{-1} : \mathcal{P}_f(\mathbb{N}) \times \mathbb{N} \longrightarrow \mathbb{N}$, where for $n \in \mathbb{N}$

$$i(n) = (e_a, b) \text{ where } \#(a, b) = n$$

and for $(e_a, b) \in \mathcal{P}_f(\mathbb{N}) \times \mathbb{N}$

$$i^{-1}(e_a, b) = \#(a, b)$$

where, for $a \in \mathbb{N}$, e_a is the finite set coded by a as defined in section 2.2.3. This indeed defines an isomorphism since both codings are bijections (see [2]). Now we define two set maps $\varphi : \mathcal{M}_f(\mathbb{N}) \times \mathbb{N} \longrightarrow \mathbb{N}$ and $\varphi^{-1} : \mathbb{N} \longrightarrow \mathcal{M}_f(\mathbb{N}) \times \mathbb{N}$ in terms of the previously defined isomorphisms:



In other words, φ is defined to be $i^{-1} \circ g^{-1}$ and φ^{-1} is $g \circ i$. Now, φ and φ^{-1} define an isomorphism in **Set** since i, i^{-1} and g, g^{-1} are both isomorphisms. Therefore, letting $\psi = \overline{\varphi}$ and $\psi^{-1} = \overline{\varphi^{-1}}$, we have that ψ, ψ^{-1} define an isomorphism of $\mathbb{N}$ and $\mathcal{M}_f(\mathbb{N}) \times \mathbb{N}$ in **MRel**.

The last part of our diagram that needs to be filled in requires us to define maps $H : \mathcal{P}(\mathcal{P}_f(\mathbb{N}) \times \mathbb{N}) \longrightarrow [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]$ and $G : [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}] \longrightarrow \mathcal{P}(\mathcal{P}_f(\mathbb{N}) \times \mathbb{N})$ such that $H \circ \Gamma(\ \overline{g^{-1}}\ ) \circ \Gamma(\psi^{-1}) = fun$ and $\Gamma(\psi) \circ \Gamma(\overline{g}) \circ G = graph$.

Consider $\mathcal{M}_f(\mathbb{N}) \times \mathbb{N}$ and $\mathcal{P}_f(\mathbb{N}) \times \mathbb{N}$. $\mathcal{M}_f(\mathbb{N}) \times \mathbb{N}$ is the exponential object $\mathbb{N} \Rightarrow \mathbb{N}$ in **MRel**. It would be nice if $\mathcal{P}_f(\mathbb{N}) \times \mathbb{N}$ was the exponential object in a category similar to **MRel** but built from $\mathcal{P}_f$ instead. If we were able to define a distributive law of $\mathcal{P}$ over $\mathcal{P}_f$ we could define a monad $\hat{\mathcal{P}}_f$ on Rel and follow the same construction as we did for **MRel** to get the coKleisli category of this $\hat{\mathcal{P}}_f$, call it **PfRel**. Then we could define the global sections functor, say $\Gamma_{\mathcal{P}_f}$ for this category, then H would arise as $\Gamma_{\mathcal{P}_f}$ of an evaluation map in **PfRel** in the same way that we had hoped H to arise from an evaluation map in **MRel** (Discussed at the beginning of this section). Unfortunately, this does not work: it is not clear whether such a distributive law exists or not. For a discussion of this, please see section 4.4.

We are however able to find a map $h : (\mathcal{P}_f(\mathbb{N}) \times \mathbb{N})\&\mathbb{N} \longrightarrow \mathbb{N} \in$ **MRel** which will serve the same purpose as the required evaluation map of **PfRel**, if **PfRel** were to exist. Considering h as a subset of $((\mathcal{P}_f(\mathbb{N}) \times \mathbb{N})\&\mathbb{N}) \times \mathbb{N}$, we define this map as follows:

$$h = \{([(1, (\{n_1, \ldots, n_k\}, n)), (2, n_1), \ldots, (2, n_k)], n) \mid n, n_i \in \mathbb{N}\}$$

This map h, is very similar to $ev_{\mathbb{N},\mathbb{N}} \in$ **MRel**, but is not quite the same: notice that here we use curly brackets $\{n_1, \ldots, n_k\}$, whereas in $ev_{\mathbb{N},\mathbb{N}}$ this would be $[n_1, \ldots, n_k]$.

We apply the global sections functor defined for **MRel** to h to get a map $\Gamma(h) : \mathcal{P}((\mathcal{P}_f(\mathbb{N}) \times \mathbb{N})\&\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$. This is a morphism in **AlgLat** where for $A \subseteq (\mathcal{P}_f(\mathbb{N}) \times \mathbb{N})\&\mathbb{N}$, $\Gamma(h)(A)$ is the following set:

$$\Gamma(h)(A) = \{n \in \mathbb{N} \mid \exists a_1, \ldots, a_k \in A \text{ s.t. } ([a_1, \ldots, a_k], n) \in h\}$$

From the above definition of h, we know that $([a_1, \ldots, a_k], n) \in h$ if and only if $a_1 = (1, (\{b_1, \ldots, b_{k-1}\}, n))$ for some $b_1, \ldots, b_{k-1} \in \mathbb{N}$ and $a_2 = (2, b_1), \ldots, a_k = (2, b_{k-1})$. Therefore

$$\Gamma(h)(A) = \{n \in \mathbb{N} \mid \exists b_1, \ldots, b_k \in \mathbb{N} \text{ s.t. } (1, (\{b_1, \ldots, b_k\}, n)), (2, b_1), \ldots, (2, b_k) \in A\}$$

