

**SECONDARY SCHOOL TEACHER'S CONCEPTIONS OF MATHEMATICAL
PROOFS AND THEIR ROLE IN THE LEARNING OF MATHEMATICS**

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Abstract

Mathematical proofs are a part of mathematics that involves thinking and reasoning, rather than computation. The conceptions of Ontario high school mathematics teachers, of what they consider to be mathematical proofs and the role proofs have in their teaching practice, were examined through the use of individual interviews (60 minutes per participant) and a focus group discussion (one 90 minute session). The transcripts were each analyzed through emergent coding before themes were formed from comparing codes across transcripts. The interpretive lens included looking at teacher beliefs on the nature of mathematics, roles of proofs, and mathematical authority. The participants distinguished their university experiences with mathematical proofs from their high school teaching experiences. They saw proofs through the Mathematical Process Expectation, Reasoning and Proving, and they also used proof-related words when describing how they would enact Reasoning and Proving. The participants valued the development of argumentation and sense-making, based on logic and reasoning, as an enduring life-skill, and outcome of school mathematics. The perspectives of the participants provided insight on how teachers inform their teaching practice with the Ontario Mathematics Curriculum. It also revealed some thoughts, desires, values, and struggles teachers may face when teaching mathematics.

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1. INTRODUCTION

This study seeks to draw out themes related to the role of mathematical proofs in Ontario secondary school mathematics. The study specifically examines high school teachers' views of the role of mathematical proofs in their teaching practice. Mathematical proofs are based on thinking and reasoning, rather than on computation (Reid & Knipping, 2010; Stylianides & Stylianides, 2009; Zaslavsky, 2005; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki-Landman, 2012), which might make them different from other components of the mathematics curriculum. In school mathematics, one of the roles of mathematical proofs is to develop students' deductive reasoning skills (Dickerson & Doerr, 2014; Knuth 2002). Furthermore, the development of reasoning skills may be a way to prepare students' cognitive capabilities for their later adult life (Dickerson & Doerr, 2014).

Teaching reasoning and proving requires the use of proof-related tasks and a suitable environment to unpack the tasks (Shilling-Traina & Stylianides, 2013). What makes a task proof-related depends on the teachers' conception of what they consider as a proof. Likewise, teachers can facilitate the creation of a learning environment that promotes discussion and argumentation as they are often viewed as the leader of the classroom (Jones & Herbst, 2012; Stylianides & Stylianides, 2009; Shilling-Traina & Stylianides, 2013; Yackel & Cobb, 1996). This leadership position allows them to influence "the roles of all actors and the location of the mathematical authority" (Steele & Rogers, 2012, p. 164). Consequently, teachers have the ability to decide where mathematical authority resides (e.g. the teacher, the textbook, a video clip, etc.), and to what extent mathematical authority will be shared with their students. By taking on some mathematical authority, students may be encouraged to engage in a creator role (of knowledge) rather than to remain only as a recipient. Engaging in a creator role is necessary for one to engage in the proving process because part of the process involves students using logic to generate a mathematical argument (Carrascal, 2015). These arguments are then subject to scrutiny, by others with mathematical authority, in which the student is required "to obtain the agreement of [others] in the interaction" (Reid & Knipping, 2010, p. 157). However, students may become confused and hesitant to create mathematical statements if they have never experienced what it is like to take on some mathematical authority.

In the Ontario Mathematics Curriculum, proofs emerge specifically through trigonometric identities in the high school university track. While the word "reasoning" appears

throughout the Grades 1 – 8, Grades 9 – 10, and Grades 11 – 12 Curriculum documents, the word “prove” only appears in the Specific Expectations for trigonometric identities. The reason may be that trigonometric identities, “an equation that is always true, regardless of the value of the variable” (Meisel, Petro, Speijer, Stewart, & Vukets, 2009), introduces students to the idea of verifying a mathematical statement as correct without the use of examples. The completion of trigonometric identities tasks often requires students to determine what algebraic manipulations to perform in order to show that one side of the mathematical statement is equal the other. However, these tasks often involve students verifying true statements rather than refuting untrue statements or creating their own statements. Nevertheless, trigonometric identities tasks can be seen as a form of proof-related tasks within the Ontario Mathematics Curriculum.

1.1 Definition of Mathematical Proofs

Past research has revealed that most teachers view the teaching of formal proofs as not necessary within the context of elementary and secondary school mathematics (Dickerson & Doerr, 2014). Instead, teachers are more accepting towards the teaching of less formal proofs (Knuth, 2002), of which there are many definitions and criteria for categorization across the research literature. This preference to accept less formal proofs may be due to possessing a less rigid structure and contain language that can be considered better suited for novices (students) who are just beginning to learn about proofs. The definitions of mathematical proofs used in this study are as follows:

Empirical proofs – consists of showing that the conjecture is true only in one or a few examples taken from a larger set of examples and assuming that the conjecture is also true in all other examples in the set. (Fiallo & Gutiérrez, 2017, p.147)

Less formal proofs – consists of correct general arguments that are perceived to be lacking in the form or vocabulary found in formal proofs. (Dickerson & Doerr, 2014, p. 712)

Formal proofs – consists of a chain of deductive statements organised without the help of specific examples and based solely on the hypothesis of the problem, axioms, definitions or accepted theorems. (Fiallo & Gutiérrez, 2017, p. 148)

Formal proofs in this study are defined to keep the characteristics that the mathematics community typically associate with a formal proof (e.g. particular structure, language, and acceptance by the community (Cabassut et al., 2012; Knuth, 2002; Reid & Knipping, 2010)). On the other hand, students often use examples as the basis to show a conjecture to be true

(empirical proof) (Cabassut et al., 2012; Dickerson & Doerr, 2014; Fiallo & Gutiérrez, 2017; Reid & Knipping, 2010; Stylianides & Stylianides, 2009). This reliance on using examples may be due to students being more comfortable using logic through concrete numbers rather than logic abstractly. For example, in Stylianides & Stylianides (2009), students were shown to determine a pattern based on a small case (small numbers) and assume its applicability to very large cases (very large numbers). This category of proofs has often been described as “not really proofs at all but arguments based on the testing of cases” (Dickerson & Doerr, 2014, p. 712; Stylianides & Stylianides, 2009). Finally, within the context of teaching and learning mathematical proofs, where students are non-experts, there often exists an area of arguments that are neither entirely Formal nor entirely Empirical (Dickerson & Doerr, 2014; Fiallo & Gutiérrez, 2017). These in-between proofs (*Less Formal Proofs*) often contain general statements (not based on examples) but lack the particular structure and mathematical language of a formal proof.

1.2 Proof in the Ontario Mathematics Curriculum

In the school setting, the learning of mathematical proofs is often associated with the development of critical thinking skills (Dickerson & Doerr, 2014; Zaslavsky 2005). In the Ontario Mathematics Curriculum, mathematical proofs are presented in two places. The first instance of proofs is as one of the seven Mathematical Processes (Mathematical Reasoning and Proving) described at the beginning of the document (Ontario, 2007, p. 17). These Mathematical Processes are to be integrated into the teaching process for each grade (1-12) in the mathematics curriculum. The other instance of proofs can be found at the university track level of senior mathematics (grade 11-12), specifically through trigonometric identities. The concept of trigonometric identities are first introduced in grade 11 (e.g. Pythagorean, quotient, and reciprocal) and expanded upon with the proving process in the grade 12 pre-calculus course (Advanced Functions). It is here in the Advanced Functions course where students assess, represent, and prove trigonometric identities through the specific curricular expectation: “prove trigonometric identities through the application of reasoning skills, using a variety of relationships, and verify identities using technology” (Ontario, 2007, p.90). This curricular expectation is the only occurrence for the word *prove* despite many occurrences for words related to reasoning and thinking throughout the Ontario Mathematics Curriculum.

1.3 Teachers' Beliefs and Conceptions

Aside from some instructional considerations, the Ontario Mathematics Curriculum does not detail how the Mathematical Processes and curricular expectations are to be transformed into lessons, leaving this detail to the teachers (Ontario, 2007, p.30). This freedom may result in a less standardized process in the way teachers interpret and implement the curricular expectations into deliverable lessons. As a result, what a student comes to know may be influenced by the teaching styles, knowledge, beliefs, and biases of their teachers.

Past research has shown a relationship between teachers' beliefs and their teaching, particularly in the subject of mathematics (Ernest, 1989; Kotelawala, 2016; Shilling-Traina & Stylianides, 2013). Beliefs are often formed over the course of learning as a student, and influence the way a teacher teaches (Ernest, 1989; Kotelawala, 2016; Nolan, 2014). In this study, *beliefs* are defined as "views that an individual holds of the subject, and what she believes is required in learning, doing, and teaching it" (Shilling-Traina & Stylianides, 2013, p.393). Also, teachers' *conceptions* towards mathematical proofs will be taken from the first portion of this definition ("views that an individual holds of the subject").

1.4 Purpose and Significance of the Study

The purpose of this study is to examine how secondary school teachers view mathematical proofs. The study will consider teachers' conceptions of mathematical proofs and what roles they give proofs in the teaching of mathematics. The following research question will guide this study: *what are some ways teachers' conceptions of mathematical proof relate to what they see as the roles of mathematical proof in school mathematics?*

This study is significant because mathematical proofs are often used to teach non-computational skills such as thinking and reasoning (Cabassut et al., 2012; Dickerson & Doerr, 2014). By studying the conceptions of school teachers towards proofs and its role in learning mathematics, we may come to understand how teachers see thinking and reasoning manifest in the learning of mathematics. We may also come to understand the values of high school mathematics teachers, what they see as important in their teaching of mathematics. As such, the way thinking and reasoning manifest in mathematics may differ from how they manifest in other subjects, which allows for the development and implementation of specific teaching strategies for the mathematics context.

This study may also provide some indications as to the role of trigonometric identities in the learning of thinking and reasoning skills. While trigonometric identities contain a connection to mathematical proofs, identities may not be the only way to learn thinking and reasoning skills. Most of the identity exercises from the books of two main textbook companies, McGraw Hill and Nelson (Wayne, Lenjosek, Meisel, & Speijer, 2008; Meisel et al., 2009) involve proving an equation that is (already) an identity, which may diminish the proof exercises to be mere algebraic manipulation tasks without the argumentation associated with mathematical proofs.

This study may aid in the development of professional development for teachers that support the role of proofs in school mathematics. For example, professional development may include exercises for teachers to examine or reflect on their conceptions towards mathematics and towards mathematical proofs. Faculties of Education, mathematics educators, and local school boards across the province may implement strategies to work with the influence teachers' conceptions can have on their pre-service teachers and implement professional development for in-service teachers. Additionally, mathematics education researchers may use the results of this study when engaging in discussions surrounding teacher-student interactions, the role of mathematical proofs in school mathematics, the influence of teachers' conceptions on their teaching, and how teacher autonomy through the curriculum can influence student learning.

2. LITERATURE REVIEW

This literature review seeks to highlight research in several areas related to teachers' conceptions towards proofs and the role of proofs in mathematics. The first section begins by considering the conceptions of proof by mathematicians and by school teachers. The second section describes the five roles for mathematical proofs as outlined by de Villiers (1990) and some connections with how the roles play out in the school mathematics context. The third portion focuses on the role and influence teachers' beliefs have on their teaching practice. I will be describing a belief framework (Ernest, 1989), a study that draws on that framework (Shilling-Traina & Stylianides, 2013), and drawing several parallels to the results of other studies related to teacher beliefs. Following this, a description of a teacher's role in the classroom, and mathematical authority, will be described using ideas from Stylianides & Stylianides (2009) and Steeles & Rogers (2012).

2.1 Conceptions of Proof

There is a consensus among mathematicians that proofs are an aspect of mathematics, but there is little consensus as to what constitutes as a proof (de Villiers, 1990; Dickerson & Doerr, 2014; Hadas, Hershkowitz, & Schwarz, 2000; Hanna & Barbeau, 2008; Kotelawala, 2016; Reiss, Hellmich, & Reiss, 2002; Steele & Rogers, 2012; Stylianides & Stylianides, 2009) due to differences in conception (Cabassut et al., 2012; Carrascal, 2015; de Villiers, 1990; Dreyfus, Nardi & Leikin, 2012; Reid & Knipping, 2010). The difference in conceptions may be related to personal values and philosophies of what mathematicians look for in a proof (Reid & Knipping, 2010). In this section, I will describe some of the ways proofs are categorized, how teachers' and mathematicians' conceptions towards proofs differ, and the possible contextual reasons for those differences.

2.1.1 What Constitutes a Proof?

In all cases, proofs should not be taken to be synonymous as formal proofs as many researchers have shown proofs to span a wide range of formal and informal arguments (Cabassut et al., 2012; Hanna & Barbeau, 2008; Dickerson & Doerr, 2014; Reid & Knipping, 2010; Reiss, Hellmich, & Reiss, 2002). For mathematical arguments to be categorized as a formal proof, the arguments typically need to have: i) sufficient rigour in argumentation, ii) organization with a particular structure and language, iii) acceptance by the mathematical community (Cabassut et

al., 2012; de Villiers, 1990; Dickerson & Doerr, 2014; Knuth, 2002; Shilling-Traina & Stylianides, 2013; Reid & Knipping, 2010). The judgement of whether a set of mathematical arguments meets these criteria lies in the mathematician, which makes it difficult to determine a general definition for a formal proof (and what is not a formal proof) (Cabassut et al., 2012).

Reid and Knipping (2010) outlined in their book, *Proof in Mathematics Education*, Lakatos' (1978) description of proofs: formal, post-formal, and pre-formal. Formal proofs bear the characteristic of being structurally articulate, possessing a sequence of symbols that allows for a "mechanical" way to verify if it is in fact a proof or not. Post-formal proofs are described as proofs about formal proofs, they are sometimes required to establish axioms or rules to situate the formal proof. On the other hand, pre-formal proofs are described as proofs which do not possess a "well-defined underlying logic" (Reid & Knipping, 2010, p. 9), relying on intuition to establish mathematical facts. While it is not uncommon to find research literature which separates proofs into different categories, there are usually two main categories, "formal" proof (with more rigid in language and form), and "less formal" proofs (less rigid but still convincing).

One reason for the existence of "less formal" proofs may be due to mathematicians often acknowledging that the validity of a mathematical conjecture can often be determined without the use of rigid language and form. There is a need to consider the idea of conviction in the validity of a set of mathematical arguments as these arguments become proofs only after acceptance by the community. Personal conviction may arise from "a combination of intuition, quasi-empirical verification and the existence of a logical (but not necessarily rigorous) proof" (de Villiers, 1990, p. 18). There are areas in literature (Cabassut et al., 2012; Dickerson & Doerr, 2014; Hanna & Barbeau, 2008; Reid & Knipping, 2010; Harel & Sowder, 2007) where researchers have defined a category to describe proofs which are separate from their own definition of formal proof, but still contain convincing reasoning. Two examples can be seen in the use of "less formal proofs" by Dickerson & Doerr (2014) and Knuth (2002) when involving students in the context of school mathematics.

In a study of high school teachers from the United States, Dickerson and Doerr (2014) observed some notable differences between their *experienced teachers* (5-10 years experience) and *veteran teachers* (more than 20 years of experience). The veteran teachers were more interested in students' understanding of key ideas than on the technical aspects of mathematical proofs. One of their participants had described, "We used to be very particular about trivial

things. You had to say ‘Measure of angle ABC equals the measure of angle DEF’, then you had to say ‘Angle ABC is congruent to angle DEF ’” (Dickerson & Doerr, 2014, p. 725). The participant described a possible reason for the shift away from technical and precise mathematical language as a way of “‘getting students to understand a little bit more about proof’” (Dickerson & Doerr, 2014, p. 725). In doing so, these veteran teachers would be viewing proof in a “less formal” way.

In a similar way, the teachers in Knuth’s (2002) study also discussed a difference between a formal and less-formal proof to be based on a difference in the use of mathematical language and structure. Formal proofs were described by teachers to be “very ritualistic in nature, tied heavily to prescribed formats and/or the use of particular language” (Knuth, 2002, p. 71). On the other hand, less formal proofs were defined more in terms of “whether the argument established the truth of its premise for all relevant cases rather than in terms of the rigor involved in the presentation of an argument” (Knuth, 2002, p. 71). Less formal proofs were seen as a precursor to formal proofs as one teacher described, “‘Is there a difference between us saying that the angles are equal as opposed to the angles are congruent? ...They need to have before the formal, the informal, where we’ll accept either one for now.’” (Knuth, 2002, p. 71). By acknowledging that students may not have developed the mastery of mathematical language and organization, a less rigorous category of proofs (less-formal) emerged.

Aside from formal and less-formal proofs, informal proofs, a third type of proofs that relies on the use of empirical testing of cases to form mathematical arguments, often emerges in the context of teaching and learning mathematical proofs (Dickerson & Doerr, 2014; Knuth, 2002; Reid & Knipping, 2012; Stylianides & Stylianides, 2009). While formal and less-formal proofs are valid for all cases within its defined parameters, informal proofs are only valid for a subset of all the cases and cannot be generalized to cover all cases (Stylianides & Stylianides, 2009). Empirical approaches can be seen in Stylianides & Stylianides (2009) when students attempted to extrapolate a pattern based on the numerical outcomes of the first few cases. When students continued to use empirical approaches for the other exercises in the study, they realized that the patterns in a subset could not be generalized to the entire set. However, the students had difficulty determining a different approach, one that was not empirical, and would instead try to choose better subsets based on their own rationale. Likewise, in Knuth (2002), all of the teachers attributed arguments that were empirically based to be informal proofs. The teachers described

the presence of empirically based arguments to be tied with the use of logic to “justify one’s mathematical actions or presents examples to support one’s claims” (Knuth, 2002, p. 72). Thus, it is possible that students use empirically based arguments as a way to logically and numerically demonstrate their understanding of mathematical concepts to their teacher.

2.2 Roles of Proof: Mathematicians and Teachers

In terms of the roles of proofs in mathematics, mathematicians have almost exclusively viewed proofs as the verification for the correctness of a mathematical conjecture (de Villiers, 1990). Likewise, school teachers have also attributed the role of verification to mathematical proofs (Cabassut et al., 2012; de Villiers, 1990). For both mathematicians and teachers, the purpose of proofs is to eliminate doubt, both personal and the doubt of others (de Villiers, 1990; Reid & Knipping, 2010). Aside from eliminating doubt, the act of proving may also build conviction (de Villiers, 1990). Both eliminating doubt and building conviction can be addressed when the correctness of a mathematical conjecture is verified. In this sense, school teachers have been described to view proofs as capable of providing absolute certainty in the correctness of a conjecture (de Villiers, 1990). Likewise, for doubt to disappear and conviction to build, leading to the acceptance of a conjecture, the idea of rigor is sometimes linked with proofs. However, de Villiers (1990) notes that in published proofs, the rigor is often not the object of scrutiny, rather the conviction in, and acceptance of, a proof is based on other factors such as the author’s authority or acceptance of the proof by other established mathematicians (Cabassut et al., 2012). Thus, de Villiers (1990) challenges the conception of only one role for proofs by proposing a model with five roles (verify, explain, systematize, discover, communicate) where in a given circumstance, verification for correctness may not be the most important.

Proofs as a means of verification, as just mentioned, refers to the verification of correctness in a mathematical conjecture (de Villiers, 1990). Verification leads to a degree of certainty that the mathematical statements are true. However, certainty in the truth of mathematical statements may not completely dispel doubt. As de Villiers (1990) points out, “proof is not necessarily a prerequisite for conviction – conviction is far more frequently a prerequisite for proof”. A proof is often created as a result of a mathematician convinced in the truth of a statement. Thus, proofs as a means of verification can be seen as a demonstration of the conviction and certainty of the mathematician who conceived the proof.

Proofs as a means of explanation refers to why something is true, as opposed to simply verifying that it is true. De Villiers (1990) describes the process of knowing *why* something is true to be associated with the idea of a psychological satisfaction. In the process of eliminating doubt, knowing why a mathematical statement is true may be more convincing than simply knowing that the mathematical statement is true. Proofs conceived with the purpose of explaining a conjecture may deepen one's understanding of the conjecture. The distinction between explanation and verification becomes most obvious when the results of a conjecture are intuitive (little doubt in correctness), but the reasons why the conjecture works are not (de Villiers, 1990). As a result, simple proofs may be used in the school context by teachers as a way to explain why a law or theorem works.

Proofs as a means to systematize refers to proofs being used to organize and systematize the results of a conjecture into a wider deductive system of definitions and theorems. De Villiers (1990) describes the importance of a systematization with several reasons. One of the reasons, “[systematization] helps with the identification of inconsistencies, circular arguments and hidden or not explicitly stated assumptions” (de Villiers, 1990, p. 20), can help build consistency in the language and representation of a conjecture such that its implications may be more easily discerned at a more global level. Knuth (2002) argues that in the case of secondary school geometry, “it is questionable whether students are cognizant of the underlying axiomatic structure” (Knuth, 2002, p. 65). He points out that students view the theorems as separate instances rather than related to an underlying axiomatic structure. This may suggest that proof as means for systematization may not exist in school mathematics in the same way (or at all) as it does for mathematicians. Nevertheless, proofs that organize and systematize may lead to the improvement of existing deductive systems which creates progress in mathematics.

Proofs as a means to discover refers to the role of proofs in the generation of new mathematics. For this role, de Villiers (1990) describes discovery as the revealing of something new (unintended) as a result of a proof being present. His argument is that by forming a proof for a mathematical conjecture, the proof may reveal other characteristics about the conjecture that can be generalized. However, this sequence of events may not always be the case. It is possible for mathematical concepts and relationships to be conceived before proofs are put in place (Hanna & Barbeau, 2008). An example of this process taking shape in the school context is through the use of dynamic geometry software in Hadas, Hershkowitz, and Schwarz (2000).

In the first activity, students were asked to use dynamic geometry software to help explain what happens to the sum of the interior angles of a polygon when the number of sides of the polygon increases. In the second activity, students were asked to measure the sum of the exterior angles of a quadrilateral and hypothesize (conjecture) what would happen to the sum of the exterior angles should the number of sides increase. After hypothesizing, they were asked to check by measuring. Although Hadas, Hershkowitz, and Schwarz (2000) designed the two tasks so their student participants would experience uncertainty and cognitive conflict between the two tasks, the first task involves students discovering the relationship between interior angles and number of polygon sides prior to making a generalization. The students may be seen as discovering the proof that *the sum of the interior angles of a polygon is $180(n-2)$, where n is the number of sides*, through examining the relationship on dynamic geometry software.

Proofs as a means of communication refers to proofs as part of an interchange of mathematical ideas between mathematicians, since mathematical arguments are addressed to a human audience (de Villiers, 1990). Many mathematicians view proofs as a social construct as a proof becomes a proof after the mathematics community accepts it as a proof (Cabassut et al., 2012; de Villiers, 1990; Knuth, 2002; Reid & Knipping, 2010). This social process of communicating results also involves “the subjective negotiation of not only the meanings of the concepts concerned, but implicitly also of the criteria for an acceptable argument” (de Villiers, 1990, p. 22). And as the outcome, proofs are often refined through identifying errors or completely rejected through the discovery of a counterexample. Knuth (2002) argues that the social nature of proofs is lacking in the practices of school mathematics where “the teacher and the textbook are the arbiters of validity” (Knuth, 2002, p. 65). For proofs to reclaim their social nature in school mathematics classrooms, the classroom norms may need to allow for mathematical discussion.

2.3 Teachers' Mathematical Beliefs

Past research studies have shown a connection between what teachers believe mathematics to be, and the way they teach mathematics (Charalambous, Panaoura, & Philippou, 2009; Ernest, 1989; Nolan, 2014). Teaching beliefs are typically formed through cumulative experiences as a student, and often influence a teacher's epistemological perspective (Ernest, 1989; Shilling-Traina & Stylianides, 2013). For example, prospective teachers entering teacher education primarily have instrumentalist or Platonist views, which reflects the nature of their

own classroom experiences as mathematics students (Carrascal, 2015; Reid & Knipping, 2010; Shilling-Traina & Stylianides, 2013; Nolan, 2014). Ernest (1989) clusters teacher beliefs into three epistemological categories: instrumentalists, Platonists, and experimentalists (problem solving) (Charalambous, Panaoura, & Philippou, 2009; Ernest, 1989; Shilling-Traina & Stylianides, 2013).

Instrumentalists view mathematics as a collection of unrelated facts and procedures, detached from specific context (Ernest, 1989). The individual is not a creator of mathematics, but a user who follows a set of directions (Shilling-Traina & Stylianides, 2013). The instrumentalist view can be seen when teachers claim there is always one best (most efficient) way to solve a problem, and the learning of mathematics is grounded in memorization and application of procedures (Shilling-Traina & Stylianides, 2013). The Platonists view mathematics as a static body of objective truths that are to be presented by the teacher and revealed to the student (Charalambous, Panaoura, & Philippou, 2009; Ernest, 1989). This view can be seen when teachers describe mathematical ability of an individual as innate or predetermined (Shilling-Traina & Stylianides, 2013). Both instrumentalist and Platonist views tend to perpetuate teaching styles which employ drill-type questions (Charalambous, Panaoura, & Philippou, 2009) with teachers performing the delivery of mathematical content to the students. In most cases, the teacher is packaging mathematics knowledge for the students, placing mathematical authority in the hands of the teacher or textbook (and not on the students) (Charalambous, Panaoura, & Philippou, 2009; Nolan, 2014; Shilling-Traina & Stylianides, 2013). In contrast, experimentalists view mathematics as a dynamic and evolving field with new mathematics constantly being invented, re-examined, and revised (Charalambous, Panaoura, & Philippou, 2009; Ernest, 1991). Under this view, the process of doing mathematics is emphasized more than the correctness of the solutions (Shilling-Traina & Stylianides, 2013). Teachers with this view will often focus on creating an environment that promotes exploration, particularly perseverance through failure (Kotelawala, 2016; Shilling-Traina & Stylianides, 2013).

