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Adela-Nicoleta COMANICI
AUTEUR DE LA THÈSE - AUTHOR OF THESIS

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Department of Mathematics
FACULTÉ, ÉCOLE, DÉPARTEMENT - FACULTY, SCHOOL, DEPARTMENT

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V.G. Leblanc
DIRECTEUR DE LA THÈSE - THESIS SUPERVISOR

CO-DIRECTEUR DE LA THÈSE - THESIS CO-SUPERVISOR

EXAMINATEURS DE LA THÈSE - THESIS EXAMINERS

Y. Bourgault

B. Dionne

S. Melkonian

W. Nagata

J.-M. De Koninck, Ph.D.

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SPIRAL WAVES ON SPHERICAL DOMAINS: A DYNAMICAL SYSTEMS APPROACH

By
Adela N. Comanici, B.Sc., M.Sc., Ph.D.
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A Thesis
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Abstract

This thesis is concerned with the dynamics and bifurcations of spiral waves in excitable media with spherical or approximately spherical geometry (e.g. cardiac tissue).

First, we study parameter-dependent systems of reaction-diffusion partial differential equations on a sphere which are equivariant under the group $SO(3)$ of all rigid rotations on a sphere. Two main types of spatial-temporal patterns that can appear in such systems are rotating waves (equilibrium in a co-rotating frame) and modulated rotating waves (periodic solution in a co-rotating frame). The transition from rotating waves to modulated rotating waves on spherical domains is explained via a supercritical Hopf bifurcation from a rotating wave and $SO(3)$ symmetry. The Baker-Campbell-Haussdorff formula in the Lie algebra $so(3)$ is used to get a formula for a primary frequency vector, as well as a formula for the periodic part associated to any modulated rotating wave obtained by a supercritical Hopf bifurcation from a rotating wave. In the resonant case, the primary frequency vector of the modulated rotating wave is generically orthogonal to the frequency vector of the initial rotating wave.

In the second part of the thesis, we study the effects of forced symmetry-breaking from $SO(3)$ to $SO(2)$ for a normally hyperbolic relative equilibrium. This is done by introducing a small $SO(2)$-equivariant perturbation into the above reaction-diffusion system. The relative equilibrium persists to a normally hyperbolic $SO(2)$-invariant manifold that is $SO(2)$-equivariant diffeomorphic to $SO(3)$. The perturbed $SO(2)$-equivariant dynamics on this manifold are studied by using the projection onto the orbit space $SO(3)/SO(2)$. Depending on the frequency vectors of the rotating waves that form the relative equilibrium, these rotating waves (up to $SO(2)$) will give either $SO(2)$-orbits of rotating waves or $SO(2)$-orbits of modulated rotating waves (if some transversality conditions hold). The

orbital stability of these solutions is established as well.

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Last, but certainly not least, I would like to thank my parents and my husband who have been a great source of encouragement through these years.

Dedications

For my parents Florica and Nicolae, my son Lucas Alexandru, my husband Daniel.

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Chapter 1

Introduction

1.1 Literature Review

The subject of this thesis is the spatio-temporal dynamics of spiral waves in excitable media with spherical or approximately spherical geometry. The study of dynamics and bifurcations of spiral waves is very important, because, for example, it may provide a clue to the cause of certain cardiac arrhythmias, which can lead to ventricular fibrillation.

There has been considerable activity in recent years, both mathematical and experimental (numerical and physical), on spiral waves and scroll waves in excitable media. Spiral waves and scroll waves are spatio-temporal patterns that have been observed in numerous physical situations, ranging from Belousov-Zhabotinsky chemical reactions [42], [88], to cardiac tissue [14], to slime-mold aggregates [25].

In 1946, Wiener and Rosenblueth [80] introduced the concept of excitable media to explain cardiac arrhythmias caused by spiral waves. They introduced the notions of refractory, excitable and excited states and described pinned spiral waves geometrically. In 1952, Turing [76] proposed a mechanism for the formation of patterns in reaction-diffusion systems. Spatio-temporal pattern formation in chemical systems was first found by Zaikin and Zhabotinsky [88]. They observed target patterns. Two years later, Winfree [81] found spiral waves in the Belousov-Zhabotinsky reaction.

There is a well-developed physical theory of rigidly rotating waves, namely the kinematical theory, which describes the movements of the spiral wave in terms of the local curvature of the wave front. The basic assumptions of the kinematical theory are a linear dependence of the normal propagation velocity on the curvature (eikonal equation) and a linear dependence of the tangential velocity at the free end of the curvature.

As we outline below, over the past few years, a mathematical theory for planar spiral waves has been developed. However, for scroll waves in $\mathbb{R}^3$ and spiral waves on spherical domains, there are more numerical results than mathematical results.

The equivariant dynamical systems approach to spiral waves was initiated by Barkley [4] who performed a numerical linear stability analysis for the basic-time periodic spiral wave solution in a reaction-diffusion system on the unbounded plane and showed evidence of a Hopf bifurcation. In particular, a simple pair of eigenvalues was shown to cross the imaginary axis while three neutral eigenvalues lie on the imaginary axis and the remainder of the spectrum is bounded into the left-half plane. Later, using an ad hoc model, Barkley [5] was the first to realize the key importance of the group $SE(2)$ of all planar translations and rotations in describing the dynamics and bifurcations of planar spiral waves. The first rigorous mathematical theory of planar spiral waves was carried out by Wulff [84]. In her thesis, Wulff studied the external periodic forcing of rotating waves which leads to modulated rotating waves or modulated travelling waves, and also proved a $SE(2)$-equivariant Hopf theorem for the bifurcation from rotating waves to modulated (rotating or travelling) waves in autonomous systems. The external periodic forcing of rotating waves was studied using a contraction mapping theorem on scales of Banach spaces, and the proof for the $SE(2)$-equivariant Hopf theorem for rotating waves was based on Liapunov-Schmidt reduction on scales of Banach spaces. In both cases, it is shown that modulated travelling waves emanate if the rotation frequency is a multiple of the external frequency, respectively the modulus of the eigenvalue leading to Hopf bifurcation. The main difficulty comes from the fact that the group $SE(2)$ is noncompact and the action of $SE(2)$ on the usual spaces of functions is not smooth. The proofs in [84] are based on the basic assumption that the linearization at the rotating wave in the co-rotating frame does not exhibit continuous spectrum near the imaginary axis.

Later, Sandstede, Scheel and Wulff [70] proved a finite-dimensional center bundle reduction theorem near a relative equilibrium Gu_0 of an infinite-dimensional vector field on a Banach space X on which acts a finite-dimensional Lie group (not necessarily compact). This generalizes Krupa's results on bifurcation from relative equilibria [46], from compact groups to noncompact groups and from finite dimensions to infinite dimensions. Using the results of [70], the Hopf bifurcation from one-armed rotating spiral wave to meandering waves can be studied [26], [70].

In the case of a one-armed rotating spiral wave, the relative equilibrium $SE(2)u_0$ is diffeomorphic to $SE(2)$, which is diffeomorphic to $\mathbb{C} \times \mathbf{S}^1$. The reduced differential equations on the center bundle $SE(2) \times \mathbb{C}$ are given by

$$\begin{aligned} \dot{p} &= e^{i\phi} f(q, \lambda), \\ \dot{\phi} &= F^{\phi}(q, \lambda), \\ \dot{q} &= F^{q}(q, \lambda), \end{aligned} \tag{1.1.1}$$

where $F^{\phi}(0,0) = \omega_{rot}$, $F^{q}(0,0) = 0$, $f(0,0) = 0$. The rotating wave u_0 corresponds to $q = 0$ at $\lambda = 0$ and to the solution $(0, \omega_{rot} t)$ for the first two differential equations in (1.1.1).

In case of a supercritical Hopf bifurcation we have $d_q F^q(0,0) = i\omega_2$, $Re\,(d_q F^q_\lambda(0,0)) \neq 0$. It follows that for any small $\lambda > 0$ there is a meandering spiral wave which becomes a drifting linear wave if $\omega_1(\lambda) = k\omega_2(\lambda)$, for some $k \in \mathbb{Z}$, where $\omega_2(\lambda)$ is the frequency that appears due to the Hopf bifurcation and $\omega_1(\lambda) = \frac{|\omega_2(\lambda)|}{2\pi} \int_0^{\frac{2\pi}{|\omega_2(\lambda)|}} F^{\phi}(q(t,\lambda),\lambda)\,dt$, and $q(t,\lambda)$ is the $\frac{2\pi}{|\omega_2(\lambda)|}$ periodic solution of the third differential equation in (1.1.1) that appears due to the supercritical Hopf bifurcation for $q = 0$ at $\lambda = 0$.

Center bundle reduction has been used to study other bifurcations of spiral and scroll waves in spatially extended systems [71].

The idea of a finite-dimensional center manifold reduction and the skew-product form of the reduced differential equations on this finite-dimensional center manifold are extended to relative periodic orbits in [47] and [71].

In [85], a G-equivariant semilinear system of parabolic equations (where G is a finite-dimensional possibly non-compact Lie group) is studied. In particular, the periodic forcing of relative equilibria and resonant periodic forcing of relative equilibria to relative periodic orbits, as well as Hopf bifurcation from relative equilibria to relative periodic orbits are treated using Liapunov-Schmidt reduction. Resonant drift phenomena are also studied. Then, these results are applied to the planar spiral waves.

For the skew-product finite-dimensional system of differential equations on the center manifold near a relative equilibrium [20], the normal form method which further simplifies the system is presented in [21]. The normal form for the case $G = SE(N)$ is obtained. Then, the known results regarding the meandering and drifting of planar spiral waves are recovered, as well as new results regarding the relative homoclinic and heteroclinic trajectories to relative equilibria of $SE(2)$-actions.

In [66], Scheel shows that for a large class of reaction-diffusion systems on the plane, m-armed spiral waves bifurcate from a homogeneous equilibrium when the latter undergoes a Hopf bifurcation. This was done using spatial dynamics.

Aswhin and Melbourne [2] have also extended to the non-compact case the results of Field [22] and Krupa [46] regarding the sharp upper bounds on the drifts associated with relative equilibria and relative periodic orbits in equivariant dynamical systems with compact Lie groups G. For the case G compact, it is known that for relative equilibria consisting of points of trivial isotropy, the drifts correspond to tori in G and generically these are maximal tori. Analogous results hold when there is a nontrivial isotropy subgroup Σ, with G replaced by $N(\Sigma)/\Sigma$, where $N(\Sigma)$ is the normalizer of Σ. Similar results are valid for relative periodic orbits, with maximal tori generalized to *Cartan subgroups*. Thus, if G is noncompact, the drifts correspond to tori or lines (unbounded copies of $\mathbb{R}$) in G and generically these are maximal tori or lines. Which of these drifts are preferred, compact or unbounded depends on G. These results were then used to obtain predictions for the drifting of the scroll solutions considered by Winfree and Strogatz [82].

In [27], it was argued that not only is Euclidean symmetry sufficient to observe meandering in physical space, it is also, in some sense, necessary. In fact, Hopf bifurcation from rotating wave to modulated rotating wave occur in many different physical systems, but the manifestation of this transition in physical space depends on the underlying symmetry group. In this sense, the translation symmetry is necessary in order to observe meandering.

The Euclidean-equivariant center bundle approach has also been used to try to explain the so called transition from rigidly rotating waves to linearly translating retracting waves. Aswhin, Melbourne and Nicol [3] proposed that this transition is a *drift bifurcation*.

In summary, the Euclidean-equivariant center bundle approach has been remarkably successful at explaining many of the experimentally observed dynamics and bifurcations of spiral waves.

However, the Euclidean symmetry is an approximation, since no physical experiment is infinite in spatial extent. There are indeed experiments on planar spiral waves which exhibit some phenomena which cannot be explained by Euclidean symmetry alone: boundary drifting [86], [89], spiral anchoring on localized inhomogeneities [14], [57] and repelling by localized inhomogeneities [57].

Boundary drifting has been observed in situations where size of the spiral core is comparable to the size of the spatial domain of the experiment. In this case, the spiral is attracted to the boundary of the domain and then drifts around the boundary in a meandering motion. The spiral waves with smaller cores do not drift around the boundary: the spiral tip goes straight through the boundary without any impediment [50]. When local inhomogeneities are present, the spiral is attracted to these inhomogeneities, migrate towards them, and then rotate around them. Some spirals are also repelled by these inhomogeneities [57].

There are also experiments [36] in which two dynamic attractors, called *entrainment attractor* and *resonance attractor* are seen, which attractor is observed depends on the initial conditions of the experiment. As has been shown in LeBlanc and Wulff [48], these phenomena are generic consequences of imperfect Euclidean symmetry. Specifically, they have studied the effects of translation symmetry breaking on normally hyperbolic relative equilibria and normally hyperbolic relative periodic orbits in general systems of Euclidean-equivariant differential equations which undergo a small perturbation that breaks the translational symmetry, while preserving rotational symmetry. For the case of normally hyperbolic relative equilibria, LeBlanc and Wulff proved that the relevant perturbed dynamics are described by a general class of ordinary differential equations on $\mathbb{C} \times \mathbf{S}^1$ of the form

$$\begin{aligned} \dot{p} &= e^{i\phi}[v + \epsilon G^p(pe^{-i\phi}, \overline{p}e^{i\phi}, \epsilon)], \\ \dot{\phi} &= \omega_{rot} + \epsilon G^\phi(pe^{-i\phi}, \overline{p}e^{i\phi}, \epsilon), \end{aligned} \tag{1.1.2}$$

where $v \in \mathbb{C}$ is constant and ω_{rot} is a real constant, G^ϕ, G^p are sufficiently smooth and $0 < \epsilon \ll 1$.

They introduce the following change of coordinates $w = pe^{-i\phi}$ and the system (1.1.2) becomes

$$\begin{aligned} \dot{w} &= v - i\omega_{rot} + \epsilon[G^p(w, \overline{w}, \epsilon) - iwG^\phi(w, \overline{w}, \epsilon)], \\ \dot{\phi} &= \omega_{rot} + \epsilon G^\phi(w, \overline{w}, \epsilon), \end{aligned} \tag{1.1.3}$$

where it was assumed that $v \neq 0$. Let

$$a_1 = G^p_w(\frac{v}{i\omega_{rot}}, -\frac{\overline{v}}{i\omega_{rot}}, 0) - iG^\phi(\frac{v}{i\omega_{rot}}, -\frac{\overline{v}}{i\omega_{rot}}, 0) - \frac{v}{i\omega_{rot}} G^\phi_w(\frac{v}{i\omega_{rot}}, -\frac{\overline{v}}{i\omega_{rot}}, 0),$$

$$h(w, \overline{w}, \epsilon) = G^p(w, \overline{w}, \epsilon) - iwG^\phi(w, \overline{w}, \epsilon),$$

$$H^w(z, \overline{z}, \epsilon) = h(z + \frac{v}{i\omega_{rot}}, \overline{z} - \frac{\overline{v}}{i\omega_{rot}}, \epsilon),$$

and

$$I(\rho) = Re[\int_0^{2\pi} e^{-ist} H^w(\rho e^{ist}, \rho e^{-ist}, 0)\, dt], \text{ where } s = -sgn(\omega_{rot}) = \pm 1.$$

If $\omega_{rot} \neq 0$, the following theorems are obtained in [48]:

Theorem 1.1.1. *There exists a sufficiently smooth branch of equilibrium points* $w(\epsilon)$ $(\epsilon \geq 0)$ *of the first equation in* (1.1.3) *such that* $w(0) = \frac{v}{i\omega_{rot}}$.
Generically, $Re(a_1) \neq 0$ *(that is the equilibrium* $w(\epsilon)$ *of the first equation in* (1.1.3) *is generically hyperbolic).*
For small $\epsilon > 0$*, the equilibrium* $w(\epsilon)$ *is locally asymptotically stable (respectively unstable) if* $Re(a_1) < 0$ *(respectively* $Re(a_1) > 0$*).*

Theorem 1.1.2. *Suppose* $\rho_0 > 0$ *is such that* $I(\rho_0) = 0$ *and* $I'(\rho_0) = \mu \neq 0$*. Then for all* $\epsilon > 0$ *small, the first equation in* (1.1.3) *has a periodic solution* $w(t, \epsilon)$ *which depends continuously on* ϵ*, and which is such that* $w(t, 0) = \rho_0 e^{-i\omega_{rot}t} + \frac{v}{i\omega_{rot}}$.
Furthermore, this periodic solution is locally asymptotically stable (respectively unstable) if $\mu < 0$ *(respectively* $\mu > 0$*).*

In [48], these theorems are then translated to the phase space (p, ϕ)—the equilibrium $w(\epsilon)$ corresponds to the $SO(2)$ orbit of a rotating wave that rotates around the origin with frequency $\omega_{rot} + O(\epsilon)$ and the periodic solution $w(t, \epsilon)$ (when it exists) corresponds to the $SO(2)$-orbit of a modulated rotating wave around the origin with the frequencies $O(\epsilon)$ and $\omega_{rot} + O(\epsilon)$.
Similar results have been obtained in [48] for the case $\omega_{rot} = 0$ (travelling waves) and for normally hyperbolic relative periodic orbits.
Both numerical and experimental work suggest that anisotropy also can lead to certain dynamical states for spiral waves which are inconsistent with Euclidean symmetry. Anisotropy can lead to phase-locked two-frequency epicycle spiral motions, and to complicated quasi-periodic meandering patterns which have overall discrete rotational symmetries. LeBlanc [49] has investigated the effects the forced rotational symmetry breaking on spiral wave dynamics. Namely, LeBlanc studied the dynamics of normally hyperbolic relative equilibria and normally hyperbolic relative periodic solutions in systems which are perturbations of $SE(2)$-equivariant systems, such that the perturbation is symmetric under translations, but its rotational symmetries are reduced to the cyclic subgroup $\mathbb{Z}_l \subset SO(2)$, where $l > 0$ is an integer.
All of these previously mentioned studies are for planar spiral waves. As mentioned earlier, spiral waves also occur in the electrical potential of cardiac tissue. The global geometry of the heart is closer to a sphere than to a plane, therefore, it would be important to study spiral wave dynamics in a context where the symmetries are that of a sphere instead of a plane.

Some numerical and physical experiments have been done for spiral waves on spherical domains (see [9], [30], [31], [32], [54], [55], [56], [75], [87], [89]). The rigidly rotating waves on a sphere have also been studied using the eikonal equation as well. The eikonal equation encapsulates the essential geometry of the spiral and simple scroll waves in three dimensions. Below we present a brief description of the main numerical and physical experiments done for spiral waves on the sphere.

In [30], numerical integration of an excitable reaction-diffusion system on a sphere is presented. The evolution of counter-rotating double spiral waves on the sphere is studied and it is shown that the tips of the spiral can either perform a meandering motion or rigidly rotate around a fixed center, depending on the system control parameter. It is observed that the rotation of the spiral wave on spherical surface is similar to that obtain on the planar surface, except that in the absence of the boundary on a spherical surface some parts of the wave can undergo self-anihilation in contrast to the spiral wave behavior on bounded planar surfaces.

In [89], the evolution of spiral waves on a circular domain and on a spherical surface is studied by numerical integration of a reaction-diffusion system. Two different asymptotic regimes are observed for both domains. The first regime is a rigid rotation of an excitation wave around the symmetry axis of the domain. The second one is a compound rotation including a drift of the rotation center of the spiral wave either along the boundary of the disk or along the equator of the sphere. In this case the shape of the wave and its rotation velocity are periodically changing in time. Simplified analytical estimates are presented to describe the rigid rotation.

In [10], the evolution of scroll waves in excitable media with spherical geometries is studied as a function of shell thickness and outer radius. There exists a critical size of the thin spherical shell below which self-sustained spiral waves cannot exist. The minimum size of the sphere that supports waves and maximum number of spiral waves that can be sustained on a sphere of a given size are determined for both regular and random initial distributions. When the inner radius is too small to support spiral waves, the filaments detach from the inner surface and form a curved filament connecting the two spiral tips in the surface. In certain parameter domains the filament is an arc of a circle that shrinks with constant shape. For parameter values close to the meandering border, the filament grows and collisions with the sphere walls lead to turbulent filament dynamics.

In [13], the authors study the ways in which the topological constraints and inhomogeneity in the excitability influence the geometry and dynamics of the spiral waves in thin spherical shells of excitable media. Heterogeneity can lead to the formation of pairs of spirals in which one spiral acts as a source and a second as a sink. Depending on the degree of inhomogeneity and the initial condition, source-sink spiral pairs may be asymptotically stable or evolve to a symmetric state of the counter-rotating spiral wave sources. Source-sink spiral wave pairs also arise on punctured spherical shells.

In [87], the dynamics of chemical spiral waves in an excitable reaction-diffusion system on a sphere is numerically investigated employing a spectral method using spherical harmonics as basis functions. Different types of spiral waves –symmetric or antisymmetric source-source or nearly antisymmetric source-nonsource– have been obtained depending on whether the medium is homogeneous or inhomogeneous, and it has been observed that the tips can either rotate steadily or change their shapes.

In [54] and [55], Belousov-Zhabotinsky chemical waves propagating on a sphere are reported. In [54], an equation describing the shape of the chemical wave is obtained (by the use of the eikonal equation) which shows agreement with experimental data, although additional effects due to the meandering of the spiral cores also become evident.

In [31], the geometrical stability of the symmetric counter-rotating spiral waves propagating on the unit sphere is studied. By the use of the eikonal equations, it is demonstrated that these solutions are stable under small perturbations normal to the wave front lying on a unit sphere.

In [9] and [32], the authors showed that stationary rotating solutions on a sphere of the eikonal equation under some boundary conditions on tips must be symmetric with respect to the equator with spiral winding out from source tips at polar points.

In [56], using the eikonal approximation to a reaction-diffusion system on a sphere, the authors prove existence of a class of counterrotating double spiral solutions which are highly asymmetric with respect to the equator. They also derive a power law, linking the angular rotation of the spiral waves with the velocity of plane waves in medium.

There is also some work on spiral waves on nonuniformly curved surfaces, for example [16]. The evolution of spiral waves in a two-dimensional excitable media with nonuniform curvature is considered. Using an analytical approach based on approximate kinematical relations, it is found that spiral waves drift at a rate proportional to the gradients of the Gaussian curvature of the surface. Also, the motion takes place in the direction which is orthogonal to the gradient and a necessary condition for drift (similar to that seen in the planar case) is the nonuniformity of the Gaussian curvature of the surface.

Regarding the mathematical analysis of the rotating waves and modulated rotating waves for dynamical systems which are equivariant under a general compact Lie group, we give below some references.

In [63], Renardy considered bifurcations from rotating waves of semilinear equations that are equivariant under a general compact Lie group and applied his results to the Laser equations. His results do not cover the resonance case. The theorems were proved using a generalized implicit function theorem on scales of Banach spaces.

In [62], Rand examined modulated rotating waves in rotating fluids and applied his results to the Taylor-Couette problem (see [12]) where modulated rotating waves, so called modulated wavy vortices occur.

In [18], a global theory for periodic solutions with symmetry was developed and rotating waves were studied as one possible phenomenon.

In [69], a model problem for the dynamics of rotating waves was investigated, namely a scalar periodically forced parabolic equation on the circle. It was proved that every bounded solution converges to a modulated rotating wave.

1.2 Overview of Thesis

Our goal in this thesis is to study the dynamics and bifurcations of the spiral waves on spherical or approximately spherical domains using equivariant dynamical systems theory and bifurcation theory, as well as the theory of Lie groups and Lie algebras. Specifically, we will study the parameterized reaction-diffusion systems

$$\frac{\partial u}{\partial t}(t,x) = D\Delta u(t,x) + F(u(t,x),\lambda,\mu) + \epsilon G(u(t,x),x,\epsilon,\lambda) \text{ on } r\mathbf{S^2}, \tag{1.2.1}$$

where $r > 0$, $r\mathbf{S^2}$ is the sphere or radius r, $\mathbf{S^2}$ is the unit sphere in $\mathbb{R}^3$, $u = (u_1,\, u_2,\, \ldots,$ $u_N)\colon r\mathbf{S^2} \to \mathbb{R}^N$ with $N \geq 1$,

$D = \begin{pmatrix} d_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & d_N \end{pmatrix}$, $d_i \geq 0$, $i = 1,\, 2,\, \ldots,\, N$ are the diffusion coefficients, $F = (F_1,\, F_2,$ $\ldots,\, F_N)$, $G = (G_1,\, G_2,\, \ldots,\, G_N)$ are sufficiently smooth functions and $|\lambda|$, $|\mu|$, $\epsilon \geq 0$ are small parameters.

We consider the group $SO(2)$ as being diffeomorphic to the following subgroup of $SO(3)$:

$\{e^{Q\theta} \mid \theta$ is in $[0, 2\pi)\}$, Q being fixed in $so(3)$ such that $|Q| = 1$. We will study the reaction-diffusion system (1.2.1) on the fractional spaces $\mathbf{Y}^\alpha$ (relative to $-D\Delta$) associated to the space

$$\mathbf{Y} = \begin{cases} \mathbf{L}^2(r\mathbf{S}^2, \mathbb{R}^N), & \text{if all } d_i > 0,\ i = 1, 2, \ldots, N; \\ \mathbf{H}^2(r\mathbf{S}^2, \mathbb{R}^N), & \text{if there exists an } i = 1, 2 \ldots,\ N \text{ such that } d_i = 0, \end{cases}$$

where $\mathbf{H}^2(r\mathbf{S}^2, \mathbb{R}^N)$ is a Sobolev space. In the presence of the parameter ϵ, we consider only the case $d_i > 0$, $i = 1, 2, \ldots, N$ and study the reaction-diffusion system (1.2.1) on the fractional spaces $\mathbf{Y}^\alpha$ (relative to $-D\Delta$) associated to the space $\mathbf{L}^2(r\mathbf{S}^2, \mathbb{R}^N)$.
The action θ of $SO(3)$ on the function space $\mathbf{Y}$ is defined and it is a smooth action on the space $\mathbf{Y}^\alpha$ for any $\alpha \in (\frac{1}{2}, 1)$.
In the case $\epsilon > 0$ we suppose that the superposition operator associated to G, denoted by $G\colon \mathbf{Y}^\alpha \times [0, \infty) \times \mathbb{R} \to \mathbf{Y}$, $G(u, \epsilon, \lambda)(x) = G(u(x), x, \epsilon, \lambda)$ is $SO(2)$-equivariant, but not $SO(3)$-equivariant with respect to the action θ of $SO(3)$ on $\mathbf{Y}$, for $\epsilon \geq 0$ and $|\lambda|$ small.
We denote the semiflow on $\mathbf{Y}^\alpha$ for any $\alpha \in (\frac{1}{2}, 1)$ associated to the reaction-diffusion system (1.2.1) by $\mathbf{\Phi}(t, u, \epsilon, \lambda, \mu)$. The reaction-diffusion system (1.2.1) is $SO(3)$-equivariant in the case $\epsilon = 0$ and only $SO(2)$-equivariant in the case $\epsilon > 0$. We fix $\alpha \in (\frac{1}{2}, 1)$.
Let $SO(3)u_0 \subset \mathbf{Y}^\alpha$ be a relative equilibrium (that is not an equilibrium) of the reaction-diffusion system (1.2.1) for $(\lambda, \mu, \epsilon) = (0, 0, 0)$ and such that the stabilizer of u_0 is $\Sigma_{u_0} = I_3$.
Let $\mathbf{\Phi}(t, u_0, 0, 0, 0) = e^{X_0 t} u_0$ and $L = D\Delta + D_u F(u_0, 0, 0) - X_0$ be the linearization at $\mathbf{\Phi}(t, u_0, 0, 0, 0) = e^{X_0 t} u_0$ for $\lambda = 0$, $\mu = 0$, $\epsilon = 0$ in the co-rotating frame.
The basic assumption is
Assumption A1 :

1. $\sigma(L) \cap \{z \in \mathbb{C} \mid Re\,(z) \geq 0\}$ is a spectral set with spectral projection P_*, $\dim(R(P_*)) < \infty$;

2. the semigroup e^{Lt} satisfies $\left|e^{Lt}|_{R(1-P_*)}\right| \leq Ce^{-\beta_0 t}$ for some $\beta_0 > 0$ and $C > 0$.

The use of the equivariant (finite-dimensional) center manifold reduction theorem near $SO(3)u_0$ (under the assumption A1) will reduce the study of the system (1.2.1) to the study of a finite-dimensional system of differential equations whose form will depend on the case $\epsilon = 0$ or $\epsilon > 0$. Our main results in this thesis are the following:

- *Case* $\epsilon = 0$ The finite-dimensional system to which the reaction-diffusion system

(1.2.1) reduces is given by

$$\begin{aligned} \dot{A} &= AX_G(q, \lambda, \mu), \\ \dot{q} &= X_N(q, \lambda, \mu), \end{aligned} \tag{1.2.2}$$

on $SO(3) \times V_*$, where V_* is a finite-dimensional space such that $R(P_*) = T_{u_0}(SO(3)u_0) \oplus V_*$, as well as the functions X_G, X_N are sufficiently smooth.

1. *Case of one parameter* λ

   - We prove that the transition from rotating waves on a sphere to modulated rotating waves is explained by a supercritical Hopf bifurcation at $q = 0$ for $\lambda = 0$ in the second differential equation in (1.2.2) ($V_* \simeq \mathbb{C}$) and by the $SO(3)$-equivariance of the reaction-diffusion system (1.2.1). Let $\omega_\lambda = \omega_{bif} + O(\lambda)$ be the frequency that appears due to this supercritical Hopf bifurcation, where $\pm i\omega_{bif}$ are the eigenvalues of $D_q X_N(0, 0)$.
   - Using the Baker-Campbell-Haussdorff formula in $so(3)$, we construct two branches of primary frequency vectors associated to the branch of modulated rotating waves or periodic solutions with period $\frac{2\pi}{|\omega_\lambda|}$ obtained by the supercritical Hopf bifurcation at $q = 0$ for $\lambda = 0$, one of the branch is such that $[\overrightarrow{X(\lambda)\frac{2\pi}{|\omega_\lambda|}}] \in D$ and $e^{X(0)\frac{2\pi}{|\omega_{bif}|}} = e^{X_0\frac{2\pi}{|\omega_{bif}|}}$, and the other branch $\overrightarrow{X^f(\lambda)}$ is such that $X^f(0) = X_0$, where D is the closed ball in $\mathbb{R}^3$ of radius π, having the antipodal points of norm π identified. Depending on the value of $|X_0|$, we have different degrees of smoothness / continuity for the branches $X(\lambda)$ and $X^f(\lambda)$ for $\lambda \geq 0$ small. Some results regarding the periodic parts $B(t, \lambda)$ (if $|X_0| = k\omega_{bif}$, for some $k \in \mathbb{Z}$) and $B^f(t, \lambda)$ (if $|X_0| \neq k\omega_{bif}$, for all $k \in \mathbb{Z}$) are also obtained (where $A(t, \lambda) = e^{X^f(\lambda)t}B^f(t, \lambda) = e^{X(\lambda)t}B(t, \lambda)$ is the solution of the initial value problem given by $\dot{A} = AX_G(q(t, \lambda), \lambda)$ and $A(0) = I_3$, where $q(t, \lambda)$ is the periodic solution that appears in the equation $\dot{q} = X_N(q, \lambda)$ by the supercritical Hopf bifurcation at $q = 0$ for $\lambda = 0$).

