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AUTOMORPHISMS OF SIMPLE ANTI-JORDAN PAIRS

By
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October 2008

A Thesis
submitted to the Faculty of Graduate and Postdoctoral Studies
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for the degree of
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Abstract

We describe the automorphism groups and involutions of certain anti-Jordan pairs. We use this to classify finite dimensional simple anti-Jordan triple systems over an algebraically closed field of characteristic zero.

Dedication

I dedicate this work to my mother and my husband.

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Introduction

Anti-Jordan pairs were introduced by Faulkner and Ferrar in [3]. We recall the definition: Let k be a commutative, associative and unital ring and let $V = (V^+, V^-)$ be a pair of k -modules with k -trilinear maps $\{\dots\}^\epsilon : V^\epsilon \times V^{-\epsilon} \times V^\epsilon \rightarrow V^\epsilon$, $(x, y, z) \rightarrow \{x, y, z\}^\epsilon =: D^\epsilon(x, y)z$, for $\epsilon = \pm$. Then V is called an *anti-Jordan pair* if for all $x, u \in V^\epsilon$, $y, v \in V^{-\epsilon}$ and $\epsilon = \pm$

$$\begin{aligned} \{x, y, x\}^\epsilon &= 0, \quad \text{and} \\ [D^\epsilon(x, y), D^\epsilon(u, v)] &= D^\epsilon(\{x, y, u\}^\epsilon, v) + D^\epsilon(u, \{y, x, v\}^{-\epsilon}). \end{aligned}$$

We note that V is an anti-Jordan pair if and only if V is a Jordan superpair in the sense of [13] with even part $V_0 = 0$. Faulkner and Ferrar [3] classified the finite dimensional simple anti-Jordan pairs over an algebraically closed field F of characteristic zero, using methods from Lie superalgebra theory. Up to isomorphism, they found the following complete list of simple finite dimensional anti-Jordan pairs:

(I) The *rectangular matrix pair* $\mathfrak{M}_{mn}(F) = (V^+, V^-) = (M_{mn}(F), M_{nm}(F))$, with product: $\{x, y, z\} = xyz - zyx$, where $x, z \in V^\epsilon$ and $y \in V^{-\epsilon}$, and $M_{mn}(F)$ is the space of all $m \times n$ matrices.

(II) The *symmetric-skew pair* $\mathfrak{ss}_n(F) = (N^+, N^-)$, a subpair of the anti-Jordan pair $\mathfrak{M}_{nn}(F)$, defined by

$$N^+ = \{x \in M_{nn}(F) : x^t = x\} \quad \text{and} \quad N^- = \{x \in M_{nn}(F) : x^t = -x\}.$$

(III) The *symplectic anti-Jordan pair*, defined as follows: Let T be a vector space with an alternating non-degenerate bilinear form $B : T \times T \rightarrow F$. Then $(T, T)_B$

is an anti-Jordan pair with product $\{x, y, z\} = B(x, y)z + B(y, z)x + B(x, z)y$ for $x, y, z \in T$.

The first purpose of the thesis is to determine the automorphism group of the anti-Jordan pairs in Examples (I), (II) and (III), see section 1.2 for the definition of an isomorphism. It is easy to see that the examples (I) and (II) can be defined over any C , where C is an associative, unital and commutative algebra over a ring k , i.e. we replace F by C and our goal is to determine the k -linear automorphisms of the anti-Jordan pairs $\mathfrak{M}_{mn}(C)$ and $\mathfrak{ss}_n(C)$. A first reduction (Lemma 2.1.13 and Lemma 3.1.3) shows that the crucial point is to determine the C -linear automorphisms of $\mathfrak{M}_{mn}(C)$ and $\mathfrak{ss}_n(C)$. This is done in Theorem 2.3.1 and Theorem 2.3.5. For the purpose of this introduction, we mention our results over fields only.

Theorem 1 (= Corollary 2.3.2 , 2.3.6). *Let F be a field of characteristic zero or characteristic $p > 3$ and such that $p \nmid m$ and $p \nmid n$.*

(i) *If $m = n$, then $\text{Aut}(\mathfrak{M}_{mn}(F)) = \{(\mu_{a,b}^+, \mu_{a,b}^-), (\eta_{a,b}^+, \eta_{a,b}^-) : a, b \in \text{GL}_m(F)\}$, where $\eta_{a,b}^+(x) = axb^{-1}$, $\eta_{a,b}^-(y) = bya^{-1}$ and $\mu_{a,b}^+(x) = ax^tb^{-1}$, $\mu_{a,b}^-(y) = -by^ta^{-1}$ for all $x \in M_{mm}(F)$ and $y \in M_{mm}(F)$.*

(ii) *If $m \neq n$, then $\text{Aut}(\mathfrak{M}_{mn}(F)) = \{(\eta_{a,b}^+, \eta_{a,b}^-) : a \in \text{GL}_m(F), b \in \text{GL}_n(F)\}$, with $(\eta_{a,b}^+, \eta_{a,b}^-)$ as defined in (i).*

Theorem 2 (= Corollary 3.2.4). *Let F be a field of characteristic zero or $p > 3$ such that p does not divide n if $p \in \{5, 7, 11\}$. Then for $n > 2$,*

$$\text{Aut}(\mathfrak{ss}_n)(F) = \{(\chi_{(\alpha,a)}^+, \chi_{(\alpha,a)}^-) : a \in \text{GL}_n(F), \alpha \in F^\times\},$$

where $\chi_{(\alpha,a)}^+(x) = \alpha axa^t$ and $\chi_{(\alpha,a)}^-(y) = (a^{-1})^t y (\alpha a)^{-1}$ for all $x \in N^+$ and $y \in N^-$.

Theorem 3 (= Corollary 4.1.3). *Let F be a field of characteristic $\neq 2$ and let T be a F -vector space with a non-degenerate alternating bilinear form $B : T \times T \rightarrow F$. Then $\text{Aut}(T, T)_B \cong \text{GSP}(T, B)$, where $\text{GSP}(T, B)$ denotes the group of invertible endomorphisms a of T , for which there exist $\alpha \in F$ such that $\alpha B(a(x), a(y)) = B(x, y)$ for all $x, y \in T$.*

To prove Theorem 1, we first show that the Tits-Kantor-Koecher algebra of $\mathfrak{M}_{mn}(F)$ is isomorphic to the Lie superalgebra

$$\mathfrak{psl}(m|n)(F) := \mathfrak{sl}(m|n)(F)/Z(\mathfrak{sl}(m|n)(F)),$$

(see Lemma 2.2.5). Also we can extend any automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(F)$ to an automorphism of the 3-graded Lie superalgebra $\mathfrak{psl}(m|n)(F)$. We then get information on $\text{Aut}(\mathfrak{M}_{mn}(F))$ by considering the induced automorphism of the even part of $\mathfrak{psl}(m|n)(F)$, making use of a theorem of Seligman which determines the automorphism group of $\mathfrak{sl}_m(F)$. To prove Theorem 2, we use similar techniques as in Theorem 1.

The second purpose of this thesis is to find the anti-Jordan triple systems associated with the simple anti-Jordan pairs in Examples (I), (II) and (III). We recall the definition: Let T be a k -module with a k -trilinear map $\langle \dots \rangle : T \times T \times T \rightarrow T$, $(x, y, z) \rightarrow \langle x, y, z \rangle =: L(x, y)z$. Then the pair $(T, \langle \dots \rangle)$ is called a *triple system*, and it is called an *anti-Jordan triple system* if for all $x, y, u, v \in T$

$$\begin{aligned} \langle x, y, x \rangle &= 0, \quad \text{and} \\ [L(x, y), L(u, v)] &= L(\langle x, y, u \rangle, v) + L(u, \langle y, x, v \rangle). \end{aligned}$$

The following lemma explains the relationship between anti-Jordan triple systems and anti-Jordan pairs with involutions, see section 1.2 for the definition of an involution.

Lemma 1 (= Lemma 1.2.1). (a) *If T is an anti-Jordan triple system then $V(T) = (T, T)$ with triple product $\{\dots\}^+ = \{\dots\}^- = \langle \dots \rangle$ is an anti-Jordan pair with involution $\eta = (\text{Id}, \text{Id})$.*

(b) *Conversely, if $V = (V^+, V^-)$ is an anti-Jordan pair with involution $\eta = (\eta^+, \eta^-)$, then $T = V^+$ together with the trilinear map $\langle \dots \rangle$ defined by $\langle x, y, z \rangle = \{x, \eta^+(y), z\}^+$ is an anti-Jordan triple system whose anti-Jordan pair $V(T) = (T, T)$ is isomorphic to V via $(\text{Id}, \eta^+) : (T, T) \rightarrow V$.*

Therefore to know anti-Jordan triple systems, it is enough to know the involutions of anti-Jordan pairs. Note that if we compose two anti-automorphisms of an anti-Jordan pair we get an automorphism of the anti-Jordan pair. Thus the involutions

of an anti-Jordan pair can be determined from the knowledge of one involution and all automorphisms. Note that in Example (II), we do not have involutions, since the dimensions of N^+ and N^- are different. Therefore we do not have anti-Jordan triple systems arising from involutions in Example (II). The following results determine the simple anti-Jordan triple systems arising from involutions in Examples (I) and (III).

Theorem 4 (=Theorem 2.4.3). (a) *Let F be a field. The following are simple finite dimensional anti-Jordan triple systems over F :*

(i) $T_1 = M_{mn}(F)$, together with the trilinear maps $\langle \dots \rangle_1$ defined by

$$\langle x, y, z \rangle_1 = xy^taz - zy^tax, \text{ where } a = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \text{ and } 2r = m.$$

(ii) $T_2 = M_{mn}(F)$, together with the trilinear maps $\langle \dots \rangle_2$ defined by

$$\langle x, y, z \rangle_2 = xby^tz - zby^tx, \text{ where } b = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \text{ and } 2r = n.$$

(iii) $T_3 = M_{nn}(F)$, together with the trilinear maps $\langle \dots \rangle_3$ defined by

$$\langle u, v, w \rangle_3 = uvw - wvu.$$

(b) *Conversely, let F be an algebraically closed field of characteristic zero or of characteristic $p > 0$, $p \neq 2, 3$ and such that $p \nmid m$ and $p \nmid n$. If T is a simple finite dimensional anti-Jordan triple system over F such that $(T, T) \cong \mathfrak{M}_{mn}(F)$, then T is isomorphic to one of the examples in (a).*

Theorem 5 (= Theorem 4.2.2). (a) *Let T be an F -vector space with an alternating, non-degenerate bilinear form $B : T \times T \rightarrow F$ and let $a \in \text{GSP}(T, B)$ such that $a^2 = \alpha \text{Id}$ for some $\alpha \in F^\times$. Then T is a simple anti-Jordan triple system with triple product $\langle x, y, z \rangle_a = \{x, a(y), z\} = B(x, a(y))z + B(a(y), z)x + B(x, z)a(y)$.*

(b) *If T is a simple finite dimensional anti-Jordan triple system over F such that $(T, T) \cong (T, T)_B$, then there exist $b \in \text{GSP}(T, B)$ such that $b^2 = \beta \text{Id}$ for some $\beta \in F^\times$ and T is the simple anti-Jordan triple system with triple product:*

$$\langle x, y, z \rangle_B = \{x, b(y), z\} = B(x, b(y))z + B(b(y), z)x + B(x, z)b(y).$$

The Theorems 4 and 5 together with Lemma 1.2.6 allow us to obtain the classification of finite dimensional simple anti-Jordan triple systems over an algebraically closed field F of characteristic zero. For the definition of a polarized triple system see section 1.2.

Theorem 6. *T is a finite dimensional simple anti-Jordan triple system over an algebraically closed field F of characteristic zero if and only if T is isomorphic to one of the following:*

- (a) *The polarized triple system associated to one of the three simple anti-Jordan pairs $\mathfrak{M}_{mn}(F)$, $\mathfrak{ss}_n(F)$ and the symplectic anti-Jordan pair $(T, T)_B$.*
- (b) *The anti-Jordan triple systems of Theorem 4 and Theorem 5.*

Chapter 1

General facts on anti-Jordan pairs and anti-Jordan triple systems

In this chapter k will be a commutative associative unital ring. All algebraic structures will be over k , unless specified otherwise. We denote by k^\times the set of invertible elements of k . Throughout this thesis we denote by $M_{mn}(R)$ the k -module of $m \times n$ matrices over R , where R is an associative and unital k -algebra.

1.1 Anti-Jordan triple systems

Let T be a k -module with a k -trilinear map $\langle \dots \rangle : T \times T \times T \rightarrow T$, $(x, y, z) \rightarrow \langle x, y, z \rangle =: L(x, y)z$. Then the pair $(T, \langle \dots \rangle)$ is called a *triple system* and it is called an *anti-Jordan triple system* if for all $x, y, u, v \in T$

$$\langle x, y, x \rangle = 0, \quad \text{and} \quad (1)$$

$$[L(x, y), L(u, v)] = L(\langle x, y, u \rangle, v) + L(u, \langle y, x, v \rangle). \quad (2)$$

Most often we call T rather than $(T, \langle \dots \rangle)$ the anti-Jordan triple system.

Let $(T_1, \langle \dots \rangle_1)$ and $(T_2, \langle \dots \rangle_2)$ be anti-Jordan triple systems over k . A linear map $\varphi : T_1 \rightarrow T_2$ is called a *homomorphism*, if

$$\varphi \langle x, y, z \rangle_1 = \langle \varphi(x), \varphi(y), \varphi(z) \rangle_2$$

for all $x, y, z \in T_1$. Isomorphisms and automorphisms are defined in the obvious way.

A submodule I of an anti-Jordan triple system T is called a *subsystem* (resp. an *ideal*) if $\langle I, I, I \rangle \subset I$ (resp. $\langle I, T, T \rangle + \langle T, I, T \rangle \subset I$). Any subsystem of an anti-Jordan triple system is again an anti-Jordan triple system. An anti-Jordan triple system T is *simple*, if $\langle T, T, T \rangle \neq 0$ and if it contains no ideals other than 0 and T .

The following lemma will be used to show that some of our examples are indeed anti-Jordan triple systems.

Lemma 1.1.1. *Let R be an associative k -algebra. Let $(T, \langle \dots \rangle')$ be a triple system over R such that*

$$(A) \quad \langle a, b, \langle c, d, e \rangle' \rangle' = \langle \langle a, b, c \rangle', d, e \rangle'.$$

Then T together with the triple product

$$\langle a, b, c \rangle = \langle a, b, c \rangle' - \langle c, b, a \rangle'$$

is an anti-Jordan triple system if and only if

$$(B) \quad \langle u, v, \langle x, y, z \rangle' \rangle' + \langle \langle z, y, x \rangle', v, u \rangle' + \langle u, \langle y, x, v \rangle', z \rangle' + \langle z, \langle v, x, y \rangle', u \rangle' \\ = \langle \langle u, y, x \rangle', v, z \rangle' + \langle z, v, \langle x, y, u \rangle' \rangle' + \langle u, \langle v, x, y \rangle', z \rangle' + \langle z, \langle y, x, v \rangle', u \rangle'.$$

In particular, (A) and (B) hold in any one of the following cases :

(i) $(T, \langle \dots \rangle')$ is an associative triple system of first kind, i.e.,

$$\langle x, y, \langle u, v, w \rangle' \rangle' = \langle \langle x, y, u \rangle', v, w \rangle' = \langle x, \langle y, u, v \rangle', w \rangle'$$

for all $x, y, u, v, w \in T$.

(ii) $(T, \langle \dots \rangle')$ is a triple system satisfying (A) and

$$(C) \quad \langle u, \langle y, x, v \rangle', z \rangle' = -\langle \langle u, v, x \rangle', y, z \rangle'.$$

for all $x, y, u, v, z \in T$.

Proof. Let $(T, \langle \dots \rangle')$ be a triple system such that (A) holds. Now consider the left hand side of (2) in the definition of an anti-Jordan triple system:

$$\begin{aligned}
[L(x, y), L(u, v)]z &= \langle x, y, \langle u, v, z \rangle \rangle - \langle u, v, \langle x, y, z \rangle \rangle \\
&= \langle x, y, \langle u, v, z \rangle' \rangle' - \langle \langle u, v, z \rangle', y, x \rangle' - \langle x, y, \langle z, v, u \rangle' \rangle' \\
&\quad + \langle \langle z, v, u \rangle', y, x \rangle' - \langle u, v, \langle x, y, z \rangle' \rangle' + \langle \langle x, y, z \rangle', v, u \rangle' \\
&\quad + \langle u, v, \langle z, y, x \rangle' \rangle' - \langle \langle z, y, x \rangle', v, u \rangle' \\
&= \langle x, y, \langle u, v, z \rangle' \rangle' - \langle u, v, \langle z, y, x \rangle' \rangle' - \langle \langle x, y, z \rangle', v, u \rangle' \\
&\quad + \langle \langle z, v, u \rangle', y, x \rangle' - \langle u, v, \langle x, y, z \rangle' \rangle' + \langle \langle x, y, z \rangle', v, u \rangle' \\
&\quad + \langle u, v, \langle z, y, x \rangle' \rangle' - \langle \langle z, y, x \rangle', v, u \rangle' \\
&= \langle x, y, \langle u, v, z \rangle' \rangle' + \langle \langle z, v, u \rangle', y, x \rangle' - \langle u, v, \langle x, y, z \rangle' \rangle' \\
&\quad - \langle \langle z, y, x \rangle', v, u \rangle'.
\end{aligned}$$

The right hand side of (2):

$$\begin{aligned}
L(\langle x, y, u \rangle, v)z + L(u, \langle y, x, v \rangle)z &= \langle \langle x, y, u \rangle, v, z \rangle + \langle u, \langle y, x, v \rangle, z \rangle \\
&= \langle \langle x, y, u \rangle', v, z \rangle' - \langle \langle u, y, x \rangle', v, z \rangle' - \langle z, v, \langle x, y, u \rangle' \rangle' \\
&\quad + \langle z, v, \langle u, y, x \rangle' \rangle' + \langle u, \langle y, x, v \rangle', z \rangle' - \langle u, \langle v, x, y \rangle', z \rangle' \\
&\quad - \langle z, \langle y, x, v \rangle', u \rangle' + \langle z, \langle v, x, y \rangle', u \rangle'.
\end{aligned}$$

Hence $[L(x, y), L(u, v)] = L(\langle x, y, u \rangle, v) + L(u, \langle y, x, v \rangle)$

$$\begin{aligned}
&\Leftrightarrow \langle u, v, \langle x, y, z \rangle' \rangle' + \langle \langle z, y, x \rangle', v, u \rangle' + \langle u, \langle y, x, v \rangle', z \rangle' + \langle z, \langle v, x, y \rangle', u \rangle' \\
&= \langle \langle u, y, x \rangle', v, z \rangle' + \langle z, v, \langle x, y, u \rangle' \rangle' + \langle u, \langle v, x, y \rangle', z \rangle' + \langle z, \langle y, x, v \rangle', u \rangle',
\end{aligned}$$

which is (B).

Note that (1) is obvious, since

$$\langle x, y, x \rangle = \langle a, b, a \rangle' - \langle a, b, a \rangle' = 0 \quad \text{for all } x, y \in T.$$

Therefore $(T, \langle \dots \rangle')$ is an anti-Jordan triple system if and only if (B) holds.

(i) Note that the left hand side of (B)

$$\begin{aligned}
&= \langle u, v, \langle x, y, z \rangle' \rangle' + \langle \langle z, y, x \rangle', v, u \rangle' + \langle u, \langle y, x, v \rangle', z \rangle' + \langle z, \langle v, x, y \rangle', u \rangle' \\
&= \langle u, \langle v, x, y \rangle', z \rangle' + \langle z, \langle y, x, v \rangle', u \rangle' + \langle \langle u, y, x \rangle', v, z \rangle' + \langle z, v, \langle x, y, u \rangle' \rangle' \\
&= \text{the right hand side of (B)}
\end{aligned}$$

Hence any associative triple system of first kind is an anti-Jordan triple system.

(ii) After applying (A) and (C), the left hand side of (B) =

$$\langle u, v, \langle x, y, z \rangle' \rangle' - \langle z, \langle v, x, y \rangle', u \rangle' - \langle u, v, \langle x, y, z \rangle' \rangle' + \langle z, \langle v, x, y \rangle', u \rangle' = 0$$

Similarly after applying (A) and (C), the right hand side of (B) = 0. Therefore (B) holds. \square

We can now give examples of anti-Jordan triple systems.

Example 1.1.2. Any associative algebra A is an associative triple system of first kind with triple product $\langle a, b, c \rangle' = abc$, hence also an anti-Jordan triple system with triple product $\langle a, b, c \rangle = abc - cba$.

Example 1.1.3. Let C be a commutative associative and unital k -algebra. Let $a \in M_{nn}(C)$ and $b \in M_{mm}(C)$ such that $a^t = \epsilon a$ and $b^t = -\epsilon b$ for $\epsilon = \pm$. Then $(M_{mn}(C), \langle \dots \rangle_{a,b})$ is an anti-Jordan triple system, where

$$\langle x, y, z \rangle_{a,b} = xay^t bz - zay^t bx.$$

We will denote the anti-Jordan triple system $(M_{mn}(C), \langle \dots \rangle_{a,b})$ by $M_{mn}(C; a, b)$.

Proof. Put $\langle x, y, z \rangle' = xay^t bz$. Note that:

$$\langle x, y, \langle u, v, z \rangle' \rangle' = xay^t buav^t bz = \langle \langle x, y, u \rangle', v, z \rangle' \text{ and}$$

$$\langle u, \langle y, x, v \rangle', z \rangle' = ua(yax^t bv)^t bz = -uav^t bxay^t bz = -\langle u, v, \langle x, y, z \rangle' \rangle'.$$

i.e. $(M_{mn}(C; a, b), \langle \dots \rangle')$ satisfies conditions (A) and (C) in part (ii) of Lemma 1.1.1. Hence $M_{mn}(C; a, b)$ is an anti-Jordan triple system. \square

Let F be a field. Recall that two matrices $a, b \in M_n(F)$ are called *cogredient* if there exists a matrix $p \in GL_n(F)$ such that $p^t a p = b$. The following is shown in [6, Sections 6.2, 6.3].

Theorem 1.1.4. *If the characteristic of F is not 2, then any two skew symmetric matrices in $M_n(F)$ are cogredient if and only if they have the same rank.*

Theorem 1.1.5. *If F is an algebraically closed field of characteristic not 2, then two symmetric matrices in $M_n(F)$ are cogredient if and only if they have the same rank.*

Lemma 1.1.6. *Let C, b, a be as in Example 1.1.3, and let $u \in GL_n(C)$ and $v \in GL_m(C)$. The map $f : M_{mn}(C; u^t a u, v^t b v) \rightarrow M_{nm}(C; b, a)$, $x \rightarrow u x^t v^t$ is an isomorphism of anti-Jordan triple systems. In particular :*

- (a) $M_{mn}(C; a, b)$ and $M_{nm}(C; b, a)$ are isomorphic.
- (b) If $C = F$ is an algebraically closed field of characteristic different from 2, and $a, d \in GL_n(F)$, $b, c \in GL_m(F)$ are such that $a^t = \epsilon a$, $d^t = \epsilon d$, $b^t = -\epsilon b$, $c^t = -\epsilon c$ for $\epsilon = \pm 1$, then $M_{mn}(F; a, b)$ and $M_{nm}(F; c, d)$ are isomorphic.

Proof. The map is obviously an isomorphism of the underlying C -modules. For $x, y, z \in M_{mn}(C; u^t a u, v^t b v)$ we have, using $(xy)^t = y^t x^t$ which holds because of commutativity of C ,

$$f\langle x, y, z \rangle_{u^t a u, v^t b v} = u x^t v^t b v y u^t a u z^t v^t - u z^t v^t b v y u^t a u x^t v^t$$

and

$$\begin{aligned} \langle f(x), f(y), f(z) \rangle_{b, a} &= \langle u x^t v^t, u y^t v^t, u z^t v^t \rangle \\ &= u x^t v^t b v y u^t a u z^t v^t - u z^t v^t b v y u^t a u x^t v^t \end{aligned}$$

Hence $f\langle x, y, z \rangle_{u^t a u, v^t b v} = \langle f(x), f(y), f(z) \rangle_{b, a}$.

- (a) If we put $u = I_n$ and $v = I_m$, then we see that $M_{mn}(C; a, b)$ and $M_{nm}(C; b, a)$ are isomorphic.

(b) From Theorem 1.1.4 and Theorem 1.1.5 we know there exist matrices $u \in \text{GL}_n(F)$ and $v \in \text{GL}_m(F)$ such that

$$u^t du = a \quad \text{and} \quad v^t cv = b.$$

Hence $M_{mn}(F; a, b)$ and $M_{nm}(F; c, d)$ are isomorphic. \square

Lemma 1.1.7. *Let $(T, \langle \dots \rangle)$ be an anti-Jordan triple system and let $f \in \text{End}_k(T)$ such that $f\langle x, f(y), z \rangle = \langle f(x), y, f(z) \rangle$ for all $x, y, z \in T$. Then T together with the triple product*

$$\langle x, y, z \rangle' = \langle x, f(y), z \rangle$$

is another anti-Jordan triple system.

Proof. It is easy to see that $\langle x, y, x \rangle' = 0$. We will show

$$[L'(x, y), L'(u, v)]z = L'(\langle x, y, u \rangle', v)z + L'(u, \langle y, x, v \rangle')z$$

for all $x, y, z, u, v \in T$.

$$\begin{aligned} [L'(x, y), L'(u, v)]z &= L'(x, y)L'(u, v)z - L'(u, v)L'(x, y)z \\ &= \langle x, y, \langle u, v, z \rangle' \rangle' - \langle u, v, \langle x, y, z \rangle' \rangle' \\ &= \langle x, f(y), \langle u, f(v), z \rangle \rangle - \langle u, f(v), \langle x, f(y), z \rangle \rangle \\ &= \langle \langle x, f(y), u \rangle, f(v), z \rangle + \langle u, f\langle y, f(x), v \rangle, z \rangle \\ &= L'(\langle x, y, u \rangle', v)z + L'(u, \langle y, x, v \rangle')z. \end{aligned}$$

\square

Example 1.1.8. Let C be a commutative associative and unital k -algebra. Let T be a C -module with an alternating bilinear form $B : T \times T \rightarrow C$. Put

$$\langle x, y, z \rangle_B = B(x, y)z + B(y, z)x + B(x, z)y \quad \text{for all } x, y, z \in T.$$

Then $(T, \langle \dots \rangle_B)$ is anti-Jordan triple system, called the *symplectic anti-Jordan triple system* (with respect to B).

Proof. Note that $B(x, y) = -B(y, x)$, $B(x, y)B(u, v) = B(u, v)B(x, y)$ and $B(x, x) = 0$ for all $x, y, u, v \in T$. Therefore $\{x, y, x\} = 0$.

We will check that $[L(x, y), L(u, v)] = L(\{x, y, u\}, v) + L(u, \{y, x, v\})$.

$$\begin{aligned}
& [L(x, y), L(u, v)]z = \{x, y, \{u, v, z\}\} - \{u, v\{x, y, z\}\} \\
& = B(u, v)\{x, y, z\} + B(u, z)\{x, y, v\} + B(v, z)\{x, y, u\} \\
& \quad - B(x, y)\{u, v, z\} - B(x, z)\{u, v, y\} - B(y, z)\{u, v, x\} \\
& = B(u, v)B(x, y)z + B(u, v)B(x, z)y + B(u, v)B(y, z)x \\
& \quad + B(u, z)B(x, y)v + B(u, z)B(x, v)y + B(u, z)B(y, v)x \\
& \quad + B(v, z)B(x, y)u + B(v, z)B(x, u)y + B(v, z)B(y, u)x \\
& \quad - B(x, y)B(u, v)z - B(x, y)B(u, z)v - B(x, y)B(v, z)u \\
& \quad - B(x, z)B(u, v)y - B(x, z)B(u, y)v - B(x, z)B(v, y)u \\
& \quad - B(y, z)B(u, v)x - B(y, z)B(u, x)v - B(y, z)B(v, x)u \\
& = B(u, z)B(x, v)y + B(u, z)B(y, v)x + B(v, z)B(x, u)y \\
& \quad + B(v, z)B(y, u)x + B(x, z)B(y, u)v + B(y, v)B(x, z)u \\
& \quad + B(x, u)B(y, z)v + B(x, v)B(y, z)u.
\end{aligned}$$

$$\begin{aligned}
& L(\{x, y, u\}, v)z + L(u, \{y, x, v\})z \\
& = \{B(x, y)u + B(x, u)y + B(y, u)x, v, z\} \\
& \quad + \{u, B(y, x)v + B(y, v)x + B(x, v)y, z\} \\
& = B(x, y)\{u, v, z\} + B(x, u)\{y, v, z\} + B(y, u)\{x, v, z\} \\
& \quad + B(y, x)\{u, v, z\} + B(y, v)\{u, x, z\} + B(x, v)\{u, y, z\} \\
& = B(x, u)B(y, v)z + B(x, u)B(y, z)v + B(x, u)B(v, z)y \\
& \quad + B(y, u)B(x, v)z + B(y, u)B(x, z)v + B(y, u)B(v, z)x \\
& \quad + B(y, v)B(u, x)z + B(y, v)B(u, z)x + B(y, v)B(x, z)u \\
& \quad + B(x, v)B(u, y)z + B(x, v)B(u, z)y + B(x, v)B(y, z)u
\end{aligned}$$

$$\begin{aligned}
& L(\{x, y, u\}, v)z + L(u, \{y, x, v\})z \\
&= B(x, u)B(y, z)v + B(x, u)B(v, z)y + B(y, u)B(x, z)v \\
&+ B(y, u)B(v, z)x + B(y, v)B(u, z)x + B(y, v)B(x, z)u \\
&+ B(x, v)B(u, z)y + B(x, v)B(y, z)u.
\end{aligned}$$

Hence $[L(x, y), L(u, v)] = L(\{x, y, u\}, v) + L(u, \{y, x, v\})$ for all $x, y, u, v \in T$.

□

Example 1.1.9. Let $(T, \langle \dots \rangle_B)$ be a symplectic anti-Jordan triple system, see Example 1.1.8. Let $f \in \text{End}(T)$ such that $B(x, y) = \alpha B(f(x), f(y))$ and $f^2 = \alpha^{-1}\text{Id}$ for some $\alpha \in C^\times$. Then T together with the triple product

$$\langle x, y, z \rangle'_B = \langle x, f(y), z \rangle_B$$

is an anti-Jordan triple system.

Proof. Note that

$$\begin{aligned}
f\langle x, f(y), z \rangle &= B(x, f(y))f(z) + B(x, z)f^2(y) + B(f(y), z)f(x) \\
&= \alpha B(f(x), f^2(y))f(z) + \alpha^{-1}\alpha B(f(x), f(z))y \\
&\quad + \alpha B(f^2(y), f(z))f(x) \\
&= B(f(x), y)(f(z) + B(f(x), f(z))y + B(y, f(z))f(x) \\
&= \langle f(x), y, f(z) \rangle.
\end{aligned}$$

Therefore $(T, \langle \dots \rangle'_B)$ is an anti-Jordan triple system by Lemma 1.1.7.