Past research has indicated both strong and weak links between professed beliefs and actual practice (Kotelawala, 2016). Considering that “[Prospective] Teachers begin teaching with over 2000 h as a mathematics student and generally a couple hundred hours at most focused on the teaching and learning of mathematics” (Kotelawala, 2016, p. 1118), it would not be

surprising for mathematics teaching beliefs to be formed during a teacher's time as a student. A study of pre-service and in-service teachers, who are settled into their profession, by Nolan (2014), described a preference for what can be considered Platonic and instrumentalist approaches in pre-service mathematics teachers. She has also observed these preferences to persist into a teacher's teaching career despite having been exposed to programs and pedagogy, during their teacher formation, which advocated for more use of an experimentalist approach. Under such conditions, the teaching of proof in such a classroom may revolve around explaining the difficult steps and "knowing that something is the case" (Carrascal, 2015, p. 4) as opposed to how it came to be.

In a different study, Shilling-Traina and Stylianides (2013) identified and attempted to change the beliefs of prospective elementary mathematics teachers through a one semester prerequisite course for the teacher certification program. Their method involved using cognitive conflict (a form of disequilibrium caused by engaging in acts of reflective thinking (Shilling-Traina & Stylianides, 2013; Zaslavsky, 2005)) to challenge existing beliefs, the use of instructional scaffolding (Stylianides & Stylianides, 2009) to guide the teachers to their own beliefs, creation of an environment for exploration to support belief change through proof-related tasks. One of the proof-related tasks which led to cognitive conflict was the Blonde Hair Problem (source from Stylianides & Stylianides, 2011). The teachers' initial reaction was that the problem was unsolvable. However, they were able to come to the answer after discussing in small groups and with some reassurance from the instructor. Cognitive conflict was created when the teachers realized the problem was solvable, by working in small groups, and within their capabilities despite being initially convinced it was unsolvable.

At the beginning of the course, Shilling-Traina and Stylianides (2013) observed 60% of participants to hold an instrumentalist view, with 90% of them as their primary view. In fact, the frequency of the instrumentalist view was double of the Platonist view, which in turn was double of the experimentalist view. Shilling-Traina and Stylianides (2013) also categorized the frequency of their codes between primary view and secondary view. The primary view contains the highest frequency of codes while the secondary view contains the second highest. At the beginning of their course, the frequency of codes related to the experimentalist view was the least represented across the participants; was not a primary view in any of the participants. By the end of the course, the responses were more evenly distributed with the experimentalist view

doubling (14% to 26%). Although primary views remained mainly instrumentalist and Platonist, the experimentalist view became the most prominent secondary view.

The lack of movement in the primary view towards an experimentalist view may be a matter of time and continued exposure to the learning environments which influenced the movement of secondary views. For instance, one participant, who maintained a primary Platonist view through the course, showed a desire to go beyond his own past classroom experiences (Platonist style) (Shilling-Traina & Stylianides, 2013). He described the learning environment, focus on explanations rather than procedures, and frequent use of group work to provide a good learning experience associated with mathematics. Another participant, who held an instrumentalist view at the start of the course, described mathematics as not a creative endeavor due to the rigid procedures. She later found the freedom of having instructors not outline correct solution paths but rather encourage their students to find their own explanations to be empowering. By the end of the course she was showing more attributes associated with the experimentalist view.

2.4 Role of the Teacher and Mathematical Authority

The learning of mathematics, and mathematical proof, can be seen as a social activity involving the interactions between the teacher and the students (Jones & Herbst, 2012; Harel, & Rabin, 2010; Otten, Engledowl, & Bleiler-Baxter, 2017; Yackel, & Cobb, 1996). These interactions form a learning environment and can involve the participation in a mathematical discourse where “meanings are being construed, discourse patterns being enacted, and interpersonal relationships being negotiated” (Otten, Engledowl, & Bleiler-Baxter, 2017, p. 113). Yackel and Cobb (1996) described the outcome of these negotiations and interactions as a socio-mathematical norm. They distinguished the difference between social norms and socio-mathematical norms with the following example.

To further clarify the subtle distinction between social norms and sociomathematical norms we offer the following examples. The understanding that students are expected to explain their solutions and their ways of thinking is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm. (Yackel, & Cobb, 1996, p. 461)

Yackel and Cobb (1996) also mentioned that teachers play a leading role in shaping the socio-mathematical norm in their classrooms. The creation and presence of a socio-mathematical norm

involves the negotiation of authority, ““a social relationship in which some people are granted the legitimacy to lead and others agree to follow”” (Otten, Engledowl, & Bleiler-Baxter, 2017, p. 113). Subsequently, authority, which is often held by teachers, can be used to influence the learning environment (Jones & Herbst, 2012; Yackel & Cobb, 1996). For example, learning environments that promote discussion and inquiry may result in norms that have students engage in “acts of challenge and justification during the process of holding each other accountable for their assertions” (Jones & Herbst, 2012, p. 264). To explore some of the factors that contribute to creating a learning environment, the role of the teacher and the idea of mathematical authority will be discussed.

2.4.1 Role of the Teacher

Yackel and Cobb (1996) and Shilling-Traina and Stylianides (2013) have both identified the role of the teacher as a “representative of the mathematical community in the classroom” (Stylianides & Stylianides, 2009, p. 317). In Stylianides and Stylianides (2009), this teacher role can be seen through their carefully designed instructional sequence. Their goal was to create multiple situations where students would experience uncertainty and perplexity in being unable to use empirical methods to validate a mathematical conjecture. The instructors would then facilitate the students to realize a need for a more secure form of validation (i.e. deductive proofs). The instructional sequence involved the creation of a learning environment that had students reflect on their own thinking while engaging in proof-related activities. In particular, the instructors would support students, with instructional scaffolding, as they resolved their uncertainty and perplexity by helping students develop “a new state of understandings that were consistent with conventional mathematical understandings. These conventional understandings were most of the time interactively constituted by the instructor and the students in the course of classroom activity” (Stylianides & Stylianides, 2009, p. 323). Conventional mathematical understanding, in their study, would refer to the understanding of mathematics that resulted from the creation of a socio-mathematical norm. By being an authority figure in this norm, the instructors were able to direct students to understand why certain approaches to a proof-related task was appropriate or inappropriate, and also lead students to become more aware of their conceptions of mathematical proofs.

The creation of a suitable learning environment for engagement in proof-related tasks, and the use of instructional scaffolding was also seen in Shilling-Traina and Stylianides (2013).

Their objective was to have a group of prospective elementary school teachers realize their own mathematical beliefs, and to possibly influence a change in those beliefs. The role of the instructors involved carrying out instructional scaffolding, which directed “students’ attention to their understandings or conceptions of a particular mathematical topic or idea” (Stylianides & Stylianides, 2009, p. 322). The use of instructional scaffolding was necessary in every step because before any changes in beliefs can occur, “individuals must first be made aware of their current beliefs” (Shilling-Traina & Stylianides, 2013, p. 395). In terms of representing the mathematical community in the classroom, Shilling-Traina and Stylianides (2013) defined the kind of student-teacher interactions that would carry out in their study.

Prospective teachers were encouraged to engage in mathematical conversations with their peers, while the instructor served as the ‘representative of the mathematical community in the classroom’ (Stylianides and Stylianides 2009, p. 317). Enactment of this role was through the instructor’s interactions with the prospective teachers when providing necessary scaffolding as they faced and attempted to resolve cognitive conflicts they experienced. (Shilling-Traina & Stylianides, 2013, p. 395)

The instructional sequences used in Stylianides and Stylianides (2009) and in Shilling-Traina and Stylianides (2013) reflect a statement, quoted by Jones and Herbst (2012), that an engaging learning environment for proofs involves teachers that “‘engage in dialogue that places responsibility for reasoning on the students, analyze student arguments, and coach students as they reason’” (Jones and Herbst, 2012, p. 265). Both studies also connected the role of an instructor as the representative of the mathematical community with the establishment of a learning environment. However, this connection was seen in an earlier study by Yackel and Cobb (1996).

Yackel and Cobb (1996) based their perspectives of learning and teaching mathematics on a model of participating in a culture, rather than a model of transmitting of knowledge. They pointed out that “in any classroom, children are well aware of the asymmetry between the teacher’s role and their role” (p. 464), which means that “the teacher’s reactions to a child’s solution can be interpreted as an implicit indicator of how it is valued mathematically” (p. 464). Yackel and Cobb (1996) described the creation of socio-mathematical norms through a case where the students had to create solution methods, but what was considered mathematically different was unknown, and had to be negotiated with the teacher. The negotiation involved the students observing the kind of responses their teacher made to their answers, and how their teacher reacted when their peers made comments on another’s answers. Through these

interactions, socio-mathematical norms emerged and influenced the way students view (and do) mathematics. In particular, the way socio-mathematical norms influence the learning of mathematics is through what aspects of the students' mathematical activities are legitimized, and what aspects are sanctioned. Once the students got a sense of what mathematical activities are acceptable, they would proceed to behave and respond in a similar way. In essence, the learning of mathematics involves students' participation in a mathematical culture where acceptable activities are heavily influenced by the teacher, who is a representative of mathematics.

2.4.2 Mathematical Authority

As leaders in the classroom, teachers have control over how mathematical authority is presented, and shared, in the teaching of mathematics (Harel & Rabin, 2010; Otten, Engledowl, & Bleiler-Baxter, 2017; Yackel & Cobb, 1996). Otten, Engledowl, and Bleiler-Baxter (2017) described authority as, “a social relationship in which some people are granted the legitimacy to lead and others agree to follow” (p. 113). And in terms of mathematical authority, “a person or object (e.g., the textbook) [has] authority over part of a proving interaction if they lead the interaction or direct the behavior of others” (Otten, Engledowl, & Bleiler-Baxter, 2017, p. 113). The presence and distribution of mathematical authority can be seen in several studies that observed the classroom interactions, between students and teachers, when engaging in proof schemes, “the mental act that a person (or community) employs to remove doubts about the truth of an assertion” (Harel & Rabin, 2010; p. 14). Otten, Engledowl, and Bleiler-Baxter (2017) described the presence of an authoritative proof scheme where the students “expect teachers to supply the claims and ‘givens’ of what is to be proved and students only expect to be responsible for completing the proof using recently learned theorems or definitions” (p. 113). A similar definition was given by Harel and Sowder, (2007) where proving with an external conviction proof scheme “depends on an authority such as a teacher or a book” (Harel & Rabin, 2010, p. 14; Harel & Sowder, 2007; p. 7). These kinds of proof schemes have mathematical authority remain with the teacher, and the authority is not shared with the students. Similarly, in Steele and Rogers (2012), the novice teacher can be seen using a proof scheme where mathematical authority resides with him or with the textbook. Likewise, mathematical authority may manifest differently in different classrooms due to the socio-mathematical norms that are in place.

In Harel and Rabin (2010), the classroom observations of two high school mathematics teachers, teaching algebra, teaching practices that contribute to an authoritative proof scheme

were noted through grounded theory. Harel and Rabin (2010) categorized their observations into three categories: i) answering students' questions, ii) responding to students' ideas, iii) lecturing. They showed two teachers (Teacher A and Teacher B) with contrasting teaching practices where Teacher A kept mathematical authority to himself, compared to Teacher B, by being more of an arbiter rather than facilitator. On the other hand, Teacher B shared some mathematical authority by encouraging student participation in discussions, and by making explicit the process of checking and justifying their answers as a communal practice.

When answering students' questions, Teacher A was observed to confirm or reject suggestions in a manner that showed a focus on procedures and how to perform tasks. Sometimes Teacher A would respond with a leading question, with the intention of directing and focusing students' ideas towards an answer desired by the teacher. In contrast, Teacher B emphasized correctness by a checking process which was presented as a public process. Correctness could be checked by any one, student or teacher. Likewise, when responding to students' ideas, if a student's answer is not one that Teacher A expects, he would reject it and respond in a way that narrows their choices. If a student's answer is correct, he confirms it without investigating the student's understanding of the answer. On the other hand, Teacher B not only determines if an answer is correct or incorrect but encourages students to find a correct answer by analyzing a flawed solution attempt. Teacher B was also seen to ask students to justify their assertions, but Teacher B would respond with justifications based on his own understanding, which was not always shared by the student. By doing so, agreement by the students to the teacher would be socially based rather than intellectually based. Finally, when lecturing, Teacher A would deliver lessons where general rules or principles are presented and illustrated with examples. He does not give any purpose, intellectual need, or motivation for learning new material other than curricular standards. Unlike Teacher A, Teacher B ensured students understood the goal and purpose of the problem and delays introducing terminology until the students see a need or usefulness in the concept being learned. However, sharing mathematical authority may not be as simple as having more student participation in the proving scheme, as seen in a case described in Otten, Engledowl, and Bleiler-Baxter (2017).

In their study, Otten, Engledowl, and Bleiler-Baxter (2017) observed the presence of mathematical authority in a case of whole-class proving through how a teacher taught three different proof tasks. In particular, they observed the presence of mathematical authority in three

phases of interactions: the initiation of a proof (who determines the claim and that it should be proved), the construction of a proof (who leads the discussion as the arguments are articulated), and the conclusion/validation of proof (who confirms the proof is complete and correct). They observed that authority over proof initiation was based on the textbook, or the teacher's enactment of the textbook tasks. Authority over proof structure and authority over proof validation both resided in the teacher as she would request her students to provide clarification of steps and justifications during proving. The researchers described the teacher-student interactions as one that followed a initiate-respond-evaluate pattern. The teacher would initiate the interaction, the students would respond, and the teacher would close and form the evaluation. This kind of interaction "allows teachers to support students' participation, it also limits the nature of their participation" (Otten, Engledowl, & Bleiler-Baxter, 2017, p. 123) in that attention may be placed on the details of the proof rather than on the arguments themselves. They speculated that the reason the teacher chose to interact with her students in such a way was because "the students were novices with regard to proving and she was serving as a more knowledgeable representative of the discipline of mathematics with regard to what counts as proof" (Otten, Engledowl, & Bleiler-Baxter, 2017, p. 123). Although the teacher in the study got students to participate in the proof tasks, the way she interacted with them resulted in mathematical authority to remain with her; not shared with the students.

Similar to Harel and Rabin (2010) and Otten, Engledowl, and Bleiler-Baxter (2017), the presence of mathematical authority can also be seen in the teachers from Steele and Rogers (2012). Steele and Rogers (2012) included a description of students' perceived roles in response to the different teaching styles of the teachers. One of the two teachers taught with a more teacher-centered approach than the other and the sharing mathematical authority may have a connection with Ernest's (1989) beliefs about mathematics. In their study, the sharing of mathematical authority by the experienced teacher (11 years of experience) resulted in students who engaged with proofs differently than when mathematical authority was not shared. The difference, as noted by Steele & Rogers (2012), between their novice teacher (pre-service teacher) and experienced teacher (11 years of experience) participants was in the learner roles adopted by the students in response to instruction. The novice teacher (pre-service) taught his lesson using steps and structures from the textbook, which limited the creating role of the students to be "providing the reasons behind a predetermined set of steps" (Steele & Rogers,

2012, p.18). In this case, mathematical authority appeared to remain predominately with the teacher, resulting in an emphasis on the procedures of proving. Also, the way the teacher and students interact is an example of the initiate-respond-evaluate interaction pattern described in Otten, Engledowl, and Bleiler-Baxter (2017). On the other hand, the experienced teacher saw proofs as a tool for communicating mathematical understanding and engaging in discussion with the mathematics community (classmates). As a result, her students responded by adopting the roles of observer, creator, and explainers of their own conjectures and proofs. These students, having been given some mathematical authority in determining and explaining their own conjectures, showed an emphasis on proving over procedures.

To connect Steele and Roger's (2012) and Harel and Rabin's (2010) studies with Ernest's (1989) three epistemologies, the novice teacher and Teacher A would appear to have taught using a Platonic approach due to their focus on proof to be based on an established set of mathematical facts and the teaching practice being teacher-centered throughout the lessons. While the experienced teacher also held some this focus, she later moved to the role of proofs as an explanatory tool in communicating mathematical knowledge. Teacher B also showed some sharing of mathematical authority with his students, allowing them first-hand experience with justifications. It is possible that the Platonic approach limited the explanatory aspect of learning proofs as the students were expected to provide reasoning for a predetermined set of steps from the textbook; reducing the possibility of psychological satisfaction (de Villiers, 1990). However, the sharing of mathematical authority appeared to have encouraged the students to engage in proofs through process, over procedure, by creating their own mathematical arguments and defending them through explanation. It is possible for the experienced teacher's students to have a greater appreciation for proofs, and to eventually develop the capabilities to evaluate proofs on their own. Likewise, the students of Teacher B may come to appreciate their learning of mathematics more than the students of Teacher A due to Teacher B attempting to share mathematical authority.

3. THEORETICAL FRAMEWORK

To address the research question, my theoretical framework consists of three components. The first component uses Ernest's (1989) three categories for teacher beliefs to look at the first part of the research question, participants' conceptions of mathematical proofs. Since a teachers' beliefs can influence their teaching, this first component can also be used to understand participants' conceptions when examining examples of their teaching. The second component uses the roles of proof from de Villiers (1990) and Knuth (2002) to look at the second part of the research question, what participants consider to be the role of proofs in school mathematics. I used the same roles of proof framework as Knuth (2002), replacing the role of proofs to systematize from de Villiers (1990) with the role of proofs to develop thinking and logic skills from Knuth (2002), because Knuth's (2002) research questions, participants (secondary school teachers), and context (high school mathematics) are all similar to mine. And finally, the last component uses Otten, Engledowl, and Bleiler-Baxter (2017) and Yackel and Cobb (1996) to look at the kind of teacher-student interactions (e.g. norms) and the locations of mathematical authority that may be related to the participants' conceptions of proofs, or to their view on the role of proofs. The inclusion of these interactions and mathematical authority is because learning of mathematics in schools is done in a social setting, often in classrooms with peers and teachers.

3.1 Teachers' Beliefs and the Teaching of Proofs

Since it is possible for teachers who possess similar knowledge of mathematics to teach in different ways (Ernest, 1989), it is important to consider teacher beliefs towards mathematics when studying teachers' instructional practices. For this framework, the three epistemologies found in Ernest (1989) are placed on a continuum based on how proofs may manifest through each of the perspectives. A continuum is used since it may be difficult to exclusively categorize a teacher's belief into one view. Also, it may be possible for a teacher to portray a primary view but also exhibit characteristics of a secondary view, as was seen in the teachers from Shilling-Traina and Stylianides (2013). The following paragraphs describe how each of the epistemologies may appear when talking about mathematical proofs.

The instrumentalist view, "mathematics is a useful but unrelated collection of facts, rules and skills" (Ernest, 1989), presents mathematics as an application of logic and numeric relationships for use in other topics. Teachers with this view may present proofs only if proofs

are required for, or are beneficial to, the understanding or useage of a mathematical concept or process being practiced by the students. Mathematical proofs may be seen as merely exercises involving logic. There may also be low personal investment on the part of students in understanding a proof, and the knowledge that the conjecture works and when it is applicable is sufficient.

The Platonist view, “mathematics as a static but unified body of knowledge, consisting of interconnecting structures and truths” (Ernest, 1989, p. 21), presents mathematics as something that exists and is to be discovered. Teachers with this view may spend more time connecting students’ prior knowledge with what is to be discovered. They may provide more opportunities for students to engage in understanding the flow of reasoning rather than simply presenting the proof. Teachers may also draw attention to the implications of a proof, and make connections to other mathematical concepts. Since the teacher often directs the students’ attention, the students may not necessarily be the primary creators and assessors of mathematical arguments in a Platonic approach to teaching proofs.

The experimentalist (problem-solving) view is that mathematics is an unfinished body of knowledge that is constantly being challenged and expanded upon (Ernest, 1989, p. 21; Shilling-Traina & Stylianides, 2013). Teachers with this view may encourage discussion and argumentation as part of learning mathematics. Mathematical proofs may be seen as exercises that involve persuasion based on logical arguments. The students would be given opportunities to create their own arguments, and assess the legitimacy of arguments constructed by their peers.

Ultimately, the influence of teachers’ mathematical beliefs may have an impact in jurisdictions where proofs are not presented as a standalone topic in the curriculum, but rather as a general theme (as the case in Ontario). Teachers may have more than one epistemological view, as seen in the participants from Shilling-Traina and Sylianides (2013), and those views may change over the course of a teacher’s career, as seen in the veteran teacher from Dickerson and Doerr (2014). As a result, teachers use their own discretion in how they approach teaching proofs in their classroom (Cabassut et al., 2012).

3.2 Role of Proofs in School Mathematics

The position of proofs and proving in the Ontario Mathematics Curriculum as part of the Mathematical Processes gives proofs a role in developing reasoning skills. Knuth (2002) mentioned teachers' perception that mathematical proofs "could be used to develop logical thinking skills that are applicable outside of mathematics classrooms" (Dickerson & Doerr, 2014, p. 712). Knuth (2002) also went on to use the five roles of proof described by de Villiers (1990) (verification, explanation, systematization, discovery, communication). However, in terms of proofs as a way to systematize, Knuth (2002) mentioned that,

It is questionable whether students are cognizant of the underlying axiomatic structure...many students view the many theorems that they are asked to prove as essentially independent of one another rather than as related by the underlying axiomatic structure. (Knuth, 2002, p. 65).

Knuth (2002) found that most of his participants attributed a role of proofs to be developing logical thinking skills. Since my study also examines secondary school teachers and the secondary school context for learning mathematics, I will be substituting *proofs as a way to systematize* with *proofs as a way to develop logical thinking skills*. A connection between Knuth's (2002) framework and my study can be found in Table 2.

Proofs as a means of verification refers to the verification of truth in mathematical conjectures (de Villiers, 1990). This role remains the same in school mathematics where proofs are used to verify the legitimacy (or illegitimacy) of mathematical arguments (Cabassut et al., 2012; Knuth, 2002). In fact, a common perception of proofs in school mathematics is being able to provide absolute certainty in the correctness of a conjecture (de Villiers, 1990). One way for this perception to emerge is in the use of textbooks or other sources (e.g. online videos) where the legitimacy of the mathematics may be assumed. For instance, the novice teacher in Steele & Rogers (2012) placed authority on the mathematics textbook being used to teach proofs. And for students, seeing a proof in a textbook (created by many authors and editors), may lead them to believe what the proof claims.

Proofs as a means of explanation refers to a capacity for proofs to explain why a mathematical argument is true (de Villiers, 1990). In school mathematics, it is more common for explanation to be used in the context of students demonstrating their understanding (Knuth, 2002). This contrasts de Villiers' (1990) description in that "the focus is not so much on an argument's illumination of the underlying mathematical concepts which determine why a

statement is true as much as it is on showing how a statement came to be true” (Knuth, 2002, p. 80). For example, showing a proof about the Pythagorean Theorem may help students understand why the theorem works, rather than simply accepting that it works because the numbers that are plugged in make the equation work.

Proofs as a means of discovery refers to the notion of generating new knowledge or information through deductive means (de Villiers, 1990). For teachers, discovery may refer to teaching mathematical knowledge that is new to students through reasoning and logic. The notion of new knowledge may be the students’ engagement in the proving process: “generate conjectures and then attempt to verify the truth of the conjectures by producing deductive proofs” (Knuth, 2002, p. 65). In the school context, ‘new’ may not necessarily mean ‘new’ in the global mathematics domain, but rather ‘new’ in terms of the students’ knowledge and conception of mathematics.

Proofs as a means of communication refers to a social component of proofs in that mathematical arguments only become proofs when the mathematical community accepts them as proofs (Knuth, 2002). Mathematicians use proofs to communicate, report, and discuss mathematical results (de Villiers, 1990). In school mathematics, the authority on accepting a proof usually comes from the teacher or from textbooks, rather than through social means (e.g. classroom discussion). However, it may be possible for the social nature of proofs to be achieved through classroom norms that support logic-based discussions as part of learning proofs as in Shilling-Traina & Stylianides (2013). For example, students may observe a pattern in the mathematics that they are doing, make a conjecture, and attempt to verify its validity through convincing their peers.

Proofs as a means to developing logical thinking skills was observed by Knuth (2002) and, for this study, will replace proofs as a means to systematize. Knuth (2002) found that the majority of his teachers (high school) associated a role of proof to developing logical thinking skills. In fact, it seems developing logical thinking skills may be the goal for teachers teaching mathematical proofs as described by one teacher: “Other than the development of reasoning skills ... I’ve never had to use proof outside of a math class. I don’t know when they might use something like that” (Knuth, 2002, p.78). Since my study also involves secondary school teachers, and the curriculum they use combines proof and reasoning, proofs as a way to develop logical thinking skills is one of the roles of proof I will be using.

