ISOMORPHISM THEOREMS IN CATEGORIES

A thesis submitted

by

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to

the Faculty of Science and Engineering

of the University of Ottawa

in partial fulfillment of the requirements

for the degree of

Master of Science

in the subject of

Mathematics

May 1972

ACKNOWLEDGEMENT

I want to thank Dr. Walter Burgess, my supervisor, who suggested the topic for this thesis, for his help throughout its development.

ABSTRACT

In this thesis, a First Isomorphism Theorem for congruence relations is formulated and proved in a very general type of category, which includes algebraic categories and varietal categories in the sense of Lawvere and Linton, respectively. Also the concept of a quotient map is considered. Moreover, a generalization of the concept of a normal subobject is given and the First Isomorphism Theorem is proved for these normal subobjects.

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INTRODUCTION

Some of the basic theorems of the theory of groups are traditionally called the "isomorphism theorems". Most of them have a natural generalization to arbitrary theories of algebraic systems where the axioms are equations (equationally defined classes of algebras). This generalization can be done by substituting for the concept of a normal subgroup that of a congruence relation in an algebraic system. A congruence relation in the algebraic system S is an equivalence relation which is congruent with the operations of S. In other words, it is an equivalence relation which is, at the same time, a subalgebra of the cartesian product $S \times S$ with the component-wise operations. In the case of groups, the two concepts give the same results because there is a one-to-one correspondence between the normal subgroups and the congruence relations of a group G, defined by $(x,y) \in R_N \Leftrightarrow xy^{-1} \in N$. But in the case of semigroups, for example, not all congruence relations are related to sub-semigroups.

The purpose of this thesis is to formulate the "First Isomorphism Theorem" in the language of categories, and to show that it holds in a very general type of category; even more general than the "algebraic categories", defined by Lawvere to replace the concept of equationally defined classes of algebras. Moreover, a definition of normal subobject is introduced, which is a generalization of the concept of normal in categories with a zero-object, and permits us to prove the theorem in terms of normal subobjects.

The first chapter presents most of the notions of category theory that are used in the sequel; i.e., order and equivalence of subobjects, and some special finite limits and colimits.

In the second chapter, the notions of congruence relation and quotient are introduced; and it is shown that if equivalent subobjects are identified, then the congruence relations in an object A form a lattice. Of course, this can be a "big" lattice in the sense that it is not necessarily a set.

The third chapter begins by introducing sufficient conditions for the main theorem and showing that they apply to the case of algebraic and varietal categories. This last kind of category being a categorical formulation of equationally defined algebraic systems with infinitary operations. Theorem 3.1 shows the existence of images under these conditions, and generalizes the "Fundamental Homomorphism Theorem", which in groups is expressed by the formula:

$$A/\mathrm{Ker}(f) \simeq \mathrm{Im}(f)$$

Theorem 3.5 corresponds to the isomorphism theorem which in groups is expressed by the formula:

$$(A/N)(M/N) \simeq A/M$$

where M and N are normal subgroups of A and $N \subset M \subset A$. Finally, Theorem 3.6 corresponds to the First Isomorphism Theorem, which in groups says that there is a lattice isomorphism between the normal subgroups of A containing a normal subgroup N and the normal subgroups of A/N.

In the fourth chapter, a definition of normal subobject is introduced which makes sense even if the category does not have a zero-object. The First Isomorphism Theorem is proved for normal subobjects, and it is shown that this definition is a generalization of the one usually given in categories with a zero-object.

The formulation of the First Isomorphism Theorem in Chapter III is a generalization of a well known theorem in algebra but it is original and does not appear elsewhere in the literature. Also Chapter IV contains only original results.

Chapter I

PRELIMINARY NOTIONS

We suppose that the definition of a category is already known. We use the symbols "$\alpha : A \to B$" or "$A \xrightarrow{\alpha} B$" to indicate that α is a map with domain A and codomain B. Given maps $\alpha : A \to B$ and $\beta : B \to C$, we use the notation $\beta\alpha$ to denote the composition $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$. The identity map of an object A will be denoted by 1_A. A map α is a monomorphism if $\alpha f = \alpha g$ implies $f = g$. It is an epimorphism if $f\alpha = g\alpha$ implies $f = g$. We say also that α is mono or α is epi, respectively. Finally, a map $\alpha : A \to B$ is an isomorphism if there exists a map $\alpha^{-1} : B \to A$ such that $\alpha^{-1}\alpha = 1_A$ and $\alpha\alpha^{-1} = 1_B$. From these definitions, we obtain easily the following properties. Suppose that the composition $\beta\alpha$ is defined, then

(1) α is mono and β is mono $\Rightarrow$ $\beta\alpha$ is mono.

(2) α is epi and β is epi $\Rightarrow$ $\beta\alpha$ is epi.

(3) $\beta\alpha$ is mono $\Rightarrow$ α is mono.

(4) $\beta\alpha$ is epi $\Rightarrow$ β is epi.

(5) α is an isomorphism $\Rightarrow$ α is mono and epi.

The converse of (5) is true in the Category of Sets, but it does not hold for arbitrary categories.

1. Subobjects.

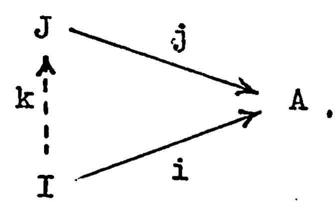
Let A be a fixed object in a category. A monomorphism with codomain A is called a *subobject of* A. In particular, 1_A is a subobject of A. We define the following relations between the subobjects of A.

DEFINITION. Let i and j be subobjects of A.

(a) i *is contained in* j, written $i \subset j$, iff there exists a map k such that $i = jk$.

(b) When this map k is an isomorphism, we say that i *is equivalent to* j, written $i \simeq j$.

This definition implies the existence of a commutative diagram



It is clear that the relation "$\simeq$" is an equivalence relation, so it divides the class of subobjects of A into equivalence classes. Moreover, the following properties can be easily obtained:

(1) $i \subset i$

(2) $i \subset j$ and $j \subset k \Rightarrow i \subset k$

(3) $i \subset j$ and $j \subset i \Leftrightarrow i \simeq j$

(1), (2), and the implication " $\Leftarrow$ " of (3) are trivial. To show the other implication of (3), observe that if $i = jk$ and $j = ik'$, then $i = ik'k$ and $j = jkk'$. Since i and j are mono, this means that $k'k = 1_I$ and $kk' = 1_J$, so $i \simeq j$. These properties express the fact that the inclusion relation "$\subset$", which is only a partial pre-order in the class of subobjects of A, becomes a partial order for the equivalence classes of subobjects by the definition:

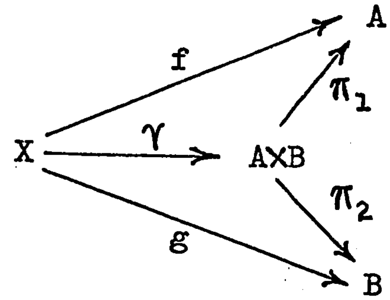
$$\bar{i} \subset \bar{j} \Leftrightarrow i \subset j$$

($\bar{i}$ and $\bar{j}$ denote the equivalence classes of i and j, respectively).

It is important to note that if $i \simeq j$, then the domains of i and j are isomorphic, because the map $k : I \to J$ is an isomorphism. However, the converse is not true, we can have an isomorphism $k : I \to J$ which does not induce an equivalence of subobjects.

2. Products.

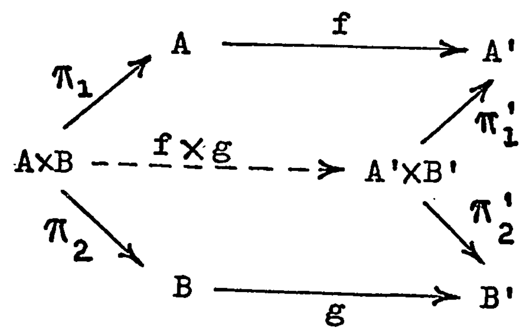
Let A and B be objects in a category. A product of A and B is a triple $(A \times B, \pi_1, \pi_2)$ where $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ are maps in the category and the following property holds. Given any pair of maps $f : X \to A$ and $g : X \to B$, there exists a unique map $\gamma : X \to A \times B$ such that $f = \pi_1\gamma$ and $g = \pi_2\gamma$. This means that the following diagram commutes.



The maps π_1 and π_2 are called projections. From this definition it follows that a triple (X, f, g) is also a product of A and B if and only if the induced map γ is an isomorphism. The "if" part is clear, to show the other direction observe that there exists a unique map $\phi : A \times B \to X$ such that $\pi_1 = f\phi$ and $\pi_2 = g\phi$. Therefore, $\pi_1 = \pi_1\gamma\phi$; but there is a unique map, namely the identity of $A \times B$, such that $\pi_1 = \pi_1 1_{A\times B}$ and $\pi_2 = \pi_2 1_{A\times B}$. This means that $\gamma\phi = 1_{A\times B}$. In the same way we show that $\phi\gamma = 1_X$.

The definition of product does not imply its existence. If every pair of objects in a category has a product, we say that the category has finite products, because we can form the product of a finite family of objects by induction.

Given a pair of maps $f : A \to A'$ and $g : B \to B'$, and products $(A \times B, \pi_1, \pi_2)$ and $(A' \times B', \pi_1', \pi_2')$, we have that the maps $f\pi_1$ and $f\pi_2$ induce a unique map $f \times g : A \times B \to A' \times B'$ making the following diagram commutative



This "product" of functions has the following property:

Consider maps $f : A \to A'$, $f' : A' \to A''$, $g : B \to B'$, and $g' : B' \to B''$, then $(f' \times g')(f \times g) = (f'f) \times (g'g)$.

This follows from the uniqueness of the map $(f'f)\times(g'g)$. The following result is also important.

LEMMA 1.1. If f and g are mono, then $f \times g$ is mono.

<u>Proof</u>. Take a pair of maps $a, b : X \to A \times B$ such that $(f \times g)a = (f \times g)b$, then $f\pi_1 a = \pi_1'(f \times g)a = \pi_1'(f \times g)b = f\pi_1 b$. Since f is mono, this implies $\pi_1 a = \pi_1 b$. Similarly, using that g is mono, we obtain that $\pi_2 a = \pi_2 b$. By definition of product there is a unique map $\gamma : X \to A \times B$ such that $\pi_1\gamma = \pi_1 a$ and $\pi_2\gamma = \pi_2 a$. Hence, $a = b = \gamma$. Q.E.D.

We shall be mainly interested in the case of the product of an object with itself. We use the notation A^2 for $A \times A$, and we will write f^2 instead of $f \times f$. With this notation we have the following

COROLLARY 1.2. If a category has finite products,

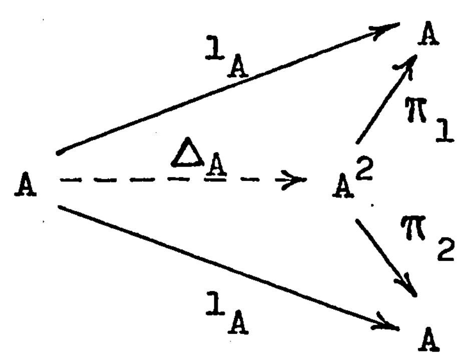
(a) $(fg)^2 = f^2 g^2$

(b) f is mono $\Rightarrow$ f^2 is mono

(c) $1_{A^2} = (1_A)^2$

Observe that the last property follows from the others. From the second property we can see that any subobject of A, $i : I \to A$, induces a subobject of A^2, $i^2 : I^2 \to A^2$. There is an important subobject of A^2 that we now define.

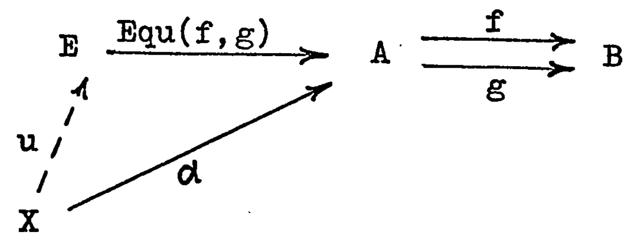
The diagonal of A, written Δ_A, is the unique map that makes the following diagram commutative.



Since 1_A is mono and $1_A = \pi_1 \Delta_A$, then Δ_A is mono.

3. Equalizers.

Let f and g be two maps with common domain and codomain f, g : A → B. An equalizer of f and g is a map Equ(f, g) : E → A such that f Equ(f, g) = g Equ(f, g), and for any other map α, the equality fα = gα implies the existence of a unique map u such that α = Equ(f, g)u.

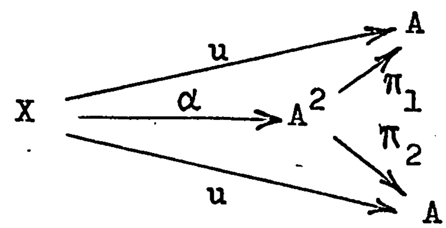


Suppose that Equ(f, g) exists; in analogy with the case of products, we can prove that map α is also an equalizer of f and g if and only if the induced map u is an isomorphism. Since the following lemma proves that an equalizer is mono, this amounts to saying that Equ(f, g) and α are equivalent subobjects of A.

LEMMA 1.3. An equalizer is mono.

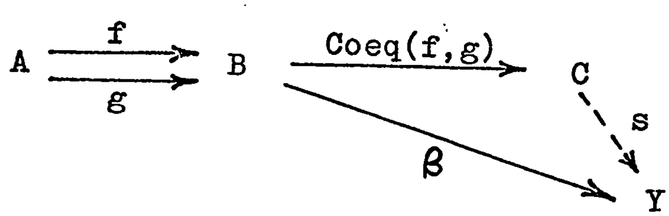
Proof. Let e = Equ(f, g) and let a and b be a pair of maps such that ea = eb. If we call α = ea = eb, then fα = gα. This implies the existence of a unique map u such that α = eu. Hence a = b = u. Q.E.D.

