ASYMPTOTIC DISTRIBUTION OF COPRIME - COORDINATE LATTICE POINTS INSIDE CERTAIN PLANE REGIONS

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Soo-Ngoh Yeo-Lim
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Special thanks should go to Mrs. S. Zayat who typed this thesis with great patience.

ABSTRACT

In this thesis we prove five theorems on the asymptotic distribution of coprime - coordinate lattice points in certain plane regions. The regions considered consist of two specific triangles and a rectangle in the first quadrant, the circular disk $u^2 + v^2 \le x$ and the elliptic disk $au^2 + bv^2 \le x$.

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CHAPTER I

INTRODUCTION

A point $(x_1, x_2, \ldots, x_n)$ in the n-dimensional Cartesian space is called a *lattice point* if its coordinates are integers. If its coordinates are coprime, we called it a *coprime-coordinate lattice point* (or *co-co lattice point* or *co^2 lattice point* for brevity). In this thesis, we shall restrict ourselves to two dimensions; so a lattice point can be identified with a Gaussian integer.

We shall study lattice points and co^2 lattice points in specific regions. Such a study can be justified as follows:

1. Several lattice point problems have a number-theoretic interpretation. For example:
   (a) The number of lattice points in the disk $u^2 + v^2 \le x$ is also equal to $\sum_{n \le x} r(n)$, where the function $r(n)$ is the number of solutions in integers of the equation $u^2 + v^2 = n$.
   A similar interpretation holds in the case of the elliptic disk $au^2 + bv^2 + cuv \le x$, where a, b, c are integers.
   (b) Dirichlet's divisor problem:
   The number of lattice points in the open first quadrant between the axes and the hyperbola $uv = x$ is

equal to $\sum_{n\le x} d(n)$, where the function $d(n)$ is the number of divisors of the integer n.

The number $N(x)$ of co^2 lattice points in the same region has a similar interpretation. Let $\delta(n)$ be the number of square-free divisors of the integer n; then

$$N(x) = \sum_{n\le x} \delta(n) .$$

(c) The number of co^2 lattice points in the shaded triangle of Figure 1.1 is equal to $\sum_{n\le x} \phi(n)$ where $\phi(n)$ is Euler's function.

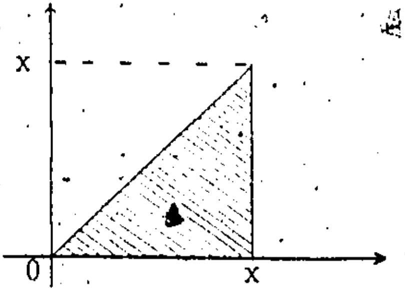

FIGURE 1.1

2. In the case of two dimensions, a lattice point (u,v) can be identified with the Gaussian integer $m = u + iv$. A lattice point is co^2 if and only if m has the property that the additive group $\mathbb{Z}[i]/(m)$ is cyclic (or equivalently, the rings $\mathbb{Z}[i]/(m)$ and $\mathbb{Z}/|m|^2$ are isomorphic).

It is known that in a sufficiently "fat" region A enclosed by a smooth curve (a curve defined by functions with sufficient derivatives), the number of lattice points is approximately equal to the area of A . As it is known that the probability of a random lattice point being a co^2 is approximately $\frac{6}{\pi^2}$, it follows that the number of co^2 lattice points in A is approximately equal to $\frac{6}{\pi^2}$ times the area of A .

The aim of this thesis is to obtain a good estimation of the error term for the difference between the actual and the approximate number of co^2 lattice points in various regions through different methods.

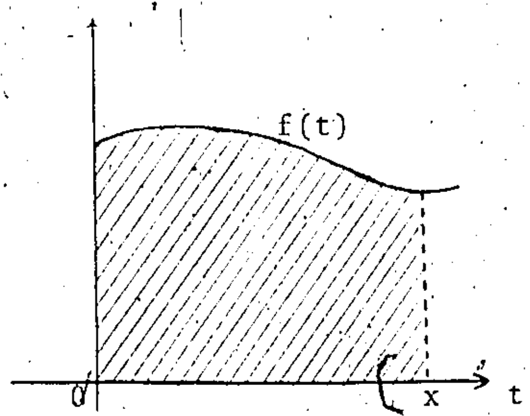

FIGURE 1.2

Let f(t) be an arbitrary continuous function defined on [0,x] such that $f(t) \geq 0$. Assuming the function f(t) is sufficiently smooth, the number of lattice points and co^2 lattice points in the shaded region of Figure 1.2 can be estimated by a suitable choice of analytic methods (Bessel functions, Fourier series, trigonometrical sums), elementary methods (trivial and non-trivial methods), Abel's method of summation or the Euler-MacLaurin sum formula. The accuracy of the results depends essentially on the properties of the function $f(t)$.

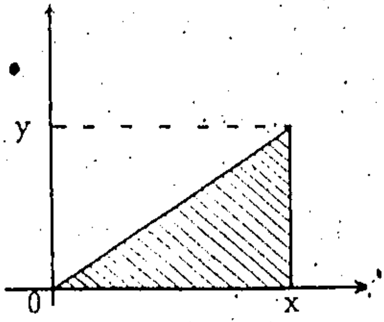

FIGURE 1.3

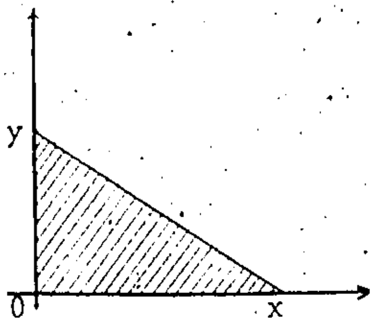

FIGURE 1.4

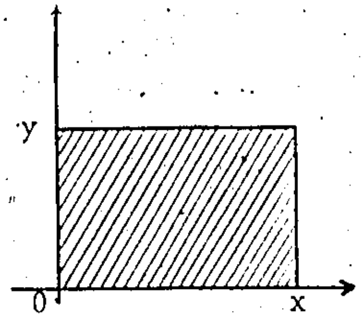

FIGURE 1.5

Apart from the general treatments, a different method based on the Möbius inversion formula is considered for the estimation of co^{2} lattice points in the shaded triangles and rectangle of Figures 1.3, 1.4 and 1.5. Due to the technical difficulties in handling the square terms, this method fails in the cases of the quarter circle and the quarter ellipse. A modified method is applied to the circular disk and the elliptic disk.

Before proceeding into the details of the thesis, let us define the notations used.

1. The symbols $[t]$ and $\{t\}$ denote the integral and fractional parts of a real number $t \geq 0$.

2. The letters m, n, d, ℓ stand for integers throughout the thesis.

3. The Order Notations:

   Let $f(x)$ and $g(x)$ be functions defined on all positive x, then

   (i) If there is a number M such that

   $$|f(x)| \leq M\,|g(x)|$$

   for sufficiently large x, then we write

   $$f(x) = O(g(x)) \quad \text{or} \quad f(x) \ll g(x) .$$

   (ii) If there exists an $\varepsilon(x)$ such that

   $$f(x) = \varepsilon(x)\, g(x)$$

   and $\varepsilon(x) \to 0$ as $x \to \infty$, then we write

   $$f(x) = o(g(x)) .$$

   (iii) If there exists an $\varepsilon(x)$ such that

   $$f(x) = g(x)(1 + \varepsilon(x))$$

   and $\varepsilon(x) \to 0$ as $x \to \infty$, then we write

   $$f(x) \sim g(x)$$

and say that $f(x)$ is asymptotically equal to $g(x)$.

4. Let n be any positive integer. We define $d(n)$ to be the number of divisors of n; $\omega(n)$ to be the number of distinct prime factors of n; and $\phi(n)$ to be the number of integers not exceeding n which are relatively prime to n (Euler's function).

5. The Möbius function $\mu(n)$ is an arithmetical function defined on natural numbers n and having the following properties:

   (i) $\mu(1) = 1$;

   (ii) $\mu(n) = 0$ if $p^2 | n$ for some prime p;

   (iii) $\mu(n) = (-1)^r$ if $n = p_1 p_2 \ldots p_r$ is a product of r distinct primes.

In the following we give three well-known lemmas which we shall later need.

LEMMA 1.1 ([7], p.103) *Let* n *be a positive integer; then*

(a) $\sum_{d|n} \mu(d) = 1$ *if* $n = 1$;

(b) $\sum_{d|n} \mu(d) = 0$ *if* $n > 1$. ∇

LEMMA 1.2 ([2], p.235)

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}. \qquad \nabla$$

LEMMA 1.3 (Möbius inversion formula) ([7], p.104)

Let f(t) *and* F(t) *be functions of the real variable* $t \geq 1$; *If these functions satisfy the relation*

$$(1-1) \qquad F(t) = \sum_{1 \leq n \leq t} f\left(\frac{t}{n}\right) ,$$

then they satisfy the "inverse" relation

$$(1-2) \qquad f(t) = \sum_{1 \leq m \leq t} \mu(m)\, F\left(\frac{t}{m}\right) .$$

Conversely, (1 - 1) *follows from* (1 - 2) . ∇

CHAPTER II

STATEMENT OF RESULTS AND OUTLINE OF METHODS

In this thesis we prove five theorems concerning the distribution of co^2 lattice points. The proofs shall be given in Chapter four.

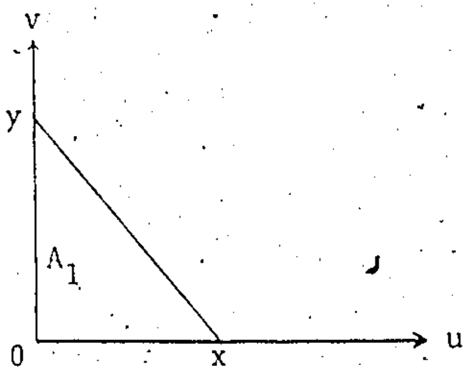

FIGURE 2.1

<u>THEOREM 2.1</u> *Let* $1 \le x \le y$. *The total number of* co^2 *lattice points in the right-angled triangle* A_1 *of Figure 2.1 is given by*

$$\frac{3}{\pi^2}xy + 0(y) + 0(x \log x) \quad . \qquad \nabla$$

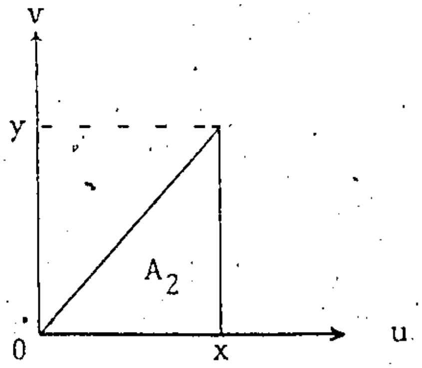

FIGURE 2.2

<u>THEOREM 2.2</u> *Let* $1 \leq x \leq y$. *The total number of* co^2 *lattice points in the right-angled triangle* A_2 *of Figure 2.2 is given by*

$$\frac{3}{\pi^2} xy + 0(y(\log x)^{\frac{2}{3}}(\log\log x)^{1+\varepsilon})$$

for any $\varepsilon > 0$. ∇

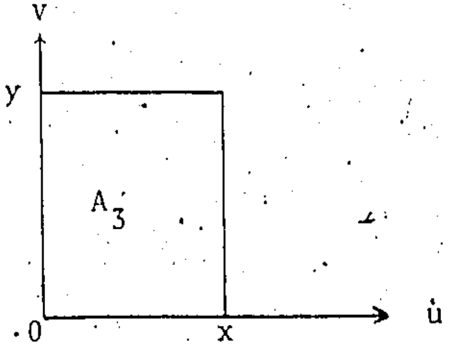

FIGURE 2.3

<u>THEOREM 2.3</u> *Let* $1 \leq x \leq y$. *The total number of* co^2 *lattice points in the rectangle* A_3 *of Figure 2.3 is given by*

$$\frac{6}{\pi^2} xy + 0(y(\log x)^{\frac{2}{3}}(\log\log x)^{1+\varepsilon})$$

for any $\varepsilon > 0$. ∇

Let $M(x) = \sum_{d \leq x} \mu(d)$ where x is any real number, $x \geq 1$ and the summation is extended over all positive integers $d \leq x$.

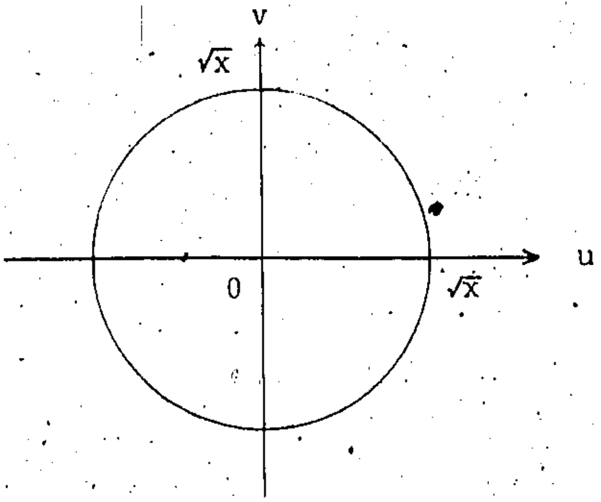

FIGURE 2.4

THEOREM 2.4 *In view of the result that* $M(x) = O(xe^{-c\sqrt{\log x}})$ [see Chapter 3, relation (3 - 3)], *the total number* $Q(x)$ *of* co^2 *lattice points in the disk* $u^2 + v^2 \leq x$ *of Figure 2.4 is given by*

$$Q(x) = \frac{6x}{\pi} + O(\sqrt{x}\ e^{-\kappa\sqrt{\log x}}) \quad ;$$

where

$$\kappa = \frac{c(1 - 2\theta)}{2\sqrt{2}(1 - \theta)} \quad ,$$

c *being the (absolute) constant of relation* (3 - 3) , *and* θ *being the constant arising in Dirichlet's divisor problem* (see Chapter 3, Lemma 3.12). ∇

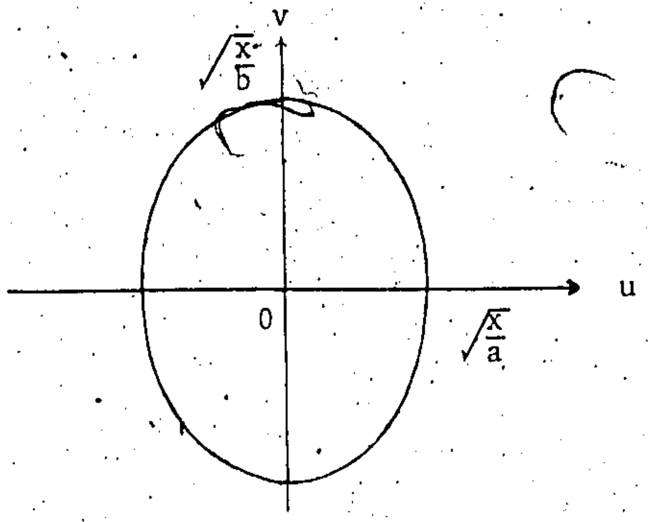

FIGURE 2.5

<u>THEOREM 2.5</u> *In view of the result that* $M(x) = O(xe^{-c\sqrt{\log x}})$ [see Chapter 3, relation (3 - 3)], *the total number* $Q(x)$ *of* co^2 *lattice points in the elliptic disk* $au^2 + bv^2 \le x$ *of Figure 2.5 is given by*

$$Q(x) = \frac{6x}{\pi\sqrt{ab}} + O(\sqrt{x}\, e^{-\kappa\sqrt{\log x}})$$

where a, b *are fixed positive integers,* $a > b$, c *is the (absolute) constant of relation* (3 - 3) *and* $\kappa = \frac{c(1-2\theta)}{2\sqrt{2}(1-\theta)}$. ∇

In the course of proving these theorems we shall need some lemmas and well-known formulae as stated in Chapter 3. One of the important lemmas leading to Theorem 2.2 states:

For any real numbers x, y $(x > 64,\ y \geq 1)$, *we have*

$$\sum_{d \leq x} \mu(d)\, r\left(\frac{x}{d}, \frac{y}{d}\right) = O(x)$$

where d *runs through all the positive integers not exceeding* x ; $r\left(\frac{x}{d}, \frac{y}{d}\right)$ *being equal to* $\sum_{n \leq \frac{x}{d}} \left\{ \frac{y}{x} n \right\}$.