Table 2		
<i>The five roles of proofs used in this study</i>		
	<u>Role of Proofs (de Villiers, 1990; Knuth, 2002)</u>	<u>My Study</u>
Verification	Verify the legitimacy of a mathematical statement	Confirm validity of a conjecture. (e.g. equivalence of a trigonometric identity)
Explanation	Explain why a mathematical statement is true	Deepen understanding of (a) concept(s). (e.g. why the Pythagorean Theorem works for any right-angled triangle)
Discovery	Generate new knowledge through deductive means	Learning something new. (e.g. exploring the relationship between the volumes of a cone and a cylinder)
Communication	Communicate mathematical thinking.	Discussion or argumentation between peers. (e.g. discussing strategies or approaches for factoring a quadratic equation)
Develop Logical Thinking Skills	Develop reasoning skills for applicability to real world.	Develop reasoning skills to carry beyond the classroom. (e.g. assessing if their answer and/or their thinking makes sense)

3.3 Mathematics Authority and the Learning of Mathematical Proofs

Classroom norms are established between the teacher and students over the duration of the mathematics course (Jones & Herbst, 2012; Yackel & Cobb, 1996). These classroom norms create a set of teacher-student interactions which influence how mathematics is practiced, as well as what roles the teacher and students have in the learning process (Yackel & Cobb, 1996). The teacher often plays the role of the mathematics authority while also being a representative for the

discipline of mathematics for their students (Jones & Herbst, 2012; Shilling-Traina & Stylianides, 2013; Stylianides & Stylianides, 2009; Yackel & Cobb, 1996; Zaslavsky et al., 2012). As the representative of the mathematics discipline, the teacher would influence “the roles of all actors and the location of mathematical authority” (Steele & Rogers, 2012, p. 164; Otten, Engledow, and Bleiler-Baxter, 2017; Yackel & Cobb, 1996). In this study, the specific aspect of teacher-student interactions being considered is the location of mathematical authority. One reason is that one of the criteria for a mathematical proof to be valid is for the proof to be accepted by the mathematical community (Cabassut et al., 2012; de Villiers, 1990; Knuth, 2002; Reid & Knipping, 2010). In other words, for someone to judge the validity of a proof, the person must be part of the mathematical community. Thus, whoever possesses mathematics authority would be part of the mathematical community. The way a teacher conceptualizes proofs may relate to how they manipulate the location of mathematical authority. Likewise, what teachers see or value as the roles of proofs may also relate to the distribution of mathematical authority with the students. It is then up to the teacher and students whether that mathematics authority is shared with the students or remains with the teacher in their classroom norm.

In relation to Ernest (1989), it is likely that the sharing of mathematical authority with students is indicative of the experimentalist approach. The reason is students are placed in a position where they are the creators of mathematical arguments and consequently, they are required to prove and show the validity of those arguments to the rest of the class (mathematics community). Under the instrumentalist approach, it is likely for the classroom norm to encourage a discussion and exchange of arguments, ideas, and feedback between students and the instructor as part of the proving process. The mathematical authority would likely remain with the teacher as he or she determines what the students need to learn and how they will learn it. It is also unlikely for the Platonic approach to share mathematical authority as students would not be seen as the creators of mathematical knowledge in the proving process. As a result, both the Platonic and the instrumentalist approaches may contain significant teacher-guided prompting and scaffolding as was displayed by the novice teacher in Steele & Rogers (2012).

3.4 Putting it Together

The way the three components of this framework relate to each other and inform the study is represented in figure 1. The research question contains two main elements, *Conceptions of Proof* and *Roles of Proof*. The first element, conceptions, involves participants to form what

they consider to be mathematical proofs while the second element focuses on what functions or purposes they give proofs in the context of high school mathematics. In the visual representation, the (elements of) the research question are boxed. They are informed by Ernest (1989), de Villiers (1990), and Knuth (2002). There is some overlap between the two parts as conceptions may reveal something about the roles of proofs and vice versa. Since this study involves teachers, the influence of their teaching experiences would be examined through mathematical authority. Mathematical authority is related to conceptions because a teacher's beliefs may influence the kind of teaching style that is used, and subsequently the presence and location of mathematical authority. The roles of proofs also relate to mathematical authority since authority is related to the roles the teacher and students have, the classroom norms, in learning mathematics.

The way the theoretical framework guided my study was in having a reference point for the research question. It helped ground the process of making sense of the codes, and understanding the meaning of the emerging themes, by understanding the similarities and differences between the participants. For example, participants who attribute the purposes of proofs in similar ways may have very different conceptions of proofs, or participants with similar conceptions have descriptions that display different distribution of mathematical authority. Additionally, the interconnectiveness of beliefs, roles, and authority allowed for one or more components to inform another, which was useful in two different circumstances. The first being related to a sense of reliability in that when a participant attributes a role of proof, the reasoning can have traces in their conceptions of proofs and in the mathematical authority of their descriptions. The second circumstance involves instances when participants comment less on conceptions or on roles, or the essence of what they are describing is not entirely clear. This situation was often seen during the focus group where the discussion would flow to different topics related to mathematics, teaching mathematics, and mathematical proofs. And sometimes a few participants would lead the conversations on a topic which meant the others did not produce as much content to analyze. The theoretical framework informed the study by relating the participants' descriptions and the emerging themes back to the elements of the research question.

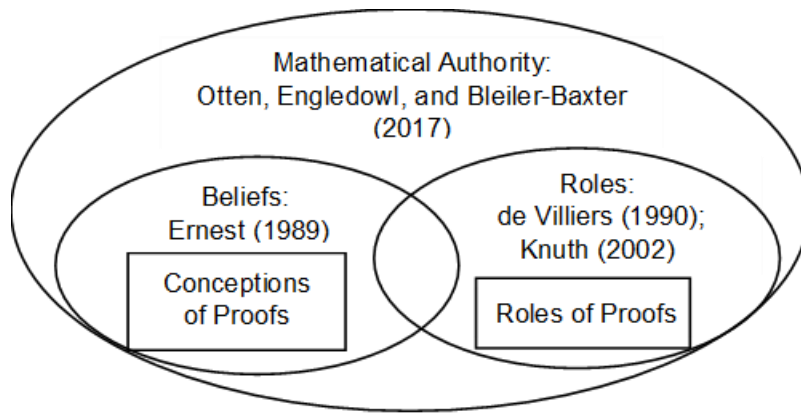


Figure 1. A visual representation of the framework that guides this study. Ernest (1989) informs conceptions while de Villiers (1990) and Knuth (2002) inform the roles of proofs in the school context. Mathematical Authority from Otten, Engledowl, and Bleiler-Baxter (2017) encompasses conceptions and roles since this study involves secondary school teachers and their teaching experiences with mathematical proofs.

4. METHODS

This study used a qualitative approach to examine teachers' beliefs and perspectives on mathematical proofs and the role proofs have in school mathematics. A qualitative approach is necessary because the nature of the research question involves participants sharing their perspectives, which are related to their lived experiences. The study also uses an interpretive framework to inform the study, and the study collects data "in a natural setting sensitive to the people and places under study" (Creswell, 2013). Each of the participants have their own set of experiences, and ways that those experiences informed their current beliefs and perspectives. It would be difficult to use a quantitative approach to describe the complexity that comes from examining beliefs and conceptions especially when participants may come from different teaching contexts. Instead, a qualitative approach would be better in helping participants draw out their perspectives.

4.1 Research Design

In determining the research design, several considerations were made in terms of suitability for the research question and practicality for a master's thesis. Methods such as a survey or a distributed task may be able to have participants respond on a wide variety of related questions but may not necessarily have participants reflect and think deeply on their beliefs. On the other hand, participant observations (classroom observations) may be able to witness participants' beliefs in action, but these observations may only be possible after several ethical considerations, especially for the high school context of this study. They may also place pressure on the teachers and on the students as their lesson would be documented. Instead, the use of interviews allows the teacher to step away, physically and mentally, from the demands of managing a classroom, while also spending some time to think and ponder about their perspectives through conversation. The use of interviews also allows the teachers some control in choosing what they wish to portray, while also allowing me (the researcher) some control over the direction and depth of the conversation. Thus, interviews have been chosen as the primary data source for this study as they allow some degree of depth and elaboration without being too intrusive.

This study used individual interviews (one 60 minute session per participant) and a focus group session (one 90 minute session with all participants) to better understand teachers' conceptions of mathematical proofs and of the roles of proofs in school mathematics. The

individual interviews used an interview guide (appendix A) to navigate the interview process while the focus group session used two verbal prompts (appendix B) and two task-based prompts (appendix C) to facilitate the discussion. The use of interviews and a focus group allows participants to portray their perspectives through the points and examples they share. What this means is that the information obtained from the participants is informed by their own experiences and perspectives of their teaching. Since the focus of the research question was on the teacher participants' views, classroom observation was not necessary. Thus, the way the participants described their classroom context, directly or through examples, was sufficient. The combination of individual interviews and a focus group were used to help participants think and reflect on their perspectives on their own and with other teacher peers.

4.1.1 Individual Interviews

Interviews are a common way to collect primary data as interviews allow the researcher to have a degree of control over what is asked, in what order, and with what wording (O'Leary, 2014). O'Leary (2014) describes interviews as "a method of data collection that involves researchers seeking open-ended answers related to a number of questions, topic areas, or themes" (p. 217). O'Leary's (2014) description fits my research study as the research focus on conceptions requires participants to speak openly about their perspectives, opinions, and experiences. I was able to access those perspectives, opinions, and experiences by creating interview questions that allowed flexibility for participants to explore how they genuinely feel about mathematical proofs and school mathematics. Interviews have also been used as primary data sources by both Knuth (2002) and Dickerson & Doerr (2014) to gather information on their (teacher) participant's conception of mathematical proofs. However, aside from conception of mathematical proofs, I also included looking at participant's perception of the role of proofs in school mathematics.

According to O'Leary's (2014) organization of interview types and formats, my study uses formal, semi-structured, and one to one interviews (O'Leary, 2014, p.218). A formal format was used to establish a sense of professionalism and develop rapport and trust as the participants would later be part of a focus group discussion. However, the participants were given the choice to choose the time and location of the interview, which helped to put them at ease. A semi-structured interview style was chosen to allow participants the freedom to expand and elaborate on ideas to allow unexpected data to emerge. Sometimes participants were asked to talk a little

bit more about an idea that popped into their head. Finally, a one to one interview was used to help build a relationship between the researcher and participant while also allowing participants to dive deeply into their thoughts. The interviews were face to face and in-person, with the researcher giving full attention to the participant, with no note taking, to create an environment that is more akin to story telling than to a job interview.

The questions that formed the interview guide were designed around the two components of the research question, conception of mathematical proofs and roles of proofs in school mathematics. The questions were piloted with an acquaintance to develop my own interview skills in addition to obtaining feedback on the quality of the questions. For example, I learned to be mentally prepared for passionate rambling as the participants may be passionate and enthusiastic about sharing their teaching and their perspectives. I also learned to be adaptive and flexible during the interview such as mixing up the order of the questions according to the participant's responses may result in a better interview flow, and less abrupt flow of thoughts. The final interview guide involved a brief warm up where the participant would talk a little bit about themselves as a teacher of mathematics before moving to talk about their conceptions towards mathematical proofs, and how it differs from mathematical reasoning. Depending on what is shared, they may be asked to provide a teaching example or expand on their opinions and perspectives of the curriculum. Prompts were used very sparingly, mainly to further develop a participant's idea or thought. For instance, by asking the participant to elaborate on their past experience as a student, the participant might have provided insight on how their past experiences influenced their current experiences. A prompt was also used when the conversation was going off topic or becoming lengthy and redundant. For the most part, the participants in this study were familiar and comfortable with the one on one interviews and there did not seem to be much anxiety or nervousness.

4.1.2 Focus Group Discussion

Focus groups have been used in qualitative research in the form of group interviews with the purpose of gathering information through the interactions within the group (Morgan, 1997). Morgan (1997) has described three uses for focus groups: focus groups as the principle source of data, focus groups as a supplementary form of data to a primary method, and focus groups as a method among multiple methods used equally. Focus groups are used as the principle source of data when the research goals align with the type of data focus groups can produce, much like any

other qualitative method (e.g. participant observations, interviews, etc.). Focus groups may also be used to supplement a quantitative method, such as to follow up on survey data or assess a program or intervention. Lastly, focus groups as part of multimethod projects typically add to other qualitative methods of data collection. Regardless of its use, focus groups need to be carefully designed and conducted in ways that suit the research needs. In my study, the focus group session was used as a data source alongside the use of individual interviews.

One focus group discussion was used, in this study, after the individual interviews were all completed as a way to have participants interact with each other and “produce data and insights that would be less accessible without the interaction found in a group” (Morgan, 1997, p. 2). The focus group allowed participants an opportunity to follow up on any ideas or thoughts that may have emerged, from the individual interview, while also reflecting and commenting on what the others have shared. Focus groups may also have “an advantage for topics that are either habit-ridden or not thought out in detail” (Morgan, 1997, p. 11). In my experience, teachers are often busy with all the tasks associated with teaching that they often do not have the time, and possibly environment, to reflect on their practice in a deep way. By holding a focus group discussion, participants were given the time and environment (with other teacher peers) to reflect on their teaching practice.

The focus group discussion used in this study was designed to be relatively unstructured with two verbal prompts and one task prompt. The two verbal prompts (“what does mathematical proofs mean to me” and “what do I see as the purpose of proving or reasoning in school mathematics”) were used to help participants recall some of their responses from the individual interviews. The first prompt was presented at the beginning to get the discussion started and the second prompt came when all of the participants have had a chance to respond and the conversation started drifting towards individual teaching examples and experiences. Although some verbal facilitation was used to encourage the quieter participants to share, generally, I sat back and let the conversations occur. There were no time allocations for the prompts and the second prompt was introduced when the participants’ conversations naturally moved towards talking about school mathematics. Since a task based prompt may disrupt the flow of ideas, the task was introduced only when the discussion slowed down with participants having exhausted their ideas.

The task-based prompt was included as a way to help participants make connections with their conceptions and teaching practice in a concrete way. The prompt task came from the idea of a task-based interview from Dickerson & Doerr's (2014) second interview, of three. In their second interview, Dickerson & Doerr (2014) asked participants to discuss whether a series of 15 mathematical arguments fit what they considered to be a proof. The prompt task used in my study (appendix C) consisted of asking participants to comment on two mathematical conjectures and the possible student responses provided. The purpose of the task was for the participants to use what they see in front of them as a springboard to make connections with their conceptions and teaching practice. As such, they were encouraged to share their opinions, including modifications. The mathematical content of the prompt task is based on the content shared by the participants in the individual interviews, and it was introduced as such. The individual interviews along with some personal experience as a high school mathematics tutor formed the basis on how I created the sample student responses.

4.2 Positionality

Gathering data on beliefs and conceptions requires a method that allows for prompts and open discussion without participants feeling intruded upon. As a person knowledgeable in mathematics, and having high school teaching experience in mathematics, I had to keep my own conceptions and perspectives in check. Additionally, Creswell (2013) mentions a need to reflect on the relationship between the interviewer and the interviewee and be conscious of power distribution. For these reasons, I approached the participants in a story telling, non-evaluative, way where I am helping to document what they have to say. When I met the participants, and before starting the individual interview, I would inform them that the intention of the study is, *wishing to learn more about how mathematics teachers think about, and teach, proofs*. I placed myself in the position of a novice wanting to learn more about teaching mathematics from my participants, whom I saw as more experienced teachers.

4.3 Participants & Data Collection

Four secondary school mathematics teachers were recruited through a network of personal acquaintances. The participants agreed to have the individual interview and focus group sessions audio recorded. Screening participants' teaching experience according to a specific number of years or number of courses was difficult and impractical. Thus, participants

were instead required to be familiar with the Ontario Mathematics Curriculum, and have some experience teaching trigonometric identities. Although not an initial requirement, the four participants of this study happened to all have university degrees in mathematics. The participants also happened to teach in the same district school board which has both urban and suburban schools. Each teacher participant will be characterized in more detail in the following chapter.

The questions and prompts used in the individual interviews were designed to gather information regarding teachers' mathematical teaching experience, conceptions towards mathematical proofs, reasoning, and their views on proofs in the context of their teaching practice. The individual interviews were in person to help establish a relationship, and the interviews were audio-recorded to preserve authenticity. Individuals were notified that the interview would cover two main topics: what, to them, is a mathematical proof, and how do they see proofs in their teaching practice. The interview guide helped facilitate the interview but was not made available to the participants to avoid rehearsed answers.

The focus group session was held a month after the last individual interview. To refresh and prepare the participants for the focus group discussion, they were reminded of the two main topics from the individual interview two weeks prior to the session. After a round of introductions, the session began with discussion on the first of two prompts, "what does mathematical proof mean to me". A week prior to the focus group session there was a provincial mathematics education conference that was held locally to which all of the participants attended. Consequently, they were able to bring into the discussion many thoughts and ideas in conjunction with their own perspectives. The second prompt, "what is the purpose of proofs in school mathematics", was used to ground participants and redirect the discussion towards mathematical proofs in the school context. This prompt also got participants talking about their teaching experience and teaching practice. The topics discussed by the participants started to branch out towards mathematics, school mathematics, and teaching in the high school setting. The introduction of the task-based prompt was delayed until the participants have started to exhaust their ideas and opinions. The prompt task was presented with the claim first, followed by each of the "proofs" or "counterproofs". When participants commented on what they saw, they made references to their teaching practice and to some of what had been said for the first

and second prompts. The prompt task served as a way to consolidate the focus group session and wrap up the thoughts and ideas shared by each participant.

Throughout the data collection process, I did not distinguish between the roles of participants as people knowledgeable in mathematics (all having university degrees) and as people who teach mathematics. Knuth (2002) made this distinction in his interviews and later reflected that, “at times this separation into two stages seemed somewhat artificial as the teachers often had trouble removing their ‘teacher hats.’” (p. 67). Instead, I chose to allow the participants to switch around their hats because their past experiences (as students or as teachers) contribute to their current conceptions of mathematical proofs and school mathematics (Ernest, 1989). Interestingly, in their individual interviews, each participant made some sort of comment, comparison, or reflection on the two different stages on their own.

4.4 Data Analysis

The transcription process was performed entirely by myself. The individual interviews were transcribed during the gap, of about a month, between the end of the last interview and the focus group session. There were two reasons for transcribing the interviews at this time. Going over the interviews, through transcription, allowed me to better understand my participants, especially what appears to inform their thinking and their views, as preparation for the focus group session. The second reason was to make use of the available time so that more time can be used for coding and analysis after the focus group interview. The transcription of the interviews was performed one participant at a time, with two passes for each, and with a break between each participant; such that I may relive the interview experience. The first pass was simply writing down the verbatim content while the second pass went over spelling and grammar, particularly the placement of periods and commas. The transcription of the focus group session was also performed with the same two passes.

The interview transcripts were analyzed one at a time, with a break in between each transcript. The reason was to allow me the time and attention to step into the thinking and thought processes of each participant. I found this immersion was necessary in order to understand how the a participant’s ideas connected with one another, and what may have been going through their mind when responding to the interview questions. It was also important for me to recognize and set aside (as much as possible) my own perspectives. The immersion also

helped me situate the participant's ideas and comments in the focus group, in terms of where those ideas or perspectives came from.

The analysis of the interview and focus group transcripts followed two main steps, the creation of codes through emergent coding, and the comparison of codes across transcripts to determine common themes. The creation of codes was done in the second readthrough, after margin notes about general content was made in the first readthrough. An example of the difference between margin notes and codes is shown in the following excerpt. The margin note for this block of text was 'defining proof'. The portion underlined was coded as *conception of proof*, the bolded sections are words that have been circled with a pencil and are coded as *proof related skills* and *equality*.

Interesting question umm...what does a proof mean...The way I explain it to students, because we started a little bit in the Grade 11 university course. It's things that we know about math, we know trig or something like that and then we're using it to prove other things that we want to know. Or to prove, show the **logical steps** why we know this is **equal** to this, and why this is true, I guess. I don't know how to elaborate more...
(Dorothy, Interview Transcript, March 9, 2019)

To facilitate the comparison of codes across transcripts, a content summary sheet and a code organization table (appendix D) was made for each transcript in the third and fourth readthroughs respectively. The purpose of these two tables was so that at a later time, I would not have to go over the entire transcript to understand the context surrounding a code. The content summary sheet consists of point form notes about the topics and ideas present in the transcript. On the other hand, the code organization table consists of the codes on one side and the associated transcript text (and page number) on the other side. I found this kind of organization helpful because sometimes the same code may have emerged from different contexts within the same transcript, or between different transcripts.

The content summary sheet and code organization tables also helped characterize each participant, in addition to determining common themes. In particular, I found that the codes from the individual interview were helpful in characterizing each participant. And thus, I decided to include a profile of each participant because the characterization seemed to be help situate, and make sense of, some of the rationale behind each participant's responses and reactions in the focus group discussion. The content that formed this section came from the individual interview codes related to 'conception of proof', 'Reasoning and Proving', and

‘teaching approach’ were used to form the background information section in reporting results. Afterwards, I compiled a series of themes from the remaining interview codes and from using the focus group codes. The themes were then displayed in the sections that follows the participant profiles. In those (theme) sections, I sometimes made references, and drew parallels, to the participants’ profiles in order to paint a richer image for the reader. To situate the themes with the research study, I also made references to topics from the theoretical framework.

During the process of organizing the codes into themes, I consulted my thesis supervisor for an additional point of view. We discussed the rationale for organizing the results into a participant profile section and a theme section. We were also careful not to deliberately separate the themes from the individual interviews from the themes from the focus group session. Instead, we made sure that both data sources were used to complement and support each other. Since the individual interviews were more structured, they tended to provide a basis for creating the participant profiles. On the other hand, the focus group session contained participant views that were similar (and often in repetition) to their individual interviews, which demonstrated a degree of consistency in their views. The selection of themes to be included in this thesis was on the basis of having connections to the individual interview transcripts and to the focus group transcripts.

4.5 Difficulties

Initially my proposed study included a classroom observation of a Grade 12 Advanced Functions lesson on mathematical proofs. Aside from the individual interviews, the teacher participant would choose a lesson on mathematical proofs for the researcher to observe and then have a post-observation interview. The purpose of the observation was to see how the teacher’s conceptions of proofs and roles of proofs (described from their individual interview) would manifest in the classroom setting. It was also to see how the students’ reaction to their teacher’s lesson. Aside from looking at teachers’ conceptions of mathematical proofs, the research question at the time asked if a need was present for learning proofs, more specifically, a need for a deductive approach (away from using numerical cases) when validating mathematical arguments. Unfortunately, it was difficult to find teachers who were both willing to participate and were teaching the Grade 12 course. Instead, a focus group discussion was used (less intrusive) in place of a classroom observation and the research question shifted to asking what

the teachers see to be the role or purpose of proofs in school mathematics (in what they teach). The participant requirements were also expanded to include all high school teachers (Grade 9 – 12) who teach mathematics.

Another challenge I faced was in the analysis of the focus group. This analysis was a challenge as the of the topics being discussed became less and less structured and the discussion became more of a conversation with ideas popping in and out. The difficulty arose in determining what topic to categorize a piece of dialogue. Since in this study the focus group discussion was the only time the participants interacted with each other, they were often observed to be thinking on the spot as they reacted to each other's comments. Sometimes these thoughts may not be complete and required some interpretation based on the participant's individual interviews or on other pieces of dialogue from the discussion.

Overall, the challenges faced over the course of this research study contained several first-hand lessons about the nature of doing qualitative research, specifically in working with human participants. Unlike chemical reagents or cell samples in a laboratory, people can refuse participation, or request additional conditions. Therefore, the research design needs to take into consideration the possibility of complications that may arise during the process of recruiting participants. One consideration is being flexible in adjusting the research design and possibly the research question itself. Another consideration is in ensuring that all interactions between the researcher and the participants (and potential participants) are cordial. A warm and friendly reception may place participants at ease and allow them to speak their mind more freely, particularly regarding sensitive topics pertaining to vulnerability. From the challenges faced over the course of my research study, I have come to understand and appreciate the characteristic of qualitative research that elevates the participants, by giving them a voice and sharing their stories.

5. RESULTS I: PARTICIPANT PROFILES

The purpose of this section is to provide the reader with information regarding each participant, particularly with respect to their perspectives on mathematical proofs, how they approach teaching proofs, and why it was important to teach proofs. The first two topics arose during the individual interviews and helped situate the participants' perspectives and responses in the focus group discussion, where the roles of proofs were more apparent. The connection between the conceptions and the mathematical authority components of the theoretical framework can be seen through the examples in this section. These connections also gave some indications of the participants' values, which is connected with what they saw as the role of mathematical proofs in school mathematics. For each participant, the three topics are characterized through four subsections: 1) what is mathematical proof, 2) reasoning and proving, 3) teaching approaches, and 4) why teach proofs. The first subsections characterize participants' conceptions of proofs in a general sense while the second subsections characterize their conceptions through a comparison between what they consider to be reasoning and to be proving. The third subsection gives a general sense of the teaching style of the participants, which displays some of the participants' values. And finally, the fourth subsection shows instances where some roles of, or purposes for teaching, proof can be interpreted from what the participants have shared.

