

# The Grothendieck gamma filtration, the Tits algebras, and the J-invariant of a linear algebraic group

by

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# Abstract

Consider a semisimple linear algebraic group  $G$  over an arbitrary field  $F$ , and a projective homogeneous  $G$ -variety  $X$ . The geometry of such varieties has been a consistently active subject of research in algebraic geometry for decades, with significant contributions made by Grothendieck, Demazure, Tits, Panin, and Merkurjev, among others.

An effective tool for the classification of these varieties is the notion of a cohomological (or alternatively, a motivic) invariant. Two such invariants are the set of Tits algebras of  $G$  defined by J. Tits [46], and the J-invariant of  $G$  defined by Petrov, Semenov, and Zainoulline [38]. Quéguiner-Mathieu, Semenov and Zainoulline discovered a connection between these invariants, which they developed in [40] through use of the second Chern class map.

The first goal of the present thesis is to extend this connection through the use of higher Chern class maps. Our main technical tool is the Steinberg basis (c.f. [44]), which provides explicit generators for the  $\gamma$ -filtration on the Grothendieck group  $K_0(X)$  in terms of characteristic classes of line bundles over  $X$ . As an application, we establish a connection between the J-invariant and the Tits algebras of a group  $G$  of inner type  $E_6$ .

The second goal of this thesis is to relate the indices of the Tits algebras of  $G$  to non-trivial torsion elements in the  $\gamma$ -filtration on  $K_0(X)$ . While the Steinberg basis provides an explicit set of generators of the  $\gamma$ -filtration, the relations are not easily computed. A tool introduced by Zainoulline in [48] called the twisted  $\gamma$ -filtration acts as a surjective image of the  $\gamma$ -filtration, with explicit sets of both generators and relations.

We use this tool to construct torsion elements in the degree 2 component of the  $\gamma$ -filtration for groups of inner type  $D_{2n}$ . Such a group corresponds to an algebra  $A$  endowed with an orthogonal involution  $\sigma$  having trivial discriminant. In the trialitarian case (i.e. type  $D_4$ ), we construct a specific element in the  $\gamma$ -filtration which detects splitting of the associated Tits algebras. We then relate the non-triviality of this element to other properties of the trialitarian triple such as decomposability and hyperbolicity.

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# Chapter 1

## Introduction

The central object of study in the present thesis is a linear algebraic group  $G$  defined over a field  $F$ . For a field extension  $K/F$ , we may consider an algebraic variety  $X$  over  $K$  such that the action of  $G$  on  $X$  is regular, i.e. there is only one orbit and all stabilizers are trivial. Such a variety  $X$  is called a  $G$ -torsor over  $K$ , or a principal homogeneous space of  $G$ .

The notion of a torsor is very convenient as these objects appear naturally when considering projective homogeneous  $G$ -varieties, such as the variety of Borel subgroups  $X = G/B$ . Torsors are also useful for classifying objects which are *twisted forms* or *locally isomorphic*, i.e. isomorphic over an algebraic closure of  $F$ . We may form a map from the set of twisted forms of an algebraic structure  $A$  over  $K$  to the set of  $\text{Aut}(A)$ -torsors over  $K$ ; the map sends a twisted form  $A'$  to the group of isomorphisms from  $A'$  to  $A \otimes_F K$ . A standard example of such a structure is a central simple algebra, which is a twisted form of a matrix algebra over a field. In this case the automorphism group is the projective linear group which can also be identified with the automorphism group of projective space.

### 1.1 The J-invariant

The geometry of projective homogeneous  $G$ -varieties has been a consistently active subject of investigation for the past number of decades, with important contributions being made by Borel, Chevalley, Demazure, Serre, Tits, and many others.

One way to study this geometry is through the Chow group  $\text{CH}^d$  of codimension  $d$  algebraic cycles modulo rational equivalence. For a semisimple linear algebraic group  $G$

and a projective  $G$ -homogeneous variety  $X$ , the free part of  $\mathrm{CH}^d(X)$  is known, but it may contain a nontrivial torsion subgroup for  $2 \leq d \leq \dim(X)$ . In the case that  $G$  is a simple and strongly inner algebraic group it was shown by Garibaldi and Zainoulline in [15] that the torsion subgroup  $\mathrm{Tors} \mathrm{CH}^2(X)$  is cyclic group of order the Dynkin index of  $G$ , and is generated by a cycle related to the Rost invariant. They extended this result to a wider class of groups, but the question remains open in the case that  $G$  is adjoint.

An invariant of  $G$ , such as type or rank, provides a tool for determining if two groups differ in some way; possessing a complete set of invariants would allow us to describe  $G$  up to isomorphism. For a group  $G$  of inner type, we will consider two different invariants of  $G$  related to Chow groups. The first invariant is the set of *Tits algebras* of  $G$ , defined by J. Tits [46]. The second is the J-invariant defined by Petrov, Semenov and Zainoulline [38], which describes the motivic behaviour of the variety of Borel subgroups of  $G$ .

Quéguiner-Mathieu, Semenov and Zainoulline discovered a connection between these two invariants, which they developed in [40] through use of the second Chern class map in the Riemann-Roch theorem without denominators. The first main result presented in this thesis is the extension of this connection through the use of higher Chern class maps, and the subsequent application of this result to a group  $G$  of inner type  $E_6$ . In fact, it follows that the degree one parameters for such a group are completely determined by the index of its associated Tits algebra.

We consider a semisimple linear algebraic group  $G$  over an arbitrary field  $F$  and its associated variety of Borel subgroups  $X = G/B$ . Let  $G_s$  (resp.  $X_s$ ) be the split form of  $G$  (resp.  $X$ ) and fix a split maximal torus  $T$  of  $G_s$ . We denote by  $T^*$  and  $\Lambda$  the character group and weight lattice of  $G_s$  respectively. We may then define the *common index* of  $G$  modulo  $p$  by

$$i_p := \gcd\{\mathrm{ind}(A_{\omega_{i_1}}^{\otimes a_1} \otimes \cdots \otimes A_{\omega_{i_s}}^{\otimes a_s}) \mid \text{at least one } a_l \text{ is coprime to } p\}$$

where the  $\omega_{i_j} \in \Lambda$  are fundamental weights of  $G$ , the  $A_{\omega_{i_j}}$  are their associated Tits algebras, and  $1 \leq s \leq \mathrm{rank}(G)$ .

Let  $I_{\mathfrak{c}} \subset \mathrm{CH}^*(X_s)$  be the ideal in the Chow group of  $X_s$  generated by the constant-free elements in the image of the characteristic map  $\mathfrak{c} : \mathbb{Z}[T^*] \rightarrow \mathrm{CH}(X_s)$ , and let  $I_{\mathrm{res}} \subset \mathrm{CH}(X_s)$  be the ideal generated by the constant-free elements in the image of the restriction map  $\mathrm{res}_{\mathrm{CH}} : \mathrm{CH}(X) \rightarrow \mathrm{CH}(X_s)$ . For any integer  $m$ , let  $I_{\mathfrak{c}}^{(m)} \subset \mathrm{CH}^m(X_s)$  and  $I_{\mathrm{res}}^{(m)} \subset \mathrm{CH}^m(X_s)$  denote the homogeneous parts of these ideals of degree  $m$ .

**Theorem.** Fix a prime  $p$  and a group  $G$  of inner type. Let  $i_p$  be the common index of  $G$  modulo  $p$ . If  $p \mid i_p$ , then  $I_{\mathrm{res}}^{(1)} = I_{\mathfrak{c}}^{(1)}$ . If  $p^2 \mid i_p$ , then  $I_{\mathrm{res}}^{(m)} = I_{\mathfrak{c}}^{(m)}$  for  $m = 2, \dots, p$ .

This theorem can be reformulated in terms of the J-invariant of  $G$  modulo  $p$ , denoted  $J_p(G) = (j_1, \dots, j_r)$ . Each  $j_l$  is bounded above by an integer  $k_l \geq 0$  (cf. [23]). Let  $J_p^{(1)}(G) = \{j_i \mid d_i = 1\}$  be the sub-tuple of  $J_p(G)$  consisting of only degree 1 parameters. We say that  $J_p^{(1)}(G) > m$  if for every index  $j_l$  such that  $k_l > m$  we have  $j_l > m$ . The following result can be seen as a generalization of Theorem 3.8 in [40].

**Corollary.** Fix a prime  $p$  and a group  $G$  of inner type. If  $p \mid i_p$ , then  $J_p^{(1)}(G) > 0$ . If  $p^2 \mid i_p$ , then  $J_p^{(1)}(G) > 1$ .

If  $G$  is of inner type  $E_6$ , then the index  $i_p$  completely determines the degree 1 component of the J-invariant for  $p = 3$ .

**Proposition.** Let  $G$  be a group of inner type  $E_6$  with Tits algebra  $A$ . Let  $J_3(G) = (j_1, j_2)$  be the J-invariant of  $G$  modulo 3. Then,

1.  $\text{ind}(A) = 1$  if and only if  $j_1 = 0$
2.  $\text{ind}(A) = 3$  if and only if  $j_1 = 1$
3.  $\text{ind}(A) \geq 9$  if and only if  $j_1 = 2$ .

## 1.2 The twisted $\gamma$ -filtration

The indices of Tits algebras can also be related to the existence of nontrivial torsion elements in the  $\gamma$ -filtration for the Grothendieck group of  $X = G/B$ . These elements are constructed using a tool called the *twisted  $\gamma$ -filtration* introduced by Zainoulline in [48].

For a semisimple linear algebraic group  $G$  over an arbitrary field  $F$ , determining torsion in  $\text{CH}^d(X)$  is a non-trivial problem and only partial results are known. For  $d = 2, 3$  and  $G$  strongly inner, we refer to [39] and [15]. The case of quadrics was considered in [27] for  $d = 2, 3, 4$ . In [25] it was found that  $\text{Tors CH}^4$  can in fact be infinitely generated. Results for arbitrary  $d$  have been obtained recently in [4] and [3] by providing upper bounds for the annihilators of  $\text{Tors CH}^d$ .

There exists a bijection between  $\text{CH}^*(X) = \bigoplus \text{CH}^d(X)$  and the topological filtration on the Grothendieck group  $K_0(X)$ . Another filtration on  $K_0(X)$ , known as the  $\gamma$ -filtration, was introduced by Grothendieck in order to approximate the topological filtration. It can be shown that the degree 2 component of this filtration  $\gamma^{2/3} K_0(X)$  surjects onto  $\text{CH}^2(X)$ , and so  $\text{Tors}(\gamma^{2/3} K_0(X))$  can be viewed as an upper bound for  $\text{Tors}(\text{CH}^2(X))$ .

The Steinberg basis of  $K_0(X)$  allows a complete description of the generators of  $\gamma^d K_0(X)$  for all  $d \geq 0$ , however not all relations in  $\gamma^{d/d+1} K_0(X)$  can be easily checked. The degree  $d$  component of the twisted  $\gamma$ -filtration can be seen as a surjective image of  $\gamma^{d/d+1} K_0(X)$  where all the relations are known.

Let  $G$  be the adjoint group  $\text{PGO}^+(A, \sigma)$ , for  $A$  a central simple algebra of degree  $2n$  with  $n \geq 4$  even, and  $\sigma$  an orthogonal involution having trivial discriminant. In this situation,  $G$  is of inner type  $D_n$ ,  $n$  even, and the Tits algebras of  $G$  are given by  $A, C_+, C_-$ , where  $C_+ \times C_-$  is the even Clifford algebra of  $(A, \sigma)$ . In the Brauer group, we have the fundamental relation  $[A] + [C_+] + [C_-] = 0$  (cf.[29]). We relabel the Tits algebras of  $G$  to be  $A, B, C$ , ordered such that  $\text{ind}(A) \leq \text{ind}(B) \leq \text{ind}(C)$ . Then by the relation in the Brauer group, we may write  $\text{ind}(A) \text{ind}(B) \geq \text{ind}(C)$ .

**1.2.1 Theorem.** *Let  $G$  be a group of inner adjoint type  $D_n$ , for  $n \geq 4$  even. Let  $A, B, C$  be the Tits algebras of  $G$  and set  $X = G/B_0$ , where  $B_0 \subset G$  is a fixed Borel subgroup. Suppose that  $1 < \text{ind}(A) \leq \text{ind}(B) \leq \text{ind}(C) \leq 2^{v_2(n)}$ . If  $\text{ind}(A) \text{ind}(B) > \text{ind}(C) \geq 4$ , then there exists a nontrivial torsion element in  $\gamma^{2/3} K_0(X)$ . Furthermore, its image in  $\gamma^{2/3} \mathfrak{G}$  is non-trivial, and its image in  $\gamma^{2/3} K_0(X_s)$  via the restriction map is trivial.*

In the degree 8 case, explicit examples of algebras with involution can be constructed such that the associated Tits algebras have indices  $\{2, 4, 4\}$  or  $\{4, 4, 4\}$  and thus give rise to such a nontrivial element.

### 1.3 An invariant of a trialitarian triple

Groups of inner type  $D_4$  are of special interest due to the additional symmetry in the Dynkin diagram. The Tits algebras  $A, B$ , and  $C$  of such a group are endowed with orthogonal involutions  $\sigma_A, \sigma_B$ , and  $\sigma_C$ , each having trivial discriminant. They possess the property that the even Clifford algebra of any one is given by the product of the other two. The set  $\{(A, \sigma_A), (B, \sigma_B), (C, \sigma_C)\}$  is called a trialitarian triple.

For a group  $G$  of inner type  $D_4$ , we have an explicit construction for an element  $\eta \in \gamma^{2/3} K_0(X)$  satisfying the hypotheses of Theorem 1.2.1. This element acts as an invariant of the trialitarian triple given by the Tits algebras of  $G$ .

We can show that  $\eta$  is trivial in  $\gamma^{2/3} K_0(X)$  if the triple contains a split algebra, and so it is natural to ask whether  $\eta$  is able to detect other properties of the triple. Again, using the study of Chow groups as motivation, we consider the notions of decomposability and hyperbolicity.

In [24], Karpenko used the  $\gamma$ -filtration to show that if a division algebra  $D$  of prime exponent is decomposable, then  $\mathrm{CH}^2(\mathrm{SB}(D))$  is torsion-free. We may ask an analogous question for algebras with involution: If the trialitarian triple associated to  $G$  contains a decomposable algebra with involution, does the element  $\eta \in \gamma^{2/3} \mathrm{K}_0(X)$  vanish?

We are able to provide an affirmative answer to this question if we replace *decomposable* with *totally decomposable*.

**1.3.1 Proposition.** *If a trialitarian triple contains a totally decomposable algebra with involution, then  $\eta$  is trivial in  $\gamma^{2/3} \mathrm{K}_0(X)$ .*

Instead of the trialitarian triple itself, we may consider the division algebras associated to the triple. That is, we denote by  $\{D_A, D_B, D_C\}$  the set of division algebras such that  $[A] = [D_A]$ ,  $[B] = [D_B]$ , and  $[C] = [D_C]$  in  $\mathrm{Br}(F)$ . By the fundamental relation in the Brauer group,  $[D_A] + [D_B] = [D_C]$ . We will define the triple to be *decomposable* if this relation holds on the level of division algebras as well, that is, if  $D_A \otimes D_B \simeq D_C$ . Using this notion of decomposability, we obtain the following result.

**1.3.2 Proposition.** *If a trialitarian triple is decomposable, then  $\eta$  is trivial in  $\gamma^{2/3} \mathrm{K}_0(X)$ .*

Finally, we investigate a property of the triple which is closely related to decomposability. If  $(A, \sigma)$  is an algebra of degree 8 with orthogonal involution having trivial discriminant, then  $\sigma$  hyperbolic implies  $A$  is totally decomposable. Thus, we can say that if a trialitarian triple contains a hyperbolic involution, the element  $\eta$  is trivial in  $\gamma^{2/3} \mathrm{K}_0(X)$ . We can expand this idea by considering not only involutions which are hyperbolic over  $F$ , but that become hyperbolic over specific field extensions of  $F$ .

**1.3.3 Proposition.** *If  $(A, \sigma)$  is hyperbolic or becomes hyperbolic over the function field of a conic, then  $\eta$  is trivial in  $\gamma^{2/3} \mathrm{K}_0(X)$ .*

## 1.4 Outline of the thesis

The thesis is organized as follows. In Chapter 2 we review the construction and properties of the Grothendieck group of an algebraic variety. The Chow group of a variety is discussed in Chapter 3, along with characteristic classes taking values in both the Chow group and the Grothendieck group. These characteristic classes are then used to define filtrations on the Grothendieck group and maps from these filtrations to the Chow group. Chapters 4 and 5 recall basic results for quadratic forms and central simple algebras with

involution. In Chapter 6 we review the notion of an algebraic group  $G$ , a projective  $G$ -homogeneous variety  $X$ , and the Tits algebras of  $G$ . Chapter 7 relates the notion of a twisted flag variety to  $G$ -torsors and cohomology. Also in this chapter, the Steinberg basis is used to provide a description of the  $\gamma$ -filtration on the Grothendieck group of a twisted flag variety. In Chapter 8 we recall the definition of the J-invariant of an algebraic group  $G$  and provide an application for groups of inner type  $E_6$ . Chapter 9 introduces the twisted  $\gamma$ -filtration and uses this tool to provide a non-trivial torsion element in the  $\gamma$ -filtration for groups of inner type  $D_{2n}$ . We conclude by exploring how this torsion element can be used to detect properties of trialitarian triples such as decomposability and hyperbolicity.

# Chapter 2

## The Grothendieck group

We begin by introducing our main objects of study, an algebraic variety  $X$  over an arbitrary field  $F$  and its Grothendieck group  $K_0(X)$ . We recall definitions of sheaves, vector bundles and projective bundles, as well as the construction of the Grothendieck group. The main references for this section are [13] and [47].

### 2.1 Preliminaries

In this section we outline the assumptions under which we will work for the entirety of the thesis. Namely, we will set  $X$  to be a smooth algebraic variety over an arbitrary field  $F$ , adding the assumption  $\text{char}(F) \neq 2$  explicitly where required. We recall the definitions of vector bundles and line bundles over  $X$ , which will be denoted by  $\mathcal{E}$  and  $\mathcal{L}$ , respectively. Finally, we define the Grothendieck group  $K_0(X)$ , which will be a central object of interest throughout the thesis.

#### 2.1.1 Algebraic varieties

Let  $X$  be a topological space (usually taken to be a variety over a field  $F$  with the Zariski topology) and let  $\mathcal{C}$  be a category, usually the category of sets, groups, abelian groups or commutative rings.

We define  $\mathcal{F}$  to be a **presheaf** with values in  $\mathcal{C}$  if it satisfies the following properties:

- For every open set  $U \subset X$ , there exists an object  $\mathcal{F}(U)$  in  $\mathcal{C}$  called the section of  $\mathcal{F}$  over  $U$  (sometimes it is denoted by  $\Gamma(U, \mathcal{F})$ ).
- For  $V \subset U$ , there exists a morphism  $\text{res}_{V,U} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$  satisfying:

1.  $\text{res}_{U,U} = \text{id}_{\mathcal{F}(U)}$
2. If  $W \subseteq V \subseteq U$ , then  $\text{res}_{W,U} = \text{res}_{W,V} \circ \text{res}_{V,U}$

sometimes we denote this restriction map simply by  $\text{res}_{V,U}(s) = s|_U$

A presheaf is a **sheaf** if it satisfies the following two additional properties:

- (local identity) If  $(U_i)$  is an open covering of  $U$  and if  $s, t \in \mathcal{F}(U)$  such that  $s|_{U_i} = t|_{U_i}$  for all  $U_i$ , then  $s = t$
- (gluing) If for each  $i$  there exists  $s_i \in \mathcal{F}(U_i)$  such that if  $U_i \cap U_j$  is nonempty we have  $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ , then there exists  $s \in \mathcal{F}(U)$  such that  $s|_{U_i} = s_i$  for all  $i$ .

A **ringed space**  $(X, \mathcal{O}_X)$  is a topological space  $X$  equipped with a sheaf of rings  $\mathcal{O}_X$  called the **structure sheaf**. We say that the space is **locally ringed** if for any open set  $U \subset X$ , the section  $\mathcal{O}_X(U)$  over  $U$  is a commutative ring and if for every  $x \in X$  the stalk  $\mathcal{O}_{X,x} = \lim_{x \in U} \mathcal{O}_X(U)$  is a local ring.

Let  $X = \text{Spec}(R)$  for a commutative ring  $R$ . Recall that for an element  $f \in R$  we define the **distinguished open set**  $D_f = \{\mathcal{P} \in \text{Spec}(R) \mid f \notin \mathcal{P}\}$ . These distinguished open sets form a basis for the topology on  $X$ . We may define a sheaf  $\mathcal{O}_X$  on these sets by setting  $\mathcal{O}_X(D_f) = R_f$ , where  $R_f$  is the localization of  $R$  at the multiplicative set  $\{1, f, f^2, f^3, \dots\}$ . Now, if  $U = \bigcup_{i \in I} D_{f_i}$ , the  $\mathcal{O}_X(U) = \lim_{i \in I} R_{f_i}$ . Furthermore, for a point  $\mathcal{P} \in \text{Spec}(R)$ , the stalk at  $\mathcal{P}$  is given by  $\mathcal{O}_{X,\mathcal{P}} = R_{\mathcal{P}}$ , the localization of  $R$  at  $\mathcal{P}$ .

An **affine scheme** is a locally ringed space which is isomorphic to  $(\text{Spec}(R), \mathcal{O}_X)$  for some commutative ring  $R$  and canonical structure sheaf  $\mathcal{O}_X$  as defined above. A **scheme** is defined to be a locally ringed space which admits a covering by open sets, such that the restriction of the structure sheaf to each of these open sets is an affine scheme. Throughout the thesis, we will use the term **(affine) algebraic variety** to denote an integral, separated (affine) scheme of finite type. We will also use the term **projective variety** to denote an algebraic variety which is isomorphic to  $(\text{Proj}(R), \mathcal{O}_X)$  for some commutative ring  $R$  with structure sheaf  $\mathcal{O}_X$ .

Let  $(X, \mathcal{O}_X)$  be an affine (or projective) algebraic variety defined over a field  $F$ . We abuse notation by using the term  $X$  to refer to the set of  $F$ -points of the variety (that is, the set of points of  $(X, \mathcal{O}_X)$  which are defined over  $F$ ). If  $K/F$  is a field extension, we denote by  $X_K$  (or  $X(K)$ ) the set of  $K$ -points of  $(X, \mathcal{O}_X)$ .

Over an algebraically closed field, an affine (resp. projective) variety can be viewed as a subset of affine (resp. projective)  $n$ -space for some  $n$ . This subset is called smooth if it contains no singular points. An affine (resp. projective) variety  $(X, \mathcal{O}_X)$  defined over an arbitrary field  $F$  is **smooth** if  $X_{\bar{F}}$  is smooth.

**2.1.1 Example.** Let  $F$  be a field with  $\text{char}(F) \neq 2$ . For  $a, b \in F^\times$ , we may define a projective variety by setting

$$X = \{[x : y] \in \mathbb{P}_F^1 \mid ax^2 + by^2 = 1\}.$$

The set of  $F$ -points of  $X$  may be empty, for instance if we take  $F = \mathbb{R}$  and  $a = b = -1$ . Over an algebraic closure  $\bar{F}$ ,  $X_{\bar{F}}$  is isomorphic to the projective line  $\mathbb{P}^1$ , and thus is called a twisted form of  $\mathbb{P}^1$ . The notion of a twisted form will play a central role in our study.

## 2.1.2 Vector bundles

An  $\mathcal{O}_X$ -**module** is a sheaf  $\mathcal{F}$  on  $X$  such that:

- For each  $U \subseteq X$  open,  $\mathcal{F}(U)$  is an  $\mathcal{O}_X(U)$ -module
- If  $V \subset U$ , the restriction map  $\text{res}_{V,U} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$  is compatible with the module structures.

**2.1.2 Example.** Let  $X = \text{Spec}(F)$ , for a field  $F$ . Then  $\mathcal{O}_X = F$ , and an  $\mathcal{O}_X$ -module is just an  $F$ -vector space.

We say that  $\mathcal{F}$  is a **free  $\mathcal{O}_X$ -module** if  $\mathcal{F}$  is isomorphic to a direct sum of copies of  $\mathcal{O}_X$ . Furthermore, we say that  $\mathcal{F}$  is **locally free** if  $X$  can be covered by open sets  $U$  for which  $\mathcal{F}|_U$  is a free  $\mathcal{O}_X$ -module. Then, we set  $\text{rank}_x(\mathcal{F}) = \text{rank}(\mathcal{F}|_U)$  for a neighbourhood  $U$  of  $x$  over which  $\mathcal{F}|_U$  is free. Since the function  $x \mapsto \text{rank}_x(\mathcal{F})$  is locally constant,  $\text{rank}(\mathcal{F})$  is a continuous function on  $X$ .

A **vector bundle**  $E$  over a ringed space  $(X, \mathcal{O}_X)$  is a locally free  $\mathcal{O}_X$ -module with a projection  $\pi : E \rightarrow X$ , whose rank is finite at every point. We write  $VB(X, \mathcal{O}_X)$  for the category of vector bundles on  $(X, \mathcal{O}_X)$ . This is an additive category since the direct sum of locally free modules is locally free.

A **section** of  $E$  is a morphism  $s : X \rightarrow E$  such that  $\pi \circ s = \text{id}_X$ . If  $E$  is determined by transition functions  $g_{ij}$ , then a section of  $E$  is determined by morphisms  $s_i : U_i \rightarrow \mathbb{A}^r$  such that  $s_i = g_{ij} \circ s_j$  on  $U_i \cap U_j$ . The **sheaf of sections** of  $E$  is a locally free sheaf  $\mathcal{E}$  of  $\mathcal{O}_X$ -modules of rank  $r$ . On the other hand, any locally free sheaf of constant rank comes from a vector bundle, which is unique up to isomorphism. We may then identify a vector bundle with its locally free sheaf of sections.

Basic operations such as direct sum  $E \oplus F$ , tensor product  $E \otimes F$ , dual bundle  $E^\vee$  and pullback  $f^*E$  (for a morphism  $f : X' \rightarrow X$ ) are defined so as to be compatible with the corresponding notions for sheaves.

A homomorphism of vector bundles  $E \rightarrow F$  over  $X$  corresponds to a homomorphism of locally free sheaves of  $\mathcal{O}_X$ -modules. Specifying such a homomorphism is equivalent to giving a section of the bundle  $\text{Hom}(E, F) = E^\vee \otimes F$ .

We define a **line bundle** to be a locally free  $\mathcal{O}_X$ -module of constant rank 1. The tensor product of line bundles is again a line bundle, giving a product structure on the set of isomorphism classes of line bundles. This product structure is associative and commutative, with identity element  $[\mathcal{O}_X]$ . We define  $\mathcal{L}^\vee := \text{Hom}(\mathcal{L}, \mathcal{O}_X)$ , which is again a line bundle, and so provides an inverse for the set of isomorphism classes of line bundles. Thus, this set has the structure of an abelian group, and is called the **Picard group**,  $\text{Pic}(X)$ , of  $X$ . A line bundle in this setting is also called an **invertible sheaf**. If  $(X, \mathcal{O}_X)$  is locally ringed, then the notions of invertible sheaves and line bundles are analogues of finitely generated projective modules and algebraic line bundles.

For  $n \in \mathbb{Z}$ , the line bundle  $\mathcal{L}^{\otimes n}$  is defined to be the  $n$ -fold tensor product of  $\mathcal{L}$  if  $n > 0$ , the  $n$ -fold tensor product of  $\mathcal{L}^\vee$  if  $n < 0$ , and the trivial line bundle  $\mathcal{O}_X$  if  $n = 0$ .

We say that a vector bundle  $E$  **splits** if it has a filtration  $E = E_0 \supset E_1 \supset \cdots \supset E_r = 0$  such that  $\mathcal{L}_i := E_{i-1}/E_i$  is an invertible sheaf for  $1 \leq i \leq r$ .

## Projective bundles

Let  $\pi : E \rightarrow X$  be a vector bundle of rank  $r$  with an open covering  $\{U_i\}$  such that  $\psi_i : \pi^{-1}(U_i) \simeq U_i \times \mathbb{A}^r$  and the composition  $\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{A}^r \rightarrow (U_i \cap U_j) \times \mathbb{A}^r$  is linear in the coordinates of  $\mathbb{A}^r$  (i.e. the cover satisfies the definition of a vector bundle). We form the **projective bundle**  $\mathbb{P}(E)$  associated to  $E$  by gluing the patches  $U_i \times \mathbb{P}^{r-1}$  along these transition functions. That is, we glue  $U_i \times \mathbb{P}^{r-1}$  to  $U_j \times \mathbb{P}^{r-1}$  along the isomorphism  $\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{P}^{r-1} \rightarrow (U_i \cap U_j) \times \mathbb{P}^{r-1}$ . There is a natural projection map  $p : \mathbb{P}(E) \rightarrow X$  sending a point  $(x, P) \mapsto x$ .

For each  $d \in \mathbb{Z}$  we can construct a line bundle  $\mathcal{O}_{\mathbb{P}(E)}(d)$  on  $\mathbb{P}(E)$  which is canonical in the sense that it is a relative version of the line bundle  $\mathcal{O}(d)$  on  $\mathbb{P}^{r-1}$ . On the section  $U_i \times \mathbb{P}^{r-1}$  of  $\mathbb{P}(E)$  we take the line bundle  $\mathcal{O}(d)$ . On the overlap  $U_i \cap U_j$ , we glue the line bundles by  $\varphi \mapsto \varphi \circ \psi_i \circ \psi_j^{-1}$ , where  $\varphi = \frac{f}{g}$  is locally a quotient of homogeneous polynomials  $f, g \in F[x_1, \dots, x_r]$  with  $\deg(f) - \deg(g) = d$ . The line bundle  $\mathcal{O}_{\mathbb{P}(E)}(1)$  is called the **canonical invertible sheaf** on  $\mathbb{P}(E)$ .

## 2.2 Construction of $K_0$

The starting point for studying the Grothendieck group  $K_0$  is the idea of the group completion of an abelian monoid. We form the **group completion** of  $M$  by first considering the free abelian group  $F(M)$  with generators  $[m]$ , for any  $m \in M$ . Now, let  $R(M)$  be the subgroup of  $F(M)$  generated by elements of the form  $[m] + [n] - [m + n]$ , and let  $M^+ := F(M)/R(M)$ . Thus,  $M^+$  is the group completion of  $M$ , with identity  $[0]$  and inverse  $[m]^{-1} = -[m]$ . We may define the map  $\gamma : M \rightarrow M^+$  by sending  $m \mapsto [m]$ .

The group completion is uniquely determined, in that this group completion map  $\gamma : M \rightarrow M^+$  satisfies the following **universal property**:

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & M^+ \\ & \searrow \phi & \downarrow \exists! \phi^+ \\ & & A \end{array}$$

**2.2.1 Example.** The most simple example of an abelian monoid is the set  $\mathbb{N}$  of natural numbers. In this case the group completion is  $\mathbb{N}^+ = \mathbb{Z}$ .

Suppose  $f : M \rightarrow N$  is a monoid map. That is,  $f(0) = 0$  and  $f(a + b) = f(a) + f(b)$  for all  $a, b \in M$ . We may consider the map  $M \rightarrow N \rightarrow N^+$ . By universality, we have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\gamma_M} & M^+ \\ f \downarrow & & \downarrow \exists! \phi^+ \\ N & \xrightarrow{\gamma_N} & N^+ \end{array}$$

Taking  $f_* : M^+ \rightarrow N^+$  to be this unique map satisfying  $f_* \circ \gamma_M = \gamma_N \circ f$ , we see that group completion is a covariant functor from the category of abelian monoids to the category of abelian groups.

We may also consider a **semiring**  $M$ . That is,  $M$  is an abelian monoid with an associative product which distributes over  $+$  and a two-sided identity element  $1$ . In other words,  $M$  is a ring without additive inverses (no subtraction). Then, the group completion  $M^+$  of  $M$  is clearly a ring, and so group completion can also be viewed as a covariant functor from the category of semirings to the category of rings, or from commutative semirings to commutative rings.

### 2.2.1 $K_0$ of a variety

If  $X$  is a smooth algebraic variety such that  $X = \text{Spec}(R)$ , there is an equivalence of categories between quasi-coherent sheaves and  $R$ -modules, so that every  $R$ -module  $M$  yields an  $\mathcal{O}_X$ -module  $\tilde{M}$ , and we have the following correspondences:

- free  $R$ -modules  $\longleftrightarrow$  free  $\mathcal{O}_X$ -modules
- finitely generated projective  $R$ -modules of rank  $n$   $\longleftrightarrow$  locally free  $\mathcal{O}_X$ -modules of rank  $n$

Thus, we have an equivalence of categories between  $VB(\text{Spec}(R), \mathcal{O}_X)$  and  $P(R) =$  (isomorphism classes of finitely generated projective  $R$ -modules,  $\oplus$ ). Defining the Grothendieck group  $K_0(X, \mathcal{O}_X) := VB(X, \mathcal{O}_X)^+$ , this equivalence provides an isomorphism  $K_0(R) \simeq K_0(\text{Spec}(R), \mathcal{O}_X)$ , where  $K_0(R) := P(R)^+$ .

Looking at the other side of things, we see that for a topological space  $X$ , we can form a locally ringed space  $X_{top} = (X, \mathcal{O}_{top})$ , where  $\mathcal{O}_{top}$  is the sheaf of ( $\mathbb{R}$ - or  $\mathbb{C}$ -valued) continuous functions on  $X$ . Then, for any open set  $U \subset X$ , we have  $\mathcal{O}_{top}(U) = C^0(U)$ .

If  $\eta : E \rightarrow X$  is a topological vector bundle of rank  $n$ , the **sheaf of continuous sections** of  $E$  is defined by  $\mathcal{E}(U) = \{s : U \rightarrow E \mid \eta \circ s = id_U\}$ . Then  $\mathcal{E}$  is a locally free  $\mathcal{O}_{top}$ -module of rank  $n$ . So we have an equivalence of categories between  $VB(X_{top})$  and  $VB(X)$ , and hence  $K_0(X_{top}) \simeq K_0(X)$ . So finally, if  $X = \text{Spec}(R)$ , then  $K_0(R) \simeq K_0((\text{Spec}(R), \mathcal{O}_X)) \simeq K_0(X)$ .

**2.2.2 Example.** Let  $X = \mathbb{P}^1$ , and recall that  $\mathbb{P}^1 \cong S^1 / \sim$ , ( $x \sim -x$ ). We have line bundles  $\mathcal{O}$  and  $\mathcal{O}(-1)$  over  $\mathbb{P}^1$  corresponding to the trivial line bundle and Möbius bundle, respectively, over  $S^1$ . Over  $\mathbb{P}^1$  every vector bundle decomposes into a finite direct sum of line bundles  $F \cong \bigoplus_{j \in \mathbb{Z}} a_j \mathcal{O}(j)$ ,  $a_j \in \mathbb{N}$ . Every line bundle  $\mathcal{O}(j)$  can be constructed via the tensor product from the two bundles  $\mathcal{O}$  and  $\mathcal{O}(-1)$ , and so  $VB(\mathbb{P}^1) \cong \mathbb{N} \oplus \mathbb{N}$ , where the terms on the RHS are generated by  $\mathcal{O}$  and  $\mathcal{O}(-1)$ . By taking the group completion, we then have  $K_0(\mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ .

# Chapter 3

## Chow groups, Chern classes and filtrations

In this chapter we recall the definition and functorial properties of the Chow group of an algebraic variety  $X$ . We discuss two characteristic classes, one taking values in the Chow group of  $X$ , the other taking values in the Grothendieck group of  $X$ . Using these characteristic classes, we define two filtrations on  $K_0(X)$  and describe their relation to  $\text{CH}^*(X)$ . All results presented here can be found in either [14] or [31].

### 3.1 The Chow group

Let  $X$  be an irreducible algebraic variety over a field  $F$ . Define a **prime divisor** on  $X$  to be a closed irreducible subvariety  $D \subset X$  of codimension 1. The local ring  $\mathcal{O}_{X,D}$  is a DVR, and we denote by  $t \in \mathcal{O}_{X,D}$  a generator of the maximal ideal. For any rational function  $f \in F(X)^\times$ , we may then write  $f = ut^n$  with  $u$  a unit in  $\mathcal{O}_{X,D}$  and define a homomorphism  $\text{ord}_D : F(X)^\times \rightarrow \mathbb{Z}$  by  $f \mapsto n$ . Essentially,  $\text{ord}_D(f)$  measures the order of vanishing of  $f$  along  $D$ . For a fixed  $f \in F(X)^\times$ , there are only finitely many prime divisors  $D$  of  $X$  such that  $\text{ord}_D(f) \neq 0$ . To see this, we first note that  $\text{ord}_D(f) \neq 0$  if and only if  $D \subset \mathcal{V}(f) := \{x \in X \mid f(x) = 0\}$ . Now since  $f \neq 0$ , then  $\mathcal{V}(f)$  is a proper closed subset, and since  $X$  is noetherian,  $\mathcal{V}(f)$  contains only finitely many closed irreducible subsets of codimension 1.