2. *Case of two parameters* λ, μ

   We suppose that a resonant Hopf bifurcation, that is $|X_0| = k\omega_{bif}$ for some $k \in \mathbb{Z}$, $k \neq 0$, occurs at $q = 0$ for $(\lambda, \mu) = (0, 0)$ in the second equation in (1.2.2) and $\pm i\omega_{bif}$ are the eigenvalues of $D_q X_N(0, 0, 0)$. This being the case, we prove that, generically, there exists either a branch of periodic solutions with period $\frac{2\pi}{|\omega_{\lambda,\mu}|}$ or a branch of modulated rotating waves having the associated primary frequency vectors orthogonal to the frequency of the initial rotating wave, $\overrightarrow{X}_0$. The fact that we have a branch of periodic solutions or a branch of modulated rotating waves depends on the function $X_G(q, \lambda, \mu)$.

- *Case* $\epsilon > 0$ We omit the parameters λ, μ.
  In this case we suppose that the relative equilibrium $SO(3)u_0$ is normally hyperbolic under the semiflow associated to the reaction-diffusion system (1.2.1) for $\epsilon = 0$.
  We study the perturbed dynamics induced by the reaction-diffusion system (1.2.1) on the $SO(2)$-invariant normally hyperbolic manifold $M(\epsilon)$ which is $SO(2)$-equivariant diffeomorphic to $SO(3)$ and persists under the $SO(2)$- equivariant perturbation G. We prove that depending on the frequency vectors of the rotating waves that form $SO(3)u_0$, these rotating waves (up to $SO(2)$) will give $SO(2)$-orbits of rotating waves or $SO(2)$-orbits of modulated rotating waves (if some transversality conditions hold). The orbital stability of these solutions is established as well.

The thesis is organized as follows. In Chapter 2, we present the basic theoretical tools used throughout the thesis, as well as the BCH formula in $so(3)$ and establish its smoothness property. This proof is based on the properties of the exponential map of $SO(3)$ and the diffeomorphism between $SO(3)$ and the real projective space $\mathbb{R}P^3$. In Chapter 3 we give the definition of the representation of $SO(3)$ on the function space $\mathbf{Y}$ and prove that this is a smooth unitary representation on $\mathbf{Y}^\alpha$ for any $\alpha \in (\frac{1}{2}, 1)$. Also, we consider a $SO(3)$-equivariant reaction-diffusion system on the sphere $r\mathbf{S^2}$ and the existence of the $SO(3)$-equivariant semiflow on $\mathbf{Y}^\alpha$, where $\alpha \in (\frac{1}{2}, 1)$, associated to this system is proved. The proof is based on the existence, uniqueness, smoothness, dependence on the initial conditions of the solutions of semilinear parabolic equations on Banach spaces, as well as the use of Sobolev embeddings. In Chapter 4 we define the notions of rotating waves and modulated rotating waves, tip position and orbital stability for solutions of the reaction-diffusion (1.2.1) and present some results related to them. In Chapter 5 we present the center manifold reduction theory for the case of a $SO(3)$-equivariant reaction-diffusion system. In Chapter 6 we carry out the proof of transition from rotating waves to modulated rotating waves and construct the branches of primary frequency vectors for modulated rotating waves or $\frac{2\pi}{|\omega_\lambda|}$-periodic solutions that appear via a supercritical Hopf bifurcation from a rotating wave, studying the smoothness properties of these branches with respect to the small parameter $\lambda \geq 0$. The resonant case is also studied. The proofs are based on the properties of differential equations on Lie groups, on the use of the BCH formula in $so(3)$, as well as on results from Lie groups and Lie algebra theory and on the geometrical interpretation of the group $SO(3)$. The study of the resonant case is based on the use of the implicit function theorem and of the BCH formula in $so(3)$. Chapter 7 is concerned with the effects of the forced symmetry-breaking from $SO(3)$ to $SO(2)$ for a normally hyperbolic relative equilibrium $SO(3)u_0$. This is done by studying $SO(2)$-equivariant reaction-diffusion systems that are small perturbations of $SO(3)$-equivariant reaction-diffusion systems. The relative equilibrium $SO(3)u_0$ persists to a normally hyperbolic $SO(2)$-invariant manifold that is $SO(2)$-equivariant diffeomorphic to $SO(3)$. The perturbed dynamics (including the orbital stability of the solutions that persist) on this manifold are studied by projection onto the orbit space $SO(3)/SO(2)$, which is diffeomorphic to the unit sphere $\mathbf{S}^2$. The proof is based on the use of the implicit function theorem and of the Poincaré map, as well as on the use of results from equivariant dynamics related to orbit space reduction. In Chapter 8, we present some results obtained with the symbolic algebra package, Maple, that visualize the transition from rotating waves to modulated rotating waves and the resonant phenomena studied in Chapter 6. In Chapter 9 we present the conclusions and future work. One appendix is included. It clarifies various computations associated to the BCH formula in $so(3)$, as well as its smoothness.

Chapter 2

Preliminaries

2.1 Implicit Function Theorem

The following theorem is a basic tool in our work.

Theorem 2.1.1 ([41],[65]). *Let X,Y,Z be Banach spaces, $U \subset X \times Y$ be an open set and $f: U \to Z$ a C^r map, $r \geq 1$. Let $z_0 = (x_0, y_0) \in U$ and $c = f(z_0)$. Suppose that the partial derivative with respect to the second variable, $D_y f(z_0): Y \to Z$, is an isomorphism. Then there exist open sets $V \subset X$ containing x_0 and $W \subset U$ containing z_0 such that, for each $x \in V$, there exists a unique $\xi(x) \in Y$ with $(x, \xi(x)) \in W$ and $f(x, \xi(x)) = c$. The map $\xi: V \to Y$, defined in this way, is of class C^r and its derivative is given by $\xi^{'}(x) = [D_y f(x, \xi(x))]^{-1} \circ D_x f(x, \xi(x))$.*

2.2 Lie Group $SO(3)$, Lie Algebra $so(3)$ and Exponential Map of $SO(3)$

Definition 2.2.1. *Let G be a group. Suppose there is a smooth manifold structure on the set G for which the maps*

$$(x, y) \to xy \text{ and } x \to x^{-1} \tag{2.2.1}$$

of $G \times G$ into G and of G into G, respectively, are both smooth. Then G, together with this smooth manifold structure and the maps defined in (2.2.1), is called a Lie group.

Definition 2.2.2. *The group $SO(3)$ is defined as the group of 3×3 real orthogonal matrices whose determinant is 1, that is*

$$SO(3) = \{A \in M_{3\times3}(\mathbb{R}) \mid AA^T = I_3,\, det(A) = 1\},$$

and it is called the special orthogonal group *or* the rotation group.

Proposition 2.2.3 ([8], [77]). *$SO(3)$ is a Lie group. As a manifold, it has dimension 3.*

Definition 2.2.4. *Let k be a field of characteristic 0. A vector space g over k is called a* Lie algebra over k *if there is a map*

$$(X, Y) \to [X, Y]$$

of $g \times g$ into g with the following properties:

1. *$(X, Y) \to [X, Y]$ is bilinear;*
2. *$[X, Y] + [Y, X] = 0$ for X, $Y \in g$;*
3. *$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for X, Y, $Z \in g$.*

For X, $Y \in g$, $[X, Y]$ is called the bracket *of X with Y.*

We will work only with Lie algebras over $k = \mathbb{R}$.

Definition 2.2.5. *The* Lie algebra g of a Lie group G *is defined as the tangent space to the identity $g = T_e(G)$, where e is the identity in G. Let X, $Y \in g$, $h_1(t)$ and $h_2(t)$ be smooth functions from $\mathbb{R}$ into G such that $h_1(0) = h_2(0) = e$, $h_1'(0) = X, h_2'(0) = Y$. The bracket on g is defined by*

$$[X, Y] = \frac{\partial^2}{\partial t \partial s}(h_1(t), h_2(s))\Big|_{t=s=0},$$

where $(h_1(t), h_2(s)) = h_1(t)h_2(s)(h_1(t))^{-1}(h_2(s))^{-1}$.

Proposition 2.2.6 ([8], [77]). *The Lie algebra $so(3)$ is the 3-dimensional vector space of 3×3 real anti-symmetric matrices, that is*

$$so(3) = \{X \in M_{3\times3}(\mathbb{R}) \mid X = -X^T\},$$

and it is denoted by $so(3)$. The bracket on $so(3)$ is $[X, Y] = XY - YX$ for X, $Y \in so(3)$.

Definition 2.2.7. *Consider a Lie group G with Lie algebra g.*
Let $l(A)B = AB$ be the left translation by A and $r(A)B = BA$ be the right translation by A, for A, $B \in G$. Then we define

$$AX = (d_A l)(X) \text{ and } XA = (d_A r)(X) \text{ for } A \in G \text{ and } X \in g. \tag{2.2.2}$$

For any $A \in G$ we consider the inner automorphism $a(A)B = ABA^{-1}$ of G. The adjoint representation of G *is the function $Ad\colon G \to GL(g)$ defined by*

$$Ad(A)X = (d_e a(A))(X) = AXA^{-1} \text{ for } X \in g, A \in G,$$

where the notations are as in (2.2.2), e is the identity element of G and $GL(g)$ is the space of all linear automorphisms of g. The adjoint representation of g *is the function $ad\colon g \to End(g)$ defined by*

$$ad(Y)X = [Y, X] \text{ for } X, Y \in g,$$

where $End(g)$ is the space of all endomorphisms of g.

Proposition 2.2.8 ([77]). *The adjoint representation of $SO(3)$ is given by*

$$Ad(A)X = AXA^{-1} \text{ for } A \in SO(3), X \in so(3).$$

The adjoint representation of $so(3)$ is given by

$$ad(X)Y = XY - YX \text{ for } X, Y \in so(3).$$

For any matrix $X \in so(3)$ one can define a vector denoted $\overrightarrow{X}$ in the following way:

$$\text{if } X = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}, \text{ then } \overrightarrow{X} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}.$$

Also, we denote $|X| = \left\|\overrightarrow{X}\right\| = \sqrt{a^2 + b^2 + c^2}$ and $\|X\| = \sqrt{2}\left\|\overrightarrow{X}\right\|$ for any $X \in so(3)$. The map $X \to \overrightarrow{X}$ is called the *hat* map and it is an isomorphism of normed spaces.

Theorem 2.2.9. *1. [64] For any X, $Y \in so(3)$,*

$$\begin{aligned} \overrightarrow{ad(X)Y} &= X\overrightarrow{Y} \text{ and} \\ \overrightarrow{(ad(X))^n Y} &= X^n \overrightarrow{Y} \text{ for any integer } n > 0. \end{aligned} \tag{2.2.3}$$

2. *For any $A \in SO(3)$ and any $X \in so(3)$, we have*

   (a)
   $$\left|AXA^{-1}\right| = |X|; \tag{2.2.4}$$

   (b) [64]
   $$AXA^{-1} = B \text{ if and only if } A\overrightarrow{X} = \overrightarrow{B}.$$

3. *For any $X \in so(3)$, we have:*

$$\begin{aligned} X^{2n} &= (-1)^{n-1}|X|^{2(n-1)}X^2 \text{ for any } n \geq 1, \\ X^{2n+1} &= (-1)^n |X|^{2n} X \text{ for any } n \geq 0. \end{aligned} \tag{2.2.5}$$

Definition 2.2.10. *A* one-parameter group *of a Lie group G (with Lie algebra g) is a homomorphism of Lie groups $\alpha\colon \mathbb{R} \to G$. The correspondence $\alpha \to \dot{\alpha}(0) \in g$ defines a canonical bijection between the set of one-parameter groups of G and g.*

Theorem 2.2.11. *Consider a Lie group G with Lie algebra g. Given a vector $X \in g$, there is only one one-parameter group $\alpha^X\colon \mathbb{R} \to G$ with $\dot{\alpha}^X(0) = X$. The map $exp\colon g \to G$, $X \to \alpha^X(1)$ is called the* exponential map *of G.*

Proposition 2.2.12 ([77])**.** *The exponential of $SO(3)$ is $exp\colon so(3) \to SO(3)$, $exp(X) = e^X$.*

Theorem 2.2.13. 1. *[59] The exponential map $exp\colon so(3) \to SO(3)$ is surjective.*

2. *[59], [64] The exponential map exp is a smooth function on $so(3)$ and its differential is given by:*

$$(d(exp))_X(Y) = e^X \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}(ad(X))^n(Y) \text{ for any } X,\, Y \in so(3)$$

   or

$$(d(exp))_X(Y) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(ad(X))^n(Y)e^X \text{ for any } X,\, Y \in so(3)$$

   (we will use the first formula later). Moreover, it is a local diffeomorphism near any $X \in so(3)$ if and only if the operator $ad(X)$ has no eigenvalues of the form $2\pi ik$ with $k \neq 0$, that is if and only if $|X| \neq 2k\pi$ for $k \in \mathbb{Z}$, $k \neq 0$.

3. *[64] $\frac{d}{dt}\left(e^{X(t)}\right) = \dot{X}(t)e^{X(t)}$ if and only if $X(t) = Xg(t)$, where $X \in so(3)$ and $g\colon \mathbb{R} \to \mathbb{R}$ is a C^1 function.*

4. *[17], [68] The Rodrigues' formula holds:*

$$e^X = I_3 + \frac{\sin|X|}{|X|}X + 2\frac{\sin^2\frac{|X|}{2}}{|X|^2}X^2 \text{ for any } X \in so(3),$$

where we take the limit when $X = O_3$.

5. *[64] The exponential map exp maps any $X \in so(3)$, $X \neq O_3$ to the right-handed rotation $A \in SO(3)$ with angle $|X|$ around $\overrightarrow{X}$.*

6. *[64] $e^X = I_3$ if and only if $|X| = 2k\pi$ for some $k \in \mathbb{Z}$.*

7. *[64] $e^X = e^Y$ if and only if either $|X| = 2k\pi = |Y|$ or $\overrightarrow{Y} = \overrightarrow{X} + \frac{2k\pi}{|X|}\overrightarrow{X}$ for some $k \in \mathbb{Z}$.*

Remark 2.2.14. 1. *If $P(t) = e^{G(t)}$, $P\colon \mathbb{R} \to SO(3)$ is a continuous T-periodic function, $G\colon \mathbb{R} \to so(3)$ is a continuous function, then G is not necessarily a periodic function. For instance, $P(t) = e^{Xt}$, $G(t) = Xt$.*

2. *If $|G(t)| < 2\pi$ and $P(t) \neq I_3$, for all $t \in [0,T]$, then G is T-periodic.*

Proof. We have $P(t+T) = P(t) \Longrightarrow e^{G(t+T)} = e^{G(t)}$. Since $P(t) \neq I_3$ for all $t \in [0,T]$, we get by using theorem 2.2.13 (7) that $|G(t+T) - G(t)| = 2k(t)\pi$ for some $k(t) \in \mathbb{Z}$. Then by using $|G(t)| < 2\pi$, for all $t \in [0,T]$, we get $k(t) = 0$ for all $t \in [0,T]$, that is $G(t+T) = G(t)$ for all $t \in [0,T]$. □

3. *If $P(t,\lambda) = e^{G(t,\lambda)}$, $P\colon \mathbb{R}\times\mathbb{R} \to SO(3)$ is a continuous T-periodic function in t for $\lambda \geq 0$ small and $G\colon \mathbb{R}\times\mathbb{R} \to so(3)$ is a continuous function such that $G(t,\lambda) = \lambda H(t,\lambda)$, then $G(t,\lambda)$ is a T-periodic function in t for $\lambda \geq 0$ small.*

Proof. We have $P(t+T,\lambda) = P(t,\lambda) \Longrightarrow e^{G(t+T,\lambda)} = e^{G(t,\lambda)} \Longrightarrow e^{\lambda H(t+T)} = e^{\lambda H(t)}$ and $\lambda H(t,\lambda)$, $\lambda H(t+T,\lambda)$ are in a neighborhood of O_3 for $\lambda \geq 0$ small. Then, by using the fact that the exponential map *exp* is a local diffeomorphism at O_3, it follows that $\lambda H(t,\lambda) = \lambda H(t+T,\lambda)$ or $G(t+T,\lambda) = G(t,\lambda)$ for $\lambda \geq 0$ small. □

2.3 Baker-Campbell-Hausdorff (BCH) Formula in $so(3)$

1. Using [68], we have that for any matrix $A \in SO(3)$, $A^2 \neq I_3$, there exists a number $\theta \in (0, \pi)$ and a matrix $V \in so(3)$ such that

$$A = e^{\theta V} = e^{(2\pi - \theta)(-V)} \text{ and } |V| = 1.$$

By using theorem 2.2.13 (7), it follows that these are the only two possible choices of a matrix $X \in so(3)$ such that $|X| < 2\pi$ and $A = e^X$. In fact (see [68]), we have

$$\theta = \arccos\left(\frac{Tr(A) - 1}{2}\right) \text{ and } V = \frac{A - A^T}{2\sin\left(\arccos\left(\frac{Tr(A)-1}{2}\right)\right)}. \tag{2.3.1}$$

Let $\alpha = \left|\frac{A-A^T}{2}\right|$. We show that $\alpha = \sin\theta$. Then we get $\theta = \sin^{-1}\alpha$, where we denote

$$\sin^{-1}\alpha = \begin{cases} \arcsin\alpha, & \text{if } A \text{ has eigenvalues with all positive real part;} \\ \pi - \arcsin\alpha, & \text{if } A \text{ has two eigenvalues with negative or zero real parts.} \end{cases}$$

(If A has eigenvalues with all positive real part, then $Tr(A) > 1$; therefore, $\theta \in [0, \frac{\pi}{2})$. If A has two eigenvalues with all negative or zero real part, then $Tr(A) \leq 1$; therefore, $\theta \in [\frac{\pi}{2}, \pi]$).
Let $A = e^B$. The eigenvalues of B are 0, $i\,|B|$ and $-i\,|B|$ and the eigenvalues of A are 1, $e^{i|B|}$ and $e^{-i|B|}$.
Then, using Rodrigues'formula for e^B and e^{-B} (theorem 2.2.13, (4)), we get

$$\alpha = \left|\frac{A - A^T}{2}\right| = \left|\frac{e^B - e^{-B}}{2}\right| = \left|\frac{\sin|B|}{|B|}B\right| = |\sin|B||.$$

and since $\theta \in (0, \pi)$, we have

$$0 < \sin(\theta) = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{Tr(A) - 1}{2}\right)^2} = \sqrt{1 - \cos^2|B|} = |\sin|B|| = \alpha.$$

2. If $A^2 = I_3$, but $A \neq I_3$, then $A = e^{\theta V} = e^{\theta(-V)}$, where $\theta = \pi$ and $(-V)^2 = V^2 = \frac{A-I_3}{2}$. By using theorem 2.2.13 (7), it follows that these are the only two possible choices of a matrix $X \in so(3)$ such that $|X| < 2\pi$ and $A = e^X$.

3. If $A = I_3$, then $\theta = 0$ and $V = O_3$, this being the only possibility of a matrix $X \in so(3)$ such that $A = e^X$ and $|X| < 2\pi$.

It is known that $SO(3)$ is diffeomorphic as a manifold to the real projective space $\mathbb{R}P^3$ (see [64]). A model for this space is the set
$D = \{\overrightarrow{y} \in \mathbb{R}^3 \mid \|\overrightarrow{y}\| \leq \pi$, whith the antipodals points of the norm $|y| = \pi$ are identified $\}$.
In fact, D is the quotient set $E/\sim$, where $\sim$ is the equivalence relation $\vec{y} \sim \vec{z}$ iff $z = -y$, $|y| = \pi$ and $E = \{\overrightarrow{y} \in \mathbb{R}^3 \mid \|\overrightarrow{y}\| \leq \pi\}$. The set D is considered with the quotient topology. Sometimes, we use $y \in so(3)$ instead of $\overrightarrow{y} \in \mathbb{R}^3$, in which case we denote the equivalence class $[y] = [\overrightarrow{y}]$. The projection map $p\colon E \to D$, $p(y) = [y]$ is smooth.
We check that there is a smooth function $d^*\colon SO(3) \to D$ such that $e^{d^*(A)} = A$. The function d^* is unique with this property (by theorem 2.2.13, (6) and (7)). We know that the exponential map exp is a local diffeomorphism at any $X \in so(3)$ such that $|X| \neq 2k\pi$ for any $k \in \mathbb{Z}$, $k \neq 0$ (theorem 2.2.13, (2)). Let us consider the following cases:

1. We know that exp is a local diffeomorphism near $X_0 = O_3$ and we choose a neighborhood $U \in so(3)$ of O_3 such that for any $X \in U$, $|X| < \pi$.
   Then, the local inverse of exp is a smooth function at I_3 given by

$$d^*(A) = \theta V = \begin{cases} \arccos\left(\frac{Tr(A)-1}{2}\right) \frac{A-A^T}{2\sin\left(\arccos\left(\frac{Tr(A)-1}{2}\right)\right)} & \text{if } A^2 \neq I_3 \text{ ;} \\ O_3, & \text{if } A = I_3, \end{cases}$$

   where $A \in exp(U)$. It follows that A is near I_3 and that $Tr(A)$ is near 3, therefore $Tr(A) \neq -1$, so $A^2 \neq I_3$ or $A = I_3$.

2. We know that exp is a local diffeomorphism near X_0 such that $0 < |X_0| < \pi$, so we choose a neighborhood U of X_0 such that for any $X \in U$, we have $0 < |X| < \pi$.
   Then, the local inverse of exp is a smooth function at e^{X_0}, given by the function

$$d^*(A) = \theta V = \arccos\left(\frac{Tr(A)-1}{2}\right) \frac{A - A^T}{2\sin(\arccos(\frac{Tr(A)-1}{2}))},$$

   where $A \in exp(U)$. For any $A \in exp(U)$ we have $A^2 \neq I_3$.

3. We know that exp is local diffeomorphism near X_0 such that $|X_0| = \pi$ and we choose a neighborhood U of X_0 such that for any $X \in U$, $|X - X_0| < \frac{\pi}{2}$.
   Then:

   (a) $x \in U$ implies $|X| < \frac{3\pi}{2}$ and $A = e^X \neq I_3$, otherwise we get $A = e^X = I_3$ and, since $|X| < \frac{3\pi}{2}$, we get $X = O_3$. But $|O_3 - X_0| = \pi > \frac{\pi}{2}$;

(b) $-X_0 \notin U$, otherwise $|-X_0 - X_0| = 2\pi > \frac{\pi}{2}$;

(c) We have that $X \in U \Longrightarrow X - \frac{2\pi}{|X|}X \notin U$, otherwise $\left|X - \frac{2\pi}{|X|}X - X_0\right| \geq 2\pi\frac{|X|}{|X|} - |X - X_0| \geq 2\pi - \frac{\pi}{2} = 3\frac{\pi}{2}$.

Since exp is a local diffeomorphism near X_0, it follows that for any $A \in exp(U)$, there exists a unique $X \in U$ such that $A = e^X$. Since $A \neq I_3$, then either $X = \theta V$ or $X = (2\pi - \theta)(-V)$, with θ and V defined at the beginning of this section in 1 and 2. Only one of θV and $(2\pi - \theta)(-V)$ is in U. So, we choose the one that is in U.

Then, this implies that the local inverse of exp is a smooth function at e^{X_0}, given by the function

$$d_1^*(A) = X = \theta V \text{ or } (2\pi - \theta)(-V)(\text{ whatever is near } X_0),$$

where $A \in exp(U)$. We define for any $X \in U$,

$$q(X) = \begin{cases} X & \text{if } |X| < \pi; \\ (-\frac{2\pi}{|X|} + 1)X & \text{if } |X| > \pi; \\ [X] & \text{if } |X| = \pi. \end{cases} \tag{2.3.2}$$

The function q is smooth from U into D and has a smooth inverse q^{-1} defined from $q(U)$ onto U by

$$q^{-1}([Y]) = \begin{cases} Y & \text{if } |Y| \leq \pi \text{ and } Y \text{ is near } X_0,\ Y \neq X_0; \\ (-\frac{2\pi}{|Y|} + 1)Y & \text{if } |Y| \leq \pi \text{ and } Y \text{ is near } -X_0,\ Y \neq -X_0; \\ X_0 & \text{if } [Y] = [X_0]. \end{cases} \tag{2.3.3}$$

It follows that the function $d^*(A) = q(d_1^*(A))$ defined by

$$d^*(A) = [\theta V] = \begin{cases} \arccos\left(\frac{Tr(A)-1}{2}\right) \frac{A - A^T}{2\sin\left(\arccos\left(\frac{Tr(A)-1}{2}\right)\right)} & \text{if } A^2 \neq I_3; \\ [\pi V](V^2 = \frac{A - I_3}{2}) & \text{if } A^2 = I_3,\ A \neq I_3 \end{cases}$$

is smooth at e^{X_0}.

Therefore, it is clear that the function d^* defined by

$$d^*(A) = [\theta V] = \begin{cases} \arccos\left(\frac{Tr(A)-1}{2}\right) \frac{A - A^T}{2\sin\left(\arccos\left(\frac{Tr(A)-1}{2}\right)\right)} & \text{if } A^2 \neq I_3; \\ [\pi V](V^2 = \frac{A - I_3}{2}) & \text{if } A^2 = I_3, A \neq I_3; \\ O_3 & \text{if } A = I_3 \end{cases} \tag{2.3.4}$$

is smooth from $SO(3)$ into D. Also, it is true that $|d^*(A)| = \theta = \arccos\left(\frac{Tr(A)-1}{2}\right)$ is continuous on $SO(3)$ and $|d^*(A)|^2$ is smooth on $SO(3)$. Also, $e^{d^*(A)} = A$ for any $A \in SO(3)$.

Theorem 2.3.1. *For any matrix X,$Y \in so(3)$, since the exponential map exp is surjective, (see theorem 2.2.13, (1)), there exists a matrix (not unique) $U \in so(3)$ such that $exp(U) = e^U = e^X e^Y$.*

We define for any X,$Y \in so(3)$, $BCH(X,Y) = d^*(e^X e^Y)$, where d^* is defined in (2.3.4). Clearly, we have $e^{BCH(X,Y)} = e^X e^Y$ and $BCH(X,Y) \in D$, for any X, $Y \in so(3)$. Taking into account that the function d^* and the exponential map exp are smooth, we get that $BCH(X,Y)$ is a smooth function from $so(3) \times so(3)$ into D.

Notation 2.3.2. *Sometimes we denote for any $A \in SO(3)$,*

$$A = e^{[X]} = \begin{cases} e^X & \text{if } |X| < \pi; \\ e^X = e^{-X} & \text{if } |X| = \pi. \end{cases}$$

Notation 2.3.3. *We denote*
$BCH_1([X],[Y]) = BCH(X,Y)$ for X, $Y \in D$ and this is well-defined since $e^X e^Y = e^{[X]}e^{[Y]}$;
$BCH_2(X,[Y]) = BCH(X,Y)$ for $X \in so(3)$, $Y \in D$ and this is well-defined since $e^X e^Y = e^X e^{[Y]}$;
$BCH_3([X],Y) = BCH(X,Y)$ for $X \in D$, $Y \in so(3)$ and this is well-defined since $e^X e^Y = e^{[X]}e^Y$.
We recall that p is the projection map from $SO(3)$ onto D. Since $BCH_1 \circ (p,p) = BCH$ and BCH and p are smooth, we get that BCH_1 is smooth from $D \times D$ into D. Since $BCH_2 \circ (id,p) = BCH$ and BCH and p are smooth, we get that BCH_2 is smooth from $so(3) \times D$ into D. Also, $BCH_3 \circ (p,id) = BCH$. Similarly, BCH_3 is smooth from $D \times so(3)$ into D. We also denote $BCH(X,Y,Z) = BCH(BCH(X,Y),Z)$ for X, Y, $Z \in D$ or X, Y, $Z \in so(3)$.
We will use the notation BCH for BCH_1, BCH_2 and BCH_3.