EXAMPLE. Consider the product (A^2, π_1, π_2), then Δ_A is an equalizer of π_1 and π_2. It is clear that $\pi_1\Delta_A = \pi_2\Delta_A = 1_A$. Moreover, if α is a map such that $\pi_1\alpha = \pi_2\alpha$, call $u = \pi_1\alpha = \pi_2\alpha$. By definition of product, α is the unique map making this diagram commutative



But we also have that $\pi_1(\Delta_A u) = 1_A u = u$ and $\pi_2(\Delta_A u) = 1_A u = u$. Hence $\alpha = \Delta_A u$. That u is the unique map such that $\alpha = \Delta_A u$ comes from the fact that Δ_A is mono.

The notion of coequalizer is dual to that of equalizer. Given a pair of maps $A \overset{f}{\underset{g}{\rightrightarrows}} B$, <u>a coequalizer of</u> f <u>and</u> g is a map $\mathrm{Coeq}(f, g) : B \to C$ such that $\mathrm{Coeq}(f, g)f = \mathrm{Coeq}(f, g)g$, and for any map β such that $\beta f = \beta g$, there exists a unique map s for which $\beta = s\,\mathrm{Coeq}(f, g)$.

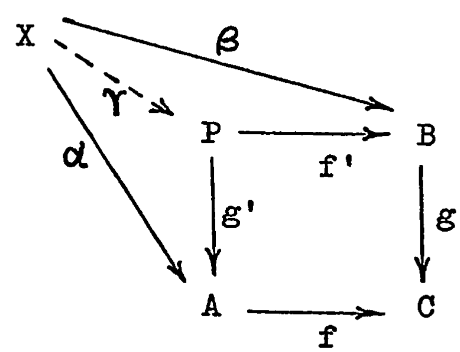


As before, we have that β is also a coequalizer of f and g if and only if the map s is an isomorphism. Moreover, a coequalizer is epi.

EXAMPLE. 1_B is a coequalizer of the pair of maps $A \overset{f}{\underset{f}{\rightrightarrows}} B$.

4. Pullbacks and Intersections.

Given a pair of maps with a common codomain, $f : A \to C$ and $g : B \to C$, consider a pair of maps $f' : P \to B$ and $g' : P \to A$ such that $fg' = gf'$. This pair is called a pullback for f and g if for any pair of maps $\beta : X \to B$ and $\alpha : X \to A$ such that $f\alpha = g\beta$, there exists a unique map $\gamma : X \to P$ for which $\beta = f'\gamma$ and $\alpha = g'\gamma$. The following diagram illustrates the situation.



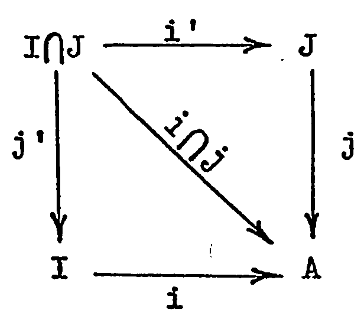
We say also that the square APBC is a pullback diagram. From the definition, it follows that the pairs (f', g') and(β, α) are both pullbacks of the pair (f, g) if and only if the induced map γ is an isomorphism.

LEMMA 1.4. In a pullback diagram as described above, if the map f is mono, then the map f' is mono.

<u>Proof</u>. Suppose that f is mono. If $f'r = f's$, then $fg'r = gf'r = gf's = fg's$ and so $g'r = g's$. Let $\beta = f'r = f's$ and $\alpha = g'r = g's$. Clearly, $f\alpha = fg'r = gf'r = g\beta$. Therefore, there exists a unique γ such that $\beta = f'\gamma$ and $\alpha = g'\gamma$. This means that $r = s = \gamma$. Q.E.D.

Observe that if both maps f and g happen to be mono, then the composite map $gf' = fg'$ will be mono. This permits us to make the following definition.

DEFINITION. Let $i : I \to A$ and $j : J \to A$ be subobjects of A. <u>An intersection of</u> i <u>and</u> j, written $i \cap j : I \cap J \to A$, is the diagonal map of a pullback diagram for i and j.



If the intersection of two subobjects exists, it is unique, up to equivalence of subobjects. The following lemma shows that the equivalence class $\overline{i \cap j}$ is a greatest lower bound or <u>infimum</u> for the equivalence classes $\overline{i}$ and $\overline{j}$.

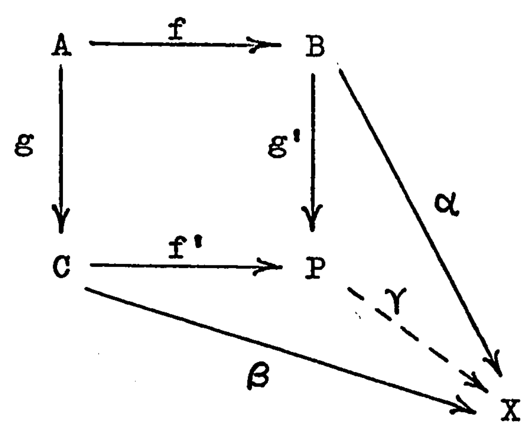
LEMMA 1.5. Suppose that the subobjects i and j of A have an intersection $i \cap j$, then

(a) $i \cap j \subset i$ and $i \cap j \subset j$

(b) for any subobject k of A, $k \subset i$ and $k \subset j \Rightarrow k \subset i \cap j$.

<u>Proof</u>. (a) is trivial. To show (b) observe that $k \subset i$ and $k \subset j$ imply $k = i\alpha = j\beta$ for certain maps α and β. By definition of pullback, there exists a unique map γ such that $\alpha = j'\gamma$ and $\beta = i'\gamma$. In particular, this means that $k = i\alpha = ij'\gamma = (i \cap j)\gamma$. Therefore $k \subset i \cap j$. Q.E.D.

The dual notion of a pullback is that of a <u>pushout</u>. The pair of maps f' and g' in the following diagram is <u>a push-out for the maps</u> f <u>and</u> g if for any pair of maps

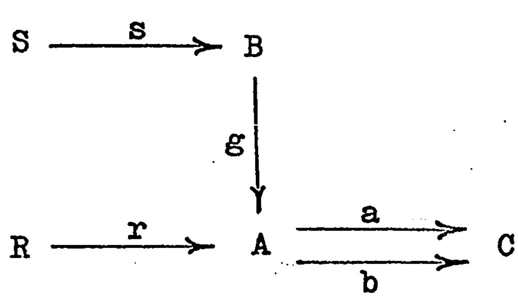


α and β such that $\alpha f = \beta g$ there exists a unique $\gamma : P \to X$ making the diagram commutative. This means $\alpha = \gamma g'$ and $\beta = \gamma f'$.

5. Categories with Finite Limits.

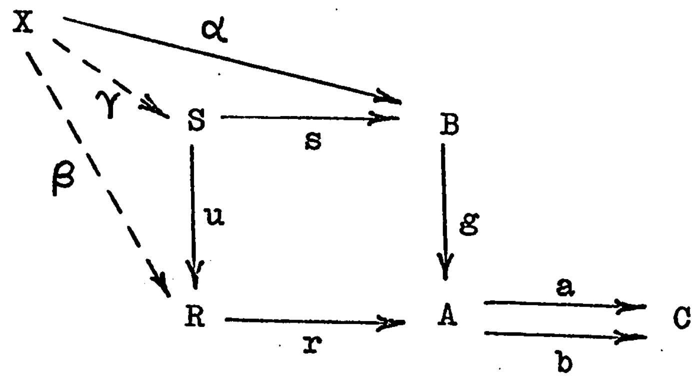
We should make the same observation for equalizers and pullbacks as we did for products. Their definition does not imply their existence. However, the three notions are examples of certain universal constructions called finite limits (see Mitchell [4]), and they are strongly related.

LEMMA 1.6. Consider the following diagram



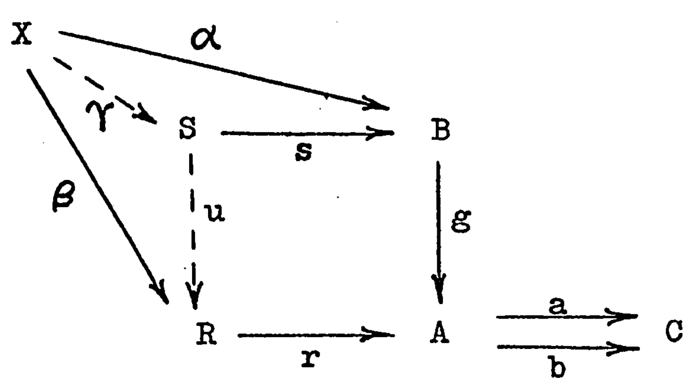
where s and g are arbitrary maps, and $r = \mathrm{Equ}(a, b)$. The left-hand square can be completed to a pullback if and only if $s = \mathrm{Equ}(ag, bg)$.

Proof. Suppose that the diagram can be completed to a pullback by a map $u : S \to R$, then we have that $(ag)s=aru=bru=(bg)s$ Moreover, if $\alpha : X \to B$ is any map such that $(ag)\alpha = (bg)\alpha$, then the map $g\alpha$ equalizes a and b. Since $r = \mathrm{Eq}(a,b)$, this implies the existence of a map $\beta : X \to R$ such that $g\alpha = r\beta$. The following diagram illustrates the situation:



By definition of pullback, α and β induce a map $\gamma : X \to S$ for which $\alpha = s\gamma$. Now, r is an equalizer and so it is mono. Therefore s is also mono (Lemma 1.4). This means that γ is the unique map such that $\alpha = s\gamma$, and so the proof that $s = \mathrm{Equ}(ag, bg)$ is completed.

Conversely, suppose that $s = \mathrm{Equ}(ag, bg)$, then $ags = bgs$. Since $r = \mathrm{Equ}(a,b)$, this means that there exists a map $u : S \to R$ such that $gs = ru$. We will show that the square RSBA, completed with u, is a pullback diagram. Consider a pair of maps α and β such that $g\alpha$ and $r\beta$.

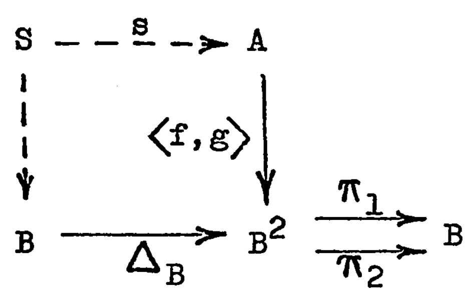


Clearly, $ag\alpha = ar\beta = br\beta = bg\alpha$. Therefore, α equalizes ag and bg, but $s = \mathrm{Equ}(ag, bg)$ and so there exists a unique map $\gamma : X \to S$ such that $\alpha = s\gamma$. We also have that $\beta = u\gamma$ because $ru\gamma = gs\gamma = g\alpha = r\beta$ and r is mono. This shows that the square RSBA is a pullback diagram. Q.E.D.

This lemma permits us to prove the following theorem which relates the existence of finite products, equalizers, and pullbacks.

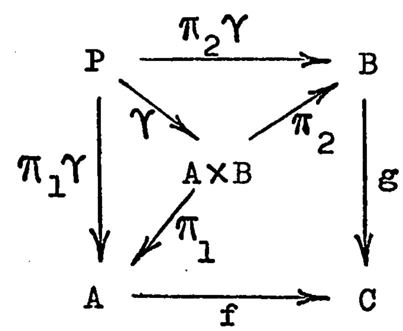
THEOREM 1.7. Let $\mathcal{A}$ be a category with finite products. $\mathcal{A}$ has pullbacks if and only if it has equalizers.

Proof. Suppose that $\mathcal{A}$ has pullbacks. Let $f, g : A \to B$ be an arbitrary pair of maps, and let $\langle f, g\rangle : A \to B^2$ be the unique map such that $f = \pi_1\langle f, g\rangle$ and $g = \pi_2\langle f, g\rangle$. Take a pullback of the maps $\langle f, g\rangle$ and the diagonal of B.



Since $\Delta_B = \mathrm{Equ}(\pi_2, \pi_2)$, by the above lemma we can conclude that $s = \mathrm{Equ}(\pi_1\langle f, g\rangle, \pi_2\langle f, g\rangle) = \mathrm{Equ}(f, g)$. This shows that $\mathcal{A}$ has equalizers.

To prove the converse it suffices to show that in the following diagram,



where γ is an equalizer of $f\pi_1$ and $g\pi_2$, the pair $(\pi_1\gamma, \pi_2\gamma)$ is a pullback of the pair (f, g). Q.E.D.

A category with finite products and equalizers (or pull-backs) is said <u>to have finite limits</u>, because it is possible to show that any kind of finite limit exists in it. Since we are interested only in the special limits already defined, we will not discuss this matter in detail (see Mitchell [4] or Pareigis [5]). The dual notion of limit is that of <u>colimit</u>, so equalizers and pushouts are <u>finite colimits</u>.

Chapter II

CONGRUENCE RELATIONS AND QUOTIENT MAPS

1. The Congruence Relation of a Map.

Consider a category with finite limits (i.e., it has products and equalizers). We can make the following definition.

DEFINITION. Let $f : A \to B$ be a map. A congruence relation of f, written Congr(f), is an equalizer of the pair $f\pi_1$ and $f\pi_2$ where π_1 and π_2 are the projections of the product A^2 into A.

Clearly, Congr(f) is a subobject of A^2. Moreover, the class of congruence relations of f coincides with the equivalence class of the subobject Congr(f). In the case of the Category of Sets, this notion coincides with the usual one of an equivalence relation in the set A. Concretely, we can choose Congr(f) such that

$$C = \{(x,y) \in A^2 | f(x) = f(y)\},$$

and Congr(f) is the inclusion map from C into A^2. In general, if the map c is a congruence relation of some map with domain A, we say that c is a congruence in A.