□

1.2 Anti-Jordan pairs

Let $V = (V^+, V^-)$ be a pair of k -modules with k -trilinear maps $\{\dots\}^\epsilon : V^\epsilon \times V^{-\epsilon} \times V^\epsilon \rightarrow V^\epsilon$, $(x, y, z) \rightarrow \{x, y, z\}^\epsilon =: D^\epsilon(x, y)z$, for $\epsilon = \pm$. Then V is called an *anti-Jordan pair* if for all $x, u \in V^\epsilon$, $y, v \in V^{-\epsilon}$ and $\epsilon = \pm$

$$\{x, y, x\}^\epsilon = 0, \quad \text{and} \quad (3)$$

$$[D^\epsilon(x, y), D^\epsilon(u, v)] = D^\epsilon(\{x, y, u\}^\epsilon, v) + D^\epsilon(u, \{y, x, v\}^{-\epsilon}). \quad (4)$$

We will usually drop the superscript ϵ and simply write $\{\dots\}$ and $D(.,.)$ for $\{\dots\}^\epsilon$ and $D^\epsilon(.,.)$. We note that V is an anti-Jordan pair if and only if V is a Jordan superpair in the sense of [13] with even part $V_0 = 0$.

A *homomorphism* of anti-Jordan pairs $V = (V^+, V^-)$ and $W = (W^+, W^-)$ over k , is a pair (η^+, η^-) with k -linear maps $\eta^\epsilon : V^\epsilon \rightarrow W^\epsilon$, $\epsilon = \pm$ such that

$$\eta^\epsilon \{x, y, z\} = \{\eta^\epsilon(x), \eta^{-\epsilon}(y), \eta^\epsilon(z)\} \quad (5)$$

holds for all $x, z \in V^\epsilon$, $y \in V^{-\epsilon}$. Isomorphisms and automorphisms are defined in the obvious way. We denote by $\text{Aut}_k(V)$ the group of all k -linear automorphisms of V .

A pair $I = (I^+, I^-)$ of submodules of an anti-Jordan pair is called a *subpair* (resp. an *ideal*) if $\{I^\epsilon, I^{-\epsilon}, I^\epsilon\} \subset I^\epsilon$ (resp. $\{I^\epsilon, V^{-\epsilon}, V^\epsilon\} + \{V^\epsilon, I^{-\epsilon}, V^\epsilon\} \subset I^\epsilon$). An anti-Jordan pair V is *simple* if it contains no ideals other than 0 and V and $\{V^\epsilon, V^{-\epsilon}, V^\epsilon\} \neq 0$, for $\epsilon = -$ or $\epsilon = +$.

The *opposite* of an anti-Jordan pair $V = (V^+, V^-)$ is the anti-Jordan pair $V^{\text{op}} = (V^-, V^+)$ with maps $(\{\dots\}^-, \{\dots\}^+)$. An *involution* of an anti-Jordan pair V is a homomorphism $\eta : V \rightarrow V^{\text{op}}$ such that $(\eta^- \circ \eta^+, \eta^+ \circ \eta^-) = \text{Id}_V$.

Lemma 1.2.1. (a) *If T is an anti-Jordan triple system, then $V(T) = (T, T)$ with triple product $\{\dots\}^+ = \{\dots\}^- = \langle \dots \rangle$ is an anti-Jordan pair with involution $\eta = (\text{Id}, \text{Id})$.*

(b) Conversely, if $V = (V^+, V^-)$ is an anti-Jordan pair with involution $\eta = (\eta^+, \eta^-)$, then $T = V^+$ together with the trilinear map $\langle \dots \rangle$ defined by $\langle x, y, z \rangle = \{x, \eta^+(y), z\}$ is an anti-Jordan triple system whose anti-Jordan pair $V(T) = (T, T)$ is isomorphic to V via $(\text{Id}, \eta^+) : (T, T) \rightarrow V$.

Proof. (a) is obvious.

(b) Conversely, let $V = (V^+, V^-)$ be an anti-Jordan pair with involution $\eta = (\eta^+, \eta^-)$. Then we will show that $T = V^+$ together with the trilinear map $\langle \dots \rangle$ defined by $\langle x, y, z \rangle = \{x, \eta^+(y), x\}$ is an anti-Jordan triple system.

Let $x, y, u, v \in T$. Then $\langle x, y, x \rangle = \{x, \eta^+(y), x\} = 0$ by (3) and

$$\begin{aligned} [L(x, y), L(u, v)] &= [D(x, \eta^+(y)), D(u, \eta^+(v))] \\ &= D(\{x, \eta^+(y), u\}, \eta^+(v)) + D(u, \{\eta^+(y), x, \eta^+(v)\}) \quad (\text{by (4)}). \end{aligned}$$

Also

$$\begin{aligned} L(\langle x, y, u \rangle, v) + L(u, \langle y, x, v \rangle) &= D(\{x, \eta^+(y), u\}, \eta^+(v)) + D(u, \eta^+\{y, \eta^+(x), v\}) \\ &= D(\{x, \eta^+(y), u\}, \eta^+(v)) + D(u, \{\eta^+(y), x, \eta^+(v)\}). \end{aligned}$$

Hence $[L(x, y), L(u, v)] = L(\langle x, y, u \rangle, v) + L(u, \langle y, x, v \rangle)$. To show that (T, T) is isomorphic to V , consider the mapping $(\text{Id}, \eta^+) : (T, T) \rightarrow V$, given by $(\text{Id}, \eta^+)(x, y) = (x, \eta^+(y))$ for all $(x, y) \in (T, T)$. We will check that (Id, η^+) is a homomorphism:

$$\begin{aligned} \text{Id}\{x, y, z\} &= \text{Id}\langle x, y, z \rangle = \{x, \eta^+(y), z\} \\ \eta^+\{x, y, z\} &= \eta^+\langle x, y, z \rangle = \eta^+\{x, \eta^+(y), z\} \\ &= \{\eta^+(x), y, \eta^+(z)\} \quad (\text{since } \eta \text{ is an involution of } V) \end{aligned}$$

for all $x, y, z \in T$. It is obvious that (Id, η^+) is bijective. Hence (Id, η^+) is an isomorphism.

Lemma 1.2.2. *Let $V = (V^+, V^-)$ and $W = (W^+, W^-)$ be anti-Jordan pairs with involutions $\eta = (\eta^+, \eta^-) : V \rightarrow V^{\text{op}}$ and $\rho = (\rho^+, \rho^-) : W \rightarrow W^{\text{op}}$. We denote by T*

and T' the anti-Jordan triple systems defined by η and ρ respectively. Thus, $T = V^+$ with triple product $\langle x, y, z \rangle = \{x, \eta^+(y), z\}$ for all $x, y, z \in T$ and $T' = W^+$ with triple product $\langle u, v, w \rangle' = \{u, \rho^+(v), w\}'$ for all $u, v, w \in T'$. Then the following are equivalent:

- (i) $f : T \rightarrow T'$ is an isomorphism of anti-Jordan triple systems.
(ii) $(f^+, f^-) = (f, \rho^+ \circ f \circ \eta^-) : V \rightarrow W$ is an isomorphism of anti-Jordan pairs.
Hence $T \cong T'$ if and only if $\rho^+ \circ \phi^+ = \phi^- \circ \eta^+$ for some isomorphism $\phi : V \rightarrow W$.

Proof. (i) \Rightarrow (ii): Let $x, z \in V^+$, and $y, v \in V^-$. Then

$$\begin{aligned} f^+\{x, v, z\} &= f\{x, \eta^+(\eta^-v), z\} = f\langle x, \eta^-v, z \rangle \\ &= \langle fx, f\eta^-v, fz \rangle' = \{fx, \rho^+f\eta^-v, z\}' \\ &= \{f^+x, f^-y, f^+z\}' \quad \text{and} \\ f^-\{y, x, v\} &= \rho^+f\eta^-\{y, x, v\} = \rho^+f\{\eta^-(y), \eta^+(x), \eta^-(v)\} \\ &= \rho^+f\langle \eta^-(y), x, \eta^-(v) \rangle = \rho^+\langle f\eta^-(y), fx, f\eta^-(v) \rangle' \\ &= \rho^+\{f\eta^-(y), \rho^+fx, f\eta^-(v)\}' = \{\rho^+f\eta^-(y), \rho^-\rho^+fx, \rho^+f\eta^-(v)\}' \\ &= \{f^-(y), f^+(x), f^-(v)\}'. \end{aligned}$$

(ii) \Rightarrow (i): Let $x, y, z \in V^+$. Then

$$\begin{aligned} f\langle x, y, z \rangle &= f\{x, \eta^+(y), z\} = \{fx, \rho^+f\eta^-\eta^+(y), fz\}' \\ &= \{fx, \rho^+f(y), fz\}' = \langle fx, f(y), fz \rangle'. \end{aligned}$$

Therefore, if $f^+ : T \rightarrow T'$ is an isomorphism of anti-Jordan triple systems, then $\rho^+ \circ f^+ = \rho^+ \circ f^+ \circ \eta^- \circ \eta^+ = f^- \circ \eta^+$.

Conversely: let $\phi = (\phi^+, \phi^-) : V \rightarrow W$ be an isomorphism of anti-Jordan pairs and $\rho^+ \circ \phi^+ = \phi^- \circ \eta^+$. Then $\phi^+ : V^+ \rightarrow W^+$ is an isomorphism of anti-Jordan triple systems, since

$$\begin{aligned} \phi^+\langle x, y, z \rangle &= \phi^+\{x, \eta^+(y), z\} = \{\phi^+x, \phi^-\eta^+(y), \phi^+z\} \\ &= \{\phi^+x, \rho^+\phi^+(y), \phi^+z\} = \langle \phi^+x, \phi^+(y), \phi^+z \rangle'. \end{aligned}$$

□

Let T be a triple system. Then T is called *polarized* if $T = T^+ \oplus T^-$ and the triple product satisfies $\langle T^\epsilon, T^\epsilon, T \rangle = 0 = \langle T, T^\epsilon, T^\epsilon \rangle$ and $\langle T^\epsilon, T^{-\epsilon}, T^\epsilon \rangle \subset T^\epsilon$, $\epsilon = \pm$. Therefore, if $x = x^+ \oplus x^-$ and similarly for $y, z \in T$, then

$$\langle x^+ \oplus x^-, y^+ \oplus y^-, z^+ \oplus z^- \rangle = \langle x^+, y^-, z^+ \rangle \oplus \langle x^-, y^+, z^- \rangle.$$

Thus we can associate to T the pair $V = (T^+, T^-)$ with triple product $\langle x^\epsilon, y^{-\epsilon}, z^\epsilon \rangle = \{x^\epsilon, y^{-\epsilon}, z^\epsilon\} \in T^\epsilon$. One can easily check for a polarized triple system $T = T^+ \oplus T^-$, that T is an anti-Jordan triple system if and only if (T^+, T^-) is an anti-Jordan pair. Conversely, if $V = (V^+, V^-)$ is an anti-Jordan pair, we can define a polarized triple system $T(V) = V^+ \oplus V^-$ by the formula above. Thus $V = (V^+, V^-)$ is an anti-Jordan pair if and only if $T(V) = V^+ \oplus V^-$ is an polarized anti-Jordan triple system.

Example 1.2.3. The *rectangular matrix pair* $\mathfrak{M}_{mn}(R) = (M_{mn}(R), M_{nm}(R)) = (V^+, V^-)$ is an anti-Jordan pair with product : $\{x, y, z\} = xyz - zyx$, where $x, z \in V^\epsilon$, $y \in V^{-\epsilon}$ and $\epsilon = \pm$.

Proof. Let $A = M_{m+n}(R)$, which we view as an anti-Jordan triple system with triple product: $\langle x, y, z \rangle = xyz - zyx$, (see Lemma 1.1.1). Put $T = T^+ \oplus T^-$, where

$$\begin{aligned} T^+ &= \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in M_{mn}(R) \right\} \\ T^- &= \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} : y \in M_{nm}(R) \right\}. \end{aligned}$$

Then T is the polarized anti-Jordan triple system. Therefore $\mathfrak{M}_{mn}(R)$ is an anti-Jordan pair with product $\{x, y, z\} = \langle x, y, z \rangle$.

Lemma 1.2.4. *Let C be a commutative, associative and unital k -algebra. Then $\mathfrak{M}_{mn}(C) \cong \mathfrak{M}_{nm}(C) = (\mathfrak{M}_{mn}(C))^{\text{op}}$.*

Proof. Define a mapping $(f^+, f^-) : \mathfrak{M}_{mn}(C) \rightarrow \mathfrak{M}_{nm}(C)$ by

$$f^+(x) = x^t, \quad f^-(y) = -y^t$$

for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$. Then it is clear that the map (f^+, f^-) is linear and bijective. We will check that (f^+, f^-) is a homomorphism of anti-Jordan pairs. Let $x, z \in M_{mn}(C)$ and $y, w \in M_{nm}(C)$, then

$$f^+\{x, y, z\} = \{x, y, z\}^t = \{x^t, -y^t, z^t\} = \{f^+(x), f^-(y), f^+(z)\}.$$

Similarly we can check that $f^-\{y, x, w\} = \{f^-(y), f^+(x), f^-(w)\}$. \square

Lemma 1.2.5. *Let $V = (V^+, V^-)$ be an anti-Jordan pair. Then the anti-Jordan triple system $T(V) = V^+ \oplus V^-$ is simple if and only if $V = (V^+, V^-)$ is simple.*

Proof. If (W^+, W^-) is an ideal of V , then $W^+ \oplus W^-$ is an ideal of $T(V)$, hence if $T(V)$ is simple then V simple. Assume now that V is simple and I is an ideal of $T(V)$. The multiplication rule of $T(V)$ shows

$$\{I \cap V^\epsilon, V^{-\epsilon}, V^\epsilon\} + \{V^\epsilon, I \cap V^{-\epsilon}, V^\epsilon\} \subset I \cap V^\epsilon,$$

i.e., $(I \cap V^+, I \cap V^-)$ is an ideal of V . If it equals V , we have $I = V$. So let us assume $(I \cap V^+, I \cap V^-) = 0$. Then, for $x = x^+ + x^- \in I$ one concludes $\langle x, V^{-\epsilon}, V^\epsilon \rangle = \{x^\epsilon, V^{-\epsilon}, V^\epsilon\} \subset I \cap V^\epsilon = 0$ and $\langle V^\epsilon, x, V^\epsilon \rangle = \{V^\epsilon, x^{-\epsilon}, V^\epsilon\} \subset I \cap V^\epsilon = 0$. Hence $(x^+, x^-) \in Z = (Z^+, Z^-)$ with

$$Z^\epsilon = \{z \in T(V) : \langle z, V^{-\epsilon}, V^\epsilon \rangle = 0 = \langle V^\epsilon, z, V^\epsilon \rangle\}$$

But Z is obviously an ideal of V , which vanishes by simplicity of V . Hence $(x^+, x^-) = 0$ and $I = 0$. \square

Lemma 1.2.6. *An anti-Jordan triple system T is simple if and only if $V(T) = (T, T)$ is simple or a direct sum of two simple ideals, $V(T) \cong (W^+, W^-) \oplus (W^-, W^+) = W \oplus W^{\text{op}}$. In the second case $T = T(W)$ is the polarized anti-Jordan triple system associated to W .*

Proof. Obviously, if $V(T)$ is simple then T is simple, since if I is an ideal of T , then (I, I) is an ideal of $V(T)$. If $V(T) = W \oplus W^{\text{op}}$ and W is a simple ideal, then $T = T(W)$ is simple by Lemma 1.2.5.

Let T be a simple anti-Jordan triple system and let $W = (W^+, W^-)$ be an ideal of $V(T)$ with $0 \neq W \neq V(T)$. Then $W^+ + W^-$ and $W^+ \cap W^-$ are ideals of T . Since T is simple and $0 \neq W \neq V(T)$, we have $W^+ \cap W^- = 0$ and $T = W^+ \oplus W^-$. Therefore from Lemma 1.2.5, we get W is simple, so W^{op} is simple too. Hence $V(T) = (T, T) \cong W \oplus W^{\text{op}}$. It then follows that $T = W^+ \oplus W^-$ is the polarized triple system. Indeed, $\langle W^\epsilon, W^{-\epsilon}, T \rangle \subset (W \cap W^{\text{op}}) = \{0\}$ and $\langle W^\epsilon, W^{-\epsilon}, W^+ \rangle \subset W^\epsilon$, since W and W^{op} are subpairs. \square

Theorem 1.2.7. *Let $\mathfrak{M}_{mn}(R) = (V^+, V^-)$ be the rectangular anti-Jordan pair, where R is an associative k -algebra and $mn > 1$.*

- (a) *Let \bar{I} be an ideal of R . Then $(M_{mn}(\bar{I}), M_{nm}(\bar{I}))$ is an ideal of $\mathfrak{M}_{mn}(R)$.*
- (b) *Conversely, if I is an ideal of $\mathfrak{M}_{mn}(R)$, then there exists an ideal \bar{I} of R such that $I = (M_{mn}(\bar{I}), M_{nm}(\bar{I}))$.*
- (c) *In particular, $\mathfrak{M}_{mn}(R)$ is simple if and only if R is simple.*

Proof. We will use the basic multiplication rule for matrix units E_{ij} :

$$\{E_{ij}, E_{pq}, E_{uv}\} = \delta_{jp}\delta_{qu}E_{iv} - \delta_{vp}\delta_{qi}E_{uj} \quad (6)$$

Since $(m, n) \neq (1, 1)$, after replacing $\mathfrak{M}_{mn}(R)$ by $\mathfrak{M}_{mn}^{\text{op}}(R)$ if $n = 1$, we may assume $n \geq 2$.

(a) It is clear that $(M_{mn}(\bar{I}), M_{nm}(\bar{I})) = (I^+, I^-)$ is a k -module. So we only need to check that

$$\{I^\epsilon, V^{-\epsilon}, V^\epsilon\} + \{V^+, I^{-\epsilon}, V^\epsilon\} \subseteq I^\epsilon, \quad \epsilon = \pm.$$

Let $a_{ij}E_{ij} \in I^\epsilon$, $b_{pq}E_{pq} \in V^{-\epsilon}$ and $c_{uv}E_{uv} \in V^\epsilon$, where $a_{ij} \in \bar{I}$, $b_{pq}, c_{uv} \in R$. Then

$$\{a_{ij}E_{ij}, b_{pq}E_{pq}, c_{uv}E_{uv}\} = a_{ij}b_{pq}c_{uv}\delta_{jp}\delta_{qu}E_{iv} - c_{uv}b_{pq}a_{ij}\delta_{vp}\delta_{qi}E_{uj}$$

Since \bar{I} is an ideal of R , $a_{ij}b_{pq}c_{uv}$, $c_{uv}b_{pq}a_{ij} \in \bar{I}$, and this gives

$$\{a_{ij}E_{ij}, b_{pq}E_{pq}, c_{uv}E_{uv}\} \in I^\epsilon.$$

By linearity of $\{\dots\}$, we get $\{I^\epsilon, V^{-\epsilon}, V^\epsilon\} \subset I^\epsilon$.

Similarly, $\{V^\epsilon, I^{-\epsilon}, V^\epsilon\} \subset I^\epsilon$. Therefore $\{I^\epsilon, V^{-\epsilon}, V^\epsilon\} + \{V^+, I^{-\epsilon}, V^\epsilon\} \subset I^\epsilon$. Hence $(M_{mn}(\bar{I}), M_{nm}(\bar{I}))$ is an ideal of \mathfrak{M}_{mn} .

(b) **Step (1).** Let $x = \sum a_{ij}E_{ij} \in I^\epsilon$. We will prove $a_{ij}E_{ij} \in I^\epsilon$ and that we can move every entry in a matrix in I^ϵ to any position in another matrix in I^ϵ .

Let $\sum a_{ij}E_{ij} \in I^-$, $E_{qp} \in V^+$, $E_{uq}, E_{pv} \in V^-$ and choose $p \neq u$ (recall $n \geq 1$). Then

$$\begin{aligned} \{\{\sum a_{ij}E_{ij}, E_{qp}, E_{uq}\}, E_{qp}, E_{pv}\} &= \{-\sum a_{pj}E_{uj}, E_{qp}, E_{pv}\} \\ &= -a_{pq}E_{uv} \in I^- \quad (\text{since } I \text{ is an ideal}). \end{aligned}$$

Therefore $a_{pq}E_{uv} \in I^-$, since I^- is an abelian group. Let $E_{vu} \in V^+$, $E_{rv} \in V^-$ and choose $r \neq u$ (recall $n \geq 1$). Then $\{a_{pq}E_{uv}, E_{vu}, E_{rv}\} = -a_{pq}E_{rv} \in I^-$. This implies $a_{pq}E_{rv} \in I^-$. Similarly, we can prove that if $\sum a_{ij}E_{ij} \in I^+$, then $a_{ij}E_{ij} \in I^+$ and we can move every entry in a matrix in I^+ to any position in another matrix in I^+ .

Step (2). Let $a_{ij}E_{ij} \in I^+$. Choose $r \neq j$. Then

$$\{E_{ri}, a_{ij}E_{ij}, E_{jk}\} = a_{ij}E_{rk} \in I^-$$

Let $b_{ij}E_{ij} \in I^-$. Choose $k \neq i$. Then

$$\{E_{ri}, b_{ij}E_{ij}, E_{jk}\} = b_{ij}E_{rk} \in I^+$$

Hence every entry of a matrix in I^ϵ is also an entry in a matrix in $I^{-\epsilon}$.

Step (3). Let $\bar{I} = \{r \in R; rE_{11} \in I^+\}$. From step (1) and step (2) it is clear that $I = (M_{mn}(\bar{I}), M_{nm}(\bar{I}))$. So we are done if we can show that \bar{I} is an ideal of R . It is clear that \bar{I} is a k -module. We will check that $R\bar{I} \subset \bar{I}$ and $\bar{I}R \subset \bar{I}$.

Since I is an ideal, $\{rE_{11}, E_{11}, aE_{12}\} = raE_{12} \in I^+$ for all $r \in R$, $a \in \bar{I}$. By step (2) we get $raE_{11} \in I^+$, which yields $ra \in \bar{I}$. Similarly, $ar \in \bar{I}$. Hence \bar{I} is an ideal of R .

(c) is obvious. □

Example 1.2.8. Let $n \in \mathbb{N}$, $n \geq 2$ and let C be a commutative unital and associative k -algebra with $\frac{1}{2} \in k$. Put

$$\begin{aligned} N^+(C) &= N^+ = \{x \in M_{nn}(C) : x^t = x\} \text{ and} \\ N^-(C) &= N^- = \{x \in M_{nn}(C) : x^t = -x\}. \end{aligned}$$

Then it is easy to check that the *symmetric-skew* pair $\mathfrak{ss}_n(C) = (N^+, N^-)$ is a subpair of anti-Jordan pair $\mathfrak{M}_{nn}(C)$, hence an anti-Jordan pair with operation

$$\{x, y, z\} = x(yz) + (yz)^t x.$$

Theorem 1.2.9. Let $\mathfrak{ss}_n(C) = (N^+, N^-)$ be as in Example 1.2.8.

- (a) Let \bar{I} be an ideal of C . Then $(N^+(\bar{I}), N^-(\bar{I}))$ is an ideal of $\mathfrak{ss}_n(C)$.
- (b) Conversely, if $n > 2$ and $I = (I^+, I^-)$ is an ideal of $\mathfrak{ss}_n(C)$, then there exists an ideal \bar{I} of C such that $I = (N^+(\bar{I}), N^-(\bar{I}))$.
- (c) In particular, $\mathfrak{ss}_n(C)$ is simple if and only if $C = F$ is a field.

Proof. Let E_{ij} denote the matrix units. Put

$$H_{pq} = E_{pq} + E_{qp} \text{ and } F_{pq} = E_{pq} - E_{qp}$$

for all p, q with $1 \leq p \neq q \leq n$.

(a) Note that $N^+(\bar{I}) = N^+ \cap M_{nn}(\bar{I})$ and $N^-(\bar{I}) = N^- \cap M_{nn}(\bar{I})$. By Theorem 1.2.7 we know that $(M_{nn}(\bar{I}), M_{nn}(\bar{I}))$ is an ideal of $\mathfrak{M}_{nn}(C)$. Hence $(N^+(\bar{I}), N^-(\bar{I})) = (M_{nn}(\bar{I}), M_{nn}(\bar{I})) \cap (N^+, N^-)$ is an ideal of $\mathfrak{ss}_n(C)$, since the intersection of an ideal and a subpair is an ideal of the subpair.

(b) **Step (1).** Let $x = \sum_{1 \leq i < j \leq n} a_{ij} F_{ij} \in I^-$. We will prove $a_{ij} F_{ij} \in I^-$ and that we can move every entry in a matrix in I^- to any position in another matrix in I^- .

Let $H_{qp} \in N^+$, $F_{vq} \in V^-$ where $1 \leq p \neq v \neq q \neq p \leq n$. Then for any u with $1 \leq u \neq q \leq n$ we have

$$\begin{aligned} \left\{ \left\{ \sum_{1 \leq i < j \leq n} a_{ij} F_{ij}, H_{qp}, F_{pv} \right\}, H_{qp}, F_{qu} \right\} &= \left\{ \sum_{i < q} a_{iq} F_{iv} + \sum_{q < j} a_{qj} F_{vj}, H_{qp}, F_{qu} \right\} \\ &= \sum_{i < q} a_{iq} \delta_{ip} F_{uv} + \sum_{q < j} a_{qj} \delta_{pj} F_{vu}. \end{aligned}$$

Therefore

$$\left\{ \left\{ \sum_{1 \leq i < j \leq n} a_{ij} F_{ij}, H_{qp}, F_{pv} \right\}, H_{qp}, F_{qu} \right\} = \begin{cases} a_{pq} F_{uv} \in I^- & \text{if } p < q, \\ a_{qp} F_{vu} \in I^- & \text{if } q < p. \end{cases}$$

Put $u = p$, then $\{a_{pq} F_{pv}, E_{vv}, F_{vq}\} = a_{pq} F_{pq} \in I^-$ for $p < q$. Hence $a_{ij} F_{ij} \in I^-$ and we can move every entry in a matrix in I^- to any position in another matrix in I^- for all $1 \leq i < j \leq n$.

Now let $x = \sum_{1 \leq i \leq n} x_{ii} E_{ii} + \sum_{1 \leq l < r \leq n} x_{lr} H_{lr} \in I^+$, $F_{pq}, F_{uq} \in I^-$ and $E_{pp}, E_{uu} \in I^+$ with $1 \leq p \neq v \neq q \neq p \leq n$ and $1 \leq u \neq q \leq n$. Then

$$\left\{ \left\{ \sum_i x_{ii} E_{ii} + \sum_{l < r} x_{lr} H_{lr}, F_{pq}, E_{pp} \right\}, F_{uq}, E_{uu} \right\} = x_{qq} H_{pu} \in I^+$$

Therefore $\{x_{qq} H_{pu}, F_{pq}, E_{qq}\} = x_{qq} H_{qu} \in I^+$, we get

$$\{x_{qq} H_{qu}, \frac{1}{2} F_{uq}, E_{qq}\} = x_{qq} E_{qq} \in I^+.$$

$$\text{Also } \left\{ \left\{ \sum_i x_{ii} E_{ii} + \sum_{l < r} x_{lr} H_{lr}, F_{pv}, E_{vv} \right\}, F_{qu}, E_{uu} \right\} = x_{pq} H_{uv} \in I^+.$$

This implies $\{x_{pq} H_{uv}, \frac{1}{2} F_{vu}, E_{uu}\} = x_{pq} E_{uu} \in I^+$. Take $u = p$, we get $x_{pq} H_{pv} \in I^+$. Therefore $\{x_{pq} H_{pv}, F_{vq}, E_{qq}\} = x_{pq} H_{pq} \in I^+$. Hence $x_{ii} E_{ii}, x_{pq} H_{pq} \in I^+$ and we can move every entry in a matrix in I^+ to any position in another matrix in I^+ for all $1 \leq i \leq n$ and $1 \leq p < q \leq n$.

Step (2). Let $x_{pq} F_{pq} \in I^-$ where $1 \leq p < q \leq n$ and $x_{pq} \in C$. Then

$$\{E_{pp}, x_{pq} F_{pq}, E_{qq}\} = x_{pq} H_{pq} \in I^+$$

Let $b_{pp}E_{pp}, b_{pq}H_{pq} \in I^+$ and $F_{pq}, F_{qp} \in I^-$ for all p, q, v with $1 \leq p \neq q \neq v \leq n$.
Then

$$\{F_{pq}, b_{pp}E_{pp}, F_{pv}\} = b_{pp}F_{vq} \in I^-,$$

$$\{F_{pq}, b_{pq}H_{pq}, F_{pv}\} = b_{pq}F_{pv} \in I^-.$$

Hence every entry of a matrix in I^ϵ is also an entry in a matrix in $I^{-\epsilon}$.

Step (3). Let $\bar{I} = \{a \in C; aE_{11} \in I^+\}$. From step (1) and step (2) it is clear that $I = (N^+(\bar{I}), N^-(\bar{I}))$. So we are done if we can show that \bar{I} is an ideal of C .

It is clear that \bar{I} is a C -module. We will check that $C\bar{I} \subset \bar{I}$ and $\bar{I}C \subset \bar{I}$. Since I is an ideal, $\{cE_{11}, F_{12}, aE_{22}\} = caH_{12} \in I^+$ for all $c \in C, a \in \bar{I}$. By step (2) we get $caE_{11} \in I^+$, which yields $ca \in \bar{I}$. Since C is commutative, we get $ac \in \bar{I}$. Hence \bar{I} is an ideal of C .

(c) is obvious. □

Example 1.2.10. For $n = 2$ the symmetric-skew pair $\mathfrak{ss}_n(C) = (N^+, N^-)$ is not simple.

Proof. Note that for $n = 2$, $N^- = \text{span}\{F_{12} = E_{12} - E_{21}\}$. Therefore for any $\alpha, \beta \in C$ we have $\{\alpha F_{12}, x, \beta F_{21}\} = \alpha\beta(F_{12}xF_{12} - F_{12}xF_{12}) = 0$ for all $x \in N^+$. Put $I^+ = N^+$ and $I^- = \{0\}$. Then $\{I^+, N^-, N^+\} \subset I^+$, and $\{N^+, I^-, N^+\} = 0 = \{I^-, N^+, N^-\} = \{N^-, I^+, N^-\}$. This implies $\{I^\epsilon, N^{-\epsilon}, N^\epsilon\} + \{N^\epsilon, I^{-\epsilon}, N^\epsilon\} \subset I^\epsilon$, for $\epsilon = \pm$. Hence (I^+, I^-) is an ideal of the symmetric-skew pair $\mathfrak{ss}_n(C)$.

Example 1.2.11. Let C be a commutative associative and unital k -algebra. Let T be a C -module with an alternating bilinear form $B : T \times T \rightarrow C$. Then T is an anti-Jordan triple system with triple product $\langle x, y, z \rangle = B(x, y)z + B(y, z)x + B(x, z)y$ (see Example 1.1.8). Hence by Lemma 1.2.1 we have $V(T) = (T, T)_B$ with triple product $\{\dots\}_B^+ = \{\dots\}_B^- = \langle \dots \rangle_B$ is an anti-Jordan pair with involution (Id, Id) , called the *symplectic anti-Jordan pair* (with respect to B).