5.A Stella

Stella has been teaching high school mathematics for about eighteen years. She did her schooling outside Ontario but did her teaching in Ontario, teaching all the Grades from 9 to 12. In addition to the Grade 12 courses of Advanced Functions and Data Management, she has taught courses in nearly all track levels: applied, academic, college, mixed, university. Stella mentioned in the focus group interview that she has a master's degree in geometry. What Stella enjoys about teaching mathematics is helping students feel good about their understanding in the subject. She mentioned finding enjoyment in teaching students in Grade 9 and promoting a positive mindset for them towards mathematics. She also enjoys showing the creativity in mathematics, having students do things beyond learning algorithms and following procedures.

5.A.1 What is Mathematical Proof?

When asked to share what she thought of the words *mathematical proof*, Stella described thinking about her mathematics degree and university experience. She specifically used the words *formal proofs* and *formalized axiomatic approaches to mathematics*, to describe the type of mathematical proofs from university. This definition corresponds with the formal definition of mathematical proof outlined in the introduction. She also mentioned that she did not see a lot of mathematical proofs in high school. Stella further talked about what she considers to be proofs and what she does not, along with the presence of empirical proofs in school mathematics.

Oh absolutely! Lots of reasoning. So, you asked what I thought about when I heard the word mathematical proofs, my mind went immediately to formal, but of course there are a lot of examples of informal, working with mathematics and reasoning and proving throughout our high school curriculum. But I think it's not rigorous the way that it is formalized when you get to post-secondary in mathematics. When we studied geometry, we do a lot of investigation and a lot of things that actually would kind of bother me as a mathematician, but I think are important as a teacher. We do a lot of what I call proof by example. Which are not proofs at all of course. (Stella, Interview Transcript, March 9, 2019)

Stella's use of the word *informal* when describing mathematical proof in the school setting, along with the use of the phrase *proof by example*, points towards the empirical proof definition (from the introduction). In sharing about her thoughts on mathematical proof and on reasoning in the school context, Stella described formal proofs to be in her university experience of mathematics and empirical proofs in school mathematics.

5.A.2 Reasoning and Proving

When asked to talk about the Mathematical Process Expectation in the curriculum, Reasoning and Proving, Stella mentioned *making sense of a problem* for reasoning and *formal structure* for proving. She described reasoning to be a forefront process in making use of mathematical language and mathematical constructs to make sense of a problem. But when she thinks about proving, "I think about a more formal structure, to start with a premise and determine that something always works" (Stella, Interview Transcript, March 9, 2019). Stella gave two examples of where she sees reasoning and proving. The first example is in Grade 9 measurement, when students are asked to double the dimensions of a rectangular prism and describe what happens to the resulting volume or surface area. The second example is in Grade

10 analytical geometry, when students are asked to classify triangles and quadrilaterals based on some observed properties. In both examples, she described a process where students would choose several examples to work through before attempting to form generalizations based on the observed patterns from their results. Also, in both examples, Stella associated generalization, when students start moving away from examples, with proving.

Aside from sense-making and a formal structure through generalization, Stella also talked about the idea of equivalence and its connection with the other two ideas. She described solving equations as a foundation in Grade 9 and Grade 10, and that “students have to verify that that solution is good, students that understand how to solve equations and understand what a solution means understand the importance, or the definition of equivalence” (Stella, Interview Transcript, March 9, 2019). She then mentioned that the process of checking for equivalence becomes formalized when students do trigonometric proofs. Instead of checking their answer through “what we would call a left side, right side check” (Stella, Interview Transcript, March 9, 2019), the student would be “proving that a result is true for all values and they’re using other properties that are true for all values of theta” (Stella, Interview Transcript, March 9, 2019). Stella added that the shift from checking specific examples to proving a result to be true for all values “is an abstract process of proving because it’s looking into a generalized result” (Stella, Interview Transcript, March 9, 2019). In her examples, Stella described a process of starting students with examples and shifting towards a generalized rule or generalized approach. For her, it seems that understanding equivalence and moving from specific examples to a generalized rule is where proofs exist in the school setting. However, while this process of generalization from examples is related to the empirical proof definition, Stella’s descriptions of trigonometric proofs and the presence of some formal structure can be a place for the less-formal proof definition to exist.

5.A.3 Teaching Approaches

When asked to describe how she would *teach* reasoning or proving, Stella elaborated on her Grade 9 measurement example and on a Grade 11 trigonometric identity example. In her measurement example, she described, in more detail, the process of starting with examples and moving to a generalized approach.

You start with a rectangular prism of dimensions 3, 4, and 5, you double all those dimensions, what happens to the volume? Or how much bigger is the volume? And students will, some students will do one example and they will, if they understand

proportional reasoning, they will divide the volumes and say, ‘oh it’s eight’. So, to that I would say, ‘okay do you think it’s always going to be eight?’ So, they try another example, often, and they would say, ‘oh, still eight’. So, the way I’m trying to push their thinking is to say well, you can’t do an infinite number of calculations, there’s an infinite number of different boxes that you could be doing. Is there a way that you could understand that the result would always be eight without using specific numbers, [...] a different strategy that you can use to help you understand that [...] the volume of the new box will always be eight times greater? (Stella, Interview Transcript, March 9, 2019)

In her description, she was the person facilitating the students’ thinking into realizing a need for a generalized approach that would account for all possible cases. In her trigonometric identity example, Stella described a similar teaching process. She would have students revisit the idea of equivalence and the understanding of the trigonometric functions on the Cartesian Plane. She described building equivalence through students picking specific values of theta (θ) to examine the relationship between $y = \sin \theta$ and $y = \cos \theta$ and recognizing that one function is a horizontal translation of the other. She then connected observing the horizontal translation, from a few examples, back to determining a generalizable rule across all values of pi (π).

Can you start to get to the idea of the cosine function being a horizontal translation of the sine function? And so, you know, so it’s true for these particular values over this domain and sine function is infinite, is there a way to generalize how you will always, can come up with some way of understanding when they will always be the same, not just looking at pi over 4, or specific values of the angle. (Stella, Interview Transcript, March 9, 2019)

In this example, Stella, again, described a process of creating a generalization through the use of examples as the starting step. In talking about her teaching, Stella has further elaborated on her views of proofs in the school context through the lens of forming generalizations.

From the two examples, measurement and trigonometric identities, Stella’s notion of sense-making and reasoning can be seen. In the measurement example, she had students connect the idea of doubling with the volume of a rectangular prism. Sense-making was also present when Stella described having the students look at the relationship between the multiple specific cases to determine an underlying trend. In the trigonometric example, the students would use their knowledge of the Cartesian Plane and graphing functions to make sense of the relationships between $y = \sin \theta$ and $y = \cos \theta$. In both examples, the students may reach the proving step once they have identified an underlying trend, and now have to form a generalization for it. In both examples, Stella seems to be the one facilitating students’ engagement in the process from

examples to generalization. At this point it is not entirely clear where Stella lies on the epistemological continuum. Teacher guidance is present, but it does not seem to dominate the teaching process since it is the students who are making the observations. However, Stella acts as a mathematical authority figure who encourages participation in the learning process.

When asked to further describe how she teaches mathematical proofs, Stella described a process that involves little direct instruction. She brought up a notion of accessibility, and how she strives to make information and mathematical knowledge accessible for all of her students. Using her measurement example, Stella described using a combination of individual and collaborative learning and formative assessment to make learning accessible. She first had students work through setting up the dimensions and determining the volume for a few rectangular prisms on their own before sharing their results with their peers.

I might ask students to do that individually first so that they have their own, at least calculation foundation of understanding, and then now each student has a different example. So, I will get two or three students working together to share their results so now they each have different examples and yet they notice that the result is the same. At that point I encourage students to collaborate on whether or not they think that result will always be true and how they might demonstrate that. (Stella, Individual Interview, March 9, 2019)

At the end of the lesson, she would ask her students to do a formative assessment in the form of an individual reflection or summary on what they understood. The formative assessment may be an exit card or a specific example, or a new question to extend the students' thinking. From her description, Stella's use of peer collaborative work points at some sharing of mathematical authority with the students, which means less teacher direction and more student investment. By having students work together to share their results, and make sense of their observed pattern, changes their roles from answering a question to creating an opinion. From detailing her teaching approach, Stella has shown an instance where there is room for students to explore and generate their own mathematical meaning. These examples of sharing mathematical authority seem to point to the presence of an experimentalist approach.

5.A.4 Why Teach Proof?

When asked why it was important to teach thinking and deductive reasoning, Stella responded with reasons that were more developmental. She revisited the idea of accessibility, though this time in terms of students being able to use abstract thinking and reasoning. Stella

mentioned, “I think students really have to have a good, quite an advanced understanding of how abstract reasoning works in mathematics to be able to access that. I don’t think that it’s accessible to most students. And most students don’t need it” (Stella, Individual Interview, March 9, 2019). Stella saw abstract thinking and reasoning as a useful skill but the formalization of those skills was not seen as a priority. If students were ready to be more formal with their thinking, then she would include more abstract thinking and reasoning in her lessons. Stella was consistent in this view for the rest of the interview and throughout the focus group.

Stella also commented on the nature of the proofs that are presented to her students as being example based.

A lot of things that actually would be, kind of bother me as a mathematician but I think are important as a teacher, [are] what I call proof by example. Which are not proofs at all of course. We ask students to look at a variety of concrete examples and to convince themselves as part of the investigative and reasoning process that a certain property might be true...We’re using a result that is known and we’re just getting comfortable with that truth I would say. (Stella, Interview Transcript, March 9, 2019)

The transcript above relates to Stella mentioning a notion of getting comfortable with the concepts of geometry by working through concrete examples. Therefore, Stella’s mention of proofs may be related to discovering new mathematical connections, explaining an existing piece of knowledge, and possibly developing thinking and sense making when connecting the worked examples with each other. For Stella, proofs may be a way for students to understand a concept by using reasoning to look at a series of tangible examples or outcomes.

5.B Mark

Mark is a retired high school mathematics teacher who started his teaching career outside Ontario. Although he was teaching computer science upon moving into Ontario, Mark mentioned his desire to teach mathematics led him to make the transition. While he did teach some of the senior mathematics courses at some point, Mark decided to mainly focus on teaching the younger, Grade 9 and Grade 10, and non-academic track courses. He felt the needs of the students in those courses were not being met and focused on the students who did not enjoy mathematics.

5.B.1 What is Mathematical Proof?

Unlike Stella, whose initial reference was her university experience, Mark's starting point to what mathematical proof meant to him was in his experience teaching a former Grade 12 course from a previous version of the Ontario Mathematics Curriculum. He described his teaching experience as, "and I always remember it as a very challenging, for a lot of students it was quite challenging. Of course, we don't do proofs anymore, in terms of mathematical proofs, in high school" (Mark, Interview Transcript, March 11, 2019). Mark also mentioned that he sees proofs in trigonometric identities, in the current curriculum, but quickly pointed out,

But that's not the power of proof, in high school math, and they're small examples of mathematical proofs. But the proving part, which is the interesting part for me, is with the overall expectations in courses, well the Process Expectations, I should say, where reasoning and proving is a big one. (Mark, Interview Transcript, March 11, 2019)

For the remainder of the interview, Mark reserved talking about mathematical proof only to describe the content from the old Grade 12 material and for trigonometric identities. He did not see proof in the Grade 9, the Grade 10, or in the non-academic courses. At this point, it is not possible to relate Mark's descriptions of mathematical proof to the three proof definitions.

5.B.2 Reasoning and Proving

When asked to compare reasoning and proving, Mark described reasoning in terms of logical sense. He described logical sense in terms of checking one's thinking and demonstrating if an answer made sense, narrowing in on the question "how do you know your answer is correct?" (Mark, Interview Transcript, March 11, 2019). Mark went on to connect logical sense with reasoning and students' over-reliance on algorithms when doing mathematics. He pointed out that without reasoning, it is hard to prove something. And in his experience, he has often seen students struggle to use reasoning or logical sense when doing mathematics.

I think of, especially kids who have struggled, the reasoning part, which I think you need to get to a proof. It's not that they're not reasonable people, they are, they're pretty good thinkers but because they're in math class, they park their reason at the door when they come in. (Mark, Interview Transcript, March 11, 2019)

Mark attributes the negligence of reasoning to an over-reliance on mathematical algorithms in the course of students' mathematical learning. He uses an example of learning two-digit multiplication to illustrate his point.

I read a study once that talked about how when students have to do two digit multiplication, then they use an algorithm, multiply this and carry this da-da-da in this column. That process is often where students sort of stop trying to think about understanding, they memorize the algorithm and go. And then it builds from there where they no longer have to think, things don't have to make sense in math class. (Mark, Interview Transcript, March 11, 2019)

Mark argued that students are not expecting things to make sense in mathematics class, that they can just follow an algorithm or shortcut and achieve success. These descriptions may tie with why he places more focus on reasoning, in his responses for the rest of the interview. The descriptions also tie with his thoughts on developing reasoning as an enduring skill that students can bring past school mathematics.

Those process expectations are in my mind more important than the mathematics in some ways. I know that very few of my students will ever use math in their lives, even in their personal lives math may not be a big deal. But the reasoning and the communication and all those other process expectations are more important, to develop thinking in the student. (Mark, Interview Transcript, March 11, 2019)

I think this excerpt describes Mark's overall interview tone quite well as he spent a large portion of the allotted time talking about the importance of including reasoning and sense making when doing mathematics. It seems he valued developing the skills associated with doing mathematics rather than the mathematics itself.

5.B.3 Teaching Approaches

Mark's views on reasoning and logical sense are related to the way he teaches mathematics in that he gives student opportunities to exercise reasoning and logical sense. He made the observation that students often get weighed down by computations, especially if they are not good at them. He then mentioned an instructional strategy that he found to be successful in getting his students engaged in mathematics without doing too much computation individually. Instead of giving a large number of questions to each student, to practice their computation, he would give a large number of questions to the class as a whole. The students would gather in small groups to work on a small portion of the questions. Mark described several benefits to utilizing this group collective learning approach. Instead of working through a large number of computations, and viewing it as a tedious task, the students would be able to pair up and work with their friends and peers. If a group was stuck or went astray, Mark did not need to intervene

since the students can adjust their thinking by looking at what their peers, in another group, are doing. Although sometimes the students would have to work through more examples, Mark generally did not need to tell them what was going on, “they kind of get it for themselves” (Mark, Interview Transcript, March 11, 2019). By using a group work approach, Mark has shifted the workload of going through examples to be shared with the class.

The way Mark described using examples spread among the students in a group collaborative effort is similar to Stella’s description of her teaching approach. Mark’s students would form their own understanding and reach a consensus on the underlying mathematical relationship from their examples, which is somewhat akin to Stella’s process of generalization. Unlike Stella, Mark specifically mentioned that he tried to avoid intervening and telling students what was going on. This deliberate avoidance means more mathematical authority is shared with his students as the students are determining, for themselves, what is going on. The rationale for his avoidance in intervening in his students’ thinking is due to the presence of social influences in the classroom. Mark shared an observation that his students “would much rather listen to their classmates than listen to me because of that social aspect. They don’t care about an old guy, what he has to say about mathematics. They care about what their peers think, and how they talk and how they present themselves” (Mark, Interview Transcript, March 11, 2019). He described that students seemed to be more impacted by what their peers said, and sometimes those words carry more weight than anything he says. So instead of trying to fight for students’ attention, Mark uses the affinity the students have for socialization to have them engage in doing mathematics. By allowing the students to discuss with each other, Mark provides students with listening and assessing roles in addition to the role of doing mathematics. Overall Mark’s teaching approach appears to connect well with an experimentalist approach as his students form the connections and understandings of the worked examples on their own while holding some mathematical authority when presenting or self-assessing their work with their peers.

5.B.4 Why Teach Proof?

Mark was quite passionate, in his interview, about the importance of having students use common sense when doing mathematics. The way he described his teaching approaches and some of the anecdotal teaching experiences placed an emphasis on the need to check one’s thinking. Mark’s strategy to implement self-checking was to have students come to a consensus through discussion. Although he did not make the connection between classroom discussion and

proofs, it was quite clear that the communication and argumentation aspects of proofs were being implemented in his teaching. Like Stella, Mark also mentioned that most of his students would not need to use mathematical proofs after high school. Instead, the skills and thinking developed through argumentation and discussion were seen to be more practical.

Like Stella, Mark also mentioned using examples as a way to have students use thinking and reasoning when learning mathematics. In his anecdotes, he emphasized the use of logic as the basis for learning mathematics rather than remembering formulas or equations. This emphasis was evident in the way he described spreading a large amount of examples across the entire class and having the students engage in argumentation. Mark pointed out that over the summer break, students would often forget what they have learned. And if they learned mathematics through only remembering formulas and equations, they would have a hard time recalling those learned concepts. He found that by having students focus on developing thinking and reasoning, they would have an easier time recalling mathematical concepts.

Those techniques are not going to keep over the summer, they're not going to last over the semester. I think of a teacher with a grade 11 class [who] wanted to introduce the cosine law, the entire class said "oh we didn't learn that in grade 10". But they did, and they did well. A girl pulled out a test, she had a great mark in that section, cosine law, none of them in the entire class remembered seeing it before. (Mark, Interview Transcript, March 11, 2019)

For Mark, thinking and reasoning seemed to be inseparable from learning mathematics, just as communication and argumentation seemed to be inseparable from the descriptions of his classrooms. Overall, it seems that Mark sees the use of thinking and reasoning as necessities when learning mathematics.

5.C Dorothy

Dorothy is a newer mathematics teacher who has been teaching for just over five years. She has been mostly teaching Grade 9 and Grade 10 but has experienced teaching Grade 11 (university track) a few times. She also has experience with the academic, applied, and essentials tracks. Dorothy mentioned that her enjoyment in teaching mathematics comes from an affinity to the topic. She found satisfaction in helping students engage with logic and having them come to a better understanding, and better opinion, towards mathematics.

5.C.1 What is Mathematical Proof?

Dorothy's response started describing mathematical proofs with what appeared to be an attempt at a definition before shifting to talking about trigonometric identities. She described proofs as "it's things that we know about math, we know trig or something like that and then we're using it to prove other things that we want to know" (Dorothy, Interview Transcript, March 9, 2019). She also described proofs to be a showing of logical steps, of why something is true, and of showing equivalence. While Stella and Mark both mentioned sense-making and reasoning, Dorothy seemed to focus more on proofs and deeper understanding of mathematics. However, her description can still be related to sense-making and reasoning.

And I feel like proofs is more about actually getting a more deeper understanding of why this is equal to this and why we're doing that stuff in the formula and why we're doing that when we're solving a problem, and stuff like that. (Dorothy, Interview Transcript, March 9, 2019)

From her descriptions of mathematical proofs, Dorothy seemed to connect proofs with a process of understanding a problem or situation based on using an established truth. The reference to trigonometry suggests the presence of structure or organization in Dorothy's definition of proof, hence a less-formal definition.

5.C.2 Reasoning and Proving

When asked to distinguish between what is a proof to what is reasoning, Dorothy commented that a proof is a manifestation of reasoning that has structure. She described that, "proof is showing your reasoning" (Dorothy, Interview Transcript, March 9, 2019). She further explained the process of how she has her students structure their proofs.

The way that I have them structure their proofs is they have to write down what they know first, and then they solve the left side and solve the right side. Show that reasoning to say that if this is equal to whatever, is equal to 1 or whatever, then so is this, with all of the steps in between. And then at the end, I don't let them just leave it the way it is, okay this is equal to this, so then make them write down left side equals right side. Sometimes Q.E.D. because that's what I was taught as well. (Dorothy, Interview Transcript, March 9, 2019)

In this excerpt, the presence of a left-side and right-side structure suggests a level of formality in Dorothy's view of proof. Dorothy went on to further distinguish reasoning and proving by using a trigonometry example. She saw proving more like a trigonometric proof where one has to use

what they know and understand to represent that the given statement is indeed equivalent. Reasoning, on the other hand, “is more like if a is true, and a and b are equal to each other, then I can put b in there to show that this is true” (Dorothy, Interview Transcript, March 9, 2019). For Dorothy, proof is more of structured reasoning where following the structure is important. She contrasted this description with reasoning where its possible to reason without proving, such as when students communicate their understanding without necessarily using proper mathematical language. At this point is it interesting to note that Dorothy’s responses on proof have been mostly based on her experience teaching trigonometric identities.

When asked to elaborate about structure and students’ communication of understanding, Dorothy talked about her teaching experiences with the earlier high school grades (Grade 9 and Grade 10). She described focusing on having students develop structure (and organization) when doing mathematics. Her reason was that showing one’s work logically and in an organized fashion leads into proofs. For her, being able to communicate one’s thinking mathematically is part of proving.

You have students who are able to say, I did this calculation and this, and they put them all in a structured order. But then there’s other students who are literally all over the page, and they’re pulling numbers out of...out of their brains or their calculators but you don’t know where they came from because they’re not showing it so I think it’s definitely important to show it, at least the logical steps in a good ordered structured to your solutions. (Dorothy, Interview Transcript, March 9, 2019)

Having students develop organization seemed to be a prevalent topic in Dorothy’s interview. She also mentioned the need for organization when teaching the applied track where she uses more manipulatives and concrete examples in her teaching.

5.C.3 Teaching Approaches

Dorothy’s teaching examples used a lot of physical objects to model and facilitate students’ understanding of mathematical concepts. Like Stella, Dorothy also used measurement and examining surface area and volume to illustrate her teaching examples. Some physical objects included filling a microwave with unit cubes to model volume, using sticky notes of the same size on different classroom objects to model surface area, and unpacking different cardboard cartons that are used to hold a dozen drinks (2×6 vs. 3×4) to explore optimization. By using common everyday objects that students may encounter (microwave and drink cases) as a starting point for students to understand the concepts of volume and surface area. Dorothy

mentioned that using concrete objects, “it gets [students] to understand what they’re doing what why they’re doing it at a deeper kind of level” (Dorothy, Interview Transcript, March 9, 2019). Her rationale for this kind of approach is to have her students understand what a formula they may see in a textbook or formula sheet really means. In all of her examples, she described providing some guidance in directing students to think about, and make connections to, the mathematical concepts present in each physical object.

Dorothy described using group collaborative work at times to get students more involved in their learning. She mentioned that the students who were confident in mathematics would be able to show off to their friends and those who were not as confident would be able to listen and follow along. She found that students tend to give up less readily when working with peers than when working individually. She also observed that students felt less scared to make mistakes on a whiteboard because mistakes can be easily erased. Whiteboards were also described to be a way Dorothy encourages discussions and sharing of mathematical authority. Dorothy mentioned implementing roles such as taking turns writing and talking so everyone in the group is involved.

Dorothy’s approach in teaching trigonometric identities contained elements of group collaborative learning and of teacher directed learning. Her example involved her, as the teacher, presenting a statement of equivalence and asking her students to *prove it to me*.

I’m like ‘okay I’m going to tell you that this thing is equal to this thing, prove it to me’. And they’re always flustered at first which is kind of funny, but then they’re like ‘well this thing has to equal this and this’ and then they follow through the steps and then kinda go from there. And a lot of the time I have them on whiteboards or groups or partners or something, so they’re a little more, instead of sitting there like ‘I don’t know how to prove this thing I don’t know what this thing means.’ And they’re a little more willing to give it a try or throw in an idea to their friend sitting beside them or whatever and then okay well we know this, here’s our tools, what can we do to prove this, to show this. (Dorothy, Interview Transcript, March 9, 2019)

The phrase *prove it to me* shows Dorothy holding mathematical authority as the students work through the ideas among themselves. The phrase (and other iterations) emerged again at later points in the interview, particularly when Dorothy mentioned giving her students trigonometric statements that are not identities, not always equivalent.

Normally I’ll tell them, this one is true so show me why. [...] But then other times they do whatever calculations and substitutions on one side and then they realize that no, this is not actually true. And some of them, even when I tell them that it’s true, will do some stuff either correctly or not, and be like ‘well left side doesn’t equal right side’. And I was like,

‘no but I told you this one actually is true so you’re not quite there yet, keep going’. Yeah there are definitely times because I want them to think about it beyond just, ‘I’m telling you that this is equal to this now show it to me.’ (Dorothy, Interview Transcript, March 9, 2019)

In this excerpt, Dorothy mentioned that she would sometimes tell her students that a statement is true, but other times she would leave it up for them to determine whether the statement is true or not. In both cases, mathematical authority resided with Dorothy as she was the one who set the criteria for whether a question had a proof or not. From the examples and descriptions Dorothy gave of her teaching, her approaches seemed to connect with a Platonic approach as her descriptions had her packaging the direction of learning for her students.

5.C.4 Why Teach Proof?

