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FILTERING AND EXTRAPOLATION TECHNIQUES
IN THE NUMERICAL SOLUTION OF
ORDINARY DIFFERENTIAL EQUATIONS

by

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A Thesis presented
to
the School of Graduate Studies
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To my parents

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INTRODUCTION

Consider an initial problem

$$y' = f(x,y), \quad y(0) = y_0 \quad (1)$$

where the vector-valued function $f(x,y)$ is Lipschitz continuous to ensure a unique solution.

Let $y_n(h)$ denote the numerical solution at $x = x_n = a + nh$ of the initial value problem (1) by any linear multistep method. For consistent and stable methods, under suitable conditions on the function $f(x,y)$ and the starting values $y_n(h)$, $n = 0, 1, \dots, k-1$, the numerical solution has an asymptotic expansion in power of h , valid as $h \rightarrow 0$ and $n \rightarrow \infty$ in such a way that $x = a + nh$ remains fixed in $(a,b]$ [Gragg 1963].

The intent of this work is to consider consistent and stable methods whose first characteristic polynomial is $\rho(z) = z^2 - 1$. These are the simplest cases among the methods whose $\rho(z)$ has more than one essential zero. The Midpoint method and Simpson's rule belong to the above category.

Chapter 1 contains definitions and well-known fundamental results. In Chapter 2 Gragg's results are simplified for the methods (of order p) under consideration and it is shown that their numerical solutions have asymptotic expansions in the form

$$y_n(h) = y(x) + \sum_{m=p}^N (e_{m1}(x) + (-1)^n e_{m2}(x)) h^m + O(h^{N+1}), \quad h \rightarrow 0. \quad (2)$$

It is also shown in Chapter 2 that for the test equation

$$y' = \lambda y, \quad y(0) = y_0 \quad (3)$$

the Midpoint method and Simpson's rule have asymptotic expansions in the form (2) with $N = p + 1$, and with

$$y(x) = y_0 e^{\lambda x},$$

$$e_{p1}(x) = (p_1 x + p_2) e^{\lambda x},$$

$$e_{p2}(x) = p_3 e^{\lambda_2 \lambda x},$$

$$e_{p+1,1}(x) = p_4 e^{\lambda x}$$

$$e_{p+1,2}(x) = p_5 e^{\lambda_2 \lambda x}$$

where $p = 2$ and $\lambda_2 = -1$ for the Midpoint method and $p = 4$ and $\lambda_2 = -1/3$ for Simpson's rule and the constants $p_1 - p_5$ are independent of h .

The main interest of Chapter 3 is to filter out the oscillating terms $(-1)^n e_{p2}(x)$ and $(-1)^n e_{p+1,2}(x)$ by means of a filter and then to increase the order of the method by extrapolation. Filtering is an averaging procedure involving the current term together with previous terms, forward terms or both.

For the Midpoint method five filters are introduced and in Theorem 3.1.a it is shown that the symmetric filter

$$p_0(E) = \frac{1}{16} (-E^{-2} + 4E^{-1} + 10 + 4E - E^2) \quad (4)$$

removes more terms from the oscillating part while keeping more accurate

the desired solution than the remaining four filters. Similarly for Simpson's rule seven filters are introduced and in Theorem 3.1.b the symmetric filter

$$P_0(E) = \frac{1}{64}(E^{-3} - 6E^{-2} + 15E^{-1} + 44 + 15E - 6E^2 + E^3)$$

is obtained with similar properties.

By using Tables (3.2.a), (3.3.a), (3.2.b) and (3.3.b) it is shown that all the filters reduce the oscillating part by a factor of h^2 while only the symmetric filters leave unchanged the exact solution and the harmless part up to h^3 and h^5 for the Midpoint method and Simpson's rule respectively and offer the possibility of extrapolation.

In Corollaries (3.1.a) and (3.1.b) the numerical value at $x_{Nn} = Nnh$ is obtained after filtering at the points $nh, 2nh, \dots, Nnh$ for both methods. Then in Lemmas (3.1.a) and (3.1.b) some additional necessary conditions are imposed on the starting values for applying extrapolation.

In Chapter 4 numerical results for both methods are quoted for both linear and non-linear problems. In each case the non-filtered solution, the filtered solution by one of the non-symmetric filters, and the filtered solution by the symmetric filter followed by extrapolation are compared.

CHAPTER 1

PRELIMINARIES

INITIAL VALUE PROBLEMS

Let $I = [a, b]$ be a closed finite interval on the x -axis and let $\mathbb{C}^l$ denote the complex l -dimensional space. Let the vector-valued functions $y(x) \in \mathbb{C}^l$ and $f = f(x, y) \in \mathbb{C}^l$ be defined and continuous on I and $D = I \times \mathbb{C}^l$ respectively. Then the system of l differential equations $y' = f(x, y)$ together with an initial condition $y(a) = \eta \in \mathbb{C}^l$ is called an initial value problem and referred to as the initial value problem (1.1),

$$y' = f(x, y), \quad y(a) = \eta. \quad (1.1)$$

Let $\| \cdot \|$ denote a vector norm on $\mathbb{C}^l$. If there exists a constant L such that the Lipschitz condition

$$\|f(x, y_1) - f(x, y_2)\| \leq L \|y_1 - y_2\| \quad (1.2)$$

holds for each $(x, y_1) \in D$ and $(x, y_2) \in D$, then there is a unique function $y \in C^1(I)$ satisfying (1.1) for $x \in I$ [see Coddington & Levinson 1955, p. 19, ch. I]. Such a function is called the solution of (1.1).

Henceforth it is assumed that the initial value problem (1.1) satisfies the condition (1.2). The intent of this work is to solve the initial value problem (1.1) numerically.

LINEAR MULTISTEP METHODS

Consider the sequence $\{x_n\}$ of grid points $x_n = a + nh$, $n = 0, 1, 2, \dots$, where h is a positive parameter. Let y_n be an approximation to the exact solution $y(x_n)$ of the initial value problem (1.1) at x_n and let $f_n = f(x_n, y_n)$. A linear relation involving y_{n+j} , $j = 0, \dots, k$, for recursively determining $\{y_n\}$ is called a linear multistep method.

The general linear multistep method may be written in the form,

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f_{n+j}, \quad (1.3)$$

where α_j, β_j are real constants with $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_0| > 0$ [Dahlquist 1956]. The positive integer k is called the stepnumber of the method and h is referred to as the steplength which is assumed to be fixed throughout the computation. If $\beta_k = 0$ the method is called explicit and otherwise it is called implicit.

For an explicit method the equation (1.3) can be solved directly for y_{n+k} ; however, for an implicit method, in general this is impossible. For an initial value problem (1.1) satisfying the condition (1.2), a unique solution for y_{n+k} exists, if $h < (L|\beta_k|)^{-1}$ and $\beta_k \neq 0$ [Lambert 1973, p. 12]. Therefore hereafter h will be chosen such that

$$0 < h < h_0 = (L|\beta_k|)^{-1}. \quad (1.4)$$

Without loss of generality, by choosing $\alpha_k = 1$ which will be assumed hereafter, the number of unknowns $\{\alpha_j, \beta_j\}$ may be reduced to $2k+1$. It is assumed that h divides $(x-a)$ and that $G_N = \{x_n | x_n = a + nh, n = 0, 1, \dots, N\}$, where $a + Nh \leq b$.

The polynomials

$$\rho(z) = \sum_{j=0}^k \alpha_j z^j \quad \text{and} \quad \sigma(z) = \sum_{j=0}^k \beta_j z^j$$

are called the generating polynomials of the linear multistep method (1.3).

Some authors term these the first and second characteristic polynomials respectively.

By introducing the shift operator E defined by

$$Ey_n = y_{n+1} \quad \text{or} \quad Ey(x) = y(x+h),$$

the linear multistep method (1.3) may be written (with the convention $E^2 y_n = E(Ey_n)$ etc.)

$$y_n = \eta_n(h), \quad n = 0, 1, 2, \dots, k-1, \quad (1.5)$$

$$\rho(E)y_n = h\sigma(E)f_n, \quad n = 0, 1, 2, \dots,$$

where $\eta_n(h)$, $n = 0, 1, 2, \dots, k-1$, are the starting values.

PROPERTIES OF MULTISTEP METHODS

CONVERGENCE

The linear multistep method (1.3) is said to be convergent for an initial value problem (1.1) satisfying the condition (1.2), if

$$\lim_{\substack{h \rightarrow 0 \\ nh = x-a}} y_n = y(x) \quad (1.6)$$

holds for all $x \in [a, b]$, and for all solutions $\{y_n\}$ of the equation (1.3) satisfying a starting condition $y_n = \eta_n(h)$ for which,

$$\lim_{h \rightarrow 0} \eta_n(h) = \eta, \quad n = 0, 1, 2, \dots, k-1.$$

LOCAL TRUNCATION ERROR

Let $y(x)$ be the solution of an initial value problem (1.1) and let the linear operators L on $C^1(I)$ be defined by

$$L[y(x);h] = \rho(E)y(x) - h\sigma(E)y'(x). \quad (1.7)$$

Then the quantity $\alpha_k^{-1}L[y(x_n);h] = L[y(x_n);h]$ is called the local truncation error of the linear multistep method (1.3) at the point x_{n+k} . It gives the error introduced when $y(x_{n+k})$ is replaced with y_{n+k} in the expression $y(x_{n+k}) - h\beta_k f(x_{n+k}, y(x_{n+k}))$ under the assumption that $y_n, y_{n+1}, \dots, y_{n+k-1}$ are exact. It also represents a measure of the extent to which the solution fails to satisfy the linear relation which determines the numerical solution.

Using the definitions of $\rho(E)$ and $\sigma(E)$, (1.7) may be written as

$$L[y(x);h] = \sum_{j=0}^k \alpha_j y(x+jh) - h \sum_{j=0}^k \beta_j y'(x+jh),$$

and if $y(x) \in C^{n+1}(I)$, a Taylor expansion about x gives

$$L[y(x);h] = \sum_{m=0}^n C_m y^{(m)}(x) h^m + O(h^{n+1}), \quad (1.8)$$

where $C_0 = \alpha_0 + \alpha_1 + \dots + \alpha_k$,

$$C_1 = \alpha_1 + 2\alpha_2 + \dots + k\alpha_k - (\beta_0 + \beta_1 + \dots + \beta_k),$$

$$C_q = \frac{1}{q!} (\alpha_1 + 2^q \alpha_2 + \dots + k^q \alpha_k) \quad (1.9)$$

$$- \frac{1}{(q-1)!} (\beta_1 + 2^{q-1} \beta_2 + \dots + k^{q-1} \beta_k), \quad q = 2, 3, \dots, N,$$

are constants independent of $y(x)$.

Let D be the operator of differentiation d/dx ; then the formal Taylor expansion

$$Ey(x) = y(x+h)$$

$$= y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \dots$$

$$= (1 + hD + \frac{h^2}{2} D^2 + \dots)y(x)$$

$$= e^{hD} y(x)$$

implies that $E = e^{hD}$ on polynomials or analytic functions.

Since

$$\begin{aligned} L[y(x); h] &= \rho(E)y(x) - hD\sigma(E)y(x) \\ &= (\rho(e^{hD}) - hD\sigma(e^{hD}))y(x), \end{aligned}$$

letting $y(x) = e^x$ gives

$$L[e^x; h] = (\rho(e^{hD}) - hD\sigma(e^{hD}))e^x,$$

and since e^x is infinitely differentiable, (1.8) can be extended to

$$L[e^x; h] = \sum_{m=0}^{\infty} C_m (hD)^m e^x.$$

These results with $z = hD$ give

$$\rho(e^z) - z\sigma(e^z) = \sum_{m=0}^{\infty} C_m z^m. \quad (1.10)$$

This is the generating function for the coefficients $\{C_m\}$.

ORDER OF THE METHOD

If

$$C_0 = C_1 = \dots = C_p = 0 \text{ and } C_{p+1} \neq 0, \quad (1.11)$$

the linear multistep step method is said to be of order p .

GLOBAL TRUNCATION ERROR

It is not always realistic to assume that the values $y_n, \dots, y_{n+k-1}$ are exact. Without that assumption the error of the linear multistep method is said to be the global truncation error. The global truncation error e_n at x_n is given by

$$e_n = y_n - y(x_n).$$

Ignoring arithmetic truncation errors, it represents the local truncation error at x_n plus the propagated effects of local truncation errors at all previous steps, including errors in the starting values.

CONSISTENCY

The linear multistep method (1.3) is said to be consistent if it is of order $p \geq 1$. From (1.9) it is clear that the method (1.3) is consistent if and only if

$$\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1), \quad (1.12)$$

i.e.

$$\alpha_0 + \alpha_1 + \dots + \alpha_k = 0,$$

$$\alpha_1 + 2\alpha_2 + \dots + k\alpha_k = \beta_0 + \dots + \beta_k$$

These may be called the α - β conditions.

THEOREM 1.1

The necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent, that the modulus of no zero of the generating polynomial $\rho(z)$ exceed 1, and that the zeros of modulus 1 be simple [Dahlquist 1956].

STABILITY

The necessary conditions mentioned above (in addition to consistency) are called the stability conditions and the linear multistep methods with that property are called zero-stable or D-stable [due to Dahlquist] or, for short, stable.

Later it will be shown that linear multistep methods with the above stability property may be subject to other types of instability, e.g. weak instability, which do not affect convergence. From hereon it is assumed that the linear multistep method (1.3) is consistent and stable.

The above analysis shows that the zeros of the generating polynomial $\rho(z)$ are of fundamental importance in the study of linear multistep methods. The zeros of $\rho(z)$ with modulus 1 are called essential and are simple by the stability requirement. The other zeros of $\rho(z)$ are of modulus less than 1 and are called non-essential. By the consistency requirement, $z = 1$ is always a zero and is referred to as the principal zero. Let $z_1 = 1$, z_i , $i = 2, 3, \dots, q$, denote all the essential zeros (necessarily distinct) and let z_i , $i = q+1, q+2, \dots, r$, denote the non-essential zeros (counting multiplicities), where $r \leq k$.