The proof of this result is given as Lemma 3.19.

Before arriving at the proofs of the lemmas and theorems, let us consider the different methods used in the development. First of all, we consider the methods used in general cases.

2.1 GENERAL METHODS

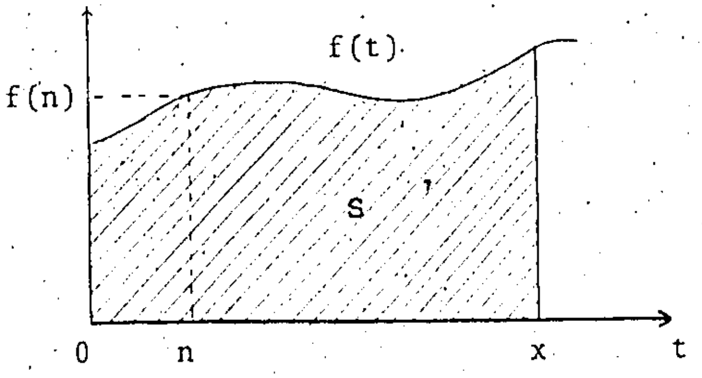

FIGURE 2.6

Let $f(t)$ be an arbitrary continuous function defined on $[0,x]$, $x \geq 1$ and $f(t) \geq 0$. Let $V(x)$ be the total number of lattice points in the area S of Figure 2.6, not counting the axes. Then

$$V(x) = \sum_{n\le x} [f(n)]$$

$$= \sum_{n\le x} f(n) - \sum_{n\le x} \{f(n)\}$$

where n runs through all positive integers not exceeding x.

If the function $f(t)$ is sufficiently smooth ($f(t)$ has sufficient derivatives), $\sum_{n\le x} f(n)$ can be evaluated by applying the Euler-MacLaurin sum formula (cf. Lemma 3.1). For example, the function $f(t) = \frac{1}{t}$ has derivatives of any order. Letting $N = [x]$, it follows that

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + \frac{C_1}{N} + \frac{C_2}{N^2} + \frac{C_3}{N^3} + \ldots + \frac{C_q}{N^q} + 0_q(N^{-q-1})$$

where $\gamma = 0.5772157 \ldots\ldots$ is Euler's constant, $C_1, C_2, \ldots, C_q$ are numerical constants, and q can be any integer greater than or equal to 1. The notation 0_q for the error term signifies that the implicit constant involved in the 0-estimation depends on q.

The methods of evaluating $\sum_{n\le x} \{f(n)\}$ may be analytic or elementary.

(1) The analytic methods consist of the systematic use of Bessel functions, Fourier series or the methods of trigonometrical sums as developed by Landau ([4]), Hardy ([3]), Littlewood ([3]), Walfisz ([12]), Vinogradov ([11]), etc.

(2) The elementary methods may be either trivial methods (straightforward methods which do not require sophistication) or non-trivial methods (see Vinogradov's elementary method described in [1]).

The application of these methods allow the lattice points to be enumerated. For the enumeration of co^2 lattice points, let $F(f(n),n)$ be the number of integers not exceeding $f(n)$ which are relatively prime to n, and let $Q(x)$ be the total number of co^2 lattice points in the area S of Figure 2.6. Then

$$Q(x) = \sum_{n\le x} F(f(n),n) \quad .$$

From Lemma 3.14, we obtain

$$Q(x) = \sum_{n\le x} (f(n) \frac{\phi(n)}{n} - \sum_{d|n} \mu(d) \{\frac{f(n)}{d}\})$$

$$= \sum_{n\le x} f(n) \frac{\phi(n)}{n} - \sum_{n\le x} \sum_{d|n} \mu(d) \{\frac{f(n)}{d}\}$$

$$= D(x) - E(x) \quad ,$$

where $D(x) = \sum_{n\le x} f(n) \frac{\phi(n)}{n}$

and $$E(x) = \sum_{n \le x} \sum_{d | n} \mu(d) \left\{ \frac{f(n)}{d} \right\} .$$

Now we have to evaluate $D(x)$ and $E(x)$.

ESTIMATION OF $D(x)$

Let $$\Psi(x) = \sum_{n \le x} \frac{\phi(n)}{n} .$$

By Abel's summation method,

$$D(x) = \sum_{n \le x} f(n) \frac{\phi(n)}{n}$$

$$= \sum_{n \le [x]-1} \Psi(n) (f(n) - f(n+1))$$

$$+ \Psi([x]) f([x]) .$$

So the term $\sum_{n \le [x]-1} \Psi(n) (f(n) - f(n+1))$ and hence $D(x)$ can be estimated if one has a good knowledge of the function $f(n)$ and the partial sums $\sum_{n} | f(n) - f(n+1) |$. Sometimes integration by parts of a Stieltjes integral can be used to obtain the results.

For example, if $f(n) = n^{\alpha}$ $(\alpha > 0)$, then by proceeding with integration by parts of a Stieltjes integral,

$$D(x) = \sum_{n \le x} n^{\alpha} \frac{\phi(n)}{n}$$

$$= \int_1^x t^{\alpha}\, d\Psi(t)$$

$$= x^{\alpha}\, \Psi(x) - \alpha \int_1^x t^{\alpha-1}\, \Psi(t)\, dt \quad .$$

It follows from Lemma 3.11 that

$$D(x) = \frac{6}{\pi^2}\, x^{1+\alpha} + x^{\alpha}\rho(x) - \frac{6}{\pi^2}\, \alpha \int_1^x t^{\alpha}\, dt$$

$$- \alpha \int_1^x t^{\alpha-1}\, \rho(t)\, dt$$

$$= \frac{6}{\pi^2}\, \frac{x^{1+\alpha}}{(1+\alpha)} + x^{\alpha}\rho(x) - \alpha \int_1^x t^{\alpha-1}\, \rho(t)\, dt$$

$$+ \text{constant} \quad .$$

Assume that

$$\rho(t) = 0\,((\log t)^{\beta})$$

where $\beta = \frac{2}{3} + \varepsilon$ for some $\varepsilon > 0$.

Then $x^{\alpha}\, \rho(x) << x^{\frac{2}{3} + \alpha + \varepsilon}$

and $\int_1^x t^{\alpha-1}\rho(t)\, dt << \int_1^x t^{\alpha-1+\beta}\, dt << x^{\alpha+\beta} << x^{\alpha + \frac{2}{3} + \varepsilon}$.

The above two estimates imply that

$$D(x) = \frac{6}{\pi^2}\, \frac{x^{1+\alpha}}{(1+\alpha)} + 0\,(x^{\alpha + \frac{2}{3} + \varepsilon}) \quad .$$

ESTIMATION OF $E(x)$

(i) For an arbitrary function $f(t)$ and any real number $x \ge 1$, we always have

$$E(x) = 0(x \log x) .$$

An elementary method gives

$$|E(x)| = \Big| \sum_{n \le x} \sum_{d | n} \mu(d) \{ \frac{f(n)}{d} \} \Big|$$

$$= \Big| \sum_{d \le x} \mu(d) \sum_{m \le \frac{x}{d}} \{ \frac{f(md)}{d} \} \Big|$$

$$\le \sum_{d \le x} \frac{x}{d} .$$

By Corollary 3.2.1, we have

$$|E(x)| \le x \log x + \gamma x + 0(1) .$$

(The result obtained in this way is crude as we are losing all information on $\mu(d)$ except for the fact that $|\mu(d)| = 1$.) In Lemmas 3.15 and 3.16, the better results

$$|E(x)| \le (\frac{6}{\pi^2} + o(1)) \, x \log x$$

and

$$|E(x)| \le (\frac{3}{\pi^2} + o(1)) \, x \log x$$

are obtained provided x is sufficiently large.

(*ii*) Depending on the special properties of $f(t)$, different methods may be appropriate. Let us write $E(x)$ as

$$E(x) = \sum_{\substack{d,\ell \\ d\ell \le x}} \mu(d)\left\{\frac{f(d\ell)}{d}\right\} .$$

(*a*) Heuristically, if f is smooth, the distribution of the fractional part $\{\frac{1}{d} f(d\ell)\}$ is close to the uniform distribution. As $\{\frac{1}{d} f(d\ell)\}$ varies uniformly between 0 and 1, its expected value is $\frac{1}{2}$ and hence the sum over a long interval is expected to be $\frac{1}{2}$ times the length of the interval; that is,

$$\sum_{\ell \le \frac{x}{d}} \left\{\frac{1}{d} f(d\ell)\right\} = \frac{x}{2d} + o(x) .$$

Consequently, one would expect to have

$$E(x) = \sum_{d \le x} \mu(d)\left(\frac{x}{2d} + o(x)\right) .$$

The right-hand side can be evaluated easily.

(*b*) If f is a linear function, then the fact that

$$f(d\ell) = df(\ell)$$

reduces $E(x)$ to

$$E(x) = \sum_{\ell \le x} \{f(\ell)\} \sum_{d \le \frac{x}{\ell}} \mu(d) .$$

For example, if $f(\ell) = \lambda\ell$, λ constant, then

$$E(x) = \sum_{\ell \le x} \{\lambda\ell\} \sum_{d \le \frac{x}{\ell}} \mu(d)$$

$$= \sum_{\ell \le x} \{\lambda\ell\} M(\frac{x}{\ell}) \quad ,$$

where

$$M(\frac{x}{\ell}) = \sum_{d \le \frac{x}{\ell}} \mu(d)$$

Applying Lemma 3.19, we conclude that

$$E(x) = 0(x) \quad .$$

2.2 A SPECIAL METHOD

In this section, an original method based on the Möbius inversion formula is considered for the purpose of enumerating co^2 lattice points.

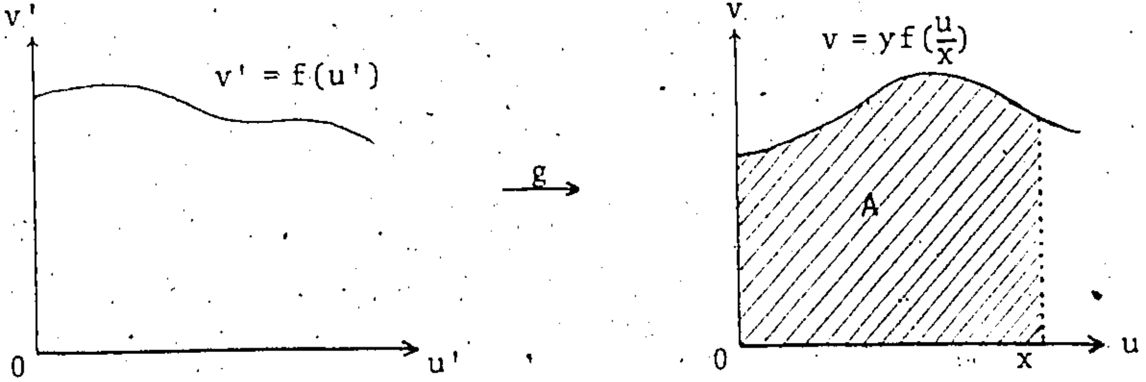

FIGURE 2.7

Let $v' = f(u')$ be a "continuous curve" in the $u'v'$ plane such that $f(u') \ge 0$ for $u' \ge 0$. Let g be a transformation from the $u'v'$ plane to the uv plane with

$$\begin{aligned} u &= g(u') = xu' \\ v &= g(v') = yv' \end{aligned}.$$

Here x and y are suitable positive real numbers. The new curve obtained after the transformation has the equation

$$v = yf\left(\frac{u}{x}\right).$$

Let A be the region under the curve $v = yf(\frac{u}{x})$ (with $0 < u \le x$) as shown in Figure 2.7. Let $G(x,y)$ and $Q(x,y)$ be respectively the total number of lattice points and co^2 lattice points in the region A. Putting $y_1 = \max_{m \le x} yf(\frac{m}{x})$, we have

$$Q(x,y) = \sum_{m \le x} \sum_{\substack{n \le yf(\frac{m}{x}) \\ (m,n)=1}} 1$$

and

$$\begin{aligned} G(x,y) &= \sum_{m \le x} \sum_{n \le yf(\frac{m}{x})} 1 \\ &= \sum_{d \le \min(x,y_1)} \sum_{\substack{m \le x \\ n \le yf(\frac{m}{x}) \\ (m,n)=d}} 1 \\ &= \sum_{d \le \min(x,y_1)} \sum_{\substack{m' \le \frac{x}{d} \\ n' \le \frac{y}{d} f(\frac{m'}{x/d}) \\ (m',n')=1}} 1 \end{aligned}$$

$$= \sum_{d \le \min(x,y_1)} Q(\tfrac{x}{d}, \tfrac{y}{d})$$

By the Möbius inversion formula for functions of two real variables, we have

$$Q(x,y) = \sum_{d \le \min(x,y_1)} \mu(d)\, G(\tfrac{x}{d}, \tfrac{y}{d}) .$$

Note that

$$G(x,y) = \sum_{m \le x} [yf(\tfrac{m}{x})]$$

$$= \sum_{m \le x} (yf(\tfrac{m}{x}) - \{yf(\tfrac{m}{x})\})$$

$$= y \sum_{m \le x} f(\tfrac{m}{x}) - \sum_{m \le x} \{yf(\tfrac{m}{x})\} .$$

Substituting this into $Q(x,y)$ we get

$$(2-1)\left\{\begin{aligned} Q(x,y) &= \sum_{d \le \min(x,y_1)} \mu(d) \Big(\tfrac{y}{d} \sum_{m \le \frac{x}{d}} f(\tfrac{md}{x}) - \sum_{m \le \frac{x}{d}} \{\tfrac{y}{d} f(\tfrac{md}{x})\}\Big) \\ &= y \sum_{d \le \min(x,y_1)} \frac{\mu(d)}{d} \sum_{m \le \frac{x}{d}} f(\tfrac{md}{x}) \end{aligned}\right.$$

$$- \sum_{d \le \min(x,y_1)} \mu(d) \sum_{m \le \frac{x}{d}} \{\tfrac{y}{d} f(\tfrac{md}{x})\} .$$

