

Heisenberg categorification and wreath Deligne category

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Abstract

We define a faithful linear monoidal functor from the partition category, and hence from Deligne's category $\text{Rep}(\mathfrak{S}_t)$, to the additive Karoubi envelope of the Heisenberg category. We show that the induced map on Grothendieck rings is injective and corresponds to the Kronecker coproduct on symmetric functions.

We then generalize the above results to any group G , the case where G is the trivial group corresponding to the case mentioned above. Thus, to every group G we associate a linear monoidal category $\mathcal{P}ar(G)$ that we call a *group partition category*. We give explicit bases for the morphism spaces and also an efficient presentation of the category in terms of generators and relations. We then define an embedding of $\mathcal{P}ar(G)$ into the group Heisenberg category associated to G . This embedding intertwines the natural actions of both categories on modules for wreath products of G . Finally, we prove that the additive Karoubi envelope of $\mathcal{P}ar(G)$ is equivalent to a wreath product interpolating category introduced by Knop, thereby giving a simple concrete description of that category.

Résumé

Nous définissons un foncteur monoïdal linéaire et fidèle de la catégorie de partition, et donc de la catégorie de Deligne $\text{Rep}(\mathfrak{S}_t)$ dans l'enveloppe de Karoubi additive de la catégorie de Heisenberg. Nous montrons que la fonction induite par l'anneau de Grothendieck appliqué à ce foncteur est injective. Cette fonction correspond plus précisément au coproduit de Kronecker sur les fonctions symétriques.

Ensuite nous généralisons les résultats ci-dessus à un groupe quelconque G , le cas où G est le groupe trivial correspondant à celui mentionné précédemment. Ainsi, nous associons à chaque groupe G une catégorie monoïdale linéaire $\mathcal{P}ar(G)$ que nous appelons *catégorie de partition associée au groupe G* . Nous donnons une base explicite de l'espace des morphismes et une présentation efficace de cette catégorie en termes de générateurs et relations. Ensuite, nous définissons un plongement de la catégorie $\mathcal{P}ar(G)$ dans la *catégorie de Heisenberg associée au groupe G* . Ce plongement commute les actions naturelles de ces catégories sur les modules du *produit en couronne* du groupe G . Enfin, nous montrons que *l'enveloppe de Karoubi additive* de la catégorie $\mathcal{P}ar(G)$ est équivalente à la *catégorie du produit en couronne interpolée* introduite par Knop, donnant ainsi une description simple et concrète de cette catégorie.

Dedication

To my sister Nyobe Audrey Ruth Sandrine.

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Introduction

This thesis is based on the papers [NLSawCR19] and [NLS20]. Compared to those papers, this thesis contains more background material and additional details of some arguments.

The *partition category* is a \mathbb{k} -linear monoidal category, depending on a parameter d in the commutative ground ring \mathbb{k} , that encodes the homomorphism spaces between tensor powers of the permutation representation of all the finite symmetric groups in a uniform way. Its additive Karoubi envelope is the category $\text{Rep}(\mathfrak{S}_d)$, introduced by Deligne in [Del07]. Deligne’s category $\text{Rep}(\mathfrak{S}_d)$ interpolates between categories of representations of symmetric groups in the sense that the category of representations of \mathfrak{S}_n is equivalent to the quotient of $\text{Rep}(\mathfrak{S}_n)$ by the tensor ideal of negligible morphisms. The endomorphism algebras of the partition category are the *partition algebras* first introduced by Martin ([Mar94]) and later, independently, by Jones ([Jon94]) as a generalization of the Temperley–Lieb algebra and the Potts model in statistical mechanics. The partition algebras are in duality with the action of the symmetric group on tensor powers of its permutation representation; that is, the partition algebras generate the commutant of this action (see [HR05, Th. 3.6] and [CSST10, Th. 8.3.13]).

When working with linear monoidal categories in practice, it is useful to have two descriptions. First, one would like to have an explicit basis for each morphism space, together with an explicit rule for the tensor product and composition of elements of these bases. Second, one wants an efficient presentation of the category in terms of generators and relations. Such a presentation is particularly useful when working with categorical actions, since one can define the action of generators and check the relations. Both descriptions exist for the partition category. Bases for morphism spaces are given in terms of *partition diagrams* with simple rules for composition and tensor product. Additionally, there is an efficient presentation, which can be summarized as the statement that the partition category is the free \mathbb{k} -linear symmetric monoidal category generated by a d -dimensional special commutative Frobenius object.

Recall the linear monoidal category $\text{Rep}(\mathfrak{S}_t)$ introduced by Deligne in [Del07]. When t is a nonnegative integer n , the category of representations of \mathfrak{S}_n is equivalent to the quotient of $\text{Rep}(\mathfrak{S}_n)$ by the tensor ideal of negligible morphisms. In [Kho14], Khovanov defined another linear monoidal category, the *Heisenberg category* $\mathcal{H}eis$, which is also motivated by the representation theory of the symmetric groups. In particular, $\mathcal{H}eis$ acts on $\bigoplus_{n \geq 0} \mathfrak{S}_n\text{-mod}$, where its two generating objects act by induction $\mathfrak{S}_n\text{-mod} \rightarrow \mathfrak{S}_{n+1}\text{-mod}$ and restriction $\mathfrak{S}_{n+1}\text{-mod} \rightarrow \mathfrak{S}_n\text{-mod}$. Morphisms in $\mathcal{H}eis$ act by natural transformations between compositions of induction and restriction functors.

Deligne's category $\text{Rep}(\mathfrak{S}_n)$ can be thought of as describing the representation theory of \mathfrak{S}_n for arbitrary n in a uniform way, but with n fixed (and not necessarily a nonnegative integer). On the other hand, the Heisenberg category goes further, allowing n to vary and describing the representation theory of all the symmetric groups at once. Thus, it is natural to expect a precise relationship between the two categories, with the Heisenberg category being larger. One main goal of this thesis is to describe such a relationship.

We describe our results in the case where t is generic. Our first main result (Theorems 3.1.1 and 3.2.2 and Remark 3.1.2) is the construction of a faithful strict linear monoidal functor

$$\Psi_t: \mathcal{P}ar(t) \rightarrow \mathcal{H}eis.$$

This functor sends t to the clockwise bubble in $\mathcal{H}eis$ and is compatible with the actions of $\mathcal{P}ar(t)$ and $\mathcal{H}eis$ on categories of modules for symmetric groups (Theorem 3.2.1). Since Deligne's category $\text{Rep}(\mathfrak{S}_t)$ is the additive Karoubi envelope of the partition category, we have an induced faithful linear monoidal functor

$$\Psi_t: \text{Rep}(\mathfrak{S}_t) \rightarrow \text{Kar}(\mathcal{H}eis),$$

where $\text{Kar}(\mathcal{H}eis)$ denotes the additive Karoubi envelope of the Heisenberg category $\mathcal{H}eis$.

The Grothendieck ring of $\text{Rep}(\mathfrak{S}_t)$ is isomorphic to the ring Sym of symmetric functions. On the other hand, the Grothendieck ring of $\mathcal{H}eis$ is isomorphic to a central reduction Heis of the universal enveloping algebra of the Heisenberg Lie algebra. This was conjectured by Khovanov in [Kho14, Conj. 1] and recently proved in [BSW18, Th. 1.1]. We thus have an induced map

$$[\Psi_t]: \text{Sym} \cong K_0(\text{Rep}(\mathfrak{S}_t)) \rightarrow K_0(\mathcal{H}eis) \cong \text{Heis}.$$

Our second main result (Theorem 4.2.4) is that this map is injective and is given by the Kronecker coproduct on Sym . We also describe the map induced by Ψ_t on the traces (or zeroth Hochschild homologies) of $\text{Rep}(\mathfrak{S}_t)$ and $\mathcal{H}eis$.

Deligne’s original paper [Del07] has inspired a great deal of further research. Of particular importance for this thesis are the generalizations of Knop and Mori. In [Kno07], Knop generalized Deligne’s construction by embedding a regular category \mathcal{A} into a family of pseudo-abelian tensor categories $\mathcal{T}(\mathcal{A}, \delta)$, which are the additive Karoubi envelope of categories $\mathcal{T}^0(\mathcal{A}, \delta)$ depending on a degree function δ . Deligne’s original construction corresponds to the case where \mathcal{A} is the category of finite boolean algebras (equivalently, the opposite of the category of finite sets). Knop’s construction, which is inspired by the calculus of relations on \mathcal{A} , has the advantage of being very general, but the disadvantage of being rather abstract. In particular, Knop does not give a presentation of his categories in terms of generators and relations.

In [Mor12], Mori generalized Deligne’s construction in a somewhat different direction. For each $d \in \mathbb{k}$, Mori defines a 2-functor \mathcal{S}_d sending a tensor category \mathcal{C} to another tensor category $\mathcal{S}_d(\mathcal{C})$, which should be thought of as a sort of interpolating wreath product functor. Morphisms are described in terms of recollements and one has a presentation using the string diagram calculus for braided monoidal categories.

In this thesis, we are interested in a setting where the constructions of Knop and Mori are closely related. This occurs when \mathcal{A} is the category of finite boolean algebras with a locally free action of a finite group G , and when \mathcal{C} is the category of representations of G . With these choices, the categories defined by Knop and Mori can both be viewed as interpolating categories for the categories of representations of the wreath products $G^n \rtimes \mathfrak{S}_n$. In fact, Mori’s interpolating category contains Knop’s as a full subcategory; see [Mor12, Rem. 4.14] for a precise statement. These “wreath Deligne categories” and other variations have been further studied in [Eti14, Har16, Ryb18, Ryb19].

Wreath products of groups and algebras appear in a surprising number of areas of mathematics, including vertex operators, the geometry of the Hilbert scheme, and categorification. In particular, to every Frobenius algebra (or, more generally, graded Frobenius superalgebra) A and choice of central charge $k \in \mathbb{Z}$, there is a *Frobenius Heisenberg category* $\mathcal{H}eis_k(A)$, introduced in [RS17, Sav19] and further studied in [BSW20]. When $k = \pm 1$, this category encodes the representation theory of *all* the wreath product algebras $A^n \rtimes \mathfrak{S}_n$, $n \in \mathbb{N}$, *simultaneously*. (For other choices of k it encodes the representation theory of more general cyclotomic quotients of the affine wreath product algebras introduced in [Sav20].) Generating objects of $\mathcal{H}eis_k(A)$ correspond to induction and restriction functors with respect to the natural embedding $G^n \rtimes \mathfrak{S}_n \hookrightarrow G^{n+1} \rtimes \mathfrak{S}_{n+1}$.

In this thesis, we are interested in the case where the Frobenius algebra A is the group algebra of a finite group G . In this case, we call $\mathcal{H}eis(G) := \mathcal{H}eis_{-1}(\mathbb{k}G)$ the *group Heisenberg category*. When G is trivial, this category is the category $\mathcal{H}eis$ and that particular case is explored in the first part of the thesis.

The second main goal of this thesis is to give a simple, explicit descriptions of wreath product analogues of partition categories and to relate these to group Heisenberg categories. First, to any group G we associate a G -partition category $\mathcal{P}ar(G)$ (Definition 5.2.6). The definition is given in terms of explicit G -partition diagrams, which form bases for the morphism spaces of the category. We then give an efficient presentation of $\mathcal{P}ar(G)$ in terms of generators and relations (Theorem 6.1.4). There is a natural categorical action of the G -partition category on tensor products of permutation representations of wreath products of G . This action can be described in terms of the generators (Theorem 6.2.1) or the bases of G -partition diagrams (Proposition 6.2.2). The action functor is full, and we give an explicit description of its kernel (Theorem 6.2.5). This gives a categorical analogue of a double centralizer property akin to Schur–Weyl duality, generalizing work of Bloss [Blo03] who defined G -colored partition algebras which are isomorphic to the endomorphism algebras in $\mathcal{P}ar(G)$.

Next, we give an explicit embedding of $\mathcal{P}ar(G)$ into the group Heisenberg category $\mathcal{H}eis(G)$ (Theorem 7.2.1), generalizing the results obtained in the first part of this thesis. This embedding intertwines the natural categorical actions of $\mathcal{P}ar(G)$ and $\mathcal{H}eis(G)$ on modules for wreath products (Theorem 7.3.1).

Finally, we prove (Theorem 8.2.5) that the group partition category $\mathcal{P}ar(G)$ is equivalent to Knop’s category $\mathcal{T}^0(\mathcal{A}, \delta)$ when \mathcal{A} is the category of finite boolean algebras with a locally free G -action (equivalently, the opposite of the category of finite sets with a free G -action). Thus, one can view $\mathcal{P}ar(G)$ as a concrete, and very explicit, realization of the wreath product interpolating categories of Knop and Mori. In particular, the calculus of G -partition diagrams is significantly simpler than the previous constructions. (Although the latter are, of course, more general.) In addition, the presentation of $\mathcal{P}ar(G)$, in terms of generators and relations, given in this thesis is considerably more efficient than the presentation given by Mori in [Mor12, Prop. 4.26]; see Remark 8.2.7 for further details.

The structure of this thesis is as follows. In Chapter 1 we give some background that will be used throughout in this thesis. In Chapter 2 we recall the definition of the partition category, Deligne’s category $\text{Rep}(\mathfrak{S}_t)$ and the Heisenberg category. We define the functor Ψ_t in Chapter 3, we show that it intertwines the natural categorical actions on categories

on modules of symmetric groups and that it is faithful for any commutative ring \mathbb{k} . In Chapter 4 we discuss the induced map on Grothendieck rings and traces. In Chapter 5 we recall some basic facts about wreath products and we define our main object of interest, the group partition category. We then give a presentation of $\mathcal{P}ar(G)$ in terms of generators and relations in Chapter 6 and we define the natural categorical action of $\mathcal{P}ar(G)$. We recall the definition of the group Heisenberg category $\mathcal{H}eis(G)$ in Chapter 7 and we define the embedding of $\mathcal{P}ar(G)$ into $\mathcal{H}eis(G)$. Then we prove that this embedding intertwines the natural categorical actions of these categories on modules for wreath products. Finally, in Chapter 8 we recall the definition of the category of finite boolean algebras and then we relate $\mathcal{P}ar(G)$ to the constructions of Knop and Mori.

New results.

Almost all the results presented in this thesis are new. In this section we summarize some of them. We let \mathbb{k} be a commutative ring and we fix $d \in \mathbb{k}$. Given any group G , we define a G -partition category $\mathcal{P}ar(G, d)$ in Definition 5.2.6. Then in Theorem 6.1.4 we give a more efficient presentation of the category $\mathcal{P}ar(G, d)$. When the group G is trivial, the category $\mathcal{P}ar(\{1_G\}, d)$ is the *partition category*. When the group G is finite, we define in Theorem 6.2.1 a categorical action of the G -partition category on the category of modules for the wreath product groups $G_n = G^n \rtimes \mathfrak{S}_n$. We denote $A_n = \mathbb{k}G_n$, the group algebra of G_n , so we can naturally identify A_n -modules and representations of G_n . We show that this action functor is full and we give an explicit description of its kernel in Theorem 6.2.5. The embedding of the category $\mathcal{P}ar(G)$ into the *group Heisenberg category* is given in Theorem 7.2.1. This embedding intertwines the natural categorical actions of $\mathcal{P}ar(G)$ and $\mathcal{H}eis(G)$ on modules for the wreath products as presented in Theorem 7.3.1. The embedding functor Ψ induces a ring homomorphism

$$[\Psi_d]: K_0(\text{Kar}(\mathcal{P}ar(G, d))) \rightarrow K_0(\text{Kar}(\mathcal{H}eis(G))), \quad [\Psi_d]([X]) = [\Psi_d(X)].$$

When the group G is trivial $\text{Kar}(\mathcal{P}ar(G, d))$ is the Deligne category $\text{Rep}(\mathfrak{S}_d)$ and the ring homomorphism $[\Psi_d]$ is injective and is given by the Kronecker coproduct on Sym as shown in Theorem 4.2.4. Finally we show in Theorem 8.2.5 that the group partition category $\mathcal{P}ar(G, d)$ is equivalent to the category $\mathcal{T}^0(\mathcal{A}, \delta)$ defined in [Kno07, Def. 3.2], where $\mathcal{A} = \mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{lf}}$ and δ is the degree function of [Kno07, (8.15)].

Future directions. The *Frobenius Heisenberg category* $\mathcal{H}eis_k(A)$ defined for any central charge $k \in \mathbb{Z}$, was associated to any Frobenius graded superalgebra A (see [RS17, Sav19]). It would be interesting to define its analogue, the partition category $\mathcal{P}ar_k(A)$ for any graded Frobenius superalgebra A and to generalize all the results mentioned above.

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Chapter 1

Preliminaries on (strict) monoidal categories

We start with some background on monoidal categories that will be use throughout this thesis. In this chapter all the categories are assumed to be *locally small*; that is for any objects $X, Y \in \mathcal{C}$, $\text{Hom}_{\mathcal{C}}(X, Y)$ is a set.

1.1 Additive and monoidal categories

Definition 1.1.1. [EGNO15, Definition 1.2.1] A category \mathcal{C} is an *additive category* if it satisfies the following axioms:

- (a) Every set $\text{Hom}_{\mathcal{C}}(X, Y)$ is equipped with the structure of an abelian group (written additively) such that composition of morphisms is biadditive with respect to this structure.
- (b) There exists a zero object $0 \in \mathcal{C}$ such that $\text{Hom}_{\mathcal{C}}(0, 0) = 0$.
- (c) For any objects $X_1, X_2 \in \mathcal{C}$ there exists a unique object (up to isomorphism) $X_1 \oplus X_2 \in \mathcal{C}$ and morphisms $p_1: X_1 \oplus X_2 \rightarrow X_1$, $p_2: X_1 \oplus X_2 \rightarrow X_2$, $i_1: X_1 \rightarrow X_1 \oplus X_2$, $i_2: X_2 \rightarrow X_1 \oplus X_2$ such that $p_1 i_1 = \text{id}_{X_1}$, $p_2 i_2 = \text{id}_{X_2}$ and $i_1 p_1 + i_2 p_2 = \text{id}_{X_1 \oplus X_2}$.

Example 1.1.2. The category of left modules over A , where A is an associative \mathbb{k} -algebra is additive.

Definition 1.1.3. A *monoidal category* is a quintuple $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$, where \mathcal{C} is a category, $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a bifunctor called the *tensor product* bifunctor, $a: (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$

is a natural isomorphism:

$$a_{X,Y,Z}: (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \quad X, Y, Z \in \mathcal{C} \quad (1.1.1)$$

called *the associative constraint* (or *associativity isomorphism*), $\mathbf{1} \in \mathcal{C}$ is an object of \mathcal{C} , and $\iota: \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ is an isomorphism, subject to the following two axioms.

1. The pentagon axiom. The diagram

$$\begin{array}{ccc}
 & ((W \otimes X) \otimes Y) \otimes Z & \\
 a_{W,X,Y} \otimes \text{id}_Z \swarrow & & \searrow a_{W \otimes X, Y, Z} \\
 (W \otimes (X \otimes Y)) \otimes Z & & (W \otimes X) \otimes (Y \otimes Z) \\
 \downarrow a_{W, X \otimes Y, Z} & & \downarrow a_{W, X, Y \otimes Z} \\
 W \otimes ((X \otimes Y) \otimes Z) & \xrightarrow{\text{id}_W \otimes a_{X, Y, Z}} & W \otimes (X \otimes (Y \otimes Z))
 \end{array}$$

is commutative for all objects $W, X, Y, Z \in \mathcal{C}$.

2. The unit axiom. The functors

$$L_1: X \mapsto \mathbf{1} \otimes X, \quad R_1: X \mapsto X \otimes \mathbf{1}, \quad \text{for any } X \in \mathcal{C}$$

of the left and right multiplication by $\mathbf{1}$ are autoequivalences of \mathcal{C} .

Example 1.1.4. (a) Any additive category (Definition 1.1.1) is monoidal, with the tensor product \otimes being the direct sum functor \oplus and the unit object $\mathbf{1}$ being the zero object.

(b) The category **Sets** of sets is a monoidal category, where the tensor product is the cartesian product and the unit object is a one element set.

(c) The category $\text{Vect}_{\mathbb{k}}$ of finite-dimensional vector spaces over a field \mathbb{k} is monoidal, where $\otimes = \otimes_{\mathbb{k}}$ and $\mathbf{1} = \mathbb{k}$.

(d) Let G be a group (not necessarily finite). The category $\text{Rep}_{\mathbb{k}}(G)$ of all representations of the group G over a field \mathbb{k} is monoidal. The tensor product of representations here being $\otimes_{\mathbb{k}}$ and the unit object is the one-dimensional trivial representation $\mathbf{1} = \mathbb{k}$.

1.2 Strict monoidal categories and string diagrams

1.2.1 Strict monoidal categories

Here we will work with a particular family of monoidal categories, called *strict monoidal categories*. The definitions that we give here can be found in [Mac71, §7.1]. The notion of string diagrams calculus for such categories is the key point of the study of those. This approach is more intuitive and play an important role in categorification. The notation 1_X will denote the identity morphism of the object X in a monoidal category \mathcal{C} .

Definition 1.2.1. A (*strict*) *monoidal category* is a category \mathcal{C} equipped with

- a bifunctor (the tensor product) $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, $(X, Y) \mapsto \otimes(X, Y) := X \otimes Y$ and
- a unit object $\mathbb{1}$,

such that, for all objects $X, Y, Z \in \mathcal{C}$, we have

- $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and
- $\mathbb{1} \otimes X = X = X \otimes \mathbb{1}$,

where for all morphisms f, g, h in \mathcal{C} , we denote $\otimes(f, g) := f \otimes g$; and we have,

- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $\text{id}_{\mathbb{1}} \otimes f = f = f \otimes \text{id}_{\mathbb{1}}$.

Remark 1.2.2. One may observe that Definition 1.2.1 is the same as Definition 1.1.3 where isomorphisms are replaced by equalities. Mac Lane's coherence theorem for monoidal categories asserts that every monoidal category is monoidally equivalent to a strict one (see [Kas95, Theorem XI.5.3.]).

Example 1.2.3. Consider the category $\text{End}(\mathcal{C})$ of *endofunctors* of a category \mathcal{C} . The objects are functors from \mathcal{C} to itself and morphisms are natural transformations. The monoidal structure \otimes is given by composition of functors and the unit object is the identity functor. So given three functors $F, G, H: \mathcal{C} \rightarrow \mathcal{C}$ in \mathcal{C} , we have

$$\otimes(F, G) := F \otimes G = FG.$$

Recall that the composition of functors is defined on any object X of \mathcal{C} and on any morphism f in \mathcal{C} , respectively by $FG(X) = F(G(X))$ and $FG(f) = F(G(f))$. It follows that the composition of functors is associative and as a consequence we get

$$F \otimes (G \otimes H) = (F \otimes G) \otimes H.$$

Moreover, the identity functor $\mathbf{1}_{\mathcal{C}}$ satisfies $\mathbf{1}_{\mathcal{C}}F = F = F\mathbf{1}_{\mathcal{C}}$. Therefore, we have

$$\mathbf{1}_{\mathcal{C}} \otimes F = F = F \otimes \mathbf{1}_{\mathcal{C}}.$$

Morphisms in this category are given by *natural transformation*. Recall that a natural transformation $\eta: F \Rightarrow G$ is a family of morphisms $\eta_X: F(X) \rightarrow G(X)$, $X \in \mathcal{C}$, such that for any morphism $f: X \rightarrow Y$ in \mathcal{C} , we have that the following diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

is commutative. Given another natural transformation $\theta: G \Rightarrow H$, the (vertical) composition of η and θ is defined for any object X by $(\theta\eta)_X = \theta_X\eta_X$. Therefore the tensor product of morphisms $\theta: F \Rightarrow G$ and $\theta': F' \Rightarrow G'$ in the category $\text{End}(\mathcal{C})$ is given by (horizontal) composition

$$\otimes(\theta, \theta') = \theta \otimes \theta': FF' \Rightarrow GG'$$

and defined by

$$\begin{array}{ccc} \begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{C} \\ \uparrow \theta & & \uparrow \theta' \\ \mathcal{C} & \xrightarrow{F} & \mathcal{C} \end{array} & \begin{array}{ccc} \mathcal{C} & \xrightarrow{G'} & \mathcal{C} \\ \uparrow \theta' & & \uparrow \theta \\ \mathcal{C} & \xrightarrow{F'} & \mathcal{C} \end{array} & = & \begin{array}{ccc} \mathcal{C} & \xrightarrow{GG'} & \mathcal{C} \\ \uparrow \theta \otimes \theta' & & \uparrow \theta \otimes \theta' \\ \mathcal{C} & \xrightarrow{FF'} & \mathcal{C} \end{array} . \end{array} \quad (1.2.1)$$

The identity morphism of an object F in $\text{End}(\mathcal{C})$ is given by

$$\text{id}_F = \text{id}_{\mathbf{1}_{\mathcal{C}}}: \mathbf{1}_{\mathcal{C}} \Rightarrow \mathbf{1}_{\mathcal{C}} = \begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathbf{1}_{\mathcal{C}}} & \mathcal{C} \\ \uparrow \text{id}_{\mathbf{1}_{\mathcal{C}}} & & \uparrow \text{id}_{\mathbf{1}_{\mathcal{C}}} \\ \mathcal{C} & \xrightarrow{\mathbf{1}_{\mathcal{C}}} & \mathcal{C} \end{array} . \quad (1.2.2)$$

It follows from the above definition that for any morphisms η, θ and γ in $\text{End}(\mathcal{C})$, we have

$$\eta \otimes (\theta \otimes \gamma) = (\eta \otimes \theta) \otimes \gamma,$$

$$\text{id}_{\mathbf{1}_{\mathcal{C}}} \otimes \theta = \theta = \theta \otimes \text{id}_{\mathbf{1}_{\mathcal{C}}}.$$

Therefore $\text{End}(\mathcal{C})$ is a strict monoidal category.

Definition 1.2.4. [Kas95, Definition XI.4.1] Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbf{1}_{\mathcal{D}})$ be two monoidal categories. A *monoidal functor* from \mathcal{C} to \mathcal{D} is a triple (F, ϕ, ψ) where $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor, $\phi: \mathbf{1}_{\mathcal{D}} \rightarrow F(\mathbf{1}_{\mathcal{C}})$ is a morphism in \mathcal{D} , and

$$\psi(X, Y): F(X) \otimes_{\mathcal{D}} F(Y) \rightarrow F(X \otimes_{\mathcal{C}} Y)$$

is a family of natural transformations indexed by all couples (X, Y) of objects of \mathcal{C} such that the diagrams

$$\begin{array}{ccc} (F(X) \otimes_{\mathcal{D}} F(Y)) \otimes_{\mathcal{D}} F(Z) & \xrightarrow{\cong} & F(X) \otimes_{\mathcal{D}} (F(Y) \otimes_{\mathcal{D}} F(Z)) \\ \psi(X, Y) \otimes \text{id}_{F(Z)} \downarrow & & \downarrow \text{id}_{F(X)} \otimes \psi(Y, Z) \\ F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} F(Z) & & F(X) \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \\ \psi(X \otimes_{\mathcal{C}} Y, Z) \downarrow & & \downarrow \psi(X, Y \otimes_{\mathcal{C}} Z) \\ F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & \xrightarrow{\cong} & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z)) \end{array} \quad (1.2.3)$$

$$\begin{array}{ccc} \mathbf{1}_{\mathcal{D}} \otimes_{\mathcal{D}} F(X) & \xrightarrow{\cong} & F(X) \\ \phi \otimes \text{id}_{F(X)} \downarrow & & \uparrow \cong \\ F(\mathbf{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} F(X) & \xrightarrow{\psi(\mathbf{1}_{\mathcal{C}}, X)} & F(\mathbf{1}_{\mathcal{C}} \otimes_{\mathcal{C}} X) \end{array} \quad (1.2.4)$$

and

$$\begin{array}{ccc} F(X) \otimes_{\mathcal{D}} \mathbf{1}_{\mathcal{D}} & \xrightarrow{\cong} & F(X) \\ \text{id}_{F(X)} \otimes \phi \downarrow & & \uparrow \cong \\ F(X) \otimes_{\mathcal{D}} F(\mathbf{1}_{\mathcal{C}}) & \xrightarrow{\psi(X, \mathbf{1}_{\mathcal{C}})} & F(X \otimes_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}) \end{array} \quad (1.2.5)$$

commute for all objects X, Y and Z in \mathcal{C} . The morphisms ϕ and ψ are called the *coherence maps*.

- When the coherence maps are invertible, the functor F is called a *strong monoidal functor*.
- When the coherence maps are identities, the monoidal functor F is called a *strict monoidal functor*.

Definition 1.2.5. A (left) action of a monoidal category \mathcal{A} on a category \mathcal{C} is a monoidal functor

$$\mathcal{A} \rightarrow \text{End}(\mathcal{C}),$$

where the category $\text{End}(\mathcal{C})$ is the category of endofunctors defined in Example 1.2.3.

We recall the following well known concepts. Consider \mathcal{C} and \mathcal{D} two categories and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor from \mathcal{C} to \mathcal{D} . The functor F induces a morphism $F_{X,Y}: \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ for any $X, Y \in \mathcal{C}$. The functor F is said to be *full* if the morphism $F_{X,Y}$ is surjective and *faithful* if $F_{X,Y}$ is injective, for every $X, Y \in \mathcal{C}$.

Later on, all the (strict) monoidal categories that we will use in this thesis will be defined in terms of generators and relations.

Definition 1.2.6. Let \mathbb{k} be a commutative ring. A \mathbb{k} -linear category \mathcal{C} is a category such that

- for any two objects X and Y of \mathcal{C} , the hom-set $\text{Hom}_{\mathcal{C}}(X, Y)$ is a \mathbb{k} -module,
- composition of morphisms is \mathbb{k} -bilinear:

$$(\alpha f + \beta g) \circ h = \alpha(f \circ h) + \beta(g \circ h),$$

$$f \circ (\alpha g + \beta h) = \alpha(f \circ g) + \beta(f \circ h),$$

for all $\alpha, \beta \in \mathbb{k}$ and morphisms f, g and h in \mathcal{C} for which the operations above make sense.

A (strict) \mathbb{k} -linear monoidal category is a (strict) monoidal category which is also \mathbb{k} -linear and such that the tensor product on morphisms is \mathbb{k} -bilinear.

Example 1.2.7. The category of modules over a commutative ring \mathbb{k} is a \mathbb{k} -linear monoidal category. In particular, the category $\text{Vect}_{\mathbb{k}}$ of finite-dimensional vector spaces over a field \mathbb{k} defined in Example 1.1.4-(c) is an additive \mathbb{k} -linear monoidal category.

A \mathbb{k} -linear (monoidal) functor is a (monoidal) functor which is also a \mathbb{k} -linear homomorphism.

Remark 1.2.8. Any \mathbb{k} -linear category \mathcal{C} can be enlarged to an additive category by taking its *additive envelope* $\text{Add}(\mathcal{C})$. The objects of $\text{Add}(\mathcal{C})$ are finite direct sums

$$\bigoplus_{i=1}^n X_i,$$

of objects in \mathcal{C} . Morphisms

$$f: \bigoplus_{i=1}^n X_i \rightarrow \bigoplus_{j=1}^m Y_j$$

are $m \times n$ matrices, where the (j, i) -entry is given by

$$f_{i,j}: X_i \rightarrow Y_j,$$

where $f_{i,j}$ is the morphism from X_i to Y_j . Composition of morphisms is given by matrix multiplication.

Interchange Law. Assume that $X_1 \xrightarrow{f} X_2$ and $Y_1 \xrightarrow{g} Y_2$ are two morphisms in the \mathbb{k} -linear strict monoidal category \mathcal{C} . Then

$$(1_{X_2} \otimes g) \circ (f \otimes 1_{Y_1}) = \otimes((1_{X_2}, g)) \circ \otimes((f, 1_{Y_1})) = \otimes((1_{X_2}, g) \circ (f, 1_{Y_1})) = \otimes((f, g)) = f \otimes g.$$

Similarly, $(f \otimes 1_{Y_2}) \circ (1_{X_1} \circ g) = f \otimes g$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} X_1 \otimes Y_1 & \xrightarrow{1_{X_1} \otimes g} & X_1 \otimes Y_2 \\ f \otimes 1_{Y_1} \downarrow & \searrow f \otimes g & \downarrow f \otimes 1_{Y_2} \\ X_2 \otimes Y_1 & \xrightarrow{1_{X_2} \otimes g} & X_2 \otimes Y_2 \end{array}$$

Remark 1.2.9. Any commutative monoid A induces a structure of (strict) monoidal category with one object $\mathbb{1}$ and $\text{End}(\mathbb{1}) = A$. Conversely, given a monoidal category with one object $\mathbb{1}$, the set $\text{End}(\mathbb{1})$ is a commutative monoid under the composition, with the identity given by $1_{\mathbb{1}}$ (see [Sav18, Example 2.2] for more details).

The commutative monoid $\text{End}(\mathbb{1})$ is called the *center* of the a monoidal category \mathcal{C} . When \mathcal{C} is also \mathbb{k} -linear the monoid $\text{End}(\mathbb{1})$ has the structure of a \mathbb{k} algebra.

Definition 1.2.10. Let \mathbb{k} be a field and \mathcal{C} be a \mathbb{k} -linear monoidal category. A *left tensor ideal* I of \mathcal{C} is a collection of subspaces $I(X, Y) \subseteq \text{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathcal{C}$ such that for all $X, Y, Z, T \in \mathcal{C}$ the following holds:

- (a) For all $\alpha \in I(X, Y)$, $\beta \in \text{Hom}_{\mathcal{C}}(Y, Z)$ and $\gamma \in \text{Hom}_{\mathcal{C}}(Z, X)$ we have $\beta \circ \alpha \in I(X, Z)$ and $\alpha \circ \gamma \in I(Z, Y)$;
- (b) for all $\alpha \in I(X, Y)$ and $\beta \in \text{Hom}_{\mathcal{C}}(Z, T)$ we have $\beta \otimes \alpha \in I(X \otimes Z, Y \otimes T)$.

Similarly one can define a *right tensor ideal*. A *tensor ideal* is a left and right tensor ideal.

When I is a tensor ideal in a \mathbb{k} -linear monoidal category \mathcal{C} , one can define a new \mathbb{k} -linear monoidal category \mathcal{C}' , the quotient of \mathcal{C} by I as follows: the objects of \mathcal{C}' are objects of \mathcal{C} ; $\text{Hom}_{\mathcal{C}'}(X, Y) := \text{Hom}_{\mathcal{C}}(X, Y)/I(X, Y)$; the composition (of morphisms) and tensor product are the same as in \mathcal{C} .

1.2.2 Karoubi envelope

Definition 1.2.11 (Idempotent completion). The *idempotent completion* (or *Karoubi envelope*) of a category \mathcal{C} is the category $\text{Kar}(\mathcal{C})$ whose objects are pairs of the form (X, e) , where X is an object of \mathcal{C} and $e: X \rightarrow X$ is an idempotent in \mathcal{C} . Morphisms are triples

$$(e, f, e'): (X, e) \rightarrow (X', e'),$$

where $f: X \rightarrow X'$ is a morphism of \mathcal{C} such that $f = e' \circ f \circ e$.

The composition of morphisms in $\text{Kar}(\mathcal{C})$ is as in \mathcal{C} . Given an object (X, e) in $\text{Kar}(\mathcal{C})$, the identity arrow of this object is (e, e, e) . The category \mathcal{C} embeds fully and faithfully in $\text{Kar}(\mathcal{C})$ via the functor

$$F: \mathcal{C} \rightarrow \text{Kar}(\mathcal{C}), \quad X \mapsto (X, 1_X).$$

Recall that an idempotent $e: X \rightarrow X$ in \mathcal{C} *splits* if there exists an object Y in \mathcal{C} and arrows $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that

$$g \circ f = e, \quad \text{and} \quad f \circ g = 1_Y.$$

In the category $\text{Kar}(\mathcal{C})$, every idempotent splits. Moreover $\text{Kar}(\mathcal{C})$ is the smallest category containing \mathcal{C} and having that property. The object Y above is called the *image* of e .

Theorem 1.2.12 ([Bor94, Proposition 6.5.9]). (Universal property) Let $F: \mathcal{C} \rightarrow \mathcal{D}$, be a linear functor from a linear category \mathcal{C} to a linear category \mathcal{D} with split idempotents. Then F extends to a functor from $\text{Kar}(\mathcal{C})$ to \mathcal{D} uniquely up to natural isomorphism.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & \nearrow \exists! & \\ \text{Kar}(\mathcal{C}) & & \end{array}$$

Example 1.2.13. Suppose R is a ring. The idempotent completion of the category of free R -modules is equivalent to the category of projective R -modules.