Now we can define $\widetilde{\Gamma(h)} : \mathcal{P}(\mathcal{P}_f(\mathbb{N}) \times \mathbb{N}) \times \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$ as follows: for $(A, B) \in \mathcal{P}(\mathcal{P}_f(\mathbb{N}) \times \mathbb{N}) \times \mathcal{P}(\mathbb{N})$, so $A \subseteq (\mathcal{P}_f(\mathbb{N}) \times \mathbb{N})$ and $B \subseteq \mathbb{N}$, then

$$\widetilde{\Gamma(h)}(A, B) = \{n \in \mathbb{N} \mid \exists b_1, \ldots, b_k \in B \text{ s.t. } (\{b_1, \ldots, b_k\}, n) \in A\}$$

Since $\widetilde{\Gamma(h)}$ is also in **AlgLat** and **AlgLat** is a cartesian closed category, we can apply the $(-)^*$ operation to $\widetilde{\Gamma(h)}$ to get $\widetilde{\Gamma(h)}^* : \mathcal{P}(\mathcal{P}_f(\mathbb{N}) \times \mathbb{N}) \longrightarrow [\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{N})]$ with $\widetilde{\Gamma(h)}^*(A) = f_A$, $A \subseteq (\mathcal{P}_f(\mathbb{N}) \times \mathbb{N})$, where $f_A \in [\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{N})]$ and for $B \subseteq \mathbb{N}$, $f_A(B) = \widetilde{\Gamma(h)}(A, B)$. Then let $H = \widetilde{\Gamma(h)}^*$

Now we define $G : [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}] \longrightarrow \mathcal{P}(\mathcal{P}_f(\mathbb{N}) \times \mathbb{N})$ to be the map in **AlgLat** such that for $F \in [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]$

$$G(F) = \{(A, n) \mid n \in F(A), A \subseteq_f \mathbb{N}\}$$

Then $H \circ G = Id_{[\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]}$. To show that from these maps we get the graph model there are a few things to check: we must show that $f_A \in$ **AlgLat**, we must also show that $G \in$ **AlgLat**, check that $H \circ G = Id_{[\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]}$ and finally, we need to verify that $H \circ \Gamma(\overline{g^{-1}}) \circ \Gamma(\psi^{-1}) = fun$ and $\Gamma(\psi) \circ \Gamma(\overline{g}) \circ G = graph$:

1. $f_A \in$ **AlgLat**:

By definition, for $A \subset (\mathcal{P}_f(\mathbb{N}) \times \mathbb{N})$, f_A is a map from $\mathcal{P}\mathbb{N}$ to $\mathcal{P}\mathbb{N}$. To show that $f_A \in$ **AlgLat**, we must show that for a directed family of subsets of the natural numbers, say $\{B_i\}$ the following equality of sets holds:

$$f_A(\bigcup B_i) = \bigcup f_A(B_i)$$

Now consider for now the left hand side of this equality:

$$\begin{aligned} f_A(\bigcup B_i) &= \widetilde{\Gamma(h)}(A, \bigcup B_i) \\ &= \{n \in \mathbb{N} \mid \exists n_1, \ldots, n_k \in \bigcup B_i \text{ s.t. } (\{n_1, \ldots, n_k\}, n) \in A\} \end{aligned}$$

Now consider the right hand side of the equality:

$$\begin{aligned} \bigcup f_A(B_i) &= \widetilde{\Gamma(h)}(A, B_i) \\ &= \bigcup \{n \in \mathbb{N} \mid \exists n_1, \ldots, n_k \in B_i \text{ s.t. } (\{n_1, \ldots, n_k\}, n) \in A\} \end{aligned}$$

Then for $n \in \mathbb{N}$

$$\begin{aligned} n \in f_A(\bigcup B_i) \quad &\text{then} \quad \exists n_1, \ldots, n_k \in \bigcup B_i \text{ s.t. } (\{n_1, \ldots, n_k\}, n) \in A \\ &\text{then} \quad \exists n_1, \ldots, n_k \exists B_1, \ldots, B_k \text{ s.t. } n_i \in B_i \text{ for } i = 1, \ldots, k \\ &\qquad\qquad \text{and } (\{n_1, \ldots, n_k\}, n) \in A \\ &\text{then} \quad \exists n_1, \ldots, n_k \exists N \text{ s.t. } n_i \in B_N \text{ for } i = 1, \ldots, k \\ &\qquad\qquad \text{and } (\{n_1, \ldots, n_k\}, n) \in A \text{ since } \{B_i\} \text{ is directed} \\ &\text{then} \quad n \in \bigcup f_A(B_i) \end{aligned}$$

and in the other direction

$$\begin{aligned} n \in \bigcup f_A(B_i) \quad &\text{then} \quad \exists N \exists n_1, \ldots, n_k \in B_N \text{ s.t. } (\{n_1, \ldots, n_k\}, n) \in A \\ &\text{then} \quad \exists n_1, \ldots, n_k \in \bigcup B_i \text{ s.t. } (\{n_1, \ldots, n_k\}, n) \in A \\ &\text{then} \quad n \in f_A(\bigcup B_i) \end{aligned}$$

and we have that $f_A(\bigcup B_i) = \bigcup f_A(B_i)$ as required.