When asked why it was important to teach proofs, Dorothy referred to her emphasis on organization and the communication of mathematical thinking. She saw the structure and communication needed to verify trigonometric identities as a metaphorical goal. Students would start developing the skills to verify identities by first forming a habit of showing their work, making a connection between their thoughts and the paper. But simply writing ideas on paper was only a start; Dorothy found herself emphasizing to her students the importance to present those ideas in a way that made sense and could be easily understood. She connected the focus on organizing thoughts with a sense of conviction in one’s answer. By organizing their thoughts, students would be able to describe how they came to a particular answer, and whether the answer was correct or incorrect. Conviction in the legitimacy of an answer and organizing mathematical thinking for better communication are both roles of proofs that Dorothy described in her interview.

Aside from organization and communication, Dorothy also placed some emphasis on presenting proofs as a way to explain a mathematical concept, or formula. During the focus group, she described using an online video which showed a rotatable right-angled triangle with square-shaped water basins attached to each triangle side. Upon turning the triangle, students would see the water from the two smaller squares fill up the larger one. Thus, the students would come to understand why the Pythagorean Theorem involved squaring the two smaller sides, and adding them together, to form the square of the hypotenuse side. Dorothy’s other anecdotes of using classroom objects or grocery store items, as mentioned in the previous subsection,

involved some aspect of helping students visualize the mathematics they were learning. Although the use of these objects were more for students to understand the problem, rather than as proofs, the element of explanation was still present.

5.D Olivia

Olivia started teaching almost twenty years ago with a variety of topics including mathematics, science, music, English, history and business. About four years into her career, Olivia transitioned to teaching mathematics exclusively. She has experience teaching all the high school Grades (9-12), several different track levels (applied, academic, college, university), and has also been an instructional coach for her school board. Olivia has a master's degree in education focused on classroom assessment in mathematics classes. She has also taught a few Teachers' College courses to prospective teachers.

5.D.1 What is Mathematical Proof?

Olivia's response to what mathematical proof means to her was based on making a conjecture and testing the conjecture. She associated mathematical proof with logic in a way that was different than argumentation in other school subject areas, such as English, or about politics. Olivia also connected mathematical proof to trigonometric identities and verifying geometrical properties in school mathematics. She expanded further on her notion of mathematical proofs by talking about representation and communication of mathematical ideas. For her, proofs was not so much about the subject but rather the conventions of setting out a thoughtful argument. In particular, Olivia states that "I see one component of proof as clarifying my own thinking about something about equivalence, then I do see a notion of proof, that proof having enough rigour to convince somebody else that that is true" (Olivia, Interview Transcript, April 5, 2019). Like Stella and Dorothy, Olivia connected equivalence with the notion of proof. And like Dorothy, Olivia also connected the notion of proof with organizing one's thoughts. From her descriptions it is clear that Olivia defines mathematical proof as having some degree of structure, but it is not entirely clear what her parameters are in terms of what is considered a proof.

5.D.2 Reasoning and Proving

When commenting on Reasoning and Proving, Olivia described that reasoning is a much broader term than proving. She characterized reasoning as "explaining what a concept is, connecting a concept to another [...] why something may have a solution or may not have a

solution” (Olivia, Interview Transcript, April 5, 2019). She also described reasoning as a skill that students can use when thinking mathematically, that computation may not be needed if one can reason through something. This description is similar to what some of the other participants (e.g. Stella and Mark) have described as sense-making, where one uses logic and reasoning to think through and understand a problem. In contrast, Olivia characterized proving as a subcategory of reasoning that “is not necessarily out there to be discovered, [proof] is there to be the result of a construction, to have thought through a process, to have convinced oneself and others of notions of equivalence” (Olivia, Interview Transcript, April 5, 2019). Olivia’s description of proof involves an investment of the person into constructing (creating) a proof. Proving would require a person to put the pieces of logic and reasoning together to form an argument.

5.D.3 Teaching Approaches

When asked to talk about how she teaches mathematical proofs, Olivia described two examples through the topic of trigonometric identities. In the first example, she mentioned that instead of simply giving her Grade 12 students a sheet that contained the double angle and compound angle identities, she would go through and demonstrate how they can be constructed.

I will demonstrate that, I will lecture and draw on the board and show how a compound angle formula can be deduced, can be constructed, and then they’re able to then extend that and then what’s interesting is that when they then construct proofs for different trig identities. They are also reasoning but it’s not just sort of blindly substituting. (Olivia, Interview Transcript, April 5, 2019)

Here the words *demonstrate* and *lecture and draw* connect with a teacher directed instruction with the teacher holding mathematical authority. Olivia also talked about the importance of teaching mathematical convention and way of thinking when teaching identities. In the second example, Olivia was responding to how she would address Grade 11 students moving algebraic terms across an equal sign when proving trigonometric identities. She talked about two different methods that she would use, with one being more direct than the other. The first method had her modelling one or two identity proofs so that the students understood what some of the conventions were. In particular, she described using a red line and gestures.

I’d draw a red line between the left side and the right side. (gestures with hand) And I would say, you can do whatever you want to whatever is on this side of the wall. You can do whatever operations that would show equivalence between each step but what you’re

doing on the left hand side here stays over here, (gestures) and on the other side of this giant thick red line (gestures) what you're doing on the right hand side of this line, this big wall cannot cross, must stay there. (Olivia, Interview Transcript, April 5, 2019)

This approach is more direct as Olivia is the one laying out how trigonometric identities are to be proven. She is also instilling in the students the conventions and what is considered acceptable mathematically. The mathematical authority resides with Olivia as the representative of mathematics. The second method involved Olivia showing and commenting on samples of student work.

I will show maybe one proof that is done correctly that maybe have, has a correct format in keeping the left and right sides apart but maybe has something that's been reduced or cancelled in a way that breaks some rules and doesn't maintain equivalence, but somehow magically ends up with the left side equalling the right side. So, I have students hunt through, sometimes, and find mistakes. And I'd usually include in that a case where a student has mixed up the left, and taken terms from the left and right side and solved it more they would be solving an equation rather than doing a proof. (Olivia, Interview Transcript, April 5, 2019)

This second method has students informing their understanding of proper conventions through examining provided examples. Rather than simply observing and accepting the given conventions, as in the first method, here the students may be using more of their own reasoning and logic to annotate the given proof. However, this second approach is still teacher directed as Olivia is the one choosing which examples to show the students.

When describing her teaching of trigonometric identities, Olivia also mentioned the use of collaborative group work. After teaching the conventions and modelling a few identities, she would have students break into groups to work through a worksheet of identities. Olivia described her worksheet to contain three categories of identities, varying in difficulty. The students would work with each other to come up with strategies and approaches for proving the identities. By allowing the students the opportunity to work through harder questions and form (and test) their own strategies, Olivia is providing her students opportunities to practice and develop their reasoning and logical skills. She also mentioned that when working in groups, "ideas can travel around if students have chosen the same question or they can see different strategies they can look around the class and go 'oh yeah, I forgot about that'" (Olivia, Interview Transcript, April 5, 2019). Olivia's description of spreading out an amount of questions through group work and allowing students to peek at each others' work is similar to Mark's description

of group collaborative work. And finally, Olivia noted the importance of having students practice mathematics in groups and as well as individually. Once her students have worked through the worksheet, Olivia would have them complete a quiz with questions of similar difficulty. She felt that by having students develop their strategies for harder questions, they would not be as frustrated when they encountered those types of questions on an assessment.

While Olivia's initial description of teaching trigonometric identities appeared to be very teacher centered, such as regarding convention, there are places where she gives some autonomy to her students. Near the end of the interview, Olivia revealed that over the course of her mathematics teaching career she has learned to be more patient and more trusting in the abilities of her students. Overall, the examples given about her teaching approaches, particularly on providing the correct conventions, connects Olivia with the Platonic approach.

5.D.4 Why Teach Proof?

Compared to the other participants, Olivia had more mentions of post-secondary mathematics, and in general, preparing students for the next level. She gave the impression that it was important for students to be exposed to deductive thinking, structure, and proofs at an earlier age because there would be a lot more emphasis on those ideas later on. For example, she mentioned in her interview,

I've had a couple years where I've taught mostly grade 12 university level math. I've had, the next year, I've had some grade 9s. That's sort of always reinforced, for me, that there should be careful attention paid to reasoning and proving in the grade 9 and 10 classes. Students do come to a more solid understanding when they're in grade 11 and 12. (Olivia, Interview Transcript, April 5, 2019)

Olivia went on to provide some brief examples in the Grade 9 and Grade 10 curriculum where she saw a connection to proofs. One such example is in the classification of shapes through analytical geometry, and the idea of a counterexample. For instance, as one example of doing a counterexample, in doing the calculations to verify that four points are the vertices of a square, if one criteria (e.g. equal side lengths) was not achieved, there would not be a need to continue. By introducing the idea of a counterexample, she thought that her students may have an easier time making sense of proofs when they were introduced in the upper grades.

5.E Putting it Together

All participants described mathematical proof as something that contained structure. They also made distinctions, between mathematical proofs and Reasoning and Proving. Furthermore, they also had their own views on what is proving and what is reasoning. The descriptions for reasoning were, overall, broader than proving but all contained some aspects of sense making and of thinking logically. Proving was more about creating a structured argument that was related to equivalence. Stella and Olivia both referred to mathematical proofs through a formalized lens before talking about proofs in the school mathematics context. On the other hand, Mark and Dorothy began with a bit of their own associations and identified a connection between mathematical proofs and trigonometric identities before moving to talk about reasoning in the school context.

Stella's response was most clear in separating the different definitions of proof. Formal proof described her university experience, and a mix of empirical proof and less formal proof described her high school teaching experience when students move from numbers to a generalized rule. While she retained mathematical authority in some points of her instruction, her use of group work with whiteboards rather than direct instruction may place her closer to the experimentalist approach. Mark's responses focused a lot on having students think mathematically, using logic and reasoning to make sense of what they are doing. To him Reasoning and Proving also involved having students evaluate if their answers were correct rather than rely on an external algorithm. His focus on using the social aspects of learning mathematics paired with white board use to have students *see for themselves* represented an experimentalist approach. Likewise, Dorothy's responses talked a lot about structure and organization of thoughts. Her mention of teaching students to prove trigonometric identities in a prescribed structured way presented proof in a more formal way. Dorothy's descriptions, however, contained a lot of teacher directed instruction, which pointed at a Platonic approach. Finally, Olivia included argumentation and construction on top of structure and as parts of proofs, but her flexibility in its representation means the proofs can be formal or less formal. Her mention of demonstrating and showing to her students the correct conventions in proving trigonometric identities connected to a Platonic approach. Some student directed learning was present when she talked about whiteboard group work, but mathematical authority resided with her as the teacher.

Since the individual interviews were all held prior to the focus group discussion, they served as a way to get to know the participants. The participants were also getting used to the researcher (myself) and the depth and breadth of the study. Over the course of the interviews the participants showed some similar ways of thinking (e.g. the use of group collaborative learning). However, the nuances of each participant's perspectives did not stand out until the focus group discussion, when they interacted with each other. Furthermore, Ernest's (1989) comment that "it is possible for two teachers to have very similar knowledge, but for one to teach mathematics with a problem-solving orientation, whilst the other has a more didactic approach" (p. 20) became evident in terms of an observed difference in what the participants valued when teaching mathematics. Thus, the participants' descriptions presented in this section serves as a way to get to know their values and perspectives.

6. RESULTS II: EMERGING THEMES

There were six main themes that emerged from the research data. The first theme involves the participants' comparisons of mathematical proofs and of Reasoning and Proving. This theme emerged primarily from the participants' responses to the first interview question, and to the first section of the focus group discussion. The participants often reflected on their past experiences of mathematical proofs as students. The second theme collected together the use of proof-related terminology, specifically structure, equivalence, and generalization. These terms emerged in various ways when participants talked about mathematical proofs in the school setting, and in particular, through their teaching experiences or teaching approaches. The third theme emerged primarily in the focus group discussion from an off-topic tangent that became a conversation about the presence of mathematical proofs in past versions of the Ontario Mathematics Curriculum. Since the conversation had participants talking about the past, and sharing past experiences, there was some connection to what participants shared for the first theme.

The fourth theme arose from participants commenting on their teaching of mathematical proofs. It involved the notion of preparing students for what they will encounter in the next level of mathematics as well as the merits of teaching a less proof-focused curriculum. The fifth theme came from a conversation about comparing argumentation in mathematics class to argumentation in other school subjects. The theme first emerged early in the focus group discussion when participants were describing their conceptions of mathematical proofs and was revisited in more detail. The participants talked about the nature of mathematics as being different than other subjects, the influence of peers in a social setting, and of the relatableness of mathematics to personal life. Finally, the sixth theme came from participants' mentions of developing sense-making and reasoning skills as part of teaching proofs. These mentions included why participants valued Reasoning and Proving, such as the idea of developing an internal dialogue when doing mathematics. In talking about reasoning, some participants mentioned students' reliance on shortcut procedures while others commented on the role of reasoning and logic in non-university track levels.

6.1 Mathematical Proof and Reasoning & Proving

A theme emerged from the individual interviews and focus group discussions which compared mathematical proofs in the context of university mathematics experience with mathematical proofs in the context of high school mathematics. All participants referred to their university experience when talking about mathematical proofs. They would then move on to talk about how different mathematics was in their university experiences compared to their high school experiences, as teachers, and sometimes as students. The essence of the difference can be seen in the following comment, “I think university math was a great shock to me, to be honest with you. That that’s what math was all about, was this very formal way of proving that this is equal to that, in terms of analysis and number theory and all of those things” (Stella, Focus Group Transcript, May 23, 2019). Despite the difference in experiences, the participants all made a connection between trigonometric identities and mathematical proofs. However, they often did not spend much time talking about trigonometric identities. Instead, they moved on to talk about the Process Expectation, Reasoning and Proving. For instance, Mark switched to talking about Reasoning and Proving right after mentioning trigonometric identities (as mentioned in section 5.B.1). Olivia also brought up Reasoning and Proving in the focus group discussion, which shifted the conversation away from trigonometric identities.

Mark: Well trig identities (Stella: uh-huh). Yeah... That’s the closest thing probably.

Olivia: I also think about things just from the Grade 9 math course about, I don’t know if the word ‘verify a proof’ is right but I think about certainly reasoning and proving when I’m asking students to prove that something is a 90 degree angle, or a right isosceles triangle. [...] Proving whether or not something is a square is sort of at a Grade 9 level. I’ve certainly done a lot more back in my university days than that, but I think that’s sort of where I go in terms of the Grade 9 course anyway, is thinking about those, not just trig identities and those kinds of things. (Focus Group Transcript, May 23, 2019)

Dorothy also described similar ideas in terms of having students prove geometrical properties in Grade 9, but it was more in the context of demonstrating their understanding. In addition to Grade 9 geometry, Stella also mentioned proving geometric properties of triangles and quadrilaterals in Grade 10. But she also mentioned that nothing was formalized in the way mathematical proofs in the university context would be. Overall, the participants’ description of mathematical proofs in their teaching was different than mathematical proofs from their

university experience. Although they would attribute certain things in their teaching with proofs, or proving, they asserted that the context and expectations are different.

6.2 *Structure, Equivalence, and Generalization*

When participants described proofs in the school context, several concepts that are related to mathematical proofs emerged. More specifically, the words *structure*, *equivalence* and *generalization*, were mentioned or indirectly referred to in the interviews and focus group discussion. There were two main ways *structure* was mentioned, the first being related to an organization of thoughts as part of communicating mathematical thinking. The second way *structure* was mentioned was through a systematic way of approaching a problem to ensure all the factors have been accounted for. On the other hand, *equivalence* was mainly mentioned as an important piece of understanding the difference between solving an equation and showing the equivalence between two statements. The participants emphasized understanding equivalence as a prerequisite to understanding trigonometric identities. Finally, *generalization* emerged mainly in Stella's interview and when empirical approaches are used to show an observed relationship. In her interview, Stella described generalization of an observed pattern as a way she saw proofs emerge in school mathematics. Since this association was touched upon in Stella's comments regarding Reasoning and Proving from her Participant Profile (5.A.2.), the following section on generalization focuses more on the use of empirical approaches by both students and teachers. The section also includes comments from the task-based prompt that compared empirical to general approaches.

6.2.1 *Structure in School Mathematics*

The concept of *structure* was mentioned when participants described mathematical proofs to contain structure. In addition to describing structure in proving trigonometric identities, structure was also brought up when participants talked about Reasoning and Proving. For instance, Dorothy emphasized having students develop organization when doing mathematics, and communicating their work using a mathematical *structure*.

I stress that in math it's very very important to show your work, so the written communication. (Dorothy, Interview Transcript, March 9, 2019)

I think the main aspect that I stress even, early on, with younger kids, and all through elementary school and all through high school is showing your work logically and in an

organized fashion, which I think definitely leads into proofs later. (Dorothy, Interview Transcript, March 9, 2019)

Aside from leading into proofs, Dorothy also had a pedagogical reason for having students show their work. She described having difficulty following students who did not have organization, and it made it difficult for her to determine the level of understanding those students have achieved. Overall, Dorothy saw a presence of structure in communicating mathematics and mathematical thinking.

On the other hand, Stella described *structure* in terms of using a systematic way of thinking to approach a problem. Stella shared an anecdote of her experience of a subtraction lesson as a volunteer in a Grade 2 mathematics class. The teacher had asked the students to think of all the ways you can take numbers away from twenty. Stella had described her thought process as very orderly but witnessed the students shouting out all kinds of answers in a disorderly, almost chaotic, manner.

Stella: And of course, me, with my adult orderly brain, (others: chuckle) I have this idea of how they're going to represent all of this. I wished I had filmed it, I really do, or taken pictures. Because on their blackboards they...the random... 'Oh I know one, twenty minus six', and then another one would be like, 'I have another one, what about twenty minus thirteen', and then they were just like all over the place. And they're written, all over, on they're individual blackboards. And I'm looking around, and this was happening everywhere in the room, and in my mind I was, what a...what's the point... (others: chuckles) why's she asking them to do this? Here's me like twenty minus one, twenty minus two (others: chuckles). My structure! There's my list, we were just going to be done and it took like an hour! (chuckles) And I don't even know if all of the students really got to the, I was like, well how do you know you've got them all? (Focus Group Transcript, May 23, 2019)

Stella described using a systematic way of representing the different subtraction cases to make sure she covered all the possibilities. Having completed her list and watching the students work through the question for an hour led to a comment on time efficiency and structure. Stella also made a comparison between her anecdotal experience with her experience teaching area optimization in Grade 9. Her students would construct different sized rectangles, in a process not unlike the Grade 2 students, which led her to ask them the same question, *how do you know you've got them all?* The way Stella described using a systematic approach to solve a problem is an aspect of structure that is related to the nature of mathematics, which will be elaborated on in a later section.

6.2.2 Equivalence in School Mathematics

The concept of *equivalence* was mentioned in the presence of trigonometric identities by several participants. They stressed the importance of understanding equivalence as an important aspect to doing mathematical proofs, in particular, trigonometric identities. Stella captured this sentiment well when she shared the following statement,

Stella: I think there's some foundations to proof that we go through in high school. So understanding equivalence is important, like understanding the meaning of the equal sign, I think, is an important foundation for at least understanding how to, you know, where we lead which is proof of trig identities. (Focus Group Transcript, May 23, 2019)

In her interview, Stella described the instructional practice she uses to introduce Grade 11 students to trigonometric identities. The teaching process involved having students see the relationship between the $y = \sin \theta$ and $y = \cos \theta$ graphs as a transformation of the other. She noted that without a good grasp of equivalence, the students would have difficulties grasping the concept of a trigonometric identity. Likewise, Dorothy and Olivia described having to facilitate the process of proving trigonometric identities by deliberately separating the left- and right-hand sides. They found that by doing the separation, students would make a distinction between showing equivalence and solving an equation. The students would also be introduced to a convention of proving mathematical statements (identities).

Another moment where participants mentioned *equivalence* was in the second task-based prompt. The comments to the prompt linked equivalence with representation and counterexamples. For instance, Stella mentioned the use of graphing calculators for students to explore and observe the idea of equivalence through the use of transformations.

Stella: You can study equivalence using transformations which students really, some students really like looking that idea of looking at that one function through the process of reflections and phase shifts, for example, could be equivalent to another function. Because we allow students the use of graphing calculators in class everyday, as Olivia was saying, the exploration process of being able to think about this a bit more intuitively and visually I think is a really...that helps them some ways. (Focus Group Transcript, May 23, 2019)

Stella pointed out that by seeing how the mathematical equations look graphically, students may have a better understanding of the parameters behind transforming functions. Likewise, for trigonometric identities, Mark mentioned how a graphing calculator becomes a handy tool. He described that students would often jump into proving a trigonometric statement without first making sure that it was an identity. And by incorporating graphing, his students are able to see a

bigger picture by observing the behaviours visually. Stella also commented that her students can sometimes be “quite traumatized if they don’t know if something is always true” (Stella, Focus Group Transcript, May 23, 2019). She pointed out that in counterproof 2 it is visually obvious that the functions are almost never equivalent, but there may be situations where the functions are almost always equivalent. And subsequently, looking for a counterexample for the latter situation can feel like searching for a needle in a haystack.

6.2.3 Generalization in School Mathematics

The concept of *generalization* was mentioned through Stella’s measurement example from her Participant Profile (5.A.2.). She described a process of students moving from an empirical approach of examining case by case to identify a common pattern, and possibly moving onto forming a rule that is generalizable across all cases. Having taught measurement recently, Dorothy affirmed Stella’s measurement example during the focus group by mentioning her own teaching experience.

Dorothy: A lot of them will automatically revert to, ‘I’ll make all the sides one and then I’ll make them two and then this is what happens’ and don’t necessarily make the connection that it’s always going to happen. What if I pick different numbers, what if my friend over here picks different numbers, how does that...they don’t really generalize any of it. (Stella: That’s true.) [...] They’re not very comfortable with the general stuff, they know that it’s true for this one because of these numbers but the general part is kind of beyond them for the most part. (Focus Group Transcript, May 23, 2019)

Like Stella, Dorothy mentioned that using cases came naturally to students and that they required a bit of a prompting to look beyond a set of cases. Although he did not mention generalization, Mark also used cases, as seen in his Participant Profile (5.B.3.), as a means to have his students realize an underlying trend. Likewise, Olivia described introducing trigonometric identities by asking her students to come up with different angle measures, which she would use to show that the mathematical statement (the identity) was always true. She would then lead her students into thinking whether the identity works for all values.

Comparing empirical and generalized approaches also emerged from participants’ comments on the task-based prompts. For instance, Olivia reflected the tendency for students to use cases and shy away from generalizations in her response to the first task-based prompt. The first task-prompt asked what would happen to the surface area of a rectangular prism when the dimensions were doubled. Olivia said, “I feel like my students might be in the proof 2 or 3 and

then many of them may not know how to get to proof 1, but if you show them proof 1, they would go, oh that makes a lot of sense” (Focus Group Transcript, May 23, 2019). Proofs 2 and 3 were designed to be examples of verifying a relationship through an empirical approach, with proof 2 using a net representation. Stella also commented that by having a different representation for proof 2, especially if the net is placed on grid paper, students may have an easier time making the transition and seeing a generalization of what would happen to the surface area. On the other hand, Dorothy commented that her students would often produce something similar to proof 3 but they would not know to show the final step of the surface area being 4 times bigger. The students may simply state that the surface area is bigger, which made Stella have to “change the question, I had to change it to how many times bigger is it, like I had to lead the witness a little bit” (Stella, Focus Group Transcript, May 23, 2019). And depending on the values chosen, students may end up with a decimal value as pointed out by Mark, “I’ve also had had 3.97 more, that sort of thing, and you’re done” (Focus Group Transcript, May 23, 2019). In addition to the examples of having to provide scaffolding for students to see the pattern of doubling dimensions, Olivia mentioned that “to get to a general case of something I think kids really have to play with numbers and values” (Focus Group Transcript, May 23, 2019). Ultimately the perception was that students needed something tangible to work with before being able to associate a pattern with a generalized rule.

6.3 Comparing Proof in Previous Versions of the Curriculum

The representation of proofs in past Curriculum versions was brought up through a brief tangent in the focus group discussion. There are two small themes that form this section, the first being participants’ recollection of their experiences with proofs in past Curriculum versions. The second theme is participants’ discussion about the level of proof representation in the different iterations of the Curriculum they had mentioned. In addition to the current version, all of the participants, except Dorothy, have taught the version that precedes the current Curriculum.

The general notion that emerged from the conversation was an observed decrease in the emphasis of proofs in each Curriculum revision. The three versions discussed were laid out in the following excerpt by Olivia,

Olivia: So, there was the 2002-ish, 2000/2002-ish revision when the OAC went away and then we had the new Curriculum 1.0 and then 2007, the senior math revised Curriculum came out 2.0. So, there’s been three that I’ve experienced, one as a student and two as a

teacher, and I would say that the direction in the Curriculum, in terms of proof, is not as strong as it was. It has gotten weaker each time, (Focus Group Transcript, May 23, 2019)

In their interviews, Stella and Mark had both mentioned seeing more proofs in a previous Curriculum. For Mark, he mentioned seeing more proofs in the former Grade 12 course, Geometry and Discrete Mathematics, from Curriculum version 1.0. For Stella, she mentioned seeing more geometry proofs involving angles and similar triangles. She specifically mentioned a change in the approach to learning mathematics. She said, “there’s not a strong foundation for a proof-based approach to mathematics in our current Curriculum. It’s much more of an investigative, problem-solving based approach to understanding mathematics”. (Stella, Interview Transcript, March 9, 2019). Olivia and Stella also recalled deriving the formulas for the different conic sections in the former Grade 11 university track course, also from Curriculum 1.0. Although Dorothy did not teach with Curriculum 1.0, she experienced it as a student. She shared that her teachers would sometimes show the students a proof but there was not an expectation for the students to construct (or reconstruct) a proof for a test. Overall, there seems to be less emphasis on proofs in the current version of the Curriculum than in earlier versions.