A **cycle** is a locally finite linear combination of irreducible closed subvarieties of  $X$ . We define the group  $Z^r(X)$  to be the free abelian group of codimension  $r$  cycles on  $X$ . Codimension 1 cycles are linear combinations of prime divisors on  $X$  and hence are

called **divisors** on  $X$ . For a subvariety  $V$  of codimension  $r$  in  $X$ , and any  $f \in F(V)^\times$ , we define a specific cycle  $[\text{div}(f)]$  on  $X$  of codimension  $r + 1$  (with support in  $V$ ) by  $[\text{div}(f)] := \sum_D \text{ord}_D(f)[D] \in Z^{r+1}(X)$ , where the sum is taken over all prime divisors  $D$  of  $V$ .

A cycle  $\alpha \in Z^r(X)$  is **rationally equivalent to zero** in  $Z^r(X)$  if there exist a finite number of subvarieties  $Y_i$  of codimension  $r - 1$  in  $X$ , and  $f_i \in F(Y_i)^\times$ , such that  $\alpha = \sum_i [\text{div}(f_i)] \in Z^r(X)$ . Now, since  $[\text{div}(f^{-1})] = -[\text{div}(f)]$ , these cycles form a subgroup of  $Z^r(X)$ , and we define  $\text{CH}^r(X)$  to be the quotient of  $Z^r(X)$  by this subgroup. With this, we define the **Chow group**  $\text{CH}^*(X) := \bigoplus_r \text{CH}^r(X)$ .

### 3.1.1 Intersection Product

Let  $Y, Z$  be closed subvarieties of  $X$  such that  $[Y] \in \text{CH}^k(X)$  and  $[Z] \in \text{CH}^l(X)$ . We define the **intersection product** by  $[Y] \cdot [Z] = [Y \cap Z]$ .

If  $X$  is smooth, then  $\text{CH}^*(X)$  is a graded ring with respect to the intersection product. The idea is that for irreducible components  $Z, Z' \subset X$ , we can “move” one such that  $Z \cap Z' = Z_1 \cup \dots \cup Z_n$  for  $Z_i$  all of the same codimension. Then we write  $[Z] \cdot [Z'] = a_1[Z_1] + \dots + a_n[Z_n]$  for certain coefficients  $a_i \in \mathbb{Z}$ . This defines a product structure on the Chow group, which is commutative and associative with unit element  $[X] \in \text{CH}^0(X)$ .

**3.1.1 Lemma** (Moving Lemma [14, Thm. 11.4]). *If  $X$  is non-singular and quasi-projective, and  $\alpha, \beta \in Z(X)$ , there exists a cycle  $\alpha' \in Z(X)$  rationally equivalent to  $\alpha$  such that  $\alpha'$  meets  $\beta$  properly.*

Thus the classes of  $\alpha, \beta$  in  $\text{CH}^*(X)$  do not depend on the choice of representatives in  $Z(X)$ , and we may assume our cycles meet properly. Thus for  $[Y] \in \text{CH}^k(X)$ ,  $[Z] \in \text{CH}^l(X)$ , the product  $[Y] \cdot [Z] = [Y \cap Z]$  lies in  $\text{CH}^{k+l}(X)$ .

### 3.1.2 Functorial Properties of CH

Sometimes it is useful to consider cycles of a given dimension instead of codimension, and so we set  $\text{CH}_i(X) := \text{CH}^{\dim X - i}(X)$ .

### Pushforward

For any proper map  $f : X \rightarrow Y$ , we may define a **pushforward** map  $f_* : \text{CH}_*(X) \rightarrow \text{CH}_*(Y)$ . For any  $Z \subset X$ ,  $f(Z) \subset Y$  closed,

$$f_*([Z]) := \begin{cases} 0 & \text{if } \dim f(Z) < \dim Z \\ d[f(Z)] & \text{if } \dim f(Z) = \dim Z \end{cases}$$

where  $d := [F(Z) : F(f(Z))] \leq \infty$ . The need for this coefficient  $d$  is explained by the need for this map to factor through rational equivalence.

**3.1.2 Example.** The map  $X \rightarrow \text{Spec}(F)$  defines a pushforward map  $\text{CH}(X) \rightarrow \text{CH}(\text{Spec}(F)) = \mathbb{Z}$ , called the **degree homomorphism**. This map is only nontrivial on  $\text{CH}_0(X)$ , since any cycle of positive dimension maps to 0 by definition of the pushforward. We have the following formula for computing the degree: let  $x \in X$  be a closed point, and set  $\deg([x]) := [F(x) : F]$ , the degree of the residue field of  $x$ . In particular,  $\deg([x]) = 1 \iff x \in X$  is a rational point.

### The Pullback (restriction map)

A flat morphism  $f : X \rightarrow Y$  defines a **pullback** map  $f^* : \text{CH}^*(Y) \rightarrow \text{CH}^*(X)$  by setting  $[Z] \mapsto [f^{-1}(Z)]$ .

**3.1.3 Example.** Let  $E/F$  be a field extension, and consider the flat morphisms  $\text{Spec}(E) \rightarrow \text{Spec}(F)$  and  $X_E \rightarrow X$ , the base change map. The pullback here, called the **restriction map**  $\text{res} : \text{CH}^*(X) \rightarrow \text{CH}^*(X_E)$  actually respects both gradings, and is a ring homomorphism if  $X$  is smooth.

We can summarize these functorial properties as follows.

**3.1.4 Theorem.** *For a smooth quasi-projective variety  $X$ , there is a unique contravariant graded ring structure on  $\text{CH}^*(X)$  such that:*

1. *It agrees with pullback of cycles.*
2. *For a proper morphism  $f : X \rightarrow Y$ ,  $f_* : \text{CH}^*(X) \rightarrow \text{CH}^*(Y)$  is a homomorphism. If  $g : Y \rightarrow Z$  is another proper morphism, then  $g_* \circ f_* = (g \circ f)_*$ .*
3. *If  $f : X \rightarrow Y$  is a proper morphism and  $\alpha \in \text{CH}^*(X), \beta \in \text{CH}^*(Y)$ , then  $f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta$ .*
4. *For subvarieties  $V, W$  of  $X$  which intersect properly, the product of  $[V]$  and  $[W]$  coincides with the product cycle  $[V] \cdot [W]$ .*

## 3.2 Chern classes in the Chow ring

Let  $X$  be a smooth quasi-projective variety over a field  $F$ . Let  $\text{Pic}(X)$  be the Picard group, consisting of isomorphism classes of invertible sheaves on  $X$ , and let  $\text{Cl}(X) := \text{CH}^1(X)$  be the divisor class group. Every divisor  $D$  on  $X$  determines an isomorphism class  $\mathcal{L}(D) \in \text{Pic}(X)$ , and conversely every invertible sheaf can be formed this way. Thus we have an isomorphism  $\text{Cl}(X) \simeq \text{Pic}(X)$  (cf. [18, II.6.16]).

For  $\mathcal{L} \in \text{Pic}(X)$ , define the **first Chern class** of  $\mathcal{L}$  in  $\text{Cl}(X)$  by  $c_1(\mathcal{L}) := [D]$ , where  $[D] \in \text{Cl}(X)$  is chosen such that  $\mathcal{O}_X(D) = \mathcal{L} \in \text{Pic}(X)$ . Then, the homomorphism  $c_1 : \text{Pic}(X) \rightarrow \text{Cl}(X)$  is inverse to the homomorphism described above.

**3.2.1 Theorem** (Projective bundle theorem, [31, 1.1.2]). *Let  $\mathcal{E}$  be a vector bundle of rank  $r + 1$  and let  $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$  be the associated projective bundle. Then  $\text{CH}(\mathbb{P}(\mathcal{E}))$  is a  $\text{CH}(X)$ -module isomorphic to  $\text{CH}(X)[H]/[H^{r+1}]$ , with  $H$  in degree 1.*

In particular, we may set  $H := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \in \text{CH}^1(\mathbb{P}(\mathcal{E}))$ , the canonical invertible sheaf on  $\mathbb{P}(\mathcal{E})$  (c.f Section 2.1.2). So, there exist unique elements  $\alpha_i \in \text{CH}^i(X)$ ,  $1 \leq i \leq r$ , such that

$$H^r - \pi^*(\alpha_1)H^{r-1} + \pi^*(\alpha_2)H^{r-2} - \cdots + (-1)^r \pi^*(\alpha_r) = 0.$$

We define the  **$i$ -th Chern class** of  $\mathcal{E}$  to be  $c_i(\mathcal{E}) := \alpha_i \in \text{CH}(X)$  for  $1 \leq i \leq r$ ,  $c_0(\mathcal{E}) = 1$  and  $c_i(\mathcal{E}) = 0$  for all  $i > r$ . With this, we define the **total Chern class** to be the formal sum  $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \cdots + c_r(\mathcal{E}) \in \text{CH}(X)$ .

The Chern classes constructed above are unique in satisfying the following set of axioms:

1.  $c_0(\mathcal{E}) = 1$  for all vector bundles  $\mathcal{E}$  on  $X$ .
2.  $c_i(\mathcal{E}) = 0$  for  $i > \text{rk } \mathcal{E}$ .
3. For any morphism  $f : X \rightarrow Y$  of smooth quasi-projective varieties,  $f^*(c_i(\mathcal{E})) = c_i(f^*(\mathcal{E}))$  for all  $i > 0$ .
4. If  $0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$  is an exact sequence of vector bundles on  $X$ , then  $c(\mathcal{E}) = c(\mathcal{E}')c(\mathcal{E}'') \in \text{CH}(X)$ .
5. For  $\mathcal{O}_X(D) \in \text{Pic}(X)$ ,  $c_1(\mathcal{O}_X(D)) = [D]$ .
6. The mapping  $\mathcal{E} \mapsto c(\mathcal{E})$  extends to a homomorphism  $c : K_0(X) \rightarrow 1 + \bigoplus_{i \geq 1} \text{CH}^i(X)$ .

**3.2.2 Remark.** Note that while the first 3 axioms hold in general for any characteristic class, the 4th axiom defines the Whitney sum formula, and can be seen as an invariant of the oriented cohomology theory which is dependent on the formal group law satisfied in this axiom (cf. [8]).

### The splitting principle

A key tool for working with Chern classes is given by the following splitting principle.

**3.2.3 Theorem** ([14, §3.2]). *For a fixed vector bundle  $\mathcal{E}$  on  $X$  of rank  $r$ , there exists a smooth, quasi-projective variety  $X'$  and a morphism  $\pi : X' \rightarrow X$  such that the pullback  $\pi^* : \text{CH}(X) \rightarrow \text{CH}(X')$  is injective and the vector bundle  $\mathcal{E}' = \pi^*(\mathcal{E})$  splits.*

*Proof.* If  $r = 1$ , there is nothing to prove. Suppose  $r > 1$  and let  $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$  be the natural projection. Since the Chow ring  $\text{CH}(\mathbb{P}(\mathcal{E}))$  is a free  $\text{CH}(X)$ -module, the pullback map  $\pi^* : \text{CH}(X) \rightarrow \text{CH}(\mathbb{P}(\mathcal{E}))$  is injective. Locally, we have a natural surjective morphism  $\pi^*\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ , whose kernel  $\mathcal{E}'$  is a vector bundle of rank  $r - 1$ . This construction can then be repeated with  $\mathcal{E}'$  on  $\mathbb{P}(\mathcal{E})$  until a line bundle is reached.  $\square$

For  $\pi : X' \rightarrow X$  as in the splitting principle, we may write  $\pi^*c(\mathcal{E}) = c(\pi^*(\mathcal{E})) = c(\mathcal{L}_1) \dots c(\mathcal{L}_r)$ . Now since the  $\mathcal{L}_i$  are line bundles,  $c(\mathcal{L}_i) = 1 + \alpha_i$ , and by the injectivity of  $\pi^*$ , we obtain  $c(\mathcal{E}) = \prod_{i=1}^r (1 + \alpha_i)$ , where  $\alpha_1, \dots, \alpha_r \in \text{CH}(X)$ , and are called the **Chern roots** of  $\mathcal{E}$  for this formal factorization. We may note that  $c_i(\mathcal{E})$  is then the  $i$ -th elementary symmetric polynomial in  $\alpha_1, \dots, \alpha_r$ , e.g.  $c_1(\mathcal{E}) = \alpha_1 + \dots + \alpha_r$  and  $c_r(\mathcal{E}) = \alpha_1 \dots \alpha_r$ . In fact, any symmetric polynomial in Chern roots determines a well-defined total Chern class (using the fact that any symmetric polynomial can be expressed uniquely as a polynomial of elementary symmetric polynomials).

We may use the splitting principle to determine the Chern classes for dual bundles and tensor products:

- If  $\mathcal{E}$  has a filtration with quotients  $\mathcal{L}_i$ , then  $\mathcal{E}^\vee$  has a filtration with quotients  $\mathcal{L}_{r-i}$ . Now, since  $c(\mathcal{L}_i^\vee) = 1 - \alpha_i$ , we have  $c(\mathcal{E}^\vee) = \prod_{i=1}^r (1 - \alpha_i)$ .
- For vector bundles  $\mathcal{E}, \mathcal{F}$  over  $X$  with  $c(\mathcal{E}) = \prod(1 + \alpha_i)$ ,  $c(\mathcal{F}) = \prod(1 + \beta_j)$ , we have  $c(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \prod_{i,j} (1 + (\alpha_i + \beta_j))$ .

## 3.3 Filtrations on $\mathbf{K}_0$

In this section we describe two filtrations on the Grothendieck group of a smooth projective variety  $X$ . We discuss the relationship between the two, as well as their relationship to the Chow group of  $X$ . The main references for this section are [14] and [31]. We note however for the  $\gamma$ -filtration we will follow the conventions of [31], which differ from those found in [14].

### 3.3.1 The topological filtration

Recall that for a smooth projective variety  $X$ , the Grothendieck group  $K_0(X)$  is generated by classes of coherent sheaves (or vector bundles) on  $X$ . We denote by  $T^d(X)$  the subgroup of  $K_0(X)$  generated by the classes of sheaves whose support has codimension  $\geq d$ . We define the **topological filtration** on  $K_0(X)$  by setting

$$T^* K_0(X) = \bigoplus_{d=0}^{\dim X} T^{d/d+1} K_0(X), \text{ where } T^{d/d+1} K_0(X) := T^d K_0(X) / T^{d+1} K_0(X).$$

There is a surjective homomorphism  $Z^d(X) \twoheadrightarrow T^{d/d+1} K_0(X)$  given by sending the class of an irreducible closed codimension  $d$  subvariety  $[V]$  to  $[\mathcal{O}_V] \in K_0(X)$ , and so we may consider the induced surjective homomorphism  $\tau : Z(X) \twoheadrightarrow T^* K_0(X)$ .

**3.3.1 Theorem.** *Let  $X$  be a smooth quasi-projective variety over an algebraically closed field.*

1. *If  $\alpha \in Z^r(X)$  and  $\beta \in Z^s(X)$  meet properly, then  $\tau(\alpha \cdot \tau(\beta)) \equiv \tau(\alpha \cdot \beta) \pmod{T^{r+s+1}(X)}$ .*
2. *For any morphism  $f : Y \rightarrow X$  of smooth quasi-projective varieties and any cycle  $\alpha \in Z^r(Y)$ ,  $\tau(f^*(\alpha)) \equiv f^*(\tau(\alpha)) \pmod{T^{r+1}(X)}$ .*
3. *If  $\alpha, \beta \in Z^r(X)$  are rationally equivalent, then  $\tau(\alpha) \equiv \tau(\beta) \pmod{T^{r+1}(X)}$ .*
4. *For any  $r, s \in \mathbb{Z}$ ,  $T^r(X) \cdot T^s(X) \subset T^{r+s}(X)$ .*

We can see that if  $[V] \in \text{CH}^d(X)$ , then  $\tau([V]) \in T^{d/d+1} K_0(X)$ , and so  $\tau$  restricts to a surjective map  $\tau_d : \text{CH}^d(X) \rightarrow T^{d/d+1} K_0(X)$ .

By the Whitney sum formula, the Chern class described in the previous section defines a group homomorphism

$$c : K_0(X) \rightarrow \text{CH}(X).$$

Taking just the  $d$ -th Chern class then gives a homomorphism

$$c_d : T^d K_0(X) \rightarrow \text{CH}^d(X).$$

By the Riemann-Roch theorem without denominators, we have that  $c_d(T^j K_0(X)) = 0$  for all  $0 < d < j$ , inducing a homomorphism

$$c_d : T^{d/d+1}(X) \rightarrow \text{CH}^d(X)$$

such that the composite  $c_d \circ \tau$  is the multiplication by  $(-1)^{d-1}(d-1)!$  ([14, Ex 15.3.6]). This result implies that  $c_d$  is an isomorphism for  $d \leq 2$  and, moreover,

**3.3.2 Lemma.** *For a smooth projective variety  $X$  over a field  $F$ ,  $c_d(T^j K_0(X)) = 0$  for all  $0 < d < j$ , and the induced homomorphism*

$$c_d: T^{d/d+1} K_0(X) \rightarrow \text{CH}^d(X)$$

*is an isomorphism over the coefficient ring  $\mathbb{Z}[\frac{1}{(d-1)!}]$  for all  $d > 0$ .*

### 3.3.2 The $\gamma$ -filtration

A second filtration on  $K_0(X)$  that we will use is the  $\gamma$ -filtration, which was introduced by Grothendieck as an approximation to the topological filtration (cf. [43, §2.3]).

Consider a smooth projective variety  $X$  over a field  $F$  and its Grothendieck group  $K_0(X)$ . For an element  $x \in K_0(X)$ , let  $\gamma(x) = \sum_{i \geq 0} \gamma_i(x)$  be a total characteristic class of  $x$ , this time with values in  $K_0(X)$  (cf. [14]). By the splitting principle, it is enough to define this characteristic class on a line bundle  $\mathcal{L}$  over  $X$ , and we follow the convention that  $\gamma_1([L]) := 1 - [L^\vee]$ , as in [31]. Using these characteristic classes, we set

$$\gamma^d K_0(X) = \langle \gamma_{i_1}(x_1) \cdot \dots \cdot \gamma_{i_m}(x_m) \mid i_1 + \dots + i_m \geq d, x_l \in K_0(X) \rangle,$$

and define the  $\gamma$ -filtration on  $K_0(X)$  by

$$\gamma^* K_0(X) = \bigoplus_{d=0}^{\dim X} \gamma^{d/d+1} K_0(X), \text{ where } \gamma^{d/d+1} K_0(X) := \gamma^d K_0(X) / \gamma^{d+1} K_0(X).$$

A result by Grothendieck and his collaborators in [43] is that  $\gamma^d K_0(X) \subseteq T^d K_0(X)$  for every  $d \geq 0$ , and they coincide for  $d \leq 2$ . Therefore, the Chern class map  $c_d$  restricted to  $\gamma^d K_0(X)$  induces a map

$$c_d: \gamma^{d/d+1} K_0(X) \rightarrow \text{CH}^d(X).$$

**3.3.3 Example.** For  $d = 1$  we have  $\gamma^{1/2} K_0(X) = T^{1/2} K_0(X)$  and  $c_1 \circ p = \text{id}_{\text{CH}^1(X)}$ , giving an isomorphism

$$c_1: \gamma^{1/2} K_0(X) \xrightarrow{\sim} \text{CH}^1(X).$$

In the previous example, we saw that the map  $c_1$  sends  $\gamma_1([L])$  to  $c_1(L)$ . For  $i = 2$  we again have  $\gamma^2 K_0(X) = T^2 K_0(X)$ , but this time  $\gamma^3 K_0(X)$  does not in general coincide with  $T^3 K_0(X)$ . We may form an exact sequence,

$$0 \rightarrow T^3 K_0(X) / \gamma^3 K_0(X) \rightarrow \gamma^{2/3} K_0(X) \rightarrow T^{2/3} K_0(X) \rightarrow 0.$$

Replacing  $T^{2/3} K_0(X)$  with  $\mathrm{CH}^2(X)$ , the map  $c_2 : \gamma^{2/3} K_0(X) \rightarrow \mathrm{CH}^2(X)$  is surjective. In addition,  $\ker(c_2) \cong T^3 K_0(X)/\gamma^3 K_0(X)$ . Note that for all  $d \geq 0$ , (cf. [26, Prop. 2.14])  $T^d K_0(X) \otimes \mathbb{Q} \cong \gamma^d K_0(X) \otimes \mathbb{Q}$ , thus  $\ker(c_2)$  is torsion.

**3.3.4 Proposition.** *The Chern class map  $c_d : \gamma^{d/d+1} K_0(X) \rightarrow \mathrm{CH}^d(X)$  is surjective over the coefficient ring  $\mathbb{Z} \left[ \frac{1}{(d-1)!} \right]$ .*

*Proof.* By Lemma 3.3.2, the map  $c_d : T^{d/d+1}(X) \rightarrow \mathrm{CH}^d(X)$  is surjective over the coefficient ring  $\mathbb{Z} \left[ \frac{1}{(d-1)!} \right]$ . Since  $\gamma^d K_0(X) \subseteq T^d K_0(X)$  for all  $d$ , we have an obvious map  $\gamma^{d/d+1} K_0(X) \rightarrow T^{d/d+1} K_0(X)$  defined by sending  $x + \gamma^{d+1} K_0(X) \mapsto x + T^{d+1} K_0(X)$ . By definition,  $c_d : \gamma^{d/d+1} K_0(X) \rightarrow \mathrm{CH}^d(X)$  is the composition of these two maps. Thus  $c_d(\gamma^{d/d+1} K_0(X) \subseteq c_d(T^{d/d+1} K_0(X))$ , and so it remains to show that  $c_d(\gamma^{d/d+1} K_0(X) \subseteq c_d(T^{d/d+1} K_0(X))$  over  $\mathbb{Z} \left[ \frac{1}{(d-1)!} \right]$ , for all  $d$ .

Consider an arbitrary element  $x \in K_0(X)$ . By the splitting principle we may assume that  $x = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$  where  $\mathcal{L}_1, \dots, \mathcal{L}_n$  are line bundles over  $X$ . By the properties of characteristic classes,  $\gamma_d(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n) = 0$  for all  $d > n$  and  $\gamma_d(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n) = s_d(\gamma_1(\mathcal{L}_1), \dots, \gamma_1(\mathcal{L}_n))$  for all  $0 < d \leq n$ , where  $s_d(\gamma_1(\mathcal{L}_1), \dots, \gamma_1(\mathcal{L}_n))$  is the  $d$ -th elementary symmetric polynomial in variables  $\gamma_1(\mathcal{L}_1), \dots, \gamma_1(\mathcal{L}_n)$ . Thus, taking the total Chern class, we have by Lemma 3.3.2

$$\begin{aligned} c(\gamma_d(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n)) &= c(s_d(\gamma_1(\mathcal{L}_1), \dots, \gamma_1(\mathcal{L}_n))) \\ &= \prod_{1 \leq j_1 < \cdots < j_d \leq n} (1 + (-1)^{d-1} (d-1)! c_1(\mathcal{L}_{j_1}) \cdots c_1(\mathcal{L}_{j_d}) + \dots) \\ &= 1 + (-1)^{d-1} (d-1)! \cdot s_d(c_1(\mathcal{L}_1), \dots, c_1(\mathcal{L}_n)) t^d + \dots \end{aligned}$$

Thus,  $c_d(\gamma_d(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n)) = (-1)^{d-1} (d-1)! s_d(c_1(\mathcal{L}_1), \dots, c_1(\mathcal{L}_n))$ . In general,

$$c_d(\gamma_d(x)) = (-1)^{d-1} (d-1)! c_d(x) \text{ for all } x \in K_0(X). \quad (3.1)$$

□

# Chapter 4

## Quadratic forms

For our purposes, we will only require quadratic forms over a field of characteristic different from 2. For simplicity, we will assume  $\text{char}(F) \neq 2$  for this entire chapter, as some definitions and results require adjustments in the case that  $\text{char}(F) = 2$ . All results presented in this chapter, as well as the corresponding results for quadratic forms over an arbitrary base field can be found in [12].

### 4.1 Bilinear and Quadratic forms

We begin by reviewing some basic definitions and structure theorems of bilinear and quadratic forms. The main purpose of this section is to provide background and context for the theory of algebras with involution in Chapter 5.

#### 4.1.1 Definitions

Let  $F$  be a field,  $V/F$  a finite-dimensional vector space.

A **bilinear form** is a map  $b : V \times V \rightarrow F$  satisfying

- $b(v + v', w) = b(v, w) + b(v', w)$
- $b(v, w + w') = b(v, w) + b(v, w')$
- $b(cv, w) = c \cdot b(v, w) = b(v, cw)$

A bilinear form is

- **symmetric** if  $b(v, w) = b(w, v)$
- **alternating** if  $b(v, v) = 0$

- **skew-symmetric** if  $b(v, w) = -b(w, v)$

Note that alternating implies skew-symmetric and that the converse holds for  $\text{char}(F) \neq 2$ . We define the dimension of a bilinear form to be the dimension of the underlying vector space.

An **isometry**  $f : b \rightarrow b'$  is a linear isomorphism  $f : V \rightarrow V'$  such that  $b'(f(v), f(w)) = b(v, w)$  for all  $v, w \in V$ . Given a bilinear form  $b$ , we may form a linear map  $l : V \rightarrow V^*$ , sending  $v \mapsto b(v, -)$ . A bilinear form is **nondegenerate** if  $b(x, y) = 0$  for all  $y \in V$  implies  $x = 0$ . Equivalently,  $b$  is nondegenerate if  $l$  is an isomorphism.

Let  $v_1, \dots, v_n$  be a basis of  $V$ . Then, any bilinear form  $b$  on  $V$  gives a matrix  $(b(v_i, v_j))_{i,j} \in M_n(F)$ . On the other hand, any matrix  $B \in M_n(F)$  gives a bilinear form. The operations are mutually inverse. Note that  $b$  is symmetric if and only if  $B$  is a symmetric matrix and  $b$  is skew-symmetric if and only if  $B$  is a skew-symmetric matrix.

Bilinear forms  $b, b'$  are isometric if and only if there exists some invertible  $A \in M_n(F)$  such that  $B' = A^T B A$  (and hence,  $\det(B)$  and  $\det(B')$  differ by a square). For a nondegenerate form  $b$ , we may define  $\det(b) := \det(B) \in F^\times / F^{\times 2}$ . Let  $W \subset V$  be a vector subspace. The restriction of  $b$  to  $W$  is a bilinear form on  $W$ , denoted  $b|_W$ , called a **subform** of  $b$ .

Related to the notion of a bilinear form is a quadratic form, defined as follows. A **quadratic form** is a map  $\varphi : V \rightarrow F$ ,  $\dim V < \infty$ , satisfying:

- $\varphi(av) = a^2\varphi(v)$  for all  $a \in F, v \in V$
- $b_\varphi : V \times V \rightarrow F$ , where  $b_\varphi(v, w) := \varphi(v+w) - \varphi(v) - \varphi(w)$  is a bilinear form (called the **polar form** of  $\varphi$ )

Again we define the **dimension** of  $\varphi$  by  $\dim(\varphi) := \dim(V)$ . The form  $b_\varphi$  is symmetric. For an arbitrary bilinear form  $b : V \times V \rightarrow F$ , the map  $\varphi_b : V \rightarrow F$ ,  $\varphi_b(v) := b(v, v)$  is a quadratic form. We have  $\varphi_{b_\varphi} = 2\varphi$ , and  $b_{\varphi_b} = b + b^t$  ( $= 2b$  if  $b$  is symmetric).

**4.1.1 Remark.** If  $\text{char}(F) \neq 2$ ,  $\varphi$  can be reconstructed from its polar form  $b_\varphi$ , so results for quadratic forms correspond to results for bilinear forms.

For a fixed basis  $v_1, \dots, v_n$  of  $V$ ,  $\varphi(x_1v_1 + \dots + x_nv_n) = \sum_{i \leq j} b_{ij}x_ix_j$ , for  $b_{ij} \in F$ . We may define an upper triangular matrix  $(b_{ij})$  by setting  $b_{ii} := \varphi(v_i)$ ,  $b_{ij} := b_\varphi(v_i, v_j)$  for  $i < j$  and  $b_{ij} = 0$  otherwise. It follows that  $\varphi = \varphi_b$  for the bilinear form  $b$  given on the basis  $v_1, \dots, v_n$  by the matrix  $(b_{ij})$ . An **isometry**  $f : \varphi \simeq \varphi'$  is a linear isomorphism of underlying vector spaces  $f : V \rightarrow V'$  such that  $\varphi'(f(v)) = \varphi(v)$  for all  $v \in V$ .

### Diagonalizable forms

Let  $b$  be an arbitrary bilinear form on  $V$ . Vectors  $v, w \in V$  are **orthogonal** if  $b(v, w) = 0$ . For a subspace  $W \subset V$ , we then define the **orthogonal complement**  $W^\perp = \{v \in V \mid b(v, w) = 0 \text{ for all } w \in W \subset V\} \subset V$ .  $W^\perp$  is also a subspace, and if  $b$  is nondegenerate,  $W^\perp = \{0\}$ . Let  $U, W \subset V$  be subspaces, then  $U$  and  $W$  are orthogonal if  $W \subset U^\perp$  (equivalently  $U \subset W^\perp$ ). If  $V = W \oplus U$  with  $W, U$  orthogonal, we write  $b = b|_W \perp b|_U$  and say that  $b$  is the (internal) **orthogonal sum** of  $b|_W$  and  $b|_U$ .

Define the **radical** of  $b$  by  $\text{rad}(b) := V^\perp$ . Note that  $b$  nondegenerate  $\iff \text{rad}(b) = 0$  (by definition). Let  $b, V, b', V'$  be bilinear forms, then the **external orthogonal sum**  $b \perp b'$  is the bilinear form on  $V \oplus V'$  defined by  $(b \perp b')((v, v'), (w, w')) = b(v, w) + b'(v', w')$ . Define  $b \perp \dots \perp b =: nb$  ( $n$  copies of  $b$ ).

**4.1.2 Proposition.** *Suppose  $b$  is nondegenerate. Then if  $b$  is alternating,  $\dim(b) = 2n$ .*

If  $b$  is symmetric, then  $b \simeq b_1 \perp \dots \perp b_n \perp b'$ , where  $\dim b_i = 1$  for all  $i$ , and  $b'$  is alternating (if  $\text{char} F \neq 2$ ,  $\dim b' = 0$ ).

Define  $\langle a \rangle : F \times F \rightarrow F$  by  $\langle a \rangle(v, w) = avw$  (or equivalently, by  $\langle a \rangle(1, 1) = a$ ). Then,  $\langle a \rangle \simeq \langle b \rangle \iff a = b = 0$ , or  $a \neq 0, b \neq 0$  and  $a \equiv b \in F^\times / (F^\times)^2$  (ie. coincide modulo squares).

The form  $\langle a_1, \dots, a_n \rangle := \langle a_1 \rangle \perp \dots \perp \langle a_n \rangle$  is called a **diagonal symmetric bilinear form**. It is nondegenerate if and only if  $a_i \neq 0$  for all  $i = 1, \dots, n$ . Set  $\det(\langle a_1, \dots, a_n \rangle) = a_1 \dots a_n \in F^\times / (F^\times)^2$ .

**4.1.3 Proposition.** *If  $b$  is symmetric, then  $b \simeq r\langle 0 \rangle \perp \langle a_1, \dots, a_n \rangle \perp$  (alternating form),  $a_i \in F^\times$ .*

We say that  $b$  is **diagonalizable** if  $b \simeq \langle a_1, \dots, a_n \rangle$  for some  $a_1, \dots, a_n \in F^\times$  and some  $r \in \mathbb{Z}$ .

**4.1.4 Remark.** In our situation, i.e.  $\text{char}(F) \neq 2$ , any symmetric bilinear form is diagonalizable.

### Isotropic forms

For a symmetric bilinear form  $b$  on  $V$ , a vector  $v$  is **anisotropic** if  $b(v, v) \neq 0$  and **isotropic** if  $v \neq 0$  but  $b(v, v) = 0$ . The form  $b$  is called anisotropic if there are no isotropic vectors in  $V$  and isotropic otherwise.

**4.1.5 Corollary.** *Any anisotropic symmetric bilinear form can be diagonalized.*

Consider quadratic forms  $\varphi, \varphi'$ . We say that  $\varphi$  and  $\varphi'$  are **similar** if  $\varphi \simeq a\varphi'$  for some  $a \in F^\times$ . A vector  $v \in V$  is **anisotropic** if  $\varphi(v) \neq 0$ , or **isotropic** if  $v \neq 0$  and  $\varphi(v) = 0$ .  $\varphi$  is **isotropic** if it has an isotropic vector, otherwise  $\varphi$  is **anisotropic** ( $V = 0 \implies \varphi$  anisotropic).

### Hyperbolic forms

For a fixed  $\lambda \in F$ , a **hyperbolic  $\lambda$ -symmetric form** is defined to be a bilinear form  $\mathbb{H}_\lambda(V) : (V \oplus V^*) \times (V \oplus V^*) \rightarrow F$  with  $\mathbb{H}_\lambda(V)((v, l), (v', l')) = l(v) + \lambda l'(v')$ . Note that  $\mathbb{H}_1(V)$  is symmetric and  $\mathbb{H}_{-1}(V)$  is alternating.

An arbitrary bilinear form is called **hyperbolic** if  $b \simeq \mathbb{H}_\lambda(V)$  for some  $\lambda \in F$  and some  $V$ . If  $V = F$ , then  $\mathbb{H}_\lambda := \mathbb{H}_\lambda(F)$  is called the **hyperbolic plane**.

A bilinear form  $b$  is isometric to the hyperbolic plane if and only if there exists  $e_1, e_2$  of  $V$  such that the matrix of  $b$  is  $\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$ . Such  $e_1, e_2$  are called a **hyperbolic pair**.

Similarly, the quadratic form  $\mathbb{H}(V) : V \oplus V^* \rightarrow F$  defined by  $v + f \mapsto f(v)$  is called the **hyperbolic quadratic form** given by  $V$ . The polar form of  $\mathbb{H}(V)$  is  $\mathbb{H}_1(V)$ , and we define the **hyperbolic plane**  $\mathbb{H} := \mathbb{H}(F)$ . Then,  $\varphi \simeq \mathbb{H}$  if and only if there exists a basis  $\{e_1, e_2\}$  of  $V$  such that  $\varphi(e_1) = 0 = \varphi(e_2)$  and  $b_\varphi(e_1, e_2) = 1$ . This basis is called a **hyperbolic pair**.

For any  $a \in F$ , we define  $\langle a \rangle : F \rightarrow F$  by  $\langle a \rangle(x) = ax^2$ . For any  $a, b \in F$ , we define  $[a, b] : F^2 \rightarrow F$  by  $[a, b](x, y) = ax^2 + xy + by^2$ .

**4.1.6 Example.**  $\mathbb{H} = [0, 0] \simeq [0, a]$  for all  $a \in F$ . The isometry is given by sending a hyperbolic pair  $\{e_1, e_2\} \mapsto \{e_1, ae_1 + e_2\}$ .

For a symmetric bilinear form  $b$  on  $V$ , a subspace  $W \subset V$  is **totally isotropic** if  $b|_W = 0$  (ie.  $b(u, v) = 0$  for all  $u, v \in W$ ). If  $b$  is nondegenerate, then  $\dim W \leq \frac{1}{2} \dim V$ , and if equality holds, we say that  $W$  is a **Lagrangian** for  $b$ . If  $b$  has a Lagrangian, then it is hyperbolic.

**4.1.7 Remark.** For an arbitrary field  $F$ , if  $b$  has a Lagrangian, it is called “metabolic”. If  $\text{char}(F) \neq 2$ , there is no difference between the terms hyperbolic and metabolic.

- an orthogonal sum of hyperbolic forms is hyperbolic
- $b$  symmetric and nondegenerate  $\implies b \perp (-b)$  on  $V \oplus V$  defined by  $(b \perp (-b))(v + u, v' + u') = b(v, v') - b(u, u')$  is hyperbolic, the Lagrangian here is given by the diagonal map  $V \hookrightarrow V \oplus V$

- a bilinear form is 2-dim hyperbolic  $\iff$  it is 2-dim non-degenerate, symmetric and isotropic: there exists  $0 \neq v \in V$  such that  $b(v, v) = 0$ , there exists  $u \in V$  such that  $b(v, u) = 1$ . Clearly the space is generated by  $u, v$  so we just need to know the value at  $u$ : let  $a := b(u, u) \in F$ . If  $a = 0$ , then  $b \simeq \mathbb{H}_1$ . If  $a \neq 0$ , then  $b \simeq \langle a, -a \rangle$ .