EXAMPLE. $\Delta_A = \mathrm{Equ}(\pi_1, \pi_2) = \mathrm{Congr}(1_A)$.

The following lemmas bring together some properties of the congruence relations.

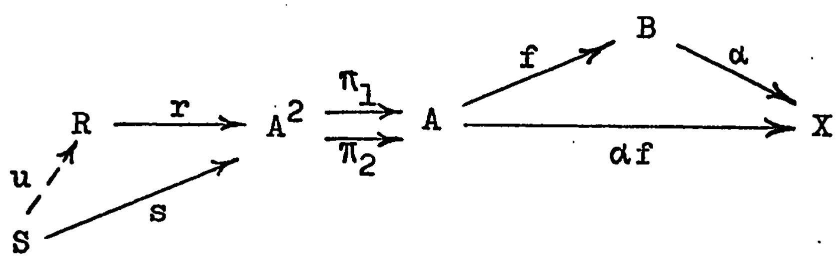
LEMMA 2.1. Let $f : A \to B$ be a map in a category with finite products, and let Congr(f) be a congruence relation of f.

(a) $\mathrm{Congr}(f) \subset \mathrm{Congr}(\alpha f)$, for any $\alpha : B \to X$.

(b) $\Delta_A \subset \mathrm{Congr}(f)$.

(c) f is mono $\Leftrightarrow \Delta_A \simeq \mathrm{Congr}(f)$.

<u>Proof</u>. (a) Consider the following diagram



where $s = \mathrm{Congr}(f)$ and $r = \mathrm{Congr}(\alpha f)$. We have $f\pi_1 s = f\pi_2 s$ and so $\alpha f\pi_1 s = \alpha f\pi_2 s$. Since $r = \mathrm{Congr}(\alpha f) = \mathrm{Equ}(\alpha f\pi_1, \alpha f\pi_2)$, this implies the existence of a unique map $u : S \to R$ such that $s = ru$. In other words, $s \subset r$.

(b) Applying the result in part (a), we have

$$\Delta_A = \mathrm{Congr}(1_A) \subset \mathrm{Congr}(f1_A) = \mathrm{Congr}(f).$$

(c) First, suppose that f is mono. Let $r = \mathrm{Congr}(f)$, then $f\pi_1 r = f\pi_2 r$ and so $\pi_1 r = \pi_2 r$. This implies that $r \subset \mathrm{Equ}(\pi_1, \pi_2) = \Delta_A$. By part (b), we already have $\Delta_A \subset r$. Hence, $\Delta_A \simeq r$.

Conversely, suppose that $\Delta_A \simeq \mathrm{Congr}(f)$. This means that Δ_A is also a congruence of f. Take a pair of maps α, $\beta : X \to A$ such that $f\alpha = f\beta$. Let $\gamma : X \to A^2$ be the unique map induced by α and β into the product A^2, with the property that $\alpha = \pi_1\gamma$ and $\beta = \pi_2\gamma$; then

$$f\pi_1\gamma = f\alpha = f\beta = f\pi_2\gamma.$$

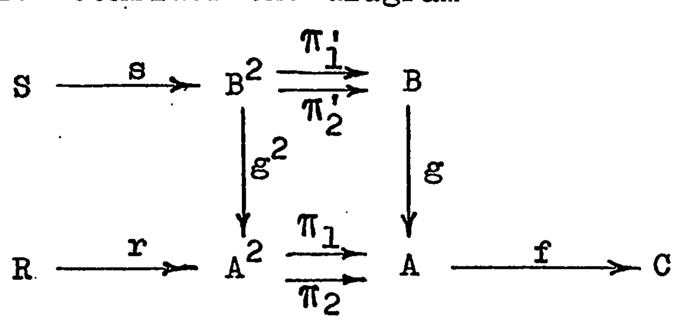
Since Δ_A is a congruence relation of f, this equation implies the existence of a map $u : X \to A$ such that $\gamma = \Delta_A u$. Therefore,

$$\alpha = \pi_1\gamma = \pi_1\Delta_A u = 1_A u = u$$

$$\beta = \pi_2\gamma = \pi_2\Delta_A u = 1_A u = u$$

This shows that $\alpha = \beta$ and so f is mono. Q.E.D.

LEMMA 2.2. Consider the diagram



where $r = \mathrm{Congr}(f)$. The left-hand square can be completed to a pullback diagram if and only if $s = \mathrm{Congr}(fg)$.

<u>Proof</u>. Since $r = \mathrm{Congr}(f) = \mathrm{Equ}(f\pi_1, f\pi_2)$, we can apply Lemma 1.6, obtaining that the square can be completed to a pullback if and only if $s = \mathrm{Equ}(f\pi_1 g^2, f\pi_2 g^2)$ $= \mathrm{Equ}(fg\pi_1', fg\pi_2') = \mathrm{Congr}(fg)$. Q.E.D.

Let $r : R \to A^2$ be a congruence in A and let $i : I \to A$ be a subobject of A; then $i^2 : I^2 \to A^2$ is a subobject of A^2. Since a category with finite limits has pullbacks, the intersection $r \cap i^2 : R \cap I^2 \to A^2$ exists. By the above lemma, the inclusion $R \cap I^2 \to I^2$ in the intersection diagram.

$$\begin{array}{ccc} R \cap I^2 & \longrightarrow & I^2 \\ \downarrow & & \downarrow{\scriptstyle i^2} \\ R & \xrightarrow{\;r\;} & A^2 \end{array}$$

is a congruence relation in I. We call it <u>the restriction of r to</u> I.

2. <u>Quotient Maps</u>.

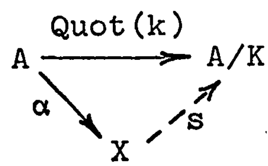
Now, we shall consider a category with finite limits which also has coequalizers. Given an arbitrary object A, we can associate to each subobject of A^2 a map with domain A in the following way.

DEFINITION. Let $k : K \to A^2$ be a subobject of A^2. <u>A quotient of</u> k, written Quot(k), is a coequalizer of $\pi_1 k$ and $\pi_2 k$, where π_1 and π_2 are the projections of A^2 into A.

If K is the domain of k, we use the notation A/K to denote the codomain of Quot(k):

$$K \xrightarrow{k} A^2 \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A \xrightarrow{\text{Quot}(k)} A/K$$

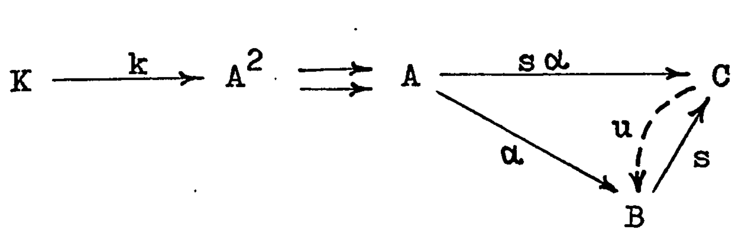
Note that Quot(k) must be epi. Moreover, $\alpha : A \to X$ is also a quotient of k if and only if there exists an isomorphism $s : X \to A/K$ making the following diagram commutative



Actually, it suffices to have a monomorphism s, because of the following lemma.

LEMMA 2.3. If $s\alpha$ is a quotient and s is mono, then s is an isomorphism (and so, α is also a quotient).

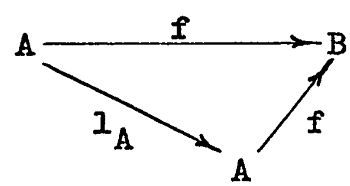
<u>Proof</u>. Consider the diagram



where $s\alpha = \text{Quot}(k)$. We have that $s\alpha\pi_1 k = s\alpha\pi_2 k$, but s is mono, so $\alpha\pi_1 k = \alpha\pi_2 k$. Since α coequalizes $\pi_1 k$ and $\pi_2 k$, and $s\alpha = \text{Coeq}(\pi_1 k, \pi_2 k)$, there exists a map $u : C \to B$ such that $\alpha = us\alpha$. Hence, $s\alpha = sus\alpha$; but $s\alpha$ is epi because it is a quotient, so $1_C = su$. Moreover, $s = 1_C s = sus$. Since s is mono, this means that $1_B = us$, completing the proof that s is an isomorphism. Q.E.D.

COROLLARY 2.4. A map f is an isomorphism if and only if it is a monomorphism and a quotient.

<u>Proof</u>. If f is an isomorphism, it is mono. Moreover, we have a commutative diagram



Since 1_A is a quotient and f is an isomorphism, this means that f is also a quotient. Conversely, suppose that f is a monomorphism and a quotient. Since $f1_A$ is a quotient and f is mono, the above lemma implies that f is an isomorphism. Q.E.D.

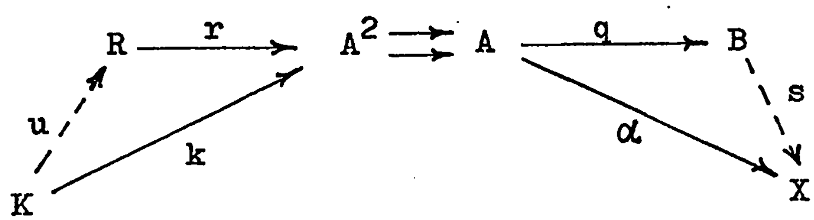
Finally, the following lemma shows the correspondence between quotient maps and congruence relations.

LEMMA 2.5. Consider a category with finite limits and coequalizers.

(a) If q is a quotient map, then $q \simeq Quot(Congr(q))$.

(b) If r is a congruence relation, then $r \simeq Congr(Quot(r))$.

<u>Proof</u>. (a) Let $q : A \to B$ be such that $q = Quot(k)$, where $k : K \to A^2$ is a subobject of A^2. Then $q\pi_1 k = q\pi_2 k$. If we take $r = Congr(q)$, the last equation implies the existence of a unique map u such that $k = ru$.



To show that $q = Quot(r)$, observe that $q\pi_1 r = q\pi_2 r$ because r is a congruence of q. Moreover, for any map α with the property that $\alpha\pi_1 r = \alpha\pi_2 r$, we obtain that $\alpha\pi_1 k = \alpha\pi_1 ru =$ $= \alpha\pi_2 ru = \alpha\pi_2 k$. Since q is a quotient of k, this means that there exists a unique map s such that $\alpha = sq$.

(b) The proof of this part is similar to that of (a).

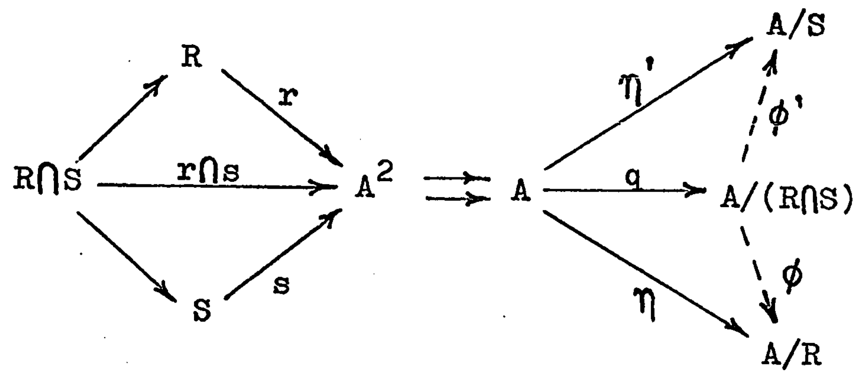
Q.E.D.

3. The Lattice of Congruences.

A partially ordered class is called a lattice if each pair of elements a and b has a least upper bound or supremum $a \vee b$, and a greatest lower bound or infimum $a \wedge b$. In a category with finite limits, the intersection of two subobjects has the properties of an infimum, and it is really an infimum if we consider the equivalence classes of subobjects instead of the subobjects themselves. However, two equivalence classes of subobjects do not necessarily have a supremum. We shall prove that the situation is better if we consider only congruence relations in a fixed object A, and we add the assumption that coequalizers exist. To show that the equivalence classes of congruences in A form a lattice in this case, we begin with the following theorem.

THEOREM 2.6. Let A be an object in a category with finite limits and coequalizers. If r and s are congruences in A, then $r \cap s$ is a congruence in A.

Proof. Take $q = \mathrm{Quot}(r \cap s)$, $\eta = \mathrm{Quot}(r)$, and $\eta' = \mathrm{Quot}(s)$. Since $r \cap s \subset r$, there exists a map ϕ such that $\eta = \phi q$. Similarly, $r \cap s \subset s$ implies that $\eta' = \phi' q$ for some map ϕ'. This is described in the following diagram



Since r and s are congruences, we have r = Congr(η) and s = Congr(η') by Lemma 2.5. Now, let c = Congr(q), then: c = Congr(q) ⊂ Congr(φq) = Congr(η) = r. Similarly, c ⊂ s and so

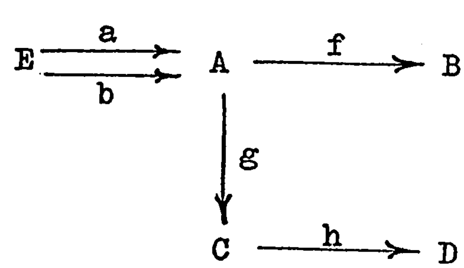
$$c \subset r \cap s$$

Moreover, $q\pi_1(r \cap s) = q\pi_2(r \cap s)$ by definition of q. Therefore, there exists a map u such that $(r \cap s) = cu$. In other words:

$$r \cap s \subset c$$

The two inclusions imply $r \cap s \simeq c = \text{Congr}(q)$. Q.E.D.

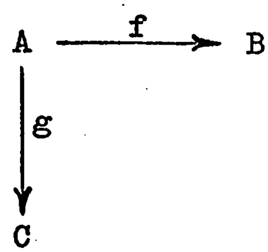
LEMMA 2.7. Consider the following diagram



where f = Coeq(a, b). The square CABD can be completed to a pushout diagram if and only if h = Coeq(ga, gb).