Lemma 1.2.12. Let C be a commutative associative and unital k -algebra with $\frac{1}{2} \in k$. Let T be a C -module with alternating bilinear form $B : T \times T \rightarrow C$ which is strongly

non-degenerate in the sense that if $0 \neq x \in T$, then $1 \in B(x, T)$. Then the symplectic anti-Jordan pair $V(T) = (T, T)_B$ is simple.

For example if C is a field of characteristic not 2 and B is a non-degenerate form in the usual sense, then the symplectic anti-Jordan pair is simple.

Proof. Let $(0, 0) \neq (I^+, I^-)$ be an ideal of the symplectic anti-Jordan pair $V(T) = (T, T)_B$. Let $0 \neq x \in I^+$, choose $y \in T$ such that $B(x, y) = 1$. Therefore $\{x, y, y\} = B(x, y)y + B(y, y)x + B(x, y)y = 2y \in I^+$. This implies $y \in I^+$, since $\frac{1}{2} \in C$. Let $z \in T$, then $\{x, y, z\} = B(x, y)z + B(y, z)x + B(x, z)y \in I^+$. We get $B(x, y)z = z \in I^+$, since $B(y, z)x, B(x, z)y \in I^+$. Hence $T = I^+$. Similarly we can prove that $I^- = T$. \square

1.3 Associative Pairs

Let $A = (A^+, A^-)$ be a pair of k -modules with k -trilinear maps $\langle \dots \rangle : A^\epsilon \times A^{-\epsilon} \times A^\epsilon \rightarrow A^\epsilon$, $(x, y, z) \rightarrow \langle x, y, z \rangle$ for $\epsilon = \pm$. Then A is called an *associative pair of first kind* (see [12, Sec. 4.1]), if

$$\langle \langle x, y, z \rangle, u, v \rangle = \langle x, \langle y, z, u \rangle, v \rangle = \langle x, y, \langle z, u, v \rangle \rangle$$

for $x, z, v \in A^\epsilon$, $y, u \in A^{-\epsilon}$ and $\epsilon \in \{+, -\}$.

We denote the left and right multiplication in A by $L(x, y)z = \langle x, y, z \rangle = R(y, z)x$. Let R_A be the k -subalgebra of $\text{End}_k(A^+) \times \text{End}_k(A^-)^{op}$ spanned by (Id, Id) and all elements $xy := (L(x, y), R(x, y))$ where $(x, y) \in (A^+, A^-)$. Let S_A be the k -subalgebra of $\text{End}_k(A^-) \times \text{End}_k(A^+)^{op}$, spanned by (Id, Id) and all elements $yx := (L(y, x), R(y, x))$ where $(x, y) \in (A^+, A^-)$. Then A^+ is a $R_A - S_A$ -bimodule for the actions $(\alpha_1, \alpha_2)x = \alpha_1(x)$ and $x(\beta_1, \beta_2) = \beta_2(x)$. Similarly A^- is a $S_A - R_A$ -bimodule for the actions $(\beta_1, \beta_2)y = \beta_1(y)$ and $y(\alpha_1, \alpha_2) = \alpha_2(y)$. The set

$$\mu_A = \begin{pmatrix} R_A & A^+ \\ A^- & S_A \end{pmatrix}$$

is an associative k -algebra with respect to the usual matrix product, called the *standard embedding* of A . It is defined in [11], see also [2].

Example 1.3.1. Let $A = \mathfrak{M}_{mn}(R) = (M_{mn}(R), M_{nm}(R)) = (A^+, A^-)$, where R is an associative unital k -algebra. Then $\mathfrak{M}_{mn}(R)$ is an associative pair of first kind with triple product

$$\langle \dots \rangle : A^\epsilon \times A^{-\epsilon} \times A^\epsilon \rightarrow A^\epsilon, (x, y, z) \rightarrow \langle x, y, z \rangle = xyz$$

for $\epsilon = \pm$.

Lemma 1.3.2. *Let $v \in M_{mn}(R)$ such that $vx = 0$ for all $x \in M_{mn}(R)$ or $yv = 0$ for all $y \in M_{nm}(R)$, then $v = 0$.*

Proof. Put $v = \sum_{1 \leq i, j \leq m} v_{ij} E_{ij}$, then $\left(\sum_{i \leq i, j \leq m} v_{ij} E_{ij} \right) E_{s1} = 0$ for all $1 \leq s \leq m$. Therefore $\sum_{1 \leq i \leq m} v_{is} E_{i1} = 0$ for all $1 \leq s \leq m$. Hence $v = 0$.

Also $E_{1s} \sum_{\substack{1 \leq i, j \leq m \\ i \neq j}} v_{ij} E_{ij} = 0$ for all $1 \leq s \leq m$. Therefore $\sum_{1 \leq j \leq m} v_{sj} E_{1j} = 0$ for all $1 \leq s \leq m$. Hence $v = 0$. \square

Lemma 1.3.3. *Let (η^+, η^-) be an automorphism of an associative pair of first kind $A = (A^+, A^-)$. Then (η^+, η^-) can be extended to an automorphism of the standard embedding μ_A of A .*

$$\begin{aligned} \text{Proof. Define } \tilde{\eta} : \mu_A \rightarrow \mu_A \text{ by } \tilde{\eta} & \begin{pmatrix} (r^+, r^-) & x^+ \\ y^- & (s^-, s^+) \end{pmatrix} \\ &= \begin{pmatrix} (\eta^+ r^+ (\eta^+)^{-1}, \eta^- r^- (\eta^-)^{-1}) & \eta^+ x^+ \\ \eta^- y^- & (\eta^- s^- (\eta^-)^{-1}, \eta^+ s^+ (\eta^+)^{-1}) \end{pmatrix} \end{aligned}$$

for $(r^+, r^-) \in R_A$, $(s^-, s^+) \in S_A$, $x^+ \in A^+$ and $y^- \in A^-$.

It is clear that $\tilde{\eta}$ is linear and bijective. We will check that $\tilde{\eta}$ is a homomorphism of algebras:

$$\begin{aligned} & \tilde{\eta} \left(\begin{pmatrix} (r_1^+, r_1^-) & x_1^+ \\ y_1^- & (s_1^-, s_1^+) \end{pmatrix} \begin{pmatrix} (r_2^+, r_2^-) & x_2^+ \\ y_2^- & (s_2^-, s_2^+) \end{pmatrix} \right) \\ &= \tilde{\eta} \begin{pmatrix} (r_1^+, r_1^-)(r_2^+, r_2^-) + x_1^+ y_2^- & (r_1^+, r_1^-) x_2^+ + x_1^+ (s_2^-, s_2^+) \\ y_1^- (r_2^+, r_2^-) + (s_1^-, s_1^+) y_2^- & y_1^- x_2^+ + (s_1^-, s_1^+) (s_2^-, s_2^+) \end{pmatrix}. \end{aligned}$$

Put $r = (\eta^+ r_1^+ (\eta^+)^{-1}, \eta^- r_1^- (\eta^-)^{-1})$, $r' = (\eta^+ r_2^+ (\eta^+)^{-1}, \eta^- r_2^- (\eta^-)^{-1})$, $s = (\eta^- s_1^- (\eta^-)^{-1}, \eta^+ s_1^+ (\eta^+)^{-1})$ and $s' = (\eta^- s_2^- (\eta^-)^{-1}, \eta^+ s_2^+ (\eta^+)^{-1})$, then

$$\begin{aligned} & \tilde{\eta} \left(\begin{pmatrix} (r_1^+, r_1^-) & x_1^+ \\ y_1^- & (s_1^-, s_1^+) \end{pmatrix} \begin{pmatrix} (r_2^+, r_2^-) & x_2^+ \\ y_2^- & (s_2^-, s_2^+) \end{pmatrix} \right) \\ &= \begin{pmatrix} r r' + \eta^+(x_1^+) \eta^-(y_2^-) & \eta^+(r_1^+ x_2^+) + \eta^+(s_2^+ x_1^+) \\ \eta^-(r_2^- y_1^-) + \eta^-(s_1^- y_2^-) & \eta^-(y_1^-) \eta^+(x_2^+) + s s' \end{pmatrix} \\ &= \begin{pmatrix} r & \eta^+ x_1^+ \\ \eta^- y_1^- & s \end{pmatrix} \begin{pmatrix} r' & \eta^+ x_2^+ \\ \eta^- y_2^- & s' \end{pmatrix} \\ &= \tilde{\eta} \begin{pmatrix} (r_1^+, r_1^-) & x_1^+ \\ y_1^- & (s_1^-, s_1^+) \end{pmatrix} \tilde{\eta} \begin{pmatrix} (r_2^+, r_2^-) & x_2^+ \\ y_2^- & (s_2^-, s_2^+) \end{pmatrix}. \end{aligned}$$

\square

Lemma 1.3.4. *Let R be an associative and unital k -algebra. Let $\mathfrak{M}_{mn} = (M_{mn}(R), M_{nm}(R)) = (V^+, V^-)$. Then the associative algebra*

$$M_{m+n, m+n}(R) = \begin{pmatrix} V^+ \cdot V^- & V^+ \\ V^- & V^- \cdot V^+ \end{pmatrix}$$

is isomorphic to $\mu_{\mathfrak{M}_{mn}}$.

Proof. Define $\psi : M_{m+n, m+n}(R) \rightarrow \mu_{\mathfrak{M}_{mn}}$ by

$$\begin{aligned} \psi & \left(\begin{array}{cc} \sum_i x_i^+ \cdot y_i^- & x^+ \\ y^- & \sum_j u_j^- \cdot v_j^+ \end{array} \right) \\ & = \left(\begin{array}{cc} \sum_i (L(x_i^+, y_i^-), R(x_i^+, y_i^-)) & x^+ \\ y^- & \sum_j (L(u_j^-, v_j^+), R(u_j^-, v_j^+)) \end{array} \right). \end{aligned}$$

The following shows that ψ is well-defined and injective.

$$\begin{aligned} \sum_i (L(x_i^+, y_i^-), R(x_i^+, y_i^-)) & = \sum_r (L(u_r^+, v_r^-), R(u_r^+, v_r^-)) \\ \Leftrightarrow \sum_i (L(x_i^+, y_i^-), R(x_i^+, y_i^-))(z^+, z^-) & = \sum_r (L(u_r^+, v_r^-), R(u_r^+, v_r^-))(z^+, z^-) \\ \Leftrightarrow \left(\sum_i x_i^+ \cdot y_i^- \right) z^+ & = \left(\sum_r u_r^+ \cdot v_r^- \right) z^+ \quad , \quad z^- \left(\sum_i x_i^+ \cdot y_i^- \right) = z^- \left(\sum_r u_r^+ \cdot v_r^- \right) \\ & \Leftrightarrow \sum_i x_i^+ \cdot y_i^- = \sum_r u_r^+ \cdot v_r^- \quad (\text{by Lemma 1.3.2}). \end{aligned}$$

It is easily verified that ψ is an algebra homomorphism. Hence it is an isomorphism. \square

Lemma 1.3.5. *Let $a \in M_{mn}(F)$ and $0 \neq b \in M_{mm}(F)$ such that $ayb = 0$ for all $y \in M_{nm}(F)$, where F is a field. Then $a = 0$.*

Proof. Let $a = \sum_{i,j} a_{ij} E_{ij}$ and $b = \sum_{r,s} b_{rs} E_{rs}$. Since $b \neq 0$, there exist $b_{pq} \neq 0$.

Then $\left(\sum_{i,j} a_{ij} E_{ij} \right) b_{pq}^{-1} E_{lp} \left(\sum_{r,s} b_{rs} E_{rs} \right) = 0$ for $1 \leq l \leq n$. This implies

$$\sum_{i,j} a_{ij} E_{ij} \sum_s b_{pq}^{-1} b_{ps} E_{ls} = 0.$$

Therefore $\sum_{i,s} a_{il} b_{pq}^{-1} b_{ps} E_{is} = 0$. Hence $a_{il} b_{pq}^{-1} b_{ps} = 0$ for all i, s . In particular for $s = q$ we get $a_{il} = 0$ and then $a = 0$. \square

Theorem 1.3.6. *Let F be a field and let $\eta = (\eta^+, \eta^-)$ be an automorphism of the associative pair $\mathfrak{M}_{mn}(F) = (M_{mn}(F), M_{nm}(F))$. Then there exist invertible matrices $a \in GL_n(F)$ and $b \in GL_m(F)$ such that $\eta^+(x) = axb^{-1}$, $\eta^-(y) = bya^{-1}$ for all $x \in M_{mn}(F)$ and $y \in M_{nm}(F)$.*

Proof. Let $\eta = (\eta^+, \eta^-)$ be an automorphism of the associative pair $\mathfrak{M}_{mn}(F)$. Then by Lemma 1.3.3 and Lemma 1.3.4, we can extend η to an automorphism $\tilde{\eta}$ of an associative algebra $M_{m+n, m+n}(F)$.

Since F is a field, $M_{m+n, m+n}(F)$ is central simple. Hence all automorphisms are inner. Therefore there exists $u \in M_{m+n, m+n}(F)$ such that

$$\tilde{\eta}(x) = u x u^{-1} \quad \text{for all } x \in M_{m+n, m+n}(F).$$

Put $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ and $u^{-1} = v = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$, then

$$\tilde{\eta} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} u_{11} x v_{21} & u_{11} x v_{22} \\ u_{21} x v_{21} & u_{21} x v_{22} \end{pmatrix}$$

and

$$\tilde{\eta} \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} u_{12} y v_{11} & u_{12} y v_{12} \\ u_{22} y v_{11} & u_{22} y v_{12} \end{pmatrix}.$$

Since $\tilde{\eta}$ leaves $M_{mn}(F)$ and $M_{nm}(F)$ invariant,

$$u_{11} x v_{21} = 0 = u_{21} x v_{22} = u_{21} x v_{21} \quad \text{for all } x \in M_{mn}(F) \quad (7)$$

and

$$u_{12} y v_{11} = 0 = u_{12} y v_{12} = u_{22} x v_{12} \quad \text{for all } y \in M_{nm}(F). \quad (8)$$

Also

$$uv = \begin{pmatrix} u_{11} v_{11} + u_{12} v_{21} & u_{11} v_{12} + u_{12} v_{22} \\ u_{21} v_{11} + u_{22} v_{21} & u_{21} v_{12} + u_{22} v_{22} \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix} \quad (9)$$

and

$$vu = \begin{pmatrix} v_{11}u_{11} + v_{12}u_{21} & v_{11}u_{12} + v_{12}u_{22} \\ v_{21}u_{11} + v_{22}u_{21} & v_{21}u_{12} + v_{22}u_{22} \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix}. \quad (10)$$

Case (i) Suppose $v_{22} = 0$:

From (9) and (10) we get, $v_{21}u_{11} = 0$, $v_{21}u_{12} = I_n$, $u_{21}v_{12} = I_n$. From (7) we have $u_{11}xv_{21} = 0$, this implies $u_{11}xv_{21}u_{12} = 0$, i.e. $u_{11}x = 0$ for all $x \in M_{mn}(F)$. Hence by Lemma 1.3.2 we get $u_{11} = 0$.

From (8) we have $u_{12}yv_{11} = 0$, this implies $v_{21}u_{12}yv_{11} = 0$. Therefore $yv_{11} = 0$ for all $y \in M_{nm}(F)$. Hence $v_{11} = 0$ by Lemma 1.3.2. Now from (10) we have $v_{12}u_{22} = 0$, this implies $u_{21}v_{12}u_{22} = 0$. Since $u_{21}v_{12} = I_n$ we get $u_{22} = 0$. Hence

$$u = \begin{pmatrix} 0 & u_{12} \\ u_{21} & 0 \end{pmatrix} \text{ and } u^{-1} = v = \begin{pmatrix} 0 & v_{12} \\ v_{21} & 0 \end{pmatrix}.$$

Also $\tilde{U}_{12} : F^n \rightarrow F^m$, $x \mapsto u_{12}x$ and $\tilde{V}_{21} : F^m \rightarrow F^n$, $y \mapsto v_{21}y$ are linear maps. Since $\tilde{U}_{12}\tilde{V}_{21} = \text{Id}_{F^m}$ and $\tilde{V}_{21}\tilde{U}_{12} = \text{Id}_{F^n}$, we get $F^m \cong F^n$, which yields $m = n$. Then

$$\tilde{\eta} \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & u_{12}yv_{12} \\ u_{21}xv_{21} & 0 \end{pmatrix}$$

But this is not possible, because this switches $M_{mn} \leftrightarrow M_{nm}$.

Case (ii) $v_{22} \neq 0$:

We know from (7) that $u_{21}xv_{22} = 0$ for all $x \in M_{mn}(F)$. Therefore $u_{21} = 0$ by Lemma 1.3.5. Then (10) yields $v_{21}u_{11} = 0$ and $v_{11}u_{11} = I_m$. This implies $v_{21} = 0$. Therefore $u_{22}v_{22} = I_n$.

From (8) we know $u_{12}yv_{11} = 0$ and $v_{22}yv_{12} = 0$. This implies $u_{12}yv_{11}u_{11} = 0$ and $v_{22}u_{22}yv_{12} = 0$. Therefore $u_{12}y = 0$ and $yv_{12} = 0$. From Lemma (1.3.2) we get $u_{12} = 0$ and $v_{12} = 0$. Hence

$$u = \begin{pmatrix} u_{11} & 0 \\ 0 & u_{22} \end{pmatrix} \text{ and } u^{-1} = v = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}.$$

Therefore $\eta^+(x) = u_{11}xv_{22}$ and $\eta^-(y) = u_{22}yv_{11}$. □

Chapter 2

Automorphisms of the anti-Jordan pair \mathfrak{M}_{mn}

Unless specified otherwise, k is a commutative associative and unital ring, R is an associative and unital k -algebra and C is a commutative associative and unital k -algebra. We denote by k^\times the set of invertible elements of k and by $\text{GL}_r(R)$ the group of all invertible matrices in $M_{rr}(R)$.

2.1 Preliminary results

Lemma 2.1.1. (a) *Let $x \in M_{nn}(C)$ and $z \in M_{mm}(C)$. Then $xyz = y$ for all $y \in M_{nm}(C)$ if and only if $x = \alpha I_n$ and $z = \alpha^{-1} I_m$ for some $\alpha \in C^\times$.*

(b) *Let $x \in M_{nn}(R)$ and $z \in M_{mm}(R)$. Then $xyz = y$ for all $y \in M_{nm}(R)$ and $zux = u$ for all $u \in M_{mn}(R)$ if and only if $x = \alpha I_n$ and $z = \alpha^{-1} I_m$ for some $\alpha \in R^\times$.*

Proof. (a) Suppose that $xyz = y$ for all $y \in M_{nm}(C)$. Write $x = \sum_{1 \leq i, j \leq n} x_{ij} E_{ij}$ and $z = \sum_{1 \leq k, l \leq m} z_{kl} E_{kl}$. Then

$$\sum_{i, j, k, l} x_{ij} E_{ij} E_{pq} z_{kl} E_{kl} = E_{pq} \quad \text{for all } 1 \leq p \leq n, 1 \leq q \leq m.$$

This implies $\sum_{i,l} x_{ip}z_{ql}E_{il} = E_{pq}$. Hence $x_{ip}z_{ql} = 0$ for $(i,l) \neq (p,q)$ and $x_{pp}z_{qq} = 1$.

So $z_{qq}x_{pp} = 1$. Therefore $x_{pp}, z_{qq} \in C^\times$.

If $n > 1$, there exist p, i such that $p \neq i$. Therefore $(i, q) \neq (p, q)$ for all q and $x_{ip}z_{qq} = 0$. Hence $x_{ip} = 0$, thus x is diagonal.

If $m > 1$, then there exist q, l such that $q \neq l$. This implies $(i, l) \neq (p, q)$ for all p, i . So in particular $x_{pp}z_{ql} = 0$, yields $z_{ql} = 0$, i.e., z is diagonal.

The equation $1 = x_{pp}z_{qq}$ for all p, q immediately implies that $x = \alpha I_n, z = \alpha^{-1} I_n$ for some $\alpha \in C^\times$. The converse is clear.

(b) Suppose that $xyz = y$ for all $y \in M_{nm}(R)$ and $zux = u$ for all $u \in M_{mn}(R)$. Put $x = \sum_{1 \leq i, j \leq n} x_{ij}E_{ij}$, $z = \sum_{1 \leq k, l \leq m} z_{kl}E_{kl}$, $y = E_{pq}$ and $u = E_{qp}$. Then a similar calculation as we did in part (a) shows that $x_{pp}, z_{qq} \in R^\times$ and $z_{kq}x_{pj} = 0$ for $(k, j) \neq (q, p)$. Therefore, as above we get that $x = \alpha I_n$ and $z = \alpha^{-1} I_m$ for some $\alpha \in R^\times$. The converse is again obvious. \square

Lemma 2.1.2. *Let $mn > 1$, and suppose for all $u, w \in M_{mn}(R)$ and some $v \in M_{nm}(R)$ we have $uvw = wvu$. Then $v = 0$.*

Proof. Let $v = \sum_{i,j} v_{ij}E_{ij}$ and $uvw = wvu$ for all $u, w \in M_{mn}$. Then

$$E_{pq} \left(\sum_{i,j} v_{ij}E_{ij} \right) E_{st} = E_{st} \left(\sum_{i,j} v_{ij}E_{ij} \right) E_{pq}$$

where $1 \leq p, s \leq m$ and $1 \leq q, t \leq n$. This implies $\sum_j v_{qj}E_{pj}E_{st} = \sum_j v_{tj}E_{sj}E_{pq}$.

Hence $v_{qs}E_{pt} = v_{tp}E_{sq}$.

Suppose $m \geq 2$. So there exist p, s such that $p \neq s$. Therefore $v_{qs} = 0 = v_{tp}$ for all q, t . This implies $v = 0$.

Suppose $n \geq 2$. Then there exist t, q such that $t \neq q$. This implies $v_{qs} = 0 = v_{tp}$ for all s, p , hence $v = 0$. \square

Lemma 2.1.3. *Let $x \in M_{mm}(R)$ and $y \in M_{nn}(R)$ such that $xz = zy$ for all $z \in M_{mn}(R)$, then $x = \alpha I_m$ and $y = \alpha I_n$, for some $\alpha \in R$.*

Proof. Suppose $xz - zy = 0$ for all $z \in M_{mn}(R)$. Put $x = \sum_{i,j} x_{ij}E_{ij}$ and $y = \sum_{r,s} y_{rs}E_{rs}$. Then $\left(\sum_{i,j} x_{ij}E_{ij}\right)E_{1t} = E_{1t}\left(\sum_{r,s} y_{rs}E_{rs}\right)$ $1 \leq t \leq n$. This implies $\sum_i x_{i1}E_{it} = \sum_s y_{ts}E_{1s}$. Therefore

$$x_{11} = y_{tt}, \quad 1 \leq t \leq n \quad \text{and} \quad y_{rs} = 0, \quad 1 \leq r \neq s \leq n.$$

Also $\left(\sum_{i,j} x_{ij}E_{ij}\right)E_{u1} = E_{u1}\left(\sum_{r,s} y_{rs}E_{rs}\right)$, $1 \leq u \leq m$. Then $\sum_i x_{iu}E_{i1} = \sum_s y_{1s}E_{us}$. Therefore $x_{uu} = y_{11}$, $1 \leq u \leq m$ and $x_{ij} = 0$, $1 \leq i \neq j \leq m$. Hence $x = \alpha I_m$ and $y = \alpha I_n$ for some $\alpha \in R$. \square

Lemma 2.1.4. *Let $m \geq 2$. Then there does not exist $a, b \in M_{mm}(C)$, $m \geq 2$ such that $ax^tb = x$ for all $x \in M_{mm}(C)$.*

Proof. Suppose there exist $a = (a_{ij}), b = \sum b_{rs}E_{rs} \in M_{mm}(C)$ such that $ax^tb = x$ for all $x \in M_{mm}(C)$. Then, evaluating the condition for $x = E_{11}$, we get

$$E_{11} = \left(\sum_{1 \leq i,j \leq m} a_{ij}E_{ij}E_{11}\right) \left(\sum_{1 \leq r,s \leq m} b_{rs}E_{rs}\right) = \sum_{1 \leq i,s \leq m} a_{i1}b_{1s}E_{is}.$$

Therefore $a_{11}b_{11} = 1$ and $a_{is}b_{1s} = 0$ for $i, s \geq 2$. This implies $a_{11}, b_{11} \in C^\times$.

Also for $x = E_{12}$ we get

$$E_{12} = \left(\sum_{1 \leq i,j \leq m} a_{ij}E_{ij}E_{21}\right) \left(\sum_{1 \leq r,s \leq m} b_{rs}E_{rs}\right) = \sum_{1 \leq i,s \leq m} a_{i2}b_{1s}E_{is}.$$

Hence $a_{12}b_{11} = 0$ and $a_{12}b_{12} = 1$. Since $b_{11} \in C^\times$, it follows that $a_{12} = 0$ and $a_{12} \in C^\times$, which is not possible. \square

Lemma 2.1.5. *Let $mn > 1$, $a, d \in GL_m(R)$ and $b, c \in GL_n(R)$. Let $\eta_{a,b}^+ : M_{mn}(R) \rightarrow M_{mn}(R)$ and $\eta_{d,c}^- : M_{nm}(R) \rightarrow M_{nm}(R)$ be linear maps, given by*

$$\eta_{a,b}^+(x) = axb^{-1}, \quad \eta_{d,c}^-(y) = cyd^{-1}.$$

- (a) Then the pair of maps $\eta_{a,b} = (\eta_{a,b}^+, \eta_{d,c}^-)$ is an automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(R) = (M_{mn}(R), M_{nm}(R))$ iff there exist $\alpha \in R^\times$ such that $c = \alpha b$ and $d = \alpha a$. Hence

$$\bigwedge := \{(\eta_{a,b}^+, \eta_{a,b}^-) : a \in \text{GL}_m(R), b \in \text{GL}_n(R)\} \subseteq \text{Aut}(\mathfrak{M}_{mn}(R))$$

is a subgroup of $\text{Aut}(\mathfrak{M}_{mn}(R))$ such that $(a, b) \rightarrow (\eta_{a,b}^+, \eta_{a,b}^-)$ induces an isomorphism

$$\text{GL}_m(R) \times \text{GL}_n(R) / \{(\alpha I_m, \alpha I_n) : \alpha \in R^\times\} \cong \bigwedge.$$

- (b) Let $m = n$, then the pair of linear maps $\eta_{a,b} = (\eta_{a,b}^+, \eta_{a,b}^-)$ is an involution of the anti-Jordan pair $\mathfrak{M}_{nn}(R) = (M_{nn}(R), M_{nn}(R))$ if and only if there exist $\alpha \in R^\times$ such that $b = \alpha a^{-1}$.

Proof. (a) Let the pair of maps $(\eta_{a,b}^+, \eta_{d,c}^-)$ be an automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(R) = (M_{mn}(R), M_{nm}(R))$. Then

$$\eta_{a,b}^+ \{x, y, z\} = \{\eta_{a,b}^+(x), \eta_{d,c}^-(y), \eta_{a,b}^+(z)\}.$$

This implies $a\{x, y, z\}b^{-1} = \{axb^{-1}, cyd^{-1}, azb^{-1}\}$. Therefore

$$x(y - b^{-1}cyd^{-1}a)z = z(y - b^{-1}cyd^{-1}a)x$$

for all $x, z \in M_{mn}(R)$. Then from Lemma 2.1.2 we get $y = b^{-1}cyd^{-1}a$ for all $y \in M_{nm}(R)$. Also

$$\eta_{d,c}^- \{y, x, w\} = \{\eta_{d,c}^-(y), \eta_{a,b}^+(x), \eta_{d,c}^-(w)\}.$$

By a similar calculation as above we get $x = d^{-1}axb^{-1}c$ for all $x \in M_{mn}(R)$. Hence $c = \alpha b$ and $d = \alpha a$ (by Lemma 2.1.1). The converse is clear.

(b) The pair of maps $(\eta_{a,b}^+, \eta_{a,b}^-)$ is an involution if and only if $\eta_{a,b}^+ \circ \eta_{a,b}^- = \text{Id}_{M_{nn}(R)}$ and $\eta_{a,b}^- \circ \eta_{a,b}^+ = \text{Id}_{M_{nn}(R)} \Leftrightarrow \eta_{a,b}^+ \circ \eta_{a,b}^-(x) = x$ and $\eta_{a,b}^- \circ \eta_{a,b}^-(y) = y$ for all $x, y \in M_{nn}(R) \Leftrightarrow \eta_{a,b}^+(bxa^{-1}) = x$ and $\eta_{a,b}^-(ayyb^{-1}) = y \Leftrightarrow abxa^{-1}b^{-1} = x$ and $bayb^{-1}a^{-1} \Leftrightarrow$ there exist $\alpha \in R^\times$ such that $b = \alpha a^{-1}$ (by Lemma 2.1.1). \square

Lemma 2.1.6. Let $a, c \in \text{GL}_n(C)$ and $b, d \in \text{GL}_m(C)$, $mn > 1$. Let $\mu_{a,b}^+ : M_{mn}(C) \rightarrow M_{nm}(C)$ and $\mu_{c,d}^- : M_{nm}(C) \rightarrow M_{mn}(C)$ be linear maps, given by

$$\mu_{a,b}^+(x) = ax^tb^{-1}, \mu_{c,d}^-(y) = -dy^tc^{-1}$$

for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$. Then

- (a) $(\mu_{a,b}^+, \mu_{c,d}^-)$ is an anti-automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C)$ iff there exist $\alpha \in C^\times$ such that $d = \alpha b$ and $c = \alpha a$. We will abbreviate $\mu_{a,b} = (\mu_{a,b}^+, \mu_{a,b}^-)$.
- (b) $\mu_{a,b}$ is an involution of the anti-Jordan pair $\mathfrak{M}_{mn}(C)$ if and only if there exist $\beta \in C^\times$ such that $-\beta a^t = a$ and $\beta b^t = b$ and $\beta^2 = 1$.

Proof. (a): Since $a, c \in GL_n(C)$ and $b, d \in GL_m(C)$, it is easy to see that the pair of linear maps $(\mu_{a,b}^+, \mu_{c,d}^-)$ is bijective. Let $x, z \in M_{mn}(C)$ and $y \in M_{nm}(C)$, then

$$\begin{aligned}
\mu_{a,b}^+\{x, y, z\} &= \{\mu_{a,b}^+(x), \mu_{c,d}^-(y), \mu_{a,b}^+(z)\} \\
\Leftrightarrow a\{x, y, z\}^t b^{-1} &= \{ax^t b^{-1}, -dy^t c^{-1}, az^t b^{-1}\} \\
\Leftrightarrow az^t y^t x^t b^{-1} - ax^t y^t z^t b^{-1} &= -ax^t b^{-1} dy^t c^{-1} az^t b^{-1} \\
&\quad + az^t b^{-1} dy^t c^{-1} ax^t b^{-1} \\
\Leftrightarrow z^t (y^t - b^{-1} dy^t c^{-1} a) x^t &= x^t (y^t - b^{-1} dy^t c^{-1} a) z^t \\
\Leftrightarrow y^t &= b^{-1} dy^t c^{-1} a \quad (\text{by Lemma 2.1.2}) \\
\Leftrightarrow d &= \alpha b, c = \alpha a \quad (\text{by Lemma 2.1.1}).
\end{aligned}$$

Similarly we can prove that $\mu_{c,d}^-\{y, x, w\} = \{\mu_{c,d}^-(y), \mu_{a,b}^+(x), \mu_{c,d}^-(w)\}$ iff there exist $\alpha \in C^\times$ such that $c = \alpha a$ and $d = \alpha b$ for all $y \in M_{nm}(C)$ and $x \in M_{mn}(C)$.