The pair of associated polynomials $\rho_i(z) = \rho(z_i z)$ and $\sigma_i(z) = \sigma(z_i z)$ are defined for each non-zero zero z_i , and, for any vector-valued function U_n , it is true that

$$\rho(E)z_i^n U_n = z_i^n \rho_i(E)U_n,$$

$$\sigma(E)z_i^n U_n = z_i^n \sigma_i(E)U_n \quad (1.13)$$

For any essential zero z_i ,

$$\lambda_i = \frac{\sigma_i(1)}{\rho_i'(1)} = \frac{\sigma(z_i)}{z_i \rho_i'(z_i)}$$

is called the growth parameter [Dahlquist 1959, p. 39]. It follows from the consistency conditions (1.12) that $\lambda_1 = 1$ and from the stability condition that $\lambda_i \neq 0$ for each i .

The quantity,

$$C = \frac{C_{p+1}}{\rho'(1)} = \frac{C_{p+1}}{\sigma(1)}$$

is called the error constant of the linear multistep method, where p is the order.

Let the associated linear operators L_i be defined by

$$L_i[U(x);h] = \rho_i(E)U(x) - h\lambda_i^{-1}\sigma_i(E)U'(x), \quad i = 1, 2, \dots, q, \quad (1.14)$$

where $U(x) \in C$ is differentiable. If $U(x) \in C^{m+1}(I)$, by using Taylor series

$$L_i[U(x);h] = \sigma_i(E) \sum_{j=2}^m a_j^{(i)} U^{(j)}(x) + O(h^{m+1}), \quad (1.15)$$

where $a_j^{(i)}$, $i = 1, 2, \dots, q$, $j = 2, 3, \dots$, are defined by the generating functions

$$A_i(z) = \frac{\rho_i(e^z)}{\sigma_i(e^z)} = \sum_{j=2}^{\infty} a_j^{(i)} z^j, \quad i = 1, 2, \dots, q, \quad (1.16)$$

and $a_1^{(i)} = \lambda_i^{-1}$. For a method of order p it is clear that

$$a_2^{(1)} = \dots = a_p^{(1)} = 0 \text{ and } C = a_{p+1}^{(1)} \neq 0. \quad (1.17)$$

CHAPTER 2

ASYMPTOTIC EXPANSIONS

2.1 THEOREMS ON ASYMPTOTIC EXPANSIONS

Under an appropriate starting procedure the global truncation error of any linear multistep method has an asymptotic expansion in integral powers of h [Gragg 1963]. Here it will be shown that, for linear multistep methods whose first characteristic polynomial $\rho(z)$ has only essential zeros such that the product of any pair of zeros is again a zero, the global truncation error has an asymptotic expansion in powers of h . A general version of this theorem may be found in [Gragg 1963, pp. 62-75] or [Stetter 1973, pp. 241-244].

The proof of the Theorem depends on the following Lemma which estimates the growth of the solution of a difference equation of the form (1.5); a proof of this Lemma may be found in [Gragg 1963, pp. 52-56].

For $h > 0$, let the vector-valued sequences $F_n(y) = F_n(y;h)$ and $\Lambda_n = \Lambda_n(h)$ be defined for $x_n \in G_N$, where $n \leq N$, and $y \in \mathbb{C}^L$. Assume there is a constant K such that,

$$\|F_n(y,h)\| \leq K \|y\|, \quad x_n \in G_N, \quad y \in \mathbb{C}^L.$$

Define the scalars γ_m , $m = 0, 1, \dots$, by

$$\frac{1}{1 + \alpha_{k-1}z + \dots + \alpha_0 z^k} = \sum_{m=0}^{\infty} \gamma_m z^m.$$

LEMMA 2.1

Let $\delta_n = \delta_n(h)$ be a solution of the difference equation

$$\rho(E)\delta_n = h\sigma(E)F_n(\delta_n) + \Lambda_n, \quad n = 0, 1, \dots, \quad (2.1)$$

such that

$$||\delta_n(h)|| \leq \delta(h), \quad n = 0, 1, \dots, k-1. \quad (2.2)$$

Then for h satisfying (1.4) and for $x_n \in G_N$

$$||\delta_n(h)|| \leq Ce^{\gamma BCK(b-a)} [\gamma A \delta(h) + \Lambda(h)],$$

where

$$A = \sum_{i=0}^k (k-i) |\alpha_i|, \quad B = \sum_{i=0}^k |\beta_i|,$$

$$C = (1 - |\beta_k|Kh)^{-1}, \quad \gamma = \sup_{m \geq 0} |\gamma_m|,$$

and

$$\Lambda(h) = \max_{x_n \in G_N} || \sum_{m=0}^{n-1} \gamma_{n-1-m} \Lambda_m(h) ||.$$

Turning to the theorem on asymptotic expansions, the assumption on starting values is that they have asymptotic expansions in powers of h which are consistent with the order of the method, i.e.

$$\eta_n(h) = y(x_n) + \sum_{m=p}^{N-1} \eta_{mn} h^m + O(h^N), \quad n = 0, 1, \dots, k-1, \quad h \rightarrow 0, \quad (2.3)$$

where η_{mn} are constant vectors.

SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

First, the system of linear differential equations

$$y' = A(x)y + b(x), \quad y(0) = y_0, \quad (2.4)$$

is considered, where $A(x)$ is an $\ell \times \ell$ square matrix and $b(x) \in \mathbb{C}^\ell$. By $A \in C^N(I)$, it is meant that every entry of A is in $C^N(I)$.

THEOREM 2.1

Let y_n denote the numerical solution of the system of differential equations (2.4) determined by the linear multistep method (1.3) and let $A(x)$ and $b(x)$

be of class $C^N(I)$. Then there exist unique vector-valued functions $e_{mi}(x)$ of class $C^{N-1-m}(I)$ such that, for h satisfying (1.4), and for $x = x_n \in G_N$,

$$y_n(h) = y(x) + \sum_{m=p}^{N-1} \left(\sum_{i=1}^k z_i^n e_{mi}(x) \right) h^m + o(h^N). \quad (2.5)$$

PROOF

In the present context, the proof of the uniqueness of the functions e_{mi} is less important and may be found in [Gragg 1963, p. 63] or [Stetter 1973, p. 241]. The relevant results describe a constructive procedure for determining the functions e_{mi} .

Let the functions $e_{mi}(x)$, $m = p, p+1, \dots, N-1$, $i = 1, 2, \dots, k$, be solutions of the differential equations,

$$e_{mi}'(a) = b_{mi0}, \quad (2.6)$$

$$e_{mi}' = \lambda_i A(x) e_{mi} + \lambda_i b_{mi}(x), \quad x \in [a, b],$$

where the functions $b_{mi}(x) \in C^{N-m}(I)$ and the initial vectors b_{mi0} are to be determined. Let

$$U_{mn} = \sum_{i=1}^k z_i^n e_{mi}(x), \quad x \in G_N,$$

and let

$$\delta_n(h) = y_n(h) - y(x) - \sum_{m=p}^{N-1} U_{mn} h^m.$$

The intent is to choose the arbitrary elements so that $\delta_n(h) \sim o(h^N)$ uniformly for h , $0 < h \leq h_0$, and for $x = x_n \in G_N$. This can be done by showing that $\delta_n(h)$ is initially small and satisfies a difference equation of the form (2.1) and then by applying the previous Lemma.

First it will be shown that $\delta_n(h)$ satisfies a difference equation of the form (2.1) with sufficiently small perturbation $\Delta_n(h)$.

Consider the linear operator L defined by

$$L = \rho(E) - h\sigma(E)A(x). \quad (2.7)$$

Thus,

$$\begin{aligned} L\delta_n &= \rho(E)\delta_n - h\sigma(E)A(x)\delta_n \\ &= L(y_n - y(x)) - L \sum_{m=p}^{N-1} u_{mn} h^m. \end{aligned} \quad (2.8)$$

And also,

$$\begin{aligned} L(y_n - y(x)) &= (\rho(E) - h\sigma(E)A(x))(y_n - y(x)) \\ &= \rho(E)y_n - h\sigma(E)(A(x)y_n + b(x)) \\ &\quad - (\rho(E)y(x) - h\sigma(E)(A(x)y(x) + b(x))) \\ &= -L[y(x); h] \end{aligned}$$

by (1.5), (1.7) and (2.4).

Since $A(x), b(x) \in C^N(I)$, it follows from (2.4) that $y(x) \in C^{N+1}(I)$ and from (1.15) and (1.17) that

$$L(y_n - y(x)) = -h\sigma(E) \sum_{m=p}^{N-1} a_{m+1}^{(1)} y^{(m+1)}(x) h^m + O(h^{N+1}). \quad (2.9)$$

From (1.13),

$$L \sum_{m=p}^{N-1} u_{mn} h^m = \sum_{m=p}^{N-1} \sum_{i=1}^k z_i^n L_i e_{mi}(x) h^m, \quad (2.10)$$

where

$$L_i(E) = \rho_i(E) - h\sigma_i(E)A(x).$$

Using the differential equations (2.6),

$$L_i e_{mi}(x) = \rho_i(E) e_{mi}(x) - h\sigma_i(E) A(x) e_{mi}(x)$$

$$= L_i[e_{mi}(x); h] + h\sigma_i(E)b_{mi}(x). \quad (2.11)$$

Since $A(x) \in C^{N-1}(I)$ and $b_{mi}(x) \in C^{N-m}(I)$, it follows that $e_{mi}(x) \in C^{N+1-m}(I)$. Thus, by (1.15)

$$L_i e_{mi}(x) = h\sigma_i(E)[b_{mi}(x) + \sum_{j=1}^{N-1-m} a_{j+1}^{(i)} e_{mi}^{(j+1)}(x)h^j + O(h^{N+1-m})]. \quad (2.12)$$

Now,

$$\begin{aligned} L \sum_{m=p}^{N-1} U_{mn} h^m &= \sum_{m=p}^{N-1} \sum_{i=1}^k z_i^n L_i e_{mi}(x) h^m \\ &= \sum_{m=p}^{N-1} \sum_{i=1}^k z_i^n \{h\sigma_i(E)[b_{mi}(x) + \sum_{j=1}^{N-1-m} a_{j+1}^{(i)} e_{mi}^{(j+1)}(x)h^j + O(h^{N+1-m})]h^m\} \\ &= h\sigma(E) \sum_{m=p}^{N-1} \sum_{i=1}^k z_i^n (b_{mi}(x) - \hat{b}_{mi}(x))h^m + O(h^{N+1}) \end{aligned}$$

by (1.13) and (2.12), where

$$\hat{b}_{mi}(x) = - \sum_{j=1}^{m-p} a_{j+1}^{(i)} e_{m-j,i}^{(j+1)}(x).$$

Thus, by (2.8)

$$\begin{aligned} -L \delta_n &= h\sigma(E) \sum_{m=p}^{N-1} \left\{ \sum_{i=1}^k z_i^n (b_{mi}(x) - \hat{b}_{mi}(x)) + a_{m+1}^{(1)} y^{(m+1)}(x) \right\} h^m + O(h^{N+1}). \end{aligned} \quad (2.13)$$

Now, if the $b_{mi}(x)$ are chosen according to

$$b_{mi}(x) = \begin{cases} \hat{b}_{mi}(x) - a_{m+1}^{(1)} y^{(m+1)}(x), & i=1, \\ \hat{b}_{mi}(x), & i=2, 3, \dots, k, \end{cases} \quad (2.14)$$

then, clearly, $L\delta_n \sim O(h^{N+1})$.

Since $A(x), b(x) \in C^N(I)$, by induction it is clear that $b_{mi}(x) \in C^{N-m}(I)$; this completes the first part of the proof. It will now be shown how to choose the arbitrary initial vectors b_{mi0} so that $\delta_n(h) \sim O(h^N)$ for $n = 0, 1, 2, \dots, k-1$.

It has been assumed that the starting values satisfy

$$\eta_n(h) - y(x_n) = \sum_{m=p}^{N-1} \eta_{mn} h^m + O(h^N), \quad n = 0, 1, \dots, k-1, \quad h \rightarrow 0. \quad (2.15)$$

Since $e_{mi}(x) \in C^{N-1-m}(I)$, using Taylor expansions,

$$e_{mi}(x) = b_{mi0} + \sum_{j=1}^{N-1-m} \frac{n^j}{j!} e_{mi}^{(j)}(a) h^j + O(h^{N-m}), \quad x = x_n.$$

Thus,

$$\sum_{m=p}^{N-1} u_{mn} h^m = \sum_{m=p}^{N-1} \sum_{i=1}^k z_i^n \left\{ b_{mi0} + \sum_{j=1}^{N-1-m} \frac{n^j}{j!} e_{mi}^{(j)}(a) h^j + O(h^{N-m}) \right\} h^m,$$

and by rearranging,

$$\sum_{m=p}^{N-1} u_{mn} h^m = \sum_{m=p}^{N-1} \sum_{i=1}^k z_i^n (b_{mi0} + \hat{b}_{mi0}(n)) h^m + O(h^N), \quad (2.16)$$

where

$$\hat{b}_{mi0}(n) = \sum_{j=1}^{m-1} \frac{n^j}{j!} e_{m-i,j}^{(j)}(a), \quad 0 \leq n \leq k, \quad 1 \leq i \leq k.$$

Now by (2.15) and (2.16)

$$\begin{aligned}\delta_n(h) &= y_n(h) - y(x) - \sum_{m=p}^{N-1} U_{mn} h^m \\ &= \sum_{m=p}^{N-1} \eta_{mn} h^m - \sum_{m=p}^{N-1} \sum_{i=1}^k z_i^n (b_{mi0} + \bar{b}_{mi0}(n)) h^m + O(h^N), \\ &= \sum_{m=p}^{N-1} (\eta_{mn} - \sum_{i=1}^k z_i^n (b_{mi0} + \bar{b}_{mi0}(n))) h^m + O(h^N).\end{aligned}$$

Thus, by choosing

$$\eta_{mn} = \sum_{i=1}^k z_i^n (b_{mi0} + \bar{b}_{mi0}(n)), \quad m = p, \dots, N-1, \quad n = 0, \dots, k-1, \quad (2.17)$$

it is clear that $\delta_n(h) \sim O(h^N)$, for $n = 0, 1, \dots, k-1$.