<u>Proof</u>. This is the dual of Lemma 1.6. Note that it holds in an arbitrary category.

COROLLARY 2.8. Consider the diagram



in a category with finite limits and coequalizers, where f is a quotient map. Then the pair (f,g) has a pushout.

<u>Proof</u>. Let f = Quot(k) = Coeq($\pi_1 k$, $\pi_2 k$). Take h : C → D such that h = Coeq($g\pi_1 k$, $g\pi_2 k$), and apply the above lemma. Q.E.D.

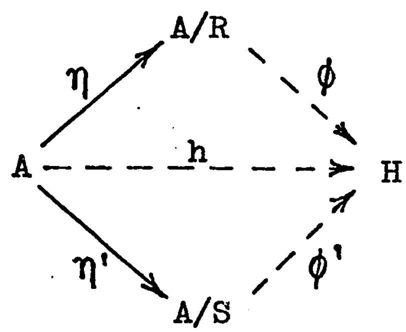
Now, we are ready to prove

THEOREM 2.9. Let A be an object in a category with finite limits and coequalizers. Then the equivalence classes of congruences in A form a lattice with respect to the inclusion between subobjects. This lattice has a first and a last element .

<u>Proof</u>. Clearly the congruences Δ_A and 1_{A^2} (or, more exactly, their equivalence classes) give the first and last elements. We

have already proved that the intersection of two congruences in A is a congruence in A; therefore, it is clear that the intersection gives the infimum in the class of all the (equivalence classes of) congruences in A. It remains to show the existence of the supremum for two equivalence classes of congruences. It will suffice to work with representatives.

Let $r : R \to A^2$ and $s : S \to A^2$ be congruences in A, and let $\eta : A \to A/R$ and $\eta' : A \to A/S$ be quotients of r and s, respectively. By Corollary 2.8 we have a pushout diagram

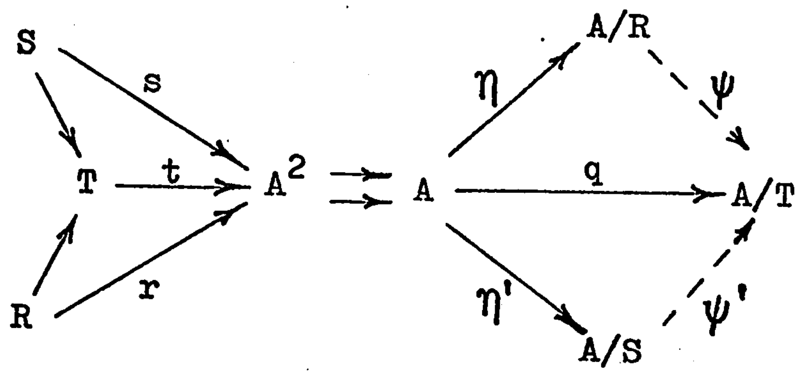


Let $h = \phi\eta = \phi'\eta'$ and define $r \vee s = \mathrm{Congr}(h)$. In the first place we have

$$r = \mathrm{Congr}(\eta) \subset \mathrm{Congr}(\phi\eta) = r \vee s$$

$$s = \mathrm{Congr}(\eta') \subset \mathrm{Congr}(\phi'\eta') = r \vee s.$$

Now, let $t : T \to A^2$ be any upper bound of r and s, and let $q : A \to A/T$ be a quotient of t. Since $r \subset t$ and $s \subset t$, there exist maps $\psi : A/R \to A/T$ and $\psi' : A/S \to A/T$ such that the following diagram commutes



Hence, $\psi\eta = q = \psi'\eta'$. By the universal property of the pushout diagram (1), this means that there exists a map $\ell : H \to A/T$ such that $\psi = \ell\phi$ and $\psi' = \ell\phi'$. Therefore, $q = \psi\eta = \ell\phi\eta = \ell h$, and so

$$r \vee s = \mathrm{Congr}(h) \subset \mathrm{Congr}(\ell h) = \mathrm{Congr}(q).$$

Since q is a quotient of t and t is a congruence, we can choose t as a congruence of q (see Lemma 2.5). So we have that $r \vee s \subset t$. This shows that $\overline{r \vee s}$ is a least upper bound for $\bar{r}$ and $\bar{s}$ with respect to all the (equivalence classes of) congruences in A.
Q.E.D.

Chapter III

FIRST ISOMORPHISM THEOREM

1. Special Categories.

In the preceding section we considered a category $\mathcal{A}$ where the following properties hold.

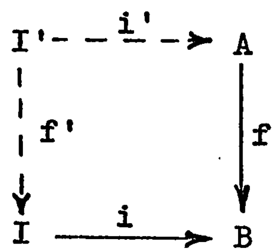
A - 1: $\mathcal{A}$ has finite limits.

A - 2: $\mathcal{A}$ has coequalizers.

These properties allow us to generalize the concepts of the congruence relation of a map and that of the quotient of a relation, and to prove some of their most trivial properties. If we want stronger results, we must add more conditions. In particular we are interested in the "First Isomorphism Theorem". In the context of Universal Algebra, this theorem states that for a given congruence relation Γ, in an algebraic system A, there is a lattice isomorphism between the congruence relations of A that contain Γ and the congruence relations of the quotient algebraic system A/Γ (cf. Pierce [6], Chap. 2, Prop. 3.3). In the category of groups, the theorem is usually stated for normal subgroups instead of congruence relations. We will show that this theorem is not characteristic of algebraic categories: in fact, the following two properties shall be sufficient for its validity.

A - 3: If f is a quotient, then $f^2 = f \times f$ is a quotient.

A - 4: If f is a quotient and i is a monomorphism, the map f' in the following pullback diagram



is also a quotient.

Observe that both axioms involve a colimit and a limit. The following are examples of categories where conditions A - 1 to A - 4 hold.

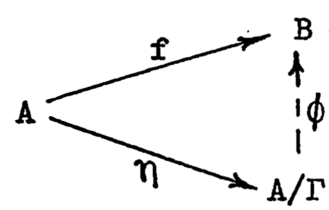
EXAMPLE 1. Category of Sets. The objects are sets and the maps are functions. For sets A and B, define $A \times B = \{(x,y) | x \in A \text{ and } y \in B\}$, $\pi_1((x,y)) = x$, and $\pi_2((x,y)) = y$. Then the triple $(A \times B, \pi_1, \pi_2)$ is a product of A and B. In fact, it is a canonical representative for the class of products of A and B. Given a pair of functions $f, g : A \to B$, define $E = \{x \in A | f(x) = g(x)\}$. Then the inclusion $E \hookrightarrow A$ is an equalizer of f and g. A coequalizer for the pair of functions $f, g : A \to B$ can be constructed in the following way. Let R be a relation in the set B, defined by

$$(b, b') \in R \iff \exists a \in A : f(a) = b \text{ and } g(a) = b'.$$

If Γ is the minimum equivalence relation in B containing R (i.e., Γ is the intersection of all the equivalence relations in B containing R), let B/Γ be the set of equivalence classes and $c : B \to B/\Gamma$ the function sending each element to its class. Then c is a coequalizer of f and g. This shows the existence of finite limits and coequalizers. To show that axioms A - 3 and A - 4 hold, it suffices to notice that the following is true in Sets:

$$f \text{ is a quotient} \Leftrightarrow f \text{ is } \underline{\text{onto}} \qquad (1)$$

(Here, "onto" has the usual meaning for functions between sets). To see this, take an arbitrary function $f : A \to B$ and define an equivalence relation in A by $\Gamma = \{(x,y) \in A^2 | f(x) = f(y)\}$. In fact, it is easy to check that the inclusion $\Gamma \hookrightarrow A^2$ is the congruence relation of f in the categorical sense. Moreover, the function $\eta : A \to A/\Gamma$, sending each element a to its equivalence class $\overline{a}$, can be seen to be a quotient of $\Gamma \hookrightarrow A^2$. Construct a commutative diagram



by defining $\phi(\overline{a}) = f(a)$. The function ϕ is well defined and is one to one, because

$$\overline{a} = \overline{a}' \Leftrightarrow (a,a') \in \Gamma \Leftrightarrow f(a) = f(a') \Leftrightarrow \phi(\overline{a}) = \phi(\overline{a}').$$

Suppose now that f is a quotient. Since ϕ is mono (in Sets, "one to one" means "mono") the equation $f = \phi\eta$ implies that ϕ is an isomorphism (Lemma 2.3). But, in Sets, an isomorphism is a bijection; therefore ϕ is onto and so f is onto. Conversely, if ϕ is onto, it must be a bijection. Therefore, ϕ is an isomorphism, but this means that f is a quotient (since η is).

Having this, it remains only to check A - 3 and A - 4 for "onto" instead of "quotient". Here, note that the function f^2 is given by $f^2((x,y)) = (f(x), f(y))$ and that any subobject of B is equivalent to a subset of B. The pullback of f and a subset of B is obtained by taking the inverse image of the subset with the obvious functions.

EXAMPLE 2. Algebraic Categories. An algebraic category is the same as an equationaly defined class of algebras in the sense of Birkhoff. From this point of view, it is a subcategory of the Category of Sets with certain special properties. We could discuss the axioms as we did for Sets, obtaining once more property (1). However, we prefer to use the definition of Algebraic Category given by Lawvere [2], because it is a special case of a more general definition, discussed in the following example.

EXAMPLE 3. Varietal Categories. This concept is defined by Linton in [3]. First we need the concept of a varietal theory. Let Card denote the category of cardinal numbers as a full subcategory of Sets. This means that we take a unique representative set for each cardinal number, and the maps are all the set-functions arising between them. Let Card^{opp} be the dual category of Card. This means that we take the same objects, but we change the direction of the maps. Since the sum of cardinals is a coproduct in Card, it becomes a product in Card^{opp}. In particular we have: $n = \prod_n 1$ for any cardinal number n. A category $\mathcal{T}$ is called a varietal theory if there exists a functor $T : \text{Card}^{\text{opp}} \to \mathcal{T}$ which is a bijection on objects and preserves arbitrary products. We can identify the objects of $\mathcal{T}$ with those of Card^{opp}. Since products are preserved, we still have $n = \prod_n 1$ in $\mathcal{T}$.

If $\mathcal{T}$ is a varietal theory, let $\text{Sets}^{\mathcal{T}}$ be the category with functors from $\mathcal{T}$ into Sets as objects and natural transformations as maps. The category of $\mathcal{T}$-algebras is the full subcategory of $\text{Sets}^{\mathcal{T}}$ obtained by taking product preserving functors only. We denote it by $\text{Sets}^{(\mathcal{T})}$. Finally, a varietal category is one which is isomorphic to a category of $\mathcal{T}$-algebras, for some varietal theory $\mathcal{T}$.

If we repeat this construction beginning with the category of finite cardinals $\mathcal{N}$, instead of Card, we will obtain algebraic theories and algebraic categories in the sense of Lawvere.

Let's recall that the objects of $\mathcal{T}$ are cardinals; therefore

1 is an object of T. For a varietal (or algebraic) category $\text{Sets}^{(T)}$, we have a "forgetful" functor $U : \text{Sets}^{(T)} \to \text{Sets}$, defined by the evaluation at 1. If A is a functor in $\text{Sets}^{(T)}$, $U(A) = A(1)$. If α is a natural transformation, $U(\alpha) = \alpha_1$. We will need the following properties of $\text{Sets}^{(T)}$ and the functor U. Pareigis proves them for algebraic categories, but the proofs can be easily generalized to varietal categories.

(1) $\text{Sets}^{(T)}$ is complete and cocomplete. This means that it has all limits and colimits (in particular, finite limits and coequalizers).

(2) U preserves limits.

(3) For any map α in $\text{Sets}^{(T)}$,

α is a coequalizer $\Leftrightarrow U(\alpha)$ is onto.

(See Pareigis [5]; theorem in Sec. 3.2 and Corollaries 3 and 4 in Sec. 3.4).

Observe that U behaves very well with respect to coequalizers, but it does not preserve colimits in general. It is easy to show that a coequalizer is a quotient of its congruence relation; therefore, from (3), we get:

(3') α is a quotient $\Leftrightarrow U(\alpha)$ is onto.

Using this last property and the fact that U preserves products, equalizers and pullbacks, we can check that A - 3 and

A - 4 hold in Sets$^{(T)}$. First, observe that for any map α, $\mathcal{U}(\alpha^2) = \mathcal{U}(\alpha)^2$, by preservation of products. Therefore, if α is a quotient, $\mathcal{U}(\alpha)$ is onto and so $\mathcal{U}(\alpha)^2 = \mathcal{U}(\alpha^2)$ is onto. Hence, α^2 is a quotient. This shows A - 3. Second, $\mathcal{U}$ must preserve the diagonal of an object, because it preserves products (and identity maps). Moreover, $\mathcal{U}$ preserves congruences (which are equalizers). This means that $\mathcal{U}$ preserves monomorphisms, because a map is mono if and only if its congruence is the diagonal. Now take a pullback diagram in Sets$^{(T)}$, where the map i is mono. Applying $\mathcal{U}$ we obtain a pullback diagram in Sets with $\mathcal{U}(i)$ mono.

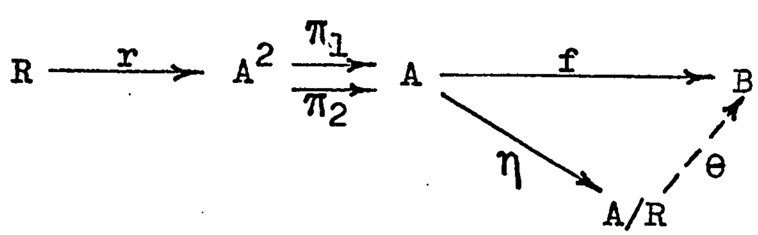
$$\begin{array}{ccc} \cdot & \xrightarrow{i'} & \cdot \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[i]{} & \cdot \end{array} \qquad\qquad \begin{array}{ccc} \cdot & \xrightarrow{\mathcal{U}(i')} & \cdot \\ {\scriptstyle \mathcal{U}(f')}\downarrow & & \downarrow{\scriptstyle \mathcal{U}(f)} \\ \cdot & \xrightarrow[\mathcal{U}(i)]{} & \cdot \end{array}$$

Suppose that f is a quotient, then $\mathcal{U}(f)$ is onto. But A - 4 holds in Sets; therefore, $\mathcal{U}(f')$ is onto and f' is a quotient. This shows A - 4.