Lemma 1.2.14. Let \mathcal{C}, \mathcal{D} be two categories and $\mathcal{C}', \mathcal{D}'$ two full subcategories of \mathcal{C} and \mathcal{D} , respectively. Assume that there exists a full functor $F: \mathcal{C} \rightarrow \mathcal{D}$. The following holds:

- (a) The restriction functor $F|_{\mathcal{C}'}: \mathcal{C}' \rightarrow \mathcal{D}$ is full.
- (b) Assume in addition that the category \mathcal{C} is additive. Then a full functor $\text{Kar}(\mathcal{C}) \rightarrow \mathcal{D}$ induces a full functor from $\mathcal{C} \rightarrow \mathcal{D}$.

Proof. (a) Obvious.

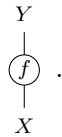
(b) Follows from $\text{Hom}_{\text{Kar}(\mathcal{C})}(X, Y) = e' \circ \text{Hom}_{\mathcal{C}}(X, Y) \circ e$, where $e: X \rightarrow X$ and $e': Y \rightarrow Y$ are idempotents of X and Y respectively.

□

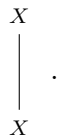
1.2.3 String diagrams

String diagrams allow one to build intuition and often make certain arguments obvious, whereas the corresponding algebraic argument can be a bit opaque. We give a quick introduction of that notion here. The reader can find a more detailed definition of this notion in [TV17, Chapter 2]. The language of string diagrams is the perfect tool for the illustration of strict monoidal categories.

Let \mathcal{C} be a \mathbb{k} -linear strict monoidal category. Using the language of string diagrams, we depict an arrow $f: X \rightarrow Y$ in \mathcal{C} by a strand with a coupon labelled f :



By convention, diagrams should be read from bottom to top. The identity map $1_X: X \rightarrow X$ is the string with no coupon:



We sometimes omit the objects labels when they are cleared or unimportant. By convention, we do not draw the identity morphism of the unit object $\mathbb{1}$.

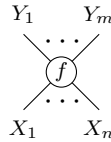
Composition of morphisms here is denoted by *vertical stacking* and tensor product of morphisms is *horizontal juxtaposition*:

$$\begin{array}{c} \circlearrowleft f \\ | \\ \circlearrowleft g \end{array} = \begin{array}{c} | \\ \circlearrowleft f \circ g \\ | \end{array} \quad \text{and} \quad \begin{array}{c} | \\ \circlearrowleft f \\ | \end{array} \otimes \begin{array}{c} | \\ \circlearrowleft g \\ | \end{array} = \begin{array}{cc} | & | \\ \circlearrowleft f & \circlearrowleft g \\ | & | \end{array}.$$

The *interchange* law defined in Section 1.2.1 becomes:

$$\begin{array}{c} | \\ \circ f \\ | \end{array} \begin{array}{c} | \\ \circ g \\ | \end{array} = \begin{array}{c} | \\ \circ f \\ | \end{array} \begin{array}{c} | \\ \circ g \\ | \end{array} = \begin{array}{c} | \\ \circ g \\ | \end{array} \begin{array}{c} | \\ \circ f \\ | \end{array} .$$

A general morphism $f: X_1 \otimes \cdots \otimes X_n \rightarrow Y_1 \otimes \cdots \otimes Y_m$ can be depicted as a coupon with n strands emanating from the bottom and m strands emanating from the top:



We draw endomorphisms $f \in \text{End}(\mathbb{1})$ of the unit object as a free-floating coupon:



(See [Sav18, Section 2] for the motivation of this notation.)

Remark 1.2.15. One can view a strict \mathbb{k} -linear monoidal category as a "two-dimensional" algebra. Besides the addition (on string diagrams), we have two "multiplications" (on morphisms): horizontal (the tensor product) and vertical (composition in the category). Therefore, as every algebra admits a *presentation* we can also define a presentation of any strict \mathbb{k} -linear monoidal category.

1.3 Examples

In the section Section 1.2.1 we have discussed the notion of strict \mathbb{k} -linear monoidal categories. In this section we give some example of such a category using the string diagram presentation. Again the arguments exposed here follow from [Sav18, Section 3]. Our generating morphisms here are diagrams and we can horizontally and vertical compose these generating morphisms (when vertical composition makes sense). The relations on generating morphisms allow us to make some local changes in our diagrams.

1.3.1 The symmetric category \mathcal{S} .

Define the strict \mathbb{k} -linear monoidal category \mathcal{S} as follows:

- one generating object \uparrow ,

- one generating morphism

$$\overleftarrow{\times} : \uparrow \otimes \uparrow \rightarrow \uparrow \otimes \uparrow,$$

- two relations

$$\begin{array}{c} \text{two strands crossing} \\ \text{with a crossing} \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array} \quad \text{and} \quad \begin{array}{c} \text{two strands crossing} \\ \text{with a crossing} \end{array} = \begin{array}{c} \text{two strands crossing} \\ \text{with a crossing} \end{array}. \quad (1.3.1)$$

One could write these relations in a more traditional algebraic manner, if so desired. More generally, the category \mathcal{S} has a universal property that allows one to see it as *the free symmetric \mathbb{k} -linear monoidal category on one object*. For example, if we let

$$s = \overleftarrow{\times} : \uparrow \otimes \uparrow \rightarrow \uparrow \otimes \uparrow,$$

then the two relations (1.3.1) become

$$s^2 = 1_{\uparrow \otimes \uparrow} \quad \text{and} \quad (s \otimes 1_{\uparrow}) \circ (1_{\uparrow} \otimes s) \circ (s \otimes 1_{\uparrow}) = (1_{\uparrow} \otimes s) \circ (s \otimes 1_{\uparrow}) \circ (1_{\uparrow} \otimes s).$$

Now, in any \mathbb{k} -linear category (monoidal or not), we have an endomorphism algebra $\text{End}(X)$ of any object X . The multiplication in this algebra is given by vertical composition. In \mathcal{S} , every object is of the form $\uparrow^{\otimes n}$ for some $n = 0, 1, 2, \dots$. An example of an endomorphism of $\uparrow^{\otimes 4}$ is

$$\begin{array}{c} \text{four strands with crossings} \\ \text{and a crossing} \end{array} + 2 \begin{array}{c} \text{four strands with crossings} \\ \text{and a crossing} \end{array}.$$

Using the relations, we see that this morphism is equal to

$$\begin{array}{c} \text{four strands with crossings} \\ \text{and a crossing} \end{array} + 2 \begin{array}{c} \text{four strands with crossings} \\ \text{and a crossing} \end{array} = \begin{array}{c} \text{four strands with crossings} \\ \text{and a crossing} \end{array} + 2 \begin{array}{c} \text{four strands with crossings} \\ \text{and a crossing} \end{array}.$$

Fix a positive integer n and recall that the group algebra $\mathbb{k}\mathfrak{S}_n$ of the symmetric group on n letters has a presentation with generators s_1, s_2, \dots, s_{n-1} (the simple transpositions) and relations. The following map

$$\mathbb{k}\mathfrak{S}_n \rightarrow \text{End}_{\mathcal{S}}(\uparrow^{\otimes n})$$

where s_i is sent to the crossing of the i -th and $(i + 1)$ -st strands (counting from right to left) is an isomorphism of algebras. So the category \mathcal{S} contains the group algebras of all of the symmetric groups.

1.3.2 Wreath product algebra category $\mathcal{W}(A)$.

Let A be an associative \mathbb{k} -algebra with basis \mathbf{B}_A . The *wreath product algebra category* $\mathcal{W}(A)$ is the strict \mathbb{k} -linear monoidal category obtained from \mathcal{S} by adding a generating morphisms

$$\uparrow_a ;$$

for any element $a \in A$, subject to the following relations:

$$\uparrow_1 = \uparrow, \quad \uparrow_{(\alpha a + \beta b)} = \alpha \uparrow_a + \beta \uparrow_b \quad \text{and} \quad \begin{array}{c} \uparrow \\ \bullet^a \\ \bullet^b \end{array} = \uparrow_{ab} \quad \text{for all } \alpha, \beta \in \mathbb{k}, a, b \in A. \quad (1.3.2)$$

We then impose the additional relation

$$\begin{array}{c} \nearrow \\ \bullet^a \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \bullet^a \\ \searrow \end{array}, \quad a \in A. \quad (1.3.3)$$

We call the closed circles appearing in the above diagrams *tokens*.

As an example of a diagrammatic proof, note that we can compose (1.3.3) on the top and bottom with a crossing to obtain

$$\begin{array}{c} \nearrow \\ \bullet^a \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \bullet^a \\ \searrow \end{array} \implies \begin{array}{c} \nearrow \\ \bullet^a \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \bullet^a \\ \searrow \end{array} \stackrel{(1.3.1)}{\implies} \begin{array}{c} \nearrow \\ \bullet^a \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \bullet^a \\ \searrow \end{array}.$$

So tokens also slide up-left through crossings. The category $\mathcal{W}(A)$ is the free symmetric \mathbb{k} -linear monoidal category generated by an object with endomorphism algebra A .

Fix a positive integer n . Let $A_n = A^{\otimes n} \rtimes \mathfrak{S}_n$ be the n -th *wreath product algebra* associated to A . As a \mathbb{k} -module,

$$A_n = A^{\otimes n} \otimes_{\mathbb{k}} \mathbb{k}\mathfrak{S}_n.$$

Multiplication is determined by

$$(a_1 \otimes \pi_1)(a_2 \otimes \pi_2) = a_1(\pi_1 \cdot a_2) \otimes \pi_1\pi_2, \quad a_1, a_2 \in A^{\otimes n}, \pi_1, \pi_2 \in \mathfrak{S}_n,$$

where $\pi_1 \cdot a_2$ denotes the natural action of $\pi_1 \in \mathfrak{S}_n$ on $a_2 \in A^{\otimes n}$ by permutation of the factors.

The wreath algebra $A_n = A^{\otimes n} \rtimes \mathfrak{S}_n$, has basis elements given by

$$\underbrace{(1, \dots, 1)}_{n-i \text{ entries}}, b, \underbrace{(1, \dots, 1)}_{i-1 \text{ entries}} s_i \cdots s_{n-1},$$

$i = 1, \dots, n$ and $b \in \mathbf{B}_A$ subject to some relations (see [Sav20, Definition 3.1]). The following map

$$A_n \rightarrow \text{End}_{\mathcal{W}(A)}(\uparrow^{\otimes n})$$

where s_i is sent to the crossing of the i -th and $(i+1)$ -st strands and $b^{(i)} := (\underbrace{1, \dots, 1}_{n-i \text{ entries}}, b, \underbrace{1, \dots, 1}_{i-1 \text{ entries}})$ to the i -th strand with a token b (counting from right to left) is an isomorphism of algebras. So the category $\mathcal{W}(A)$ contains all the wreath algebras A_n for any $n \in \mathbb{N}$.

Note that when $A = \mathbb{k}$, we get $\mathcal{W}(\mathbb{k}) = \mathcal{S}$ and $A_n = \mathbb{k}\mathfrak{S}_n$.

1.4 Pivotal categories and categorification

1.4.1 Duality

Definition 1.4.1. A *pairing* between two objects X and Y of a monoidal category \mathcal{C} is a morphism $\omega: X \otimes Y \rightarrow \mathbb{1}$. A pairing ω is *nondegenerate* if there is a morphism $\Omega: \mathbb{1} \rightarrow Y \otimes X$ in \mathcal{C} such that

$$(\text{id}_Y \otimes \omega)(\Omega \otimes \text{id}_Y) = \text{id}_Y \quad \text{and} \quad (\omega \otimes \text{id}_X)(\text{id}_X \otimes \Omega) = \text{id}_X. \quad (1.4.1)$$

The morphism Ω is determined by ω uniquely and we call it the *coevaluation* of ω .

A *right dual* of an object X of a monoidal category \mathcal{C} is a pair (X^\vee, ev_X) , where X^\vee is an object of \mathcal{C} and $\text{ev}_X: X \otimes X^\vee \rightarrow \mathbb{1}$ is a nondegenerate pairing.

Similarly, one can define the *left dual* ${}^\vee X$ of an object X by simply switching the order of objects in the tensor product of the above definition. By convention ${}^\vee \mathbb{1} = \mathbb{1} = \mathbb{1}^\vee$.

When they exist, the right and left dual of a specific object are unique up to unique isomorphism preserving the evaluation pairing. A monoidal category where each object has a right (resp. left) dual is called a *right (resp. left) rigid category*. A *rigid monoidal category* is a left and right rigid category.

Lemma 1.4.2 ([TV17, §1.6.3]). Let $X, Y \in \mathcal{C}$ be two objects in a rigid monoidal category. We have

$$X^\vee \otimes Y^\vee \cong (Y \otimes X)^\vee \quad \text{and} \quad {}^\vee X \otimes {}^\vee Y \cong {}^\vee(Y \otimes X).$$

Proof. We show the first isomorphism; the proof of the second one (dealing with left dual) is similar. By definition of the objects X^\vee, Y^\vee and $(X \otimes Y)^\vee$, there exist nondegenerate morphisms $\text{ev}_X: X \otimes X^\vee \rightarrow \mathbb{1}$ (with coevaluation map coev_X), $\text{ev}_Y: Y \otimes Y^\vee \rightarrow \mathbb{1}$ (with

coevaluation map coev_Y) and $\text{ev}_{X \otimes Y}: X \otimes Y \otimes (X \otimes Y)^\vee \rightarrow \mathbb{1}$ (with coevaluation map $\text{coev}_{X \otimes Y}$) respectively, each satisfying (1.4.1). We define the isomorphism

$$(\text{id}_{X^\vee \otimes Y^\vee} \otimes \text{ev}_{X \otimes Y})(\text{id}_{X^\vee} \otimes \text{coev}_Y \otimes \text{id}_{X \otimes (Y \otimes X)^\vee})(\text{coev}_X \otimes \text{id}_{(Y \otimes X)^\vee}): (Y \otimes X)^\vee \rightarrow X^\vee \otimes Y^\vee,$$

with the inverse given by the following isomorphism defined in [TV17, §1.6.3]

$$(\text{id}_{(Y \otimes X)^\vee} \otimes \text{ev}_Y)(\text{id}_{(Y \otimes X)^\vee \otimes Y} \otimes \text{ev}_X \otimes \text{id}_{Y^\vee})(\text{coev}_{Y \otimes X} \otimes \text{id}_{X^\vee \otimes Y^\vee}): X^\vee \otimes Y^\vee \rightarrow (Y \otimes X)^\vee.$$

□

It follows from the definition of left and right duals that ${}^\vee(X^\vee) \cong X \cong ({}^\vee X)^\vee$, for any object $X \in \mathcal{C}$. Note that the right dual of a morphism $f: X \rightarrow Y$ in a rigid monoidal category \mathcal{C} is a morphism $f^\vee: Y^\vee \rightarrow X^\vee$ defined by

$$f^\vee = (\text{id}_{X^\vee} \otimes \text{ev}_Y)(\text{id}_{X^\vee} \otimes f \otimes \text{id}_{Y^\vee})(\text{coev}_X \otimes \text{id}_{Y^\vee}).$$

(The left dual ${}^\vee f: {}^\vee Y \rightarrow {}^\vee X$ in \mathcal{C} can be defined similarly.)

Following [TV17, § 2.1.3] and [Sav18, § 4] we now present the above definition with the approach of string diagrams. Suppose a strict monoidal category has at least two objects \uparrow and \downarrow . A morphism $\mathbb{1} \rightarrow \downarrow \otimes \uparrow$ is represented by the string diagram

$$\cup \uparrow : \mathbb{1} \rightarrow \downarrow \otimes \uparrow.$$

We may decorate the cup with some symbol (tokens) if we have more than one such morphism. We call this map the *unit*. The fact that the bottom of the diagram is empty space indicates that the domain of this morphism is the unit object $\mathbb{1}$. Similarly, the *counit* morphism $\uparrow \otimes \downarrow \rightarrow \mathbb{1}$ can be represented by

$$\cap \downarrow : \uparrow \otimes \downarrow \rightarrow \mathbb{1}.$$

We say that \downarrow is *right dual* to \uparrow (and \uparrow is *left dual* to \downarrow) if we have morphisms

$$\cup \uparrow : \mathbb{1} \rightarrow \downarrow \otimes \uparrow. \quad \text{and} \quad \cap \downarrow : \uparrow \otimes \downarrow \rightarrow \mathbb{1}$$

such that

$$\downarrow \cup \uparrow = \downarrow \quad \text{and} \quad \uparrow \cap \downarrow = \uparrow. \quad (1.4.2)$$

Remark 1.4.3. Consider the 2-category of categories, where the 0-cells are categories, the 1-cells are functors between categories and 2-cells are natural transformations between functors (see [EGNO15, §2.12] for a more general definition of this notion). When we view the two objects \uparrow and \downarrow as 1-cells in a 2-category of categories, the relation (1.4.2) above is precisely the formulation of the fact that the unit and counit turn the functor \uparrow into a functor right adjoint to the functor \downarrow (and \downarrow into a functor left adjoint to \uparrow). Recall that two functors are *biadjoint* if they are both left and right adjoint to each other. The relation (1.4.3) below combined with (1.4.2) turn the 2-cells \uparrow and \downarrow into biadjoint pairs.

If \uparrow and \downarrow are both left and right dual to each other, then, in addition to (1.4.2), we also have morphisms

$$\smile : \mathbb{1} \rightarrow \uparrow \otimes \downarrow \quad \text{and} \quad \frown : \downarrow \otimes \uparrow \rightarrow \mathbb{1}$$

such that

$$\begin{array}{c} \uparrow \\ \smile \\ \downarrow \end{array} = \uparrow \quad \text{and} \quad \begin{array}{c} \downarrow \\ \frown \\ \uparrow \end{array} = \downarrow. \quad (1.4.3)$$

If \uparrow and \downarrow are both left and right dual to each other, then we may form closed diagrams of the form

$$\begin{array}{c} \circlearrowright \\ f \end{array}, \quad f \in \text{End}(\uparrow). \quad (1.4.4)$$

Such closed diagrams live in the center $\text{End}(\mathbb{1})$ of the category (see Remark 1.2.9). In the category $\text{Vect}_{\mathbb{k}}$, such a diagram corresponds to $\text{tr}(f)$, the usual trace of the linear map f .

1.4.2 Pivotal categories

Consider a rigid strict monoidal category \mathcal{C} and let X^\vee be the right dual of an object $X \in \mathcal{C}$. We denote the identity endomorphisms of X and X^\vee by upward and downward strands labeled X , respectively:

$$1_X = \begin{array}{c} \uparrow \\ X \end{array} \quad \text{and} \quad 1_{X^\vee} = \begin{array}{c} \downarrow \\ X \end{array}.$$

Hence, X^\vee been right dual to X means that we have morphisms

$$\begin{array}{c} X \\ \smile \end{array} : \mathbf{1} \rightarrow X^\vee \otimes X \quad \text{and} \quad \begin{array}{c} \frown \\ X \end{array} : X \otimes X^\vee \rightarrow \mathbf{1} \quad (1.4.5)$$

such that

$$\begin{array}{c} X \\ \smile \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ X \end{array} \quad \text{and} \quad \begin{array}{c} \frown \\ \uparrow \\ X \end{array} = \begin{array}{c} \uparrow \\ X \end{array}. \quad (1.4.6)$$

Suppose X and Y have right duals X^\vee and Y^\vee , respectively. Then every morphism

$$\begin{array}{c} Y \\ \uparrow \\ \textcircled{f} \\ \downarrow \\ X \end{array} \in \text{End}(X, Y) \quad \text{has right mate} \quad \begin{array}{c} X \\ \downarrow \\ \textcircled{f} \\ \downarrow \\ Y \end{array} \in \text{End}(Y^\vee, X^\vee).$$

We can then define a functor

$$\mathcal{R}: \mathcal{C} \rightarrow \mathcal{C}, \quad X \mapsto X^\vee.$$

This functor maps a morphism f in \mathcal{C} to his right mate f^\vee . The functor \mathcal{R} is a strong *contravariant* monoidal functor (see [TV17, § 1.6.3]). In particular, the functor \mathcal{R} induces a monoid anti-automorphism $\mathcal{R}_X: \text{End}(X) \rightarrow \text{End}(X^\vee)$ for any $X \in \mathcal{C}$. When \mathcal{C} is strict \mathbb{k} -linear monoidal, \mathcal{R}_X is an algebra anti-automorphism.

Likewise, if ${}^\vee X$ and ${}^\vee Y$ are *left* dual to X and Y , respectively, then every

$$\begin{array}{c} Y \\ \uparrow \\ \textcircled{f} \\ \downarrow \\ X \end{array} \in \text{End}(X, Y) \quad \text{has left mate} \quad \begin{array}{c} X \\ \downarrow \\ \textcircled{f} \\ \downarrow \\ Y \end{array} \in \text{End}({}^\vee Y, {}^\vee X).$$

This gives another strong contravariant monoidal endofunctor $\mathcal{L}: \mathcal{C} \rightarrow \mathcal{C}$.

Assume that the right dual of any two objects $X \in \mathcal{C}$ satisfies $(X^\vee)^\vee = X$. It follows that X^\vee is also *left* dual to X for every object X .

Definition 1.4.4 (Strict pivotal category). A strict monoidal category \mathcal{C} is a *strict pivotal category* if every object X has a right dual X^\vee with (fixed) morphisms (1.4.5) satisfying (1.4.6) and the following three additional conditions:

- (a) For all objects X and Y in \mathcal{C} ,

$$(X^\vee)^\vee = X, \quad (X \otimes Y)^\vee = Y^\vee \otimes X^\vee, \quad \mathbb{1}^\vee = \mathbb{1}.$$

- (b) For all objects X and Y in \mathcal{C} , we have

$$X \otimes Y \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} = \begin{array}{c} XY \\ \curvearrowright \\ \curvearrowright \end{array} \quad \text{and} \quad \begin{array}{c} X \otimes Y \\ \curvearrowright \\ \curvearrowright \end{array} = \begin{array}{c} XY \\ \curvearrowright \\ \curvearrowright \end{array}.$$

(c) For every morphism $f: X \rightarrow Y$ in \mathcal{C} , its right and left mates are equal:

$$\begin{array}{c} X \\ \downarrow \\ \text{---} \circlearrowleft (f) \text{---} \\ \downarrow \\ Y \end{array} = \begin{array}{c} X \\ \downarrow \\ \text{---} \circlearrowright (f) \text{---} \\ \downarrow \\ Y \end{array} . \tag{1.4.7}$$

Remark 1.4.5. (a) The condition (b) in the above definition gives the compatibility of the tensor product with duality data.

(b) If \mathcal{C} is strict pivotal, and we have a morphism

$$\begin{array}{c} Y \\ \uparrow \\ \text{---} \circlearrowleft (f) \text{---} \\ \downarrow \\ X \end{array} \in \text{Hom}_{\mathcal{C}}(X, Y) ,$$

we will denote by

$$\begin{array}{c} X \\ \downarrow \\ \text{---} \circlearrowleft (f) \text{---} \\ \downarrow \\ Y \end{array} := \begin{array}{c} X \\ \downarrow \\ \text{---} \circlearrowleft (f) \text{---} \\ \downarrow \\ Y \end{array} = \begin{array}{c} X \\ \downarrow \\ \text{---} \circlearrowright (f) \text{---} \\ \downarrow \\ Y \end{array} .$$

(c) A strict monoidal category \mathcal{C} defined in terms of generators and relations is pivotal when $(X^\vee)^\vee = X$ for each generating objects and the right and left mates of each generating morphisms are equal.

As explained in [TV17, §2.4], in a strict pivotal category, isotopic string diagrams represent the same morphism. This allows us to use geometric intuition and topological arguments when studying such categories.

Example 1.4.6. Categories where morphisms consist of planar diagrams *up to isotopy* are strict pivotal. This is the case for the Heisenberg categories defined in [Kho14, CL12, RS17, LS13], the categorified quantum group of [KL11] and the categories defined in Section 1.5.

We give more examples in the following two sections.

1.5 Temperley–Lieb category $\mathcal{TL}(\delta)$

One of the more common examples of a strict pivotal category is the *Temperley–Lieb category* $\mathcal{TL}(\delta)$, $\delta \in \mathbb{k}$. The definition of this category can be given in two manners: an algebraic

definition with bases of morphisms spaces and by a presentation with generators and relations using string diagrams. We will call a *simple graph* an undirected graph containing no loops or multiple edges.

1.5.1 Definition of the category $\mathcal{TL}(\delta)$

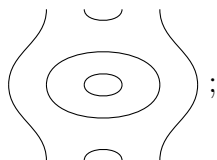
A *Temperley–Lieb diagram* of type $\binom{m}{n}$ is a simple graph having two horizontal rows with m vertices in the top row and n vertices in the bottom row pairwise linked by edges (within the horizontal rows) with no intersections between them. (The edges drawn within those horizontal rows are drawn *up to isotopy*.) The following diagram is an example of a Temperley–Lieb diagram of type $\binom{6}{4}$:

$$T_1 = \left(\begin{array}{c} \text{---} \cup \text{---} \\ \text{---} \cup \text{---} \end{array} \right).$$

Given two Temperley–Lieb diagrams T_1 and T_2 of type $\binom{m}{n}$ and $\binom{k}{m}$ respectively; we can compose them to obtain a Temperley–Lieb diagram of type $\binom{k}{n}$ that we denote by $T_2 \star T_1$. The composition is defined by stacking the diagram T_2 in the top of T_1 (identifying the bottom row of m vertices of T_2 with the top row of m vertices of T_1), by composing paths (up to isotopy) and then removing the connected component (circles) in the middle row of the stacking diagrams. We denote by $\alpha(T_1, T_2)$, the components in this middle.

Example 1.5.1. Below is the composition of two Temperley–Lieb diagrams, T_1 of type $\binom{6}{4}$ given above and T_2 of type $\binom{4}{6}$ given by

$$T_2 = \left(\begin{array}{c} \text{---} \cup \text{---} \\ \text{---} \cup \text{---} \end{array} \right).$$

After stacking the diagram T_2 at the top T_1 , we get: 

so that,

$$T_2 \star T_1 = \left| \begin{array}{c} \text{---} \cup \text{---} \\ \text{---} \cup \text{---} \end{array} \right|.$$

Here we have $\alpha(T_1, T_2) = 2$.

Definition 1.5.2. The *Temperley–Lieb category* $\mathcal{TL}(\delta)$ is a \mathbb{k} -linear monoidal category, where objects are non-negative integers \mathbb{N} . The sets of morphisms $\text{Hom}(n, m)$ between two given objects m and n is the set of formal \mathbb{k} -linear combinations of Temperley–Lieb diagrams of type $\binom{m}{n}$ when $n + m$ is even, and zero otherwise.

The monoidal structure is given by addition on \mathbb{N} and the unit object is zero. On morphisms the monoidal structure is given by horizontal juxtaposition of Temperley–Lieb diagrams, extended by linearity. The usual (vertical) composition of two Temperley–Lieb diagrams T, T' when it makes sense is given by

$$T \circ T' = \delta^{\alpha(T, T')} T \star T',$$

extended by linearity.

Example 1.5.3. The composition of the Temperley–Lieb diagrams T_1 and T_2 defined in Example 1.5.1 is given by:

$$T_2 \circ T_1 = \delta^{\alpha(T_2, T_1)} T_2 \star T_1 = \delta^2 \left| \begin{array}{c} \cup \\ \cap \end{array} \right|.$$

Lemma 1.5.4. The category $\mathcal{TL}(\delta)$ is strict pivotal.

Proof. For any objects ℓ, m, n of $\mathcal{TL}(\delta)$, the monoidal structure of $\mathcal{TL}(\delta)$ defined above gives:

$$(\ell \otimes m) \otimes n = \ell \otimes (m \otimes n)$$

$$m \otimes 0 = m = 0 \otimes m.$$

Likewise, we can easily check those axioms on morphisms. It follows that the Temperley–Lieb category is strict. Moreover, for any objects n in $\mathcal{TL}(\delta)$ the maps

$$\eta = \begin{array}{c} \dots \cup \dots \\ \cup \end{array} : 0 \rightarrow n \otimes n, \quad \text{and} \quad \varepsilon = \begin{array}{c} \cup \\ \dots \cap \dots \end{array} : n \otimes n \rightarrow 0,$$

satisfy

$$(\text{id}_n \otimes \varepsilon)(\eta \otimes \text{id}_n) = \text{id}_{2n} \quad \text{and} \quad (\varepsilon \otimes \text{id}_n)(\text{id}_n \otimes \eta) = \text{id}_{2n}.$$

Therefore for any n we have $n^\vee = n$ and so $(n^\vee)^\vee = n$, $(n \otimes m)^\vee = m^\vee \otimes n^\vee$ and $0^\vee = 0$. The axioms (b) and (c) of Definition 1.4.4 follow easily. \square

For $1 \in \mathbb{N}$, the endomorphism algebra $\text{End}_{\mathcal{TL}(\delta)}(1^{\otimes n})$ is the *Temperley–Lieb algebra* $TL_n(\delta)$. Below we give a presentation of the category $\mathcal{TL}(\delta)$.

1.5.2 Presentation of the category $\mathcal{TL}(\delta)$

The category $\mathcal{TL}(\delta)$ can also be presented as a strict \mathbb{k} -linear monoidal category on one generating object X and two generating morphisms,

$$\cup : \mathbb{1} \rightarrow X \otimes X, \quad \cap : X \otimes X \rightarrow \mathbb{1},$$

satisfying the following relations:

$$\begin{array}{c} \cup \\ \cap \end{array} = \begin{array}{c} | \\ | \end{array} = \begin{array}{c} \cap \\ \cup \end{array}. \quad (1.5.1)$$

We also impose the relation

$$\bigcirc = \delta. \quad (1.5.2)$$

Note that the generating object X given above can be identified with $1 \in \mathbb{N}$. Moreover, the generating morphisms with (1.5.1) make this object self-dual. Therefore in order to show that the category $\mathcal{TL}(\delta)$ is pivotal it suffices to show that the right and left mates of the generating morphisms are equal. Indeed, we have

$$\mathcal{R}(\cup) = \cap = \mathcal{L}(\cup) \quad \text{and} \quad \mathcal{R}(\cap) = \cup = \mathcal{L}(\cap).$$

1.6 Brauer category $\mathcal{B}(\delta)$

The *Brauer category* is also a strict pivotal \mathbb{k} -linear monoidal category which contains $\mathcal{TL}(\delta)$ as a subcategory. We give the definition and a presentation of this category below.

1.6.1 Definition of the category $\mathcal{B}(\delta)$

We call a *Brauer diagram* of type $\binom{m}{n}$ is a simple graph having two horizontal rows with m vertices in the top row and n vertices in the bottom row pairwise linked by edges (within the horizontal rows). (The edges drawn within those horizontal rows are drawn *up to isotopy*.) The main difference between a Brauer and a Temperley–Lieb diagrams is that the intersection of edges are allowed in a Brauer diagram. The following diagram is an example of a Brauer diagram of type $\binom{6}{4}$:



Definition 1.6.1. The *Brauer category* $\mathcal{B}(\delta)$ is a \mathbb{k} -linear monoidal category, where objects are non-negative integers \mathbb{N} . The set of morphisms $\text{Hom}(n, m)$ between two given objects m and n is the set of formal \mathbb{k} -linear combinations of diagrams of Brauer diagrams of type $\binom{m}{n}$ when $n + m$ is even and zero otherwise.

The monoidal structures are the same as in the category $\mathcal{TL}(\delta)$ and the unit object is zero. The usual (vertical) composition is given by vertical stacking as explained in Example 1.5.1.

When $X = 1 \in \mathbb{N}$, the endomorphism algebra $\text{End}_{\mathcal{B}(\delta)}(X^{\otimes n})$ is the *Brauer algebra* $\mathcal{B}_n(\delta)$. We will give below a presentation of the category $\mathcal{B}(\delta)$ in the next section.

Lemma 1.6.2. The category $\mathcal{B}(\delta)$ is strict pivotal.

Proof. The proof of this statement is similar to the proof we did in Lemma 1.5.4. □

1.6.2 Presentation of the category $\mathcal{B}(\delta)$

The presentation of the category $\mathcal{B}(\delta)$ is obtained from the one of $\mathcal{TL}(\delta)$ given in Section 1.5.2, but with one more generating morphism, that will call *crossing*,

$$\times : X \otimes X \rightarrow X \otimes X,$$

and additional relations:

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}, \quad \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} = \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array}; \tag{1.6.1}$$

$$\begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} = \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array}, \quad \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} = \begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array}. \tag{1.6.2}$$

Since

$$\mathcal{R}(\times) = \times = \mathcal{L}(\times),$$

it follows from the arguments presented in Section 1.5.2 that $\mathcal{B}(\delta)$ is also strict pivotal.

1.7 (De)categorification functors

Roughly speaking the *categorification* is a process that consists of lifting structure to a higher categorical level. Whereas, *decategorification* consists of doing the same procedure but in the

other direction. Thus decategorification is a procedure of reducing the complexity of a given mathematical structure and forgetting some information. The notion of categorification is most of the time better explained by the notion of decategorification. A bit more formally, a decategorification can be thought of as a map

$$(n - 1)\text{-category} \xleftarrow{\mathcal{D}} n\text{-category}$$

simplifying an n -categorical structure into an $(n - 1)$ -categorical structure.

Categorification addresses the following: given a specific $(n - 1)$ -category A , find an n -category B such that $A \cong \mathcal{D}(B)$. It turns out that the map \mathcal{D} above plays a key role in the process of (de)categorification. In the following two sections we give two examples of such map.

1.7.1 The Grothendieck ring

Let \mathcal{C} be an additive \mathbb{k} -linear category and $\text{Iso}_{\mathbb{Z}}(\mathcal{C})$ denote the free abelian group generated by isomorphism classes of objects of \mathcal{C} . Denote by $[X]_{\cong}$ the isomorphism class of an object $X \in \mathcal{C}$. Consider J , the subgroup generated by the elements

$$[X \oplus Y]_{\cong} - [X]_{\cong} - [Y]_{\cong}, \quad X, Y \text{ objects of } \mathcal{C}. \quad (1.7.1)$$

Definition 1.7.1. The *split Grothendieck group* of \mathcal{C} is

$$K_0(\mathcal{C}) := \text{Iso}_{\mathbb{Z}}(\mathcal{C})/J.$$

In other words, the split Grothendieck group of any additive category \mathcal{C} , is simply the free abelian group generated by isomorphisms classes of his objects satisfying the relation (1.7.1).

When \mathcal{C} is an additive \mathbb{k} -linear *monoidal* category, then $K_0(\mathcal{C})$ is a ring with multiplication given by

$$[X]_{\cong} \cdot [Y]_{\cong} = [X \otimes Y]_{\cong}$$

for objects X and Y of \mathcal{C} (with the multiplication extended to all of $K_0(\mathcal{C})$ by linearity). The ring axioms follow from Definition 1.2.1.

The functor \mathcal{D} here is K_0 defined from a 1-category (category defined as usual) to a 0-category (groups or rings). The following example is discussed in [Sav18, Example 5.2] and [BHLv17, §1.2].

Example 1.7.2 (deategorification of $\text{Vect}_{\mathbb{k}}$). Let $\text{Vect}_{\mathbb{k}}$ be the category in Example 1.2.7. Up to isomorphism, every vector space is determined uniquely by its dimension. Thus

$$\text{Iso}_{\mathbb{Z}}(\text{Vect}_{\mathbb{k}}) = \text{span}_{\mathbb{Z}}\{[\mathbb{k}^n]_{\cong} : n \in \mathbb{N}\}.$$

For an n -dimensional vector space V , we have

$$V \cong \mathbb{k}^{\oplus n},$$

and so $[V]_{\cong} = n[\mathbb{k}]_{\cong}$ in $K_0(\text{Vect}_{\mathbb{k}})$. Therefore, we have the following isomorphism

$$K_0(\text{Vect}_{\mathbb{k}}) \xrightarrow{\cong} \mathbb{Z}, \quad \sum_{i=1}^n a_i [V_i]_{\cong} \mapsto \sum_{i=1}^n a_i \dim V_i. \quad (1.7.2)$$

Since, for finite-dimensional vector spaces U and V , we have $\dim(U \otimes V) = (\dim U)(\dim V)$, the isomorphism (1.7.2) is one of rings. So, the ring \mathbb{Z} is a deategorification of the category $\text{Vect}_{\mathbb{k}}$.

1.7.2 The trace

There is another common (forgetful) functor \mathcal{D} used for (de)categorification which is defined for any \mathbb{k} -linear category. The following definition is given in [BHLv17, §3.2]

Definition 1.7.3. Let \mathcal{C} be a \mathbb{k} -linear category. The *trace*, or *zeroth Hochschild homology*, of \mathcal{C} is the \mathbb{k} -module

$$\text{Tr}(\mathcal{C}) := \left(\bigoplus_X \text{End}_{\mathcal{C}}(X) \right) / \text{span}_{\mathbb{k}}\{f \circ g - g \circ f\},$$

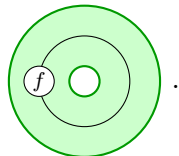
where the sum is over all objects X of \mathcal{C} , and f and g run through all pairs of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow X$ in \mathcal{C} .