Altogether by choosing the arbitrary elements so as to satisfy (2.14) and (2.17), it is seen that $\delta_n(h)$ satisfies a difference equation of the form (2.1) with

$$F_n(\delta_n; h) = A(x) \delta_n \quad \text{and} \quad \delta(h) \leq \bar{a} h^N$$

for a suitable constant $\bar{a}$. Furthermore the perturbation

$$||\Delta_n(h)|| < \Lambda(h) \leq \bar{b} h^{n+1}$$

for a suitable constant $\bar{b}$.

Thus, by Lemma 2.1

$$||y_n(h) - y(x) - \sum_{m=p}^{N-1} U_{mn} h^m|| \leq M h^N$$

for some M , uniformly for h , $0 < h \leq h_0$ and for $x = x_n \in G_N$, and hence the theorem is proved. Q.E.D.

Next, the asymptotic expansion of the numerical solution of the test equation

$$y' = \lambda y, \quad y(0) = y_0 \quad (2.18)$$

is examined, where $\lambda \in \mathbb{C}$.

2.2 ASYMPTOTIC EXPANSION FOR THE MIDPOINT METHOD

The Midpoint method is given by

$$y_n = \eta_n(h), \quad n = 0, 1,$$

$$y_{n+2} = y_n + 2hf_{n+1}, \quad n = 0, 1, 2, \dots$$

Thus, the generating polynomials are

$$\rho(z) = z^2 - 1, \quad \sigma(z) = 2z,$$

and the zeros of $\rho(z)$,

$$z_1 = 1 \quad \text{and} \quad z_2 = -1,$$

satisfy the requirement for the Theorem 2.1.

The growth parameters are $\lambda_1 = 1$ and $\lambda_2 = -1$, and according to (1.16), the coefficients of the generating function are

$$a_j^{(1)} = -a_j^{(2)} = \begin{cases} 0 & j \text{ even,} \\ \frac{1}{j!} & j \text{ odd, } j > 2 \end{cases} \quad (2.19)$$

The order of the method is 2, and N is set to 4, i.e. an asymptotic expansion of the form

$$\begin{aligned} y_n(h) &= y(x) + \sum_{m=2}^3 \left(\sum_{i=1}^2 z_i^n e_{mi}(x) \right) h^m + O(h^4) \\ &= y(x) + (e_{21}(x) + (-1)^n e_{22}(x)) h^2 \\ &\quad + (e_{31}(x) + (-1)^n e_{32}(x)) h^3 + O(h^4), \quad x = x_n, \end{aligned} \quad (2.20)$$

will be obtained.

According to (2.6) the first pair of differential equations to be solved are

$$e'_{21} = \lambda e_{21} + b_{21}(x), \quad e_{21}(0) = b_{210},$$

and

(2.21)

$$e'_{22} = -\lambda e_{22} - b_{22}(x), \quad e_{22}(0) = b_{220}.$$

The starting values are assumed to have the asymptotic expansion

$$\eta_n(h) = y(x_n) + \sum_{m=2}^3 \eta_{mn} h^m + O(h^4), \quad n = 0, 1. \quad (2.22)$$

By solving (2.17), since $\hat{b}_{mi0}(0) = 0$,

$$b_{m10} = \frac{1}{2}[-\hat{b}_{m10}(1) + \hat{b}_{m20}(1) + \eta_{m0} + \eta_{m1}], \quad m = 2, 3,$$

and

(2.23)

$$b_{m20} = \frac{1}{2}[\hat{b}_{m10}(1) - \hat{b}_{m20}(1) + \eta_{m0} - \eta_{m1}], \quad m = 2, 3,$$

are obtained.

The definition of $\hat{b}_{mi0}(n)$ shows that $\hat{b}_{210}(1) = \hat{b}_{220}(1) = 0$ since e_{11} and e_{12} are assumed to be 0. Therefore (2.23) gives

$$b_{210} = \frac{1}{2}[\eta_{20} + \eta_{21}],$$

(2.24)

$$b_{220} = \frac{1}{2}[\eta_{20} - \eta_{21}].$$

Again the definition of $\hat{b}_{mi}(x)$ shows that $\hat{b}_{21}(x) = \hat{b}_{22}(x) = 0$, and therefore (2.14) gives

$$b_{21}(x) = -a_3^{(1)} y^{(3)}(x) = -\frac{\lambda^3}{6} y_0 e^{\lambda x}$$

and

(2.25)

$$b_{22}(x) = 0,$$

since the exact solution is $y(x) = y_0 e^{\lambda x}$.

Solving (2.21) by the aid of (2.24) and (2.25) the solutions are

$$e_{21}(x) = \left[-\frac{\lambda}{6} y_0 x + \frac{1}{2} (\eta_{20} + \eta_{21}) \right] e^{\lambda x}$$

and

$$e_{22}(x) = \frac{1}{2} [\eta_{20} - \eta_{21}] e^{-\lambda x}.$$

The next pair of differential equations to be solved are

$$e'_{31} = \lambda e_{31} + b_{31}(x), \quad e_{31}(0) = b_{310},$$

and

(2.26)

$$e'_{32} = -\lambda e_{32} - b_{32}(x), \quad e_{32}(0) = b_{320}.$$

The definitions of $\hat{b}_{mi}(x)$ and (2.19) give

$$\hat{b}_{31}(x) = \hat{b}_{32}(x) = 0,$$

and then (2.14) implies

$$b_{31}(x) = b_{32}(x) = 0.$$

(2.27)

The equations (2.23) with $e_{21}(x)$, $e_{22}(x)$ give

$$b_{310} = \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right],$$

(2.28)

$$b_{320} = \frac{1}{2} \left[-\frac{\lambda^3}{6} y_0 + \lambda \eta_{20} + \eta_{30} - \eta_{31} \right].$$

Using (2.27) and (2.28) the solution of (2.26) are

$$e_{31}(x) = \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda x}$$

and

$$e_{32}(x) = \frac{1}{2} \left[-\frac{\lambda^3}{6} y_0 + \lambda \eta_{20} + \eta_{30} - \eta_{31} \right] e^{-\lambda x}.$$

Thus, the required asymptotic expansion is

$$\begin{aligned} y_n(h) = & y_0 e^{\lambda x} \\ & + \left\{ \left[-\frac{\lambda^3}{6} y_0 x + \frac{1}{2} (\eta_{20} + \eta_{21}) \right] e^{\lambda x} + \frac{(-1)^n}{2} [\eta_{20} - \eta_{21}] e^{-\lambda x} \right\} h^2 \\ & + \left\{ \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda x} \right. \\ & \left. + \frac{(-1)^n}{2} \left[-\frac{\lambda^3}{6} y_0 + \lambda \eta_{20} + \eta_{30} - \eta_{31} \right] e^{-\lambda x} \right\} h^3 \\ & + O(h^4), \quad x = x_n \in G_N. \end{aligned} \quad (2.29)$$

2.3 ASYMPTOTIC EXPANSION FOR SIMPSON'S RULE (MILNE'S CORRECTOR)

Simpson's rule is,

$$y_n = \eta_n(h), \quad n = 0, 1,$$

$$y_{n+2} = y_n + \frac{h}{3} (f_{n+2} + 4f_{n+1} + f_n), \quad n = 0, 1, 2, \dots \quad (2.30)$$

Thus the generating polynomials are

$$\rho(z) = z^2 - 1, \quad \sigma(z) = \frac{1}{3}(z^2 + 4z + 1),$$

and, as in the case of the Midpoint method, the zeros of $\rho(z)$ satisfy the requirement of Theorem 2.1. The growth parameters are $\lambda_1 = 1$ and $\lambda_2 = -1/3$

and according to (1.16) the coefficients of the generating functions are

$$\begin{aligned} a_2^{(1)} = a_3^{(1)} = a_4^{(1)} = 0, \quad a_5^{(1)} = -\frac{1}{180}, \quad a_6^{(1)} = 0, \\ a_2^{(2)} = 0, \quad a_3^{(2)} = -2, \quad a_4^{(2)} = 0, \quad a_5^{(2)} = -\frac{23}{20}, \quad a_6^{(2)} = 0. \end{aligned} \quad (2.31)$$

The order of the method is 4 and N is set to 6, i.e. an asymptotic expansion of the form

$$\begin{aligned} y_n(h) &= y(x) + \sum_{m=4}^5 \left(\sum_{i=1}^2 z_i^n e_{mi}(x) \right) h^m + O(h^6) \\ &= y(x) + (e_{41}(x) + (-1)^n e_{42}(x)) h^4 \\ &\quad + (e_{51}(x) + (-1)^n e_{52}(x)) h^5 + O(h^6), \quad x = x_n \in G_N \end{aligned} \quad (2.32)$$

will be obtained.

According to (2.6) the first pair of differential equations to be solved are

$$e'_{41} = \lambda e_{41} + b_{41}(x), \quad e_{41}(0) = b_{410},$$

and

$$e'_{42} = -\frac{\lambda}{3} e_{42} - \frac{1}{3} b_{42}(x), \quad e_{42}(0) = b_{420}.$$

The starting values are assumed to have the asymptotic expansion

$$\eta_n(h) = y(x_n) + \sum_{m=4}^5 \eta_{mn} h^m + O(h^6), \quad n = 0, 1, \quad h \rightarrow 0. \quad (2.34)$$

The values of b_{m10} , b_{m20} , $m = 4, 5$, will be given by (2.23) and

$\hat{b}_{410}(1) = \hat{b}_{420}(1) = 0$ by the definition of $\hat{b}_{mi0}(n)$ and hence (2.23) gives

$$b_{410} = \frac{1}{2}[\eta_{40} + \eta_{41}], \quad (2.35)$$

$$b_{420} = \frac{1}{2}[\eta_{40} - \eta_{41}].$$

Since e_{mi} , $m \leq 3$, $i = 1, 2$, are assumed to be zero, $\hat{b}_{41}(x) = \hat{b}_{42}(x) = 0$, and therefore (2.14) gives

$$b_{41}(x) = -a_5^{(1)} y^{(5)}(x) = \frac{\lambda^5}{180} y_0 e^{\lambda x}, \quad (2.36)$$

$$b_{42}(x) = 0.$$

Solving (2.33) with (2.35) and (2.36) gives

$$e_{41}(x) = \left[\frac{\lambda^5}{180} y_0 x + \frac{1}{2}(\eta_{40} + \eta_{41}) \right] e^{\lambda x},$$

$$e_{42}(x) = \frac{1}{2}[\eta_{40} - \eta_{41}] e^{-\lambda x/3}.$$

The next pair of differential equations to be solved are

$$e'_{51} = \lambda e_{51} + b_{51}(x), \quad e_{51}(0) = b_{510}$$

and

$$e'_{52} = -\frac{\lambda}{3} e_{52} - \frac{1}{3} b_{52}(x), \quad e_{52}(0) = b_{520}.$$

(2.37)

Clearly $\hat{b}_{51}(x) = \hat{b}_{52}(x) = 0$ by the definition of $\hat{b}_{mi}(x)$ and (2.31), and hence

$$b_{51}(x) = b_{52}(x) = 0 \quad (2.38)$$

by (2.14). The equation (2.23) with $e_{41}(x)$ and $e_{42}(x)$ gives

$$b_{510} = \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{1}{3} \lambda \eta_{41} + \eta_{50} + \eta_{51} \right], \quad (2.39)$$

$$b_{520} = \frac{1}{2} \left[\frac{\lambda^5}{180} y_0 + \frac{2}{3} \lambda \eta_{40} + \frac{1}{3} \lambda \eta_{41} + \eta_{50} - \eta_{51} \right].$$

Using (2.38) and (2.39), the solutions of (2.37) are

$$e_{51}(x) = \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{1}{3} \lambda \eta_{41} + \eta_{50} + \eta_{51} \right] e^{\lambda x},$$

$$e_{52}(x) = \frac{1}{2} \left[\frac{\lambda^5}{180} y_0 + \frac{2}{3} \lambda \eta_{40} + \frac{1}{3} \lambda \eta_{41} + \eta_{50} - \eta_{51} \right] e^{-\lambda x/3}.$$

Thus, the required asymptotic expansion is

$$y_n(h) = y_0 e^{\lambda x}$$

$$+ \left\{ \left[\frac{\lambda^5}{180} y_0 x + \frac{1}{2} (\eta_{40} + \eta_{41}) \right] e^{\lambda x} + \frac{(-1)^n}{2} [\eta_{40} - \eta_{41}] e^{-\lambda x/3} \right\} h^4$$

$$+ \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{1}{3} \lambda \eta_{41} + \eta_{50} + \eta_{51} \right] e^{\lambda x} \right. \quad (2.40)$$

$$\left. + \frac{(-1)^n}{2} \left[\frac{\lambda^5}{180} y_0 + \frac{2}{3} \lambda \eta_{40} + \frac{1}{3} \lambda \eta_{41} + \eta_{50} - \eta_{51} \right] e^{-\lambda x/3} \right\} h^5$$

$$+ O(h^6), \quad x = x_n \in G_N.$$

SYSTEM OF NON-LINEAR DIFFERENTIAL EQUATIONS

In the case $f(x,y)$ is nonlinear in y , the above analysis for a system of linear differential equations may be generalized to show that when $N \leq 2p$, the situation is the same as if the differential equations were linear, except that $A(x)$ is now the Jacobian matrix of $f(x,y)$. If the product of each pair of essential zeros is again a zero of $\rho(z)$, then even if $N > 2p$ the situation is as in the case of a system linear differential equations [Gragg 1963, p.77].

CHAPTER 3

FILTERING AND EXTRAPOLATING

In addition to the assumption made at the beginning of Chapter 2, that all the zeros of the polynomial $\rho(z)$ are essential and hence distinct, here it is also assumed that $\rho(z)$ and $\sigma(z)$ have no common factors, a condition which is satisfied by the two methods under consideration.

3.1 WEAK INSTABILITY

Consider the application of a k -step linear multistep method with the above properties to the initial value problem

$$y' = \lambda y, \quad y(0) = 1, \quad (3.1)$$

where λ is a complex constant.

The corresponding difference equation is

$$\rho(E)y_n = h\lambda\sigma(E)y_n \quad (3.2)$$

Let z_{jh} , $j = 1, 2, \dots, k$, be zeros of the polynomial $\pi(z) = \rho(z) - h\lambda\sigma(z)$.