EXAMPLE 4. Consider a full sub-category of a varietal (or algebraic) category which is closed with respect to sub-algebras, epimorphic images, and finite products. For example, take the following subcategories of the Category of Groups. (a) Finite Groups. (b) Finitely Generated Abelian Groups. (c) Torsion Groups. Notice that these categories are not algebraic because they are not closed with respect to arbitrary products.

2. Canonical Factorization and Images.

Given an arbitrary map f, in a category with finite limits and coequalizers, let η = Quot(Congr(f)). Then we have a factorization $f = \theta\eta$ where θ is uniquely determined. This is just a direct consequence of the definitions of congruence and quotient. The following diagram illustrates the situation.



If r = Congr(f), then $f\pi_1 r = f\pi_2 r$; but η = Quot(r) = Coeq($\pi_1 r$, $\pi_2 r$). So there exists a unique θ making the diagram commutative. We call this the canonical factorization of f.

In general, we cannot say much about the map θ. However, we will prove that if properties A - 3 and A - 4 hold in the category, then the map θ is actually a subobject of B. Moreover, it will be also true that the only factorization of f by a quotient map and a monomorphism is essentially, the canonical factorization.

This fact constitutes the substance of the "Fundamental Homomorphism Theorem". It has been proved for algebraic categories (Cohn [1]). Obviously, our generalization applies to a more general kind of category. Before proving it, we introduce a new definition.

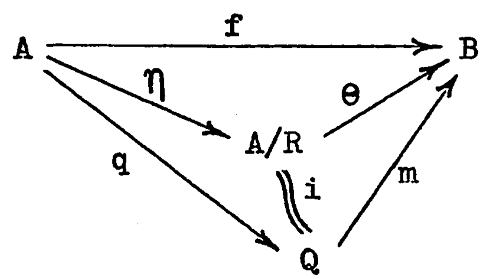
DEFINITION. Let $f : A \to B$ be a map in an arbitrary category. An <u>image of</u> f is a subobject of B, Im(f), for which there exists a factorization $f = \mathrm{Im}(f)\alpha$, and that is included in any other subobject of B with the same property.

Obviously, any two images of f are equivalent subobjects of B. If every map in a category has an image, we say that the category <u>has images</u>. If the map α is epi, we say that it <u>has epimorphic images</u>. The following theorem shows that a category where properties A - 1 to A - 4 hold has epimorphic images. In fact, the images will be given by the canonical factorization.

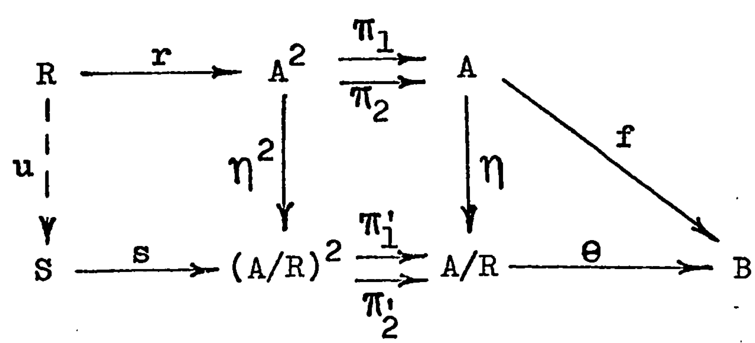
THEOREM 3.1. (Fundamental Homomorphism Theorem). Let A be a category satisfying A - 1 to A - 4. If f is an arbitrary map, r = Congr(f), and η = Quot(r), let θ be the unique map in the canonical factorization f = $\theta\eta$. Then we have:

(a) θ is an image of f.

(b) Given any factorization f = mq, where q is a quotient and m is mono, there exists an isomorphism i such that q = iη and θ = mi. So the following diagram commutes.

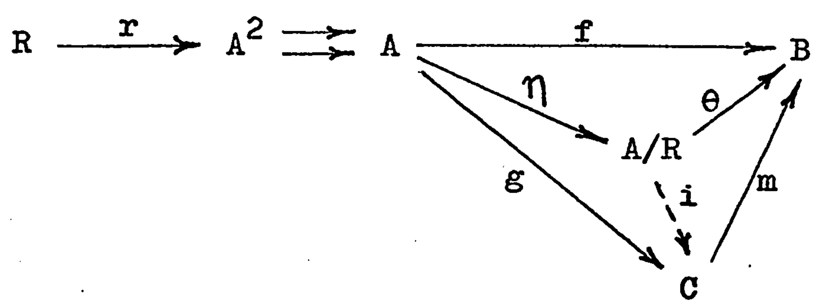


Proof. (a) First we must show that θ is mono. To do this we show that any congruence relation of θ is equivalent (as subobject) to the diagonal of A/R. Consider the diagram



where $r = \mathrm{Congr}(f)$, $\eta = \mathrm{Quot}(r)$, and $s = \mathrm{Congr}(\theta)$. By Lemma 2.2, the left-hand square can be completed to a pullback by a map u. Since η is a quotient, η^2 is a quotient by A - 3, and so u is also a quotient by A - 4. Therefore, u is epi. On the other hand $\eta\pi_1 r = \eta\pi_2 r$; hence, $\pi_1^! su = \pi_1^! \eta^2 r = \eta\pi_1 r = \eta\pi_2 r = \eta_2^! \eta^2 r = \pi_2^! su$. Since u is epi, this means that $\pi_1^! s = \pi_2^! s$. Moreover, we have seen that $\Delta_{A/R}$ is an equalizer of the projections $\pi_1^!$ and $\pi_2^!$. Then the last equation implies that $s \subset \Delta_{A/R}$. To conclude, observe that s is a congruence, and so $\Delta_{A/R} \subset s$; therefore, $s \simeq \Delta_{A/R}$.

To show that $\theta = \mathrm{Im}(f)$ consider the diagram



where η and r have the same meaning as before, $f = mg$, and m is mono. Clearly, $mg\pi_1 r = f\pi_1 r = f\pi_2 r = mg\pi_2 r$. Hence, $g\pi_1 r = g\pi_2 r$. Since $\eta = \mathrm{Quot}(r)$, this last equality implies the existence of a map $i : A/R \to C$ such that $g = i\eta$. We have also that $\theta = mi$ because $mi\eta = mg = f = \theta\eta$ and η is epi. Therefore, $\theta \subset m$.

(b) To prove the uniqueness, observe that in the above diagram the map i is mono because θ is mono and $\theta = mi$. Therefore, if g is a quotient, the equality $g = i\eta$ implies that i must be an isomorphism (Lemma 2.3), Q.E.D.

The commutativity of the diagram in part (b) of this theorem means that $\theta \approx m$. Therefore, both maps represent the same equivalence class of subobjects of B, and both are images of f. Moreover, the maps η and q are both quotients of $r = \text{Congr}(f)$. It is in this sense that the factorization is unique. In the sequel, we will refer to Theorem 3.1 by FHT.

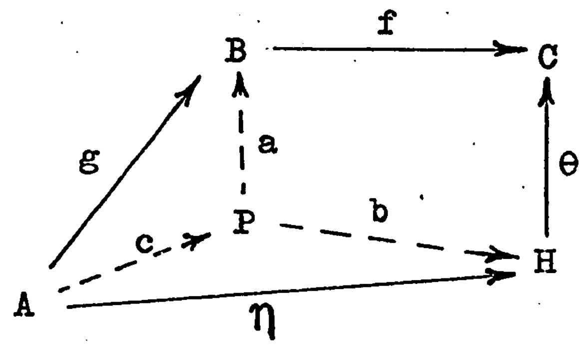
We have seen that the class of quotient maps, in a category with finite limits and coequalizers, is a subclass of the class of epimorphisms. Moreover, a map is an isomorphism if and only if it is a quotient and a monomorphism. Therefore, the concept of a quotient map resembles that of being <u>onto</u> in the category of sets. The FHT allows us to extend this analogy.

COROLLARY 3.2. Let f and g be maps in a category satisfying A - 1 to A - 4.

(a) If f and g are quotients, then fg is a quotient.

(b) If fg is a quotient, then f is a quotient.

Proof. (a) By the FHT, the composite map fg has a canonical factorization $fg = \theta\eta$ where θ is mono and η is a quotient. Let the pair of maps a and b be a pull-back for f and θ.

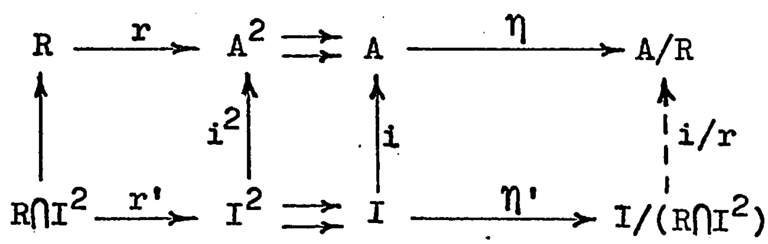


Since θ is mono, a is mono. Moreover, there exists a unique map $c : A \to P$ making the diagram commutative. In particular, $g = ac$. Since g is a quotient by hypothesis, and a is mono, a must be an isomorphism (Lemma 2.3). Hence, $fa = \theta b$ implies $f = \theta b a^{-1}$. But f is a quotient and θ is mono; therefore, using Lemma 2.3 once more, θ must be an isomorphism. This fact and the equation $fg = \theta\eta$, where η is a quotient, imply that fg is a quotient.

(b) Suppose that fg is a quotient. The canonical factorization of f, $f = \theta\eta$, gives a factorization of fg, $fg = \theta\eta g$. Since fg is a quotient and θ is mono, θ must be an isomorphism. Therefore, the equation $f = \theta\eta$ implies that f is a quotient. Q.E.D.

Let us recall that if $r : R \to A^2$ is a congruence in A and $i : I \to A$ is a subobject, we can obtain a congruence in I by taking the restriction $r' : R \cap I^2 \to I^2$ of r to I (see Section 1.4). The following corollary shows that $I/(R \cap I^2)$ can be "included" in a natural way into A/R.

COROLLARY 3.3. Let $r : R \to A^2$ be a congruence in A and $r' : R \cap I^2 \to I^2$ its restriction to I, where $i : I \to A$ is a subobject of A. If η and η' are quotients of r and r', respectively, there exists a unique monomorphism i/r making the following diagram commutative.



<u>Proof</u>. The existence of the map comes from the fact that the left-hand square is a pullback. Hence, $r' = \mathrm{Congr}(\eta i)$ by Lemma 2.2, and there is a canonical factorization of ηi, $\eta i = (i/r)\eta'$. By the FHT, i/r must be mono. Q.E.D.

To simplify our discussion, we introduce the notion of image of a subobject. Let $f : A \to B$ be a map and $i : I \to A$ a subobject. An image of the composition fi will be called "<u>image of</u> i <u>by</u> f", and it will be denoted by $f_*(i) : f_*(I) \to B$.

Obviously, $f_*(i)$ is uniquely determined by i up to equivalence of subobjects of B. Observe that in the special case of the above Corollary, $i/r = \eta_*(i)$. Given a subobject m of B, we can also define an "<u>inverse image of</u> m <u>by</u> f" as the subobject of A induced by a pullback of f and m. We use the notation $f^{-*}(m)$ to denote this subobject. The following properties are trivial consequences of these definitions.

(1) $i \subset i' \Rightarrow f_*(i) \subset f_*(i')$

(2) $m \subset m' \Rightarrow f^{-*}(m) \subset f^{-*}(m')$

(3) $i \subset f^{-*}(f_*(i))$

(4) $f_*(f^{-*}(m)) \subset m$

Let us show (1), for example. By definition of $f_*(i')$, we have a factorization $fi' = f_*(i')\alpha$ for a certain map α. Moreover, $i \subset i'$ means that $i = i'k$ for some k. Hence, $fi = fi'k = f_*(i')\alpha k$. Since $f_*(i) = \mathrm{Im}(fi)$ and the map $f_*(i')$ is mono, this implies that $f_*(i) \subset f_*(i')$.

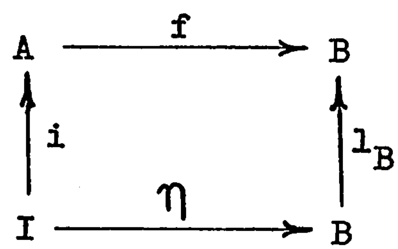
From property (1), we can see that if $i \simeq i'$, then $f_*(i) \simeq f_*(i')$. This means that f_* can be considered as a well defined function sending the equivalence class of a subobject of A to the equivalence class of its image by f. Obviously, it is an inclusion preserving function. Similarly, f^{-*} can be considered as an inclusion preserving function going in the other direction. The following theorem characterizes the quotient maps in terms of these induced functions.

THEOREM 3.4. Let $f : A \to B$ be a map in a category with properties A - 1 to A - 4. Consider f_* and f^{-*} as functions between the partially ordered classes of (equivalence classes of) subobjects of A and B. Then

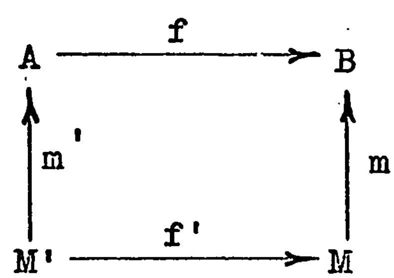
$$f \text{ is a quotient} \Leftrightarrow f_* \text{ is onto.}$$

Moreover, if this is the case, $f_*(f^{-*}(\overline{m})) = \overline{m}$ for any subobject of B.