2. $G \in$ **AlgLat**:

   To verify this we must show that $G(\bigvee F_i) = \bigvee G(F_i)$, $\{F_i\}$ a directed family and $F_i \in [\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{N})]$. Notice that $\bigvee F_i(A)$, $A \subseteq \mathbb{N}$ is defined to be $\bigvee(F_i(A))$. Then

$$\begin{aligned} G(\bigvee F_i) &= \{(A, n) \mid n \in (\bigvee F_i)(A), A \subseteq_f \mathbb{N}\} \\ &= \{(A, n) \mid n \in \bigvee(F_i(A)), A \subseteq_f \mathbb{N}\} \end{aligned}$$

and

$$\bigvee(G(F_i)) = \bigvee\{(A,n) \mid n \in F_i(A), A \subseteq_f \mathbb{N}\}$$

Suppose that $(A,n) \in G(\bigvee F_i)$, then $n \in \bigvee(F_i(A))$ where $A \subseteq_f \mathbb{N}$. This means that there is an i such that $n \in F_i(A)$, which means that $(A,n) \in G(F_i)$. Therefore $(A,n) \in \bigvee(G(F_i))$. Now suppose that $(A,n) \in \bigvee(G(F_i))$, then for some i, $(A,n) \in G(F_i)$. This, by definition, means that $n \in F_i(A)$ and $A \subseteq_f \mathbb{N}$, which means that $n \in \bigvee F_i(A)$. Therefore we get that $(A,n) \in G(\bigvee F_i)$, as required. This shows that $G(\bigvee F_i) = \bigvee G(F_i)$ and therefore $G \in$ **AlgLat**.

3. $H \circ G = Id_{[\mathcal{P}(\mathbb{N}),\mathcal{P}(\mathbb{N})]}$:

   Let $F \in [\mathcal{P}(\mathbb{N}),\mathcal{P}(\mathbb{N})]$, then

$$\begin{aligned} H \circ G &= (\widetilde{\Gamma(h)}^* \circ G)F \\ &= \widetilde{\Gamma(h)}^*(G(F)) \\ &= f_{G(F)} \end{aligned}$$

   To check that $H \circ G = Id_{[\mathcal{P}(\mathbb{N}),\mathcal{P}(\mathbb{N})]}$ we must show that $f_{G(F)} = F$. Let $U \subseteq \mathbb{N}$, then

$$\begin{aligned} f_{G(F)}(U) &= \widetilde{\Gamma(h)}(G(F),U) \\ &= \{n \in \mathbb{N} \mid \exists n_1,\ldots,n_k \in U \text{ s.t. } (\{n_1,\ldots,n_k\},n) \in G(F)\} \end{aligned}$$

   and, using the definition of $G(F)$, we also know that $(\{n_1,\ldots,n_k\},n) \in G(F)$ if and only if $n \in F(\{n_1,\ldots,n_k\})$. Now [2], tells us that a map $F : \mathcal{P}\mathbb{N} \longrightarrow \mathcal{P}\mathbb{N}$ in **AlgLat** can be defined by how it acts on finite sets, more precisely, $F(U) = \bigcup\{F(A) | A \subseteq_f U\}$. Therefore

$$\begin{aligned} n \in f_{G(F)}(U) &\text{ iff } \exists n_1,\ldots,n_k \in U \text{ s.t. } n \in F(\{n_1,\ldots,n_k\}) \\ &\text{ iff } \exists A \subseteq_f U \text{ s.t. } n \in F(A) \\ &\text{ iff } n \in \bigcup\{F(A)|A \subseteq_f U\} \\ &\text{ iff } n \in F(U) \end{aligned}$$

   Therefore $f_{G(F)}(U) = F(U)$ for every $U \subseteq \mathbb{N}$, and hence $f_{G(F)} = F$ and $H \circ G = Id_{[\mathcal{P}(\mathbb{N}),\mathcal{P}(\mathbb{N})]}$ as required.

4. $H \circ \Gamma(\,\overline{g^{-1}}\,) \circ \Gamma(\psi^{-1}) = fun$:

   First notice the following

$$\begin{aligned} H \circ \Gamma(\,\overline{g^{-1}}\,) \circ \Gamma(\psi^{-1}) &= H \circ \Gamma(\,\overline{g^{-1}} \circ \psi^{-1}) \\ &= H \circ \Gamma(\,\overline{g^{-1}} \circ \overline{\varphi^{-1}}) \\ &= H \circ \Gamma(\,\overline{g^{-1} \circ \varphi^{-1}}) \\ &= H \circ \Gamma(\,\overline{g^{-1} \circ g \circ i}\,) \\ &= H \circ \Gamma(\,\overline{i}\,) \end{aligned}$$

   We now must show that $H \circ \Gamma(\,\overline{i}\,) = fun$. First, we recall that $H = \widetilde{\Gamma(h)}^*$, therefore, for $A \subseteq \mathbb{N}$, $H \circ \Gamma(\,\overline{i}\,) = \widetilde{\Gamma(h)}^* \circ \Gamma(\,\overline{i}\,)$, which is, by definition of $\widetilde{\Gamma(h)}^*$, the map $f_{\Gamma(\,\overline{i}\,)(A)} : \mathcal{P}\mathbb{N} \longrightarrow \mathcal{P}\mathbb{N}$. To show that $H \circ \Gamma(\,\overline{i}\,) = fun$, we must now show that for every $B \subseteq \mathbb{N}$, $f_{\Gamma(\overline{i})(A)}(B) = fun(A)(B)$:

$$\begin{aligned} f_{\Gamma(\overline{i})(A)}(B) &= \widetilde{\Gamma(h)}(\Gamma(\,\overline{i}\,)(A), B) \\ &= \{m \in \mathbb{N} \mid \exists n_1, \ldots, n_k \in B.(\{n_1, \ldots, n_k\}, m) \in \Gamma(\,\overline{i}\,)(A)\} \\ &= \{m \in \mathbb{N} \mid \exists n_1, \ldots, n_k \in B.\exists a_1, \ldots, a_l \in A. \\ &\qquad ([a_1, \ldots, a_l], (\{n_1, \ldots, n_k\}, m)) \in \overline{i}\} \\ &= \{m \in \mathbb{N} \mid \exists n_1, \ldots, n_k \in B.\exists a \in A.([a], (\{n_1, \ldots, n_k\}, m)) \in \overline{i}\} \\ &= \{m \in \mathbb{N} \mid \exists n_1, \ldots, n_k \in B.\exists a \in A.i(a) = (\{n_1, \ldots, n_k\}, m)\} \\ &= \{m \in \mathbb{N} \mid \exists e_n \subseteq B.\exists a \in A.i(a) = (e_n, m)\} \\ &= \{m \in \mathbb{N} \mid \exists e_n \subseteq B.\exists a \in A.\#(n, m) = a\} \\ &= \{m \in \mathbb{N} \mid \exists e_n \subseteq B.\#(n, m) \in A\} \\ &= fun(A)(B) \end{aligned}$$