**4.1.8 Lemma.** *If  $b$  is nondegenerate and symmetric, then every isotropic vector is in a hyperbolic plane.*

**4.1.9 Proposition.** *Any hyperbolic bilinear form  $b$  is isometric to an orthogonal sum of  $n$  hyperbolic planes, and  $\det(b) = (-1)^n$ .*

**4.1.10 Theorem** (Bilinear Witt decomposition). *Let  $b$  be a nondegenerate symmetric bilinear form on  $V$ . There exist subspaces  $V_1, V_2$  of  $V$  such that  $b = b|_{V_1} \perp b|_{V_2}$  with  $b|_{V_1}$  anisotropic,  $b|_{V_2}$  hyperbolic. The form  $b|_{V_1}$  is unique up to isometry and is called the **anisotropic part** of  $b$ . The **Witt index** of  $b$  is  $\frac{\dim(b|_{V_2})}{2}$ .*

**4.1.11 Corollary** (Witt cancellation for anisotropic forms). *Suppose  $b_1 \perp b \simeq b_2 \perp b$ , for  $b, b_1, b_2$  nondegenerate symmetric bilinear forms, with  $b_1, b_2$  anisotropic. Then  $b_1 \simeq b_2$ .*

For a quadratic form  $\varphi : V \rightarrow F$  and a subspace  $W \subset V$ , the restriction  $\varphi|_W : W \rightarrow F$  is called a **subform** of  $\varphi$ .  $W \subset V$  is a **totally isotropic subspace** if  $\varphi|_W = 0$ .

Define the **orthogonal sum**  $\varphi \perp \varphi' : V \oplus V' \rightarrow F$  by  $v + v' \mapsto \varphi(v) + \varphi(v')$ . Note that  $V$  and  $V'$  are orthogonal with respect to  $b_{\varphi \perp \varphi'}$ . Suppose  $V = U \oplus W$  with  $U \subset W^\perp$ . Then  $\varphi \simeq \varphi|_U \perp \varphi|_W$ .

For any  $\varphi$ , the **radical** is defined to be the linear subspace  $\text{rad}(\varphi) := \{v \in \text{rad } b_\varphi \mid \varphi(v) = 0\} \subset \text{rad } b_\varphi$ . (In  $\text{char} \neq 2, \text{rad } \varphi = \text{rad } b_\varphi$ ). We say that  $\varphi$  is **regular** if  $\text{rad } \varphi = \{0\}$  (we will see in the next section that nondegenerate will mean something slightly different for quadratic forms).

## 4.1.2 Extension of Scalars

Let  $K/F$  be a field extension of  $F$ , and let  $V_K := V \otimes_F K$  be the  $K$ -vector space obtained from an  $F$ -vector space  $V$  by scalar extension. A bilinear form  $b : V \times V \rightarrow F$  gives a bilinear form  $b_K : V_K \times V_K \rightarrow K$  uniquely determined by  $b_K|_{V \times V} = b$ . Similarly, a quadratic form  $\varphi : V \rightarrow F$  gives a quadratic form  $\varphi_K : V_K \rightarrow K$ , uniquely determined by  $\varphi_K|_V = \varphi$ , and  $b_{\varphi_K}|_V = b_\varphi$ .

**4.1.12 Lemma.** *For  $\varphi/F$ , the following are equivalent:*

1.  $\varphi_K$  regular for all  $K/F$
2.  $\varphi_K$  regular for some algebraically closed  $K/F$
3.  $\varphi$  regular and  $\dim(\text{rad}(b_\varphi)) \leq 1$

If  $\varphi$  satisfies the conditions of the above lemma, we say it is **nondegenerate**. If  $\varphi, \psi$  are nondegenerate, then  $\varphi \perp \psi$  is nondegenerate as well.

**4.1.13 Remark.** Note that for  $\text{char}(F) \neq 2$ , these notions coincide; however if  $\text{char}(F) = 2$ , an odd-dimensional quadratic form can be regular over  $F$ , but degenerate in general.

**4.1.14 Example.** A quadratic form  $\varphi$  defines a projective quadric  $X_\varphi \subset \mathbb{P}(V)$  by setting  $\varphi = 0$ . Then  $X_\varphi$  is smooth if and only if  $\varphi$  is nondegenerate. For example,  $\langle a \rangle$  nondeg.  $\iff a \neq 0$ , and  $[a, b]$  nondeg.  $\iff 1 - 4ab \neq 0$  (this follows from the association  $b_\varphi \leftrightarrow \begin{bmatrix} 2a & 1 \\ 1 & 2b \end{bmatrix}$ , so for  $\text{char}(F) \neq 2$ , this is just the determinant).

**4.1.15 Lemma.** *Let  $\varphi$  be regular and  $v \in V$  isotropic. Then there exists  $u \in V$  such that  $v, u$  is a hyperbolic pair.*

**4.1.16 Proposition.** *For a quadratic form  $\varphi$  on  $V$ , if  $W \subset V$  is a subspace such that  $b_\varphi|_W$  is non-degenerate, then  $\varphi \simeq \varphi|_W \perp \varphi|_{W^\perp}$ . (In particular  $V = W \oplus W^\perp$ .)*

As in the setting of bilinear forms, we define a form  $\langle a \rangle : F \rightarrow F$  by  $x \mapsto ax^2$ , for  $a \in F$ . We then call the form  $\langle a_1, \dots, a_n \rangle := \langle a_1 \rangle \perp \dots \perp \langle a_n \rangle, a_i \in F$  a **diagonal quadratic form**. We say  $\varphi$  is **diagonalizable** if  $\varphi \simeq \langle a_1, \dots, a_n \rangle$ .

**4.1.17 Remark.** Under our restriction that  $\text{char} F \neq 2$ , any  $\varphi$  is diagonalizable. In general however,  $\varphi$  is diagonalizable if and only if  $\varphi = \varphi_b$  for some symmetric bilinear form  $b$ .

**4.1.18 Proposition.** *Suppose  $\varphi$  is nondegenerate.  $\varphi$  is hyperbolic if and only if it contains a totally isotropic subspace of dimension  $\frac{1}{2} \dim \varphi$ .*

**4.1.19 Proposition** (Witt Decomposition Theorem). *If  $\varphi$  is regular, then  $\varphi \simeq \varphi_{an} \perp m\mathbb{H}$  for some anisotropic  $\varphi_{an}$  and some  $m \geq 0$ . Moreover,  $\varphi_{an}$  is determined by  $\varphi$  up to isomorphism. Thus,  $m$  is determined by  $\varphi$  as well. We call  $i(\varphi) := m$  the **Witt index** of  $\varphi$ , and  $\varphi_{an}$  the **anisotropic part** of  $\varphi$ .*

The proof of the Witt decomposition theorem uses the following two results.

**4.1.20 Proposition** (Witt Extension Theorem). *Let  $W, W' \subset V$  be subspaces with  $\text{rad } b_\varphi \cap W = \{0\} = \text{rad } b_\varphi \cap W'$ . Then any isometry  $f : W \simeq W'$  can be extended to an isometry  $V \simeq V$ .*

**4.1.21 Proposition** (Witt Cancellation Theorem). *Suppose  $\varphi \perp \psi \simeq \varphi' \perp \psi$  and  $b_\psi$  nondegenerate. Then  $\varphi \simeq \varphi'$ .*

## 4.2 Clifford Algebras

Clifford algebras will play an important role in the main results of this thesis. As such, we recall here some basic definitions and properties, all of which can be found in [12].

### 4.2.1 Quaternion Algebras

For a field  $F$  with separable closure  $F_s$ , a finite dimensional  $F$ -algebra  $S$  is called **étale** if  $S \otimes_F F_s$  is isomorphic to the  $F_s$ -algebra  $F_s^n = F_s \times \cdots \times F_s$  for some  $n \geq 1$ . Étale algebras are direct products of finite separable field extensions of  $F$ . For any integer  $n \geq 1$ , we denote by  $Et_n(F)$  the groupoid whose objects are  $n$ -dimensional étale  $F$ -algebras and whose morphisms are  $F$ -algebra isomorphisms. Étale algebras of dimension 2 are called **quadratic étale algebras**.

Let  $L/F$  be a quadratic étale algebra. If  $\text{char } F \neq 2$ , then  $L = F_a := F[t]/(t^2 - a)$  for some  $a \in F^\times$ . We construct an  $F$ -algebra  $Q = L \oplus Lj$ , where  $j^2 = b$  for some  $b \in F^\times$ . Generators of  $Q$  are given by  $i := t, j$ , with relations  $i^2 = a, j^2 = b, ij = -ji$ . The algebra  $Q$  is called a **quaternion algebra** and is usually denoted by  $Q = (a, b)_F$  or  $Q = \left(\begin{smallmatrix} a & b \\ & F \end{smallmatrix}\right)$ .

**4.2.1 Remark.** By dimension arguments, we can see that for  $a, b \in F^\times$ , the quaternion algebra  $(a, b)_F$  is either a division algebra (if  $a, b \notin F^{\times 2}$ ) or is isomorphic to  $M_2(F)$  (if either  $a \in F^{\times 2}$  or  $b \in F^{\times 2}$ ). We will abuse notation slightly: if  $a \in F^{\times 2}$ , then by  $F_a = F[t]/(t^2 - a)$ , we mean  $F_a = F \times F$ .

### 4.2.2 Definition of Clifford algebra

Let  $F$  be of arbitrary characteristic. For an  $F$ -vector space  $V$ , we define the **tensor algebra** of  $V$  by:

$$\mathcal{T}(V) := \coprod_{n \geq 0} V^{\otimes n} = F \oplus V \oplus V^{\otimes 2} \oplus \dots \quad (4.1)$$

$\mathcal{T}(V)$  is an  $F$ -algebra which is  $\mathbb{Z}$ -graded, non-commutative, and associative with unit  $1 \in F$ . A quadratic form  $\varphi$  on  $V$  gives a 2-sided ideal  $I \in \mathcal{T}(V)$ , generated by differences  $v \otimes v - \varphi(v)$ ,  $v \in V$ . We define the **Clifford algebra** of  $\varphi$  by  $C(\varphi) := \mathcal{T}(V)/I$ . It is a  $\mathbb{Z}/2\mathbb{Z}$ -graded  $F$ -algebra, and so decomposes as  $C(\varphi) = C_0(\varphi) \oplus C_1(\varphi)$ , where  $C_0(\varphi)$  is a subalgebra, called the **even Clifford algebra** of  $\varphi$ . For  $u, v \in V \subseteq \mathcal{T}(V)$ ,

$$b_\varphi(v, u) = \varphi(v + u) - \varphi(v) - \varphi(u) = (v + u)^2 - v^2 - u^2 = vu + uv \in C(\varphi).$$

In particular,  $b_\varphi(v, u) = 0 \iff vu = -uv \in C(\varphi)$ .

**4.2.2 Example.** Let  $\varphi : V \rightarrow F$  be the zero map, i.e.  $\varphi(v) = 0$  for all  $v \in V$ . then  $C(\varphi) = \wedge(V)$ , the exterior algebra of  $V$ .

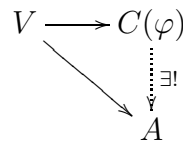
We have  $\dim C(\varphi) = 2^n$ , where  $n = \dim V$  for any  $\varphi$ , and  $\dim C_0(\varphi) = 2^{n-1}$ .

- $\dim V = 0$ :  $C(\varphi) = F = C_0(\varphi)$ .
- $\dim V = 1$ :  $\varphi = \langle a \rangle$  for some  $a \in F$ , and so  $C_0(\varphi) = F, C(\varphi) = F[t]/(t^2 - a)$ .
- $\dim V = 2$  and  $\varphi$  is nondegenerate: for  $\text{char} F \neq 2$ ,  $\varphi = \langle a, b \rangle$  with  $a, b \in F^\times$ , and  $C(\varphi)$  is the quaternion algebra  $(a, b)_F$  over  $F$  given by  $i^2 = a, j^2 = b, ij = -ji$ .

If  $\text{char} F \neq 2$ , then the even Clifford algebra  $C_0(\langle a, b \rangle)$  has generator  $ij$ , with  $(ij)^2 = -ab$ , so  $C_0(\langle a, b \rangle) = F_{-ab}$ . Therefore,  $C_0(\langle 1, -a \rangle) = F_a$ .

### 4.2.3 Properties of Clifford algebras

The Clifford algebra of a quadratic form  $\varphi$  satisfies the following **universal property**: Let  $V \rightarrow C(\varphi)$  be the canonical  $F$ -linear map. For any  $F$ -algebra  $A$  with  $f : V \rightarrow A$  an  $F$ -linear map such that  $f(v)^2 = \varphi(v)$  for all  $v \in V$ , there exists a unique  $F$ -algebra homomorphism  $C(\varphi) \rightarrow A$  such that the triangle commutes.



Next, we define the **canonical involution on  $C(\varphi)$** . Let  $C(\varphi)^{op}$  be the opposite algebra (the same vector space as  $C(\varphi)$ , with opposite multiplication, ie.  $ab \in C(\varphi)^{op}$  is  $ba \in C(\varphi)$ ). The map  $V \rightarrow C(\varphi)^{op}$  is the same linear map as for  $C(\varphi)$ , this gives an  $F$ -algebra homomorphism  $C(\varphi) \simeq C(\varphi)^{op}$  by sending  $v_1 \cdots v_n \mapsto v_n \cdots v_1$  for all  $v_1, \dots, v_n \in V$ . Involutions in general will be discussed in Chapter 5.

**4.2.3 Proposition.** For any  $\varphi$ ,

1.  $C_0(a\varphi) \simeq C_0(\varphi)$  for all  $a \in F^\times$
2.  $C(\varphi) \simeq C_0(\langle -1 \rangle \perp \varphi)$

**4.2.4 Proposition.** For  $\varphi$  with  $\dim \varphi \geq 2$  even, the following are equivalent:

1.  $\varphi$  is nondegenerate
2.  $C(\varphi)$  is a central simple algebra
3.  $C_0(\varphi)$  is a separable algebra
4.  $C_0(\varphi)$  is a separable algebra and  $Z(C_0(\varphi)) =: Z(\varphi)$  is a quadratic étale algebra

**4.2.5 Proposition.** For  $\varphi$  with  $\dim \varphi \geq 3$  odd, the following are equivalent:

1.  $\varphi$  is nondegenerate
2.  $C_0(\varphi)$  is a central simple algebra

The definition of a central simple algebra will be given in Chapter 5.

**4.2.6 Lemma.** Let  $\varphi$  be nondegenerate, with  $\dim \varphi \geq 2$  even. Then for all  $x \in Z(C_0(\varphi)) =: Z(\varphi)$ ,  $y \in C_1(\varphi)$ , one has  $yx = \bar{x}y$ , where  $\bar{x}$  is the image of  $x$  under the only non-trivial automorphism of  $Z(\varphi)$ .

**4.2.7 Corollary.** For  $\varphi$  nondegenerate,  $\dim \varphi \geq 2$ , if  $a \in F^\times$  is a norm for  $Z(\varphi)$ , then  $C(a\varphi) \simeq C(\varphi)$ .

## 4.2.4 Discriminant of a quadratic form

Let  $\varphi$  be a nondegenerate quadratic form of even dimension. Define the **discriminant**  $\text{disc}(\varphi) := [Z(\varphi)] \in Et_2(F)$  (with multiplicative notation induced by the  $\star$  operation). if  $\dim \varphi \geq 2$ . If  $\dim \varphi = 0$ , then set  $\text{disc}(\varphi) := 1$ .

**4.2.8 Example.** For  $\text{char} F \neq 2$  and  $a, b \in F^\times$ ,  $\text{disc}(\langle a, b \rangle) = F_{-ab} \in Et_2(F)$ . If we identify  $Et_2(F) = F^\times / (F^\times)^2$ , this is also  $-ab$ .

**4.2.9 Example.** Suppose  $\dim \varphi = 2 = \dim \psi$  nondegenerate. Then  $\text{disc}(\varphi) = \text{disc}(\psi) \iff \varphi \sim \psi$ .  $\text{disc}(\varphi) = 1 \iff \varphi$  is the hyperbolic plane.

**4.2.10 Lemma.**  $\text{disc}(\varphi \perp \psi) = \text{disc}(\varphi) \text{disc}(\psi)$  for all  $\varphi, \psi$  nondegenerate even-dimensional forms.

**4.2.11 Corollary.** For  $\text{char} F \neq 2$  and  $a_1, \dots, a_n \in F^\times$ , the discriminant of  $\langle a_1, \dots, a_n \rangle$  is given by  $\text{disc}(\langle a_1, \dots, a_n \rangle) = (-1)^n a_1 \cdots a_n \in F^\times / (F^\times)^2$ .

# Chapter 5

## Central simple algebras with involution

For this chapter, we continue to restrict our study to a field  $F$  of characteristic not equal to 2. Furthermore, we consider all algebras to be associative and finite dimensional. The main reference for this chapter is [29]; a more concise review of the material is given in [45].

### 5.1 Central simple algebras

We begin by recalling basic definitions, properties and examples of central simple algebras. We will assume that all algebras are associative and finite-dimensional over  $F$ .

#### 5.1.1 Definitions and examples

An algebra  $A$  is **central** over  $F$  if  $Z(A) = F$  and is **simple** if it has no two-sided ideals except  $\{0\}$  and  $A$  (that is,  $A$  is simple as a ring).

**5.1.1 Example.** Examples of central simple algebras include:  $F$ , the base field itself; quaternion algebras; cyclic algebras; division algebras with centre  $F$ ; and algebras of the form  $M_r(D)$ , where  $D$  is a central division algebra over  $F$ .

**5.1.2 Proposition.** *Let  $A, B$  be central simple algebras over  $F$ , and  $L/F$  a field extension. Then*

- $A_L = A \otimes_F L$  is a central simple  $L$ -algebra

- $A \otimes_F B$  is a central simple  $F$ -algebra

**5.1.3 Example.**  $(a, b)_F \otimes (c, d)_F$  is called a **biquaternion algebra**.

From now on, we let  $A$  be a central simple algebra over  $F$ .

**5.1.4 Theorem (Wedderburn).** *There exists a central division algebra  $D$  over  $F$ , uniquely defined up to isomorphism, such that  $A \simeq M_r(D)$  for some  $r \in \mathbb{N}$ .*

We say that  $A$  is **split** if  $D = F$ , ie.  $A \simeq M_r(F)$  for some  $r \in \mathbb{N}$ .

**5.1.5 Corollary.**  $M_r(D) \simeq M_s(D')$  for  $D, D'$  division algebras over  $F$  if and only if  $D \simeq D'$  and  $r = s$ .

**5.1.6 Remark.** Let  $I$  be a nontrivial minimal right ideal in  $A$ . Then, if  $A = D$ ,  $I = D$  as well, since  $D$  has no nontrivial ideals. On the other hand, if  $A = M_r(D)$  for some  $r > 1$ , then  $I$  has non-trivial entries only in the  $i$ th row (for some  $i$ ). In either case,  $I$  is a simple  $A$ -module. By Schur's lemma,  $\text{End}_A(I) = D$  is division, and we have a natural map  $A \simeq \text{End}_D(I)$ . Thus, the division part of  $A$  is given explicitly by looking at minimal modules.

**5.1.7 Theorem.** *Any central simple algebra over an algebraically closed field is split.*

*Proof.* Let  $D$  be a central division algebra over an algebraically closed field  $\bar{F}$ , and let  $d \in D$ . Then  $\bar{F}(d) = \bar{F}$  because  $\bar{F}(d)/\bar{F}$  is a finite field extension of an algebraically closed field (and must therefore be trivial). So,  $D = \bar{F}$  implies  $A = M_n(\bar{F})$ , so  $A$  is split.  $\square$

**5.1.8 Corollary.** *For any field  $F$  and any central simple algebra  $A$ ,  $A_{\bar{F}} = A \otimes_F \bar{F}$  is split.*

We define the **degree** of  $A$  by  $\deg(A) := \sqrt{\dim_F(A)} \in \mathbb{N}$  and the **index** of  $A$  by  $\text{ind}(A) := \deg(D)$ , where  $A \simeq M_r(D)$ .

## 5.1.2 The Brauer group

We say that  $A, B$  are **Brauer equivalent**, and write  $A \sim_{\text{Br}} B$  (or simply  $A \sim B$ ), if there exist  $p, q \in \mathbb{N}$  such that  $M_p(A) \simeq M_q(B)$ . Equivalently, there exists a central division algebra  $D$  and integers  $r, s \geq 1$  such that  $A \simeq M_r(D)$  and  $B \simeq M_s(D)$ .

**5.1.9 Example.**  $M_r(D) \sim_{\text{Br}} D$ .

With this, we define the set

$$\begin{aligned} \text{Br}(F) &:= \{\text{c.s.a.'s over } F\} / \sim_{\text{Br}} \\ &= \{\text{isomorphism classes of central division algebra over } F\} \end{aligned}$$

The tensor product of central simple algebras endows  $\text{Br}(F)$  with the structure of an abelian group. Thus, the **Brauer group** is defined to be  $(\text{Br}(F), \otimes)$ , and is usually written with additive notation:

- $[A] + [B] := [A \otimes_F B]$
- $0 := [F]$
- $-[A] = [A^{op}]$

The last point follows from the isomorphism  $A \otimes A^{op} \rightarrow \text{End}_F(A)$ , sending  $a \otimes b^{op} \mapsto (x \mapsto axb^{op})$ , which must be injective by centrality, and thus bijective by a dimension argument.

**5.1.10 Example.** The Brauer group of an algebraically closed field is trivial.  $\text{Br}(\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$ , generated by  $(-1, -1)_{\mathbb{R}}$ , the Hamiltonians.

The **exponent** of  $A$ , denoted  $\exp(A)$  (sometimes called the **period** of  $A$ ), is the order of  $[A]$  in  $\text{Br}(F)$ . We note that  $\exp(A) = 1$  if and only if  $A \simeq_{\text{Br}} 0$  (ie.  $A$  is split).

**5.1.11 Proposition.** *For any c.s.a.  $A$ ,  $\exp(A) \mid \text{ind}(A)$  and they have the same prime factor.*

**5.1.12 Lemma.** *For a quaternion division algebra  $Q/F$ ,  $\exp(Q) = 2$ .*

*Proof.* Consider the **canonical involution**  $\text{can} : Q \rightarrow Q$  given by  $x + yi + zj + tij \mapsto x - yi - zj - tij$ . We have  $\text{can}(qq') = \text{can}(q')\text{can}(q)$  and  $\text{can}^2 = \text{id}$ . In particular,  $\text{can}$  induces  $Q \simeq Q^{op}$ , by sending  $x \mapsto \text{can}(x)^{op}$ . (Note that  $x \mapsto x^{op}$  is not an isomorphism, but  $\text{can}$  is an anti-automorphism, so twisting by  $\text{can}$  gives an isomorphism.) So,  $[Q] = [Q^{op}] = -[Q] \implies 2[Q] = 0$ . Therefore,  $\exp(A) = 2$  if and only if there exists an anti-automorphism on  $A$ .  $\square$

**5.1.13 Lemma.** *Let  $A = Q_1 \otimes Q_2$ , a biquaternion algebra. Then*

- $\deg(A) = 4$
- $\text{ind}(A) = 1, 2 \text{ or } 4$  ( $\deg(D) \mid \deg(A)$ )
- $\exp(A) = 1 \text{ or } 2$  (in this case  $\exp < \deg$ )

### 5.1.3 Structure theorems

**5.1.14 Theorem** (Skolem-Noether theorem, [29, 1.4]). *Let  $A$  be a central simple algebra over  $F$ .*

1. *Every automorphism of  $A$  is inner.*
2. *For any simple subalgebra  $B \subset A$  any map  $B \rightarrow A$  can be extended to an inner automorphism of  $A$ .*

**5.1.15 Theorem** (Double centralizer theorem, [29, 1.5]). *Let  $A$  be a central simple algebra over  $F$ . Assume that  $B \subset A$  is a simple subalgebra and let  $Z(B) = L \supset F$ . Then, the centralizer of  $B$  in  $A$ , denoted  $Z_A(B)$ , is a central simple algebra over  $L$ . Moreover,  $Z_A(Z_A(B)) = B$  and  $\dim_F(B) \cdot \dim(Z_A(B)) = \dim_F(A)$ .*

**5.1.16 Example.** Let  $A = Q_1 \otimes_F Q_2$ . Then  $Z_A(Q_1) = Q_2$  and  $Z_A(Q_2) = Q_1$ .

**5.1.17 Corollary.** *Let  $B \subset A$  be a central simple subalgebra of  $A$ . Then,  $A \simeq B \otimes_F Z_A(B)$ .*

Note that this result is important for the decomposition of a central simple algebra, as it tells us that we only need to find one part of the decomposition to give the remaining part. We have the following application of these theorems.

**5.1.18 Proposition.** *Every degree 2 central simple algebra over  $F$  is a quaternion algebra.*

*Proof.* If  $A \simeq M_2(F)$ , then  $A \simeq (1, 1)_F$ . So suppose  $A$  is not split. Then by dimension,  $A \simeq D$  for some division algebra  $D$ . Let  $F \neq K \subset A$  such that  $K/F$  is a field extension of degree 2. There exists an element  $i \in A \setminus F$  with  $i^2 = a \in F^\times$  such that  $K = F(i) \neq A$  (since  $F(i)$  is commutative and  $A$  is not), and  $[F(i) : F] = 2$ . By Theorem 5.1.14, there exists  $j \in A$  such that  $\text{Inn}(j)|_K = \text{can}$ , the canonical involution on  $K$ , i.e..  $jij = \text{can}(i) = -i$ . Then  $\text{Inn}(j^2)|_K = \text{id}$  since  $\text{can}^2 = \text{id}$ , so  $j^2 \in Z_A(K)$  and by Theorem 5.1.15,  $Z_A(K) = K$ . So,  $\text{can}(j)^2 = jj^2j^{-1} = j^2$  and so  $j^2 \in F^\times$ , i.e.  $j^2 = b$ .  $\square$

### 5.1.4 Splitting fields

We say that  $L/F$  is a **splitting field** for  $A$  if  $A \otimes_F L \simeq M_n(L)$ .

**5.1.19 Example.** For  $Q = (a, b)_F$ ,  $F_a = F(\sqrt{a})$  is a splitting field: There is an isomorphism  $Q \otimes_F F_a \simeq M_2(F_a)$  defined by mapping

$$i \mapsto \begin{bmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{bmatrix}, \quad j \mapsto \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix}$$

**5.1.20 Example.** Consider again the quaternion algebra  $Q = (a, b)_F$  and let  $F_Q$  be the function field of the conic  $\langle 1, -a, -b \rangle$ . Then  $F_Q = F[s, t]/(s^2 - at^2 - b)$ , and note that  $0 = (s + ti + j)(s - ti - j) \in Q \otimes_F F_Q$ . So,  $Q$  is not a division algebra over  $F_Q$  and we must have  $Q \otimes_F F_Q \simeq M_2(F_Q)$ . Note however that  $F$  is **quadratically closed** in  $F_Q$ : that is, for all  $s \in F$ ,  $\sqrt{s} \in F_Q$ .

We define a **commutative subfield**  $K$  of a central simple  $F$ -algebra  $A$  to be a field extension  $K/F$  such that  $K \subset A$ . We say that a commutative subfield  $K$  is **maximal** if for any commutative subfield  $K'$  with  $K \subseteq K' \subset A$ , we have  $K = K'$ . The Double centralizer theorem has the following corollary:

**5.1.21 Corollary.** *Let  $K \subset A$  be a commutative subfield. Then,  $Z_A(K)$  is Brauer equivalent to  $A \otimes_F K$ .*

**5.1.22 Example.** Let  $A = Q_1 \otimes_F Q_2$  and  $K \subset Q_1$ . Then  $Q_{1K}$  is split, and  $Z_A(K) = K \otimes_F Q_2 = Q_{2K} \implies [A_K] = [Q_{2K}]$ .

**5.1.23 Theorem.** *Let  $D/F$  be a central division algebra. Any maximal commutative subfield is a splitting field, and has degree equal to  $\deg(D)$ .*

*Proof.* Pick  $L \subset D$  maximal. Then  $Z_D(L) = L$  (for all  $d \in Z_D(L)$ ,  $L(d)$  is a commutative subfield of  $D$  containing  $L$ , so  $d \in L$  by maximality of  $L$ ). Then,  $D_L \sim Z_D(L) \implies D_L$  is split.  $\square$

Any division algebra contains a maximal subfield which is separable over the base field, so any central simple algebra admits a **Galois splitting field**. This leads to the definition of the **reduced trace** and **reduced norm**, using Galois descent. We set

$$\mathrm{Trd}_A(a) := \mathrm{tr}(\varphi(a \otimes 1)) \in F^\times \tag{5.1}$$

$$\mathrm{Nrd}_A(a) := \det(\varphi(a \otimes 1)) \in F^\times, \tag{5.2}$$

where  $\varphi : A \otimes L \simeq M_n(L)$  for a splitting field  $L$ .

Recall that minimal right ideals in  $M_r(D)$  are of the form

$$I = \begin{bmatrix} 0 & \dots & 0 \\ * & \dots & * \\ 0 & \dots & 0 \end{bmatrix} \leftarrow \text{(non-zero entries in the } i\text{-th row only)}$$

and so  $\dim(I) = rd^2$ , where  $d = \text{ind}(A) = \text{deg}(D)$ . The **reduced dimension** of a right ideal  $I$  in  $A$  is defined to be  $\text{rdim}_F(I) := \frac{\dim_F(I)}{\text{deg}(A)}$ . For example, if  $I$  is a minimal right ideal, then  $\text{rdim}_F(I) = \frac{rd^2}{rd} = d$ . Thus,  $A$  has ideals of  $\text{rdim} = 1$  if and only if  $A \sim 0$  (i.e.  $A$  is split).

**5.1.24 Example.** The **Severi-Brauer variety** of  $A$ , denoted  $SB(A)$ , is the variety of right ideals of reduced dimension 1. This variety has a rational point if and only if  $A$  is split. The function field  $F_A := F(SB(A))$  is a splitting field for  $A$ . We also note that  $F$  is quadratically closed in  $F_A$ .

## 5.2 Involutions

Central simple algebras of exponent two are endowed with extra structure, in the form of involutions. In this section we recall properties of involutions including kind and type, and provide a partial classification of algebras with involution for small degrees.

### 5.2.1 Definitions and Examples

For a central simple algebra  $A$ , a map  $\sigma : A \rightarrow A$  is called an **involution** if  $\sigma(xy) = \sigma(y)\sigma(x)$  (i.e. is an anti-automorphism) and  $\sigma^2 = \text{id}$ . Recall that we defined the canonical involution on a quaternion algebra in Section 5.1.2. We define an **algebra with involution** to be a pair  $(A, \sigma)$ . We say that  $(A, \sigma)$  and  $(B, \tau)$  are **isomorphic** as algebras with involution if there exists an isomorphism  $\varphi : A \rightarrow B$  such that  $\varphi \circ \sigma = \tau \circ \varphi$ .

Note that while  $\sigma$  is not in general an algebra automorphism, it induces an automorphism on the base field. From now on, we will set  $F := Z(A)^\sigma$ . Then, either  $Z(A) = F$ , in which case we say  $\sigma$  is  **$F$ -linear** (or an involution of the **first kind**), or  $Z(A) = F_a := F(\sqrt{a})$ , in which case we say  $\sigma$  is  $F_a/F$  **semi-linear** or **unitary** (or an involution of the **second kind**). In both cases, we say  $(A, \sigma)$  is an algebra with involution **over**  $F$  (note that this is a slight change in the base field).

**5.2.1 Remark.** Assume  $\sigma$  is  $K/F$  unitary. For all  $L/F$ ,  $\sigma$  induces an involution  $\sigma \otimes \text{id}$  on  $A \otimes_F L$ . But,  $A \otimes_F L$  is not always simple. In fact  $Z(A \otimes_F L) = K \otimes_F L$ , which is

either a quadratic field extension of  $L$  or isomorphic to  $L \times L$ , which may not be a field (e.g.  $Z(A \otimes_F K) \simeq K \times K$ ). If  $A(A \otimes_F L) \simeq L \times L$ , then  $(A \otimes_F L, \sigma \otimes \text{id}) \simeq (A \times A^{op}, ex)$ , where  $ex$  is the exchange involution. So,  $(A \times A^{op}, ex)$  for a central simple algebra  $A$  over  $F$  is also called an algebra with involution over  $F$ .

If  $(A, \sigma)$  is an algebra with involution over  $F$ , then  $(A_L, \sigma_L)$  is an algebra with involution over  $L$ .

**5.2.2 Example.** Consider the quaternion algebra  $Q = (a, b)_F$ , and let  $q = x + yi + zj + tij \in Q$ . Then by (5.1),  $\text{Trd}(q) = 2x$ . Recall that the canonical involution on  $Q$  maps  $q = x + yi + zj + tij \mapsto \text{can}(q) = x - yi - zj - tij$ , and so  $\text{can}(q) = \text{Trd}(q) - q$ . The involution  $\text{can}$  is indeed canonical in the sense that it doesn't depend on the choice of basis. We define a **pure quaternion** to be an element  $q \in Q$  having  $\text{Trd}(q) = 0$ , and denote by  $Q^0 \subset Q$  the set of pure quaternions in  $Q$ . Pick an element  $q \in Q^0$ . Then,  $\sigma_q := \text{Inn}(q) \circ \text{can}$  is also an involution of  $Q$ .

**5.2.3 Proposition.** *Any  $F$ -linear involution on  $Q$  is either  $\text{can}$  or  $\sigma_q$  for some pure quaternion  $q \in Q^0$ .*

*Proof.* Let  $\sigma$  be an  $F$ -linear involution, then  $\sigma \circ \text{can} = \text{Inn}(q)$  for some  $q$ . Then,  $\sigma^2 = \text{can}^2 = \text{id}$  implies  $\text{can}(q) = \pm q$ . If  $\text{can}(q) = q$ , then  $q \in F$  implies  $\sigma = \text{Inn}(q) \circ \text{can} = \text{can}$ . If  $\text{can}(q) = -q$ , then  $\sigma = \text{Inn}(q) \circ \text{can} = \sigma_q$ .  $\square$

## Isotropy and hyperbolicity

Taking the viewpoint that central simple algebras with involution are generalizations of quadratic forms, we may also extend the properties of isotropy and hyperbolicity to this setting.

A right ideal  $I \subset A$  is called **isotropic** (with respect to the involution  $\sigma$ ) if  $\sigma(x)x = 0$  for all  $x \in I$ . The algebra with involution  $(A, \sigma)$ , or the involution  $\sigma$  itself, is called **isotropic** if  $A$  contains a nonzero isotropic ideal.

An algebra with involution  $(A, \sigma)$ , or the involution  $\sigma$  itself, is called **hyperbolic** if  $A$  contains an isotropic ideal  $I$  of dimension  $\dim_F I = \frac{1}{2} \dim_F A$  (for  $\text{char}(F) \neq 2$ ).

## 5.2.2 The type of an involution

For the split case  $A = M_n(F)$ , we have the obvious involution  $t : A \rightarrow A$  given by the transpose map. The Skolem-Noether theorem implies that any  $F$ -linear involution on

$M_n(F)$  is given by  $\text{ad}_B : M \mapsto B^{-1}M^tB$  for some  $B$  such that  $B^t = \pm B$  (symmetric or skew-symmetric), where  $\text{ad}_B$  is the adjoint of the underlying bilinear map. So, we have a one-to-one correspondence between isomorphism classes of  $F$ -linear involutions and symmetric or skew-symmetric bilinear forms of rank  $n$ , up to similarity.

We say that  $\text{ad}_B$  is **orthogonal** if  $B$  is symmetric (i.e.  $B^t = B$ ) and  $\text{ad}_B$  is **symplectic** if  $B$  is skew-symmetric (i.e.  $B^t = -B$ ). We define the **type of an involution**  $\sigma$  on  $A$  to be the type of  $\sigma_L$  for any  $L/F$  which splits  $A$  (i.e.  $A_L \simeq M_n(L)$ ).

Alternatively, we may characterize the type of an involution as follows. For any involution  $\sigma$  on a central simple algebra  $A$ , we define the sets of symmetric and skew-symmetric elements respectively:

$$\begin{aligned}\text{Sym}(A, \sigma) &= \{x \in A \mid \sigma(x) = x\} \\ \text{Skew}(A, \sigma) &= \{x \in A \mid \sigma(x) = -x\}\end{aligned}$$

For involutions of the first type,  $\text{Sym}(A, \sigma)$  and  $\text{Skew}(A, \sigma)$  are both  $F$ -vector spaces. Furthermore, we have a decomposition  $(A, \sigma) = \text{Sym}(A, \sigma) \oplus \text{Skew}(A, \sigma)$ , due to the fact that every element  $x \in A$  decomposes as  $x = \frac{1}{2}(x + \sigma(x)) + \frac{1}{2}(x - \sigma(x))$ .