Theorem 2.3.4. *The BCH formula in $so(3)$ has the form*

$$BCH(X,Y) = [\alpha X + \beta Y + \gamma[X,Y]] \text{ for } X,\ Y \in so(3), \tag{2.3.5}$$

where

$$\alpha = k(X,Y)h_\alpha(X,Y),\ \beta = k(X,Y)h_\beta(X,Y),\ \gamma = k(X,Y)h_\gamma(X,Y),$$

and

$$e = \cos\frac{|X|}{2}\cos\frac{|Y|}{2} - \sin\frac{|X|}{2}\sin\frac{|Y|}{2}\cos(\angle(\vec{X},\vec{Y})),$$

$$a_1 = \sin\frac{|X|}{2}\cos\frac{|Y|}{2},\ a = a_1 e,$$

$$b_1 = \sin\frac{|Y|}{2}\cos\frac{|X|}{2},\ b = b_1 e,$$

$$c_1 = \sin\frac{|X|}{2}\sin\frac{|Y|}{2},\ c = c_1 e,$$

$$d_1 = \sqrt{a_1^2 + b_1^2 + 2a_1 b_1\cos(\angle(\vec{X},\vec{Y})) + c_1^2(\sin(\angle(\vec{X},\vec{Y}))^2},$$

$$d = d_1\,|e|\,,$$

where $\angle(\vec{X},\vec{Y})$ *is the angle between the two vectors* $\overrightarrow{X}$ *and* $\overrightarrow{Y}$*,*

$$h_\alpha(X,Y) = \begin{cases} \frac{a_1}{|X|} & \text{if } X \neq O_3; \\ \cos\frac{|Y|}{2} & \text{if } X = O_3, \end{cases}$$

$$h_\beta(X,Y) = \begin{cases} \frac{b_1}{|Y|} & \text{if } Y \neq O_3; \\ \cos\frac{|X|}{2} & \text{if } Y = O_3, \end{cases}$$

$$h_\gamma(X,Y) = \begin{cases} \frac{c_1}{|X||Y|} & \text{if } X \neq O_3, Y \neq O_3; \\ \frac{\sin\frac{|Y|}{2}}{|Y|} & \text{if } X = O_3, Y \neq O_3; \\ \frac{\sin\frac{|X|}{2}}{|X|} & \text{if } Y = O_3, X \neq O_3; \\ 1 & \text{if } Y = O_3, X = O_3, \end{cases}$$

$$k(X,Y) = \begin{cases} s\frac{\arcsin(d)}{d_1} & \text{if } (e^X e^Y)^2 \neq I_3,\ e^X e^Y \text{ has eigenvalues with positive real parts;} \\ s\frac{\pi - \arcsin(d)}{d_1} & \text{if } (e^X e^Y)^2 \neq I_3,\ e^X e^Y \text{ has two eigenvalues with negative} \\ & \quad \text{or zero real parts;} \\ \pi & \text{if } (e^X e^Y)^2 = I_3,\ e^X e^Y \neq I_3; \\ s & \text{if } e^X e^Y = I_3, \end{cases}$$

where $s = sgn(e) = \begin{cases} 1, & \text{if } e > 0; \\ -1, & \text{if } e < 0, \end{cases}$ *for any* $(X,Y) \in so(3)\times so(3)$*, such that* $(e^X e^Y)^2 \neq I_3$ *or* $e^X e^Y = I_3$*. The functions* α^2*,* β^2*,* γ^2 *are smooth on* $so(3)\times so(3)$ *and* $|\alpha|$*,* $|\beta|$*,* $|\gamma|$ *are continuous on* $so(3)\times so(3)$*. Also, the function* BCH *is smooth from* $so(3)\times so(3)$ *into* D *(as we have already seen). The proof of this theorem is in appendix* A.

2.4 Action of a Lie Group on a Manifold and on a Banach Space

Definition 2.4.1. *Let M be a smooth manifold or a Banach space and G be a Lie group. We call* left action *of G on M a function $\theta\colon G \times M \to M$ such that*

1. *$\theta(e,p) = p$, for all $p \in M$, where e is the identity element in G;*
2. *$\theta(A_1, \theta(A_2, p)) = \theta(A_1 A_2, p)$ for all A_1, $A_2 \in G$, $p \in M$.*

We denote $Ap = \theta(A,p)$.
The action θ is called transitive *if for any x, $y \in M$ one can find $A \in G$ such that $y = Ax$.*

Definition 2.4.2. *Let V be a normed space and G be a Lie group. We call* representation *of G on V a function $T\colon G \to GL(V)$ such that*

1. *the map $A \to T(A)p$ is continuous from G to V for each $p \in V$;*
2. *$T(e) = Id$;*
3. *$T(A_1A_2) = T(A_1)T(A_2)$ for all A_1, $A_2 \in G$,*

where $GL(V)$ is the space of linear bounded invertible operators on V.

If V is a Hilbert space with scalar inner product $(\cdot,\cdot)$, then we say that the representation T of G on V is unitary *if $(T(A)p, T(A)q) = (p,q)$ for all $A \in G$ and all p, $q \in V$.*

Theorem 2.4.3 ([29]). *To each representation T we can associate an action $\theta\colon G \times V \to V$, $\theta(A,p) = T(A)p$ for $A \in G$ and $p \in V$. This kind of action is called* linear.

Definition 2.4.4. *Let θ be the action of M on G as defined in Definition* 2.4.1.
The orbit *of the action θ on $p \in M$ is the set $Gp = \{\theta(A,p) \mid A \in G\}$.*
A set $N \subset M$ is called invariant*under G if $\{\theta(A,p) \mid A \in G, p \in N\} \subset N$.*
The isotropy subgroup *or* stabilizer *of $p \in M$ is the subgroup $\Sigma_p = \{A \in G \mid \theta(A,p) = p\}$.*

Definition 2.4.5. *Let g be a Lie algebra and V be a normed space. We call* representation *of g on V a function $\pi\colon g \to End(V)$ such that*

1. *π is linear;*
2. *$\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$ for any X, $Y \in g$,*

where $End(V)$ is the space of endomorphisms of V. If V is finite-dimensional, then the above definition is equivalent to saying that π is a homomorphism of g into $End(V)$.

Definition 2.4.6. *Let T be the representation of G on V as defined in (2.4.2) and $1 \leq r \leq \infty$. A vector $p \in V$ is called C^r if the function $A \to T(A)p$ from G to V is C^r. Let V^{sr} be the space of C^r vectors. We define*

$$T'(X)p = \left(\frac{d}{dt}T(exp(tX))p\right)\Big|_{t=0} \quad \text{for } X \in g, p \in V^{sr}.$$

Then, $T'(X) \in End(V^{sr})$ and $T' : X \to T'(X)$ is a representation of g on V^{sr} (see [78]) and is called the differential *of the representation T. Sometimes we denote $T'(X) = X$. If B is a basis of g, then $T'(X)$, for any $X \in B$ is called an* infinitesimal generator *of g. The representation T of G on V is called C^r if $V^{sr} = V$.*

Definition 2.4.7. *Let θ be the action of G on M as defined in 2.4.1 and $0 \leq r \leq \infty$. The action θ of G on M is called C^r if θ is a C^r map. If θ is differentiable, then the* differential *of the action θ is defined as follows: for any $X \in g$ and $p \in M$, $Xp = \left(\frac{d}{dt}\theta(exp(Xt), p)\right)|_{t=0}$.*

2.5 Dynamical Systems and Poincaré Map

Definition 2.5.1. *Let M be a smooth manifold or a normed space and $0 \leq r \leq \infty$. A* vector field *X of class C^r on M is a C^r map $X\colon M \to TM$ which associates a vector $X(p) \in T_pM$ to each point $p \in M$ (if M is a normed space, then $T_pM = M$). Let M be a smooth manifold. A C^r* flow *on M is a C^r function $\Phi\colon \mathbb{R} \times M \to M$ such that*

1. *$\Phi(0,p) = p$ for any $p \in M$;*

2. *$\Phi(t+s,p) = \Phi(t, \Phi(s,p))$ for any $p \in M$ and any $t,\ s \in \mathbb{R}$.*

Let M be a normed space. A C^r semiflow *on M is a C^r function $\Phi\colon [0,\infty) \times M \to M$ such that*

1. *$\Phi(0,p) = p$ for any $p \in M$;*

2. *$\Phi(t+s,p) = \Phi(t, \Phi(s,p))$ for any $p \in M$ and any $t,\ s \in [0,\infty)$.*

We denote $\Phi_t(p) = \Phi(t,p)$.

Definition 2.5.2. *Let M and N be two diffeomorphic smooth manifolds, $\mathbf{\Phi}$ a flow on M and $\mathbf{\Psi}$ a flow on N. We say that Ψ and Φ are* topologically equivalent *if there exists a homeomorphism $h\colon M \to N$ such that $h \circ \Phi_t = \Psi_t \circ h$ for all $t \in \mathbb{R}$.*
In case M and N are normed spaces, we have a similar definition for semiflows but $t \geq 0$ only.

Definition 2.5.3. *A subset $N \subset M$ is called* semiflow invariant *if $\{\mathbf{\Phi}(t,x) \mid x \in N, t \geq 0\} \subset N$. We have a similar definition for flows.*

Theorem 2.5.4 ([41], [83]). *Let M be a compact smooth manifold or a normed space and $0 \leq r \leq \infty$. To each C^r vector field X on M we can associate a C^r flow $\mathbf{\Phi}$ on M, if M is a manifold, or a C^r semiflow $\mathbf{\Phi}$, if M is a normed space.*

Definition 2.5.5. 1. *Let N be a compact smooth submanifold of a smooth manifold M, $1 \leq r \leq \infty$ and $f\colon M \to M$ a C^r diffeomorphism such that $f(N) = N$. We say that N is* normally hyperbolic *under f if*

(a) *for each $p \in M$, there exist closed subspaces N_p^u and N_p^s, such that $T_pM = N_p^u \oplus N_p^c \oplus N_p^s$ with $N_p^c = T_pN$;*

(b) *for each $p \in M$, if we denote $p_1 = f(p)$, $D_pf\colon N_p^\alpha \to N_{p_1}^\alpha$ for $\alpha = u$, c, s;*

(c) *there is a Riemannnian metric on TM and $n_0 \geq 0$ such that for all $p \in M$, $k \in [0,1]$ and $n \geq n_0$,*

$$\inf\{|D_pf^n x^u| \mid x^u \in N_p^u, |x^u| = 1\} > \frac{\lambda^n}{C} \left\| D_pf^n|_{N_p^c} \right\|^k,$$

$$C\mu^n \inf\{|D_pf^n x^c| \mid x^c \in N_p^c, |x^c| = 1\} > \left\| D_pf^n|_{N_p^s} \right\|^k$$

for some constants $\mu \in (0,1)$, $\lambda > 1$, $C > 0$.

Let $\mathbf{\Phi}$ be a C^r flow on M such that N is flow invariant. We say that N is normally hyperbolic *under the flow $\mathbf{\Phi}$ if there is a $t > 0$ such that N is normally hyperbolic under $\mathbf{\Phi}_t$.*

2. *Let X be a Banach space (with norm $\|\cdot\|$), $1 \leq r \leq \infty$ and $\mathbf{\Phi}$ a C^r semiflow on X and $M \subset X$ a compact semiflow invariant manifold. We say that M is* normally hyperbolic *under the flow $\mathbf{\Phi}$ if*

(a) for each $m \in M$ there is a decomposition

$$X = X^c_m \oplus X^u_m \oplus X^s_m$$

of closed subspaces with $X^c_m = T_m M$;

(b) for each $m \in M$ and $t \geq 0$, if we denote $m_1 = \Phi(t, m)$,

$$D_m \Phi_t|_{X^\alpha_m} : X^\alpha_m \to X^\alpha_{m_1} \text{ for } \alpha = c, u, s$$

and $D_m \Phi_t|_{X^u_m}$ is an isomorphism from X^u_m onto $X^u_{m_1}$;

(c) there exists $t_0 \geq 0$ and $\lambda \leq 1$ such that for all $t \geq t_0$,

$$\lambda \inf\{|D_m \Phi_t x^u| \mid x^u \in X^u_m, |x^u| = 1\} > \max\{1, \left\|D_m \Phi_t|_{X^c_m}\right\|\},$$

$$\lambda \min\{1, \inf\{|D_m \Phi_t x^c| \mid x^c \in X^c_m, |x^c| = 1\}\} > \left\|D_m \Phi_t|_{X^s_m}\right\|.$$

Definition 2.5.6. *Let M be a smooth manifold, Φ a C^r flow on M and $\Phi(., p_0)$ a periodic solution, where $0 \leq r \leq \infty$. Let $\gamma = \{\Phi(t, p_0) \mid t \in \mathbf{R}\}$. Through the point $p_0 \in \gamma$ we consider a section Σ transversal to the vector field X associated to the flow Φ. The orbit through p_0 returns to intersect Σ at time τ, where τ is the period of γ. By the continuity of the flow Φ, the orbit through a point $p \in \Sigma$ sufficiently close to p_0 returns to intersect Σ at a time close to τ. Thus, if $V \subset \Sigma$ is a sufficiently small neighborhood of p_0, we can define a map $P \colon V \to \Sigma$ which associates to each point $p \in V$ the first point $P(p)$ of the orbit of p returning to intersect Σ. This map is called the* Poincaré map *associated to the orbit γ and section Σ. This map is a C^r diffeomorphism from a neighborhood of p in Σ onto an open neighborhood of p in Σ.*

Definition 2.5.7. *Let X be a topological space, and $U \subset X$. U is called a* residual *set if it is the intersection of a countable number of sets each of which are open and dense in X. A property of a map (respectively vector field) is said to be C^r* generic *if the set of maps (resp. vector fields) having that property contains a residual subset in the C^r topology (the topology induced on the space of C^r functions by the measure of distance between two elements of the space of C^r functions).*

Definition 2.5.8. *Let M and N be two smooth manifolds, $S \subset N$ be a smooth submanifold and let $f \colon M \to N$ be a C^r function with $1 \leq r \leq \infty$. We say that f is* transversal *to S at a point $p \in M$ if either $f(p) \notin S$ or $d_p f(T_p M) + T_{f(p)} S = T_{f(p)} N$. We say that f is* transversal *to S if f is transversal to S at each point $p \in M$.*

Theorem 2.5.9 (Thom, [60]). *Suppose that M is a compact smooth manifold, N is a smooth manifold, $S \subset N$ is a closed smooth submanifold and $1 \leq r \leq \infty$. The set of maps $f \in C^r(M, N)$ that are transversal to S is open and dense.*

Definition 2.5.10. *Let M be a smooth manifold, X a vector field on M and Φ the flow associated to X. The ω-*limit* set of a point $p \in M$, denoted $\omega(p)$, is the set of those points $q \in M$ for which there exists a sequence $t_n \to \infty$ with $\Phi(t_n, p) \to q$. The ω-*limit* set of the orbit of p is $\omega(p)$.*

Definition 2.5.11. *Let M be a smooth manifold, X a vector field on M and Φ the flow associated to X. The α-*limit* set of a point $p \in M$, denoted $\alpha(p)$, is the set of those points $q \in M$ for which there exists a sequence $t_n \to -\infty$ with $\Phi(t_n, p) \to q$. The α-*limit* set of the orbit of p is $\alpha(p)$.*

The Poincaré-Bendixson theorem on a sphere for ω-limit sets is:

Theorem 2.5.12 (Poincaré-Bendixson on $\mathbf{S}^2$, [41], [60], [83]). *Let X be a smooth vector field on the unit sphere $\mathbf{S}^2$ with a finite number of singularities. Take $p \in \mathbf{S}^2$ and let $\omega(p)$ be the ω-limit set of p. Then one of the following possibilities holds:*

1. *$\omega(p)$ is an equilibrium of the flow associated to X;*
2. *$\omega(p)$ is a closed orbit of the flow associated to X;*
3. *$\omega(p)$ consists of equilibria $p_1, \ldots, p_n$ and regular orbits of the flow associated to X, such that if $\gamma \subset \omega(p)$ then $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$.*

A similar theorem is valid for α-limit sets.

Definition 2.5.13. *Consider a Riemannian manifold M or a normed space X, and Φ a flow on M or a semiflow on X. In the case of a Riemannian manifold M, we denote by d the Riemann metric on M. In the case of a normed space X, d is defined by $d(x, y) = \|x - y\|$, where $\|\cdot\|$ is the norm on X. Let $x_0 \in M$ or X.*

1. *$\Phi(t, x_0)$ is said to be* stable *if, for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, for any $y_0 \in M$ (respectively X) such that $d(x_0, y_0) < \delta$, then $d(\Phi(t, x_0), \Phi(t, y_0)) < \epsilon$ for any $t > 0$;*

2. $\Phi(t, x_0)$ *is said to be* asymptotically stable *if it is stable and if there exists a constant* $b > 0$ *such that, if* $d(x_0, y_0) < b$ *for any* $y_0 \in M$ *(respectively* X*), then* $\lim_{t\to\infty} d(\Phi(t, x_0), \Phi(t, y_0)) = 0$*;*

3. $\Phi(t, x_0)$ *is said to be* unstable *if it is not stable;*

4. *Suppose that* x_0 *is an equilibrium for* Φ *and let us denote by* f *the vector field associated to* Φ*. Then we call* x_0 hyperbolic *if the linearization* $D_{x_0} f$ *of* f *at* x_0 *has no eigenvalues with zero real part.*

The term "bifurcation" is extremely general.

Definition 2.5.14. *Let* $\Phi(t, x, \epsilon)$ *be a flow (respectively semiflow) and* x_0 *be an equilibrium for this flow (respectively semiflow) at* $\epsilon = 0$*. We say that the flow (respectively semiflow) undergoes a* bifurcation *at* $\epsilon = 0$ *if the flow (respectively semiflow) for* ϵ *near* 0 *and* x *near* x_0 *is not topologically equivalent to the flow (respectively semiflow) at* $\epsilon = 0$ *near* $x = x_0$*.*

Definition 2.5.15. *Let* M *be a compact smooth manifold of dimension* n *and let* $N \geq 1$ *be an integer. When* $k \geq 0$ *is an integer, the* Sobolev space $\mathbf{H^k}(M, \mathbb{R}^N)$ *is defined as follows:*

$$\mathbf{H^k}(M, \mathbb{R}^N) = \{u \colon M \to \mathbb{R}^N \mid u \in \mathbf{L^2}(M, \mathbb{R}^N) \text{ and } D^\alpha u \in \mathbf{L^2}(M, \mathbb{R}^N) \text{ for } |\alpha| \leq k\},$$

where $D^\alpha u$ *is interpreted a priori as a tempered distribution.*
The norm on the Sobolev space $\mathbf{H^k}(M, \mathbb{R}^N)$ *is* $\|u\|_{\mathbf{H^k}, \mathbb{R}^N} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{\mathbf{L^2}, \mathbb{R}^N}$*.*

Theorem 2.5.16 ([73]). *Let* M *be a compact smooth manifold of dimension* n *and* $u \in \mathbf{H^s}(M, \mathbb{R}^N)$ *for* $s \geq 0$, $N \geq 1$, s, $N \in \mathbb{Z}$*. Then*

1. $u \in C(M, \mathbb{R}^N)$ *provided* $s > \frac{n}{2}$*;*

2. $u \in C^k(M, \mathbb{R}^N)$ *provided* $s > \frac{n}{2} + k$*.*

Also, $C^\infty(M, \mathbb{R}^N)$ *is dense in* $\mathbf{H^s}(M, \mathbb{R}^N)$ *and in* $\mathbf{L^2}(M, \mathbb{R}^N)$*.*

2.6 Equivariant Dynamical Systems

Definition 2.6.1. *Let* M *be a smooth manifold or a Banach space and* G *a Lie group acting on* M*. If* M *is a manifold, then the action of* G *on* TM *is defined by*

$$(A, X) \to AX = \left(\frac{d}{dt}(Ag(t))\right)\bigg|_{t=0} \text{ for } A \in G,\ X \in T_p(M) \text{ and } p \in M,$$

where $g(t)$ *is a differentiable map from* $\mathbb{R}$ *into* M *satisfying* $g(0) = p$, $\dot{g}(0) = X$*.*

1. *A vector field X on M is called G-*equivariant *if*

$$X(Ap) = AX(p) \text{ for all } A \in G, p \in M.$$

2. *A semiflow (respectively flow) Φ on M is called G-*equivariant *if*

$$\Phi(t, Ap) = A\Phi(t,p) \text{ for all } A \in G,\ p \in M \text{ and } t \in [0,\infty)\ (\text{respectively } t \in \mathbb{R}).$$

Definition 2.6.2. *Let G be a Lie group acting on a smooth manifold M. A function $f\colon M \to \mathbb{R}$ is called G-* invariant *if*

$$f(Ap) = f(p) \text{ for all } A \in G,\ p \in M.$$

Theorem 2.6.3 ([29]). *The flow (respectively semiflow) associated to an G-equivariant vector field is G-equivariant.*

Theorem 2.6.4 ([23], [33], [34], [35]). *Let G be a compact Lie group acting on a smooth manifold M and $1 \leq r \leq \infty$. Assume that $X\colon M \to TM$ is a G-equivariant C^r smooth vector field on M. Let $N \subset M$ be a compact smooth submanifold which is invariant under the flow Φ corresponding to X and G-invariant. Assume that N is normally hyperbolic under the flow Φ. Let $H \subset G$ be a subgroup and $Y\colon M \to TM$ be a H-equivariant C^r vector field on M with $\|X - Y\|_{C^r} \leq \epsilon$. Then, if $\epsilon > 0$ is sufficiently small, there exists a unique sufficiently smooth manifold N_ϵ near N which is invariant under the flow Φ_ϵ corresponding to Y and H-invariant. Moreover, there exists a C^r diffeomorphism $T_\epsilon\colon N \to N_\epsilon$ which is H-equivariant.*

2.7 Hopf Bifurcation

Theorem 2.7.1 (Hopf Bifurcation, [28], [83]). *Consider the ordinary differential equations*

$$\dot{z} = g(z,\mu), z \in \mathbb{R}^n, \mu \in \mathbb{R}, \tag{2.7.1}$$

where $g \in C^k$ $(k \geq 5)$ on an open set U containing the fixed point $(0,0)$ of g, that is $g(0,0) = 0$. Suppose that $D_z g(0,0)$ has two purely imaginary eigenvalues with the remaining $n-2$ eigenvalues having nonzero real parts. After performing center manifold reduction and normal forms transformation, the vector field (2.7.1) has the following form

$$\begin{aligned} \dot{x} &= \alpha(\mu)x - \omega(\mu)y + (a(\mu)x - b(\mu)y)(x^2+y^2) + O(|x|^5, |y|^5), \\ \dot{y} &= \omega(\mu)x + \alpha(\mu)y + (b(\mu)x + a(\mu)y)(x^2+y^2) + O(|x|^5, |y|^5), \end{aligned} \tag{2.7.2}$$

where $\lambda(\mu) = \alpha(\mu) \pm i\omega(\mu)$ *are the eigenvalues of* $D_z g(0,\mu)$ *satisfying* $\alpha(0) = 0, \omega(0) \neq 0$ *(we suppose* $g(0,\mu) = 0$ *for* $|\mu|$ *small). In polar coordinates* (r,θ) *we have,*

$$\begin{aligned} \dot{r} &= \alpha(\mu)r + a(\mu)r^3 + O(r^5), \\ \dot{\theta} &= \omega(\mu) + b(\mu)r^2 + O(r^4). \end{aligned} \tag{2.7.3}$$

After expanding in Taylor series the coefficients in (2.7.3), we get

$$\begin{aligned} \dot{r} &= \alpha'(0)\mu r + a(0)r^3 + O(\mu^2 r, \mu r^3, r^5), \\ \dot{\theta} &= \omega(0) + \omega'(0)\mu + b(0)r^2 + O(\mu^2, \mu r^2, r^4). \end{aligned} \tag{2.7.4}$$

Suppose that $\alpha'(0) \neq 0$ *and* $a(0) \neq 0$*. Then, for sufficiently small* $|\mu|$*, one of the following cases holds:*

1. *Case 1.(subcritical)* $\alpha'(0) > 0$, $a(0) > 0$
   The origin is an unstable equilibrium for $\mu > 0$ *and an asymptotically stable equilibrium for* $\mu < 0$*. Furthermore, there is an unstable periodic solution for* $\mu < 0$ *of period* $\frac{2\pi}{|\omega(0)|} + O(\mu)$ *and amplitude* $O(\sqrt{|\mu|})$*;*

2. *Case 2.(supercritical)* $\alpha'(0) > 0$, $a(0) < 0$
   The origin is an asymptotically stable equilibrium for $\mu < 0$ *and an unstable equilibrium for* $\mu > 0$*. Furthermore, there is an asymptotically stable periodic solution for* $\mu > 0$ *of period* $\frac{2\pi}{|\omega(0)|} + O(\mu)$ *and amplitude* $O(\sqrt{|\mu|})$*;*

3. *Case 3.(subcritical)* $\alpha'(0) < 0$, $a(0) > 0$
   The origin is an unstable equilibrium for $\mu < 0$ *and an asymptotically stable equilibrium for* $\mu > 0$*. Furthermore, there is an unstable periodic solution for* $\mu > 0$ *of period* $\frac{2\pi}{|\omega(0)|} + O(\mu)$ *and amplitude* $O(\sqrt{|\mu|})$*;*

4. *Case 4.(supercritical)* $\alpha'(0) < 0$, $a(0) < 0$
   The origin is an asymptotically stable equilibrium for $\mu > 0$ *and an unstable equilibrium for* $\mu < 0$*. Furthermore, there is an asymptotically stable periodic solution for* $\mu < 0$ *of period* $\frac{2\pi}{|\omega(0)|} + O(\mu)$ *and amplitude* $O(\sqrt{|\mu|})$*.*

2.8 Proofs of some Theorems of Chapter 2

Proof of Theorem 2.2.9. We first prove (2.2.5).

Let $X = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$.

Then $X^2 = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}\begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix} = \begin{pmatrix} -a^2-b^2 & bc & ac \\ bc & -a^2-c^2 & ab \\ ac & ab & -b^2-c^2 \end{pmatrix}$

and

$X^3 = X^2X = \begin{pmatrix} -a^2-b^2 & bc & ac \\ bc & -a^2-c^2 & ab \\ ac & ab & -b^2-c^2 \end{pmatrix}\begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix} =$

$= -(a^2+b^2+c^2)\begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$.

Thus, $X^3 = -|X|^2 X$ and the general statement is easy to get (it is also stated in [53]).
To prove (2.2.4), we use the relation $\overrightarrow{Ad(A)(X)} = A\overrightarrow{X}$ for $A \in SO(3)$ and $X \in so(3)$.
We have $|AXA^{-1}|^2 = \left\|\overrightarrow{AXA^{-1}}\right\|^2 = \left\|\overrightarrow{Ad(A)(X)}\right\|^2 = \left\|A\overrightarrow{X}\right\|^2 = (A\overrightarrow{X})^T(A\overrightarrow{X}) = \overrightarrow{X}^TA^TA\overrightarrow{X} = \left\|\overrightarrow{X}\right\|^2 = |X|^2$. $\square$

Chapter 3

Reaction-diffusion Systems on $r\mathrm{S}^2$

We consider a reaction-diffusion system of the form

$$\frac{\partial u}{\partial t}(t,x) = D\Delta u(t,x) + F(u(t,x)) \text{ on } r\mathbf{S}^2, \tag{3.0.1}$$

where $r > 0$, $r\mathbf{S}^2$ is the sphere of radius r, $\mathbf{S}^2$ is the unit sphere in $\mathbb{R}^3$, $u = (u_1,\, u_2,\, \ldots,\, u_N)\colon r\mathbf{S}^2 \longrightarrow \mathbb{R}^N$ with $N \geq 1$,

$D = \begin{pmatrix} d_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & d_N \end{pmatrix}$ with $d_i \geq 0$ for $i = 1,\, 2,\, \ldots,\, N$ are the diffusion coefficients and $F = (F_1,\, F_2,\, \ldots,\, F_N)\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a C^{k+2} function with $0 \leq k \leq \infty$. We have

$D\Delta u = \begin{pmatrix} d_1\Delta u_1 \\ d_2\Delta u_2 \\ \vdots \\ d_N\Delta u_N \end{pmatrix}$. We study the reaction-diffusion system (3.0.1) on the function space

$$\mathbf{Y} = \begin{cases} \mathbf{L}^2(r\mathbf{S}^2, \mathbb{R}^N) & \text{if } d_i > 0 \text{ for } i = 1,\, 2,\, \ldots,\, N; \\ \mathbf{H}^2(r\mathbf{S}^2, \mathbb{R}^N) & \text{if there exists } i \in \{1,\, 2\, \ldots,\, N\} \text{ such that } d_i = 0. \end{cases}$$

Some well-known examples of reaction-diffusion systems on the plane and on $\mathbb{R}^3$ in the literature are:

1. The FitzHugh-Nagumo model
2. The Oregonator model

3. The Ginzburg-Lindau equation

Some of these examples have been studied on a sphere in $\mathbb{R}^3$ as well (FitzHugh-Nagumo and Oregonator).

3.1 Existence and Uniqueness of Solutions

Throughout this section the main references are [39], [61].

Definition 3.1.1. *An* analytic semigroup *on a Banach space X is a family of continuous linear operators on X, $\{T(t)\}_{t\geq 0}$, satisfying*

1. $T(0) = I, T(t)T(s) = T(t+s)$ *for* $t \geq 0,\ s \geq 0$;
2. $T(t)x \to x$ *as* $t \to 0+$ *for each* $x \in X$;
3. $t \to T(t)x$ *is real analytic on* $(0, \infty)$ *for each* $x \in X$.

The infinitesimal generator L *of this semigroup is defined by*

$$Lx = \lim_{t\to 0+} \frac{1}{t}(T(t)x - x),$$

its domain $D(L)$ consisting of all $x \in X$ for which this limit (in X) exists. We usually write $T(t) = e^{Lt}$.

Definition 3.1.2. *We call a linear operator A in a Banach space X a* sectorial operator *if it is a closed densely defined operator such that, for some $\phi \in (0, \frac{\pi}{2})$ and some $M \geq 1$ and $a \in \mathbb{R}$, the sector*

$$S_{a,\phi} = \{\lambda \in \mathbb{C} \mid \phi \leq |arg(\lambda - a)| \leq \pi, \lambda \neq a\}$$

is in the resolvent set of A and

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda - a|}$$

for all $\lambda \in S_{a,\phi}$. We recall that the domain of definition of A is denoted by $D(A)$ and the range of A by $R(A)$.

Definition 3.1.3. *Suppose A is a sectorial operator in a Banach space X such that $Re\,\sigma(A) > 0$, where $\sigma(A)$ is the spectrum of A. Then, for any $\alpha > 0$ we define*

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At}\, dt$$

and

$$A^{\alpha} = (A^{-\alpha})^{-1}.$$

Also $A^0 = I$. These are called fractional powers *of A .*

Definition 3.1.4. *Suppose A is a sectorial operator in a Banach space X and suppose that there exists a real number a such that $A_1 = A + aI$ satisfies $Re\,\sigma(A_1) > 0$.*
For each $\alpha \geq 0$, we define

$$X^{\alpha} = D(A_1^{\alpha}) \text{ with graph norm } \|x\|_{\alpha} = \|A_1^{\alpha} x\|, x \in X^{\alpha}.$$

These spaces are called fractional spaces *of X relative to A. Different choices of a give equivalent norms on X^{α}.*

Theorem 3.1.5 ([24], [38], [39]). *Let M be a compact smooth manifold of dimension n and A be a sectorial operator in $X = \mathbf{L^2}(M)$ such that the fractional spaces of X relative to A are defined and $D(A) \subset \mathbf{H^2}(M)$. Then for $\alpha \in (\frac{1}{2}, 1)$, $X^{\alpha} \subset C(M)$.*

Definition 3.1.6. *If A is a linear operator in a Banach space X, and $\sigma(A)$ denotes its spectrum, a set $\sigma \subset \sigma(A) \cup \{\infty\} \equiv \widehat{\sigma(A)}$ is a* spectral set *if both σ and $\widehat{\sigma(A)} \backslash \sigma$ are closed in the extended plane $\mathbb{C} \cup \{\infty\}$. The* spectral projection *associated to the bounded spectral set σ is defined by the integral*

$$P_* = \int_{\gamma} (\lambda I - A)^{-1}\, d\lambda,$$

where γ is a closed positively oriented Jordan curve in $\mathbb{C}$ and σ is in the interior of γ.