(b): Since the map $(\mu_{a,b}^+, \mu_{a,b}^-)$ is an anti-isomorphism, $\mu_{a,b}$ is an involution of the anti-Jordan pair $\mathfrak{M}_{mn}(C)$ if and only if $\mu_{a,b}^+ \circ \mu_{a,b}^- = \text{Id}_{M_{nm}(C)}$ and $\mu_{a,b}^- \circ \mu_{a,b}^+ = \text{Id}_{M_{mn}(C)}$
 $\Leftrightarrow \mu_{a,b}^+ \circ \mu_{a,b}^-(y) = y$ for all $y \in M_{nm}(C)$ and $\mu_{a,b}^- \circ \mu_{a,b}^+(x) = x$ for all $x \in M_{mn}(C)$
 $\Leftrightarrow \mu_{a,b}^+(-by^t a^{-1}) = y$ for all $y \in M_{nm}(C)$ and $\mu_{a,b}^-(ax^t b^{-1}) = x$ for all $x \in M_{mn}(C)$
 $\Leftrightarrow -a(a^t)^{-1} y b^t b^{-1} = y$ for all $y \in M_{nm}(C)$ and $-b(b^t)^{-1} x a^t a^{-1} = x$ for all $x \in M_{mn}(C)$
 \Leftrightarrow there exist $\beta \in C^\times$ such that $a = -\beta a^t$ and $b = \beta b^t$ (by Lemma 2.1.1). Now $a = -\beta a^t$ and $a^t = -\beta a$ yields $a = \beta^2 a$. So $\beta^2 = 1$. \square

Proposition 2.1.7. (a) Let $\zeta = (\zeta^+, \zeta^-)$ be a C -linear automorphism of the anti-Jordan pair $(\mathfrak{M}_{1n}(C))$, $n > 1$. Then there exist $a \in GL_n(C)$ such that $\zeta^+ = \eta_{1C,a}^+$ and $\zeta^- = \eta_{1C,a}^-$. Hence $\zeta^+(x) = xb$ and $\zeta^-(y) = by$ and

$$\{(\eta_{1C,b}^+, \eta_{1C,b}^-) : b \in GL_n(C)\} = \text{Aut}(\mathfrak{M}_{1n}(C)) \cong GL_n(C).$$

(b) Let $\xi = (\xi^+, \xi^-)$ be an C -linear anti-automorphism of the anti-Jordan pair $\mathfrak{M}_{1n}(C)$, $n > 1$. Then there exist $a \in \text{GL}_n(C)$ such that $\xi^+ = \mu_{1,a}^+$ and $\xi^- = \mu_{1,a}^-$.

(c) Let $\nu = (\nu^+, \nu^-)$ be an C -linear involution of the anti-Jordan pair $\mathfrak{M}_{1n}(C)$, $n > 1$. Then there exist $a \in \text{GL}_n(C)$ such that $\nu^+ = \mu_{1,a}^+$, $\nu^- = \mu_{a,1}^-$ and $a^t = -a$.

Proof. (a) Let $\zeta = (\zeta^+, \zeta^-) \in \text{Aut}(\mathfrak{M}_{1n}(C))$, then there exist $a, b \in \text{GL}_n(C)$ such that $\zeta^+(x) = xa = \eta_{1_C, a^{-1}}^+(x)$ and $\zeta^-(y) = by = \eta_{1_C, b}^-(y)$. Then by Lemma 2.1.5 we get $b = a^{-1}$. But $\{(\eta_{1_C, b}^+, \eta_{1_C, b}^-) : b \in \text{GL}_n(C)\} \subseteq \text{Aut}(\mathfrak{M}_{1n}(C))$. Therefore

$$\{(\eta_{1_C, b}^+, \eta_{1_C, b}^-) : b \in \text{GL}_n(C)\} = \text{Aut}(\mathfrak{M}_{1n}(C)).$$

(b) Let $\xi = (\xi^+, \xi^-)$ be an anti-automorphism of the anti-Jordan pair $\mathfrak{M}_{1n}(C)$. By Lemma 2.1.6 we know that $\mu^+ : M_{1n} \rightarrow M_{n1}$, $\mu^+(x) = x^t$ and $\mu^- : M_{n1} \rightarrow M_{1n}$, $\mu^-(y) = -y^t$ is an anti-automorphism. Hence $\xi \circ \mu = (\xi^- \circ \mu^+, \xi^+ \circ \mu^-)$ is an automorphism of the anti-Jordan pair $\mathfrak{M}_{1n}(C)$. Therefore by part (a), there exist $a \in \text{GL}_n(C)$ such that

$$\xi^- \circ \mu^+(x) = xa^{-1} \text{ and } \xi^+ \circ \mu^-(x) = ay,$$

for all $x \in M_{1n}(C)$ and $y \in M_{n1}(C)$. This implies $\xi^-(x^t) = xa^{-1}$ and $\xi^+(-y^t) = ay$, i.e. $\xi^+(x) = \mu_{a,1}^+(x) = ax^t$ and $\xi^-(y) = \mu_{a,1}^-(y) = -y^t a^{-1}$.

(c) Let $\nu = (\nu^+, \nu^-)$ is an anti-automorphism of the anti-Jordan pair $\mathfrak{M}_{1n}(C)$. Therefore by part (b), we get there exist $a \in \text{GL}_n(C)$ such that

$$\nu^+ = \mu_{a,1}^+(x) = ax^t, \nu^- = \mu_{a,1}^-(y) = -y^t a^{-1}.$$

Hence by Lemma 2.1.6 we get $a^t = -a$. □

Remark: In the rest of this section we will determine $\text{Aut}(\mathfrak{M}_{mn}(C))$ and the involutions of $\mathfrak{M}_{mn}(C)$ where $m, n \geq 2$, i.e. $m \geq 2$ and $n \geq 2$.

It is easy to check that the map $\xi : \text{GL}_m(C) \times \text{GL}_m(C) \rightarrow \text{GL}_m(C) \times \text{GL}_m(C)$ defined by

$$\xi(a, b) = ((b^t)^{-1}, (a^t)^{-1})$$

for all $a, b \in \mathrm{GL}_m(C)$ is an automorphism. Now define an action of $H = \mathbb{Z}_2$ on $K = \mathrm{GL}_m(C) \times \mathrm{GL}_m(C)$ by

$$\bar{0} \cdot (a, b) = (a, b), \quad \bar{1} \cdot (a, b) = \xi(a, b) = ((b^t)^{-1}, (a^t)^{-1}).$$

Then this is an action of the group \mathbb{Z}_2 because $\bar{1} \cdot (\bar{1} \cdot (a, b)) = (\bar{1} + \bar{1}) \cdot (a, b) = \bar{0} \cdot (a, b) = (a, b)$ for all $a, b \in \mathrm{GL}_m(C)$. Let G be the set of all ordered pairs (k, h) with $h \in H = \mathbb{Z}_2$ and $k \in K = \mathrm{GL}_m(C) \times \mathrm{GL}_m(C)$ and define the following multiplication on G :

$$(k_1, h_1)(k_2, h_2) = (k_1(h_1 \cdot k_2), h_1 + h_2).$$

This multiplication makes $G = (\mathrm{GL}_m(C) \times \mathrm{GL}_m(C)) \rtimes \mathbb{Z}_2$ into a group.

Lemma 2.1.8. *Let C be a commutative, associative and unital k -algebra. For $a, b \in \mathrm{GL}_m(C)$, $m \geq 2$, define $\eta_{a,b}$ as in Lemma 2.1.5 and $\mu_{a,b}$ as in Lemma 2.1.6. Then $\eta_{a,b}$ and $\mu_{a,b}$ are automorphisms of the anti-Jordan pair $\mathfrak{M}_{mm}(C)$. Moreover, the map $\tau : (\mathrm{GL}_m(C) \times \mathrm{GL}_m(C)) \rtimes \mathbb{Z}_2 \rightarrow \mathrm{Aut}(\mathfrak{M}_{mm}(C))$ defined by*

$$\tau(a, b, \bar{0}) = (\eta_{a,b}^+, \eta_{a,b}^-) \quad \text{and} \quad \tau(a, b, \bar{1}) = (\mu_{a,b}^+, \mu_{a,b}^-)$$

for all $a, b \in \mathrm{GL}_m(C) \times \mathrm{GL}_m(C)$, is a group homomorphism with

$$\ker(\tau) = \{(\alpha I_m, \alpha I_m, 0) : \alpha \in C^\times\}$$

and $\tau(\mathrm{GL}_m(C) \times \mathrm{GL}_m(C) \rtimes \mathbb{Z}_2) = \{\mu_{a,b}, \eta_{a,b} : a, b \in \mathrm{GL}_m(C)\} = \tilde{\Lambda}$.

Proof. Since $m = n$, every anti-automorphism of $\mathfrak{M}_{mm}(C)$ is an automorphism. Hence from Lemma 2.1.6 and Lemma 2.1.5 we get that $\mu_{a,b}$ and $\eta_{a,b}$ are automorphisms of $\mathfrak{M}_{mm}(C)$. Let $a, b, c, d \in \mathrm{GL}_m(C)$ and $x, y \in \mathfrak{M}_{mm}(C)$. We will check that τ is a group homomorphism:

$$\begin{aligned} \tau((a, b, \bar{0})(c, d, \bar{0})) &= \tau(ac, bd, \bar{0}) = \eta_{ac, bd} \quad \text{and} \\ \tau(a, b, \bar{0})\tau(c, d, \bar{0})(x, y) &= \eta_{a,b}\eta_{c,d}(x, y) = (acx(bd)^{-1}, bdy(ac)^{-1}) \\ &= \eta_{ac, bd}(x, y), \\ \tau((a, b, \bar{0})(c, d, \bar{1})) &= \tau(ac, bd, \bar{1}) = \mu_{ac, bd} \quad \text{and} \\ \tau(a, b, \bar{0})\tau(c, d, \bar{1})(x, y) &= \eta_{a,b}\mu_{c,d}(x, y) = (acx^t(bd)^{-1}, -bdy^t(ac)^{-1}) \\ &= \mu_{ac, bd}(x, y), \end{aligned}$$

$$\begin{aligned}
\tau((a, b, \bar{1})(c, d, \bar{0})) &= \tau(a(d^t)^{-1}, b(c^t)^{-1}, \bar{1}) = \mu_{a(d^t)^{-1}, b(c^t)^{-1}} \quad \text{and} \\
\tau(a, b, \bar{1})\tau(c, d, \bar{0})(x, y) &= \mu_{a,b}\eta_{c,d}(x, y) = (a(d^t)^{-1}x^t c^t b^{-1}, -b(c^t)^{-1}y^t d^t a^{-1}) \\
&= \mu_{a(d^t)^{-1}, b(c^t)^{-1}}(x, y), \\
\tau((a, b, \bar{1})(c, d, \bar{1})) &= \tau(a(d^t)^{-1}, b(c^t)^{-1}, \bar{0}) = \eta_{a(d^t)^{-1}, b(c^t)^{-1}} \quad \text{and} \\
\tau(a, b, \bar{1})\tau(c, d, \bar{1})(x, y) &= (\mu_{a,b}^+, \mu_{a,b}^-)(\mu_{c,d}^+, \mu_{c,d}^-)(x, y) \\
&= (a(d^t)^{-1}x^t c^t b^{-1}, b(c^t)^{-1}y^t d^t a^{-1}) = \eta_{a(d^t)^{-1}, b(c^t)^{-1}}(x, y).
\end{aligned}$$

Hence τ is a group homomorphism. Note that

$$\begin{aligned}
\tau(a, b, \bar{0})(x, y) = (x, y) &\Leftrightarrow (\eta_{a,b}^+, \eta_{a,b}^-)(x, y) = (x, y) \\
&\Leftrightarrow (axb^{-1}, bya^{-1}) = (x, y) \Leftrightarrow axb^{-1} = x, bya^{-1} = y.
\end{aligned}$$

Therefore by Lemma 2.1.1 there exist $\alpha \in C^\times$ such that $x = \alpha I_m = y$. Now

$$\begin{aligned}
\tau(a, b, \bar{1})(x, y) = (x, y) &\Leftrightarrow (\mu_{a,b}^+, \mu_{a,b}^-)(x, y) = (x, y) \\
&\Leftrightarrow (ax^t b^{-1}, -by^t a^{-1}) = (x, y) \Leftrightarrow ax^t b^{-1} = x, -by^t a^{-1} = y.
\end{aligned}$$

From Lemma 2.1.4 we get that there does not exist $a, b \in \text{GL}_m(C)$ such that

$$ax^t b^{-1} = x \quad \text{and} \quad -by^t a^{-1} = y$$

for all $x, y \in M_{mm}(C)$. Hence $\ker(\tau) = \{(\alpha I_m, \alpha I_m, 0) : \alpha \in C^\times\}$. □

Lemma 2.1.9. *Let $\mathfrak{M}_{mn}(C) = (M_{mn}(C), M_{nm}(C)) = (V^+, V^-)$. Then*

$$\{f \in \text{End}_k(V^+) : [f, D(x, y)] = 0, \text{ for all } (x, y) \in (V^+, V^-)\} = C\text{Id}_{V^+}.$$

Proof. Suppose that f lies in the left hand side. Note that for all $z \in V^+$,

$$[f, D(x, y)] = 0 \Leftrightarrow f\{x, y, z\} = \{x, y, f(z)\}.$$

This implies $f\{E_{ij}, E_{pq}, E_{rs}\} = \{E_{ij}, E_{pq}, f(E_{rs})\}$ for all i, j, p, q, r, s . Then for $1 \leq q \neq i \leq m$, we get $\{E_{is}, E_{sq}, E_{qs}\} = \delta_{ss}\delta_{qq}E_{is} - \delta_{ss}\delta_{qi}E_{qs} = E_{is}$. This implies

$$f(E_{is}) = f\{E_{is}, E_{sq}, E_{qs}\} = \{E_{is}, E_{sq}, f(E_{qs})\}.$$

We know $f(E_{qs}) = \sum_{a,b} c_{ab}E_{ab}$, therefore

$$\begin{aligned}
f(E_{is}) &= \{E_{is}, E_{sq}, \sum_{a,b} c_{ab}E_{ab}\} \\
&= \sum_{a,b} c_{ab}\{E_{is}, E_{sq}, E_{ab}\} \\
&= \sum_{a,b} c_{ab}(\delta_{qa}E_{ib} - \delta_{bs}\delta_{qi}E_{as}) \\
&= \sum_b c_{qb}E_{ib} \quad (\text{since } q \neq i).
\end{aligned}$$

Since $E_{is} = -\{E_{qs}, E_{sq}, E_{is}\}$, we get

$$\begin{aligned}
f(E_{is}) &= -\{E_{qs}, E_{sq}, f(E_{is})\} \\
&= -\{E_{qs}, E_{sq}, \sum_b c_{qb}E_{ib}\} \\
&= c_{qs}E_{is} \quad 1 \leq i \neq q \leq m, 1 \leq s \leq n.
\end{aligned} \tag{i}$$

It now follows

$$f(E_{qs}) = \{E_{qs}, E_{si}, f(E_{is})\} = \{E_{qs}, E_{si}, c_{qs}E_{is}\} = c_{qs}E_{qs}. \tag{ii}$$

$$f(E_{ip}) = -\{E_{qp}, E_{sq}, f(E_{is})\} = -\{E_{qp}, E_{sq}, c_{qs}E_{is}\} = c_{qs}E_{ip}. \tag{iii}$$

From Equation (i), (ii) and (iii) we get $f(E_{1s}) = c_{2s}E_{1s}$, $f(E_{1s}) = c_{1s}E_{1s}$ and $f(E_{1p}) = c_{21}E_{1p}$ for all p, s with $1 \leq p, s \leq n$. Therefore $c_{21} = c_{2s} = c_{1s}$ for all s . From Equation (i) we have $f(E_{is}) = c_{1s}E_{is} = c_{21}E_{is}$ for all i, s with $2 \leq i \leq m$ and $1 \leq s \leq n$.

Hence $f(E_{is}) = \alpha E_{is}$ for all $1 \leq i \leq m$ and $1 \leq s \leq n$ with some $\alpha \in C$. Now for arbitrary $c \in C$ we have $cE_{is} = \{cE_{is}, E_{sq}, E_{qs}\}$ for any $i \neq q$. Therefore $f(cE_{is}) = \{cE_{is}, E_{sq}, f(E_{qs})\} = \{cE_{is}, E_{sq}, \alpha E_{qs}\} = \alpha cE_{is}$. Hence $f = \alpha \text{Id}_{V^+}$. The inclusion from right to left is clear. \square

Corollary 2.1.10. *Let $\mathfrak{M}_{mn}(C) = (M_{mn}(C), M_{nm}(C)) = (V^+, V^-)$. Let (f^+, f^-) be an C -linear anti-automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C)$.*

(a) *If $f^+\{x, y, z\} = \{f^+(z), y^t, x^t\}$ for all $x, z \in V^+$ and $y \in V^-$, then there exists $\alpha \in C^\times$ such that $f^+(x) = \alpha x^t$ for all $x \in V^+$.*

(b) *If $f^-\{y, x, w\} = \{f^-(w), x^t, y^t\}$ for all $y, w \in V^-$ and $x \in V^+$, then there exists $\beta \in C^\times$ such that $f^-(y) = \beta y^t$ for all $y \in V^-$.*

Proof. (a): Define $g \in \text{End}_C(V^+)$ by $g = (f^+(x))^t$. Then

$$g\{x, y, z\} = (f^+\{x, y, z\})^t = \{f^+(z), y^t, x^t\}^t = \{x, y, g(z)\}.$$

Therefore by Lemma 2.1.9 we get there exists $\alpha \in C$ such that $g = \alpha \text{Id}_V^+$, i.e., $f^+(x) = \alpha x^t$ for all $x \in V^+$. Since f^+ is invertible, $\alpha \in C^\times$.

(b) follows similarly . □

Lemma 2.1.11. (a) Let $\phi \in \text{Aut}_k(C)$. Define $\tilde{\phi}^+ : M_{mn}(C) \rightarrow M_{mn}(C)$ and $\tilde{\phi}^- : M_{nm}(C) \rightarrow M_{nm}(C)$ by

$$\tilde{\phi}^+(x) := (\phi(x_{ij})), \quad \tilde{\phi}^-(y) := (\phi(y_{ij})),$$

for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$. Then $(\tilde{\phi}^+, \tilde{\phi}^-) \in \text{Aut}_k(\mathfrak{M}_{mn}(C))$ and the map $\text{Aut}_k(C) \rightarrow \text{Aut}_k(\mathfrak{M}_{mn}(C)), \phi \mapsto \tilde{\phi}$ is a group monomorphism.

(b) The group $\text{Aut}_C(\mathfrak{M}_{mn}(C))$ of all C -linear automorphisms of $\mathfrak{M}_{mn}(C)$ is a normal subgroup of $\text{Aut}_k(\mathfrak{M}_{mn}(C))$ and

$$\text{Aut}_k(\mathfrak{M}_{mn}(C)) = \text{Aut}_C(\mathfrak{M}_{mn}(C)) \rtimes (\text{Aut}_k(C))^\sim.$$

Proof. (a) is immediate.

(b) Let $\zeta = (\zeta^+, \zeta^-) \in \text{Aut}_k(\mathfrak{M}_{mn}(C))$ and let $f \in \text{End}_k(C)$ such that $f = c\text{Id}$ for some $c \in C$. Then $(\zeta^+ \circ c\text{Id} \circ (\zeta^+)^{-1})\{x, y, z\} = \zeta^+ \circ c\text{Id}\{(\zeta^+)^{-1}(x), (\zeta^-)^{-1}(y), (\zeta^+)^{-1}(z)\} = \zeta^+\{(\zeta^+)^{-1}(x), (\zeta^-)^{-1}(y), c\text{Id} \circ (\zeta^+)^{-1}(z)\}$. This implies

$$(\zeta^+ \circ c\text{Id} \circ (\zeta^+)^{-1})\{x, y, z\} = \{x, y, (\zeta^+ \circ c\text{Id} \circ (\zeta^+)^{-1})(z)\}$$

for all $x, z \in M_{mn}(C)$ and $y \in M_{nm}(C)$. Hence $\zeta^+ \circ c\text{Id} \circ (\zeta^+)^{-1} = \varphi(c)\text{Id}$ by Lemma 2.1.9 for some map $\varphi : C \rightarrow C$. This implies $\varphi(c)\text{Id}\zeta^+(x) = \zeta^+ \circ c\text{Id} \circ (\zeta^+)^{-1}\zeta^+(x) = \zeta^+(cx)$ for all $x \in M_{mn}(C)$ and $c \in C$. It is obvious that φ is k -linear.

The same argument applied to $(\zeta^+)^{-1} \circ c\text{Id} \circ \zeta^+$ proves the existence of a k -linear map $\psi : C \rightarrow C$ such that $\psi(c)\text{Id} = (\zeta^+)^{-1} \circ c\text{Id} \circ \zeta^+$ for all $c \in C$. Then $\varphi(\psi(c))\text{Id} = \zeta^+ \circ (\psi(c)\text{Id}) \circ (\zeta^+)^{-1} = \zeta^+ \circ (\zeta^+)^{-1} \circ c\text{Id} \circ \zeta^+ \circ (\zeta^+)^{-1} = \text{Id}_C$. Similarly $\psi \circ \varphi = \text{Id}_C$. So φ is bijective. Also $\varphi(c_1c_2) = \zeta^+ \circ (c_1c_2)\text{Id} \circ (\zeta^+)^{-1} = \zeta^+ \circ c_1\text{Id} \circ (\zeta^+)^{-1} \circ \zeta^+ \circ c_2\text{Id} \circ (\zeta^+)^{-1} = \varphi(c_1)\varphi(c_2)$ for all $c_1, c_2 \in C$. Therefore $\varphi \in \text{Aut}_k(C)$.

Similarly we can prove that $\zeta^- \circ c\text{Id} \circ (\zeta^-)^{-1} = \varphi_1(c)\text{Id}$, where $\varphi_1 \in \text{Aut}_k(C)$. Then $\varphi_1(c)\text{Id}\zeta^-(y) = \zeta^- \circ c\text{Id} \circ (\zeta^-)^{-1}\zeta^-(y) = \zeta^-(cy)$ for all $y \in M_{nm}(C)$. Note that $\zeta^+(c\{x, y, z\}) = \zeta^+\{x, cy, z\}$, since $c\{x, y, z\} = \{x, cy, z\}$. Therefore

$$\varphi(c)\{\zeta^+(x), \zeta^-(y), \zeta^+(z)\} = \{\zeta^+(x), \varphi_1(c)\zeta^-(y), \zeta^+(x)\}.$$

This implies $(\varphi(c) - \varphi_1(c))\{\zeta^+(x), \zeta^-(y), \zeta^+(z)\} = 0$ for all $x, z \in M_{mn}(C)$ and $y \in M_{nm}(C)$. Hence $\varphi(c) = \varphi_1(c)$. Therefore $\zeta^-(cy) = \varphi(c)\text{Id}\zeta^-(y)$.

Let $(\sigma^+, \sigma^-) \in \text{Aut}_C(\mathfrak{M}_{mn}(C))$. Let $c \in C$ and $x \in M_{mn}(C)$, then

$$\begin{aligned} \zeta^+ \circ \sigma^+ \circ (\zeta^+)^{-1}(cx) &= \zeta^+ \circ \sigma^+(\varphi^{-1}(c)(\zeta^+)^{-1}(x)) = \zeta^+(\varphi^{-1}(c)\sigma^+(\zeta^+)^{-1}(x)) \\ &= \varphi(\varphi^{-1}(c))\zeta^+ \circ \sigma^+ \circ (\zeta^+)^{-1}(x) = c\zeta^+ \circ \sigma^+ \circ (\zeta^+)^{-1}(x). \end{aligned}$$

Similarly $\zeta^- \circ \sigma^- \circ (\zeta^-)^{-1}(cx) = c\zeta^- \circ \sigma^- \circ (\zeta^-)^{-1}(x)$. Hence $\text{Aut}_C(\mathfrak{M}_{mn}(C))$ is a normal subgroup of $\text{Aut}_k(\mathfrak{M}_{mn}(C))$.

Now consider $\zeta = (\zeta^+, \zeta^-) \in \text{Aut}_k(\mathfrak{M}_{mn}(C))$. Define $\varphi \in \text{Aut}_k(C)$ as in the first part of the proof and note that $\tilde{\varphi} \in \text{Aut}_k(\mathfrak{M}_{mn}(C))$. Therefore

$$(\eta^+, \eta^-) = (\zeta^+ \circ (\tilde{\varphi}^+)^{-1}, \zeta^- \circ (\tilde{\varphi}^-)^{-1}) \in \text{Aut}_k(\mathfrak{M}_{mn}(C)).$$

We check that (η^+, η^-) is C -linear. Indeed,

$$\zeta^+ \circ (\tilde{\varphi}^+)^{-1}(cx) = \zeta^+(\varphi^{-1}(c)(\tilde{\varphi}^+)^{-1}(x)) = \varphi(\varphi^{-1}(c))\zeta^+((\tilde{\varphi}^+)^{-1}(x)) = \zeta^+ \circ (\tilde{\varphi}^+)^{-1}(x).$$

Similarly we can check that $\zeta^- \circ (\tilde{\varphi}^-)^{-1}(cx) = c\zeta^- \circ (\tilde{\varphi}^-)^{-1}(x)$. Therefore $(\eta^+, \eta^-) \in \text{Aut}_k(\mathfrak{M}_{mn}(C))$ and $\zeta^+ = \eta^+ \circ \tilde{\varphi}$ and $\zeta^- = \eta^- \circ \tilde{\varphi}$. Hence

$$\text{Aut}_k(\mathfrak{M}_{mn}(C)) = \text{Aut}_C(\mathfrak{M}_{mn}(C)) \rtimes (\text{Aut}_k(C))^\sim.$$

□

In the rest of the chapter we will consider only C -linear maps.

Lemma 2.1.12. *Let (f^+, f^-) be an automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C) = (M_{mn}(C), M_{nm}(C))$. Then there does not exist $\alpha \in C$ such that*

(a) $x^t y^t + f^-(y) f^+(x) = \alpha I_n$ and $f^+(x) f^-(y) - xy = \alpha I_m$ holds for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$, or

(b) $y^t x^t + f^+(x) f^-(y) = \alpha I_m$ and $f^-(y) f^+(x) - yx = \alpha I_n$ holds for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$, or

(c) $x^t y^t + f^-(y) f^+(x) = \alpha I_n$ and $f^+(x) f^-(y) + y^t x^t = \alpha I_m$ holds for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$.

Proof. (a) : Evaluating the condition for $x = 0$ shows $\alpha = 0$. Thus

$$x^t y^t + f^-(y) f^+(x) = 0 \quad (11)$$

and

$$f^+(x) f^-(y) - xy = 0 \quad (12)$$

To distinguish $E_{ij} \in M_{mn}(C)$ and $E_{pq} \in M_{nm}(C)$ we will (sometimes) write E_{ij}^{mn} and E_{pq}^{nm} . Let $x = E_{ij}^{mn}$ and $y = E_{ri}^{nm}$ such that $j \neq r$. We have

$$f^+\{E_{ij}, E_{ri}, E_{ir}\} = f^+(E_{ij}) f^-(E_{ri}) f^+(E_{ir}) - f^+(E_{ir}) f^-(E_{ri}) f^+(E_{ij}).$$

But $\{E_{ij}, E_{ri}, E_{ir}\} = 0 - E_{ir} E_{ri} E_{ij} = -E_{ij}^{mn}$, while using (12) and (11) to get

$$f^+(E_{ij}) f^-(E_{ri}) f^+(E_{ir}) = E_{ij}^{mn} E_{ri}^{nm} f^+(E_{ir}) = 0 \text{ and}$$

$$-f^+(E_{ir}) f^-(E_{ri}) f^+(E_{ij}) = f^+(E_{ir}) E_{ji}^{nm} E_{ir}^{mn} = f^+(E_{ir}) E_{jr}^{nn}.$$

Hence $-f^+(E_{ij}) = f^+(E_{ir}) E_{jr}^{nn}$. Multiply both sides by $f^-(E_{ji}) \in M_{nm}(F)$, to get $-f^-(E_{ji}) f^+(E_{ij}) = f^-(E_{ji}) f^+(E_{ir}) E_{jr}^{nn}$ and apply (11) to obtain

$$E_{ji}^{nm} E_{ij}^{mn} = -E_{ri}^{nm} E_{ij}^{mn} E_{jr}^{nn},$$

whence $E_{jj}^{nn} = -E_{rr}^{nn}$, a contradiction.

(b) follows from (a) by exchanging m and n , and $+/-$.

(c) Evaluating the condition for $x = 0$ shows $\alpha = 0$. Thus

$$x^t y^t + f^-(y) f^+(x) = 0 \quad (13)$$

and

$$f^+(x) f^-(y) + y^t x^t = 0 \quad (14)$$

holds for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$. Note that

$$f^+(x)f^-(y)f^+(x) = f^+(x)f^-(y)f^+(x).$$

Therefore $f^+(x)x^ty^t = y^tx^tf^+(x)$ (by equation (13) and (14)). This implies

$$(f^+(x)x^t)y^t = y^t(x^tf^+(x))$$

for all $y \in M_{nm}(C)$. Hence from Lemma 2.1.3, we get

$$f^+(x)x^t = \beta I_m \text{ and } x^tf^+(x) = \beta I_n.$$

We have $x^t(y^tz^t) = -x^tf^+(x)f^-(y) = \beta f^-(y)$, so $xyx \in C(f^-(y))^t$ for all $x \in M_{mn}(C)$. Now

$$E_{ii} = E_{ii}(E_{11} + E_{22})E_{ii} \in C(f^-(E_{11} + E_{22}))^t$$

for $i = 1, 2$ is a contradiction. □

Lemma 2.1.13. *Let (f^+, f^-) be an anti-isomorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C) = (M_{mn}(C), M_{nm}(C))$. Then there does not exist $\alpha \in C$ such that*

(a) $f^-(y)f^+(x) - xy = \alpha I_m$ and $f^+(x)f^-(y) - yx = \alpha I_n$ holds for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$, or

(b) $x^ty^t + f^+(x)f^-(y) = \alpha I_n$ and $f^-(y)f^+(x) - xy = \alpha I_m$ holds for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$, or

(c) $y^tx^t + f^-(y)f^+(x) = \alpha I_m$ and $f^+(x)f^-(y) - yx = \alpha I_n$ holds for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$.

Proof. Note that for f^ϵ as in the lemma, the maps $g^\epsilon(x) = \epsilon f^\epsilon(x)^t$ defines an automorphism of $\mathfrak{M}_{mn}(C)$. Applying the map $z \rightarrow -z^t$ to the equations above shows that the claims follow from Lemma 2.1.12. More precisely, (a), (b) and (c) reduce to (c), (b) and (a) of Lemma 2.1.12 respectively. □

2.2 Superalgebras

Let V be a k -module and let Γ be an abelian group. A Γ -gradation of the k -module V is a family $(V_\gamma)_{\gamma \in \Gamma}$ of k -submodules of V such that

$$V = \bigoplus_{\gamma \in \Gamma} V_\gamma.$$

The module V is said to be Γ -graded if it is equipped with a Γ -gradation. An element of V is called *homogeneous* of degree γ , $\gamma \in \Gamma$, if it is an element of V_γ . In case $\Gamma = \mathbb{Z}_2$ the elements of $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are also called *even* (resp. *odd*).