**5.2.4 Proposition** ([45, Prop. 2.1]). *Let  $A$  be a central simple  $F$ -algebra of degree  $n$  and let  $\sigma$  be an  $F$ -linear involution on  $A$ .*

1. *If  $\sigma$  is orthogonal, then*

$$\dim_F(\text{Sym}(A, \sigma)) = \frac{n(n+1)}{2} \text{ and } \dim_F(\text{Skew}(A, \sigma)) = \frac{n(n-1)}{2}.$$

2. *If  $\sigma$  is symplectic, then*

$$\dim_F(\text{Sym}(A, \sigma)) = \frac{n(n-1)}{2} \text{ and } \dim_F(\text{Skew}(A, \sigma)) = \frac{n(n+1)}{2}.$$

**5.2.5 Example.** Let  $Q$  be a quaternion  $F$ -algebra and  $\text{can}$  its canonical involution.

$$\begin{aligned}\text{Sym}(Q, \text{can}) &= F & (\dim_F = 1) \\ \text{Skew}(Q, \text{can}) &= Fi \oplus Fj \oplus Fij = Q^0 & (\dim_F = 3).\end{aligned}$$

Thus, by Proposition 5.2.4,  $\text{can}$  is symplectic. Now, let  $\sigma_i := \text{Inn}(i) \circ \text{can}$ .

$$\begin{aligned}\text{Sym}(Q, \sigma_i) &= F \oplus Fj \oplus Fij & (\dim_F = 3) \\ \text{Skew}(Q, \sigma_i) &= Fi & (\dim_F = 1).\end{aligned}$$

In this case, Proposition 5.2.4 shows that  $\sigma_i$  is orthogonal.

In fact, examples 5.2.2 and 5.2.5 can be generalized to central simple algebras of larger degree.

**5.2.6 Proposition** ([29], Prop. 2.7). *Let  $A$  be a central simple algebra over  $F$  and let  $\sigma$  be an  $F$ -linear involution on  $A$ .*

1. *For every unit  $u \in A^\times$  such that  $\sigma(u) = \pm u$ , the map  $\text{Inn}(u) \circ \sigma$  is an  $F$ -linear involution on  $A$ .*
2. *Conversely, for every  $F$ -linear involution  $\tau$  on  $A$ , there exists some  $u \in A^\times$ , uniquely determined up to  $F^\times$ , such that  $\tau = \text{Inn}(u) \circ \sigma$  and  $\sigma(u) = \pm u$ .*
3. *Suppose that  $\tau = \text{Inn}(u) \circ \sigma$  for  $u \in A^\times$  with  $\sigma(u) = \pm u$ . If  $\text{char}(F) \neq 2$ , then  $\tau$  and  $\sigma$  are of the same type if and only if  $\sigma(u) = u$ .*

### 5.2.3 Clifford algebra of an orthogonal involution

In Prop. 4.2.3 we saw that for a quadratic form  $q : V \rightarrow F$ , the even Clifford algebra  $C_0(\lambda q) \simeq C_0(q)$  for all  $\lambda \in F^\times$ . So,  $C_0(q)$  is an invariant of  $\text{ad}_q$ , the adjoint involution on  $\text{End}_F(V)$  with respect to  $b_q$ . This is not true however for the full Clifford algebra.

**5.2.7 Lemma.** *If  $b_q(v, w) = 0$  for some  $v, w \in V$ , then  $vw = -wv \in C(q)$ .*

This follows from the fact that if  $v, w$  are orthogonal, then  $(v + w)^2 = q(v + w) = q(v) + q(w) = v^2 + w^2$ .

**5.2.8 Example.** Let  $q = \langle a_1, a_2, a_3 \rangle$  with an orthogonal basis  $\{e_1, e_2, e_3\}$  such that  $e_i^2 = a_i, i = 1, 2, 3$ . Then  $e_1e_2, e_1e_3 \in C_0(q)$  generate a quaternion algebra, so  $(-a_1a_2, -a_1a_3)_F \simeq C_0(q)$  (by dimension arguments). The canonical involution gives  $\text{can}(e_1e_2) = \text{can}(e_2)\text{can}(e_1) = e_2e_1 = -e_1e_2$ , so the canonical involution here is the same as the canonical involution for the quaternion algebra.

We can expand this notion to a more general case. Let  $(A, \sigma)$  be a central simple algebra with orthogonal involution. Let  $\underline{A}$  denote  $A$  viewed as an  $F$ -vector space. Let  $\mathcal{T}(\underline{A})$  denote the tensor algebra of  $\underline{A}$  (c.f. (4.1)). We denote by  $\mu : \underline{A} \otimes \underline{A} \rightarrow \underline{A}$  the multiplication map  $\mu(\underline{a} \otimes \underline{b}) = \underline{ab}$ , and let  $\mathcal{J}_1(\sigma)$  be the ideal generated by elements of the form  $u - \mu(u)$  for all  $u \in \underline{A} \otimes \underline{A}$  such that  $\sigma \otimes \sigma(u) = u$ . Next, let  $\mathcal{J}_2(\sigma)$  be the ideal generated by elements of the form  $\underline{s} - \text{Trd}_A(s)$  for all  $s \in A$  such that  $\sigma(s) = s$ . Finally, we define the **Clifford algebra of  $(A, \sigma)$**  to be the quotient

$$C(A, \sigma) = \frac{\mathcal{T}(\underline{A})}{\mathcal{J}_1(\sigma) + \mathcal{J}_2(\sigma)}.$$

**5.2.9 Remark.** If  $A$  is split, then  $(A, \sigma)$  is of the form  $\text{Ad}_q := (\text{End}_F(V), \text{ad}_q) = (M_n(F), \text{ad}_B)$  for a symmetric quadratic form  $q$  having matrix  $B$ . In this case,  $C(A, \sigma) = C_0(q)$ , as one would expect.

We may may associate to  $(A, \sigma)$  the algebra with involution  $(C(A, \sigma), \underline{\sigma})$ , where  $C(A, \sigma)$  is the Clifford algebra of  $(A, \sigma)$  and  $\underline{\sigma}$  is the involution on  $\mathcal{T}(\underline{A})$  induced by  $\sigma$ , namely,  $\underline{\sigma}(a_1 \otimes \cdots \otimes a_r) = \underline{\sigma}(a_r) \otimes \cdots \otimes \underline{\sigma}(a_1)$ . We have

- $C((A, \sigma)_L) = (C(A, \sigma), \underline{\sigma})_L$
- $(C(\text{Ad}_q), \underline{\text{ad}}_q) = (C_0(q), \text{can})$

**5.2.10 Theorem** ([29, Prop. 15.1]). *Let  $(A, \sigma)/F$  be a central simple algebra with orthogonal involution. If  $3 \leq \deg(A) \leq 6$ , then  $(A, \sigma)$  is uniquely determined by its Clifford algebra  $(C(A, \sigma), \underline{\sigma})$ .*

## 5.2.4 Decomposability

We say that a central simple  $F$ -algebra  $A$  is **decomposable** if  $A \simeq B \otimes C$  for  $B, C$  central simple  $F$ -algebras. If  $\deg(A) = 2^s$  and  $A \simeq \bigotimes_{i=1}^s Q_i$  for  $Q_i$  quaternion, we say that  $A$  is **totally decomposable**.

**5.2.11 Example.** If  $\deg(A) = 4$  and  $\exp(A) = 2$ ,  $A \simeq Q_1 \otimes Q_2$ , i.e.  $A$  is a biquaternion algebra. If  $\deg(A) = 8$  and  $\exp(A) = 2$ , then  $A$  is decomposable if and only if  $A \simeq Q_1 \otimes Q_2 \otimes Q_3$ . In this case, we say that  $A$  is a **triquaternion algebra**.

The notion of decomposability can be extended to algebras with involution, and is convenient for the classification of involutions on algebras of small degree. Let  $(A, \sigma_A)$  and  $(B, \sigma_B)$  be central simple  $F$ -algebras with involution. We say that  $(A, \sigma_A)$  is **decomposable** as an algebra with involution and write  $(A, \sigma_A) \supset (B, \sigma_B)$  if  $A$  contains a  $\sigma_A$ -stable subalgebra isomorphic to  $B$ , and over which the involution induced by  $\sigma_A$  is conjugate to  $\sigma_B$ . By the double centralizer theorem,  $(A, \sigma_A) \supset (B, \sigma_B) \iff (A, \sigma_A) \simeq (B, \sigma_B) \otimes (C, \sigma_C)$  for some central simple  $F$ -algebra with involution  $(C, \sigma_C)$ .

If  $\deg(A) = 2^s$ , we say that  $(A, \sigma)$  is **totally decomposable** as an algebra with involution if  $(A, \sigma) \simeq \prod_{i=1}^s (Q_i, \sigma_i)$  with  $Q_i$  quaternion and  $\sigma_i$  an  $F$ -linear involution.

## 5.2.5 Classification

### Degree 2

In example 5.2.5 we saw that the canonical involution  $can$  on a quaternion algebra  $Q/F$  is of symplectic type. It follows from Proposition 5.2.6 that  $can$  is the unique symplectic involution on  $Q$ .

Let  $\sigma$  be an orthogonal involution on an even degree algebra  $A/F$ , then  $Z(C(A, \sigma)) = F[X]/(X^2 - \delta)$  for some  $\delta \in F^\times/F^{\times 2}$ , called the **discriminant** of  $\sigma$ , denoted by  $\delta =: \text{disc}(\sigma)$ . Define  $\text{disc}(\sigma_L)$  to be the image of  $\delta$  under the map  $F^\times/F^{\times 2} \rightarrow L^\times/L^{\times 2}$ .

**5.2.12 Proposition.** *Two orthogonal involutions on a quaternion algebra are isomorphic if and only if they have the same discriminant.*

*Proof.* Let  $\sigma_{q_1}, \sigma_{q_2}$  be orthogonal involutions on a quaternion  $F$ -algebra  $Q$ . By Prop. 5.2.3,  $\sigma_{q_i} = \text{Inn}(q_i) \circ can$ , and since  $q_1, q_2 \in Q^0$ , so  $\text{disc}(\sigma_{q_i}) = q_i^2$ . Thus, it remains to show that  $q_1^2 = q_2^2$  implies  $\sigma_{q_1} \simeq \sigma_{q_2}$ . If  $q_1^2 = q_2^2$ , then  $F(q_1) \simeq F(q_2) \hookrightarrow Q$  is an algebra homomorphism sending  $q_1 \mapsto q_2$ . By Theorem 5.1.14, there exists  $x \in Q$  such that  $xq_1x^{-1} = q_2$ . We may write this as  $q_2 = xq_1x^{-1}can(x)^{-1}can(x)$ . Then, using the fact that for all  $x \in Q, x \cdot can(x) = \text{Nrd}_Q(x)$ , we see that  $q_2 = xq_1 \cdot can(x) \cdot \text{Nrd}_Q(x)^{-1}$ , and hence  $\sigma_{q_2} \simeq \sigma_{q_1}$ .  $\square$

### Degree 3 (odd degree in general)

Let  $\sigma$  be an  $F$ -linear involution on a central simple algebra  $A$  of odd degree. In this case,  $\sigma$  must be of orthogonal type and  $A$  must be split. The first part of this statement follows from the fact that every alternating form on a vector space of odd dimension is singular. The second is a consequence of the fact that  $\text{ind}(A)$  must be a power of 2, and that  $\text{ind}(A) \mid \text{deg}(A)$  (cf.[29, 2.8]).

We have one-to-one correspondences between the following:

- $\{(A, \sigma) \text{ with } \text{deg}(A) = 2n + 1 \text{ and } \sigma \text{ orthogonal involution}\} / \text{isomorphism}$
- $\{q \text{ quadratic form with } \text{dim}(q) = 2n + 1\} / \text{similarity}$
- $\{q' \text{ quadratic form with } \text{dim}(q') = 2n + 1 \text{ and } \text{disc}(q') = 1\} / \text{isomorphism}$

### Degree 4

We consider first the case of symplectic involutions. In this situation we have a correspondence between algebras of degree 4 with symplectic involution and isometry classes of quadratic forms having dimension 5 and discriminant 1.

For an algebra  $(A, \tau)$  of degree 4 with symplectic involution,  $\dim_F \text{Sym}(A, \tau) = 6$ . Then,

$$A_\tau^0 := \{x \in \text{Sym}(A, \tau) \mid \text{Trd}_A(x) = 0\}$$

has dimension 5.

**5.2.13 Proposition.** *For all  $x \in A_\tau^0$ ,  $x^2 \in F$ , so  $s_\tau : A_\tau^0 \rightarrow F$ , defined by  $x \mapsto x^2$  is a quadratic form of dimension 5.*

**5.2.14 Theorem (Albert).** *Let  $A$  be a central simple  $F$ -algebra of degree 4, exponent 2.*

1.  *$A$  is isomorphic to a tensor product of 2 quaternion algebras,  $A \simeq Q_1 \otimes Q_2$ .*
2. *For any symplectic involution  $\tau$ ,  $(A, \tau)$  also decomposes as  $(Q_1, \text{can}) \otimes (Q_2, \text{can})$ , ie.  $(A, \tau)$  is totally decomposable.*

Turning next to orthogonal involutions, we have a correspondence between algebras of degree 4 with orthogonal involution and quaternion algebras over a quadratic étale extension of  $F$ . Consider  $K/F$  a quadratic étale extension with Galois group  $\{1, \iota\}$ . For a quaternion  $K$ -algebra  $Q$ , we define  ${}^tQ$  by setting  ${}^tQ \simeq Q$  as a ring, with scalar multiplication given by  $\lambda \cdot {}^t a = \iota(\lambda) \cdot a$ . We define the **norm**  $N_{K/F}(Q) := (Q \otimes_K {}^tQ)^{sw}$  where  $sw({}^t a \otimes b) := {}^t b \otimes a$  is the **switch map**. Note that in the special case that  $K = F \times F$ ,  $Q = Q_1 \times Q_2$  with  $Q_i$  a quaternion  $F$ -algebra. Then  $N_{Z/F}(Q) = Q_1 \otimes Q_2$ . The tensor product  ${}^t \text{can} \otimes \text{can}$  induces an involution on  $N_{K/F}(Q)$ , giving an algebra of degree 4 with orthogonal involution.

**5.2.15 Theorem ([30]).** *Let  $(A, \sigma)$  be a degree 4 algebra with orthogonal involution. Then  $(A, \sigma)$  decomposes as a product of quaternion algebras if and only if  $\text{disc}(\sigma) = 1$ .*

In fact, there is a unique decomposition as  $(Q_1, \tau_1) \otimes (Q_2, \tau_2)$  with  $\tau_1, \tau_2$  symplectic, which is obtained by taking  $Q_1, Q_2$  to be the components of  $C(A, \sigma)$ . From this, we may construct many decompositions of the form  $(H_1, \sigma_1) \otimes (H_2, \sigma_2)$ , for  $\sigma_1, \sigma_2$  orthogonal. These are obtained through twisting  $\tau_1$  and  $\tau_2$  by different elements.

### Degree 8 (trialitarian triples)

Let  $(A, \sigma)$  be an algebra of degree 8 with orthogonal involution  $\sigma$  having  $\text{disc}(\sigma) = 1$ . In this case, we have  $C(A, \sigma) = (C_+, \sigma_+) \times (C_-, \sigma_-)$ , where  $C_+, C_-$  are both of degree 8 and  $\sigma_+, \sigma_-$  are both orthogonal involutions. A triple  $((A, \sigma_A), (B, \sigma_B), (C, \sigma_C))$  of degree 8 algebras with orthogonal involutions is called **trialitarian** if  $C(A, \sigma_A) = (B, \sigma_B) \times (C, \sigma_C)$ .

**5.2.16 Theorem** ([29, 42.A]). *If  $((A, \sigma_A), (B, \sigma_B), (C, \sigma_C))$  is a trialitarian triple, then the Clifford algebra of any element of the triple is the direct product of the other two.*

Algebras with orthogonal involution having trivial discriminant satisfy conditions called the **fundamental relations**. These are relations in the Brauer group, the origin of which we will explore in Chapter 7 (see Example 7.3.3).

Let  $A$  be an algebra with orthogonal involution  $\sigma$ . Suppose  $\text{disc}(\sigma) = 1$  and denote by  $C_+, C_-$  the two components of the even Clifford algebra of  $(A, \sigma)$ . The fundamental relations in the case  $\text{deg}(A) \equiv 0 \pmod{4}$  are given by:

$$2[A] = 2[C_+] = 2[C_-] = 0 \quad (5.3)$$

$$[A] + [C_+] + [C_-] = 0. \quad (5.4)$$

So if  $((A, \sigma_A), (B, \sigma_B), (C, \sigma_C))$  is a trialitarian triple, then  $[A] + [B] + [C] = 0 \in \text{Br}(F)$ .

By the fundamental relations defined above, we see immediately that a trialitarian triple containing a split component can be described very explicitly.

**5.2.17 Lemma.** *Let  $((A, \sigma_A), (B, \sigma_B), (C, \sigma_C))$  be a trialitarian triple. If  $A \sim 0$ , then  $(B, \sigma_B) \simeq (C, \sigma_C)$ .*

*Proof.*  $A \sim 0$  implies  $\sigma_A = \text{ad}_q$ . Then  $(C_0(q), \text{can}) = (B, \sigma_B) \times (C, \sigma_C)$ , and we have an explicit description of  $C_0(q)$ : If  $q = \langle a_1, \dots, a_7 \rangle$ , then  $C_0(q) = Q_1 \otimes Q_2 \otimes Q_3 \otimes (F \times F)$  with an involution defined by  $\text{can} \otimes \sigma_2 \otimes \text{can}$ .  $\square$

So, a trialitarian triple with a split component is of the form  $(\text{Ad}_q, (B, \sigma_B), (B, \sigma_B))$ , where  $\text{Ad}_q := (\text{End}_F(V), \text{ad}_q)$ . Furthermore, a trialitarian triple with two split components must be of the form  $(\text{Ad}_\pi, \text{Ad}_\pi, \text{Ad}_\pi)$ , for  $\pi$  a 3-fold Pfister form (i.e.  $\pi \simeq \langle 1, a_1 \rangle \otimes \langle 1, a_2 \rangle \otimes \langle 1, a_3 \rangle$  for some  $a_1, a_2, a_3 \in F$ ).

**5.2.18 Theorem** ([29] Thm. 42.11). *Let  $(A, \sigma)$  be an algebra of degree 8 with orthogonal involution  $\sigma$ . Then,  $(A, \sigma)$  is totally decomposable if and only if  $\text{disc}(\sigma) = 1$  and  $C_0(A, \sigma)$  has a split component.*

**5.2.19 Remark.** Algebras of degree 8 and exponent 2 are not always decomposable. In [2], Amitsur, Rowen and Tignol produced an example of an algebra  $A$  of degree 8, exponent 2, such that  $A \supset F(\sqrt{a}, \sqrt{b}, \sqrt{c})$  but it does not contain any quaternion subalgebra.

# Chapter 6

## Linear algebraic groups

In this chapter we recall the definition of a linear algebraic group, which can be described as an algebraic variety endowed with an additional group structure. We describe the classification of these groups via root systems and Dynkin diagrams. The main references for this chapter are [19] and [29].

### 6.1 Definitions and Examples

Let  $F$  be an arbitrary field, and set  $F^n = F \times \cdots \times F = \mathbb{A}^n$ . We will denote by  $\bar{F}$  the algebraic closure of  $F$ . A **linear algebraic group** over  $F$  is an affine algebraic variety  $G$  over  $F$  equipped with two morphisms:

- $i : G \times G \rightarrow G$ , defined by  $(x, y) \mapsto xy$  (multiplication)
- $j : G \rightarrow G$ , defined by  $x \mapsto x^{-1}$  (inverse)

subject to the traditional group laws.

**6.1.1 Remark.** As described in Section 2.1.1, we use the notation  $G(K)$  or  $G_K$  to denote the set of  $K$ -points of  $G$  as an algebraic variety, for any field extension  $K/F$ .

Given a field extension  $K/F$ , every point  $g \in G(K)$  gives rise to a morphism  $\varphi_g : G(K) \rightarrow G(K)$ , defined by  $x \mapsto xg$  (translation), which is an isomorphism of algebraic varieties. Note that all geometric properties which are valid at one point will be valid at all points.

**6.1.2 Proposition** ([6, Prop. I.1.2]). *An algebraic group  $G$  over a field  $F$  is smooth as a variety.*

**6.1.3 Remark.** The notion of smoothness of an algebraic group can also be defined in a more subtle way, for example, we may say that  $G$  is smooth if and only if  $G_{\bar{F}}$  is *regular* (c.f. [34, 7.4]). By Cartier's theorem ([9]), if  $\text{char}(F) = 0$ , then all algebraic groups over  $F$  are smooth in this sense as well. Throughout this thesis, we will assume that all linear algebraic groups are smooth with respect to both definitions. While it may be needlessly cautious, we will emphasize the smoothness requirement where it is absolutely necessary.

A morphism of algebraic groups is defined to be a morphism of varieties which is also a homomorphism of groups. Two groups  $G, G'$  are **isomorphic** as linear algebraic groups if there exists an isomorphism of varieties  $\varphi : G \rightarrow G'$  such that  $\varphi(g_1 g_2) = \varphi(g_1) \varphi(g_2)$ . Note that a bijective morphism is not enough, an inverse is required in order to have an isomorphism of varieties.

**6.1.4 Example.** Examples of linear algebraic groups include:

1. The **multiplicative group**  $\mathbb{G}_m$ , with  $\mathbb{G}_m(K) = K^\times = K \setminus \{0\} \leftrightarrow \{(x, y) \in \mathbb{A}_K^2 \mid xy = 1\}$  for any field extension  $K/F$ .
2. The **additive group**  $\mathbb{G}_a$ , with  $\mathbb{G}_a(K) = \mathbb{A}_K^1$  for any field extension  $K/F$ . The group operation on  $\mathbb{G}_a(K)$  is given by  $(x, y) \mapsto x + y$ .
3. The **general linear group**  $\text{GL}_n$ , with  $\text{GL}_n(K) = \{A \in M_n(K) \mid \det(A) \in F^\times\}$  for any field extension  $K/F$ . The operation on  $\text{GL}_n(K)$  is given by matrix multiplication.
4. The direct product  $G_1 \times G_2$  of linear algebraic groups.
5. In the Zariski topology, every closed subgroup of  $G$  is also an algebraic group.

Suppose we break down  $G$  into its irreducible components  $G_1, \dots, G_i$  as an affine algebraic variety, i.e.  $G = G_1 \cup \dots \cup G_i$ . The identity element  $e \in G$  lives in a unique irreducible component of  $G$ , which we call the **connected component** of  $G$ , and denote by  $G^0$ .  $G^0$  is a normal closed subgroup of  $G$  of finite index. In general, we say that  $G$  is **connected** if it is irreducible as an affine algebraic variety. A connected non-trivial linear algebraic group is called **simple** if it contains no non-trivial closed connected normal subgroups over the algebraic closure  $\bar{F}$  of  $F$ .  $G$  is called **semisimple** if  $G \neq 1$ ,  $G$  is connected, and  $G_{\bar{F}}$  has no nontrivial solvable connected normal subgroups. Even more generally, we say that  $G$  is **reductive** if the unipotent radical of  $G_{\bar{F}}$  is trivial. All simple and semisimple groups are reductive, but the converse does not hold. For example,  $\text{GL}_n$  is reductive, but not simple nor semisimple.

A linear algebraic group  $T$  is called a **torus** of rank  $n$  if it becomes isomorphic to  $n$  copies of  $\mathbb{G}_m$  over an algebraic closure of  $F$ . That is,  $T_{\bar{F}} \simeq \mathbb{G}_{m\bar{F}} \times \dots \times \mathbb{G}_{m\bar{F}}$ . If this

isomorphism is defined over the base field  $F$ , we say that  $T$  is **split**. A reductive group  $G$  is then called **split** if there exists a split torus  $T$  which is contained in  $G$  as a subgroup, and which is maximal with respect to inclusion among tori contained in  $G$ . We also use the notion of a maximal torus to define the rank of  $G$ : We say that  $G$  has rank  $n$  if  $G_{\bar{F}}$  contains a split maximal torus of rank  $n$ .

**6.1.5 Example.** For any  $n \geq 1$ ,  $\mathrm{GL}_n$  is a split linear algebraic group, containing the split maximal torus  $D_n$  of diagonal  $n \times n$  matrices with nonzero determinant. On the other hand, suppose  $A$  is a nontrivial central division  $F$ -algebra. Then  $\mathrm{Aut}(A) = \mathrm{PGL}(A)$  is a non-split linear algebraic group over  $F$ .

**6.1.6 Remark.** It turns out that  $G$  is a linear algebraic group over  $F$  if and only if  $G$  is a closed subgroup of  $\mathrm{GL}(n, F)$  for some  $n$ . This explains the reasoning behind the term “linear algebraic group” and also shows that over an algebraically closed field, all linear algebraic groups are necessarily split.

## 6.2 Root systems

In this section we introduce the machinery needed for the classification of linear algebraic groups. In particular we recall definitions from the theory of Lie algebras, such as roots and weights. The results presented in this section can be found in [20].

### 6.2.1 Definition of a root system

Throughout this section, we will fix a finite-dimensional  $\mathbb{R}$ -vector space  $V$  together with a positive definite symmetric bilinear form  $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ . For a fixed vector  $v \in V$ , we define the **reflection**  $s_v : V \rightarrow V$  to be a linear map sending  $w \mapsto w - 2\frac{(w,v)}{(v,v)} \cdot v$  for any  $w \in V$ . We note that  $s_v$  sends  $v \mapsto -v$  and fixes all points in the **reflecting hyperplane**  $P_v := \{w \in V \mid (w, v) = 0\}$ .

A subset  $\Phi \subset V$  is called a (reduced) **root system** in  $V$  if it satisfies the following axioms:

- (R1)  $\Phi$  is finite, spans  $V$  and does not contain 0.
- (R2) If  $\alpha \in \Phi$ , then  $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ .
- (R3) If  $\alpha, \beta \in \Phi$ , then  $s_\alpha(\beta) \in \Phi$ .
- (R4) If  $\alpha, \beta \in \Phi$ , then  $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

Let  $\Phi$  be a root system in  $V$ . We denote by  $W(\Phi)$  the subgroup of  $GL(V)$  generated by the reflections  $s_\alpha$  for  $\alpha \in \Phi$ . By (R3),  $W(\Phi)$  permutes  $\Phi$ , and so we may identify  $W(\Phi)$  with a subgroup of the symmetric group on  $\Phi$ . In particular, since  $\Phi$  is finite by (R1),  $W(\Phi)$  is also finite. The group  $W(\Phi)$  is called the **Weyl group** of  $\Phi$ .

We define the **rank**  $\text{rank}(\Phi)$  of a root system  $\Phi$  to be the dimension of the  $\mathbb{R}$ -vector space  $V$ . A subset  $\Pi \in \Phi$  is called a **base** of  $\Phi$  if:

- (B1)  $\Pi$  is a basis of  $V$
- (B2) Each root  $\beta \in \Phi$  can be written as  $\beta = \sum_{\alpha \in \Pi} k_\alpha \alpha$  with integer coefficients  $k_\alpha$  all nonnegative or all nonpositive.

The elements of  $\Pi$  are called **simple roots**. Since  $\Pi$  is a basis of  $V$ ,  $|\Pi| = \text{rank}(\Phi)$ , and the expression in (B2) for  $\beta$  is unique. If  $\Pi = \{\alpha_1, \dots, \alpha_n\}$ , then the corresponding reflection  $s_i := s_{\alpha_i}$  generate  $W(\Phi)$ , and are called **simple reflections**.

### 6.2.2 Dynkin diagrams

The root system axiom (R4) severely limits the possible angles  $\theta$  found between a pair of roots  $\alpha, \beta \in \Phi$ . In fact, if  $\alpha \neq \pm\beta$  we must have  $\theta \in \{\pi/2, \pi/3, 2\pi/3, \pi/4, 3\pi/4, \pi/6, 5\pi/6\}$ . We may use this classification to construct a graph called a **Dynkin diagram**  $\mathcal{D}(\Phi)$ :

1. Draw  $\text{rank}(\Phi)$  vertices, in one-to-one correspondence with the simple roots of  $\Pi$ .
2. Given  $\alpha_i, \alpha_j \in \Pi$ ,  $i \neq j$ , connect the corresponding vertices with the following number of edges:
  - (a) 1 if  $\theta = 2\pi/3$
  - (b) 2 if  $\theta = 3\pi/4$
  - (c) 3 if  $\theta = 5\pi/6$
  - (d) 0 otherwise
3. If the number of edges between  $\alpha_i, \alpha_j$  is greater than 1, add an arrow pointing towards  $\alpha_i$  if  $(\alpha_i, \alpha_i) < (\alpha_j, \alpha_j)$  and towards  $\alpha_j$  otherwise.

The Dynkin diagram  $\mathcal{D}(\Phi)$  does not depend on the choice of  $\Pi \subset \Phi$  and determines  $\Phi$  uniquely. In fact, we can recover the Weyl group  $W(\Phi)$  from  $\mathcal{D}(\Phi)$  in the following way: Let  $n_{ij}$  denote the number of edges between vertices  $\alpha_i, \alpha_j$ , for  $i \neq j$ . Then,

$$W(\Phi) = \{s_1, \dots, s_n \mid (s_i)^2 = 1, (s_i s_j)^{n_{ij}+2} = 1 \text{ if } n_{ij} < 3 \text{ and } (s_i s_j)^6 = 1 \text{ if } n_{ij} = 3\}.$$

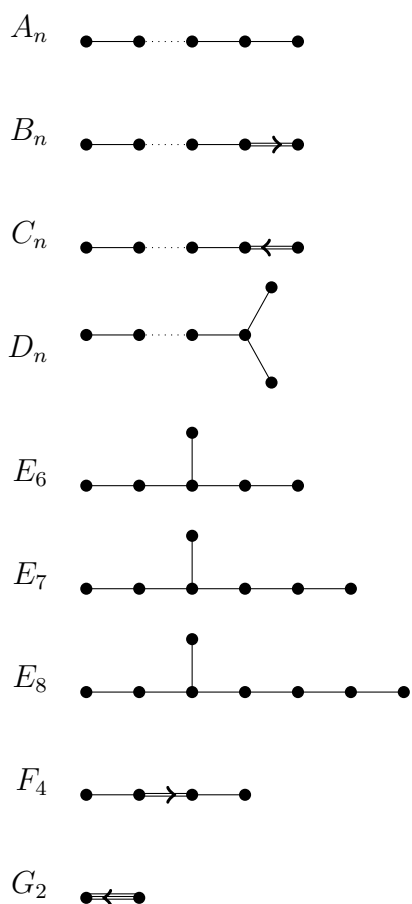
A root system  $\Phi$  is called **irreducible** if it cannot be partitioned into the union of two proper subsets  $\Phi_1, \Phi_2$  such that  $(\alpha_1, \alpha_2) = 0$  for all  $\alpha_1 \in \Phi_1, \alpha_2 \in \Phi_2$ . It follows that  $\Phi$  is irreducible if and only if any base  $\Pi \subset \Phi$  cannot be partitioned in the same way (cf. [20, 10.4]).

By the definition of the Dynkin diagram, it is clear that  $\Phi$  is irreducible if and only if  $\mathcal{D}(\Phi)$  is connected (in the graph-theoretical sense). Suppose  $\mathcal{D}(\Phi)$  is not connected, and let  $\Pi = \Pi_1 \cup \dots \cup \Pi_t$  be a partition of  $\Pi$  corresponding to the connected components of  $\mathcal{D}(\Phi)$ . The  $\Pi_i$  are thus pairwise orthogonal, and we have  $V = \text{Span}(\Pi_1) \oplus \dots \oplus \text{Span}(\Pi_t)$ . The  $\mathbb{Z}$ -linear combinations of  $\Pi_i$  which are roots form a root system in  $V_i := \text{Span}(\Pi_i)$ , which we denote by  $\Phi_i$ . Each vector subspace  $V_i$  is  $W(\Phi)$ -invariant and so each root  $\alpha \in \Phi$  lies in precisely one of the  $V_i$ .

**6.2.1 Proposition** ([20, Prop. 11.3]). *A root system  $\Phi$  in  $V$  decomposes (uniquely) as the unions of irreducible root systems  $\Phi_i$  in subspaces  $V_i$  of  $V$  such that  $V = V_1 \oplus \dots \oplus V_t$ .*

The upshot of introducing Dynkin diagrams is that they allow for a complete classification of irreducible root systems. By Prop. 6.2.1, this classification is then enough to classify root systems in general.

**6.2.2 Theorem** ([20, 11.4]). *If  $\Phi$  is an irreducible root system of rank  $n$ , its Dynkin diagram is one of the following:*



**6.2.3 Remark.** We refer to Dynkin diagrams of type  $A_n, B_n, C_n, D_n$  as **classical**, while Dynkin diagrams of type  $E_6, E_7, E_8, F_4, G_2$  are said to be **exceptional**.

### 6.2.3 Root and weight lattices

Let  $\Phi$  be a root system in  $V$ . For each  $\alpha \in \Phi$ , there is a linear map  $\alpha^\vee : V \rightarrow \mathbb{R}$  defined by  $v \mapsto 2\frac{(v, \alpha)}{(\alpha, \alpha)}$ . This map is called the **coroot** of  $\alpha$  and we denote by  $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$  the set of coroots of  $\Phi$ . The set of coroots forms a root system in the dual vector space  $V^* := \text{Hom}(V, \mathbb{R})$ , which we call the **dual** root system of  $\Phi$ .

We define a **weight** of  $\Phi$  to be a vector  $\lambda \in V$  such that  $\alpha^\vee(\lambda) \in \mathbb{Z}$  for all  $\alpha \in \Phi$ . We denote by  $\Lambda$  the set of all weights of  $\Phi$ . Since the coroot map is linear,  $\Lambda$  is a lattice in  $V$  which contains  $\Phi$ . We call  $\Lambda$  the **weight lattice** of  $\Phi$ .

Similarly, we may define another lattice in  $V$  called the **root lattice** of  $\Phi$ , which is denoted by  $\Lambda_r$ , and is generated by  $\Phi$  (equivalently, is generated by  $\Pi$ ). By definition,

$\Lambda_r \subset \Lambda$ , and their quotient  $\Lambda/\Lambda_r$  is a finite abelian group called the **fundamental group** (cf. [20, 13.1]).

**6.2.4 Proposition** ([19, A.8]). *The fundamental group  $\Lambda/\Lambda_r$  for an irreducible root system has the following form:*

- $\mathbb{Z}/(n+1)\mathbb{Z}$  for type  $A_n$
- $\mathbb{Z}/2\mathbb{Z}$  for types  $B_n, C_n, E_7$
- $\mathbb{Z}/4\mathbb{Z}$  for type  $D_n, n$  odd
- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  for type  $D_n, n$  even
- $\mathbb{Z}/3\mathbb{Z}$  for type  $E_6$
- $0$  for types  $E_8, F_4, G_2$

## 6.3 Classification of split simple algebraic groups

In this section, we associate a root system to a split simple linear algebraic group  $G$  over an arbitrary field  $F$ , thus providing a classification of split simple linear algebraic groups. Results presented here can be found in [29] or [35].

### 6.3.1 The Lie algebra of an algebraic group

Let  $F[\varepsilon] := F[X]/(X^2)$  be the ring of dual numbers. We have a  $F$ -algebra homomorphisms  $F \rightarrow F[\varepsilon] \rightarrow F$  defined by sending  $a \mapsto a + 0\varepsilon$  in the first map and then  $a + b\varepsilon \mapsto a$  in the second. For a reductive group  $G$  over  $F$ , we have corresponding morphisms  $G \rightarrow G_{F[\varepsilon]} \rightarrow G$ . We define the **Lie algebra**  $\mathfrak{g}$  of  $G$  to be the kernel of the second map,  $\mathfrak{g} := \ker(G_{F[\varepsilon]} \rightarrow G)$ .

**6.3.1 Example.** Suppose  $G = \mathrm{GL}(n, F)$ . Then,

$$G_{F[\varepsilon]} = \mathrm{GL}(n, F[\varepsilon]) = \{A + B\varepsilon \mid A \in \mathrm{GL}(n, F), B \in M_n(F)\}.$$

Thus,  $\mathfrak{g} = \{I_n + B\varepsilon \mid B \in M_n(F)\} \simeq M_n(F)$ . We follow the convention that for an  $F$ -vector space  $V$ , the Lie algebra of  $\mathrm{GL}(V)$  is denoted by  $\mathfrak{gl}_V$ .

Let  $V$  be an  $F$ -vector space and let  $V[\varepsilon] := F[\varepsilon] \otimes V \simeq V \oplus V\varepsilon$ . Then,

$$\mathrm{End}_{F[\varepsilon]}(V[\varepsilon]) = \{f + g\varepsilon \mid f, g \in \mathrm{End}_F(V)\},$$

where  $f + g\varepsilon$  acts on  $V[\varepsilon]$  by

$$(f + g\varepsilon)(x + y\varepsilon) = f(x) + (f(y) + g(x))\varepsilon.$$

With this, we have

$$\mathrm{GL}_V(F[\varepsilon]) = \mathrm{GL}(V[\varepsilon]) = \{f + g\varepsilon \mid f \in \mathrm{Aut}(V), g \in \mathrm{End}(V)\},$$

and so  $\mathfrak{gl}_V = \{\mathrm{id}_V + g\varepsilon \mid g \in \mathrm{End}(V)\} \simeq \mathrm{End}(V)$ .