The theory of complex analysis includes the results that, for each sufficiently small $\epsilon > 0$, there exists $\delta > 0$ such that for $0 \leq h < \delta$ the equation $\pi(z) = 0$ has exactly as many zeros in each of the disks $|z - z_j| < \epsilon$ ($j = 1, 2, \dots, k$) as the equation $\rho(z) = 0$ [Ahlfors 1966, p. 131]. Therefore the distinctness of the zeros $z_1, \dots, z_k$ of $\rho(z)$ implies the distinctness of the zeros $z_{1h}, \dots, z_{kh}$ of $\pi(z)$.

Thus the solution of the equation (3.2) may be written

$$y_n = \sum_{j=1}^k A_j z_{jh}^n,$$

where the constants A_j , $j = 1, 2, \dots, k$, are complex.

It is also obvious that, for small values of h , the zero z_{jh} of $\pi(z)$ approaches one of the zeros $z_1, \dots, z_k$ of $\rho(z)$; therefore

$$z_{jh} \rightarrow z_j, \quad h \rightarrow 0, \quad (3.3)$$

and, in particular,

$$z_{1h} \rightarrow z_1 = 1.$$

It is convenient to write $A_1 = 1 + C_1$ and $A_j = C_j$, $j = 2, \dots, k$, where the constants C_j , $j = 1, 2, \dots, k$, depend on the linear multistep method itself and the starting values $\eta_n(h)$, $n = 0, 1, \dots, k-1$. Then the solution of the equation (3.2) becomes

$$y_n = z_{1h}^n + \sum_{j=1}^k C_j z_{jh}^n. \quad (3.4)$$

It is shown in [Henrici 1962, pp. 235-239] that z_{jn} may be written

$$z_{jh} = z_j [1 + \lambda_j \lambda h + O(h^2)].$$

Since $\rho(z_j) = 0$, and since

$$\pi(z_{jh}) = \rho(z_{jh}) - h\lambda\sigma(z_{jh})$$

$$= \rho\{z_j [1 + \lambda_j \lambda h + O(h^2)]\} - h\lambda\sigma\{z_j [1 + \lambda_j \lambda h + O(h^2)]\} = 0,$$

Taylor series about z_j gives

$$\lambda_j = \frac{\sigma(z_j)}{z_j \rho'(z_j)}, \quad j = 1, \dots, k.$$

These are the growth parameters introduced in Chapter 1.

The main interest is the asymptotic behaviour of z_{jh}^n when $h \rightarrow 0$ and $n \rightarrow \infty$ in such a way that $nh = x$ remains constant.

The approximation

$$\begin{aligned} [1 + \lambda_j \lambda h + O(h^2)]^n &= \exp \{xh^{-1} \ln[1 + \lambda_j \lambda h + O(h^2)]\}, \\ &= \exp \{xh^{-1} [\lambda_j \lambda h + O(h^2)]\}, \\ &= \exp (\lambda_j \lambda x) + O(h), \end{aligned}$$

with $z_j = e^{i\phi_j}$, gives

$$z_{jh}^n = e^{in\phi_j} [e^{\lambda_j \lambda x} + O(h)], \quad h \rightarrow 0.$$

In particular $z_{1h}^n = e^{\lambda \lambda x} + O(h)$, which corresponds to the exact solution.

If $\text{Re}(\lambda x) < 0$ and if some λ_j is negative, the solution which is close to the exact solution (i.e. z_{1h}^n) will be dominated by the terms in (3.4) with negative λ_j 's. This will result in an inaccurate numerical solution for the initial value problem (3.1), a phenomenon called weak instability. For the following discussion the term z_{1h}^n , the terms with negative growth parameters λ_j 's and the rest of the terms in the equation (3.4) will be called the "desired solutions", "oscillating part" and the "harmless part" respectively.

Both the Midpoint method and Simpson's rule have negative growth parameters. In Chapter 2 it was shown that the numerical solutions obtained by these methods have asymptotic expansions in integral powers of h , whose coefficients contain oscillating terms corresponding to the negative growth parameters. Now it is clear that these terms cause this so-called weak instability.

In this chapter this weak instability will be treated by means of a filter which will reduce the oscillating part by a factor of a power of h while keeping the desired solution and the harmless part unchanged up to a power of h . Then the order of the method will be increased by extrapolation.

Filtering is an averaging procedure involving the current term together with previous terms, forward terms or both. If $y_n(h)$ is the unfiltered solution, then the filtered solution $y_n^*(h)$ may be written as

$$y_n^*(h) = P(E)y_n(h), \quad (3.5)$$

where $P(E)$ is the filtering operator expressed in terms of the shift operator E ,

$$P(E) = \sum_{j=-r_1}^{r_2} a_j E^j, \quad (3.6)$$

the a_j 's being real and r_1 and r_2 being non-negative integers.

Let $s = h\lambda$; then elementary calculus gives

$$P(E)e^{\lambda x} = e^{\lambda x} P(e^s),$$

$$P(E)xe^{\lambda x} = e^{\lambda x} [xP(e^s) + h \frac{\partial}{\partial s} P(e^s)], \quad (3.7)$$

$$P(E)(-1)^n e^{-\mu \lambda x} = (-1)^n e^{-\mu \lambda x} p(-e^{-\mu s}), \text{ where } x = x_n \in G_N.$$

It may be noted that s and h have the same order of magnitude.

3.2 DESIGN AND CONSTRUCTION OF FILTERS

(a) THE MIDPOINT METHOD

In Chapter 2 it was shown that the Midpoint solution has the asymptotic expansion,

$$\begin{aligned} y_n(h) = & y_0 e^{\lambda x} \\ & + \left\{ \left[-\frac{\lambda^3}{6} y_0 x + \frac{1}{2} (\eta_{20} + \eta_{21}) \right] e^{\lambda x} + \frac{(-1)^n}{2} [\eta_{20} - \eta_{21}] e^{-\lambda x} \right\} h^2 \\ & + \left\{ \left[\frac{1}{2} \left[-\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda x} + \frac{(-1)^n}{2} \left[-\frac{\lambda^3}{6} y_0 + \lambda \eta_{20} + \eta_{30} - \eta_{31} \right] e^{-\lambda x} \right] \right\} h^3 \\ & + O(h^4), \quad x = x_n \in G_N, \end{aligned} \quad (3.8)$$

for the test equation (2.18) with starting values (2.22). Thus, by (3.5) and (3.7) after application of the filter (3.6)

$$\begin{aligned} y_n^*(h) = & y_0 e^{\lambda x} p(e^s) \\ & + \left\{ -\frac{\lambda^3}{6} y_0 e^{\lambda x} [x p(e^s) + h \frac{\partial}{\partial s} p(e^s)] + \frac{1}{2} [\eta_{20} + \eta_{21}] e^{\lambda x} p(e^s) \right. \\ & \left. + \frac{(-1)^n}{2} [\eta_{20} - \eta_{21}] e^{-\lambda x} p(-e^{-s}) \right\} h^2 \end{aligned}$$

$$\begin{aligned}
 & + \left\{ \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \lambda_{30} + \lambda_{31} \right] e^{\lambda x} P(e^s) \right. \\
 & + \frac{(-1)^n}{2} \left[-\frac{\lambda^3}{6} y_0 + \lambda \eta_{20} + \eta_{30} - \eta_{31} \right] e^{-\lambda x} P(-e^{-s}) \} h^3 \\
 & + O(h^4), \quad x = x_n \in G_N.
 \end{aligned} \tag{3.9}$$

The above equation shows that in order for the solution to be kept unchanged up to h^N for a positive integer N , it is sufficient that the operator P satisfies

$$P(e^s) = 1 + O(s^{N+1}), \tag{3.10}$$

and in order for the oscillating part to be reduced by a factor of h^M for a positive integer M it is also sufficient that P satisfies

$$P(-e^{-s}) = O(s^M). \tag{3.11}$$

The equation (3.10) gives

$$\frac{\partial}{\partial s} P(e^s) = O(s^N). \tag{3.12}$$

Thus,

$$y_n^*(h) = y_0 e^{\lambda x} [1 + O(s^{N+1})]$$

$$+ \left\{ -\frac{\lambda^3}{6} y_0 e^{\lambda x} [x(1 + O(s^{N+1})) + h O(s^N)] \right.$$

$$+ \frac{1}{2} [\eta_{20} + \eta_{21}] e^{\lambda x} [1 + O(s^{N+1})] + \frac{(-1)^n}{2} [\eta_{20} - \eta_{21}] e^{-\lambda x} O(s^M) \} h^2$$

$$\begin{aligned}
 & + \left\{ \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda x} [1 + O(s^{N+1})] \right. \\
 & + \frac{(-1)^n}{2} \left[-\frac{\lambda^3}{6} y_0 + \lambda \eta_{20} + \eta_{30} - \eta_{31} \right] e^{-\lambda x} O(s^M) \Big\} h^3 \\
 & + O(h^4), \quad x = x_n \in G_N.
 \end{aligned} \tag{3.13}$$

The term $\left\{ -\frac{\lambda^3}{6} y_0 e^{\lambda x} [x(1 + O(s^{N+1}))] \right\}$ in (3.13) shows that this procedure is unable to improve the order of the method. It suggests taking $N = 2$. In order to apply extrapolation it is necessary to reduce the oscillating part up to $O(h^4)$, which suggests taking $M = 2$. Thus,

$$P(e^s) = 1 + O(s^3), \tag{3.14}$$

$$P(-e^{-s}) = O(s^2).$$

Equations (3.14) consist of five linear equations which can be solved for at most five unknowns (i.e. for any five a_j 's). This set of unknowns is generally not unique and may be written

$$\{a_{\lambda-2}^{(\lambda)}, a_{\lambda-1}^{(\lambda)}, a_{\lambda}^{(\lambda)}, a_{\lambda+1}^{(\lambda)}, a_{\lambda+2}^{(\lambda)}\}, \quad \lambda = -2, -1, 0, 1, 2,$$

where, for example, the case $\lambda = -2$ is the case $r_1 = 4, r_2 = 0$, and the case $\lambda = -1$ is the case $r_1 = 3, r_2 = 1$, and so on. Now elementary linear algebra shows that, in all the above cases, the systems of linear equations are linearly independent and can be solved uniquely for the five unknowns. In each case, this result yields a filter involving five terms.

A general filter may be written

$$P_{\lambda}(E) = \sum_{j=-2}^2 a_{\lambda+j}^{(\lambda)} E^j, \quad \lambda = -2, -1, 0, 1, 2. \tag{3.15}$$

THEOREM 3.1.a

Among the above five filters the filter P_0 is symmetric and for that filter $\{P_0(e^s) - 1\}$ and $P_0(-e^{-s})$ contain only even powers of s (i.e., are both even functions of s).

PROOF

The five equations (3.14) are, for each λ ($= -2, -1, 0, 1, 2$),

$$\sum_{j=-2}^2 a_{\lambda+j}^{(\lambda)} = 1,$$

$$\sum_{j=-2}^2 \frac{(j)^z}{z!} a_{\lambda+j}^{(\lambda)} = 0, \quad z = 1, 2,$$

(3.16)

$$\sum_{j=-2}^2 (-1)^j a_{\lambda+j}^{(\lambda)} = 0,$$

$$\sum_{j=-2}^2 (-1)^j (-j) a_{\lambda+j}^{(\lambda)} = 0.$$

It is clear that, if $\{a_{\lambda+j}^{(\lambda)}\}_{j=-2}^2$ is a solution of (3.16), then so is

$\{a_{-\lambda-j}^{(-\lambda)}\}_{j=-2}^2$ with $a_{\lambda+j}^{(\lambda)} = a_{-\lambda-j}^{(-\lambda)}$. When $\lambda = 0$, $a_j^{(0)} = a_{-j}^{(0)}$ leads to the symmetric filter

$$P_0(E) = \frac{1}{16}(-E^{-2} + 4E^{-1} + 10 + 4E - E^2). \quad (3.17)$$

Table 3.1.a contains coefficients $a_{\ell+j}^{(\ell)}$ of all five filters. The rest of the theorem is clear from the fact that $a_j^{(0)} = a_{-j}^{(0)}$ and from (3.15). Q.E.D.

TABLE 3.1.a

Coefficients of five-term filters for the Midpoint method

FILTER	$16a_{\ell-2}^{(\ell)}$	$16a_{\ell-1}^{(\ell)}$	$16a_{\ell}^{(\ell)}$	$16a_{\ell+1}^{(\ell)}$	$16a_{\ell+2}^{(\ell)}$
P_{-2}	3	-4	-6	12	11
P_{-1}	-1	0	6	8	3
P_0	-1	4	10	4	-1
P_1	3	8	6	0	-1
P_2	11	12	-6	-4	3

Let

$$P_{\ell}(e^s) = 1 + \alpha_3 s^3 + \alpha_4 s^4 + \alpha_5 s^5 + \alpha_6 s^6 + \alpha_7 s^7 + O(s^8)$$

and

$$P_{\ell}(-e^{-s}) = \beta_2 s^2 + \beta_3 s^3 + \beta_4 s^4 + \beta_5 s^5 + \beta_6 s^6 + \beta_7 s^7 + O(s^8).$$

Tables 3.2.a and 3.3.a give values of the coefficients α and β , respectively, for all five filters.

TABLE 3.2.a

Coefficients α_j ($3 \leq j \leq 7$) of Taylor expansion of $\{P_\lambda(e^s)-1\}$ about 0.

FILTER	$4\alpha_3$	$16\alpha_4$	$8\alpha_5$	$96\alpha_6$	$480\alpha_7$
P_{-2}	-2	15	-8	75	-216
P_{-1}	1	-3	1	-6	13
P_0	0	-1	0	0	0
P_1	-1	-3	-1	-1	-13
P_2	2	15	8	75	216

TABLE 3.3.a

Coefficients β_j ($2 \leq j \leq 7$) of Taylor expansion of $P_\lambda(-e^{-s})$ about 0.

FILTER	$2\beta_2$	$4\beta_3$	$48\beta_4$	$8\beta_5$	$1440\beta_6$	$480\beta_7$
P_{-2}	3	10	123	16	1851	340
P_{-1}	0	3	9	1	90	13
P_0	-1	0	-5	0	-17	0
P_1	0	-3	9	-1	90	-13
P_2	3	-10	123	-16	1851	-340

Finally, it is seen that these five filters (Table 3.1.a) are the filters introduced by M. Iri in 1964 [Iri 1964]. But his suggestion is to use the filter P_{-2} . The above analysis shows the advantage of using the symmetric filter P_0 rather than using any other, necessarily non-symmetric, filter. Now the application of the symmetric filter P_0 is considered.