Theorem 3.1.7 ([39], [61]).

1. *If A is a sectorial operator in a Banach space X, then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t \geq 0}$ and the converse is also true ([24]).*

2. *Let A be a sectorial operator in a Banach space X such that $Re\,\sigma(A) > 0$. Then, for any $\alpha > 0$, $A^{-\alpha}$ is a bounded linear operator on X which is one-to-one and satisfies $A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$ whenever $\alpha > 0$, $\beta > 0$. Moreover, A^{α} is closed densely defined and $D(A^{\alpha}) \subset D(A^{\beta})$ for $\alpha \geq \beta \geq 0$.*

3. *Let A be a sectorial operator in a Banach space X such that $Re\,\sigma(A_1) > 0$, where $A_1 = A + aI$, $a \in \mathbb{R}$. Then, X^α is a Banach space with graph norm $\|\cdot\|_\alpha$ for any $\alpha \geq 0$ and for any $\alpha \geq \beta \geq 0$, X^α is a dense subspace of X^β with continuous inclusion. If A has compact resolvent, then the inclusion $X^\alpha \subset X^\beta$ is compact when $\alpha > \beta \geq 0$.*

Consider the system of semilinear parabolic differential equation

$$\frac{du}{dt} = -Au + f(t, u, \mu), \text{ on } \mathbf{Y}, \tag{3.1.1}$$

where we assume that A is a sectorial operator in a Banach space $\mathbf{Y}$ so that the fractional powers of $A_1 = A + aI$ are well defined and the spaces $\mathbf{Y}^\alpha = D(A_1^\alpha)$ with the graph norm $\|x\|_\alpha = \|A_1^\alpha x\|$ are defined for $\alpha \geq 0$ (that is $Re\,\sigma(A_1) > 0$), where $a \in \mathbb{R}$. We also assume that f maps some open set $U \subset \mathbb{R} \times \mathbf{Y}^\alpha \times \mathbb{R}^n$ into $\mathbf{Y}$ for some $\alpha \in [0, 1)$, and f is continuous in t and μ, and locally Lipschitz in u on U, where $n \geq 1$.

Example 3.1.8. *The reaction-diffusion system (3.0.1) is an example of the abstract differential equation (3.1.1), where*

$$\mathbf{Y} = \begin{cases} \mathbf{L}^2(r\mathbf{S}^2, \mathbb{R}^N) & \text{if } d_i > 0 \text{ for } i \in \{1, 2, \ldots, N\}; \\ \mathbf{H}^2(r\mathbf{S}^2, \mathbb{R}^N) & \text{if there exists an } i = 1, 2, \ldots, N \text{ such that } d_i = 0. \end{cases}$$

the linear part A is given by $A = \begin{pmatrix} -d_1\Delta \\ -d_2\Delta \\ \vdots \\ -d_N\Delta \end{pmatrix}$ *with $d_i \geq 0$ for $i = 1, 2, \ldots, N$, and the nonlinearity f in (3.1.1) is the nonlinearity F from (3.0.1).*

Definition 3.1.9. 1. *A* classical solution *of the initial value problem*

$$\begin{aligned} \frac{du}{dt} &= -Au + f(t, u, \mu),\ t > 0, \\ u(0) &= u_0 \text{ for } (0, u_0, \mu) \in U, \end{aligned} \tag{3.1.2}$$

is a continuous function $u(., u_0, \mu)\colon [0, t_0) \longrightarrow \mathbf{Y}$ such that $u(0, u_0, \mu) = u_0$ and on $(0, t_0)$ we have $(t, u(t, u_0, \mu), \mu) \in U$, $u(t, u_0, \mu) \in D(A)$, $\frac{\partial u}{\partial t}(t, u_0, \mu)$ exists, $t \longrightarrow f(t, u(t, u_0, \mu), \mu)$ is continuous, $\int_0^{t_0} \|f(t, u(t, u_0, \mu), \mu)\|\, dt < \infty$, and the first differential equation from (3.1.2) is satisfied on $(0, t_0)$.

2. *A classical solution $u(.,u_0,\mu)$ of (3.1.2) is called a C^k* solution *$(1 \le k \le \infty)$ if $u(t,u_0,\mu)$ is C^k in (t,u_0,μ) such that $(t,0,u_0,\mu) \in [0,t_0) \times U$.*

Definition 3.1.10. *Consider the integral equation*

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(s,u(s),\mu)\, ds \text{ for } (0,u_0,\mu) \in U. \tag{3.1.3}$$

If $u(.,u_0,\mu)$ is a continuous function from $[0,t_0)$ into $\mathbf{Y}^\alpha$, such that on $(0,t_0)$ we have $(t,u(t,u_0,\mu),\mu) \in U$ and $\int_0^\rho \|f(s,u(s,u_0,\mu),\mu)\|\ ds < \infty$ for some $\rho > 0$, and if the integral equation (3.1.3) holds for $t \in [0,t_0)$, then $u(.,u_0,\mu)$ is called a mild solution *of the initial value problem (3.1.2).*

Theorem 3.1.11 ([39]). *Consider the semilinear parabolic differential equation (3.1.1). We assume that A is a sectorial operator in a Banach space $\mathbf{Y}$ so that the fractional powers of $A_1 = A + aI$ are well defined and the spaces $\mathbf{Y}^\alpha = D(A_1^\alpha)$ with the graph norm $\|x\|_\alpha = \|A_1^\alpha x\|$ are defined for $\alpha \ge 0$ (that is $Re\,\sigma(A_1) > 0$), where $a \in \mathbb{R}$. We also assume that f maps some open set $U \subset \mathbb{R} \times \mathbf{Y}^\alpha \times \mathbb{R}^n$ into $\mathbf{Y}$ for some $\alpha \in [0,1)$, and f is continuous in t and μ, and locally Lipschitz in u on U, where $n \ge 1$.*
Then, for any $(0,u_0,\mu) \in U$ there exists a unique mild solution $u(.,u_0,\mu)$ of the initial value problem (3.1.2) on $[0,t_0)$. If f is C^k where $1 \le k \le \infty$, then $u(.,u_0,\mu)$ is a C^k classical solution.

Definition 3.1.12 ([73]). *The* Laplacian *on the sphere $r\mathbf{S}^2$ is the closed densely defined operator in $\mathbf{L}^2(r\mathbf{S}^2,\mathbb{R})$ given by*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2},$$

where $u \in \mathbb{C}^\infty(r\mathbf{S}^2,\mathbb{R})$, $(x,y,z) \in r\mathbf{S}^2$ and $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$ with $\phi \in [0,\pi]$ and $\theta \in [0,2\pi)$.

Definition 3.1.13. *A* surface spherical harmonic *f is a function $f\colon \mathbf{S}^2 \to \mathbb{C}$, $f \ne 0$, such that $\Delta(f) = \lambda f$ for some $\lambda \in \mathbb{C}$, and where Δ is the Laplacian defined in 3.1.12 for $r = 1$.*

Let us define the *Legendre polynomials* P_n by $2^n n! P_n(x) = \frac{d^n}{dx^n}[(x^2-1)^n]$ for $n = 0, 1, 2, \ldots$ and the *associated Legendre functions* $P_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m P_n(x)}{dx^m}$ for $|m| \le n$ and $n = 0, 1, 2, \ldots$.

Theorem 3.1.14 ([72], [74]). *1. The surface spherical harmonics are given by* $f_n^m(\theta, \phi) = e^{im\phi} P_n^m(\cos\theta)$ *for* $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$, $n = 0, 1, 2, \ldots$ *and* $|m| \leq n$, *and they form an orthogonal basis of* $\mathbf{L}^2(\mathbf{S}^2, \mathbb{C})$.

2. *Every* C^2 *function* $f \colon \mathbf{S}^2 \to \mathbb{C}$ *can be expanded in an absolutely and uniformly convergent series of surface spherical harmonics. Let* $\{Y_n^m \colon |m| \leq n, n = 0, 1, 2, \ldots\}$ *be an orthonormal basis for* $\mathbf{L}^2(\mathbf{S}^2, \mathbb{C})$ *formed by surface spherical harmonics, then*

$$f(\theta, \phi) = \sum_{n \geq 0} \sum_{|m| \leq n} \widehat{f}(n, m) Y_n^m(\theta, \phi), \text{ where } \widehat{f}(n, m) = \int_{\mathbf{S}^2} f \overline{Y_n^m} \sin\theta \, d\theta d\phi.$$

3. *Let us define for* $n = 0, 1, 2, \ldots$ *and* $g \in SO(3)$ *a* $(2n+1) \times (2n+1)$ *matrix* $A_n(g)$ *where the entries are defined by* $Y_n^m(gx) = \sum_{|k| \leq n} (A_n)_{m,k} Y_n^k(x)$ *for* $|m| \leq n$. *Then, for any* $n = 0, 1, 2, \ldots$, $A_n(g)$ *defines an irreducible unitary representation of* $SO(3)$. *Moreover, any function* $f \in \mathbf{L}^2(SO(3), \mathbb{C})$ *with respect to Haar measure has a Fourier series expansion, converging in the* $\mathbf{L}^2$ *norm,*

$$f(g) = \sum_{n=0}^{\infty} (2n+1) Tr[\widehat{f}(n) A_n(g)], \text{ where } \widehat{f}(n) = \int_{SO(3)} f(g) \overline{A_n(g)} \, dg$$

and dg *is the Haar measure on* $SO(3)$.

Theorem 3.1.15 ([39], [73], [84]). *The Laplacian* Δ *as an operator in* $\mathbf{L}^2(r\mathbf{S}^2, \mathbb{R})$ *has the spectrum formed by a countable set* $\{\lambda_n\}_{n \geq 0}$ *of zero and negative eigenvalues, and finite dimensional generalized eigenspaces, and* $D(\Delta) = \mathbf{H}^2(r\mathbf{S}^2, \mathbb{R})$. *Also, there exists an analytic semigroup* $\{e^{t\Delta}\}_{t \geq 0}$ *in* $\mathbf{L}^2(r\mathbf{S}^2, \mathbb{R})$ *with the infinitesimal generator* Δ.
Therefore, $-\Delta$ *is a sectorial operator in* $\mathbf{Y}$, *with* $N = 1$ *in the definition of the space* $\mathbf{Y}$, *and for any* $\alpha \in (\frac{1}{2}, 1)$, *we have*

$$\frac{\partial}{\partial x}(I - \Delta)^{-\alpha} \in \mathcal{L}(\mathbf{L}^2(r\mathbf{S}^2, \mathbb{R})),$$
$$\frac{\partial}{\partial y}(I - \Delta)^{-\alpha} \in \mathcal{L}(\mathbf{L}^2(r\mathbf{S}^2, \mathbb{R})) \text{ and}$$
$$\frac{\partial}{\partial z}(I - \Delta)^{-\alpha} \in \mathcal{L}(\mathbf{L}^2(r\mathbf{S}^2, \mathbb{R})),$$

where $\mathcal{L}(\mathbf{L}^2(r\mathbf{S}^2, \mathbb{R}))$ *is the space of bounded linear operators on* $\mathbf{L}^2(r\mathbf{S}^2, \mathbb{R})$.

Theorem 3.1.16. *Let $F\colon \mathbb{R}^N \to \mathbb{R}^N$ be a C^{k+2} function such that $F(0) = 0$, where $0 \leq k \leq \infty$. Let $\mathbf{Y}^\alpha = D((I - D\Delta)^\alpha)$, where $\alpha \in (\frac{1}{2}, 1)$. The reaction-diffusion system (3.0.1) defines a sufficiently smooth local semiflow $\mathbf{\Phi}$ on the function space $\mathbf{Y}^\alpha$. The flow is C^{k+2} if $\mathbf{Y} = \mathbf{L^2}(r\mathbf{S^2}, \mathbb{R}^N)$ and C^k if $\mathbf{Y} = \mathbf{H^2}(r\mathbf{S^2}, \mathbb{R}^N)$. Namely, for any $u_0 \in \mathbf{Y}^\alpha$, let $u(t, u_0)$ be the sufficiently smooth solution (C^{k+2} or C^k respectively) of the initial value problem given by the reaction-diffusion system (3.0.1) and by the initial condition $u(0) = u_0$, defined on the maximal interval of existence $I(u_0) = [0, t_0(u_0))$. Let $W = \{(t, u_0) \in [0, \infty) \times \mathbf{Y}^\alpha \mid t \in I(u_0)\}$. Then, the local semiflow $\mathbf{\Phi}\colon W \to \mathbf{Y}^\alpha$ is defined by $\mathbf{\Phi}(t, u_0) = u(t, u_0)$, for any $(t, u_0) \in W$.*

3.2 Action of $SO(3)$ on $\mathbf{Y}^\alpha$

Definition 3.2.1. *The* action *of $SO(3)$ on $\mathbb{R}^3$ is given by the function $\theta^R\colon SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$, defined by*

$$\theta^R(A, x) = Ax \text{ where } A \in SO(3) \text{ and } x \in \mathbb{R}^3. \tag{3.2.1}$$

Definition 3.2.2.

1. *The* action *of $SO(3)$ on the function space $\mathbf{Y}$ is given by the function $\theta\colon SO(3) \times \mathbf{Y} \to \mathbf{Y}$, defined by*

   $$\theta(A, u)(x) = u(A^{-1}x) \text{ where } A \in SO(3), u \in \mathbf{Y} \text{ and } x \in r\mathbf{S^2}. \tag{3.2.2}$$

   We denote $\theta(A, u) = Au$.

2. *The* representation *T of $SO(3)$ on $\mathbf{Y}$ is given by the function $T\colon SO(3) \to GL(\mathbf{Y})$, defined by*

   $$T(A)u(x) = u(A^{-1}x) \text{ where } A \in SO(3), u \in \mathbf{Y}, x \in r\mathbf{S^2}. \tag{3.2.3}$$

Theorem 3.2.3.

1. *The function θ^R defined in (3.2.1) and restricted to $r\mathbf{S^2}$ is a transitive left action of the group $SO(3)$ on $r\mathbf{S^2}$.*

2. *The function T defined by (3.2.3) is a unitary representation of the group $SO(3)$ on the function space $\mathbf{Y}$. The function θ defined by (3.2.2) is a left action of the group $SO(3)$ on $\mathbf{Y}$.*

Theorem 3.2.4. *The Laplacian operator $\Delta\colon \mathbf{Y} \to \mathbf{Y}$, where $N = 1$ in the definition of the space $\mathbf{Y}$, is $SO(3)$-equivariant with respect to the action θ defined by (3.2.2). The restriction of the representation T defined by (3.2.3) to $\mathbf{Y}^\alpha = D((I - D\Delta)^\alpha)$, where $\alpha \geq 0$ is a unitary* *representation of $SO(3)$ on $\mathbf{Y}^\alpha$. The restriction of the action θ defined by (3.2.2) to $\mathbf{Y}^\alpha$, where $\alpha \geq 0$ is also a left action of $SO(3)$ on $\mathbf{Y}^\alpha$.*

The left action of $SO(2) \subset SO(3)$ on $\mathbf{Y}$ is defined as the restriction of the action θ defined by (3.2.2) to $SO(2)$. We will use the same notation for this action.

Theorem 3.2.5. *Let*

$$Z_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; Z_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } Z_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The infinitesimal generators of the representation T defined by (3.2.3) are bounded operators on $\mathbf{Y}^\alpha = D((I - D\Delta)^\alpha)$ for any $\alpha \in (\frac{1}{2}, 1)$, and they are given by

1. $T'(Z_1) = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$;
2. $T'(Z_2) = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$;
3. $T'(Z_3) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$.

Theorem 3.2.6. *The restriction of the representation T defined in (3.2.3) to $\mathbf{Y}^\alpha = D((I - D\Delta)^\alpha)$, where $\alpha \in (\frac{1}{2}, 1)$, is smooth.*

3.3 $SO(3)$-Equivariance of a Reaction-diffusion System on $r\mathbf{S}^2$

Theorem 3.3.1. *Let $F\colon \mathbb{R}^N \to \mathbb{R}^N$ be a C^{k+2} function such that $F(0) = 0$ and $1 \leq k \leq \infty$. Let $\mathbf{Y}^\alpha = D((I - D\Delta)^\alpha)$, where $\alpha \in (\frac{1}{2}, 1)$. The local semiflow $\mathbf{\Phi}$ given in Theorem 3.1.16 is $SO(3)$-equivariant with respect to the action θ defined in (3.2.2) restricted to $\mathbf{Y}^\alpha$.*

3.4 Parameter-dependent Reaction-diffusion Systems on $r\mathbf{S}^2$

We consider a parameter-dependent reaction-diffusion system of the form

$$\frac{\partial u}{\partial t}(t, x) = D\Delta u(t, x) + F(u(t, x), \lambda) \text{ on } r\mathbf{S}^2, \tag{3.4.1}$$

where $r > 0$, $r\mathbf{S}^2$ is the sphere of radius r, $\mathbf{S}^2$ is the unit sphere in $\mathbb{R}^3$, $u = (u_1, u_2, \ldots, u_N)\colon r\mathbf{S}^2 \to \mathbb{R}^N$ with $N \geq 1$,

$D = \begin{pmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_N \end{pmatrix}$, $d_i \geq 0$, $i = 1, 2, \dots, N$ are the diffusion coefficients and $F = (F_1, F_2, \dots, F_N)\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ a C^{k+2} function with $0 \leq k \leq \infty$. We study the reaction-diffusion system (3.4.1) on the function space

$$\mathbf{Y} = \begin{cases} \mathbf{L}^2(r\mathbf{S}^2\mathbb{R}^N), & \text{if } d_i > 0 \text{ for } i = 1, 2 \dots, N; \\ \mathbf{H}^2(r\mathbf{S}^2, \mathbb{R}^N) & \text{if there exists an } i \in \{1, 2, \dots, N\} \text{ such that } d_i = 0. \end{cases}$$

Theorem 3.4.1. *Let F be a C^{k+2} function such that $F(0, \lambda) = 0$ for $|\lambda|$ small, and $1 \leq k \leq \infty$. Let $\mathbf{Y}^\alpha = D((I - D\Delta)^\alpha)$, where $\alpha \in (\frac{1}{2}, 1)$. The reaction-diffusion system* (3.4.1) *defines a sufficiently smooth (C^{k+2} if $\mathbf{Y} = \mathbf{L}^2(r\mathbf{S}^2, \mathbb{R}^N)$ and C^k if $\mathbf{Y} = \mathbf{H}^2(r\mathbf{S}^2, \mathbb{R}^N)$) parameter-dependent local semiflow $\mathbf{\Phi}\colon W \subset [0, \infty) \times \mathbf{Y}^\alpha \times (-\lambda_0, \lambda_0) \to \mathbf{Y}^\alpha$ for $\lambda_0 > 0$ small. The local semiflow $\mathbf{\Phi}$ is $SO(3)$-equivariant with respect to the action θ defined in* (3.2.2) *restricted to $\mathbf{Y}^\alpha$, where W is defined as in Theorem* 3.1.16.

Remark 3.4.2. *The theorem* 3.4.1 *is valid for more than one parameter as well.*

3.5 Examples

Some examples of reaction-diffusion systems are the following:

Example 3.5.1 (The FitzHugh-Nagumo model).

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{\epsilon}(-\frac{u^3}{3} + u - v) + D_u \Delta u, \\ \frac{\partial v}{\partial t} &= \epsilon(u - \alpha v + \beta) + D_v \Delta v, \end{aligned}$$

where u and v are two scalar fields, D_u, D_v are the constant diffusion coefficients.
The parameters α and β characterize the local dynamics and, hence, the local excitability. There are other variants of the system (3.5.1). *The system* (3.5.1) *or a variant of the system* (3.5.1) *on a sphere was studied in [30] and [89]. Also, it was studied on a spherical shell in [10] and [13].*
This system is also described in [44].

Example 3.5.2 (The Oregonator model).

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u \Delta u + \frac{1}{\epsilon}(u - u^2 - fv\frac{u-q}{u+q}),\\ \frac{\partial v}{\partial t} &= D_v \Delta v + u - v,\end{aligned}$$

where u is the 'propagator' variable, v is the 'recovery' variable, and D_u, D_v are diffusion coefficients. The parameters $\frac{1}{\epsilon}$ measures ratio of the excitation rate to the recovery rate, f measures the threshold for excitation relative to the amplitude of uniform excitation. This model was studied on a sphere with radius $R = 30$ in [87].

Example 3.5.3 (The Ginzburg-Lindau equation).

$$\frac{\partial A}{\partial t} = A + (1 + ib)\Delta A - (1 + ic) |A|^2 A,$$

where $A = A_1 + iA_2$ is a complex function of time t and space $(x, y) \in \mathbb{R}^2$, the linear parameters b and c characterize linear and nonlinear dispersion. If $b \neq c$, spiral waves are observed [58].

3.6 Proofs of some Theorems of Chapter 3

Proof of Theorem 3.1.16. By theorem 3.1.15, we have that $-D\Delta$ is a sectorial operator on $\mathbf{Y}$.

That F is a sufficiently smooth operator from $\mathbf{Y}^\alpha$ into $\mathbf{Y}$ for any $\alpha \in (\frac{1}{2}, 1)$ is deducted as in [84], because the Sobolev embedding theorems used in [84] are valid for $r\mathbf{S}^2$ (see Theorems 2.5.16 and 3.1.5) and $r\mathbf{S}^2$ is also compact and connected. The superposition operator F is C^{k+2} if $\mathbf{Y} = \mathbf{L^2}(r\mathbf{S^2}, \mathbb{R}^N)$ and C^k if $\mathbf{Y} = \mathbf{H^2}(r\mathbf{S^2}, \mathbb{R}^N)$ (see [84]).

Then, Theorem 3.1.11 implies that the local semiflow $\mathbf{\Phi}(t, u)$ on $\mathbf{Y}^\alpha$ is sufficiently smooth for any $\alpha \in (\frac{1}{2}, 1)$, where the degree of smoothness is as indicated in the statement of the theorem. $\square$

Proof of Theorem 3.2.3. *(1)* We prove that θ^R is a left action.

1. $\theta^R(AB, x) = ABx$ and $\theta^R(A, \theta^R(B, x)) = \theta^R(A, Bx) = A(Bx) = ABx$ for any A,$B \in SO(3), x \in \mathbb{R}^3$;

2. $\theta^R(I_3, x) = I_3 x = x$ for any $x \in \mathbb{R}^3$.

If $x \in r\mathbf{S}^2$, we clearly have $Ax \in r\mathbf{S}^2$.
The transitivity of θ^R on the sphere $r\mathbf{S}^2$ is proved in [79].
(2) Suppose that $\mathbf{Y} = \mathbf{L^2}(r\mathbf{S^2}, \mathbb{R}^N)$ and $u \in \mathbf{Y}$, $A \in SO(3)$.
We check that the function T is well-defined.
We have $\int_{r\mathbf{S}^2} \|(T(A)u)(x)\|^2 \, dx = \int_{r\mathbf{S}^2} \left\|u(A^{-1}x)\right\|^2 \, dx = \int_{r\mathbf{S}^2} \|u(y)\|^2 \, dy < \infty$, that is $T(A)u \in \mathbf{Y}$.
It is clear that $T(A)$ is linear and invertible for any $A \in SO(3)$, the inverse of $T(A)$ being $T(A^{-1})$.
Since $\|T(A)u\|_{\mathbf{L}^2,\mathbb{R}^N} = \|u\|_{\mathbf{L}^2,\mathbb{R}^N}$ for any $u \in \mathbf{Y}$, $T(A)$ is an isometry on $\mathbf{Y}$ (see also [73]) for any $A \in SO(3)$. Therefore, $T(A)$ is bounded on $\mathbf{Y}$ for any $A \in SO(3)$.
Moreover, T is a representation of $SO(3)$ on the function space $\mathbf{Y}$ because

1. $T(I_3)u(x) = u(I_3^{-1}x) = u(x)$ for any $x \in r\mathbf{S}^2$ and $u \in \mathbf{Y}$;

2. For A, $B \in SO(3)$, $x \in r\mathbf{S}^2$ and $u \in \mathbf{Y}$, we have $T(AB)u(x) = u((AB)^{-1}x) = u(B^{-1}A^{-1}x) = (T(B)u)(A^{-1}x) = T(A)(T(B)u(x))$. Then, $(T(A)T(B))u(x) = u(B^{-1}A^{-1}x)$.

Suppose that $\mathbf{Y} = \mathbf{H^2}(r\mathbf{S^2}, \mathbb{R}^N)$ and $u \in \mathbf{Y}$, $A \in SO(3)$.
We have that $u \in C^\infty \Longrightarrow T(A)u \in C^\infty$ and then using the density of C^∞ in $\mathbf{H^2}(r\mathbf{S^2}, \mathbb{R}^N)$ (see Theorem 2.5.16), it follows that $T(A)u \in \mathbf{Y}$. The same result can be obtained if we take into account the characterization of the Sobolev space $\mathbf{H^2}(r\mathbf{S^2}, \mathbb{R})$ by spherical Fourier transform (see [51], [72], [74]) and Theorem 3.1.14. From this, we also get that $T(A)$ is an isometry on $\mathbf{Y}$ for any $A \in SO(3)$.
If $\mathbf{Y} = \mathbf{L^2}(r\mathbf{S^2}, \mathbb{R}^N)$ or $\mathbf{Y} = \mathbf{H^2}(r\mathbf{S^2}, \mathbb{R}^N)$, it follows from the continuity of the action θ^R, the fact that $T(A)$ is an isometry on $\mathbf{Y}$ and the density of $C^\infty(r\mathbf{S^2}, \mathbb{R}^N)$ in $\mathbf{Y}$ (see Theorem 2.5.16), that the function $A \to T(A)u$ is continuous from $SO(3)$ into $\mathbf{Y}$ for any $u \in \mathbf{Y}$.
Since T is a representation of $SO(3)$ on $\mathbf{Y}$, it is clear that θ is a left action of $SO(3)$ on $\mathbf{Y}$.
By theorem 2.4.3, θ is a linear action. $\square$

Proof of Theorem 3.2.4. The $SO(3)$-equivariance of the operator $\Delta \colon \mathbf{Y} \to \mathbf{Y}$ (with $N = 1$ in the definition of $\mathbf{Y}$) is proved in [15], [72], [73] or [74].
Therefore, the operator $D\Delta \colon \mathbf{Y} \to \mathbf{Y}$ is $SO(3)$-equivariant. From the $SO(3)$-equivariance of the operator $D\Delta \colon \mathbf{Y} \to \mathbf{Y}$ it follows that $\mathbf{Y}^\alpha$ is invariant under the representation T and that T is also a unitary representation of $SO(3)$ on $\mathbf{Y}^\alpha$. Moreover, it follows that θ is a left action of $SO(3)$ on $\mathbf{Y}^\alpha$. $\square$

Proof of Theorem 3.2.5. From [72] or [74], we get the form of the infinitesimal generators $T^{'}(Z_1), T^{'}(Z_2), T^{'}(Z_3)$.

Let $\mathbf{Y} = \mathbf{L^2}(r\mathbf{S^2}, \mathbb{R}^N)$.

From Theorem 3.1.15, the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are bounded from $\mathbf{Y}^\alpha$ into $\mathbf{Y}$ for any $\alpha \in (\frac{1}{2}, 1)$. Hence, since $(x, y, z) \in r\mathbf{S^2}$, it follows that the infinitesimal generators $T^{'}(Z_1)$, $T^{'}(Z_2)$, $T^{'}(Z_3)$ are bounded from $\mathbf{Y}^\alpha$ into $\mathbf{Y}$ for any $\alpha \in (\frac{1}{2}, 1)$.

Let $\mathbf{Y} = \mathbf{H^2}(r\mathbf{S^2}, \mathbb{R}^N)$.

We know that $D(\Delta) = \mathbf{H}^2(r\mathbf{S}^2, \mathbb{R}) \subset D(\frac{\partial}{\partial x}), D(\frac{\partial}{\partial y}), D(\frac{\partial}{\partial z})$. So, the operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ are bounded on $\mathbf{H}^2(r\mathbf{S}^2, \mathbb{R}^N)$. As before, it results that the infinitesimal generators $T^{'}(Z_1)$, $T^{'}(Z_2)$, $T^{'}(Z_3)$ are bounded on $\mathbf{Y}$. $\square$

Proof of Theorem 3.2.6. Since $\{Z_1, Z_2, Z_3\}$ is a basis of $so(3)$, Theorem 3.2.5 implies that for any $X \in so(3)$, $T^{'}(X)$ is a bounded operator on $\mathbf{Y}^\alpha$ (with range $\mathbf{Y}^\alpha$ by [78]) for any $\alpha \in (\frac{1}{2}, 1)$. Thus, it follows that T is a smooth representation of $SO(3)$ on $\mathbf{Y}^\alpha$. $\square$

Proof of Theorem 3.3.1. The $SO(3)$-equivariance of the operator $D\Delta\colon \mathbf{Y} \to \mathbf{Y}$ results from $SO(3)$-equivariance of the Laplacian operator (see Theorem 3.2.4). Since $\mathbf{Y}^\alpha$ is $SO(3)$-invariant (see the proof of Theorem 3.2.4), it follows that the operator $D\Delta\colon \mathbf{Y}^\alpha \to \mathbf{Y}$ is $SO(3)$-equivariant.