The application of Equation (2 - 1) to the regions A_1, A_2, A_3 of Figures 2.1, 2.2, 2.3, respectively, leads to Theorems 2.1, 2.2 and 2.3. For the circular disk and the elliptic disk in the first quadrant, this application fails because the term $\sum_{m\le\frac{x}{d}} f(\frac{md}{x})$ in Equation (2 - 1) gives rise to an error term which is as big as the main term. Instead, a similar method is applied *directly* to the circular disk and the elliptic disk from which we derive Theorems 2.4 and 2.5. The details of this method and the proofs are given in Chapter 4.

CHAPTER III

TOOLS AND LEMMAS

In this chapter, lemmas are introduced for the preparation of Chapter IV as well as a supplement to Chapter II.

LEMMA 3.1 (Euler-MacLaurin Sum Formula) ([8], p. 14.)
Let $f(x)$ *be a real-valued function defined on an interval, with continuous derivatives up to order* q . *If* a *and* b *are positive integers* $(a < b)$ *at which* f *is defined, then*

$$\sum_{n=a+1}^{b} f(n) = \int_a^b f(x)\,dx + \sum_{r=1}^{q} (-1)^r \frac{B_r}{r!} \{f^{(r-1)}(b) - f^{(r-1)}(a)\} + R_q ,$$

where the remainder term is

$$R_q = \frac{(-1)^{-q-1}}{q!} \int_a^b B_q(x - [x])\, f^{(q)}(x)\,dx ,$$

B_r *being the Bernoulli number*

$$B_r = \sum_{j=1}^{r} \frac{1}{j+1} \sum_{\ell=1}^{j} (-1)^\ell \binom{j}{\ell} \ell^r$$

and $B_q(x)$ *being the Bernoulli function*

$$B_q(x) = q \sum_{j \geq 0} A_{q-1,j} \binom{x}{j+1} + B_q .$$

where $$A_{q-1,j} = \sum_{m=0}^{j} (-1)^m \binom{j}{m} (j-m)^{q-1} \quad . \qquad \nabla$$

REMARK. It is known that $B_1 = -\frac{1}{2}$ and $B_{2k+1} = 0$ for $k \geq 1$. The first eight Bernoulli numbers of even index are

$B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$,

$B_{12} = -\frac{691}{2730}$, $B_{14} = \frac{6}{7}$, $B_{16} = -\frac{3617}{510}$.

LEMMA 3.2 ([7], p.98.) *If* $g(t)$ *is a non-increasing function,* $g(t) > 0$ *for all* $t > 0$, *then*

$$\sum_{1 \leq n \leq x} g(n) = \int_1^x g(t)\,dt + C + O(g(x)) \, ,$$

where x *is real,* $x \geq 1$ *; and* C *is a constant depending only on the function* $g(t)$. ∇

COROLLARY 3.2.1 ([7], p.99.)

$$\sum_{1 \leq n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right) \, ,$$

where $\gamma = 0.5772157 \ldots$ *is Euler's constant.* ∇

In Chapters III and IV, we shall use the following abbreviations.

Prime Number Theorem : PNT

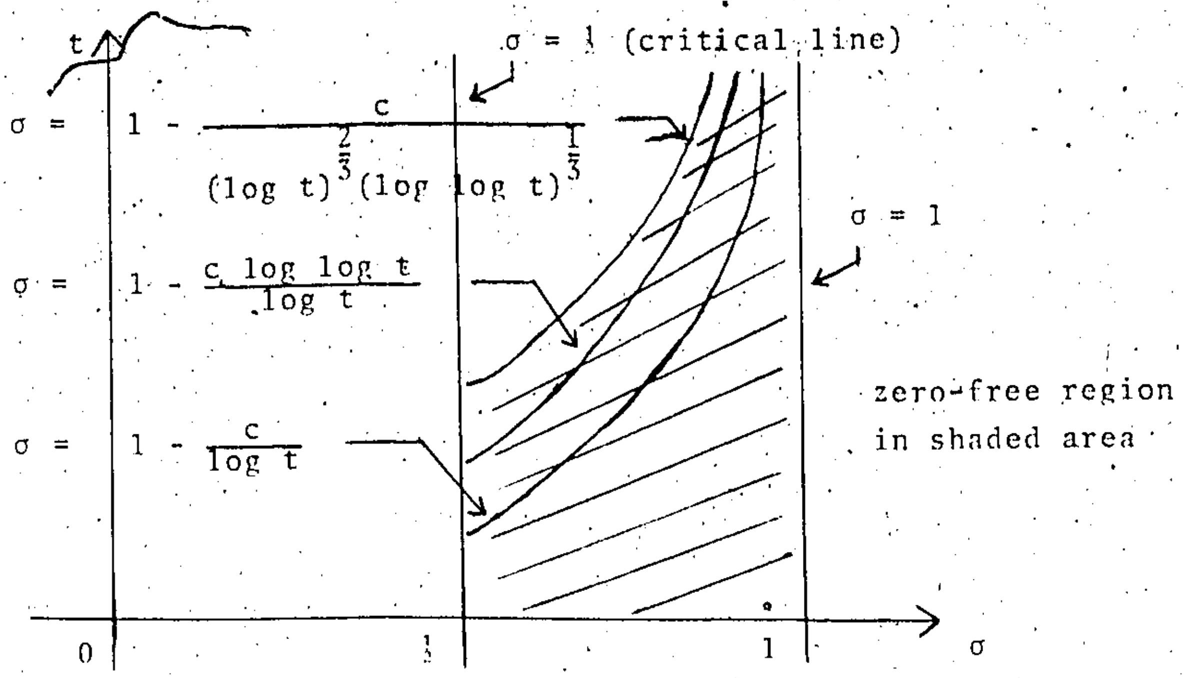

FIGURE 3.1

Zero-free regions of $\zeta(s)$ in the critical strip	O-estimates of $M(x) = \sum_{d\le x}\mu(d)$
$\zeta(s)$ has no zero on $\sigma = 1$ $(s = \sigma + it)$	$M(x) = o(x)$
$\zeta(s)$ has no zero in $\sigma > 1 - \frac{c}{\log t}$ $(t\ge t_0)$	$M(x) = O(xe^{-c\sqrt{\log x}})$
$\zeta(s)$ has no zero in $\sigma > 1 - \frac{c\log\log t}{\log t}$ $(t\ge t_0)$	$M(x) = O(xe^{-c\sqrt{\log x\log\log x}})$
$\zeta(s)$ has no zero in $\sigma > 1-c(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}}$ $(t\ge t_0)$	$M(x) = O(x\exp(-c\log^{\frac{3}{5}}x\log^{-\frac{1}{5}}\log x))$
$\zeta(s)$ has no zero in $\sigma > \frac{1}{2}$	$M(x) = O(x^{\frac{1}{2}+\frac{1}{\log\log x}}) = O(x^{\frac{1}{2}+\varepsilon})$

TABLE 3.1

Riemann Hypothesis : RH
De la Vallée Poussin's
form of the PNT : DVP
Richert-Walfisz' form
of the PNT : RW

Several results are known for the value of $M(x)$:

(i) (3 - 1) $M(x) = o(x)$

is equivalent to PNT (equivalent to the existence of two zero-free lines $\sigma = 0$ and $\sigma = 1$ of Riemann's function $\zeta(s)$, $s = \sigma + it$; [4], pp. 588 - 590).

(ii) (3 - 2) $M(x) = O(x^{\frac{1}{2}+\varepsilon})$,

for any $\varepsilon > 0$, is equivalent to the RH (equivalent to the statement that all the zeros of $\zeta(s)$ lie on the line $\sigma = \frac{1}{2}$; [10], pp. 315 - 316).

(iii) (3 - 3) $M(x) = O(xe^{-c\sqrt{\log x}})$,

c being an (absolute) constant, is due to DVP (equivalent to the existence of some zero-free regions of $\zeta(s)$; see Figure 3.1; [12], p. 147, Relation 11).

(*iv*) The best known result is that

$$M(x) = O\left(xe^{-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}}\right), \tag{3 - 4}$$

c being an (absolute) constant. This is due to RW (equivalent to the existence of some zero-free regions of $\zeta(s)$; see Figure 3.1; [12], p. 191).

We now derive the following lemmas from (i), (ii) and (iii).

<u>LEMMA 3.3</u> *Let* d *be a positive integer, then*

$$\sum_{1\le d\le x} \frac{\mu(d)}{d^2} = \begin{cases} \frac{6}{\pi^2} + O(\frac{1}{x}) & \text{trivial} \\ \frac{6}{\pi^2} + o(\frac{1}{x}) & \text{PNT} \\ \frac{6}{\pi^2} + O_\varepsilon(x^{-\frac{3}{2}+\varepsilon}) \text{ (for any } \varepsilon>0) & \text{RH} \\ \frac{6}{\pi^2} + O(\frac{1}{x}\, e^{-c\sqrt{\log x}}) & \text{DVP} \end{cases}$$

c *being the (absolute) constant of relation* (3 - 3) . ∇

<u>*PROOF.*</u> To prove the equalities, we rewrite

$\sum_{1\le d\le x} \frac{\mu(d)}{d^2}$ as

$$\sum_{1\le d\le x} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d=[x]+1}^{\infty} \frac{\mu(d)}{d^2}$$

since $\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$

By applying Lemma 3.2,

$$\sum_{d=[x]+1}^{\infty} \frac{\mu(d)}{d^2} \ll \sum_{d=[x]+1}^{\infty} \frac{1}{d^2} \ll \int_x^{\infty} \frac{1}{t^2}\,dt \ll \frac{1}{x} \quad ;$$

hence the first result follows immediately.

Next, integration by parts of a Stieltjes integral gives

$$(3-5)\quad \begin{cases} \displaystyle\sum_{d>x} \frac{\mu(d)}{d^2} = \int_x^{\infty} \frac{dM(t)}{t^2} \\[2ex] \displaystyle\qquad = -\frac{M(x)}{x^2} + 2\int_x^{\infty} \frac{M(t)}{t^3}\,dt \ . \end{cases}$$

Substituting (3 - 1), (3 - 2), (3 - 3) into (3 - 5) we obtain the required results.

Q.E.D.

LEMMA 3.4

$$\sum_{1\le n\le x} |\mu(n)| = \frac{6}{\pi^2}x + R(x) \ ,$$

where

$$R(x) = \begin{cases} O(\sqrt{x}) & \text{trivial} \\ o(\sqrt{x}) & \text{PNT} \\ O_\varepsilon(x^{\frac{2}{5}+\varepsilon}) \quad (\text{for any } \varepsilon > 0) & \text{RH} \\ O(\sqrt{x}e^{-c_1\sqrt{\log x}}) & \text{DVP} \end{cases}$$

and $c_1 = \frac{c}{2\sqrt{2}}$, c *being the (absolute) constant of relation* (3 - 3). ∇

PROOF. Let $Q(x) = \sum_{1 \le n \le x} |\mu(n)|$. For any real number x,

$$[x] = \sum_{1 \le n \le x} 1 .$$

Since any integer $n \ge 1$ can be expressed uniquely as the product of a square term d^2 and a square-free term q , that is, $n = d^2 q$, we can write

$$[x] = \sum_{\substack{d,q \\ d^2 q \le x \\ |\mu(q)|=1}} 1 = \sum_{d \le \sqrt{x}} \sum_{\substack{q \le \frac{x}{d^2} \\ |\mu(q)|=1}} 1 = \sum_{d \le \sqrt{x}} Q\left(\frac{x}{d^2}\right) .$$

By the Möbius inversion formula, we obtain

$$Q(x) = \sum_{d \le \sqrt{x}} \mu(d) \left[\frac{x}{d^2}\right]$$

$$= x \sum_{d \le \sqrt{x}} \frac{\mu(d)}{d^2} - \sum_{d \le \sqrt{x}} \mu(d) \left\{\frac{x}{d^2}\right\} .$$

By the trivial result in Lemma 3.3 and the estimate

$$\sum_{d \le \sqrt{x}} \mu(d) \left\{\frac{x}{d^2}\right\} \ll \sqrt{x} ,$$

we have proved that

$$\sum_{1 \le n \le x} |\mu(n)| = \frac{6}{\pi^2} x + O(\sqrt{x}) .$$

To achieve better results for $R(x)$, rewrite

$$Q(x) = \sum_{d\le\sqrt{x}} \mu(d)\left[\frac{x}{d^2}\right] = \sum_{d\le\sqrt{x}} \mu(d) \sum_{m\le\frac{x}{d^2}} 1 = \sum_{\substack{m,d\\ md^2\le x}} \mu(d) .$$

Let $\xi = \xi(x)$, $1 < \xi < \sqrt{x}$ and $\xi \to \infty$.