The assignment  $G \mapsto \mathfrak{g}$  is a functor, allowing us to use this construction when considering field extensions of  $F$ . A representation  $r : G \rightarrow \mathrm{GL}_V$  of  $G$  on  $V$  defines a homomorphism  $dr : \mathfrak{g} \rightarrow \mathrm{End}(V)$  by sending an element  $x \in G_{F[\varepsilon]}$  to  $r(\varepsilon) \in \mathrm{Aut}(V[\varepsilon])$ . If  $x \in \mathfrak{g}$ , then  $r(x) = \mathrm{id}_V + g\varepsilon$  for some  $g \in \mathrm{End}(V)$  and  $dr(x) = g$ .

Conjugation defines an action of  $G(F[\varepsilon])$  on itself which preserves  $\mathfrak{g}$  and hence induces an action of  $G(F[\varepsilon]) \times \mathfrak{g} \rightarrow \mathfrak{g}$  which can be extended to an action  $G(A) \times \mathfrak{g}_A \rightarrow \mathfrak{g}_A$  for any  $F$ -algebra  $A$ . Thus, we have a representation  $\mathrm{Ad} : G \rightarrow \mathrm{GL}_{\mathfrak{g}}$  of  $G$  on the vector space  $\mathfrak{g}$ .

**6.3.2 Example.** Suppose  $G = \mathrm{GL}(V)$ . The action of  $G(A)$  on  $\mathfrak{g}_A$  is given by

$$(x, y) \mapsto x \circ y \circ x^{-1} : \mathrm{GL}(V_A) \times \mathrm{End}(V_A) \rightarrow \mathrm{End}(V_A).$$

**6.3.3 Proposition** ([35, 2.5]). *Let  $H$  be an algebraic subgroup of a connected algebraic group  $G$  and let  $\mathfrak{h}$  be the Lie algebra of  $H$ . If  $\mathfrak{h} = \mathfrak{g}$  then  $H = G$ .*

### 6.3.2 The root system of a split semisimple algebraic group

We assume throughout this section that  $G$  is a semisimple group over  $F$ . We begin by fixing a split maximal torus  $T \subset G$ , and denoting by  $T^* := \mathrm{Hom}(T, \mathbb{G}_m)$  the **character group** of  $T$ . Since  $T$  is split,  $T^* \simeq \mathbb{Z}^n$ , where  $n$  is the rank of  $G$ . For  $x \in T$ ,  $\mathrm{Inn}(x)$  is an automorphism of  $G$ , and so  $\mathrm{Ad}(x)$  is an automorphism of the associated Lie algebra  $\mathfrak{g}$ . Now, since  $\mathrm{Ad}$  is a morphism of algebraic groups,  $\mathrm{Ad}(T)$  is a diagonalizable subgroup of  $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ . For a character  $\alpha \in T^*$ , we define a subspace of  $\mathfrak{g}$  by

$$\mathfrak{g}_\alpha := \{v \in \mathfrak{g} \mid \mathrm{Ad}(t)v = \alpha(t) \cdot v \text{ for all } t \in T\}.$$

If  $\mathfrak{g}_\alpha \neq 0$ , then  $\alpha$  is called a **root** of  $G$  with respect to  $T$ . We thus obtain a decomposition  $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\chi} \mathfrak{g}_\chi$ , where  $\mathfrak{g}_0$  where  $\chi$  runs through the roots of  $G$  (with respect to  $T$ ). The roots form a finite subset of  $T^*$  which we denote by  $\Phi(G, T)$ .

**6.3.4 Example.** Let  $G = \mathrm{SL}(2, F)$  and define  $T$  to be the set of diagonal matrices of the form  $\mathrm{diag}(x, x^{-1})$ ,  $x \in F^\times$ . Then  $T^* = \mathbb{Z}\chi$  where  $\chi$  is the character sending  $\mathrm{diag}(x, x^{-1}) \mapsto x$ . The Lie algebra is given by  $\mathfrak{g} = \{A \in M_2(F) \mid \mathrm{tr}(A) = 0\}$  and  $T$  acts on  $\mathfrak{g}$  by conjugation. The roots are given by  $\alpha = 2\chi$  and  $-\alpha = -2\chi$ .

**6.3.5 Theorem** ([29, 25.1]). *For a split semisimple linear algebraic group  $G$  and split maximal torus  $T$ , the set  $\Phi(G, T) \subset T^*$  satisfies axioms (R1)-(R4) and thus defines a root system in  $T^*$ .*

We define the **Weyl group**  $W(G, T)$  of  $G$  with respect to  $T$  to be the quotient  $W(G, T) := N_G(T)/T$ , where  $N_G(T)$  is the normalizer of  $T$  in  $G$ . For a field extension  $K/F$ , we define  $W(G_K, T_K) := N_{G_K}(T_K)/T_K$ . Note that  $W(G_K, T_K) = W(G, T)$  for any field extension, in particular,  $W(G_{\bar{F}}, T_{\bar{F}}) = W(G, T)$ . Thus,  $W(G, T) = W(\Phi(G, T))$  in the sense of an abstract root system.

**6.3.6 Remark.** If we consider the root lattice  $\Lambda_r$  and weight lattice  $\Lambda$  of  $\Phi(G, T)$  as an abstract root system, then we have inclusions  $\Lambda_r \subseteq T^* \subseteq \Lambda$ .

We have seen in Theorem 6.2.2 that all root systems can be classified by their Dynkin diagrams. Thus, we can extend this classification to split simple linear algebraic groups by using the root system structure described above.

**6.3.7 Theorem.** *If  $T$  and  $T'$  are split maximal tori for a simple linear algebraic group  $G$  over  $F$ , then  $T$  and  $T'$  are conjugate by an element of  $G$ .*

**6.3.8 Example.** Let  $G = \mathrm{SL}_2$ . If  $T$  is a split maximal torus in  $G$ , then there is a basis of  $V$  under which  $T(F) = D_n(F)$ , the set of diagonal matrices with entries in  $F$ . The result follows, since any two bases of  $V$  are conjugate by an element in  $\mathrm{GL}(V)$ .

This result is one of the cornerstones of classification of semisimple linear algebraic groups, and shows that the root system associated to  $G$  depends only on the group  $G$  itself, and not on the choice of split maximal torus. Furthermore, the following result shows that over a given field  $F$ , there is precisely one split simple linear algebraic group (up to isomorphism) for each irreducible root system.

**6.3.9 Theorem** ([29, 25.5]). *Each irreducible root system described in Theorem 6.2.2 occurs as a root system of a split simple linear algebraic group. Furthermore, every isomorphism of root systems  $\Phi(G, T) \rightarrow \Phi(G', T')$  arises from an isomorphism of linear algebraic groups  $G \rightarrow G'$  which sends  $T \rightarrow T'$ .*

In fact, given an irreducible abstract root system  $\Phi$  and an intermediate lattice  $\Lambda_r \subseteq \Lambda$ , there exists a split simple linear algebraic group  $G$  and split maximal torus  $T$  such that  $\Phi(G) \simeq \Phi$  and  $T^* \simeq \Lambda$ . If  $T^* \simeq \Lambda_r$ , then  $G$  is called **adjoint** and denoted  $G^{\text{ad}}$ . If  $T^* \simeq \Lambda$ ,  $G$  is called **simply connected** and denoted  $G^{\text{sc}}$ .

A surjective morphism  $\varphi : H \rightarrow G$  of semisimple linear algebraic groups over  $F$  is called an **isogeny** if the kernel of  $\varphi_K : H_K \rightarrow G_K$  is finite for any field extension  $K/F$ . Two semisimple linear algebraic groups  $G, G'$  over  $F$  are called **isogenous** if there exists a linear algebraic group  $H$  over  $F$  and isogenies  $\varphi : H \rightarrow G, \varphi' : H \rightarrow G'$ .

**6.3.10 Remark.** Two split linear algebraic groups over  $F$  are isogenous if and only if they have the same Dynkin diagrams. For any split simple linear algebraic group  $G$ , there exists a simply connected group  $G^{\text{sc}}$  and an adjoint group  $G^{\text{ad}}$  with isogenies  $G^{\text{sc}} \rightarrow G \rightarrow G^{\text{ad}}$ . The group  $G^{\text{sc}}$  is usually referred to as the **simply connected cover of  $G$** .

We conclude this section by summarizing the classification of split simple linear algebraic groups of classical type defined over a field  $F$  (cf. [29, §25-26]):

- $A_n$ :  $G = \text{SL}_{n+1} / \mu_l$ , where  $l$  divides  $n + 1$ . For  $l = n + 1$ , we obtain  $G^{\text{ad}} = \text{PGL}_{n+1}$ , while for  $l = 1$  we obtain  $G^{\text{sc}} = \text{SL}_{n+1}$ .
- $B_n$ :  $G^{\text{ad}} = \text{O}_{2n+1}^+$  and  $G^{\text{sc}} = \text{Spin}_{2n+1}$ .
- $C_n$ :  $G^{\text{ad}} = \text{PGSp}_{2n+1}$  and  $G^{\text{sc}} = \text{Sp}_{2n+1}$ .
- $D_n$ : If  $n$  is odd:  $G^{\text{ad}} = \text{PGO}_{2n}^+$ ,  $G^{\text{sc}} = \text{Spin}_{2n}$ , and  $G = \text{O}_{2n}^+$  otherwise.  
If  $n$  is even:  $G^{\text{ad}} = \text{PGO}_{2n}^+$ ,  $G^{\text{sc}} = \text{Spin}_{2n}$ , and  $G = \text{HSpin}_{2n}^\pm$  or  $\text{O}_{2n}^+$  otherwise.

## 6.4 Projective homogeneous varieties

Let  $G$  be a smooth linear algebraic group. If  $N \subset G$  is a closed normal subgroup of  $G$ , then we may take the quotient  $G/N$ . While this process is defined for any  $N$  (cf. [42]), we will restrict to the case that  $N$  is also smooth. Under this assumption, the quotient  $G/N$  has the structure of a linear algebraic group. In this section, we recall the definition of a projective homogeneous variety, and how these varieties relate to quotients of linear algebraic groups.

### 6.4.1 Borel and parabolic subgroups

For elements  $x, y \in G$ , we denote by  $(x, y)$  the **commutator**  $xyx^{-1}y^{-1}$ . If  $A, B$  are subgroups of  $G$ , then the subgroup of  $G$  generated by all  $(x, y), x \in A, y \in B$  will be denoted by  $(A, B)$ .

Recall that the **derived series** of an abstract group  $G$  is defined by setting  $D^0(G) = G$  and  $D^{i+1}(G) = (D^i(G), D^i(G))$  for  $i \geq 0$ .  $G$  is called **solvable** if its derived series terminates in  $e$ . In the case that  $G$  is an algebraic group,  $D^i(G)$  is a closed normal subgroup of  $G$ , which is connected if  $G$  is.

A **Borel subgroup** of  $G$  is a closed connected solvable subgroup which cannot be properly included in another (and therefore is automatically closed). The Borel subgroups of  $G$  and  $G^0$  coincide, so when discussing Borel subgroups we may assume that  $G$  is connected.

**6.4.1 Example.** The group  $B$  of  $n \times n$  nonsingular upper triangular matrices is a Borel subgroup of  $\mathrm{GL}_n(F)$  and contains the maximal torus  $T$  consisting of  $n \times n$  non-singular diagonal matrices.

Clearly a connected solvable subgroup of largest possible dimension in  $G$  is a Borel subgroup, but it is not immediately obvious that every Borel subgroup has the same dimension. To obtain this, we use the following result.

**6.4.2 Theorem** ([6]). *Let  $B$  be any Borel subgroup of a smooth linear algebraic group  $G$ . Then  $G/B$  is a projective variety and all other Borel subgroups in  $G$  are conjugate to  $B$ .*

**6.4.3 Corollary.** *The maximal tori of  $G$  are given by the maximal tori of the Borel subgroups of  $G$ , and are all conjugate.*

It follows that if  $T, T'$  are maximal tori of  $G$ , then  $\dim(T) = \dim(T')$ . We define the **rank** of  $G$  to be the dimension of a maximal torus of  $G$ . For example,  $\mathrm{SL}(n, F)$  has rank  $n - 1$ .

We define a closed subgroup  $P$  of  $G$  to be **parabolic** if and only if  $G/P$  is projective. Equivalently, a subgroup  $P \subset G$  is parabolic if and only if  $P$  contains a Borel subgroup of  $G$ . We will see in the next section that  $G/B$  is in fact the largest “homogeneous space” for  $G$  having the structure of a projective variety.

## 6.4.2 Projective homogeneous varieties

Let  $G$  be a smooth linear algebraic group over  $F$ . We define an **action** of  $G$  on a variety  $X$  over  $F$  to be a morphism  $\varphi : G \times X \rightarrow X$ ,  $(g, x) \mapsto g \cdot x$  of varieties such that

1.  $g_2 \cdot (g_1 \cdot x) = (g_1 g_2) \cdot x$  for all  $g_1, g_2 \in G_{\bar{F}}, x \in X_{\bar{F}}$ .
2.  $e \cdot x = x$  for all  $x \in X_{\bar{F}}$ .

We say that  $G$  **acts transitively** if for all  $x_1, x_2 \in X_{\bar{F}}$  if there exists  $g \in G_{\bar{F}}$  such that  $g \cdot x_1 = x_2$ . A variety on which  $G$  acts transitively is called a **homogeneous  $G$ -variety**, or a  **$G$ -homogeneous space**.

We fix a split simple linear algebraic group  $G$ , a split maximal torus  $T$  and a set of simple roots  $\Pi$  corresponding to the root system  $\Phi(G, T)$ . Since  $G$  is smooth, the set of **projective homogeneous  $G$ -varieties** is in one-to-one correspondence with the set of subsets of  $\Pi$ . In fact, this correspondence does not depend on the choice of  $T$ , or even the isogeny class of  $G$ .

We may extend this correspondence to the set of parabolic subgroups of  $G$  in the following way. Fixing a Borel subgroup  $B \supset T$ , we define  $P_J := \langle B, s_\alpha \mid \alpha \in J \rangle$ , for a subset  $J \subseteq \Pi$ . By definition,  $P_J$  is a parabolic subgroup of  $G$ , and all parabolic subgroups are conjugate to precisely one group of this type.

**6.4.4 Proposition** ([6]). *Let  $G$  be a smooth split simple linear algebraic group with a split maximal torus  $T$  and set of simple roots  $\Pi$  for the root system  $\Phi(G, T)$ .  $X$  is a homogeneous  $G$ -variety if and only if  $X \simeq G/P_J$  for a subset  $J \subseteq \Pi$ .*

**6.4.5 Example.** Let  $G$  be a split group of type  $A_n$ . By  $P_{i_1, \dots, i_r}$  we denote the parabolic subgroup  $P_J$  of the complementary subset  $J = \Pi \setminus \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$ . Then,  $G/P_1 \simeq G/P_n \simeq \mathbb{P}^n$ . More generally,  $G/P_i \simeq G/P_{n-i} \simeq \text{Gr}(i, n+1)$  where the **Grassmannian variety**  $\text{Gr}(i, n+1)$  is the variety of all  $i$ -dimensional linear subspaces of  $\mathbb{A}^{n+1}$ .

If  $J = \emptyset$ , then  $P_\emptyset$  is a Borel subgroup of  $G$ , and  $G/P_\emptyset$  is called the **variety of Borel subgroups** or the **complete flag variety** of  $G$ . These varieties play a central role in the following chapters. We conclude this chapter with a short explanation for the term “flag variety”.

### Flag varieties

Given a vector space  $V$  over a field  $F$ , we define a **flag** to be a sequence of increasing subspaces  $0 = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_k = V$ . If we set  $d_i = \dim(V_i)$ , then we have

$d_i < d_{i+1}$  and  $0 < d_1 < \cdots < d_k = n$ , and so  $k \leq n$ . If  $d_i = i$  for each  $i = 1, \dots, k$  then the sequence is called a **complete flag**. Otherwise, it is called a **partial flag**. Any partial flag can be obtained from a complete flag by deleting subspaces, and conversely, any partial flag can be completed by inserting subspaces.

We call  $(d_1, \dots, d_k)$  the **signature** of a flag and define the **flag variety**  $\text{Fl}(d_1, \dots, d_k, F)$  to be the set of flags in a vector space  $V$  of dimension  $n = d_k$  over  $F$  having signature  $(d_1, \dots, d_k)$ .

If we fix an ordered basis of  $V$  to identify it with  $F^n$ , then we may associate a **standard flag** to this basis by letting the  $i$ th subspace be spanned by the first  $i$  vectors of the basis. Relative to this basis, the **stabilizer** of this flag is the group of  $n \times n$  non-singular upper triangular matrices, which we denote by  $B_n$ . We may write the complete flag variety as  $\text{GL}_n(F)/B_n$ .

Generalizing this notion, we may think along the lines that Borel subgroups are stabilizers of complete flags, and that parabolic subgroups are stabilizers of partial flags. Thus, for a linear algebraic group  $G$  and Borel subgroup  $B$ ,  $G/B$  is a complete flag variety (and in turn,  $G/P$  is a partial flag variety for any parabolic subgroup  $P$ ).

# Chapter 7

## Torsors and twisted forms

In this chapter we consider linear algebraic groups which are not split over a field  $F$ . We show that such groups can be described using the language of torsors and twisted forms. Since we will be using both split and non-split objects, we will use the subscript  $s$  to denote a split form. For example, a split linear algebraic group will be denoted by  $G_s$ , while an arbitrary linear algebraic group will be denoted simply by  $G$ .

### 7.1 $G$ -Torsors

The goal of this section is to define a  $G$ -torsor for a linear algebraic group, and to relate the set of isomorphism classes of  $G$ -torsors to Galois cohomology. The main reference for this section is [29].

#### 7.1.1 Preliminaries

To define the notion of a torsor, we begin with a linear algebraic group  $G$  over a base field  $F$ . We define a  $G$ -**torsor** over  $F$  to be a variety  $Y$  over  $F$  together with a structure map  $f : Y \rightarrow F$  and a right action of  $G$  on  $Y$  defined on points by  $(x, g) \mapsto x \cdot g$  such that the map  $Y \times_F G \rightarrow Y \times_F Y$  defined on points by  $(x, g) \mapsto (x, x \cdot g)$  is an isomorphism of varieties. In other words, the action of  $G$  on  $Y$  preserves the fibres of  $f$  and the action on each geometric fibre (fibre considered over  $\bar{F}$ ) is faithful and transitive.

Torsors appear naturally when considering projective homogeneous  $G$ -varieties. Torsors are also useful in giving interpretations of cohomology groups, and for classifying objects which are **locally isomorphic**, that is, isomorphic over an algebraic closure.

**7.1.1 Remark.** What we have defined to be a  $G$ -torsor over  $F$  may also be referred to as an  $F$ -torsor under  $G$  or a **principal homogeneous space of  $G$  over  $F$** .

Two  $G$ -torsors  $Y$  and  $Y'$  are **isomorphic** if there exists an isomorphism  $\varphi : Y \rightarrow Y'$  of  $F$ -varieties such that  $\varphi(x \cdot g) = \varphi(x) \cdot g$  for any  $x \in Y_{\bar{F}}$ ,  $g \in G_{\bar{F}}$ . We note that  $G$  itself is a  $G$ -torsor when equipped with the right action of  $G$  by translation. A  $G$ -torsor is called **trivial** if it is isomorphic to  $G$  over  $F$ .

**7.1.2 Proposition.** *A  $G$ -torsor  $Y$  over  $F$  is trivial if and only if the set of  $F$ -points of  $Y$  is non-empty.*

*Proof.* Suppose  $Y \simeq G$ . Since the set of  $F$ -points of  $G$  is non-empty, the set of  $F$ -points of  $Y$  must be as well. On the other hand, suppose  $x_0 \in Y$  is an  $F$ -point. The map  $G \rightarrow Y$  given by  $g \mapsto x_0 \cdot g$  gives the required isomorphism.  $\square$

For a field extension  $K/F$ , and a  $G$ -torsor  $Y$  over  $F$ , we may define a  $G_K$ -torsor over  $K$  by  $Y_K := Y \times_F K$ . Now, suppose  $H \subset G$  is a normal subgroup. Setting  $G' = G/H$ , we may define a  $G'$ -torsor  $Y'$  by taking the quotient of  $Y$  by the action of the subgroup  $H$ .

## 7.1.2 Galois Cohomology

For a field  $F$ , we denote by  $\Gamma := \text{Gal}(\bar{F}/F)$  the absolute Galois group of  $F$ . That is,  $\Gamma$  is the projective limit of  $\text{Gal}(K/F)$  where  $K$  ranges over all finite Galois field extensions of  $F$ . Let  $M$  be a discrete group endowed with a continuous left action of  $\Gamma$  on  $M$  which is compatible with the group structure of  $M$ . By this, we mean that  $\delta(ab) = \delta(a)\delta(b)$  for all  $\delta \in \Gamma$ ,  $a, b \in M$ .

We define the **degree 0 cohomology set** with coefficients in  $M$  by

$$H^0(\Gamma, M) = M^\Gamma = \{a \in M \mid \delta(a) = a \text{ for all } \delta \in \Gamma\}.$$

**7.1.3 Example.** For a linear algebraic group  $G$  over  $F$ ,  $H^0(\Gamma, G_{\bar{F}}) = G_{\bar{F}}$ .

We define a **cocycle** to be a continuous map  $\Gamma \rightarrow M$ ,  $\delta \mapsto c_\delta$  such that  $c_{\delta\mu} = c_\delta \delta(c_\mu)$  for any  $\delta, \mu \in \Gamma$ . As in the setting of Chow groups (cf. Chapter 3), the set of cocycles is denoted by  $Z^1(\Gamma, M)$ . The **degree 1 cohomology set** with coefficients in  $M$  is denoted by  $H^1(\Gamma, M)$  and is defined to be the quotient of  $Z^1(\Gamma, M)$  by the following equivalence relation.

$$c \sim c' \text{ if there exists } a \in M \text{ such that } c'_\delta = a^{-1}c_\delta\delta(a) \text{ for any } \delta \in \Gamma.$$

A cocycle  $c$  is called **trivial** if there exists  $a \in M$  such that  $c_\delta = a^{-1}\delta(a)$  for any  $\delta \in \Gamma$ . The class of the trivial cocycle defines a distinguished element for  $H^1(\Gamma, M)$ .

**7.1.4 Example.** Consider the case where  $\delta(a) = a$  for all  $\delta \in \Gamma$  and all  $a \in M$ , i.e.  $\Gamma$  has trivial action. Then,  $H^0(\Gamma, M) = M$  and  $Z^1(\Gamma, M)$  is the set of continuous homomorphisms from  $\Gamma$  to  $M$ . In  $H^1(\Gamma, M)$ , we have  $[f] = [g]$  if and only if there exists  $a \in M$  such that  $f = a^{-1}ga$ , so  $H^1(\Gamma, M)$  is simply  $\text{Hom}_{\text{cont}}(\Gamma, M)$  modulo conjugation by  $M$ .

We follow the convention that  $H^1(F, M) := H^1(\Gamma, M_{\bar{F}})$ . We have the following useful correspondence.

**7.1.5 Theorem** ([29, §28]). *For a linear algebraic group  $G$  over an arbitrary field  $F$ , there exists a functorial bijection from the set of isomorphism classes of  $G$ -torsors over  $F$  to  $H^1(F, \text{Aut}(G))$ .*

**7.1.6 Remark.** By functorial, we mean that the bijection is compatible with base changes, and with morphisms of algebraic groups. Furthermore, the bijection sends the trivial torsor to the class of the trivial cocycle.

Recall that we have a map  $G \rightarrow \text{Aut}(G)$  given by sending an element  $g$  to the inner automorphism  $x \mapsto gxg^{-1}$ . The image of this map is called the group of **inner automorphisms**. If the class  $[\xi]$  lies in the image of the induced map  $H^1(F, G) \rightarrow H^1(F, \text{Aut}(G))$ , then we say that  $\xi$  is of **inner type**. We have yet another map  $H^1(F, G^{\text{sc}}) \rightarrow H^1(F, G)$  induced by the isogeny  $G^{\text{sc}} \rightarrow G$ . If an inner cocycle  $[\xi]$  lies in the image of this map as well, we say that  $\xi$  is of **strongly inner type**.

**7.1.7 Example.** The cohomology groups  $H^d(F, M) := H^d(\Gamma, M)$  can be defined for any degree  $d$  and abelian torsion group  $M$ . For example, the Brauer group of  $F$  is isomorphic to  $H^2(F, \mathbb{Q}/\mathbb{Z})$ .

## 7.2 Twisted Forms

In this section, we extend the correspondence between the set  $G$ -torsors and  $H^1(F, \text{Aut}(G))$  to include the set of twisted forms of  $G$ . This leads to the statement that every non-split smooth simple linear algebraic group  $G$  can be associated to a cocycle  $[\xi] \in H^1(F, \text{Aut}(G_s))$ , where  $G_s$  is a split simple linear algebraic group of the same type as  $G$ .

### 7.2.1 Definitions and examples

Let  $G, G'$  be linear algebraic groups defined over a field  $F$ .  $G'$  is called a **twisted form** of  $G$  if there is an isomorphism of algebraic groups  $G_{\bar{F}} \simeq G'_{\bar{F}}$ . Since all simple linear algebraic groups become split over an algebraically closed field, a simple group  $G'$  is a twisted form of  $G_s$ , for some split simple linear algebraic group  $G_s$ . For a fixed field  $F$ , the set of isomorphism classes of twisted forms of  $G_s$  defined over  $F$  forms a pointed set which is isomorphic to  $H^1(F, \text{Aut}(G_s))$  by general descent theory. This means that we may characterize a twisted form of  $G_s$  by its corresponding cocycle  $\xi \in Z^1(F, \text{Aut}(G_s))$ . To emphasize this correspondence, we denote this twisted form by  $G' = {}_{\xi}G_s$ .

**7.2.1 Remark.** The notion of a twisted form is not limited to algebraic groups. For any algebraic structure defined over a field  $F$ , we may define a **twisted form** of  $A$  to be an algebraic structure  $B$ , of the same type as  $A$ , such that  $A$  and  $B$  become isomorphic over  $\bar{F}$ . For example, let  $A$  be a central simple  $F$ -algebra of degree  $n$ . Since  $A_{\bar{F}} \simeq M_n(\bar{F})$ , we may say that central simple algebras are twisted forms of matrix algebras.

In accordance with our previous terminology, we say that  ${}_{\xi}G_s$  is of **inner type** (resp. of **strongly inner type**) if  $\xi$  is of inner type (resp. strongly inner type).

Consider a projective homogeneous  ${}_{\xi}G_s$ -variety  $X$  over  $F$ . By definition,  $X_{\bar{F}} \simeq G_s/P_J \times_F \bar{F}$  for a parabolic subgroup  $P_J \subset G_s$ . Thus,  $X$  corresponds to a subset  $J \subset \Pi$  of the Dynkin diagram of  $G_s$ . To highlight this correspondence, denote this twisted projective homogeneous variety by  $X = {}_{\xi}(G_s/P_J)$ .

**7.2.2 Example.** Let  $G$  be a non-split group of inner type  $A_n$  and let  $X$  be a projective homogeneous  $G$ -variety. Recall from Example 6.4.5 that  $G_s/P_1 \simeq G_s/P_n \simeq \mathbb{P}^n$ . So, if  $X = {}_{\xi}(G_s/P_1)$  or  ${}_{\xi}(G_s/P_n)$  for some  $\xi \in Z^1(F, \text{Aut}(G_s))$ , then  $X$  is a twisted form of the projective space  $\mathbb{P}^n$ . Such a variety is called a **Severi-Brauer variety** over  $F$ . If  $X = {}_{\xi}(G_s/P_i)$ , the  $X$  is a twisted form of the Grassmannian variety  $\text{Gr}(i, n+1)$  and is called a **generalized Severi-Brauer variety** over  $F$ .

**7.2.3 Example.** For the projective linear group  $\text{PGL}_{n+1}$  over a field  $F$ , the set  $H^1(F, \text{PGL}_{n+1})$  is isomorphic to the set of isomorphism classes of central simple  $F$ -algebras of degree  $n+1$ . Since  $\text{PGL}_{n+1} \simeq \text{Aut}(G_s)$  for a split group  $G_s$  of type  $A_n$ , we have  $H^1(F, \text{PGL}_{n+1}) \simeq H^1(F, \text{Aut}(G_s))$ , and so a Severi-Brauer variety  $X$  of dimension  $n$  over  $F$  can be identified with the variety of right ideals in a central simple  $F$ -algebra  $A$  of degree  $n+1$ . To highlight this correspondence, we denote this variety by  $X = \text{SB}(A)$ .

## 7.3 The Tits algebras of $G$

Tits algebras were first introduced by J. Tits in [46], and the definitions and results below are consistent with this original paper. However, the examples we present in this section can be found in [29], and we will use this as the main reference for Tits algebras.

For a split simple linear algebraic group  $G_s$  over a field  $F$ , we fix a maximal split torus  $T$  and a root system  $\Phi = \Phi(G_s)$ . Given a basis  $\Pi \subset \Phi$ , the **cone of dominant weights** is a subset  $\Lambda_+ \subset \Lambda$  defined by

$$\Lambda_+ = \{\lambda \in \Lambda \mid s_\alpha(\lambda) \geq 0 \text{ for all } \alpha \in \Pi\}.$$

We can define a partial ordering on  $\Lambda$  by  $\lambda \geq \mu$  if  $\lambda - \mu$  is a non-negative linear combination of elements of  $\Pi$ . Let  $\rho : G_s^{\text{sc}} \times_F \bar{F} \rightarrow \text{GL}(V)$  be an irreducible representation of the simply connected cover  $G_s^{\text{sc}}$  of  $G_s$ . The set of weights of  $\rho$  is a finite subset in  $\text{Hom}(\Lambda, \mathbb{G}_m)$  and contains a maximal element  $\lambda \in \Lambda_+$ , which we call the **highest weight** of  $\rho$ . There is a one-to-one correspondence between  $T^* \cap \Lambda_+$  and isomorphism classes of irreducible representations of  $G_s^{\text{sc}}$  given by associating to each representation its highest weight (cf. [29, §27]).

Returning to our general setting, let  $G$  be a (not necessarily split) smooth simple linear algebraic group over  $F$ . We fix a maximal torus  $T \subset G$  and a root system  $\Phi = \Phi(G_{\bar{F}})$ . We define an **algebra representation** to be morphism  $\rho : G^{\text{sc}} \rightarrow \text{GL}_1(A)$  for a central simple  $F$ -algebra  $A$ . Two algebra representations  $\rho : G^{\text{sc}} \rightarrow \text{GL}_1(A)$  and  $\rho' : G^{\text{sc}} \rightarrow \text{GL}_1(A')$  are isomorphic if there exists an isomorphism  $\phi : A \simeq A'$  such that  $\rho' = \phi \circ \rho$ . Extending  $\rho$  to  $\rho_{\bar{F}} : G_{\bar{F}}^{\text{sc}} \rightarrow \text{GL}_1(A_{\bar{F}}) = \text{GL}_n(\bar{F})$  defines a usual representation; we say that  $\rho$  is **irreducible** if  $\rho_{\bar{F}}$  is an irreducible representation. If  $\rho$  is irreducible, then the highest weight of  $\rho$  is the highest weight of  $\rho_{\bar{F}}$ .

Recall that the Galois group  $\Gamma = \text{Gal}(\bar{F}/F)$  acts naturally on  $\Lambda$ . This action however does not preserve either  $\Pi$  or  $\Lambda_+$ . We define another action of  $\Gamma$  on  $\Lambda$ , called the  **$\star$ -action**. For any  $\gamma \in \Gamma$ , there exists a unique element  $w_\gamma$  in the Weyl group  $W(G)$  such that  $w_\gamma(\gamma(\Pi)) = \Pi$ . We define  $\gamma \star \lambda := w_\gamma(\gamma(\lambda))$ . This action does preserve  $\Pi$  and  $\Lambda_+$ . Let  $\Lambda_+^\Gamma$  be the subgroup of  $\Lambda_+$  fixed by this action of  $\Gamma$ .

**7.3.1 Theorem** ([46, 3.3]). *Let  $G$  be a simple linear algebraic group over a field  $F$  and let  $G^{\text{sc}}$  be its simply connected cover. The map associating an irreducible algebra representation of  $G^{\text{sc}}$  to its highest weight induces a one-to-one correspondence between isomorphism classes of irreducible algebra representations of  $G^{\text{sc}}$  and  $\Lambda_+^\Gamma$ .*

Hence, we can associate to each weight  $\lambda \in \Lambda_+^\Gamma$  a central simple  $F$ -algebra denoted by  $A_\lambda$  and called the **Tits algebra** associated to  $\lambda$ . For any  $\lambda, \mu \in \Lambda_+^\Gamma$ , we have  $[A_{\lambda+\mu}] = [A_\lambda] + [A_\mu] \in \text{Br}(F)$  and  $[A_\lambda] = 0 \in \text{Br}(F)$  for any  $\lambda \in \Lambda_r$ . Thus, the map  $\Lambda_+^\Gamma \rightarrow \text{Br}(F)$  defined by  $\lambda \mapsto [A_\lambda]$  can be extended to a group morphism  $\beta : (\Lambda/\Lambda_r)^\Gamma \rightarrow \text{Br}(F)$ .

**7.3.2 Example.** Consider a group  $G$  of inner type  $A_n$ . We have  $G^{\text{sc}} = \text{SL}_1(A)$  for some central simple  $F$ -algebra of degree  $n+1$ . By Proposition 6.2.4,  $\Lambda/\Lambda_r \simeq \mathbb{Z}/(n+1)\mathbb{Z}$  and the  $\star$ -action of  $\Gamma$  is trivial. The morphism  $\beta : \Lambda/\Lambda_r \rightarrow \text{Br}(F)$  is given by  $\bar{\omega}_i \mapsto [A^{\otimes i}]$ .

**7.3.3 Example.** Consider a group  $G$  of inner type  $D_n$ . Then,  $G^{\text{sc}} = \text{Spin}^+(A, \sigma)$  for a central simple  $F$ -algebra of degree  $2n$  and an orthogonal involution  $\sigma$  having trivial discriminant. By Proposition 6.2.4,  $\Lambda/\Lambda_r$  is of order 4, and its structure depends on the parity of  $n$ :

- For  $n$  odd,  $\Lambda/\Lambda_r \simeq \mathbb{Z}/4\mathbb{Z}$  and is generated by  $\bar{\omega}_{n-1}$ . We have relations  $\bar{\omega}_1 = 2\bar{\omega}_{n-1}$  and  $\bar{\omega}_n = 3\bar{\omega}_{n-1}$ . The Tits algebras associated to these weights are  $A_{\bar{\omega}_1} = A$ ,  $A_{\bar{\omega}_{n-1}} = C^+(A, \sigma)$ . and  $A_{\bar{\omega}_n} = C^-(A, \sigma)$ . By the group relations in  $\Lambda/\Lambda_r$ , we obtain the fundamental relations  $[A] = 2[C^+] = 2[C^-]$ ,  $2[A] = 0$  in the Brauer group.
- For  $n$  even,  $\Lambda/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  with generators  $\bar{\omega}_{n-1}$  and  $\bar{\omega}_n$  and relations  $\bar{\omega}_1 = \bar{\omega}_{n-1} + \bar{\omega}_n$ . As in the odd case, the Tits algebras are given by  $A_{\bar{\omega}_1} = A$ ,  $A_{\bar{\omega}_{n-1}} = C^+(A, \sigma)$ . and  $A_{\bar{\omega}_n} = C^-(A, \sigma)$ . This time, however, we obtain the fundamental relations  $[A] = [C^+] + [C^-]$ ,  $2[A] = 2[C^+] = 2[C^-] = 0$ .

Recall that  $G$  is a twisted form of a split group  $G_s$ , and is characterized by a cocycle  $\xi \in Z^1(F, \text{Aut}(G_s))$ . Recall that if  $G$  is of inner type, then  $\xi$  is given by the image of a cocycle in  $Z^1(F, G_s)$ , and if  $G$  is of strongly inner type,  $\xi$  is given by the image of a cocycle in  $Z^1(F, G_s^{\text{sc}})$ . We can further classify the group  $G$  by saying that  $G$  is a twisted form of  $G_s$  **by means of a  $G'$ -torsor** if the cocycle  $\xi$  defining  $G$  lies in the image of  $Z^1(F, G') \rightarrow Z^1(F, G_s)$ .

**7.3.4 Example.** Let  $G$  be a group of inner type  $A_n$ . If  $G$  is adjoint, then  $G = {}_\xi G_s$  for some  $\xi \in Z^1(F, \text{PGL}_n)$ . Thus,  $G$  is a twisted form of  $A_n$  by means of a  $\text{PGL}_n$ -torsor.