5. $\Gamma(\psi) \circ \Gamma(\overline{g}) \circ G = graph$:

   Notice

$$\begin{aligned} \Gamma(\psi) \circ \Gamma(\overline{g}) \circ G &= \Gamma(\psi \circ \overline{g}) \circ G \\ &= \Gamma(\overline{\varphi} \circ \overline{g}) \circ G \\ &= \Gamma(\overline{\varphi \circ g}) \circ G \\ &= \Gamma\big(\overline{i^{-1} \circ g^{-1} \circ g}\big) \circ G \\ &= \Gamma\big(\overline{i^{-1}}\big) \circ G \end{aligned}$$

It remains to show that $\Gamma(\overline{i^{-1}}) \circ G = graph$. Let $F \in [\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{N})]$, then

$$
\begin{aligned}
(\Gamma(\overline{i^{-1}}) \circ G)F &= \Gamma(\overline{i^{-1}})(G(F)) \\
&= \{m \in \mathbb{N} \mid \exists (A_1, a_1), \ldots, (A_k, a_k) \in G(F). \\
&\qquad ([(A_1, a_1), \ldots, (A_k, a_k)], m) \in \overline{i^{-1}}\} \\
&= \{m \in \mathbb{N} \mid \exists (A, a) \in G(F).([(A, a)], m) \in \overline{i^{-1}}\} \\
&= \{m \in \mathbb{N} \mid \exists (A, a) \in G(F).i^{-1}((A, a)) = m\} \\
&= \{m \in \mathbb{N} \mid \exists (A, a).a \in F(A) \text{ and } i^{-1}((A, a)) = m\} \\
&= \{m \in \mathbb{N} \mid \exists (A, a).a \in F(A) \text{ and } \#(n, a) = m \text{ where } A = e_n\} \\
&= \{m \in \mathbb{N} \mid \exists (e_n, a).a \in F(e_n) \text{ and } \#(n, a) = m\} \\
&= \{\#(n, a) \mid a \in F(e_n)\} \\
&= graph(F)
\end{aligned}
$$

This discussion is summarized in the following proposition.

Proposition 4.2.1. *The $\mathcal{P}\omega$-model arises from* **MRel**. *This happens by applying of the global sections functor Γ, which lifts a particular isomorphism $\psi : \mathbb{N} \Rightarrow \mathbb{N} \longrightarrow \mathbb{N}$ (defined from coding functions) to* **AlgLat** *as in the figure on the top of p.70.*

4.3 Retractions in MRel give retractions in AlgLat

In the previous section, we were able to describe the graph model from objects and maps in **MRel** using the global sections functor. In fact, we can generalize this result to the following:

Proposition 4.3.1. *Given a retraction $(U \Rightarrow U) \lhd U$ in* **MRel**, *we get a retraction $[\mathcal{P}U, \mathcal{P}U] \lhd \mathcal{P}(U)$ in* **AlgLat**.

Notice that here we are not aiming for a specific retraction pair as we did in section 4.2, therefore we will not encounter the same difficulty of having to move away from finite multisets into finite power sets. The proof of this proposition is similar to the work in section 4.2, with a few subtle differences.

Proof. The idea here is to follow what we had initiallly hoped to do in section 4.2, replacing $\mathbb{N}$ for U, but were not able to do, since the graph model is given in terms of codings of finite sets. Following this process, we have that since Γ is a functor, it preserves retractions. That is to say that if $U \Rightarrow U \lhd U$ via **MRel** maps $\varphi : U \longrightarrow U \Rightarrow U$ and $\psi : U \Rightarrow U \longrightarrow U$, then $\mathcal{P}(U \Rightarrow U) \lhd \mathcal{P}(U)$ via maps $\Gamma(\varphi) : \mathcal{P}(U) \longrightarrow \mathcal{P}(U \Rightarrow U)$ and $\Gamma(\psi) : \mathcal{P}(U \Rightarrow U) \longrightarrow \mathcal{P}(U)$, $\Gamma(\varphi) \circ \Gamma(\psi) = Id_{\mathcal{P}(U \Rightarrow U)}$. Consider the following diagram:

$$\mathcal{P}(U) \underset{\psi}{\overset{\Gamma(\varphi)}{\rightleftarrows}} \mathcal{P}(U \Rightarrow U) \underset{G}{\overset{H}{\rightleftarrows}} [\mathcal{P}U, \mathcal{P}U]$$

To get the required retraction, we need maps $H : \mathcal{P}(U \Rightarrow U) \longrightarrow [\mathcal{P}(U), \mathcal{P}(U)]$ and $G : [\mathcal{P}(U), \mathcal{P}(U)] \longrightarrow \mathcal{P}(U \Rightarrow U)$ such that $H \circ G = Id_{[\mathcal{P}(U),\mathcal{P}(U)]}$. Then

$$\begin{aligned} H \circ \Gamma(\varphi) \circ \Gamma(\psi) \circ G &= H \circ Id_{\mathcal{P}(U \Rightarrow U)} \circ G \\ &= H \circ G \\ &= Id_{[\mathcal{P}U, \mathcal{P}U]} \end{aligned}$$

as required. To define H we start with the evaluation map $ev_{U,U} : (U \Rightarrow U)\&U \longrightarrow U$ in **MRel** which we will denote ev for the rest of this section. Recall that

$$\begin{aligned} ev_{A,A} &= \{([(1, ([a_1, \ldots, a_k], n)), (2, a_1), \ldots, (2, a_k)], n) \mid a_i, n \in A\} \\ &\cong \{(([(\alpha, n)], \alpha), n) \mid \alpha \in \mathcal{M}_f(A), n \in A\} \end{aligned}$$