We let $[f] \in \text{Tr}(\mathcal{C})$ denote the class of an endomorphism $f \in \text{End}_{\mathcal{C}}(X)$.

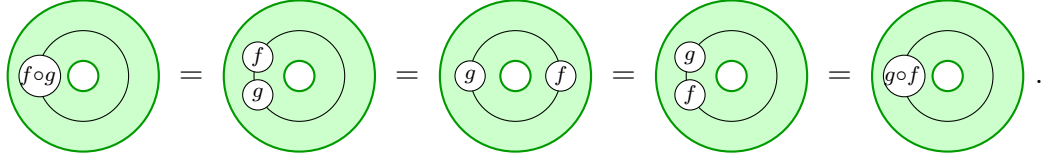
If the category \mathcal{C} is strict pivotal, we can think of the trace as consisting of diagrams on an annulus. In particular, if



is an endomorphism in \mathcal{C} , then we picture $[f]$ as



The fact that $[f \circ g] = [g \circ f]$ in $\text{Tr}(\mathcal{C})$ then corresponds to the fact we can slide diagrams around the annulus:



See also [TV17, §2.6] for more details of this approach.

If \mathcal{C} is a \mathbb{k} -linear monoidal category, then $\text{Tr}(\mathcal{C})$ is a ring, with multiplication given by

$$[f] \cdot [g] = [f \otimes g].$$

It follows that the trace gives us another method of *deategorification*. To categorify a \mathbb{k} -algebra R can mean to find a \mathbb{k} -linear monoidal category \mathcal{C} such that $\text{Tr}(\mathcal{C}) \cong R$ as \mathbb{k} -algebras. The example below justifies the use of the term *trace*.

Example 1.7.4 ([Sav18, §5.3]). Consider the category $\text{Vect}_{\mathbb{k}}$ defined in Example 1.2.7. We have an isomorphism

$$g: V \xrightarrow{\cong} \mathbb{k}^n.$$

For $1 \leq a \leq n$, define the inclusion and projection maps

$$\begin{aligned} i_a: \mathbb{k} &\rightarrow \mathbb{k}^n, & \alpha &\mapsto (0, \dots, 0, \alpha, 0, \dots, 0), \\ p_a: \mathbb{k}^n &\rightarrow \mathbb{k}, & (\alpha_1, \dots, \alpha_n) &\mapsto \alpha_a. \end{aligned}$$

Note that

$$p_b \circ i_a = \delta_{a,b} 1_{\mathbb{k}} \quad \text{and} \quad \sum_{a=1}^n i_a \circ p_a = 1_{\mathbb{k}^n}.$$

Now suppose $f: V \rightarrow V$ is a linear map. For $1 \leq a, b \leq n$, define

$$f_{a,b} = p_a \circ g \circ f \circ g^{-1} \circ i_b: \mathbb{k} \rightarrow \mathbb{k}.$$

Then we have

$$f = g^{-1} \circ g \circ f \circ g^{-1} \circ g = \sum_{a,b=1}^n g^{-1} i_a f_{a,b} p_b g.$$

Hence, in $\text{Tr}(\text{Vect}_{\mathbb{k}})$, we have

$$[f] = \sum_{a,b=1}^n [g^{-1} i_a f_{a,b} p_b g] = \sum_{a,b=1}^n [f_{a,b} p_b g g^{-1} i_a] = \sum_{a,b=1}^n [f_{a,b} p_b i_a] = \sum_{a=1}^n [f_{a,a}]. \quad (1.7.3)$$

So the class of $[f]$ is equal to the sum of classes of endomorphisms of \mathbb{k} given by its diagonal components in some basis. It follows that we have an isomorphism of rings

$$\mathrm{Tr}(\mathrm{Vect}_{\mathbb{k}}) \cong \mathbb{k}, \quad [f] \mapsto \mathrm{tr}(f). \quad (1.7.4)$$

In particular, $\mathrm{Vect}_{\mathbb{k}}$ is a trace categorification of the field \mathbb{k} .

Proposition 1.7.5 ([BGHL14, Proposition 3.2]). Let \mathcal{C} be a \mathbb{k} -linear linear category. The map

$$\mathrm{Tr}(\mathcal{C}) \rightarrow \mathrm{Tr}(\mathrm{Kar}(\mathcal{C})),$$

induced from the faithful functor $\mathcal{C} \hookrightarrow \mathrm{Kar}(\mathcal{C})$ is a bijective \mathbb{k} -module homomorphism.

1.7.3 The Chern character

There is a nice relationship between the split Grothendieck group of a category and the trace of that category, as we now explain. Suppose \mathcal{C} is an additive \mathbb{k} -linear category.

Lemma 1.7.6 ([BGHL14, Lem. 3.1]). If $f: X \rightarrow X$ and $g: Y \rightarrow Y$ are morphisms in \mathcal{C} , then

$$[f \oplus g] = [f] + [g]$$

in $\mathrm{Tr}(\mathcal{C})$.

Proof. We have $f \oplus g = (f \oplus 0) + (0 \oplus g): X \oplus Y \rightarrow X \oplus Y$. Thus

$$[f \oplus g] = [f \oplus 0] + [0 \oplus g].$$

Let

$$i: X \rightarrow X \oplus Y \quad \text{and} \quad p: X \oplus Y \rightarrow X$$

denote the obvious inclusion and projection. Then

$$[f \oplus 0] = [i f p] = [p i f] = [f].$$

Similarly, $[0 \oplus g] = [g]$. This completes the proof. \square

If X and Y are objects of \mathcal{C} , then we have

$$1_{X \oplus Y} = 1_X \oplus 1_Y.$$

Thus, by Lemma 1.7.6, we have

$$[1_{X \oplus Y}] = [1_X \oplus 1_Y] = [1_X] + [1_Y].$$

It follows that we have a well-defined map of abelian groups

$$h_{\mathcal{C}}: K_0(\mathcal{C}) \rightarrow \mathrm{Tr}(\mathcal{C}), \quad h_{\mathcal{C}}([X]_{\cong}) = [1_X]. \quad (1.7.5)$$

Definition 1.7.7. The map $h_{\mathcal{C}}$ is called the *Chern character map*.

Remark 1.7.8. (a) If \mathcal{C} is an additive strict \mathbb{k} -linear monoidal category, then $h_{\mathcal{C}}$ is a homomorphism of rings.

(b) In general, the Chern character map may not be injective and it may not be surjective. See, for example, [BGHL14, Examples 8–10]. However, it is an isomorphism under some additional conditions.

Definition 1.7.9. A \mathbb{k} -linear category \mathcal{C} is called *semisimple* if it has finite direct sums, idempotents split (i.e. \mathcal{C} has subobjects), and there exist objects X_i , $i \in I$, such that $\text{Hom}_{\mathcal{C}}(X_i, X_j) = \delta_{i,j}\mathbb{k}$ (such objects are called simple) and such that for any two objects V and W in \mathcal{C} , the natural composition map

$$\bigoplus_{i \in I} \text{Hom}_{\mathcal{C}}(V, X_i) \otimes \text{Hom}_{\mathcal{C}}(X_i, W) \rightarrow \text{Hom}_{\mathcal{C}}(V, W)$$

is an isomorphism.

Proposition 1.7.10 ([Sav18, Proposition 5.4]). If \mathcal{C} is a semisimple additive \mathbb{k} -linear category, then the map

$$h_{\mathcal{C}} \otimes 1: K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{k} \rightarrow \text{Tr}(\mathcal{C})$$

is an isomorphism.

Chapter 2

The partition category, the Deligne's category $\text{Rep}(\mathfrak{S}_t)$ and the Heisenberg category

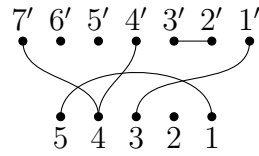
In this chapter, we recall the definition of *Deligne's category* $\text{Rep}(\mathfrak{S}_t)$ as the additive Karoubi envelope of the *partition category*. Then we give the definition of the Heisenberg category, which is the *group Heisenberg category* $\mathcal{H}\text{eis}(G)$ (defined in Section 7.1) in the particular case where G is the trivial group.

2.1 The partition category and Deligne's category $\text{Rep}(\mathfrak{S}_t)$

In this section we recall the definition and some important facts about one of our main objects of study. We refer the reader to [Lau12, Sav18] for a brief treatment of the language of string diagrams and strict linear monoidal categories suited to the current work and introduced in the previous chapter. For an object X of a category, we will denote the identity morphism on X by 1_X .

For $m, \ell \in \mathbb{N}$, a *partition* of type $\binom{\ell}{m}$ is a partition of the set $\{1, \dots, m, 1', \dots, \ell'\}$. The elements of the partition will be called *blocks*. We depict such a partition as a graph with ℓ vertices in the top row, labelled $1', \dots, \ell'$ from *right to left*, and m vertices in the bottom row, labelled $1, \dots, m$ from *right to left*. (We choose the right-to-left numbering convention to better match with the Heisenberg category later.) We draw edges so that the blocks are the connected components of the graph. For example, the partition

$\{\{1, 5\}, \{2\}, \{3, 1'\}, \{4, 4', 7'\}, \{2', 3'\}, \{5'\}, \{6'\}\}$ of type $\binom{7}{5}$ is depicted as follows:



Note that different graphs can correspond to the same partition since only the connected components of the graph are relevant.

From now on, we will omit the labels of the vertices when drawing partition diagrams. We write $D: m \rightarrow \ell$ to indicate that D is a partition of type $\binom{\ell}{m}$. We denote the unique partition diagrams of types $\binom{1}{0}$ and $\binom{0}{1}$ by

$$\uparrow : 0 \rightarrow 1 \quad \text{and} \quad \downarrow : 1 \rightarrow 0.$$

Given two partitions $D': m \rightarrow \ell$, $D: \ell \rightarrow k$, one can stack D on top of D' to obtain a diagram $\begin{matrix} D \\ D' \end{matrix}$ with three rows of vertices. We let $\alpha(D, D')$ denote the number of components

containing only vertices in the middle row of $\begin{matrix} D \\ D' \end{matrix}$. Let $D \star D'$ be the partition of type $\binom{k}{m}$ with the following property: vertices are in the same block of $D \star D'$ if and only if the corresponding vertices in the top and bottom rows of $\begin{matrix} D \\ D' \end{matrix}$ are in the same block.

Recall that \mathbb{k} is a commutative ring and fix $t \in \mathbb{k}$. The *partition category* $\mathcal{P}ar(t)$ is the strict \mathbb{k} -linear monoidal category whose objects are nonnegative integers and, given two objects m, ℓ in $\mathcal{P}ar(t)$, the morphisms from m to ℓ are \mathbb{k} -linear combinations of partitions of type $\binom{\ell}{m}$. The vertical composition is given by

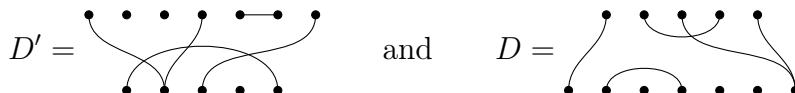
$$D \circ D' = t^{\alpha(D, D')} D \star D'$$

for composable partition diagrams D, D' , and extended by linearity. The bifunctor \otimes is given on objects by

$$\otimes : \mathcal{P}ar(t) \times \mathcal{P}ar(t) \rightarrow \mathcal{P}ar(t), \quad (m, n) \mapsto m + n.$$

The tensor product on morphisms is given by horizontal juxtaposition of diagrams, extended by linearity.

For example, if



then

$$\begin{array}{c} D \\ D' \end{array} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \end{array}, \quad D \star D' = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \end{array}, \quad \text{and} \quad D \circ D' = t^2 \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \end{array}.$$

The partition category is denoted $\text{Rep}_0(\mathfrak{S}_t)$ in [Del07] and $\text{Rep}_0(\mathfrak{S}_t; \mathbb{k})$ in [CO11].

For a linear monoidal category \mathcal{C} , we let $\text{Kar}(\mathcal{C})$ denote its additive Karoubi envelope, that is, the idempotent completion of its additive envelope $\text{Add}(\mathcal{C})$. Then $\text{Kar}(\mathcal{C})$ is again naturally a linear monoidal category. *Deligne's category* $\text{Rep}(\mathfrak{S}_t)$ is the additive Karoubi envelope of $\mathcal{P}ar(t)$. (See [Del07, §8] and [CO11, §2.2].)

The following proposition gives a presentation of the partition category.

Proposition 2.1.1. As a \mathbb{k} -linear monoidal category, the *partition category* $\mathcal{P}ar(t)$ is generated by the object 1 and the morphisms

$$\mu = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} : 2 \rightarrow 1, \quad \delta = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} : 1 \rightarrow 2, \quad s = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} : 2 \rightarrow 2, \quad \eta = \begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} : 0 \rightarrow 1, \quad \varepsilon = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} : 1 \rightarrow 0,$$

subject to the following relations:

$$\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}, \quad \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}, \quad \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \end{array}, \quad (2.1.1)$$

$$\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \end{array}, \quad \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \end{array}, \quad (2.1.2)$$

$$\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}, \quad \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \end{array}, \quad \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}, \quad \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \end{array}, \quad (2.1.3)$$

$$\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}, \quad \begin{array}{c} \bullet \\ \diamond \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}, \quad \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} = t1_0. \quad (2.1.4)$$

In fact, one only needs one of the equalities in the first string of equalities in (2.1.1). The other then follows using the first relation in (2.1.4) and the first relation in (2.1.3). The reader who prefers a more traditional algebraic formulation of the above presentation of $\mathcal{P}ar(t)$ can find this in [Com16, Th. 2.1].

Proof. This result is proved in [Com16, Th. 2.1]. While it is assumed throughout [Com16] that \mathbb{k} is a field of characteristic not equal to 2, these restrictions are not needed in the proof of [Com16, Th. 2.1]. The essence of the proof is noting that $\mathcal{P}ar(t)$ is isomorphic to the

category obtained from the \mathbb{k} -linearization of a skeleton of the category $2\mathcal{Cob}$ of 2-dimensional cobordisms by factoring out by the second and third relations in (2.1.4). Then the result is deduced from the presentation of $2\mathcal{Cob}$ described in [Koc04, §1.4]. \square

Definition 2.1.2 ([Koc04, 3.6.8]). A *Frobenius object* in a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is an object F equipped with four maps:

$$\mu = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} : F \otimes F \rightarrow F, \quad \delta = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} : F \rightarrow F \otimes F, \quad \uparrow : \mathbb{1} \rightarrow F, \quad \varepsilon = \downarrow : F \rightarrow \mathbb{1},$$

satisfying the unit and counit axioms:

$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ | \\ \bullet \end{array}, \quad \begin{array}{c} \bullet \\ | \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array},$$

as well as the *Frobenius relation*:

$$\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}.$$

The relations (2.1.1) are then equivalent to the statement that $(1, \mu, \eta, \delta, \varepsilon)$ is a Frobenius object. Relations (2.1.2) and (2.1.3) are precisely the statement that s equips $\mathcal{P}ar(t)$ with the structure of a symmetric monoidal category (see, for example, [Koc04, §1.3.27, §1.4.35]). Then the relations (2.1.4) are precisely the statements that the Frobenius object 1 is commutative, special, and of dimension t , respectively. Thus, Proposition 2.1.1 states that $\mathcal{P}ar(t)$ is the free \mathbb{k} -linear symmetric monoidal category generated by a t -dimensional special commutative Frobenius object.

The endomorphism algebra $P_k(t) := \text{End}_{\mathcal{P}ar(t)}(k)$ is called the *partition algebra*. We have a natural algebra homomorphism

$$\mathbb{k}\mathfrak{S}_k \rightarrow P_k(t), \tag{2.1.5}$$

mapping $\tau \in \mathfrak{S}_k$ to the partition with blocks $\{i, \tau(i)'\}$, $1 \leq i \leq k$.

Let $V = \mathbb{k}^n$ be the permutation representation of \mathfrak{S}_n and let $\mathbf{1}_n$ denote the one-dimensional trivial \mathfrak{S}_n -module. As explained in [Com16, §2.4], there is a strong monoidal functor

$$\Phi_n : \mathcal{P}ar(n) \rightarrow \mathfrak{S}_n\text{-mod} \tag{2.1.6}$$

defined on generators by setting $\Phi_n(1) = V$ and

$$\begin{aligned} \Phi_n(\mu) : V \otimes V &\rightarrow V, & v_i \otimes v_j &\mapsto \delta_{i,j} v_i, \\ \Phi_n(\eta) : \mathbf{1}_n &\rightarrow V, & 1 &\mapsto \sum_{i=1}^n v_i, \end{aligned}$$

$$\begin{aligned} \Phi_n(\delta): V &\rightarrow V \otimes V, & v_i &\mapsto v_i \otimes v_i, \\ \Phi_n(\varepsilon): V &\rightarrow \mathbf{1}_n, & v_i &\mapsto 1, \\ \Phi_n(s): V \otimes V &\rightarrow V \otimes V, & v_i \otimes v_j &\mapsto v_j \otimes v_i. \end{aligned}$$

The proposition below is a generalization of the duality property of the partition algebra stated in [HR05, Th. 3.6] and [CSST10, Th. 8.3.13].

Proposition 2.1.3. (a) The functor Φ_n is full.

(b) The induced map

$$\mathrm{Hom}_{\mathcal{P}ar(n)}(k, \ell) \rightarrow \mathrm{Hom}_{\mathfrak{S}_n}(V^{\otimes k}, V^{\otimes \ell})$$

is an isomorphism if and only if $k + \ell \leq n$.

Proof. This is proved in [Com16, Th. 2.3]. While it is assumed throughout [Com16] that \mathbb{k} is a field of characteristic not equal to 2, that assumption is not needed in the proof of [Com16, Th. 2.3]. Comes made use of the isomorphisms $\mathrm{Hom}_{\mathcal{P}ar(n)}(k, \ell) \cong \mathrm{Hom}_{\mathcal{P}ar(n)}(k + \ell, 0)$ and $\mathrm{Hom}_{\mathfrak{S}_n}(V^{\otimes k}, V^{\otimes \ell}) \cong \mathrm{Hom}_{\mathfrak{S}_n}(V^{\otimes k + \ell}, \mathbf{1}_n)$; these allow one to reduce to the particular case of partition diagrams $D: k \rightarrow 0$ of type $\binom{0}{k}$. For such a partition diagram D , he defined the bases f_D and x_D of $\mathrm{Hom}_{\mathfrak{S}_n}(V^{\otimes k}, \mathbf{1}_n)$ and $\mathrm{Hom}_{\mathcal{P}ar(n)}(k, 0)$ respectively (D having at most n parts) such that $\Phi_n(x_D) = f_D$. The result then follows from the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{P}ar(n)}(k, \ell) & \longrightarrow & \mathrm{Hom}_{\mathcal{P}ar(n)}(k + \ell, 0) \\ \Phi_n \downarrow & & \downarrow \Phi_n \\ \mathrm{Hom}_{\mathfrak{S}_n}(V^{\otimes k}, V^{\otimes \ell}) & \longrightarrow & \mathrm{Hom}_{\mathfrak{S}_n}(V^{\otimes k + \ell}, \mathbf{1}_n) \end{array}$$

When $k = \ell$, the current proposition reduces to a statement about the partition algebra; see [HR05, Th. 3.6]. □

A more general result followed by a more general proof will be given later in this thesis (see Theorem 6.2.5).

2.2 The Heisenberg category

In this section we define the Heisenberg category originally introduced by Khovanov in [Kho14]. This is the central charge -1 case of a more general Heisenberg category described

in [MS18, Bru18]. We give here the efficient presentation of this category described in [Bru18, Rem. 1.5(2)].

The *Heisenberg category* $\mathcal{H}eis$ is the additive envelope of the strict \mathbb{k} -linear monoidal category generated by two objects \uparrow, \downarrow , (we use horizontal juxtaposition to denote the tensor product) and morphisms

$$\begin{array}{c} \nearrow \searrow : \uparrow \uparrow \rightarrow \uparrow \uparrow, \quad \cup : \mathbf{1} \rightarrow \downarrow \uparrow, \quad \cap : \uparrow \downarrow \rightarrow \mathbf{1}, \quad \cup : \mathbf{1} \rightarrow \uparrow \downarrow, \quad \cap : \downarrow \uparrow \rightarrow \mathbf{1}, \end{array}$$

where $\mathbf{1}$ denotes the unit object, subject to the relations

$$\begin{array}{c} \begin{array}{cc} \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array}, & \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array}, \end{array} \end{array} \quad (2.2.1)$$

$$\begin{array}{c} \begin{array}{cc} \begin{array}{c} \cup \\ \cup \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array}, & \begin{array}{c} \cap \\ \cap \end{array} = \begin{array}{c} \downarrow \\ \downarrow \end{array}, \end{array} \end{array} \quad (2.2.2)$$

$$\begin{array}{c} \begin{array}{ccc} \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array}, & \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} = \begin{array}{c} \downarrow \\ \uparrow \end{array} - \begin{array}{c} \cup \\ \cap \end{array}, & \begin{array}{c} \cap \\ \cap \end{array} = 0, \quad \begin{array}{c} \circlearrowleft \\ \circlearrowleft \end{array} = \mathbf{1}_{\mathbf{1}}. \end{array} \end{array} \quad (2.2.3)$$

Here the left and right crossings are defined by

$$\begin{array}{c} \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} := \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array}, \quad \begin{array}{c} \searrow \nearrow \\ \nearrow \searrow \end{array} := \begin{array}{c} \searrow \nearrow \\ \nearrow \searrow \end{array}. \end{array}$$

The category $\mathcal{H}eis$ is strictly pivotal, meaning that morphisms are invariant under isotopy (see [Bru18, Th. 1.3(ii),(iii)]). The relations (2.2.3) imply that

$$\downarrow \uparrow \cong \uparrow \downarrow \oplus \mathbf{1}. \quad (2.2.4)$$

In addition, we have the following *bubble slide* relations (see [Kho14, p. 175], [Bru18, (13), (19)]):

$$\begin{array}{c} \begin{array}{c} \circlearrowleft \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \circlearrowleft \end{array} + \begin{array}{c} \uparrow \end{array} \quad \text{and} \quad \begin{array}{c} \circlearrowleft \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \circlearrowleft \end{array} - \begin{array}{c} \downarrow \end{array}. \end{array} \quad (2.2.5)$$

The first relations in (2.2.5) can be shown as follows:

$$\begin{array}{c} \begin{array}{c} \circlearrowleft \\ \uparrow \end{array} \stackrel{(2.2.3)}{=} \begin{array}{c} \uparrow \\ \circlearrowleft \end{array} + \begin{array}{c} \uparrow \end{array} \stackrel{(2.2.3)}{=} \begin{array}{c} \uparrow \\ \circlearrowleft \end{array} + \begin{array}{c} \uparrow \end{array}. \end{array}$$

The second relation can be shown similarly. We can define downward crossings

$$\begin{array}{c} \begin{array}{c} \searrow \nearrow \\ \nearrow \searrow \end{array} := \begin{array}{c} \searrow \nearrow \\ \nearrow \searrow \end{array}$$

and then we have

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \quad \text{for all possible orientations of the strands} \quad (2.2.6)$$

(see [Kho14, p. 175], [Bru18, (20)]).

For $1 \leq i \leq k-1$, let $s_i \in \mathfrak{S}_k$ denote the simple transposition of i and $i+1$. We have natural algebra homomorphisms

$$\mathbb{k}\mathfrak{S}_k \rightarrow \text{End}_{\mathcal{H}eis}(\uparrow^k) \quad \text{and} \quad \mathbb{k}\mathfrak{S}_k \rightarrow \text{End}_{\mathcal{H}eis}(\downarrow^k), \quad (2.2.7)$$

where s_i is mapped to the crossing of strands i and $i+1$, numbering strands *from right to left*.

Let $\mathcal{H}eis_{\uparrow\downarrow}$ denote the full \mathbb{k} -linear monoidal subcategory of $\mathcal{H}eis$ generated by $\uparrow\downarrow$. It follows immediately from (2.2.5) that

$$\circlearrowleft \begin{array}{c} \uparrow \\ \downarrow \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array} \circlearrowright.$$

In other words, the clockwise bubble is strictly central in $\mathcal{H}eis_{\uparrow\downarrow}$. Thus, fixing $t \in \mathbb{k}$, we can define $\mathcal{H}eis_{\uparrow\downarrow}(t)$ to be the quotient of $\mathcal{H}eis_{\uparrow\downarrow}$ by the additional relation

$$\circlearrowleft = t1_{\mathbb{1}}. \quad (2.2.8)$$

For additive categories \mathcal{C}_i , $i \in I$, the direct product category $\prod_{i \in I} \mathcal{C}_i$ has objects $(X_i)_{i \in I}$, where $X_i \in \mathcal{C}_i$. Morphisms $(X_i)_{i \in I} \rightarrow (Y_i)_{i \in I}$ are $(f_i)_{i \in I}$, where $f_i \in \text{Hom}_{\mathcal{C}_i}(X_i, Y_i)$, with componentwise composition. The direct sum category $\bigoplus_{i \in I} \mathcal{C}_i$ is the full subcategory of $\prod_{i \in I} \mathcal{C}_i$ on objects $(X_i)_{i \in I}$ where all but finitely many of the X_i are the zero object.

We now recall the action of $\mathcal{H}eis$ on the category of \mathfrak{S}_n -modules first defined by Khovanov [Kho14, §3.3]. We begin by defining a strong \mathbb{k} -linear monoidal functor

$$\Theta: \mathcal{H}eis \rightarrow \prod_{m \in \mathbb{N}} \left(\bigoplus_{n \in \mathbb{N}} (\mathfrak{S}_n, \mathfrak{S}_m)\text{-bimod} \right).$$

The tensor product structure on the codomain is given by the usual tensor product of bimodules, where we define the tensor product $M \otimes N$ of $M \in (\mathfrak{S}_n, \mathfrak{S}_m)$ -bimod and $N \in (\mathfrak{S}_k, \mathfrak{S}_\ell)$ -bimod to be zero when $m \neq k$. We adopt the convention that \mathfrak{S}_0 is the trivial group, so that $\mathfrak{S}_0\text{-mod}$ is the category of \mathbb{k} -vector spaces. For $0 \leq m, k \leq n$, let ${}_k(n)_m$ denote $\mathbb{k}\mathfrak{S}_n$, considered as an $(\mathfrak{S}_k, \mathfrak{S}_m)$ -bimodule. We will omit the subscript k or m when

$k = n$ or $m = n$, respectively. We denote tensor product of such bimodules by juxtaposition. For instance $(n)_{n-1}(n)$ denotes $\mathbb{k}\mathfrak{S}_n \otimes_{n-1} \mathbb{k}\mathfrak{S}_n$, considered as an $(\mathfrak{S}_n, \mathfrak{S}_n)$ -bimodule, where we write \otimes_m for the tensor product over $\mathbb{k}\mathfrak{S}_m$. We adopt the convention that $s_i s_{i+1} \cdots s_j = 1$ when $i > j$. Then the elements

$$g_i = s_i s_{i+1} \cdots s_{n-1}, \quad i = 1, \dots, n, \quad (2.2.9)$$

form a complete set of left coset representatives of \mathfrak{S}_{n-1} in \mathfrak{S}_n .

On objects, we define

$$\Theta(\uparrow) = ((n)_{n-1})_{n \geq 1}, \quad \Theta(\downarrow) = ({}_{n-1}(n))_{n \geq 1}.$$

On the generating morphisms, we define

$$\begin{aligned} \Theta \left(\begin{array}{c} \nearrow \times \searrow \\ \nwarrow \end{array} \right) &= \left((n)_{n-2} \rightarrow (n)_{n-2}, g \mapsto g s_{n-1} \right)_{n \geq 2}, \\ \Theta \left(\begin{array}{c} \cup \\ \uparrow \end{array} \right) &= \left((n-1) \rightarrow {}_{n-1}(n)_{n-1}, g \mapsto g \right)_{n \geq 1}, \\ \Theta \left(\begin{array}{c} \cap \\ \downarrow \end{array} \right) &= \left((n)_{n-1}(n) \rightarrow (n), g \otimes h \mapsto gh \right)_{n \geq 1}, \\ \Theta \left(\begin{array}{c} \uparrow \\ \cup \end{array} \right) &= \left((n) \rightarrow (n)_{n-1}(n), g \mapsto \sum_{i=1}^n g_i \otimes g_i^{-1} g = \sum_{i=1}^n g g_i \otimes g_i^{-1} \right)_{n \geq 1}, \\ \Theta \left(\begin{array}{c} \downarrow \\ \cap \end{array} \right) &= \left({}_{n-1}(n)_{n-1} \rightarrow (n-1), g \mapsto \begin{cases} g & \text{if } g \in \mathfrak{S}_{n-1}, \\ 0 & \text{if } g \in \mathfrak{S}_n \setminus \mathfrak{S}_{n-1} \end{cases} \right)_{n \geq 1}. \end{aligned}$$

One can then compute that

$$\begin{aligned} \Theta \left(\begin{array}{c} \nearrow \times \searrow \\ \nwarrow \times \swarrow \end{array} \right) &= \left({}_{n-1}(n)_{n-1} \rightarrow (n-1)_{n-2}(n-1), \begin{cases} g s_{n-1} h \mapsto g \otimes h, & g, h \in \mathfrak{S}_{n-1}, \\ g \mapsto 0, & g \in \mathfrak{S}_{n-1} \end{cases} \right)_{n \geq 2}, \\ \Theta \left(\begin{array}{c} \nearrow \times \searrow \\ \nwarrow \times \swarrow \end{array} \right) &= \left((n-1)_{n-2}(n-1) \rightarrow {}_{n-1}(n)_{n-1}, g \otimes h \mapsto g s_{n-1} h \right)_{n \geq 2}, \\ \Theta \left(\begin{array}{c} \nearrow \times \searrow \\ \nwarrow \times \swarrow \end{array} \right) &= \left({}_{n-2}(n) \rightarrow {}_{n-2}(n), g \mapsto s_{n-1} g \right)_{n \geq 2}. \end{aligned}$$

Restricting to $\mathcal{H}eis_{\uparrow\downarrow}$ yields a functor, which we denote by the same symbol,

$$\Theta: \mathcal{H}eis_{\uparrow\downarrow} \rightarrow \bigoplus_{m \in \mathbb{N}} (\mathfrak{S}_m, \mathfrak{S}_m)\text{-bimod.}$$

Recall that $\mathbf{1}_n$ denotes the one-dimensional trivial \mathfrak{S}_n -module. Then the functor $-\otimes_n \mathbf{1}_n$ of tensoring on the right with $\mathbf{1}_n$ gives a functor

$$\bigoplus_{m \in \mathbb{N}} (\mathfrak{S}_m, \mathfrak{S}_m)\text{-bimod} \xrightarrow{-\otimes_n \mathbf{1}_n} \mathfrak{S}_n\text{-mod.}$$

Here we define $M \otimes_n \mathbf{1}_n = 0$ for $M \in (\mathfrak{S}_m, \mathfrak{S}_m)\text{-bimod}$, $m \neq n$. Consider the composition

$$\mathcal{H}eis_{\uparrow\downarrow} \xrightarrow{\Theta} \bigoplus_{m \in \mathbb{N}} (\mathfrak{S}_m, \mathfrak{S}_m)\text{-bimod} \xrightarrow{- \otimes_n \mathbf{1}_n} \mathfrak{S}_n\text{-mod}.$$

It is straightforward to verify that the image of the relation (2.2.8) under this composition holds in $\mathfrak{S}_n\text{-mod}$ with $t = n$. Therefore, the composition factors through $\mathcal{H}eis_{\uparrow\downarrow}(n)$ to give us our action functor:

$$\Omega_n : \mathcal{H}eis_{\uparrow\downarrow}(n) \rightarrow \mathfrak{S}_n\text{-mod}. \quad (2.2.10)$$

Note that the functor Ω_n is not monoidal, since the functor $- \otimes_n \mathbf{1}_n$ is not.

Chapter 3

Existence of the embedding functor, actions and faithfulness

In this chapter we show that the Deligne's category is a subcategory of the additive Karoubi envelope of the Heisenberg category, giving an explicit faithful functor. Next we show that the actions of partition and Heisenberg categories on the category $\mathfrak{S}_n\text{-mod}$, $n \in \mathbb{N}$, are compatible with that functor. A more general result will be given in Chapter 7.

3.1 Existence of the embedding functor

In this section we define a functor from the partition category to the Heisenberg category. We will later show, in Theorem 3.2.2 that this functor is faithful. As we will see in Section 3.2, the existence of this functor arises from the fact that the composition $\text{Ind}_{n-1}^n \circ \text{Res}_{n-1}^n$ of the induction functor $\text{Ind}_{n-1}^n: \mathfrak{S}_{n-1}\text{-mod} \rightarrow \mathfrak{S}_n\text{-mod}$ and the restriction functor $\text{Res}_{n-1}^n: \mathfrak{S}_n\text{-mod} \rightarrow \mathfrak{S}_{n-1}\text{-mod}$ is naturally isomorphic to the functor of tensoring with the permutation module of \mathfrak{S}_n .

Theorem 3.1.1. There is a strict linear monoidal functor $\Psi_t: \mathcal{P}ar(t) \rightarrow \mathcal{H}eis_{\uparrow\downarrow}(t)$ defined on objects by $k \mapsto (\uparrow\downarrow)^k$ and on generating morphisms by

$$\begin{aligned} \mu = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} &\mapsto \begin{array}{c} \uparrow \\ \diagdown \quad \diagup \\ \downarrow \end{array}, & \delta = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} &\mapsto \begin{array}{c} \downarrow \\ \diagup \quad \diagdown \\ \uparrow \end{array}, & s = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} &\mapsto \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \downarrow \quad \downarrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \\ \diagup \quad \diagdown \\ \downarrow \quad \downarrow \end{array}, \\ \eta = \begin{array}{c} \uparrow \\ \bullet \end{array} &\mapsto \begin{array}{c} \uparrow \\ \cup \end{array}, & \varepsilon = \begin{array}{c} \downarrow \\ \bullet \end{array} &\mapsto \begin{array}{c} \downarrow \\ \cap \end{array}. \end{aligned}$$

Proof. It suffices to prove that the functor Ψ_t preserves the relations (2.1.1) to (2.1.4). Since the objects \uparrow and \downarrow are both left and right dual to each other, the fact that Ψ_t preserves

the relations (2.1.1) corresponds to the well-known fact that when X and Y are objects in a monoidal category that are both left and right dual to each other, then XY is a Frobenius object. Alternatively, one easily can verify directly that Ψ_t preserves the relations (2.1.1). This uses only the isotopy invariance in $\mathcal{H}eis$ (i.e. the fact that $\mathcal{H}eis$ is strictly pivotal).

To verify the first relation in (2.1.2), we compute the image of the left-hand side. Since left curls in $\mathcal{H}eis_{\uparrow\downarrow}(t)$ are zero by (2.2.3), this image is

$$\Psi_t(s) \circ \Psi_t(s) = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \stackrel{\substack{(2.2.1) \\ (2.2.3)}}{=} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} + \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \stackrel{(2.2.3)}{=} 1_{(\uparrow\downarrow)^2}.$$

Next we verify the second relation in (2.1.2). First we compute

$$\Psi_t(1_1 \otimes s) \circ \Psi_t(s \otimes 1_1) = \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \\ \text{Diagram 11} \\ \text{Diagram 12} \end{array}.$$

Thus, using the fact that left curls are zero and counterclockwise bubbles are 1_1 by (2.2.3), we have

$$\Psi_t(s \otimes 1_1) \circ \Psi_t(1_1 \otimes s) \circ \Psi_t(s \otimes 1_1) = \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \\ \text{Diagram 15} \\ \text{Diagram 16} \\ \text{Diagram 17} \\ \text{Diagram 18} \\ \text{Diagram 19} \\ \text{Diagram 20} \end{array}.$$

Similarly,

$$\Psi_t(1_1 \otimes s) \circ \Psi_t(s \otimes 1_1) \circ \Psi_t(1_1 \otimes s) = \begin{array}{c} \text{Diagram 21} \\ \text{Diagram 22} \\ \text{Diagram 23} \\ \text{Diagram 24} \\ \text{Diagram 25} \end{array}.$$

Hence it follows from (7.1.5) that Ψ_t preserves the second relation in (2.1.2).