**7.3.5 Example.** Let  $G$  be a group of inner type  $D_n$  for  $n$  even. Then  $\Lambda/\Lambda_r = \{0, \bar{\omega}_{n-1}\} \oplus \{0, \bar{\omega}_n\}$ .

- If  $G$  is adjoint, we have  $\Lambda/T^* = \Lambda/\Lambda_r$ , and  $G$  is twisted by means of a  $\text{PGO}_{2n}^+$ -torsor.

- If  $G$  is simply connected,  $\Lambda/T^* = 0$ , and  $G$  is twisted by means of a  $\text{Spin}_{2n}$ -torsor.
- If  $G$  is neither adjoint nor simply connected, then  $\Lambda/T^*$  is of order 2:
  - If  $\Lambda/T^*$  is the quotient of  $\Lambda/\Lambda_r$  by the diagonal subgroup  $\{0, \bar{\omega}_{n-1} + \bar{\omega}_n\}$ , then  $G$  is twisted by means of an  $\text{O}_{2n}^+$ -torsor.
  - if  $\Lambda/T^*$  is the quotient of  $\Lambda/\Lambda_r$  by either  $\{0, \bar{\omega}_{n-1}\}$  or  $\{0, \bar{\omega}_n\}$ , then  $G$  is twisted by means of an  $\text{HSpin}_{2n}$ -torsor.

For a group  $G$  of inner type, we can restrict the map  $\beta : \Lambda/\Lambda_r \rightarrow \text{Br}(F)$  to take into account the cocycle  $\xi$ . We define the **Tits map** of  $G$  by  $\beta_\xi : \Lambda/T^* \rightarrow \text{Br}(F)$  by composing  $\beta$  with the map  $\Lambda/T^* \rightarrow \Lambda/\Lambda_r$ .

## 7.4 Twisted Flag Varieties

In this section, we introduce the Steinberg basis of  $K_0(G_s/B)$ , first defined by Steinberg in [44]. For a split simple smooth linear algebraic group  $G_s$  and Borel subgroup  $B \subset G_s$ , the Steinberg basis provides an explicit set of generators for the  $\gamma$ -filtration on  $K_0(G_s/B)$ . After constructing this set of generators, we describe how the Steinberg basis can also be useful when considering a twisted form of  $G_s$ , due to a result of Panin [36].

### 7.4.1 The Steinberg basis

Let  $G_s$  be a split simple smooth linear algebraic group of rank  $n$  over an arbitrary field  $F$ . We fix a split maximal torus  $T \subset G_s$  and a Borel subgroup  $B \supset T$ . Let  $T^*$  be the character group of  $T$ ,  $\{\alpha_1, \dots, \alpha_n\}$  a set of simple roots with respect to  $B$  and  $\{\omega_1, \dots, \omega_n\}$  the respective set of fundamental weights. We have the relation  $\Lambda_r \subset T^* \subset \Lambda$ , where  $\Lambda_r$  and  $\Lambda$  denote the root lattice and weight lattice respectively. Consider the simply connected cover  $G_s^{\text{sc}}$  of  $G_s$  with corresponding Borel subgroup  $B^{\text{sc}}$  and maximal split torus  $T^{\text{sc}}$ .

For  $G_s^{\text{sc}}$ , we have  $\Lambda = \text{Hom}(T^{\text{sc}}, \mathbb{G}_m)$ , and given any  $\lambda \in \Lambda$ , we can lift  $\lambda : T^{\text{sc}} \rightarrow \mathbb{G}_m$  uniquely to a homomorphism  $\lambda : B^{\text{sc}} \rightarrow \mathbb{G}_m$ . Fixing a 1-dimensional  $F$ -vector space  $V_1$ , we define the quotient

$$G_s^{\text{sc}} \times^{B^{\text{sc}}} V_1 = G_s^{\text{sc}} \times V_1 / \{(g, v) \sim (g \cdot b, \lambda(b)^{-1} \cdot v)\}.$$

Now, the projection map  $G_s^{\text{sc}} \times^{B^{\text{sc}}} V_1 \rightarrow G_s^{\text{sc}}/B^{\text{sc}}$  defines a line bundle  $\mathcal{L}(\lambda)$  over  $G_s^{\text{sc}}/B^{\text{sc}} = G_s/B$ , the variety of Borel subgroups of  $G_s$  (cf. [11, §1.5]).

We define the integral group ring  $\mathbb{Z}[T^*]$  whose elements are linear combinations  $\sum_i a_i e^{\alpha_i}$ , for  $a_i \in \mathbb{Z}$  and  $\alpha_i \in T^*$ . Let

$$\mathbf{c} : \mathbb{Z}[T^*] \rightarrow K_0(G_s/B)$$

be the **characteristic map**, defined by sending  $e^\lambda$  to the class of the associated line bundle  $[\mathcal{L}(\lambda)]$  for  $\lambda \in \Lambda$ . Note that while  $K_0(G_s/B)$  does not depend on the isogeny class of  $G_s$ , the image of this map does [43]. In particular, the simply connected cover  $G_s^{\text{sc}}$  provides a surjection

$$\mathbf{c}^{\text{sc}} : \mathbb{Z}[\Lambda] \twoheadrightarrow K_0(G_s/B).$$

The Weyl group  $W := W(G_s, T)$  (cf. Chapter 6) acts on weights via simple reflections  $s_{\alpha_i}$ ,  $i = 1, \dots, n$ . That is

$$s_{\alpha_i}(\lambda) = \lambda - \alpha_i^\vee(\lambda)\alpha_i, \text{ for } \lambda \in \Lambda.$$

So, for each element  $w \in W$  we may define a corresponding weight  $\rho_w \in \Lambda$  (cf. [44, §2.1]) in the following explicit way:

$$\rho_w := \sum_{\{i \in 1, \dots, n \mid w^{-1}(\alpha_i) < 0\}} w^{-1}(\omega_i).$$

In particular if  $w = s_{\alpha_i}$ , then  $\rho_w = s_{\alpha_i}(\omega_i) = \omega_i - \alpha_i$ .

Since  $W$  acts trivially on  $\Lambda/\Lambda_r$ , we have

$$\bar{\rho}_w = \sum_{\{i \in 1, \dots, n \mid w^{-1}(\alpha_i) < 0\}} \bar{\omega}_i \in \Lambda/T^*,$$

where  $\bar{\rho}_w$  denotes the class of  $\rho_w \in \Lambda$  modulo  $T^*$ .

By the characteristic map  $\mathbf{c}^{\text{sc}}$ , we may associate to each  $w \in W$  the class of the associated line bundle  $g_w := \mathbf{c}(e^{\rho_w}) = [\mathcal{L}(\rho_w)]$ . These elements form a  $\mathbb{Z}$ -basis of  $K_0(G_s/B)$  which we call the **Steinberg basis**.

Since the Steinberg basis provides an explicit set of generators for  $K_0(G_s/B)$ , it can also be used to give explicit generators for the  $\gamma$ -filtration on  $K_0(G_s/B)$ .

**7.4.1 Lemma.** *For a split simple linear algebraic group  $G_s$  and Borel subgroup  $B \subset G_s$ ,*

$$\gamma^{d/d+1} K_0(G_s/B) = \langle \gamma_1(g_{w_1}) \cdot \dots \cdot \gamma_1(g_{w_d}) \mid w_1, \dots, w_d \in W \rangle.$$

### 7.4.2 The restriction map

Consider the twisted form  $G/B$  of  $G_s/B$  by means of a  $G_s$ -torsor  $\xi \in Z^1(F, G_s)$ . In general the group  $K_0(G/B)$  is not generated by classes of line bundles. Hence,  $\gamma^d K_0(G/B)$  does not have a nice set of generators as in the split case.

For a field extension  $K/F$  and a variety  $Y$  over  $F$  we have a restriction map  $\text{res}: K_0(Y) \rightarrow K_0(Y \times_F K)$ . In particular, taking the variety  $\xi(G_s/B)$  and a splitting field  $L$  of  $\xi$ ,  $K_0(\xi(G_s/B) \times_F L) \simeq K_0(G_s/B \times_F L) \simeq K_0(G_s/B)$ , so we may consider the restriction map

$$\text{res} : K_0(\xi(G_s/B)) \rightarrow K_0(G_s/B).$$

The image of this map can be given in terms of the Steinberg basis, and involves the indices of the Tits algebras of  $G$  (cf. Section 7.3).

**7.4.2 Theorem** ([36]). *Let  $G_s$  be a split smooth simple linear algebraic group over  $F$ , let  $G$  be an inner twisted form of  $G_s$  by means of  $\xi \in Z^1(F, \text{Aut}(G_s))$ , and let  $G/B$  be the variety of Borel subgroups of  $G$ . Consider a splitting field  $L/F$  and the corresponding restriction map  $\text{res} : K_0(G/B) \rightarrow K_0(G_s/B)$ . Then,  $\text{im}(\text{res}) = \langle \text{ind}(A_w)g_w \rangle_{w \in W}$ , where  $[A_w]$  is the Brauer class of the Tits algebra of  $G$ , given by  $\beta_\xi(\bar{\rho}_w)$ .*

**7.4.3 Remark.** If  $G$  is of strongly inner type, then the restriction map is an isomorphism.

# Chapter 8

## The J-invariant

For a linear algebraic group  $G$ , we have seen in Chapter 6 that the Tits algebras of  $G$  provide an invariant of the group. In this chapter we introduce a second invariant of  $G$ , called the J-invariant, which was defined by Petrov, Semenov and Zainoulline [38], and describes the motivic behaviour of the variety of Borel subgroups of  $G$ .

Quéguiner-Mathieu, Semenov and Zainoulline discovered a connection between these two invariants, which they developed in [40], through use of the second Chern class map in the Riemann-Roch theorem without denominators. Here we extend this connection through use of higher Chern class maps and apply the result to groups of inner type  $E_6$ . The results of this chapter are original to the author, and can be found in [21]

### 8.1 Preliminaries

Let  $G$  be an inner twisted form of a split simple linear algebraic group  $G_s$  corresponding to a cocycle  $\xi \in Z^1(F, G_s)$ . We fix a maximal split torus  $T_s$  of  $G_s$  and a Borel subgroup  $B_s \supset T_s$ . Let  $T$  and  $B$  be the corresponding torus and Borel subgroup of  $G$ , respectively. We will consider, as in Chapter 7, the projective homogeneous variety  $X = G/B$ , and corresponding split variety  $X_s$ .

We recall the definitions of the  $\gamma$ -filtration on  $K_0(X)$  and the Chow group  $\text{CH}(X)$  from Chapter 3. The  $d$ -th Chern class map defines a group homomorphism

$$c_d : \gamma^{d/d+1} K_0(X_s) \rightarrow \text{CH}^d(X_s),$$

which is surjective over  $\mathbb{Z} \left[ \frac{1}{(d-1)!} \right]$  by Proposition 3.3.4.

Since characteristic classes commute with restriction maps, we have the following commutative diagram.

$$\begin{array}{ccc} \gamma^{d/d+1} K_0(X) & \xrightarrow{\text{res}_\gamma} & \gamma^{d/d+1} K_0(X_s) \\ c_d \downarrow & & \downarrow c_d \\ CH^d(X) & \xrightarrow{\text{res}_{CH}} & CH^d(X_s). \end{array}$$

**8.1.1 Proposition.** *Let  $\{g_w\}_{w \in W}$  be the Steinberg basis of  $K_0(X_s)$ . Consider the composition*

$$\phi_d: \gamma^{d/d+1} K_0(X) \xrightarrow{\text{res}_\gamma} \gamma^{d/d+1} K_0(X_s) \xrightarrow{c_d} CH^d(X_s).$$

*The image of  $\phi_d$  is generated by  $\text{ind}(A_w)c_1(g_w)$  for all  $w \in W$ . In general, the image of  $\phi_d$  is generated by the elements*

$$(d-1)! \binom{\text{ind}(A_{w_1})}{i_1} \cdots \binom{\text{ind}(A_{w_m})}{i_m} c_1(g_{w_1})^{i_1} \cdots c_1(g_{w_m})^{i_m}$$

*where  $i_1 + \cdots + i_m = d$  for all  $w_1, \dots, w_m \in W$ .*

*Proof.* Recall that by Theorem 7.4.2,  $\text{res}(K_0(X)) = \langle \text{ind}(A_w)g_w \rangle$ . Combining this result with the definition of the  $\gamma$ -filtration on  $K_0(X_s)$ , we can see that the image of  $\text{res}_\gamma$  is generated by products

$$\text{res}_\gamma(\gamma^{d/d+1} K_0(X)) = \langle \gamma_{i_1}(\text{ind}(A_{w_1})g_{w_1}) \cdots \gamma_{i_m}(\text{ind}(A_{w_m})g_{w_m}) \mid i_1 + \cdots + i_m = d \rangle,$$

where  $w_1, \dots, w_m \in W$ .

Since the  $g_w$ 's are classes of line bundles, we have total characteristic classes

$$\gamma(\text{ind}(A_w)g_w) = (1 + \gamma_1(g_w))^{\text{ind}(A_w)} = \sum_{k=1}^{\text{ind}(A_w)} \binom{\text{ind}(A_w)}{k} \gamma_1(g_w)^k. \quad (8.1)$$

Consider an element in  $\text{im}(\text{res}_\gamma)$  of the form  $x = \gamma_{i_1}(\text{ind}(A_{w_1})g_{w_1}) \cdots \gamma_{i_m}(\text{ind}(A_{w_m})g_{w_m})$  such that  $i_1 + \cdots + i_m = d$  for some  $w_1, \dots, w_m \in W$ . By (8.1)

$$\begin{aligned} x &= \gamma_{i_1}(\text{ind}(A_{w_1})g_{w_1}) \cdots \gamma_{i_m}(\text{ind}(A_{w_m})g_{w_m}) \\ &= \binom{\text{ind}(A_{w_1})}{i_1} \cdots \binom{\text{ind}(A_{w_m})}{i_m} \gamma_1(g_{w_1})^{i_1} \cdots \gamma_1(g_{w_m})^{i_m}. \end{aligned}$$

Taking Chern classes, we use Lemma 3.3.2 along with (3.1) to obtain

$$c_d(x) = (-1)^{d-1} (d-1)! \binom{\text{ind}(A_{w_1})}{i_1} \cdots \binom{\text{ind}(A_{w_m})}{i_m} c_1(g_{w_1})^{i_1} \cdots c_1(g_{w_m})^{i_m}.$$

□

## 8.2 The common index

If  $G^{\text{sc}}$  is the simply connected cover of  $G$ , the degree 1 characteristic map defines an isomorphism

$$\mathbf{c}_{\text{sc}}^{(1)} : \Lambda \xrightarrow{\sim} \text{CH}^1(X_s),$$

such that the cycles  $h_i = c_1(\mathcal{L}(\omega_i))$ ,  $i = 1, \dots, n$  form a  $\mathbb{Z}$ -basis of  $\text{CH}^1(X_s)$ . We define the **degree 1 characteristic map** of  $G$  to be the restriction of this isomorphism to the character group  $T^*$  of  $G_s$

$$\mathbf{c}^{(1)} : T^* \rightarrow \text{CH}^1(X_s),$$

so that if  $\lambda = \sum_{i=1}^n a_i \omega_i$ , then  $\mathbf{c}^{(1)}(\lambda) = c_1(\mathcal{L}(\lambda)) = \sum_{i=1}^n a_i h_i$ . In general, we extend this map to define the **characteristic map**

$$\mathbf{c} : S^*(T^*) \rightarrow \text{CH}(X_s).$$

We denote by  $\pi : \text{CH}^*(X_s) \rightarrow \text{CH}^*(G_s)$  the pull-back induced by the natural projection  $G_s \rightarrow X_s$ . By [17, Section 4, Rem. 2],  $\pi$  is surjective and its kernel is given by the ideal  $I_{\mathbf{c}} \subset \text{CH}^*(X_s)$  generated by the constant-free elements in the image of the characteristic map. In particular, we have  $I_{\mathbf{c}}^{(1)} = \text{im}(\mathbf{c}^{(1)})$ , and

$$\text{CH}^1(G_s) \simeq \text{CH}^1(X_s)/(\text{im}(\mathbf{c}^{(1)})) \simeq \Lambda/T^*.$$

Given a prime  $p$ , set  $\text{Ch}(X_s) = \text{CH}(X_s) \otimes \mathbb{F}_p$ . Taking  $\mathbb{F}_p$ -coefficients, we have

$$\text{Ch}^1(G_s) \simeq \text{Ch}^1(X_s)/(\text{im}(\mathbf{c}^{(1)})) \simeq \Lambda/T^* \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

It is known (cf. [23]) that  $\text{Ch}(X_s)/I_{\mathbf{c}}$  is isomorphic (as an  $\mathbb{F}_p$ -algebra) to a truncated polynomial ring of the form

$$\text{Ch}(X_s)/I_{\mathbf{c}} \cong \mathbb{F}_p[x_1, \dots, x_r]/(x_1^{p^{k_1}}, \dots, x_r^{p^{k_r}})$$

for integers  $r$  and  $k_i \geq 0$  for  $i = 1, \dots, r$ , which are dependent on the group  $G_s$ . For each  $i$ , we denote by  $d_i$  be the degree of the generator  $x_i$ . The number of generators of degree 1 is given by the dimension  $s$  of the  $\mathbb{F}_p$ -vector space  $\Lambda/T^* \otimes_{\mathbb{Z}} \mathbb{F}_p$ .

Since  $\omega_1, \dots, \omega_n$  generate  $\Lambda$  we may choose a minimal set  $\{i_1, \dots, i_s\} \subset \{1, \dots, n\}$  such that the classes of  $\omega_{i_1}, \dots, \omega_{i_s}$  generate  $\Lambda/T^* \otimes_{\mathbb{Z}} \mathbb{F}_p$ . Then,  $h_{i_l} = c_1(\mathcal{L}(\omega_{i_l}))$ ,  $l = 1, \dots, s$  generate  $\text{Ch}^1(X_s)$  and so we may set  $x_l = \pi(h_{i_l})$ ,  $l = 1, \dots, s$  to be the generators of  $\text{Ch}^1(G_s)$ .

In fact, this definition of the generators  $x_1, \dots, x_s$  can be simplified using properties of the Steinberg basis. For  $i = 1, \dots, n$ , we define  $g_i := \mathcal{L}(\omega_i) - \mathcal{L}(\alpha_i)$ , the Steinberg element corresponding to the simple reflection  $s_{\alpha_i}$ . Then,  $c_1(\mathcal{L}(\alpha_i)) \in \text{im}(\mathbf{c}^{(1)})$  implies  $c_1(\mathcal{L}(\omega_i)) \in \ker(\pi)$  for all  $i = 1, \dots, n$ . So, for each  $l = 1, \dots, s$ , we have

$$\pi(h_{i_l}) = \pi(c_1(\mathcal{L}(\omega_{i_l}))) = \pi(c_1(g_{i_l})) + \pi(c_1(\mathcal{L}(\alpha_{i_l}))) = \pi(c_1(g_{i_l})).$$

Thus, we may take the generators of  $\text{Ch}^1(G_s)$  to be  $x_l = \pi(c_1(g_{i_l}))$  for  $l = 1, \dots, s$ .

For the non-split group  $G$ , we recall the definition of the Tits map  $\beta_\xi : \Lambda/T^* \rightarrow \text{Br}(F)$  from Section 7.3. Let  $H$  be the subgroup in  $\text{Br}(F)$  generated by the classes of the Tits algebras of  $G$ . Since  $\Lambda/T^*$  is a finite abelian group,  $H$  is a finite abelian group as well. We let  $[A_i] := \beta_\xi(\bar{\omega}_i)$ , and define the **common index** of  $\xi$  modulo  $p$  by

$$i_p := \gcd\{\text{ind}(A_{i_1}^{\otimes a_1} \otimes \dots \otimes A_{i_s}^{\otimes a_s}) \mid \text{at least one } a_l \text{ is coprime to } p\}.$$

**8.2.1 Example.** Let  $G$  be a group of inner type  $E_6$ . In this case,  $H$  is a cyclic group generated by a Tits algebra with index  $3^d$  with  $0 \leq d \leq 3$  [46, 6.4.1]. Therefore, by the definition of the common index, we have  $i_3 = 3^d$  as well.

**8.2.2 Example.** Let  $G$  be an inner twisted form of  $\text{PGO}_{4m}^+$ ,  $m \geq 2$ . That is,  $G$  is an adjoint group of inner type  $D_n$ ,  $n \geq 4$  even. In this case,  $\Lambda/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  and we may define  $H = \{0, [A_1], [A_{n-1}], [A_n]\}$ , where  $A_i := \beta_\xi(\bar{\omega}_i)$ . By the relations in  $H$  (induced by the group structure of  $\Lambda/\Lambda_r$ ), we have  $i_2 = \min\{\text{ind}(A_1), \text{ind}(A_{n-1}), \text{ind}(A_n)\}$ .

Let  $I_{\text{res}} \subset \text{CH}(X_s)$  be the ideal generated by the constant-free elements in the image of the restriction map  $\text{res}_{\text{CH}} : \text{CH}(X) \rightarrow \text{CH}(X_s)$ . For any integer  $m$ , we denote by  $I_{\text{res}}^{(m)} \subset \text{CH}^m(X)$  the homogeneous part of  $I_{\text{res}}$  of degree  $m$ . Cycles lying in the image of the restriction map are called **rational cycles**. In [28, 6.4(1)] it was shown that if  $G$  is an inner twisted form of  $G_s$ , then all cycles lying in the image of the characteristic map are rational. In the following theorem, we will show that under specific hypotheses on the common index  $i_p$ , any rational cycle of degree at most  $p$  also lies in the image of the characteristic map.

**8.2.3 Theorem.** *Fix a prime  $p$  and a group  $G$  of inner type. Let  $i_p$  be the common index of  $\xi$  modulo  $p$ . If  $p \mid i_p$ , then  $I_{\text{res}}^{(1)} = I_c^{(1)}$ . If  $p^2 \mid i_p$ , then  $I_{\text{res}}^{(m)} = I_c^{(m)}$  for  $m = 2, \dots, p$ .*

*Proof.* Since  $I_c \subseteq I_{\text{res}}$ , it suffices to prove that  $I_{\text{res}}^{(1)} \subseteq I_c^{(1)}$  for  $p \mid i_p$ , and that  $I_{\text{res}}^{(m)} \subseteq I_c^{(m)}$ ,  $m = 1, \dots, p$  for  $p^2 \mid i_p$ . By Proposition 3.3.4 and the commutative diagram above,

we see that for any  $i \geq 0$ ,  $\text{im}(\text{res}_{\text{CH}}^{(d)}) = c_d(\text{im}(\text{res}_\gamma^{(d)}))$  over the coefficient ring  $\mathbb{Z}\left[\frac{1}{(d-1)!}\right]$ .

We begin first with the case  $m = 1$  and assume  $p \mid i_p$ . By the definition of  $I_{\text{res}}$ , we have  $I_{\text{res}}^{(1)} = \text{im}(\text{res}_{\text{CH}}^{(1)})$ . To show that  $I_{\text{res}}^{(1)} \subseteq I_c^{(1)}$ , we must show that for any  $w \in W$ , the element  $\text{ind}(A_w)c_1(g_w)$  belongs (after tensoring with  $\mathbb{F}_p$ ) to  $I_c^{(1)} = \text{im}(\mathbf{c}^{(1)})$ . Recall that  $g_w = \mathcal{L}(\rho_w)$ , and that we may write  $\rho_w = \sum_{i=1}^n a_i \omega_i$ . Taking the total Chern class, we have

$$c(g_w) = c(\mathcal{L}(\omega_1)^{\oplus a_1} \oplus \cdots \oplus \mathcal{L}(\omega_n)^{\oplus a_n}) = 1 + \left( \sum_{i=1}^n a_i c_1(\mathcal{L}(\omega_i)) \right) + \dots,$$

and hence,

$$c_1(g_w) = \sum_{i=1}^n a_i c_1(\mathcal{L}(\omega_i)) = \sum_{i=1}^n a_i (c_1(g_i) + c_1(\mathcal{L}(\alpha_i))) \equiv \sum_{l=1}^s a_{i_l} c_1(g_{i_l}) \pmod{\text{im}(\mathbf{c}^{(1)})},$$

for our chosen subset  $\{i_1, \dots, i_s\} \subset \{1, \dots, n\}$ . If all  $a_{i_l} \in \mathbb{Z}$  are divisible by  $p$ , we are done. So we assume at least one  $a_{i_l}$  is coprime to  $p$ .

Applying the Tits map  $\beta_\xi$  to the class of  $\rho_w$ , we get

$$\beta_\xi(\bar{\rho}_w) = \beta_\xi \left( \sum_{l=1}^s a_{i_l} \bar{\omega}_{i_l} \right) = \bigotimes_{l=1}^s \beta_\xi(\bar{\omega}_{i_l})^{\otimes a_{i_l}} = \bigotimes_{l=1}^s [A_{\bar{\omega}_{i_l}}]^{\otimes a_{i_l}}$$

By the assumptions that at least one of the  $a_{i_l}$  is coprime to  $p$  and that  $p \mid i_p$ , we have  $\beta_\xi(\bar{\rho}_w) \in H \setminus \{1\}$ . Thus,  $p \mid \text{ind}(\beta_\xi(\bar{\rho}_w))$ , and so  $\text{ind}(A_w)c_1(g_w) = 0 \in \text{Ch}^1(X)$ .

For the case  $m > 1$  we work under the hypothesis that  $p^2 \mid i_p$ , and proceed by induction. We assume that the result  $I_{\text{res}}^{(m')} \subseteq I_c^{(m')}$  holds for all  $m' < m$ . It can be seen that

$$I_{\text{res}}^{(m)} = \left( \bigoplus_{j=1}^{m-1} \text{CH}^{m-j}(X) \cdot \text{im}(\text{res}_{\text{CH}}^{(j)}) \right) \oplus \text{im}(\text{res}_{\text{CH}}^{(m)}).$$

By the inductive hypothesis,  $\text{im}(\text{res}_{\text{CH}}^{(j)}) \subseteq I_{\text{res}}^{(j)} \subseteq I_c^{(j)}$  for  $1 \leq j \leq m-1$ , which implies that  $\text{CH}^{m-j}(X) \cdot \text{im}(\text{res}_{\text{CH}}^{(j)}) \subseteq I_c^{(m)}$  for  $1 \leq j \leq m-1$ . It remains to show that  $\text{im}(\text{res}_{\text{CH}}^{(m)}) \subseteq I_c^{(m)}$ .

By Proposition 8.1.1 we know that  $\text{im}(\text{res}_{\text{CH}}^{(m)}) = c_m(\text{im}(\text{res}_\gamma^{(m)}))$ , and is generated by elements of the form

$$a = (m-1)! \binom{\text{ind}(A_{w_1})}{i_1} \cdots \binom{\text{ind}(A_{w_k})}{i_k} c_1(g_{w_1})^{i_1} \cdots c_1(g_{w_k})^{i_k},$$

where  $i_1 + \dots + i_k = m$ , and  $w_1, \dots, w_k \in W$ .

If  $i_l < m$  for all  $l = 1, \dots, k$ , then  $\binom{\text{ind}(A_{w_l})}{i_l} c_1(g_{w_l})^{i_l} \in I_{\mathfrak{c}}^{(i_l)}$  by the inductive hypothesis. Thus,  $a \in I_{\mathfrak{c}}^{(i_1)} \dots I_{\mathfrak{c}}^{(i_k)} \subseteq I_{\mathfrak{c}}^{(m)}$ . If, on the other hand,  $a$  is of the form  $a = \binom{\text{ind}(A_w)}{m} c_1(g_w)^m$ , then we apply the previous argument. Namely, we have  $\rho_w = \sum_{i=1}^n a_i \omega_i$ , which implies

$$(c_1(g_w))^m = \left( \sum_{i=1}^n a_i c_1(\mathcal{L}(\omega_i)) \right)^m = \left( \sum_{i=1}^n a_i (c_1(g_i) + c_1(\mathcal{L}(\alpha_i))) \right)^m.$$

Again,  $c_1(\mathcal{L}(\alpha_i)) \in \text{im}(\mathfrak{c}^{(1)})$  implies  $c_1(\mathcal{L}(\alpha_i)) \in I_{\mathfrak{c}}^{(1)}$  for all  $i = 1, \dots, n$ , and so all terms in the above expansion divisible by some  $c_1(\mathcal{L}(\alpha_i))$  are contained in  $I_{\mathfrak{c}}^{(m)}$ . So, we may write

$$(c_1(g_w))^m = \left( \sum_{l=1}^s a_{i_l} c_1(g_{i_l}) \right)^m \pmod{I_{\mathfrak{c}}^{(m)}}.$$

If  $a_{i_l}$  is divisible by  $p$  for all  $l = 1, \dots, s$ , then we are done, so we assume that at least one  $a_{i_l}$  is coprime to  $p$ . This ensures that  $i_p \mid \text{ind}(\beta_{\xi}(\bar{\rho}_w))$ , and so  $v_p(\text{ind}(A_w)) \geq v_p(i_p)$ . It is clear that for any  $b \in \mathbb{Z}_{>0}$  if  $p^2 \mid b$  then  $p \mid \binom{b}{l}$  for all  $1 \leq l \leq p$ . Thus, under the hypothesis that  $p^2 \mid i_p$ , we have  $p \mid \binom{\text{ind}(A_w)}{m}$ , and so  $\binom{\text{ind}(A_w)}{m} (c_1(g_w))^m = 0$  in  $\text{Ch}^m(X)$ .  $\square$

In the case  $p = 2$  and  $G$  an orthogonal group, this is proven by Quéguiner-Mathieu, Semenov and Zainoulline in [40].

### 8.3 The $J$ -invariant

As in the previous section, let  $X_s$  be the variety of Borel subgroups of a split simple linear algebraic group  $G_s$ . Fix a split maximal torus  $T \subset G_s$  and denote by  $T^*$  the group of characters of  $G_s$ . Let  $I_{\mathfrak{c}}$  be the ideal generated by the constant-free elements in the image of the characteristic map  $\mathfrak{c} : S^*(T^*) \rightarrow \text{CH}(X_s)$ .

Recall that for a fixed prime  $p$ ,  $\text{Ch}(X_s)/I_{\mathfrak{c}} \simeq \mathbb{F}_p[x_1, \dots, x_r]/(x_1^{p^{k_1}}, \dots, x_r^{p^{k_r}})$ . Let  $d_i$  be the degree of the generator  $x_i$ . We will impose a well-ordering on the monomials  $x_1^{m_1} \dots x_r^{m_r}$  known as the **DegLex order** [38]. For ease of notation, we denote the monomial  $x_1^{m_1} \dots x_r^{m_r}$  by  $x^M$ , where  $M$  is the  $r$ -tuple of integers  $(m_1, \dots, m_r)$ , and set  $|M| = \sum_{i=1}^r d_i m_i$ . Given two  $r$ -tuples  $M = (m_1, \dots, m_r)$  and  $N = (n_1, \dots, n_r)$ , we say that  $x^M \leq x^N$  (or equivalently  $M \leq N$ ) if either  $|M| < |N|$ , or  $|M| = |N|$  and  $m_i \leq n_i$  for the greatest  $i$  such that  $m_i \neq n_i$ .

Consider the restriction map  $\text{res}_{\text{CH}} : \text{CH}^*(X) \rightarrow \text{CH}^*(X_s)$ . As in the previous section, we let  $I_{\text{res}}$  denote the ideal generated by the constant-free elements in the image of  $\text{res}_{\text{CH}}$ . In [28, Thm. 6.4(1)], it is shown that  $I_{\text{res}} \supseteq I_{\mathfrak{c}}$  and that there always exists a cocycle  $\xi$  over some field extension  $K$  of  $F$  such that  $I_{\text{res}} = I_{\mathfrak{c}}$ . Such a  $\xi$  is called a **generic torsor**.

This inclusion induces surjections  $\text{CH}(X_s)/I_{\mathfrak{c}} \twoheadrightarrow \text{CH}(X_s)/I_{\text{res}}$  and  $\text{Ch}(X_s)/I_{\mathfrak{c}} \twoheadrightarrow \text{Ch}(X_s)/I_{\text{res}}$ . For each  $1 \leq i \leq r$ , we define  $j_i$  to be the smallest integer such that  $I_{\text{res}}$  contains an element  $a$  of the form

$$a = x_i^{p^{j_i}} + \sum_{x^M < x_i^{p^{j_i}}} c_M x^M, \quad c_M \in \mathbb{F}_p.$$

While  $r, d_i$  and  $k_i$  for  $i = 1, \dots, r$  depend only on the group  $G_s$ , the values  $j_1, \dots, j_r$  depend also on the choice of  $\xi$ , or equivalently, of the non-split group  $G$ . Thus given the DegLex ordering defined above, we have a well-defined  $r$ -tuple  $J_p(G) = (j_1, \dots, j_r)$ , called the **J-invariant** of  $G$ . We note that for any choice of  $G$ ,  $0 < j_i < k_i$  for all  $i = 0, \dots, r$ .

Let  $J_p^{(1)}(G) = \{j_i \mid d_i = 1\}$  be the sub-tuple of  $J_p(G)$  consisting of only degree 1 parameters. We say that  $J_p^{(1)}(G) > m$  if for every index  $j_l$  such that  $k_l > m$  we have  $j_l > m$ .

We can now reformulate the first theorem in terms of the J-invariant to obtain the following result, which can be seen as a generalization of Theorem 3.8 in [40].

**8.3.1 Theorem.** *Fix a prime  $p$  and a group  $G$  of inner type. If  $p \mid i_p$ , then  $J_p^{(1)}(G) > 0$ . If  $p^2 \mid i_p$ , then  $J_p^{(1)}(G) > 1$ .*

*Proof.* Let  $\pi : \text{Ch}(X_s) \rightarrow \text{Ch}(G_s)$  be the pullback induced by the natural projection  $G_s \twoheadrightarrow X_s$ . We consider the composite map  $\pi \circ \text{res}_{\text{CH}} : \text{Ch}(X) \rightarrow \text{Ch}(X_s) \rightarrow \text{Ch}(G_s)$ . We begin with the assumption that  $p \mid i_p$ .

Let  $R_{\xi} = \text{im}(\pi \circ \text{res}_{\text{CH}})$ . Then  $\pi(a) \in R_{\xi}^{(1)}$  implies that  $a \in \text{im}(\text{res}_{\text{CH}}^{(1)}) = I_{\mathfrak{c}}^{(1)}$  by Theorem 8.2.3. Thus,  $\pi(a) = 0 \in \text{Ch}^1(X_s)/I_{\mathfrak{c}}$  and so  $R_{\xi}^{(1)} = \{0\}$ . Let  $x_1, \dots, x_s$  be generators of degree 1 in  $\text{Ch}^1(G_s)$ . By the definition of the J-invariant,  $j_1$  is the smallest non-negative integer  $m$  such that  $x_1^{p^m} \in R_{\xi}$ . Since  $x_1$  is non-trivial, we must have  $x_1^{p^0} = x_1 \notin R_{\xi}^{(1)}$  by the above argument, and so  $j_1 > 0$ .

The same argument applies for the remaining generators. Let  $1 < i \leq s$ , then if  $x^M < x_i$ ,  $x^M = x_j$  for some  $j < i$ . Since  $x_i + a_{i-1}x_{i-1} + \dots + a_1x_1$  is non-trivial for any  $a_1, \dots, a_{i-1} \in \mathbb{F}_p$ , it cannot belong to  $R_{\xi}^{(1)}$ . Therefore  $j_i > 0$ , and so  $J_p^{(1)}(G) > 0$ .

Under the hypothesis that  $p^2 \mid i_p$ , we have the inclusion  $\text{im}(\text{res}_{\text{CH}}^{(p)}) \subset I_{\text{res}}^{(p)}$  and by Theorem 8.2.3,  $I_{\text{res}}^{(p)} = I_{\mathfrak{c}}^{(p)}$ . Again,  $R_{\xi}^{(p)} = \text{im}(\pi \circ \text{res}_{\text{CH}}^{(p)}) = \{0\}$ . To show that

$J_p^{(1)}(G) > 1$ , we begin with the generator  $x_1$ . If  $k_1 \leq 1$  we are done, so suppose  $k_1 > 1$ . Then,  $x_1^{p^1} = x_1^p \in \text{Ch}^p(G_s)$  is non-trivial, and so  $x^p \notin R_\xi^{(p)}$  implies  $j_1 > 1$ . Again, we extend the argument for the remaining generators. Suppose that  $k_i > 1$  for some  $1 < i \leq s$ . Then, the element

$$\pi(a) = x_i^p + \sum_{(x^M < x_i^p) \cap (|M|=p)} a_M x^M$$

is non-trivial for any  $a_M \in \mathbb{F}_p$  and hence  $\pi(a) \notin R_\xi^{(p)}$ , and so  $j_i > 1$ . Thus  $J_p^{(1)}(G) > 1$ .  $\square$

## 8.4 Application

We now apply the results of the previous section to some  $E_6$  varieties. For this, we will require the following result concerning the possible values of the  $J$ -invariant.