6.4 Proofs and Its Influence on Teaching

A theme emerged surrounding the way the participants talked about teaching proofs and the notion of the teacher laying out the conventions, derivations, and ways of thinking. In their interviews, Olivia, and to a lesser extent, Dorothy talked about teaching in a way that prepared students for the next level of mathematics. For Olivia, she talked about preparing students for post-secondary mathematics while for Dorothy, the descriptions were more about helping Grade 9 and Grade 10 students prepare for trigonometric identities in Grades 11 and 12. A brief conversation also emerged in the focus group discussion that commented on a teacher-driven nature of teaching proofs in the previous version of the Curriculum. The conversation was followed by Stella making a comment on the merits of focusing on the underlying thinking skills and reasoning skills associated with mathematical proofs.

6.4.1 Preparing for the Next Step

In both her interview and during the focus group discussion, Olivia brought up a concern about a perceived gap between a high school experience of mathematics and a university experience of mathematics. From her own experience with a previous version of the Curriculum

(OAC), she felt there was a gap going into university. She described, “in my university [experience], all the numbers were gone, you really had to understand equivalence and have strategies for how to work on these proofs. Sometimes it was done with very little instruction” (Olivia, Interview Transcript, April 5, 2019). And with each Curriculum revision focusing less and less on proofs, she felt that this gap would have widened even more. Olivia’s concern may be a reason for her to emphasize and implement more teacher-directed instruction earlier on. For instance, she commented on the first task-based prompt by saying that she would lecture and demonstrate proof 1. The students may then explore the validity of proof 1 by testing cases until they were convinced. Other examples were mentioned in Olivia’s Participant Profile (5.D.3.) when she described lecturing and demonstrating the compound angle identity, and also when showing Grade 11 students to keep the left and right sides separate when verifying an identity. Preparing students for post-secondary mathematics is connected to the way Olivia teaches proofs.

Although she was not preparing students for university, Dorothy’s descriptions of her teaching methods exhibited a similar trend of preparation. Dorothy worked mostly with Grade 9 and Grade 10 students, preparing them for the mathematics in Grade 11 and Grade 12. In her interview, Dorothy described an important aspect of proofs to begin teaching earlier on is being able to show one’s work with organization. In particular, she talked about structuring one’s thoughts into logical steps as seen in section 6.2.1. For her, the focus was on having students convey their thinking through organized logical steps. Dorothy also described trigonometric identities to contain structure and organization which could be a reason for her emphasis on developing organization in earlier Grades. By implementing logical organization at an earlier stage, students will be able to form habits and conventions for doing mathematics by the time they encounter mathematics that is more abstract. Like Dorothy, Olivia also mentioned a preparation based on logical structure and organization for Grade 9 and Grade 10 students. It seems that for Dorothy and Olivia, the presence of proofs at a higher level was influential to their teaching as they saw a need to prepare their students.

6.4.2 Focusing on Process Expectations

When participants talked about their teaching experience with previous versions of the Curriculum, their instructional experiences were mostly teacher-based. In the focus group discussion, Olivia shared an experience with a fire drill going off during the derivations of conic

sections from a previous Curriculum (1.0). She specifically mentioned that it was already difficult to get the students engaged. Stella pointed out that teacher-based instruction was a trend with teaching proofs and other generalized equations. She illustrated this trend by sharing a similar experience, with the current Curriculum (2.0), in seeing a lack of student engagement when her colleagues constructed the sine law, or the quadratic formula, from a general case.

Stella: Well that's the thing with reasoning, that's the thing with mathematical proof. The burden of all of that is on the teacher now. [...] But the expectation for the students is to just follow along. There's no expectation for them to do any of those constructions or derivations themselves, and so of course they don't. There's only really three people that are listening to me during those talks. (Focus Group Transcript, May 23, 2019)

Stella was specifically referring to the Specific Expectation 3.2 on page 90 of the Ontario Curriculum Document (Ontario, 2007), where students are not required to reproduce the development of the compound angle formulas from a general case. Later on in the discussion, Stella followed up the mention of this teacher-based trend by pointing out some merits to teaching the current Curriculum, which is less proof based. She described that the current Curriculum places more emphasis on investigations and problem solving. She used an example of focusing on her Grade 11 students' thought process when they were verifying trigonometric identities. She said, "so, I think I'm not so much honouring the QED at the end as much as trying to honour the, the process expectation of seeing them live in the act of doing that" (Stella, Focus Group Transcript, May 23, 2019). Mark even mentioned in his interview a freeing feeling of placing emphasis on the Process Expectations rather than only on the Specific Expectations. He described that students would often forget formulas and concepts, but it was hard for them to forget how to think. Overall, Stella emphasized that the focus of a less proof based curriculum was not so much on being able to reproduce formulas or proper conventions as it was on the logic and reasoning that was used in the process.

6.5 Why Proofs in Mathematics Class?

In talking about mathematical proofs and the school context, a theme emerged based on the following question, "how is that different in math than other subject areas?". (Olivia, Focus Group Transcript, May 23, 2019). The context surrounding the question was asking what was special, or different, about reasoning and argumentation in mathematics class, compared to what some students may experience in their other school subjects. It is an interesting theme as it

relates to Ernest (1989) as he said, “teachers' views of the nature of mathematics may also be compounded with additional constructs, such as views of the relationship between different subject matter areas, for example. Is mathematics entirely distinct from other disciplines?” (p. 21). The theme is unpacked in three parts, the first part explores statements about a nature of mathematics as being objective and established mathematical truths as being applicable to other situations beyond the current situation. The second part brings up a need to consider the social aspects of a learning atmosphere and its influence the perception of what is true. The last section comments on the relativeness of mathematics to everyday life through a brief exchange in the focus group discussion.

Some participants had perceived mathematics to be based on objective truths. Objectivity was mentioned in the focus group discussion when proving in mathematics was compared with argumentation in English.

Dorothy: I think, between math and other subjects, is that a lot of other subjects have opinions and things but math is, in theory at least, inherently true, so we can prove something is true all the time because of the structure and the way we go about solving these things. Whereas if we're talking about debating in English or history or something, it's opinions and it's very subjective and it's not so much objective. And it might be true for this person or in this situation or something, but in math... “a” squared plus “b” squared equals “c” squared is always true for a right-angled triangle and that's a thing we know and can apply to other situations. And it's not just a one-time thing. (Focus Group Transcript, May 23, 2019)

In this excerpt, Dorothy mentioned that proving in mathematics contains a level of objectivity where once a truth is established, it can be completely applied to a different context. She also expressed a similar point in her interview where she related proofs to the use of prior known relationships as a basis for proving a mathematical statement; show equivalence. Another example of the mention of objectivity and mathematics is in an observation shared by Mark. He described that from working through examples, his students would come to the same answer eventually and agree on its correctness. There was not a need to address personal beliefs, or opinions, as “everyone kind of moved that way where you can accept that this is the solution, or this is the result and we're good with that” (Mark, Focus Group Transcript, May 23, 2019). There may be students who are still a little doubtful, who may not have fully grasped the understanding yet, but for the most part, a truth had been established that is accepted by

everyone. Finally, the applicability of mathematics was also mentioned by Olivia. She brought up that mathematics is not dependent on context, which is unlike other topics.

Olivia: When I'm talking to these mathematicians, it's almost like the actual problem doesn't matter. Engineers will work on a problem and the context matters because it's something that's getting built or something will break. When you're doing comedy, the context matters, the reading the room. But in math, it's almost like you were saying, the geometry, the topic that you're studying doesn't matter. [...] So that to me is different than other areas because I don't see that a lot in other places. (Focus Group Transcript, May 23, 2019)

The perception here is that mathematical truths do not depend on their context. Olivia also made a point of saying that mathematical proofs are not dependent on one's charisma or personality but on the way the arguments and statements have been laid out. She demonstrated this point by showing everyone a picture of visual proof of the Pythagorean Theorem that was represented through two images. This independence of context and opinion is what makes mathematics objective and applicable in ways not found in other school subjects. Together, Dorothy, Olivia, and Mark, described a nature of mathematics as containing a degree of objectivity.

However, in response to Olivia's statement of mathematical proofs being independent from subjective influence, Mark brought up a social experiment he had conducted in the past. He had a Grade 9 mathematics class answer a series of multiple-choice questions by moving to the corner of the room that they thought was the correct answer. What he noticed was that the students tended to move to the same corner as whomever was perceived to be the 'smart guy'. His experiment pointed out that social factors can override students' sense of mathematical correctness. For example, students lacking confidence (or interest) in mathematics may tend to go where their friends, are rather than where the correct answer is. Students may also feel less secure being in a corner with fewer numbers, and instead, feel more secure by being with the more credible students. While students may convince each other, rightly or wrongly, the teacher may also convince their students of the correctness (or incorrectness) of an answer. Thus, students with incorrect answers can be convinced of what the correct answer is. Mark commented on this social interaction by saying, "I think that is the beauty and that's why discussions in mathematics, in my mind, are so important. In terms of everybody getting to the same, accepting the same result eventually" (Mark, Focus Group Transcript, May 23, 2019). Indeed, the correct answer is still the correct answer despite a majority choice on something

different. Although mathematical correctness is objective, its perception can be influenced by social factors, as was seen in Mark's social experiment.

The topic of relatableness emerged when participants commented on how there are always some students who lag behind their peers. For whatever the reason may be, these students have not formed the mathematical understanding to the same level as their peers. And the participants described that what often happened was a sense of giving up, and simply accepting a concept without fully understanding it. Olivia commented that for these students, mathematics may appear to be something that is done to them. She specifically compared her teaching experience of other subjects to say,

Olivia: I feel like in science, when I was teaching chemistry, it resembled, it was simpler, but it resembled the work of what a chemist may do. We did some synthesis, we did different kinds of reactions, we purified things, we looked at a mass of something after a reaction and we can explain that with theory. [...] So, what does a musician do, they play an instrument, they composed a song, that resembles what I did as a music teacher. I feel like in math, the work of a mathematician, is solving new and current problems, is not the work that we do. (Focus Group Transcript, May 23, 2019)

Olivia described a disconnect between what mathematicians do and what school students learning mathematics do. She went on to describe this disconnect as a possible reason for students to not be as involved (or interested) in mathematics than other school subjects. She described how students are able to immerse themselves in a Shakespearean play by relating and connecting the themes of love, forbidden love, and family feud to their own lives. But on the other hand, "when we're asking students in Grade 10 to define the circumcenter of a triangle, that connection to the work of mathematicians...is not really there...and then connections to their own lives aren't really there" (Olivia, Focus Group Transcript, May 23, 2019). The themes of shapes, algebra, and intersection of lines are harder to connect with their own lives. For the students, the process of doing mathematics then becomes an abstract practice that involves thinking and logic. On the other hand, Stella responded to Olivia's examples by bringing up a point about students' cognitive readiness. She said, "different students will be at different places in terms of seeing generalization, seeing structural elements" (Stella, Focus Group Transcript, May 23, 2019). Opposite to lagging behind, Stella used an example of her experience with a Grade 9 student who went beyond her peers in classifying quadrilaterals based on diagonals. The expectation was to investigate the relationships between diagonals and quadrilaterals, but

this student went beyond establishing a relationship and attempted to get to a generalizable rule. Although the connection between quadrilateral classification and her personal life was not really mentioned, the point Stella made was that this student formed relatedness by seeing the possibility for a generalizable rule to emerge; and decided to pursue that generalization. Stella commented that the other students may have reached where this student was, but it would require a deeper level of thinking and probing that many had not developed yet. In this brief conversation between Olivia and Stella, both students that lag behind, and students that push forward, were mentioned alongside the notion of forming a relationship with mathematics.

6.6. Reasoning and Sense-Making

The theme of reasoning and sense-making emerged from all participants having made a connection between reasoning and making sense of a problem or solution. For Stella, the sense-making involved a process of understanding a mathematical problem, whereas Olivia related sense-making to a habitual process of doing mathematics. On the other hand, Mark talked about checking whether an answer made sense, if it was reasonable. And Dorothy mentioned having students use reasoning to dig deeper in understanding of the underlying mathematical concepts in a problem. The topics *logic skills* and *internal dialogue* were found in the participants' descriptions. Following the participants' descriptions, an example of the absence of reasoning is presented as it became a conversational reference point for some of the other themes in the focus group discussion. Finally, reasoning and sense-making was also brought up by Stella and by Dorothy in relation to the non-academic/university track courses.

To begin, Stella spoke of sense-making, and its connection with the other Process Expectations, when she described reasoning in her interview. Her interview description focused more on interpreting and understanding the problem but later in the focus group discussion, Stella mentioned a more holistic application of reasoning in the overall process of doing mathematics.

Reasoning, I think, just makes use of math language and math constructs to make sense of a problem. It's part of modelling, it's part of understanding the problem, [...] It overlaps, I think, with other math processes as well, it overlaps with selecting tools and sequencing. Because if you're selecting, as you're reasoning through a problem, you're deciding what tools you're going to use to help you understand and make sense of a problem. If you're looking at a right triangle, one of the reasonable ways to find the side lengths may be to use Pythagoras, it might be to use Trigonometry. (Stella, Interview Transcript, March 9, 2019)

Stella linked reasoning with another Process Expectation, Selecting Tools and Computational Strategies, to describe making sense of a problem. She also extended this notion of making sense of a problem to making sense of the world by emphasizing the applicability of the Process Expectations beyond school mathematics. Her desire is for students to be able to form an internal dialogue in which skills such as reasoning, thinking, and problem solving can be used to reflect on how the different pieces of a problem (or solution) fit together and whether it made sense. Likewise, Olivia also mentioned sense-making when characterizing reasoning. But instead of mentioning Process Expectations, Olivia's description connected more with forming an internal dialogue as part of doing mathematics.

I might be thinking to myself, going through thought processes of, like I said, is there a solution, is there not a solution. Should the answer to this be a big number or a small number, does it make sense that what I've come up with is quadratic here, does it make sense that what I've come up with is linear here. And I think that I'm always in mathematics myself, and with my students trying to work on that sense of reasoning, to be always asking what came before this, what comes after this. (Olivia, Interview Transcript, April 5, 2019)

Here Olivia made a connection between sense-making and logic when she described her engagement with reasoning about a solution. In the focus group discussion, she described a holistic aspect to sense-making by having students 'play with' trigonometric equations and graphs to learn what kind of outcomes to expect when different types of values are put in. Both Stella and Olivia related sense-making to the process of doing mathematics. Olivia emphasized the prevalence of reasoning in the overall process while Stella talked more about the underlying Process Expectations.

Connecting reasoning and making sense of a solution or outcome of a mathematical process were seen in Mark's and Dorothy's interviews. Mark brought importance for students to form a habit of checking if their answer makes sense. More specifically, he mentioned that "they need some technique for asking themselves if [their] answer [is] reasonable" (Mark, Interview Transcript, March 11, 2019). The phrase *asking themselves* can be related to the internal dialogue mentioned by Stella and detailed by Olivia. He had also pointed out that although students often make mistakes, "they're pretty good thinkers" (Mark, Interview Transcript, March 11, 2019). Although there are often answers at the back of the textbook, Mark leaned more towards having the students develop a process that was more personal to themselves. Dorothy, on the other

hand, talked about having students understand why the mathematics they did led to their answers. Her focus was on having students go beyond remembering mathematical formulas to understanding the mathematical relationships that formed the formulas. To do so, she described drawing students' attention to using reasoning and logic to construct new relationships, using prior knowledge as the starting point. By trying to get students to think about their mathematical processes, Dorothy may also be trying to instill a habit of internal dialogue. In summary, Mark and Dorothy described the use of logic and reasoning to make sense of an outcome. Mark described having student use thinking and reasoning to check for correctness while Dorothy described having students achieve a deeper understanding of a mathematical relationship through reasoning.

The next part of the theme on reasoning and sense-making originated from an anecdote from Mark's interview, which he also shared in the focus group discussion. Mark mentioned that he often observed that his students approach mathematics without using reasoning. He attributed the phenomenon as being related to an over-reliance on mathematical algorithms and shortcuts over the course of a student's mathematics learning.

Mark: What I'm interested in is how reason is lost in high school math. Reason that kids had when they were younger, it's just gone out the window because they've been struggling to sort of make sense of things and come up with some interesting, perhaps creative ideas. The reason is, I mean some of the stuff is incredible. I mentioned an example of the student from last semester I was tutoring, was solving an equation and he finished with two equals three and was quite happy with it. Somebody in Grade 3 wouldn't buy that, like equivalence, they would know that they were different. But [the] student in Grade 9 academic, who's been working hard to make sense of stuff, has gone through all the steps as far as he's concerned, he's done everything he thought he should do, and that's the result, and so therefore it's good. (Focus Group Transcript, May 23, 2019)

Mark's amazement came from the Grade 9 student accepting a result from his procedure that was blatantly illogical. It is possible that the procedure itself was erroneous or inappropriately applied, but Mark had expected the student to catch the flawed process through realizing the unreasonableness of his result. He further expressed that reasoning, and also proving, cannot happen under the idea that doing mathematics does not have to make sense. An overreliance on a shortcut without deeper understanding can also be seen when Dorothy described her students' engagement with the Pythagorean Theorem. In the focus group discussion, Dorothy shared that

she questioned her students about which side of a right-angled triangle is the hypotenuse and why it is the longest side. She found that some of them understood that the hypotenuse was the longest side all the time, and the reasons for it. But other students would simply accept that it is always longest because, “‘I’ve been told that the one across from the little square, that one is the longest’” (Dorothy, Focus Group Transcript, May 23, 2019), and end their thought process. While the students’ response to Dorothy’s example can be unpacked in many ways, acceptance of a shortcut can be seen. Right angled triangles are usually drawn with the right angle labelled, but when a right angle is implied (e.g. through geometric relationships), students would need to employ some degree of sense-making and reasoning. As expressed by Mark, and further illustrated through Dorothy’s example, students have been observed to approach mathematics without using reasoning, and instead, rely on some form of shortcut.

6.6.1 Non-University Track Levels

Only Stella and Dorothy made specific references to other track levels outside of the academic/university track. Both participants associated mathematics outside of the university track to emphasize the use of reasoning through concrete experiences. For instance, Stella described the presence of a concrete experiential aspect of mathematics that is present in those courses.

Our Grade 11 workplace course, for example, is a very practical math course that involve budgeting, taxation, but the focus is on decision making. So, thinking about what is reasonable and reasoning through problems that involve number sense and money and proportional reasoning. Those are really important to be able to reason with because they involve our real world and being able to understand what is reasonable when you’re talking about money when you’re talking about how much tax you should pay on an item of clothing. (Stella, Interview Transcript, March 9, 2019)

By exploring practical topics, such as budgeting and taxation, students engage in reasoning and sense-making through developing knowledge that is applicable to their lives. Stella further argued that reasoning may be more prevalent in these non-university courses as a result of the greater emphasis on the experiential aspect of mathematics. Similarly, Dorothy mentioned students having an easier time relating to the mathematics being taught through the use of concrete examples and tasks, particularly those that involve the students experiencing the concepts for themselves. For example, she described the use of sticky notes and small centimeter cubes to help students conceptualize surface area and volume. She saw students

engage in reasoning when they were discussing how much cardboard was needed to construct a box to carry drinks.

So, a lot of them went straight for, ‘well there are six rectangles, boom boom boom, here we go, there’s the answer’. And then other ones were, ‘well if she’s asking how much cardboard, what about the little flaps that are glued onto the other sides, what about the ends that have flaps that overlap each other,’ and all these much more complex shapes. Some of them were asking, ‘well can we open up the box and measure that way’. And some of them were really nitty picking the little, ‘there’s a triangle’, ‘the corner’s cut off of part of the box’, [...] So there’s definitely an aspect of reasoning in there and it’s just a matter of getting it out of them. (Dorothy, Interview Transcript, March 9, 2019)

The experiential engagement can be seen when students analyzed the drink boxes, which can be found at a local grocery store. The amount of reasoning in what to include, and what to exclude, in their calculations for surface area can be extended to thinking about the manufacturing process of making a box. Thus, reasoning in the non-university track is related to a practical and experiential form of mathematics that uses concrete examples to help students apply mathematics to the real world.

6.7. Putting it Together

It was clear from the individual interview and focus group discussions that the participants saw mathematical proof in two ways. The first view was associated with their university experience of proof, a description that was more mathematical. The other view was associated with their experiences teaching high school mathematics, a description that was more pedagogical. It was through this second view that the participants talked about their conceptions and perspectives about mathematical proofs. The participants also used proof-related words to describe some overall mathematical processes and skills that they wish for their students to learn. From the use of these words, the roles the participants attributed to proofs emerged. The participants also reflected on the presence of those processes and skills over the last few curriculum revisions and how, as a result, their teaching has developed over time. Some of the nuanced similarities and differences in teaching styles emerged (e.g. mathematical authority). In talking about the role, or purpose, of proofs, the participants discussed what argumentation meant in terms of school mathematics in comparison to other school subjects. The participants’ comments in this theme provided insight on their perspective on the nature of mathematics. And finally, the participants emphasized the importance to develop enduring skills that students can

carry past school mathematics. They described the skills as sense-making to understand a situation and reasoning to think through a situation logically. Overall, the participants appeared to have communicated through their position as mathematics teachers, allowing their teaching experience to inform their comments and responses.

7. DISCUSSION

The discussion is broken down into three main sections, mirroring the Theoretical Framework chapter. The first section reflects on the connections between the emergent themes and the participants' conceptions of mathematical proofs (first component of the research question). Participants recognized mathematical proofs, in their mathematical form, being absent from school mathematics. Instead they generally had a pedagogical approach to their conceptions of proofs in the school context. The second section reflects on the connections between the emergent themes and what the participants attribute to be the role of proofs in school mathematics (second component of the research question). Most of the participants' descriptions of proofs were through the enactment of Reasoning and Proving. They also brought up some social factors of the classroom learning environment. The specific categories of proofs described by de Villiers (1990) and Knuth (2002) were not mentioned to the participants. Thus, the roles participants associated with proofs were interpreted through the way they talked about proofs and their mathematical teaching practice. Finally, the last section reflects on how the mathematical beliefs and mathematical authority emerged from the teaching examples shared by the participants. Since this study was not focused on examining teaching practice, mathematical beliefs and authority can only be interpreted as far as the examples and descriptions the participants chose to share. However, beliefs and authority gave insight into how participants came to their conceptions of proofs and what roles were present in their descriptions.

The way the three sections of the discussion relate to each other and to the theoretical framework is represented in figure 2. When participants talked about beliefs or roles directly, the connections to the research question was clear. But when participants talked about their teaching, or if it was not obvious that they were expressing beliefs or roles, it was necessary to look at their examples through the lenses of beliefs, of roles, and of mathematical authority. From their examples, the participants' conceptions and attributed roles could be interpreted. The values and participant profiles also aided in this interpretation.

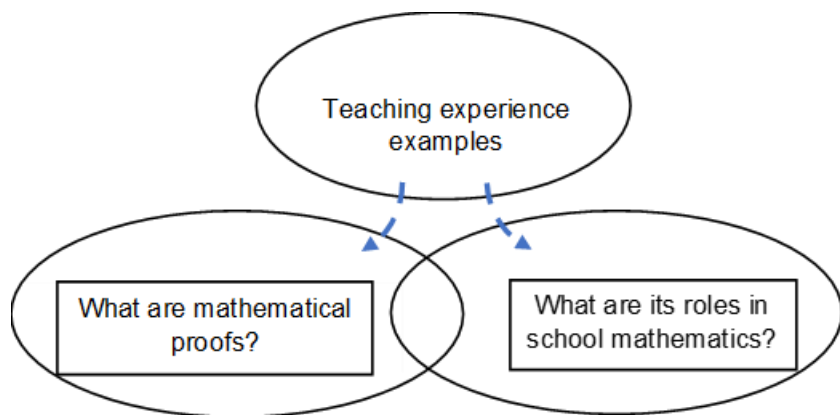


Figure 2. A visual representation of participants’ responses. Sometimes the participants would talk directly about one of the two parts of the research question, but other times they would talk about their teaching experience by providing an example. The connection between the teaching examples and the research question would have to be interpreted through beliefs, authority, and roles of proof.