Then

$$(3-6)\quad \begin{cases} Q(x) = \displaystyle\sum_{d\le\xi} \mu(d)\left[\frac{x}{d^2}\right] + \sum_{m\le\frac{x}{\xi^2}} \sum_{\xi<d\le\sqrt{\frac{x}{m}}} \mu(d) \\[2ex] \quad = x \displaystyle\sum_{d\le\xi} \frac{\mu(d)}{d^2} - \sum_{d\le\xi} \mu(d)\left\{\frac{x}{d^2}\right\} \\[2ex] \quad + \displaystyle\sum_{m\le\frac{x}{\xi^2}} M\left(\sqrt{\frac{x}{m}}\right) - M(\xi)\left[\frac{x}{\xi^2}\right] \end{cases}$$

Next, we have to choose ξ suitably according to the assumptions (3 - 2) and (3 - 3) .

Under the assumption (3 - 2), i.e., $M(x) = 0(x^{\frac{1}{2}+\varepsilon})$, we have from (3 - 6) that

$$Q(x) = \frac{6}{\pi^2} x + \cancel{0(x\xi}^{-\frac{3}{2}+\varepsilon}) + 0(\xi)$$

$$+ 0\left(x^{\frac{1}{4}+\frac{\varepsilon}{2}} \sum_{m\le\frac{x}{\xi^2}} \frac{1}{m^{\frac{1}{4}+\frac{\varepsilon}{2}}}\right)$$

$$= \frac{6}{\pi^2} x + 0(x\xi^{-\frac{3}{2}+\varepsilon}) + 0(\xi) .$$

To determine ξ, let

$$x\xi^{-\frac{3}{2}+\varepsilon} \sim \xi .$$

This gives $\xi \sim x^{\frac{2}{5-2\varepsilon}}$.

Hence

$$\sum_{1\le n\le x} |\mu(n)| = \frac{6}{\pi^2}x + O(x^{\frac{2}{5-2\varepsilon}}) = \frac{6}{\pi^2}x + O(x^{\frac{2}{5}+\varepsilon_1})$$

where $\varepsilon_1 > 0$ goes to 0 as ε goes to 0 .

In the case of assumption (3 - 3) , i.e., $M(x) = O(xe^{-c\sqrt{\log x}})$, we obtain from (3 - 6) that

$$Q(x) = \frac{6}{\pi^2}x + O(\frac{x}{\xi}e^{-c\sqrt{\log \xi}}) + O(\xi)$$

$$+ O(\sqrt{x}\, e^{-c\sqrt{\log \xi}} \sum_{m\le \frac{x}{\xi^2}} \frac{1}{\sqrt{m}})$$

$$= \frac{6}{\pi^2}x + O(\frac{x}{\xi}e^{-c\sqrt{\log \xi}}) + O(\xi) .$$

To determine ξ , let

$$\frac{x}{\xi}e^{-c\sqrt{\log \xi}} \sim \xi .$$

That is, $x \sim \xi^2 e^{c\sqrt{\log \xi}}$

or, equivalently, $x = \xi^2 e^{c\sqrt{\log \xi}}(1 + o(1))$.

Taking logarithms on both sides,

$$\log x = 2\log\xi(1 + \frac{c}{2}\frac{1}{\sqrt{\log\xi}} + \circ(\frac{1}{\log\xi}))$$

since $$\log(1 + \circ(1)) = \circ(1) .$$

This gives

$$\log\xi = \frac{1}{2}(\log x)(1 + \frac{c}{2}\frac{1}{\sqrt{\log\xi}} + \circ(\frac{1}{\log\xi}))^{-1} .$$

By the binomial expansion and successive substitution for $\log\xi$, we have

$$\log\xi = \frac{1}{2}(\log x)(1 - \frac{c}{2}\frac{1}{\sqrt{\log\xi}} + \circ(\frac{1}{\log\xi}) + \frac{c^2}{4}\frac{1}{\log\xi})$$

$$= \frac{1}{2}(\log x)(1 - \frac{c}{2}\frac{\sqrt{2}}{\sqrt{\log x}}(1 - \frac{c}{4\sqrt{\log\xi}} + 0(\frac{1}{\log\xi}))^{-1}$$

$$+ \frac{c^2}{4}\cdot\frac{2}{\log x}(1 - \frac{c}{2}\frac{1}{\sqrt{\log\xi}} + 0(\frac{1}{\log\xi}))^{-1} + \circ(\frac{1}{\log\xi}))$$

$$= \frac{1}{2}(\log x)(1 - \frac{c}{\sqrt{2}}\frac{1}{\sqrt{\log x}}(1 - \frac{c}{4}\frac{\sqrt{2}}{\sqrt{\log x}}\frac{1}{(1+0(\frac{1}{\sqrt{\log x}}))} + 0(\frac{1}{\log x}))$$

$$+ \frac{c^2}{2}\frac{1}{\log x}(1 + 0(\frac{1}{\sqrt{\log x}})) + \circ(\frac{1}{\log x}))$$

$$= \frac{1}{2}(\log x)(1 - \frac{c}{\sqrt{2}}\frac{1}{\sqrt{\log x}}(1 + \frac{c}{2\sqrt{2}}\frac{(1 + 0(\frac{1}{\log x}))}{\sqrt{\log x}(1 + 0(\frac{1}{\sqrt{\log x}}))}$$

$$+ O(\frac{1}{\log x})) + \frac{c^2}{2}\frac{1}{\log x}(1 + o(1)) + o(\frac{1}{\log x}))$$

$$= \frac{1}{2}\log x - \frac{c}{2\sqrt{2}}\sqrt{\log x} - \frac{c^2}{8} + \frac{c^2}{4} + o(1)$$

$$= \frac{1}{2}\log x - \frac{c}{2\sqrt{2}}\sqrt{\log x} + \frac{c^2}{8} + o(1) .$$

Hence,

$$\xi \sim \sqrt{x}\, e^{-\frac{c}{2\sqrt{2}}\sqrt{\log x}}\, e^{\frac{c^2}{8}} .$$

This gives

$$O(\xi) = O(\sqrt{x}\, e^{-\frac{c}{2\sqrt{2}}\sqrt{\log x}}) .$$

On the other hand, we get

$$\sqrt{\log \xi} = \sqrt{\frac{\log x}{2}}(1 - \frac{c}{\sqrt{2}}\frac{1}{\sqrt{\log x}} + \frac{c^2}{4}\frac{1}{\log x} + o(\frac{1}{\log x}))^{\frac{1}{2}}$$

$$= \sqrt{\frac{\log x}{2}}(1 - \frac{c}{2\sqrt{2}}\frac{1}{\sqrt{\log x}} + O(\frac{1}{\log x}))$$

$$= \sqrt{\frac{\log x}{2}} - \frac{c}{4} + O(\frac{1}{\sqrt{\log x}})$$

and

$$-c\sqrt{\log \xi} = -\frac{c}{\sqrt{2}}\sqrt{\log x} + \frac{c^2}{4} + O(\frac{1}{\sqrt{\log x}}) .$$

Therefore,

$$e^{-c\sqrt{\log \xi}} = e^{-\frac{c}{\sqrt{2}}\sqrt{\log x}}\, e^{\frac{c^2}{4}}\, e^{O\left(\frac{1}{\sqrt{\log x}}\right)}$$

$$= e^{-\frac{c}{\sqrt{2}}\sqrt{\log x}}\, e^{\frac{c^2}{4}}\left(1 + O\left(\frac{1}{\sqrt{\log x}}\right)\right)$$

and

$$O\left(\frac{x}{\xi}\, e^{-c\sqrt{\log \xi}}\right) = O\left(\sqrt{x}\, e^{-\frac{c}{2\sqrt{2}}\sqrt{\log x}}\right)$$

Consequently, we have proved that

$$\sum_{1 \le n \le x} |\mu(n)| = \frac{6}{\pi^2}x + O\left(\sqrt{x}\, e^{-c_1\sqrt{\log x}}\right)$$

where $c_1 = \frac{c}{2\sqrt{2}}$

For the proof that $R(x) = o(x)$, see [4], pp. 606-609.

Q.E.D.

LEMMA 3.5

$$\sum_{1 \le n \le x} \frac{|\mu(n)|}{n} = \frac{6}{\pi^2}\log x + G(x)$$

where

$$G(x) = \begin{cases} O(\frac{1}{\sqrt{x}}) & \text{trivial} \\ o(\frac{1}{\sqrt{x}}) & \text{PNT} \\ O_\varepsilon(x^{-\frac{3}{5}+\varepsilon}) \quad (\text{for any } \varepsilon > 0) & \text{RH} \\ O(\frac{1}{\sqrt{x}} \cdot e^{-c_1\sqrt{\log x}}) & \text{DVP} \end{cases}$$

Here c_1 *is the constant described in Lemma* 3.4 ∇

<u>*PROOF*</u>. From Lemma 3.4, we have,

$$Q(x) = \sum_{1 \le n \le x} |\mu(n)| = \frac{6}{\pi^2} x + R(x) .$$

By the integration by parts of a Stieltjes integral,

$$\sum_{1 \le n \le x} \frac{|\mu(n)|}{n} = \int_1^x \frac{dQ(t)}{t}$$

$$= \frac{Q(x)}{x} + \int_1^x \frac{Q(t)}{t^2} dt$$

$$= \frac{6}{\pi^2} + \frac{R(x)}{x} + \frac{6}{\pi^2} \log x + \int_1^x \frac{R(t)}{t^2} dt$$

$$= \frac{6}{\pi^2} \log x + \frac{6}{\pi^2} + \int_1^\infty \frac{R(t)}{t^2} dt + \frac{R(x)}{x} - \int_x^\infty \frac{R(t)}{t^2} dt$$

(the last two terms giving the error).

By substituting in the different estimations of $R(x)$ as given in Lemma 3.4, we obtain the required results for $G(x)$.

Q.E.D.

LEMMA 3.6 $\sum_{1\le d\le x} \mu(d)\left[\frac{x}{d}\right] = 1$. ∇

PROOF. By the Möbius inversion formula, the proof follows from the fact that $[x] = \sum_{d\le x} 1$.

Q.E.D.

LEMMA 3.7 (E. Landau's thesis)

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d} = 0 \qquad \nabla$$

This is equivalent to PNT.

REMARK. It is interesting to note that from the fact that $\sum_{d\ge 1} \frac{\mu(d)}{d}$ converges, it follows that necessarily $\sum_{d\ge 1} \frac{\mu(d)}{d} = 0$ ([4], pp. 583-584).

LEMMA 3.8

$$\sum_{1\le d\le x} \frac{\mu(d)}{d} = \begin{cases} O(1) & \text{trivial} \\ o(1) & \text{PNT} \\ O_\varepsilon(x^{-\frac{1}{2}+\varepsilon}) \quad (\text{for any } \varepsilon > 0) & \text{RH} \\ O_\varepsilon(e^{-(c-\varepsilon)\sqrt{\log x}}) \quad (\text{for any } \varepsilon > 0) & \text{DVP} \end{cases}$$

c *being the (absolute) constant of relation* (3 - 3) . ∇

PROOF. By replacing $[\frac{x}{d}] = \frac{x}{d} - \{\frac{x}{d}\}$ in Lemma 3.6 and applying the trivial result (the first one), we obtain

$$\sum_{1\le d\le x} \frac{\mu(d)}{d} = O(1) \quad .$$

For the proof that $\sum_{1\le d\le x} \frac{\mu(d)}{d} = o(1)$, see [4] (pp. 591 - 593).

To obtain the other two results, integration by parts of a Stieltjes integral is used. Thus

$$\sum_{1\le d\le x} \frac{\mu(d)}{d} = \int_1^x \frac{dM(t)}{t}$$

$$= \frac{M(x)}{x} + \int_1^x \frac{M(t)}{t^2}\,dt$$

$$= \frac{M(x)}{x} + \int_1^\infty \frac{M(t)}{t^2}\,dt - \int_x^\infty \frac{M(t)}{t^2}\,dt \quad .$$

Assuming $M(x) = O(x^{\frac{1}{2}+\varepsilon})$, we have

$$\sum_{1\le d\le x} \frac{\mu(d)}{d} = O(x^{-\frac{1}{2}+\varepsilon})$$

since $\int_1^\infty \frac{M(t)}{t^2}\,dt$ converges for $\varepsilon < \frac{1}{2}$ in this case. Next, assuming $M(x) = O(xe^{-c\sqrt{\log x}})$, we get

$$\sum_{1\le d\le x} \frac{\mu(d)}{d} = O(e^{-c\sqrt{\log x}}) + \kappa - \int_x^\infty \frac{M(t)}{t^2}\,dt ,$$

where $$\kappa = \int_1^\infty \frac{M(t)}{t^2}\,dt .$$

As we note that $e^{-c\sqrt{\log t}} = O(\frac{1}{\log^2 t})$, it is clear that κ converges absolutely since $\int_2^\infty \frac{dt}{t\log^2 t}$ does. Thus κ is a finite constant.

Now we will evaluate $\int_x^\infty \frac{M(t)}{t^2}\,dt$ by a change of variable. We know that

$$\int_x^\infty \frac{M(t)}{t^2}\,dt \ll \int_x^\infty \frac{e^{-c\sqrt{\log t}}}{t}\,dt .$$

Letting $u = \log t$, we get

$$\int_x^\infty \frac{M(t)}{t^2}\,dt \ll \int_{\log x}^\infty e^{-c\sqrt{u}}\,du .$$

To evaluate $\int_{\log x}^{\infty} e^{-c\sqrt{u}}\,du$, we integrate by parts:

$$\int_{\log x}^{\infty} e^{-c\sqrt{u}}\,du = \int_{\log x}^{\infty} \left(-\frac{c}{2\sqrt{u}}\, e^{-c\sqrt{u}}\right)\left(-\frac{2}{c}\sqrt{u}\right) du$$

$$= \frac{2}{c}\sqrt{\log x}\;\; e^{-c\sqrt{\log x}} + \frac{1}{c}\int_{\log x}^{\infty} e^{-c\sqrt{u}}\,\frac{du}{\sqrt{u}}$$

$$= \frac{2}{c}\sqrt{\log x}\;\; e^{-c\sqrt{\log x}} + 0\left(\frac{1}{\sqrt{\log x}}\right)\int_{\log x}^{\infty} e^{-c\sqrt{u}}\,du\ .$$

Thus,

$$\left(1 + 0\left(\frac{1}{\sqrt{\log x}}\right)\right)\int_{\log x}^{\infty} e^{-c\sqrt{u}}\,du = \frac{2}{c}\sqrt{\log x}\;\; e^{-c\sqrt{\log x}}\ .$$

That is,

$$\int_{\log x}^{\infty} e^{-c\sqrt{u}}\,du = \left(1 + 0\left(\frac{1}{\sqrt{\log x}}\right)\right)\frac{2}{c}\sqrt{\log x}\;\; e^{-c\sqrt{\log x}}\ .$$

Hence,

$$\sum_{1\le d\le x} \frac{\mu(d)}{d} = 0\left(e^{-c\sqrt{\log x}}\right) + 0\left(e^{-c\sqrt{\log x}}\;\sqrt{\log x}\right)$$

$$+ \kappa$$

$$= \kappa + 0\left(e^{-c\sqrt{\log x}}\sqrt{\log x}\right)\ .$$

Here $\kappa = 0$ because the rest of the terms go to zero as x goes to infinity.