We then apply the global sections functor Γ to ev to get the map $\Gamma(ev) : \mathcal{P}((U \Rightarrow U)\&U) \longrightarrow \mathcal{P}(U)$ in the category of algebraic lattices, where for $A \subseteq (U \Rightarrow U)\&U$

$$\Gamma(ev)(A) = \{u \in U \mid \exists a_1, \ldots, a_k \in A \text{ s.t. } ([a_1, \ldots, a_k], n) \in ev\}$$

Since Γ preserves products, we get a map $\widetilde{\Gamma}(ev) : \mathcal{P}(U \Rightarrow U) \times \mathcal{P}(U) \longrightarrow \mathcal{P}(U)$ in **AlgLat**, where for $(A, B) \in \mathcal{P}(U \Rightarrow U) \times \mathcal{P}(U)$, so $A \subseteq U \Rightarrow U = \mathcal{M}_f(U) \times U$ and $B \subseteq U$, then

$$\widetilde{\Gamma(ev)}(A, B) = \{u \in U \mid \exists b_1, \ldots, b_k \in B \text{ s.t. } ([b_1, \ldots, b_k], u) \in A\}$$

Now since this is a map in **AlgLat** and **AlgLat** is a cartesian closed category, we can form the map $\widetilde{\Gamma(ev)}^* : \mathcal{P}(U \Rightarrow U) \longrightarrow [\mathcal{P}U, \mathcal{P}U]$, where for $A \subseteq U \Rightarrow U$, $\widetilde{\Gamma(h)}^*(A) = f_A$. Here $f_A \in [\mathcal{P}U, \mathcal{P}U]$ and for $B \subseteq \mathbb{N}$, $f_A(B) = \widetilde{\Gamma(h)}(A, B)$. Let $H = \widetilde{\Gamma(ev)}^*$. To define G we will need to use the map $q_A : \mathcal{M}_f(A) \longrightarrow \mathcal{P}(A)$, for A a set. For

$\alpha \in \mathcal{M}_f(A)$, with say $\alpha = \{(a_1, m(a_1)), \ldots, (a_k, m(a_k))\}$, $q(\alpha) = \{a_1, \ldots, a_k\}$. For a map $F \in [\mathcal{P}U, \mathcal{P}U]$, let

$$G(F) = \{(\alpha, u) \mid u \in F(q(\alpha)), \alpha \in \mathcal{M}_f(U)\}$$

To finish the proof we must check that f_A and G are both maps in **AlgLat**, these two verifications are omitted since they follow almost exactly the proof in 4.2. Finally, we must check that $H \circ G = Id_{[\mathcal{P}U,\mathcal{P}U]}$. Let $F \in [\mathcal{P}U, \mathcal{P}U]$, then

$$\begin{aligned}(H \circ G)F &= H(G(F)) \\ &= \widetilde{\Gamma(ev)}^*(G(F)) \\ &= f_{G(F)}\end{aligned}$$

Therefore, to show that $H \circ G = Id_{[\mathcal{P}U,\mathcal{P}U]}$, we must show that $f_{G(F)} = F$. Let $B \subseteq U$, then

$$f_{G(F)}(B) = \{u \in U \mid \exists b_1, \ldots, b_k \in B \text{ s.t. } ([b_1, \ldots, b_k], u \in G(F)\}$$

Then

$$\begin{aligned} n \in f_{G(F)}(B) \quad &\text{iff} \quad \exists b_1, \ldots, b_k \in B \text{ s.t. } ([b_1, \ldots, b_k], u) \in G(F) \\ &\text{iff} \quad \exists b_1, \ldots, b_k \in B \text{ s.t. } u \in F(q([b_1, \ldots, b_k])) \\ &\text{iff} \quad \exists b_1, \ldots, b_{k'} \in B \text{ s.t. } u \in F(\{b_1, \ldots, b_{k'}\}) \\ &\text{iff} \quad \exists B' \subseteq_f B \text{ s.t. } u \in F(B') \\ &\text{iff} \quad u \in F(B) \text{ since } F(B) = \cup\{F(B') \mid B' \subseteq_f B\} \end{aligned}$$

Hence $f_{G(F)} = F$ and therfore $H \circ G = Id_{[\mathcal{P}U,\mathcal{P}U]}$. □

Remark 4.3.2. Applying the above to $\mathbb{N}$ will not yield the same retraction pair used in the graph model. This is why we could not use this proposition to show that we can get the graph model from **MRel**.

4.4 Investigating the existence or non-existence of PfRel

In section 4.2, when trying to find suitable maps to get the $\mathcal{P}\omega$-model from **MRel** using the global sections functor, we noticed that if there were a category analog to **MRel** but built from the finite power set functor, call it **PfRel**, and if this category **PfRel** did indeed have analogous structure to that of **MRel**, then one of the maps we were looking for, namely $H : \mathcal{P}(\mathcal{P}_f\mathbb{N} \times \mathbb{N}) \longrightarrow [\mathcal{P}\mathbb{N}, \mathcal{P}\mathbb{N}]$, would arise from an evaluation map from within **PfRel**. This section will discuss whether or not this category exists and if it can be constructed as **MRel** was. In other words, whether or not this **PfRel** is the coKleisli category of some monad.