Since $(T(A)F(u))(x) = F(u)(A^{-1}x) = F(u(A^{-1}x)) = F(T(A)u(x)) = F(T(A)u)(x)$ for any $A \in SO(3)$, $x \in r\mathbf{S}^2$, $u \in \mathbf{Y}^\alpha$, the superposition operator $F\colon \mathbf{Y}^\alpha \to \mathbf{Y}$ is $SO(3)$-equivariant. Thus, the local semiflow $\mathbf{\Phi}$ obtained in theorem 3.1.16 is $SO(3)$-equivariant. $\square$

Proof of Theorem 3.4.1. Recall that $-D\Delta$ is a sectorial operator on $\mathbf{Y}$ (see the proof of Theorem 3.1.16). That the superposition operator F is a sufficiently smooth operator from $\mathbf{Y}^\alpha \times \mathbb{R}$ into $\mathbf{Y}$ for any $\alpha \in (\frac{1}{2}, 1)$ is proved as in Theorem 3.1.16. Using Theorem 3.1.11, we get that the local semiflow $\mathbf{\Phi}$ is sufficiently smooth on $\mathbf{Y}^\alpha$ for any $\alpha \in (\frac{1}{2}, 1)$.

We proved in Theorem 3.3.1 that the local semiflow $\mathbf{\Phi}$ is $SO(3)$-equivariant. $\square$

Chapter 4

Rotating Waves and Modulated Rotating Waves on $r\mathbb{S}^2$

Let us consider the reaction-diffusion system (3.0.1). Throughout this chapter we fix $\alpha \in (\frac{1}{2}, 1)$. Also, we will make references to chapter 5.

4.1 Rotating Waves

Definition 4.1.1. *Let $u_0 \in \mathbf{Y}^\alpha$ be such that the stabilizer of u_0 is $\Sigma_{u_0} = I_3$. The orbit of u_0 under the action θ defined in (3.2.2) is called a* relative equilibrium *for (3.0.1) if there exists a matrix $X_0 \in so(3)$ such that*

$$\Phi(t, u_0) = e^{X_0 t} u_0 \text{ for any } t \geq 0. \tag{4.1.1}$$

Sometimes, when $SO(3)u_0$ is a relative equilibrium, we call u_0 a relative equilibrium. *If $X_0 \neq O_3$, then any solution of the reaction-diffusion system (3.0.1) of the form $A\Phi(., u_0)$, where $A \in SO(3)$ is called a* rotating wave *for (3.0.1).*

There are other definitions of relative equilibria for an equivariant dynamical system, but in the case of the compact group $SO(3)$ it can be proved that they are equivalent to the one given above. The dynamics on relative equilibria in G-equivariant dynamical systems, where G is a compact Lie group, contain either equilibria or periodic solutions (this is the generic case) since the maximal torus in $SO(3)$ is $SO(2)$ (see [22]).

Definition 4.1.2. *Let $\mathbf{\Phi}(., Au_0)$ be a rotating wave as in Definition* 4.1.1, *where $A \in SO(3)$. Then, the vector $\overrightarrow{AX_0A^{-1}}$ is called the* frequency vector *of the rotating wave. If the vector $\frac{1}{|X_0|}\overrightarrow{AX_0A^{-1}}$ is in the north hemisphere, then $\omega_0 = |X_0|$ is called the* frequency *of the rotating wave. Otherwise, $-\omega_0$ is called the* frequency *of the rotating wave.*

Remark 4.1.3. 1. *Because the action θ of $SO(3)$ defined in (3.2.2) restricted to $\mathbf{Y}^\alpha$ is smooth, then the relative equilibrium defined in Definition* 4.1.1 *is a sufficiently smooth manifold in $\mathbf{Y}^\alpha$ diffeomorphic to $SO(3)$ since $\Sigma_{u_0} = I_3$.*

2. *Let $u_0 \in \mathbf{Y}^\alpha$ be a relative equilibrium which is not an equilibrium for (3.0.1) such that the stabilizer of u_0 is $\Sigma_{u_0} = I_3$. Then, the rotating waves in $SO(3)u_0$ have their anti-symmetric matrices associated with the frequency vectors on the same adjoint orbit.*

4.2 Modulated Rotating Waves

Definition 4.2.1. *Let $u_0 \in \mathbf{Y}^\alpha$ be such that the stabilizer of u_0 is $\Sigma_{u_0} = I_3$. The set defined by $\{A\mathbf{\Phi}(t, u_0) \mid A \in SO(3), t \in \mathbb{R}_+\}$ is called a* relative periodic orbit *for* (3.0.1) *if it is not a relative equilibrium and there exist a number $T > 0$ and a matrix $X_0 \in so(3)$ such that*

$$\mathbf{\Phi}(T, u_0) = e^{X_0 T}u_0 \text{ and } \mathbf{\Phi}(t, u_0) \notin SO(3)u_0 \text{ for any } t \in (0, T). \tag{4.2.1}$$

If $|X_0|\,T \neq 2k\pi$ for any $k \in \mathbb{Z}$, then any solution of the reaction-diffusion system (3.0.1) of the form $A\mathbf{\Phi}(., u_0)$, with $A \in SO(3)$ is called a modulated rotating wave *for (3.0.1).*

There are other definitions of relative periodic orbit for an equivariant dynamical system, but it can be proved that they are equivalent to the one given above.

Remark 4.2.2. 1. *Because the action θ of $SO(3)$ on $\mathbf{Y}^\alpha$ defined in (3.2.2) is smooth, then the relative periodic orbit defined in Definition* 4.2.1 *is a sufficiently smooth manifold in $\mathbf{Y}^\alpha$.*

2. $$\mathbf{\Phi}(T_1, u_0) = e^{X_0 T_1}u_0 \text{ if and only if } T_1 \in T\mathbb{Z}.$$

Proof. By the definition of a semiflow and the $SO(3)$-equivariance of the semiflow $\mathbf{\Phi}(t, u_0)$, it follows that $\mathbf{\Phi}(kT, u_0) = e^{X_0 kT}u_0$ for any $k \in \mathbb{Z}$.

Let $T_1 \in \mathbb{R}$ be such that $\Phi(T_1, u_0) \in SO(3)u_0$ and let $T_1 = kT + r$, where $r \in [0, T)$ and $k \in \mathbb{Z}$.

We have $\Phi(T_1, u_0) = \Phi(kT+r, u_0) = \Phi(r, \Phi(kT, u_0)) = \Phi(r, e^{X_0 kT}u_0) = e^{X_0 kT}\Phi(r, u_0)$. Thus, $\Phi(r, u_0) \in SO(3)u_0$ and $r \in [0, T)$. By the definition of T, we get that $r = 0$. So, $T_1 = kT$. $\square$

3. *If $|X_0| = \frac{2k\pi}{T}$ for some $k \in \mathbb{Z}$, then $\Phi(t, u_0)$ is a T-periodic solution of the reaction-diffusion (*3.0.1*).*

4. *There are periodic solutions of the reaction-diffusion system (*3.0.1*) that are modulated rotating waves in the sense of Definition* 4.2.1*. We show that their period is an integer multiple of the corresponding T from Definition* 4.2.1 *and $|X_0|\,T \in 2\pi\mathbb{Q}$.*

   Proof. Let T_1 be the period of $\Phi(t, u_0)$, that is $\Phi(T_1, u_0) = u_0$. By the proof of statement 2 above, we get that $T_1 = kT$ since $\Phi(T_1, u_0) = u_0 \in SO(3)u_0$. Also, we have $\Phi(0, u_0) = u_0 = \Phi(kT, u_0) = e^{X_0 kT}u_0$ and since $\Sigma_{u_0} = I_3$, it follows that $e^{-kX_0T} = I_3$ and, by Theorem 2.2.13 (6), we get $|X_0|\,kT = 2l\pi$ for some $l \in \mathbb{Z}$, $l \geq 0$. Thus, $|X_0|\,T \in 2\pi\mathbb{Q}$. $\square$

5. *If $\Phi(., u_0)$ is a periodic solution, then $A\Phi(., u_0)$ is a periodic solution for any $A \in SO(3)$. If $\Phi(., u_0)$ is a modulated rotating wave, then $A\Phi(., u_0)$ is a modulated rotating wave for any $A \in SO(3)$.*

Definition 4.2.3. *Let $\Phi(., Au_0)$ be a modulated rotating wave as in Definition* 4.2.1*, where $A \in SO(3)$. Then, the vector $\overrightarrow{AX_0A^{-1}}$ is called a* primary frequency vector *of the modulated rotating wave and the number T is called the* secondary period *of the modulated rotating wave. If the vector $\frac{1}{|X_0|}\overrightarrow{AX_0A^{-1}}$ is in the north hemisphere, then $\omega_0 = |X_0|$ is called the* primary frequency *of the modulated rotating wave. Otherwise, $-\omega_0$ is called the* primary frequency *of the modulated rotating wave.*

Remark 4.2.4. 1. *If $\Phi(t, Au_0)$ is a T-periodic solution of the reaction-diffusion system (*3.0.1*) such that*

$$\Phi(T, Au_0) = Au_0 \text{ and } \Phi(t, Au_0) \notin SO(3)u_0 \text{ for all } t \in (0, T), \tag{4.2.2}$$

then a primary frequency vector of $\Phi(t, Au_0)$ can be any vector $\overrightarrow{X_1}$ in $\mathbb{R}^3$ with $|X_1| = \frac{2k\pi}{T}$ for $k \in \mathbb{Z}$.

2. *A primary frequency vector $\overrightarrow{X_1}$ of a modulated rotating wave $\Phi(t, Au_0)$ is unique up to an integer multiple of $\frac{2\pi}{|X_1|T}\overrightarrow{X_1}$.*

Theorem 4.2.5. *Let $u_0 \in \mathbf{Y}^\alpha$ be a relative equilibrium which is not an equilibrium for the reaction-diffusion system* (3.0.1) *and such that $\Sigma_{u_0} = I_3$. Suppose that Theorems* 5.1.2 *and* 5.2.1 *hold. Let $\Phi(., u_1)$ be a modulated rotating wave as defined in Definition* 4.2.1 *with $u_1 \in M^{cu}_{u0}$ and $\Sigma_{u_1} = I_3$, where M^{cu}_{u0} is defined in Theorem* 5.1.2. *Let $T > 0$ and $X_1 \in so(3)$ be as in Definition* 4.2.1 *for the modulated rotating wave $\Phi(., u_1)$. Then there exists a periodic function $R\colon \mathbb{R}_+ \to \mathbf{Y}^\alpha$ of period $T \neq \frac{2\pi}{|X_1|}$ such that $R(0) = u_1$ and*

$$\Phi(t, u_1) = e^{X_1 t} R(t).$$

Obviously, for any $A_0 \in SO(3)$

$$\Phi(t, A_0 u_1) = e^{A_0 X_1 A_0^{-1} t} R^{A_0}(t),$$

where $R^{A_0}(t)$ is a T-periodic function such that $R^{A_0}(0) = A_0 u_1$.

Remark 4.2.6. *Let $SO(3)\Phi(., u_0) \subset \mathbf{Y}^\alpha$ be a relative periodic orbit for* (3.0.1) *such that the stabilizer of u_0 is $\Sigma_{u_0} = I_3$. Then the modulated rotating waves in $SO(3)\Phi(., u_0)$ have the anti-symmetric matrices associated with the primary frequency vectors on the same adjoint orbit (if we consider the primary frequency vectors as unique up to a multiple of $\frac{2k\pi}{T}$, $k \in \mathbb{Z}$ of the corresponding unit primary frequency vector).*

4.3 Tip Position

Definition 4.3.1. *For $0 \leq k \leq \infty$, a C^k function $x_{tip}\colon \mathbf{Y}^\alpha \to r\mathbf{S}^2$ that is $SO(3)$-equivariant is called a* tip position *function. By the* tip *of $u \in \mathbf{Y}^\alpha$, we understand the point $x_{tip}(u) \in r\mathbf{S}^2$.*

Remark 4.3.2. *If the function x_{tip} is defined only on an open subset of $\mathbf{Y}^\alpha$, then the function x_{tip} is called a* local tip position *function.*

Theorem 4.3.3. *For the reaction-diffusion system* (3.0.1)*, consider the* tip motion *function $x_{tip}(\Phi(t, u_0))$, where x_{tip} is a tip position function and $\Phi(t, u_0)$ is the solution of the reaction-diffusion system* (3.0.1) *with the initial condition $u(0) = u_0$.*

1. *For a rotating wave $\Phi(t, u_0)$, the tip motion $x_{tip}(\Phi(t, u_0))$ is a circle on the sphere $r\mathbf{S}^2$ with the center on the line having the direction of the frequency vector of $\Phi(t, u_0)$, and this is independent of the choice of the tip position function.*

2. *For a modulated rotating wave $\Phi(t, u_0)$, the tip motion $x_{tip}(\Phi(t, u_0))$ has the property that $x_{tip}(\Phi(kT, u_0))$ for $k \in \mathbb{Z}$ are points of a circle on the sphere $r\mathbf{S}^2$ with the center on the line having the direction of the primary frequency vector of $\Phi(t, u_0)$; this is independent of the choice of the tip position function.*

Remark 4.3.4. *For a modulated rotating wave, the tip motion $x_{tip}(\Phi(t, u_0))$ may describe an epicycle-like motion. The existence of well-defined petals, as well as the size of petals depends on the choice of the tip position function.*

Let $u_0 \in \mathbf{Y}^\alpha$ be a relative equilibrium that is not an equilibrium for the reaction-diffusion system (3.0.1) such that $\Sigma_{u_0} = I_3$. Suppose that Theorems 5.1.2 and 5.2.1 hold.

Definition 4.3.5. *Let us consider a C^k local tip position function x_{tip} on $M^{cu}_{u_0}$, where $1 \leq k \leq \infty$. Any C^k function $x^*_{tip}\colon V_* \to r\mathbf{S}^2$ is called a* tip position *function on V_*, where V_* and $M^{cu}_{u_0}$ are defined in Theorems* 5.1.2 *and* 5.2.1.

Theorem 4.3.6 ([21]). *The connection between a C^1 local tip position function x_{tip} on $M^{cu}_{u_0}$ and a C^1 position tip function x^*_{tip} on V_* is given by :*

$$x_{tip}(A\Psi(q)) = Ax^*_{tip}(q), \tag{4.3.1}$$

*where A and q are the coordinates on $SO(3) \times V_*$. Let $u^* \in M^{cu}_{u_0}$. Then the tip motion $x(t) = x_{tip}(\Phi(t, u^*))$ of the solution $\Phi(t, u^*)$ of the reaction-diffusion system (*3.0.1*) with the initial condition $u(0) = u^*$ satisfies the differential equation:*

$$\dot{x}(t) = A(t)\widetilde{X}(q(t)), \tag{4.3.2}$$

*where $\widetilde{X}(q) = X_G(q)x^*_{tip}(q) + D_q x^*_{tip}(X_N(q))$, and $A(t)$ and $q(t)$ are the solutions of the differential equations (*5.2.1*) given in Theorem* 5.2.1 *associated to $\Phi(t, u^*)$.*

Example 4.3.7 (Examples of tip position function). *For simplicity, we consider the local tip position function on V_* defined by $x^*_{tip}(q) = x_0$ for any $q \in V_*$, where $x_0 \in r\mathbf{S}^2$ is fixed.*

4.4 Stability of Rotating Waves and Modulated Rotating Waves

Definition 4.4.1. *Let $u_0 \in \mathbf{Y}^\alpha$ such that $\Phi(t, u_0)$ is a rotating wave for the reaction-diffusion system (*3.0.1*). Then it is* linearly orbitally stable *if $E^{cu} = T_{u_0}(SO(3)u_0)$ (see Theorems* 5.1.2 *and* 5.2.1 *for the definition of E^{cu}).*

Remark 4.4.2. *The same concept can be given for a modulated rotating wave.*

Definition 4.4.3.

1. *A solution $\Phi(., u_0)$, with $u_0 \in \mathbf{Y}^\alpha$, of the reaction-diffusion system (*3.0.1*) is* orbitally stable *if for any $\epsilon > 0$ there is some $\rho = \rho(\epsilon) > 0$ such that for each $u \in \mathbf{Y}^\alpha$ with $\|u - u_0\| \leq \rho$ and for all $t \geq 0$,*

$$\inf_{A \in SO(3)} \|A\Phi(t, u) - \Phi(t, u_0)\| \leq \epsilon.$$

2. *A solution $\Phi(., u_0)$, with $u_0 \in \mathbf{Y}^\alpha$, of the reaction-diffusion system (*3.0.1*) is* orbitally asymptotically stable *if it is stable and if there is some $\rho > 0$ such that for all $u \in \mathbf{Y}^\alpha$ with $\|u - u_0\| \leq \rho$ we have*

$$\inf_{A \in SO(3)} \|A\Phi(t, u) - \Phi(t, u_0)\| \to 0 \text{ as } t \to \infty.$$

3. *A solution $\Phi(., u_0)$, with $u_0 \in \mathbf{Y}^\alpha$ of the reaction-diffusion system (*3.0.1*) is* orbitally unstable *if it is not orbitally stable.*

Let u_0 be a relative equilibrium that is not an equilibrium for the reaction-diffusion system (3.0.1) and such that $\Sigma_{u_0} = I_3$. Suppose that Theorems 5.1.2 and 5.2.1 hold. Let $\Phi(., u_1)$ be a modulated rotating wave as defined in Definition 4.2.1 with $u_1 \in M^{cu}_{u_0}$ and $\Sigma_{u_1} = I_3$, where $M^{cu}_{u_0}$ is defined in Theorem 5.1.2. By Theorem 5.2.1, $\Phi(., u_0)$ corresponds to 0 and $\Phi(., u_1)$ corresponds to $q_1(t)$, where 0 and $q_1(t)$ are solutions of the second differential equation in (5.2.1).

Theorem 4.4.4.

1. *$\Phi(., u_0)$ is orbitally stable (respectively unstable) if 0 is stable (respectively unstable) in the second differential equation of (*5.2.1*).*

2. *$\Phi(., u_1)$ is orbitally stable (respectively unstable) if $q_1(t)$ is stable (respectively unstable) in the second differential equation of (*5.2.1*).*

4.5 Proofs of some Theorems of Chapter 4

Proof of Theorem 4.2.5. Since $\Phi(t, u_1)$ is a modulated rotating wave for the reaction-diffusion system (3.0.1), there exist a number $T > 0$ and a matrix $X_1 \in so(3)$ such that $\Phi(T, u_1) = e^{X_1 T} u_1$ and $\Phi(t, u_1) \notin SO(3)u_1$ for all $t \in (0, T)$.

By Theorem 5.2.1, there exist two functions $A(t)$ and $q(t)$ such that the differential equations (5.2.1) are satisfied and $\Phi(t, u_1) = A(t)\Psi(q(t))$. Let us denote by $q_1(t) = \Psi(q(t))$, where the function Ψ is defined in the proof of Theorem 5.2.1 (see [20] and [70]). Also we have $\Phi(0, u_1) = A(0)q_1(0) = A(0)\Psi(q(0)) = u_1$.

Therefore, one has $A(T)q_1(T) = e^{X_1 T} A(0)q_1(0) => e^{-X_1 T} A(T)q_1(T) = A(0)q_1(0)$ and also $\Phi(t + T, u_1) = \Phi(t, \Phi(T, u_1)) = \Phi(t, e^{X_1 T} u_1) = e^{X_1 T}\Phi(t, u_1)$, for all $t \in [0, \infty)$. We get $e^{-X_1(t+T)} A(t + T)q_1(t + T) = e^{-X_1 t} A(t)q_1(t)$ for all $t \in [0, \infty)$. If we define the function $R(t) = e^{-X_1 t} A(t)q_1(t)$ for $t \in [0, \infty)$, then this function is T-periodic, $R(0) = A(0)q_1(0) = u_1$ and

$$\Phi(t, u_1) = A(t)q_1(t) = A(t)A(t)^{-1}e^{X_1 t}R(t) = e^{X_1 t}R(t) \text{ for } t \in [0, \infty).$$

The other assertion of the theorem is obvious. □

Proof of Theorem 4.3.3. Let $\Phi(t, u_0) = e^{X_0 t} u_0$ be a rotating wave of the reaction-diffusion system (3.0.1). Using the $SO(3)$-equivariance of the tip position function, we have $x_{tip}(\Phi(t, u_0)) = x_{tip}(e^{X_0 t} u_0) = e^{X_0 t} x_{tip}(u_0)$.

Let $\Phi(t, u_0)$ be a modulated rotating wave of the reaction-diffusion system (3.0.1), such that $\Phi(T, u_0) = e^{X_0 T} u_0$ and $\Phi(t, u_0) \notin SO(3)u_0$ for $t \in (0, T)$. For any n integer, using the $SO(3)$-equivariance and the definition of the semiflow, we get $\Phi(nT, u_0) = e^{X_0 nT} u_0$. Therefore, using the $SO(3)$-equivariance of the tip position function, it follows that $x_{tip}(\Phi(nT, u_0)) = x_{tip}(e^{X_0 nT} u_0) = e^{X_0 nT} x_{tip}(u_0)$. □

Proof of Theorem 4.4.4. This is a result of the definition of orbital stability (respectively orbitally instability) and Theorems 5.1.2 and 5.2.1. □

Chapter 5

Center Manifold Reduction for Relative Equilibria (Rotating Waves on $r\mathbb{S}^2$)

Let us consider the reaction-diffusion system (3.0.1). Throughout this chapter we fix $\alpha \in (\frac{1}{2}, 1)$ and and assume that F is of class C^{k+2} with $k \geq 3$.

5.1 The Center Manifold Reduction

Theorem 5.1.1 ([85]). *Let $u_0 \in \mathrm{Y}^\alpha$ be a relative equilibrium that is not an equilibrium for (3.0.1) and such that the stabilizer of u_0 is $\Sigma_{u_0} = I_3$. Let L be the linearization of the right-hand side of (3.0.1) with respect to the rotating wave $\Phi(t, u_0) = e^{X_0 t}u_0$ in the co-rotating frame, that is*

$$L = D\Delta + D_u F(u_0) - X_0.$$

Then the operator L has eigenvalues 0, $\pm i\,|X_0|$ due to the $SO(3)$-equivariance of the reaction-diffusion system (3.0.1).

Theorem 5.1.2. *Let $u_0 \in \mathrm{Y}^\alpha$ be a relative equilibrium that is not an equilibrium for (3.0.1) and such that the stabilizer of u_0 is $\Sigma_{u_0} = I_3$. Let L be the linearization of the right-hand side of (3.0.1) with respect to the rotating wave $\Phi(t, u_0) = e^{X_0 t}u_0$ in the co-rotating frame, that is*

$$L = D\Delta + D_u F(u_0) - X_0.$$

Suppose that:

1. *the spectrum $\sigma(L)$ of L is such that $\sigma(L) \cap \{z \in \mathbb{C} \mid Re\,(z) \geq 0\}$ is a spectral set with spectral projection P_*, and $dim(R(P_*)) < \infty$;*

2. *the semigroup e^{Lt} satisfies $\left|e^{Lt}|_{R(1-P_*)}\right| \leq Ce^{-\beta_0 t}$ for some $\beta_0 > 0$ and $C > 0$.*

Then there exists a $SO(3)$-invariant sufficiently smooth manifold M^{cu}_{u0} in $\mathbf{Y}^\alpha$ that contains $SO(3)u_0$ and is locally invariant under the semiflow Φ_t for any $t \geq 0$. This manifold is locally exponentially attracting and contains all the solutions which stay close to the $SO(3)u_0$ for all backward times.

Definition 5.1.3. *The manifold M^{cu}_{u0} in Theorem 5.1.2 is called a* center manifold *for the relative equilibrium u_0.*

5.2 The Differential Equations on the Center Manifold

Theorem 5.2.1. *Let $u_0 \in \mathbf{Y}^\alpha$ be a relative equilibrium that is not an equilibrium for (3.0.1) and such that the stabilizer of u_0 is $\Sigma_{u_0} = I_3$. Let L be the linearization of the right-hand side of (3.0.1) with respect to the rotating wave $\Phi(t, u_0) = e^{X_0 t}u_0$ in the co-rotating frame, that is*

$$L = D\Delta + D_u F(u_0) - X_0.$$

Suppose that:

1. *$\sigma(L) \cap \{z \in \mathbb{C} \mid Re\,(z) \geq 0\}$ is a spectral set with spectral projection P_*, and $dim(R(P_*)) < \infty$;*

2. *the semigroup e^{Lt} satisfies $\left|e^{Lt}|_{R(1-P_*)}\right| \leq Ce^{-\beta_0 t}$ for some $\beta_0 > 0$ and $C > 0$.*

Let V_* be the orthogonal complement of $T_{u_0}(SO(3)u_0)$ in $E^{cu} = R(P_*)$.
Then the center manifold M^{cu}_{u0} given by Theorem 5.1.2 is diffeomorphic to $SO(3) \times V_*$. Furthermore, there exist sufficiently smooth functions $X_G \colon V_* \to so(3)$ and $X_N \colon V_* \to V_*$ such that any solution of

$$\begin{aligned} \dot{A} &= AX_G(q), \\ \dot{q} &= X_N(q), \end{aligned} \tag{5.2.1}$$

on $SO(3) \times V_*$ *corresponds to a solution of the reaction-diffusion system* (3.0.1) *on* M^{cu}_{u0} *under the diffeomorphic identification. Also,* $X_G(0) = X_0$, $X_N(0) = 0$ *and* $\sigma(D_u X_N(0)) = \sigma(Q_* L|_{V_*})$, *where* Q_* *is the projection onto* V_* *along* $T_{u_0}(SO(3)u_0)$.

Remark 5.2.2. *We say that the vector field* (5.2.1) *is the* pull-back *of the vector field on* M^{cu}_{u0} *to* $SO(3) \times V_*$. *This vector field is in skew-product form.*
For $\mathbf{Y} = \mathbf{L}^2(r\mathbf{S}^2, \mathbb{R}^N)$, *we have that* M^{cu}_{u0} *is a* C^{k+1} *manifold and* X_N, X_G *are* C^k *functions (see [20] and [70]).*
For $\mathbf{Y} = \mathbf{H}^2(r\mathbf{S}^2, \mathbb{R}^N)$, *we have that* M^{cu}_{u0} *is a* C^{k-1} *manifold and* X_N, X_G *are* C^{k-2} *functions (see [20] and [70]).*

5.3 Center Manifold Reduction for Parameter-dependent Reaction-diffusion Systems on $r\mathbf{S}^2$

Let us consider the reaction-diffusion system (3.4.1).

Theorem 5.3.1. *Let* $u_0 \in \mathbf{Y}^\alpha$ *be a relative equilibrium that is not an equilibrium for* (3.4.1) *at* $\lambda = 0$ *and such that the stabilizer of* u_0 *is* $\Sigma_{u_0} = I_3$. *Let* L *be the linearization of the right-hand side of* (3.4.1) *with respect to the rotating wave* $\mathbf{\Phi}(t, u_0, 0) = e^{X_0 t}u_0$ *at* $\lambda = 0$ *in the co-rotating frame, that is*

$$L = D\Delta + D_u F(u_0, 0) - X_0.$$

Suppose that:

1. $\sigma(L) \cap \{z \in \mathbb{C} \mid Re(z) \geq 0\}$ *is a spectral set with spectral projection* P_*, *and* $dim(R(P_*)) < \infty$;
2. *the semigroup* e^{Lt} *satisfies* $\left|e^{Lt}|_{R(1-P_*)}\right| \leq Ce^{-\beta_0 t}$ *for some* $\beta_0 > 0$ *and* $C > 0$.

Then there exists a sufficiently smooth parameter-dependent center manifold $M^{cu}_{u0}(\lambda)$ *for the relative equilibrium* u_0.

Theorem 5.3.2. *Let* $u_0 \in \mathbf{Y}^\alpha$ *be a relative equilibrium that is not an equilibrium for* (3.4.1) *at* $\lambda = 0$ *and such that the stabilizer of* u_0 *is* $\Sigma_{u_0} = I_3$. *Let* L *be the linearization of the right-hand side of* (3.4.1) *with respect to the rotating wave* $\mathbf{\Phi}(t, u_0, 0) = e^{X_0 t}u_0$ *at* $\lambda = 0$ *in the co-rotating frame, that is*

$$L = D\Delta + D_u F(u_0, 0) - X_0.$$

Suppose that:

1. *$\sigma(L) \cap \{z \in \mathbb{C} \mid Re(z) \geq 0\}$ is a spectral set with spectral projection P_*, and $dim(R(P_*)) < \infty$;*

2. *the semigroup e^{Lt} satisfies $\left|e^{Lt}|_{R(1-P_*)}\right| \leq Ce^{-\beta_0 t}$ for some $\beta_0 > 0$ and $C > 0$.*

Let V_* be the orthogonal complement of $T_{u_0}(SO(3)u_0)$ in $E^{cu} = R(P_*)$.
Then the center manifold $M^{cu}_{u_0}(\lambda)$ given by Theorem 5.3.1 is diffeomorphic to $SO(3) \times V_*$ for $|\lambda|$ small. Furthermore, there exist sufficiently smooth functions $X_G\colon V_* \times \mathbb{R} \to so(3)$ and $X_N\colon V_* \times \mathbb{R} \to V_*$ such that any solution of

$$\begin{aligned} \dot{A} &= AX_G(q,\lambda), \\ \dot{q} &= X_N(q,\lambda), \end{aligned} \tag{5.3.1}$$

on $SO(3) \times V_*$ corresponds to a solution of the reaction-diffusion system (3.4.1) on $M^{cu}_{u_0}(\lambda)$ under the diffeomorphic identification for $|\lambda|$ small. Also, $X_G(0,0) = X_0$, $X_N(0,0) = 0$ and $\sigma(D_u X_N(0,0)) = \sigma(Q_* L|_{V_*})$, where Q_* is the projection onto V_* along $T_{u_0}(SO(3)u_0)$.