Let ε be any positive constant. Then

$$\log\sqrt{\log x} \le \varepsilon\sqrt{\log x}$$

for x sufficiently large. Thus, we have proved that

$$\sum_{1\le d\le x} \frac{\mu(d)}{d} = O(e^{-(c-\varepsilon)\sqrt{\log x}}) .$$

Q.E.D.

An important and very difficult lemma is the following:

LEMMA 3.9 (See [9].) *Let*

$$\gamma(x) = \sum_{1\le n\le x} \frac{\mu(n)}{n} \{ \frac{x}{n} \}.$$

Then $\gamma(x) = O((\log x)^{\frac{2}{3}}(\log\log x)^{1+\varepsilon})$

for any $\varepsilon > 0$. ∇

REMARK. It is obvious from Lemma 3.5 that

$$\gamma(x) = O(\log x) .$$

LEMMA 3.10 $$\sum_{1 \le n \le x} \phi(n) = \frac{3}{\pi^2} x^2 + \sigma(x)$$

where $$\sigma(x) = \begin{cases} O(x \log x) \\ O(x(\log x)^{\frac{2}{3}} (\log \log x)^{1+\varepsilon}) \end{cases}$$

for any $\varepsilon > 0$. ∇

LEMMA 3.11 $$\Psi(x) = \sum_{1 \le n \le x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + \rho(x)$$

where $$\rho(x) = \begin{cases} O(\log x) \\ O((\log x)^{\frac{2}{3}} (\log \log x)^{1+\varepsilon}) \end{cases}$$

for any $\varepsilon > 0$. ∇

The above two statements follow from Lemmas 1.2, 3.3, 3.6 and 3.9.

LEMMA 3.12 (Cf. [5], Ch. 8.) *Let* A(N) *be the total number of lattice points in and on the circle of radius* $\sqrt{N}$ *; then*

$$A(N) = \pi N + O(N^{\theta}) ,$$

where $\frac{1}{4} < \theta < \frac{1}{3}$. ∇

REMARK. Through the work of Hardy and Landau ([5], Satz 542), we know that $\theta \neq \frac{1}{4}$.

<u>LEMMA 3.13</u> (Cf. [6].) *Let* $A(x)$ *be the total number of lattice points in and on the ellipse* $au^2 + bv^2 = x$ *where* a,b *are positive integers,* $a > b$. *Then*

$$A(x) = \alpha x + 0(x^{\theta})$$

where αx *equals the area of the ellipse and* $\frac{1}{4} < \theta < \frac{1}{3}$; ∇

<u>LEMMA 3.14</u> . *Let* n *be any positive integer and* $x \geq 0$. *If* $F(x,n)$ *is the number of integers not exceeding* x *which are relatively prime to* n , *then*

$$F(x,n) = x\frac{\phi(n)}{n} - \sum_{d|n} \mu(d) \{\frac{x}{d}\} .$$ ∇

<u>PROOF</u>. From the definition of $F(x,n)$, we have

$$F(x,n) = \sum_{\substack{1\leq k\leq x \\ (k,n)=1}} 1 .$$

Rearranging the sum and applying Lemma 1.1, we have

$$\begin{aligned} F(x,n) &= \sum_{1\leq k\leq x} \sum_{d|(k,n)} \mu(d) \\ &= \sum_{1\leq k\leq x} \sum_{\substack{d|k \\ d|n}} \mu(d) \\ &= \sum_{\substack{d|n \\ 1\leq d\leq x}} \mu(d) \sum_{\substack{1\leq k\leq x \\ k\equiv 0 \pmod d}} 1 \end{aligned}$$

That is,

$$(3-7) \qquad F(x,n) = \sum_{\substack{d|n \\ 1\le d\le x}} \mu(d) \left[\frac{x}{d}\right] .$$

<u>CASE 1.</u> If $x \ge n$, we can drop the condition $d \le x$ in $(3-7)$. Thus,

$$F(x,n) = \sum_{d|n} \mu(d) \left[\frac{x}{d}\right]$$

$$= x \sum_{d|n} \frac{\mu(d)}{d} - \sum_{d|n} \mu(d) \left\{\frac{x}{d}\right\} .$$

It follows from Lemma 1.2 that

$$F(x,n) = x\frac{\phi(n)}{n} - \sum_{d|n} \mu(d) \left\{\frac{x}{d}\right\} ,$$

or

$$F(x,n) = x\frac{\phi(n)}{n} + 0(2^{\omega(n)}) .$$

<u>CASE 2.</u> If $x \le n$, we write

$$F(x,n) = F(x+n,n) - \phi(n) .$$

Now $x+n \ge n$. Applying the result of case 1, we obtain

$$F(x,n) = (x+n)\frac{\phi(n)}{n} - \sum_{d|n} \mu(d)\left\{\frac{x+n}{d}\right\} - \phi(n)$$

$$= x\frac{\phi(n)}{n} - \sum_{d|n} \mu(d) \left\{\frac{x}{d}\right\}$$

$$= x\frac{\phi(n)}{n} + 0(2^{\omega(n)})$$

because $d|n$. Therefore, in general, for any $x \geq 0$,

$$F(x,n) = x\frac{\phi(n)}{n} - \sum_{d|n} \mu(d) \{ \frac{x}{d} \} .$$

Q.E.D.

REMARK. The behaviour of

$$R(n,x) = - \sum_{d|n} \mu(d) \{\frac{x}{d}\}$$

may be difficult to study. However, we can compute the mean value of $R(n,x)$ in the interval $[0,n]$. First note that

$$\int_0^n R(n,x)dx = - \sum_{d|n} \mu(d) \int_0^n \{\frac{x}{d}\} dx .$$

Now $\{ \frac{x}{d} \} = \frac{x}{d}$ if $0 \leq x < d$,

and $\int_0^d \{ \frac{x}{d} \} dx = \frac{1}{d} \int_0^d x\, dx = \frac{d}{2}$.

Therefore $\int_0^n \{\frac{x}{d}\} dx = \frac{n}{d} \frac{d}{2} = \frac{n}{2}$

Consequently,

the mean value of $R(n,x) = - \frac{n}{2} \sum_{d|n} \mu(d)$.

$$= \begin{cases} -\frac{1}{2} & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

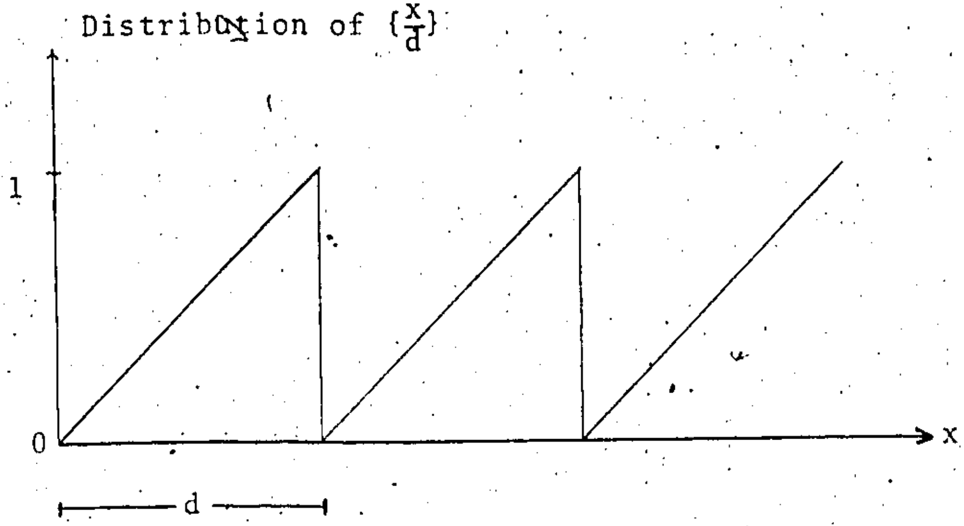

FIGURE 3.2

LEMMA 3.15 *For x sufficiently large,*

$$|E(x)| \leq \left(\frac{6}{\pi^2} + o(1)\right) x \log x \quad ,$$

where $E(x) = \sum_{n \leq x} \sum_{d|n} \mu(d) \{\frac{f(n)}{d}\}$

and $f(n) \geq 0$. ∇

PROOF. Let $n = md$. Then

$$|E(x)| = \left| \sum_{1 \leq d \leq x} \mu(d) \sum_{m \leq \frac{x}{d}} \{\frac{f(md)}{d}\} \right| \leq \sum_{1 \leq d \leq x} |\mu(d)| \frac{x}{d} \; .$$

Applying Lemma 3.5, we obtain

$$|E(x)| \leq \left(\frac{6}{\pi^2} + o(1)\right) x \log x$$

for sufficiently large x . Q.E.D.

LEMMA 3.16 *For x sufficiently large,*

$$|E(x)| \le \left(\frac{3}{\pi^2} + o(1)\right) x \log x \quad . \qquad \nabla$$

PROOF. Rearrange the summations in $E(x)$ and let $n = md$. Then

$$|E(x)| = \left| \sum_{1 \le d \le x} \mu(d) \sum_{m \le \frac{x}{d}} \left\{\frac{f(md)}{d}\right\} \right|$$

$$= \left| \sum_{\substack{1 \le d \le x \\ \mu(d)=1}} \sum_{m \le \frac{x}{d}} \left\{\frac{f(md)}{d}\right\} - \sum_{\substack{1 \le d \le x \\ \mu(d)=-1}} \sum_{m \le \frac{x}{d}} \left\{\frac{f(md)}{d}\right\} \right|$$

$$\le \max\left(\sum_{\substack{1 \le d \le x \\ \mu(d)=1}} \sum_{m \le \frac{x}{d}} \left\{\frac{f(md)}{d}\right\}, \sum_{\substack{1 \le d \le x \\ \mu(d)=-1}} \sum_{m \le \frac{x}{d}} \left\{\frac{f(md)}{d}\right\} \right)$$

$$\le \max\left(x \sum_{\substack{1 \le d \le x \\ \mu(d)=1}} \frac{1}{d} , \quad x \sum_{\substack{1 \le d \le x \\ \mu(d)=-1}} \frac{1}{d}\right)$$

since $|A - B| \le \max(A,B)$ if $A \ge 0$ and $B \ge 0$.

Let
$$P(x) = \sum_{\substack{1 \le d \le x \\ \mu(d)=1}} \frac{1}{d}$$

and
$$N(x) = \sum_{\substack{1 \le d \le x \\ \mu(d)=-1}} \frac{1}{d} \quad ;$$

then by Lemmas 3.5 and 3.8,

$$P(x) - N(x) = \sum_{1 \le d \le x} \frac{\mu(d)}{d} = O(1) .$$

and $$P(x) + N(x) = \sum_{1\le d\le x} \frac{|\mu(d)|}{d} = \frac{6}{\pi^2}\log x + 0(\frac{1}{\sqrt{x}}) .$$

By adding and subtracting, we obtain that

$$P(x) = \frac{3}{\pi^2}\log x + 0(\frac{1}{\sqrt{x}}) = \frac{3}{\pi^2}\log x + o(\log x)$$

and $$N(x) = \frac{3}{\pi^2}\log x + 0(\frac{1}{\sqrt{x}}) = \frac{3}{\pi^2}\log x + o(\log x) .$$

Hence for sufficiently large x,

$$|E(x)| \le (\frac{3}{\pi^2} + o(1))\, x \log x .$$

Q.E.D.

LEMMA 3.17 *For constants* α *and* β *such that* $\alpha > 0$, *and* $x \le y$,

$$x(\log y)^{\alpha}\,(\log\log y)^{\beta} << y(\log x)^{\alpha}\,(\log\log x)^{\beta}$$

for sufficiently large x . ∇

PROOF. Let $\rho(x) = (\log x)^{\alpha}\,(\log\log x)^{\beta}$.

For $x \ge x_0$, $\rho(x)$ is a concave increasing function and $\rho(x) \nearrow \infty$.

Let $t = \frac{y}{x}$; then $t \ge 1$. Note that t here is not a constant but depends on x and y .

Assume that $x \geq x_0$. We want to show that

$$\rho(tx) << t\rho(x) .$$

Let us consider two cases:

CASE 1. If $x_0 \leq t \leq x$, then

$$\rho(tx) = (\log x + \log t)^{\alpha} (\log(\log x + \log t))^{\beta}$$

$$\leq (2\log x)^{\alpha} (\log(2\log x))^{\beta}$$

$$<< (\log x)^{\alpha} (\log 2 + \log\log x)^{\beta}$$

$$= (\log x)^{\alpha} (\log\log x)^{\beta} (1 + \frac{\log 2}{\log\log x})^{\beta}$$

$$\sim (\log x)^{\alpha} (\log\log x)^{\beta} .$$

Therefore, $\rho(tx) << \rho(x)$.

This implies that

$$\rho(tx) << t\rho(x)$$

CASE 2. If $t > x \geq x_0$, then

$$\begin{aligned} \rho(tx) &\leq \rho(t^2) \\ &= (2\log t)^{\alpha} (\log 2 + \log\log t)^{\beta} \\ &<< (\log t)^{\alpha} (\log\log t)^{\beta} \\ &\lneq t \\ &\leq t\rho(x) . \end{aligned}$$

Thus we have proved the lemma. Q.E.D.

LEMMA 3.18 *If f is an increasing function and the condition* $\frac{f(\lambda x)}{f(x)} \to 1$ *as* $x \to \infty$ *is always true for* $\lambda \geq 1$, λ *constant, then*

$$x \sim y \Longrightarrow f(x) \sim f(y) \quad .$$

[One says that f is a "slowly varying" function.] ∇

PROOF. For x, y large enough, suppose that $x \leq y$; then $x \leq y \leq 2x$.