By Tables 3.2.a and 3.3.a

$$\begin{aligned}
 y_n^*(h) &= y_0 e^{\lambda x} [1 + O(s^4)] \\
 &+ \left\{ -\frac{\lambda^3}{6} y_0 e^{\lambda x} [x(1 + O(s^4)) + h O(s^3)] \right. \\
 &+ \frac{1}{2} [\eta_{20} + \eta_{21}] e^{\lambda x} (1 + O(s^4)) + \frac{(-1)^n}{2} [\eta_{20} - \eta_{21}] e^{-\lambda x} O(s^2) \} h^2 \\
 &+ \left\{ \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda x} (1 + O(s^4)) \right. \\
 &+ \frac{(-1)^n}{2} \left[-\frac{\lambda^3}{6} y_0 + \eta_{20} + \eta_{30} - \eta_{31} \right] e^{-\lambda x} O(s^2) \} h^3 + O(h^4) \\
 &= y_0 e^{\lambda x} + \{ e^{\lambda x} [-\frac{\lambda^3}{6} y_0 x + \frac{1}{2} (\eta_{20} + \eta_{21})] \} h^2 \\
 &+ \left\{ \frac{e^{\lambda x}}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] \right\} h^3 + O(h^4), \quad x = x_n \in G_N.
 \end{aligned} \tag{3.19}$$

Now it is clear that the symmetric filter reduces the oscillating part by a factor of h^2 and keeps unchanged the harmless part up to $O(h^4)$. Table 3.2.a shows that the other four filters do not have this latter property since $\alpha_3 \neq 0$.

COROLLARY 3.1.a

Let $y_M^{n*}(h)$ denote the numerical value at $x_M = Mh$ by the Midpoint method after one filtering at every n steps with the last filtering at x_M .

Then for fixed positive integers n and m ($< n-1$), and for any positive integer N ,

$$y_{Nn+m}^{n(*)}(h) = y_0 e^{\lambda(Nx+mh)} + \left\{ \left[-\frac{\lambda^3}{6} y_0 (Nx+mh) + \frac{1}{2}(\eta_{20} + \eta_{21}) \right] e^{\lambda(Nx+mh)} \right\} h^2 \\ + \left\{ \left[\frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda(Nx+mh)} \right\} h^3 + o(h^4), \right. \\ x = x_n \in G_N. \quad (3.20)$$

PROOF

The proof of the corollary depends on mathematical induction on N and it is assumed that (3.20) is true for all positive integers $n \leq q$.

Thus, by (3.20)

$$y_{qn+m}^{n(*)}(h) = y_0 e^{\lambda(qx+mh)} + \left\{ \left[-\frac{\lambda^3}{6} y_0 (qx+mh) + \frac{1}{2}(\eta_{20} + \eta_{21}) \right] e^{\lambda(qx+mh)} \right\} h^2 \\ + \left\{ \left[\frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda(qx+mh)} \right\} h^3 + o(h^4), \right. \\ x = x_n \in G_N. \quad (3.21)$$

and

$$y_{qn+m+1}^{n(*)}(h) = y_0 e^{\lambda(qx+mh+h)} \\ + \left\{ \left[-\frac{\lambda^3}{6} y_0 (qx+mh+h) + \frac{1}{2}(\eta_{20} + \eta_{21}) \right] e^{\lambda(qx+mh+h)} \right\} h^2 \\ + \left\{ \left[\frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda(qx+mh+h)} \right\} h^3 \\ + o(h^4), \quad x = x_n \in G_N. \quad (3.22)$$

Now a new initial value problem

$$y^{*'} = \lambda y^*, \quad y^*(0) = y_0^*,$$

can be started at $(qx+mh)$ with the starting values $y_{qn+m}^{n(*)}(h)$ and $y_{qn+m+1}^{n(*)}(h)$. If the starting values $y_{qn+m}^{n(*)}(h)$ and $y_{qn+m+1}^{n(*)}(h)$ have asymptotic expansions in the form.

$$y_{qn+m+i}^{n(*)} = y^*(ih) + \sum_{j=2}^3 \eta_{ji}^* h^j + O(h^4), \quad i = 0, 1,$$

then by (3.21) and (3.22)

$$y_0^* = y^*(0) = y_0 e^{\lambda(qx+mh)},$$

$$\eta_{20}^* = \left[-\frac{\lambda^3}{6} y_0(qx+mh) + \frac{1}{2}(\eta_{20} + \eta_{21}) \right] e^{\lambda(qx+mh)},$$

$$\eta_{30}^* = \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda(qx+mh)}, \quad (3.24)$$

$$\eta_{21}^* = \left[-\frac{\lambda^3}{6} y_0(qx+mh+h) + \frac{1}{2}(\eta_{20} + \eta_{21}) \right] e^{\lambda(qx+mh+h)},$$

$$\eta_{31}^* = \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda(qx+mh+h)}, \quad x = x_n \in G_N.$$

It is clear that the first filtered value of the new initial value problem at the next n^{th} step is the same as the $(q+1)^{\text{st}}$ filtered value of the original initial value problem (i.e. $y_{(q+1)n+m}^{n(*)}(h)$).

Therefore, considering the new initial value problem, by (3.19)

$$\begin{aligned}
 y_{(q+1)n+m}^{n(*)}(h) &= y_0^* e^{\lambda x} + \left\{ \left[-\frac{\lambda^3}{6} y_0^* x + \frac{1}{2}(\eta_{20}^* + \eta_{21}^*) \right] e^{\lambda x} \right\} h^2 \\
 &+ \left\{ \frac{1}{2} \left[\frac{\lambda^3}{6} y_0^* - \lambda \eta_{20}^* + \eta_{30}^* + \eta_{31}^* \right] e^{\lambda x} \right\} h^3 + O(h^4), \\
 x &= x_n \in G_N.
 \end{aligned} \tag{3.25}$$

By substituting (3.24) into (3.25) and rearranging,

$$\begin{aligned}
 y_{(q+1)n+m}^{n(*)}(h) &= y_0 e^{\lambda((q+1)x+mh)} \\
 &+ \left\{ \left[-\frac{\lambda^3}{6} y_0((q+1)x+mh) + \frac{1}{2}(\eta_{20} + \eta_{21}) \right] e^{\lambda((q+1)x+mh)} \right\} h^2 \\
 &+ \left\{ \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda((q+1)x+mh)} \right\} h^3 \\
 &+ O(h^4), \quad x = x_n \in G_N.
 \end{aligned} \tag{3.26}$$

and the corollary is proved by induction.

Q.E.D.

When $m = 0$ the equation (3.20) becomes

$$\begin{aligned}
 y_{Nn}^{n(*)}(h) &= y_0 e^{\lambda N x} + \left\{ \left[-\frac{\lambda^3}{6} y_0 N x + \frac{1}{2}(\eta_{20} + \eta_{21}) \right] e^{\lambda N x} \right\} h^2 \\
 &+ \left\{ \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 - \lambda \eta_{20} + \eta_{30} + \eta_{31} \right] e^{\lambda N x} \right\} h^3 + O(h^4),
 \end{aligned} \tag{3.27}$$

and $y_{Nn}^{n(*)}(h)$ gives the value at $x_{Nn} = Nnh = Nx$ after filtering at the points $x, 2x, \dots, Nx$.

(b) SIMPSON'S RULE

This analysis is similar to the analysis done in (a). In Chapter 2 it was shown that Simpson's solution has the asymptotic expansion,

$$y_n(h) = y_0 e^{\lambda x}$$

$$+ \left\{ \left[-\frac{\lambda^5}{180} y_0 x + \frac{1}{2} (\eta_{40} + \eta_{41}) \right] e^{\lambda x} + \frac{(-1)^n}{2} [\eta_{40} - \eta_{41}] e^{-\lambda x/3} \right\} h^4$$

$$+ \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{\lambda}{3} \eta_{41} + \eta_{50} + \eta_{51} \right] e^{\lambda x} \right.$$

$$\left. + \frac{(-1)^n}{2} \left[-\frac{\lambda^5}{180} y_0 + \frac{2}{3} \lambda \eta_{40} + \frac{\lambda}{3} \eta_{41} + \eta_{50} - \eta_{51} \right] e^{-\lambda x/3} \right\} h^5$$

$$+ O(h^6), \quad x = x_n \in G_N,$$

for the test equation (2.18) with the starting values (2.34). Thus, by (3.5) and (3.7) after application of the filter

$$y_n^*(h) = y_0 e^{\lambda x} P(e^S) + \left\{ \frac{\lambda^5}{180} y_0 e^{\lambda x} \left[x P(e^S) + h \frac{\partial}{\partial s} P(e^S) \right] \right.$$

$$\left. + \frac{1}{2} [\eta_{40} + \eta_{41}] e^{\lambda x} P(e^S) + \frac{(-1)^n}{2} [\eta_{40} - \eta_{41}] e^{-\lambda x/3} P(-e^{-S/3}) \right\} h^4$$

$$+ \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{\lambda}{3} \eta_{41} + \eta_{50} + \eta_{51} \right] e^{\lambda x} P(e^S) \right.$$

$$\left. + \frac{(-1)^n}{2} \left[-\frac{\lambda^5}{180} y_0 + \frac{2}{3} \lambda \eta_{40} + \frac{\lambda}{3} \eta_{41} + \eta_{50} - \eta_{51} \right] e^{-\lambda x/3} P(-e^{-S/3}) \right\} h^5$$

$$+ O(h^6), \quad x = x_n \in G_N.$$

(3.28)

When the conditions (3.10) - (3.12) are applied to this case the equation similar to (3.13) becomes

$$\begin{aligned}
 y_0^*(h) &= y_0 e^{\lambda x} (1 + O(s^{N+1})) \\
 &+ \left\{ \frac{\lambda^5}{180} y_0 e^{\lambda x} [x(1 + O(s^{N+1})) + h O(s^N)] \right. \\
 &+ \frac{1}{2} [\eta_{40} + \eta_{41}] e^{\lambda x} (1 + O(s^{N+1})) + \frac{(-1)^n}{2} [\eta_{40} - \eta_{41}] e^{-\lambda x/3} O(s^M) \} h^4 \\
 &+ \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{\lambda}{3} \eta_{41} + \eta_{50} + \eta_{51} \right] e^{\lambda x} (1 + O(s^{N+1})) \right. \\
 &+ \frac{(-1)^n}{2} \left[\frac{\lambda^5}{180} y_0 + \frac{2}{3} \lambda \eta_{40} + \frac{\lambda}{3} \eta_{41} + \eta_{50} - \eta_{51} \right] e^{-\lambda/3} O(s^M) \} h^5 \\
 &+ O(h^6), \quad x = x_n \in G_N.
 \end{aligned} \tag{3.29}$$

The term $\left\{ \frac{\lambda^5}{180} y_0 e^{\lambda x} [1 + O(s^{N+1})] \right\} h^4$ in (3.29) shows that this procedure is unable to improve the order of the method. It suggests taking $N = 4$. In order to apply extrapolation it is necessary to reduce the oscillating part up to $O(h^6)$. This suggests taking $M = 2$. Thus, the equations similar to (3.14) are

$$\begin{aligned}
 P(e^s) &= 1 + O(s^5), \\
 P(-e^{-s/3}) &= O(s^2).
 \end{aligned} \tag{3.30}$$

Equation (3.30) consists of seven linear equations which can be solved for at most seven unknowns (i.e. for any seven a_j 's). This set of unknowns is generally not unique and may be written,

$$\{a_{\lambda-3}^{(\lambda)}, a_{\lambda-2}^{(\lambda)}, a_{\lambda-1}^{(\lambda)}, a_{\lambda}^{(\lambda)}, a_{\lambda+1}^{(\lambda)}, a_{\lambda+2}^{(\lambda)}, a_{\lambda+3}^{(\lambda)}\} \quad \lambda = -3, -2, -1, 0, 1, 2, 3,$$

where, for example, the case $\lambda = -3$ is the case $r_1 = 6, r_2 = 0$ and the case $\lambda = -2$ is the case $r_1 = 5, r_2 = 1$ and so on. Now elementary linear algebra shows that, in all the above cases, the systems of linear equations are linearly independent and can be solved uniquely for seven unknowns. In each case, result yields a filter involving seven terms.

A general filter may be written

$$P_{\lambda}(E) = \sum_{j=-3}^3 a_{\lambda+j}^{(\lambda)} E^j, \quad \lambda = -3, -2, -1, 0, 1, 2, 3. \quad (3.31)$$

THEOREM 3.1.b

Among the above seven filters, the filter P_0 is symmetric and for that filter $\{P_0(e^s) - 1\}$ and $P_0(-e^{-s/3})$ contains only even powers of s (i.e., are both even functions of s).

PROOF

The proof of the theorem is similar to that of Theorem 3.1.a. The symmetric filter is

$$P_0(E) = \frac{1}{64}(E^{-3} - 6E^{-2} + 15E^{-1} + 44 + 15E - 6E^2 + E^3). \quad (3.32)$$

Tables (3.1.b), contains the values of the coefficients for all seven filters.

Q.E.D.

TABLE 3.1.b

Coefficients of seven-term filters of Simpson's rule.

FILTER	$64a_{\ell-3}^{(\ell)}$	$64a_{\ell-2}^{(\ell)}$	$64a_{\ell-1}^{(\ell)}$	$64a_{\ell}^{(\ell)}$	$64a_{\ell+1}^{(\ell)}$	$64a_{\ell+2}^{(\ell)}$	$64a_{\ell+3}^{(\ell)}$
P_{-3}	5	-18	15	20	-45	30	57
P_{-2}	-3	10	-5	-20	35	42	5
P_{-1}	1	-2	-5	20	39	14	-3
P_0	1	-6	15	44	15	-6	1
P_1	-3	14	39	20	-5	-2	1
P_2	5	42	35	-20	-5	10	-3
P_3	57	30	-45	20	15	-18	5

Let $P_{\ell}(e^s) = 1 + \alpha_5 s^5 + \alpha_6 s^6 + \alpha_7 s^7 + \alpha_8 s^8 + O(s^9)$

and

$$P_{\ell}(-e^{-s/3}) = \beta_2 s^2 + \beta_3 s^3 + \beta_4 s^4 + \beta_5 s^5 + \beta_6 s^6 + \beta_7 s^7 + O(s^8).$$

Tables (3.2.b) and (3.3.b) give the values of the coefficients α and β , respectively, for all seven filters.