Remark 5.3.3. *Theorems 5.3.1 and 5.3.2 are valid for more than one parameter.*

5.4 Proofs of Theorems of Chapter 5

Proof. All the proofs of the theorems in this chapter results from applying the theorems given in [20] and [70] to the sectorial operator $A = -D\Delta$, the compact Lie group $G = SO(3)$ and the function space $\mathbf{Y}$ chosen in chapter 3.
We also use that the representation T of $SO(3)$ is unitary and smooth on $\mathbf{Y}^\alpha$ for any $\alpha \in (\frac{1}{2}, 1)$ by Theorems 3.2.3 and 3.2.6, and the superposition operator F is a sufficiently smooth operator from $\mathbf{Y}^\alpha$ or $\mathbf{Y}^\alpha \times [0,\infty)$ into $\mathbf{Y}$ for any $\alpha \in (\frac{1}{2}, 1)$. $\square$

Chapter 6

Transition from Rotating Waves to Modulated Rotating Waves on $r\mathbb{S}^2$

Throughout this chapter we fix $\alpha \in (\frac{1}{2}, 1)$.
Let $X_0^1 = \frac{1}{|X_0|} X_0$. There exist $X_1, X_2 \in so(3)$ such that the set $\{\overrightarrow{X}_0^1, \overrightarrow{X}_1, \overrightarrow{X}_2\}$ is an orthonormal basis in $\mathbb{R}^3$ satisfying

$$\overrightarrow{X}_0^1 \times \overrightarrow{X}_1 = \overrightarrow{X}_2, \overrightarrow{X}_1 \times \overrightarrow{X}_2 = \overrightarrow{X}_0^1 \text{ and } \overrightarrow{X}_2 \times \overrightarrow{X}_0^1 = \overrightarrow{X}_1.$$

Since $so(3)$ is isomorphic to $\mathbb{R}^3$ and $\overrightarrow{[X,Y]} = \overrightarrow{X} \times \overrightarrow{Y}$, $\{X_0^1, X_1, X_2\}$ is a basis of the Lie algebra $so(3)$ such that

$$[X_0^1, X_1] = X_2, [X_1, X_2] = X_0^1 \text{ and } [X_2, X_0^1] = X_1.$$

Definition 6.0.1. *Let M be a smooth manifold, X a normed space or the empty set and $Y \colon X \times [0, \lambda_0) \to M$ for $\lambda_0 > 0$ small. We say that the function Y is CS on $X \times [0, \lambda_0)$ if the function $Z \colon X \times [0, \epsilon_0) \to M$ defined by*

$$Z(x, \epsilon) = Y(x, \epsilon^2),$$

is smooth on $X \times [0, \epsilon_0)$, where $\epsilon_0 = \sqrt{\lambda_0}$.
We say that Y is C^k-CS if Z is C^k, where $k \in \mathbb{Z}$, $k \geq 1$. We say that Y is sufficiently *CS if Z is sufficiently smooth.*

6.1 Hopf Bifurcation Theorem to Modulated Rotating Waves and the Bifurcation Diagram

Let us consider the reaction-diffusion system (3.4.1). Throughout this section we assume that F is of class C^{k+2} with $k \geq 3$.

Theorem 6.1.1 (Hopf Bifurcation Theorem for Rotating Waves on a Sphere). *Let $u_0 \in \mathbf{Y}^\alpha$ be a relative equilibrium that is not an equilibrium for (3.4.1) at $\lambda = 0$ and such that the stabilizer of u_0 is $\Sigma_{u_0} = I_3$.*
Let L be the linearization of the right-hand side of (3.4.1) with respect to the rotating wave $\mathbf{\Phi}(t, u_0, 0) = e^{X_0 t} u_0$ at $\lambda = 0$ in the co-rotating frame, that is

$$L = D\Delta + D_u F(u_0, 0) - X_0.$$

Suppose that:

1. *$\sigma(L) \cap \{z \in \mathbb{C} \mid Re(z) \geq 0\}$ is a spectral set with spectral projection P_*, and $dim(R(P_*)) = 5$;*
2. *the semigroup e^{Lt} satisfies $\left|e^{Lt}|_{R(1-P_*)}\right| \leq Ce^{-\beta_0 t}$ for some $\beta_0 > 0$ and $C > 0$.*

Then Theorems 5.3.1 and 5.3.2 can be applied.
Suppose that a supercritical Hopf bifurcation with eigenvalues $\pm i\omega_{bif}$ takes place in the second differential equation of (5.3.1) in V_* at $q = 0$ for $\lambda = 0$. Namely,

1. *$X_N(0, 0) = 0$;*
2. *$D_q X_N(0, 0)$ has eigenvalues $\pm i\omega_{bif}$; without loss of generality, we assume that $X_N(0, \lambda) = 0$ for $|\lambda|$ small;*
3. *$D_q X_N(0, \lambda)$ has the eigenvalues $\alpha(\lambda) \pm i(\omega_{bif} + \beta(\lambda))$ with $\alpha(0) = \beta(0) = 0$ such that $\alpha'(0) > 0$;*
4. *$a(0) < 0$, where $a(0)$ is defined in Theorem 2.7.1.*

Let us denote by $q(t, \lambda)$ the stable periodic solution near $q = 0$ that appears for $\lambda > 0$ small due to the supercritical Hopf bifurcation. Let $T(\lambda) = \frac{2\pi}{|\omega_\lambda|}$ be the period of the function $q(t, \lambda)$, where $\omega_\lambda = \omega_{bif} + O(\lambda)$ for $\lambda \geq 0$ small.
Then, there exists a sufficiently CS branch $\mathbf{\Phi}(t, u_\lambda, \lambda)$ such that $\mathbf{\Phi}(t, u_0, 0) = e^{X_0 t} u_0$ and

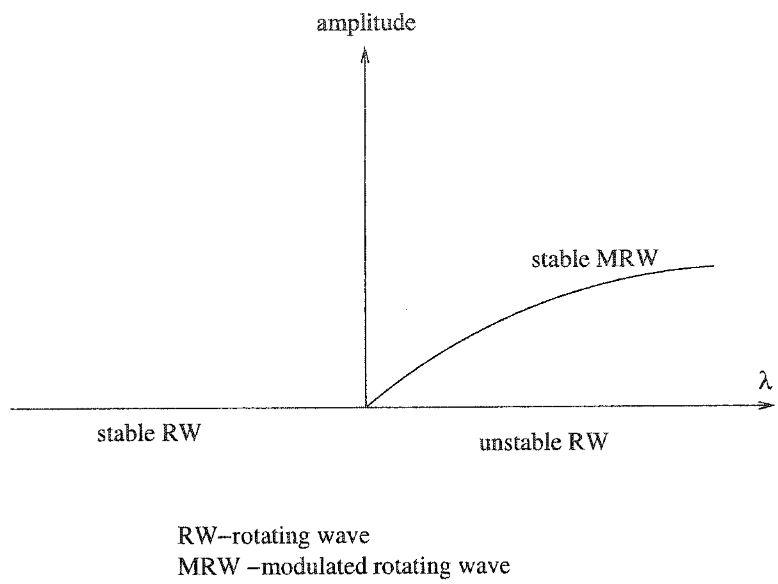

Figure 6.1.1: Bifurcation Diagram for Nonresonant Hopf Bifurcation

for $\lambda > 0$ small, $\Phi(t, u_\lambda, \lambda)$ is either an orbitally stable modulated rotating wave with a primary frequency vector $\overrightarrow{X(\lambda)}$ and the secondary frequency ω_λ, or an orbitally stable periodic solution with the period $\frac{2\pi}{|\omega_\lambda|}$.

The bifurcation diagram is illustrated in Figure (6.1.1).

6.2 The Primary Frequency Formula

Let us consider the reaction-diffusion system (3.4.1).
Throughout this section, we will suppose that $u_0 \in \mathbf{Y}^\alpha$ is a relative equilibrium that is not an equilibrium for (3.4.1) at $\lambda = 0$ and such that the stabilizer of u_0 is $\Sigma_{u_0} = I_3$.
Suppose that Theorem 6.1.1 holds.
Let us define

$$X^G(t, \lambda) = \begin{cases} X_G(q(t,\lambda), \lambda) & \text{if } \lambda > 0 \text{ and } t \in [0, \infty); \\ X_0 & \text{if } \lambda = 0 \text{ and } t \in [0, \infty), \end{cases} \tag{6.2.1}$$

where $X_G(q, \lambda)$ is defined in Theorem 5.3.2 and $q(t, \lambda)$ is the periodic solution in V_* which appears due to the supercritical Hopf bifurcation at $q = 0$. Let $q(t, 0) = 0$ for all $t \in \mathbf{R}$. Then, it follows that $X^G(t, \lambda) = X_G(q(t, \lambda), \lambda)$ for $t \in [0, \infty)$ and $\lambda \geq 0$ small.

Since $q(t,\lambda) = \lambda^{\frac{1}{2}} r(t,\lambda)$ given in Theorem 2.7.1 is sufficiently CS and $X_G(q,\lambda)$ is sufficiently smooth, it follows that the function X^G is sufficiently CS.
Writing $q(t,\lambda) = \lambda^{\frac{1}{2}} r(t,\lambda)$, we have

$$\begin{aligned} X^G(t,\lambda) &= x_0(t,\lambda)X_0^1 + \lambda^{\frac{1}{2}} x_1(t,\lambda)X_1 + \lambda^{\frac{1}{2}} x_2(t,\lambda)X_2, \\ x_0(t,\lambda) &= |X_0| + \lambda^{\frac{1}{2}} x_{01}(t) + \lambda x_{02}(t,\lambda) \end{aligned} \tag{6.2.2}$$

for $t \in [0,\infty)$ and $\lambda \geq 0$ small. The functions $X^G(t,\lambda)$, $x_0(t,\lambda)$, $x_1(t,\lambda)$, $x_2(t,\lambda)$ are $\frac{2\pi}{|\omega_\lambda|}$-periodic in t for $\lambda \geq 0$ small.
Let $A(t,\lambda)$ for $t \in [0,\infty)$ and $\lambda \geq 0$ small be the solution of the initial value problem

$$\begin{aligned} \dot{A} &= AX^G(t,\lambda), \\ A(0) &= I_3, \end{aligned} \tag{6.2.3}$$

where $X^G(t,\lambda)$ is defined in (6.2.1).
From the proofs of Theorems 6.1.1 and 6.5.1, a primary frequency vector $\overrightarrow{X(\lambda)}$ of $\Phi(t,u_\lambda,\lambda)$ is given by $A(\frac{2\pi}{|\omega_\lambda|},\lambda) = e^{X(\lambda)\frac{2\pi}{|\omega_\lambda|}}$ for $\lambda > 0$ small.

Theorem 6.2.1. *Suppose the hypotheses of Theorem 6.1.1 are satisfied. Then, there exists a sufficiently CS branch $X(\lambda)$ with $\lambda \geq 0$ such that $\overrightarrow{X(\lambda)}$ is a primary frequency vector of $\Phi(t,u_\lambda,\lambda)$ for $\lambda > 0$ small and $X(0)$ is such that $\Phi(\frac{2\pi}{|\omega_{bif}|},u_0,0) = e^{X(0)\frac{2\pi}{|\omega_{bif}|}}u_0$. Also, there exists a branch $X^f(\lambda)$ for $\lambda \geq 0$ such that $\overrightarrow{X^f(\lambda)}$ is a primary frequency vector of $\Phi(t,u_\lambda,\lambda)$ for $\lambda > 0$ small and $X^f(0) = X_0$. Moreover,*

1. *if $|X_0| \neq k\omega_{bif}$ for all $k \in \mathbb{Z}$, then the branch $X^f(\lambda)$ for $\lambda \geq 0$ is sufficiently CS and*
$$X^f(\lambda) = (|X_0| + O(\lambda^{\frac{1}{2}}))X_0^1 + O(\lambda^{\frac{1}{2}})X_1 + O(\lambda^{\frac{1}{2}})X_2 \text{ for small } \lambda \geq 0. \tag{6.2.4}$$
(The branch $\Phi(t,u_\lambda,\lambda)$ contains only modulated rotating waves for $\lambda \geq 0$ small).

2. *if $|X_0| = k\omega_{bif}$ for some $k \in \mathbb{Z}$, $k \neq 0$, then $|X^f(\lambda)|$ for $\lambda \geq 0$ is continuous and*
$$\left|X^f(\lambda)\right| = |X_0| + O(\lambda^{\frac{1}{4}}) \text{ for small } \lambda \geq 0. \tag{6.2.5}$$

Remark 6.2.2. *If $|X_0| = k\omega_{bif}$ for some $k \in \mathbb{Z}$, $k \neq 0$, it may be possible that the branch $X^f(\lambda)$ be discontinuous at $\lambda = 0$ or/and at any $\lambda > 0$ such that $|X(\lambda)| = 0$.*

In the rest of the section we give a way of constructing $X(\lambda)$ and $X^f(\lambda)$ for $\lambda \geq 0$ using the BCH formula in $so(3)$ given by Theorem 2.3.4.

Theorem 6.2.3. *Suppose that the hypotheses of Theorem 6.1.1 are satisfied. Let us consider the following initial value problem:*

$$\begin{aligned} \vec{\dot{Z}} &= \left[I_3 + \tfrac{1}{2}Z + \left(\tfrac{1}{|Z|^2} - \tfrac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|} \right) Z^2 \right] \overrightarrow{X^G(t,\lambda)}, \\ Z(0) &= O_3. \end{aligned} \tag{6.2.6}$$

Then

1. *there exists a positive integer n independent of λ such that the initial value problem (6.2.6) has a unique sufficiently CS solution $Z(t,\lambda)$ on $t \in [0, \frac{2\pi}{n|\omega_\lambda|}]$ and $\lambda \geq 0$ small;*

2. $$Z(t,\lambda) = \left(|X_0|\, t + \lambda^{\frac{1}{2}} \int_0^t x_{01}(s)\, ds + \lambda x_{03}(t,\lambda) \right) X_0^1 + \lambda^{\frac{1}{2}} z_1(t,\lambda) X_1 + \lambda^{\frac{1}{2}} z_2(t,\lambda) X_2 \tag{6.2.7}$$

   for any $t \in [0, \frac{2\pi}{n|\omega_\lambda|}]$ and $\lambda \geq 0$ small;

3. *for any $i = 1, 2, \ldots, n-1$ the initial value problem*

   $$\begin{aligned} \vec{\dot{Z}} &= \left[I_3 + \tfrac{1}{2}Z + \left(\tfrac{1}{|Z|^2} - \tfrac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|} \right) Z^2 \right] \overrightarrow{X^G(t,\lambda)}, \\ Z(i\tfrac{2\pi}{n|\omega_\lambda|}) &= O_3 \end{aligned} \tag{6.2.8}$$

   has a unique sufficiently CS solution on $[i\frac{2\pi}{n|\omega_\lambda|}, (i+1)\frac{2\pi}{n|\omega_\lambda|}]$ and $\lambda \geq 0$ small.

Theorem 6.2.4. *Suppose that the hypotheses of Theorem 6.1.1 are satisfied. Then,*

1. *there exists a function $Z(t,\lambda)$ such that $[Z(t,\lambda)]$ is sufficiently CS for $t \in [0,\infty)$ and $\lambda \geq 0$ small, and $A(t,\lambda) = e^{Z(t,\lambda)}$ for $t \in [0,\infty)$ and $\lambda \geq 0$ small, where $A(t,\lambda)$ is the solution of the initial value problem (6.2.3).*

2. *the function $Z(t,\lambda)$ is defined by*

   (a)

   $$[Z(t,\lambda)] = [Z^0(t,\lambda)] = BCH(Z_1(\frac{2\pi}{n\,|\omega_\lambda|},\lambda), Z_2(2\frac{2\pi}{n\,|\omega_\lambda|},\lambda), \ldots, Z_i(t,\lambda)) \tag{6.2.9}$$

   for all $t \in [(i-1)\frac{2\pi}{n|\omega_\lambda|}, i\frac{2\pi}{n|\omega_\lambda|}]$ and $\lambda \geq 0$ small, where $i = 1, 2, \ldots, n$ and Z_i is the solution of the following initial value problem on the interval $[(i-1)\frac{2\pi}{n|\omega_\lambda|}, i\frac{2\pi}{n|\omega_\lambda|}]$:

   $$\begin{aligned} \vec{\dot{Z}} &= \left[I_3 + \tfrac{1}{2}Z + \left(\tfrac{1}{|Z|^2} - \tfrac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|} \right) Z^2 \right] \overrightarrow{X^G(t,\lambda)}, \\ Z((i-1)\tfrac{2\pi}{n|\omega_\lambda|}) &= O_3. \end{aligned} \tag{6.2.10}$$

(b) On the interval $[i\frac{2\pi}{|\omega_\lambda|},(i+1)\frac{2\pi}{|\omega_\lambda|}]$ *for any integer* $i \geq 1$ *and* $\lambda \geq 0$ *small, we have*

$$[Z(t,\lambda)] = [Z^i(t,\lambda)] = BCH(Z^0(\frac{2\pi}{|\omega_\lambda|},\lambda), Z^{i-1}(t-\frac{2\pi}{|\omega_\lambda|},\lambda)). \tag{6.2.11}$$

Theorem 6.2.5. *Suppose that the hypotheses of Theorem* 6.1.1 *are satisfied. The branch* $\overrightarrow{X(\lambda)}$ *for* $\lambda \geq 0$ *from Theorem* 6.2.1 *is defined then by:*

1. *if* $|X_0| \neq \frac{2k+1}{2}\omega_{bif}$ *for any* $k \in \mathbb{Z}$, *then*

$$X(\lambda) = \frac{|\omega_\lambda|}{2\pi}Z(\frac{2\pi}{|\omega_\lambda|},\lambda) \text{ for small } \lambda \geq 0, \tag{6.2.12}$$

where $Z(t,\lambda)$ *is given in Theorem* 6.2.4*;*

2. *if* $|X_0| = \frac{2k+1}{2}\omega_{bif}$ *for some* $k \in \mathbb{Z}$, *then*

$$X(\lambda) = \frac{|\omega_\lambda|}{2\pi}q^{-1}(Z(\frac{2\pi}{|\omega_\lambda|},\lambda)) \text{ for small } \lambda \geq 0, \tag{6.2.13}$$

where q^{-1} *is defined in* (2.3.3) *for* $Z(\frac{2\pi}{|\omega_{bif}|},0) = \pi X_0^1$.

Then, the branch $\overrightarrow{X^f(\lambda)}$ *for* $\lambda \geq 0$ *from Theorem* 6.2.1 *is defined by:*

1. *if* $|X_0| \neq k\omega_{bif}$ *for all* $k \in \mathbb{Z}$, *then* $X^f(0) = X(0)$.
   Let $|X_0|T(0) = \alpha_0 + 2k\pi$, *where* $\alpha_0 \in [0,2\pi)$, $k \in \mathbb{Z}$, $k \geq 0$, *and* $T(0) = \frac{2\pi}{|\omega_{bif}|}$.
   If $\alpha_0 \in [0,\pi]$, *then we define* $X^f(\lambda) = \frac{|X(\lambda)|+k|\omega_\lambda|}{|X(\lambda)|}X(\lambda)$ *for* $\lambda > 0$ *small.*
   If $\alpha_0 \in (\pi,2\pi)$, *then we define* $X^f(\lambda) = \frac{-|X(\lambda)|+(k+1)|\omega_\lambda|}{|X(\lambda)|}(-X(\lambda))$ *for* $\lambda > 0$ *small.*

2. *if* $|X_0| = k\omega_{bif}$ *for some* $k \in \mathbb{Z}$, $k \neq 0$, *then we define:*
   $X^f(0) = X_0$;
   if $|X(\lambda)| \neq 0$, *then we define* $X^f(\lambda) = \frac{|X(\lambda)|+k|\omega_\lambda|}{|X(\lambda)|}X(\lambda)$ *for* $\lambda > 0$ *small;*
   if $|X(\lambda)| = 0$, *then we define* $X^f(\lambda) = X(\lambda) + k\,|\omega_\lambda|\,Q(\lambda)$, *where* $Q(\lambda) \in so(3)$, $|Q(\lambda)| = 1$ *for* $\lambda > 0$ *small.*

All the statements given in Theorem 6.2.1 *are valid for* $X(\lambda)$ *and* $X^f(\lambda)$ *with* $\lambda \geq 0$ *small.*

By Theorems 4.2.5, 5.3.2 and 6.5.1, for $\lambda > 0$ small, we can associate $A(t,\lambda) = e^{X(\lambda)t}B(t,\lambda) = e^{X^f(\lambda)t}B^f(t,\lambda)$ to each modulated rotating wave or periodic solution of period $\frac{2\pi}{|\omega_\lambda|}$ obtained by Theorem 6.1.1, where $B(t,\lambda)$, $B^f(t,\lambda)$ are $\frac{2\pi}{|\omega_\lambda|}$-periodic in t for $\lambda > 0$ small. Also, let $B(t,0)$ and $B^f(t,0)$ be such that $A(t,0) = e^{X(0)t}B(t,0) = e^{X^f(0)t}B^f(t,0)$, that is $B^f(t,0) = I_3$.

Theorem 6.2.6. *Suppose that the hypotheses of Theorem* 6.1.1 *are satisfied. Then*

1. *if* $|X_0| \neq k\omega_{bif}$ *for all* $k \in \mathbb{Z}$, *we have*

$$B^f(t,\lambda) = e^{BCH(-X^f(\lambda)t, Z(t,\lambda))} \text{ for } t \in [0,\infty) \text{ and } \lambda \geq 0 \text{ small} \tag{6.2.14}$$

is sufficiently CS, where $Z(t,\lambda)$ *is given in Theorem* 6.2.4.
If we define $[Per^f(t,\lambda)] = BCH(-X^f(\lambda)t, Z(t,\lambda))$, *then* $Per^f(t,\lambda)$ *is* $\frac{2\pi}{|\omega_\lambda|}$*-periodic in* t *and sufficiently CS function for* $t \in [0,\infty)$ *and* $\lambda \geq 0$ *small. Moreover,* $Per^f(t,\lambda) = \lambda^{\frac{1}{2}}Y(t,\lambda)$ *for* $t \in [0,\infty)$ *and* $\lambda \geq 0$ *small.*

2. *if* $|X_0| = k\omega_{bif}$ *for some* $k \in \mathbb{Z}$, $k \neq 0$, *then*

$$B(t,\lambda) = e^{BCH(-X(\lambda)t, Z(t,\lambda))} \text{ for } t \in [0,\infty) \text{ and } \lambda \geq 0 \text{ small} \tag{6.2.15}$$

is sufficiently CS, where $Z(t,\lambda)$ *is given in Theorem* 6.2.4.
If we define $[Per(t,\lambda)] = BCH(-X(\lambda)t, Z(t,\lambda))$, *then* $[Per(t,\lambda)]$ *is a sufficiently CS function for* $t \in [0,\infty)$ *and* $\lambda \geq 0$ *small.*

6.3 Resonant Drift Phenomena for Modulated Rotating Waves on $r\mathbf{S}^2$

Let us consider the reaction-diffusion system:

$$\frac{\partial u}{\partial t}(t,x) = D\Delta u(t,x) + F(u(t,x),\lambda,\mu) \text{ on } r\mathbf{S}^2, \tag{6.3.1}$$

where $r > 0$, $r\mathbf{S}^2$ is the sphere of radius r, $\mathbf{S}^2$ is the unit sphere in $\mathbb{R}^3$, $u = (u_1, u_2, \dots, u_N)\colon r\mathbf{S}^2 \to \mathbb{R}^N$ with $N \geq 1$,
$D = \begin{pmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_N \end{pmatrix}$ with $d_i \geq 0$ for $i = 1, 2, \dots, N$ are the diffusion coefficients and $F = (F_1, F_2, \dots, F_N)\colon \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^N$ are sufficiently smooth functions such that $F(0,\lambda,\mu) = 0$ for $|\lambda|$, $|\mu|$ small.
We study the reaction-diffusion system (6.3.1) on the function space

$$\mathbf{Y} = \begin{cases} \mathbf{L}^2(r\mathbf{S}^2, \mathbb{R}^N) & \text{if } d_i > 0 \text{ for } i = 1, 2, \dots, N; \\ \mathbf{H}^2(r\mathbf{S}^2, \mathbb{R}^N) & \text{if there exists an } i \in \{1, 2, \dots, N\} \text{ such that } d_i = 0. \end{cases}$$

The superposition operator associated to F, denoted by $F\colon \mathbf{Y}^\alpha \times \mathbb{R} \times \mathbb{R} \to \mathbf{Y}$ is sufficiently smooth and $SO(3)$-equivariant for any λ, μ (see the proof of Theorem 3.4.1). Let $\mathbf{\Phi}(t, u, \lambda, \mu)$ be the $SO(3)$-equivariant sufficiently smooth local semiflow defined as in the Theorem 3.4.1.

Remark 6.3.1. *The exponential map $exp\colon so(3) \to SO(3)$ is not a local diffeomorphism near $X_1 \in so(3)$ such that $|X_1| = 2k\pi$, where $k \in \mathbb{Z}$, $k \geq 1$. Let $X_1 \in so(3)$ with $|X_1| = 2k\pi$ for k fixed. Then there exists $X_2 \in so(3)$ such that the associated vector $\overrightarrow{X}_2$ is orthogonal to the vector $\overrightarrow{X}_1$, $|X_2|$ is close to $2k\pi$ and e^{X_2} is close to I_3 in $SO(3)$.*

Definition 6.3.2. *Let M be a smooth manifold, X a normed space or the empty set and $Y\colon X \times [0, \lambda_0) \times (-\mu_0, \mu_0) \to M$ for $\lambda_0 > 0$, $\mu_0 > 0$ small. We say that the function Y is CS_λ on $X \times [0, \lambda_0) \times (-\mu_0, \mu_0)$ if the function $Z\colon X \times [0, \epsilon_0) \times (-\mu_0, \mu_0) \to M$ defined by*

$$Z(x, \epsilon, \mu) = Y(x, \epsilon^2, \mu), \quad \textit{where } \epsilon_0 = \sqrt{\lambda_0}$$

is smooth on $X \times [0, \epsilon_0) \times (-\mu_0, \mu_0)$.
We say that Y is C^k-CS_λ if Z is C^k, where $k \in \mathbb{Z}$, $k \geq 1$. We say that Y is sufficiently CS_λ if Z is sufficiently smooth.

Theorem 6.3.3 (Resonance Case). *Let $u_0 \in \mathbf{Y}^\alpha$ be a relative equilibrium that is not an equilibrium for (6.3.1) at $(\lambda, \mu) = (0, 0)$ and such that the stabilizer of u_0 is $\Sigma_{u_0} = I_3$.*
Let L be the linearization of the right-hand side of (6.3.1) with respect to the rotating wave $\mathbf{\Phi}(t, u_0, 0, 0) = e^{X_0 t} u_0$ at $(\lambda, \mu) = (0, 0)$ in the co-rotating frame, that is

$$L = D\Delta + D_u F(u_0, 0, 0) - X_0.$$

Suppose that:

1. *$\sigma(L) \cap \{z \in \mathbb{C} \mid Re(z) \geq 0\}$ is a spectral set with spectral projection P_*, and $dim(R(P_*)) = 5$;*

2. *the semigroup e^{Lt} satisfies $\left|e^{Lt}|_{R(1-P_*)}\right| \leq Ce^{-\beta_0 t}$ for some $\beta_0 > 0$ and $C > 0$.*

Then Theorems 5.3.1 and 5.3.2 with parameters λ, μ can be applied.
Suppose that a supercritical Hopf bifurcation with eigenvalues $\pm i\omega_{bif}$ takes place in the second differential equation of (5.3.1) in V_* at $q = 0$ for $(\lambda, \mu) = (0, 0)$, that is:

1. *$X_N(0, 0, 0) = 0$;*

2. $D_q X_N(0,0,0)$ *has eigenvalues* $\pm i\omega_{bif}$; *without loss of generality, we assume that* $X_N(0,\lambda,\mu) = 0$ *for* $|\lambda|$, $|\mu|$ *small;*

3. $D_q X_N(0,\lambda,\mu)$ *has the eigenvalues* $\alpha(\lambda,\mu) \pm i(\omega_{bif} + \beta(\lambda,\mu)$ *with* $\alpha(0,0) = \beta(0,0) = 0)$ *and* $\alpha_\lambda(0,0) > 0$. *This implies that* $\alpha(\lambda_H(\mu),\mu) = 0$ *for some sufficiently smooth curve* $\lambda = \lambda_H(\mu)$ *with* $\lambda_H(0) = 0$. *This curve represents the Hopf points and without loss of generality, we suppose that* $\lambda_H(\mu) = 0$ *for* $|\mu|$ *small;*

4. $a(0,0) < 0$, *where* $a(0,0)$ *is defined Theorem* 2.7.1.

Suppose that $|X_0| = k\omega_{bif}$ *for some* $k \in \mathbb{Z}$, $k \neq 0$.
Let us denote by $q(t,\lambda,\mu)$ *the stable periodic solution near* $q = 0$ *that appears for* $\lambda > 0$ *small and* $|\mu|$ *small due to the supercritical Hopf bifurcation. Let us define the sufficiently* CS_λ *function*

$$X^G(t,\lambda,\mu) = \begin{cases} X_G(q(t,\lambda,\mu),\lambda,\mu) & \text{if } \lambda > 0,\ |\mu| \text{ small and } t \in [0,\infty); \\ X_G(0,\lambda,\mu) & \text{if } \lambda = 0,\ |\mu| \text{ small and } t \in [0,\infty), \end{cases} \tag{6.3.2}$$

where $X_G(q,\lambda,\mu)$ *is defined in Theorem* 5.3.2.
Let

$$X_G(q,\lambda,\mu) = x_0(q,\lambda,\mu)X_0^1 + x_1(q,\lambda,\mu)X_1 + x_2(q,\lambda,\mu)X_2.$$

For $\lambda > 0$ *small and* $|\mu|$ *small, let* $T(\lambda,\mu) = \frac{2\pi}{|\omega_{\lambda,\mu}|}$ *be the period of the function* $q(t,\lambda,\mu)$, *where* $\omega_{\lambda,\mu} = \omega_{bif} + O(\mu) + \lambda s(\lambda,\mu)$ *for* $\lambda \geq 0$ *small and* $|\mu|$ *small.*
By Theorem 6.1.1, *there exists a sufficiently CS branch* $\mathbf{\Phi}(t,u_{\lambda,\mu},\lambda,\mu)$ *for* $\lambda \geq 0$ *small and* $|\mu|$ *small such that for* $|\mu|$ *small,* $\mathbf{\Phi}(t,u_{0,\mu},0,\mu) = e^{X_G(0,0,\mu)t}u_{0,\mu}$ *and such that for* $\lambda > 0$ *small and* $|\mu|$ *small,* $\mathbf{\Phi}(t,u_{\lambda,\mu},\lambda,\mu)$ *is an orbitally stable modulated rotating wave or periodic solution of period* $\frac{2\pi}{|\omega_{\lambda,\mu}|}$. *Let* $\overrightarrow{X(\lambda,\mu)}$ *be the sufficiently CS branch such that for* $|\mu|$ *small,* $e^{X(0,\mu)\frac{2\pi}{|\omega_{0,\mu}|}} = e^{X_G(0,0,\mu)\frac{2\pi}{|\omega_{0,\mu}|}}$, *and for* $\lambda > 0$ *small and* $|\mu|$ *small,* $\overrightarrow{X(\lambda,\mu)}$ *is a primary frequency vector corresponding to* $\mathbf{\Phi}(t,u_{\lambda,\mu},\lambda,\mu)$.
Then,

1. *if* $(x_0)_\mu(0,0,0) \neq k(\omega_{0,\mu})'|_{\mu=0}$, *there exists a sufficiently CS curve* $\mu = \mu(\lambda)$ *for* $\lambda \geq 0$ *small such that* $\mu(0) = 0$, *and there exists a sufficiently CS branch of solutions* $\mathbf{\Phi}(t,u_\lambda,\lambda,\mu(\lambda))$ *for* $\lambda > 0$ *small, such that* $\mathbf{\Phi}(t,u_\lambda,\lambda,\mu(\lambda))$ *is either a branch of orbitally stable modulated rotating waves with a primary frequency vector* $\overrightarrow{X(\lambda,\mu(\lambda))}$ *orthogonal to* $\overrightarrow{X}_0$ *or a branch of orbitally stable periodic solutions with period* $\frac{2\pi}{|\omega_{\lambda,\mu(\lambda)}|}$.