7.1 Conceptions of Proofs

While the conceptions of proofs held by the participants varied when compared with each other in terms of differences in values, all of the conceptions had pedagogical considerations. The first theme highlighted that the participants saw a difference between mathematical proofs, as described and used by mathematicians, and Reasoning and Proving, which is rooted in the context of school mathematics. Subsequently, the way participants talked about proofs, and Reasoning and Proving, in the school context often took into account the classroom setting and the students’ cognitive abilities. This picture is similar to the way Cabassut et al. (2012) described that mathematics educators “try to devise a learning path from a cognitive state in which an individual learner is able to construct argumentations with some deductive components to a state in which the learner manages to understand and develop mathematical proofs in their proper sense” (Cabassut et al., 2012, 171). Cabassut et al. (2012) saw this learning path as a common theme with the conceptions of mathematical proofs among mathematics educators. In our case, the extent of what is considered ‘proper’ varied. For example, Olivia established a learning path through lecturing and directing students’ attention to the conventions and method of proving trigonometric identities. What is proper to her may be for students to construct trigonometric proofs through arranging logic in a particular format or structure. It may also be to enable her students to partake in argumentation, using the conventions taught, when working out

different examples and difficulties with their peers. Since Olivia brought up post-high school preparations, for her students and from her personal experience, in both her interview and in the focus group discussion, it is likely that for her, proper can also refer to a form of proofs that is more formal than some of the other participants.

Another instance of this learning path was seen in Stella's description of teaching measurement. By starting from tangible examples and scaffolding students' thinking into recognizing a pattern, she helped students construct the argument for a more generalized rule. In this case, proper may be to base arguments on observed patterns on something general such as the formula for the volume of a rectangular prism. Finally, although Mark did not mention proofs when he described his teaching, his strategy of having students engage in argumentation with very little teacher input had a similar effect. By having students work through many examples collectively there was opportunity for discussion and argumentation among themselves. What is proper could be based on the process of students convincing one another, until a consensus on the observed pattern was reached. Through these examples, the way the participants conceptualized proofs can be seen in what they shared. For some, it was easy to see the pedagogical considerations in terms of cognitive development of reasoning and logic. And for others, the value was placed on providing students with the appropriate tools and format to do mathematics.

7.1.1 Conceptions and Definition of Mathematical Proofs

Just like the way mathematical proofs were described in three levels in the Introduction, the way the participants described mathematical proofs also had three levels. *Formal proofs* were attributed to participants' university and post-high school experience of mathematics. Olivia and Stella described these proofs as having moved away from numbers and cases to have generalization across n th dimensions with rigorous logical structure, and symbolism. All participants agreed that those types of proofs are absent in school mathematics. Instead, the most prevalent type of proofs in school mathematics from the participants' examples are *informal proofs*. Mathematically, informal proofs are not really proofs as they rely on the use of empirical cases for the basis of validity (Stylianides & Stylianides, 2009). In my study, the large majority of teaching examples shared by participants were in the nature of informal proofs. These examples involved an empirical approach, having students work through many cases, until a pattern emerged, and an argument can be made. Stella's and Mark's examples from the previous

paragraph illustrate this empirical approach but Dorothy's use of manipulatives also resembled an empirical approach. She mentioned having students quantify, using smaller counting objects (e.g. unit cubes, sticky-notes), and understand the concepts of volume and of surface area. The counting units may serve as a tangible tool for students to create their own cases, albeit in a more experiential way. For instance, using sticky-notes to determine the surface area for different soft drink cases, or using dry beans to compare the volumes of different shapes.

Finally, *semi-formal proofs* emerged through instances where participants' descriptions included the presence of some form of generalization or of specific structure. In particular, generalization and structure emerged most prominently when participants talked about trigonometric identities. They would often mention the use of some kind of procedure, or prescribed process, and the importance of the understanding of equivalence. For instance, the introduction of proper procedures to students was used by Dorothy and by Olivia to detach students from thinking of trigonometric identities as being similar to solving (trigonometric) equations for an unknown variable. Also, the importance of understanding equivalence was mentioned by Stella when referring to transformations of functions as well as to trigonometric identities. The notion was to have students see past specific cases or values and see equivalence across all cases. By shifting students' thinking towards approaches that are generalized and structured, the participants are essentially moving towards a more formalized form of proofs. The degree of formality would depend on the extent of a participant's use, and acceptance, of mathematical language. An example of what the use of mathematical language looks like can be seen from the teachers in Dickerson & Doerr (2014). However, since the use of mathematical language was not the focus of this study, the overall categorization would still be within the umbrella of semi-formal proofs, separate from participants' experience of formal proofs from their university mathematics.

7.1.2 Conceptions and Curriculum

Conceptions of proofs also emerged when participants talked about the Curriculum, and specifically, when the use of proof-related words were used. Through the comparison of previous Ontario Curricula, the participants seemed to have shifted their perception from proofs being in the content of mathematics to being in the processes of doing mathematics. The shift allows for more emphasis on developing thinking and reasoning skills, which, as Dickerson & Doerr (2014) had pointed out, students can apply beyond school mathematics. For example,

Dorothy's emphasis on organization in order to communicate understanding can be related to how formal proofs possess structure to convey its argument. Organization based on logical structure can also be an applicable skill outside mathematics, particularly when one needs to present arguments based on logic. But in the context of mathematical arguments or generalized statements, the word structure was used to describe the organization or arrangement of ideas.

Another example of proof-related words is in Stella's goal of generalization through her use of scaffolding in her measurement example. Her descriptions leaned towards probing the students into seeing a need for a generalized rule instead of relying on testing different cases. Although students may not always be required to create a generalizable rule across all possible cases in their daily lives, recognizing and characterizing a pattern may be a useful life skill. Interestingly, Stella's described teaching process bore similarities to a simplified version of the instructional process from Stylianides & Stylianides (2009), where instructional scaffolding was also used. In both situations, the students were prompted into recognizing a need for a deductive (generalized) approach.

Being possibly the last remaining area in the current Curriculum to contain proofs, trigonometric identities was a topic where the concept of equivalence emerged. The difference between equivalence and equal was something that the participants saw a lot of their students struggle with. And in response, Stella talked about equivalence through the idea of transforming the sine function to be equivalent to the cosine function. Furthermore, Mark connected equivalence to representation by using graphing as a practical way to check if a trigonometric statement was an identity. The inclusion of a visual representation was described in Dreyfus, Nardi & Leikin (2012) to be something mathematicians include in their own reasoning and analytical processes. They specifically mentioned that, "illustrations and gestures can be closer to the cognitive processes students need to carry out in order to develop understanding" (p. 194). Graphing may be a way to help students form familiarity and connection with the other types of functions they have worked with (and graphed) throughout high school mathematics. Although showing students a mathematical concept through multiple representations is another of the seven Process Expectations, it is being used to support the understanding of equivalence, a proof-related word.

Ultimately the presence of a curriculum with less proof content allows for emphasis to be placed on the proof-related skills that can be applied beyond school mathematics. Yet it can be

interesting to note that the way the participants reflected and discussed the past and current curricula may be a result of maturing as teachers. They may have developed a better understanding of the underlying concepts and skills, and their nuances, as outlined in the curriculum. But as some of them mentioned, the reality is that most students will not encounter mathematical proofs in the formal form, but rather the proof-related skills that they have learned will be applied elsewhere in their lives.

7.1.3 Conceptions and the Influence of Social Factors

The influence of school context on the participants' conceptions of proofs can be seen when social factors were brought up by Mark in his social experiment. Social factors add an additional dimension to mathematical learning such that learning occurs not just through a process of active construction, but also through "a process of acculturation into the mathematical practices of wider society" (Yackel & Cobb, 1996, p. 460). Aside from mentioning that the students were in Grade 9, Mark did not mention when or how the experiment was carried out. However, the way the students behaved, by congregating with the perceived 'smart guy', showed a social norm that was not mathematical, and was instead based on some sort of hierarchal social structure. The students relied on the 'smart guy' as an external authority rather than draw on their own mathematical abilities. This outcome is similar to the way Yackel & Cobb (1996) described intellectual autonomy.

Students who are intellectually autonomous in mathematics are aware of, and draw on, their own intellectual capabilities when making mathematical decisions and judgments as they participate in these practices (Kamii, 1985). These students can be contrasted with those who are intellectually heteronomous and who rely on the pronouncements of an authority to know how to act appropriately. (p. 473)

Even though the correct answer may have been achievable from the students' mathematical abilities, they instead chose to rely on the 'smart guy'. The way Mark described his teaching approach in his interview almost seemed like a response to the outcome of his social experiment. By having students work through examples in a collaborative manner, and distancing himself, Mark gave the impression of a sociomathematical norm in which the students drive mathematical discussions through sharing ideas. Instead of relying on an external source of authority, the students would be expected to learn mathematical relationships by thinking for themselves. In his interview, Mark also mentioned that he would go back to the group exercises whenever students got stuck or forgot a concept. By going back to group exercises, Mark would reinforce the

notion that overcoming difficulties in mathematics involved looking back at prior knowledge and drawing on the students' own intellectual capabilities. In summary, Mark demonstrated an awareness of the possible influences the social and sociomathematical norms of his classroom have on his students' learning of mathematics.

On the other hand, a different sociomathematical norm can be seen in how Olivia teaches her students how to prove trigonometric identities. Instead of distancing herself and encouraging students to think through and come to a consensus on their own, she directed their attention right from the beginning. She demonstrated her role as a representative of the mathematical community by showing her students what is an acceptable way to prove an identity, thus establishing a different sociomathematical norm. Her students would then engage in collaborative group work to explore the more complicated and difficult problems under the knowledge of what is acceptable. Olivia's descriptions gave rise to a set of sociomathematical norms that were different from what arose from Mark's descriptions, which is to be expected. In fact, social factors are also present in the academic mathematical community. De Villiers (1990) pointed out that mathematicians often have to consider the audience that receives their proofs, and typically only publish "those parts of their arguments which they deem important for the sake of conviction," (p. 19). Likewise, as mathematics representatives, teachers may influence the type of sociomathematical norms their students are exposed to. However, just as norms differ across participants and their examples, they may also differ across examples of the same participant, merely due to a difference in content or a difference in audience. Olivia may see trigonometric identities as an area where she needed to take more of a lead compared to a different topic in the course. And Mark may encounter a class of students who are not as willing to engage in discussion as his other classes. Sociomathematical norms may also differ between courses, track levels, and Grades, as teachers would have a different level of expectation from their students.

Aside from sociomathematical norms, Stella brought up a different social consideration. In her interview, Stella briefly pondered the possible influence cognitive and social development of maturing teenagers might have on their ability to use reasoning and logic. Her mention of *accessible* learning touched on a point of reaching students where they are at, developmentally. She described creating a learning environment where students are used to collaborative group work with whiteboards. For her, this group work set up was a way for students to work through

many examples and see an emerging pattern, and possibly a need for a generalizable rule. It is similar to how Shilling-Traina & Stylianides (2013) described a suitable environment for proofs to be a necessary component for their participants to engage in proof-like activities. Like Mark, Stella mentioned that one of the main reasons she does very little direct teaching is because she recognized the developmental gap between her students and herself. Her students do not have the level of mathematical thinking that she does, from experience as a mathematician and as a mathematics teacher. And by providing opportunities for peer group work, her students can work with each other, who have similar levels of cognitive development as themselves. Overall, the learning mathematics in the school context contains social and developmental factors that teachers need to consider in addition to their conceptions of proofs.

7.1.4 Conceptions and Mathematical Beliefs

The connection between a teacher's mathematical beliefs and their experiences as mathematics students was mentioned in literature (e.g. Ernest (1989) and Shilling-Traina & Stylianides (2013)) and can also be seen in a few of the participants' discussions. For example, Olivia described experiencing a gap, as a student, moving from high school mathematics to university mathematics. And during the focus group discussion, she referenced her past experience when identifying a need to engage high school students in more proofs and ways of thinking about mathematics that is similar to the university context. Alternatively, at the end of her interview, Stella described how she is getting a better sense of what it means to be a mathematics teacher. In particular, she described learning the difference in thinking about mathematics as a mathematician to thinking about it as a high school teacher. For her, the underlying processes and skills, which she values now as a teacher, were not so apparent when she was learning mathematics as a student. For some participants, the influence of their past mathematical experiences on their current beliefs are less subtle than others.

The way Olivia described engaging students in proofs (trigonometric identities) and Reasoning and Proving resembled a Platonic approach to mathematics. She would describe using proof like language with her Grade 9 and Grade 10 students when they verified the properties of triangle and quadrilaterals. Furthermore, she mentioned introducing and facilitating the idea of a counterexample in Grade 9. Many of her descriptions involved her delivering the mathematical skills and concepts for her students rather than have them figure it out for themselves. She also used the words *direct* and *lecture* at certain times to emphasize the

importance of communicating a particular way of thinking or doing mathematics, which resembles an instrumentalist approach. For example, in the first task-prompt, she mentioned that she would lecture proof 1 to her students because proofs 2 and 3 were not deemed to be fully acceptable for her. Olivia was also the one of the participants who pointed out the decrease in the amount of mathematical proofs over each Curriculum revision. It is possible that the reasons for Olivia's examples to exhibit a Platonic and an instrumentalist approach is due to her experiences as a student and in managing the gap between high school and university mathematics.

The way Dorothy and Stella described their teaching also resembled a Platonic approach to mathematics. They both described facilitating students' learning through scaffolding towards a desired goal. For Dorothy, she approached teaching Grade 11 students the process of proving trigonometric identities by facilitating their prior knowledge of triangle relations and primary trigonometric ratios. Similarly, Stella described returning to, and building from, the concept of equivalence through the use of concrete examples. Both participants built and progressed their students' understandings towards an intended goal. However, unlike Olivia's examples, the examples shared by Dorothy and by Stella did not contain the same vigour for having students do mathematics in a particular, "correct", way. The reason may be due to Dorothy and Stella progressing students' understanding towards forming the appropriate mathematical conventions whereas Olivia starts by setting the conventions, deemed necessary, in place. The use of group work by Stella and by Dorothy were described in a more investigative way, with more student exploration, rather than in a practice or consolidatory way, where students refine their understanding. The presence of an investigative notion bears some resemblance to an experimentalist approach as the students are actively creating mathematical thoughts and ideas, followed by examining and revising those ideas. However, those group work examples also contained teacher facilitation similar to when Dorothy and Stella talked about teaching trigonometric identities.

The way Mark described his teaching was different than the others and held some resemblance to an experimentalist approach to mathematics. Unlike the others, he described using a more hands-off approach, where students learn mathematics through group work and discussing to reach a consensus. In his interview, Mark mentioned having a more teacher-directed approach in the past, but he surrendered the approach in favour of using the social environment of the classroom as part of his teaching method. Mark's use of group collaborative

work was different from the other participants in that he described maintaining a distance away from students' thinking. He described how he did not need to interject on his students' work as they are able to look at the work of their peers around them to self-evaluate their own approaches. Thus, his students practice the roles of creating, assessing, and revising their mathematical thinking, which are characteristics of an experimentalist approach to mathematics.

Since mathematical beliefs were compared based on what the participants shared, it is likely for everyone to exhibit a completely different approach when the examples are viewed in real time, in the classroom. For instance, Olivia's apparent use of lecturing and directed instruction may actually be seen by her students as clarifying the mathematical parameters for them to engage in mathematical investigation in an experimental way. Likewise, in the eyes of his students, or of an outside observer, Mark may actually be interjecting more than he described when facilitating his students' mathematical discussions. And finally, the participants may use a different teaching approach when enacting other Process Expectations, such as *Problem Solving* or *Selecting Tools and Computational Strategies*, and for other mathematical content outside of proofs.

7.2 Role of Proofs in School Mathematics

The following sub-sections make the connections between the roles of proofs, as defined earlier in the theoretical framework, with their presence in the research data. Since the five roles were not made explicit to the participants during data collection, the following sub-sections will highlight some of the characteristics from participants' data that fit each of the five roles. However, since there were often overlaps, where one example contained aspects from more than a single role, general comparisons were made when relating proofs to one of the five roles. Additionally, smaller roles, which were referred to only once or twice, were categorized more generally in one of the five roles. For example, representing mathematical thinking would fall under communication, but may also have aspects of explanation or developing thinking and logic skills.

7.2.1 Proofs and Verification

Proofs as a means to verify a mathematical claim was mentioned to be a major characteristic of mathematical proofs in the mathematical sense. However, everyone agreed that verification was not the only role for proofs. In the school context, verifying mathematical

claims typically referred to verifying trigonometric identities. Some participants talked about giving students trigonometric statements without knowledge of whether they were true across all cases; if they were identities. The students would then have to verify whether the statements were identities or provide a counterexample to show otherwise. The way participants talked about proof representation pushed the boundary of what is considered an acceptable verification (or counterexample). For example, Mark and Stella commented on the use of graphing calculators, and on the graphical representation in counterproof 2 from the second task prompt. They saw graphing as a way for students to explore and visualize the verification of a statement. Whether they accepted a visual proof was not explored, but graphing was seen as a complementary tool to help students get a sense of what their mathematical statement meant. The mathematical statements could be a trigonometric relationship, but it could also be a polynomial inequality or a transformation of functions, all of which can be verified graphically.

Aside from verifying trigonometric identities, verification of a mathematical claim can also be seen when participants talked about the use of a generalized rule, such as a formula. In Stella's measurement example, the observed pattern of the volume of a rectangular prism increasing by a factor of 8 can be verified by going back to the formula for volume. Likewise, Dorothy described having students form the association that right-angled triangles have a special relationship based on their lengths. And this association, the Pythagorean Theorem, holds true for all right-angled triangles, making it possible to use the Theorem as a way to verify a triangle as having a right angle. Although the methods are usually empirical, the creation of generalized rules from an emerging pattern can be an example of proofs as a means to verify mathematical claims.

7.2.2 Proofs and Explanation

Proofs as a means to explain a mathematical claim or relationship appeared mostly in instances where the participants focused on deepening students' understanding of *how* a theorem or formula came to be. Knuth (2002) described something similar from his own participants, "The focus is not so much on an argument's illumination of the underlying mathematical concepts which determine why a statement is true as much as it is on showing how a statement came to be true" (p. 80). In my study, the main instances of proofs as a means to explain came about when Olivia and Dorothy talked about the Pythagorean Theorem. They described representing the Pythagorean Theorem through different ways to help students understand a bit

about the inner workings of the Theorem. In discussing mathematical proofs, Olivia shared a picture that consisted of two images where students can visually construct the Pythagorean Theorem from the geometric relationships present. In Olivia's case, the students would access the Theorem by navigating through geometry. The students would construct the Theorem for themselves, thus gaining an understanding how the Theorem came to be. On the other hand, Dorothy's video, described in 5.C.4, involved an explanation based on the squares of the sides. She used this interactive proof to show students why the Pythagorean Theorem contained symbols that were squared. In both cases, the participants described using other representations as a way to help students deepen their understanding of the Pythagorean Theorem by understanding how it works.

7.2.3 Proofs and Communication

Proofs as a means to communicate a mathematical claim emerged based on the social dimension of argumentation and corresponding with peers. Like de Villiers' (1990) description of proofs as an interchange of mathematical ideas between mathematicians, an interchange of mathematical ideas among the students was also present from participants' descriptions. For example, Dorothy described emphasizing to her Grade 9 and Grade 10 students the importance of showing their work when answering mathematical questions as a form of communicating their understanding. This association of communication is pedagogically based because a student's work can be an insight into their thought processes. Dorothy connected showing one's work with the notion of structure and organization in one's thinking. In essence, Dorothy is having her students develop their logical thinking skills which also lead into proofs, in the form of trigonometric identities later in high school. By showing their work and developing written structure, the students are practicing the interchange of mathematical ideas with Dorothy, their teacher.

The interchange of mathematical ideas also emerged in the context of collaborative group work. For example, Stella described learning in her class as a general process where students work on an example individually before sharing their results with their peers. She intended for the act of sharing to allow students an opportunity to assess each other's work and reflect on their own mathematical thinking. And ultimately, for students to be exposed to many worked examples rather than working through the examples themselves. The exposure to many worked examples would then allow for patterns and ideas to emerge, possibly leading to a generalization.

Stella's connection to communication starts from the individual and expands to the whole class realizing the presence of a pattern.

Communication in a group setting also includes social factors, as students interact with each other. For instance, the two examples from the discussion on sociomathematical norms, Olivia and Mark, also demonstrated the role of proofs as a way to communicate with peers. In Olivia's case, the communication aspect of proofs came after establishing the sociomathematical norm of what is an acceptable proof of a trigonometric identity. Her descriptions of leaving her students to work on problems of varying levels of complexity and difficulty places them in an environment where they are able to convince each other of the validity of their work. Furthermore, the students can be making their arguments, and ideas, mathematical by following the sociomathematical norms that were established. In Mark's case, his students were able to demonstrate more social relations than Olivia's students through his sociomathematical norms of doing mathematics. Knuth (2002) argued that the social nature of proofs is lacking in school mathematics when the mathematical authority for accepting a proof lies only on the teacher or the textbook, rather than through social means. However, Mark's decision to distance himself from his students' argumentation allows them to bear more mathematical authority. Bearing this authority can allow the students to engage doing mathematics in a way that is similar to mathematicians. The product of the students' argumentation is a consensus that is accepted by everyone, just as the product of mathematical argumentation becomes accepted as a proof once the community of mathematicians accepts it as such. Therefore, proofs and argumentation as a way to interchanging mathematical ideas seemed to emerge naturally through group work when students are given the opportunity to interact and share ideas.

7.2.4 Proofs and Discovery

Proofs as a means to discover new knowledge emerged whenever participants talked about developing deeper understanding of a mathematical concept. In fact, nearly everything the participants shared that was related to teaching methods or teaching experiences can be linked to students learning new mathematical knowledge. It is important to note two definitions, the participants associated proofs in the school context to Reasoning and Proving (i.e. proofs include reasoning), and what is considered new is in terms of the students' knowledge of mathematics. The participants often talked about using reasoning to bridge between students' prior knowledge of mathematics and the new mathematical concept being introduced in the classroom. And often

times, the use of reasoning was encouraged for the purposes of creating knowledge that was meaningful to the student (e.g. help better understand a concept). This act of creating knowledge can be based on group work and student-based argumentation, as the case with Mark. It can also be through a more teacher-guided approach such as reasoning through pattern recognition to determine a need for a generalizable rule, seen with Stella. And it can also be a directly teacher facilitated, such as Dorothy questioning students on the mathematical rationale for their answers, or Olivia's students using her conventions to approach and verify more difficult and complicated trigonometric identities. Every one of the participants' examples can be interpreted to contain elements of discovering new mathematical knowledge, possibly because the participants are mathematics teachers whose role is to facilitate learning mathematics. Ultimately, the creation of new mathematical knowledge is arguably a prevalent aspect of mathematics in the school setting. And with the current Ontario Mathematics Curriculum mandating the integration of the seven Process Expectations across all courses and grades, Reasoning and Proving would be naturally linked with discovering new knowledge.

7.2.5 Proofs and Developing Thinking and Logic Skills

Proofs as a means to develop thinking and logic skills seemed to be a common association due to participants associating mathematical proofs with Reasoning and Proving. Just like the teacher participants from Knuth (2002), my participants also talked about developing thinking and logical skills for application in the real world. They talked about thinking and logical skills in terms of reasoning and sense-making. And all of them mentioned that most of their students will likely not encounter proofs, in their mathematical form, past high school. Instead, the students will carry the mathematical thinking associated with doing proofs, which includes (but are not limited to) thinking, reasoning, and sense-making, onto their next stage in life. Thus, the participants described spending effort in developing these enduring skills, alongside mathematical content. In a way, reasoning and sense-making appeared to be an emphasis when participants talked about Reasoning and Proving.

The way the participants described reasoning, sense-making, and forming an *internal dialogue* brings up the idea of developing students to become independent thinkers. The notion of the *internal dialogue* cannot exist if students are reliant on an external source for mathematical conduct, in other words, being unable to think about mathematics on their own. In fact, Knuth (2002) interpreted that, for his teachers, "proof provides students with an opportunity

to become mathematically independent thinkers” (p. 81). He referred to the process of proofs being an opportunity for students to generate and validate conjectures (both their own and their peer’s) based on mathematical thinking. In my study, the participants saw the use of reasoning and sense-making as part of students’ process of validating their thinking. Ultimately, by developing and encouraging students to use thinking and reasoning, the participants are encouraging the development of independent mathematical thinking.

7.2.6 Summary

In summary, *Verification* emerged as being described as the most mathematical of the five roles as it was brought up and commented on in the focus group discussion. Aside from when defining proofs, verification was characterized the most when participants talked about trigonometric identities. It was also characterized through the use of a generalized rule/formula, since both identities and generalization were seen by the participants to resemble mathematical proofs the most. *Explanation* emerged when participants talked about helping students’ understanding of how a mathematical concept worked. It was characterized by the participants’ mention of using multiple representations to demonstrate a concept, Theorem, or formula. *Communication* emerged when participants talked about the Reasoning and Proving through the use of group collaborative learning. It was characterized through the types of sociomathematical norms that were related to argumentation. It was also characterized through the way Dorothy described students communicating understanding, and through the way Stella described the process of pattern recognition. *Discovery* emerged when participants talked about students generating new mathematical knowledge, and it was characterized everywhere, whenever participants talked about their teaching. Finally, *development of thinking and logic skills* emerged as reasoning and sense-making. It was characterized by the participants’ mention of developing of enduring life skills and independent mathematical thinking.