Hence $f(x) \leq f(y) \leq f(2x)$.

Now $f(x) \sim f(2x)$.

This implies that

$$f(x) \sim f(y) \quad .$$

Q.E.D.

EXAMPLES.

(1) $x \sim y \Longrightarrow \log x \sim \log y$;

(2) $x \sim y \Longrightarrow x^{\alpha} \sim y^{\alpha}$ for any real α .

Next, let us consider the expression

$$\sum_{d \leq x} \mu(d) \sum_{m \leq \frac{x}{d}} \{ \frac{y}{x} m \} \quad .$$

It is obvious that

$$\sum_{d\le x}\mu(d)\sum_{m\le\frac{x}{d}}\{\tfrac{y}{x}m\} = 0(x\log x) .$$

Unfortunately, this gives an estimation which is not suitable for our purposes in dealing with the error term in Theorem 2.2. Instead of this, the sharper result

$$\sum_{d\le x}\mu(d)\sum_{m\le\frac{x}{d}}\{\tfrac{y}{x}m\} = 0(x)$$

is given in the following lemma.

LEMMA 3.19. *For any real numbers* x, y $(x > 64,\ y \ge 1)$, we have

$$\sum_{1\le d\le x}\mu(d)\, r(\tfrac{x}{d}, \tfrac{y}{d}) = 0(x) ,$$

where

$$r(\tfrac{x}{d}, \tfrac{y}{d}) = \sum_{1\le n\le\frac{x}{d}}\{\tfrac{y}{x}n\} .$$ ∇

PROOF. Let $T(x,y) = \sum_{1\le d\le x}\mu(d)\, r(\tfrac{x}{d}, \tfrac{y}{d})$.

Rearrange the sum and note that

$$M(\tfrac{x}{n}) = 0\left[\frac{\frac{x}{n}}{(\log\frac{x}{n})^2}\right]^{\dagger} , \qquad \text{or,}$$

---

† Although $M(\frac{x}{n}) = 0\left[\frac{\frac{x}{n}}{(\log\frac{x}{n})^2}\right]$ is weaker than $M(\frac{x}{n}) = 0(\frac{x}{n}\, e^{-c\sqrt{\log\frac{x}{n}}})$, it suffices for our present purpose.

(3 - 8) $$|M(\tfrac{x}{n})| \le c \frac{\frac{x}{n}}{(\log \frac{x}{n})^2}, \quad c \text{ being a constant.}$$

Then $$T(x,y) = \sum_{\substack{n,d \\ nd \le x}} \mu(d) \{ \tfrac{y}{x} n \}$$

$$= \sum_{n \le x} \{\tfrac{y}{x} n\} M(\tfrac{x}{n}) .$$

The assumption $x > 64$ ensures that $\sqrt{x} < \frac{x}{8}$. Thus we can split the sum in $T(x,y)$ into three parts as follows:

$$T(x,y) = A + B + D ,$$

where $$A = \sum_{n \le \sqrt{x}} \{\tfrac{y}{x} n\} M(\tfrac{x}{n}) ,$$

$$B = \sum_{\sqrt{x} < n \le \frac{x}{8}} \{\tfrac{y}{x} n\} M(\tfrac{x}{n}) ,$$

$$D = \sum_{\frac{x}{8} < n \le x} \{\tfrac{y}{x} n\} M(\tfrac{x}{n}) .$$

Next, we evaluate A, B and D separately.

To evaluate D :

We split the sum in such a way that the value of $M(\frac{x}{n})$ can be estimated.

$$D = \sum_{\frac{x}{2}<n\leq x} \{\tfrac{y}{x}n\}M(\tfrac{x}{n}) + \sum_{\frac{x}{3}<n\leq\frac{x}{2}} \{\tfrac{y}{x}n\}M(\tfrac{x}{n})$$

$$+ \sum_{\frac{x}{4}<n\leq\frac{x}{3}} \{\tfrac{y}{x}n\}M(\tfrac{x}{n}) + \sum_{\frac{x}{5}<n\leq\frac{x}{4}} \{\tfrac{y}{x}n\}M(\tfrac{x}{n})$$

$$+ \sum_{\frac{x}{6}<n\leq\frac{x}{5}} \{\tfrac{y}{x}n\}M(\tfrac{x}{n}) + \sum_{\frac{x}{7}<n\leq\frac{x}{6}} \{\tfrac{y}{x}n\}M(\tfrac{x}{n})$$

$$+ \sum_{\frac{x}{8}<n\leq\frac{x}{7}} \{\tfrac{y}{x}n\}M(\tfrac{x}{n})$$

$$= \sum_{\frac{x}{2}<n\leq x} \{\tfrac{y}{x}n\} - \sum_{\frac{x}{4}<n\leq\frac{x}{3}} \{\tfrac{y}{x}n\} - \sum_{\frac{x}{5}<n\leq\frac{x}{4}} \{\tfrac{y}{x}n\}$$

$$- 2\sum_{\frac{x}{6}<n\leq\frac{x}{5}} \{\tfrac{y}{x}n\} - \sum_{\frac{x}{7}<n\leq\frac{x}{6}} \{\tfrac{y}{x}n\} - 2\sum_{\frac{x}{8}<n\leq\frac{x}{7}} \{\tfrac{y}{x}n\}$$

$$= O(x) \quad .$$

Hence, $D = O(x)$.

To evaluate A :

$$A = \sum_{n\leq\sqrt{x}} \{\tfrac{y}{x}n\}\ M(\tfrac{x}{n}) \quad .$$

From $n \leq \sqrt{x}$, it follows that

$$\log\frac{x}{n} \geq \frac{1}{2}\log x \quad .$$

So that by (3 - 8) ,

$$|M(\tfrac{x}{n})| \leq 4c\,\frac{x}{n(\log x)^2}$$

~~Now,~~

$$|A| \leq \sum_{n\leq\sqrt{x}} 4c\,\frac{x}{n(\log x)^2}$$

$$= \frac{4cx}{(\log x)^2} \sum_{n\leq\sqrt{x}} \frac{1}{n}$$

By Corollary 3.2.1, we obtain

$$|A| \leq \frac{4cx}{(\log x)^2}\left(\log\sqrt{x} + \gamma + 0(\tfrac{1}{\sqrt{x}})\right)$$

$$\leq \frac{2cx}{\log x} + \frac{4cx}{(\log x)^2}\left(\gamma + 0(\tfrac{1}{\sqrt{x}})\right)$$

Thus $A = 0(\frac{x}{\log x})$;

this implies $A = 0(x)$.

To evaluate B :

$$B = \sum_{\sqrt{x}<n\leq\frac{x}{8}} \{\tfrac{y}{x}n\}\, M(\tfrac{x}{n})$$

and $|B| \leq c\,x \sum_{\sqrt{x}<n\leq\frac{x}{8}} \frac{1}{n(\log\frac{x}{n})^2}$

For fixed x, differentiating the function $\frac{1}{t(\log\frac{x}{t})^2}$ with respect to t, we have by a simple computation that

$$\frac{d}{dt}\left(\frac{1}{t(\log\frac{x}{t})^2}\right) = \frac{(2 - \log\frac{x}{t})}{t^2(\log\frac{x}{t})^3} < 0 \qquad \text{if} \quad t < \frac{x}{e^2} .$$

To ensure that $\frac{x}{8} > \sqrt{x}$, we must have $x > 64$. Since $\frac{x}{8} < \frac{x}{e^2}$, the function $\frac{1}{t(\log\frac{x}{t})^2}$ is strictly decreasing on $(\sqrt{x}, \frac{x}{8}]$ for fixed $x > 64$. Therefore, by applying Corollary 3.2.1,

$$\sum_{\sqrt{x}<n\le\frac{x}{8}} \frac{1}{n(\log\frac{x}{n})^2} = \int_{\sqrt{x}}^{\frac{x}{8}} \frac{dt}{t(\log\frac{x}{t})^2} + 0\left(\frac{1}{\sqrt{x}}\right) .$$

Putting $u = \frac{x}{t}$, the sum becomes

$$\sum_{\sqrt{x}<n\le\frac{x}{8}} \frac{1}{n(\log\frac{x}{n})^2} = \int_{8}^{\sqrt{x}} \frac{du}{u(\log u)^2} + 0\left(\frac{1}{\sqrt{x}}\right)$$

$$= \frac{1}{\log 8} - \frac{2}{\log x} + 0\left(\frac{1}{\sqrt{x}}\right) .$$

Therefore $B = 0(x)$. Consequently, we have proved that

$$T(x,y) = 0(x) .$$

Q.E.D.

CHAPTER IV

PROOFS OF THEOREMS

We now proceed to the proofs of Theorems 2.1, 2.2, 2.3, 2.4 and 2.5.

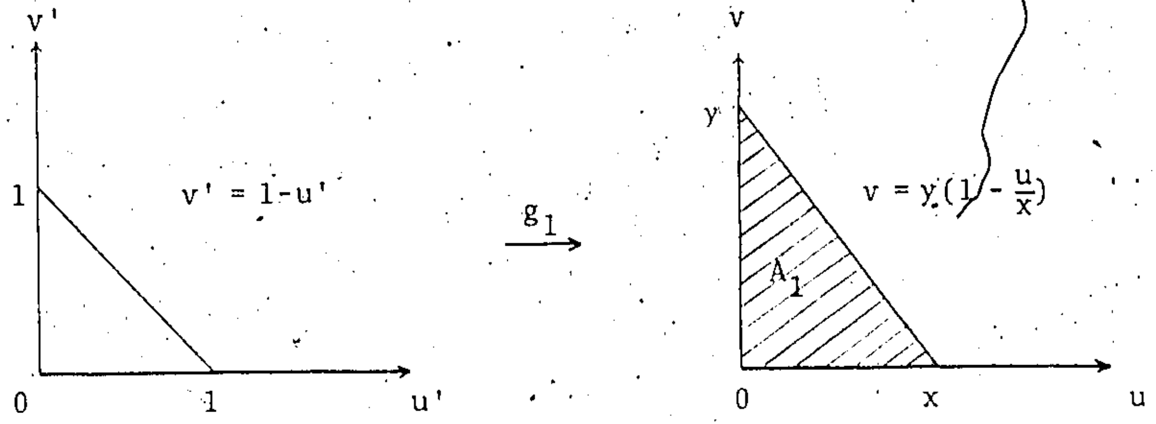

FIGURE 4.1

PROOF OF THEOREM 2.1

Let $$g_1 : \begin{cases} g_1(u') = u = xu' \\ g_1(v') = v = yv' \end{cases}$$

be a transformation which maps the line $v' = 1-u'$ into the line $v = y(1 - \frac{u}{x})$. Let $Q(x,y)$ be the total number of co^2 lattice points in the triangle A_1 of Figure 4.1. Applying Formula (2 - 1), we obtain

$$Q(x,y) = y \sum_{d \le x} \frac{\mu(d)}{d} \sum_{m \le \frac{x}{d}} \left(1 - \frac{md}{x}\right)$$

$$- \sum_{d \le x} \mu(d) \sum_{m \le \frac{x}{d}} \left\{\frac{y}{d}\left(1 - \frac{md}{x}\right)\right\} .$$

$$= y \sum_{d \le x} \frac{\mu(d)}{d}\left(\left[\frac{x}{d}\right] - \frac{d}{2x}\left[\frac{x}{d}\right]\left(1 + \left[\frac{x}{d}\right]\right)\right)$$

$$- \sum_{d \le x} \mu(d)\, r_1\left(\frac{x}{d}, \frac{y}{d}\right)$$

$$= y \sum_{d \le x} \frac{\mu(d)}{d}\left[\frac{x}{d}\right] - \frac{y}{2x} \sum_{d \le x} \mu(d)\left[\frac{x}{d}\right]$$

$$- \frac{y}{2x} \sum_{d \le x} \mu(d)\left[\frac{x}{d}\right]^2 - \sum_{d \le x} \mu(d)\, r_1\left(\frac{x}{d}, \frac{y}{d}\right) ,$$

where $$r_1\left(\frac{x}{d}, \frac{y}{d}\right) = \sum_{m \le \frac{x}{d}} \left\{\frac{y}{d}\left(1 - \frac{md}{x}\right)\right\} .$$

Substituting $\left[\frac{x}{d}\right] = \frac{x}{d} - \left\{\frac{x}{d}\right\}$, we find that

$$Q(x,y) = \frac{xy}{2} \sum_{d \le x} \frac{\mu(d)}{d^2} - \frac{y}{2x} \sum_{d \le x} \mu(d)\left[\frac{x}{d}\right]$$

$$- \frac{y}{2x} \sum_{d \le x} \mu(d) \left\{\frac{x}{d}\right\}^2 - \sum_{d \le x} \mu(d)\, r_1\left(\frac{x}{d}, \frac{y}{d}\right) .$$

Now,

$$\sum_{d\le x} \mu(d)\,\{\tfrac{x}{d}\}^2 \ll \sum_{d\le x} 1 \ll x \quad ;$$

from Corollary 3.2.1,

$$\sum_{d\le x} \mu(d)\, r_1(\tfrac{x}{d}, \tfrac{y}{d}) \ll x \sum_{d\le x} \frac{1}{d} \ll x \log x \quad .$$

Together with Lemmas 3.3 and 3.6, we conclude that

$$Q(x,y) = \frac{3}{\pi^2}\,xy + O(y) + O(x \log x) \quad .$$

Q.E.D.