**8.4.1 Lemma.** *If  $d_i = 1$ , then  $j_i \leq \max\{v_p(\text{ind } A)\}$ , where  $A$  runs through all Tits algebras of  $G$ . If there exists a Tits algebra  $A$  of  $G$  with  $p \mid \text{ind}(A)$ , then  $j_i > 0$  for at least one generator  $x_i$  having  $d_i = 1$ .*

*Proof.* For the first statement, we note that for any  $w \in W$ ,  $c_1(\mathcal{L}(\rho_w))^{p^b} \in I_{\text{res}}$ , where  $b = v_p(\text{ind}(A_{\rho_w}))$  [40, Lemma 1.12]. Since  $\beta_\xi(\bar{\rho}_i) = \beta_\xi(\bar{\omega}_i)$  for  $i = 1, \dots, n$ , we have  $\text{ind}(A_{\bar{\rho}_i}) = \text{ind}(A_{\bar{\omega}_i})$  for  $l = 1, \dots, s$ . Letting  $b_l = v_p(\text{ind}(A_{\bar{\omega}_i}))$ , we then have  $c_1(g_{i_l})^{p^{b_l}} \in I_{\text{res}}$ , and hence  $j_l \leq v_p(\text{ind}(A_{\bar{\omega}_i}))$ . For proof of the second statement, see [37, Prop. 4.2].  $\square$

**8.4.2 Example.** Let  $G$  be a simple group of inner type  $B$ ,  $C$ ,  $E_6$ , or  $E_7$ . Then, the subgroup  $H \subset \text{Br}(F)$  generated by the Tits algebras of  $G$  is a cyclic group of order  $p = 2$  or  $3$ , and  $J_p^{(1)}(G)$  consists of a single integer  $j_1$  [7, Chap. 6]. As a consequence of Theorem 8.3.1 and Lemma 8.4.1,  $j_1 = 0$  if and only if all of the Tits algebras of  $G$  are split.

Recall that if  $G$  is a group of inner type  $E_6$ , it has one Tits algebra  $A$  of index  $3^d$  for some  $0 \leq d \leq 3$ . Consider the  $J$ -invariant of  $G$  modulo  $p = 3$ . We note that  $\text{Ch}(G)$  has precisely two generators  $x_1$  and  $x_2$ , with  $d_1 = 1$  and  $d_2 = 4$ , where the 3-power relations are defined by  $k_1 = 2$  and  $k_2 = 1$  [23, Table II]. Thus  $J_3(G) = (j_1, j_2)$ , where  $0 \leq j_1 \leq 2$  and  $0 \leq j_2 \leq 1$ . These values are independent of the characteristic of the base field.

**8.4.3 Proposition.** *Let  $G$  be a group of inner type  $E_6$  with Tits algebra  $A$ . Let  $J_3(G) = (j_1, j_2)$  be the  $J$ -invariant of  $G$  modulo 3. Then,*

1.  $\text{ind}(A) = 1$  if and only if  $j_1 = 0$
2.  $\text{ind}(A) = 3$  if and only if  $j_1 = 1$
3.  $\text{ind}(A) \geq 9$  if and only if  $j_1 = 2$ .

*Proof.* We first note that  $J_3(G) = (0, 0)$  if and only if  $G$  splits by a field extension of degree coprime to 3 [38, Corollary 6.7]. For the second case, suppose first that  $j_1 = 1$ . By Corollary 8.3.1 this implies that  $\text{ind } A = 1$  or 3. However, by the first case,  $j_1 \neq 0$  implies  $\text{ind } A \neq 1$  and so  $\text{ind } A = 3$ . Conversely, suppose  $\text{ind } A = 3$ . Then  $v_3(\text{ind } B) \leq 1$  for all  $[B] \in H$ . By Lemma 8.4.1,  $j_1 \leq \max(v_3(\text{ind } B)) = 1$ . Again by the first case,  $\text{ind } A \neq 1$  implies  $j_1 \neq 0$  and so  $j_1 = 1$ .

Suppose  $\text{ind } A = 27$ , then for any splitting field  $K/F$  of  $A$ ,  $27 \mid [K : F]$ . By [38, Prop. 6.6], we must then have  $3 \leq j_1 + j_2$ . Since  $j_1 \leq k_1 = 2$  and  $j_2 \leq k_2 = 1$ , the only possible value for the  $J$ -invariant is  $J_3(G) = (2, 1)$ .  $\square$

# Chapter 9

## The twisted $\gamma$ -filtration

### 9.1 Introduction

Let  $G_s$  be a split simple linear algebraic group over a field  $F$ . As in the previous chapters, we let  $G$  be an inner twisted form of  $G_s$ , by means of a cocycle  $\xi \in Z^1(F, G_s)$ . We fix a split maximal torus  $T_s \subset G_s$  and Borel subgroup  $B_s \supset T_s$ . Let  $T$  and  $B$  be the corresponding torus and Borel subgroup of  $G$ , respectively. We let  $X = G/B$  be the variety of Borel subgroups of  $G$ , and similarly we let  $X_s = G_s/B_s$  be the corresponding split variety.

As we saw in Chapter 7, the  $\gamma$ -filtration on the Grothendieck group of  $X$  acts as an approximation to the Chow groups  $\mathrm{CH}^*(X)$  of algebraic cycles modulo rational equivalence. In particular, we have a surjection  $\gamma^{2/3} K_0(X) \twoheadrightarrow \mathrm{CH}^2(X)$  (cf. [14, Ex. 15.3.6]), and so the torsion subgroup of  $\gamma^{2/3} K_0(X)$  may be viewed as an upper bound for the torsion subgroup of  $\mathrm{CH}^2(X)$ .

Determining torsion in  $\mathrm{CH}^d(X)$  is a non-trivial problem, and only partial results are known. For  $d = 2, 3$  and  $G$  strongly inner, we refer to [39] and [15]. The case of quadrics was considered in [27] for  $d = 2, 3, 4$ . The  $\gamma$ -filtration was used in [26], and in [25] it was found that  $\mathrm{Tors} \mathrm{CH}^4$  can be infinitely generated. Recent results in [4] and [3] provide upper bounds for the annihilators of  $\mathrm{Tors} \mathrm{CH}^d$  for arbitrary  $d$ .

In Section 9.2 we introduce the twisted  $\gamma$ -filtration, first defined in [48]. Sections 9.3 and 9.4 contain original results by the author, constructing a torsion element in  $\gamma^{2/3} K_0(X)$  for groups of type  $D_n$ . A version of these results can be found in [22]. Section 9.5 also consists of original results, relating this torsion element to properties of triality triples.

## 9.2 Defining the twisted $\gamma$ -filtration

For the split group  $G_s$  of rank  $n$ , we fix a set of simple roots  $\alpha_1, \dots, \alpha_n$  and a corresponding set of fundamental weights  $\omega_1, \dots, \omega_n$ . We denote by  $\Lambda_r$  and  $\Lambda$  the root lattice and weight lattice of  $G_s$ , respectively (cf. Chapter 6 for definitions).

We recall that  $K_0(X_s)$  is generated by classes of line bundles, and has a  $\mathbb{Z}$ -basis  $\{g_w\}_{w \in W}$  indexed by elements of the Weyl group of  $G_s$ . The basis element  $g_w$  is constructed by first defining a weight

$$\rho_w = \sum_{\{i \in 1, \dots, n \mid w^{-1}(\alpha_i) < 0\}} w^{-1}(\omega_i).$$

Then, we set  $g_w$  to be the image of  $\rho_w$  under the characteristic map  $\mathbf{c} : \mathbb{Z}[T^*] \rightarrow K_0(X_s)$ . This map is defined by sending  $e^\lambda$  to the class of the associated line bundle  $[\mathcal{L}(\lambda)]$  for  $\lambda \in \Lambda$ . Note that while  $X_s$  (and therefore  $K_0(X_s)$ ) does not depend on the isogeny class of  $G_s$ , the image of this map does. In particular, if  $G_s$  is simply connected then the characteristic map is surjective [43]. In particular, we denote by  $g_i$  the Steinberg element corresponding to the simple reflection  $s_{\alpha_i}$ , and note that  $\rho_i = \omega_i - \alpha_i$ .

In Chapter 7, we saw that the restriction map  $\text{res} : K_0(X) \hookrightarrow K_0(X_s)$  is injective, and the image is given by the sub lattice  $\langle \text{ind}(A_w)g_w \rangle$ , where  $[A_w] \in \text{Br}(F)$  is the class of the Tits algebra associated to the weight  $\rho_w$  via the Tits map  $\beta_\xi : \Lambda/T^* \rightarrow \text{Br}(F)$  (cf. Chapter 7).

Since characteristic classes commute with restrictions, the restriction map on the degree  $d$  component of the  $\gamma$ -filtration is well defined

$$\text{res}_\gamma : \gamma^d K_0(X) \rightarrow \gamma^d K_0(X_s),$$

and is generated by terms of the following form:

$$\text{res}(\gamma^d K_0(X)) = \left\langle \prod_{j=1}^m \binom{\text{ind}(A_{w_j})}{n_j} \gamma_1(g_{w_j})^{n_j} \mid n_1 + \dots + n_m \geq d, w_j \in W \right\rangle. \quad (9.1)$$

Unfortunately, this description is not as practical to work with as one would hope. One problem is that expressing the tensor product of two Steinberg elements as a linear combination of Steinberg elements is a non-trivial task (especially when  $W$  is large). Therefore, for each degree  $d$ , we are left with an explicit (albeit large) set of generators, but no reasonable set of relations.

The **twisted  $\gamma$ -filtration** was introduced in [48] as a tool for identifying non-trivial elements in  $\gamma^{2/3} K_0(X)$  more easily. Specifically, the degree  $d$  component of the twisted

$\gamma$ -filtration is a surjective image of the subgroup  $\gamma^{d/d+1} K_0(X)$ , with well-defined and manageable sets of both generators and relations. Returning to the characteristic map and the canonical surjection  $\Lambda \rightarrow \Lambda/T^*$ , we have the following diagram,

$$\begin{array}{ccccc} \mathbb{Z}[\Lambda] & \xrightarrow{\mathbf{c}} & K_0(X_s) & \xrightarrow{\cong} & \mathbb{Z}[\Lambda]/\ker(\mathbf{c}) \\ \downarrow & & \searrow q & & \downarrow \\ \mathbb{Z}[\Lambda/T^*] & \longrightarrow & & & \overline{\mathbb{Z}[\Lambda/T^*]/\ker(\mathbf{c})} \end{array}$$

which allows us to define the quotient ring

$$\mathfrak{G}_s := \mathbb{Z}[\Lambda/T^*]/\overline{\ker(\mathbf{c})},$$

and the composite map  $q : K_0(X_s) \rightarrow \mathfrak{G}_s$ , which is a surjective ring homomorphism. Observe that if  $G$  is simply connected then  $\mathfrak{G}_s \simeq \mathbb{Z}$ .

**9.2.1 Lemma** ([48, 3.3]). *The ideal  $\overline{\ker(\mathbf{c})} \subset \mathbb{Z}[\Lambda/T^*]$  is generated by the elements  $d_i(1 - e^{\bar{\omega}_i})$ ,  $i = 1, \dots, n$ , where  $d_i$  is the number of elements in the  $W$ -orbit of the fundamental weight  $\omega_i$ .*

The  $\gamma$ -filtration can also be defined on the ring  $\mathbb{Z}[\Lambda]$ , by setting

$$\gamma^d \mathbb{Z}[\Lambda] = \langle (1 - e^{\lambda_1}) \dots (1 - e^{\lambda_k}) \mid k \geq d \rangle.$$

With this description, we can see that  $\gamma^d K_0(X_s) = \mathbf{c}(\gamma^d \mathbb{Z}[\Lambda])$  and so we may define the  $\gamma$ -filtration on the ring  $\mathfrak{G}_s$  by setting  $\gamma^d \mathfrak{G}_s := q(\gamma^d K_0(X_s))$ , for  $d \geq 0$ .

At this point, we have a filtration on  $\mathfrak{G}_s$  which agrees with the  $\gamma$ -filtration on  $K_0(X_s)$ . We can extend this idea to the non-split case using the restriction map  $\text{res} : K_0(X) \rightarrow K_0(X_s)$ . For  $d \geq 0$ , we define

$$\gamma^d \mathfrak{G} := q(\text{res}(\gamma^d K_0(X)))$$

and let  $\gamma^{d/d+1} \mathfrak{G} = \gamma^d \mathfrak{G} / \gamma^{d+1} \mathfrak{G}$ . The associated graded ring  $\gamma^* \mathfrak{G} := \bigoplus_{d \geq 0} \gamma^{d/d+1} \mathfrak{G}$  is called the  $\gamma$ -invariant of  $G$  (or equivalently, the  $\gamma$ -invariant of  $\xi$ ). There exists a surjective ring homomorphism  $\gamma^* K_0(X) \rightarrow \gamma^* \mathfrak{G}$ , so we may provide a set of generators of the twisted  $\gamma$ -filtration, corresponding to (9.1).

**9.2.2 Theorem.** *The degree  $d$  component of the twisted  $\gamma$ -filtration is generated as follows:*

$$\gamma^d \mathfrak{G} = \left\langle \prod_{j=1}^m \binom{\text{ind}(A_{w_j})}{n_j} (1 - e^{-\bar{\rho}_{w_j}})^{n_j} \mid n_1 + \dots + n_m \geq d, w_j \in W \right\rangle$$

*Proof.* By definition,  $g_w = \mathbf{c}(\rho_w)$  and  $\gamma_1(g_w) = \mathbf{c}(1 - e^{-\rho_w})$ . By the commutative diagram above,  $q(\gamma_1(g_w)) = 1 - e^{-\bar{\rho}_w}$ . Therefore, applying the map  $q$  to the generators of  $\text{res}(\gamma^d K_0(X))$  from (9.1) gives the desired result.  $\square$

**9.2.3 Example.** Suppose  $\Lambda/T^* = \langle \sigma \rangle$  has order 2. The corresponding Tits algebra  $A_\sigma := \beta_\xi(\sigma)$  has index  $2^r$  for some  $r \geq 0$ . Setting  $y = 1 - e^\sigma$ , we have  $\mathfrak{G}_s \simeq \mathbb{Z}[y]/(y^2 - 2y, dy)$ , where  $d$  is the greatest common divisor of the  $d_i$  such that  $\bar{\omega}_i = \sigma$ . These values were computed in [48]:

$B_n$ :  $\bar{\omega}_i = 0$  for all  $i < n$ , so  $d = d_n = 2^n$ .

$C_n$ :  $\bar{\omega}_i = 0$  for  $i$  even, so  $d = \text{gcd}\{d_1, d_3, \dots\}$ .

$E_7$ :  $\bar{\omega}_1 = \bar{\omega}_3 = \bar{\omega}_4 = \bar{\omega}_6 = 0$ , so  $d = \text{gcd}\{d_2, d_5, d_7\} = 8$ .

If  $G$  is of type  $D_n$ , then  $\Lambda/T^*$  may also be of order 2. These cases will be considered in the following section.

**9.2.4 Example.** Suppose  $G$  is of inner adjoint type  $B_n, C_n$  or  $E_7$ , so that  $\Lambda/T^*$  is of order 2, as in the previous example. We set  $y := 1 - e^\sigma$  and note that for any  $w \in W$ ,  $\bar{\rho}_w \in \{0, \sigma\}$ . By Theorem 9.2.2 we thus have

$$\gamma^d \mathfrak{G} = \left\langle \prod_{j=1}^m \binom{\text{ind}(A_\sigma)}{n_1} y^{n_j} \mid n_1 + \dots + n_m \geq d \right\rangle,$$

Since  $y^2 = 2y \in \mathfrak{G}_s$ , and  $\text{ind}(A_\sigma) = 2^r$ , we may rewrite this as

$$\gamma^d \mathfrak{G} = \left\langle \frac{2^r}{(2^r, m)} 2^{m-1} y \mid m \geq d \right\rangle,$$

where we denote by  $(a, b)$  the greatest common divisor of integers  $a$  and  $b$ . So, if  $\text{ind}(A_\sigma) = 1$ , then  $\gamma^d \mathfrak{G} = \langle 2^{d-1} y \rangle$ . If, on the other hand  $\text{ind}(A_\sigma) > 1$ , then we have

$$\begin{aligned} \gamma^1 \mathfrak{G} &= \langle 2^r y \rangle \\ \gamma^2 \mathfrak{G} &= \langle 2^r y \rangle \\ \gamma^3 \mathfrak{G} &= \begin{cases} \langle 2^{r+1} y \rangle & \text{if } r \geq 2 \\ \langle 2^{r+2} y \rangle & \text{if } r = 1 \end{cases} \end{aligned}$$

So, we observe that  $\gamma^{1/2} \mathfrak{G}$  is trivial. Adding in the relation  $dy \equiv 0$ , we obtain a complete description of  $\gamma^{2/3} \mathfrak{G}$ .

$$\gamma^{2/3} \mathfrak{G} = \begin{cases} \langle 2y \rangle / \langle y^2 - 2y, (d, 8)y \rangle & \text{if } \text{ind}(A_\sigma) = 2 \\ \langle 2^r y \rangle / \langle y^2 - 2y, (d, 2^{r+1})y \rangle & \text{if } \text{ind}(A_\sigma) = 2^r \geq 4 \end{cases}$$

### 9.3 Groups of type $D_n$ , $n$ even

Let  $G_s$  be the adjoint group  $\text{PGO}_{2n}^+$ , for  $n \geq 4$  even. That is,  $G_s$  is of type  $D_n$ , for  $n$  even. Let  $G$  be an inner twisted form of  $G_s$ , by means of a cocycle  $\xi \in Z^1(F, G_s)$ . As we saw in Chapter 6,  $\Lambda/T^* = \Lambda/\Lambda_r = \{0, \bar{\omega}_1, \bar{\omega}_{n-1}, \bar{\omega}_n\} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ . We denote by  $C_+$  and  $C_-$  the Tits algebras corresponding to  $\bar{\omega}_{n-1}$  and  $\bar{\omega}_n$  respectively, and denote by  $A$  the Tits algebra corresponding to  $\bar{\omega}_1$ . We recall that the fundamental relations in  $\text{Br}(F)$  for this group are as follows:

$$\begin{aligned} 2[A] &= 2[C_+] = 2[C_-] = 0 \\ [A] + [C_+] + [C_-] &= 0. \end{aligned}$$

In order to simplify some calculations, we label the nontrivial elements of  $\Lambda/T^*$  by  $\sigma_a, \sigma_b, \sigma_c$  such that  $\text{ind}(A_{\sigma_a}) \leq \text{ind}(A_{\sigma_b}) \leq \text{ind}(A_{\sigma_c})$ . The fundamental relations then provide the following inequality:

$$\text{ind}(A_{\sigma_a}) \text{ind}(A_{\sigma_b}) \geq \text{ind}(A_{\sigma_c}). \quad (9.2)$$

We define  $y_a := 1 - e^{\sigma_a}$  and  $y_b := 1 - e^{\sigma_b}$  to obtain

$$\mathfrak{G}_s \cong \mathbb{Z}[y_a, y_b] / (y_a^2 - 2y_a, y_b^2 - 2y_b, d_a y_a, d_b y_b, d_c(y_a + y_b - y_a y_b)), \quad (9.3)$$

where  $d_a, d_b, d_c$  are determined by the number of elements in the  $W$ -orbit of the three corresponding fundamental weights.

**9.3.1 Example.** If  $n = 4$  (ie.  $G_s = \text{PGO}_8^+$ ), then  $d_a = d_b = d_c = 8$ , and so

$$\mathfrak{G}_s \cong \mathbb{Z}[y_a, y_b] / (y_a^2 - 2y_a, y_b^2 - 2y_b, 8y_a, 8y_b).$$

To compute  $\gamma^{2/3}\mathfrak{G}$  in this case, we begin by making note of 4 cases, which depend on the indices of the Tits algebras. The arguments used are independent of the choice of generator, so without loss of generality we consider the generator  $y_a$  and its associated index  $\text{ind}(A_{\sigma_a})$ .

- I: Suppose  $\text{ind}(A_{\sigma_a}) = 1$ . Then,  $y_a^2 \equiv 2y_a \in \gamma^2\mathfrak{G}$ , and  $y_a^3 \equiv 4y_a \in \gamma^3\mathfrak{G}$ . Thus  $\gamma^{2/3}\mathfrak{G}$  has a summand isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ .
- II: Suppose  $\text{ind}(A_{\sigma_a}) = 2$ . Then  $\binom{2}{2}y_a^2 \equiv 2y_a \in \gamma^2\mathfrak{G}$ , and  $\binom{2}{2}\binom{2}{1}y_a^3 \equiv 8y_a \in \gamma^3\mathfrak{G}$ . This time,  $\gamma^{2/3}\mathfrak{G}$  has a summand isomorphic to a quotient of  $\mathbb{Z}/4\mathbb{Z}$ , dependent on  $d_a$ .

- III: Suppose  $4 \mid \text{ind}(A_{\sigma_a})$  and  $\text{ind}(A_{\sigma_a}) \mid d_a$ . Then  $\binom{\text{ind}(A_{\sigma_a})}{2} y_a^2 \equiv \text{ind}(A_{\sigma_a}) y_a \in \gamma^2 \mathfrak{G}$ , and  $\binom{\text{ind}(A_{\sigma_a})}{4} y_a^4 \equiv 2 \text{ind}(A_{\sigma_a}) y_a \in \gamma^4 \mathfrak{G} \subseteq \gamma^3 \mathfrak{G}$ . So,  $\gamma^{2/3} \mathfrak{G}$  has a summand which is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ .
- IV: Suppose  $d_a \mid \text{ind}(A_{\sigma_a})$ . In this situation we have  $\binom{\text{ind}(A_{\sigma_a})}{r} y_a^r \mid \text{ind}(A_{\sigma_a}) y_a \equiv 0$  for all  $r \geq 2$ , and so the summand is trivial.

For  $y_a + y_b - y_a y_b$ , the results are slightly different for  $\text{ind}(A_{\sigma_c}) \geq 4$ . We have  $\text{ind}(A_{\sigma_c})(y_a + y_b - y_a y_b) \in \gamma^2 \mathfrak{G}$  as before; however, in  $\gamma^4 \mathfrak{G}$  we have the additional element  $\binom{\text{ind}(A_{\sigma_a})}{2} \binom{\text{ind}(A_{\sigma_b})}{2} y_a^2 y_b^2 \equiv \text{ind}(A_{\sigma_a}) \text{ind}(A_{\sigma_b}) y_a y_b$ . By the relation (9.2), we have a non-trivial multiple of  $y_a + y_b - y_a y_b$  only if this inequality is strict.

**9.3.2 Lemma.** *Let  $\gamma^{2/3} \mathfrak{G} = G_a \oplus G_b \oplus G_c$ . Then we have*

$$\begin{aligned}
 G_a &\simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_a}) = 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_a}) = 2 \text{ and } d_a = 4 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_a}) = 2 \text{ and } 8 \mid d_a \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_a}) \geq 4 \text{ and } 2 \text{ind}(A_{\sigma_a}) \mid d_a \\ 0 & \text{if } d_a \mid \text{ind}(A_{\sigma_a}) \end{cases} \\
 G_b &\simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_b}) = 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_b}) = 2 \text{ and } d_b = 4 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_b}) = 2 \text{ and } 8 \mid d_b \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_b}) \geq 4 \text{ and } 2 \text{ind}(A_{\sigma_b}) \mid d_b \\ 0 & \text{if } d_b \mid \text{ind}(A_{\sigma_b}) \end{cases} \\
 G_c &\simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_c}) = 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_c}) = 2 \text{ and } d_c = 4 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_c}) = 2 \text{ and } 8 \mid d_c \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_c}) \geq 4, 2 \text{ind}(A_{\sigma_c}) \mid d_c \text{ and } \text{ind}(A_{\sigma_a}) \text{ind}(A_{\sigma_b}) > \text{ind}(A_{\sigma_c}) \\ 0 & \text{if } \text{ind}(A_{\sigma_c}) \geq 4 \text{ and } \text{ind}(A_{\sigma_a}) \text{ind}(A_{\sigma_b}) = \text{ind}(A_{\sigma_c}) \\ 0 & \text{if } d_c \mid \text{ind}(A_{\sigma_c}) \end{cases}
 \end{aligned}$$

**9.3.3 Example.** Let  $G_s = \text{HSpin}_{2n}$  be a half-spin group of rank  $n \geq 4$ , and suppose  $G$  is an inner twisted form of  $G_s$ . Then,  $G$  is of type  $D_n$  for  $n$  even, and we have  $\Lambda/T^* = \langle \sigma \rangle$  where  $\sigma$  is of order 2 [29].

This corresponds to taking the quotient of  $\Lambda/\Lambda_r \simeq \langle \sigma_a \rangle \oplus \langle \sigma_b \rangle$  modulo one of the generators, e.g.  $\sigma_b \equiv 0$ . In this situation, we set  $y_b \equiv 0$  in (9.3) and obtain

$$\mathfrak{G}_s \simeq \mathbb{Z}[y_a]/(y_a^2 - 2y_a, dy_a),$$

where  $d = \min\{d_a, d_b, d_c\} = 2^{v_2(n)+1}$ . Thus we may describe the twisted  $\gamma$ -filtration of  $G$  by taking the quotient of our previous description of  $\gamma^{2/3}\mathfrak{G}_s$  for an inner twisted form of  $\text{PGO}_{2n}^+$  by  $\langle 2^{v_2(n)+1}y_a, y_b \rangle$ . We obtain the following description of  $\gamma^{2/3}\mathfrak{G}$ , which corresponds to the result given in [48, Example 4.8].

$$\gamma^{2/3}\mathfrak{G}_s \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_a}) = 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_a}) = 2 \text{ and } d = 4 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_a}) = 2 \text{ and } 8 \mid d \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \text{ind}(A_{\sigma_a}) \geq 4 \text{ and } 2 \text{ind}(A_{\sigma_a}) \mid d \\ 0 & \text{if } d \mid \text{ind}(A_{\sigma_a}) \end{cases}$$

## 9.4 Constructing a torsion element

Given a split group  $G_s$  over a field  $F$  of arbitrary characteristic, we fix a split maximal torus  $T_s$  and Borel subgroup  $B_s \supset T_s$ . We fix a set of simple roots  $\{\alpha_1, \dots, \alpha_n\}$  and corresponding fundamental weights  $\{\omega_1, \dots, \omega_n\}$ . Denote by  $\Lambda_r, \Lambda$  the root and weight lattices of  $G_s$ , respectively, and note that we have  $\Lambda_r \subset T^* \subset \Lambda$ , where  $T^*$  is the character group of  $G_s$ . We denote by  $X_s = G_s/B_s$  the variety of Borel subgroups of  $G_s$ .

As we saw in Chapter 6, the quotient lattice  $\Lambda/T^*$  is a finite group of order 2, 3, 4 (or cyclic of order  $m \geq 2$  for groups of type  $A_n$ ). It follows that we have at most 2 generators for  $\Lambda/T^*$  (only one in all cases other than  $G_s$  of type  $D_n$ ,  $n$  even). We will set  $\langle \sigma_a, \sigma_b \rangle = \Lambda/T^*$ , identifying  $\sigma_a = \sigma_b$  if the group is cyclic.

We may choose pre-images for  $\sigma_a, \sigma_b$  in  $\Lambda$ , and it is convenient to choose them to be Steinberg weights  $\rho_a, \rho_b$  (again, with  $\rho_a = \rho_b$  if  $\sigma_a = \sigma_b$ ). Finally, we let  $g_a = [\mathcal{L}(\rho_a)], g_b = [\mathcal{L}(\rho_b)]$  be the corresponding Steinberg elements in  $K_0(X_s)$ .

Let  $G$  be an inner twisted form of  $G_s$ , given by the class of a cocycle  $\xi \in Z^1(F, G_s)$  and define  $B_0 \subset G$  to be the Borel subgroup of  $G$  corresponding to  $B_s \subset G_s$ . We define  $X = G/B_0$  and recall that the image of the restriction map  $\text{res} : K_0(X) \hookrightarrow K_0(X_s)$  is given by  $\text{im}(\text{res}) = \langle \text{ind}(A_{\rho_w})g_w \rangle$ .

Now, we define the following element in  $K_0(X_s)$ :

$$\mu := \gamma_1(g_a^2)\gamma_1(g_b^2) + \gamma_2(4g_a) + \gamma_2(4g_b) - \gamma_2(4g_ag_b) \quad (9.4)$$

By Chern class operations, we may rewrite  $\mu$  as

$$\mu = 2\gamma_1(g_a)^2\gamma_1(g_b) + 2\gamma_1(g_a)\gamma_1(g_b)^2 - \gamma_1(g_a)^2\gamma_1(g_b)^2.$$

Note that if  $\Lambda/T^* = \langle \sigma_a \rangle$ , this reduces to  $\mu = 4\gamma_1(g_a)^3 - \gamma_1(g_a)^4$ .

**9.4.1 Example.** If  $G$  has trivial Tits algebras, then  $\mu \in \gamma^3 K_0(X)$ . This is not true in general, in fact it can happen that  $\mu \notin K_0(X)$  at all.

**9.4.2 Proposition.** *Let  $G$  be a simple group of inner type  $B_n, C_n$  or  $D_n$  ( $n$  even). Fix a Borel subgroup  $B_0 \subset G$ , set  $X = G/B_0$  and denote by  $X_s$  the split variety corresponding to  $X$ . There exists an integer  $t \geq 1$ , dependent on the group type, isogeny class, and indices of the Tits algebras of  $G$ , such that*

1.  $t\mu \in \gamma^2 K_0(X)$
2.  $\text{res}(t\mu) \in \gamma^3 K_0(X_s)$
3. There exists some integer  $r \geq t$  such that  $r\mu \in \gamma^3 K_0(X)$

*Proof.* For the case  $\Lambda/T^* \simeq \mathbb{Z}/\mathbb{Z}$ , we refer to [48, Prop. 5.1]. In particular, we may take  $t = \max\{1, \text{ind}(A_\sigma)/8\}$  and  $r = 8$ . We consider here the case that  $\Lambda/T^* = \langle \sigma_a \rangle \oplus \langle \sigma_b \rangle$ , with  $\sigma_c := \sigma_a + \sigma_b$ . For  $\lambda, \lambda' \in \Lambda$ , we have  $[\mathcal{L}(\lambda + \lambda')] = [\mathcal{L}(\lambda)][\mathcal{L}(\lambda')]$ . In particular,  $[\mathcal{L}(2\rho_a)] = g_a^2$ ,  $[\mathcal{L}(2\rho_b)] = g_b^2$  and  $[\mathcal{L}(\rho_a + \rho_b)] = g_a g_b$ . By [16, Cor. 5.5],  $\mathfrak{c}(\mathbb{Z}[T^*]) \subset K_0(X)$ , so we have  $\gamma_1(g_a^2)\gamma_1(g_b^2) \in \gamma^2 K_0(X)$ . By characteristic class operations, we have

$$\gamma_1(g_a^2)\gamma_1(g_b^2) = \gamma_1(g_a)^2\gamma_1(g_b)^2 - 2\gamma_1(g_a)^2\gamma_1(g_b) - 2\gamma_1(g_a)\gamma_1(g_b)^2 + 4\gamma_1(g_a)\gamma_1(g_b) \quad (9.5)$$

$$\gamma_1(g_a g_b) = \gamma_1(g_a) + \gamma_1(g_b) - \gamma_1(g_a)\gamma_1(g_b) \quad (9.6)$$

Since  $\beta_\xi(\bar{\rho}_a + \bar{\rho}_b) = [A_{\sigma_c}]$ , and we have chosen  $\sigma_a, \sigma_b, \sigma_c$  such that  $\text{ind}(A_{\sigma_a}), \text{ind}(A_{\sigma_b}) \mid \text{ind}(A_{\sigma_c})$ , we have

$$\gamma_2(\text{ind}(A_{\sigma_c})g_a), \gamma_2(\text{ind}(A_{\sigma_c})g_b), \gamma_2(\text{ind}(A_{\sigma_c})g_a g_b) \in \gamma^2 K_0(X).$$

With this, we set  $t := \max\{\frac{\text{ind}(A_{\sigma_c})}{4}, 1\}$  and define  $\eta \in \gamma^2 K_0(X)$  by

$$\eta = t\gamma_1(g_a^2)\gamma_1(g_b^2) + \gamma_2(\text{ind}(A_{\sigma_c})g_a) + \gamma_2(\text{ind}(A_{\sigma_c})g_b) - \gamma_2(\text{ind}(A_{\sigma_c})g_a g_b) \quad (9.7)$$

Combining (9.5), (9.6) and (9.7), we obtain

$$\begin{aligned} \eta &= t[\gamma_1(g_a)^2\gamma_1(g_b)^2 - 2\gamma_1(g_a)^2\gamma_1(g_b) - 2\gamma_1(g_a)\gamma_1(g_b)^2 + 4\gamma_1(g_a)\gamma_1(g_b)] \\ &\quad + \frac{\text{ind}(A_{\sigma_c})}{2} [\gamma_1(g_a)^2 + \gamma_1(g_b)^2 - (\gamma_1(g_a) + \gamma_1(g_b) - \gamma_1(g_a)\gamma_1(g_b))^2] \\ &= t[2\gamma_1(g_a)^2\gamma_1(g_b) + 2\gamma_1(g_a)\gamma_1(g_b)^2 - \gamma_1(g_a)^2\gamma_1(g_b)^2] \\ &= t\mu \end{aligned}$$

The statement that  $\text{res}(t\mu) \in \gamma^3 K_0(X_s)$  follows immediately from (9.4). For the final statement, we consider an element  $\eta' \in \gamma^2 K_0(X)$  defined as follows:

$$\begin{aligned} \eta' &= 2\gamma_1(\text{ind}(A_{\sigma_c})g_a)^2\gamma_1(\text{ind}(A_{\sigma_c})^2g_b) + 2\gamma_1(\text{ind}(A_{\sigma_c})^2g_a)\gamma_1(\text{ind}(A_{\sigma_c})g_b)^2 \\ &\quad - \gamma_1(\text{ind}(A_{\sigma_c})g_a)^2\gamma_1(\text{ind}(A_{\sigma_c})g_b)^2 \\ &= 2\text{ind}(A_{\sigma_c})^4\gamma_1(g_a)^2\gamma_1(g_b) + 2\text{ind}(A_{\sigma_c})^4\gamma_1(g_a)\gamma_1(g_b)^2 - \text{ind}(A_{\sigma_c})^4\gamma_1(g_a)^2\gamma_1(g_b)^2 \\ &= \text{ind}(A_{\sigma_c})^4\mu. \end{aligned}$$

Setting  $r := \text{ind}(A_{\sigma_c})^4$ , we have  $r \geq t$  and  $r\mu \in \gamma^3 K_0(X)$ .  $\square$

Through use of the twisted  $\gamma$ -filtration, we can now provide criteria for when  $t\mu \notin \gamma^3 K_0(X)$ , and hence is a nontrivial torsion element in  $\gamma^{2/3} K_0(X)$ . We begin by computing the image of  $t\mu$  in  $\gamma^{2/3}\mathfrak{G}$ .

- If  $\Lambda/T^* \simeq \mathbb{Z}/2\mathbb{Z}$ , then  $q(t\mu) = \begin{cases} 4y & \text{if } \text{ind}(A_\sigma) \leq 2 \\ \text{ind}(A_\sigma)y & \text{if } \text{ind}(A_\sigma) \geq 4 \end{cases}$
- If  $\Lambda/T^* \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  then  $q(t\mu) = \begin{cases} 4y_a y_b & \text{if } \text{ind}(A_{\sigma_c}) \leq 2 \\ \text{ind}(A_{\sigma_c})y_a y_b & \text{if } \text{ind}(A_{\sigma_c}) \geq 4 \end{cases}$

To check the non triviality of  $t\mu \in \gamma^{2/3} K_0(X)$ , it suffices to show that the subgroup  $\langle q(t\mu) \rangle \subset \gamma^{2/3}\mathfrak{G}$  is nontrivial. This is dependent on the type of  $G$  as well as its Tits algebras.

**9.4.3 Proposition** ([48, Prop. 5.1]). *Suppose  $\Lambda/T^* = \langle \sigma \rangle$  is of order 2. If  $8 \leq \text{ind}(A_\sigma) < 2^{v_2(n)+1}$ , then  $t\mu$  is a nontrivial element of  $\gamma^{2/3} K_0(X)$  of order 2. Furthermore,  $\text{res}(t\mu)$  is trivial in  $\gamma^{2/3} K_0(X_s)$  and  $q(t\mu)$  is nontrivial in  $\gamma^{2/3}\mathfrak{G}$ .*

Suppose now that  $\Lambda/T^* \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ . The integers  $d_a, d_b, d_c$  appearing in the definition of the twisted  $\gamma$ -filtration in (9.3) belong to the set  $\{2^{v_2(n)+1}, 2^{n-1}\}$ . For  $n \geq 3$ , it can be seen that  $2^{v_2(n)+1} \mid 2^{n-1}$ . So, in order to preserve our labelling such that  $\text{ind}(A_{\sigma_a}) \leq \text{ind}(A_{\sigma_b}) \leq \text{ind}(A_{\sigma_c})$ , we set  $d_a = d_b = d_c = 2^{v_2(n)+1}$ .