Let A be an algebra over k , i.e., A is a k -module together with a bilinear multiplication $A \times A \rightarrow A$, $(a, b) \rightarrow ab$. The algebra A is said to be Γ -graded if the underlying vector space of A is Γ -graded and $A_\gamma A_\delta \subset A_{\gamma+\delta}$ for $\delta, \gamma \in \Gamma$. A \mathbb{Z}_2 -graded algebra is called a *superalgebra*. A superalgebra S is said to be \mathbb{Z} -graded if we are given a family $(S_j)_{j \in \mathbb{Z}}$ of \mathbb{Z}_2 -graded subspaces of S such that

$$S = \bigoplus_{j \in \mathbb{Z}} S_j \quad \text{and}$$

$$S_i S_j \subset S_{i+j} \quad \text{for all } i, j \in \mathbb{Z}.$$

A \mathbb{Z} -gradation $(S_j)_{j \in \mathbb{Z}}$ is said to be *consistent* with the \mathbb{Z}_2 -gradation of a superalgebra S if

$$S_{\bar{0}} = \bigoplus_{j \in \mathbb{Z}} S_{2j} \quad , \quad S_{\bar{1}} = \bigoplus_{j \in \mathbb{Z}} S_{2j+1}.$$

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a superalgebra whose multiplication is denoted by a square bracket $[,]$. This implies in particular that

$$[L_\alpha, L_\beta] \subset L_{\alpha+\beta} \quad \text{for all } \alpha, \beta \in \mathbb{Z}_2.$$

We call L a *Lie superalgebra* if the multiplication satisfies the following identities:

$$[A, B] = -(-1)^{\alpha\beta} [B, A] \tag{15}$$

$$(-1)^{\gamma\alpha} [A, [B, C]] + (-1)^{\alpha\beta} [B, [C, A]] + (-1)^{\beta\gamma} [C, [A, B]] = 0 \tag{16}$$

for all $A \in L_\alpha, B \in L_\beta, C \in L_\gamma; \alpha, \beta, \gamma \in \mathbb{Z}_2$.

A \mathcal{Z} -grading of Lie superalgebra L is a (not necessarily consistent) \mathbb{Z} -grading $L = \bigoplus_{i \in \mathbb{Z}} L_i$ of L such that $L_i = 0$ for $i \notin \{\pm 1, 0\}$ and $L_i = (L_i \cap L_{\bar{0}}) \oplus (L_i \cap L_{\bar{1}})$ for $i \in \{0, \pm 1\}$.

A Lie superalgebra is called \mathcal{Z} -graded if it has a \mathcal{Z} -grading. A \mathcal{Z} -graded Lie superalgebra $L = L_1 \oplus L_0 \oplus L_{-1}$ will be called *Jordan \mathcal{Z} -graded* if

(i) $[L_1, L_{-1}] = L_0$, and

(ii) there exist a Jordan superpair (see [13] for the definition of a Jordan superpair) on (L_1, L_{-1}) whose Jordan triple product is related to the Lie product by

$$\{x, y, z\} = [[x, y], z] \text{ for all } x, z \in L_{\pm 1}, y \in L_{\mp 1}.$$

In this case, $V = (L_1, L_{-1})$ will be called the *associated Jordan superpair*.

Example 2.2.1. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an associative superalgebra over k . Then A^- is a Lie superalgebra with the supercommutator $[a, b] = ab - (-1)^{|a||b|}ba$, where $|x|$ denotes the degree of x . For example, let $A = A_{\bar{0}} \oplus A_{\bar{1}} =: M_{p|q}(R)$, where R is an associative k -algebra and

$$A_{\bar{0}} = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix} : a_1 \in M_{pp}(R), a_4 \in M_{qq}(R) \right\}$$

$$A_{\bar{1}} = \left\{ \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix} : a_2 \in M_{pq}(R), a_3 \in M_{qp}(R) \right\}$$

Then $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is an associative superalgebra, with respect to matrix multiplication and hence A^- is a Lie superalgebra.

Proof. Note that

$$\begin{aligned} [A, B] &= AB - (-1)^{\alpha\beta} BA \\ &= -(-1)^{\alpha\beta} (BA - (-1)^{\alpha\beta} AB) = -(-1)^{\alpha\beta} [B, A] \end{aligned}$$

for all $A \in A_\alpha$, $B \in A_\beta$; $\alpha, \beta \in \mathbb{Z}_2$. It remains to check (16). Let $A, C \in A_{\bar{0}}$ and $B \in A_{\bar{1}}$. Then

$$\begin{aligned} [A, [B, C]] &= ABC - BCA - ACB + CBA, \\ [B, [C, A]] &= BCA - CAB - BAC + ACB, \\ [C, [A, B]] &= CAB - ABC - CBA + BAC. \end{aligned}$$

Hence $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$. Similarly, for $A \in A_\alpha$, $B \in A_\beta$, $C \in A_\gamma$, $\alpha, \beta, \gamma \in \mathbb{Z}_2$, we can prove that $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$. Hence (16) holds.

Example 2.2.2. Let R be an associative k -algebra. We define $\mathfrak{sl}(m|n)(R)$ or $\mathfrak{sl}(m|n)$ for short by $\mathfrak{sl}(m|n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \in M_{mm}(R), b \in M_{mn}(R), c \in M_{nm}(R), \right.$
 $\left. d \in M_{nn}(R), (\text{tr}(a) - \text{tr}(d)) \in [R, R] \right\}$

where $[R, R]$ is the space of the supercommutators of R .

Put

$$\begin{aligned} L_1 &= \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in M_{mn}(R) \right\} \\ L_0 &= \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in M_{mm}(R), d \in M_{nn}(R), \text{tr}(a) - \text{tr}(d) \in [R, R] \right\} \\ L_{-1} &= \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} : c \in M_{nm}(R) \right\}. \end{aligned}$$

Thus $L_{\bar{0}} = L_0$, $L_{\bar{1}} = L_1 \oplus L_{-1}$ and $\mathfrak{sl}(m|n) = L_{\bar{0}} \oplus L_{\bar{1}} = L_1 \oplus L_0 \oplus L_{-1}$, is a 3-graded Lie superalgebra, with the usual supercommutator product. In fact, it is a subalgebra of $M_{m|n}(R)^-$.

Proof. To show that $\mathfrak{sl}(m|n)$ is a Lie superalgebra, it is enough to check that $\mathfrak{sl}(m|n)$ is closed under $[\cdot, \cdot]$, since any graded subspace of $M_{m|n}(R)^-$ which is closed under $[\cdot, \cdot]$ is a Lie superalgebra. It is obvious that $\mathfrak{sl}(m|n)$ is closed, since $[L_\epsilon, L_\epsilon] = 0$, $[L_\epsilon, L_{-\epsilon}] \subset L_0$, $[L_\epsilon, L_0] \subset L_\epsilon$ and $[L_0, L_0] \subset L_0$; $\epsilon = \pm 1$.

Lemma 2.2.3. *Let R be a associative and unital k -algebra. Let $u \in \mathrm{GL}_m(R)$ and $v \in \mathrm{GL}_n(R)$. Then the map*

$$\tilde{\theta}_{u,v} : \mathfrak{sl}(m|n)(R) \rightarrow \mathfrak{sl}(m|n)(R), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} u^{-1}au & u^{-1}bv \\ v^{-1}cu & v^{-1}dv \end{pmatrix}$$

is an automorphism of the Lie superalgebra $\mathfrak{sl}(m|n)(R)$. Hence

$$\tilde{\Omega} = \{\tilde{\theta}_{u,v} : u \in \mathrm{GL}_m(R), v \in \mathrm{GL}_n(R)\} \subseteq \mathrm{Aut}(\mathfrak{sl}(m|n)(R)).$$

Moreover, $\tilde{\Omega}$ is a subgroup of $\mathrm{Aut}(\mathfrak{sl}(m|n)(R))$ such that $(u, v) \rightarrow (\tilde{\theta}_{u,v})$ induces an isomorphism $\mathrm{GL}_m(R) \times \mathrm{GL}_n(R) / \{(\alpha I_m, \alpha I_n) : \alpha \in R^\times\} \cong \tilde{\Omega}$.

Proof. It is easy to see that $\tilde{\theta}_{u,v}$ is linear and bijective. Therefore we will check that $\tilde{\theta}_{u,v}$ is a homomorphism of Lie superalgebras:

$$\begin{aligned} & \tilde{\theta}_{u,v} \left[\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right] \\ &= \tilde{\theta}_{u,v} \begin{pmatrix} [a_1, a_2] + b_1c_2 + b_2c_1 & a_1b_2 - b_2d_1 + b_1d_2 - a_2b_1 \\ d_1c_2 - c_2a_1 + c_1a_2 - d_2c_1 & [d_1, d_2] + c_1b_2 + c_2b_1 \end{pmatrix} \\ &= \begin{pmatrix} u^{-1}([a_1, a_2] + b_1c_2 + b_2c_1)u & u^{-1}(a_1b_2 - b_2d_1 + b_1d_2 - a_2b_1)v \\ v^{-1}(d_1c_2 - c_2a_1 + c_1a_2 - d_2c_1)u & v^{-1}([d_1, d_2] + c_1b_2 + c_2b_1)v \end{pmatrix} \\ &= \left[\begin{pmatrix} u^{-1}a_1u & u^{-1}b_1v \\ v^{-1}c_1u & v^{-1}d_1v \end{pmatrix}, \begin{pmatrix} u^{-1}a_2u & u^{-1}b_2v \\ v^{-1}c_2u & v^{-1}d_2v \end{pmatrix} \right] \\ &= \left[\tilde{\theta}_{u,v} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \tilde{\theta}_{u,v} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right]. \end{aligned}$$

Let (u, v) be in the kernel of the map $(u, v) \rightarrow \tilde{\theta}_{u,v}$, then

$$\tilde{\theta}_{u,v} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This implies $\begin{pmatrix} u^{-1}au & u^{-1}bv \\ v^{-1}cu & v^{-1}dv \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Therefore $u = \alpha I_m$ and $v = \alpha I_n$ by Lemma 2.1.3. \square

Lemma 2.2.4. (a) Let $1/2 \in k$ and let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a 3-graded Lie superalgebra over k , where $L_{\bar{0}} = L_0$ and $L_{\bar{1}} = L_1 \oplus L_{-1}$. Then the pair of spaces (L_1, L_{-1}) with the trilinear map

$$\{x, y, z\} = [[x, y], z], \quad x, z \in L_{\epsilon}, y \in L_{-\epsilon}, \epsilon = \pm 1,$$

is an anti-Jordan pair.

(b) Conversely, we can associate to every anti-Jordan pair $V = (V^+, V^-)$ a 3-graded Lie superalgebra. Namely, let $\text{IDer}V = \text{span}\{\delta(x, y) = (D^+(x, y), D^-(y, x)) \in \text{End}(V^+) \times \text{End}(V^-), \text{ for all } x, y \in V\}$. We define

$$\mathfrak{K}(V) = V^+ \oplus \text{IDer}V \oplus V^-$$

with product

$$\begin{aligned} [x^+ \oplus c \oplus x^-, y^+ \oplus d \oplus y^-] &= (c^+y^+ - d^+x^+) \oplus ([c, d] + \delta(x^+, y^-) \\ &\quad - \delta(y^+, x^+)) \oplus (c^-y^- - d^-x^-) \end{aligned}$$

where $x^{\epsilon}, y^{\epsilon} \in V^{\epsilon}$, $\epsilon = \pm$ and $c = (c^+, c^-)$, $d = (d^+, d^-) \in \text{IDer}V$.

Then $\mathfrak{K}(V)$ is a 3-graded Lie superalgebra with $L_0 = L_{\bar{0}}$ and $L_{\bar{1}} = L_1 \oplus L_{-1}$, whose associated anti-Jordan pair is V .

Proof. To verify (3) we apply the Jacobi identity :

$$\{x, y, x\} = [[x, y], x] = [x, [y, x]] + [y, [x, x]].$$

By using the property of super anticommutativity and $[x, x] = 0$, we have $[[x, y], x] = -[[x, y], x]$. This yields $[[x, y], x] = 0$, since $1/2 \in k$. Hence (3) holds.

To verify (4) we also apply the Jacobi identity and super anticommutativity :

$$\begin{aligned}
[D(x, y), D(u, v)]z &= \{x, y, \{u, v, z\}\} - \{u, v, \{x, y, z\}\} \\
&= [[x, y], [[u, v], z]] - [[u, v], [[x, y], z]] \\
&= [[x, y], [[u, v], z]] + [[x, y], [z, [u, v]]] \\
&\quad - [z, [[x, y], [u, v]]] \\
&= [[x, y], [[u, v], z]] - [[x, y], [[u, v], z]] \\
&\quad - [z, [[x, y], [u, v]]] \\
&= -[z, [[x, y], [u, v]]] \\
&\quad \text{for all } x, u, z \in L_\epsilon, y, v \in L_{-\epsilon}, \epsilon = \pm 1
\end{aligned}$$

$$\begin{aligned}
D(\{x, y, u\}, v)z + D(u, \{y, x, v\})z &= \{[[x, y], u], v, z\} + \{u, [[y, x], v], z\} \\
&= [[[[x, y], u], v], z] + [[u, [[y, x], v]], z] \\
&= [[x, y], [u, v], z] + [[u, [v, [x, y]]], z] \\
&\quad + [[u, [[y, x], v]], z] \\
&= -[z, [[x, y], [u, v]]] - [[u, [[y, x], v]], z] \\
&\quad + [[u, [[y, x], v]], z] \\
&= -[z, [[x, y], [u, v]]] \\
&\quad \text{for all } x, u, z \in L_\epsilon, y, v \in L_{-\epsilon}, \epsilon = \pm 1
\end{aligned}$$

(b) It is known that $\mathfrak{K}(V)$ is a Lie superalgebra [13]. Hence (4) holds. \square

One calls $\mathfrak{K}(V)$ the Tits-Kantor-Koecher superalgebra associated to V .

Lemma 2.2.5. $\mathfrak{sl}(m|n)(R) = L_1 \oplus L_0 \oplus L_{-1}$ is a Jordan 3-graded Lie superalgebra with associated Jordan superpair $\mathfrak{M}_{mn}(R)$. Hence the Tits-Kantor-Koecher superalgebra $\mathfrak{K}(\mathfrak{M}_{mn}(R))$ is isomorphic to

$$\mathfrak{psl}(m|n)(R) := \mathfrak{sl}(m|n)(R)/Z(\mathfrak{sl}(m|n)(R)).$$

Proof. We know that $\mathfrak{sl}(m|n) = L_1 \oplus L_0 \oplus L_{-1}$ is a 3-graded Lie superalgebra (see Example 2.2.2). Also for all $x, z \in L_1$ and $y \in L_{-1}$, we have $[[x, y], z] = [xy + yx, z] =$

$xyz - zyx = \{x, y, z\}$ (since $L_\epsilon L_\epsilon = 0$, $\epsilon = \pm 1$). We will check that $[L_1, L_{-1}] = L_0$. Let E_{ij} denote the matrix units. Then

$$\begin{aligned} L_1 &= \text{span} \{E_{ij} : 1 \leq i \leq m, m+1 \leq j \leq m+n\} \\ L_{-1} &= \text{span} \{E_{ji} : 1 \leq i \leq m, m+1 \leq j \leq m+n\} \\ L_0 &= \text{span} \{E_{ii} + E_{j,j} : 1 \leq i \leq m, m+1 \leq j \leq m+n\} \\ &\quad \cup \text{span} \{E_{ij} : 1 \leq i \neq j \leq m\} \cup \text{span} \{E_{ij} : m+1 \leq i \neq j \leq m+n\}. \end{aligned}$$

Note that

$$\begin{aligned} [E_{ij}, E_{ji}] &= E_{ii} + E_{jj} \quad 1 \leq i \leq m, m+1 \leq j \leq m+n \\ [E_{ij}, E_{jk}] &= E_{ik} \quad 1 \leq i \neq k \leq m, j = m+1 \\ [E_{ij}, E_{ri}] &= E_{rj} \quad i = 1, m+1 \leq j \neq r \leq m+n \end{aligned}$$

Therefore the spanning set for L_0 is contained in $[L_1, L_{-1}]$, so $L_0 \subseteq [L_1, L_{-1}]$. But $[L_1, L_{-1}] \subseteq L_0$, which yields $[L_1, L_{-1}] = L_0$. Hence $\mathfrak{sl}(m|n)(R)$ is a Jordan 3-graded Lie superalgebra with associated Jordan superpair $\mathfrak{M}_{mn}(R)$. Therefore $\mathfrak{K}(\mathfrak{M}_{mn}) \cong \mathfrak{psl}(m|n)(R)$ by [4, Lemma 2.4]. \square

Lemma 2.2.6. *Let R be a unital and associative k -algebra. Then*

$$Z(\mathfrak{gl}_m(R)) = \{x \in \mathfrak{gl}_m(R) : [x, \mathfrak{gl}_m(R)] = 0\} = Z(R)I_m.$$

Proof. It is easy to see that $Z(\mathfrak{gl}_m(R)) \supseteq Z(R)I_m$. We will check that $Z(\mathfrak{gl}_m(R)) \subseteq Z(R)I_m$. We may assume $m > 1$, let $x = (x_{rs}) \in Z(\mathfrak{gl}_m(R))$ and $a \in R$. In particular for all i, j with $i \neq j$ we have $(\sum x_{rs}E_{rs})aE_{ij} = aE_{ij}(\sum x_{rs}E_{rs})$, i.e. $\sum_r x_{ri}aE_{rj} = \sum_s ax_{js}E_{is}$. This implies $x_{ri} = 0$ for $r \neq i$ and $x_{ii}a = ax_{jj}$ for $i \neq j$. Therefore $x = \alpha I_m$, $\alpha \in Z(R)$. \square

Lemma 2.2.7. *Let $L = \mathfrak{sl}(m|n)(C)$ be as in Example 2.2.2, suppose $m \geq 1$ and $n \geq 1$. Then*

$$Z(L_0) = \left\{ \begin{pmatrix} \alpha I_m & 0 \\ 0 & \beta I_n \end{pmatrix} : \alpha m = \beta n, \alpha, \beta \in C \right\}.$$

Proof. Let $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in Z(L_0)$, where $x \in M_{mm}(C)$ and $y \in M_{nn}(C)$ such that $\text{tr}(x) = \text{tr}(y)$. In particular for $a \in M_{mm}(C)$ and $d = \text{tr}(a)E_{11}$ we get $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in L_0$, so $[x, a] = 0$ for all $x \in M_{mm}(C)$. Hence $a = \alpha I_m$ for some $\alpha \in C$ by Lemma 2.2.6. Similarly, one proves $d = \beta I_n$ for some $\beta \in C$. Since $\text{tr}(a) = \text{tr}(d)$, we obtained $\alpha m = \beta n$. Thus

$$Z(L_0) \subseteq \left\{ \begin{pmatrix} \alpha I_m & 0 \\ 0 & \beta I_n \end{pmatrix} : \alpha m = \beta n, \alpha, \beta \in C \right\}.$$

The converse is clear. □

Lemma 2.2.8. *Let $L = \mathfrak{sl}(m|n)(C)$, then*

$$Z(L) = \{z \in L : [z, L] = 0\}$$

is contained in L_0 and has the following description:

$$Z(L) = \{\alpha I_{m+n} : \alpha \in C, \text{ and } \alpha m = \alpha n\} \subseteq Z(L_0),$$

$$\text{i.e. } Z(L) = \{a I_{m+n} : a \in A\}, \text{ where } A = \{\alpha \in C : \alpha(m - n) = 0\}.$$

Hence $Z(L) \neq 0$ if and only if $(m - n)1_C$ is a zero divisor in C or zero. Moreover:

(i) *If $m1_C, n1_C \in C^\times$, then*

$$\begin{aligned} L_0 &= \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in M_{mm}(C), d \in M_{nn}(C), \text{tr}(a) = \text{tr}(d) \right\} \\ &= \left\{ \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix} : a' \in M_{mm}(F), d' \in M_{nn}(C), \text{tr}(a') = 0 = \text{tr}(d') \right\} \\ &\quad \oplus C \begin{pmatrix} \frac{1}{m} I_m & 0 \\ 0 & \frac{1}{n} I_n \end{pmatrix} \\ &= \mathfrak{sl}_m(C) \oplus \mathfrak{sl}_n(C) \oplus Z(L_0), \quad \text{where } Z(L_0) = C \begin{pmatrix} \frac{1}{m} I_m & 0 \\ 0 & \frac{1}{n} I_n \end{pmatrix}. \end{aligned}$$

Hence $L_0/Z(L) = \mathfrak{sl}_m(C) \oplus \mathfrak{sl}_n(C) \oplus \frac{C}{A} \begin{pmatrix} \frac{1}{m}I_m & 0 \\ 0 & \frac{1}{n}I_n \end{pmatrix}$ and

$$Z(L_0/Z(L)) = \frac{C}{A} \begin{pmatrix} \frac{1}{m}I_m & 0 \\ 0 & \frac{1}{n}I_n \end{pmatrix}.$$

(ii) Let $m1_C = n1_C \in C^\times$. Then $Z(L) = Z(L_0)$ and hence

$$L_0/Z(L) = L_0/Z(L_0) \cong \mathfrak{sl}_m(C) \oplus \mathfrak{sl}_m(C).$$

Proof. Since $L = L_1 \oplus L_0 \oplus L_{-1}$ is a finite \mathbb{Z} -graded algebra, the center of L is graded too, i.e., $Z(L) = (Z(L) \cap L_{-1}) \oplus (Z(L) \cap L_0) \oplus (Z(L) \cap L_1)$. Suppose $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in Z(L) \cap L_1$. Then

$$0 = \left[\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \right] = \begin{pmatrix} bc & 0 \\ 0 & cb \end{pmatrix}$$

whence $bc = 0$ for all $c \in M_{nm}(C)$, which implies $b = 0$. Hence $Z(L) \cap L_1 = \{0\}$. Similarly $Z(L) \cap L_{-1} = \{0\}$.

Let $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in Z(\mathfrak{sl}(m|n)) \cap L_0$. Then

$$0 = \left[\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \right]$$

which implies $dc = ca$ for all $c \in M_{nm}(C)$. Therefore by Lemma 2.1.3, we get $a = \alpha I_m$ and $d = \alpha I_n$ for some $\alpha \in C$. Since $\text{tr}(a) = \text{tr}(d)$ we get $\alpha n = \alpha m$, or $\alpha(m - n) = 0$. Hence

$$Z(L) = \{\alpha I_{m+n} : \alpha \in C, \text{ and } \alpha m = \alpha n\} \subseteq Z(L_0).$$

(i) Let $a \in \mathfrak{gl}_m(C)$. Then $\text{tr}(a - (\frac{1}{m}\text{tr}(a))I_m) = 0$ and so

$$a = (a - (\frac{1}{m}\text{tr}(a))I_m) + \frac{1}{m}\text{tr}(a)I_m.$$

This implies $a \in \mathfrak{sl}_m(C) + s_m(C)$, where $\mathfrak{sl}_m(C)$ is the Lie algebra of trace zero matrices and $s_m(C)$ denotes the scalar matrices. Let $x \in \mathfrak{sl}_m(C) \cap s_m(C)$. Then $x = \alpha I_m$ for

some $\alpha \in C$. Therefore $\text{tr}(x) = \alpha m = 0$. Since $m1_C \in C^\times$ we get $x = 0$. Hence $gl_m(C) = \mathfrak{sl}_m(C) \oplus s_m(C)$. Similarly we can prove that $gl_n(C) = \mathfrak{sl}_n(C) \oplus s_n(C)$.

Let $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathfrak{sl}(m|n)$. Then by what we just showed, we have

$$a = s\left(\frac{1}{m}\right)I_m \oplus a' \quad \text{and} \quad d = t\frac{1}{n}I_n \oplus d'$$

where $d' \in \mathfrak{sl}_n(C)$, $a' \in \mathfrak{sl}_m(C)$ and $s, t \in C$. Since $\text{tr}(a) = \text{tr}(d)$ we get $s = t$. Hence

$$L_0 = \mathfrak{sl}_m(C) + \mathfrak{sl}_n(C) + Z(L_0), \quad \text{where } Z(L_0) = C \begin{pmatrix} \frac{1}{m}I_m & 0 \\ 0 & \frac{1}{n}I_n \end{pmatrix}.$$

Let $0 = \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix} + \begin{pmatrix} c\frac{1}{m}I_m & 0 \\ 0 & c\frac{1}{n}I_n \end{pmatrix}$ where $c \in C$. Then

$$0 = a' + c\frac{1}{m}I_m \quad \text{and} \quad 0 = d' + c\frac{1}{n}I_n.$$

This implies $0 = \text{tr}(c\frac{1}{m}I_m)$ and $0 = \text{tr}(c\frac{1}{n}I_n)$. Hence $c = 0$. Therefore the sum is direct. It is clear that

$$L_0/Z(L) = \mathfrak{sl}_m(C) \oplus \mathfrak{sl}_n(C) \oplus \frac{C}{A} \begin{pmatrix} \frac{1}{m}I_m & 0 \\ 0 & \frac{1}{n}I_n \end{pmatrix}.$$

For the last claim it is easy to see that $Z(L_0/Z(L)) \supseteq \frac{C}{A} \begin{pmatrix} \frac{1}{m}I_m & 0 \\ 0 & \frac{1}{n}I_n \end{pmatrix}$.

To prove the other inclusion, we can assume that

$$\begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix} \in Z(L_0/Z(L)) \cap (\mathfrak{sl}_m(C) \oplus \mathfrak{sl}_n(C)).$$

Then

$$\left[\begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right] = \begin{pmatrix} [a', b] & 0 \\ 0 & [b', b] \end{pmatrix} = 0$$

for all $a \in \mathfrak{sl}_m(C)$ and $b \in \mathfrak{sl}_n(C)$, i.e. $a' \in Z(\mathfrak{sl}_m(C))$ and $b' \in Z(\mathfrak{sl}_n(C))$. Hence $a' = 0$ and $b' = 0$. Therefore

$$Z(L_0/Z(L)) = \frac{C}{A} \begin{pmatrix} \frac{1}{m}I_m & 0 \\ 0 & \frac{1}{n}I_n \end{pmatrix}.$$

(ii) Let $m1_C \in C^\times$. Then

$$Z(L_0) = \left\{ \begin{pmatrix} \beta \frac{n}{m} I_m & 0 \\ 0 & \beta I_n \end{pmatrix} : \beta \in C \right\}.$$

Since $m1_C = n1_C$, we get

$$Z(L_0) = C \begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix} = Z(L).$$

□

Let $\zeta = (\zeta^+, \zeta^-)$ be an automorphism of an anti-Jordan pair V . For $T = (T^+, T^-) \in \text{End}V^+ \times \text{End}V^-$ we define

$$\zeta.T = \zeta.(T^+, T^-) = (\zeta^+ T^+ (\zeta^+)^{-1}, \zeta^- T^- (\zeta^-)^{-1}).$$

If $T = \delta(x, y) = (D^+(x, y), D^-(y, x))$, then

$$\zeta.T = (\zeta^+ D^+(x, y) (\zeta^+)^{-1}, \zeta^- D^-(y, x) (\zeta^-)^{-1}) = \delta(\zeta^+ x, \zeta^- y). \quad (17)$$

Lemma 2.2.9. (a) Let $\zeta = (\zeta^+, \zeta^-)$ be an automorphism of an anti-Jordan pair $V = (V^+, V^-)$ over k . Then

$$\tilde{\zeta} : \mathfrak{K}(V) \rightarrow \mathfrak{K}(V), \quad x^+ \oplus d \oplus x^- \mapsto \zeta^+(x^+) \oplus \zeta.d \oplus \zeta^-(x^-)$$

is an automorphism of $\mathfrak{K}(V)$, where $\zeta.\delta(x, y) = \delta(\zeta^+(x), \zeta^-(y))$.

If $Z(\mathfrak{K}(V)_0) \subseteq \{(\alpha \text{Id}, \beta \text{Id}) : \alpha, \beta \in k\}$ then the extended automorphism $\tilde{\zeta}$ fixes the center of $\mathfrak{K}(V)_0$.

(b) Let $\zeta = (\zeta^+, \zeta^-)$ be a automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C) = (M_{mn}(C), M_{nm}(C))$. If $Z(\mathfrak{K}(V)_0) \subseteq \{(c_1 \text{Id}, c_2 \text{Id}) : c_1, c_2 \in C\}$ then the extended automorphism $\tilde{\zeta}$ fixes the center of $\mathfrak{K}(V)_0$.

Proof. (a) It is clear that $\tilde{\zeta}$ is well defined, i.e., $\zeta.d \in \text{IDer}V$. We will check $\tilde{\zeta}$ is a Lie superalgebra homomorphism:

$$\begin{aligned} \tilde{\zeta}[x^+ \oplus c \oplus x^-, y^+ \oplus d \oplus y^-] &= \tilde{\zeta}((c^+ y^+ - d^+ x^+) \oplus ([c, d] + \delta(x^+, y^-) \\ &\quad + \delta(y^+, x^+)) \oplus (c^- y^- - d^- x^-)) \\ &= \zeta^+(c^+ y^+ - d^+ x^+) \oplus \zeta.[c, d] + \zeta.\delta(x^+, y^-) \\ &\quad + \zeta.\delta(y^+, x^+) \oplus \zeta^-(c^- y^- - d^- x^-) \end{aligned}$$

and

$$\begin{aligned}
& [\tilde{\zeta}(x^+ \oplus c \oplus x^-), \tilde{\zeta}(y^+ \oplus d \oplus y^-)] \\
&= [\zeta^+(x^+) \oplus \zeta \cdot c \oplus \zeta^-(x^-), \zeta^+(y^+) \oplus \zeta \cdot d \oplus \zeta^-(y^-)] \\
&= (\zeta \cdot c)^+ \zeta^+(y^+) - (\eta \cdot d)^+ \eta^+(x^+) \oplus [\zeta \cdot c, \zeta \cdot d] \\
&\quad + \delta(\zeta^+ x^+, \zeta^- y^-) + \delta(\eta^+ y^+, \zeta^- x^-) \\
&\quad \oplus (\zeta \cdot c)^- \zeta^-(y^-) - (\zeta \cdot d)^- \zeta^-(x^-) \\
&= \zeta^+ c^+ (\eta^+)^{-1} \eta^+(y^+) - \eta^+ d^+ (\zeta^+)^{-1} \zeta^+(x^+) \oplus \\
&\quad [(\zeta^+ c^+ (\zeta^+)^{-1}, \zeta^- c^- (\zeta^-)^{-1}), (\zeta^+ d^+ (\zeta^+)^{-1}, \zeta^- d^- (\zeta^-)^{-1})] \\
&\quad + \eta \cdot \delta(y^+, x^-) + \zeta \cdot \delta(x^+, y^-) \oplus \zeta^- c^- (\zeta^-)^{-1} \zeta^-(y^-) \\
&\quad \quad - \zeta^- d^- (\zeta^-)^{-1} \zeta^-(x^-) \\
&= \zeta^+ c^+(y^+) - \zeta^+ d^+(x^+) \oplus [(\zeta^+ c^+ (\zeta^+)^{-1}, \zeta^+ d^+ (\eta^+)^{-1}), \\
&\quad [\zeta^- c^- (\zeta^-)^{-1}, \zeta^- d^- (\zeta^-)^{-1}]] + \zeta \cdot \delta(y^+, x^-) + \zeta \cdot \delta(x^+, y^-) \\
&\quad \oplus \zeta^- c^- (y^-) - \zeta^- d^- (x^-) \\
&= \zeta^+ c^+(y^+) - \zeta^+ d^+(x^+) \oplus \zeta \cdot [c, d] + \zeta \cdot \delta(y^+, x^-) \\
&\quad + \zeta \cdot \delta(x^+, y^-) \oplus \zeta^- c^- (y^-) - \zeta^- d^- (x^-)
\end{aligned}$$

Therefore $\tilde{\zeta}([x^+ \oplus c \oplus x^-, y^+ \oplus d \oplus y^-]) = [\tilde{\zeta}(x^+ \oplus c \oplus x^-), \tilde{\zeta}(y^+ \oplus d \oplus y^-)]$

Hence $\tilde{\zeta}$ is a Lie superalgebra homomorphism. Let $x^+ \oplus c \oplus x^- \in \text{Ker } \tilde{\zeta}$. Then

$$\tilde{\zeta}(x^+ \oplus c \oplus x^-) = 0 \Rightarrow \zeta^+(x^+) \oplus \zeta \cdot c \oplus \zeta^-(x^-) = 0$$

since the sum is direct. So $x^+ = 0$, $c = 0$, and $x^- = 0$ shows that $\tilde{\zeta}$ is injective.