TABLE 3.2.b

Coefficients α_j ($5 \leq j \leq 8$) of Taylor expansion of $\{P_{\ell}(e^s)-1\}$ about 0.

FILTER	$16\alpha_5$	$64\alpha_6$	$192\alpha_7$	$256\alpha_8$
P_{-3}	-3	35	-165	245
P_{-2}	2	-15	50	-55
P_{-1}	-2	3	-7	5
P_0	0	1	0	1
P_1	2	3	7	5
P_2	-2	-15	-50	-55
P_3	3	35	165	245

TABLE 3.3.b

Coefficients β_j ($2 \leq j \leq 7$) of Taylor expansion of $P_\lambda(-e^{-s/3})$ about 0.

FILTER	$36\beta_2$	$108\beta_3$	$972\beta_4$	$3888\beta_5$	$2099520\beta_6$	$2099520\beta_7$
P_{-3}	15	35	150	217	34905	8935
P_{-2}	5	10	35	42	6035	1210
P_{-1}	-1	1	2	3	313	61
P_0	-3	0	-3	0	-141	0
P_1	-1	-1	2	-3	313	-61
P_2	5	-10	35	-42	6035	-1210
P_3	15	-35	150	-217	34905	-8935

Finally, it is seen that these seven filters (Table 3.1.b) are the filters introduced by M. Iri in [Iri 1964]. But his suggestion is to use the filter P_{-3} . The above analysis shows the advantage of using the symmetric filter P_0 rather than any other, necessarily non-symmetric, filter. Now the application of the symmetric filter P_0 is considered.

By tables (3.2.b) and (3.3.b),

$$y_n^*(h) = y_0 e^{\lambda x} [1 + O(s^6)]$$

$$+ \left\{ \frac{\lambda^5}{180} y_0 e^{\lambda x} [x(1 + O(s^6)) + h O(s^5)] + \frac{1}{2} [\eta_{40} + \eta_{41}] e^{\lambda x} (1 + O(s^6)) \right.$$

$$+ \frac{(-1)^n}{2} [\eta_{40} - \eta_{41}] e^{-\lambda x/3} O(s^2) h^4$$

$$+ \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{\lambda}{3} \eta_{41} + \eta_{50} + \eta_{51} \right] e^{\lambda x} (1 + O(s^6)) \right.$$

$$+ \frac{(-1)^n}{2} \left[\left[\frac{\lambda^5}{180} y_0 + \frac{2}{3} \lambda \eta_{40} + \frac{\lambda}{3} \eta_{41} + \eta_{50} - \eta_{51} \right] e^{-\lambda x/3} O(s^2) h^5 \right.$$

$$+ O(h^6),$$

$$\begin{aligned}
 &= y_0 e^{\lambda x} + \left\{ \left(-\frac{\lambda}{180} y_0 x + \frac{1}{2} (\eta_{40} + \eta_{41}) \right) e^{\lambda x} \right\} h^4 \\
 &\quad + \left\{ \frac{1}{2} \left(-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{\lambda}{3} \eta_{41} + \eta_{50} + \eta_{51} \right) e^{\lambda x} \right\} h^5 \\
 &\quad + O(h^6), \quad x = x_n \in G_N. \tag{3.33}
 \end{aligned}$$

Now it is clear that the symmetric filter reduces the oscillating part by a factor of h^2 and keeps unchanged the harmless part up to $O(h^6)$. Table 3.2.b shows that the other six filters do not have this latter property since $\alpha_5 \neq 0$.

COROLLARY 3.1.b

Let $y_M^{n(*)}(h)$ denotes the numerical value at $x_M = Mh$ by Simpson's rule after one filtering at every n steps with the last filtering at x_M . Then for fixed positive integers n and m ($< n-1$), and for any positive integer N ,

$$\begin{aligned}
 y_{Nh+m}^{n(*)}(h) &= y_0 e^{\lambda(Nx+mh)} \\
 &\quad + \left\{ \left[\frac{\lambda^5}{180} y_0 (Nx+mh) + \frac{1}{2} (\eta_{40} + \eta_{41}) \right] e^{\lambda(Nx+mh)} \right\} h^4 \\
 &\quad + \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{\lambda}{3} \eta_{41} + \eta_{50} + \eta_{51} \right] e^{\lambda(Nx+mh)} \right\} h^5 \\
 &\quad + O(h^6), \quad x = x_n \in G_N. \tag{3.34}
 \end{aligned}$$

PROOF

Again mathematical induction on N is used here and it is assumed that (3.34) is true for all positive integers $\leq q$.

Thus, by (3.34)

$$\begin{aligned} y_{qn+m}^{n(*)}(h) &= y_0 e^{\lambda(qx+mh)} \\ &+ \left\{ \left[\frac{\lambda^5}{180} y_0 (qx+mh) + \frac{1}{2} (\eta_{40} + \eta_{41}) \right] e^{\lambda(qx+mh)} \right\} h^4 \\ &+ \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{\lambda}{3} \eta_{41} + \eta_{50} + \eta_{51} \right] e^{\lambda(qx+mh)} \right\} h^5 \\ &+ O(h^6), \quad x = x_n \in G_N, \end{aligned} \quad (3.35)$$

and

$$\begin{aligned} y_{qn+m+1}^{n(*)}(h) &= y_0 e^{\lambda(qx+mh+h)} \\ &+ \left\{ \left[\frac{\lambda^5}{180} y_0 (qx+mh+h) + \frac{1}{2} (\eta_{40} + \eta_{41}) \right] e^{\lambda(qx+mh+h)} \right\} h^4 \\ &+ \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{\lambda}{3} \eta_{41} + \eta_{50} + \eta_{51} \right] e^{\lambda(qx+mh+h)} \right\} h^5 \\ &+ O(h^6), \quad x = x_n \in G_N. \end{aligned} \quad (3.36)$$

Now a new initial value problem

$$y^*{}' = \lambda y^*, \quad y^*(0) = y_0^*,$$

can be started at $(qx+mh)$ with starting values $y_{qn+m}^{n(*)}(h)$ and $y_{qn+m+1}^{n(*)}(h)$.

If the starting values $y_{qn+m}^{n(*)}(h)$ and $y_{qn+m+1}^{n(*)}(h)$ have asymptotic expansions in the form

$$y_{qn+m+i}^{n(*)}(h) = y^*(ih) + \sum_{j=4}^5 \eta_{ji}^* h^j + O(h^6), \quad i = 0, 1,$$

then by (3.35) and (3.36)

$$y_0^* = y^*(0) = y_0 e^{\lambda(qx+mh)}$$

$$\eta_{40}^* = \left[\frac{\lambda}{180} y_0^5 (qx+mh) + \frac{1}{2} (\eta_{40}^* + \eta_{41}^*) \right] e^{\lambda(qx+mh)} \quad (3.37)$$

$$\eta_{50}^* = \frac{1}{2} \left[-\frac{\lambda}{180} y_0^5 - \frac{2}{3} \lambda \eta_{40}^* - \frac{\lambda}{3} \eta_{41}^* + \eta_{50}^* + \eta_{51}^* \right] e^{\lambda(qx+mh)}$$

$$\eta_{41}^* = \left[\frac{\lambda}{180} y_0^5 (qx+mh+h) + \frac{1}{2} (\eta_{40}^* + \eta_{41}^*) \right] e^{\lambda(qx+mh+h)}$$

$$\eta_{51}^* = \frac{1}{2} \left[-\frac{\lambda}{180} y_0^5 - \frac{2}{3} \lambda \eta_{40}^* - \frac{\lambda}{3} \eta_{41}^* + \eta_{50}^* + \eta_{51}^* \right] e^{\lambda(qx+mh+h)}, \quad x = x_n \in G_N.$$

It is clear that the first filtered value of the new initial value problem at the next n^{th} step is the same as the $(q+1)^{\text{st}}$ filtered value of the original initial value problem (i.e. $y_{(q+1)n+m}^{n(*)}(h)$).

Therefore, considering the new initial value problem, by (3.33)

$$\begin{aligned} y_{(q+1)n+m}^{n(*)}(h) &= y_0^* e^{\lambda x} + \left\{ \left[\frac{\lambda}{180} y_0^* x + \frac{1}{2} (\eta_{40}^* + \eta_{41}^*) \right] e^{\lambda x} \right\} h^4 \\ &\quad + \left\{ \left[-\frac{\lambda}{180} y_0^* - \frac{2}{3} \lambda \eta_{40}^* - \frac{\lambda}{3} \eta_{41}^* + \eta_{50}^* + \eta_{51}^* \right] e^{\lambda x} \right\} h^5 \\ &\quad + O(h^6), \quad x = x_n \in G_N. \end{aligned} \quad (3.38)$$

By substituting (3.37) into (3.38) and rearranging,

$$\begin{aligned}
 y_{(q+1)n+m}^{n(*)}(h) &= y_0 e^{\lambda((q+1)x+mh)} \\
 &+ \left\{ \left[\frac{\lambda^5}{180} y_0 ((q+1)x+mh) + \frac{1}{2} (\eta_{40} + \eta_{41}) \right] e^{\lambda((q+1)x+mh)} \right\} h^4 \\
 &+ \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{\lambda}{3} \eta_{41} + \eta_{50} + \eta_{51} \right] e^{\lambda(q+1)x+mh} \right\} h^5 \\
 &+ O(h^6), \quad x = x_n \in G_N,
 \end{aligned} \tag{3.39}$$

then the corollary follows by induction.

Q.E.D.

When $m = 0$ the equation (3.34) becomes

$$y_{Nn}^{n(*)}(h) = y_0 e^{\lambda N x} + \left\{ \left(\frac{\lambda^5}{180} y_0 N x + \frac{1}{2} (\eta_{40} + \eta_{41}) \right) e^{\lambda N x} \right\} h^4 \tag{3.40}$$

$$+ \left\{ \frac{1}{2} \left(-\frac{\lambda^5}{180} y_0 - \frac{2}{3} \lambda \eta_{40} - \frac{\lambda}{3} \eta_{41} + \eta_{50} + \eta_{51} \right) e^{\lambda N x} \right\} h^5 + O(h^6),$$

and $y_{Nn}^{n(*)}(h)$ gives the value at $x_{Nn} = Nnh = Nx$ after filtering at the points $x, 2x, \dots, Nx$.

3.3. EXTRAPOLATION

For any initial value problem, let $y(x;h)$ denote the numerical solution at $x = a + nh$, computed with steplength h . Suppose that $y(x;h)$ has an asymptotic expansion in the form

$$y(x;h) = y(x) + A_p h^p + A_{p+1} h^{p+1} + \dots + A_N h^N + O(h^{N+1}), \quad h \rightarrow 0, \tag{3.41}$$

where $y(x)$ is the exact solution of the initial value problem at the point x , and the constants $A_p, A_{p+1}, \dots, A_N$ are independent of h .

Let $y_i^{(0)} = y(x; h/2^i)$, $i = 0, 1, 2, \dots$.

Then

$$y_0^{(0)} = y(x) + A_p h^p + A_{p+1} h^{p+1} + \dots + A_N h^N + O(h^{N+1}), \quad (3.42)$$

$$y_1^{(0)} = y(x) + A_p (h/2)^p + A_{p+1} (h/2)^{p+1} + \dots + A_N (h/2)^N + O(h^{N+1})$$

and

$$y_2^{(0)} = y(x) + A_p (h/4)^p + A_{p+1} (h/4)^{p+1} + \dots + A_N (h/4)^N + O(h^{N+1}).$$

By taking the linear combination

$$y_1^{(1)} = \frac{2^p y_1^{(0)} - y_0^{(0)}}{2^p - 1} = y(x) - \frac{A_{p+1}}{2(2^p - 1)} h^{p+1} + \dots + \frac{A_N}{(2^p - 1)} \left(\frac{2^p}{2^N} - 1 \right) h^N + O(h^{N+1}),$$

the lowest power, h^p is eliminated in $y_1^{(1)}$, and it is clear that $y_1^{(1)}$ gives a better approximation to $y(x)$ than either $y_0^{(0)}$ or $y_1^{(0)}$.

Similarly taking the linear combination

$$y_2^{(1)} = \frac{2^p y_2^{(0)} - y_0^{(0)}}{2^p - 1} = y(x) - \frac{A_{p+1}}{2(2^p - 1)} \left(\frac{h}{2} \right)^{p+1} + \dots + \frac{A_N}{(2^p - 1)} \left(\frac{2^p}{2^N} - 1 \right) \left(\frac{h}{2} \right)^N + O(h^{N+1})$$

the lowest power $\left(\frac{h}{2}\right)^p$ is eliminated in $y_2^{(1)}$, and it is also clear that $y_2^{(1)}$ gives a better approximation to $y(x)$ than either $y_1^{(0)}$ or $y_2^{(0)}$.

Again the linear combination

$$y_2^{(2)} = \frac{2^{p+1}y_2^{(1)} - y_1^{(1)}}{2^{p+1} - 1} = y(x) + O(h^{p+2})$$

eliminates the lowest power h^{p+1} in $y_2^{(2)}$ and clearly $y_2^{(2)}$ is a better approximation to $y(x)$ than either $y_1^{(1)}$ or $y_2^{(1)}$.

This approximation procedure, by eliminating successively higher powers of h can be continued as far as terms involving h^N ; it can be summarized by the following algorithm:

$$y_n^{(0)} = y(x; h/2^n), \quad n = 0, 1, 2, \dots, \quad (3.43)$$

$$y_n^{(m)} = \frac{2^{m+p-1}y_n^{(m-1)} - y_{n-1}^{(m-1)}}{2^{m+p-1} - 1}, \quad m = 1, 2, \dots, n = m, m+1, \dots,$$

or by the following diagram

$y_0^{(0)}$				
$y_1^{(0)}$	$y_1^{(1)}$			
$y_2^{(0)}$	$y_2^{(1)}$	$y_2^{(2)}$		
$y_3^{(0)}$	$y_3^{(1)}$	$y_3^{(2)}$	$y_3^{(3)}$	
$y_4^{(0)}$	$y_4^{(1)}$	$y_4^{(2)}$	$y_4^{(3)}$	$y_4^{(4)}$

But this algorithm is precisely the Neville procedure for polynomial extrapolation of the original data to the value $h = 0$. It is thus a process of polynomial "extrapolation to the limit" which is justified by the asymptotic expansion (3.41).