First, we try to define a category analogous to to **MRel** but using the finite power set functor mimicking the structure of **MRel**. We will, for now, call this **PfRel**. Let the objects of **PfRel** be sets and for sets X, Y, an arrow $X \xrightarrow{f} Y \in$ **PfRel** corresponds to a relation from $\mathcal{P}_f(X)$ to Y. Therefore f can be thought of as a subset of $\mathcal{P}_f(X) \times Y$. For a set A, the identity map $Id_X : X \longrightarrow X$ will the following subset of $\mathcal{P}_f(X) \times X$:

$$Id_X = \{(\{x\}, x) \mid x \in X\}$$

For sets X, Y, Z, and maps $f : X \longrightarrow Y, g : Y \longrightarrow Z \in$ **PfRel**, define $g \circ f$ to be the following subset of $\mathcal{P}_f(X) \times Z$:

$$g \circ f = \{(A, z) \mid \exists (A_1, y_1), \ldots, (A_k, y_k) \in f. A = A_1 \cup \cdots \cup A_k \text{ and } (\{y_1, \ldots, y_k\}, z) \in g\}$$

We will now check if this category is indeed a category:

First, we must check that composition in **PfRel** is associative: Let $f : X \longrightarrow Y, g : Y \longrightarrow Z, h : Z \longrightarrow W \in$ **PfRel**, we must show that $h \circ (g \circ f) = (h \circ g) \circ f$.

$$\begin{aligned} h \circ (g \circ f) &= \{(A, w) \mid \exists (A_1, z_1), \ldots, (A_k, z_k) \in g \circ f. A = \cup A_i \text{ and } (\{z_1, \ldots, z_k\}, w) \in h\} \\ (h \circ g) \circ f &= \{(A, w) \mid \exists (A_1, y_1), \ldots, (A_{k,y} k) \in f. A = \cup A_i \text{ and } (\{y_1, \ldots, y_k\}, w) \in g \circ h\} \end{aligned}$$

Now consider the following

$$\begin{aligned} (A, w) \in h \circ (g \circ f) \quad &\text{iff} \quad \exists (A_1, z_1), \ldots, (A_k, z_k) \in g \circ f \text{ s.t. } A = \cup A_i \\ &\qquad \text{and } (\{z_1, \ldots, z_k\}, w) \in h \\ &\text{iff} \quad \exists (A_i^1, y_i^1), \ldots, (A_i^{l_i}, y_i^{l_i}) \in f \text{ s.t. } A_i = \textstyle\bigcup_{j=1}^{l_i} A_i^j \\ &\qquad \text{and } (\{y_i^1, \ldots, y_i^{l_i}\}, z_i) \in g \text{ for } i = 1, \ldots k\ , \\ &\qquad \text{and } A = \cup A_i,\ (\{z_1, \ldots, z_k\}, w) \in h \\ &\text{iff} \quad \exists (A_i^1, y_i^1), \ldots, (A_i^{l_i}, y_i^{l_i}) \in f \text{ for } i = 1, \ldots k \text{ s.t. } A = \textstyle\bigcup_{i=1}^{k} \bigcup_{j=1}^{l_i} A_i^j \\ &\qquad \text{and } (\{y_1^1, \ldots, y_1^{l_1}, \ldots, y_k^1, \ldots, y_k^{l_k}\}, w) \in h \circ g \\ &\text{iff} \quad (A, w) \in (h \circ g) \circ f \end{aligned}$$

Therefore $h \circ (g \circ f) = (h \circ g) \circ f$ as required, Now, we check that for a map $f : X \longrightarrow Y \in$ **PfRel**, $f \circ Id_X = f = Id_Y \circ f$:

By definition of composition we have the following

$$\begin{aligned} f \circ Id_X &= \{(A, y) \mid \exists (A_1, x_1), \ldots, (A_k, x_k) \in Id_X . A = \cup A_i \text{ and } (\{x_1, \ldots, x_k\}, y) \in f\} \\ &= \{(A, y) \mid \exists x_1, \ldots, x_k \in A . A = \{x_1, \ldots, x_k\} \text{ and } (A, y) \in f\} \\ &= f \end{aligned}$$

as required, but we now check composition with the identity on the left. Here we run into a problem.

$$Id_Y \circ f = \{(A, y) \mid \exists (A_1, y_1), \ldots, (A_k, y_k) \in f . A = \cup A_i \text{ and } (\{y_1, \ldots, y_k\}, y) \in Id_B\}$$

Suppose now that we have a map f such that $(\{x_1\}, y), (\{x_2\}, y) \in f$ for $x_1, x_2 \in X$ and $y \in Y$. Then there exists elements in f, $((\{x_1\}, y), (\{x_2\}, y) \in f$, such that $A = \{x_1\} \cup \{x_2\}$ and $(\{y, y\}, y) \in Id_B$. Notice that $(\{y, y\}, y) \in Id_B$ since $\{y, y\} = \{y\}$ and $(\{y\}, y) \in Id_B$. Hence the pair $(\{x_1, x_2\}, y) \in Id_Y \circ f$, but $(\{x_1, x_2\}, y) \notin f$. Therefore $Id_Y \circ f \neq f$. We have that composition is associative but the the identity morphism as defined above is not a true identity. **PfRel**, as defined, is therefore a semi-category, not a category.