2. *the branch* $X(\lambda, \mu(\lambda))$ *is sufficiently* CS *for* $\lambda \geq 0$ *small and* $X(\lambda, \mu(\lambda)) = O(\lambda^{\frac{k}{2}})X_1 + O(\lambda^{\frac{k}{2}})X_2$ *for* $\lambda \geq 0$ *small.*

3. $B(t, \lambda, \mu(\lambda)) = e^{BCH(-X(\lambda,\mu(\lambda))t, Z(t,\lambda,\mu(\lambda)))}$ *for* $t \in [0, \infty)$ *and* $\lambda \geq 0$ *small, where* B *and* Z *have the same meaning as in Theorem* 6.2.6. *If we define* $Per^{res}(t, \lambda, \mu(\lambda)) = BCH(-X(\lambda, \mu(\lambda))t, Z(t, \lambda, \mu(\lambda)))$, *then* $B(t, \lambda, \mu(\lambda))$ *and* $[Per^{res}(t, \lambda, \mu(\lambda))]$ *are sufficiently* CS *functions for* $t \in [0, \infty)$ *and* $\lambda \geq 0$ *small.*

4. *there exists a branch* $X^f(\lambda, \mu(\lambda))$ *with* $\lambda \geq 0$ *that is discontinuous at* $\lambda = 0$, *but* $|X^f(\lambda, \mu(\lambda))|$ *is continuous for* $\lambda \geq 0$ *small and* $|X^f(\lambda, \mu(\lambda))| = |X_0| + O(\lambda^{\frac{k}{4}})$ *for* $\lambda \geq 0$ *small.*

Remark 6.3.4. *The branch* $\mu = \mu(\lambda)$ *for* $\lambda \geq 0$ *small can be found by the use of the BCH formula in* $so(3)$.

Theorem 6.3.5. *The parameter space in the case of the resonant Hopf bifurcation is illustrated in Figure (*6.3.1*).*

The results in Sections 6.1, 6.2 and 6.3 show that a rotating wave on a sphere generically undergoes a transition to quasi-periodic meandering (modulated rotating wave) at a Hopf bifurcation. Furthermore, the primary frequency vector of the modulated rotating wave is determined by both the critical Hopf eigenvalues and the frequency of the rotating wave undergoing the bifurcation. In particular, resonances between the critical Hopf eigenvalue and the frequency of the rotating wave undergoing the bifurcation lead to orthogonal drift. While we have concentrated on the case of Hopf bifurcation, it should be mentioned that we would obtain similar results for *periodic forcing* of a rotating wave: equation (6.2.3) represents the dynamical equations of a periodically forced rotating wave. In a spherical heart, for example, one could alter the dynamics of a rotating spiral wave simply by applying a (weak) periodic forcing (e.g. pacing), thereby inducing meandering. By an appropriate choice of the forcing frequency, one could then control the direction of the primary frequency vector (i.e. steer the spiral tip to another location). Of course, this assumes perfect spherical symmetry. As we will see in the next chapter, geometrical imperfections can also influence the dynamics of rotating waves.

6.4 Examples

Throughout this section, we let $\epsilon = \sqrt{\lambda}$.

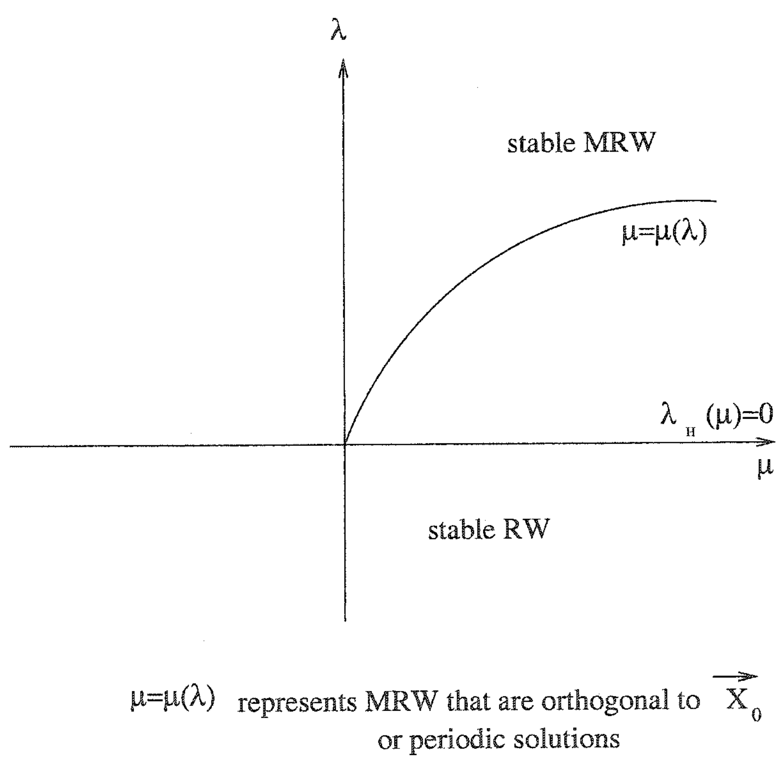

Figure 6.3.1: Parameter Space for Resonant Hopf Bifurcation

Example 6.4.1 (Hopf bifurcation to modulated rotating waves). *Let $g(t,\lambda)$ be a sufficiently smooth periodic function of period $\frac{2\pi}{|\omega_{bif}+\lambda|}$ such that $g(0,\lambda)=0$ for $\lambda \geq 0$ small. For any $t \in [0,\infty)$ and $\lambda \geq 0$ small, we define*

$$X^G(t,\lambda) = (2\epsilon X_1+2\epsilon X_2+\epsilon X_0)\dot{g}(t,\lambda)+e^{-(2\epsilon X_1+2\epsilon X_2+\epsilon X_0)g(t,\lambda)}(X_0+\epsilon X_1)e^{(2\epsilon X_1+2\epsilon X_2+\epsilon X_0)g(t,\lambda)}.$$

It is clear that $X^G(t,\lambda)$ is a sufficiently CS, $\frac{2\pi}{|\omega_{bif}+\lambda|}$-periodic function such that $X^G(t,0)=X_0$. Then, the initial value problem (6.2.3) has the solution

$$A(t,\lambda) = e^{(X_0+\epsilon X_1)t}e^{(2\epsilon X_1+2\epsilon X_2+\epsilon X_0)g(t,\lambda)}.$$

Clearly, $\overrightarrow{X}_0+\epsilon\overrightarrow{X}_1$ is not orthogonal to $\overrightarrow{X}_0$.

Example 6.4.2 (Example 1. Resonant drift phenomena for modulated rotating waves). *Let $\omega_{bif} = |X_0|$ and $g(t,\lambda)$ be a sufficiently smooth periodic function of period $\frac{2\pi}{|\omega_{bif}+\lambda|}$ such that $g(0,\lambda)=0$ for $\lambda \geq 0$ small.*
For any $t \in [0,\infty)$ and $\lambda \geq 0$ small, we define

$$X^G(t,\lambda) = (X_0+\epsilon X_2)\left(\frac{|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\lambda}}+\lambda\dot{g}(t,\lambda)\right)+\epsilon e^{-(X_0+\epsilon X_2)\left(\frac{|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\lambda}}t+\lambda g(t,\lambda)\right)}X_1 \cdot e^{(X_0+\epsilon X_2)\left(\frac{|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\lambda}}t+\lambda g(t,\lambda)\right)}. \quad (6.4.1)$$

It is clear that $X^G(t,\lambda)$ is a sufficiently CS, $\frac{2\pi}{|\omega_{bif}+\lambda|}$- periodic function such that $X^G(t,0)=X_0$. Then, the initial value problem (6.2.3) has the solution

$$A(t,\lambda) = e^{\epsilon X_1 t}e^{(X_0+\epsilon X_2)\left(\frac{|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\lambda}}t+\lambda g(t,\lambda)\right)}. \quad (6.4.2)$$

Clearly, $\epsilon\overrightarrow{X}_1$ is orthogonal to $\overrightarrow{X}_0$.

Example 6.4.3 (Example 2. Resonant drift phenomena for modulated rotating waves). *Let $\omega_{bif} = |X_0|$ and $g(t,\lambda)$ be a sufficiently smooth periodic function of period $\frac{2\pi}{|\omega_{bif}+\lambda|}$ such that $g(0,\lambda)=0$ for $\lambda \geq 0$ small.*
For any $t \in [0,\infty)$ and $\lambda \geq 0$ small, we define

$$X^G(t,\lambda) = (X_0+\epsilon X_2)\left(\frac{|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\lambda}}+\lambda\dot{g}(t,\lambda)\right)+\epsilon e^{-(X_0+\epsilon X_2)\left(\frac{|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\lambda}}t+\lambda g(t,\lambda)\right)} \cdot (X_0+X_1)e^{(X_0+\epsilon X_2)\left(\frac{|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\lambda}}t+\lambda g(t,\lambda)\right)}. \quad (6.4.3)$$

It is clear that $X^G(t,\lambda)$ is a sufficiently CS, $\frac{2\pi}{|\omega_{bif}+\lambda|}$-periodic function such that $X^G(t,0) = X_0$. Then, the initial value problem (6.2.3) has the solution

$$A(t,\lambda) = e^{\epsilon(X_0+X_1)t} e^{(X_0+\epsilon X_2)\left(\frac{|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\lambda}}t+\lambda g(t,\lambda)\right)}. \tag{6.4.4}$$

Clearly, $\epsilon(\overrightarrow{X}_0 + \overrightarrow{X}_1)$ is not orthogonal to $\overrightarrow{X}_0$.

Example 6.4.4 (Example 3. Resonant drift phenomena for modulated rotating waves). *Let $\omega_{bif} = \frac{1}{k}|X_0|$ for $k \in \mathbb{Z}$, $k \neq 0$ and $g(t,\lambda,\mu)$ be a sufficiently smooth periodic function of period $\frac{2\pi}{|\omega_{bif}+\lambda|}$ such that $g(0,\lambda,\mu) = 0$ for $\lambda \geq 0$, $|\mu|$ small.*
For any $t \in [0,\infty)$, $\lambda \geq 0$ small and $|\mu|$ small, we define

$$X^G(t,\lambda,\mu) = (X_0+\mu X_1)\left(\frac{k|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\mu^2}} + \lambda\dot{g}(t,\lambda,\mu)\right)$$
$$+\epsilon^k e^{-(X_0+\mu X_1)\left(\frac{k|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\mu^2}}t+\lambda g(t,\lambda,\mu)\right)}((\epsilon-\mu)X_0+X_1+X_2)e^{(X_0+\mu X_1)\left(\frac{k|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\mu^2}}t+\lambda g(t,\lambda,\mu)\right)}. \tag{6.4.5}$$

It is clear that $X^G(t,\lambda,\mu)$ is a sufficiently CS_λ, $\frac{2\pi}{|\omega_{bif}+\lambda|}$ -periodic function such that $X^G(t,0,\mu) = X_0+\mu X_1$. Then, the initial value problem (6.2.3) for two parameters λ, μ has the solution

$$A(t,\lambda,\mu) = e^{\epsilon^k((\epsilon-\mu)X_0+X_1+X_2)t} e^{(X_0+\mu X_1)\left(\frac{k|\omega_{bif}+\lambda|}{\sqrt{|X_0|^2+\mu^2}}t+\lambda g(t,\lambda,\mu)\right)}. \tag{6.4.6}$$

Clearly, for $\mu = \sqrt{\lambda}$, we have orthogonality, that is $\epsilon^k(\overrightarrow{X}_1 + \overrightarrow{X}_2)$ is orthogonal to $\overrightarrow{X}_0$.

6.5 Proofs of some Theorems of Chapter 6

Proof of Theorem 6.1.1. We apply Theorems 5.3.1 and 5.3.2. Then, it is sufficient to study the differential equations (5.3.1).
The supercritical Hopf bifurcation in V^* for the second differential equation in (5.3.1) at $q = 0$ for $\lambda = 0$ implies that there exists a unique sufficiently CS branch of periodic solutions $q(t,\lambda)$ for $\lambda > 0$ near $q = 0$.
Recall that we have denoted by $T(\lambda) = \frac{2\pi}{|\omega_\lambda|}$ the period of $q(t,\lambda)$, where $\omega_\lambda = \omega_{bif} + O(\lambda)$ is a sufficiently smooth function for $\lambda \geq 0$ small.

If we substitute $q(t,\lambda)$ in the first of the differential equations (5.3.1), we get the differential equation

$$\dot{A} = AX^G(t,\lambda), \tag{6.5.1}$$

where we recall that the function $X^G(t,\lambda)$ is sufficiently CS and $\frac{2\pi}{|\omega_\lambda|}$-periodic for $t \in [0,\infty)$ and $\lambda \geq 0$ small (and it is defined by (6.2.2)).
We consider the following initial value problem on $SO(3)$:

$$\begin{aligned} \dot{A} &= AX^G(t,\lambda), \\ A(0) &= I_3, \end{aligned} \tag{6.5.2}$$

because any solution of the differential equation (6.5.1) with $A(0) = A_0$, where $A_0 \in SO(3)$ is given by $A_0A^*(t,\lambda)$ and $A^*(t,\lambda)$ is the solution of the initial value problem (6.5.2).
For $\lambda > 0$ small, by applying Theorem 6.5.1 (whose statement and proof are given below), we get

$$A(t,\lambda) = e^{X(\lambda)t}B(t,\lambda) \text{ for } t \in [0,\infty) \text{ and } \lambda \geq 0 \text{ small },$$

where $B(t,\lambda)$ has period $T(\lambda)$ and $B(0,\lambda) = I_3$.
Consequently, by using Theorems 5.3.1 and 5.3.2, there exists a solution $\Phi(t,u_\lambda,\lambda)$ of the reaction-diffusion system (3.4.1) of the form:

$$\Phi(t,u_\lambda,\lambda) = A(t,\lambda)\Psi(q(t,\lambda)) = e^{X(\lambda)t}B(t,\lambda)\Psi(q(t,\lambda)), \text{ where } t \in [0,\infty) \text{ and } \lambda > 0 \text{ small },$$

with $\Psi(q(0,\lambda)) = u_\lambda$ and Ψ appears in the proof of Theorem 5.3.2 (see [20] and [70]). If we write

$$q_1(t,\lambda) = B(t,\lambda)\Psi(q(t,\lambda)),$$

then the function $q_1(t,\lambda)$ is $T(\lambda)$-periodic in t, since $B(t,\lambda)$ and $q(t,\lambda)$ are $T(\lambda)$-periodic in t. Also, we get

$$\Phi(t,u_\lambda,\lambda) = e^{X(\lambda)t}q_1(t,\lambda) \text{ and } \Phi(T(\lambda),u_\lambda,\lambda) = e^{X(\lambda)T(\lambda)}u_\lambda.$$

We check that, at least generically, $\Phi(t,u_\lambda,\lambda) \notin SO(3)u_\lambda$ for any $t \in (0,T(\lambda))$ and $\lambda > 0$ small.
Suppose that there exists a $T_1(\lambda) \in (0,T(\lambda))$ such that $\Phi(T_1(\lambda),u_\lambda,\lambda) \in SO(3)u_\lambda$ and for $t \in (0,T_1(\lambda))$ we have $\Phi(t,u_\lambda,\lambda) \notin SO(3)u_\lambda$. Let $T(\lambda) = lT_1(\lambda) + r$, where $r \in [0,T_1(\lambda))$ and $l \in \mathbb{Z}$.
Let $\Phi(T_1(\lambda),u_\lambda,\lambda) = e^{Y(\lambda)T_1(\lambda)}u_\lambda$, where $Y(\lambda) \in so(3)$.

Using the definition and $SO(3)$-equivariance of the semiflow Φ, we get $\Phi(T(\lambda), u_\lambda, \lambda) = \Phi(r, \Phi(lT_1(\lambda), u_\lambda, \lambda)) = e^{lY(\lambda)T_1(\lambda)}\Phi(r, u_\lambda, \lambda)$.
Therefore, $\Phi(r, u_\lambda, \lambda) = e^{-lY(\lambda)T_1(\lambda)}e^{X(\lambda)T(\lambda)}u_\lambda \in SO(3)u_\lambda$ and $r \in [0, T(\lambda))$. It follows that $r = 0$ and so $T(\lambda) = lT_1(\lambda)$. Thus, $\Phi(T(\lambda), u_\lambda, \lambda) = e^{X(\lambda)T(\lambda)}u_\lambda = e^{Y(\lambda)T(\lambda)}u_\lambda$ and $e^{X(\lambda)T(\lambda)} = e^{Y(\lambda)T(\lambda)}$ because $\Sigma_{u_\lambda} = I_3$.
Also, we get $\Phi(t, u_\lambda, \lambda) = e^{X(\lambda)t}q_1(t, \lambda) = e^{Y(\lambda)t}e^{-Y(\lambda)t}e^{X(\lambda)t}B(t, \lambda)\Psi(q(t, \lambda))$.
Let $B_1(t, \lambda) = e^{-Y(\lambda)t}e^{X(\lambda)t}B(t, \lambda)$. $B_1(t, \lambda)$ is $T(\lambda)$-periodic, because $B(t, \lambda)$ is $T(\lambda)$-periodic and $e^{X(\lambda)T(\lambda)} = e^{Y(\lambda)T(\lambda)}$. Thus,

$$\Phi(t, u_\lambda, \lambda) = e^{Y(\lambda)t}B_1(t, \lambda)\Psi(q(t, \lambda)).$$

Then $\Phi(T_1(\lambda), u_\lambda, \lambda) = e^{Y(\lambda)T_1(\lambda)}u_\lambda = e^{Y(\lambda)T_1(\lambda)}B_1(T_1(\lambda), \lambda)\Psi(q(T_1(\lambda), \lambda))$, and it follows that $\Psi(q(T_1(\lambda), \lambda)) = (B(T_1(\lambda), \lambda))^{-1}u_\lambda$.
We have that u_λ is close to u_0 and $SO(3)u_0 \cap \Psi(V_*) = \{u_0\}$, therefore, generically, $SO(3)u_\lambda \cap \Psi(V^*) = \{u_\lambda\}$. Since $\Psi(q(T_1(\lambda), \lambda)) = (B(T_1(\lambda), \lambda))^{-1}u_\lambda \in \Psi(V_*) \cap SO(3)u_\lambda$, it follows that $B(T_1(\lambda), \lambda)u_\lambda = u_\lambda$ and, since $\Sigma_{u_\lambda} = I_3$, $B_1(T_1(\lambda), \lambda) = I_3$ and $\Psi(q(T_1(\lambda), \lambda)) = u_\lambda = \Psi(q(0, \lambda))$. Because Ψ is a local diffeomorphism (see [20] and [70]) and $q(t, \lambda) = \lambda^{\frac{1}{2}}r(t, \lambda)$, it results that $q(T_1(\lambda), \lambda) = q(0, \lambda)$. We know that $q(t, \lambda)$ is a solution of the differential equation $\dot{q} = X_N(q, \lambda)$. Therefore, it follows that $q(t, \lambda)$ is $T_1(\lambda)$-periodic. But it has the period $T(\lambda)$. That is, $T(\lambda) = T_1(\lambda)$ contradicting the fact that $T_1(\lambda) \in (0, T(\lambda))$.
If $|X(\lambda)|\, T(\lambda) = 2k\pi$ for some $k \in \mathbb{Z}$, then $\Phi(t, u_\lambda, \lambda)$ is a periodic solution with the period $T(\lambda)$.
Otherwise, $\Phi(t, u_\lambda, \lambda)$ is a modulated rotating wave with a primary frequency vector $\overrightarrow{X(\lambda)}$.
By theorem 4.4.4, the orbital stability of the modulated rotating wave or of the periodic solution obtained above is the same as the stability of the periodic solution $q(t, \lambda)$.
Therefore, since the periodic solution $q(t, \lambda)$ for $\lambda > 0$ is stable, then the corresponding modulated rotating wave or periodic solution is orbitally stable as well.
Since $A(t, \lambda)$ and $q(t, \lambda)$ are sufficiently CS and Ψ is a local diffeomorphism, we get that $\Phi(t, u_\lambda, \lambda)$ is sufficiently CS. The same result can be inferred if we take into account that $u_\lambda = \Psi(q(0, \lambda))$. $\square$

Theorem 6.5.1. *Consider the initial value problem*

$$\begin{aligned} \dot{A} &= AX^G(t, \lambda), \\ A(0) &= I_3, \end{aligned} \tag{6.5.3}$$

where $X^G(t,\lambda)$ defined by (6.3.2) is sufficiently CS and $\frac{2\pi}{|\omega_\lambda|}$-periodic in t for $\lambda > 0$ small. Then, there exists a sufficiently CS solution $A(t,\lambda) = e^{X(\lambda)t}B(t,\lambda)$ of the initial value problem (6.5.3), where $X(\lambda) \in so(3)$ and $B(t,\lambda)$ is a $\frac{2\pi}{|\omega_\lambda|}$-periodic function such that $B(0,\lambda) = I_3$.

Proof of Theorem 6.5.1. Let us omit the parameter λ.

Since $SO(3)$ is a compact manifold and $X^G(.,\lambda)$ is a sufficiently CS function, the initial value problem (6.5.3) has a unique sufficiently CS solution that is globally defined. Let $A^*(t)$ be this solution. Since the exponential map exp: $so(3) \to SO(3)$ is surjective, there exists a matrix $X \in so(3)$ such that $A^*(T) = e^{XT}$.

Let us make the following change of variable $B = e^{-Xt}A$.

Then the first equation of (6.5.3) becomes:

$$\dot{B} = -e^{-Xt}XA(t) + e^{-Xt}\dot{A}(t), \tag{6.5.4}$$

or since $\dot{A} = AX^G(t)$, we get

$$\dot{B} = -e^{-Xt}XA(t) + e^{-Xt}AX^G(t), \tag{6.5.5}$$

or since $A = e^{Xt}B$ it follows that

$$\dot{B} = -XB(t) + B(t)X^G(t). \tag{6.5.6}$$

The initial condition $A(0) = I_3$ becomes $B(0) = I_3$.

Let us consider now the initial value problem

$$\begin{aligned} \dot{B} &= -XB + BX^G(t), \\ B(0) &= I_3. \end{aligned} \tag{6.5.7}$$

The initial value problem (6.5.7) has a unique global solution since $SO(3)$ is a compact manifold.

Since $A^*(t)$ is the solution of the initial value problem (6.5.3), it follows that $B^*(t) = e^{-Xt}A^*(t)$ is a solution of (6.5.7) and $B^*(T) = I_3$.

Let us define

$$C^*(t) = B^*(t+T).$$

We will check that C^* is another solution of (6.5.7). We have $C^*(0) = B^*(T) = I_3$ and for any $t \in [0, \infty)$,

$$\frac{d}{dt}\left(C^*(t)\right) = \frac{d}{dt}\left(B^*(t+T)\right) = \frac{dB^*}{dt}(t+T) = -XB^*(t+T) + B^*(t+T)X^G(t+T)$$
$$= -XB^*(t+T) + B^*(t+T)X^G(t) = -XC^*(t) + C^*(t)X^G(t), \tag{6.5.8}$$

where we have used the fact that X^G is a T-periodic function.
Therefore, $B^*(t) = C^*(t)$ for $t \in [0, \infty)$, that is, B^* is T-periodic.
So, $A^*(t) = e^{Xt}B^*(t)$ for $t \in [0, \infty)$.
Taking into account the parameter $\lambda > 0$, since $X^G(t, \lambda)$ is sufficiently CS for $t \in [0, \infty)$ and $\lambda > 0$, we have that $A(t, \lambda)$ is sufficiently CS for $t \in [0, \infty)$ and $\lambda > 0$ small. If we work with $\lambda \geq 0$, then $A(t, \lambda)$ is sufficiently CS for $t \in [0, \infty)$ and $\lambda \geq 0$ small. $\square$

Proof of Theorem 6.2.1. We recall that $T(\lambda) = \frac{2\pi}{|\omega_\lambda|}$ for $\lambda \geq 0$ small.
From the proof of Theorems 6.1.1 and 6.5.1, a primary frequency vector $\overrightarrow{X(\lambda)}$ of $\Phi(t, u_\lambda, \lambda)$ is given by $A(T(\lambda), \lambda) = e^{X(\lambda)T(\lambda)}$ for $\lambda > 0$ small, where $A(t, \lambda)$ is the solution of the initial value problem (6.5.3).
Also, the initial value problem (6.5.3) for $\lambda = 0$ has the solution $A(t, 0) = e^{X_0 t}$.
From Section 2.3, we get that $X(\lambda)$ for $\lambda \geq 0$ small can be defined as follows:

1. If $(A(T(\lambda), \lambda))^2 \neq I_3$, then

$$X(\lambda) = \arccos\left(\frac{Tr(A(T(\lambda), \lambda)) - 1}{2}\right)\frac{1}{T(\lambda)}\frac{A(T(\lambda), \lambda) - (A(T(\lambda), \lambda))^T}{2\sin(\arccos(\frac{Tr(A(T(\lambda), \lambda)) - 1}{2}))}. \tag{6.5.9}$$

   Also, $|X(\lambda)| = \arccos(\frac{Tr(A(T(\lambda), \lambda)) - 1}{2})\frac{1}{T(\lambda)}$.

2. If $(A(T(\lambda), \lambda))^2 = I_3$, but $A(T(\lambda), \lambda) \neq I_3$, then $X(\lambda) \in so(3)$ is such that $(X(\lambda))^2 = \frac{A(T(\lambda), \lambda) - I_3}{2}\frac{\pi^2}{(T(\lambda))^2}$.
   Also, $|X(\lambda)| = \frac{\pi}{T(\lambda)}$ and $Tr(A(T(\lambda), \lambda)) = -1$.

3. If $A(T(\lambda), \lambda) = I_3$, we have $X(\lambda) = O_3$.
   Then, $|X(\lambda)| = 0$ and $Tr(A(T(\lambda), \lambda)) = 3$.

Let us denote

$$\theta(\lambda) = \arccos\left(\frac{Tr(A(T(\lambda), \lambda)) - 1}{2}\right) = |X(\lambda)|\,T(\lambda) \text{ and}$$

$$V(\lambda)=\begin{cases}\frac{X(\lambda)T(\lambda)}{\theta(\lambda)} & \text{if } \theta(\lambda)\neq 0;\\ O_3 & \text{if } \theta(\lambda)=0.\end{cases}$$

Since $A(t,\lambda)$ is sufficiently CS, $T(\lambda)$ is sufficiently smooth and arccos is continuous, it follows that the function $\theta(\lambda)$ is continuous and $|V(\lambda)|=1$ if $\theta(\lambda)\neq 0$ for small $\lambda\geq 0$.
We have $X(\lambda)=\frac{\theta(\lambda)}{T(\lambda)}V(\lambda)$ and $X(\lambda)T(\lambda)=\theta(\lambda)V(\lambda)$ for small $\lambda\geq 0$.
The branch $[X(\lambda)T(\lambda)]\in D$ is the only branch such that $A(T(\lambda),\lambda)=e^{X(\lambda)T(\lambda)}$ for $\lambda\geq 0$, where D is defined in Section 2.3.
From Section 2.3, we know that $[\theta(\lambda)V(\lambda)]=d^*(A(T(\lambda),\lambda))$, where d^* is defined in (2.3.4). Since d^* is smooth and $A(T(\lambda),\lambda)$ is sufficiently CS, then $[\theta(\lambda)V(\lambda)]$ is sufficiently CS. Therefore, we get that the branch $[X(\lambda)T(\lambda)]$ for $\lambda\geq 0$ is sufficiently CS.
Let $|X_0|\,T(0)=\alpha_0+2k\pi$ for $k\in\mathbb{Z}$, $k\geq 0$, $\alpha_0\in[0,2\pi)$.
We have $A(T(0),0)=e^{X_0T(0)}=e^{X(0)T(0)}$. By the definition of $\theta(0)$, we get that $\theta(0)=\arccos(\frac{Tr(A(T(0),0))-1}{2})=\arccos(\cos(|X_0|\,T(0)))=\arccos(\cos\alpha_0)$, therefore

$$\theta(0)=\begin{cases}\alpha_0, & \text{if } \alpha_0\in[0,\pi];\\ 2\pi-\alpha_0, & \text{if } \alpha_0\in(\pi,2\pi).\end{cases}$$

1. If $|X_0|\neq\frac{2k+1}{2}\omega_{bif}$ for any $k\in\mathbb{Z}$, then $\alpha_0\neq\pi$, that is $\theta(0)<\pi$. Therefore, since $\theta(\lambda)$ is continuous, $\theta(\lambda)=|X(\lambda)|\,T(\lambda)<\pi$ for $\lambda\geq 0$ small. Thus, $X(\lambda)T(\lambda)$ is sufficiently CS and, since $T(\lambda)$ is sufficiently smooth, it follows that $X(\lambda)$ is sufficiently CS.