3.4 APPLICATION OF EXTRAPOLATION TO THE FILTERED SOLUTION

(a) THE MIDPOINT METHOD

The filtered solutions by the Midpoint method may be extrapolated twice at the point $Nx = Nnh$. Thus, with the notation in Corollary 3.1.a, $y_{Nn}^{n(*)}(h)$, $y_{2Nn}^{n(*)}(\frac{h}{2})$, and $y_{4Nn}^{n(*)}(\frac{h}{4})$ have to be computed with steplengths h , $h/2$ and $h/4$ respectively.

Suppose that the Midpoint method is used with the following starting values,

$$\eta_n(h/2^i) = y(hn/2^i) + \sum_{m=2}^3 \eta_{mn}^{(i)}(h/2^i)^m + O(h^4), \quad i = 0, 1, 2. \quad (3.44)$$

Clearly $i = 0, 1, 2$ are relevant to the steplengths h , $h/2$ and $h/4$ respectively. Since the initial condition is exact

$$\eta_{mn}^{(i)} = 0, \quad n=0, \quad m=2, 3, \quad i=0, 1, 2. \quad (3.45)$$

Thus, by (3.27), (3.44), and (3.45)

$$\begin{aligned} y_{Nn}^{n(*)}(h) &= y_0 e^{\lambda Nx} + \left\{ \left[-\frac{\lambda^3}{6} y_0^{(0)} + \frac{1}{2} \eta_{21}^{(0)} \right] e^{\lambda Nx} \right\} h^2 \\ &\quad + \left\{ \left[\frac{1}{2} \left[\frac{\lambda^3}{6} y_0^{(0)} + \eta_{31}^{(0)} \right] e^{\lambda Nx} \right\} h^3 + O(h^4), \end{aligned} \quad (3.46.0)$$

$$y_{2Nn}^{n(*)}(h/2) = y_0 e^{\lambda Nx} + \left\{ \left[-\frac{\lambda^3}{6} y_0^{Nx} + \frac{1}{2} \eta_{21}^{(1)} \right] e^{\lambda Nx} \right\} (h/2)^2 \quad (3.46.1)$$

$$+ \left\{ \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 + \eta_{31}^{(1)} \right] e^{\lambda Nx} \right\} (h/2)^3 + o(h^4),$$

$$y_{4Nn}^{n(*)}(h/4) = y_0 e^{\lambda Nx} + \left\{ \left[-\frac{\lambda^3}{6} y_0^{Nx} + \frac{1}{2} \eta_{21}^{(2)} \right] e^{\lambda Nx} \right\} (h/4)^2 \quad (3.46.2)$$

$$+ \left\{ \frac{1}{2} \left[\frac{\lambda^3}{6} y_0 + \eta_{31}^{(2)} \right] e^{\lambda Nx} \right\} (h/4)^3 + o(h^4),$$

where $x = nh = 2n(h/2) = 4n(h/4)$.

With the above notations it is clear that

$$y_{Nn}^{n(*)}(h) = y(Nx, h) = y_0^{(0)},$$

$$y_{2Nn}^{n(*)}(h/2) = y(Nx, h/2) = y_1^{(0)},$$

$$y_{4Nn}^{n(*)}(h/4) = y(Nx, h/4) = y_2^{(0)}, \quad x = nh.$$

LEMMA 3.1.a

A necessary and sufficient condition for applying extrapolation to the filtered solution at $Nx = Nnh$ by the Midpoint method is that

$$\eta_{21}^{(0)} = \eta_{21}^{(1)} = \eta_{21}^{(2)}, \quad \eta_{31}^{(0)} = \eta_{31}^{(1)} = \eta_{31}^{(2)}, \quad (3.47)$$

in (3.44).

PROOF

The proof is very clear by examining coefficients of $(h/2^i)^m$,

$i = 0, 1, 2, \quad m = 2, 3$ in (3.46).

Q.E.D.

NOTE

It is not always possible to realize the condition (3.47) unless the exact solution is known. Therefore to obtain appropriate starting values an one-step method of order 4 is suggested. Otherwise an one-step method of lower order could be used with appropriate extrapolations. They would give $\eta_{21}^{(0)} = \eta_{21}^{(1)} = \eta_{21}^{(2)} = 0$ and $\eta_{31}^{(0)} = \eta_{31}^{(1)} = \eta_{31}^{(2)} = 0$.

(b) SIMPSON'S RULE

The filtered solution by Simpson's rule may be extrapolated twice at the point $Nx = Nnh$. Thus, with the notation in Corollary 3.1.b, $y_{Nn}^{n(*)}(h)$, $y_{2Nn}^{n(*)}(h/2)$, and $y_{4Nn}^{n(*)}(h/4)$ have to be computed with steplengths h , $h/2$, $h/4$ respectively.

Suppose that Simpson's rule is used with the starting values

$$\eta_n(h/2^i) = y(hn/2^i) + \sum_{m=4}^5 \eta_{mn}^{(i)}(h/2^i)^m + O(h^6), \quad i = 0, 1, 2. \quad (3.48)$$

Clearly $i = 0, 1, 2$ are relevant to the steplengths h , $h/2$ and $h/4$ respectively. Since the initial condition is exact,

$$\eta_{mn}^{(i)} = 0, \quad n \neq 0, \quad m = 4, 5, \quad i = 0, 1, 2. \quad (3.49)$$

Thus, by (3.40), (3.48) and (3.49)

$$y_{Nn}^{n(*)}(h) = y_0 e^{\lambda Nx} + \left\{ \left[-\frac{\lambda^5}{180} y_0 Nx + \frac{1}{2} \eta_{41}^{(0)} \right] e^{\lambda Nx} \right\} h^4 \quad (3.50.0)$$

$$+ \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{\lambda}{3} \eta_{41}^{(0)} + \eta_{51}^{(0)} \right] e^{\lambda Nx} \right\} h^5 + O(h^6)$$

$$\begin{aligned}
 y_{2Nn}^{n(*)}(h/2) &= y_0 e^{\lambda N x} + \left\{ \left[\frac{\lambda^5}{180} y_0 N x + \frac{1}{2} \eta_{41}^{(1)} \right] e^{\lambda N x} \right\} (h/2)^4 \\
 &+ \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{\lambda}{3} \eta_{41}^{(1)} + \eta_{51}^{(1)} \right] e^{\lambda N x} \right\} (h/2)^5 + O(h^6),
 \end{aligned}
 \tag{3.50.1}$$

and

$$\begin{aligned}
 y_{4Nn}^{n(*)}(h/4) &= y_0 e^{\lambda N x} + \left\{ \left[\frac{\lambda^5}{180} y_0 N x + \frac{1}{2} \eta_{41}^{(2)} \right] e^{\lambda N x} \right\} (h/4)^4 \\
 &+ \left\{ \frac{1}{2} \left[-\frac{\lambda^5}{180} y_0 - \frac{\lambda}{3} \eta_{41}^{(2)} + \eta_{51}^{(2)} \right] e^{\lambda N x} \right\} (h/4)^5 + O(h^6),
 \end{aligned}
 \tag{3.50.2}$$

where $x = nh \triangleq 2n(h/2) = 4n(h/4)$.

LEMMA 3.1.b

A necessary and sufficient condition for applying extrapolation to the filtered solution at $Nx = Nnh$ by Simpson's rule is that

$$\eta_{41}^{(0)} = \eta_{41}^{(1)} = \eta_{41}^{(2)}, \quad \eta_{51}^{(0)} = \eta_{51}^{(1)} = \eta_{51}^{(2)}
 \tag{3.51}$$

in (3.41).

PROOF

The proof is very clear by examining the coefficients of $(h/2^i)^m$,
 $i = 0, 1, 2$, and $m = 4, 5$ in (3.50). Q.E.D.

NOTE

It is not always possible to realize the condition (3.51) unless the exact solution is known. Therefore to obtain appropriate starting values, an one-step method of order 6 is suggested. Otherwise an one-step method of lower order could be used with appropriate extrapolations. They would give $\eta_{41}^{(0)} = \eta_{41}^{(1)} = \eta_{41}^{(2)} = 0$ and $\eta_{51}^{(0)} = \eta_{51}^{(1)} = \eta_{51}^{(2)} = 0$.

CHAPTER 4

NUMERICAL RESULTS

In this Chapter the preceding results are illustrated by their application to several initial value problems, specifically, a linear test equation, a non-linear equation, two systems of linear equations and the echinodome equation.

The test equation and the first system of linear equations were solved by both the Midpoint method and Milne's method. The non-linear single equation was solved by the Midpoint method and the echinodome equation was solved by Milne's method.

In all examples results based on one of the non-symmetric filters are shown together with results based on the corresponding symmetric filter followed by extrapolation. The formulae

$$P_{-2}(E) = \frac{1}{16} (11 + 12E^{-1} - 6E^{-2} - 4E^{-3} + 3E^{-4})$$

and

$$P_{-3}(E) = \frac{1}{64} (57 + 30E^{-1} - 45E^{-2} + 20E^{-3} + 15E^{-4} - 18E^{-5} + 5E^{-6})$$

define the non-symmetric filters used with the Midpoint method and Milne's method respectively.

In every case filters were applied at every 10 steps at steplength $h = 0.1$ in the first four cases and $h = 1$ in the last case. In Milne's method the corrector (Simpson's rule) was iterated three times.

All computations were done in double precision on the AMDAHL 470 V8 computer at the University of Ottawa.

EXAMPLE 1

The test equation is

$$y' = -y, \quad y(0) = 1.$$

The exact solution is $y(x) = e^{-x}$. Results by the Midpoint method and Milne's method are given in Tables 4.1.a and 4.1.b respectively.

EXAMPLE 2

The non-linear single equation is

$$y' = 1 - y^2, \quad y(0) = 0.$$

The exact solution is $y(x) = \tanh x$. This equation was solved by the Midpoint method and results are shown in Table 4.2. This example was taken from [Iri 1964] and used the non-symmetric filter introduced and used in that paper for the same method. It was inserted here to compare Iri's filtered solution with the symmetrically filtered and extrapolated solution.

EXAMPLE 3

The second order differential equation

$$y'' + xy' + y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

was converted to a system of two linear equations

$$y_1' = y_2, \quad y_1(0) = 0,$$

$$y_2' = -y_1 - xy_2, \quad y_2(0) = 1,$$

by letting $y = y_1$, $y' = y_2$. The exact solution is

$$y(x) = \left(-\frac{x^2}{2}\right) \int_0^x e^{t^2/2} dt.$$

This system was solved by Milne's method and the integral $\int_0^x \exp(t^2/2) dt$ was obtained by the IMSL subroutine DCADRE. Results are shown in Table 4.3. This example was also taken from [Iri 1964] and used the non-symmetric filter introduced and used in that paper for the same method.

The purpose of this example is to illustrate the well-known weak instability of Milne's method and its correction by filtering; moreover the error in the nonoscillatory filtered solution is further reduced by extrapolation.

EXAMPLE 4

The system of 4 linear equations is

$$y' = Ay, \quad y(0) = (1, 0, 1, 0)^T,$$

where

$$A = \begin{bmatrix} -4 & 2 & 0 & 2 \\ -4 & 1 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -1 & 1 & -1 & 2 \end{bmatrix}.$$

The exact solution is

$$y_1 = e^{-t} \cos t - 3e^{-t} \sin t,$$

$$y_2 = -3e^{-t} \sin t,$$

$$y_3 = e^{-t} \cos t - 2e^{-t} \sin t,$$

$$y_4 = -2e^{-t} \sin t.$$

This example was taken from [Milne and Reynolds 1960] and results are shown in Tables 4.4.a and 4.4.b for the Midpoint method and Milne's method respectively.

EXAMPLE 4

The echinodome equation

$$y' + \frac{y}{x} = \epsilon z, \quad \epsilon > 0 \text{ small},$$

$$z' = \frac{y}{\sqrt{1-y^2}},$$

was solved with steplength $h = 1$ by Milne's method with the symmetric filter P_0 and the extrapolation (SE) and the non-symmetric filter P_{-3} . It was also solved by the explicit improved/modified Euler method (EIME). The initial conditions were $y(0.001) = 1000$ and $z(0.001) = 0.000\ 009\ 68$. Finally an approximate solution with error $0.000\ 001$ was obtained for comparison purposes. This example was taken from [Sofoluwe et al., 1981]. Near the singularity of the solution, EIME gave a better solution than SE, while otherwise SE gave a better solution. Results are shown in Table

4.5.

In tables 4.1 to 4.5 the following notation is used:

- x - value of the independent variable
- TR - exact (true) solution at x
- NU - numerical solution at x without filtering
- NS - solution at x by the numerical method with the non-symmetric filter
- SE - solution at x by the numerical method with the symmetric filter followed by extrapolation.
- E%(NS) - percentage error at x by NS
- E%(SE) - percentage error at x by SE
- APPROX - approximate value to 10^{-6}
- EIME - numerical solution by the explicit improved/modified Euler method

In Tables 4.4.a and 4.4.b the four values indicated with each value of x from top to bottom are the values of the components y_1 , y_2 , y_3 and y_4 respectively. In Table 4.5 the pair of values are y and z from top to bottom.