To verify the first relation in (2.1.3), we compute

$$\Psi_t(s) \circ (1_1 \otimes \eta) = \begin{array}{c} \text{Diagram 26} \\ \text{Diagram 27} \end{array} \stackrel{\substack{(2.2.1) \\ (2.2.3)}}{=} \begin{array}{c} \text{Diagram 28} \\ \text{Diagram 29} \end{array} = \Psi_t(\eta \otimes 1_1).$$

To verify the second relation in (2.1.3), we first compute

$$\Psi_t(s \otimes 1_1) \circ \Psi_t(1_1 \otimes s) = \begin{array}{c} \text{Diagram 1} \\ + \\ \text{Diagram 2} \\ + \\ \text{Diagram 3} \\ + \\ \text{Diagram 4} \end{array} \downarrow$$

Then, using (2.2.3), we have

$$\Psi_t(1_1 \otimes \mu) \circ \Psi_t(s \otimes 1_1) \circ \Psi_t(1_1 \otimes s) = \begin{array}{c} \text{Diagram 5} \\ + \\ \text{Diagram 6} \end{array} = \Psi_t(s) \circ \Psi_t(\mu \otimes 1_1).$$

The proofs of the second and third relations in (2.1.3) are analogous.

Finally, to verify the relations (2.1.3), we compute

$$\begin{aligned} \Psi_t(\mu) \circ \Psi_t(s) &= \begin{array}{c} \text{Diagram 7} \\ + \\ \text{Diagram 8} \end{array} \stackrel{(2.2.3)}{=} \begin{array}{c} \text{Diagram 9} \\ = \Psi_t(\mu), \end{array} \\ \Psi_t(\mu) \circ \Psi_t(\delta) &= \begin{array}{c} \text{Diagram 10} \end{array} \stackrel{(2.2.3)}{=} \begin{array}{c} \text{Diagram 11} \\ = \Psi_t(1_1), \end{array} \\ \Psi_t(\varepsilon) \circ \Psi_t(\eta) &= \begin{array}{c} \text{Diagram 12} \end{array} \stackrel{(2.2.8)}{=} t1_1 = \Psi_t(t1_0). \quad \square \end{aligned}$$

As an example that will be used later, we compute

$$\Psi_t \left(\begin{array}{c} \bullet \\ \square \\ \bullet \end{array} \right) = \Psi_t \left(\begin{array}{c} \bullet \\ \times \\ \bullet \end{array} \right) = \begin{array}{c} \text{Diagram 13} \\ \downarrow \end{array} \quad (3.1.1)$$

Note that this is the second term in $\Psi_t(s)$ from Theorem 3.1.1.

Remark 3.1.2. There are two natural ways to enlarge the codomain of the functor Ψ_t to the entire Heisenberg category \mathcal{Heis} (or a suitable quotient), rather than the category $\mathcal{Heis}_{\uparrow\downarrow}(t)$. The obstacle to this is that the clockwise bubble is not central in \mathcal{Heis} and so the relation (2.2.8) is not well behaved there. We continue to suppose that \mathbb{k} is a commutative ring.

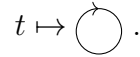
- (a) We can define \mathcal{Par} to be the \mathbb{k} -linear *partition category with bubbles*, which has the same presentation as in Proposition 2.1.1, but without the last relation in (2.1.4). Free floating blocks (i.e. blocks not containing any vertices at the top or bottom of a diagram) are strictly central “bubbles”. They are strictly central since

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \stackrel{(2.1.2)}{=} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \stackrel{(2.1.3)}{=} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \bullet$$

The category $\mathcal{P}ar(t)$ is obtained from $\mathcal{P}ar$ by specializing the bubble at t . Then we have a \mathbb{k} -linear monoidal functor $\mathcal{P}ar \rightarrow \mathcal{H}eis$ (factoring through $\mathcal{H}eis_{\uparrow\downarrow}$) mapping the bubble of $\mathcal{P}ar$ to the clockwise bubble

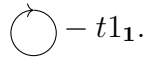


This is equivalent to considering $\mathcal{P}ar(t)$ over the ring $\mathbb{k}[t]$ and $\mathcal{H}eis$ over \mathbb{k} and viewing Ψ_t as a \mathbb{k} -linear monoidal functor $\mathcal{P}ar(t) \rightarrow \mathcal{H}eis$ with



(Then t is the “bubble” in the partition category.) We refer to this setting by saying that t is *generic*.

- (b) If $t \in \mathbb{k}$, let \mathcal{I} denote the left tensor ideal of $\mathcal{H}eis$ generated by



Then Ψ_t induces a \mathbb{k} -linear functor

$$\mathcal{P}ar(t) \rightarrow \mathcal{H}eis/\mathcal{I}.$$

Note, however, that this induced functor is no longer monoidal. Rather, it should be thought of as an action of $\mathcal{P}ar(t)$ on the quotient $\mathcal{H}eis/\mathcal{I}$.

As noted in Section 2.1, the partition category is the free \mathbb{k} -linear symmetric monoidal category generated by an t -dimensional special commutative Frobenius object. Thus, Theorem 3.1.1 implies that $\uparrow\downarrow$, together with certain morphisms, is a special commutative Frobenius object in the Heisenberg category. Note, however, that neither the Heisenberg category nor $\mathcal{H}eis_{\uparrow\downarrow}(t)$ is symmetric monoidal.

3.2 Actions and faithfulness

Consider the standard embedding of \mathfrak{S}_{n-1} in \mathfrak{S}_n , and hence of $\mathbb{k}\mathfrak{S}_{n-1}$ in $\mathbb{k}\mathfrak{S}_n$. Recall that we adopt the convention that $\mathbb{k}\mathfrak{S}_n = \mathbb{k}$ when $n = 0$. We have the natural induction and restriction functors

$$\text{Ind}_{n-1}^n: \mathfrak{S}_{n-1}\text{-mod} \rightarrow \mathfrak{S}_n\text{-mod}, \quad \text{Res}_{n-1}^n: \mathfrak{S}_n\text{-mod} \rightarrow \mathfrak{S}_{n-1}\text{-mod}.$$

If we let B denote $\mathbb{k}\mathfrak{S}_n$, considered as an $(\mathfrak{S}_n, \mathfrak{S}_{n-1})$ -bimodule, then we have

$$\mathrm{Ind}_{n-1}^n \mathrm{Res}_{n-1}^n(M) = B \otimes_{n-1} M, \quad M \in \mathfrak{S}_n\text{-mod},$$

where we recall that \otimes_{n-1} denotes the tensor product over $\mathbb{k}\mathfrak{S}_{n-1}$. We will use the unadorned symbol \otimes to denote tensor product over \mathbb{k} . As before, we denote the trivial one-dimensional \mathfrak{S}_n -module by $\mathbf{1}_n$.

Recall the coset representatives $g_i \in \mathfrak{S}_n$ defined in (2.2.9). In particular, we have

$$g_i^{-1}g_j \in \mathfrak{S}_{n-1} \iff i = j. \quad (3.2.1)$$

Let $V = \mathbb{k}^n$ be the permutation \mathfrak{S}_n -module with basis v_1, \dots, v_n . Then we have

$$B \otimes_{n-1} \mathbf{1}_{n-1} = \mathrm{Ind}_{n-1}^n(\mathbf{1}_{n-1}) \cong V \quad \text{as } \mathfrak{S}_n\text{-modules}. \quad (3.2.2)$$

Furthermore, the elements $g_i \otimes_{n-1} 1$, $1 \leq i \leq n$, form a basis of $B \otimes_{n-1} \mathbf{1}_{n-1}$ and the isomorphism (3.2.2) is given explicitly by

$$B \otimes_{n-1} \mathbf{1}_{n-1} \xrightarrow{\cong} V, \quad g_i \otimes_{n-1} 1 \mapsto v_i = g_i v_n.$$

More generally, define

$$B^k := \underbrace{B \otimes_{n-1} B \otimes_{n-1} \cdots \otimes_{n-1} B}_{k \text{ factors}}.$$

Then we have an isomorphism of \mathfrak{S}_n -modules

$$\begin{aligned} \beta_k: V^{\otimes k} &\xrightarrow{\cong} B^k \otimes_{n-1} \mathbf{1}_{n-1}, \\ v_{i_k} \otimes \cdots \otimes v_{i_1} &\mapsto g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1, \quad 1 \leq i_1, \dots, i_k \leq n, \end{aligned}$$

extended by linearity, with inverse map

$$\begin{aligned} \beta_k^{-1}: B^k \otimes_{n-1} \mathbf{1}_{n-1} &\xrightarrow{\cong} V^{\otimes k}, \\ \pi_k \otimes \cdots \otimes \pi_1 \otimes 1 &\mapsto (\pi_k v_n) \otimes (\pi_k \pi_{k-1} v_n) \otimes \cdots \otimes (\pi_k \cdots \pi_1 v_n), \quad \pi_1, \dots, \pi_k \in \mathfrak{S}_n, \end{aligned}$$

extended by linearity.

Theorem 3.2.1. Fix $n \in \mathbb{N}$, and recall the following functors from (2.1.6), (2.2.10), and Theorem 3.1.1:

$$\begin{array}{ccc} \mathcal{P}ar(n) & \xrightarrow{\Psi_n} & \mathcal{H}eis_{\uparrow\downarrow}(n) \\ & \searrow \Phi_n & \downarrow \Omega_n \\ & & \mathfrak{S}_n\text{-mod} \end{array} \quad (3.2.3)$$

The morphisms β_k , $k \in \mathbb{N}$, give a natural isomorphism of functors $\Omega_n \circ \Psi_n \cong \Phi_n$.

Proof. Since the β_k are isomorphisms, it suffices to verify that they define a natural transformation. For this, we check the images of a set of generators of $\mathcal{P}ar(n)$. Since the functor Ω_n is not monoidal, we need to consider generators of $\mathcal{P}ar(n)$ as a \mathbb{k} -linear category. Such a set of generators is given by

$$1_k \otimes x \otimes 1_j, \quad k, j \in \mathbb{N}, \quad x \in \{\mu, \delta, s, \eta, \epsilon\}.$$

See, for example, [Liu18, Th. 5.2].

Let $j \in \{1, 2, \dots, n-1\}$. We compute that

$$\beta_{k-1}^{-1} \circ (\Omega_n \circ \Psi_n(1_{k-j-1} \otimes \mu \otimes 1_{j-1})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes k-1}$$

is the \mathfrak{S}_n -module map given by

$$\begin{aligned} v_{i_k} \otimes \cdots \otimes v_{i_1} &\mapsto g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\ &\stackrel{(3.2.1)}{\mapsto} \delta_{i_j, i_{j+1}} g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_{j+3}}^{-1} g_{i_{j+2}} \otimes g_{i_{j+2}}^{-1} g_{i_j} \otimes g_{i_j}^{-1} g_{i_{j-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\ &\mapsto \delta_{i_j, i_{j+1}} v_{i_k} \otimes \cdots \otimes v_{i_{j+2}} \otimes v_{i_j} \otimes \cdots \otimes v_{i_1}. \end{aligned}$$

This is precisely the map $\Phi_n(1_{k-j-1} \otimes \mu \otimes 1_{j-1})$.

Similarly, we compute that

$$\beta_{k+1}^{-1} \circ (\Omega_n \circ \Psi_n(1_{k-j} \otimes \delta \otimes 1_{j-1})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes k+1}$$

is the \mathfrak{S}_n -module map given by

$$\begin{aligned} v_{i_k} \otimes \cdots \otimes v_{i_1} &\mapsto g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\ &\mapsto g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_{j+1}}^{-1} g_{i_j} \otimes 1 \otimes g_{i_j}^{-1} g_{i_{j-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\ &= g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_{j+1}}^{-1} g_{i_j} \otimes g_{i_j}^{-1} g_{i_j} \otimes g_{i_j}^{-1} g_{i_{j-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\ &\mapsto v_{i_k} \otimes \cdots \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_j} \otimes v_{i_{j-1}} \otimes \cdots \otimes v_{i_1}. \end{aligned}$$

This is precisely the map $\Phi_n(1_{k-j} \otimes \delta \otimes 1_{j-1})$.

Now let $j \in \{1, \dots, n\}$. We compute that

$$\beta_{k+1}^{-1} \circ (\Omega_n \circ \Psi_n(1_{k-j} \otimes \eta \otimes 1_j)) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes k+1}$$

is the map

$$v_{i_k} \otimes \cdots \otimes v_{i_1} \mapsto g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1$$

$$\begin{aligned}
& \mapsto \sum_{m=1}^n g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_{j+2}}^{-1} g_{i_{j+1}} \otimes g_{j+1}^{-1} g_m \otimes g_m^{-1} g_j \otimes g_{i_j}^{-1} g_{i_{j-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\
& \mapsto \sum_{m=1}^n v_{i_k} \otimes \cdots \otimes v_{i_{j+1}} \otimes v_m \otimes v_{i_j} \otimes \cdots \otimes v_{i_1}.
\end{aligned}$$

This is precisely the map $\Phi_n(1_{k-j} \otimes \eta \otimes 1_j)$.

We also compute that

$$\beta_{k-1}^{-1} \circ (\Omega_n \circ \Psi_n(1_{k-j} \otimes \varepsilon \otimes 1_{j-1})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes k-1}$$

is the map

$$\begin{aligned}
v_{i_k} \otimes \cdots \otimes v_{i_1} & \mapsto g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\
& \mapsto g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_{j+2}}^{-1} g_{i_{j+1}} \otimes g_{i_{j+1}}^{-1} g_{i_{j-1}} \otimes g_{i_{j-1}}^{-1} g_{i_{j-2}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\
& \mapsto v_{i_k} \otimes \cdots \otimes v_{i_{j+1}} \otimes v_{i_{j-1}} \otimes \cdots \otimes v_{i_1}.
\end{aligned}$$

This is precisely the map $\Phi_n(1_{k-j} \otimes \varepsilon \otimes 1_{j-1})$.

It remains to consider the generator s . Let $j \in \{1, 2, \dots, n-1\}$. Define the elements $x, y \in \text{End}_{\mathcal{H}eis}(\uparrow \downarrow \uparrow \downarrow)$ by

$$x = \begin{array}{c} \curvearrowright \\ \curvearrowleft \\ \curvearrowright \\ \curvearrowleft \end{array}, \quad y = \begin{array}{c} \uparrow \\ \cup \\ \downarrow \\ \cap \\ \downarrow \end{array}. \quad (3.2.4)$$

Note that

$$x = x_3 \circ x_2 \circ x_1,$$

where

$$x_1 = \begin{array}{c} \uparrow \\ \curvearrowright \\ \downarrow \end{array}, \quad x_2 = \begin{array}{c} \curvearrowright \\ \curvearrowleft \\ \curvearrowright \\ \curvearrowleft \end{array}, \quad x_3 = \begin{array}{c} \uparrow \\ \curvearrowright \\ \downarrow \end{array}.$$

Suppose $i, j \in \{1, \dots, n\}$ and $h, h' \in \mathbb{k}\mathfrak{S}_n$. We first compute the action of $\Theta(x)$ and $\Theta(y)$ on the element

$$\alpha = h g_i \otimes g_i^{-1} g_j \otimes g_j^{-1} h' \in (n)_{n-1} (n)_{n-1} (n).$$

If $i = j$, then $g_i^{-1} g_j = 1$, and so $\Phi_n(x_1)(\alpha) = 0$. Now suppose $i < j$. Then we have

$$g_i^{-1} g_j = s_{n-1} \cdots s_i s_j \cdots s_{n-1} = s_{j-1} \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_i.$$

Hence

$$\Theta(x_1)(\alpha) = h g_i s_{j-1} \cdots s_{n-2} \otimes s_{n-2} \cdots s_i g_j^{-1} h' \in (n)_{n-2} (n).$$

Thus

$$\Theta(x_2 \circ x_1)(\alpha) = h g_i s_{j-1} \cdots s_{n-1} \otimes g_i^{-1} g_j^{-1} h' \in (n)_{n-2} (n),$$

and so

$$\begin{aligned}
\Theta(x)(\alpha) &= hg_i s_{j-1} \cdots s_{n-1} \otimes s_{n-1} \otimes g_i^{-1} g_j^{-1} h' \\
&= hg_j g_i s_{n-1} \otimes s_{n-1} \otimes s_{n-2} \cdots s_{j-1} g_i^{-1} h' \\
&= hg_j \otimes g_i s_{n-2} \cdots s_{j-1} \otimes g_i^{-1} h' \\
&= hg_j \otimes g_j^{-1} g_i \otimes g_i^{-1} h'.
\end{aligned}$$

The case $i > j$ is similar, Suppose $i > j$. Then we have

$$g_i^{-1} g_j = s_{n-1} \cdots s_i s_j \cdots s_{n-1} = s_j \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_{i-1}.$$

Hence

$$\Theta(x_1)(\alpha) = hg_i s_j \cdots s_{n-2} \otimes s_{n-2} \cdots s_{i-1} g_j^{-1} h' \in (n)_{n-2}(n).$$

Thus

$$\Theta(x_2 \circ x_1)(\alpha) = hg_i g_j \otimes s_{n-1} \cdots s_{i-1} g_j^{-1} h' \in (n)_{n-2}(n),$$

and so

$$\begin{aligned}
\Theta(x)(\alpha) &= hg_i g_j \otimes s_{n-1} \otimes s_{n-1} \cdots s_{i-1} g_j^{-1} h' \\
&= hg_j s_{i-1} \cdots s_{n-2} \otimes s_{n-1} \otimes s_{n-1} g_j^{-1} g_i^{-1} h' \\
&= hg_j \otimes s_{i-1} \cdots s_{n-2} g_j^{-1} \otimes g_i^{-1} h' \\
&= hg_j \otimes g_j^{-1} g_i \otimes g_i^{-1} h'.
\end{aligned}$$

giving

$$\Theta(x)(hg_i \otimes g_i^{-1} g_j \otimes g_j^{-1} h') = \begin{cases} 0 & \text{if } i = j, \\ hg_j \otimes g_j^{-1} g_i \otimes g_i^{-1} h' & \text{if } i \neq j. \end{cases}$$

We also easily compute that

$$\Theta(y)(hg_i \otimes g_i^{-1} g_j \otimes g_j^{-1} h') = \begin{cases} hg_i \otimes 1 \otimes g_i^{-1} h' & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus, for all $i, j \in \{1, \dots, n\}$, we have

$$\Theta(x + y)(hg_i \otimes g_i^{-1} g_j \otimes g_j^{-1} h') = hg_j \otimes g_j^{-1} g_i \otimes g_i^{-1} h'. \quad (3.2.5)$$

It now follows easily that

$$\beta_k^{-1} \circ (\Omega_n \circ \Psi_n (1_{k-j-1} \otimes s \otimes 1_{j-1})) \circ \beta_k = \beta_k^{-1} \circ \Omega_n \left(1_{\uparrow\downarrow}^{\otimes(k-j-1)} \otimes (x + y) \otimes 1_{\uparrow\downarrow}^{\otimes(j-1)} \right) \circ \beta_k$$

is the map given by

$$v_{i_k} \otimes \cdots \otimes v_{i_1} \mapsto v_{i_k} \otimes \cdots \otimes v_{i_{j+2}} \otimes v_{i_j} \otimes v_{i_{j+1}} \otimes v_{i_{j-1}} \otimes \cdots \otimes v_{i_1},$$

which is precisely the map $\Phi_n(1_{k-j-1} \otimes s \otimes 1_{j-1})$. \square

Theorem 3.2.2. The functor Ψ_t is faithful.

We start by giving a proof under the assumption that \mathbb{k} is a commutative ring of characteristic zero using Theorem 3.2.1 and Proposition 2.1.3. A more general proof will follow directly.

Proof. It suffices to show that given $k, \ell \geq 0$ the linear map

$$\Psi_t(k, \ell): \text{Hom}_{\mathcal{P}ar(t)}(k, \ell) \rightarrow \text{Hom}_{\mathcal{H}eis_{\uparrow\downarrow}(t)}((\uparrow\downarrow)^k, (\uparrow\downarrow)^\ell)$$

is injective. Consider $\mathcal{P}ar(t)$ over $\mathbb{k}[t]$ and suppose

$$f = \sum_{i=1}^m a_i(t) f_i \in \ker(\Psi_t(k, \ell))$$

for some $a_i \in \mathbb{k}[t]$ and partition diagrams f_i . Choose $n \geq k + \ell$ and evaluate at $t = n$ to get

$$f_n = \sum_{i=1}^m a_i(n) f_i \in \text{Hom}_{\mathcal{P}ar(n)}(k, \ell).$$

(Here $\mathcal{P}ar(n)$ is a \mathbb{k} -linear category.) Theorem 3.2.1 implies that $\Phi_n(f_n) = 0$. Then Proposition 2.1.3 implies $f_n = 0$. Since the partition diagrams form a basis for the morphisms spaces in $\mathcal{P}ar(n)$, we have $a_i(n) = 0$ for all i . Since this holds for all $n \geq k + \ell$, we have $a_i = 0$ for all i . (Here we use that the characteristic of \mathbb{k} is zero.) Hence $f = 0$ and so Ψ_t is faithful. \square

Now we give a general proof over any commutative ring suggested by Christopher Ryba.

We say a partition diagram is a *permutation* if it is the image of an element of \mathfrak{S}_k , $k \in \mathbb{N}$, under the map (2.1.5). We say a partition diagram is *tensor-planar* if it is a tensor product (horizontal juxtaposition) of partition diagrams each consisting of a single block. Note that every tensor-planar partition diagram is planar (i.e. can be represented as a graph without edge crossings inside of the rectangle formed by its vertices) but the converse is false.

Every partition diagram D can be factored as a product $D = D_1 \circ D_2 \circ D_3$, where D_1 and D_3 are permutations and D_2 is tensor-planar. Furthermore, we may assume that D_1 and

D_3 are compositions of simple transpositions that only transpose vertices in different blocks (since transposing vertices in the same block has no effect). The number of blocks in D is clearly equal to the number of blocks in D_2 . For example, the partition diagram

$$D = \begin{array}{c} \bullet & \bullet & \bullet & \bullet \\ \curvearrowright & \curvearrowright & \curvearrowright & \curvearrowright \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

has four blocks and decomposition $D = D_1 \circ D_2 \circ D_3$, where

$$D_1 = \begin{array}{c} \bullet & \bullet & \bullet & \bullet \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \bullet & \bullet & \bullet & \bullet \end{array}, \quad D_2 = \begin{array}{c} \bullet & \bullet & \bullet & \bullet \\ \curvearrowright & \curvearrowright & \curvearrowright & \curvearrowright \\ \bullet & \bullet & \bullet & \bullet \end{array} = \downarrow \otimes \begin{array}{c} \bullet & \bullet \\ \curvearrowright & \curvearrowright \\ \bullet & \bullet \end{array} \otimes \begin{array}{c} \bullet & \bullet \\ \curvearrowright & \curvearrowright \\ \bullet & \bullet \end{array} \otimes \begin{array}{c} \bullet & \bullet \\ \downarrow & \downarrow \\ \bullet & \bullet \end{array}, \quad D_3 = \begin{array}{c} \bullet & \bullet & \bullet & \bullet \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

For $n, k, \ell \in \mathbb{N}$, let $\text{Hom}_{\mathcal{P}ar}^{\leq n}(k, \ell)$ denote the subspace of $\text{Hom}_{\mathcal{P}ar}(k, \ell)$ spanned by partition diagrams with at most n blocks. Composition respects the corresponding filtration on morphism spaces.

Note that objects of the category $\mathcal{H}eis$ are sequences of up and down arrows and the length of such an object is this number of arrows. Moreover $\text{Hom}_{\mathcal{H}eis}(Q, Q') = 0$ when the number of \uparrow s in Q plus the number of \downarrow s in Q' is not equal to the number of \downarrow s in Q plus the number of \uparrow s in Q' .

Proposition 3.2.3 ([Kho14, Proposition 5]). For any objects $Q, Q' \in \mathcal{H}eis$ of length k and ℓ respectively, $k, \ell \in \mathbb{Z}_{\geq 0}$, a basis of the \mathbb{k} -module $\text{Hom}_{\mathcal{H}eis}(Q, Q')$ is given by the set $B(k, \ell)$, which is the set of planar diagrams obtained in the following manner:

- The object Q and Q' are written at the bottom and top (respectively) of the plane $\mathbb{R} \times [0, 1]$.
- The arrows of Q and Q' are matched by oriented segments embedded in the strip in such a way that their orientations match the signs (that is, they start at either a \uparrow of Q or a \downarrow of Q' , and end at either a \downarrow of Q or a \uparrow of Q'), each segments intersect at most once and no triple intersections are allowed.
- Any number of dots may be placed on each interval near its out endpoint (i.e. between its out endpoint and any intersections with their intervals).
- In the rightmost region of the diagram, a finite number of clockwise disjoint nonnested circles with any number of hollow dots may be drawn.

Recall the bases of the morphism spaces of $\mathcal{H}eis$ given in Proposition 3.2.3. For any such basis element X in $\text{Hom}_{\mathcal{H}eis_{\uparrow\downarrow}}((\uparrow\downarrow)^k, (\uparrow\downarrow)^\ell)$, define the *block number* of X to be number of distinct closed (possibly intersecting) loops in the diagram

$$\curvearrowright^{\otimes \ell} \circ X \circ \curvearrowleft^{\otimes k}.$$

For $n \in \mathbb{N}$, let $\text{Hom}_{\mathcal{H}\text{eis}_{\uparrow\downarrow}}^{\leq n}((\uparrow\downarrow)^k, (\uparrow\downarrow)^\ell)$ denote the subspace of $\text{Hom}_{\mathcal{H}\text{eis}_{\uparrow\downarrow}}((\uparrow\downarrow)^k, (\uparrow\downarrow)^\ell)$ spanned by basis elements with block number at most n . Composition respects the resulting filtration on morphism spaces.

The image under Ψ_t of tensor-planar partition diagrams (writing the image in terms of the aforementioned bases of the morphism spaces of $\mathcal{H}\text{eis}$) is particularly simple to describe. Since each tensor-planar partition diagram is a tensor product of single blocks, consider the case of a single block. Then, for example, we have

$$\Psi_t(\text{two dots with a curved arrow between them}) = \text{a larger curved arrow with two smaller curved arrows below it} \quad \text{and} \quad \Psi_t(\text{two dots with a horizontal line between them}) = \text{a diagram with two dots, a horizontal line, and several curved arrows around it}.$$

The general case is analogous. (In fact, the images of all planar partition diagrams are similarly easy to describe.) In particular, if D is a tensor-planar partition diagram with n blocks, then $\Psi_t(D)$ is a planar diagram with block number n .

Let

$$\xi = \text{a square diagram with four dots at the corners} \in P_2(t) \quad \text{and} \quad \xi_i = 1_{k-i-1} \otimes \xi \otimes 1_{i-1} \in P_k(t) \quad \text{for } 1 \leq i \leq k-1.$$

For a permutation partition diagram $D: k \rightarrow k$, let $T(D)$ be the planar diagram (a morphism in $\mathcal{H}\text{eis}_{\uparrow\downarrow}$) defined as follows: Write $D = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_r}$ as a reduced word in simple transpositions and let

$$T(D) = \Psi_t(s_{i_1} - \xi_{i_1}) \circ \Psi_t(s_{i_2} - \xi_{i_2}) \circ \cdots \circ \Psi_t(s_{i_r} - \xi_{i_r}).$$

(See (4.2.5).) It follows from the braid relations (7.1.5) that $T(D)$ is independent of the choice of reduced word for D .

Proposition 3.2.4. Suppose $D: k \rightarrow \ell$ is a partition diagram with n blocks. Write $D = D_1 \circ D_2 \circ D_3$, where D_2 is a tensor-planar partition diagram and D_1 and D_3 are compositions of simple transpositions that only transpose vertices in different blocks. Then

$$\Psi_t(D) - T(D_1) \circ \Psi_t(D_2) \circ T(D_3) \in \text{Hom}_{\mathcal{H}\text{eis}_{\uparrow\downarrow}}^{\leq n-1}((\uparrow\downarrow)^k, (\uparrow\downarrow)^\ell).$$

Proof. We have $\Psi_t(D) = \Psi_t(D_1) \circ \Psi_t(D_2) \circ \Psi_t(D_3)$. As noted above, D_2 has n blocks and $\Psi_t(D_2)$ has block number n . Suppose $1 \leq j < \ell$. If $D': k \rightarrow \ell$ is a partition diagram with n blocks such that j' and $(j+1)'$ lie in different blocks, then $\xi_j \circ D'$ has $n-1$ blocks. It follows that

$$\Psi_t(s_j) \circ \Psi_t(D') - \Psi_t(s_j - \xi_j) \circ \Psi_t(D') \in \text{Hom}_{\mathcal{H}\text{eis}_{\uparrow\downarrow}}^{\leq n-1}((\uparrow\downarrow)^k, (\uparrow\downarrow)^\ell).$$

Similarly,

$$\Psi_t(D') \circ \Psi_t(s_j) - \Psi_t(D') \circ \Psi_t(s_j - \xi_j) \in \text{Hom}_{\mathcal{H}eis_{\uparrow\downarrow}}^{\leq n-1}((\uparrow\downarrow)^k, (\uparrow\downarrow)^\ell)$$

for any $1 \leq j < k$ such that j and $j + 1$ lie in different blocks of D' . The result then follows by writing D_1 and D_3 as reduced words in simple transpositions. \square

Corollary 3.2.5. The functor Ψ_t is faithful over an arbitrary commutative ring \mathbb{k} .

Proof. It is clear that, in the setting of Proposition 3.2.4, $T(D_1) \circ \Psi_t(D_2) \circ T(D_3)$ is uniquely determined by D . Indeed, D is the partition diagram obtained from $T(D_1) \circ \Psi_t(D_2) \circ T(D_3)$ by replacing each pair $\uparrow\downarrow$ by a vertex and each strand by an edge. Furthermore, the diagrams of the form $T(D_1) \circ \Psi_t(D_2) \circ T(D_3)$ are linearly independent by [Kho14, Prop. 5]. The result then follows by a standard triangularity argument. \square

Since any faithful linear monoidal functor induces a faithful linear monoidal functor on additive Karoubi envelopes, we obtain the following corollary.

Corollary 3.2.6. The functor Ψ_t induces a faithful linear monoidal functor from Deligne's category $\text{Rep}(\mathfrak{S}_t)$ to the additive Karoubi envelope $\text{Kar}(\mathcal{H}eis_{\uparrow\downarrow}(t))$ of $\mathcal{H}eis_{\uparrow\downarrow}(t)$.

Note that Ψ_t is *not* full. This follows immediately from the fact that the morphism spaces in $\mathcal{P}ar(t)$ are finite-dimensional, while those in $\mathcal{H}eis_{\uparrow\downarrow}$ are infinite-dimensional (see Proposition 3.2.3), as follows from the explicit basis described in [Kho14, Prop. 5] (see also [BSW18, Th. 6.4]).

Chapter 4

Grothendieck rings

In this chapter, we show that the induced map obtained by taking the additive Karoubi envelope and then the Grothendieck ring to the faithful functor Ψ_t in Theorem 3.1.1 and Theorem 3.2.2 is injective and is precisely the *Kronecker coproduct* on symmetric functions. Throughout in this chapter, \mathbb{k} is a field of characteristic zero. We consider $\mathcal{P}ar(t)$ over the ground ring $\mathbb{k}[t]$ and $\mathcal{H}eis$ over \mathbb{k} . We can then view Ψ_t as a \mathbb{k} -linear functor $\mathcal{P}ar(t) \rightarrow \mathcal{H}eis$ as noted in Remark 3.1.2(a).

4.1 Grothendieck ring of the Deligne category

For an additive linear monoidal category \mathcal{C} , we let $K_0(\mathcal{C})$ denote its split Grothendieck ring defined in 1.7.1. The multiplication in $K_0(\mathcal{C})$ is given by $[X][Y] = [X \otimes Y]$, where $[X]$ denotes the class in $K_0(\mathcal{C})$ of an object X of \mathcal{C} .

Recall that Deligne's category $\text{Rep}(\mathfrak{S}_t)$ is the additive Karoubi envelope $\text{Kar}(\mathcal{P}ar(t))$ of the partition category. The additive monoidal functor Ψ_t of Theorem 3.1.1 induces a ring homomorphism

$$[\Psi_t]: K_0(\text{Rep}(\mathfrak{S}_t)) \rightarrow K_0(\text{Kar}(\mathcal{H}eis)), \quad [\Psi_t]([X]) = [\Psi_t(X)]. \quad (4.1.1)$$

The main result of this section (Theorem 4.2.4) is a precise description of this homomorphism.

Let \mathcal{Y} denote the set of Young diagrams $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$. (We avoid the terminology *partition* here to avoid confusion with the partition category.) For a Young diagram $\lambda \in \mathcal{Y}$, we let $|\lambda|$ denote its size (i.e. the sum of its parts). Let Sym denote the ring of symmetric functions with integer coefficients. Then Sym has a \mathbb{Z} -basis

given by the Schur functions s_λ , $\lambda \in \mathcal{Y}$. We have

$$\mathrm{Sym}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Sym} \cong \mathbb{Q}[p_1, p_2, \dots] = \bigoplus_{\lambda \in \mathcal{Y}} \mathbb{Q}p_\lambda, \quad (4.1.2)$$

where p_n denotes the n -th power sum and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$ for a Young diagram $\lambda = (\lambda_1, \dots, \lambda_k)$.

The *infinite-dimensional Heisenberg Lie algebra* \mathfrak{h} is the Lie algebra over \mathbb{Q} generated by $\{p_n^\pm, c : n \geq 1\}$ subject to the relations

$$[p_m^-, p_n^-] = [p_m^+, p_n^+] = [c, p_n^\pm] = 0, \quad [p_m^+, p_n^-] = \delta_{m,n}nc.$$

The central reduction $U(\mathfrak{h})/(c+1)$ of its universal enveloping algebra can also be realized as the *Heisenberg double* $\mathrm{Sym}_{\mathbb{Q}} \#_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}$ with respect to the bilinear Hopf pairing

$$\langle -, - \rangle : \mathrm{Sym}_{\mathbb{Q}} \times \mathrm{Sym}_{\mathbb{Q}}, \quad \langle p_m, p_n \rangle = \delta_{m,n}n.$$

By definition, $\mathrm{Sym}_{\mathbb{Q}} \#_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}$ is the vector space $\mathrm{Sym}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}$ with associative multiplication given by

$$(e \otimes f)(g \otimes h) = \sum_{(f),(g)} \langle f_{(1)}, g_{(2)} \rangle e g_{(1)} \otimes f_{(2)} h,$$

where we use Sweedler notation for the usual coproduct on $\mathrm{Sym}_{\mathbb{Q}}$ determined by

$$p_n \mapsto p_n \otimes 1 + 1 \otimes p_n. \quad (4.1.3)$$

Comparing the coefficients appearing in [BS17, Th. 5.3] to [BS17, (2.2)], we see that the pairing of two complete symmetric functions is an integer. (Note that our p_n^\pm are denoted p_n^\mp in [BS17].) We can therefore restrict $\langle -, - \rangle$ to obtain a biadditive form $\langle -, - \rangle : \mathrm{Sym} \otimes_{\mathbb{Z}} \mathrm{Sym} \rightarrow \mathbb{Z}$. The corresponding Heisenberg double

$$\mathrm{Heis} := \mathrm{Sym} \#_{\mathbb{Z}} \mathrm{Sym}$$

is a natural \mathbb{Z} -form for $U(\mathfrak{h})/(c+1) \cong \mathrm{Sym}_{\mathbb{Q}} \#_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}$. For $f \in \mathrm{Sym}$ we let f^- and f^+ denote the elements $f \otimes 1$ and $1 \otimes f$ of Heis , respectively.

Recall the algebra homomorphisms (2.1.5) and (2.2.7), which we use to view elements of $\mathbb{k}\mathfrak{S}_k$ as endomorphisms in the partition and Heisenberg categories. In particular, the homomorphisms (2.2.7) induce a natural algebra homomorphism

$$\mathbb{k}\mathfrak{S}_k \otimes_{\mathbb{k}} \mathbb{k}\mathfrak{S}_k \rightarrow \mathrm{End}_{\mathcal{H}\mathrm{eis}}(\uparrow^k \downarrow^k). \quad (4.1.4)$$

We will use this homomorphism to view elements of $\mathbb{k}\mathfrak{S}_k \otimes \mathbb{k}\mathfrak{S}_k$ as elements of $\text{End}_{\mathcal{H}\text{eis}}(\uparrow^k \downarrow^k)$.

One can deduce explicit presentations of Heis (see [BS17, §5] and [LRS18, Appendix A]), but we will not need such presentations here. Important for our purposes is that

$$s_\lambda^+ s_\mu^-, \quad \lambda, \mu \in \mathcal{Y},$$

is a \mathbb{Z} -basis for Heis, and that there is an isomorphism of rings

$$\text{Heis} \xrightarrow{\cong} K_0(\text{Kar}(\mathcal{H}\text{eis})), \quad s_\lambda^+ s_\mu^- \mapsto [(\uparrow^{|\lambda|} \downarrow^{|\mu|}, e_\lambda \otimes e_\mu)], \quad \lambda, \mu \in \mathcal{Y}, \quad (4.1.5)$$

where e_λ is the Young symmetrizer corresponding to the Young diagram λ . We adopt the convention that $e_\emptyset = 1$ and $s_\emptyset = 1$, where \emptyset denotes the empty Young diagram of size 0. The isomorphism (4.1.5) was conjectured in [Kho14, Conj. 1] and proved in [BSW18, Th. 1.1]. Via the isomorphism (4.1.5), we will identify $K_0(\text{Kar}(\mathcal{H}\text{eis}))$ and Heis in what follows.