Our attempt at defining **PfRel** directly has failed. Taking a different approach, we would like to construct a category following the steps taken to construct **MRel**, but now using $\mathcal{P}_f$ instead of $\mathcal{M}_f$. We know from Example 2.3.9 that we can define η and μ such that $(\mathcal{P}_f, \eta, \mu)$ is a monad on **Set**. To construct a category from this monad as we did for **MRel**, we would need a distributive law $l : \mathcal{P}_f \mathcal{P} \longrightarrow \mathcal{P}\mathcal{P}_f$ of the power set functor over the finite power set functor. With this we would be able to lift $(\mathcal{P}_f, \eta, \mu)$ to a monad on **Rel**, say $(\hat{\mathcal{P}}_f, \eta, \mu)$ and since **Rel** is self-dual, a comonad $(\hat{\mathcal{P}}_f, \epsilon, \delta)$ on **Rel**. Then we could form the coKleisli category of this comonad $coKl(\hat{\mathcal{P}}_f)$. If so, then we would have that this category $coKl(\hat{\mathcal{P}}_f)$ is a cartesian closed category since it is clear that $\mathcal{P}_f$ also satisfies the Seely isomorphism:

$$\mathcal{P}_f(A \& B) \cong \mathcal{P}_f(A) \times \mathcal{P}_f(B)$$

where & stands for disjoint union, the categorical product of **Rel**.

Suppose we could define a distributive law of the power set functor over the finite power set functor and suppose the coKleisli category of the finite power set functor lifted to **Rel**, $coKl(\hat{\mathcal{P}}_f)$, had analogous structure to **MRel**, i.e. the structure defined for **PfRel**. Then the H map needed in section 4.2 would arise from an evaluation map in **PfRel** using a global sections functor for **PfRel**.

A distributive law of the power set functor over the finite power set functor is a natural transformation $l : \mathcal{P}_f \circ \mathcal{P} \longrightarrow \mathcal{P} \circ \mathcal{P}_f$ by the family

$$\{l_X : \mathcal{P}_f\mathcal{P}(X) \longrightarrow \mathcal{P}\mathcal{P}_f(X) \mid X \text{ a set}\}$$

such that the following four diagrams commute:

$$\begin{CD}
\mathcal{P}_f\mathcal{P}\mathcal{P}(X) @>{l_{\mathcal{P}(X)}}>> \mathcal{P}\mathcal{P}_f\mathcal{P}(X) @>{\mathcal{P}(l_X)}>> \mathcal{P}\mathcal{P}\mathcal{P}_f(X)\\
@V{\mathcal{P}_f(\mu^{\mathcal{P}}_X)}VV @. @VV{\mu^{\mathcal{P}}_{\mathcal{P}_f(X)}}V\\
\mathcal{P}_f\mathcal{P}(X) @>>{l_X}> \cdots @>>> \mathcal{P}\mathcal{P}_f(X)
\end{CD}$$

$$\begin{CD}
\mathcal{P}_f\mathcal{P}_f\mathcal{P}(X) @>{\mathcal{P}_f(l_X)}>> \mathcal{P}_f\mathcal{P}\mathcal{P}_f(X) @>{l_{\mathcal{P}_f(X)}}>> \mathcal{P}\mathcal{P}_f\mathcal{P}_f(X)\\
@V{\mu^{\mathcal{P}_f}_{\mathcal{P}(X)}}VV @. @VV{\mathcal{P}(\mu^{\mathcal{P}_f}_X)}V\\
\mathcal{P}_f\mathcal{P}(X) @>>{l_X}> \cdots @>>> \mathcal{P}\mathcal{P}_f(X)
\end{CD}$$

$$\begin{array}{ccc}
 & \mathcal{P}_f(X) & \\
\swarrow{\scriptstyle \mathcal{P}_f(\eta^{\mathcal{P}}_X)} & & {\scriptstyle \eta^{\mathcal{P}}_{\mathcal{P}_f(X)}}\searrow \\
\mathcal{P}_f\mathcal{P}(X) & \xrightarrow[l_X]{} & \mathcal{P}\mathcal{P}_f(X)
\end{array}$$

$$\begin{array}{ccc}
 & \mathcal{P}(X) & \\
\swarrow{\scriptstyle \eta^{\mathcal{P}_f}_{\mathcal{P}(X)}} & & {\scriptstyle \mathcal{P}(\eta^{\mathcal{P}_f}_X)}\searrow \\
\mathcal{P}_f\mathcal{P}(X) & \xrightarrow[l_X]{} & \mathcal{P}\mathcal{P}_f(X)
\end{array}$$

where X is a set.

Suppose we define l as follows: for a set X, let $l_X : \mathcal{P}_f\mathcal{P}(X) \longrightarrow \mathcal{P}\mathcal{P}_f(X)$, where $l_X(\{A_1, \ldots, A_k\}) = \{\{a_1, \ldots, a_k\} \mid a_i \in A_i\}$, $A_i \subseteq X$. To show that this is a distributive law, we need to show that it is a natural transformation and that it makes the four diagrams above commute. As defined, l is a natural transformation: Let $f : X \longrightarrow Y \in$ **Set**, we must show that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}_f\mathcal{P}(X) & \xrightarrow{l_X} & \mathcal{P}\mathcal{P}_f(X) \\ {\scriptstyle \mathcal{P}_f\mathcal{P}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{P}\mathcal{P}_f(f)} \\ \mathcal{P}_f\mathcal{P}(Y) & \xrightarrow{l_Y} & \mathcal{P}\mathcal{P}_f(Y) \end{array}$$