Let $x^+ \oplus c \oplus x^- \in \mathfrak{K}(V)$. Then $\tilde{\zeta}((\zeta^+)^{-1}(x^-) \oplus \zeta^{-1} \cdot c \oplus (\zeta^-)^{-1}(x^+)) = x^+ \oplus c \oplus x^-$, since $\zeta = (\zeta^+, \zeta^-)$ is an automorphism. Therefore $\tilde{\zeta}$ is bijective. Hence $\tilde{\zeta} \in \text{Aut}(\mathfrak{K}(V))$.

Let $T = (\alpha \text{Id}, \beta \text{Id}) \in Z(\mathfrak{K}(V)_0)$, where $\alpha, \beta \in k$. Then

$$\tilde{\zeta} \cdot T = (\zeta^+(\alpha \text{Id})(\zeta^+)^{-1}, \zeta^-(\beta \text{Id})(\zeta^-)^{-1}) = T.$$

Similarly we can prove (b). □

Lemma 2.2.10. Let $\frac{1}{n} \in C$ and suppose I is an ideal of the Lie algebra $\mathfrak{sl}_n(C)$. Then there exists an ideal \bar{I} of C such that $I = \mathfrak{sl}_n(\bar{I})$. Conversely, $\mathfrak{sl}_n(\bar{I})$ is an ideal of $\mathfrak{sl}_n(C)$ for any ideal \bar{I} of C . Moreover, every ideal of $\mathfrak{sl}_n(C)$ is perfect in the sense that $I = [I, I]$.

Proof. It is clear that $\mathfrak{sl}_n(\bar{I})$ is an ideal of $\mathfrak{sl}_n(C)$ for any ideal \bar{I} of C . Note that

$$\mathfrak{sl}_n(C) = \text{span}(\{E_{ii} - E_{i+1, i+1} : 1 \leq i \leq n-1\} \cup \{E_{ij} : 1 \leq i \neq j \leq n\}).$$

Case 1. ($n = 2$):

Step 1: Let $x = (c_{11}(E_{11} - E_{22}) + \sum_{1 \leq i \neq j \leq 2} c_{ij}E_{ij}) \in I$. We will prove $c_{12}E_{12}, c_{21}E_{21}, c_{11}(E_{11} - E_{22}) \in I$ and that we can move every entry in a matrix in I to any position in another matrix in I . Then for $1 \leq u \neq v \leq 2$, we get $[[x, E_{uv}], \frac{-1}{2}E_{uv}] = c_{vu}E_{uv} \in I$. Therefore $[c_{vu}E_{uv}, E_{vu}] = c_{vu}(E_{uu} - E_{vv}) \in I$. This implies

$$[c_{vu}(E_{uu} - E_{vv}), \frac{-1}{2}E_{vu}] = c_{vu}E_{vu} \in I.$$

Since $x \in I$ and $(c_{12}E_{12} + c_{21}E_{21}) \in I$, we get $x - \sum_{1 \leq i \neq j \leq 2} c_{ij}E_{ij} = c_{11}(E_{11} - E_{22}) \in I$.

Step 2: Let $\bar{I} = \{a \in C : aE_{12} \in I\}$. Then for any $c \in C$ we get

$$[ca(E_{11} - E_{22}), \frac{1}{2}E_{12}] = caE_{12} \in I.$$

This implies $ca \in \bar{I}$. Therefore \bar{I} is an ideal of C . Note that every element of I is a sum of commutators, i.e. $[\frac{1}{2}(E_{uu} - E_{vv}), aE_{uv}] = aE_{uv}$ and $[E_{12}, aE_{21}] = a(E_{11} - E_{22})$ for all $a \in \bar{I}$. Therefore every ideal of $\mathfrak{sl}_2(C)$ is perfect.

Case 2. ($n > 2$):

Step 1: Let $x = (\sum_{i=1}^{n-1} c_{ii}(E_{ii} - E_{i+1, i+1}) + \sum_{1 \leq i \neq j \leq n} c_{ij}E_{ij}) \in I$. We will prove $c_{ij}E_{ij}, c_{ii}(E_{ii} - E_{i+1, i+1}) \in I$ and that we can move every entry in a matrix in I to any position in another matrix in I . Then for $1 \leq p \neq u \neq v \neq p \leq n$, we get

$$[[[x, E_{uv}], E_{up}], -E_{pv}] = [-c_{vu}E_{up} - c_{pu}E_{uv}, -E_{pv}] = c_{vu}E_{uv} \in I.$$

Therefore $[c_{vu}E_{uv}, E_{vu}] = c_{vu}(E_{uu} - E_{vv}) \in I$. This implies $[c_{vu}(E_{uu} - E_{vv}), -E_{vp}] = c_{vu}E_{vp} \in I$ and $[c_{vu}(E_{uu} - E_{vv}), E_{up}] = c_{vu}E_{up} \in I$. Therefore $[c_{vu}E_{vp}, E_{pu}] = c_{vu}E_{vu} \in I$. Therefore

$$[E_{ru}, c_{vu}E_{up}] = c_{vu}E_{rp} \in I \text{ for } 1 \leq r \neq u \neq p \neq r \leq n \text{ and}$$

$$[E_{sv}, c_{vu}E_{vp}] = c_{vu}E_{sp} \in I \text{ for } 1 \leq s \neq v \neq p \neq s \leq n.$$

Hence $[c_{vu}E_{rp}, E_{pr}] = c_{vu}(E_{rr} - E_{pp}) \in I$ for $1 \leq r \neq u \neq p \neq r \leq n$ and $[c_{vu}E_{sp}, E_{ps}] = c_{vu}(E_{ss} - E_{pp}) \in I$ for $1 \leq s \neq v \neq p \neq s \leq n$. This implies $[c_{vu}(E_{rr} - E_{pp}), -E_{pu}] = c_{vu}E_{pu} \in I$ and $[c_{vu}(E_{ss} - E_{pp}), -E_{pv}] = c_{vu}E_{pv} \in I$.

Since $x \in I$ and $\sum_{1 \leq i \neq j \leq n} c_{ij}E_{ij} \in I$, we get

$$x - \sum_{1 \leq i \neq j \leq n} c_{ij}E_{ij} = \sum_{i=1}^{n-1} c_{ii}(E_{ii} - E_{i+1,i+1}) \in I.$$

This implies $[\sum_{i=1}^{n-1} c_{ii}(E_{ii} - E_{i+1,i+1}), E_{1n}] = (c_{11} + c_{n-1,n-1})E_{1n} \in I$, and

$$\left[\sum_{i=1}^{n-1} c_{ii}(E_{ii} - E_{i+1,i+1}), E_{1,u} \right] = (c_{11} - c_{u,u} + c_{u-1,u-1})E_{1,u} \in I, \quad 2 \leq u \leq n-1.$$

Therefore $[(c_{11} - c_{u,u} + c_{u-1,u-1})E_{1,u}, E_{u,n}] = (c_{11} - c_{u,u} + c_{u-1,u-1})E_{1,n} \in I$. (*)

Then $\frac{1}{n}(\sum_{u=2}^{n-1}(c_{11} - c_{u,u} + c_{u-1,u-1}) + c_{11} + c_{n-1,n-1})E_{1n} = c_{11}E_{1n} \in I$. This implies

$$(c_{11} + c_{n-1,n-1} - c_{11})E_{1,n} = c_{n-1,n-1}E_{1,n} \in I.$$

Hence from (*) we get $c_{ii}E_{1n} \in I$, for $2 \leq i \leq n-1$.

Step 2: Let $\bar{I} = \{a \in C : aE_{12} \in I\}$. Then $[caE_{13}, E_{32}] = caE_{12} \in I$ for any $c \in C$. This implies $ca \in \bar{I}$. Therefore \bar{I} is an ideal of C . Hence $I = \mathfrak{sl}_n(\bar{I})$.

Note that $[E_{vv} - E_{uu}, aE_{vp}] = aE_{vp}$ and $[E_{uv}, aE_{vu}] = a(E_{uu} - E_{vv})$ for all $a \in \bar{I}$. Therefore every ideal of $\mathfrak{sl}_n(C)$ is perfect. \square

Lemma 2.2.11. *Let $\frac{1}{n}, \frac{1}{m} \in C$ and let I_1, I_2 be ideals of C . Then $\mathfrak{sl}_m(I_1) \oplus \mathfrak{sl}_n(I_2)$ is a perfect ideal of $L = \mathfrak{sl}_m(C) \oplus \mathfrak{sl}_n(C)$. Conversely, any perfect ideal of L has this form.*

Proof. Let I_1, I_2 ideals of C . Then $\mathfrak{sl}_m(I_1)$ and $\mathfrak{sl}_n(I_2)$ are perfect ideals of $\mathfrak{sl}_m(C)$ and $\mathfrak{sl}_n(C)$ respectively by Lemma 2.2.10. Hence $\mathfrak{sl}_m(I_1) \oplus \mathfrak{sl}_n(I_2)$ is a perfect ideal of L .

Conversely, let I be an perfect ideal of L . Let $\pi_1 : \mathfrak{sl}_m(C) \oplus \mathfrak{sl}_n(C) \rightarrow \mathfrak{sl}_m(C)$ and similarly $\pi_2 : \mathfrak{sl}_m(C) \oplus \mathfrak{sl}_n(C) \rightarrow \mathfrak{sl}_n(C)$ be the canonical projections. Then π_1 and π_2 are algebra epimorphisms, hence $\pi_1(I)$ is an ideal of $\mathfrak{sl}_m(C)$ and $\pi_2(I)$ is an ideal

of $\mathfrak{sl}_n(C)$. By Lemma 2.2.10 there exist ideals I_1, I_2 of C , such that $\pi_1(I) = \mathfrak{sl}_m(I_1)$ and $\pi_1(I) = \mathfrak{sl}_n(I_2)$. We know that $\pi_i(I)$ is a perfect ideal of L . Therefore

$$\pi_1(I) = \pi_1([I, L]) = [\pi_1(I), \pi_1(L)] = [\pi_1(I), \mathfrak{sl}_m(C)]$$

and $\pi_2(I) = [\pi_2(I), \mathfrak{sl}_n(C)]$. Also $I = [I, I] = [I, \mathfrak{sl}_m(C) \oplus \mathfrak{sl}_n(C)] = [I, \mathfrak{sl}_m(C)] \oplus [I, \mathfrak{sl}_n(C)] = [\pi_1(I), \mathfrak{sl}_m(C)] \oplus [\pi_2(I), \mathfrak{sl}_n(C)] = [\mathfrak{sl}_m(I_1), \mathfrak{sl}_m(C)] \oplus [\mathfrak{sl}_n(I_2), \mathfrak{sl}_n(C)] = \mathfrak{sl}_m(I_1) \oplus \mathfrak{sl}_n(I_2)$. \square

Proposition 2.2.12. *Let $\frac{1}{n}, \frac{1}{m} \in C$ and let $\varphi \in \text{Aut}_C(L)$ where $L = L_1 \oplus L_2$ and $L_1 = \mathfrak{sl}_m(C)$ and $L_2 = \mathfrak{sl}_n(C)$. Suppose C is connected if $m = n$. Then either $\varphi(L_i) = L_i$ for $i = 1, 2$ or $\varphi(L_1) = L_2, \varphi(L_2) = L_1$ and $m = n$.*

Recall C is *connected* if and only if C is not a proper direct sum of two ideals.

Proof. Since $\varphi \in \text{Aut}_C(L)$ and L_1, L_2 are perfect ideals, we have $L = \varphi(L_1) \oplus \varphi(L_2)$, where $\varphi(L_i)$ are perfect ideals of L . By Lemma 2.2.11 we therefore have $\varphi(L_1) = \mathfrak{sl}_m(I_1) \oplus \mathfrak{sl}_n(J_1)$ and $\varphi(L_2) = \mathfrak{sl}_m(I_2) \oplus \mathfrak{sl}_n(J_2)$ where I_1, I_2, J_1, J_2 are ideals of C . Also

$$L = \varphi(L_1) \oplus \varphi(L_2) = (\mathfrak{sl}_m(I_1) \oplus \mathfrak{sl}_n(J_1)) \oplus (\mathfrak{sl}_m(I_2) \oplus \mathfrak{sl}_n(J_2)).$$

Hence $\mathfrak{sl}_m(C) = \mathfrak{sl}_m(I_1) \oplus \mathfrak{sl}_m(I_2)$ and similarly for $\mathfrak{sl}_n(C)$. By considering the (12)-position of this decomposition, one sees that

$$C = I_1 \oplus I_2 = J_1 \oplus J_2. \tag{a}$$

We also know $\mathfrak{sl}_m(C)$ and $\mathfrak{sl}_n(C)$ are free C -modules of rank $m^2 - 1$ and $n^2 - 1$ respectively. Therefore $\varphi(L_1) = I$ is a free C -module of rank $m^2 - 1$. Hence, for every maximal ideal M of C , $I/M I$ is a F -vector space of dimension $m^2 - 1$ where $F = C/M$.

Now $F = C/MC = I_1/M I_1 \oplus I_2/M I_2$ shows that $I_1/M I_1 = 0$ or F . Moreover, if $I_1/M I_1 = 0$, then $I_1 = M I_1 \subset M$, while if $I_1/M I_1 = F$, then $I_2/M I_2 = 0, I_2 \subset M$, and $I_1 \not\subset M$, since $M \neq C$. Similarly, $J_1/M J_1 = 0$ or F .

Let $m \neq n$ and suppose $I_1 \neq C$, equivalently $I_2 \neq 0$. Let M_2 be a maximal ideal of C . Then $M = I_1 \oplus M_2$ is a maximal ideal of C , and

$$I/M I = \mathfrak{sl}_m(I_1/M I_1) \oplus \mathfrak{sl}_n(J_1/M J_1) = \{0\} \oplus \mathfrak{sl}_n(J_1/M J_1).$$

with $J_1/MJ_1 = 0$ or F . Thus, I/IM has dimension 0 or $n^2 - 1$, a contradiction. Therefore $I_1 = C$ follows, so $I = \varphi(\mathfrak{sl}_m(C)) = \mathfrak{sl}_m(C)$. Similarly $J_1 \neq \{0\}$, yields a contradiction, so $C = J_2$ and $\varphi(\mathfrak{sl}_n(C)) = \mathfrak{sl}_n(C)$.

Suppose $m = n$. Then C is connected, i.e. in (a) we have $I_1 = \{0\}$ or $I_2 = \{0\}$.

Case (i): If $I_1 = \{0\}$, then $\varphi(L_1) = L_2$. Again by connectedness $J_1 = \{0\}$ or $J_2 = \{0\}$. If $J_1 = \{0\}$, then $\varphi(L_2) = 0 \oplus \mathfrak{sl}_n(C) = L_2 = \varphi(L_1)$, which is a contradiction. Hence $C = J_1$ and $J_2 = \{0\}$ follows, i.e. $\varphi(L_2) = L_1$.

Case (ii): If $I_2 = \{0\}$, then $\varphi(L_1) = L_1$. Again by connectedness $J_1 = \{0\}$. Then $\varphi(L_2) = L_2$. □

Remark: If $m = n$ and $C = I_1 \oplus I_2$ is a nontrivial decomposition, then

$$L = \mathfrak{sl}_m(C) \oplus \mathfrak{sl}_m(C) = \mathfrak{sl}_m(I_1) \oplus \mathfrak{sl}_m(I_2) \oplus \mathfrak{sl}_m(I_1) \oplus \mathfrak{sl}_m(I_2),$$

and there exists an automorphism of L , exchanging the first and third, and fixing the second and fourth summands. So the proposition is not true in this case.

For easier reference we recall:

Theorem 2.2.13. ([14, Theorem 1]) *Let L be the Lie algebra of $n \times n$ matrices of trace zero over F , where F is a field of characteristic zero or $p > 3$ such that p does not divide n if $p \in \{5, 7, 11\}$. Then every automorphism of L is of the form $x \rightarrow a^{-1}xa$ or of the form $x \rightarrow -a^{-1}x^t a$, for some invertible matrix a .*

Theorem 2.2.14. ([10, Theorem 5.19]) *Let C be a unital, commutative and associative algebra over a field k of characteristic 0. Assume that C is a unique factorization domain. Then every automorphism of $L = \mathfrak{sl}_n(C)$, $n > 1$ is of the form $x \rightarrow a^{-1}xa$ or of the form $x \rightarrow -a^{-1}x^t a$, for some invertible matrix a .*

2.3 Automorphisms and involutions of the anti-Jordan pair \mathfrak{M}_{mn}

Theorem 2.3.1. *Suppose that every C -linear automorphism of $\mathfrak{sl}_q(C)$ is of the form $a \mapsto uau^{-1}$ or $a \mapsto -ua^t u^{-1}$ for some $u \in \mathrm{GL}_q(C)$, where $q = n$ or m . Let $m1_C \neq n1_C$, $m, n \geq 2$ and suppose $m1_C, n1_C \in C^\times$.*

(a') *Then $\mathrm{Aut}(\mathfrak{M}_{mn})(C) = \Lambda \cong \mathrm{GL}_m(C) \times \mathrm{GL}_n(C) / \{(\gamma I_m, \gamma I_n) : \gamma \in C^\times\}$, see Lemma 2.1.5 for the definition of Λ .*

(b') *Let $\xi = (\xi^+, \xi^-)$ be an anti-automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C) = (M_{mn}(C), M_{nm}(C))$. Then there exist matrices $a \in \mathrm{GL}_n(C)$ and $b \in \mathrm{GL}_m(C)$ such that $\xi^+ = \mu_{a,b}^+(x) = ax^t b^{-1}$, $\xi^- = \mu_{a,b}^-(y) = -by^t a^{-1}$ for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$.*

(c') *Let $\nu = (\nu^+, \nu^-)$ be an involution of the anti-Jordan pair $\mathfrak{M}_{mn}(C)$. Then there exist matrices $a \in \mathrm{GL}_n(C)$, $b \in \mathrm{GL}_m(C)$ and $\beta \in C^\times$ such that $(\nu^+, \nu^-) = (\mu_{a,b}^+, \mu_{a,b}^-)$, $a = -\beta a^t$ and $b = \beta b^t$ and $\beta^2 = 1$.*

Proof. (a') Let $\zeta = (\zeta^+, \zeta^-)$ be an automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C) = (M_{mn}(C), M_{nm}(C))$. Then by Lemma 2.2.9 we can extend ζ to an automorphism $\tilde{\zeta}$ of $\mathfrak{K}(V)$ which by Lemma 2.2.5 is isomorphic to $\mathfrak{psl}(m|n)(C) := \mathfrak{sl}(m|n)/Z(\mathfrak{sl}(m|n))$. We will identify $\mathfrak{sl}(m|n)/Z(\mathfrak{sl}(m|n))$ with $\mathfrak{K}(V)$. By Lemma 2.2.8 we have

$$Z(\mathfrak{K}(V)_0) = \frac{C}{A} \begin{pmatrix} \frac{1}{m}I_m & 0 \\ 0 & \frac{1}{n}I_n \end{pmatrix}.$$

Note that $\begin{pmatrix} \frac{1}{m}I_m & 0 \\ 0 & \frac{1}{n}I_n \end{pmatrix}$ acts on (V^+, V^-) by $(\frac{1}{m} - \frac{1}{n})(\mathrm{Id}, -\mathrm{Id})$. Hence by part (b) of Lemma 2.2.9, the extended automorphism $\tilde{\zeta}$ fixes the center pointwise. Also, since $\mathfrak{sl}_m(C)$ and $\mathfrak{sl}_n(C)$ are perfect, $[\mathfrak{K}(V)_0, \mathfrak{K}(V)_0] = \mathfrak{sl}_m(C) \oplus \mathfrak{sl}_n(C)$ and is stabilized by $\tilde{\zeta}$. Hence by Proposition 2.2.12 we have $\tilde{\zeta}(\mathfrak{sl}_m(C)) = \mathfrak{sl}_m(C)$ and $\tilde{\zeta}(\mathfrak{sl}_n(C)) = \mathfrak{sl}_n(C)$. Therefore $\tilde{\zeta} : \mathfrak{psl}(m|n)(C) \rightarrow \mathfrak{psl}(m|n)(C)$ is given by

$$\tilde{\zeta} \begin{pmatrix} a & x \\ y & d \end{pmatrix} = \begin{pmatrix} \tilde{\zeta}(a') + (\gamma + A)\frac{1}{m}I_m & \zeta^+(x) \\ \zeta^-(y) & \tilde{\zeta}(d') + (\gamma + A)\frac{1}{n}I_n \end{pmatrix}$$

where $a = a' + (\gamma + A)\frac{1}{m}I_m$ and $d = d' + (\gamma + A)\frac{1}{n}I_n$ such that $a' \in \mathfrak{sl}_m(C)$, $d' \in \mathfrak{sl}_n(C)$ and $\gamma \in C$.

By assumption there exist $u \in \mathrm{GL}_m(C)$ and $v \in \mathrm{GL}_n(C)$ such that for all $a' \in \mathfrak{sl}_m(C)$ and $d' \in \mathfrak{sl}_n(C)$ one of the following four cases holds:

$$\text{case (i): } \tilde{\zeta}(a') = ua'u^{-1}, \tilde{\zeta}(d') = vd'v^{-1}$$

$$\text{case (ii): } \tilde{\zeta}(a') = ua'u^{-1}, \tilde{\zeta}(d') = -vd'^t v^{-1}$$

$$\text{case (iii): } \tilde{\zeta}(a') = -ua'^t u^{-1}, \tilde{\zeta}(d') = vd'v^{-1}$$

$$\text{case (iv): } \tilde{\zeta}(a') = -ua'^t u^{-1}, \tilde{\zeta}(d') = -vd'^t v^{-1}$$

We know from Lemma 2.2.3 that $\tilde{\theta}_{u,v}$ is an automorphism of $\mathfrak{sl}(m|n)$ for any $u \in \mathrm{GL}_m(C)$ and $v \in \mathrm{GL}_n(C)$. Since $\tilde{\theta}_{u,v}$ leaves the center of $\mathfrak{sl}(m|m)$ invariant, it induces an automorphism of $\mathfrak{psl}(m|n)$. Hence $\phi = \tilde{\theta}_{u,v} \circ \tilde{\zeta}$ is also an automorphism of $\mathfrak{sl}(m|n)/Z(\mathfrak{sl}(m|n))$. Let $x, z \in M_{mn}(C)$ and $y \in M_{nm}(C)$.

$$\text{In case (i) we have } \phi \begin{pmatrix} a & x \\ y & d \end{pmatrix} = \begin{pmatrix} a & u^{-1}\zeta^+(x)v \\ v^{-1}\zeta^-(y)u & d \end{pmatrix} \text{ and}$$

$$\begin{aligned} \phi \begin{pmatrix} 0 & \{x, y, z\} \\ 0 & 0 \end{pmatrix} &= \phi \left[\left[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \right], \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \right] \\ &= \phi \left[\begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}, \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \right] \\ &= \left[\phi \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}, \begin{pmatrix} 0 & \phi^+(z) \\ 0 & 0 \end{pmatrix} \right]. \end{aligned}$$

Since $\phi \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix} = \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}$, this implies

$$\phi^+\{x, y, z\} = \left[\begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}, \begin{pmatrix} 0 & \phi^+(z) \\ 0 & 0 \end{pmatrix} \right] = \{x, y, \phi^+(z)\}.$$

Therefore from Lemma 2.1.9, we get that $\phi^+ = \alpha \mathrm{Id}$, $\alpha \in C$. Since ϕ^+ is bijective, we get $\alpha \in C^\times$. Similarly $\phi^- = \beta \mathrm{Id}$, $\beta \in C^\times$. Then $\phi^+\{x, y, z\} = \{\phi^+(x), \phi^-(y), \phi^+(z)\} = \alpha^2\beta\{x, y, z\}$, which implies $\alpha\beta = 1$. Hence $\phi^+(x) = u^{-1}\zeta^+(x)v = \alpha x$ and $\phi^-(y) =$

$\beta y = v^{-1}\zeta^-(y)u$, which implies $\zeta^+(x) = (\alpha u)xv^{-1}$ and $\zeta^-(y) = vy(\alpha u)^{-1}$. Therefore

$$\text{Aut}(\mathfrak{M}_{mn}(C)) = \bigwedge.$$

In case (ii) we have $\phi \begin{pmatrix} a & x \\ y & d \end{pmatrix} = \begin{pmatrix} a & u^{-1}\zeta^+(x)v \\ v^{-1}\zeta^+(y)u & -d^t \end{pmatrix}$.

Since $\phi \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix} = \begin{pmatrix} xy & 0 \\ 0 & -x^t y^t \end{pmatrix}$, we get with a similar calculation as in case (i) that

$$\begin{aligned} \phi \begin{pmatrix} 0 & \{x, y, z\} \\ 0 & 0 \end{pmatrix} &= \left[\begin{pmatrix} xy & 0 \\ 0 & -x^t y^t \end{pmatrix}, \begin{pmatrix} 0 & \phi^+(z) \\ 0 & 0 \end{pmatrix} \right] \\ \{\phi^+(x), \phi^-(y), \phi^+(z)\} &= xy\phi^+(z) + \phi^+(z)x^t y^t \\ (\phi^+(x)\phi^-(y) - xy)\phi^+(z) &= \phi^+(z)(\phi^-(y)\phi^+(x) + x^t y^t). \end{aligned}$$

From Lemma 2.1.3, we get $\phi^+(x)\phi^-(y) - xy = \alpha I_m$ and $\phi^-(y)\phi^+(x) + x^t y^t = \alpha I_n$ for some $\alpha \in C$. Then from Lemma 2.1.12 we know there does not exist $\alpha \in C$ which satisfies these conditions.

In case (iii), since $\phi \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix} = \begin{pmatrix} -y^t x^t & 0 \\ 0 & yx \end{pmatrix}$, we get with a similar calculation as in case (ii) that $\phi^+(x)\phi^-(y) + y^t x^t = \alpha I_m$ and $\phi^-(y)\phi^+(x) - xy = \alpha I_n$ for some $\alpha \in C$. With a similar reasoning as in case (ii) we get that this is not possible.

In case (iv), since $\phi \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix} = \begin{pmatrix} -y^t x^t & 0 \\ 0 & -x^t y^t \end{pmatrix}$. We get with a similar calculation as in case (ii) that $\phi^+(x)\phi^-(y) + y^t x^t = \alpha I_m$ and $\phi^-(y)\phi^+(x) + x^t y^t = \alpha I_n$, for some $\alpha \in C$. But by Lemma 2.1.12 there does not exist such an $\alpha \in C$.

(b') Let $\xi = (\xi^+, \xi^-)$ be an anti-automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C) = (M_{mn}(C), M_{nm}(C))$. By Lemma 2.1.6 we know that $\mu^+ : M_{mn} \rightarrow M_{nm}$, $\mu^+(x) = x^t$ and $\mu^- : M_{nm} \rightarrow M_{mn}$, $\mu^-(y) = -y^t$ is an anti-automorphism. Hence $\xi \circ \mu = (\xi^- \circ \mu^+, \xi^+ \circ \mu^-)$ is an automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C)$. Therefore by part (a'), there exist $a \in \text{GL}_n(C)$ and $b \in \text{GL}_m(C)$ such that

$$\xi^- \circ \mu^+(x) = axb^{-1} \text{ and } \xi^+ \circ \mu^-(x) = bya^{-1},$$

for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$. This implies $\xi^-(x^t) = axb$ and $\xi^+(-y^t) = bya^{-1}$, i.e. $\xi^+(x) = \mu_{a,b}^+(x) = ax^tb^{-1}$ and $\xi^-(y) = \mu_{a,b}^-(y) = -by^ta^{-1}$.

(c') Since $\nu = (\nu^+, \nu^-)$ is an anti-automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C) = (M_{mn}(C), M_{nm}(C))$. Therefore by part (b'), we get there exist $a \in GL_n(C)$ and $b \in GL_m(C)$ such that

$$\nu^+ = \mu_{a,b}^+(x) = ax^tb^{-1}, \nu^- = \mu_{a,b}^-(y) = -by^ta^{-1}.$$

Hence by Lemma 2.1.6 there exist $\beta \in C^\times$ such that $-\beta a^t = a$, $\beta b^t = b$ and $\beta^2 = 1$. □

Corollary 2.3.2. *Let $m \neq n \geq 2$ and suppose one of the following:*

- (i) *C is a unital, commutative and associative algebra over a field k of characteristic 0 and C is a unique factorization domain, or*
- (ii) *C is a field of characteristic zero or characteristic $p > 3$ and such that $p \nmid m$ and $p \nmid n$.*

Then $\text{Aut}(\mathfrak{M}_{mn}(C)) = \Lambda$.

Proof. In case (i) by Theorem 2.2.14 and in case (ii) by Theorem 2.2.13, there exist $u \in GL_q(C)$ such that every automorphism of $\mathfrak{sl}_q(C)$ is of the form $a \mapsto uau^{-1}$ or $a \mapsto -ua^tu^{-1}$, where $q = n$ or m . Therefore it follows from Theorem 2.3.1 that $\text{Aut}(\mathfrak{M}_{mn}(C)) = \Lambda$. □

Corollary 2.3.3. *Let m, n and C be as in Corollary 2.3.2. Suppose $\xi = (\xi^+, \xi^-)$ is an anti-automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C) = (M_{mn}(C), M_{nm}(C))$. Then there exist matrices $a \in GL_n(C)$ and $b \in GL_m(C)$ such that $\xi^+ = \mu_{a,b}^+(x) = ax^tb^{-1}$, $\xi^- = \mu_{a,b}^-(y) = -by^ta^{-1}$ for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$.*

Conversely, let $a \in GL_n(C)$ and $b \in GL_m(C)$ and $\mu_{a,b} = (\mu_{a,b}^+, \mu_{a,b}^-)$ defined by $\mu_{a,b}^+(x) = ax^tb^{-1}$, $\mu_{a,b}^-(y) = -by^ta^{-1}$ for all $x \in M_{mn}(C)$ and $y \in M_{nm}(C)$. Then $\mu = (\mu^+, \mu^-)$ is an anti-automorphism of the anti-Jordan pair $\mathfrak{M}_{mn}(C)$.