Recall that there is an isomorphism of Hopf algebras

$$\bigoplus_{n=0}^{\infty} K_0(\mathfrak{S}_n\text{-mod}) \cong \text{Sym}, \quad [\mathbb{k}\mathfrak{S}_n e_\lambda] \mapsto s_\lambda. \quad (4.1.6)$$

The product on $\bigoplus_{n=0}^{\infty} K_0(\mathfrak{S}_n\text{-mod})$ is given by

$$[M] \cdot [N] = \left[\text{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} (M \boxtimes N) \right], \quad M \in \mathfrak{S}_m\text{-mod}, \quad N \in \mathfrak{S}_n\text{-mod},$$

while the coproduct (4.1.3) is given by

$$[K] \mapsto \bigoplus_{n+m=k} \left[\text{Res}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_k} K \right], \quad K \in \mathfrak{S}_k\text{-mod}.$$

In addition to the coproduct (4.1.3), there is another well-studied coproduct on $\text{Sym}_{\mathbb{Q}}$, the *Kronecker coproduct*, which is given by

$$\Delta_{\text{Kr}}: \text{Sym}_{\mathbb{Q}} \rightarrow \text{Sym}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \text{Sym}_{\mathbb{Q}}, \quad \Delta_{\text{Kr}}(p_\lambda) = p_\lambda \otimes p_\lambda.$$

It is dual to the Kronecker (or internal) product on $\text{Sym}_{\mathbb{Q}}$. Restriction to Sym gives a coproduct

$$\Delta_{\text{Kr}}: \text{Sym} \rightarrow \text{Sym} \otimes_{\mathbb{Z}} \text{Sym}. \quad (4.1.7)$$

The fact that the restriction of Δ_{Kr} to Sym lands in $\text{Sym} \otimes_{\mathbb{Z}} \text{Sym}$ is implied by the following categorical interpretation of the Kronecker coproduct. The diagonal embedding $\mathfrak{S}_n \rightarrow \mathfrak{S}_n \times \mathfrak{S}_n$ extends by linearity to an injective algebra homomorphism

$$d: \mathbb{k}\mathfrak{S}_n \rightarrow \mathbb{k}\mathfrak{S}_n \otimes_{\mathbb{k}} \mathbb{k}\mathfrak{S}_n. \quad (4.1.8)$$

Under the isomorphism (4.1.6), the functor

$$\mathfrak{S}_n\text{-mod} \rightarrow (\mathfrak{S}_n \times \mathfrak{S}_n)\text{-mod}, \quad M \mapsto \text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_n \times \mathfrak{S}_n}(M),$$

corresponds precisely to Δ_{Kr} after passing to Grothendieck groups. (See [Lit56].)

Now view the Kronecker coproduct as a linear map

$$\Delta_{\text{Kr}}: \text{Sym} \rightarrow \text{Sym} \#_{\mathbb{Z}} \text{Sym} = \text{Heis}. \quad (4.1.9)$$

It is clear that the map (4.1.7) is a ring homomorphism with the product ring structure on $\text{Sym} \otimes_{\mathbb{Z}} \text{Sym}$. In fact, it turns out that we also have the following.

Lemma 4.1.1. The map (4.1.9) is an injective ring homomorphism.

Proof. We prove the result over \mathbb{Q} ; then the statement follows by restriction to Sym . By (4.1.2), it suffices to prove that $\Delta_{\text{Kr}}(p_n)$ and $\Delta_{\text{Kr}}(p_m)$ commute for $n, m \in \mathbb{N}$. Since, for $n \neq m$,

$$\Delta_{\text{Kr}}(p_n)\Delta_{\text{Kr}}(p_m) = p_n^+ p_n^- p_m^+ p_m^- = p_m^+ p_m^- p_n^+ p_n^- = \Delta_{\text{Kr}}(p_m)\Delta_{\text{Kr}}(p_n),$$

we see that Δ_{Kr} is a ring homomorphism. It is clear that it is injective. \square

4.2 Explicit description of the functor ψ_t

Our first step in describing the map (4.1.1) is to decompose the objects $(\uparrow\downarrow)^k$ appearing in the image of Ψ_t . Recall that the Stirling number of the second kind $\left\{ \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\}$, $k, \ell \in \mathbb{N}$, counts the number of ways to partition a set of k labelled objects into ℓ nonempty unlabelled subsets. These numbers are given by

$$\left\{ \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\} = \frac{1}{\ell!} \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} (\ell - i)^k$$

and are determined by the recursion relation

$$\left\{ \begin{smallmatrix} k+1 \\ \ell \end{smallmatrix} \right\} = \ell \left\{ \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} k \\ \ell-1 \end{smallmatrix} \right\} \quad \text{with} \quad \left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1 \quad \text{and} \quad \left\{ \begin{smallmatrix} k \\ 0 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 0 \\ k \end{smallmatrix} \right\} = 0, \quad k > 0.$$

Lemma 4.2.1. In Heis , we have

$$(\uparrow\downarrow)^k \cong \bigoplus_{\ell=1}^k (\uparrow^{\ell}\downarrow^{\ell})^{\oplus \left\{ \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\}}. \quad (4.2.1)$$

In particular, since $\left\{ \begin{smallmatrix} k \\ k \end{smallmatrix} \right\} = 1$, the summand $\uparrow^k\downarrow^k$ appears with multiplicity one.

Proof. First note that repeated use of the isomorphism (2.2.4) gives

$$\uparrow\downarrow\uparrow^k\downarrow^k \cong \uparrow^{k+1}\downarrow^{k+1} \oplus (\uparrow^k\downarrow^k)^{\oplus k}. \quad (4.2.2)$$

We now prove the lemma by induction on k . The case $k = 1$ is immediate. Suppose the result holds for some $k \geq 1$. Then we have

$$\begin{aligned} (\uparrow\downarrow)^{k+1} &\cong (\uparrow\downarrow) \left(\bigoplus_{\ell=1}^k (\uparrow^\ell\downarrow^\ell)^{\oplus \{\ell\}^k} \right) \stackrel{(4.2.2)}{\cong} \bigoplus_{\ell=1}^k \left((\uparrow^{\ell+1}\downarrow^{\ell+1})^{\oplus \{\ell\}^k} + (\uparrow^\ell\downarrow^\ell)^{\oplus \ell \{\ell\}^k} \right) \\ &\cong \bigoplus_{\ell=1}^{k+1} (\uparrow^\ell\downarrow^\ell)^{\oplus (\ell \{\ell\}^k + \{\ell-1\}^k)} \cong \bigoplus_{\ell=1}^{k+1} (\uparrow^\ell\downarrow^\ell)^{\oplus \{\ell\}^{k+1}}. \quad \square \end{aligned}$$

Recall that, under (4.1.4), for each Young diagram λ of size k , we have the idempotent

$$d(e_\lambda) \in \text{End}_{\mathcal{H}eis}(\uparrow^k\downarrow^k),$$

where d is the map (4.1.8). Recall also the definition $P_k(t) = \text{End}_{\mathcal{P}ar(t)}(k)$ of the partition algebra. Let

$$\xi = \begin{array}{|c|c|} \hline \square & \\ \hline \end{array} \in P_2(t) \quad \text{and} \quad \xi_i = 1_{k-i-1} \otimes \xi \otimes 1_{i-1} \in P_k(t) \quad \text{for } 1 \leq i \leq k-1.$$

It is straightforward to verify that the intersection of $P_k(t)$ with the tensor ideal of $\mathcal{P}ar(t)$ generated by ξ is equal to the ideal (ξ_1) of $P_k(t)$ generated by ξ_1 . Denote this ideal by $P_k^\xi(t)$.

As noted in [CO11, Lem. 3.1(2)], we have an isomorphism

$$P_k(t)/P_k^\xi(t) \cong \mathbb{k}\mathfrak{S}_k, \quad a + P_k^\xi(t) \mapsto a, \quad a \in \mathbb{k}\mathfrak{S}_k, \quad (4.2.3)$$

where we view elements of $\mathbb{k}\mathfrak{S}_k$ as elements of $P_k(t)$ via the homomorphism (2.1.5). This observation allows one to classify the primitive idempotents in $P_k(t)$ by induction on k . This classification was first given by Martin in [Mar96].

Proposition 4.2.2. For $k > 0$, the primitive idempotents in $P_k(t)$, up to conjugation, are in bijection with the set of Young diagrams $\lambda \in \mathcal{Y}$ with $0 < |\lambda| \leq k$. Furthermore:

- (a) Under this bijection, idempotents lying in $P_k^\xi(t)$ correspond to Young diagrams λ with $0 < |\lambda| < k$.
- (b) For each Young diagram λ of size k , we can choose a primitive idempotent $f_\lambda \in P_k(t)$ corresponding to λ so that $f_\lambda + P_k^\xi(t)$ maps to the Young symmetrizer e_λ under the isomorphism (4.2.3).

Proof. This follows as in the proof of [CO11, Th. 3.4]. Note that since t is generic,

$$\eta \circ \varepsilon = \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}$$

is not an idempotent in $P_1(t)$. Thus, the argument proceeds as in the $t = 0$ case in [CO11, Th. 3.4]. \square

For $\lambda \in \mathcal{Y}$, define the indecomposable object of $\text{Rep}(\mathfrak{S}_t)$

$$L(\lambda) := (|\lambda|, f_\lambda).$$

Proposition 4.2.3. Fix an integer $k \geq 0$. The map

$$\lambda \mapsto L(\lambda), \quad \lambda \in \mathcal{Y},$$

gives a bijection from the set of $\lambda \in \mathcal{Y}$ with $0 \leq |\lambda| \leq k$ to the set of nonzero indecomposable objects in $\text{Rep}(\mathfrak{S}_t)$ of the form (m, e) with $m \leq k$, up to isomorphism. Furthermore

- (a) If $\lambda \in \mathcal{Y}$ with $0 < |\lambda| \leq k$, then there exists an idempotent $e \in P_k(t)$ with $(k, e) \cong L(\lambda)$.
- (b) We have that $(0, 1_0)$ is the unique object of the form (m, e) that is isomorphic to $L(\emptyset)$.

Proof. This follows as in the proof of [CO11, Lem. 3.6]. Again, our assumption that t is generic implies that we proceed as in the $t = 0$ case of [CO11, Lem. 3.6]. \square

We are now ready to prove the main result of this section.

Theorem 4.2.4. The homomorphism $[\Psi_t]$ of (4.1.1) is injective and its image is

$$[\Psi_t](K_0(\text{Rep}(\mathfrak{S}_t))) = \Delta_{\text{Kr}}(\text{Sym}) \subseteq \text{Heis}, \quad (4.2.4)$$

where Heis is identified with $K_0(\mathcal{H}\text{eis})$ as in (4.1.5).

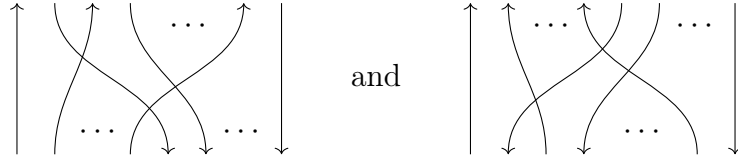
Proof. For $k \in \mathbb{N}$, let $\text{Rep}_k(\mathfrak{S}_t)$ denote the full subcategory of $\text{Rep}(\mathfrak{S}_t)$ containing the objects of the form (k, e) . By Proposition 4.2.3, $\text{Rep}_k(\mathfrak{S}_t)$ is also the full subcategory of $\text{Rep}(\mathfrak{S}_t)$ containing the objects of the form (m, e) , $m \leq k$.

We prove by induction on k that the restriction of $[\Psi_t]$ to $K_0(\text{Rep}_k(\mathfrak{S}_t))$ is injective, and that

$$[\Psi_t](K_0(\text{Rep}_k(\mathfrak{S}_t))) = \Delta_{\text{Kr}}(\text{Sym}_{\leq k}),$$

where $\text{Sym}_{\leq k}$ denotes the subspace of Sym spanned by symmetric functions of degree $\leq k$. The case $k = 0$ is immediate.

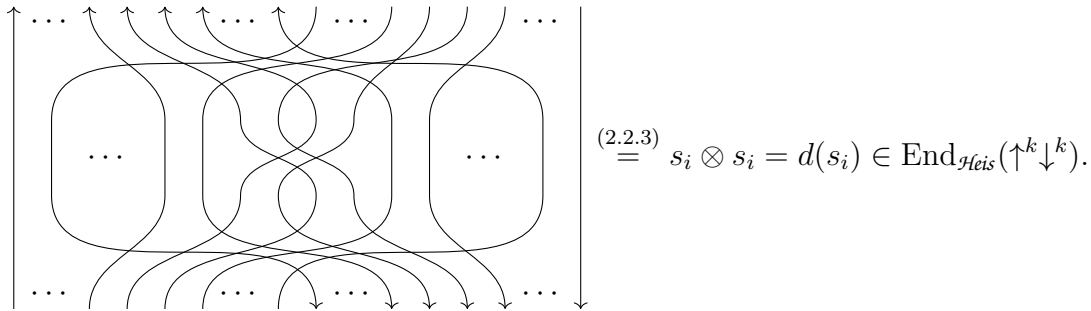
Suppose $k \geq 1$. The components $\uparrow^k \downarrow^k \rightarrow (\uparrow \downarrow)^k$ and $(\uparrow \downarrow)^k \rightarrow \uparrow^k \downarrow^k$ of the isomorphism (4.2.1) are



respectively. For $i = 1, \dots, k - 1$, consider the morphism (we use (3.1.1) here)

$$\Psi_t(s_i - \xi_i) = \left(\uparrow \downarrow \cdots \uparrow \downarrow \begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \\ \downarrow \end{array} \uparrow \downarrow \cdots \uparrow \downarrow \right) \in \text{End}_{\mathcal{H}eis}((\uparrow \downarrow)^k). \quad (4.2.5)$$

Under the isomorphism (4.2.1), this corresponds to



It follows that, for any Young diagram λ of size k , we have $\Psi_t(e_\lambda) - d(e_\lambda) \in \Psi_t(P_k^\xi(t))$. Since $f_\lambda - e_\lambda \in P_k^\xi(t)$ by Proposition 4.2.2, this implies that

$$\Psi_t(f_\lambda) - d(e_\lambda) = (\Psi_t(f_\lambda) - \Psi_t(e_\lambda)) + (\Psi_t(e_\lambda) - d(e_\lambda)) \in \Psi_t(P_k^\xi(t)).$$

Thus, by Proposition 4.2.2 and the induction hypothesis, we have

$$[\Psi_t(L(\lambda))] - \Delta_{\text{Kr}}(s_\lambda) = [\Psi_t(L(\lambda))] - [\uparrow^k \downarrow^k, d(e_\lambda)] \in \Delta_{\text{Kr}}(\text{Sym}_{\leq (k-1)}).$$

Since the s_λ with $|\lambda| = k$ span the space of degree k symmetric functions, we are done. \square

As an immediate corollary of Theorem 4.2.4, we recover the following result of [Del07, Cor. 5.12]. (The T_n of [Del07] correspond to the complete symmetric functions.)

Corollary 4.2.5. We have an isomorphism of rings $K_0(\text{Rep}(\mathfrak{S}_t)) \cong \text{Sym}$.

The Grothendieck ring is one method of decategorification. Another is the trace, or zeroth Hochschild homology (see Definition 1.7.3). We refer the reader to [BGHL14] for details. The functor Ψ_t induces a ring homomorphism on traces. We conclude with a brief discussion of this induced map. First, note that the trace of a category is isomorphic to the trace of its additive Karoubi envelope. (See [BGHL14, Prop. 3.2].) Thus, $\text{Tr}(\mathbf{Par}(t)) \cong \text{Tr}(\text{Rep}(\mathfrak{S}_t))$.

In addition, our assumption that t is generic (in particular, $t \notin \mathbb{N}$) implies that $\text{Rep}(\mathfrak{S}_t)$ is semisimple. (See [Del07, Th. 2.18].) It follows from Proposition 1.7.10 that the Chern character map

$$h: K_0(\text{Rep}(\mathfrak{S}_t)) \rightarrow \text{Tr}(\text{Rep}(\mathfrak{S}_t))$$

is an isomorphism. Hence $\text{Tr}(\mathcal{P}ar(t)) \cong \text{Tr}(\text{Rep}(\mathfrak{S}_t)) \cong \text{Sym}$ by Corollary 4.2.5. On the other hand, the trace of the Heisenberg category was computed in [CLLS18, Th. 1] and shown to be equal to a quotient of the W-algebra $W_{1+\infty}$ by a certain ideal I . This quotient contains the Heisenberg algebra Heis and the Chern character map induces an injective ring homomorphism

$$\text{Heis} \cong K_0(\mathcal{H}eis) \rightarrow \text{Tr}(\mathcal{H}eis) \cong W_{1+\infty}/I.$$

It follows that the functor Ψ_t induces an injective ring homomorphism

$$\text{Sym} \cong \text{Tr}(\text{Rep}(\mathfrak{S}_t)) \rightarrow \text{Tr}(\mathcal{H}eis) \cong W_{1+\infty}/I,$$

and the image of this map is $\Delta_{\text{Kr}}(\text{Sym}) \subseteq \text{Heis} \subseteq W_{1+\infty}/I$.

Chapter 5

Wreath products and group partition categories

This chapter marks the beginning of the generalization of the notions explored in the previous chapters. We recall the definition of the *Wreath product algebra* and we give the associated permutation representation. Then we introduce the notion of *group partition category*.

5.1 Wreath products

Fix a commutative ground ring \mathbb{k} and a group G with identity element 1_G . We use an unadorned tensor product \otimes to denote the tensor product over \mathbb{k} . We will often define linear maps on tensor products by specifying the images of simple tensors; such maps are always extended by linearity.

For $n \geq 1$, the symmetric group \mathfrak{S}_n acts on G^n by permutation of the factors, where we *number factors from right to left*:

$$\pi \cdot (g_n, \dots, g_1) = (g_{\pi^{-1}(n)}, \dots, g_{\pi^{-1}(1)}).$$

The *wreath product group* $G_n := G^n \rtimes \mathfrak{S}_n$ has underlying set $G^n \times \mathfrak{S}_n$, and multiplication

$$(\mathbf{g}, \pi)(\mathbf{h}, \sigma) = (\mathbf{g}(\pi \cdot \mathbf{h}), \pi\sigma), \quad \mathbf{g}, \mathbf{h} \in G^n, \pi, \sigma \in \mathfrak{S}_n.$$

We identify G^n and \mathfrak{S}_n with the subgroups $G^n \times \{1_{\mathfrak{S}_n}\}$ and $\{1_G\} \times \mathfrak{S}_n$, respectively, of $G^n \rtimes \mathfrak{S}_n$. Hence we write $\mathbf{g}\pi$ for (\mathbf{g}, π) .

Let $A := \mathbb{k}G$ be the group algebra of G . Then the group algebra $A_n := \mathbb{k}G_n$ is isomorphic to the *wreath product algebra* $A^{\otimes n} \rtimes \mathfrak{S}_n$, which is isomorphic to $A^{\otimes n} \otimes \mathbb{k}\mathfrak{S}_n$ as a \mathbb{k} -module,

with multiplication given by

$$(\mathbf{a} \otimes \pi)(\mathbf{b} \otimes \sigma) = \mathbf{a}(\pi \cdot \mathbf{b}) \otimes \pi\sigma, \quad \mathbf{a}, \mathbf{b} \in A^{\otimes n}, \quad \pi, \sigma \in \mathfrak{S}_n.$$

We adopt the convention that G_0 is the trivial group, so that $A_0 = \mathbb{k}$.

For $1 \leq i \leq n-1$, let $s_i \in \mathfrak{S}_n$ be the simple transposition of i and $i+1$. The elements

$$\pi_i := s_i s_{i+1} \cdots s_{n-1}, \quad i = 1, \dots, n,$$

mentioned before in (2.2.9), form a complete set of left coset representatives of \mathfrak{S}_{n-1} in \mathfrak{S}_n .

Here we adopt the convention that $\pi_n = 1_{\mathfrak{S}_n}$.

There is an injective group homomorphism

$$G_{n-1} \hookrightarrow G_n, \quad (g_{n-1}, \dots, g_1) \mapsto (1_G, g_{n-1}, \dots, g_1).$$

This induces an embedding of the wreath product algebra A_{n-1} into A_n . For $g \in G$ and $1 \leq i \leq n$, define

$$g^{(i)} := (\underbrace{1_G, \dots, 1_G}_{n-i \text{ entries}}, g, \underbrace{1_G, \dots, 1_G}_{i-1 \text{ entries}}) \in G^n.$$

Then the set

$$\{g^{(i)}\pi_i = \pi_i g^{(n)} : 1 \leq i \leq n, \quad g \in G\} \quad (5.1.1)$$

is a complete set of left coset representatives of G_{n-1} in G_n . Hence it is a basis for A_n as a right A_{n-1} -module.

We adopt the convention that $\mathbf{g} = (g_n, \dots, g_1)$ and $\mathbf{h} = (h_n, \dots, h_1)$. That is, g_i denotes the i -th component of \mathbf{g} (counting from right to left, as usual), and similarly for h_i . This convention applies to all boldface letters denoting elements of G^n for some $n \in \mathbb{N}$.

Definition 5.1.1. The *permutation representation* of A_n is the \mathbb{k} -module $V = A^n$, with action given by

$$\mathbf{g}\pi \cdot (a_n, \dots, a_1) = (g_n a_{\pi^{-1}(n)}, \dots, g_1 a_{\pi^{-1}(1)}), \quad (5.1.2)$$

$\mathbf{g} \in G^n$, $\pi \in \mathfrak{S}_n$, $a_n, \dots, a_1 \in A$, extended by linearity. In other words, \mathfrak{S}_n acts by permuting the entries of elements of A^n , while G^n acts by componentwise multiplication.

For $i = 1, \dots, n$, define $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in V$, where the 1 appears in the i -th position counting from right to left. Then (5.1.2) implies that

$$\mathbf{g}\pi \cdot (ae_i) = g_{\pi(i)} a e_{\pi(i)}, \quad \mathbf{g} \in G^n, \quad \pi \in \mathfrak{S}_n, \quad a \in A, \quad 1 \leq i \leq n. \quad (5.1.3)$$

The set

$$\mathbf{B}_k := \{g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} : g_1, \dots, g_k \in G, 1 \leq i_1, \dots, i_k \leq n\} \quad (5.1.4)$$

is a basis for $V^{\otimes k}$.

In the sequel, we write \otimes_n for \otimes_{A_n} (as in Section 2.2).

Lemma 5.1.2. For any A_n -module W , we have an isomorphism of A_n -modules

$$A_n \otimes_{n-1} W \rightarrow V \otimes W, \quad \gamma \otimes w \mapsto \gamma e_n \otimes \gamma w, \quad \gamma \in A_n, w \in W,$$

with inverse

$$V \otimes W \rightarrow A_n \otimes_{n-1} W, \quad g e_i \otimes w \mapsto g^{(i)} \pi_i \otimes \pi_i^{-1} (g^{-1})^{(i)} w, \quad 1 \leq i \leq n, g \in G, w \in W.$$

Proof. The first map is well-defined since A_{n-1} acts trivially on e_n , and it is clearly a homomorphism of A_n -modules. It is straightforward to verify that the second map is the inverse of the first. \square

Denote by $\mathbf{1}_n$ the trivial one-dimensional A_n -module. Let B denote A_n , considered as an (A_n, A_{n-1}) -bimodule, and, for $k \geq 1$, define

$$B^k := \underbrace{B \otimes_{n-1} B \otimes_{n-1} \cdots \otimes_{n-1} B}_{k \text{ factors}}.$$

Corollary 5.1.3. For $k \geq 1$, we have an isomorphism of A_n -modules

$$\begin{aligned} \beta_k: V^{\otimes k} &\xrightarrow{\cong} B^k \otimes \mathbf{1}_n, \\ g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} &\mapsto g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g_k^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1, \end{aligned}$$

with inverse map

$$\begin{aligned} \beta_k^{-1}: B^k \otimes \mathbf{1}_n &\xrightarrow{\cong} V^{\otimes k}, \\ \gamma_k \otimes \cdots \otimes \gamma_1 \otimes 1 &\mapsto (\gamma_k e_n) \otimes (\gamma_k \gamma_{k-1} e_n) \otimes \cdots \otimes (\gamma_k \cdots \gamma_1 e_n), \quad \gamma_k, \dots, \gamma_1 \in G_n. \end{aligned}$$

5.2 Group partition categories

We continue to fix a group G and a commutative ring \mathbb{k} . For $k, l \in \mathbb{N} = \mathbb{Z}_{\geq 0}$, a *partition of type $\binom{l}{k}$* is a partition of the set $X_k^l := \{1, \dots, k, 1', \dots, l'\}$. A G -*partition of type $\binom{l}{k}$* is a pair (P, \mathbf{g}) , where P is a partition of type $\binom{l}{k}$ and $\mathbf{g} = (g_1, \dots, g_k, g_{1'}, \dots, g_{l'}) \in G^{X_k^l}$. We define a *part* of (P, \mathbf{g}) to be a part of the partition P .

Let (P, \mathbf{g}) and (P, \mathbf{h}) be G -partitions of type $\binom{l}{k}$, with $P = \{P_1, \dots, P_r\}$. We say these G -partitions are *equivalent*, and we write $(P, \mathbf{g}) \sim (P, \mathbf{h})$, if there exist $t_1, \dots, t_r \in G$ such that, for each $i = 1, \dots, r$, we have

$$h_a = t_i g_a \quad \text{for every } a \in P_i.$$

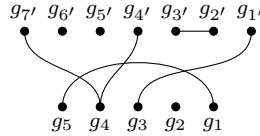
This clearly defines an equivalence relation on the set of G -partitions of type $\binom{l}{k}$. We let $[P, \mathbf{g}]$ denote the equivalence class of (P, \mathbf{g}) .

We depict the G -partition (P, \mathbf{g}) of type $\binom{l}{k}$ as a graph with l vertices in the top row, labelled $g_{1'}, \dots, g_{l'}$ from *right to left*, and k vertices in the bottom row, labelled g_1, \dots, g_k from *right to left*. (We will always number vertices from right to left.) We draw edges so that the parts of the partition are the connected components of the graph.

For example, the equivalence class of the G -partition (P, \mathbf{g}) of type $\binom{7}{5}$ with

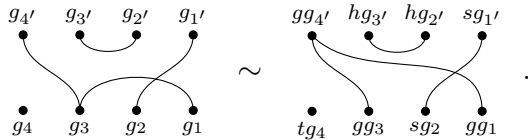
$$P = \{\{1, 5\}, \{2\}, \{3, 1'\}, \{4, 4', 7'\}, \{2', 3'\}, \{5'\}, \{6'\}\}$$

can be depicted as follows:



We call this a *G -partition diagram*. Forgetting the labels, we obtain a *partition diagram* for P . Note that different G -partition diagrams can correspond to the same G -partition since only the connected components of the graph are relevant, and similarly for partition diagrams. Two G -partition diagrams are equivalent if their graphs have the same connected components and the vertex labels of one are obtained from those of the other by, for each connected component, multiplying the labels in that component on the left by the same element of G .

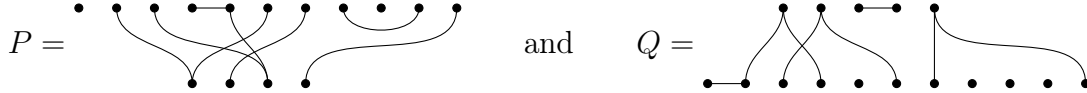
Example 5.2.1. For $g, h, s, t \in G$, the following G -partition diagrams of type $\binom{4}{4}$ are equivalent:



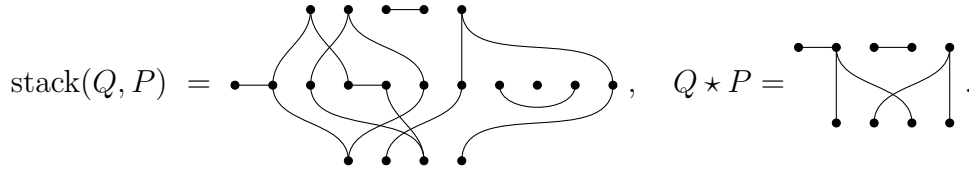
Suppose P is a partition of type $\binom{l}{k}$ and Q is a partition of type $\binom{m}{l}$. As in Section 2.1, we can stack the partition diagram of Q on top of the partition diagram of P and identify the middle row of vertices to obtain a diagram $\text{stack}(Q, P)$ with three rows of vertices. We

define $Q \star P$ to be the partition of type $\binom{k}{m}$ defined as follows: vertices are in the same part of $Q \star P$ if and only if the corresponding vertices in the top and bottom row of $\text{stack}(Q, P)$ are in the same connected component. We let $\alpha(Q, P)$ denote the number of connected components containing only vertices in the middle row of $\text{stack}(Q, P)$.

Example 5.2.2. If



then $\alpha(P, Q) = 2$ and



Suppose (P, \mathbf{g}) and (Q, \mathbf{h}) are G -partitions of types $\binom{l}{k}$ and $\binom{m}{l}$, respectively. We define $\text{stack}((Q, \mathbf{h}), (P, \mathbf{g}))$ to be the graph $\text{stack}(Q, P)$ with vertices labeled by elements of G as follows: vertices in the top and bottom rows are labeled as in the top and bottom rows of Q and P , respectively, while the i -th vertex in the middle row is labelled by the product $g_i h_i^{-1}$. (As usual, we label vertices from right to left.) We say that the pair $((Q, \mathbf{h}), (P, \mathbf{g}))$ is *compatible* if any two vertices in the middle row of $\text{stack}((Q, \mathbf{h}), (P, \mathbf{g}))$ that are in the same connected component of Q have the same label. If $((Q, \mathbf{h}), (P, \mathbf{g}))$ is compatible, we define $\mathbf{h} \star_{Q, P} \mathbf{g}$ to be the element $\mathbf{t} \in G^{X_k^m}$ where

- for $1 \leq i \leq m$, $t_{i'} = g h_{i'}$, where g is the common label of vertices in the middle row that are in the same connected component of Q as the i -th vertex of the top row of Q (i.e. the vertex labeled by $h_{i'}$), where we adopt the convention $g = 1_G$ if there are no such vertices;
- for $1 \leq i \leq k$, $t_i = g_i$.

Lemma 5.2.3. Suppose that $((Q, \mathbf{h}), (P, \mathbf{g}))$ and $((Q, \mathbf{h}'), (P, \mathbf{g}'))$ are compatible, that $(Q, \mathbf{h}) \sim (Q, \mathbf{h}')$, and that $(P, \mathbf{g}) \sim (P, \mathbf{g}')$. Then $(Q \star P, \mathbf{h} \star_{Q, P} \mathbf{g}) \sim (Q \star P, \mathbf{h}' \star_{Q, P} \mathbf{g}')$.

Proof. By transitivity, it suffices to consider the case where $\mathbf{h} = \mathbf{h}'$ and the case where $\mathbf{g} = \mathbf{g}'$. Suppose $\mathbf{h} = \mathbf{h}'$ and consider a connected component Y of $\text{stack}(Q, P)$. Then Y is a union of some connected components of Q and some connected components P_1, \dots, P_r

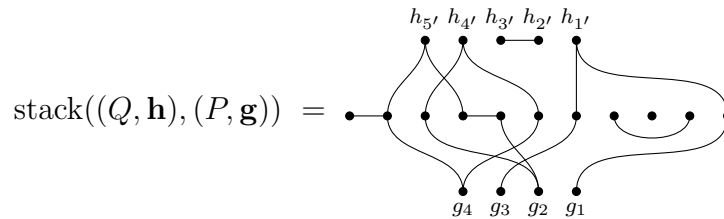
of P . (The cases where Y is a single connected component of Q or a single connected component of P are straightforward, so we assume Y is a union of a positive number of connected components of P and a positive number of connected components of Q .) Since $(P, \mathbf{g}) \sim (P, \mathbf{g}')$, there exist $v_1, \dots, v_r \in G$ such that $g'_a = v_i g_a$ for all $a \in P_i, i \in \{1, 2, \dots, r\}$. Now, the fact that $((Q, \mathbf{h}), (P, \mathbf{g}))$ and $((Q, \mathbf{h}), (P, \mathbf{g}'))$ are compatible and that P_1, \dots, P_r are in the same connected component of $\text{stack}(Q, P)$ implies that $v_1 = v_2 = \dots = v_r$. Thus, if $\mathbf{t} = \mathbf{h} \star_{Q,P} \mathbf{g}$ and $\mathbf{t}' = \mathbf{h} \star_{Q,P} \mathbf{g}'$, we have $t'_a = v_1 t_a$ for all vertices $a \in Y$. Since this holds for each connected component Y of $\text{stack}(Q, P)$, we have $(Q \star P, \mathbf{h} \star_{Q,P} \mathbf{g}) \sim (Q \star P, \mathbf{h} \star_{Q,P} \mathbf{g}')$, as desired. The case where $\mathbf{g} = \mathbf{g}'$ is analogous. \square

We say that the pair $([Q, \mathbf{h}], [P, \mathbf{g}])$ is compatible if there exist representatives (Q, \mathbf{h}') and (P, \mathbf{g}') of the equivalence classes $[Q, \mathbf{h}]$ and $[P, \mathbf{g}]$ such that $((Q, \mathbf{h}'), (P, \mathbf{g}'))$ is compatible. Whenever we refer to a compatible pair $([Q, \mathbf{h}], [P, \mathbf{g}])$, we assume that $((Q, \mathbf{h}), (P, \mathbf{g}))$ is a compatible pair of representatives. By Lemma 5.2.3, we can define

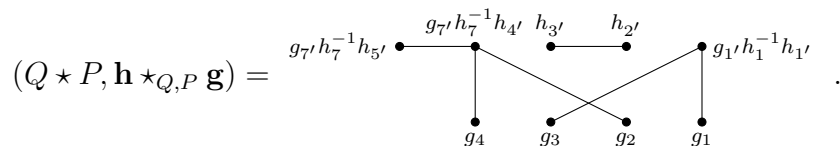
$$[Q, \mathbf{h}] \star [P, \mathbf{g}] := [Q \star P, \mathbf{h} \star_{Q,P} \mathbf{g}] \tag{5.2.1}$$

for a compatible pair $((Q, \mathbf{h}), (P, \mathbf{g}))$, and this definition is independent of our choice of a compatible pair of representatives.

Example 5.2.4. If P and Q are as in Example 5.2.2, then



where the vertices in the middle row are labeled $g_1 h_1^{-1}, g_2 h_2^{-1}, g_3 h_3^{-1}, g_4 h_4^{-1}, g_5 h_5^{-1}, g_6 h_6^{-1}, g_7 h_7^{-1}, g_8 h_8^{-1}, g_9 h_9^{-1}, g_{10} h_{10}^{-1}, g_{11} h_{11}^{-1}$ from right to left. The pair $((Q, \mathbf{h}), (P, \mathbf{g}))$ is compatible if and only if $g_1 h_1^{-1} = g_5 h_5^{-1}, g_6 h_6^{-1} = g_9 h_9^{-1}$, and $g_7 h_7^{-1} = g_8 h_8^{-1} = g_{10} h_{10}^{-1} = g_{11} h_{11}^{-1}$. If the pair is compatible, then



where we adopt the convention that unlabelled vertices implicitly carry the label 1_G .

In the next chapter we will give a presentation of the category $\mathcal{P}ar(G, d)$ and its categorical action on the category $A_n\text{-mod}$ of A_n modules. In fact, the category $\mathcal{P}ar(G, d)$ is a generalization of $\mathcal{P}ar(d)$ and its action on the category $A_n\text{-mod}$ is an extension of the action $\mathcal{P}ar(d)$ on the category $\mathfrak{S}_n\text{-mod}$.

Chapter 6

Presentation and categorical action

In this chapter we give another definition of the G -partition category in terms of generators and relations. Then we define the action of any G -partition on the category of modules for the associated wreath product groups.

6.1 Presentation

In this section we give a presentation of group partition categories by generators and relations. This representation is inspired by the one of $\mathcal{P}ar(d)$ given in Section 2.1 with one more generating morphism and some relations including the elements of the group G . We use the usual string diagram calculus for monoidal categories. We denote the unit object of a monoidal category by $\mathbf{1}$ and the identity morphism of an object X by 1_X .