Taking this into consideration, we can restate the  $\gamma$ -filtration as

$$\gamma^d \mathfrak{G} = \left\langle \binom{\text{ind}(A_{\sigma_a})}{i} \binom{\text{ind}(A_{\sigma_b})}{j} y_a^i y_b^j \mid i + j \geq d \right\rangle / \langle y_a^2 - 2y_a, y_b^2 - 2y_b, d_a y_a, d_b y_b \rangle.$$

We consider generators for  $d = 2, 3$  prior to taking the quotient by  $\langle y_a^2 - 2y_a, y_b^2 - 2y_b, d_a y_a, d_b y_b \rangle$ .

$$\gamma^2 \mathfrak{G} = \begin{cases} \langle 2y_a, 2y_b \rangle & \text{if } \text{ind}(A_{\sigma_a}), \text{ind}(A_{\sigma_b}) \leq 2 \\ \langle 2y_a, \text{ind}(A_{\sigma_b})y_b \rangle & \text{if } \text{ind}(A_{\sigma_a}) \leq 2, \text{ind}(A_{\sigma_b}) \geq 4 \\ \langle \text{ind}(A_{\sigma_a})y_a, \text{ind}(A_{\sigma_b})y_b \rangle & \text{if } \text{ind}(A_{\sigma_a}), \text{ind}(A_{\sigma_b}) \geq 4 \end{cases}$$

$$\gamma^3 \mathfrak{G} = \begin{cases} \langle 4y_a, 4y_b, 2y_a y_b \rangle & \text{if } \text{ind}(A_{\sigma_a}), \text{ind}(A_{\sigma_b}) \leq 2 \\ \langle 4y_a, 2 \text{ind}(A_{\sigma_b})y_b, \text{ind}(A_{\sigma_b})y_a y_b \rangle & \text{if } \text{ind}(A_{\sigma_a}) \leq 2, \text{ind}(A_{\sigma_b}) \geq 4 \\ \langle 2 \text{ind}(A_{\sigma_a})y_a, 2 \text{ind}(A_{\sigma_b})y_b, \\ \quad \text{ind}(A_{\sigma_a}) \text{ind}(A_{\sigma_b})y_a y_b \rangle & \text{if } \text{ind}(A_{\sigma_a}), \text{ind}(A_{\sigma_b}) \geq 4 \end{cases} \quad (9.8)$$

With this description, we can provide criteria for when  $q(t\mu) \notin \gamma^3 \mathfrak{G}$ , and hence  $t\mu$  is nontrivial in  $\gamma^{2/3} K_0(X)$ .

**9.4.4 Theorem.** *Let  $\Lambda/T^* = \langle \sigma_a \rangle \oplus \langle \sigma_b \rangle$ , and set  $\sigma_c = \sigma_a + \sigma_b$ . Suppose that  $1 < \text{ind}(A_{\sigma_a}) \leq \text{ind}(A_{\sigma_b}) \leq \text{ind}(A_{\sigma_c}) \leq 2^{v_2(n)}$ . If  $\text{ind}(A_{\sigma_a}) \text{ind}(A_{\sigma_b}) > \text{ind}(A_{\sigma_c}) \geq 4$ , then  $\text{ind}(A_{\sigma_c})\mu$  is a nontrivial torsion element in  $\gamma^{2/3} K_0(X)$ . Furthermore, its image in  $\gamma^{2/3} \mathfrak{G}$  via the map  $q$  is non-trivial, and its image in  $\gamma^{2/3} K_0(X_s)$  via  $\text{res}$  is trivial.*

*Proof.* We can see immediately from (9.8) that  $q(t\mu) \in \gamma^3 \mathfrak{G}$  if  $\text{ind}(A_{\sigma_c}) \leq 2$  or if  $\text{ind}(A_{\sigma_c}) = \text{ind}(A_{\sigma_a}) \text{ind}(A_{\sigma_b})$ . By the relations  $d_a y_a \equiv 0$  and  $d_b y_b \equiv 0$  in the twisted  $\gamma$ -filtration,  $\text{ind}(A_{\sigma_c}) > 2^{v_2(n)}$  implies  $q(t\mu) \equiv 0$ . We must therefore check that  $q(\text{ind}(A_{\sigma_c})\mu) \notin \gamma^3 \mathfrak{G}$  in the remaining cases. By the hypothesis that  $\text{ind}(A_{\sigma_c}) \leq 2^{v_2(n)}$ , we have  $q(t\mu) = \text{ind}(A_{\sigma_c})y_a y_b \notin \gamma^3 \mathfrak{G}$ , and hence  $t\mu \notin \gamma^3 K_0(X)$ . The final statements follow from the proof of 9.4.2.  $\square$

**9.4.5 Example.** Let  $G$  be of inner type  $D_4$ . Then, the indices which satisfy the necessary hypotheses are  $2 \leq \text{ind}(A_{\sigma_a}) \leq 4$ ,  $\text{ind}(A_{\sigma_b}) = \text{ind}(A_{\sigma_c}) = 4$ .

We conclude this section by presenting a concrete example for a group  $G$  of inner type  $D_4$  satisfying the requirements for the non-triviality of  $t\mu$ , as described in Theorem 9.4.4. Specifically, we consider the case that  $\text{ind}(A_{\sigma_a}) = 2$  and  $\text{ind}(A_{\sigma_b}) = \text{ind}(A_{\sigma_c}) = 4$ . This example was first constructed in [40], using the notion of direct sums of algebras with involution introduced by Dejaiffe (cf. [10]).

**9.4.6 Example.** Let  $K = F(x, y, z, t)$  be a function field in 4 variables over a field  $F$ , and consider the quaternion algebras over  $K$  defined by

$$Q_1 = (x, zt), \quad Q_2 = (y, zt), \quad Q_3 = (xy, t), \quad Q_4 = (xy, z).$$

Let  $(A, \tau_A)$  be the direct sum of  $(Q_1, \bar{\phantom{a}}) \otimes (Q_3, \bar{\phantom{a}})$  and  $(Q_2, \bar{\phantom{a}}) \otimes (Q_4, \bar{\phantom{a}})$ . Denote by  $(B, \tau_B)$  the component of  $\mathcal{C}(A, \tau_A)$  Brauer equivalent to  $(Q_1, \bar{\phantom{a}}) \otimes (Q_3, \bar{\phantom{a}}) \sim (x, t) \otimes (y, z)$ , and by  $(C, \tau_C)$  the component of  $\mathcal{C}(A, \tau_A)$  Brauer equivalent to  $(Q_1, \bar{\phantom{a}}) \otimes (Q_4, \bar{\phantom{a}}) \sim (x, z) \otimes (y, t)$ . Thus  $\text{ind}(A) = 2$ ,  $\text{ind}(B) = 4$  and  $\text{ind}(C) = 4$ , as required.

## 9.5 Properties of trialitarian triples

The torsion element constructed in the previous section can be considered as an invariant of the associated trialitarian triple. The motivation for this idea is due to work of Karpenko in [24] and [26] relating the decomposability of a central simple algebra to torsion in the Chow group of its associated Severi-Brauer variety. We present here the results of Karpenko and provide an analogue for the case of algebras with involution.

### 9.5.1 Motivation

Let  $A$  be a central simple algebra over a field  $F$ . We associate to  $A$  the Severi-Brauer variety  $\text{SB}(A)$ , which is a twisted form of projective space. The algebra  $A$  is split if and only if  $\text{SB}(A)$  has a point over  $F$ .

In Chapter 5 we recalled that an algebra  $A$  of exponent 2 is called **totally decomposable** if  $A \simeq Q_1 \otimes \cdots \otimes Q_s$  for quaternion algebras  $Q_1, \dots, Q_s$ . We saw that any central simple algebra of degree 2 is isomorphic to either  $M_2(F)$  or a division quaternion algebra  $Q$ . So, by the definition of decomposability, any central simple algebra of degree 2 is totally decomposable. We consider next the case that  $\text{deg}(A) = 4$ .

**9.5.1 Proposition** (Albert's Theorem, [1, p. 369]). *Any central simple  $F$ -algebra of degree 4 and exponent 2 can be written in the form  $A = Q_1 \otimes Q_2$ , where  $Q_1, Q_2$  are (possibly split) quaternion  $F$ -algebras.*

Let  $A$  be a central simple  $F$ -algebra of degree 8 and exponent 2. In this case,  $A$  is not necessarily decomposable. However, by dimension arguments, if  $A$  is decomposable, then it is totally decomposable. Examples of indecomposable algebras of degree 8 were first constructed by Amitsur, Rowen and Tignol in 1979 [2]. Therefore, for an algebra  $A$  of degree 8 and exponent 2, one would like to provide a method for determining whether  $A$  is decomposable.

As we can detect splitting of a central simple algebra by the existence of an  $F$ -point on  $\text{SB}(A)$ , we can also use  $\text{SB}(A)$  to detect decomposability of the algebra. We first note

that if  $\text{ind}(A) < \text{deg}(A)$ , then  $A \simeq M_r(D)$  with  $r > 1$  and we have a decomposition  $A \simeq M_r(F) \otimes D$ . Thus, when discussing indecomposable algebras, we will begin by restricting ourselves to the case of division algebras.

**9.5.2 Theorem** ([26, 5.3]). *Let  $D$  be a division algebra of prime exponent over a field  $F$ . If  $D$  decomposes, then  $\text{CH}^2(\text{SB}(D))$  is torsion-free.*

For central simple algebras  $A, A'$ , if  $[A'] = [A] \in \text{Br}(F)$ , then  $\text{Tors CH}^2(\text{SB}(A')) \simeq \text{Tors CH}^2(\text{SB}(A))$  (cf. [26]), and so we can extend this result to any central simple algebra  $A$  having prime exponent. Namely, if  $A = M_r(D)$  and  $D$  is decomposable, then  $\text{CH}^2(\text{SB}(A))$  is torsion-free.

Karpenko used this result in the following way. He began by explicitly constructing a codimension 2-cycle on  $\text{SB}(D)$  for  $D$  a generic algebra of index 8 and exponent 2. Then, by showing that this cycle is a nontrivial torsion element in  $\text{CH}^2(\text{SB}(D))$ , he obtains the result that the algebra  $D$  is indecomposable. The goal of this application is to produce an analogue of this result, using the torsion element  $t\mu$  described in the previous chapter.

## 9.5.2 Extension to algebras with involution

We must first extend the notion of decomposability to algebras with involution. Let  $(A, \sigma_A), (B, \sigma_B)$  be central simple  $F$ -algebras with involution. We say that  $(A, \sigma_A) \supset (B, \sigma_B)$  if  $A$  contains a  $\sigma_A$ -stable subalgebra isomorphic to  $B$ , and over which the involution induced by  $\sigma_A$  is conjugate to  $\sigma_B$ . By the double centralizer theorem,  $(A, \sigma_A) \supset (B, \sigma_B) \iff (A, \sigma_A) \simeq (B, \sigma_B) \otimes (C, \sigma_C)$  for some algebra with involution  $(C, \sigma_C)$ . This is what we will call a **decomposition of algebras with involution**. We say that  $(A, \sigma)$  is **totally decomposable** if  $(A, \sigma) \simeq (Q_1, \sigma_1) \otimes \cdots \otimes (Q_s, \sigma_s)$  with  $Q_i$   $\sigma$ -stable quaternion algebras and  $\sigma_i$  the restriction of  $\sigma$  to  $Q_i$ .

Again, if  $\text{deg}(A) = 2$ , then for any involution  $\sigma$  of the first kind,  $(A, \sigma)$  is totally decomposable by definition. In the case that  $\text{deg}(A) = 4$ , we have the following results.

**9.5.3 Theorem** ([29, 16.16], [30, 5.2]). *Let  $A$  be a central simple  $F$ -algebra of degree 4 and exponent 2.*

- For  $\sigma$  symplectic,  $(A, \tau) \simeq (Q_1, \bar{\phantom{\sigma}}) \otimes (Q_2, \bar{\phantom{\sigma}})$ .
- For  $\sigma$  orthogonal,  $(A, \sigma)$  is totally decomposable  $\iff \text{disc}(\sigma) = 1$ .

Consider an algebra  $A$  of degree 8, together with an orthogonal involution  $\sigma$  having trivial discriminant. As in the central simple algebra case, if  $(A, \sigma) \supset (Q, \bar{\phantom{\sigma}})$ , then

$(A, \sigma)$  is totally decomposable, since we are left with a degree 4 algebra with symplectic involution. Similarly, if  $(A, \sigma) \supset (Q, \tau)$  with  $\tau$  orthogonal and  $\text{disc}(\tau) = 1$ , then  $(A, \sigma)$  is totally decomposable.

In the spirit of our motivating case, we would like to detect the decomposition of these algebras by considering an algebraic variety corresponding to  $(A, \sigma)$ . As we saw in Chapter 6, algebras of degree  $2n$  with orthogonal involution having trivial discriminant can be associated to smooth linear algebraic groups of inner adjoint type  $D_n$ . In particular, algebras of degree 8 with orthogonal involution having trivial discriminant correspond to the trialitarian group  $D_4$ . While we have a direct correspondence  $(A, \sigma) \rightsquigarrow G = \text{PGO}^+(A, \sigma)$ , we may also consider the correspondence defined via the Tits algebras of  $G$ .

For a group of inner adjoint type  $D_4$ , the Tits algebras are given by a trialitarian triple  $\{(A, \sigma_A), (B, \sigma_B), (C, \sigma_C)\}$ . Recall that each algebra in this set is of degree 8, each involution is orthogonal with trivial discriminant, and furthermore, the Clifford algebra of any member of the triple decomposes as the product of the other two. We say that the triple is *ordered by indices* if  $\text{ind}(A) \leq \text{ind}(B) \leq \text{ind}(C)$ .

The algebraic variety which we will consider in place of the Severi-Brauer variety is the projective homogeneous variety  $X = G/B$ , where  $B$  is a Borel subgroup of  $G = \text{PGO}^+(A, \sigma)$ . In place  $\text{CH}^2(\text{SB}(A))$ , we consider  $\gamma^{2/3} K_0(X)$ , as in previous chapters.

Theorem 9.4.4 in Chapter 9 provides us with a torsion element  $\eta : \text{ind}(C)\mu \in \gamma^{2/3} K_0(X)$ , where

$$\mu = 2\gamma_1(g_a)^2\gamma_1(g_b) + 2\gamma_1(g_a)\gamma_1(g_b)^2 - \gamma_1(g_a)^2\gamma_1(g_b)^2,$$

as in (9.4). The following result describes the conditions for the triviality of  $\eta$  in  $\gamma^{2/3} K_0(X)$ .

**9.5.4 Theorem.** *Fix  $F$  with  $\text{char}(F) \neq 2$  and let  $G$  be a smooth linear algebraic group over  $F$  with Borel subgroup  $B \subset G$ . Suppose  $G$  is of inner adjoint type  $D_n$  for  $n \geq 4$  even. Let  $A, B, C$  be the Tits algebras of  $G$ , with  $\text{ind}(A) \leq \text{ind}(B) \leq \text{ind}(C) \leq n$ . There exists a torsion element  $\eta := \text{ind}(C)\mu$  in  $\gamma^{2/3} K_0(X)$  which is trivial if:*

1.  $\text{ind}(A) = 1$
2.  $\text{ind}(A)\text{ind}(B) = \text{ind}(C)$ , or
3.  $\text{ind}(C) \leq 2$

*and nontrivial otherwise.*

Note that these three conditions for triviality are not mutually exclusive. For instance, if the Tits algebras have indices  $(1, 2, 2)$ , all three conditions are satisfied.

**9.5.5 Corollary.** *Consider a trialitarian triple  $\{(A, \sigma_A), (B, \sigma_B), (C, \sigma_C)\}$ , ordered by indices. If the indices of the algebras are either  $(2, 4, 4)$  or  $(4, 4, 4)$ , then  $\mu$  is non-trivial.*

### 9.5.3 Results

In Theorem 9.5.4, we saw that there are 3 conditions under which the torsion element  $\eta = \text{ind}(C)\mu$  is trivial in  $\gamma^{2/3} K_0(X)$ . For each of these conditions, we describe a class of trialitarian triples which satisfies the given hypotheses.

#### Case I:

**9.5.6 Lemma** ([29]). *If  $\deg(A) = 8$  and  $\sigma$  is orthogonal, then  $(A, \sigma)$  is totally decomposable if and only if  $\text{disc}(\sigma) = 1$  and at least one of  $C_+, C_-$  is split.*

It follows that a trialitarian triple contains a decomposable algebra if and only if it contains a split algebra. Thus, we have an analogue to the motivating case of Severi-Brauer varieties.

**9.5.7 Proposition.** *If a triple contains a totally decomposable algebra, then  $\eta = \text{ind}(C)\mu$  is trivial in  $\gamma^{2/3} K_0(X)$ .*

So the idea of decomposability explaining the triviality of our element does not extend past Case I. The notion of decomposability as an algebra with involution is much stronger than in the case of central simple algebras. For instance we can have the situation that  $(A, \sigma) \not\cong (Q, \bar{\phantom{\sigma}})$ , but  $A \supset Q$ . So we can still look at decomposability in the sense of central simple algebras.

#### Case II:

Recall that the motivating result for Severi-Brauer varieties depended not on the central simple algebra  $A$  itself, but rather on the underlying division algebra  $D$ , with  $A \simeq M_r(D)$ .

For a trialitarian triple  $\{(A, \sigma_A), (B, \sigma_B), (C, \sigma_C)\}$ , let  $D_A, D_B, D_C$  denote the division algebras corresponding to  $A, B, C$  respectively.

By the relation in the Brauer group, we have  $[D_C] = [D_A \otimes D_B]$ .

$$\begin{aligned}
\text{ind}(A) \text{ind}(B) = \text{ind}(C) &\iff \text{deg}(D_A) \text{deg}(D_B) = \text{deg}(D_C) \\
&\iff \text{deg}(D_A) \text{deg}(D_B) = \text{deg}(D_A \otimes D_B) \\
&\iff D_A \otimes D_B \simeq D_C.
\end{aligned}$$

**9.5.8 Proposition.** *If  $D_C$  decomposes as  $D_A \otimes D_B$ , then  $\eta = \text{ind}(C)\mu$  is trivial in  $\gamma^{2/3} K_0(X)$ .*

**9.5.9 Remark.** Note that asking the question of whether  $D_C$  is decomposable *in general* does not yield a helpful answer, since by Albert's theorem, any division algebra of degree 4 is decomposable.

### Case III:

We've exhausted our options for decomposability, but case 3 still remains. So, we will turn to a property which is closely related to decomposability.

Let  $(A, \sigma)$  be an algebra with an  $F$ -linear involution. If there exists an idempotent  $e \in A$  (i.e.  $e^2 = e$ ) such that  $\sigma(e) = 1 - e$ , then  $\sigma$  is called **hyperbolic** (cf. Chapter 5).

**9.5.10 Lemma** ([29]). *Let  $A$  be an  $F$ -algebra with orthogonal involution. If  $\text{deg}(A) = 8$ , then  $\sigma$  hyperbolic  $\implies A$  totally decomposable.*

**9.5.11 Proposition.** *If a trialitarian triple contains a hyperbolic involution, then one of the algebras must be split, and hence  $\eta = \text{ind}(C)\mu$  is trivial in  $\gamma^{2/3} K_0(X)$ .*

This recovers Case I, and we can extend this idea to apply to Case III. Hyperbolicity is a property which is retained over field extensions and so it is reasonable to look at behaviour of the involutions over some field extensions of  $F$ . Namely, we consider an extension to  $F_Q$ , the function field of the conic  $\text{SB}(Q)$  for a quaternion algebra  $Q$ .

**9.5.12 Theorem** ([41]). *Consider  $(A, \sigma)$  with  $\text{deg}(A) = 8$  and  $\sigma$  orthogonal. If  $\sigma_{F_Q}$  is hyperbolic for a quaternion  $F$ -algebra  $Q$ , then there exists a quaternion  $F$ -algebra  $Q'$  with  $\text{ind}(Q \otimes Q') \leq 2$  and providing the following relations in the Brauer group:*

$$[A] = [Q \otimes Q'], \quad [C_+] = [Q], \quad [C_-] = [Q'].$$

It follows that a triple containing such an involution must contain a split algebra or have indices of the form  $(2, 2, 2)$ .

**9.5.13 Proposition.** *If a trialitarian triple contain an involution which becomes hyperbolic over the function field of a conic, then  $\mu$  is trivial in  $\gamma^{2/3} K_0(G/B)$ .*

# Chapter 10

## Application to cohomological invariants

For a field  $F$ , consider an algebraic structure  $A$  over  $F$  (e.g.  $F$ -algebra, quadratic form), and a field extension  $K/F$ . Recall that an algebraic structure  $B$  over  $K$  is called a twisted form of  $A$  if  $B_{\bar{F}} \simeq A_{\bar{F}}$ . We may form a map from the set of twisted forms of  $A$  defined over  $K$  to the set of  $G$ -torsors over  $K$  for some linear algebraic group  $G$ . This map is defined by first setting  $G = \text{Aut}_F(A)$ , and then mapping a twisted form  $B$  to the group of isomorphisms from  $B$  to  $A_K$ , that is,  $\text{Isom}(B, A_K)$ .

**10.0.14 Example.** Let  $q$  be a nondegenerate quadratic form over  $F$  of dimension  $n$ . Then,  $\text{Aut}(q) = \text{O}_n$ , the orthogonal group. This map gives a bijection between the isomorphism classes of quadratic forms over  $F$  of dimension  $n$  and the set of  $\text{O}_n$ -torsors over  $F$ .

**10.0.15 Example.** Consider the set  $M_n(F)$  of  $n \times n$  matrices with coefficients in  $F$ . In this case,  $\text{Aut}(M_n(F)) = \text{PGL}_n$ . As we saw in Chapter 5, twisted forms of matrix algebras are central simple algebras, so we have a bijection between the set of central simple  $F$ -algebras,  $\text{CSA}_n(F)$ , and  $\text{PGL}_n$ -torsors over  $F$ . By noting that  $\text{PGL}_n$  is also the automorphism group of projective space  $\mathbb{P}^{n-1}$ , we obtain a further bijection between  $\text{CSA}_n(F)$  and Severi-Brauer varieties of dimension  $n-1$ , which are twisted forms of  $\mathbb{P}^{n-1}$  over  $F$ .

Next, we consider a map  $u_F$  from the set of  $G$ -torsors over  $F$  to the Galois cohomology group  $H^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$ . This map is called a **cohomological invariant** of  $G$  of degree  $d$ . An invariant is said to be **normalized** if the the trivial torsor is mapped to zero.

**10.0.16 Example.** If we take again  $G = O_n$ , the discriminant defines a map  $\text{disc} : \text{Quad}_n(F) \rightarrow F^\times/F^{\times n}$ , which is a degree 1 invariant. For  $G = \text{PGL}_n$ , there is a degree 2 invariant  $\text{CSA}_n(F) \rightarrow \text{Br}(F)$  defined by sending  $A \mapsto [A]$ .

If we fix both  $G$  and  $d$ , then the set of normalized invariants forms a group, which we denote by  $\text{Inv}^d(G, d-1)_{\text{norm}}$ . Determining the group of invariants turns out to be a highly non-trivial problem. If the set of invariants is complete, then it would describe the group  $G$  up to isomorphism. Of course, an incomplete set of invariants can still be quite useful.

A normalized invariant  $a$  is called **decomposable** if it is given by a cup-product with an invariant of degree 2 (i.e. a class in the Brauer group). We denote the subgroup of decomposable invariants by  $\text{Inv}^3(G, 2)_{\text{dec}}$ . The quotient group  $\text{Inv}^3(G, 2)_{\text{norm}}/\text{Inv}^3(G, 2)_{\text{dec}}$  is denoted by  $\text{Inv}^3(G, 2)_{\text{ind}}$  and is called the group of **indecomposable invariants**.

In [32], Merkurjev used results from motivic cohomology to compute the group  $\text{Inv}^3(G_s, 2)_{\text{ind}}$  for any split adjoint group  $G_s$ . The result was then extended in [5] to all split simple groups.

The notion of a semi-decomposable invariant was introduced by Merkurjev, Neshitov and Zainouline in [33] and is defined as follows. An invariant  $a \in \text{Inv}^3(G, 2)_{\text{norm}}$  is **semi-decomposable** if for any field extension  $K/F$  and a  $G$ -torsor  $Y$  over  $K$ , there exists a finite set  $I$  such that

$$a(T) = \sum_{i \in I} \phi_i \cup b_i(Y) \quad \text{for some } \phi_i \in K^\times \text{ and } b_i \in \text{Inv}^2(G, 1)_{\text{norm}}.$$

The set of semi-decomposable invariants form a subgroup of  $\text{Inv}^3(G, 2)_{\text{norm}}$ , which we denote by  $\text{Inv}^3(G, 2)_{\text{sdec}}$ . By definition,  $\text{Inv}^3(G, 2)_{\text{dec}} \subseteq \text{Inv}^3(G, 2)_{\text{sdec}} \subseteq \text{Inv}^3(G, 2)_{\text{norm}}$ .

If  $G$  is a split semisimple linear algebraic group and  $X^v$  is a *versal*  $G$ -flag, it is shown in [33] that there is a short exact sequence

$$0 \rightarrow \frac{\text{Inv}^3(G, 2)_{\text{sdec}}}{\text{Inv}^3(G, 2)_{\text{dec}}} \rightarrow \text{Inv}^3(G, 2)_{\text{ind}} \rightarrow \text{Tor}(\text{CH}^2(X^v)) \rightarrow 0.$$

In particular, if  $G$  is simple, then  $\text{Inv}^3(G, 2)_{\text{sdec}} = \text{Inv}^3(G, 2)_{\text{dec}}$ , and so there is an isomorphism  $\text{Inv}^3(G, 2)_{\text{ind}} \simeq \text{Tor}(\text{CH}^2(X^v))$ .

In the specific case that  $X^v$  is a versal  $G$ -flag, it can be shown that the topological and  $\gamma$ -filtrations on  $K_0(X^v)$  coincide. Thus, for simple groups, we have

$$\text{Tor}(\gamma^{2/3} K_0(X^v)) \simeq \text{Tor}(T^{2/3} K_0(X^v)) \simeq \text{Tor}(\text{CH}^2(X^v)) \simeq \text{Inv}^3(G, 2)_{\text{ind}}. \quad (10.1)$$

Therefore, providing a nontrivial torsion element in  $\gamma^{2/3} K_0(X^v)$  would in turn provide an indecomposable invariant of degree 2 for  $G$ . If  $X^v$  is a versal flag, then the Tits algebras of  $G$  are of maximal index. For a group of inner type  $D_n$ ,  $n$  even, this means that  $\text{ind}(A) = \text{ind}(B) = \text{ind}(C) = 2^{v_2(n)+1}$ ,

Unfortunately, for the element  $\eta$  produced in Chapter 9, we cannot yet say whether  $\eta$  is in fact nontrivial in  $\gamma^{2/3} K_0(X^v)$ . This is due to the fact that the twisted  $\gamma$ -filtration degenerates when any of the algebras have index  $2^{v_2(n)+1}$ . We can however ask the following questions:

- Q1: If  $X$  is a twisted flag variety by means of an arbitrary torsor, what is the relation between  $\text{Tors}(\gamma^{2/3} K_0(X))$  and  $\text{Inv}^3(G, 2)_{\text{ind}}$ ?
- Q2: Can we retain any of the isomorphisms in (10.1) if we relax the requirement that  $X^v$  be a versal flag?
- Q3: If the map  $\text{Tors}(\gamma^{2/3} K_0(X)) \rightarrow \text{Tors}(\text{CH}^2(X))$  is not injective, can we describe the lost information in terms of some invariant of  $G$ ? That is, does  $\text{Tors}(\gamma^{2/3} K_0(X))$  correspond to a group of *conditional invariants*?

While Q1 remains completely unanswered at this time, this thesis provides some intuition for approaching Q2 and Q3. In Chapter 8 we mentioned that a cocycle  $\xi \in Z^1(F, G_s)$  is called *generic* if  $I_c = I_{\text{res}}$ . A versal torsor is therefore generic, and we can go one step further in weakening this condition. If  $G$  is defined by a generic cocycle, then for the appropriate prime  $p$ , the J-invariant  $J_p(G)$  is maximal, i.e.  $J_p(G) = (k_1, \dots, k_r)$ .

**10.0.17 Example.** In [40] it was shown that if  $G$  is an adjoint group of inner type  $D_4$ , then  $J_2(G)$  is maximal under the conditions that all three Tits algebras of  $G$  have index at least 4. On the other hand, if  $G$  corresponds to a versal torsor, then all three Tits algebras must have index 8. If  $X$  is the variety of Borel subgroups of  $G$ , then by Theorem 9.4.4 there exists a nontrivial torsion element in  $\gamma^{2/3} K_0(X)$  in the case that  $G$  has maximal J-invariant, but does not correspond to a versal torsor. An interesting class of torsors to consider for Q2 may be the class of torsors corresponding to groups with maximal J-invariant.

The invariant  $\eta$  described in Chapter 9 was also shown to detect some information about the group  $G$  (and hence about the associated torsor). Namely,  $\eta$  is able to detect information about the indices of the Tits algebras of  $G$ . When considering conditions 1 and 3 in Theorem 9.5.4, we use arguments of triality, which do not extend to groups

of higher rank. Proposition 9.5.8 however is not restricted to the trialitarian case. The result holds for groups of inner type  $D_n$  for all  $n \geq 4$  even.

**10.0.18 Proposition.** *Let  $G$  be an adjoint groups of inner type  $D_n$ , for  $n \geq 4$  even, and let  $X$  be the associated variety of Borel subgroups. Let  $A, B, C$  be the Tits algebras of  $G$ , ordered by indices, and let  $D_A, D_B, D_C$  be division algebras such that  $[A] = [D_A], [B] = [D_B]$ , and  $[C] = [D_C]$  in  $\text{Br}(F)$ . If  $D_A \otimes D_B \simeq D_C$  as division algebras, then the element  $\eta = \text{ind}(C)\mu$  is trivial in  $\gamma^{2/3} K_0(X)$ .*

If  $G$  corresponds to a versal torsor, then  $\text{ind}(A) = \text{ind}(B) = \text{ind}(C) = 2^{v_2(n)+1}$ , and so by dimension arguments we must have  $D_A \otimes D_B \not\simeq D_C$ . While we cannot say for certain that  $\eta$  is non-trivial in this situation, the result is still encouraging. This element may correspond to some condition invariant which detects decomposition in this manner.

# Bibliography

- [1] A. A. Albert. Normal division algebras of degree four over an algebraic field. *Trans. Amer. Math. Soc.*, 34:363–372, 1932.
- [2] S. A. Amitsur, L. H. Rowen, and J.-P. Tignol. Division algebras of degree 4 and 8 with involution. *Israel J. Math.*, 33(2):133–148, 1979.
- [3] S. Baek. On the torsion of Chow groups of Severi-Brauer varieties. preprint available at <http://arxiv.org/abs/1206.2704>.
- [4] S. Baek, K. Zainoulline, and C. Zhong. On the torsion of Chow groups of twisted Spin-flags. *Math. Res. Lett.*, 20(4):601–614, 2013.
- [5] H. Bermudez and A. Ruoizzi. Degree 3 cohomological invariants of groups that are neither simply connected nor adjoint. preprint available at <http://arxiv.org/abs/1305.2899>.
- [6] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [7] N. Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.
- [8] B. Calmès, V. Petrov, and K. Zainoulline. Invariants, torsion indices and oriented cohomology of complete flags. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(3):405–448 (2013), 2013.
- [9] P. Cartier. Groupes algébriques et groupes formels. In *Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962)*, pages 87–111. Librairie Universitaire, Louvain; GauthierVillars, Paris, 1962.

- [10] I. Dejaiffe. Somme orthogonale d'algèbres à involution et algèbre de Clifford. *Comm. Algebra*, 26(5):1589–1612, 1998.
- [11] P. Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, 43:273–307, 1974.
- [12] R. Elman, N. Karpenko, and A. S. Merkurjev. *The algebraic and geometric theory of quadratic forms*, volume 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.
- [13] E. M. Friedlander. An introduction to  $K$ -theory. In *Some recent developments in algebraic K-theory*, volume 23 of *ICTP Lect. Notes*, pages 1–77. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2008.
- [14] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.
- [15] S. Garibaldi and K. Zainoulline. The gamma-filtration and the Rost invariant. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, DOI: 10.1515/crelle-2012-0114, February 2013.
- [16] S. Gille and K. Zainoulline. Equivariant pretheories and invariants of torsors. *Transform. Groups*, 17(2):471–498, 2012.
- [17] A. Grothendieck. *Séminaire C. Chevalley; 2e année: 1958. Anneaux de Chow et applications*. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958.
- [18] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
- [19] J. E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.
- [20] J. E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978. Second printing, revised.

- [21] C. Junkins. The J-invariant and Tits algebras for groups of inner type  $E_6$ . *Manuscripta Math.*, 140(1-2):249–261, 2013.
- [22] C. Junkins. Twisted gamma filtration and algebras with orthogonal involution. *Cent. Eur. J. Math.*, 12(3):421–428, 2014.
- [23] V. G. Kac. Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups. *Invent. Math.*, 80(1):69–79, 1985.
- [24] N. Karpenko. Torsion in  $\mathrm{CH}^2$  of Severi-Brauer varieties and indecomposability of generic algebras. *Manuscripta Math.*, 88(1):109–117, 1995.
- [25] N. Karpenko. Order of torsion in  $\mathrm{CH}^4$  of quadrics. *Doc. Math.*, 1:No. 02, 57–65 (electronic), 1996.
- [26] N. Karpenko. Codimension 2 cycles on Severi-Brauer varieties. *K-Theory*, 13(4):305–330, 1998.
- [27] N. Karpenko and A. S. Merkurjev. Chow groups of projective quadrics. *Algebra i Analiz*, 2(3):218–235, 1990.
- [28] N. Karpenko and A. S. Merkurjev. Canonical  $p$ -dimension of algebraic groups. *Adv. Math.*, 205(2):410–433, 2006.
- [29] M.-A. Knus, A. S. Merkurjev, M. Rost, and J.-P. Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.
- [30] M.-A. Knus, R. Parimala, and R. Sridharan. Involutions on rank 16 central simple algebras. *J. Indian Math. Soc.*, 57:142–151, 1991.
- [31] M. Levine and F. Morel. *Algebraic cobordism*. Springer Monographs in Mathematics. Springer, Berlin, 2007.
- [32] A. S. Merkurjev. Degree three cohomological invariants of semisimple groups. to appear in the Journal of the European Mathematical Society.
- [33] A.S. Merkurjev, A. Neshitov, and K. Zainoulline. Invariants of degree 3 and torsion in the Chow group of a versal flag. preprint available at <http://arxiv.org/abs/1312.0842>.

- [34] J. S. Milne. Basic theory of affine group schemes, 2012. Available at [www.jmilne.org/math/](http://www.jmilne.org/math/).
- [35] J. S. Milne. Reductive groups, 2012. Available at [www.jmilne.org/math/](http://www.jmilne.org/math/).
- [36] I. A. Panin. On the algebraic  $K$ -theory of twisted flag varieties. *K-Theory*, 8(6):541–585, 1994.
- [37] V. Petrov and N. Semenov. Generically split projective homogeneous varieties. *Duke Math. J.*, 152(1):155–173, 2010.
- [38] V. Petrov, N. Semenov, and K. Zainoulline.  $J$ -invariant of linear algebraic groups. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(6):1023–1053, 2008.
- [39] E. Peyre. Galois cohomology in degree three and homogeneous varieties. *K-Theory*, 15(2):99–145, 1998.
- [40] A. Quéguiner-Mathieu, N. Semenov, and K. Zainoulline. The  $J$ -invariant, Tits algebras and triality. *J. Pure Appl. Algebra*, 216(12):2614–2628, 2012.
- [41] A. Quéguiner-Mathieu and J.-P. Tignol. Algebras with involution that become hyperbolic over the function field of a conic. *Israel J. Math.*, 180:317–344, 2010.
- [42] SGA3. *Schémas en groupes. I: Propriétés générales des schémas en groupes*. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151. Springer-Verlag, Berlin, 1970.
- [43] SGA6. *Théorie des intersections et théorème de Riemann-Roch*. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre.
- [44] R. Steinberg. On a theorem of Pittie. *Topology*, 14:173–177, 1975.
- [45] J.-P. Tignol. Central simple algebras, involutions, and quadratic forms. Lecture notes, National Taiwan University, Taiwan, 1993.
- [46] J. Tits. Représentations linéaires irréductibles d’un groupe réductif sur un corps quelconque. *J. Reine Angew. Math.*, 247:196–220, 1971.

- [47] C. A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic  $K$ -theory.
- [48] K. Zainoulline. Twisted gamma filtration of a linear algebraic group. *Compos. Math.*, 148(5):1645–1654, 2012.