Let $\{A_1, \ldots, A_k\} \in \mathcal{P}_f\mathcal{P}(X)$ then we have

$$\begin{aligned} (\mathcal{P}\mathcal{P}_f(f) \circ l_X)\{A_1, \ldots, A_n\} &= \mathcal{P}\mathcal{P}_f(f)(\{\{a_1, \ldots, a_n\} \mid a_i \in A_i\}) \\ &= \{\mathcal{P}_f(f)(\{a_1, \ldots, a_n\}) \mid a_i \in A_i\}) \\ &= \{\{f(a_1), \ldots, f(a_n)\}) \mid a_i \in A_i\}) \\ &= \{\{b_1, \ldots, b_n\}) \mid b_i \in f[A_i]\}) \\ &= l_Y(\{f[A_1], \ldots, f[A_n]\}) \\ &= l_Y(\{\mathcal{P}(f)(A_1), \ldots, \mathcal{P}(f)(A_n)\}) \\ &= (l_Y \circ \mathcal{P}_f\mathcal{P}(f))\{A_1, \ldots, A_n\} \end{aligned}$$

as required. But when checking if the four diagrams commute we run into some problems, namely with the multiplicative rules: Consider the first diagram. Let X be a set with at least three distinct elements $x, y, z \in X$. Define $U, V \subseteq X$ as follows

$$U = \{\{x, y\}, \{z\}\}, V = \{\{x\}, \{y, z\}\}$$

Then $\{U, V\} \in \mathcal{P}_f\mathcal{P}\mathcal{P}(X)$. In one direction, we have the following

$$\begin{aligned} (l_X \circ \mathcal{P}_f(\mu_X^{\mathcal{P}})(\{U, V\}) &= l_X(\{\mu_X^{\mathcal{P}}(U), \mu_X^{\mathcal{P}}(V)\}) \\ &= l_X(\{\{x, y, z\}, \{x, y, z\}\}) \\ &= l_X(\{\{x, y, z\}\}) \\ &= \{\{x_1\} \mid x_1 \in \{x, y, z\}\} \\ &= \{\{x\}, \{y\}, \{z\}\} \\ & \quad LHS \end{aligned}$$

Now, in the other direction we have

$$\begin{aligned}
(\mu^{\mathcal{P}}_{\mathcal{P}_f(X)} \circ \mathcal{P}(l_X) \circ l_{\mathcal{P}(X)})(\{U, V\}) &= (\mu^{\mathcal{P}}_{\mathcal{P}_f(X)} \circ \mathcal{P}(l_X))(\{\{x_1, x_2\} \mid x_1 \in U, x_2 \in V\}) \\
&= (\mu^{\mathcal{P}}_{\mathcal{P}_f(X)} \circ \mathcal{P}(l_X))(\Big\{\{\{x, y\}, \{x\}\}, \{\{x, y\}, \{y, z\}\}, \dots \Big\}) \\
&= (\mu^{\mathcal{P}}_{\mathcal{P}_f(X)}(\Big\{l_X(\{\{x, y\}, \{x\}\}), l_X(\{\{x, y\}, \{y, z\}\}), \dots \Big\}) \\
&= (\mu^{\mathcal{P}}_{\mathcal{P}_f(X)}(\{\{\{x_1, x_2\} \mid x_1 \in \{x, y\}, x_2 \in \{x\}\}, \dots\} \\
&= (\mu^{\mathcal{P}}_{\mathcal{P}_f(X)})\{\{\{x\}, \{y, x\}\}, \dots\} \\
&= \{\{x\}, \{y, x\}, \dots\} \\
&\quad RHS
\end{aligned}$$

This shows that the RHS contains at least one element, $\{y, x\}$, that is not in the LHS. Therefore the first diagram does not commute for every set X, and so l cannot be a distributive law of the finite power set monad over the power set monad.

Another attempt at defining a distributive law, taken from [10], is to define l as follows: For X a set,

$$\begin{aligned}
l_X(\{A_1, \dots, A_k\}) &= \{\{b_1, \dots, b_l\} \mid \forall j = 1, \dots, l \exists 0 \leq i \leq k.b_j \in A_i \\
&\qquad \text{and } \forall i = 1, \dots, k \exists 0 \leq j \leq l.b_j \in A_i\}
\end{aligned}$$

But, it is shown in [10] that this does not yield a distributive law of monads, it is something weaker: a distributive law of endofunctor $\mathcal{P}_f$ over monad $\mathcal{P}$. This paper also goes on to build a semi-category from a Kleisli construction using this distributive law of endofunctors.

The failure of these two trial distributive laws of monads in no way proves that there is no such distributive law. It is commonly believed that there is no distributive law of the power set monad over the finite power set monad, although, there is no proof of this found in the literature.

5 Conclusions

This thesis has presented the background needed to understand the details of the construction of **MRel** as presented in [7]. This included a detailed treatment of finite multisets, an introduction to the lambda calculus and its models and a detailed presentation of the construction of the category in question. With this understanding of the construction of **MRel**, we are able to investigate several aspects of its structure. **MRel** is a cartesian closed category with a reflexive object, we are therefore able to model the untyped lambda calculus in **MRel**. Another aspect of its structure is that the terminal object **1** is not a generator: no of **MRel** has enough points. Combining these two facts, we have an example of a model of the untyped lambda calculus built from an object without enough points. In [7] and in this thesis, it is shown that the hypothesis that U has enough points is unnecessary: in **MRel** no object has enough points and yet there is still a reflexive object that models the lambda calculus. We have gone on to show that we are able to build a well known model, the graph model, a model whose reflexive object does have enough points, from objects and morphisms in **MRel**. This reinforces the fact that it is not required to have enough points to be able to model the lambda calculus. We have also shown that, in general, a reflexive object in **MRel** gives rise to a reflexive object in **AlgLat**.

This research could be extended by taking a closer look at the existence or non-existence of a distributive law of the power set monad over the finite power set monad, possibly finding a proof that there is no such distributive law. Another possible direction would be to define a congruence relation on maps in **MRel** in such a way that taking the quotient category might give us a category similar to what we would like **PfRel** to be.

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