Definition 6.1.1. Let $\mathcal{P}ar(G)$ be the strict \mathbb{k} -linear monoidal category with one generating object V , where we denote

$$| := 1_V,$$

and generating morphisms

$$\begin{aligned} \frown : V \otimes V \rightarrow V, \quad \smile : V \rightarrow V \otimes V, \quad \times : V \otimes V \rightarrow V \otimes V, \\ \circ : \mathbf{1} \rightarrow V, \quad \wp : V \rightarrow \mathbf{1}, \quad g \blacklozenge : V \rightarrow V, \quad g \in G, \end{aligned}$$

subject to the following relations:

$$\begin{aligned} \frown \circ = |, \quad \smile \circ = | = \smile, \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \end{array}, \end{aligned} \tag{6.1.1}$$

$$\begin{array}{c} \text{⌘} = | \quad |, \quad \text{⌘} = \text{⌘}, \end{array} \quad (6.1.2)$$

$$\begin{array}{c} \text{⌘} = \text{⌘} \quad |, \quad \text{⌘} = \text{⌘} \quad |, \quad \text{⌘} = \text{⌘}, \quad \text{⌘} = \text{⌘}, \end{array} \quad (6.1.3)$$

$$\begin{array}{c} \text{⌘} = \text{⌘}, \quad g \text{⌘} h = \delta_{g,h} \text{⌘} \quad | \quad g, \end{array} \quad (6.1.4)$$

$$\begin{array}{c} g \text{⌘} h = gh \text{⌘}, \quad 1 \text{⌘} = |, \quad g \text{⌘} = \text{⌘} \quad | \quad g, \quad g \text{⌘} = g \text{⌘} \quad | \quad g, \quad g \text{⌘} = \text{⌘}. \end{array} \quad (6.1.5)$$

Remark 6.1.2. The relations (6.1.1) are equivalent to the statement that $(\mathbf{V}, \text{⌘}, \text{⌘}, \text{⌘}, \text{⌘})$ is a Frobenius object (see, for example, [Koc04, Prop. 2.3.24]). Relations (6.1.2) and (6.1.3) and the third relation in (6.1.5) are precisely the statement that ⌘ equips $\mathcal{P}ar(G)$ with the structure of a symmetric monoidal category (see, for example, [Koc04, §1.3.27, §1.4.35]). Then the first relation in (6.1.4) is the statement that the Frobenius object \mathbf{V} is commutative. When $g = h = 1_G$, the second relation in (6.1.4) is the statement that the Frobenius object \mathbf{V} is special.

We refer to the morphisms $g \text{⌘}$ as *tokens*, and the open dots in ⌘ and ⌘ as *pins*. We call ⌘ a *merge*, ⌘ a *split*, and ⌘ a *crossing*. Define *cups* and *caps* by

$$\cup := \text{⌘} \quad \text{and} \quad \cap := \text{⌘}.$$

Proposition 6.1.3. The following relations hold in $\mathcal{P}ar(G)$:

$$\begin{array}{c} \text{⌘} = | \quad \text{⌘}, \quad \text{⌘} = | \quad \text{⌘}, \quad \text{⌘} = |, \quad \text{⌘} = \text{⌘}, \end{array} \quad (6.1.6)$$

$$\text{⌘} = | = \text{⌘}, \quad (6.1.7)$$

$$\text{⌘} = \text{⌘} = \text{⌘}, \quad \text{⌘} = \text{⌘} = \text{⌘}, \quad (6.1.8)$$

$$\text{⌘} = \text{⌘} = \text{⌘}, \quad \text{⌘} = \text{⌘} = \text{⌘}, \quad \text{⌘} = \text{⌘} = \text{⌘}, \quad (6.1.9)$$

$$g \text{⌘} = \text{⌘} \quad | \quad g^{-1}, \quad g \text{⌘} = \text{⌘} \quad | \quad g^{-1}, \quad \text{⌘} \quad | \quad g = g \text{⌘}, \quad g \text{⌘} = \text{⌘}, \quad (6.1.10)$$

$$g \text{⌘} = \text{⌘} \quad | \quad g^{-1}, \quad \text{⌘} \quad | \quad g = g^{-1} \text{⌘}, \quad g \text{⌘} = \text{⌘} \quad | \quad g, \quad g \text{⌘} = \text{⌘} \quad | \quad g^{-1}, \quad \text{⌘} \quad | \quad g = g^{-1} \text{⌘}. \quad (6.1.11)$$

Proof. The first two relations in (6.1.6) follow from the first two relations in (6.1.3) by composing on the top and bottom, respectively, with the crossing and using (6.1.2). Then the third relation in (6.1.6) follows from the first relation in (6.1.1) using the first relation in (6.1.4) and the first relation in (6.1.6). (The fourth relation in (6.1.6) will be proved below.)

The relations (6.1.7) follow from the fourth and fifth equalities in (6.1.1) after placing pins on the merges and splits. The first equality in (6.1.8) follows from the fifth equality in

(6.1.1) after placing a pin on the bottom-left of both diagrams involved in the equality. The remaining equalities in (6.1.8) are proved similarly.

Starting with the third relation in (6.1.3), adding a pin to the top-right strand, a crossing to the two rightmost strands at the bottom, and using the second relation in (6.1.3) and the first relation in (6.1.2), we obtain the relation

$$\curvearrowright = \curvearrowleft .$$

Adding a strand on the left (i.e. tensoring on the left with 1_V), then adding a cup to the two leftmost bottom strands and using (6.1.7), yields the first equality in (6.1.9). The second equality in (6.1.9) is proved similarly. The remaining relations in (6.1.9) follow from placing pins at the top of the morphisms in first relation in (6.1.1) and the third relation in (6.1.6) and from placing pins at the bottom of the morphisms in the second two equalities in (6.1.1).

Now the fourth relation in (6.1.6) follows from rotating the first relation in (6.1.4) using the cups and caps, together with (6.1.2) and (6.1.8).

To prove the first relation in (6.1.10), we compute

$$g \downarrow \cup = g \downarrow \circlearrowleft = g \downarrow \circlearrowright^{g^{-1}} = g \downarrow \circlearrowleft^{g^{-1}} = g \downarrow \circlearrowright = g \downarrow \circlearrowleft^{g^{-1}} .$$

Then we prove the second relation in (6.1.10) as follows:

$$g \downarrow \curvearrowright \stackrel{(6.1.7)}{=} \curvearrowright \downarrow g = \curvearrowright \downarrow g^{-1} \stackrel{(6.1.7)}{=} \curvearrowright \downarrow g^{-1} .$$

The third relation in (6.1.10) is obtained from the third relation in (6.1.5) by composing with a crossing and using (6.1.2).

Finally, the relations in (6.1.11) are obtained from the fourth relation in (6.1.5) by attaching the appropriate cups and caps, then using (6.1.8) and (6.1.10). \square

The relation (6.1.7) implies that the object V is self-dual. It follows from Proposition 6.1.3 that the cups and caps endow $\mathcal{P}ar(G)$ with the structure of a strict pivotal category: we have an isomorphism of strict monoidal categories

$$*: \mathcal{P}ar(G) \rightarrow (\mathcal{P}ar(G)^{\text{op}})^{\text{rev}} ,$$

where op denotes the opposite category and rev denotes the reversed category (switching the order of the tensor product). This isomorphism is the identity on objects and, for a general morphism f represented by a single string diagram, the morphism f^* is given by rotating the diagram through 180° .

Moreover, we have that morphisms are invariant under isotopy, except that we must use (6.1.10) when we slide tokens over cups and caps. In addition, it follows from the first two relations in (6.1.3) and (6.1.6) that

$$\begin{array}{c} \circ \\ | \\ \circ \end{array} = \begin{array}{c} | \\ \circ \end{array}. \quad (6.1.12)$$

In other words, the morphism $\begin{array}{c} \circ \\ | \\ \circ \end{array}$ is strictly central.

Theorem 6.1.4. Let $d \in \mathbb{k}$. As a \mathbb{k} -linear monoidal category, $\mathcal{P}ar(G, d)$ is isomorphic to the quotient of $\mathcal{P}ar(G)$ by the relation

$$\begin{array}{c} \circ \\ | \\ \circ \end{array} = d. \quad (6.1.13)$$

Proof. Let $\mathcal{P}ar'(G, d)$ denote the quotient of $\mathcal{P}ar(G)$ by the additional relation (6.1.13). We define a functor $F: \mathcal{P}ar'(G, d) \rightarrow \mathcal{P}ar(G, d)$ as follows: On objects, define $F(\mathbf{V}^{\otimes k}) = k$, $k \in \mathbb{N}$. We define F on the generating morphisms by

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \mapsto \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}, \quad \begin{array}{c} \diagdown \\ \diagup \end{array} \mapsto \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}, \quad \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \mapsto \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}, \quad \circ \mapsto \bullet, \quad \varphi \mapsto \bullet, \quad g \downarrow \mapsto \begin{array}{c} \bullet \\ | \\ \bullet \\ g \end{array} = \begin{array}{c} g^{-1} \\ \bullet \\ | \\ \bullet \end{array},$$

where, by convention, unlabelled vertices in G -partition diagrams carry the label 1_G . Note that the image of \downarrow is the unique G -partition diagram of type $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (i.e. the vertex there is in the top row), while the image of φ is the unique G -partition diagram of type $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. It is straightforward to verify that the relations (6.1.1) to (6.1.5) and (6.1.13) are preserved by F , so that F is well defined.

Indeed, we have

$$\begin{aligned} F\left(\begin{array}{c} \circ \\ | \\ \circ \end{array}\right) &= F(\varphi) \circ F(\downarrow) \\ &= (\bullet) \circ (\bullet) \\ &= \bullet \\ &= d \quad (\text{by Definition 5.2.6}). \end{aligned}$$

Therefore, (6.1.13) is satisfied. Moreover, one can show the first relation in (6.1.1) as follows:

$$\begin{aligned} F\left(\begin{array}{c} | \\ \diagup \quad \diagdown \\ \circ \end{array}\right) &= F(\begin{array}{c} \diagup \\ \diagdown \end{array}) \circ F(\downarrow \otimes \downarrow) \\ &= \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \circ (\downarrow \otimes \bullet) \\ &= \downarrow \quad (\text{by Definition 5.2.6}). \end{aligned}$$

The second relation in (6.1.1) is shown similarly. For the second equality of the third relation, we have:

$$\begin{aligned} F \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) &= F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) \circ F \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right) \\ &= \left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \end{array} \right) \circ \left(\begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \end{array} \right) \\ &= \begin{array}{c} \square \\ \hline \square \end{array} \quad (\text{by Definition 5.2.6}); \end{aligned}$$

and

$$\begin{aligned} F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) &= F \left(\begin{array}{c} \square \\ \hline \diagup \quad \diagdown \end{array} \right) \circ F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) \\ &= \left(\begin{array}{c} \square \\ \hline \bullet \quad \bullet \end{array} \right) \circ \left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \end{array} \right) \\ &= \begin{array}{c} \square \\ \hline \square \end{array} \quad (\text{by Definition 5.2.6}). \end{aligned}$$

The first equality in the third relation (6.1.1) can be shown similarly. We show (6.1.2) as follows.

$$\begin{aligned} F \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) &= F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) \circ F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) \\ &= \left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \end{array} \right) \circ \left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \end{array} \right) \\ &= \begin{array}{c} \square \\ \hline \square \end{array} \quad (\text{by Definition 5.2.6}). \end{aligned}$$

The second relation in (6.1.2) is a direct consequence of the composition of the G -partition diagram defined in Definition 5.2.6. We also have

$$\begin{aligned} F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) &= F \left(\begin{array}{c} \square \\ \hline \bullet \quad \bullet \end{array} \right) \circ F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) \\ &= \left(\begin{array}{c} \square \\ \hline \bullet \quad \bullet \end{array} \right) \circ \left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \end{array} \right) \\ &= \begin{array}{c} \bullet \quad \bullet \\ \hline \square \end{array} \quad (\text{by Definition 5.2.6}). \end{aligned}$$

This shows the second relation in (6.1.3). For the last relation we have,

$$\begin{aligned} F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) &= F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) \circ F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) \\ &= \left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \end{array} \right) \circ \left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \end{array} \right) \\ &= \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \end{array} \quad (\text{by Definition 5.2.6}); \end{aligned}$$

and

$$F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) = F \left(\begin{array}{c} \square \\ \hline \diagdown \quad \diagup \end{array} \right) \circ F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) \circ F \left(\begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right)$$

$$\begin{aligned}
 &= (\downarrow \otimes \times) \circ (\times \otimes \downarrow) \circ (\downarrow \otimes \vee) \\
 &= \text{diagram} \quad (\text{by Definition 5.2.6}) .
 \end{aligned}$$

The first equation in (6.1.4) can be showed with the same arguments used in the above proof.

For the second one, we have

$$\begin{aligned}
 F \left(\text{diagram} \right) &= F(\wedge) \circ F(g \downarrow \otimes \downarrow h) \circ F(\vee) \\
 &= (\text{diagram}) \circ \left(\begin{matrix} \downarrow & \downarrow \\ g & h \end{matrix} \right) \circ (\text{diagram}) \\
 &= \delta_{g^{-1}, h^{-1}} (\text{diagram}) \circ \left(\begin{matrix} g^{-1} & h^{-1} \\ \text{diagram} \end{matrix} \right) \quad (\text{by Definition 5.2.6}) \\
 &= \delta_{g^{-1}, h^{-1}} \begin{matrix} g^{-1} \\ \downarrow \\ \bullet \end{matrix} \\
 &= \delta_{g, h} \begin{matrix} \bullet \\ \downarrow \\ g \end{matrix} .
 \end{aligned}$$

The relations in (6.1.5) are shown similarly. Indeed,

$$\begin{aligned}
 F \left(\begin{matrix} g \downarrow \\ h \downarrow \end{matrix} \right) &= F(g \downarrow) \circ F(h \downarrow) \\
 &= \left(\begin{matrix} g^{-1} \\ \downarrow \\ \bullet \end{matrix} \right) \circ \left(\begin{matrix} h^{-1} \\ \downarrow \\ \bullet \end{matrix} \right) \\
 &= \begin{matrix} h^{-1} g^{-1} \\ \downarrow \\ \bullet \end{matrix} \quad (\text{by Definition 5.2.6}) \\
 &= \begin{matrix} (gh)^{-1} \\ \downarrow \\ \bullet \end{matrix} \\
 &= F(gh \downarrow) .
 \end{aligned}$$

The others relations in (6.1.5) can be shown similarly and the last one follows from:

$$\begin{aligned}
 F \left(\begin{matrix} g \downarrow \\ \circ \end{matrix} \right) &= F(g \downarrow) \circ F(\circ) \\
 &= \left(\begin{matrix} g^{-1} \\ \downarrow \\ \bullet \end{matrix} \right) \circ (\bullet) \\
 &= \bullet \quad (\text{by Convention 5.2.5}) \\
 &= F(\circ) .
 \end{aligned}$$

Since F is clearly bijective on objects, it remains to show that it is full and faithful. Since Theorem 6.1.4 is known to hold in the case where G is the trivial group (see [Com16, Th. 2.1], or Proposition 2.1.1 for a diagrammatic treatment), it follows from the existence of the functor (5.2.2) that any partition diagram $[P] = [P, \mathbf{1}]$ is in the image of F . Thus, by (5.2.3) so is an arbitrary equivalence class $[P, \mathbf{g}]$ of G -partitions. Hence F is full.

It remains to prove that F is faithful. To do this, it suffices to show that

$$\dim \operatorname{Hom}_{\mathcal{P}ar'(G,d)}(\mathbf{V}^{\otimes k}, \mathbf{V}^{\otimes l}) \leq \dim \operatorname{Hom}_{\mathcal{P}ar(G,d)}(k, l) \quad \text{for all } k, l \in \mathbb{N}.$$

We do this by showing that every morphism of $\mathcal{P}ar'(G, d)$ obtained from the generators by composition and tensor product can be reduced to a scalar multiple of a standard form, with the standard forms being in natural bijection with the number of G -partition diagrams.

We first introduce *star diagrams* $S_a^b \in \operatorname{Hom}_{\mathcal{P}ar'(G,d)}(\mathbf{V}^{\otimes a}, \mathbf{V}^{\otimes b})$ for $(a, b) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ as follows. Define

$$S_1^0 := \circlearrowleft, \quad S_0^1 := \circlearrowright, \quad S_1^1 := |.$$

Then define general star diagrams recursively by

$$\begin{aligned} S_a^{b+1} &:= (1_{\mathbf{V}^{\otimes(b-1)}} \otimes \vee) \circ S_a^b \quad \text{for } b \geq 1, \\ S_{a+1}^b &:= S_a^b \circ (1_{\mathbf{V}^{\otimes(a-1)}} \otimes \wedge) \quad \text{for } a \geq 1. \end{aligned}$$

For example, we have

$$S_2^0 = \circlearrowleft = \cap, \quad S_0^3 = \circlearrowright, \quad S_3^4 = \begin{array}{c} | \\ | \\ | \\ \diagup \\ \diagdown \\ | \end{array}.$$

Recall from (2.1.5) that every permutation $\pi \in \mathfrak{S}_k$ gives rise to a partition of type $\binom{k}{k}$ with parts $\{i, \pi(i)'\}$, $1 \leq i \leq k$. Fixing a reduced decomposition for π induces a decomposition of the corresponding partition diagram as a composition of tensor products of the generator \times and identity morphisms. We fix such a decomposition for each permutation, writing D_π for the corresponding element of $\mathcal{P}ar'(G, d)$. For example, if we choose the reduced decomposition $s_1 s_2 s_1$ for the permutation $(13) \in \mathfrak{S}_3$, we have

$$D_{(13)} = \begin{array}{c} \times \\ \times \\ \times \end{array}.$$

Now, fix a representative (P, \mathbf{g}) of each equivalence class $[P, \mathbf{g}]$ of G -partitions. Then, for each such representative, fix a *standard decomposition*

$$[P, \mathbf{g}] = \begin{array}{c} g_1' \\ \vdots \\ g_2' g_1' \\ \vdots \end{array} \circ F(D_\pi) \circ F(S) \circ F(D_\sigma) \circ \begin{array}{c} \vdots \\ \vdots \\ g_k \\ \vdots \\ g_2 \\ \vdots \\ g_1 \end{array}, \quad (6.1.14)$$

where $\pi \in \mathfrak{S}_l$, $\sigma \in \mathfrak{S}_k$, and S is a tensor product of star diagrams. Then define

$$y_{P,\mathbf{g}} := \left(g_{l'}^{-1} \downarrow \cdots g_{2'}^{-1} \downarrow g_{1'}^{-1} \downarrow \right) \circ D_\pi \circ S \circ D_\sigma \circ \left(g_k \downarrow \cdots g_{2'} \downarrow g_{1'} \downarrow \right). \quad (6.1.15)$$

Hence $F(y_{P,\mathbf{g}}) = [P, \mathbf{g}]$

To complete the proof that F is faithful, it remains to show that any morphism in $\mathcal{P}ar'(G, d)$ that is obtained from the generators by tensor product and composition is equal to a scalar multiple of $y_{P,\mathbf{g}}$ for some chosen representative (P, \mathbf{g}) . As noted above, Theorem 6.1.4 holds for the partition category, which is the case where $G = \{1\}$ is the trivial group. Now, if we ignore tokens, the relations (6.1.1) to (6.1.4) and (6.1.13) correspond to the relations in the $G = \{1\}$ case, except for the fact that the second relation in (6.1.4) gives zero when $g \neq h$. It follows that every morphism in $\mathcal{P}ar'(G, d)$ obtained from the generators by tensor product and composition is equal to a (potentially zero) scalar multiple of a morphism obtained from some $D_\pi \circ S \circ D_\sigma$ by adding tokens (since, ignoring tokens, this can be done in the partition category). Then, since the string diagram for $D_\pi \circ S \circ D_\sigma$ is a tree (i.e. contains no cycles), one can use relations (6.1.5), (6.1.10), and (6.1.11) to move all tokens to the ends of strings and combine them into a single token at each endpoint. This yields a diagram of the form (6.1.15), except that the tokens may not correspond to our chosen representative of the equivalence class of G -partitions. However, we then use the relations (6.1.5), (6.1.10), and (6.1.11) to adjust the tokens at the endpoints so that we obtain the chosen representative. \square

In the context of Remark 6.1.2, (6.1.13) is the statement that the Frobenius object \mathbf{V} has dimension d . From now on, we will identify $\mathcal{P}ar(G, d)$ with the quotient of $\mathcal{P}ar(G)$ by the relation (6.1.13) via the isomorphism of Theorem 6.1.4. In particular, we will identify the object k of $\mathcal{P}ar(G, d)$ with the object $\mathbf{V}^{\otimes k}$ of $\mathcal{P}ar(G)$, for $k \in \mathbb{N}$.

Remark 6.1.5. Suppose we define $\mathcal{P}ar(G, d)$ as in Definition 5.2.6, but over the ring $\mathbb{k}[d]$, so that d is an indeterminate. It then follows from Theorem 6.1.4 that $\mathcal{P}ar(G, d)$ is isomorphic to $\mathcal{P}ar(G)$ as a \mathbb{k} -linear monoidal category. Under this isomorphism, $d1_1$ corresponds to \circlearrowleft .

For $k \in \mathbb{N}$, we define the G -partition algebra

$$P_k(G, d) := \text{End}_{\mathcal{P}ar(G, d)}(\mathbf{V}^{\otimes k}). \quad (6.1.16)$$

When we do not wish to make G explicit, we call these *group partition algebras*. These algebras appeared in [Blo03], where they are called *G -colored partition algebras*. A diagrammatic description of these algebras, different from that of this thesis, is given in [Blo03, §6.2].

6.2 Categorical action

In this section we assume that the group G is finite of order $|G|$. We define a categorical action of the G -partition category on the category of representations of the wreath product groups $G_n = G^n \rtimes \mathfrak{S}_n$. We first describe this action by giving the action of the generators, and then describe the action of an arbitrary G -partition diagram. Recall that $A_n = \mathbb{k}G_n$ is the group algebra of G_n , and so we can naturally identify A_n -modules and representations of G_n .

Theorem 6.2.1. For $n \in \mathbb{N}$, we have a strong \mathbb{k} -linear monoidal functor $\Phi_n: \mathcal{Par}(G, n|G|) \rightarrow A_n\text{-mod}$ given as follows. On objects, Φ_n is determined by $\Phi_n(V) = V$. On generating morphisms, Φ_n is given by

$$\begin{aligned} \Phi_n(\frown): V \otimes V &\rightarrow V, & ge_i \otimes he_j &\mapsto \delta_{g,h} \delta_{i,j} ge_i, \\ \Phi_n(\vee): V &\rightarrow V \otimes V, & ge_i &\mapsto ge_i \otimes ge_i, \\ \Phi_n(\bowtie): V \otimes V &\rightarrow V, & v \otimes w &\mapsto w \otimes v, \\ \Phi_n(\delta): \mathbf{1}_n &\rightarrow V, & 1 &\mapsto \sum_{g \in G} \sum_{i=1}^n ge_i, \\ \Phi_n(\varphi): V &\rightarrow \mathbf{1}_n, & ge_i &\mapsto 1, \\ \Phi_n(g \blacktriangleright): V &\rightarrow V, & he_i &\mapsto hg^{-1}e_i, \end{aligned}$$

for $g, h \in G$, $1 \leq i, j \leq n$, $v, w \in V$.

Proof. We must show that the action preserves the relations (6.1.1) to (6.1.5).

Relations (6.1.1): To prove the first three equalities in (6.1.1), we compute

$$\begin{aligned} \Phi_n(\frown) \circ \Phi_n(| \otimes \delta)(ge_i) &= \sum_{h \in G} \sum_{j=1}^n \Phi_n(\frown)(ge_i \otimes he_j) = \sum_{h \in G} \sum_{j=1}^n \delta_{g,h} \delta_{i,j} ge_i = ge_i, \\ \Phi_n(| \otimes \varphi) \circ \Phi_n(\vee)(ge_i) &= \Phi_n(| \otimes \varphi)(ge_i \otimes ge_i) = ge_i = \Phi_n(\varphi \otimes |) \circ \Phi_n(\vee)(ge_i). \end{aligned}$$

To prove the third relation, we compute

$$\begin{aligned} \Phi_n(\vee) \circ \Phi_n(\frown)(ge_i \otimes he_j) &= \delta_{g,h} \delta_{i,j} \Phi_n(\vee)(ge_i) = \delta_{g,h} \delta_{i,j} (ge_i \otimes ge_i), \\ \Phi_n(\frown \otimes |) \circ \Phi_n(| \otimes \vee)(ge_i \otimes he_j) &= \Phi_n(\frown \otimes |)(ge_i \otimes he_j \otimes he_j) = \delta_{g,h} \delta_{i,j} (ge_i \otimes ge_i), \\ \Phi_n(| \otimes \frown) \circ \Phi_n(\vee \otimes |)(ge_i \otimes he_j) &= \Phi_n(| \otimes \frown)(ge_i \otimes ge_i \otimes he_j) = \delta_{g,h} \delta_{i,j} (ge_i \otimes ge_i), \end{aligned}$$

concluding that the three maps are identical.

Relations (6.1.2): The relations (6.1.2) are straightforward. To prove the second relation, we see that the left-hand side is sent to the map given by

$$\begin{aligned} & \Phi_n(\times \otimes |) \circ \Phi_n(| \otimes \times) \circ \Phi_n(\times \otimes |)(v \otimes w \otimes z) \\ &= \Phi_n(\times \otimes |) \circ \Phi_n(| \otimes \times)(w \otimes v \otimes z) = \Phi_n(\times \otimes |)(w \otimes z \otimes v) = z \otimes w \otimes v \end{aligned}$$

while the right-hand side is sent to the map given by

$$\begin{aligned} & \Phi_n(| \otimes \times) \circ \Phi_n(\times \otimes |) \circ \Phi_n(| \otimes \times)(v \otimes w \otimes z) \\ &= \Phi_n(| \otimes \times) \circ \Phi_n(\times \otimes |)(v \otimes z \otimes w) = \Phi_n(| \otimes \times)(z \otimes v \otimes w) = z \otimes w \otimes v. \end{aligned}$$

Relations (6.1.3): To prove the first relation in (6.1.3), we compute

$$\Phi_n(\times) \circ \Phi_n(| \otimes \flat)(ge_i) = \sum_{h \in G} \sum_{j=1}^n \Phi_n(\times)(ge_i \otimes he_j) = \sum_{h \in G} \sum_{j=1}^n (he_j \otimes ge_i),$$

and

$$\Phi_n(\flat \otimes |)(ge_i) = \sum_{h \in G} \sum_{j=1}^n (he_j \otimes ge_i).$$

Similarly, to prove the second relation, we calculate

$$\Phi_n(| \otimes \uparrow) \circ \Phi_n(\times)(ge_i \otimes he_j) = \Phi_n(| \otimes \uparrow)(he_j \otimes ge_i) = he_j = \Phi_n(\uparrow \otimes |)(ge_i \otimes he_j).$$

The proof of the last two equalities in (6.1.3) are similar; we only check the last one. We have

$$\begin{aligned} & \Phi_n(| \otimes \times) \circ \Phi_n(\times \otimes |) \circ \Phi_n(| \otimes \vee)(ge_i \otimes he_j) = \Phi_n(| \otimes \times) \circ \Phi_n(\times \otimes |)(ge_i \otimes he_j \otimes he_j) \\ &= \Phi_n(| \otimes \times)(he_j \otimes ge_i \otimes he_j) = he_j \otimes he_j \otimes ge_i \end{aligned}$$

and

$$\Phi_n(\vee \otimes |) \circ \Phi_n(\times)(ge_i \otimes he_j) = \Phi_n(\vee \otimes |)(he_j \otimes ge_i) = he_j \otimes he_j \otimes ge_i.$$

Relations (6.1.4): To prove the first relation in (6.1.4), we compute

$$\Phi_n(\wedge) \circ \Phi_n(\times)(ge_i \otimes he_j) = \Phi_n(\wedge)(he_j \otimes ge_i) = \delta_{g,h} \delta_{i,j} ge_i = \Phi_n(\wedge)(ge_i \otimes he_j).$$

For the second relation, we have

$$\begin{aligned} \Phi_n \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \circ \Phi_n \left(\begin{array}{c} g \downarrow \\ \otimes \\ h \downarrow \end{array} \right) \circ \Phi_n \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) (ke_i) &= \Phi_n \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \circ \Phi_n \left(\begin{array}{c} g \downarrow \\ \otimes \\ h \downarrow \end{array} \right) (ke_i \otimes ke_i) \\ &= \Phi_n \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) (kg^{-1}e_i \otimes kh^{-1}e_i) = \delta_{g,h}(kg^{-1}e_i) = \delta_{g,h}\Phi_n \left(\begin{array}{c} g \downarrow \end{array} \right) (ke_i). \end{aligned}$$

Relations (6.1.5): The relations (6.1.5) are straightforward to verify. For the first relation, we have

$$\Phi_n \left(\begin{array}{c} g \downarrow \end{array} \right) \circ \Phi_n \left(\begin{array}{c} h \downarrow \end{array} \right) (ke_i) = \Phi_n \left(\begin{array}{c} g \downarrow \end{array} \right) (kh^{-1}e_i) = kh^{-1}g^{-1}e_i = \Phi_n \left(\begin{array}{c} gh \downarrow \end{array} \right) (ke_i).$$

The second relation is clear. For the third relation, we have:

$$\begin{aligned} \Phi_n \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) \circ \Phi_n \left(\begin{array}{c} g \downarrow \\ | \\ \otimes \\ | \\ g \downarrow \end{array} \right) (he_i \otimes ke_j) &= \Phi_n \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) (hg^{-1}e_i \otimes ke_j) = ke_j \otimes hg^{-1}e_i \\ &= \Phi_n \left(\begin{array}{c} | \\ \otimes \\ g \downarrow \end{array} \right) (ke_j \otimes he_i) = \Phi_n \left(\begin{array}{c} | \\ \otimes \\ g \downarrow \end{array} \right) \circ \Phi_n \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) (he_i \otimes ke_j). \end{aligned}$$

The fourth and fifth relations are clear. \square

If (P, \mathbf{g}) is a G -partition of type $\binom{l}{k}$, then $\Phi_n([P, \mathbf{g}]) \in \text{Hom}_{G_n}(V^{\otimes k}, V^{\otimes l})$ is uniquely described by its matrix coefficients:

$$\Phi_n([P, \mathbf{g}])(h_k e_{i_k} \otimes \cdots \otimes h_1 e_{i_1}) = \sum_{\substack{h_{1'}, \dots, h_{l'} \in G \\ 1 \leq i_{1'}, \dots, i_{l'} \leq n}} M(P, \mathbf{g})_{h_1, \dots, h_k, i_1, \dots, i_k}^{h_{1'}, \dots, h_{l'}, i_{1'}, \dots, i_{l'}} h_{l'} e_{i_{l'}} \otimes \cdots \otimes h_{1'} e_{i_{1'}}. \quad (6.2.1)$$

The matrix $M(P, \mathbf{g})$ depends only on the equivalence class $[P, \mathbf{g}]$ of (P, \mathbf{g}) .

Proposition 6.2.2. Suppose (P, \mathbf{g}) is a G -partition of type $\binom{l}{k}$. Then $M(P, \mathbf{g})_{h_1, \dots, h_k, i_1, \dots, i_k}^{h_{1'}, \dots, h_{l'}, i_{1'}, \dots, i_{l'}} = 1$ if $i_a = i_b$ and $h_a g_a^{-1} = h_b g_b^{-1}$ for all $a, b \in X_k^l$ in the same part of P . Otherwise, $M(P, \mathbf{g})_{h_1, \dots, h_k, i_1, \dots, i_k}^{h_{1'}, \dots, h_{l'}, i_{1'}, \dots, i_{l'}} = 0$.

Proof. This follows from a straightforward computation using the definition of Φ_n in Theorem 6.2.1 and the isomorphism described in Theorem 6.1.4, writing each component of P as a composition of tokens, merges, splits, and crossings as in (6.1.14). Indeed, by Theorem 6.2.1 the result is true on generating morphisms of the category $\mathcal{Par}(G, n|G|)$. Furthermore we can identify any G -partition (P, \mathbf{g}) with his corresponding decomposition (6.1.15), given in the proof of Theorem 6.1.4:

$$[P, \mathbf{g}] = \left(\begin{array}{c} g_1^{-1} \downarrow \\ \cdots \\ g_{2'}^{-1} \downarrow \\ \cdots \\ g_{1'}^{-1} \downarrow \end{array} \right) \circ D_\pi \circ S \circ D_\sigma \circ \left(\begin{array}{c} g_k \downarrow \\ \cdots \\ g_{2'} \downarrow \\ \cdots \\ g_{1'} \downarrow \end{array} \right). \quad (6.2.2)$$

The result follows by applying consecutively the morphisms in the above composition on any element of the form

$$h_k e_{i_k} \otimes \cdots \otimes h_a e_{i_a} \otimes \cdots \otimes h_b e_{i_b} \otimes \cdots \otimes h_1 e_{i_1}.$$

\square

Recall the basis \mathbf{B}_k for $V^{\otimes k}$ from (5.1.4). Given a G -partition (P, \mathbf{g}) of type $\binom{0}{k}$, let $O_{P, \mathbf{g}}$ denote the set of all $h_k e_{i_k} \otimes \cdots \otimes h_1 e_{i_1} \in \mathbf{B}_k$ such that

- $i_a = i_b$ if and only if a, b are in the same part of the partition P , and
- $h_a g_a^{-1} = h_b g_b^{-1}$ for all a, b in the same part of the partition P .

Then we have $O_{P, \mathbf{g}} = O_{P, \mathbf{h}}$ if and only if $(P, \mathbf{g}) \sim (P, \mathbf{h})$, and so we can define $O_{[P, \mathbf{g}]} := O_{P, \mathbf{g}}$. The G_n -orbits of \mathbf{B}_k are the $O_{[P, \mathbf{g}]}$ with $[P, \mathbf{g}]: \mathbf{V}^{\otimes k} \rightarrow \mathbf{1}$ having at most n parts (i.e. P is a partition of $\{1, \dots, k\}$ having at most n parts).

For $[P, \mathbf{g}]: \mathbf{V}^{\otimes k} \rightarrow \mathbf{1}$, let $f_{[P, \mathbf{g}]}: \mathbf{V}^{\otimes k} \rightarrow \mathbb{k}$ be the \mathbb{k} -linear map determined on the basis \mathbf{B}_k by

$$f_{[P, \mathbf{g}]}(g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1}) = \begin{cases} 1 & \text{if } g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} \in O_{[P, \mathbf{g}]}, \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma is now immediate.

Lemma 6.2.3. The set of all $f_{[P, \mathbf{g}]}$ with $[P, \mathbf{g}]: \mathbf{V}^{\otimes k} \rightarrow \mathbf{1}$ having at most n parts is a basis for $\text{Hom}_{G_n}(V^{\otimes k}, \mathbf{1}_n)$.

For partitions P, Q of X_k^l , we write $P \leq Q$ when Q is coarser than P . Thus, $P \leq Q$ if and only if every part of P is a subset of some part of Q . We write $(P, \mathbf{g}) \leq (Q, \mathbf{h})$ and $[P, \mathbf{g}] \leq [Q, \mathbf{h}]$ whenever $P \leq Q$.