<u>Proof</u>. Suppose that f_* is onto. Then 1_B must be the image by f of some subobject of A. So we have a commutative diagram, where η must be a quotient by the FHT.



Therefore, $fi = 1_B\eta = \eta$ is a quotient, and so f is a quotient (Corollary 3.2). To prove the other direction assume that f is a quotient. Let $m : M \to B$ be a subobject of B and $m'=f^{-*}(m)$. The map f' in the pullback diagram



must be a quotient (A - 4). Therefore, by uniqueness of the canonical factorization in the FHT, we have that m is an image of fm'. Hence, $m = f_*(m') = f_*(f^{-*}(m))$. Q.E.D.

3. The First Isomorphism Theorem.

Let L_1 and L_2 be two partially ordered sets. A function $\Phi : L_1 \longrightarrow L_2$ is called an order isomorphism if Φ is a bijection (i.e., Φ is one to one and onto) with the property:

$$\ell \leq \ell' \Longleftrightarrow \Phi(\ell) \leq \Phi(\ell') \qquad (1)$$

This means that an order isomorphism preserves all the properties defined purely in terms of the order relation. In particular, if L_1 and L_2 are lattices, we have that

$$\Phi(\ell \wedge \ell') = \Phi(\ell) \wedge \Phi(\ell')$$

$$\Phi(\ell \vee \ell') = \Phi(\ell) \vee \Phi(\ell')$$

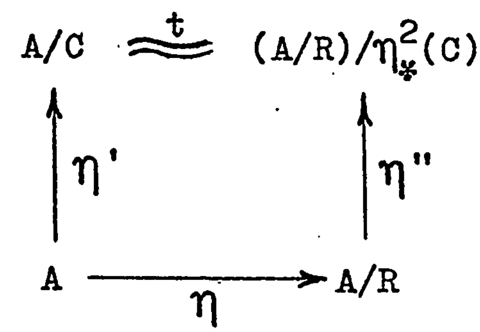
In this case we call Φ a lattice isomorphism. Observe that to prove that Φ is an order isomorphism it is sufficient to check that Φ is onto and has property (1). The fact that the correspondence is one to one follows from (1).

In Chapter II, we proved that the equivalence classes of congruences in A form a lattice. Note that if r is a fixed congruence on A, the equivalence classes of congruences containing $\bar{r}$ are a sublattice of this lattice. We conclude with the following theorems which generalize two of the "classical isomorphism theorems".

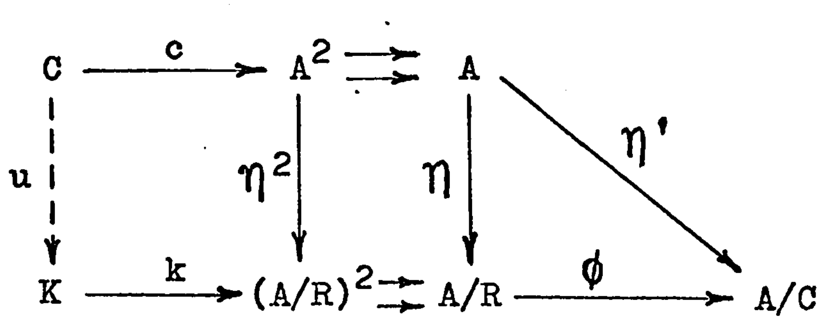
THEOREM 3.5. Consider a category satisfying properties A - 1 to A - 4. Let $r : R \to A^2$ be a congruence in A with quotient $\eta : A \to A/R$. If $c : C \to A^2$ is a congruence in A such that $r \subset c$ and $\eta_*^2(c)$ is an image of c by $\eta^2 : A^2 \to (A/R)^2$, we have

(a) $\eta_*^2(c)$ is a congruence in A/R.

(b) If η' and η'' are quotients of c and $\eta_*^2(c)$ respectively, there exists an isomorphism t such that the following diagram commutes



Proof. (a) Let $\eta' : A \to A/C$ be a quotient of c. Since $r \subset c$, there exists a map $\phi : A/R \to A/C$ such that $\eta' = \phi\eta$. We will show that $\eta_*^2(c)$ is a congruence of this map ϕ. To do this, consider the following diagram where $k = \text{Congr}(\phi)$.



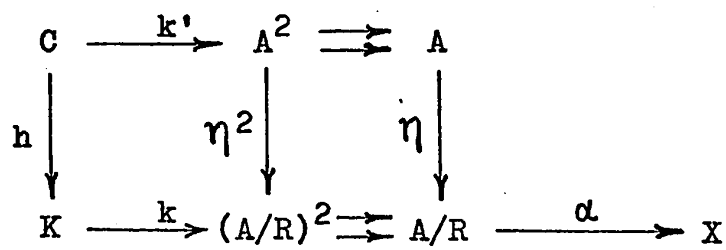
Since c is a congruence, it is a congruence of its quotient. Hence, $c = \mathrm{Congr}(\eta') = \mathrm{Congr}(\bar{\phi}\eta)$. By Lemma 2.2, this means that the left hand square can be completed to a pullback by some map u. Since η is a quotient, u is also a quotient (A - 3 and A - 4). Therefore, we have a factorization of $\eta^2 c$ into a monomorphism k and a quotient u. By the uniqueness of the FHT, this implies that $k = \mathrm{Im}(\eta^2 c) = \eta^2_*(c)$.

(b) We have just proved that $\eta^2_*(c)$ is a congruence of the map ϕ in the above diagram. But ϕ is a quotient because $\eta' = \phi\eta$ is a quotient (Corollary 3.2). Therefore, ϕ must be a quotient of its congruence $\eta^2_*(c)$. This means that for any other quotient of $\eta^2_*(c)$, $\eta'' : A/R \to (A/R)/\eta^2_*(C)$, there exists an isomorphism $t : (A/R)/\eta^2_*(C) \to A/C$ such that $\phi = t\eta''$. Hence, $\eta' = \phi\eta = t\eta''\eta$. Q.E.D.

THEOREM 3.6. (First Isomorphism Theorem). Consider a category satisfying A - 1 to A - 4. Let $r : R \to A^2$ be a congruence in A with quotient $\eta : A \to A/R$. Then η^2_* gives a lattice isomorphism between the lattice of (equivalence classes of) congruences in A which contain r, and the lattice of (equivalence classes of) congruences in A/R.

<u>Proof</u>. We have just shown that η^2_* sends congruences in A containing r to congruences in A/R. It is also true that

the inverse image by η^2 of a congruence in A/R is a congruence in A containing r. To see this let k be a congruence in A/R, namely k = Congr(α) for some α. If k' is an inverse image of k by η^2, the left hand square in the following diagram



is a pullback. Therefore, k' = Congr($\alpha\eta$) by Lemma 2.2; moreover, r = Congr(η) $\subset$ Congr($\alpha\eta$) = k'. This shows that k' is a congruence in A containing r. Actually, we have shown that η_*^2 is onto with respect to congruences, because the map h in the above diagram must be a quotient and so, by the uniqueness of factorizations in the FHT, k = Im(η^2k') = η_*^2(k').

To complete the proof that η_*^2 gives a lattice isomorphism it remains to show that if c_1 and c_2 are congruences in A such that $c_1 \supset r$ and $c_2 \supset r$, then

$$c_1 \subset c_2 \Leftrightarrow \eta_*^2(c_1) \subset \eta_*^2(c_2)$$

The implication "$\Longrightarrow$" is always true. Suppose now that $\eta_*^2(c_1) \subset \eta_*^2(c_2)$. Let η_1'' and η_2'' be quotients of $\eta_*^2(c_1)$ and

$\eta^2_*(c_2)$, respectively. Then there exists a map ϕ such that $\eta''_2 = \phi\eta''_1$. On the other hand, by the commutativity of the diagram in part (b) of Theorem 3.5, we have that $\eta''_1\eta$ is a quotient of c_1 and similarly $\eta''_2\eta$ is a quotient of c_2. Since c_1 and c_2 are congruences, this means that

$$c_1 = \mathrm{Congr}(\eta''_1\eta) \subset \mathrm{Congr}(\phi\eta''_1\eta) = \mathrm{Congr}(\eta''_2\eta) = c_2.$$

Therefore, we have shown the other implication. Q.E.D.

Observe that from the proof of this theorem we can conclude that the function $(\eta^2)^{-*}$, sending a congruence in A/R to its inverse image in A, gives also a lattice isomorphism. If $c \supset r$, then $(\eta^2)^{-*}(\eta^2_*(c)) = c$.

Chapter IV

NORMAL SUBOBJECTS

1. Congruences Generated by Squares.

Let $\mathcal{A}$ be a category with finite limits and coequalizers. Consider an object A and a subobject of the product A^2, $s : S \to A^2$. We can define a "minimal" congruence in A which contains s in the following way.

DEFINITION. $\Gamma(s) = \text{Congr}(\text{Quot}(s))$. Intuitively, $\Gamma(s)$ is the congruence relation generated by s as the following lemma shows.

LEMMA 4.1. $s \subset \Gamma(s)$; and for any congruence relation r in A, if $s \subset r$, then $\Gamma(s) \subset r$.

Proof. By definition of quotient, we have $\text{Quot}(s)\pi_1 s = \text{Quot}(s)\pi_2 s$. Since $\Gamma(s)$ is a congruence of Quot(s), this means that $s \subset \Gamma(s)$. Let r be a congruence in A such that $s \subset r$. Let $\eta = \text{Quot}(s)$ and $\eta' = \text{Quot}(r)$; since $s \subset r$, there exists a map ϕ such that $\eta' = \phi\eta$. Hence,

$$\Gamma(s) = \text{Congr}(\eta) \subset \text{Congr}(\phi\eta) = \text{Congr}(\eta') = r,$$

where the last equality holds because r is a congruence (and so it is congruence of its own quotient). Q.E.D.

The following corollary is trivial.

COROLLARY 4.2. $s \subset t \Rightarrow \Gamma(s) \subset \Gamma(t)$.

We are interested in congruences generated by squares, i.e., congruences of the form $\Gamma(i^2)$ where $i : I \to A$ is a subobject of A. Consider the following examples.

EXAMPLES.

(1) Category of Sets. Let A be a set with a subset $I \subseteq A$, then $\Gamma(I^2) = I^2 \cup \Delta_A$. In other words, it is the equivalence relation in A whose equivalence classes are I itself and the singletons {a} where $a \notin I$. This example shows that not every congruence relation in A is of the form $\Gamma(I^2)$, for some $I \subseteq A$, because an equivalence relation can have more than one non-trivial equivalence class (i.e., a class with two or more elements).

(2) Category of Semigroups. Consider the semigroup defined in A = {0, 1, 2, 3} by the table

•	0	1	2	3
0	0	1	0	1
1	1	1	1	1
2	2	3	2	3
3	3	3	3	3

The semigroup operation in A^2 is given by $(a,b)\cdot(a',b') = (a\cdot a', b\cdot b')$. Let us construct $\Gamma(I^2)$ where I is the subsemigroup of A, I = {0,1}. First, $\Gamma(I^2)$ must contain the diagonal Δ_A and the square I^2. Since $(2,2) \in \Delta_A$, $(1,0) \in I^2$, and $\Gamma(I^2)$ is a sub-semigroup of A^2, we have that $(2,2)\cdot(1,0) = (2\cdot 1,\ 2\cdot 0) = (3,2) \in \Gamma(I^2)$. Besides, $\Gamma(I^2)$ is an equivalence relation, and so $(3,2) \in \Gamma(I^2)$ implies $(2,3) \in \Gamma(I^2)$. In this way we get a subset $S \subseteq \Gamma(I^2)$ represented in the following graph.

	(3,0)	(3,1)	(3,2)	(3,3)
	(2,0)	(2,1)	(2,2)	(2,3)
S →	(1,0)	(1,1)	(1,2)	(1,3)
	(0,0)	(0,1)	(0,2)	(0,3)

Clearly, S is an equivalence relation in A with equivalence classes $I = \{0,1\}$ and $J = \{2,3\}$. Moreover, it is easy to check that S is a sub-semigroup of A^2. This means that S is a congruence relation containing the square I^2. Hence, $\Gamma(I^2) \subseteq S$, and so $\Gamma(I^2) = S$. If we construct $\Gamma(J^2)$ where $J = \{2,3\}$ is also a sub-semigroup of A, we find that $\Gamma(J^2) = \Gamma(I^2)$, in spite of the fact that $I \cap J = \phi$.

(3) Category of Groups. Let A be a group with a subgroup $G \subseteq A$. If $\overline{G}$ denotes the normal closure of G in A, it is well known that we can define a congruence relation in A by

$$(x,y) \in R_G \Leftrightarrow xy^{-1} \in \overline{G}$$

We can show that $\Gamma(G^2) = \Gamma(\overline{G}^2) = R_{\overline{G}}$. To see this, observe that in Groups any congruence is of the form R_N for some normal subgroup N. (If R is the congruence relation of a homomorphism $f : A \to B$, then $R = R_N$ where N is the kernel of f). Therefore, $\Gamma(G^2) = R_N$ for some normal subgroup N. Let e be the identity of the group A. If $x \in G$, then $(x,e) \in G^2$. But $G^2 \subseteq \Gamma(G^2) = R_N$; hence, $xe^{-1} = x \in N$. This shows that $G \subseteq N$. Since N is normal, this implies that $\overline{G} \subseteq N$ and so $R_{\overline{G}} \subseteq R_N$. On the other hand, $G^2 \subseteq \overline{G}^2 \subseteq R_{\overline{G}}$. By Lemma 4.1 and Corollary 4.2, this gives:

$$\Gamma(G^2) \subseteq \Gamma(\overline{G}^2) \subseteq R_{\overline{G}} \subseteq R_N = \Gamma(G^2).$$

Hence,

$$\Gamma(G^2) = \Gamma(\overline{G}^2) = R_{\overline{G}}.$$

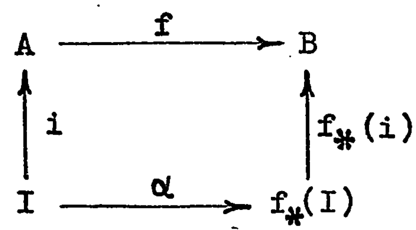
Since any congruence is of the form R_N for some normal subgroup N, and $\Gamma(N^2) = R_{\overline{N}} = R_N$, we obtain that in the Category of Groups any congruence is generated by a square (compare this with the case of Sets in example (1)).