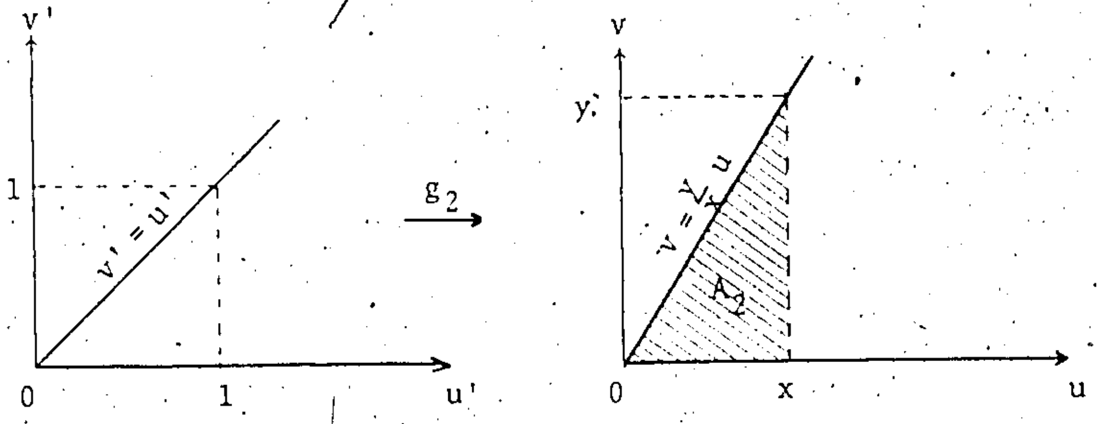

FIGURE 4.2

PROOF OF THEOREM 2.2

Let
$$g_2 : \begin{cases} g_2(u') = u = xu' \\ g_2(v') = v = yv' \end{cases}$$

be a transformation which maps the line $v' = u'$ into the line $v = \frac{y}{x}u$. Let $Q(x,y)$ be the total number of co^2

lattice points in the triangle A_2 of Figure 4.2. Applying Formula (2 - 1), we obtain

$$Q(x,y) = y \sum_{d \le x} \frac{\mu(d)}{d} \sum_{m \le \frac{x}{d}} \frac{md}{x} - \sum_{d \le x} \mu(d) \sum_{m \le \frac{x}{d}} \{\tfrac{y}{x} m\}$$

$$= \frac{y}{x} \sum_{d \le x} \left(\frac{\mu(d)}{2} [\tfrac{x}{d}](1 + [\tfrac{x}{d}])\right) - \sum_{d \le x} \mu(d) \sum_{m \le \frac{x}{d}} \{\tfrac{y}{x} m\} .$$

Letting $[\frac{x}{d}] = \frac{x}{d} - \{\frac{x}{d}\}$ and applying Lemma 3.17, we have

$$Q(x,y) = \frac{y}{2x} \sum_{d \le x} \mu(d) [\tfrac{x}{d}] + \frac{xy}{2} \sum_{d \le x} \frac{\mu(d)}{d^2}$$

$$- y \sum_{d \le x} \frac{\mu(d)}{d} \{\tfrac{x}{d}\} + \frac{y}{2x} \sum_{d \le x} \mu(d) \{\tfrac{x}{d}\}^2 + O(x) .$$

From Lemmas 3.3, 3.6, 3.9 and the estimate

$$\sum_{d \le x} \mu(d) \{\tfrac{x}{d}\}^2 \ll x ,$$

we obtain the required result:

$$Q(x,y) = \frac{3}{\pi^2} xy + \frac{y}{2x} + O(y) + O(x) + y\, O((\log x)^{\frac{2}{3}} (\log \log x)^{1+\varepsilon})$$

$$= \frac{3}{\pi^2} xy + O(y(\log x)^{\frac{2}{3}} (\log \log x)^{1+\varepsilon}) .$$

Q.E.D.

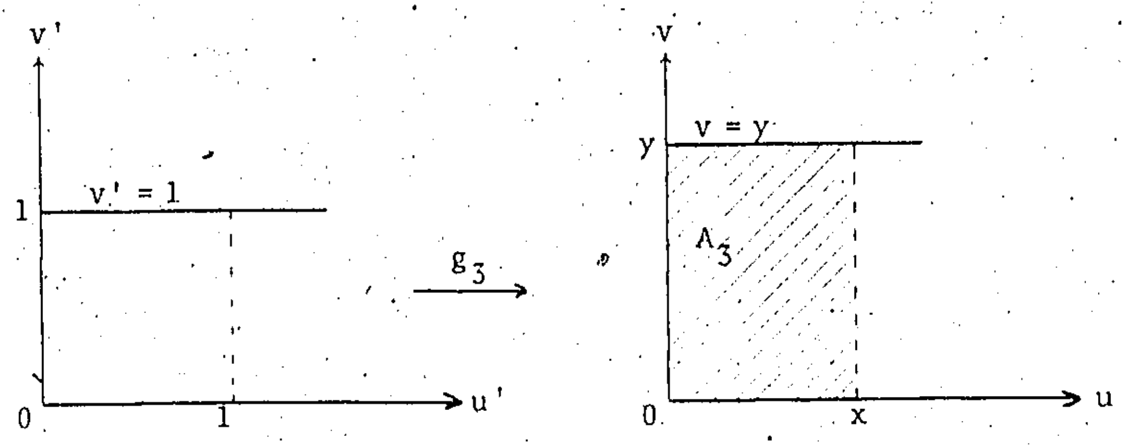

FIGURE 4.3

PROOF OF THEOREM 2.3

Let
$$g_3 : \begin{cases} g_3(u') = u = xu' \\ g_3(v') = v = yv' \end{cases}$$

be a transformation which maps the line $v' = 1$ into the line $v = y$. Let $Q(x,y)$ be the total number of co^2 lattice points in the rectangle A_3 of Figure 4.3. Then applying Formula (2 - 1), we have

$$Q(x,y) = y \sum_{d\le x} \frac{\mu(d)}{d} \sum_{m\le\frac{x}{d}} 1 - \sum_{d\le x} \mu(d) \sum_{m\le\frac{x}{d}} \{\tfrac{y}{d}\}$$

$$= y \sum_{d\le x} \frac{\mu(d)}{d} [\tfrac{x}{d}] - \sum_{d\le x} \mu(d) \{\tfrac{y}{d}\} [\tfrac{x}{d}]$$

$$= xy \sum_{d\le x} \frac{\mu(d)}{d^2} - y \sum_{d\le x} \frac{\mu(d)}{d} \{\tfrac{x}{d}\}$$

$$- x \sum_{d\le x} \frac{\mu(d)}{d} \{\tfrac{y}{d}\} + \sum_{d\le x} \mu(d) \{\tfrac{y}{d}\} \{\tfrac{x}{d}\} .$$

Now, $\sum_{d\le x} \frac{\mu(d)}{d}\{\frac{y}{d}\} = \sum_{d\le y} \frac{\mu(d)}{d}\{\frac{y}{d}\} - \sum_{x<d\le y} \frac{\mu(d)}{d}\{\frac{y}{d}\}$

$$= O((\log y)^{\frac{2}{3}}(\log\log y)^{1+\varepsilon}) + O(\frac{y}{x}) + O(\frac{1}{x})$$

due to Lemma 3.9 and the estimate

$$\sum_{x<d\le y} \frac{\mu(d)}{d}\{\frac{y}{d}\} << \sum_{x<d\le y} \frac{1}{d} = \sum_{d\le y} \frac{1}{d} - \sum_{d\le x} \frac{1}{d}$$

$$= \log \frac{y}{x} + O(\frac{1}{x})$$

$$<< \frac{y}{x} + O(\frac{1}{x}) \quad .$$

Next, $\sum_{d\le x} \mu(d)\{\frac{x}{d}\}\{\frac{y}{d}\} << x$.

With the above estimates and Lemmas 3.3, 3.9 and 3.17, we obtain

$$Q(x,y) = \frac{6}{\pi^2}xy + O(y(\log x)^{\frac{2}{3}}(\log\log x)^{1+\varepsilon})$$

$$+ O(x(\log y)^{\frac{2}{3}}(\log\log y)^{1+\varepsilon})$$

$$= \frac{6}{\pi^2}xy + O(y(\log x)^{\frac{2}{3}}(\log\log x)^{1+\varepsilon}) .$$

Q.E.D.

PROOF OF THEOREM 2.4

Let $A(x)$ and $Q(x)$ be the total number of lattice points and co^2 lattice points in the circular disk $u^2+v^2 \leq x$. Then

$$Q(x) = \sum_{\substack{u,v \\ 1\leq u^2+v^2\leq x \\ (u,v)=1}} 1 \quad ,$$

and

$$A(x) - 1 = \sum_{\substack{u,v \\ 1\leq u^2+v^2\leq x}} 1 \quad . \tag{4 - 1}$$

The origin has been removed from $A(x)$ because the g.c.d. of $(0,0)$ is not defined. Rewriting (4 - 1), we have

$$A(x) - 1 \left\{ \begin{aligned} &= \sum_{1\leq d\leq\sqrt{x}} \sum_{\substack{u,v \\ 1\leq u^2+v^2\leq x \\ (u,v)=d}} 1 \\ &= \sum_{1\leq d\leq\sqrt{x}} \sum_{\substack{u',v' \\ 1\leq u'^2+v'^2\leq \frac{x}{d^2} \\ (u',v')=1}} 1 \quad \text{where} \quad \begin{aligned} u' &= \tfrac{u}{d} \\ v' &= \tfrac{v}{d} \end{aligned} \\ &= \sum_{1\leq d\leq\sqrt{x}} Q\left(\frac{x}{d^2}\right) \quad . \end{aligned} \right. \tag{4 - 2}$$

Applying the Möbius inversion formula to (4 - 2), we get

$$Q(x) = \sum_{1 \le d \le \sqrt{x}} \mu(d)\left(A\left(\frac{x}{d^2}\right) - 1\right)$$

$$= \sum_{1 \le d \le \sqrt{x}} \mu(d)\ B\left(\frac{x}{d^2}\right) ,$$

where $B\left(\frac{x}{d^2}\right) = A\left(\frac{x}{d^2}\right) - 1$

$$= \sum_{1 \le n \le \frac{x}{d^2}} f(n)$$

and $f(n) = \sum_{n=u^2+v^2} 1$

Now, let $\xi = \xi(x)$ be some parameter, to be chosen later, such that $1 < \xi < \sqrt{x}$ and $\xi \to \infty$. Note that $\xi = o(\sqrt{x})$.

Then

$$(4 - 3) \left\{ \begin{aligned} Q(x) &= \sum_{1 \le d \le \xi} \mu(d)\ B\left(\frac{x}{d^2}\right) + \sum_{\xi < d \le \sqrt{x}} \mu(d) \sum_{1 \le n \le \frac{x}{d^2}} f(n) \\ &= \sum_{1 \le d \le \xi} \mu(d)\ B\left(\frac{x}{d^2}\right) + \sum_{n < \frac{x}{\xi^2}} f(n) \sum_{\substack{d \le \sqrt{\frac{x}{n}} \\ \xi < d \le \sqrt{x}}} \mu(d) \\ &= \sum_{1 \le d \le \xi} \mu(d)\ B\left(\frac{x}{d^2}\right) + \sum_{n < \frac{x}{\xi^2}} f(n)\ M\left(\sqrt{\frac{x}{n}}\right) - M(\xi)\ B\left(\frac{x}{\xi^2}\right) \end{aligned} \right.$$

since $1 \le n < \frac{x}{\xi^2} \implies \sqrt{x} \ge \sqrt{\frac{x}{n}} > \xi$.

Let us write

$$M(x) = 0(xe^{-\delta(\log x)}) .$$

From the different estimates (3 - 2), (3 - 3) and (3 - 4) mentioned for the value of $M(x)$, it is clear that $\delta(\log x)$ can be assumed to be an increasing function tending to $+\infty$, and $\delta(\log x) \geq c\sqrt{\log x}$.

Applying Lemma 3.12 to (4 - 3), we get

$$Q(x) = \sum_{1\leq d\leq \xi} \mu(d) \left(\pi\frac{x}{d^2} + 0\left(\frac{x^\theta}{d^{2\theta}}\right)\right)$$

$$+ 0\left(\sum_{n<\frac{x}{\xi^2}} f(n)\sqrt{\frac{x}{n}}\, e^{-\delta(\log\sqrt{\frac{x}{n}})}\right)$$

$$= \frac{6x}{\pi} - \pi x \sum_{d>\xi} \frac{\mu(d)}{d^2} + 0\left(x^\theta \sum_{d\leq\xi} \frac{1}{d^{2\theta}}\right) + 0(P) ,$$

where

$$P = \sum_{n<\frac{x}{\xi^2}} f(n)\sqrt{\frac{x}{n}}\; e^{-\delta(\log\sqrt{\frac{x}{n}})} .$$

Now

$$\pi x \sum_{d>\xi} \frac{\mu(d)}{d^2} = \pi x \int_\xi^{+\infty} \frac{dM(t)}{t^2}$$

$$= -\pi x \frac{M(\xi)}{\xi^2} + 2\pi x \int_\xi^{+\infty} \frac{M(t)}{t^3}\, dt$$

$$= O\left(\frac{x}{\xi} e^{-\delta(\log\xi)}\right) + O\left(xe^{-\delta(\log\xi)} \int_{\xi}^{+\infty} \frac{dt}{t^{2}}\right)$$

$$= O\left(\frac{x}{\xi} e^{-\delta(\log\xi)}\right) ;$$

so it follows from Corollary 3.2.1 that

$$O\left(x^{\theta} \sum_{d\le\xi} \frac{1}{d^{2\theta}}\right) = O\left(x^{\theta} \xi^{1-2\theta}\right) .$$

Thus,

$$Q(x) = \frac{6x}{\pi} + O\left(x^{\theta}\xi^{1-2\theta}\right) + O\left(\frac{x}{\xi} e^{-\delta(\log\xi)}\right) + O(P) .$$

Next, we shall estimate P. In the expression of P, we have $n < \frac{x}{\xi^{2}}$; this implies $\sqrt{\frac{x}{n}} > \xi$. Therefore, $e^{-\delta(\log\sqrt{\frac{x}{n}})} \le e^{-\delta(\log\xi)}$ since $\delta(t)$ can be assumed to be an increasing function. Hence,

$$|P| \le e^{-\delta(\log\xi)} \sqrt{x} \sum_{n<\frac{x}{\xi^{2}}} \frac{f(n)}{\sqrt{n}}$$

$$= e^{-\delta(\log\xi)} \sqrt{x} \int_{1}^{\frac{x}{\xi^{2}}} \frac{dB(t)}{\sqrt{t}}$$

$$= e^{-\delta(\log\xi)} \sqrt{x} \left(\frac{\xi}{\sqrt{x}} B\left(\frac{x}{\xi^{2}}\right) + \frac{1}{2} \int_{1}^{\frac{x}{\xi^{2}}} t^{-\frac{3}{2}} B(t)\, dt \right) ;$$

this can be computed accurately. Using the weaker estimate $B(t) = O(t)$, we obtain that

$$|P| \leq e^{-\delta(\log\xi)}\sqrt{x} \cdot O\left(\frac{\sqrt{x}}{\xi}\right)$$

$$= O\left(\frac{x}{\xi}\, e^{-\delta(\log\xi)}\right) .$$

Therefore,

$$Q(x) = \frac{6x}{\pi} + O(x^{\theta}\, \xi^{1-2\theta}) + O\left(\frac{x}{\xi}\, e^{-\delta(\log\xi)}\right) .$$

Now we have to compare $x^{\theta}\, \xi^{1-2\theta}$ and $\frac{x}{\xi}\, e^{-\delta(\log\xi)}$ in order to minimize the error term.