It follows from Proposition 6.2.2 that

$$\Phi_n([P, \mathbf{g}]) = \sum_{[Q, \mathbf{h}] \geq [P, \mathbf{g}]} f_{[Q, \mathbf{h}]} \quad \text{for all } [P, \mathbf{g}]: \mathbf{V}^{\otimes k} \rightarrow \mathbf{1}. \quad (6.2.3)$$

Thus, if we define a new basis $\{x_{[P, \mathbf{g}]}: [P, \mathbf{g}]: \mathbf{V}^{\otimes k} \rightarrow \mathbf{V}^{\otimes l}\}$ of $\text{Hom}_{\text{Par}(G, n|G)}(\mathbf{V}^{\otimes k}, \mathbf{V}^{\otimes l})$ recursively by

$$x_{[P, \mathbf{g}]} = [P, \mathbf{g}] - \sum_{[Q, \mathbf{h}] > [P, \mathbf{g}]} x_{[Q, \mathbf{h}]}, \quad (6.2.4)$$

then a straightforward argument by induction gives us the following result:

Lemma 6.2.4. We have,

$$\Phi_n(x_{[P, \mathbf{g}]}) = f_{[P, \mathbf{g}]} \quad \text{for any } G\text{-partition } (P, \mathbf{g}) \text{ of type } \binom{0}{k}. \quad (6.2.5)$$

Proof. To see this, we proceed by induction on the number of parts of a G -partition. Assume that the G -partition (P, \mathbf{g}) has n parts. When $n = 1$, we have

$$x_{[P, \mathbf{g}]} = [P, \mathbf{g}] \implies \Phi_n(x_{[P, \mathbf{g}]}) = \Phi_n([P, \mathbf{g}]) = f_{[P, \mathbf{g}]}.$$

Assume that the result holds for any G -diagram with m parts, $m < n$, that is

$$\Phi_n(x_{[Q, \mathbf{g}]}) = f_{[Q, \mathbf{g}]} \quad \text{for any } G\text{-diagram } (Q, \mathbf{g}) \text{ with } m \text{ parts, } m < n.$$

We have by definition of $x_{[P, \mathbf{g}]}$:

$$x_{[P, \mathbf{g}]} = [P, \mathbf{g}] - \sum_{[Q, \mathbf{h}] > [P, \mathbf{g}]} x_{[Q, \mathbf{h}]}.$$

Note that the G -partitions (Q, \mathbf{h}) such that $[Q, \mathbf{h}] > [P, \mathbf{g}]$ are G -partitions with fewer than n parts. Therefore,

$$\begin{aligned} \Phi_n(x_{[P, \mathbf{g}]}) &= \Phi_n([P, \mathbf{g}]) - \sum_{[Q, \mathbf{h}] > [P, \mathbf{g}]} \Phi_n(x_{[Q, \mathbf{h}]}) \\ &= \Phi_n([P, \mathbf{g}]) - \sum_{[Q, \mathbf{h}] > [P, \mathbf{g}]} f_{[Q, \mathbf{h}]} \\ &= f_{[P, \mathbf{g}]} + \sum_{[Q, \mathbf{h}] > [P, \mathbf{g}]} f_{[Q, \mathbf{h}]} - \sum_{[Q, \mathbf{h}] > [P, \mathbf{g}]} f_{[Q, \mathbf{h}]} \\ &= f_{[P, \mathbf{g}]} \end{aligned}$$

□

In particular, $\Phi_n(x_{[P, \mathbf{g}]}) = 0$ if $[P, \mathbf{g}] : k \rightarrow 0$ has more than n parts.

Theorem 6.2.5. (a) The functor Φ_n is full.

(b) The kernel of the induced map

$$\mathrm{Hom}_{\mathcal{P}ar(G, n|G|)}(\mathbf{V}^{\otimes k}, \mathbf{V}^{\otimes l}) \rightarrow \mathrm{Hom}_{G_n}(V^{\otimes k}, V^{\otimes l})$$

is the span of all $x_{[P, \mathbf{g}]}$ with $[P, \mathbf{g}] : \mathbf{V}^{\otimes k} \rightarrow \mathbf{V}^{\otimes l}$ having more than n parts. In particular, this map is an isomorphism if and only if $k + l \leq n$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{P}ar(G, n|G|)}(\mathbf{V}^{\otimes k}, \mathbf{V}^{\otimes l}) & \longrightarrow & \mathrm{Hom}_{\mathcal{P}ar(G, n|G|)}(\mathbf{V}^{\otimes(k+l)}, \mathbf{1}) \\ \Phi_n \downarrow & & \downarrow \Phi_n \\ \mathrm{Hom}_{G_n}(V^{\otimes k}, V^{\otimes l}) & \longrightarrow & \mathrm{Hom}_{G_n}(V^{\otimes(k+l)}, \mathbf{1}_n) \end{array}$$

where the horizontal maps are adjunction isomorphisms arising from the fact that \mathbf{V} is a self-dual object in $\mathcal{P}ar(G, n|G|)$ and that V is a self-dual object in the category of A_n -modules.

(We refer the reader to the proof of [Com16, Th. 2.3] for more details of these adjunctions in the special case $G = \{1\}$. The argument is the same in the case of general G .) It follows from Lemma 6.2.3 and (6.2.5) that the right-hand vertical map is surjective and its kernel is the span of all $x_{[P, \mathbf{g}]}$ with $[P, \mathbf{g}]: \mathbf{V}^{\otimes(k+l)} \rightarrow \mathbf{1}$ having more than n parts. Since the adjunction isomorphisms preserve the number of parts of G -partitions, as well as the partial order on G -partitions, the result follows. \square

When $G = \{1\}$ is the trivial group, Theorem 6.2.5 reduces to [Com16, Th. 2.3]. In general, Theorem 6.2.5 is a categorical generalization of the double centralizer property [Blo03, Th. 6.6]. More precisely, recall the G -partition algebras from (6.1.16). The functor Φ_n induces an algebra homomorphism

$$P_k(G, n|G|) \rightarrow \text{End}_{G_n}(V^{\otimes k}).$$

Theorem 6.2.5 implies that this homomorphism is surjective, and is an isomorphism when $n \geq 2k$. When the characteristic of \mathbb{k} does not divide $n|G| = |G_n|$, so that A_n is semisimple, the Double Centralizer Theorem implies that A_n generates $\text{End}_{P_k(G, n|G|)}(V^{\otimes k})$. Hence G_n and $P_k(G, n|G|)$ generate the centralizers of each other in $\text{End}_{\mathbb{k}}(V^{\otimes k})$.

Chapter 7

The group Heisenberg category, the embedding functor and compatibility of categorical actions

In this chapter, we give the definition of the *group Heisenberg category* and then we generalize the notions introduced in Chapter 3.

7.1 The group Heisenberg category

In this section we recall a special case of the Frobenius Heisenberg category. We are interested in the special case of central charge -1 , where this category was first defined in [RS17]. Furthermore, we will specialize to the case where the Frobenius algebra is the group algebra of a finite group G . We follow the presentation in [Sav19], referring the reader to that paper for proofs of the statements made here.

Definition 7.1.1. The *group Heisenberg category* $\mathcal{H}eis(G)$ associated to the finite group G is the strict \mathbb{k} -linear monoidal category generated by two objects \uparrow, \downarrow , and morphisms

$$\begin{aligned} \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} : \uparrow\uparrow \rightarrow \uparrow\uparrow, \quad g \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \end{array} : \uparrow \rightarrow \uparrow, \quad g \in G, \\ \cup : \mathbf{1} \rightarrow \downarrow\uparrow, \quad \cap : \uparrow\downarrow \rightarrow \mathbf{1}, \quad \cup : \mathbf{1} \rightarrow \uparrow\downarrow, \quad \cap : \downarrow\uparrow \rightarrow \mathbf{1}, \end{aligned}$$

subject to the relations

$$\begin{array}{c} \uparrow \\ \diagdown \\ \diagup \\ \uparrow \end{array} = \uparrow\uparrow, \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \searrow \\ \searrow \\ \nearrow \end{array}, \quad \begin{array}{c} g \\ \uparrow \\ \bullet \\ h \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ gh \\ \uparrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \end{array} = \uparrow, \quad \begin{array}{c} \nearrow \\ \searrow \\ \bullet \\ \nearrow \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \bullet \\ \searrow \\ \nearrow \\ \searrow \end{array}, \quad (7.1.1)$$

$$\begin{aligned}
 & \uparrow \cup \uparrow = \uparrow, \quad \downarrow \cap \downarrow = \downarrow, \quad (7.1.2) \\
 & \text{left crossing} = \uparrow \downarrow, \quad \text{right crossing} = \downarrow \uparrow - \sum_{g \in G} \begin{array}{c} \text{cup} \\ \uparrow \quad \uparrow \\ \text{token } g \end{array} \begin{array}{c} \text{cap} \\ \downarrow \quad \downarrow \\ \text{token } g^{-1} \end{array}, \quad \text{crossing} = 0, \quad \text{loop} = \delta_{g,1} \mathbf{1}. \quad (7.1.3)
 \end{aligned}$$

Here the left and right crossings are defined by

$$\begin{array}{c} \diagdown \\ \diagup \end{array} := \begin{array}{c} \text{cup} \\ \diagdown \quad \diagup \end{array}, \quad \begin{array}{c} \diagup \\ \diagdown \end{array} := \begin{array}{c} \text{cap} \\ \diagup \quad \diagdown \end{array}.$$

The objects \uparrow and \downarrow are both left and right dual to each other. Furthermore, the cups and caps endow $\mathcal{H}eis(G)$ with the structure of a strict pivotal category, meaning that morphisms are invariant under isotopy. We define downwards crossings and downward tokens by

$$\begin{array}{c} \diagdown \\ \diagup \end{array} := \begin{array}{c} \text{cup} \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \text{cap} \\ \diagup \quad \diagdown \end{array}, \quad \begin{array}{c} \bullet \\ \downarrow \end{array} := \begin{array}{c} \text{cup} \\ \bullet \quad \downarrow \end{array} = \begin{array}{c} \text{cap} \\ \downarrow \quad \bullet \end{array}. \quad (7.1.4)$$

It follows that,

$$\begin{array}{c} \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \quad \text{for all possible orientations of the strands,} \quad (7.1.5)$$

and we that tokens slide over cups and caps:

$$\begin{array}{c} \bullet \\ \downarrow \end{array} \cup \uparrow = \cup \begin{array}{c} \bullet \\ \uparrow \end{array}, \quad \begin{array}{c} \bullet \\ \downarrow \end{array} \cap \downarrow = \cap \begin{array}{c} \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \bullet \\ \downarrow \end{array} \cap \downarrow = \cap \begin{array}{c} \bullet \\ \downarrow \end{array}, \quad \cup \begin{array}{c} \bullet \\ \downarrow \end{array} = \begin{array}{c} \bullet \\ \downarrow \end{array} \cup \uparrow. \quad (7.1.6)$$

(One should compare this to the first two relations in (6.1.10).) Because of this, we will sometimes place tokens at the critical point of cups and caps, since there is no ambiguity. In addition it follows from (2.2.1), (2.2.2), and (7.1.6) that

$$\begin{array}{c} \bullet \\ \downarrow \end{array} \cap \downarrow = \cap \begin{array}{c} \bullet \\ \downarrow \end{array}, \quad \begin{array}{c} \bullet \\ \downarrow \end{array} \cap \downarrow = \cap \begin{array}{c} \bullet \\ \downarrow \end{array} \quad \text{for all possible orientations of the strands.} \quad (7.1.7)$$

We will use (7.1.6) and (7.1.7) frequently without mention. Note how tokens multiply on downward strands (using (2.2.1), (7.1.4), and (7.1.6)):

$$\begin{array}{c} \bullet \\ \downarrow \\ \bullet \\ \downarrow \end{array} = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \\ \downarrow \end{array}. \quad (7.1.8)$$

7.2 The embedding functor

In this section we define an explicit embedding of the group partition category into the group Heisenberg category. We assume throughout this section that G is a finite group.

Theorem 7.2.1. There is a faithful strict \mathbb{k} -linear monoidal functor $\Psi: \mathcal{P}ar(G) \rightarrow \mathcal{H}eis(G)$ defined on objects by $\mathbf{V} \mapsto \uparrow \otimes \downarrow$ and on generating morphisms by

$$\begin{aligned} \text{triple junction} &\mapsto \text{curved triple junction}, & \text{Y-junction} &\mapsto \text{curved Y-junction}, & \text{crossing} &\mapsto \text{crossing} + \sum_{g \in G} g^{-1} \left[\text{crossing with } g \text{ dots} \right] g^{-1}, \\ \text{cup} &\mapsto \text{cup}, & \text{cap} &\mapsto \text{cap}, & g \downarrow &\mapsto g \uparrow \downarrow g^{-1}, \quad g \in G. \end{aligned}$$

The proof of Theorem 7.2.1 will occupy the remainder of this section. We break the proof into two parts, first showing that Ψ is well defined, and then that it is faithful.

Proposition 7.2.2. The functor Ψ is well defined.

Proof. It suffices to show that the images of the generating morphisms of the partition category $\mathcal{P}ar(G)$ satisfy relations (6.1.1) to (6.1.5).

Relations (6.1.1): These relations are easy to check, using isotopy invariance in $\mathcal{H}eis(G)$.

We check some of them, leaving the verification of the others to the reader. For the first relation in (6.1.1), we have

$$\begin{aligned} \Psi \left(\text{triple junction with cup} \right) &= \Psi \left(\text{triple junction} \right) \circ \Psi \left(\text{cup} \right) \\ &= \text{curved triple junction} \circ \left(\uparrow \downarrow \otimes \text{cup} \right) \\ &= \text{curved triple junction with cup} \\ &\stackrel{(7.1.2)}{=} \uparrow \downarrow. \end{aligned}$$

For the second equality of the third relation, we have:

$$\begin{aligned} \Psi \left(\text{triple junction with Y-junction} \right) &= \Psi \left(\text{triple junction} \right) \circ \Psi \left(\text{Y-junction} \right) \\ &= \left(\text{curved triple junction} \right) \circ \left(\text{curved Y-junction} \right) \\ &= \text{curved triple junction with Y-junction}; \end{aligned}$$

and

$$\begin{aligned} \Psi \left(\text{triple junction with crossing} \right) &= \Psi \left(\text{triple junction} \right) \circ \Psi \left(\text{crossing} \right) \\ &= \left(\text{curved triple junction} \right) \circ \left(\text{crossing} \right) \end{aligned}$$

$$\begin{aligned}
 &= \text{Diagram 1} \\
 &= \text{Diagram 2}
 \end{aligned}$$

The last equality above follows from isotopy of the diagrams.

Relations (6.1.2): Using the third relation in (2.2.3), which says that left curls are equal to zero, we have

$$\begin{aligned}
 \Psi(\times) \circ \Psi(\times) &= \text{Diagram 1} + \sum_{g,h \in G} h^{-1}g^{-1} \cdot \text{Diagram 2} \cdot g^{-1}h^{-1} \\
 &\stackrel{(7.1.3)}{=} \text{Diagram 3} + \sum_{g \in G} \text{Diagram 4} \stackrel{(7.1.3)}{=} \text{Diagram 5} = \Psi(|),
 \end{aligned}$$

proving the first relation in (6.1.2). To prove the second relation in (6.1.2), we compute

$$\begin{aligned}
 \Psi(| \otimes \times) \circ \Psi(\times \otimes |) &= \text{Diagram 1} + \sum_{g \in G} g^{-1} \cdot \text{Diagram 2} \cdot g^{-1} \\
 &+ \sum_{g \in G} g^{-1} \cdot \text{Diagram 3} \cdot g^{-1} + \sum_{g,h \in G} g^{-1} \cdot \text{Diagram 4} \cdot h^{-1}g^{-1} \cdot \text{Diagram 5} \cdot h^{-1}. \quad (7.2.1)
 \end{aligned}$$

Therefore, using the fact that left curls are zero, we compute

$$\begin{aligned}
 &\Psi(| \otimes \times) \circ \Psi(\times \otimes |) \circ \Psi(| \otimes \times) \\
 &\stackrel{(7.1.3)}{=} \stackrel{(7.1.7)}{=} \text{Diagram 1} + \sum_{g \in G} g^{-1} \cdot \text{Diagram 2} \cdot g^{-1} + \sum_{g \in G} g^{-1} \cdot \text{Diagram 3} \cdot g^{-1} \\
 &+ \sum_{g \in G} g^{-1} \cdot \text{Diagram 4} \cdot g^{-1} + \sum_{g,h,t \in G} g^{-1} \cdot \text{Diagram 5} \cdot h^{-1}g^{-1} \cdot \text{Diagram 6} \cdot h^{-1} \cdot (ht)^{-1} \\
 &\stackrel{(7.1.1)}{=} \stackrel{(7.1.3)}{=} \text{Diagram 7} + \sum_{g \in G} g^{-1} \cdot \text{Diagram 8} \cdot g^{-1} + \sum_{g \in G} g^{-1} \cdot \text{Diagram 9} \cdot g^{-1}
 \end{aligned}$$

Relations (6.1.5): For the third relation in (6.1.5), we compute

$$\begin{aligned} \Psi(\text{cross}) \circ \Psi(g \downarrow |) &= \text{cross} + \sum_{h \in G} h^{-1}g \downarrow \begin{array}{c} \curvearrowright h \\ \downarrow \\ g^{-1}h \\ \downarrow \\ h^{-1} \end{array} \\ &= \text{cross} + \sum_{t \in G} t^{-1} \downarrow \begin{array}{c} \downarrow \\ gt \\ \downarrow \\ t \end{array} t^{-1}g^{-1} = \Psi(| \downarrow g) \circ \Psi(\text{cross}), \end{aligned}$$

where we let $t = g^{-1}h$. The other relations in (6.1.5) are straightforward to verify. For the first relation, we have

$$\Psi(g \downarrow) \circ \Psi(h \downarrow) \stackrel{(7.1.1)}{=} gh \downarrow \downarrow h^{-1}g^{-1} = gh \downarrow \downarrow (gh)^{-1} = \Psi(gh \downarrow).$$

The second relation is clear. We check the fourth relation:

$$\Psi(g \downarrow \downarrow g) \circ \Psi(\text{Y}) \stackrel{(7.1.1)}{=} \begin{array}{c} \downarrow \\ g \\ \downarrow \\ g^{-1} \end{array} \text{Y} = \Phi_n(\text{Y}) \circ \Phi_n(g \downarrow).$$

The fifth relation follows immediately from (7.1.6). □

We now wish to show that Ψ is faithful. Our approach is inspired by the proof of Theorem 3.2.2 which deals with the case where G is the trivial group.

In what follows, we will identify a permutation $\pi \in \mathfrak{S}_k$ with the partition of type $\binom{k}{k}$ with parts $\{i, \pi(i)'\}$, $1 \leq i \leq k$. Recall that, for $1 \leq i < j \leq k$, the pair (i, j) is an *inversion* in $\pi \in \mathfrak{S}_k$ if $\pi(i) > \pi(j)$. Suppose Q is a partition of type $\binom{l}{k}$. We say that a permutation $P \in \mathfrak{S}_l$ is a *left shuffle* for Q if there is no inversion (i, j) in P such that vertices i' and j' lie in the same connected component of Q . Intuitively, P is a left shuffle for Q if it does not change the relative order of vertices in each component. Similarly, we say that a permutation $P \in \mathfrak{S}_k$ is a *right shuffle* for Q if there is no inversion (i, j) in P^{-1} such that vertices i and j lie in the same component of Q . For example, if

$$P = \begin{array}{ccc} \bullet & \bullet & \bullet \\ & \times & \\ \bullet & \bullet & \bullet \end{array} \quad \text{and} \quad Q = \begin{array}{ccc} \bullet & \bullet & \bullet \\ | & | & | \\ \bullet & \bullet & \bullet \end{array},$$

then P is a left shuffle for Q but not a right shuffle for Q . But if we consider

$$Q = \begin{array}{ccc} \bullet & \bullet & \bullet \\ & \diagdown & / \\ \bullet & \bullet & \bullet \end{array},$$

then P is a left and right shuffle for Q .

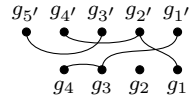
We say a partition diagram is *tensor-planar* if it is a tensor product (horizontal juxtaposition) of partition diagrams consisting of a single connected component. Note that every

tensor-planar partition diagram is planar (i.e. can be represented as a graph without edge crossings inside of the rectangle formed by its vertices) but the converse is false.

Every equivalence class $[P, \mathbf{g}]$ of G -partitions can be factored as a product

$$[P, \mathbf{g}] = \begin{array}{c} g_{l'} \\ \vdots \\ g_{2'} g_{1'} \\ \vdots \\ g_k \quad g_2 \quad g_1 \end{array} \circ [P_1] \circ [P_2] \circ [P_3] \circ \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \quad (7.2.2)$$

where P_2 is tensor-planar, P_1 is a left shuffle for P_2 , and P_3 is a right shuffle for P_2 . (See (6.1.14).) The number of connected components in P is equal to the number of connected components in P_2 . For example, the G -partition diagram



has four connected components and decomposition

$$\begin{array}{c} g_{5'} \quad g_{4'} \quad g_{3'} \quad g_{2'} \quad g_{1'} \\ \vdots \\ g_4 \quad g_3 \quad g_2 \quad g_1 \end{array} = \begin{array}{c} g_{5'} \quad g_{4'} \quad g_{3'} \quad g_{2'} \quad g_{1'} \\ \vdots \\ g_4 \quad g_3 \quad g_2 \quad g_1 \end{array} \circ [P_1] \circ [P_2] \circ [P_3] \circ \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array},$$

where

$$P_1 = \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \begin{array}{c} \times \\ \times \\ \times \\ \times \\ \times \end{array} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}, \quad P_2 = \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} = \begin{array}{c} \otimes \\ \otimes \\ \otimes \\ \otimes \\ \otimes \end{array}, \quad P_3 = \begin{array}{c} \times \\ \times \\ \times \\ \times \\ \times \end{array}$$

For $n, k, l \in \mathbb{N}$, let $\text{Hom}_{\text{Par}(G)}^{\leq n}(\mathbb{V}^{\otimes k}, \mathbb{V}^{\otimes l})$ denote the subspace of $\text{Hom}_{\text{Par}(G)}(\mathbb{V}^{\otimes k}, \mathbb{V}^{\otimes l})$ spanned by G -partition diagrams with at most n connected components. Composition respects the corresponding filtration on morphism spaces.

Recall the bases of the morphism spaces of $\mathcal{H}eis(G)$ given in [BSW20, Thm. 7.2]. (The category $\mathcal{H}eis(G)$ is $\mathcal{H}eis_{-1}(\mathbb{k}G)$ in the notation of [BSW20].) For any such basis element f in $\text{Hom}_{\mathcal{H}eis(G)}((\uparrow\downarrow)^{\otimes k}, (\uparrow\downarrow)^{\otimes l})$, define the *block number* of f to be number of distinct closed (possibly intersecting) loops in the diagram

$$\curvearrowright^{\otimes l} \circ f \circ \curvearrowleft^{\otimes k}.$$

For $n \in \mathbb{N}$, let $\text{Hom}_{\mathcal{H}eis(G)}^{\leq n}((\uparrow\downarrow)^{\otimes k}, (\uparrow\downarrow)^{\otimes l})$ denote the subspace of $\text{Hom}_{\mathcal{H}eis(G)}((\uparrow\downarrow)^{\otimes k}, (\uparrow\downarrow)^{\otimes l})$ spanned by basis elements with block number at most n . Composition respects the resulting filtration on morphism spaces.

The image under Ψ of tensor-planar partition diagrams (writing the image in terms of the aforementioned bases of the morphism spaces of $\mathcal{H}eis(G)$) is particularly simple to describe. Since each tensor-planar partition diagram is a tensor product of single connected

components, consider the case of a single connected component. Then, for example, we have

$$\Psi(\text{diagram with two dots and a curved arrow}) = \text{diagram with two dots and two curved arrows} \quad \text{and} \quad \Psi(\text{diagram with four dots in a square}) = \text{diagram with four dots and four curved arrows}.$$

The general case is analogous. (In fact, the images of all planar partition diagrams are similarly easy to describe.) In particular, if P is a tensor-planar partition diagram with n connected components, then $\Psi(P)$ is a planar diagram with block number n .

For $i = 1, \dots, k-1$, consider the morphism

$$\begin{aligned} \chi_i &:= \Psi\left(1_{\mathbb{V}}^{\otimes(k-i-1)} \otimes \left(\text{diagram with two dots and a crossing} - \text{diagram with four dots in a square}\right) \otimes 1_{\mathbb{V}}^{\otimes(i-1)}\right) \\ &= \begin{array}{c} \uparrow \downarrow \cdots \uparrow \downarrow \text{diagram with two crossings} \uparrow \downarrow \cdots \uparrow \downarrow \end{array} \in \text{End}_{\mathcal{H}\text{eis}(G)}((\uparrow\downarrow)^{\otimes k}). \end{aligned} \quad (7.2.3)$$

For a permutation partition diagram $P: k \rightarrow k$, let $T(P)$ be the morphism in $\mathcal{H}\text{eis}(G)$ defined as follows: Write $D = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_r}$ as a reduced word in simple transpositions and let

$$T(P) = \chi_{i_1} \circ \chi_{i_2} \circ \cdots \circ \chi_{i_r}.$$

It follows from the braid relations (7.1.5) that $T(P)$ is independent of the choice of reduced word for P .

Lemma 7.2.3. Suppose (P, \mathbf{g}) is a G -partition diagram with decomposition (7.2.2). Then

$$\Psi([P, \mathbf{g}]) - \begin{array}{c} g_{1'} \quad g_{2'} g_{1'} \\ \vdots \cdots \vdots \vdots \end{array} \circ [T(P_1)] \circ [P_2] \circ [T(P_3)] \circ \begin{array}{c} \vdots \cdots \vdots \vdots \\ g_k \quad g_2 \quad g_1 \end{array} \in \text{Hom}_{\mathcal{H}\text{eis}(G)}^{\leq n-1}((\uparrow\downarrow)^{\otimes k}, (\uparrow\downarrow)^{\otimes l}).$$

Proof. The case where $\mathbf{g} = \mathbf{1}$ is Proposition 3.2.4. Since composition with G -partition diagrams of the form

$$\begin{array}{c} g_{1'} \quad g_{2'} g_{1'} \\ \vdots \cdots \vdots \vdots \end{array} \quad \text{and} \quad \begin{array}{c} \vdots \cdots \vdots \vdots \\ g_k \quad g_2 \quad g_1 \end{array}$$

does not change the block number, the general case follows. \square

Proof of Theorem 7.2.1. Since Ψ is well defined by Proposition 7.2.2, it remains to show it is faithful. As in Remark 6.1.5, we view $\mathcal{P}\text{ar}(G)$ as $\mathcal{P}\text{ar}(G, d)$ over the ring $\mathbb{k}[d]$. Note that the image of Ψ is contained in the full monoidal subcategory $\mathcal{H}\text{eis}_{\uparrow\downarrow}(G)$ of $\mathcal{H}\text{eis}(G)$ generated by the object $\uparrow \otimes \downarrow$. It follows from the defining relations of $\mathcal{H}\text{eis}(G)$ that

$$\text{diagram with a circle and a vertical arrow} = \text{diagram with a vertical arrow and a circle}.$$

In other words, the clockwise bubble is strictly central in $\mathcal{H}eis_{\uparrow\downarrow}(G)$. Let $\mathcal{H}eis_{\uparrow\downarrow}(G, d)$ be the quotient of $\mathcal{H}eis_{\uparrow\downarrow}(G)$, defined over $\mathbb{k}[d]$, by the additional relation

$$\bigcirc = d1_{\mathbf{1}}.$$

Then, as in Remark 6.1.5, $\mathcal{H}eis_{\uparrow\downarrow}(G)$, defined over \mathbb{k} , is isomorphic as a \mathbb{k} -linear category to $\mathcal{H}eis_{\uparrow\downarrow}(G, d)$. In other words, we can view the clockwise bubble of $\mathcal{H}eis_{\uparrow\downarrow}(G)$ and the morphism \circlearrowright of $\mathcal{P}ar(G)$, both of which are strictly central, as elements of the ground ring.

Now, it is clear that, in the setting of Lemma 7.2.3,

$$\begin{array}{c} g_{i'} \\ \vdots \\ g_{2'} g_{1'} \\ \vdots \end{array} \circ [T(P_1)] \circ [P_2] \circ [T(P_3)] \circ \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ g_k \\ \vdots \\ g_2 \\ \vdots \\ g_1 \end{array} \quad (7.2.4)$$

is uniquely determined by $[P, \mathbf{g}]$. Indeed, P is the partition diagram obtained from $T(P_1) \circ \Psi(P_2) \circ T(P_3)$ by replacing each pair $\uparrow\downarrow$ by a vertex and each strand by an edge. Furthermore, the diagrams of the form (7.2.4) are linearly independent by [BSW20, Thm. 7.2]. The result then follows by a standard triangularity argument. \square

7.3 Compatibility of categorical actions

We continue to assume that G is a finite group. The group Heisenberg category acts naturally on the direct sum of the categories $A_n\text{-mod}$, $n \in \mathbb{N}$. In this section, we recall this action and show that it is compatible with the embedding of the group partition category into the group Heisenberg category and the action of the group partition category described in Theorem 6.2.1.

For $0 \leq m, k \leq n$, let ${}_k(n)_m$ denote A_n , considered as an (A_k, A_m) -bimodule. We will omit the subscript k or m when $k = n$ or $m = n$, respectively. We denote tensor product of such bimodules by juxtaposition. For instance $(n)_{n-1}(n)$ denotes $A_n \otimes_{n-1} A_n$, considered as an (A_n, A_n) -bimodule, where we write \otimes_m for the tensor product over A_m . As explained in [RS17, §7], we have a strong \mathbb{k} -linear monoidal functor

$$\Theta: \mathcal{H}eis(G) \rightarrow \prod_{m \in \mathbb{N}} \left(\bigoplus_{n \in \mathbb{N}} (A_n, A_m)\text{-bimod} \right)$$

given by

$$\Theta(\bowtie) = ((n)_{n-2} \rightarrow (n)_{n-2}, x \mapsto x s_{n-1})_{n \geq 2},$$

$$\begin{aligned}
\Theta(\cup) &= ((n-1) \rightarrow_{n-1} (n)_{n-1}, x \mapsto x)_{n \geq 1}, \\
\Theta(\cap) &= ((n)_{n-1} (n) \rightarrow (n), x \otimes y \mapsto xy)_{n \geq 1}, \\
\Theta(\uparrow) &= ((n) \rightarrow (n)_{n-1} (n), x \mapsto x \sum_{1 \leq i \leq n, h \in G} h^{(i)} \pi_i \otimes \pi_i^{-1} (h^{-1})^{(i)})_{n \geq 1}, \\
\Theta(\downarrow) &= \left((n-1) (n)_{n-1} \rightarrow (n-1), \mathbf{g}\pi \mapsto \begin{cases} (g_{n-1}, g_{n-2}, \dots, g_1)\pi & \text{if } \pi \in \mathfrak{S}_{n-1}, g_n = 1_G \\ 0 & \text{otherwise} \end{cases} \right)_{n \geq 1}, \\
\Theta(\uparrow g) &= ((n)_{n-1} \rightarrow (n)_{n-1}, x \mapsto x (g^{-1})^{(n)})_{n \geq 1}, \\
\Theta(\times) &= ((n-1) (n)_{n-1} \rightarrow (n-1)_{n-2} (n-1), \\
&\quad \mathbf{g}\pi \mapsto \begin{cases} (g_{n-1}, \dots, g_1)\sigma_1 \otimes g_n^{(n-1)}\sigma_2 & \text{if } \pi = \sigma_1 s_{n-1} \sigma_2 \text{ for } \sigma_1, \sigma_2 \in \mathfrak{S}_{n-1}, \\ 0 & \text{if } \pi \in \mathfrak{S}_{n-1} \end{cases})_{n \geq 2}, \\
\Theta(\otimes) &= ((n-1)_{n-2} (n-1) \rightarrow_{n-1} (n)_{n-1}, x \otimes y \mapsto x s_{n-1} y)_{n \geq 2}, \\
\Theta(\otimes) &= ((n-2) (n) \rightarrow_{n-2} (n), x \mapsto s_{n-1} x)_{n \geq 2}, \\
\Theta(g \downarrow) &= ((n-1) (n) \rightarrow_{n-1} (n), x \mapsto (g^{-1})^{(n)} x)_{n \geq 1}.
\end{aligned}$$

As noted in the proof of Theorem 7.2.1, the image of Ψ lies in the full monoidal subcategory $\mathcal{H}eis_{\uparrow\downarrow}(G)$ of $\mathcal{H}eis(G)$ generated by $\uparrow \otimes \downarrow$. For $n \in \mathbb{N}$, consider the composition

$$\Omega_n: \mathcal{H}eis_{\uparrow\downarrow}(G) \xrightarrow{\Theta} \bigoplus_{m \in \mathbb{N}} (A_m, A_m)\text{-bimod} \xrightarrow{- \otimes_{A_n} \mathbf{1}_n} \bigoplus_{m \in \mathbb{N}} A_m\text{-mod},$$

where we declare $M \otimes_{A_n} \mathbf{1}_n = 0$ for $M \in (A_m, A_m)\text{-bimod}$ with $m \neq n$. The functor Ω_n is \mathbb{k} -linear, but no longer monoidal.

Theorem 7.3.1. Consider the functors:

$$\begin{array}{ccc}
\mathcal{P}ar(G) & \xrightarrow{\Psi} & \mathcal{H}eis_{\uparrow\downarrow}(G) \\
& \searrow \Phi_n & \downarrow \Omega_n \\
& & A_n\text{-mod}
\end{array} \quad (7.3.1)$$

The isomorphisms β_k , $k \in \mathbb{N}$, defined in Corollary 5.1.3, give a natural isomorphism of functors $\Omega_n \circ \Psi \cong \Phi_n$.

Proof. Since the β_k are isomorphisms of A_n -modules it suffices to show that they determine a natural transformation between the given functors. Therefore, following the argument used in the proof of Theorem 3.2.1 we need to check on elements of the form

$$1_{V^{\otimes k}} \otimes x \otimes 1_{V^{\otimes j}}, \quad k, j \in \mathbb{N}, \quad x \in \{ h \downarrow, \times, \vee, \wedge, \circ, \circ : h \in G \}.$$

Tokens: For $h \in G$,

$$\beta_k^{-1} \circ (\Omega_n \circ \Psi (1_{V^{\otimes(k-j-1)}} \otimes h \downarrow \otimes 1_{V^{\otimes(j-1)}})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes k}$$

is the A_n -module map given by

$$\begin{aligned} g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} &\mapsto g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g_k^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1 \\ &\mapsto g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g_k^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \otimes \pi_{i_{j+1}}^{-1} (g_{j+1}^{-1})^{(i_{j+1})} g_1^{(i_j)} \pi_{i_j} (h^{-1})^{(n)} \\ &\quad \otimes h^{(n)} \pi_{i_j}^{-1} (g_j^{-1})^{(i_j)} g_1^{i_{j-1}} \pi_{i_{j-1}} \otimes \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1 \\ &\mapsto g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g_k^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \otimes \pi_{i_{j+1}}^{-1} (g_{j+1}^{-1})^{(i_{j+1})} (g_j h^{-1})^{(i_j)} \pi_{i_j} \\ &\quad \otimes \pi_{i_j}^{-1} (h g_j^{-1})^{(i_j)} g_1^{i_{j-1}} \pi_{i_{j-1}} \otimes \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1 \\ &\mapsto g_k e_{i_k} \otimes \cdots \otimes g_{j+1} e_{i_{j+1}} \otimes g_j h^{-1} e_{i_j} \otimes g_{j-1} e_{i_{j-1}} \otimes \cdots \otimes g_1 e_{i_1}. \end{aligned}$$

This is precisely the map $\Phi_n(1_{V^{\otimes(k-j-1)}} \otimes h \downarrow \otimes 1_{V^{\otimes(j-1)}})$.

Merge: For $g, h \in G$ and $1 \leq i, j \leq n$,

$$\pi_i^{-1} (g^{-1})^{(i)} h^{(j)} \pi_j = (g^{-1})^{(n)} h^{(\pi_i^{-1}(j))} \pi_i^{-1} \pi_j. \quad (7.3.2)$$

We have $\pi_i^{-1} \pi_j \in \mathfrak{S}_{n-1}$ if and only if $i = j$, in which case (7.3.2) is equal to $(g^{-1}h)^{(n)}$. Thus, the composition

$$\beta_{k-1}^{-1} \circ (\Omega_n \circ \Psi (1_{V^{\otimes(k-j-1)}} \otimes \frown \otimes 1_{V^{\otimes(j-1)}})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes(k-1)}$$

is the A_n -module map given by

$$\begin{aligned} g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} &\mapsto g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g_k^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \otimes \pi_{i_{j+1}}^{-1} (g_{j+1}^{-1})^{(i_{j+1})} g_j^{(i_j)} \pi_{i_j} \otimes \\ &\quad \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1 \\ &\mapsto \delta_{i_j, i_{j+1}} \delta_{g_j, g_{j+1}} g_k e_{i_k} \otimes \cdots \otimes g_{j+2} e_{i_{j+2}} \otimes g_j e_{i_j} \otimes \cdots \otimes g_1 e_{i_1}. \end{aligned}$$

This is precisely the map $\Phi_n(1_{V^{\otimes(k-j-1)}} \otimes \frown \otimes 1_{V^{\otimes(j-1)}})$.

Split: The composition

$$\beta_{k+1}^{-1} \circ (\Omega_n \circ \Psi (1_{V^{\otimes(k-j)}} \otimes \vee \otimes 1_{V^{\otimes(j-1)}})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes(k+1)}$$

is the A_n -module map given by

$$g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} \mapsto g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g_k^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \otimes \pi_{i_{j+1}}^{-1} (g_{j+1}^{-1})^{(i_{j+1})} g_j^{(i_j)} \pi_{i_j} \otimes 1$$

$$\begin{aligned} & \otimes \pi_{i_j}^{-1} (g_j^{-1})^{(i_j)} g_{j-1}^{(i_{j-1})} \otimes \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1 \\ & \mapsto g_k e_{i_k} \otimes \cdots \otimes g_{j+1} e_{i_{j+1}} \otimes g_j e_{i_j} \otimes g_j e_{i_j} \otimes g_{j-1} e_{i_{j-1}} \otimes \cdots \otimes g_1 e_{i_1}. \end{aligned}$$

This is precisely the map $\Phi_n (1_{V^{\otimes(k-j)}} \otimes \text{Y} \otimes 1_{V^{\otimes(j-1)}})$.