Corollary 2.3.4. *Let m, n and C be as in Corollary 2.3.2. Let $\nu = (\nu^+, \nu^-)$ be an involution of the anti-Jordan pair $\mathfrak{M}_{mn}(C) = (M_{mn}(C), M_{nm}(C))$. Then there exist matrices $a \in GL_n(C)$, $b \in GL_m(C)$ and $\beta \in C^\times$ such that*

Case (i): $(\nu^+, \nu^-) = (\mu_{a,b}^+, \mu_{a,b}^-)$, $a = -\beta a^t$ and $b = \beta b^t$ and $\beta^2 = 1$.

Case (ii): $(\nu^+, \nu^-) = (\mu_{a,b}^+, \mu_{a,b}^-)$, $a = -\epsilon a^t$ and $b = \epsilon b^t$ and $\epsilon = \pm 1$.

The proofs of Corollary 2.3.3 and Corollary 2.3.4 are analogous to the proof of Corollary 2.3.2.

Theorem 2.3.5. *Suppose that every automorphism of $\mathfrak{sl}_m(C)$ is of the form $a \mapsto uau^{-1}$ or $a \mapsto -ua^t u^{-1}$ for some $u \in \mathrm{GL}_m(C)$. Let $m1_C \in C^\times$.*

(a) *Recall the subgroup $\tilde{\Lambda}$ of $\mathrm{Aut}(\mathfrak{M}_{mm}(C))$ defined in Lemma 2.1.8. Then $\tilde{\Lambda}$ coincides with the subgroup of the automorphism group of $\mathfrak{M}_{mm}(C)$ whose extended automorphism to $\mathfrak{K}(V)$ leaves the two factors $\mathfrak{sl}_m(C)$ of $\mathfrak{K}(V)_0$ invariant or exchanges them.*

(b) *Let $\nu = (\nu^+, \nu^-) \in \tilde{\Lambda}$ be an involution of the anti-Jordan pair $\mathfrak{M}_{mm}(C)$. Then there exist matrices $a, b \in \mathrm{GL}_m(C)$ and $\alpha, \beta \in C^\times$ such that for all $x, y \in \mathfrak{M}_{mm}(C)$,*

(i') $\nu^+(x) = \eta_{a,\alpha a}^+ = ax(\alpha a)^{-1}$, $\nu^- = \eta_{a,\alpha a}^-(y) = \alpha a y a^{-1}$ or,

(ii') $(\nu^+, \nu^-) = (\mu_{a,b}^+, \mu_{a,b}^-)$, where $a = -\beta a^t$ and $b = \beta b^t$ and $\beta^2 = 1$.

Proof. (a) Let $\zeta = (\zeta^+, \zeta^-)$ be an automorphism of the anti-Jordan pair $\mathfrak{M}_{mm}(C) = (\mathfrak{M}_{mm}(C), \mathfrak{M}_{mm}(C))$. Then by Lemma 2.2.9 we can extend ζ to an automorphism $\tilde{\zeta}$ of $\mathfrak{K}(V)$, which by Lemma 2.2.5 is isomorphic to $\mathfrak{psl}(m|m)$. We will identify $\mathfrak{psl}(m|m)$ with $\mathfrak{K}(V)$, which implies $\mathfrak{K}(V)_0 = \mathfrak{sl}_m(C) \oplus \mathfrak{sl}_m(C)$ by Lemma 2.2.8. Suppose $\tilde{\zeta}$ leaves the factors invariant.

Thus, $\tilde{\zeta} : \mathfrak{psl}(m|m) \rightarrow \mathfrak{psl}(m|m)$ is given by

$$\tilde{\zeta} \begin{pmatrix} a & x \\ y & d \end{pmatrix} = \begin{pmatrix} \tilde{\zeta}(a) & \zeta^+(x) \\ \zeta^-(y) & \tilde{\zeta}(d) \end{pmatrix}$$

where $a, d \in \mathfrak{sl}_m(C)$. Therefore, by Theorem 2.2.13, there exist $u, v \in \mathrm{GL}_n(C)$ such that for all $a, d \in \mathfrak{sl}_n(C)$ one of the following four cases holds:

case (i): $\tilde{\zeta}(a) = uau^{-1}$, $\tilde{\zeta}(d) = vdv^{-1}$,

case (ii): $\tilde{\zeta}(a) = uau^{-1}$, $\tilde{\zeta}(d) = -vd^t v^{-1}$,

case (iii): $\tilde{\zeta}(a) = -ua^t u^{-1}$, $\tilde{\zeta}(d) = vdv^{-1}$,

case (iv): $\tilde{\zeta}(a) = -ua^t u^{-1}$, $\tilde{\zeta}(d) = -vd^t v^{-1}$.

We know from Lemma 2.2.3 that $\tilde{\theta}_{u,v}$ is an automorphism of $\mathfrak{sl}(m|m)$ for any $u, v \in \mathrm{GL}_m(C)$. Since $\tilde{\theta}_{u,v}$ leaves the center of $\mathfrak{sl}(m|m)$ invariant, it induces an automorphism of $\mathfrak{psl}(m|m)$. Hence $\phi = \tilde{\theta}_{u,v} \circ \tilde{\zeta}$ is also an automorphism of $\mathfrak{psl}(m|m)$.

In case (i) a similar calculation as in case (i) of Theorem 2.3.1, yields $\zeta^+(x) = axb^{-1}$ and $\zeta^-(y) = bya^{-1}$ for all $x, y \in M_{mm}(C)$, i.e., $(\zeta^+, \zeta^-) \in \tilde{\Lambda}$.

In case (ii), (iii) and (iv) similar calculations as in case (i) together with Lemma 2.1.12 yield a contradiction.

Before we consider the case where $\tilde{\zeta}$ exchanges the two factors, we look at the map $\psi: (x, y) \rightarrow (x^t, -y^t)$. The map ψ is an automorphism of $\mathfrak{M}_{mm}(C)$, see Lemma 1.2.4. We claim that the automorphism $\tilde{\psi}|_{\mathfrak{K}(V)_0}$ exchanges the two factors $\mathfrak{sl}_m(C)$. Indeed, for $x \in V^+$ and $y \in V^-$ we have $\left[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \right] = \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}$, hence

$$\begin{aligned} \tilde{\psi} \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix} &= \left[\begin{pmatrix} 0 & \psi^+(x) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \psi^-(y) & 0 \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} 0 & x^t \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -y^t & 0 \end{pmatrix} \right] \\ &= - \begin{pmatrix} (yx)^t & 0 \\ 0 & (xy)^t \end{pmatrix}. \end{aligned}$$

Let $x = E_{12}$ and $y = E_{22}$. Then $xy = E_{12}$, $yx = 0$, $x^t y^t = 0$ and $y^t x^t = E_{21}$. This implies

$$\tilde{\psi} \begin{pmatrix} E_{12} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -E_{21} \end{pmatrix}.$$

So it follows that $\tilde{\psi}|_{\mathfrak{K}(V)_0}$ exchanges the two ideals of $\mathfrak{K}(V)_0$. Thus it follows $\tilde{\Lambda}$ coincides with the subgroup of the automorphism group of $\mathfrak{M}_{mm}(C)$ whose extended automorphism to $\mathfrak{K}(V)$ leaves the two factors $\mathfrak{sl}_m(C)$ of $\mathfrak{K}(V)_0$ invariant or exchanges them.

(b) Let $\nu = (\nu^+, \nu^-) \in \tilde{\Lambda}$ be an involution of the anti-Jordan pair $\mathfrak{M}_{mm}(C) = (M_{mm}(C), M_{mm}(C)) = (V^+, V^-)$. Since $V^+ = V^-$, therefore $\nu = (\nu^+, \nu^-)$ is an automorphism of $\mathfrak{M}_{mm}(C)$. Hence by part (a) we get there exist matrices $a, b \in \mathrm{GL}_m(C)$ such that

(i') $\nu^+(x) = \eta_{a,b}^+(x) = axb^{-1}$, $\nu^-(y) = \eta_{a,b}^-(y) = bya^{-1}$ for all $x, y \in M_{mm}(C)$, or

(ii') $\nu^+(x) = \mu_{a,b}^+(x) = ax^tb^{-1}$, $\nu^-(y) = \mu_{a,b}^-(y) = -by^ta^{-1}$ for all $x, y \in M_{mm}(C)$.

In case (i') by Lemma 2.1.5 we get $\nu = (\nu^+, \nu^-)$ is an involution of the anti-Jordan pair $\mathfrak{M}_{mm}(C) = (M_{mm}(C), M_{mm}(C))$ if and only if $b = \alpha a$. In case (ii') by using Lemma 2.1.6 we obtained $\nu = (\nu^+, \nu^-)$ is an involution of the anti-Jordan pair $\mathfrak{M}_{mm}(C)$ if and only if there exist $\beta \in C^\times$ such that $a = -\beta a^t$ and $b = \beta b^t$ and $\beta^2 = 1$. \square

Corollary 2.3.6. *Let $m \geq 2$ and suppose one of the following:*

- (i) C is a unital, commutative and associative algebra over a field k of characteristic 0 which is a unique factorization domain, or
- (ii) C a field of characteristic zero or characteristic $p > 3$ and such that p does not divide m .

Then $\text{Aut}(\mathfrak{M}_{mm}(C)) = \tilde{\Lambda}$.

Proof. In case (i) by Theorem 2.2.14 and in case (ii) by Theorem 2.2.13, there exists $u \in \text{GL}_t(C)$ such that every automorphism of $\mathfrak{sl}_m(C)$ is of the form $a \mapsto uau^{-1}$ or $a \mapsto -ua^tu^{-1}$. Any automorphism of the anti-Jordan pair $V = \mathfrak{M}_{mm}(C)$ extended to $\mathfrak{K}(V(F))$ is an automorphism of $\mathfrak{K}(V)_0 \cong \mathfrak{sl}_m(F) \oplus \mathfrak{sl}_m(C)$. Since C is connected, by Proposition 2.2.12, there are two cases, either the automorphism leaves the factors invariant or exchanges the factor. Therefore $\text{Aut}(\mathfrak{M}_{mm}(C)) = \tilde{\Lambda}$ by Theorem 2.3.5. \square

Corollary 2.3.7. *Let m and C be as in Corollary 2.3.6. Let $\nu = (\nu^+, \nu^-)$ be an involution of the anti-Jordan pair $\mathfrak{M}_{mm}(C) = (M_{mm}(C), M_{mm}(C))$. $m \geq 2$, Then there exist matrices $a, b \in \text{GL}_m(C)$ and $\alpha, \beta \in C^\times$ such that for all $x, y \in M_{mm}(C)$,*

(i) $\nu^+(x) = \eta_{a,\alpha a}^+ = ax(\alpha a)^{-1}$, $\nu^- = \eta_{a,\alpha a}^-(y) = \alpha a y a^{-1}$ or,

(ii) $(\nu^+, \nu^-) = (\mu_{a,b}^+, \mu_{a,b}^-)$, $a = -\beta a^t$ and $b = \beta b^t$ and $\beta^2 = 1$. or,

(iii) $(\nu^+, \nu^-) = (\mu_{a,b}^+, \mu_{a,b}^-)$, where $a = -\epsilon a^t$, $b = \epsilon b^t$ and $\epsilon = \pm 1$.

Conversely, the maps in (i), (ii) and (iii) above are involutions.

Remark: If an involution of type (iii) exists then we get $a^t = -a \in \text{GL}_m(C)$ or $b^t = -b \in \text{GL}_m(C)$, so m is even.

2.4 Anti-Jordan triple systems

Lemma 2.4.1. *Let F be an algebraically closed field of characteristic different from 2. Let $T_1 = (M_{nn}(F), \langle \dots \rangle_1)$ and $T_2 = (M_{nn}(F), \langle \dots \rangle_2)$ be two anti-Jordan triple systems, defined by $\langle x, y, z \rangle_1 = \{x, y, z\}$ and $\langle x, y, z \rangle_2 = \alpha\{x, aya, z\}$ respectively, where $a \in GL_n(F)$ and $\alpha \in F^\times$. Then $T_1 \cong T_2$.*

Proof. We know from [1, Theorem 8.32] that there exist a matrix $p \in GL_n(F)$ and $\beta \in F$ such that $a = p^2$ and $\beta^2 = \alpha$. Define $f : M_{nn}(F) \rightarrow M_{nn}(F)$ by

$$f(x) = \beta^{-1}p^{-1}xp^{-1}$$

Then obviously f is bijective. Also

$$f\langle x, y, z \rangle_1 = \beta^{-1}p^{-1}\{x, y, z\}p^{-1}$$

and

$$\begin{aligned} \langle f(x), f(y), f(z) \rangle_2 &= \langle \beta^{-1}p^{-1}xp^{-1}, \beta^{-1}p^{-1}yp^{-1}, \beta^{-1}p^{-1}zp^{-1} \rangle_2 \\ &= \alpha\beta^{-3}\{p^{-1}xp^{-1}, ap^{-1}yp^{-1}a, p^{-1}zp^{-1}\} \\ &= \beta^{-1}p^{-1}xp^{-1}ap^{-1}yp^{-1}ap^{-1}zp^{-1} \\ &\quad - \beta^{-1}p^{-1}zp^{-1}ap^{-1}yp^{-1}ap^{-1}xp^{-1} \\ &= \beta^{-1}p^{-1}xyzp^{-1} - \beta^{-1}p^{-1}zyxp^{-1} \\ &= \beta^{-1}p^{-1}\{x, y, z\}p^{-1}. \end{aligned}$$

Therefore

$$f\langle x, y, z \rangle_1 = \langle f(x), f(y), f(z) \rangle_2.$$

Hence T_1 and T_2 are isomorphic. \square

Lemma 2.4.2. *Let η_1 and η_2 be two involutions of the anti-Jordan pair $\mathfrak{M}_{mn}(F) = (M_{mn}(F), M_{mn}(F))$, where F is an algebraically closed field of characteristic different from two, given by invertible symmetric matrices a_1, a_2 and skew symmetric matrices c_1, c_2 or skew symmetric matrices a_1, a_2 and symmetric matrices c_1, c_2 , as in Lemma 2.1.6. Then the anti-Jordan triple systems T_1 and T_2 , defined by $\langle x, y, z \rangle_1 = \{x, a_1y^t c_1, z\}$ and $\langle x, y, z \rangle_2 = \{x, a_2y^t c_2, z\}$ respectively, are isomorphic.*

Proof. From Theorem 1.1.4 and Theorem 1.1.5 we know there exist matrices $p \in \text{GL}_n(F)$ and $q \in \text{GL}_m(F)$ such that $p^t a_1 p = a_2$ and $q^t c_1 q = c_2$. Define

$$f : M_{mn}(F) \rightarrow M_{mn}(F) \text{ by } f(x) = q^{-1} x (p^{-1})^t.$$

Then obviously f is bijective. Also

$$f\langle x, y, z \rangle_1 = f\{x, a_1 y^t c_1, z\} = q^{-1}(\{x, a_1 y^t c_1, z\}(p^{-1})^t)$$

and

$$\begin{aligned} \langle f(x), f(y), f(z) \rangle_2 &= \langle q^{-1} x (p^{-1})^t, q^{-1} y (p^{-1})^t, q^{-1} z (p^{-1})^t \rangle_2 \\ &= \{q^{-1} x (p^{-1})^t, a_2 p^{-1} y^t (q^{-1})^t c_2, q^{-1} z (p^{-1})^t\} \\ &= q^{-1} x (p^{-1})^t a_2 p^{-1} y^t (q^{-1})^t c_2 q^{-1} z (p^{-1})^t \\ &\quad - q^{-1} z (p^{-1})^t a_2 p^{-1} y^t (q^{-1})^t c_2 q^{-1} x (p^{-1})^t \\ &= q^{-1} x a_1 y^t c_1 z (p^{-1})^t - q^{-1} z a_1 y^t c_1 x (p^{-1})^t \\ &= q^{-1} \{x, a_1 y^t c_1, z\} (p^{-1})^t. \end{aligned}$$

Therefore

$$f\langle x, y, z \rangle_1 = \langle f(x), f(y), f(z) \rangle_2.$$

Hence T_1 and T_2 are isomorphic.

We summarize the results so far obtained in the following theorem.

Theorem 2.4.3. (a) *Let F be a field. The following are simple finite dimensional anti-Jordan triple systems over F :*

(i) $T_1 = M_{mn}(F)$, together with the trilinear maps $\langle \dots \rangle_1$ defined by

$$\langle x, y, z \rangle_1 = xy^t az - zy^t ax, \text{ where } a = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \text{ and } 2r = m.$$

(ii) $T_2 = M_{mn}(F)$, together with the trilinear maps $\langle \dots \rangle_2$ defined by

$$\langle x, y, z \rangle_2 = xby^t z - zby^t x, \text{ where } b = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \text{ and } 2r = n.$$

(iii) $T_3 = M_{nn}(F)$, together with the trilinear maps $\langle \dots \rangle_3$ defined by

$$\langle u, v, w \rangle_3 = uvw - wvu.$$

(b) Conversely, let F be an algebraically closed field of characteristic zero or of characteristic $p > 0$, $p \neq 2, 3$ and such that $p \nmid m$ and $p \nmid n$. If T is a simple finite dimensional anti-Jordan triple system over F such that $(T, T) \cong \mathfrak{M}_{mn}(F)$, then T is isomorphic to one of the examples in (a).

Proof. We know from Lemma 2.1.6 that $(\mu_{I,a}^+, \mu_{I,a}^-)$, $(\mu_{b,I}^+, \mu_{b,I}^-)$ and $(\eta_{I,I}^+, \eta_{I,I}^-)$ are involutions of the anti-Jordan pair $\mathfrak{M}_{mn}(F)$. Therefore by Lemma 1.2.1 we get that T_1 , T_2 and T_3 are anti-Jordan triple systems with triple product:

$$\langle x, y, z \rangle_1 = \langle x, \mu_{I,a}^+(y), z \rangle = xy^taz - zy^tax,$$

$$\langle x, y, z \rangle_2 = \langle x, \mu_{b,I}^+(y), z \rangle = xby^tz - zby^tx,$$

$$\langle u, v, w \rangle_3 = \langle x, \eta_{I,I}^+(y), z \rangle = uvw - wvu.$$

Also $(T_1, T_1) \cong \mathfrak{M}_{mn}(F)$, $(T_2, T_2) \cong \mathfrak{M}_{mn}(F)$ and $(T_3, T_3) \cong \mathfrak{M}_{nn}(F)$ by part (b) of Lemma 1.2.1. From Theorem 1.2.7 we know $\mathfrak{M}_{mn}(F)$ is simple. Hence T_1 , T_2 and T_3 are simple anti-Jordan triple system by Lemma 1.2.6.

Conversely, let T be a simple anti-Jordan triple system over F such that $(T, T) \cong \mathfrak{M}_{mn}(F)$. Then $T = M_{mn}(F)$ as a vector space and T has triple product:

$$\langle x, y, s \rangle = \{x, \nu^+(y), z\}$$

for some involution (ν^+, ν^-) by Lemma 1.2.1. If $m \neq n$, then by Corollary 2.3.4 there exist matrices $a^t = a \in \text{GL}_n(F)$ and $b^t = -b \in \text{GL}_m(F)$ or $a^t = -a \in \text{GL}_n(F)$ and $b^t = b \in \text{GL}_m(F)$ such that $\nu^+ = \mu_{a,b}^+(x) = ax^tb^{-1}$, $\nu^- = \mu_{b,a}^-(y) = -by^ta^{-1}$ for all $x \in M_{mn}(F)$ and $y \in M_{nm}(F)$. Hence from Lemma 2.4.2, we get $T \cong T_1$ or $T \cong T_2$.

If $m = n$, then by Corollary 2.3.7 there exist $a, b \in \text{GL}_m(F)$ and $\alpha \in F^\times$ such that for all $x, y \in M_{mm}(F)$,

(i) $\nu^+(x) = \eta_{a,\alpha a}^+ = ax(\alpha a)^{-1}$, $\nu^- = \eta_{a,\alpha a}^-(y) = \alpha a y a^{-1}$ or,

(ii) $\nu^+ = \mu_{a,b}^+(x) = ax^t b^{-1}$, $\nu^- = \mu_{a,b}^-(y) = -by^t a^{-1}$, where $a^t = a$ and $b^t = -b$ or $a^t = -a$ and $b^t = b$.

In case(i) by Lemma 2.4.1 we get $T \cong T_3$ and in case (ii) Lemma 2.4.2 yields $T \cong T_1$ or $T \cong T_2$. □

Chapter 3

Automorphisms of the anti-Jordan pair \mathfrak{ss}_n

In this chapter all algebraic structures are defined over a commutative associative and unital ring k with $\frac{1}{2} \in k$. We denote by C a commutative associative and unital k -algebra and we use the abbreviation $N^+ = N^+(C) = \{x \in M_{nn}(C) : x^t = x\}$ and $N^- = N^-(C) = \{y \in M_{nn}(C) : y^t = -y\}$, $n \geq 2$.

3.1 Preliminary results

- Lemma 3.1.1.** (a) Let $x \in N^+$ such that $xy = 0$ for all $y \in N^-$. Then $x = 0$.
 (b) Let $a \in M_{nn}(C)$. If $ax = -xa^t$ for all $x \in N^+$. Then $a = 0$.
 (c) Let $w, z \in M_{nn}(C)$. Then $wxz = x$ for all $x \in N^+$ if and only if $w = \alpha I_n$ and $z = \alpha^{-1} I_n$ for some $\alpha \in C^\times$.

Proof. (a) Put $x = \sum_i x_{ii} E_{ii} + \sum_{l \neq j} x_{lj} (E_{lj} + E_{jl})$. Then, for all p, q with $1 \leq p < q \leq n$, $\left(\sum_i x_{ii} E_{ii} + \sum_{l < j} x_{lj} (E_{lj} + E_{jl}) \right) (E_{pq} - E_{qp}) = 0$. This implies

$$x_{pp} E_{pq} - x_{qq} E_{qp} + \sum_{l < p} x_{lp} E_{lq} - \sum_{l < q} x_{lq} E_{lp} + \sum_{p < j} x_{pj} E_{jq} - \sum_{q < j} x_{qj} E_{jp} = 0.$$

Therefore $x_{ii} = 0$ and $x_{lj} = 0$ for $1 \leq i \leq n$ and $1 \leq l < j \leq n$. Hence $x = 0$.

(b) Suppose $ax = -xa^t$ for all $x \in N^+$. Write $a = \sum_{i \neq j} a_{ij} E_{ij} + \sum_i a_{ii} E_{ii}$. Then, for all i, j with $1 \leq i \neq j \leq n$ we have

$$\left(\sum_{i \neq j} a_{ij} E_{ij} + \sum_i a_{ii} E_{ii} \right) E_{pp} = -E_{pp} \left(\sum_{i \neq j} a_{ij} E_{ji} + \sum_i a_{ii} E_{ii} \right).$$

This implies $\sum_{i \neq p} a_{ip} E_{ip} + a_{pp} E_{pp} = - \sum_{i \neq p} a_{ip} E_{pi} - a_{pp} E_{pp}$. Therefore

$$2a_{pp} = 0 \text{ for } 1 \leq p \leq n \text{ and } a_{ip} = 0 \text{ for } 1 \leq i \neq p \leq n.$$

Since $\frac{1}{2} \in C$, $a_{pp} = 0$ for $1 \leq p \leq n$. Hence $a = 0$.

(c) Suppose that $wxz = x$ for all $x \in N^+$. Put $w = (w_{ij})_{1 \leq i, j \leq n}$ and $z = (z_{kl})_{1 \leq k, l \leq n}$. Then for all p with $1 \leq p \leq n$,

$$E_{pp} = (w_{ij})_{1 \leq i, j \leq n} E_{pp} (z_{kl})_{1 \leq k, l \leq n} = \sum_i \sum_l w_{ip} z_{pl} E_{il}.$$

This implies $w_{pp} z_{pp} = 1$ for $1 \leq p \leq n$ and $w_{ip} = 0 = z_{pl}$ for $1 \leq i \neq p \leq n$ and $1 \leq l \neq p \leq n$. Hence w and z are diagonal matrices. Therefore for $1 \leq p \neq q \leq n$ we have

$$E_{pq} + E_{qp} = \sum_i w_{ii} E_{ii} (E_{pq} + E_{qp}) \sum_k z_{kk} E_{kk} = w_{pp} z_{qq} E_{pq} + w_{qq} z_{pp} E_{qp}.$$

This implies $w_{pp} z_{qq} = 1$ and $w_{qq} z_{pp} = 1$ for $1 \leq p \neq q \leq n$. But $w_{pp} z_{pp} = 1$. Hence $w = \alpha I_n$ and $z = \alpha^{-1} I_n$ for some $\alpha \in C^\times$. The converse is easy to see. \square

Lemma 3.1.2. *Let $\mathfrak{ss}_n(C) = (N^+, N^-)$. Then*

$$(i) \{f \in \text{End}_k(N^+) : [f, D(x, y)] = 0, \text{ for all } (x, y) \in (N^+, N^-)\} = C\text{Id}_{N^+}.$$

$$(ii) \{g \in \text{End}_k(N^-) : [g, D(y, x)] = 0, \text{ for all } (x, y) \in (N^+, N^-)\} = C\text{Id}_{N^-},$$

for $n > 2$.

Proof. (i) Note that

$$[f, D(x, y)] = 0 \Leftrightarrow f\{x, y, z\} = \{x, y, f(z)\}.$$

Let E_{pq} denote the matrix units. Put $H_{pq} = E_{pq} + E_{qp}$ and $F_{pq} = E_{pq} - E_{qp}$ for all p, q with $1 \leq q \neq p \leq n$, then $\{E_{pp}, \frac{1}{2}(F_{pq}), H_{qp}\} = E_{pp}$ and $\{E_{pp}, F_{pq}, E_{qq}\} = H_{qp}$. Therefore $f(H_{qp}) = \{E_{pp}, F_{pq}, f(E_{qq})\}$. Put

$$f(E_{qq}) = \sum_i z_{ii} E_{ii} + \sum_{l < r} z_{lr} (H_{lr}).$$

Then

$$f(H_{qp}) = \{E_{pp}, F_{pq}, \sum_i z_{ii} E_{ii} + \sum_{l \leq r} z_{lr} (H_{lr})\}.$$

This implies $f(H_{qp}) = \sum_{r=1}^n z_{qr} (H_{pr})$. Hence

$$\begin{aligned} f(E_{pp}) &= \{E_{pp}, \frac{1}{2}(F_{pq}), f(H_{qp})\}, \\ &= \{E_{pp}, \frac{1}{2}(F_{pq}), \sum_{r=1}^n z_{qr} (H_{pr})\} \\ &= z_{qq} E_{pp}. \end{aligned}$$

Therefore $f(E_{pp}) = z_{11} E_{pp} = \alpha E_{pp}$, $2 \leq p \leq n$. This implies

$$\begin{aligned} f(H_{qp}) &= -\{E_{qq}, F_{pq}, f(E_{pp})\} \\ &= \{E_{qq}, F_{pq}, \alpha E_{pp}\} \\ &= \alpha (H_{qp}). \end{aligned}$$

Also

$$\begin{aligned} f(E_{11}) &= \{E_{11}, F_{1p}, f(H_{p1})\} \\ &= \{E_{11}, F_{1p}, \alpha (H_{p1})\} \\ &= \alpha E_{11}. \end{aligned}$$

This proves $f(H_{pq}) = \alpha H_{pq}$ for $1 \leq p \neq q \leq n$ and $f(E_{pp}) = \alpha E_{pp}$ for $1 \leq p \leq n$ with $\alpha \in C$. For arbitrary $c \in C$ we have $\{cE_{pp}, F_{pq}, E_{qq}\} = cH_{qp}$. Therefore $g(cH_{pq}) =$

$\{cE_{pp}, F_{pq}, g(E_{qq})\} = \{E_{pp}, F_{pq}, \alpha E_{qq}\} = \alpha cH_{qp}$. Similarly $g(cE_{pp}) = \alpha cE_{pp}$. Hence $f = \alpha \text{Id}$, $\alpha \in C$.

(ii) Let $n > 2$. Then for all p, q, v with $1 \leq p \neq v \neq q \leq n$, we have

$$\{F_{pq}, H_{vq}, F_{vq}\} = F_{pq}.$$

This implies $g(F_{pq}) = \{F_{pq}, H_{qv}, g(F_{vq})\}$. Put

$$g(F_{vq}) = \sum_{l < r} z_{lr}(F_{lr}).$$

Then

$$\begin{aligned} g(F_{pq}) &= \{F_{pq}, H_{qv}, \sum_{l < r} z_{lr}(F_{lr})\} \\ &= \sum_{l < r} z_{lr}(E_{pv}(F_{lr}) + (F_{lr})E_{vp}) \\ &= \sum_{v < r} z_{vr}(F_{pr}) + \sum_{l < v} z_{lv}(F_{lp}). \end{aligned}$$

Hence

$$\begin{aligned} g(F_{pq}) &= -\{F_{vq}, H_{qv}, g(F_{pq})\} \\ &= -\{F_{vq}, H_{qv}, \sum_{v < r} z_{vr}(F_{pr})\} - \{F_{vq}, H_{qv}, \sum_{l < v} z_{lv}(F_{lp})\} \\ &= -\sum_{v < r} z_{vr}((E_{vv} - E_{qq})(F_{pr}) - (F_{pr})(E_{qq} - E_{vv})) \\ &\quad - \sum_{l < v} z_{lv}((E_{vv} - E_{qq})(F_{lp}) - (F_{lp})(E_{qq} - E_{vv})) \\ &= z_{vq}(F_{pq}) \quad 1 \leq p \neq v < q \leq n. \end{aligned} \tag{a}$$

Since $F_{pv} = \{F_{vq}, E_{qq}, F_{pq}\}$ and $F_{vq} = \{F_{vp}, E_{pp}, F_{pq}\}$. We get

$$g(F_{pv}) = \{F_{vq}, E_{qq}, g(F_{pq})\} \quad \text{and} \quad g(F_{vq}) = \{F_{pv}, E_{pp}, g(F_{pq})\}.$$

Therefore

$$g(F_{pv}) = -\{F_{qv}, E_{qq}, z_{vq}(F_{pq})\} = z_{vq}(F_{pv}) \tag{b}$$

and

$$g(F_{vq}) = \{F_{vp}, E_{pp}, z_{vq}(F_{pq})\} = z_{vq}(F_{vq}). \quad (c)$$

for $1 \leq p \neq v < q \leq n$.