MIDPOINT METHOD

THE TEST EQUATION

$$Y' = -Y, \quad Y(0) = 1$$

X	TR	NU	NS	SE	E%(NS)	E%(SE)
1.0	0.3678794D+00	0.3686655D+00	0.3686870D+00	0.3678794D+00	0.2195D+00	-0.8443D-05
2.0	0.1353353D+00	0.1363251D+00	0.1359384D+00	0.1353353D+00	0.4457D+00	-0.1129D-05
3.0	0.4978707D-01	0.5152470D-01	0.5012209D-01	0.4978707D-01	0.6729D+00	-0.3073D-05
4.0	0.1831564D-01	0.2248770D-01	0.1848061D-01	0.1831564D-01	0.9007D+00	0.3523D-06
5.0	0.6737947D-02	0.1778836D-01	0.6814023D-02	0.6737948D-02	0.1129D+01	0.8628D-05
6.0	0.2478752D-02	0.3234076D-01	0.2512412D-02	0.2478753D-02	0.1358D+01	0.2179D-04
7.0	0.9118820D-03	0.8189445D-01	0.9263562D-03	0.9118823D-03	0.1587D+01	0.3986D-04
8.0	0.3354626D-03	0.2200797D+00	0.3415586D-03	0.3354628D-03	0.1817D+01	0.6285D-04
9.0	0.1234098D-03	0.5964497D+00	0.1259367D-03	0.1234099D-03	0.2048D+01	0.9078D-04

TABLE 4.1 - a

MILNE'S METHOD

THE TEST EQUATION

$$Y' = Y, Y(0) = 1$$

X	TR	NU	NS	SE	E%(NS)	E%(SE)
1.0	0.3678794D+00	0.3678793D+00	0.3678802D+00	0.3678794D+00	0.2099D-03	-0.1705D-09
2.0	0.1353353D+00	0.1353351D+00	0.1353358D+00	0.1353353D+00	0.4062D-03	-0.2282D-09
3.0	0.4978707D-01	0.4978694D-01	0.4978737D-01	0.4978707D-01	0.6025D-03	-0.2723D-09
4.0	0.1831564D-01	0.1831553D-01	0.1831579D-01	0.1831564D-01	0.7988D-03	-0.3164D-09
5.0	0.6737947D-02	0.6737824D-02	0.6738014D-02	0.6737947D-02	0.9951D-03	-0.3605D-09
6.0	0.2478752D-02	0.2478598D-02	0.2478782D-02	0.2478752D-02	0.1191D-02	-0.4046D-09
7.0	0.9118820D-03	0.9116742D-03	0.9118946D-03	0.9118820D-03	0.1388D-02	-0.4485D-09
8.0	0.3354626D-03	0.3351761D-03	0.3354679D-03	0.3354626D-03	0.1584D-02	-0.4924D-09
9.0	0.1234098D-03	0.1230115D-03	0.1234120D-03	0.1234098D-03	0.1780D-02	-0.5363D-09

TABLE 4.1 - b

THE MIDPOINT METHOD
THE FIRST ORDER NON-LINEAR DIFFERENTIAL EQUATION
 $Y' = 1 - Y^2$, $Y(0) = 0$

X	TR	NU	NS	SE	E%(NS)	E%(SE)
1.0	0.7615942D+00	0.7623121D+00	0.7617390D+00	0.7615940D+00	0.1902D-01	-0.2108D-04
2.0	0.9640276D+00	0.9660661D+00	0.9635591D+00	0.9640277D+00	-0.4859D-01	0.1158D-04
3.0	0.9950548D+00	0.1011244D+01	0.9949010D+00	0.9950548D+00	-0.1545D-01	0.8098D-06
4.0	0.9993293D+00	0.1115472D+01	0.9992963D+00	0.9993293D+00	-0.3301D-02	0.1205D-06
5.0	0.9999092D+00	0.1700712D+01	0.9999032D+00	0.9999092D+00	-0.5968D-03	-0.5188D-07
6.0	0.9999877D+00	0.8508360D+00	0.9999868D+00	0.9999877D+00	-0.9595D-04	-0.1407D-07
7.0	0.9999983D+00	-0.7466921D+00	0.9999982D+00	0.9999983D+00	-0.1334D-04	-0.4279D-08
8.0	0.9999998D+00	-0.9637723D+00	0.9999998D+00	0.9999998D+00	-0.1311D-05	-0.8482D-09
9.0	0.1000000D+01	-0.9774040D+00	0.1000000D+01	0.1000000D+01	0.6193D-07	-0.1803D-09

TABLE 4.2

MILNE'S METHOD
THE SECOND ORDER NON-LINEAR DIFFERENTIAL EQUATION

$$Y'' + X*Y' + Y = 0, \quad Y(0) = 0, \quad Y'(0) = 1$$

X	TR	NU	NS	SE	E%(NS)	E%(SE)
1.0	0.7247779D+00	0.7247770D+00	0.7247774D+00	0.7247784D+00	-0.6553D-04	0.6756D-04
2.0	0.6399877D+00	0.6399839D+00	0.6400018D+00	0.6399881D+00	0.2215D-02	0.6326D-04
3.0	0.3931665D+00	0.3931657D+00	0.3931698D+00	0.3931669D+00	0.8315D-03	0.8736D-04
4.0	0.2703963D+00	0.2703958D+00	0.2703987D+00	0.2703963D+00	0.8955D-03	-0.5021D-05
5.0	0.2092460D+00	0.2092453D+00	0.2092479D+00	0.2092458D+00	0.9171D-03	-0.1160D-03
6.0	0.1717503D+00	0.1717489D+00	0.1717518D+00	0.1717501D+00	0.8887D-03	-0.1326D-03
7.0	0.1459725D+00	0.1459658D+00	0.1459740D+00	0.1459725D+00	0.1029D-02	0.1033D-04
8.0	0.1270530D+00	0.1270015D+00	0.1270540D+00	0.1270527D+00	0.8034D-03	-0.2156D-03
9.0	0.1125373D+00	0.1122026D+00	0.1125382D+00	0.1125371D+00	0.8661D-03	-0.1531D-03

TABLE 4.3

MIDPOINT METHOD

THE SYSTEM OF FOUR LINEAR DIFFERENTIAL EQUATIONS

X	TR	NU	NS	SE	E%(NS)	E%(SE)
1.0	-0.7299135D+00	-0.7254533D+00	-0.7228631D+00	-0.7299123D+00	-0.9659D+00	-0.1601D-03
	-0.9286796D+00	-0.9239962D+00	-0.9220024D+00	-0.9286785D+00	-0.7190D+00	-0.1198D-03
	-0.4203536D+00	-0.4174546D+00	-0.4155290D+00	-0.4203528D+00	-0.1148D+01	-0.1897D-03
	-0.6191198D+00	-0.6159974D+00	-0.6146683D+00	-0.6191190D+00	-0.7190D+00	-0.1198D-03
3.0	-0.7036668D-01	-0.8022512D-01	-0.7116530D-01	-0.7036672D-01	0.1135D+01	0.6525D-04
	-0.2407785D-01	-0.3361703D-01	-0.2265426D-01	-0.2107785D-01	0.7479D+01	-0.5913D-05
	-0.6334073D-01	-0.6901945D-01	-0.6361388D-01	-0.6334077D-01	0.4312D+00	0.7314D-04
	-0.1405190D-01	-0.2241135D-01	-0.1510284D-01	-0.1405190D-01	0.7479D+01	-0.5913D-05
5.0	0.2129484D-01	0.1177594D+00	0.2078201D-01	0.2129489D-01	-0.2408D+01	0.2004D-03
	0.1938354D-01	0.9694409D-01	0.1905105D-01	0.1938357D-01	-0.1715D+01	0.1298D-03
	0.1483366D-01	0.8544474D-01	0.1443166D-01	0.1483370D-01	-0.2710D+01	0.2311D-03
	0.1292236D-01	0.6462939D-01	0.1270070D-01	0.1292238D-01	-0.1715D+01	0.1298D-03
7.0	-0.1109813D-02	-0.8468583D-01	-0.9958020D-03	-0.1109831D-02	-0.1027D+02	0.1644D-02
	-0.1797283D-02	0.1485971D+00	-0.1678252D-02	-0.1797300D-02	-0.6623D+01	0.9377D-03
	-0.5107186D-03	-0.1342182D+00	-0.4363846D-03	-0.5107312D-03	-0.1455D+02	0.2473D-02
	-0.1198188D-02	0.9906471D-01	-0.1118835D-02	-0.1198200D-02	-0.6623D+01	0.9377D-03
9.0	-0.2650208D-03	-0.4846765D+01	-0.2645275D-03	-0.2650188D-03	-0.1861D+00	-0.7646D-03
	-0.1525784D-03	-0.5220308D+01	-0.1587105D-03	-0.1525754D-03	0.4019D+01	-0.1967D-02
	-0.2141613D-03	-0.3106662D+01	-0.2116240D-03	-0.2141603D-03	-0.1185D+01	-0.4790D-03
	-0.1017189D-03	-0.3480206D+01	-0.1058070D-03	-0.1017169D-03	0.4019D+01	-0.1967D-02

TABLE 4.4 - a

MILNE'S METHOD
THE SYSTEM OF FOUR LINEAR DIFFERENTIAL EQUATIONS

X	TR	NU	NS	SE	E%(NS)	E%(SE)
4.0	-.7299135D+00 -.9286796D+00 -.4203536D+00 -.6191198D+00	-.7299128D+00 -.9286800D+00 -.4203529D+00 -.6191199D+00	-.7299198D+00 -.9286817D+00 -.4203593D+00 -.6191210D+00	-.7299135D+00 -.9286796D+00 -.4203536D+00 -.6191198D+00	0.8550D-03 0.2197D-03 0.1352D-02 0.2075D-03	0.1988D-08 0.2525D-07 0.2742D-07 0.9456D-08
3.0	-.7036668D-01 -.2107785D-01 -.6334073D-01 -.1405190D-01	-.7036694D-01 -.2107785D-01 -.6334097D-01 -.1405195D-01	-.7036131D-01 -.2107292D-01 -.6333698D-01 -.1404861D-01	-.7036668D-01 -.2107785D-01 -.6334073D-01 -.1405190D-01	-.0.7627D-02 -.0.2341D-01 -.0.5923D-02 -.0.2346D-01	0.1011D-06 0.1556D-06 0.2675D-07 0.6091D-07
5.0	0.2129484D-01 0.1938354D-01 0.1483366D-01 0.1292236D-01	0.2129655D-01 0.1938570D-01 0.1483469D-01 0.1292365D-01	0.2129442D-01 0.1938284D-01 0.1483348D-01 0.1292188D-01	0.2129484D-01 0.1938354D-01 0.1483366D-01 0.1292236D-01	-.0.1972D-02 -.0.3645D-02 -.0.1230D-02 -.0.3747D-02	0.1725D-06 0.2488D-06 0.1635D-06 0.9421D-07
7.0	-.1109813D-02 -.1797283D-02 -.5107186D-03 -.1198188D-02	-.1109340D-02 -.1795543D-02 -.5107065D-03 -.1197231D-02	-.1109974D-02 -.1797381D-02 -.5108484D-03 -.1198252D-02	-.1109813D-02 -.1797283D-02 -.5107186D-03 -.1198188D-02	0.1448D-01 0.5444D-02 0.2542D-01 0.5297D-02	0.2285D-06 0.3456D-06 0.6095D-06 0.1315D-06
9.0	-.2650208D-03 -.1525784D-03 -.2141613D-03 -.1017189D-03	-.2690344D-03 -.1541162D-03 -.2174355D-03 -.1029442D-03	-.2649839D-03 -.1525412D-03 -.2141367D-03 -.1016939D-03	-.2650208D-03 -.1525784D-03 -.2141613D-03 -.1017189D-03	-.0.1391D-01 -.0.2439D-01 -.0.1152D-01 -.0.2459D-01	0.3554D-06 0.5020D-06 0.1804D-06 0.1877D-06

TABLE 4.4 - b

MILNE'S METHOD

THE ECHINODOME EQUATION

X	APPROX	EIME	NU	NS	SE
10.	0.100048504D+04	0.100048506D+04	0.100055302D+04	0.100054694D+04	0.100048961D+04
10.	0.967831305D-01	0.967864553D-01	0.960642965D-01	0.991203714D-01	0.968869933D-01
20.	0.100195463D+04	0.100195476D+04	0.100205093D+04	0.100203285D+04	0.100195995D+04
20.	0.193698179D+00	0.193711240D+00	0.191063478D+00	0.194856914D+00	0.193750834D+00
30.	0.100445505D+04	0.100445545D+04	0.100457670D+04	0.100454364D+04	0.100446084D+04
30.	0.290901261D+00	0.290930816D+00	0.287344875D+00	0.291687338D+00	0.290937250D+00
40.	0.100807212D+04	0.100807309D+04	0.100822052D+04	0.100816896D+04	0.100807829D+04
40.	0.388549381D+00	0.388602895D+00	0.384373148D+00	0.389154676D+00	0.388577372D+00
50.	0.101294818D+04	0.101295025D+04	0.101312732D+04	0.101305258D+04	0.101295470D+04
50.	0.486817866D+00	0.486903989D+00	0.482165211D+00	0.487319638D+00	0.486841348D+00
60.	0.101931746D+04	0.101932156D+04	0.101953462D+04	0.101942966D+04	0.101932435D+04
60.	0.585912419D+00	0.586041789D+00	0.580866892D+00	0.586349767D+00	0.585933161D+00
70.	0.102758630D+04	0.102759433D+04	0.102785497D+04	0.102770759D+04	0.102759362D+04
70.	0.686090635D+00	0.686277421D+00	0.680706771D+00	0.686486455D+00	0.686109672D+00
80.	0.103855104D+04	0.103856773D+04	0.103890034D+04	0.103868490D+04	0.103855897D+04
80.	0.787707408D+00	0.787972917D+00	0.782023249D+00	0.788076820D+00	0.787725429D+00

TABLE 4.5

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ABSTRACT

The accuracy of a numerical solution of an initial value problem

$$y' = f(x, y) \quad y(0) = y_0$$

obtained by a linear multistep method of order p whose first characteristic polynomial is $\rho(z) = z^2 - 1$ may be improved by the combined application of processes of filtering and extrapolation. These processes are derived and justified by means of asymptotic expansions

$$y_n(h) = y(x) + \sum_{m=p}^{N-1} (e_{m1}(x) + (-1)^n e_{m2}(x)) h^m + O(h^N)$$

of the global analytic truncation error in the numerical solution. The combined process is illustrated by numerical solutions based upon the Midpoint and Milne-Simpson integration formulae applied, among others, to the linearized test equation

$$y' = \lambda y, \quad y(0) = y_0.$$

The filter used is shown to be more effective than one proposed by M. Iri [A stabilizing device for unstable numerical solutions of ordinary differential equations - Design principle and application of a "filter", Inf. Proc. Japan 4 (1964), pp. 65-73] and makes possible the application of two steps of polynomial extrapolation [see W. Gragg, Repeated Extrapolation to the Limit in the Numerical Solution of Ordinary Differential Equations, Ph.D. Thesis, UCLA, 1963].