Returning to the general situation, we will show that the property of being generated by a square is preserved by the lattice isomorphism of the First Isomorphism Theorem. This is a corollary of the following two more general lemmas.

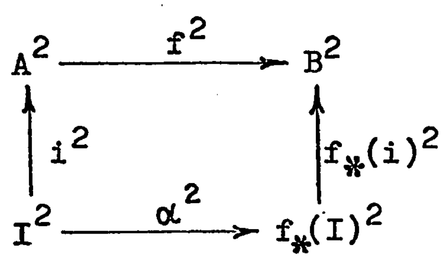
LEMMA 4.3. Let f be a map in a category with properties A - 1 to A - 4. For any subobject i of A we have:

$$f_*^2(i^2) \simeq f_*(i)^2.$$

<u>Proof</u>. By the FHT we have a commutative diagram



where $f_*(i)$ is the image of i by f, and α is a quotient. By squaring, we obtain a commutative diagram



where $f_*(i)^2$ is mono (since $f_*(i)$ is mono) and α^2 is a quotient (by A - 3). Hence, by the uniqueness of the canonical factorization, we have that $f_*(i)^2$ is the image by f^2 of i^2. Q.E.D.

LEMMA 4.4. Assume properties A - 1 to A - 4. Let $r : R \to A^2$ be a congruence in A with quotient $\eta : A \to A/R$. If $s : S \to A^2$ is a subobject of A^2 such that $\Gamma(s) \supset r$, then

$$\eta_*^2(\Gamma(s)) \approx \Gamma(\eta_*^2(s)).$$

<u>Proof</u>. Since $\Gamma(s) \supset r$, $\eta_*^2(\Gamma(s))$ is a congruence in A/R by the First Isomorphism Theorem. Moreover, $\eta_*^2(s) \subset \eta_*^2(\Gamma(s))$ because $s \subset \Gamma(s)$. Therefore, by Lemma 4.1 we have

$$\Gamma(\eta_*^2(s)) \subset \eta_*^2(\Gamma(s)).$$

To see the other inclusion, let $\omega = \Gamma(\eta_*^2(s))$. By the First Isomorphism Theorem, the congruence ω is the image by η^2 of a unique congruence c in A such that $c \supset r$; namely $c = (\eta^2)^{-*}(\omega)$, the inverse image of ω by η^2. Since $\eta_*^2(s) \subset \Gamma(\eta_*^2(s)) = \omega$ and inverse images preserve inclusion, we have

$$s \subset (\eta^2)^{-*}(\eta_*^2(s)) \subset (\eta^2)^{-*}(\omega) = c,$$

where the first inclusion is a general property of images and inverse images. Since c is a congruence, this means that $\Gamma(s) \subset c$. Taking images, we obtain

$$\eta_*^2(\Gamma(s)) \subset \eta_*^2(c) = \omega = \Gamma(\eta_*^2(s)). \quad \text{Q.E.D.}$$

COROLLARY 4.5. Assume properties A - 1 to A - 4. Let $r : R \to A^2$ be a congruence in A with a quotient $\eta : A \to A/R$. Let $i : I \to A$ be a subobject of A such that $\Gamma(i^2) \supset r$. Then

$$\eta_*^2(\Gamma(i^2)) \approx \Gamma(\eta_*(i)^2).$$

2. Normal Subobjects.

DEFINITION. Let n be a subobject of A. n is normal in A if, and only if, for any subobject i of A, $i^2 \subset \Gamma(n^2)$ implies $i \subset n$.

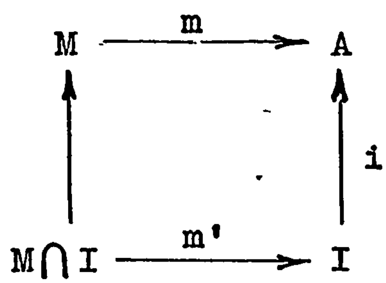
We will see that this definition is not as artificial as it looks. In particular, it means that n^2 is the maximum square generating the congruence $\Gamma(n^2)$. We will show that the First Isomorphism Theorem, which is given in Groups in terms of normal subgroups, is also valid for our normal subobjects. More significant is the fact that if we add to our assumptions (properties A - 1 to A - 4) the existence of a zero-object, this definition of normal subobject coincides with that usually given for categories with zero-object; i.e., n is normal in A if n is the inverse image of zero through some map with domain A (see Mitchell [4]). In analogy with the case of groups, we can prove from the definition the following facts.

LEMMA 4.6. If m and n are normal subobjects of A, then $m \cap n$ is normal in A.

Proof. Since $m \cap n \subset m$, then $(m \cap n)^2 \subset m^2$ and $\Gamma((m \cap n)^2) \subset \Gamma(m^2)$. Similarly, $\Gamma((m \cap n)^2) \subset \Gamma(n^2)$. Let i be a subobject of A such that $i^2 \subset \Gamma((m \cap n)^2)$, then $i^2 \subset \Gamma(m^2)$

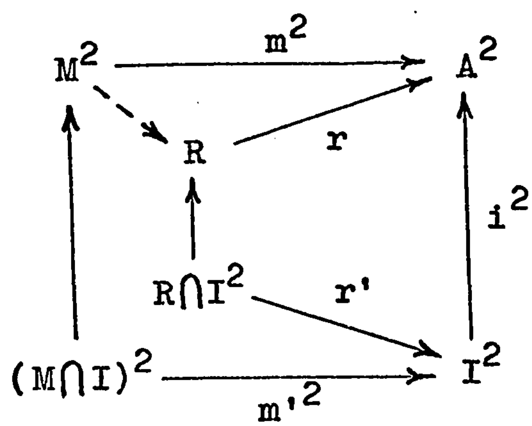
and $i^2 \subset \Gamma(n^2)$. Since m and n are normal, this means $i \subset m$ and $i \subset n$; hence, $i \subset m \cap n$. Q.E.D.

LEMMA 4.7. Let $m : M \to A$ be normal in A and let $i : I \to A$ be an arbitrary subobject. Then $m' : M \cap I \to I$, the subobject of I induced by the intersection of i and m,



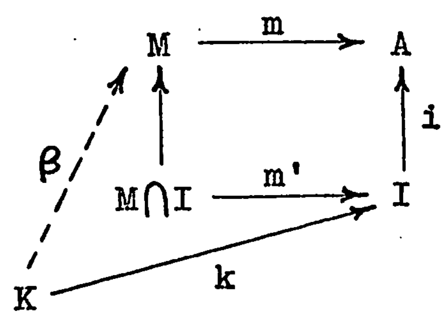
is normal in I.

Proof. Let $r = \Gamma_A(m^2)$. Consider the following commutative diagram



where the inner square is an intersection diagram for r and i^2, and the dotted arrow results from the fact that $m^2 \subset \Gamma_A(m^2) = r$. Since the exterior square commutes and the inner one is a pullback, there exists a map $\gamma : (M \cap I)^2 \to R \cap I^2$ such that $m'^2 = r'\gamma$. In other words, $m'^2 \subset r'$. But we know that r', the restriction of r to I, is a congruence in I; hence, $\Gamma_I(m'^2) \subset r'$.

Suppose now that $k : K \to I$ is a subobject of I such that $k^2 \subset \Gamma_I(m'^2)$; then $k^2 \subset r'$. This means that $k^2 = r'\alpha$ for some α; hence, $(ik)^2 = i^2k^2 = i^2r'\alpha = (r \cap i^2)\alpha$. Thus, $(ik)^2 \subset (r \cap i^2) \subset r = \Gamma_A(m^2)$. Since m is normal in A, we conclude that $ik \subset m$. This means that $ik = m\beta$ for some β, and the following diagram commutes.



Since the inner square is a pullback, we have $k \subset m'$. Q.E.D.

EXAMPLE. 1_A is trivially normal in A.

EXAMPLE. (Category of Groups). In the third example of the first part of this chapter, we showed that for a normal subgroup N, $\Gamma(N^2) = R_{\overline{N}} = R_N$. Moreover, $G^2 \subseteq \Gamma(N^2)$ implies $G \subseteq N$, because

$$x \in G \Rightarrow (x,e) \in G^2 \subseteq \Gamma(N^2) = R_N \Rightarrow xe^{-1} = x \in N.$$

Therefore, normal in the sense of groups implies normal in our categorical sense. Conversely, let G be normal in our sense. Since $\Gamma(G^2) = R_{\overline{G}}$ and $\overline{G}^2 \subseteq R_{\overline{G}}$, we have $\overline{G}^2 \subseteq \Gamma(G^2)$ and so $\overline{G} \subseteq G$. Therefore, $G = \overline{G}$ is normal in the sense of groups.

3. First Isomorphism Theorem for Normal Subobjects.

Before proving this theorem we need the following lemma which shows how normality is preserved, under certain circumstances, by images and inverse images.

LEMMA 4.8. Let A be an object in a category with properties A - 1 to A - 4. Let $r : R \to A^2$ be a congruence in A with quotient $\eta : A \to A/R$.

(a) If n is a normal subobject of A such that $\Gamma(n^2) \supset r$, then $\eta_*(n)$ is normal in A/R.

(b) If n and m are normal subobjects of A with $\Gamma(n^2) \supset r$ and $\Gamma(m^2) \supset r$, then

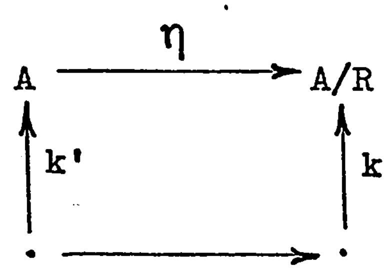
$$n \subset m \Leftrightarrow \eta_*(n) \subset \eta_*(m)$$

(c) If u is a normal subobject of A/R and $u' = \eta^{-*}(u)$ is an inverse image of u by η, then u' is normal in A.

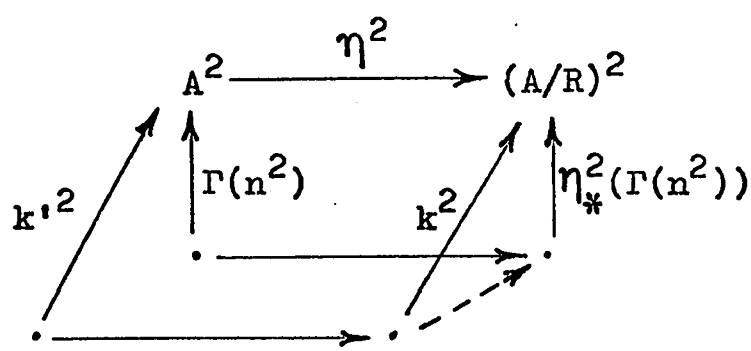
Proof. (a) Suppose that n is normal in A and $\Gamma(n^2) \supset r$.

By Corollary 4.5, we have that $\Gamma(\eta_*(n)^2) \simeq \eta_*^2(\Gamma(n^2))$. Since $\Gamma(n^2) \supset r$, we can apply the First Isomorphism Theorem and so $\Gamma(n^2)$ is the inverse image of $\eta_*^2(\Gamma(n^2))$.

Suppose now that k is a subobject of A/R such that $k^2 \subset \Gamma(\eta_*(n)^2)$. If k' is an inverse image of k by η, we have a commutative diagram



and $k = \eta_*(k')$, because η is a quotient. This diagram induces another one:



where the dotted arrow comes from the fact that $k^2 \subset \Gamma(\eta_*(n)^2) \simeq \eta_*^2(\Gamma(n^2))$. Since the inner square is a pullback, there exists α such that $k'^2 = \Gamma(n^2)\alpha$ and so

$k'^2 \subset \Gamma(n^2)$. But n is normal; therefore, $k' \subset n$. Taking images we obtain $k = \eta_*(k') \subset \eta_*(n)$. This shows that $\eta_*(n)$ is normal in A/R.

(b) $n \subset m \Rightarrow \eta_*(n) \subset \eta_*(m)$ is always true. Suppose now that $\eta_*(n) \subset \eta_*(m)$, then $\eta_*(n)^2 \subset \eta_*(m)^2$ and $\Gamma(\eta_*(n)^2) \subset \Gamma(\eta_*(m)^2)$. Using Corollary 4.5 we obtain: $\eta_*^2(\Gamma(n^2)) \subset \eta_*^2(\Gamma(m^2))$. Since $\Gamma(n^2)$ and $\Gamma(m^2)$ contain r, we can apply the First Isomorphism Theorem obtaining $\Gamma(n^2) \subset \Gamma(m^2)$. Therefore, $n^2 \subset \Gamma(m^2)$ and $n \subset m$ because m is normal in A.

(c) Let u be normal in A/R, and let u' be an inverse image of u by η. Since η is a quotient, we have that $u = \eta_*(u')$. To show that u' is normal in A, let k be a subobject of A such that $k^2 \subset \Gamma(u'^2)$; then $\eta_*^2(k^2) \subset \eta_*^2(\Gamma(u'^2)) \simeq \Gamma(\eta_*(u')^2) \simeq \Gamma(u^2)$. Moreover, $\eta_*^2(k^2) \simeq \eta_*(k)^2$ by Lemma 4.3; hence, $\eta_*(k)^2 \subset \Gamma(u^2)$. Since u is normal, this means that $\eta_*(k) \subset u$, and so

$$k \subset \eta^{-*}(\eta_*(k)) \subset \eta^{-*}(u) = u'. \quad \text{Q.E.D.}$$

The above lemma gives us almost all the elements to prove a First Isomorphism Theorem for normal subobjects.