Let $$x^{\theta}\, \xi^{1-2\theta} \sim \frac{x}{\xi}\, e^{-\delta(\log\xi)} .$$

This implies that

$$x \sim \xi^{2}\, e^{\frac{\delta(\log\xi)}{(1-\theta)}}$$

or, equivalently,

$$x \quad \xi^{2}\, e^{\frac{\delta(\log\xi)}{(1-\theta)}}\,(1 + o(1)) .$$

Taking logarithms on both sides, we get

$$(4-4) \qquad \log x = 2\log\xi + \frac{\delta(\log\xi)}{(1-\theta)} + o(1) .$$

For simplicity, let $\rho = \log x$ and $\eta = \log\xi$.

Then (4 - 4) becomes

$$\rho = 2\eta + \frac{\delta(\eta)}{(1-\theta)} + o(1) \quad .$$

It follows that

$$(4 - 5) \quad \begin{cases} \eta = \frac{\rho}{2} - \frac{\delta(\eta)}{2(1-\theta)} + o(1) \\ \quad = \frac{\rho}{2} - \frac{1}{2(1-\theta)}\, \delta\left(\frac{\rho}{2} - \frac{\delta(\eta)}{2(1-\theta)} + o(1)\right) + o(1) \quad . \end{cases}$$

In the case of results (3 - 3) and (3 - 4) we have $\delta(t) = o(t)$. Thus, $\eta \le \frac{\rho}{2}$ implies that $\delta(\eta) \le \delta(\frac{\rho}{2}) = o(\rho)$. Substituting this into (4 - 5) , we get

$$\eta = \frac{\rho}{2} + o(\rho) \quad .$$

Hence $\quad \eta = \frac{\rho}{2} - \frac{1}{2(1-\theta)}\, \delta(\frac{\rho}{2} + o(\rho)) + o(1) \quad .$

Since $\delta(t)$ is a slowly varying function as defined in Lemma 3.18, we have that

$$\frac{\rho}{2} + o(\rho) \sim \frac{\rho}{2} \quad ;$$

this implies that

$$\delta(\frac{\rho}{2} + o(\rho)) \sim \delta(\frac{\rho}{2}) \quad ;$$

that is,

$$\delta(\frac{\rho}{2} + o(\rho)) = (1 + o(1))\, \delta(\frac{\rho}{2})$$

$$= \delta(\frac{\rho}{2}) + o(\delta(\frac{\rho}{2})) \quad .$$

Therefore,

$$\eta = \frac{\rho}{2} - \frac{1}{2(1-\theta)} \delta(\tfrac{\rho}{2}) + o(\delta(\tfrac{\rho}{2})) .$$

From this we obtain

$$\log \xi = \frac{\log x}{2} - \frac{1}{2(1-\theta)} \delta(\frac{\log x}{2}) + o(\delta(\frac{\log x}{2}))$$

or, equivalently,

$$\xi = \sqrt{x}\, e^{-\frac{1}{2(1-\theta)} \delta(\frac{\log x}{2}) + o(\delta(\frac{\log x}{2}))} .$$

In the case of $\delta(\log x) = c\sqrt{\log x}$, one notes that

$$\begin{aligned} x^{\theta}\, \xi^{1-2\theta} &= \sqrt{x}\, e^{-\frac{(1-2\theta)}{2(1-\theta)} \frac{c}{\sqrt{2}} \sqrt{\log x} + o(\sqrt{\log x})} \\ &\sim \sqrt{x}\, e^{-\frac{(1-2\theta)}{2(1-\theta)} \frac{c}{\sqrt{2}} \sqrt{\log x}} \end{aligned}$$

and

$$\frac{x}{\xi} e^{-\delta(\log \xi)} = \sqrt{x}\, e^{\frac{1}{2(1-\theta)} \frac{c}{\sqrt{2}} \sqrt{\log x} + o(\sqrt{\log x})}\, e^{-c\sqrt{\log \xi}} .$$

Now,

$$\begin{aligned} \sqrt{\log \xi} &= \sqrt{\frac{\log x}{2}} \left(1 - \frac{c}{\sqrt{2}(1-\theta)} \frac{1}{\sqrt{\log x}} + o(\frac{1}{\sqrt{\log x}})\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{\log x}{2}} \left(1 - \frac{c}{2\sqrt{2}(1-\theta)} \frac{1}{\sqrt{\log x}} + O(\frac{1}{\log x}) + o(\frac{1}{\sqrt{\log x}})\right) \\ &= \sqrt{\frac{\log x}{2}} - \frac{c}{4(1-\theta)} + o(1) . \end{aligned}$$

Thus,

$$e^{-c\sqrt{\log\xi}} \sim e^{-\frac{c}{\sqrt{2}}\sqrt{\log x}}\; e^{\frac{c^2}{4(1-\theta)}}$$

and $$\frac{x}{\xi}\, e^{-\delta(\log\xi)} \sim \sqrt{x}\, e^{-\frac{(1-2\theta)}{2(1-\theta)}\frac{c}{\sqrt{2}}\sqrt{\log x}}\; e^{\frac{c^2}{4(1-\theta)}} .$$

Consequently, when $\delta(\log x) = c\sqrt{\log x}$ [which is equivalent to the assumption that $M(x) = O(x\, e^{-c\sqrt{\log x}})$], we have

$$Q(x) = \frac{6x}{\pi} + O(\sqrt{x}\, e^{-\frac{(1-2\theta)}{2(1-\theta)}\frac{c}{\sqrt{2}}\sqrt{\log x}}) .$$

Thus we have proved Theorem 2.4.

Q.E.D.

<u>REMARK</u>. Under the assumption that $M(x) = O(x^{\frac{1}{2}+\varepsilon})$, we have

$$\delta(\log x) = (\tfrac{1}{2} - \varepsilon)\log x .$$

Substituting this into (4 - 5), we get

$$\eta = \frac{\rho}{2} - \frac{(\frac{1}{2} - \varepsilon)\eta}{2(1-\theta)} + o(1) ,$$

that is, $$\eta = \frac{\rho}{2}\;\frac{1}{1 + \frac{(\frac{1}{2}-\varepsilon)}{2(1-\theta)}} + o(1)$$

$$= \frac{2(1-\theta)}{5-4\theta-2\varepsilon}\,\rho + o(1)$$

It follows that

$$\log \xi = \frac{2(1-\theta)}{5-4\theta-2\varepsilon}\,\log x + o(1)$$

or, equivalently,

$$\xi \sim x^{\frac{2(1-\theta)}{5-4\theta-2\varepsilon}} .$$

Hence,

$$O(x^{\theta}\xi^{1-2\theta}) = O(x^{\frac{2-\theta-2\varepsilon\theta}{5-4\theta-2\varepsilon}})$$

and

$$O(\frac{x}{\xi}\,e^{-\delta(\log\xi)}) = O(x\xi^{-\frac{3}{2}+\varepsilon})$$

$$= O(x^{\frac{2-\theta-2\varepsilon\theta}{5-4\theta-2\varepsilon}}) .$$

Finally, we obtain

$$Q(x) = \frac{6x}{\pi} + O(x^{\frac{2-\theta-2\varepsilon\theta}{5-4\theta-2\varepsilon}})$$

$$= \frac{6x}{\pi} + O(x^{\frac{2-\theta}{5-4\theta}+\varepsilon})$$

under the assumption that $M(x) = O(x^{\frac{1}{2}+\varepsilon})$:

PROOF OF THEOREM 2.5

Let $A(x)$ and $Q(x)$ be the total number of lattice points and co^2 lattice points in the elliptic disk $au^2+bv^2 \le x$. By a procedure analogous to the methods of the proof of Theorem 2.4, one can obtain

$$Q(x) = \sum_{1 \le d \le \sqrt{\frac{x}{a}}} \mu(d)\, B\left(\frac{x}{d^2}\right),$$

where

$$B\left(\frac{x}{d^2}\right) = A\left(\frac{x}{d^2}\right) - 1 = \sum_{n \le \frac{x}{d^2}} f(n)$$

and

$$f(n) = \sum_{\substack{u,v \\ au^2+bv^2=n}} 1 .$$

Let $\xi = \xi(x)$ be some parameter, to be chosen later, such that $1 < \xi < \sqrt{\frac{x}{a}}$ and $\xi \to \infty$, $\xi = o(\sqrt{x})$. Assuming that $M(x) = O(xe^{-c\sqrt{\log x}})$ and applying Lemma 3.13 to $B(\frac{x}{d^2})$, we obtain

$$Q(x) = \sum_{1 \le d \le \xi} \mu(d) \left[\frac{\pi x}{d^2\sqrt{ab}} + O\left(\frac{x^\theta}{d^{2\theta}}\right)\right]$$

$$+ \sum_{\xi < d \le \sqrt{\frac{x}{a}}} \mu(d) \sum_{n \le \frac{x}{d^2}} f(n)$$

$$= \frac{6x}{\pi\sqrt{ab}} + O\left(\frac{x}{\xi} e^{-\delta(\log\xi)}\right)$$

$$+ O(x^{\theta}\ \xi^{1-2\theta}) + R \quad ,$$

where
$$R = \sum_{n < \frac{x}{\xi^2}} f(n) \sum_{\substack{d \le \sqrt{\frac{x}{n}} \\ \xi < d \le \sqrt{\frac{x}{a}}}} \mu(d)$$

$$= \sum_{n < \frac{x}{\xi^2}} f(n) \sum_{\xi < d \le \min\left(\sqrt{\frac{x}{a}},\ \sqrt{\frac{x}{n}}\right)} \mu(d)$$

$$= - M(\xi)\ B\left(\frac{x}{\xi^2}\right) + \sum_{n < \frac{x}{\xi^2}} f(n)\ M\left(\min\left(\sqrt{\frac{x}{a}},\ \sqrt{\frac{x}{n}}\right)\right) .$$

Now,

$$\min\left(\sqrt{\frac{x}{a}}, \sqrt{\frac{x}{n}}\right) = \begin{cases} \sqrt{\frac{x}{n}} & \text{if } n \ge a \\ \sqrt{\frac{x}{a}} & \text{if } n \le a \end{cases} .$$

[The condition $n < \frac{x}{\xi^2}$ can be ignored here because $\frac{x}{\xi^2} \to \infty$] .

Thus taking $B(x) = O(x)$, one sees that

$$R = O\left(\frac{x}{\xi} e^{-\delta(\log\xi)}\right) + \sum_{n \le a} f(n)\ M\left(\sqrt{\frac{x}{a}}\right)$$

$$+ \sum_{a \le n < \frac{x}{\xi^2}} f(n)\ M\left(\sqrt{\frac{x}{n}}\right)$$

$$= O(\frac{x}{\xi}\, e^{-\delta(\log\xi)}) + O(\sqrt{x}\, e^{-\delta(\log\sqrt{\frac{x}{a}})})$$

$$+ O(\sqrt{x}\, e^{-\delta(\log\xi)} \sum_{a\le n<\frac{x}{\xi^2}} \frac{f(n)}{\sqrt{n}})$$

$$= O(\frac{x}{\xi}\, e^{-\delta(\log\xi)}) + O(\sqrt{x}\, e^{-\delta(\log\sqrt{\frac{x}{a}})}) .$$

Therefore,

$$Q(x) = \frac{6x}{\pi\sqrt{ab}} + O(\frac{x}{\xi}\, e^{-\delta(\log\xi)})$$

$$+ O(x^{\theta}\, \xi^{1-2\theta}) + O(\sqrt{x}\, e^{-\delta(\log\sqrt{\frac{x}{a}})}) .$$

When $\delta(\log\xi) = c\sqrt{\log\xi}$, we obtain

$$\sqrt{x}\, e^{-c\sqrt{\log\sqrt{\frac{x}{a}}}} = \sqrt{x}\, e^{-\frac{c}{\sqrt{2}}\sqrt{\log x - \log a}}$$

$$= \sqrt{x}\, e^{-\frac{c}{\sqrt{2}}\sqrt{\log x}\sqrt{1 - \frac{\log a}{\log x}}}$$

$$= \sqrt{x}\, e^{-\frac{c}{\sqrt{2}}\sqrt{\log x}}\ (1 + O(\frac{1}{\log x}))$$

$$= \sqrt{x}\, e^{-\frac{c}{\sqrt{2}}\sqrt{\log x}}\ e^{-\frac{c}{\sqrt{2}}\, O(\frac{1}{\sqrt{\log x}})}$$

$$\sim \sqrt{x}\, e^{-\frac{c}{\sqrt{2}}\sqrt{\log x}}$$

Hence,

$$O(\sqrt{x}\, e^{-\delta(\log\sqrt{\frac{x}{a}})}) = O(\sqrt{x}\, e^{-\frac{c}{\sqrt{2}}\sqrt{\log x}}) .$$

Equating the orders of magnitude of $\frac{x}{\xi}\, e^{-\delta(\log\xi)}$ and $x^{\theta}\xi^{1-2\theta}$

we obtain (for the corresponding ξ)

$$\frac{x}{\xi}\, e^{-\delta(\log\xi)} = O(x^{\theta}\xi^{1-2\theta}) = O(\sqrt{x}\, e^{-\frac{c(1-2\theta)}{2\sqrt{2}(1-\theta)}\sqrt{\log x}}) .$$

Since $\min(\frac{c}{\sqrt{2}}, \frac{c(1-2\theta)}{2\sqrt{2}(1-\theta)}) = \frac{c(1-2\theta)}{2\sqrt{2}(1-\theta)} = \kappa$, say,

we conclude that

$$Q(x) = \frac{6x}{\pi\sqrt{ab}} + O(\sqrt{x}\, e^{-\kappa\sqrt{\log x}}) .$$

Hence we have proved Theorem 2.5 .

Q.E.D.

REMARK. For the elliptic disk $au^2 + bv^2 + cuv \le x$, where a,b and c are integers with a,b positive, the enumeration of co^2 lattice points can be done in a similar way, although the proof is quite tedious.

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