Unit pin: The composition

$$\beta_{k+1}^{-1} \circ (\Omega_n \circ \Psi (1_{V^{\otimes(k-j)}} \otimes \circlearrowleft \otimes 1_{V^{\otimes j}})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes(k+1)}$$

is the map

$$g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} \mapsto \sum_{h \in G} \sum_{i=1}^n g_k e_{i_k} \otimes \cdots \otimes g_{j+1} e_{i_{j+1}} \otimes h e_i \otimes g_j e_{i_j} \otimes \cdots \otimes g_1 e_{i_1},$$

which is equal to the map $\Phi_n (1_{V^{\otimes(k-j)}} \otimes \circlearrowleft \otimes 1_{V^{\otimes j}})$.

Counit pin: The composition

$$\beta_{k-1}^{-1} \circ (\Omega_n \circ \Psi (1_{V^{\otimes(k-j)}} \otimes \circlearrowright \otimes 1_{V^{\otimes(j-1)}})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes(k-1)}$$

is the map

$$g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} \mapsto g_k e_{i_k} \otimes \cdots \otimes g_{j+1} e_{i_{j+1}} \otimes g_{j-1} e_{i_{j-1}} \otimes \cdots \otimes g_1 e_{i_1},$$

which is equal to the map $\Phi_n (1_{V^{\otimes(k-j)}} \otimes \circlearrowright \otimes 1_{V^{\otimes(j-1)}})$.

Crossing: Define the elements $f, f' \in \text{End}_{\mathcal{H}eis}(\uparrow\downarrow\uparrow\downarrow)$ by

$$f = \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}, \quad f' = \sum_{g \in G} g^{-1} \begin{array}{c} \uparrow \quad \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \quad \downarrow \\ \downarrow \quad \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \quad \uparrow \\ g^{-1} \end{array}. \quad (7.3.3)$$

Note that

$$f = f_3 \circ f_2 \circ f_1,$$

where

$$f_1 = \uparrow \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \downarrow, \quad f_2 = \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array}, \quad f_3 = \uparrow \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \downarrow.$$

Suppose $i, j \in \{1, \dots, n\}$ and $x, y \in A_n$. We first compute the action of $\Theta(f)$ and $\Theta(f')$ on

$$\begin{aligned} \alpha &= x g_1^{(i)} \pi_i \otimes \pi_i^{-1} (g_1^{-1})^{(i)} g_2^{(j)} \pi_j \otimes \pi_j^{-1} (g_2^{-1})^{(j)} y \\ &= x g_1^{(i)} \pi_i \otimes (g_1^{-1})^{(n)} g_2^{(\pi_i^{-1}(j))} \pi_i^{-1} \pi_j \otimes \pi_j^{-1} (g_2^{-1})^{(j)} y \in (n)_{n-1} (n)_{n-1} (n), \end{aligned}$$

where $x, y \in A_n$. If $i = j$, then $\pi_i^{-1}\pi_j = 1_{\mathfrak{S}_n}$, and so $\Omega_n(f_1)(\alpha) = 0$. Now suppose $i < j$ so that

$$\pi_i^{-1}\pi_j = s_{n-1} \cdots s_i s_j \cdots s_{n-1} = s_{j-1} \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_i.$$

Thus

$$\Theta(f_1)(\alpha) = x g_1^{(i)} \pi_i g_2^{(\pi_i^{-1}(j))} s_{j-1} \cdots s_{n-2} \otimes (g_1^{-1})^{(n-1)} s_{n-2} \cdots s_i \pi_j^{-1} (g_2^{-1})^{(j)} y \in (n)_{n-2}(n).$$

Hence

$$\Theta(f_2 \circ f_1)(\alpha) = x g_1^{(i)} \pi_i g_2^{(\pi_i^{-1}(j))} \pi_{j-1} \otimes (g_1^{-1})^{(n)} \pi_i^{-1} \pi_j^{-1} (g_2^{-1})^{(j)} y \in (n)_{n-2}(n),$$

and so

$$\begin{aligned} \Theta(f)(\alpha) &= x g_1^{(i)} \pi_i g_2^{(\pi_i^{-1}(j))} \pi_{j-1} \otimes s_{n-1} \otimes (g_1^{-1})^{(n)} \pi_i^{-1} \pi_j^{-1} (g_2^{-1})^{(j)} y \\ &= x g_2^{(j)} \pi_j g_1^{(i)} s_i \cdots s_{n-2} \otimes s_{n-1} \otimes s_{n-2} \cdots s_{j-1} \pi_i^{-1} (g_2^{-1})^{(j)} (g_1^{-1})^{(i)} y \\ &= x g_2^{(j)} \pi_j \otimes g_1^{(i)} \pi_i s_{n-2} \cdots s_{j-1} (g_2^{-1})^{(j-1)} \otimes \pi_i^{-1} (g_1^{-1})^{(i)} y \\ &= x g_2^{(j)} \pi_j \otimes \pi_j^{-1} g_1^{(i)} \pi_i (g_2^{-1})^{(j-1)} \otimes \pi_i^{-1} (g_1^{-1})^{(i)} y \\ &= x g_2^{(j)} \pi_j \otimes \pi_j^{-1} (g_2^{-1})^{(j)} g_1^{(i)} \pi_i \otimes \pi_i^{-1} (g_1^{-1})^{(i)} y. \end{aligned}$$

The case $i > j$ is similar, Suppose $i > j$. Then we have

$$\pi_i^{-1}\pi_j = s_{n-1} \cdots s_i s_j \cdots s_{n-1} = s_j \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_{i-1}.$$

So,

$$\Theta(f_1)(\alpha) = x g_1^{(i)} \pi_i g_2^{(j)} s_j \cdots s_{n-2} \otimes (g_1^{-1})^{(n-1)} s_{n-2} \cdots s_{i-1} \pi_j^{-1} (g_2^{-1})^{(j)} y \in (n)_{n-2}(n).$$

Hence

$$\Theta(f_2 \circ f_1)(\alpha) = x g_1^{(i)} \pi_i g_2^{(j)} \pi_j \otimes (g_1^{-1})^{(n)} \pi_{i-1}^{-1} \pi_j^{-1} (g_2^{-1})^{(j)} y \in (n)_{n-2}(n),$$

and thus

$$\begin{aligned} \Theta(f)(\alpha) &= x g_1^{(i)} \pi_i g_2^{(j)} \pi_j \otimes s_{n-1} \otimes (g_1^{-1})^{(n)} \pi_{i-1}^{-1} \pi_j^{-1} (g_2^{-1})^{(j)} y \\ &= x g_1^{(i)} g_2^{(j)} \pi_i \pi_j \otimes s_{n-1} \otimes \pi_{i-1}^{-1} \pi_j^{-1} (g_1^{-1})^{(i)} (g_2^{-1})^{(j)} y \\ &= x g_2^{(j)} g_1^{(i)} \pi_j s_{i-1} \cdots s_{n-2} \otimes s_{n-1} \otimes s_{n-2} \cdots s_j \pi_i^{-1} (g_2^{-1})^{(j)} (g_1^{-1})^{(i)} y \\ &= x g_2^{(j)} \pi_j \otimes g_1^{(i-1)} s_{i-1} \cdots s_{n-2} \pi_j^{-1} (g_2^{-1})^{(j)} \otimes \pi_i^{-1} (g_1^{-1})^{(i)} y \\ &= x g_2^{(j)} \pi_j \otimes g_1^{(i-1)} \pi_j^{-1} \pi_i (g_2^{-1})^{(j)} \otimes \pi_i^{-1} (g_1^{-1})^{(i)} y \end{aligned}$$

$$= xg_2^{(j)} \pi_j \otimes \pi_j^{-1} (g_2^{-1})^{(j)} g_1^{(i)} \pi_i \otimes \pi_i^{-1} (g_1^{-1})^{(i)} y.$$

giving

$$\Theta(f)(\alpha) = \begin{cases} 0 & \text{if } i = j, \\ xg_2^{(j)} \pi_j \otimes \pi_j^{-1} (g_2^{-1})^{(j)} g_1^{(i)} \pi_i \otimes \pi_i^{-1} (g_1^{-1})^{(i)} y & \text{if } i \neq j. \end{cases}$$

We also compute that

$$\Theta(f')(\alpha) = \begin{cases} xg_2^{(j)} \pi_j \otimes \pi_j^{-1} (g_2^{-1})^{(j)} g_1^{(i)} \pi_i \otimes \pi_i^{-1} (g_1^{-1})^{(i)} y & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus, for all $i, j \in \{1, \dots, n\}$, we have

$$\Theta(f + f')(\alpha) = xg_2^{(j)} \pi_j \otimes \pi_j^{-1} (g_2^{-1})^{(j)} g_1^{(i)} \pi_i \otimes \pi_i^{-1} (g_1^{-1})^{(i)} y.$$

Therefore we have that

$$\begin{aligned} \beta_k^{-1} \circ \left(\Omega_n \circ \Psi \left(1_{\mathbb{V}^{\otimes(k-j-1)}} \otimes \times \otimes 1_{\mathbb{V}^{\otimes(j-1)}} \right) \right) \circ \beta_k \\ = \beta_k^{-1} \circ \left(\Omega_n \left(1_{\uparrow\downarrow}^{\otimes(k-j-1)} \otimes (f + f') \otimes 1_{\uparrow\downarrow}^{\otimes(j-1)} \right) \right) \circ \beta_k \end{aligned}$$

is the map

$$g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} \mapsto g_k e_{i_k} \otimes \cdots \otimes g_{j+2} e_{i_{j+2}} \otimes g_j e_{i_j} \otimes g_{j+1} e_{i_{j+1}} \otimes g_{j-1} e_{i_{j-1}} \otimes \cdots \otimes g_1 e_{i_1},$$

which is precisely the map $\Phi_n \left(1_{\mathbb{V}^{\otimes(k-j-1)}} \otimes \times \otimes 1_{\mathbb{V}^{\otimes(j-1)}} \right)$. \square

Note that we can use this result Theorem 7.3.1 to show that the functor Ψ is faithful under the characteristic zero hypothesis (see the proof of Theorem 3.2.2).

Chapter 8

Finite boolean algebras and interpolating categories

In this last chapter we give the connection between the group partition category defined in Chapter 5 and generalized versions of Deligne’s category introduced by Knop in [Kno07] and Mori in [Mor12].

8.1 Category of (finite) boolean algebras

Most of the notions introduced in this section can be found in [GH09]. The theory of boolean algebras is related to linear algebra and gives a connection between abstract algebra and computational science. Here, we introduce the *category of boolean algebras*.

8.1.1 Boolean algebras

Definition 8.1.1. A *boolean algebra* is a non-empty set A , together with two binary operations \vee and \wedge (on A), a unary operation \neg and two distinguished elements $0, 1$ satisfying the following axioms:

$$\neg 0 = 1, \quad \neg 1 = 0, \quad (8.1.1)$$

$$a \wedge 0 = 0, \quad a \vee 1 = 1, \quad (8.1.2)$$

$$a \wedge 1 = a, \quad a \vee 0 = a, \quad (8.1.3)$$

$$a \wedge (\neg a) = 0, \quad a \vee (\neg a) = 1, \quad (8.1.4)$$

$$\neg(\neg a) = a, \quad (8.1.5)$$

$$a \wedge a = a, \quad a \vee a = a, \quad (8.1.6)$$

$$(a \wedge b)^c = (\neg a) \vee (\neg b), \quad \neg(a \vee b) = \neg a \wedge \neg b, \quad (8.1.7)$$

$$a \wedge b = b \wedge a, \quad a \vee b = b \vee a, \quad (8.1.8)$$

$$(a \wedge b) \wedge d = a \wedge (b \wedge d), \quad (a \vee b) \vee d = a \vee (b \vee d), \quad (8.1.9)$$

$$a \wedge (b \vee d) = (a \wedge b) \vee (a \wedge d), \quad a \vee (b \wedge d) = (a \vee b) \wedge (a \vee d). \quad (8.1.10)$$

Example 8.1.2. Let X be an arbitrary set. The power set $(\text{Pow}(X), \cap, \cup, \neg, \emptyset, X)$ of the class of all subsets of X is a boolean algebra.

Given two nonempty disjoint sets X and Y , consider the set

$$\text{Pow}(X) \times \text{Pow}(Y) = \{(A, B) : A \subseteq X \text{ and } B \subseteq Y\}.$$

There exists a bijection from $\text{Pow}(X) \times \text{Pow}(Y)$ into $\text{Pow}(X \sqcup Y)$ given by

$$\text{Pow}(X) \times \text{Pow}(Y) \longrightarrow \text{Pow}(X \sqcup Y), \quad (A, B) \mapsto A \sqcup B, \quad (8.1.11)$$

where the symbol " \sqcup " denotes the disjoint union of two sets. The above map is clearly surjective and it not difficult to show that it is injective.

The following result was shown by Huntington in [Hun04].

Proposition 8.1.3. A boolean algebra is a non-empty set A , together with the operations and distinguished elements defined in Definition 8.1.1 satisfying *the identity laws* (8.1.3), *the complement laws* (8.1.4), *the commutative laws* (8.1.8) and *the distributive laws* (8.1.10).

The operations \wedge , \vee and \neg are respectively called *meet*, *join* and *complement*. The distinguished elements 0 and 1 are called, the *zero* and the *unit* respectively.

A *boolean subalgebra* A of a boolean algebra B is a nonempty subset of B such that A , with the distinguished elements and operations of B (restricted to A) is a boolean algebra.

Lemma 8.1.4. Let B be a boolean algebra. For $a, b \in B$ we have,

$$a \wedge (a \vee b) = a \quad \text{and} \quad a \vee (a \wedge b) = a. \quad (8.1.12)$$

Proof. We have,

$$a \wedge (a \vee b) \stackrel{(8.1.3)}{=} (a \vee 0) \wedge (a \vee b) \stackrel{(8.1.10)}{=} a \vee (0 \wedge b) = a \vee 0 = a.$$

The proof of the second equality is similar. □

Lemma 8.1.5 ([GH09, Lemma 7.1]). Let B be a boolean algebra. We have for any $a, b \in B$

$$a \wedge b = a \text{ if and only if } a \vee b = b.$$

Proof. The result follows from Lemma 8.1.4. □

One can define a binary operation \leq in every boolean algebra B as follows: We write

$$(a \leq b \text{ or } b \geq a) \text{ if and only if } (a \wedge b = a \text{ or equivalently } a \vee b = b).$$

Lemma 8.1.6 ([GH09, Lemma 7.2]). The relation \leq is a partial order.

8.1.2 Boolean homomorphism

Let A and B be two boolean algebras. A homomorphism between B and A is a structure-preserving mapping between B and A in the sense of the definition below.

Definition 8.1.7. A *boolean algebra homomorphism* f from B to A is a function from B to A such that for all $b, b' \in B$ we have

$$f(b \wedge b') = f(b) \wedge f(b') \tag{8.1.13}$$

$$f(b \vee b') = f(b) \vee f(b'), \tag{8.1.14}$$

$$f(\neg b) = \neg f(b). \tag{8.1.15}$$

A homomorphism from B to A is also called *A-valued homomorphism on B*.

Example 8.1.8. Let X and Y be two nonempty sets and consider the boolean algebras $\text{Pow}(X)$ and $\text{Pow}(Y)$ given in Example 8.1.2. Every boolean algebra homomorphism $\text{Pow}(X) \rightarrow \text{Pow}(Y)$ is of the form f^{-1} for some set map $f: Y \rightarrow X$.

Remark 8.1.9. Let B and A be two boolean algebras and $f: B \rightarrow A$ be a boolean homomorphism. We have

- (a) $f(1) = 1$ and $f(0) = 0$;
- (b) $\text{im}(f)$ is a boolean subalgebra of A .

The notions of *monomorphism*, *epimorphism*, *isomorphism* and *automorphism* are defined similarly to the ones commonly known.

Denote $\bar{m} = \{0, 1, \dots, m - 1\}$, $m \geq 1$.

Corollary 8.1.10 ([GH09, Corollary 15.1]). Any finite boolean algebra B is isomorphic to the boolean algebra $\text{Pow}(\bar{n})$ for some positive integer n .

In the sequel, we will identify each finite boolean algebra to the boolean algebra $\text{Pow}(X)$; where X is a finite set with n elements.

The category of finite boolean algebras is a category where objects are finite boolean algebras and morphisms are homomorphisms of boolean algebras. We denote this category by $\mathcal{FinBoolAlg}$.

Let \mathcal{FinSet} denotes the category of finite sets. We have the following result.

Corollary 8.1.11. The categories $\mathcal{FinSet}^{\text{op}}$ and $\mathcal{FinBoolAlg}$ are equivalent.

Proof. Define the functor

$$F: \mathcal{FinSet}^{\text{op}} \rightarrow \mathcal{FinBoolAlg}, \quad X \mapsto \text{Pow}(X).$$

On morphisms, the functor F maps a morphism $\phi \in \text{Hom}_{\mathcal{FinSet}^{\text{op}}}(X, Y)$ to the morphism $\phi^{-1} \in \text{Hom}_{\mathcal{FinBoolAlg}}(\text{Pow}(Y), \text{Pow}(X))$. It follows from the definition of F on objects that it is essentially surjective. Now for any objects X and Y in $\mathcal{FinSet}^{\text{op}}$, consider the map

$$\gamma: \text{Hom}_{\mathcal{FinSet}^{\text{op}}}(X, Y) \rightarrow \text{Hom}_{\mathcal{FinBoolAlg}}(\text{Pow}(Y), \text{Pow}(X)), \quad \phi \mapsto \phi^{-1}.$$

The map γ is trivially onto and one-to-one. Thus the functor F is full and faithful. \square

Remark 8.1.12. Let $\text{Pow}(X)$ be a finite boolean algebra with $|X| = n$. Any permutation $\pi: X \rightarrow X$ acts on $\text{Pow}(X)$ as follows: if $A = \{a_1, \dots, a_k\} \in \text{Pow}(X)$, then

$$\pi \cdot A = \{\pi(a_1), \dots, \pi(a_k)\}. \tag{8.1.16}$$

Thus we can associate an automorphism of $\text{Pow}(X)$ to any permutation $\pi \in \mathfrak{S}_X$ such that the map

$$\mathfrak{S}_X \rightarrow \text{Aut}(\text{Pow}(X))$$

is a group homomorphism. Therefore (8.1.16) defines a natural action of \mathfrak{S}_X on $\text{Pow}(X)$. If we enumerate the elements of X , where $|X| = n$; then \mathfrak{S}_n acts on $\text{Pow}(X)$.

Definition 8.1.13. Suppose a group G acts on a set X . Then we have the corresponding action of G on the finite boolean algebra $\text{Pow}(X)$ given by the group homomorphism

$$\begin{aligned} \theta: G &\rightarrow \text{Aut}(\text{Pow}(X)), \\ g &\mapsto g \cdot A := \{g \cdot a : a \in A\} . \end{aligned}$$

Definition 8.1.14. We say that an action of a group G on a boolean algebra is *locally free* when every element of the boolean algebra is a union of elements on which G acts freely.

Remark 8.1.15. One may observe that if a group G acts freely on a G -set X , then the action of G on $\text{Pow}(X)$ is *locally free*. Hence by Corollary 8.1.11, the category of finite boolean algebras with locally free G -actions is equivalent to the opposite category of the finite set Set^{op} with free G -actions.

8.2 Interpolating categories

We assume throughout this section that G is a finite group. In [Kno07], Knop generalized the work [Del07] of Deligne by embedding a regular category \mathcal{A} into a family of pseudo-abelian tensor categories $\mathcal{T}(\mathcal{A}, \delta)$, which are the additive Karoubi envelope of categories $\mathcal{T}^0(\mathcal{A}, \delta)$ depending on a degree function δ . Deligne’s original construction corresponds to the case where \mathcal{A} is the category of finite boolean algebras.

As we now explain, the group partition category $\mathcal{P}ar(G, d)$ is equivalent to $\mathcal{T}^0(\mathcal{A}, \delta)$, where \mathcal{A} is the category of finite boolean algebras with a locally free G -action and δ is a degree function depending on d . In this way, $\mathcal{P}ar(G, d)$ can be viewed as a concrete realization (including explicit bases of morphisms spaces) of the category $\mathcal{T}^0(\mathcal{A}, \delta)$, whose definition is rather abstract. Moreover, Theorem 6.1.4 can be viewed as giving an efficient presentation of Knop’s category. On the other hand, the equivalence of $\mathcal{P}ar(G, d)$ and $\mathcal{T}^0(\mathcal{A}, \delta)$ allows us to deduce from Knop’s work several important properties of $\mathcal{P}ar(G, d)$.

By Definition 8.1.13, an action of a group G on the boolean algebra $\text{Pow}(X)$ is a group homomorphism from G to the automorphism group of $\text{Pow}(X)$ in $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg$. It follows from the axioms of a boolean algebra that this action is uniquely determined by the action of G on singletons or, equivalently, by a G -action on the set X . In this way, the category $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)$ of finite boolean algebras with G -actions (with morphisms being homomorphisms of boolean algebras that intertwine the G -actions) is equivalent to the opposite of the category of finite G -sets.

Let $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{lf}}$ denote the category of finite boolean algebras with locally free G -actions and $\mathcal{F}in\mathcal{S}et(G)_{\text{free}}^{\text{op}}$ the opposite of the category of finite sets with free G -action. Recall from Remark 8.1.15 the following equivalence of those two categories:

$$\mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{lf}} \simeq \mathcal{F}in\mathcal{S}et(G)_{\text{free}}^{\text{op}}. \tag{8.2.1}$$

The category $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{lf}}$ is regular, exact, and Malcev, using the definitions of these concepts given in [Kno07].

Knop's definition of the category $\mathcal{T}^0(\mathcal{A}, \delta)$ involves the diagram [Kno07, (3.2)]:

$$\begin{array}{ccccc}
 & & r \times_y s & & \\
 & \swarrow & \downarrow & \searrow & \\
 & r & s \circ r & s & \\
 \swarrow & & \downarrow & & \searrow \\
 x & & y & & z,
 \end{array} \tag{8.2.2}$$

where $x, y, z \in \mathcal{A}$, r is a subobject of $x \times y$, s is a subobject of $y \times z$, and $s \circ r$ is the image of the natural surjective map $r \times_y s \rightarrow x \times z$. To relate Knop's construction to the G -partition category, we consider the diagram (8.2.2) in the case where $\mathcal{A} = \mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{lf}}$.

First recall from Section 8.1 that the product $\text{Pow}(X) \times \text{Pow}(Y)$ is isomorphic to $\text{Pow}(X \sqcup Y)$. Our next goal is to interpret the subobjects r, s in (8.2.2) as G -partition diagrams.

Every finite free G -set is isomorphic to one of the form $X \times G$, where X is a finite set (indexing the G -orbits), with G action given by

$$g \cdot (x, h) = (x, hg^{-1}), \quad x \in X, \quad g, h \in G.$$

Thus, by (8.2.1), every element of $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{lf}}$ is isomorphic to one of the form $\text{Pow}(X \times G)$. Moreover, every element is, in fact, isomorphic to $\text{Pow}(\{1, 2, \dots, r\} \times G)$ for some $r \in \mathbb{N}$. (We adopt the convention that $\{1, 2, \dots, r\} = \emptyset$ when $r = 0$.)

Define the natural projection map $p_X : X \times G \rightarrow X$. For a morphism $\varphi : \text{Pow}(\{1, 2, \dots, r\} \times G) \rightarrow \text{Pow}(X \times G)$, define

$$\begin{aligned}
 P_i^\varphi &:= p_X \circ \varphi(\{(i, 1_G)\}) \subseteq X, \quad 1 \leq i \leq r, \\
 \lambda^\varphi &:= \bigcup_{i=1}^r \varphi(\{(i, 1_G)\}) \in G^X,
 \end{aligned}$$

and set $\vec{P}^\varphi = (P_1^\varphi, \dots, P_r^\varphi)$. Here we use the formal definition of an element of G^X , the set of functions $X \rightarrow G$, as a subset of $X \times G$. Let

$$\text{Par}_r(X) := \{(P_1, \dots, P_r) \in \text{Pow}(X)^r : \bigcup_{i=1}^r P_i = X, P_i \neq \emptyset, P_i \cap P_j = \emptyset \text{ for all } 1 \leq i, j \leq r\}.$$

In other words $\text{Par}_r(X)$ is the set of all r -tuples of nonempty disjoint sets whose union is X . For $x, y \in \mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{lf}}$, let $\text{Mon}(x, y)$ denote the set of monomorphisms $x \rightarrow y$ in $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{lf}}$.

Lemma 8.2.1. The map

$$\text{Mon}(\text{Pow}(\{1, 2, \dots, r\} \times G), \text{Pow}(X \times G)) \rightarrow \text{Par}_r(X) \times G^X, \quad \varphi \mapsto (\vec{P}^\varphi, \lambda^\varphi), \quad (8.2.3)$$

is a bijection.

Proof. It follows from the definition of morphisms in $\mathcal{FinBoolAlg}$ that $\bigcup_{i=1}^r P_i^\varphi = X$ and $P_i^\varphi \cap P_j^\varphi = \emptyset$ for $i \neq j$. Moreover, since $\varphi \in \text{Mon}(\text{Pow}(\{1, 2, \dots, r\} \times G), \text{Pow}(X \times G))$, the induced map $X \times G \rightarrow \{1, \dots, r\} \times G$ is onto, hence it follows that $P_i^\varphi \neq \emptyset$ for all $1 \leq i \leq r$. Therefore $\vec{P}^\varphi \in \text{Par}_r(X)$.

For $(\vec{P}, \lambda) \in \text{Par}_r(X) \times G^X$, define $\varphi'_{\vec{P}, \lambda}: \{1, 2, \dots, r\} \times G \rightarrow \text{Pow}(X \times G)$ by

$$\varphi'_{\vec{P}, \lambda}(i, g) = \{(x, \lambda(x)g) : x \in P_i\} \subseteq X \times G.$$

This induces a map $\varphi_{\vec{P}, \lambda}: \text{Pow}(\{1, 2, \dots, r\} \times G) \rightarrow \text{Pow}(X \times G)$. We check first that, $\varphi_{\vec{P}, \lambda} \in \text{Mon}(\text{Pow}(\{1, 2, \dots, r\} \times G), \text{Pow}(X \times G))$. For any $(i, g) \in \{1, 2, \dots, r\} \times G$,

$$\varphi_{\vec{P}, \lambda}(\{(i, g)\}) = \{\varphi'_{\vec{P}, \lambda}(i, g)\} = \{(x, \lambda(x)g) : x \in P_i\}.$$

It follows that $\varphi_{\vec{P}, \lambda}(\{(i, g)\}) \neq \emptyset$ and $\varphi_{\vec{P}, \lambda}(\{(i, g)\}) \cap \varphi_{\vec{P}, \lambda}(\{(j, h)\}) = \emptyset$ for any $(i, g), (j, h) \in \{1, 2, \dots, r\} \times G$ such that $(i, g) \neq (j, h)$. Furthermore,

$$\bigcup_{\substack{i=1, \dots, r \\ g \in G}} \varphi_{\vec{P}, \lambda}(\{(i, g)\}) = \bigcup_{\substack{i=1, \dots, r \\ g \in G}} \{(x, \lambda(x)g) : x \in P_i\} = X \times G.$$

The last equality above is a consequence of the fact that the P_i 's form a partition of X . Hence $\varphi_{\vec{P}, \lambda}$ maps distinct elements of $\text{Pow}(\{1, \dots, r\} \times G)$ to distinct elements of $\text{Pow}(X \times G)$.

Next we verify that the map $(\vec{P}, \lambda) \mapsto \varphi_{\vec{P}, \lambda}$ is inverse to (8.2.3). We first show that $P_i^{\varphi_{\vec{P}, \lambda}} = P_i$ and $\lambda^{\varphi_{\vec{P}, \lambda}} = \lambda$ for every $i \in \{1, \dots, r\}$. We have for every $i \in \{1, \dots, r\}$

$$P_i^{\varphi_{\vec{P}, \lambda}} = p_X \circ \varphi_{\vec{P}, \lambda}(\{(i, 1_G)\}) = p_X \circ \varphi'_{\vec{P}, \lambda}(i, 1_G) = p_X \{(x, \lambda(x)) : x \in P_i\} = P_i;$$

and

$$\lambda^{\varphi_{\vec{P}, \lambda}} = \bigcup_{i=1}^r \varphi_{\vec{P}, \lambda}(\{(i, 1_G)\}) = \bigcup_{i=1}^r \{\varphi'_{\vec{P}, \lambda}(i, 1_G)\} = \bigcup_{i=1}^r \{(x, \lambda(x)) : x \in P_i\} = \lambda.$$

The last equality above is a consequence of the fact that the P_i 's form a partition of X and viewing a function $X \rightarrow G$ as a subset of $X \times G$. Therefore $P^{\varphi_{\vec{P}, \lambda}} = P$ and $\lambda^{\varphi_{\vec{P}, \lambda}} = \lambda$, so

$(\vec{P}, \lambda) \mapsto \varphi_{\vec{P}, \lambda}$ is right inverse to (8.2.3). To show that the map $(\vec{P}, \lambda) \mapsto \varphi_{\vec{P}, \lambda}$ is left inverse to (8.2.3), we need to show that

$$\varphi_{\vec{P}, \lambda^\varphi}(\{(i, g)\}) = \varphi(\{(i, g)\}) \iff \varphi'_{\vec{P}, \lambda^\varphi}(i, 1_G) = \varphi(\{(i, 1_G)\})$$

for every $(i, g) \in \{1, \dots, r\} \times G$. By definition of $\varphi_{\vec{P}, \lambda^\varphi}$, we have

$$\varphi'_{\vec{P}, \lambda^\varphi}(i, 1_G) = \{(x, \lambda^\varphi(x)) : x \in P_i^\varphi\} = \varphi(\{(i, g)\}).$$

(The last equality follows from the definition of λ^φ and P_i^φ .) Therefore $\varphi = \varphi_{\vec{P}, \lambda^\varphi}$ and it follows that the map $(\vec{P}, \lambda) \mapsto \varphi_{\vec{P}, \lambda}$ is the two-sided inverse of (8.2.3). \square

The proof of the following lemma is straightforward.

Lemma 8.2.2. The automorphism group of $\text{Pow}(\{1, 2, \dots, r\} \times G)$ in $\mathbf{FinBoolAlg}(G)_{\text{if}}$ is the wreath product $G_r = G^r \rtimes \mathfrak{S}_r$, where the action is determined by its action on elements of $\{1, 2, \dots, r\} \times G$ as follows:

$$(\mathbf{g}, \pi) \cdot (i, h) := (\pi(i), g_{\pi(i)}h), \quad \pi \in \mathfrak{S}_r, \quad i = \{1, 2, \dots, r\}, \quad h \in G, \quad \mathbf{g} \in G^r. \quad (8.2.4)$$

We also have an action of G_r on $\text{Pow}_r(X) \times G^X$ given by

$$(\mathbf{g}, \pi) \cdot ((P_1, \dots, P_r), (h_x)_{x \in X}) := ((P_{\pi^{-1}(1)}, \dots, P_{\pi^{-1}(r)}), (g_{\pi(i_x)}h_x)_{x \in X}), \quad (8.2.5)$$

where $i_x \in \{1, 2, \dots, r\}$ is determined by $x \in P_{i_x}$. The following lemma is also a straightforward verification.

Lemma 8.2.3. The bijection (8.2.3) intertwines the actions (8.2.4) and (8.2.5).

Let $\text{Par}(X)$ denote the set of partitions of X . For $\vec{P} = (P_1, \dots, P_r) \in \text{Par}_r(X)$, let $P = \{P_1, \dots, P_r\} \in \text{Par}(X)$ denote the corresponding partition of X . For $P = \{P_1, \dots, P_r\} \in \text{Par}(X)$, define an equivalence relation \sim_P on G^X as follows: $(g_x)_{x \in X} \sim_P (h_x)_{x \in X}$ if and only if there exist $t_1, \dots, t_r \in G$ such that $g_x = t_i h_x$ for all $x \in P_i$.

Corollary 8.2.4. The subobjects of $\text{Pow}(X \times G)$ are naturally enumerated by the set

$$\bigsqcup_{P \in \text{Par}(X)} G^X / \sim_P. \quad (8.2.6)$$

Now consider the diagram (8.2.2) with $x = \text{Pow}(\{1, 2, \dots, k\} \times G)$, $y = \text{Pow}(\{1, 2, \dots, l\} \times G)$, and $z = \text{Pow}(\{1, 2, \dots, m\} \times G)$. When $X = X_k^l \cong \{1, 2, \dots, k\} \sqcup \{1, 2, \dots, l\}$, the set (8.2.6) can be naturally identified with the equivalence classes of G -partitions of type $\binom{l}{k}$. Thus we can view r and s as equivalence classes of G -partitions $[P, \mathbf{g}]$ and $[Q, \mathbf{h}]$, respectively. We then leave it to the reader to verify that $r \times_y s$ exists if and only if the pair $([Q, \mathbf{g}], [P, \mathbf{h}])$ is compatible. If this pair is compatible, then $r \times_y s$ is the equivalence class of $\text{stack}((Q, \mathbf{h}), (P, \mathbf{g}))$ and $s \circ r = [Q \star P, \mathbf{h} \star_{Q,P} \mathbf{g}]$. Thus we have the following result.

Theorem 8.2.5. The G -partition category $\mathcal{P}ar(G, d)$ is equivalent to the category $\mathcal{T}^0(\mathcal{A}, \delta)$ defined in [Kno07, Def. 3.2], where $\mathcal{A} = \mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{if}}$ and δ is the degree function of [Kno07, (8.15)] with the t there equal to d .

Proof. This follows from the above discussion and [Kno07, Ex. 2, p. 596]. \square

Let $\text{Kar}(\mathcal{P}ar(G, d))$ be the additive Karoubi envelope (also known as the pseudo-abelian completion) of $\mathcal{P}ar(G, d)$. Let $\mathcal{N}(G, d)$ be the tensor radical (also known as the tensor ideal of negligible morphisms) of $\text{Kar}(\mathcal{P}ar(G, d))$.

Corollary 8.2.6. Suppose \mathbb{k} is a field of characteristic zero.

- (a) The category $\text{Kar}(\mathcal{P}ar(G, d))/\mathcal{N}(G, d)$ is a semisimple (hence abelian) category.
- (b) We have $\mathcal{N}(G, d) = 0$ if and only if $d \notin \mathbb{N}|G|$.
- (c) If $\mathcal{N}(G, d) = 0$, then the simple objects of $\text{Kar}(\mathcal{P}ar(G, d))$ are naturally parameterized by the set of N -tuples of Young diagrams, where N is the number of isomorphism classes of simple G -modules.
- (d) If $d = n|G|$, then $\text{Kar}(\mathcal{P}ar(G, d))/\mathcal{N}(G, d)$ is equivalent to the category of $\mathbb{k}G_n$ -modules.

Proof. (a) This follows from [Kno07, Th. 6.1(i)].

(b) This follows from [Kno07, Ex. 2, p. 596].

(c) By parts (iii) and (iv) of [Kno07, Th. 6.1], the simple objects of $\text{Kar}(\mathcal{P}ar(G, d))$ are in bijection with the simple modules of the automorphism groups of objects of $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{if}}$ which, by Lemma 8.2.2, are precisely the wreath products G_n , $n \in \mathbb{N}$. The statement then follows from the classification of irreducible modules of wreath product groups. (See, for example, [RS17, Prop. 4.3].)

(d) This is explained in [Kno07, Ex. 2, p. 606]. □

Remark 8.2.7. Deligne’s construction has also been generalized by Mori [Mor12], who defined, for each $d \in \mathbb{k}$, a 2-functor \mathcal{S}_d sending a tensor category \mathcal{C} to another tensor category $\mathcal{S}_d(\mathcal{C})$, which should be thought of as a sort of interpolating wreath product functor. When \mathcal{C} is the category $G\text{-mod}$ of G -modules, $\mathcal{S}_t(G\text{-mod})$ can also be thought of as a family of interpolating categories for modules of the wreath products G_n , $n \in \mathbb{N}$. Mori’s interpolating category contains Knop’s as a full subcategory; see [Mor12, Rem. 4.14]. Mori gives a presentation of his categories, the relations of which can be found in [Mor12, Prop. 4.26]. The presentation of Definition 6.1.1 is considerably more efficient. For example, $\mathcal{P}ar(G)$ has just *one* generating object, whereas Mori’s category (before taking the additive Karoubi envelope) has a generator for each representation of G . In addition, the presentation of [Mor12] includes as generating morphisms *all* morphisms in the category $G\text{-mod}$, whereas the presentation of Definition 6.1.1 only includes a morphism for each element of the group (the tokens).

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