THEOREM 4.9. (F.I.T. for normal subobjects) Consider a category with properties A - 1 to A - 4. Let $i : I \to A$ be a fixed subobject of A, $\Gamma(i^2) : \Gamma(I^2) \to A^2$ the congruence generated by its square, and $\eta : A \to A/\Gamma(I^2)$ a quotient of $\Gamma(i^2)$. Then η_* induces an order isomorphism between the class of (equivalence classes of) normal subobjects of A containing i, and the class of (equivalence classes of) normal subobjects of $A/\Gamma(I^2)$.

<u>Proof</u>. Let $r = \Gamma(i^2)$. If n is a normal in A such that $n \supset i$, then $n^2 \supset i^2$, and so $\Gamma(n^2) \supset \Gamma(i^2) = r$. Therefore, applying parts (a) and (b) of Lemma 4.8, we have that η_* sends normals in A containing i to normals in A/R, and the equivalence

$$n \subset m \Leftrightarrow \eta_*(n) \subset \eta_*(m)$$

holds for normals containing i. This means that we have an order isomorphism <u>into</u> the normal subobjects of A/R. It remains to prove that the correspondence is <u>onto</u>. By part (c) of the same lemma, any normal subobject of A/R is image by η of a normal subobject of A (namely, its inverse image); everything reduces to proving that this inverse image contains i.

Let u be normal in A/R and $u' = \eta^{-*}(u)$ its inverse image. We have:

$$\eta_*(i)^2 \subset \Gamma(\eta_*(i)^2) \simeq \eta_*^2(\Gamma(i^2)) = \eta_*^2(r).$$

By the First Isomorphism Theorem, the image of r must be the minimum congruence in A/R, i.e., the diagonal $\Delta_{A/R}$. Since any congruence contains the diagonal, this means that

$$\eta_*(i)^2 \subset \eta_*^2(r) = \Delta_{A/R} \subset \Gamma(u^2).$$

But u is normal; therefore $\eta_*(i) \subset u$. Taking inverse images:

$$i \subset \eta^{-*}(\eta_*(i)) \subset \eta^{-*}(u) = u'. \quad \text{Q.E.D.}$$

Observe that in this theorem the subobject i does not need to be normal.

The First Isomorphism Theorem for Groups becomes an application of Theorem 4.9. However, in Groups, the correspondence induced by η_* is also an order isomorphism if we drop the condition of normality, and we consider all the subgroups of A containing I and all the subgroups of $A/\Gamma(I^2)=A/R_I$. The following example shows that this is not the general case, even for algebraic categories.

EXAMPLE. (Category of Monoids). Consider the commutative monoid defined in A = {0, 1, 2, 3} by the table

$\cdot$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	1	1
3	3	3	1	1

The sub-monoids of A are:

$$S_1 = \{0\},\ S_2 = \{0,1\},\ S_3 = \{0,1,3\},\ S_4 = A$$

Let us consider $\Gamma(S_2{}^2)$. It is represented as a subset of A^2

	(3,0)	(3,1)	(3,2)	(3,3)
	(2,0)	(2,1)	(2,2)	(2,3)
$\Gamma(S_2{}^2) \rightarrow$	(1,0)	(1,1)	(1,2)	(1,3)
	(0,0)	(0,1)	(0,2)	(0,3)

Clearly, the given subset contains $S_2{}^2 = \{0,1\}^2$. It is an equivalence relation with equivalence classes $\overline{01} = \{0,1\}$ and $\overline{23} = \{2,3\}$. Moreover, it is easy to check that it is a sub-monoid of A^2. This means that it is really a congruence relation in the monoid A. To see that it coincides with $\Gamma(S_2{}^2)$, observe that $\Gamma(S_2{}^2)$ must contain the diagonal Δ_A. Since $(2,2) \in \Delta_A$ and $(0,1) \in S_2{}^2$, then $(2,3) = (0,1)\cdot(2,2) \in \Gamma(S_2{}^2)$. By symmetry, $(3,2) \in \Gamma(S_2{}^2)$.

Let $\Gamma = \Gamma(S_2{}^2)$. The quotient monoid A/Γ is given by the table

$\cdot$	$\overline{01}$	$\overline{23}$
$\overline{01}$	$\overline{01}$	$\overline{23}$
$\overline{23}$	$\overline{23}$	$\overline{01}$

and we have the following correspondence between sub-monoids of A containing S_2 and sub-monoids of A/Γ.

$$\begin{array}{l} A = \{0,1,2,3\} \mapsto \{\overline{01}, \overline{23}\} = A/\Gamma \\ S_3 = \{0,1,3\} \mapsto \{\overline{01}, \overline{23}\} \\ S_2 = \{0,1\} \mapsto \{\overline{01}\} \end{array}$$

Obviously, it is not one to one. The reason for this is that $S_3 = \{0,1,3\}$ is not a normal sub-monoid of A, because $\Gamma(\{0,1,3\}^2) = A^2$ and $A \not\subseteq \{0,1,3\}$. The other sub-monoids involved are normal in A. A is trivially normal, and S_2 is normal because the only "square" sub-monoids contained in $\Gamma(S_2{}^2)$ are $S_2{}^2$ and $S_1{}^2 = \{0\}^2$, but $S_1 \subseteq S_2$.

4. Normality in Categories With Zero.

The Category of Groups is an example of a category with a zero object. Consider the trivial group $\{e\}$. For any group G there exists a unique homomorphism $h : \{e\} \to G$; namely, the one sending e to the identity of G. Similarly, there exists a unique homomorphism $h' : G \to \{e\}$. We can generalize this definition.

DEFINITION. Let $\mathcal{A}$ be a category. An object 0 in $\mathcal{A}$ is a zero-object if for any object A in $\mathcal{A}$, there exists a unique map $A \to 0$ and a unique map $0 \to A$.

It can be proved from the definition that if 0 is a zero-object in $\mathcal{A}$, $0'$ will be also a zero-object if and only if the unique map $0 \to 0'$ is an isomorphism. Moreover, it is easy to see that $A \to 0$ is always epi and $0 \to A$ is a subobject of A. Let us define $0_A = (0 \to A)$, then $0_A \subset i$ for any sub-object i of A because $(0 \to I \overset{i}{\to} A) = (0 \to A)$. Finally, observe that $0^2 \simeq 0$. In terms of congruences and quotients, we have the following properties.

LEMMA 4.10. Let $\mathcal{A}$ be a category with finite limits, coequalizers, and a zero-object. Then for any object A in $\mathcal{A}$,

(a) $1_{A^2} = \mathrm{Congr}(A \to 0)$

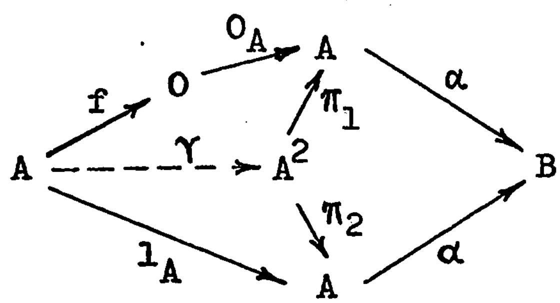
(b) $(A \to 0) = \mathrm{Quot}(1_{A^2})$

(c) $\Gamma(0_A{}^2) = \Delta_A$

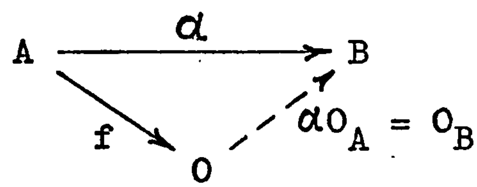
(d) 0_A is normal in A.

Proof. (a) Let $f = (A \to 0)$. There is a unique map $A^2 \to 0$; hence $f\pi_1 1_{A^2} = f\pi_2 1_{A^2}$. Therefore, $1_{A^2} \subset \mathrm{Congr}(f)$. But it is trivial that $\mathrm{Congr}(f) \subset 1_{A^2}$; hence, $\mathrm{Congr}(f) \simeq 1_{A^2}$.

(b) We have to show that $f = (A \to 0)$ is the coequalizer of π_1 and π_2. We already have $f\pi_1 = f\pi_2$. Suppose now that $\alpha : A \to B$ is a map such that $\alpha\pi_1 = \alpha\pi_2$. Let γ be the unique map making the following diagram commutative.



Then $\alpha 0_A f = \alpha\pi_1\gamma = \alpha\pi_2\gamma = \alpha 1_A = \alpha$. Therefore, we have a factorization



where the map $0 \to B$ is unique. This shows that $f = \text{Coeq}(\pi_1, \pi_2) = \text{Quot}(1_{A^2})$.

(c) Since $0^2 \simeq 0$ (as objects), we have $0_A{}^2 \simeq 0_{A^2}$ (as subobjects of A^2). But any sub-object of A^2 contains 0_{A^2}; hence, $0_A{}^2 \subset \Delta_A$. Since Δ_A is the minimum congruence in A, this means that $\Gamma(0_A{}^2) \simeq \Delta_A$.

(d) Consider a sub-object $i : I \to A$. Let π_1 and π_2 be the projections of A^2 and π_1', π_2' the projections of I^2. Suppose that $i^2 \subset \Delta_A$. We must show that $i \subset 0_A$. If $i^2 \subset \Delta_A$, then $i^2 = \Delta_A\alpha$ for some α; therefore,

$$i\pi_1' = \pi_1 i^2 = \pi_1 \Delta_A \alpha = \alpha = \pi_2 \Delta_A \alpha = \pi_2 i^2 = i\pi_2'.$$

Since $(I \to 0) = \mathrm{Coeq}(\pi_1', \pi_2')$ by part (b) of this theorem, the equality $i\pi_1' = i\pi_2'$ implies that i can be factored through $I \to 0$. So we have $(I \overset{i}{\to} A) = (I \to 0 \to A)$. But $(0 \to A) = 0_A$; therefore, $i \subset 0_A$. Q.E.D.

Let $f : A \to B$ be a map. A sub-object of A is called a <u>kernel of</u> f, written Ker(f), if it is the inverse image by f of 0_B. Clearly, Ker(f) determines a unique class of sub-objects of A. We want to show that a sub-object is normal if, and only if, it is a kernel of some map. First, we prove the following theorem which shows the relation between the kernel and the congruence relation of a map.

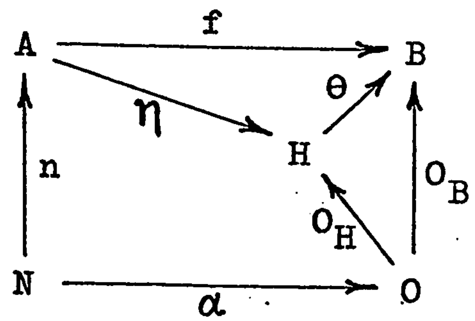
THEOREM 4.11. Let $f : A \to B$ be an arbitrary map in a category with properties A - 1 to A - 4 and a zero-object. Let $n = \mathrm{Ker}(f)$ and $r = \mathrm{Congr}(f)$. Then

(a) n is normal in A

(b) $\Gamma(n^2) \subset r$

(c) If r is generated by a square, $\Gamma(n^2) \approx r$.

<u>Proof</u>. (a) Let $f = \theta\eta$ be a canonical factorization of f. Since $n = f^{-*}(0_B)$, we have a commutative diagram



(1)

where the exterior rectangle is a pullback, and the triangle HBO commutes by the uniqueness of the map $0 \to B$. Since θ is mono, it is clear that the rectangle NAHO also commutes and is actually a pullback. Therefore, $n = \eta^{-*}(0_H)$. Since 0_H is normal in H, we can apply part (c) of Lemma 4.8 to obtain that n is normal in A.

(b) Let π_1 and π_2 be the projections of A^2 and π_1', π_2' the projections of N^2. Using diagram (1), we have $f\pi_1 n^2 = fn\pi_1' = 0_B\alpha\pi_1'$. Similarly, $f\pi_2 n^2 = fn\pi_2' = 0_B\alpha\pi_2'$. But $\alpha\pi_1' = \alpha\pi_2' = (N^2 \to 0)$; therefore, $f\pi_1 n^2 = f\pi_2 n^2$. By definition of r, this means that $n^2 \subset r$. Hence, $\Gamma(n^2) \subset r$.

(c) Let $r = \Gamma(i^2)$ where i is a sub-object of A. In diagram (1) of part (a), we have $\eta = \mathrm{Quot}(r)$ and $n = \eta^{-*}(0_H)$. By the First Isomorphism Theorem for normal sub-objects (Theorem 4.9), $n \supset i$. Therefore, $n^2 \supset i^2$ and $\Gamma(n^2) \supset \Gamma(i^2) = r$. From this and part (b), we obtain $\Gamma(n^2) \simeq r$. Q.E.D.

As a corollary of this theorem we have a characterization of normal sub-objects.

THEOREM 4.12. Let A be an object in a category with properties A - 1 to A - 4 and a zero object. Then n is a normal sub-object of A if and only if n is a kernel.

<u>Proof</u>. Part (a) of Theorem 4.11 shows that a kernel is normal. To show the other direction suppose that n is normal in A. Let $r = \Gamma(n^2)$ and let η be a quotient of r. Then $r = \mathrm{Congr}(\eta)$. By part (a) of Theorem 4.11, $n' = \mathrm{Ker}(\eta)$ is normal. By part (c) of the same theorem $\Gamma(n'^2) \simeq r = \Gamma(n^2)$. Since n and n' are both normal, this means that $n \subset n'$ and $n' \subset n$. Hence $n \simeq n' = \mathrm{Ker}(\eta)$. This shows that n is a kernel. Q.E.D.

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