7.3 Mathematical Authority

When talking about their conceptions of proof or about their teaching experiences, the examples and opinions that participants provided contained small snippets of mathematical authority. These snippets also revealed possible connections between the roles of proof and the participants’ conceptions of proof (how the two components of the research question inform each other). The participants all talked about collaborative group work at some point, but each in their

own ways. From these descriptions the presence of mathematical authority can be characterized and described. However, it is important to note that the mathematical authority described in this section are in terms of the participants' perspective and are based on the examples they chose to share. Therefore, mathematical authority is discussed in terms of its presence in what the participants have shared.

The way group work was described by each participant brought out a different manifestation of mathematical authority. For example, some of Dorothy's use of hands-on activities allowed for students to take on some mathematical authority. The example where students used sticky notes to model surface area on classroom objects gave the opportunity for students to choose the objects and create a mathematical representation that they would then share with their peers. The opportunity for exercising additional roles is similar to the students from the second teacher in Steele & Rogers (2012). If Dorothy chose the objects or stuck the sticky notes herself, she would limit her students to fewer roles, and also place more emphasis on herself; what she modelled. Likewise, Stella's description of facilitating student learning also had the possibility of sharing mathematical authority with her students. When elaborating on the idea of accessible learning, she included having students share their worked examples with each other and encouraged collaboration. For collaboration in a group to occur, the individuals need to have some sort of trust or recognition of each other. And this recognition can be where mathematical authority emerges. Ultimately, Dorothy's descriptions and Stella's descriptions both contained less obvious instances of mathematical authority being shared. Olivia and Mark, on the other hand, provided examples where mathematical authority was easier to see.

The way Olivia described teaching trigonometric identities revealed a glimpse of where mathematical authority resides in her classroom. As a representative of mathematics, Olivia directed the sociomathematical norm of doing trigonometric identities by demonstrating to her students on the whiteboard. In the individual interview, she described modelling a few identity proofs and mentioned drawing a thick red line between the left and right sides because she anticipated students mixing the sides. Although her students are given the opportunity to work through a worksheet of identities through group work, the basis for what is considered an acceptable proof had been established by their teacher, Olivia. Therefore, in this case, mathematical authority resided with Olivia and not with the students.

The way Mark described enacting Reasoning and Proving involved having his students take on some mathematical authority. Although Mark did not talk about teaching trigonometric identities, he described observing that students' peers often have a greater impact, than himself, in attaining each other's attention. And subsequently, they can have a greater impact on their learning of mathematics. Rather than fight for their attention, Mark shared mathematical authority with his students by encouraging them to discuss their work and share their mathematical thinking. Like the students of the experienced teacher from Steel & Rogers (2012), the sharing of authority grants the students access to several roles such as creator, validator, observer, and explainer. Through those roles, students are able to exercise a degree of autonomy in determining and understanding the underlying mathematical relationships for themselves.

Just like mathematical beliefs, the emergence of mathematical authority through the participants' examples and descriptions of their teaching is subjective to interpretation. For the purposes of this study, looking at mathematical authority helped provide some basis for connecting participants' conceptions of proofs to their role in school mathematics. For example, Stella's instructional process where students work individually before working with peers connected the role of communication with her perspective of school mathematics involving an investigative process. Her belief that instruction needs to be accessible to everyone, and that proofs involved a process of generalization colored communication through the context of group work and collaboration. Through group work, the students would be exposed to their peer's ways of thinking and understanding which may be different than interacting with the teacher, who thinks with more mathematical experience. In another case, the role of communication was colored differently in Olivia's examples. Her perspective of a gap between high school level and university level mathematics placed authority on herself as the teacher, and representative of the domain of mathematics. Her beliefs led to a focus on communication in terms of using proper mathematical conventions and representations to share one's thinking. Overall, the mathematical authority lens helped situate and understand the nuances between the participants' conceptions and the way they attribute to be the role of proofs. However, since all the participants' teaching examples were anecdotal, recollections, or hypotheticals, exploring mathematical authority would require a more in-depth examination of the actual teaching practices of the participants.

7.4 Putting it Together

Overall the participants conceptualized mathematical proofs through its underlying mathematical processes. The participants talked about mathematical proofs in terms of argumentation, thinking, and logic. They also often referred to their teaching experiences and described how those experiences has informed their current views of mathematical proofs. To answer the research question, the roles that the participants attribute to the purpose of learning mathematical proofs in high school vary depending on what they believe to be important for students to carry forward, past their high school mathematics courses. Among other things such as preparation for post-secondary mathematics, the participants all placed importance on the development of reasoning and logic skills along with the ability to reflect on whether something makes sense logically. The purpose for mathematical proofs also seemed to be to contain an aspect of providing students with the skills and abilities to think about problems in the real world in a mathematical way, while also being able to assess their own logic without the need for an external source.

8. CONCLUSION

The connection between teachers' conceptions and their teaching, as noted in Ernest (1989), was observed in the participants through the ways their conceptions of mathematical proofs related to the purpose they saw for teaching proofs. Overall, the participants seemed to view proofs in terms of its related skills and characteristics. There are two possible explanations for this view. The first point that the participants saw school mathematics as being more investigative and example based whereas mathematics in university would be more abstract and generalized through deductive means (*formal proofs*). This distinction can be seen in the way the participants talked about mathematical proofs, This group of participants all have university degrees in mathematics, which made it possible for them to make the comparison between the two settings. Although there may be instances for students to reach a generalized rule, the process is usually facilitated and using concrete examples (*informal proofs*). The second point is that the participants have identified that most of their students may not necessarily encounter formal mathematical proofs after high school (i.e. not everyone will be taking mathematics courses, especially with proofs, in university). Instead, students will more often be using the mathematical skills that they have developed through doing school mathematics. And as a result, the participants described mathematical proofs through the lens of the Mathematical Process of Reasoning and Proving.

The roles that the participants attributed to mathematical proofs were based through their view of Reasoning and Proving, in addition to their beliefs. Reasoning was viewed as a forefront in doing mathematics and it was widely associated with using logic and sense-making to understand a problem or to check one's thinking (and answer). Proving was seen when the reasoning was accompanied by a particular organizational structure and associated with a generalized statement. Proving was often associated with trigonometric identities and other examples that involved the idea of equivalence. Overall these skills were seen to develop students' sense of logical reasoning, and of communicating their thinking, when presented with an argument or a problem. Aside from their views of Reasoning and Proving, the participants' own beliefs, accumulated from their past experiences, also influenced what they saw to be the purpose for mathematical proofs in school mathematics. An example was seen in the discussion about the past mathematics curriculum where one participant saw the value in the current, reformed, curriculum while another saw the decrease in the emphasis on proofs as a possible

issue. These differences appeared to inform the way they talked about teaching mathematical proofs, and to the roles that they attributed.

8.1 Implications of this Study

This study has revealed that the teacher participants place importance on incorporating Mathematical Processes in their teaching. The participants saw value in the current Curriculum despite commenting on the decrease emphasis on mathematical proofs. The value they saw came through the notion of having students leave their class at the end of the school year being able to think and reason logically. The nature of mathematics explored in the focus group discussion also gave rise to the idea that argumentation in mathematics involved logic and structure, which may not be the way argumentation manifests in other school subjects. This group of participants valued underlying mathematical processes over mathematical content since in their eyes, mathematical content can be recalled more easily with well developed thinking and reasoning skills. Perhaps the key to increasing mathematics performance lies in developing those underlying processes and instilling habits of sense-making and internal dialogue.

Some participants felt at ease with the current Curriculum but others felt that more proofs could be incorporated. Olivia's concern of a post-secondary gap originated, and has persisted, from her own experiences as a student. Those experiences appeared to influence the way she talked about her conceptions of proofs and the way she would teach proofs. While Olivia's personal experiences and concerns are all legitimate, can influence her teaching practice, they may also be used as a source of reflection and professional development. In this study, several participants commented about how they enjoyed spending some time to reflect on their beliefs and on their teaching practice as a result of being part of a research study. And as the focus group discussion progressed, the participants seem to find a sense of harmony as their concerns are addressed through discussion. Perhaps professional development can incorporate more opportunities for colleagues to enter into discussion about the perspectives of the topics they are teaching.

Trigonometric identities were mentioned as an area of high school mathematics which contained many aspects of mathematical proofs. However, there were other examples of concepts that can be made to resemble a proof, albeit in an informal way. Stella's measurement example was an example of how investigating a property through examples led to observing a pattern and its eventual generalization. Her example gives rise to the idea that one can simply

expose students to aspects of proofs by adjust the way a concept is presented. She reflected this idea in her comments on the first prompt task and how the language of the claim was very “proofy”. Perhaps by nuancing the presentation of mathematical concepts, teacher may develop students’ mathematical skills in different ways. And thus, a single concept can be presented in different ways such that each of the seven Mathematical Processes can be incorporated. The implications of this freedom to manipulate the presentation of concepts allows teachers to choose how to present mathematics to their students.

8.2 Looking Forward

In talking about how they would teach mathematical proofs, the participants revealed their beliefs, values, and gave some indication on how mathematical authority would look in their class. For this study, the concepts of beliefs and mathematical authority helped link participants’ teaching examples with their conceptions of proofs and with what they saw to be the roles of proofs. However, since this study focused only on the participants’ perspectives, it did not include alternative perspectives such as a researcher’s perspective or the participants’ students’ perspectives. It may be interesting to observe the participants (and their classes) teach a few lessons involving mathematical proof or include the students in the data collection process. Through these other perspectives, the sociomathematical norms would be visible and it can better inform the location of mathematical authority in addition to the connection between mathematical beliefs and teaching practice.

The roles of proofs in this study were categorized under five umbrella sections. However, with the differences in beliefs and values in the participants, it may be interesting to observe what kinds of subgroups exist within one or two of the roles. For example, proofs as a way to discover new mathematics was used in this study to describe any moment students learned something new. What is considered new may be subjective to the individual to include realizing a connection between the current problem and an existing concept, (e.g. using quadratic factoring to help verify a trigonometric identity). Also, the different roles may have some overlap with each other depending on the interpretation of the proof. For example, a teacher may present a proof with the intention to explain a concept but the students saw the proof as a way to organize mathematical thinking. Perhaps there are roles that are emphasized more under the sociomathematical norm present in the classroom. Likewise, perhaps the roles are layered in such a way that there is one role that is always present (e.g. developing mathematical reasoning).

In the end, the way roles exist in the learning of mathematics can be further explored beyond the five used in this study.

It may also be interesting to interview teachers who do not have any experience with mathematical proofs at the university level. In Ontario, teachers may teach a subject in high school once they have accumulated a certain number of university level courses. Thus, it is possible for a teacher to have achieved the required number of mathematics courses but did not take any proof-based or proof heavy courses. Participants who do not have experience with mathematical proofs may also conceptualize proofs in school mathematics through Reasoning and Proving, since it contains the word proving. However, their values may differ in that structure, generalizability, and discussion may not be as much of an emphasis as it was with this group of participants.

8.3 Personal Reflections

As someone who did not experience formal mathematical proofs in university, I was curious as to how mathematics teachers think about non-computational mathematics. This curiosity was the basis for choosing mathematical proofs and school mathematics as the topics for this study. I entered the study viewing mathematics as a school subject that contained computational accuracy, structured communication through symbols, and logic in assessing the reasonableness of an answer. I also saw mathematical proofs solely in terms of trigonometric identities (as I did not have other exposures). In fact, the experience of learning to verify identities in high school was quite memorable as it did not involve numbers, which confused many of my classmates at the time. Even after I understood how to verify trigonometric identities, it was still just an exercise, the concepts of equivalence or generalization were not made apparent. To me, verifying trigonometric identities was a logic exercise involving algebra and substitutions. And, I wondered how teachers who taught mathematics thought of trigonometric identities and mathematical proofs.

When applying Ernest's (1989) belief framework, I see myself as having approached mathematics with a Platonist approach but also with several elements of an instrumentalist approach. And upon reflection, most of my former teachers exhibited similar approaches where the relationships between mathematical concepts were prepackaged and the delivery was mostly teacher directed. After the data collection process and thinking about the emergent themes, I now wonder what it would be like to approach mathematics from an experimentalist approach

and use teaching strategies such as problem-based learning, or student led discussions, which I had not really attributed to school mathematics.

The concepts of mathematical authority and sociomathematical norms were new ideas for me to reflect on my own teaching practice. I was able to relate to Mark in terms of finding difficulty maintaining students' attention during a lesson, and to Olivia in terms of thinking about what students will encounter at the next level of mathematics. And hearing the discussion about curriculum values and the importance of underlying mathematical processes placed a sense of enlightenment. The result of doing this research study has broadened my perspectives on what school mathematics entails while also deepening my understanding of how mathematics teachers view and interpret the Ontario Mathematics Curriculum. In a way, conducting the interviews and being present in the focus group discussion has been a source of professional development for me.

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APPENDIX A: INTERVIEW GUIDE

Warm-up

1. How many years have you been teaching mathematics?
2. Have you taught the Grade 12 Advanced Functions course before? If so, how many times?

Questions

1. What is a “mathematical proof” to you? Could you provide an example to go with your description?
2. Proving and reasoning are part of the 7 Mathematical Processes that are suggested as part of every math course. Do you see proving and reasoning as the same things or are they different?
3. Do you teach mathematical proofs in the Grade 12 Advanced Functions course? If so, at what points in the courses does this occur?
4. How well do you think proofs are represented in the Ontario Curriculum? Why?
 - a. Do you think trigonometric identities is a good way to teach proof? Why?
 - b. Do you think there are other topics or other ways for students to engage in proofs? What is an example?
5. Do you think it is important to do mathematical proofs as part of learning mathematics? When and how do you think they should be taught?
6. How do you approach the teaching of proof (or reasoning) in your classes? Why this approach?
7. Other comments?

APPENDIX B: FOCUS GROUP GUIDE

Warm-up

1. To begin, let us go around and introduce ourselves.
 - What interests me about mathematical proofs as a mathematics teacher?
 - What has drawn me to be part of this research study?

Prompting Questions

1. What does “mathematical proofs” mean to me?
2. What do I see as the purpose of proving or reasoning in school mathematics?

APPENDIX C: TASK-BASED PROMPTS

Prompt 1: Doubling Dimensions and Surface Area

Claim: Doubling each side of a rectangular prism results in a surface area that is 4 times the original.

“Proof” 1:

$$SA_1 = 2lw + 2lh + 2wh$$

$$SA_2 = 2(2l \cdot 2w) + 2(2l \cdot 2h) + 2(2w \cdot 2h)$$

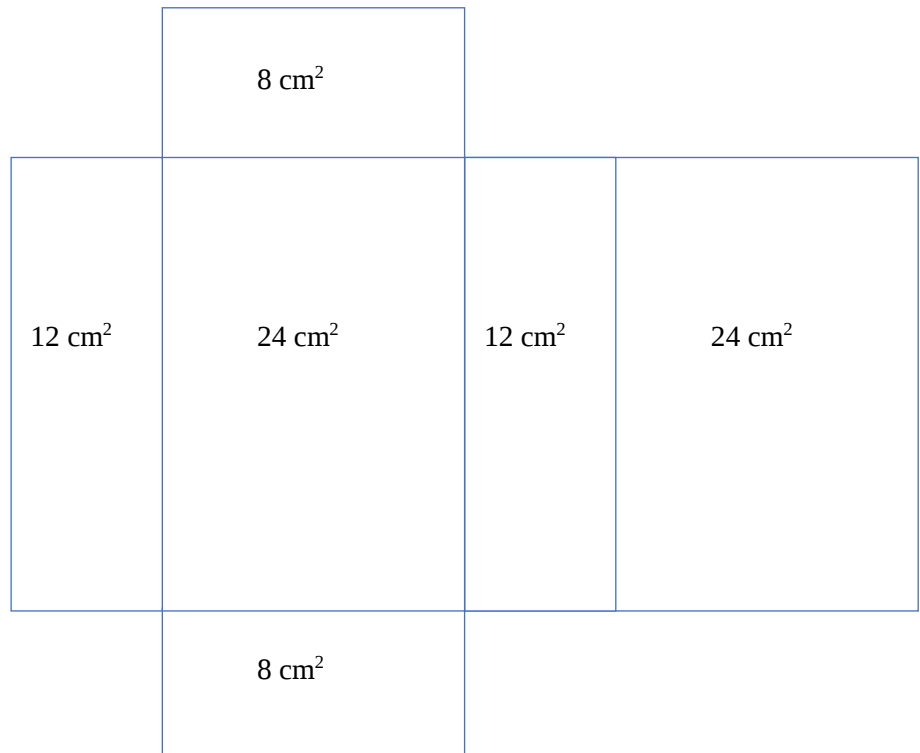
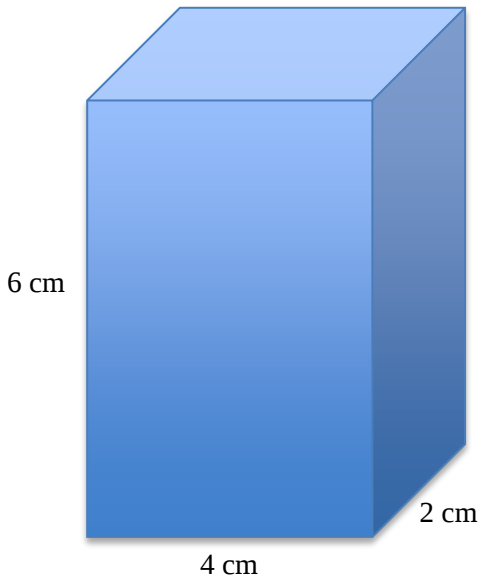
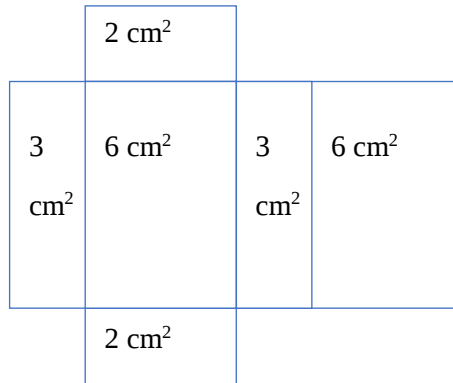
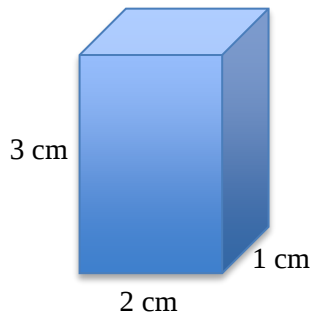
$$SA_2 = 2(4lw + 4lh + 4wh)$$

$$SA_2 = 8lw + 8lh + 8wh$$

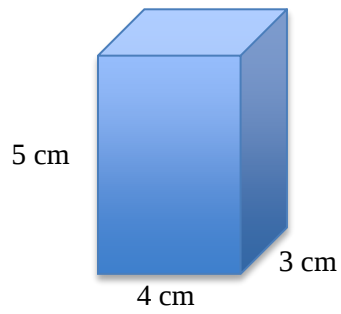
$$SA_2 = 4SA_1$$

$\therefore$ Doubling each side gives a surface area that is 4 times the original

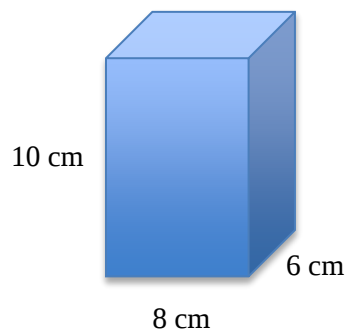
“Proof” 2:



“Proof” 3:



$$\begin{aligned} SA &= 2(lw + lh + wh) \\ &= 2(4 \cdot 3 + 4 \cdot 5 + 3 \cdot 5) \\ &= 2(12 + 20 + 15) \\ &= 2(47) \\ &= 94 \text{ cm}^2 \end{aligned}$$



$$\begin{aligned} SA &= 2(lw + lh + wh) \\ &= 2(8 \cdot 6 + 8 \cdot 10 + 6 \cdot 10) \\ &= 2(48 + 80 + 60) \\ &= 2(188) \\ &= 376 \text{ cm}^2 \end{aligned}$$

Prompt 2: Non-equivalent Trigonometric Statement

Claim: $\cos 2x = 2 \sin x \cos x$ for all values of x

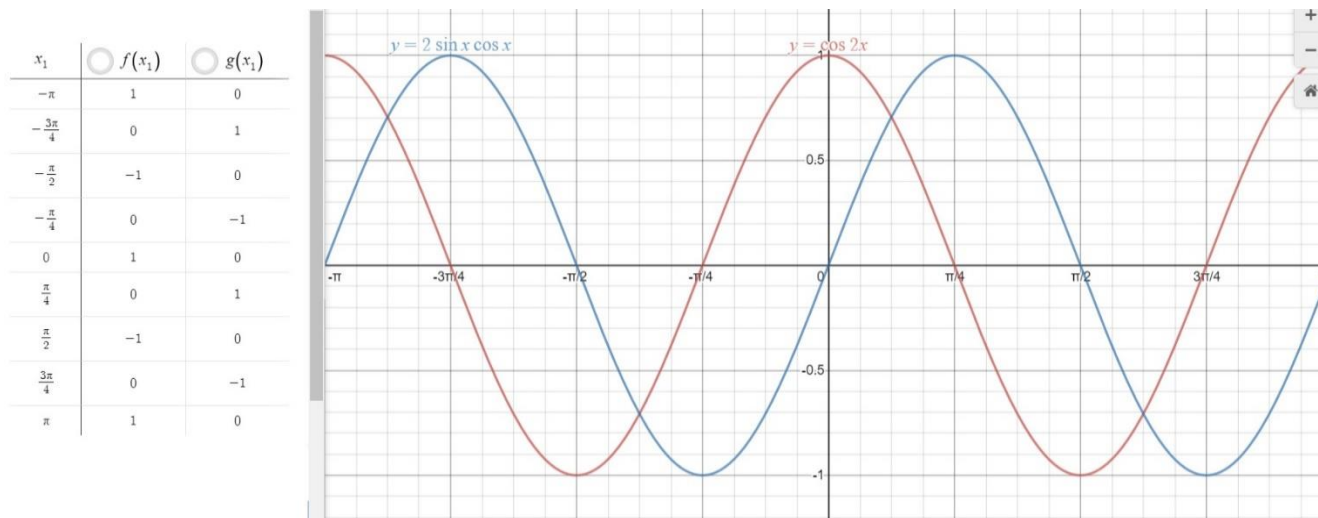
“Counterproof 1”:

$$\text{Let } x = \pi/2$$

$$\begin{array}{l|l} \text{L.S.} & \text{R.S.} \\ \hline = \cos 2\left(\frac{\pi}{2}\right) & = 2 \sin \frac{\pi}{2} \cos \frac{\pi}{2} \\ = \cos \pi & = 2(1)(0) \\ = -1 & = 0 \end{array}$$

$$L.S. \neq R.S.$$

“Counterproof 2”:



APPENDIX D: SAMPLE SUMMARY TABLES

Content Summary Sheet

- mathematician background
- disparity → views accessibility important
 - ↳ experiential w/ HS then move to formalized generalizations.
- reasoning vs. proving → proving (uni math), not very present in current curriculum
 - ↳ proving (not in HS math) → maybe in idea of equivalence and generalization by example first
 - ↳ reasoning (everywhere) → informal proofs
 - ↳ showing understanding
 - ↳ make sense of math
 - ↳ modelling practical math
 - ↳ concrete rather than abstract.
- teaching approach → teacher ~~is~~ guided, peer involvement
 - ↳ specific examples to generalizations.
 - ↳ making math accessible to general audience.
 - ↳ Plutonic ↔ Experimental → experiential learning
- Curriculum → less proof, more ~~experi~~ investigative.

Code Organization Table

Code	Quote
Teaching approach	"I like to share with students... the creativity in mathematics... beyond perform algorithms (pg 1)
development differentiated	"It's trying to reach all students where they're at... (pg 3)
	"I will teach those particular students, those more formal reasoning... (pg 3)
	"But there are some students who are ready... (pg 3)
example based	"The way that I've been teaching math... (pg 5)
	"So that's probably what I do, starting with one example... (pg 5)
peer based learning	"I do very little upfront direct teaching... (pg 6)
	"At the end of that lesson, or process, ... (pg 6)
	"One is not necessarily better than the other... (pg 6)
conception of proofs	"In fact, I don't think very much about my... (pg 1)
	"So informal... so you asked... (pg 1)
	"proof by example. which are not proofs... (pg 1)
	"when I think about proofing, I think about a more formal structure... (pg 2)
	"Again, the focus is on using specific examples... And that's getting, breaking (pg 2)
	"So I would say that I do formally teach proving... I think in grade 9+10 (pg 3)
"well I see proofs in trigonometry, I see proofs in analytic... (pg 4)	
curriculum	"I think proving, I don't really think the idea really exists in our math curriculum except... (pg 2)
	"We do a little bit of that in gr 9+10... (pg 2)
	"in gr 10 you'll see it as generalization of... (pg 2)
	"And doing what we call a left side... (pg 3)
	"In gr 9+10 address as informal reasoning... (pg 3)
	"If we're thinking about formal proofs, that's not emphasized at all... (pg 3)