From Equations (a), (b) and (c) we have $g(F_{1q}) = z_{2q}F_{1q}$, $g(F_{12}) = z_{2q}F_{12}$ and $g(F_{1q}) = z_{1q}(F_{1q})$ respectively for all q with $3 \leq q \leq n$. This implies $z_{23} = z_{13} = z_{1q} = z_{2q}$ for all q with $4 \leq q \leq n$. Therefore $g(F_{1q}) = z_{13}F_{1q}$, for $2 \leq q \leq n$. Hence equation (a) yields,

$$g(F_{pq}) = z_{1q}F_{pq} = z_{13}F_{pq} \quad \text{for } 2 \leq p < q \leq n.$$

Therefore $g(F_{pq}) = \beta F_{pq}$ for $1 \leq p < q \leq n$ with some $\beta \in C$. For arbitrary $c \in C$ we have $\{cF_{pq}, H_{vq}, F_{vq}\} = cF_{pq}$ for any v with p, q, v distinct. Hence $g(cF_{pq}) = \{cF_{pq}, H_{vq}, g(F_{vq})\} = \{cF_{pq}, H_{vq}, \beta F_{vq}\} = c\beta F_{pq}$. This proves $g = \beta \text{Id}_{N^-}$. The converse is clear. \square

Lemma 3.1.3. (a) Let $\phi \in \text{Aut}_k(C)$. Define $\tilde{\phi}^+ : N^+(C) \rightarrow N^+(C)$ and $\tilde{\phi}^- : N^-(C) \rightarrow N^-(C)$ by

$$\tilde{\phi}^+(x) := \phi((x_{ij})) = (\phi(x_{ij})), \quad \tilde{\phi}^-(y) := \phi((y_{ij})) = (\phi(y_{ij})),$$

for all $x \in N^+(C)$ and $y \in N^-(C)$. Then $(\tilde{\phi}^+, \tilde{\phi}^-) \in \text{Aut}_k(\mathfrak{ss}_n(C))$ and the map $\text{Aut}_k(C) \rightarrow \text{Aut}_k(\mathfrak{ss}_n(C)), \phi \mapsto \tilde{\phi}$ is a group monomorphism.

(b) $\text{Aut}_C(\mathfrak{ss}_n(C))$ is a normal subgroup of $\text{Aut}_k(\mathfrak{ss}_n(C))$ and

$$\text{Aut}_k(\mathfrak{ss}_n(C)) = \text{Aut}_C(\mathfrak{ss}_n(C)) \rtimes (\text{Aut}_k(C))^\sim.$$

Proof. (a) is immediate.

(b) can be proved as we proved (b) in Lemma 2.1.11 by using Lemma 3.1.2.

Example 3.1.4. We define $\mathfrak{ssl}(n|n)(C)$ or $\mathfrak{ssl}(n|n)$ for short by

$$\mathfrak{ssl}(n|n) = \left\{ \begin{pmatrix} a & x \\ y & -a^t \end{pmatrix} : a \in \mathfrak{sl}_n(C), (x, y) \in (N^+, N^-) \right\}.$$

Put

$$\begin{aligned} L_1 &= \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x^t = x \in M_{nn}(C) \right\}, \\ L_0 &= \left\{ \begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix} : a \in \mathfrak{sl}_n(C) \right\}, \\ L_{-1} &= \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} : y^t = -y \in M_{nn}(C) \right\}. \end{aligned}$$

Then $\mathfrak{ssl}(n|n) = L_{\bar{0}} \oplus L_{\bar{1}} = L_1 \oplus L_0 \oplus L_{-1}$, is a 3-graded Lie superalgebra, with respect to the usual super commutator product. In fact, it is easy to verify that it is a subalgebra of the Lie superalgebra of $M_{n|n}(C)^-$:

$$\begin{aligned} & \left[\begin{pmatrix} a & x_1 \\ y_1 & -a^t \end{pmatrix}, \begin{pmatrix} b & x_2 \\ y_2 & -b^t \end{pmatrix} \right] \\ &= \begin{pmatrix} [a, b] + x_1 y_2 + x_2 y_1 & a x_2 + x_2 a^t - x_1 b^t - b x_1 \\ y_1 b + b^t y_1 - a^t y_2 - y_2 a & -([a, b] + x_1 y_2 + x_2 y_1)^t \end{pmatrix}. \end{aligned}$$

□

Lemma 3.1.5. $\mathfrak{ssl}(n|n)(C) = L_1 \oplus L_0 \oplus L_{-1}$ is a Jordan 3-graded Lie superalgebra with associated Jordan superpair $\mathfrak{ss}_n(C)$.

Proof. We know that $\mathfrak{ssl}(n|n) = L_1 \oplus L_0 \oplus L_{-1}$ is a 3-graded Lie superalgebra (see Example 3.1.4). Also for all $x, z \in L_1$ and $y \in L_{-1}$, we have $[[x, y]z] = [xy + yx, z] = xyz - zyx = \{x, y, z\}$ (since $[L_\epsilon, L_\epsilon] = 0$, $\epsilon = \pm$). We will check that $[L_1, L_{-1}] = L_0$. Since we already know $[L_1, L_{-1}] \subseteq L_0$, we need to show that $L_0 \subseteq [L_1, L_{-1}]$, i.e., every matrix $\begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix}$ is of the form

$$\sum_i \left[\begin{pmatrix} 0 & x_i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y_i & 0 \end{pmatrix} \right] = \sum_i \begin{pmatrix} x_i y_i & 0 \\ 0 & -(x_i y_i)^t \end{pmatrix}.$$

Therefore it is enough to show that every $a \in \mathfrak{sl}_n$ is of the form $a = \sum_i x_i y_i$ for $x_i \in N^+$ and $y_i \in N^-$. Let $a \in \mathfrak{sl}_n(C)$. Then

$$\begin{aligned} a &= \sum_{u=1}^{n-1} a_{uu}(E_{uu} - E_{u+1,u+1}) + \sum_{1 \leq i \neq j \leq n} a_{ij} E_{ij} \\ &= \sum_{u=1}^{n-1} a_{uu}(E_{u,u+1} + E_{u+1,u})(E_{u+1,u} - E_{u,u+1}) + \sum_{1 \leq i \neq j \leq n} a_{ij} E_{ii}(E_{ij} - E_{ji}). \end{aligned}$$

Hence $L_0 \subseteq [L_1, L_{-1}]$. □

Lemma 3.1.6. *Let $u \in \text{GL}_n(C)$. Then the map $\tilde{\theta}_u : \mathfrak{ssl}(n|n)(C) \rightarrow \mathfrak{ssl}(n|n)(C)$,*

$$\begin{pmatrix} a & x \\ y & -a^t \end{pmatrix} \mapsto \begin{pmatrix} uau^{-1} & uxu^t \\ (u^{-1})^t y u^{-1} & -(uau^{-1})^t \end{pmatrix}$$

is an automorphism of the Lie superalgebra $\mathfrak{ssl}(n|n)(C)$.

Proof. It is easy to see that $\tilde{\theta}_u$ is linear and bijective with $(\tilde{\theta}_u)^{-1} = \tilde{\theta}_{u^{-1}}$. Therefore it remains to check that $\tilde{\theta}_u$ is a homomorphism of Lie superalgebras:

$$\begin{aligned} &\tilde{\theta}_u \left[\begin{pmatrix} a & x_1 \\ y_1 & -a^t \end{pmatrix}, \begin{pmatrix} b & x_2 \\ y_2 & -b^t \end{pmatrix} \right] \\ &= \tilde{\theta}_u \begin{pmatrix} [a, b] + x_1 y_2 + x_2 y_1 & ax_2 + x_2 a^t - x_1 b^t - bx_1 \\ y_1 b + b^t y_1 - a^t y_2 - y_2 a & -([a, b] + x_1 y_2 + x_2 y_1)^t \end{pmatrix} \\ &= \begin{pmatrix} u([a, b] + x_1 y_2 + x_2 y_1)u^{-1} & u(ax_2 + x_2 a^t - x_1 b^t - bx_1)u^t \\ (u^{-1})^t (y_1 b + b^t y_1 - a^t y_2 - y_2 a)u^{-1} & -(u^{-1})^t ([a, b] + x_1 y_2 + x_2 y_1)^t u^t \end{pmatrix} \\ &= \left[\begin{pmatrix} uau^{-1} & ux_1 u^t \\ (u^{-1})^t y_1 u^{-1} & -(uau^{-1})^t \end{pmatrix}, \begin{pmatrix} ubu^{-1} & ux_2 u^t \\ (u^{-1})^t y_2 u^{-1} & -(ubu^{-1})^t \end{pmatrix} \right] \\ &= \left[\tilde{\theta}_u \begin{pmatrix} a & x_1 \\ y_1 & -a^t \end{pmatrix}, \tilde{\theta}_u \begin{pmatrix} b & x_2 \\ y_2 & -b^t \end{pmatrix} \right]. \end{aligned}$$

□

Proposition 3.1.7. *Let $L = \mathfrak{ssl}(n|n)(C)$. Then the center $Z(L) = 0$. Hence*

$$\mathfrak{K}(\mathfrak{ss}_n(C)) \cong L.$$

Proof. As we know that $L = L_1 \oplus L_0 \oplus L_{-1}$ is a finite \mathbb{Z} -graded algebra, the center of L is graded too, i.e.,

$$Z(L) = (Z(L) \cap L_{-1}) \oplus (Z(L) \cap L_0) \oplus (Z(L) \cap L_1).$$

Suppose $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in Z(L) \cap L_1$. Then

$$0 = \left[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \right] = \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}$$

whence $xy = 0$ for all $y \in N^-$, which implies $x = 0$, by part (a) of Lemma 3.1.1. Hence $Z(L) \cap L_1 = \{0\}$. Similarly $Z(L) \cap L_{-1} = \{0\}$.

Let $\begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix} \in Z(\mathfrak{ssl}(n|n)) \cap L_0$. Then

$$0 = \left[\begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right].$$

This implies $ax = -xa^t$ for all $x \in N^+$. Therefore by part (b) of Lemma 3.1.1, we get $a = 0$. Hence $Z(L) = 0$.

Since L is a 3-graded Lie superalgebra with associated Jordan superpair $\mathfrak{ss}_n(C)$, we get $\mathfrak{K}(\mathfrak{ss}_n(C)) \cong L$ by [4, Lemma 2.4]. \square

3.2 Automorphisms

Lemma 3.2.1. *For $a \in \mathrm{GL}_n(C)$ and $\alpha \in C^\times$, the map $\chi_{(\alpha,a)} = (\chi_{(\alpha,a)}^+, \chi_{(\alpha,a)}^-)$, where*

$$\chi_{(\alpha,a)}^+(x) = \alpha axa^t, \quad \chi_{(\alpha,a)}^-(y) = (a^{-1})^t y (\alpha a)^{-1}$$

is an automorphism of the anti-Jordan pair $\mathfrak{ss}_n(C) = (N^+, N^-)$, see Example 1.2.8. Moreover, the map $(\alpha, a) \mapsto \chi_{(\alpha,a)}$ induces an isomorphism from

$$C^\times \times \mathrm{GL}_n(C) / \{(\beta^2, \beta^{-1}I_n) : \beta \in C^\times\}$$

onto the subgroup $\Gamma := \{(\chi_{(\alpha,a)}^+, \chi_{(\alpha,a)}^-) : a \in \mathrm{GL}_n(C), \alpha \in C^\times\}$ of $\mathrm{Aut}(\mathfrak{ss}_n(C))$.

Proof. It is easy to see that the pair of linear maps $(\chi_{(\alpha,a)}^+, \chi_{(\alpha,a)}^-)$ is bijective, since $a \in \text{GL}_n(C)$ and $\alpha \in C^\times$. We will check that $(\chi_{(\alpha,a)}^+, \chi_{(\alpha,a)}^-)$ is an automorphism of the anti-Jordan pair $\mathfrak{ss}_n(C)$. Let $x, z \in N^+$ and $y \in N^-$, then $\chi_{(\alpha,a)}^+\{x, y, z\} = \alpha a\{x, y, z\}a^t = \{\alpha a x a^t, (a^{-1})^t y (\alpha a)^{-1}, \alpha a z a^t\}$, i.e. $\chi_{(\alpha,a)}^+\{x, y, z\} = \{\chi_{(\alpha,a)}^+(x), \chi_{(\alpha,a)}^-(y), \chi_{(\alpha,a)}^+(z)\}$. Similarly we can prove that

$$\chi_{(\alpha,a)}^-\{y, x, w\} = \{\chi_{(\alpha,a)}^-(y), \chi_{(\alpha,a)}^+(x), \chi_{(\alpha,a)}^-(w)\}$$

for all $x \in N^+$ and $y, w \in N^-$.

Next we will check that $\chi : C^\times \times \text{GL}_n(C) \rightarrow \text{Aut}(\mathfrak{ss}_n(C))$ is a homomorphism:

$$\chi_{(\alpha,a)}^+ \circ \chi_{(\beta,b)}^+(x) = \alpha a \beta b x b^t a^t = \alpha \beta a b x (ab)^t = \chi_{(\alpha\beta, ab)}^+(x).$$

Similarly $\chi_{(\alpha,a)}^- \circ \chi_{(\beta,b)}^-(x) = \chi_{(\alpha\beta, ab)}^-$. Therefore $\chi(\alpha, a) \circ \chi(\beta, b) = \chi(\alpha\beta, ab)$. Finally we determine $\text{Ker}(\chi)$. Thus, let $\chi(\alpha, a) = \text{Id}$. Therefore $\alpha a x a^t = x$ for all $x \in N^+$. Then by part (c) of Lemma 3.1.1, there exists $\beta \in C^\times$ such that $\alpha a = \beta I_n$ and $a^t = \beta^{-1} I_n$. Therefore $\alpha^{-1} \beta^2 a = a$. This implies $\alpha = \beta^2$. Hence $(\alpha, a) = (\beta^2, \beta^{-1} I_n)$. Conversely, for $(\alpha, a) = (\beta^2, \beta^{-1} I_n)$ with $\beta \in C^\times$ we get

$$\chi_{(\alpha,a)}^+(x) = \beta^2 \beta^{-1} x \beta^{-1} = x \quad \text{and} \quad \chi_{(\alpha,a)}^-(y) = \beta y \beta^2 \beta^{-1} = y.$$

□

Lemma 3.2.2. *There does not exist $(f^+, f^-) \in \text{Aut}(\mathfrak{ss}_n(C))$ such that*

$$f^+(x) f^-(y) = yx$$

for all $(x, y) \in (N^+, N^-)$, for $n > 2$.

Proof. Suppose there exist $(f^+, f^-) \in \text{Aut}(\mathfrak{ss}_n(C))$ such that

$$f^+(x) f^-(y) = yx$$

for all $(x, y) \in (N^+, N^-)$. Put $H_{pq} = E_{pq} + E_{qp}$ and $F_{pq} = E_{pq} - E_{qp}$, for $1 \leq p \neq q \leq n$. Then for all p, q, r with $1 \leq p \neq q \neq r \leq n$, we have

$$\begin{aligned} f^+(H_{pq}) &= \{f^+(H_{pq}), f^-(F_{qr}), f^+(H_{qr})\} \\ &= f^+(H_{pq}) f^-(F_{qr}) f^+(H_{qr}) - f^+(H_{qr}) f^-(F_{qr}) f^+(H_{pq}) \\ &= (F_{qr})(H_{pq}) f^+(H_{qr}) - (F_{qr})(H_{qr}) f^+(H_{pq}) \\ &= -E_{rp} f^+(H_{qr}) - (E_{qq} - E_{rr}) f^+(H_{pq}). \end{aligned}$$

Multiplying both sides by $f^-(F_{pq})$, we get

$$f^+(H_{pq})f^-(F_{pq}) = -E_{rp}f^+(H_{qr})f^-(F_{pq}) - (E_{qq} - E_{rr})f^+(H_{pq})f^-(F_{pq}).$$

This implies $(F_{pq})(H_{pq}) = -E_{rp}(F_{pq})(H_{qr}) - (E_{qq} - E_{pp})(F_{pq})(H_{pq})$. Hence $E_{pp} - 2E_{qq} = -E_{rr}$, which is a contradiction, since $p \neq q \neq r$. \square

Theorem 3.2.3. *Suppose that every automorphism of $\mathfrak{sl}_n(C)$ is of the form $a \mapsto uau^{-1}$ or $a \mapsto -ua^t u^{-1}$ for some $u \in \mathrm{GL}_n(C)$. Then $\mathrm{Aut}(\mathfrak{ss}_n)(C) = \Gamma$ as introduced in Lemma 3.2.1.*

Proof. Let $\zeta = (\zeta^+, \zeta^-)$ be an automorphism of the anti-Jordan pair $\mathfrak{ss}_n(C) = (N^+, N^-)$. Then by Lemma 2.2.9 we can extend ζ to an automorphism $\tilde{\zeta}$ of $\mathfrak{K}(\mathfrak{ss}_n)$ which by Proposition 3.1.7 is isomorphic to $\mathfrak{ssl}(n|n)(C)$. We will identify $\mathfrak{ssl}(n|n)(C)$ with $\mathfrak{K}(\mathfrak{ss}_n)$. Hence $\tilde{\zeta} : \mathfrak{ssl}(n|n)(C) \rightarrow \mathfrak{ssl}(n|n)(C)$ is given by

$$\tilde{\zeta} \begin{pmatrix} a & x \\ y & -a^t \end{pmatrix} = \begin{pmatrix} \tilde{\zeta}(a) & \zeta^+(x) \\ \zeta^-(y) & -(\tilde{\zeta}(a))^t \end{pmatrix}.$$

In particular $\tilde{\zeta}|_{\mathfrak{ssl}(n|n)_0}$ is an automorphism of $\mathfrak{ssl}(n|n)_0 \cong \mathfrak{sl}_n(C)$. Hence there exists $u \in \mathrm{GL}_n(C)$ such that for all $a \in \mathfrak{sl}_n(C)$ one of the following two cases holds:

case(i): $\tilde{\zeta}(a) = uau^{-1}$,

case(ii): $\tilde{\zeta}(a) = -ua^t u^{-1}$.

We know from Lemma 3.1.6 that $\tilde{\theta}_{u^{-1}}$ is an automorphism of $\mathfrak{ssl}(n|n)(C)$ for any $u \in \mathrm{GL}_n(C)$. Hence $\phi = \tilde{\theta}_{u^{-1}} \circ \tilde{\zeta}$ is also an automorphism of $\mathfrak{ssl}(n|n)(C)$. Let $x, z \in N^+(C)$ and $y \in N^-(C)$. In case (i) we have

$$\phi \begin{pmatrix} a & x \\ y & -a^t \end{pmatrix} = \begin{pmatrix} a & u^{-1}\zeta^+(x)(u^{-1})^t \\ u^t\zeta^-(y)u & -a^t \end{pmatrix}.$$

Therefore

$$\begin{aligned}
\phi^+\{x, y, z\} &= \phi \left[\left[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \right], \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \right] \\
&= \phi \left[\begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}, \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \right] \\
&= \left[\phi \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}, \begin{pmatrix} 0 & \phi^+(z) \\ 0 & 0 \end{pmatrix} \right].
\end{aligned}$$

Since $\phi \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix} = \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}$, we get

$$\phi^+\{x, y, z\} = \left[\begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}, \begin{pmatrix} 0 & \phi^+(z) \\ 0 & 0 \end{pmatrix} \right] = \{x, y, \phi^+(z)\}.$$

From Lemma 3.1.2, we get that $\phi^+ = \alpha \text{Id}$, $\alpha \in C^\times$. Similarly $\phi^- = \beta \text{Id}$, $\beta \in C^\times$. Then $\phi^+\{x, y, z\} = \{\phi^+(x), \phi^-(y), \phi^+(z)\} = \alpha^2\beta\{x, y, z\}$, which implies $\alpha = \alpha^2\beta$ and so $\alpha\beta = 1$. Therefore

$$\phi^+(x) = u^{-1}\zeta^+(x)(u^{-1})^t = \alpha x \quad \text{and} \quad \phi^-(y) = \beta y = u^t\zeta^-(y)u,$$

i.e. $\zeta^+(x) = \chi_{(\alpha, u)}^+(x) = \alpha uxu^t$ and $\zeta^-(y) = \chi_{(\alpha, u)}^-(y) = (u^{-1})^t y (\alpha u)^{-1}$. Hence $\text{Aut}(\mathfrak{ss}_n)(C) \subseteq \Gamma$. But $\text{Aut}(\mathfrak{ss}_n)(C) \supseteq \Gamma$ by Lemma 3.2.1.

In case (ii) we have $\phi \begin{pmatrix} a & x \\ y & -a^t \end{pmatrix} = \begin{pmatrix} -a^t & u^{-1}\zeta^+(x)(u^{-1})^t \\ u^t\zeta^-(y)u & a \end{pmatrix}$. Note that

$$\begin{aligned}
\phi^+\{x, y, z\} &= \phi \left[\left[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \right], \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \right] \\
&= \phi \left[\begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}, \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \right] \\
&= \left[\phi \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}, \begin{pmatrix} 0 & \phi^+(z) \\ 0 & 0 \end{pmatrix} \right].
\end{aligned}$$

Since $\phi \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix} = \begin{pmatrix} -(xy)^t & 0 \\ 0 & xy \end{pmatrix}$, this implies

$$\phi^+\{x, y, z\} = \left[\begin{pmatrix} yx & 0 \\ 0 & xy \end{pmatrix}, \begin{pmatrix} 0 & \phi^+(z) \\ 0 & 0 \end{pmatrix} \right] = yx\phi^+(z) - \phi^+(z)xy.$$

Therefore $(\phi^+(x)\phi^-(y) - yx)\phi^+(z) = -\phi^+(z)(\phi^+(x)\phi^-(y) - yx)^t$ for all $z \in N^+$. By part (b) of Lemma 3.1.1 we get $\phi^+(x)\phi^-(y) - yx = 0$ for all $(x, y) \in (N^+, N^-)$. Hence by Lemma 3.2.2 we know there does not exist such a (ϕ^+, ϕ^-) for $n > 2$. \square

Corollary 3.2.4. *Let F be a field of characteristic zero or $p > 3$ such that p does not divide n if $p \in \{5, 7, 11\}$. Then for $n > 2$, $\text{Aut}(\mathfrak{ss}_n)(F) = \Gamma$, as introduced in Lemma 3.2.1.*

Proof. By Theorem 2.2.13, there exists $u \in \text{GL}_n(F)$ such that every automorphism of $\mathfrak{sl}_n(F)$ is of the form $a \mapsto uau^{-1}$ or $a \mapsto -ua^t u^{-1}$. Hence by Theorem 3.2.3 we get $\text{Aut}(\mathfrak{ss}_n)(F) = \Gamma$. \square

Chapter 4

Anti-Jordan triple systems obtained from the symplectic anti-Jordan pair

In this chapter, let F be a field with $\frac{1}{2} \in F$.

4.1 Isomorphisms and involutions of the symplectic anti-Jordan pair

Lemma 4.1.1. *Let C be a commutative, associative and unital k -algebra. Let T be a C -module with alternating bilinear forms $B : T \times T \rightarrow C$ and $B' : T \times T \rightarrow C$. Let $a \in \text{GL}_C(T)$ and $\alpha \in C^\times$ such that $B(x, y) = \alpha B'(a(x), a(y))$ for all $x, y \in T$. Define $f^+(x) = a(x)$ and $f^-(y) = \alpha a(y)$. Then $(f^+, f^-) : (T, T)_B \rightarrow (T, T)_{B'}$ is an isomorphism of symplectic anti-Jordan pairs, see Example 1.2.11.*

Proof. It is obvious that the pair of linear maps (f^+, f^-) is bijective. Also

$$\begin{aligned} f^+\{x, y, z\} &= B(x, y)a(z) + B(x, z)a(y) + B(y, z)a(x) \\ &= \alpha B'(a(x), a(y))a(z) + \alpha B'(a(y), a(z))a(x) + \alpha B'(a(x), a(z))a(y), \end{aligned}$$

since $B(x, y) = \alpha B'(a(x), a(y))$. Hence $f^+\{x, y, z\} = \{f^+(x), f^-(y), f^+(z)\}$. Similarly $f^-\{x, y, z\} = \{f^-(x), f^+(y), f^-(z)\}$.

Theorem 4.1.2. *Let T be an F -vector space with non-degenerate alternating bilinear forms $B : T \times T \rightarrow F$ and $B' : T \times T \rightarrow F$. Then $f = (f^+, f^-)$ is an isomorphism of symplectic anti-Jordan pairs $(T, T)_B$ and $(T, T)_{B'}$ if and only if there exist $a \in \text{GL}_F(T)$ and $\alpha \in F^\times$ such that*

- (i) $B(x, y) = \alpha B'(a(x), a(y))$ for all $x, y \in T$ and
- (ii) $f^+(x) = a(x)$, $f^-(y) = \alpha a(y)$.

Proof. Let f be an isomorphism of symplectic anti-Jordan pairs. Since f^+ and f^- are invertible, there exist $a, b \in \text{GL}_F(T)$ such that

$$f^+ : T \rightarrow T, x \rightarrow a(x) \quad \text{and} \quad f^- : T \rightarrow T, y \rightarrow b(y).$$

Since f is an automorphism of symplectic anti-Jordan pairs,

$$f^+\{x, y, z\} = \{f^+(x), f^-(y), f^+(z)\}, \quad \text{i.e. } a(\{x, y, z\}) = \{a(x), b(y), a(z)\}.$$

This implies

$$\begin{aligned} (*) \quad & B(x, y)a(z) + B(x, z)a(y) + B(y, z)a(x) \\ & = B'(a(x), b(y))a(z) + B'(a(x), a(z))b(y) + B'(b(y), a(z))a(x). \end{aligned}$$

Put $x = y$ in (*). Then

$$B'(a(x), b(x))a(z) + B'(a(x), a(z))b(x) + B'(b(x), a(z))a(x) = 2B(x, z)a(x).$$

Evaluating this equation for $B'(-, a(z))$, yields

$$2B'(b(x), a(z))B'(a(x), a(z)) = 2B(x, z)B'(a(x), a(z)).$$

This implies $B'(b(x), a(z)) = B(x, z)$ for all x, z with $B'(a(x), a(z)) \neq 0 \in F$. After a base field extension we can assume that F is an infinite field. Hence the set

$$\{(x, z) \in T \times T : B(a(x), a(z)) \neq 0\}$$

is Zariski-dense. Thus we can assume $B'(b(x), a(z)) = B(x, z)$ for all $(x, z) \in T \times T$. Then from (*) we get $B(x, z)a(y) = B'(a(x), a(z))b(y)$. Since B is non-degenerate, there exist (x, z) such that $B(x, z) = 1$, then $\beta^{-1}b^{-1}a(y) = y$ for all $y \in T$, where $\beta = B'(a(x), a(z))$. Hence $\beta^{-1}b^{-1}a = \text{Id}$, yields $b = \alpha a$, where $\alpha = \beta^{-1}$. The converse is clear (see Lemma 4.1.1). \square

We denote by $\text{GSP}(T, B)$ the group of invertible endomorphisms a of T , for which there exists $\alpha \in F$ such that $\alpha B(a(x), a(y)) = B(x, y)$ for all $x, y \in T$, where B is a non-degenerate alternating bilinear form $T \times T \rightarrow F$. Since B is non-degenerate, $\alpha \in F$ is uniquely determined and called the *multiplier* of a .

Corollary 4.1.3. *Let T be a F -vector space with a non-degenerate alternating bilinear form $B : T \times T \rightarrow F$. Then*

$$\text{Aut}(T, T)_B \cong \text{GSP}(T, B).$$

Corollary 4.1.4. *A pair (ν^+, ν^-) of F -linear maps is an involution of the symplectic anti-Jordan pair $(T, T)_B$ if and only if there exists $a \in \text{GSP}(T, B)$ with multiplier $\alpha \in F^\times$ such that $\nu^+(x) = a(x)$, $\nu^-(y) = \alpha a(y)$ and $a^2 = \alpha^{-1}\text{Id}$.*

Proof. Let $\nu = (\nu^+, \nu^-)$ be an involution of the symplectic anti-Jordan pair $(T, T)_B$. Then by Theorem 4.1.2, we have that there exists $a \in \text{GSP}(T, B)$ such that $\nu^+(x) = a(x)$, $\nu^-(y) = \alpha a(y)$. Since $x = \nu^+ \circ \nu^-(x) = \alpha a^2(x)$, it follows that $a^2 = \alpha^{-1}\text{Id}$, for some $\alpha \in F^\times$. The converse is trivial. \square

4.2 Anti-Jordan triple system

Proposition 4.2.1. *Let ν_1 and ν_2 be two involutions of symplectic anti-Jordan pairs $(T, T)_B$ and $(T, T)_{B'}$ respectively, where B and B' are as in Theorem 4.1.2, given by two invertible endomorphisms $a \in \text{GSP}(T, B)$, and $b \in \text{GSP}(T, B')$, as in Corollary 4.1.4. Then the anti-Jordan triple systems T and T' , defined by $\langle x, y, z \rangle_B = \{x, a(y), z\}_B$ and $\langle x, y, z \rangle_{B'} = \{x, b(y), z\}_{B'}$ respectively, are isomorphic if and only if there exist $c \in \text{GL}_F(T)$ and $\alpha \in F^\times$ such that $b = \alpha c a c^{-1}$ and $B(x, y) = \alpha B'(c(x), c(y))$ for all $x, y \in T$.*

Proof. From Theorem 4.1.2 we know that $(f^+, f^-) : (T, T)_B \rightarrow (T, T)_{B'}$ is an isomorphism of anti-Jordan pairs if and only if there exist $c \in \text{GL}_F(T)$ and $\alpha \in F^\times$ such that $B(x, y) = \alpha B'(c(x), c(y))$ for all $x, y \in T$ and $f^+(x) = c(x)$, $f^-(y) = \alpha c(y)$. Therefore from Lemma 1.2.2 we get $T \cong T'$ if and only if $\nu_2^+ \circ f^+ = f^- \circ \nu_1^+$ if and only if $b = \alpha c a c^{-1}$. \square

Theorem 4.2.2. (a) *Let T be an F -vector space with an alternating non-degenerate bilinear form $B : T \times T \rightarrow F$ and let $a \in \text{GSP}(T, B)$ such that $a^2 = \alpha \text{Id}$ for some $\alpha \in F^\times$. Then T is a simple anti-Jordan triple system with triple product $\langle x, y, z \rangle_B = \{x, a(y), z\} = B(x, a(y))z + B(a(y), z)x + B(x, z)a(y)$.*

(b) *If T is a simple finite dimensional anti-Jordan triple system over F such that $(T, T) \cong (T, T)_B$, then there exists $b \in \text{GSP}(T, B)$ such that $b^2 = \beta \text{Id}$ for some $\beta \in F^\times$, and T is the simple anti-Jordan triple system with triple product:*

$$\langle x, y, z \rangle_B = \{x, b(y), z\} = B(x, b(y))z + B(b(y), z)x + B(x, z)b(y).$$

Proof. (a): From Lemma 1.2.12 we know that the symplectic anti-Jordan pair (T, T) is simple. Hence it follows from Lemma 1.2.6 that T is simple.

(b): Let T be a simple anti-Jordan triple system over F such that $(T, T) \cong (T, T)_B$. Then T has triple product:

$$\langle x, y, z \rangle = \{x, \nu^+(y), z\}$$

for some involution (ν^+, ν^-) by Lemma 1.2.1. Then by Corollary 4.1.4, there exists $b \in \text{GSP}(T, B)$ with multiplier $\beta \in F^\times$ such that $\nu^+(x) = b(x)$, $\nu^-(y) = \beta^{-1}b(y)$ and $b^2 = \beta \text{Id}$. \square

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