2. If $|X_0|=\frac{2k+1}{2}\omega_{bif}$ for some $k\in\mathbb{Z}$, then $\alpha_0=\pi$, that is $\theta(0)=\pi$, and since $\theta(\lambda)$ is continuous for $\lambda\geq 0$ small, we have $\frac{\pi}{2}<\theta(\lambda)=|X(\lambda)|\,T(\lambda)<\frac{3\pi}{2}$.
We check that $(\pi X_0^1)^2=(X(0)T(0))^2$.
It follows by using definition of $X(0)$, the relation $A(T(0),0)=e^{X_0T(0)}$ and the Rodrigues' formula that
$$(X(0))^2=\frac{A(T(0),0)-I_3}{2}\frac{\pi^2}{(T(0))^2}=\frac{e^{X_0T(0)}-I_3}{2}\frac{\pi^2}{(T(0))^2}$$
or
$$(X(0))^2=\frac{\sin^2\frac{|X_0T(0)|}{2}}{|X_0|^2\,(T(0))^2}(X_0T(0))^2\frac{\pi^2}{(T(0))^2}=(\frac{\pi}{T(0)\,|X_0|}X_0)^2.$$
Thus, $(\pi X_0^1)^2=(X(0)T(0))^2$. Since $|\pi X_0^1|=|X(0)T(0)|=\pi$, it follows that $[\pi X_0^1]=[X(0)T(0)]$. Since $[X(\lambda)T(\lambda)]$ is sufficiently CS, $[X(\lambda)T(\lambda)]$ is close to $[\pi X_0^1]$ for $\lambda\geq 0$ small.

Then, for $\lambda \geq 0$ small, we define $X_1(\lambda) = \frac{1}{T(\lambda)} q^{-1}([X(\lambda)T(\lambda)])$, where q^{-1} is defined in (2.3.3) for πX_0^1 (Section 2.3). It is clear that $e^{X_1(\lambda)T(\lambda)} = e^{X(\lambda)T(\lambda)}$ for $\lambda \geq 0$ small. Since q^{-1} is smooth and $[X(\lambda)T(\lambda)]$ is sufficiently CS, $T(\lambda)$ is sufficiently smooth, it follows that $X_1(\lambda)$ is sufficiently CS. Thus, we relabel $X_1(\lambda)$ to $X(\lambda)$.
In both cases, it results that $|X(\lambda)|^2$ for $\lambda \geq 0$ small is sufficiently CS.
Using Taylor formula for $\epsilon \to \left|X(\epsilon^2)\right|^2$ around $\epsilon = 0$ up to the term of order ϵ, and then taking $\epsilon = \lambda^{\frac{1}{2}}$, we get $|X(\lambda)|^2 = |X(0)|^2 + O(\epsilon) = |X(0)|^2 + O(\lambda^{\frac{1}{2}})$ for $\lambda \geq 0$ small. Also, $|X(\lambda)|$ is continuous for $\lambda \geq 0$.

The branch $X^f(\lambda)$ is constructed below.

1. case $|X_0| \neq k\omega_{bif}$ for $k \in \mathbb{Z}$;
   Then $\alpha_0 > 0$, that is $\theta(0) > 0$. Hence, since $\theta(\lambda)$ is continuous, it follows that $\theta(\lambda) > 0$ for $\lambda \geq 0$ small.
   If $\alpha_0 \in [0, \pi]$, then we define $X^f(\lambda) = \frac{\theta(\lambda)+2k\pi}{\theta(\lambda)} X(\lambda)$ for $\lambda \geq 0$ small.
   If $\alpha_0 \in (\pi, 2\pi)$, then we define $X^f(\lambda) = \frac{2(k+1)\pi-\theta(\lambda)}{\theta(\lambda)}(-X(\lambda))$ for $\lambda \geq 0$ small.

2. case $|X_0| = k\omega_{bif}$, for some $k \in \mathbb{Z}$, $k \neq 0$;
   Then $\alpha_0 = \theta(0) = 0$.
   We define $X^f(0) = X_0$.
   If $\theta(\lambda) \neq 0$, then we define $X^f(\lambda) = \frac{\theta(\lambda)+2k\pi}{\theta(\lambda)} X(\lambda)$ for $\lambda > 0$ small.
   If $\theta(\lambda) = 0$, then we define $X^f(\lambda) = X(\lambda) + \frac{2k\pi}{T(\lambda)} Q(\lambda)$, where $Q(\lambda) \in so(3)$, $|Q(\lambda)| = 1$ and $\lambda > 0$ is small.

We check now the statement regarding continuity versus smoothness of $X^f(\lambda)$.

1. Case $|X_0| \neq k\omega_{bif}$ for any $k \in \mathbb{Z}$.
   Since $X(\lambda)$ is sufficiently CS,$T(\lambda)$ is sufficiently smooth and $\theta(\lambda) > 0$ for $\lambda \geq 0$ small, we get that $\theta(\lambda) = |X(\lambda)|\, T(\lambda)$ is sufficiently CS for $\lambda \geq 0$ small.
   Since $\theta(\lambda) > 0$ for $\lambda \geq 0$ small, $\theta(\lambda)$ and $X(\lambda)$ are sufficiently CS, it follows that $X^f(\lambda)$ is sufficiently CS for $\lambda \geq 0$ small.
   We check that $X^f(0) = X_0$.

   (a) If $\alpha_0 \in (0, \pi)$, then $\alpha_0 = \theta(0)$ and $|X_0|\, T(0) = \theta(0) + 2k\pi$.
   By the definition of $X(\lambda)$ for $\lambda = 0$, $A(T(0), 0) = e^{X_0 T(0)}$ and Rodrigues' formula (Theorem 2.2.13, (4)), we have that $X(0) = \frac{A(T(0),0)-A(T(0),0)^{-1}}{2\sin\theta(0)} \frac{\theta(0)}{T(0)} =$

$\frac{e^{X_0T(0)}-e^{-X_0T(0)}}{2\sin\theta(0)}\frac{\theta(0)}{T(0)} = \frac{\sin(|X_0|T(0))}{\sin\theta(0)}\frac{\theta(0)}{|X_0|T(0)}X_0$. Thus, $X(0) = \frac{\theta(0)}{|X_0|T(0)}X_0$ and $X^f(0) = \frac{\theta(0)+2k\pi}{\theta(0)}\frac{\theta(0)}{|X_0|T(0)}X_0 = X_0$.

(b) If $\alpha_0 = \pi$, then $\alpha_0 = \theta(0) = \pi$ and $|X_0|T(0) = (2k+1)\pi$, and by the definition of $X_1(\lambda)$ for $\lambda = 0$, we have $X_1(0) = \frac{1}{T(0)}q^{-1}([X(0)T(0)]) = \frac{1}{T(0)}q^{-1}([\pi X_0^1]) = \frac{1}{T(0)}\pi X_0^1$. Then, since we have relabeled $X_1(0)$ to $X(0)$, $X(0) = \frac{1}{T(0)}\pi X_0^1$. Thus, $X^f(0) = \frac{\theta(0)+2k\pi}{\theta(0)}X(0) = \frac{\theta(0)+2k\pi}{\pi}\frac{\pi}{|X_0|T(0)}X_0 = X_0$.

(c) If $\alpha_0 \in (\pi, 2\pi)$, then as for $\alpha \in (0,\pi)$, we get $X^f(0) = X_0$.

Using the Taylor formula for $\epsilon \to X^f(\epsilon^2)$ around $\epsilon = 0$ up to the term of order ϵ and then taking $\epsilon = \lambda^{\frac{1}{2}}$, we get $X^f(\lambda) = X^f(0) + O(\epsilon)X_0^1 + O(\epsilon)X_1 + O(\epsilon)X_2 = X_0 + O(\lambda^{\frac{1}{2}})X_0^1 + O(\lambda^{\frac{1}{2}})X_1 + O(\lambda^{\frac{1}{2}})X_2$ for $\lambda \geq 0$ small, thus we get the formula (6.2.4).

We have only modulated rotating waves because $|X_0|T(0) \neq 2k\pi$ for any $k \in \mathbb{Z}$ implies $|X(\lambda)|T(\lambda) \neq 2k\pi$ for any $k \in \mathbb{Z}$ and for $\lambda \geq 0$ small by the continuity of $\theta(\lambda) = |X(\lambda)|T(\lambda)$.

2. In the case $|X_0| = k\omega_{bif}$ for some $k \in \mathbb{Z}$, $k \neq 0$, we have $\theta(0) = 0 = |X(0)|$. We have $\left|X^f(0)\right| = |X_0| = k\omega_{bif} = \frac{2k\pi}{T(0)}$.
For $\lambda > 0$ small such that $\theta(\lambda) \neq 0$, we have $\left|X^f(\lambda)\right| = |X(\lambda)| + 2k\pi\frac{|X(\lambda)|}{\theta(\lambda)} = |X(\lambda)| + \frac{2k\pi}{T(\lambda)}$.
For $\lambda > 0$ small such that $\theta(\lambda) = 0$, we have $|X(\lambda)| = 0$. Then, $\left|X^f(\lambda)\right| = \frac{2k\pi}{T(\lambda)}|Q(\lambda)| = \frac{2k\pi}{T(\lambda)}$.
So, for $\lambda \geq 0$ small, we get $\left|X^f(\lambda)\right| = |X(\lambda)| + \frac{2k\pi}{T(\lambda)}$.
Since $|X(\lambda)|^2 = |X(0)|^2 + O(\lambda^{\frac{1}{2}})$ and $|X(0)| = 0$, we get $|X(\lambda)| = O(\lambda^{\frac{1}{4}})$ and then $\left|X^f(\lambda)\right| = O(\lambda^{\frac{1}{4}}) + \frac{2k\pi}{T(0)} + O(\lambda) = O(\lambda^{\frac{1}{4}}) + |X_0|$ for $\lambda \geq 0$ small, thus we get the formula (6.2.5).
Also, $\left|X^f(\lambda)\right|$ is continuous since $X(\lambda)$ and $T(\lambda)$ are continuous for $\lambda \geq 0$ small.

By using Theorem 2.2.13 ((6) and (7)) and $\theta(\lambda) = |X(\lambda)|T(\lambda)$, we have $e^{X^f(\lambda)T(\lambda)} = e^{X(\lambda)T(\lambda)}$ for $\lambda \geq 0$ small. □

Proof of Theorem 6.2.3. We will use the following result (see [37]):

Lemma 6.5.2. *Let $g(t,u)$ be a continuous function on an open connected set $[a_1, b_1) \times \mathbb{R}^+ \subset \Omega \subset \mathbb{R}^2$ and such that the initial value problem for the scalar equation $\dot{u} = g(t,u)$*

has a unique solution $u(t) \geq 0$ on $t \in [a_1, b_1)$. If $f\colon [a_1, b_1) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $\|f(t,x)\| \leq g(t, \|x\|)$ for $t \in [a_1, b_1)$ and $x \in \mathbb{R}^n$, then the solutions of $\dot{x} = f(t,x)$, $\|x(a_1)\| \leq u(a_1)$ exists on $[a_1, b_1)$ and $\|x(t)\| \leq u(t)$ for $t \in [a_1, b_1)$.

There exists a constant $M > 0$ independent of $\lambda \geq 0$ small such that $\left|X^G(t,\lambda)\right| < M$ for $t \in [0, \infty)$ and $\lambda \geq 0$ small. For $\lambda \geq 0$ small and any $|Z| \leq \pi$, we have

$$\begin{aligned} &\left\| \left[I_3 + \frac{1}{2} Z + \left(\frac{1}{|Z|^2} - \frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|} \right) Z^2 \right] \overrightarrow{X^G(t,\lambda)} \right\| \leq \\ &\left[\|I_3\| + \frac{1}{2}\|Z\| + \left(\frac{1}{|Z|^2} + \frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|} \right) \|Z\|^2 \right] M, \end{aligned} \tag{6.5.10}$$

$$\begin{aligned} &\left\| \left[I_3 + \frac{1}{2} Z + \left(\frac{1}{|Z|^2} - \frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|} \right) Z^2 \right] \overrightarrow{X^G(t,\lambda)} \right\| \leq \\ &M \left[\sqrt{3} + \frac{1}{2}\sqrt{2}|Z| + \left(\frac{1}{|Z|^2} + \frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|} \right) 2|Z|^2 \right] \end{aligned} \tag{6.5.11}$$

or, if we use the fact that the function $x \to \frac{x \cos x}{\sin x}$ is decreasing on $[0, \frac{\pi}{2}]$,

$$\begin{aligned} &\left\| \left[I_3 + \frac{1}{2} Z + \left(\frac{1}{|Z|^2} - \frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|} \right) Z^2 \right] \overrightarrow{X^G(t,\lambda)} \right\| \leq \\ &M \left[\sqrt{3} + \frac{1}{2}\sqrt{2}|Z| + (2 + 2 \cdot 1) \right] \leq M(6 + |Z|). \end{aligned} \tag{6.5.12}$$

The initial value problem

$$\begin{aligned} \dot{a} &= M(6 + a), \\ a(0) &= 0 \end{aligned} \tag{6.5.13}$$

has solution on an maximal interval $[0, b_{max}]$, where b_{max} is defined such that $a(t) \leq \pi$ on $[0, b_{max}]$; b_{max} is independent of λ.

Therefore, using Lemma 6.5.2 with $a_1 = 0$, it follows that the sufficiently CS solution $Z(t, \lambda)$ of the initial value problem (6.2.6) is defined for any $t \in [0, b_{max}]$ and $\lambda \geq 0$ small, and that $|Z(t,\lambda)| \leq \pi$ for any $t \in [0, b_{max}]$ and $\lambda \geq 0$ small.

If we choose a positive integer $n > 0$ such that $n > \frac{T(\lambda)}{b_{max}}$, then the initial value problem (6.2.6) has a solution $Z(t, \lambda)$ defined and sufficiently CS on $t \in [0, \frac{T(\lambda)}{n}]$ and $\lambda \geq 0$ small. Since $T(\lambda) = T(0) + O(\lambda)$, then we can choose n independent of λ for $\lambda \geq 0$ small, where $T(0) = \frac{2\pi}{|\omega_{bif}|}$.

This ends the proof of (1) of Theorem 6.2.3.

To prove the formula (6.2.7), we write the Taylor expansion of $Z^*(t,\epsilon) \stackrel{\text{def}}{=} Z(t,\epsilon^2)$:

$$\begin{aligned} Z^*(t,\epsilon) &= z_{00}(t,\epsilon)X_0^1 + z_{11}(t,\epsilon)X_1 + z_{22}(t,\epsilon)X_2, & (6.5.14)\\ z_{00}(t,\epsilon) &= Z_0(t) + Z_1(t)\epsilon + \epsilon^2 Z_2(t,\epsilon). & (6.5.15)\end{aligned}$$

If we substitute (6.5.14), (6.5.15) and (6.2.2) in the first of the differential equations given by (6.2.6) with $\lambda = \epsilon^2$, identify the orders of ϵ in both sides and then take $\epsilon = \lambda^{\frac{1}{2}}$, we get the formula (6.2.7) for $Z(t,\lambda)$. For the sake of simplicity, we omit ϵ.

The result is

$$\begin{aligned} \dot{z}_{00}X_0^1 + \dot{z}_{11}X_1 + \dot{z}_{22}X_2 = [I_3 + \frac{z_{00}}{2}X_0^1 + \frac{z_{11}}{2}X_1 + \frac{z_{22}}{2}X_2 \\ + \left(\frac{1}{z_{00}^2 + z_{11}^2 + z_{22}^2} - \frac{\cos\frac{\sqrt{z_{00}^2+z_{11}^2+z_{22}^2}}{2}}{2\sqrt{z_{00}^2 + z_{11}^2 + z_{22}^2}\sin\frac{\sqrt{z_{00}^2+z_{11}^2+z_{22}^2}}{2}}\right) \\ \cdot(z_{00}X_0^1 + z_{11}X_1 + z_{22}X_2)^2]\overrightarrow{x_0X_0^1 + \epsilon x_1X_1 + \epsilon x_2X_2}. \end{aligned} \tag{6.5.16}$$

Since Z^* is sufficiently smooth, we get $Z^*(t,\epsilon) = Z^*(t,0) + \epsilon H(t,\epsilon) = |X_0|\,tX_0^1 + \epsilon H(t,\epsilon)$. Therefore, $z_{11}(t,\epsilon) = \epsilon z_1(t,\epsilon)$ and $z_{22}(t,\epsilon) = \epsilon z_2(t,\epsilon)$. In the right-hand side of (6.5.16), the coefficient of X_0^1 is $x_0(t,\epsilon) + \epsilon^2 k(t,\epsilon) = |X_0| + \epsilon x_{01}(t) + \epsilon^2 x_{02}(t,\epsilon) + \epsilon^2 k(t,\epsilon)$ and thus $\dot{z}_{00} = |X_0| + \epsilon x_{01}(t) + \epsilon^2 k_1(t,\epsilon)$, where x_0, x_{01}, x_{02} are defined in (6.2.2). Therefore,

$$z_{00}(t,\lambda) = |X_0|\,t + \epsilon\int_0^t x_{01}(s)\,ds + \epsilon^2 x_{03}(t,\epsilon).$$

This gives the formula (6.2.7) after we relabel $z_1(t,\lambda^{\frac{1}{2}})$, $z_2(t,\lambda^{\frac{1}{2}})$ and $x_{03}(t,\lambda^{\frac{1}{2}})$ to $z_1(t,\lambda)$, $z_1(t,\lambda)$ and $x_{03}(t,\lambda)$. This ends the proof of (2) of Theorem 6.2.3.

We have that the initial value problem

$$\begin{aligned} \dot{a} &= M(6+a), \\ a(0) &= 0 \end{aligned} \tag{6.5.17}$$

has the solution a defined on $[0, b_{max}]$ with $a(t) \leq \pi$ on $[0, b_{max}]$. Let $t_0 = \frac{T(\lambda)}{n} \in [0, b_{max}]$. Consider the initial value problem

$$\begin{aligned} \dot{a} &= M(6+a), \\ a(t_0) &= 0 \end{aligned} \tag{6.5.18}$$

The solution to this problem satisfies $a(t) \leq \pi$ for any $t \in [t_0, 2t_0]$.
We repeat the previous argument for $t_0 = i\frac{T(\lambda)}{n}$ for $i = 2, 3, \ldots, \quad n-1$. From this and Lemma 6.5.2 with $a_1 = t_0 = i\frac{T(\lambda)}{n}$ for $i = 2, 3, \ldots, \quad n-1$, we get the last conclusion of Theorem 6.2.3. □

Proof of Theorem 6.2.4. Recall that $T(\lambda) = \frac{2\pi}{|\omega_\lambda|}$ for $\lambda \geq 0$ small.

Part 1 On the interval $[0, T(\lambda)]$:

Consider the initial value problem

$$\begin{aligned} \dot{A} &= AX^G(t,\lambda), \\ A(0) &= I_3. \end{aligned} \tag{6.5.19}$$

Let us make the change of variable $A = e^Z$ near I_3 in the initial value problem (6.5.19).

Then, using Theorem 2.2.13 (2) we get the following initial value problem in Z:

$$\begin{aligned} e^Z \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}(adZ)^n \dot{Z} &= e^Z X^G(t,\lambda), \\ Z(0) &= O_3, \end{aligned} \tag{6.5.20}$$

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}(adZ)^n \dot{Z} &= X^G(t,\lambda), \\ Z(0) &= O_3, \end{aligned} \tag{6.5.21}$$

$$\begin{aligned} \overrightarrow{\textstyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}(adZ)^n \dot{Z}} &= \overrightarrow{X^G(t,\lambda)}, \\ Z(0) &= O_3, \end{aligned} \tag{6.5.22}$$

or, using again Theorem 2.2.9 (1) we get

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} Z^n \overrightarrow{\dot{Z}} &= \overrightarrow{X^G(t,\lambda)}, \\ Z(0) &= O_3. \end{aligned} \tag{6.5.23}$$

Using Theorem 2.2.9 (3), we have that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} Z^n = \left(I_3 + \frac{\cos|Z| - 1}{|Z|^2} Z + \frac{|Z| - \sin|Z|}{|Z|^3} Z^2 \right). \tag{6.5.24}$$

(6.5.24) will proved later.

If we put (6.5.24) into (6.5.23), we get

$$\begin{aligned} \left(I_3 + \frac{\cos|Z|-1}{|Z|^2} Z + \frac{|Z|-\sin|Z|}{|Z|^3} Z^2 \right) \overrightarrow{\dot{Z}} &= \overrightarrow{X^G(t,\lambda)}, \\ Z(0) &= O_3. \end{aligned} \tag{6.5.25}$$

We will prove later that for any $|Z| < 2\pi$,

$$\left(\frac{\cos|Z|-1}{|Z|^2}Z+\frac{|Z|-\sin|Z|}{|Z|^3}Z^2\right)^{-1}=I_3+\frac{1}{2}Z+\left(\frac{1}{|Z|^2}-\frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|}\right)Z^2. \quad (6.5.26)$$

Since we are looking for $|Z| \leq \pi < 2\pi$, the system (6.5.25) becomes

$$\begin{aligned} \overrightarrow{\dot{Z}} &= \left[I_3+\tfrac{1}{2}Z+\left(\tfrac{1}{|Z|^2}-\tfrac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|}\right)Z^2\right]\overrightarrow{X^G(t,\lambda)}, \\ Z(0) &= O_3, \end{aligned} \quad (6.5.27)$$

where we are looking for a solution Z such that $|Z| \leq \pi$ on some interval $[0, t_0(\lambda)]$. We now prove (6.5.26). It is enough to check that

$$\left(I_3+\frac{\cos|Z|-1}{|Z|^2}Z+\frac{|Z|-\sin|Z|}{|Z|^3}Z^2\right)\left[I_3+\frac{1}{2}Z+\left(\frac{1}{|Z|^2}-\frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|}\right)Z^2\right]=I_3. \quad (6.5.28)$$

Using Theorem 2.2.9 (3), that is $Z^3 = -|Z|^2 Z$ and $Z^4 = -|Z|^2 Z^2$, we have

$$\begin{aligned}
&\left(I_3+\frac{\cos|Z|-1}{|Z|^2}Z+\frac{|Z|-\sin|Z|}{|Z|^3}Z^2\right)\left[I_3+\frac{1}{2}Z+\left(\frac{1}{|Z|^2}-\frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|}\right)Z^2\right] \\
&=I_3+\frac{1}{2}Z+\left(\frac{1}{|Z|^2}-\frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|}\right)Z^2+\frac{\cos|Z|-1}{2|Z|^2}Z^2 \\
&+\frac{\cos|Z|-1}{|Z|^2}Z+\frac{\cos|Z|-1}{|Z|^2}\left(\frac{1}{|Z|^2}-\frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|}\right)Z^3+\frac{|Z|-\sin|Z|}{|Z|^3}Z^2+\frac{|Z|-\sin|Z|}{2|Z|^3}Z^3 \\
&+\frac{|Z|-\sin|Z|}{|Z|^3}\left(\frac{1}{|Z|^2}-\frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|}\right)Z^4 \\
&=I_3+\left[\frac{1}{2}+\frac{\cos|Z|-1}{|Z|^2}-\frac{\cos|Z|-1}{|Z|^2}+\frac{\cos|Z|-1}{|Z|^2}\frac{|Z|\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}}-\frac{1}{2}+\frac{\sin|Z|}{2|Z|}\right]Z \\
&\left[\frac{1}{|Z|^2}-\frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|}+\frac{\cos|Z|-1}{2|Z|^2}+\frac{|Z|-\sin|Z|}{|Z|^3}-\frac{|Z|-\sin|Z|}{|Z|}\frac{1}{|Z|^2}+\right. \\
&\left.+\frac{|Z|-\sin|Z|}{|Z|}\frac{\cos\frac{|Z|}{2}}{2\sin\frac{|Z|}{2}|Z|}\right]Z^2 \\
&=I_3+\left[\frac{-\sin^2\frac{|Z|}{2}}{|Z|}\frac{\cos\frac{|Z|}{2}}{\sin\frac{|Z|}{2}}+\frac{\sin|Z|}{2|Z|}\right]Z+\left[\frac{\cos^2\frac{|Z|}{2}}{|Z|^2}-\frac{2\sin\frac{|Z|}{2}\cos\frac{|Z|}{2}}{2|Z|^2}\frac{\cos\frac{|Z|}{2}}{\sin\frac{|Z|}{2}}\right]Z^2=I_3.
\end{aligned} \quad (6.5.29)$$

The proof of (6.5.24) follows by using Theorem 2.2.9 (3):

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} Z^n &= I_3 + \sum_{k=1}^{\infty} \frac{(-1)^{2k}}{(2k+1)!} Z^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+2)!} Z^{2k+1} \\
&= I_3 + \left(-\frac{1}{2!} + \frac{1}{4!} |Z|^2 - \frac{1}{6!} |Z|^4 + \dots \right) Z \\
&+ \left(\frac{1}{3!} - \frac{1}{5!} |Z|^2 + \frac{1}{7!} |Z|^4 - \dots \right) Z^2 \\
&= I_3 + \frac{\cos|Z| - 1}{|Z|^2} Z + \frac{|Z| - \sin|Z|}{|Z|^3} Z^2,
\end{aligned} \tag{6.5.30}$$

where we have used the Taylor expansions for *sine* and *cosine*.
We apply Theorem 6.2.3 to get $A(t,\lambda) = e^{Z_1(t,\lambda)}$, for any $t \in [0, \frac{T(\lambda)}{n}]$ and $\lambda \geq 0$ small, where $Z_1(t,\lambda)$ is the sufficiently CS solution of (6.2.6). We make the change of variable $B = (A(\frac{T(\lambda)}{n}, \lambda))^{-1} A$. We get that the solution of the initial value problem (6.5.19) is given by

$$A(t,\lambda) = e^{Z_1(\frac{T(\lambda)}{n},\lambda)} B(t,\lambda),$$

where $B(t,\lambda)$ is the solution of the initial value problem

$$\begin{aligned}
\dot{B} &= B X^G(t,\lambda), \\
B(\tfrac{1}{n} T(\lambda)) &= I_3.
\end{aligned} \tag{6.5.31}$$

By using the above argument and Theorem 6.2.3, we have that $B(t,\lambda) = e^{Z_2(t,\lambda)}$ for any $t \in [\frac{T(\lambda)}{n}, 2\frac{T(\lambda)}{n}]$ and $\lambda \geq 0$ small, where $Z_2(t,\lambda)$ is the sufficiently CS solution of (6.2.10) for $i = 2$.
If we continue this, we get for $\lambda \geq 0$ small and any $t \in [(i-1)\frac{T(\lambda)}{n}, i\frac{T(\lambda)}{n}]$,

$$A(t,\lambda) = e^{Z^0(t,\lambda)},$$

where $Z^0(t,\lambda)$ is given by

$$e^{Z^0(t,\lambda)} = e^{Z_1(\frac{T(\lambda)}{n},\lambda)} e^{Z_2(2\frac{T(\lambda)}{n},\lambda)} \dots e^{Z_i(t,\lambda)} \tag{6.5.32}$$

for any $t \in [(i-1)\frac{T(\lambda)}{n}, i\frac{T(\lambda)}{n}]$ and small $\lambda \geq 0$ for $i = 1, 2, \dots, n$, where Z_i is the solution of the initial value problem (6.2.10) on the interval $[(i-1)\frac{T(\lambda)}{n}, i\frac{T(\lambda)}{n}]$.
Then $[Z(t,\lambda)]$ is defined by formula (6.2.9).

Also, from (6.2.7) in Theorem 6.2.3 we get

$$Z_i(t,\lambda) = \left(|X_0|\,(t-(i-1)\frac{T(\lambda)}{n}) + \lambda^{\frac{1}{2}} \int_{(i-1)\frac{T(\lambda)}{n}}^{t} x_{01}(s)\,ds + \lambda x_{03}^{i}(t,\lambda) \right) X_0^1$$
$$+ \lambda^{\frac{1}{2}} z_1^i(t,\lambda) X_1 + \lambda^{\frac{1}{2}} z_2^i(t,\lambda) X_2 \quad (6.5.33)$$

for any $t \in [(i-1)\frac{T(\lambda)}{n}, i\frac{T(\lambda)}{n}]$ and small $\lambda \geq 0$ for $i = 1, 2, \ldots, n$.

Because the BCH formula in $so(3)$ is smooth from $so(3) \times so(3)$ into D (see Theorem 2.3.4) and $Z_i(t,\lambda)$ for $i = 1, 2, \ldots, n$ are sufficiently CS, then $[Z(t,\lambda)]$ is sufficiently CS on $[0, T(\lambda)]$ and $\lambda \geq 0$ small.

Part 2 On the interval $[iT(\lambda), (i+1)T(\lambda)]$ for any integer $i \geq 0$.

Since

$$A(t+T(\lambda),\lambda) = e^{X(\lambda)T(\lambda)} A(t,\lambda) = A(T(\lambda),\lambda)A(t,\lambda) = e^{Z^0(T(\lambda),\lambda)} e^{Z^0(t,\lambda)},$$

it is easy to see that we can define for any $t \in [T(\lambda), 2T(\lambda)]$ and $\lambda \geq 0$ small,

$$[Z^1(t,\lambda)] = BCH(Z^0(T(\lambda),\lambda), Z^0(t-T(\lambda),\lambda)).$$

Because the BCH formula in $so(3)$ is smooth from $so(3) \times so(3)$ into D, $Z^0(t,\lambda)$ is sufficiently CS and $T(\lambda)$ is sufficiently smooth, then $[Z(t,\lambda)]$ is sufficiently CS for any $t \in [T(\lambda), 2T(\lambda)]$ and for $\lambda \geq 0$ small.

We then repeat the above argument.

It is clear that the function $[Z(t,\lambda)]$ is sufficiently CS for $t \in [0,\infty)$ and $\lambda \geq 0$ small, and $A(t,\lambda) = e^{Z(t,\lambda)}$. □

Proof of Theorem 6.2.5. If we define

$$X(\lambda) = \frac{1}{T(\lambda)} Z(T(\lambda),\lambda) \text{ for small } \lambda \geq 0, \quad (6.5.34)$$

then $A(T(\lambda),\lambda) = e^{X(\lambda)T(\lambda)}$.

Since $[Z(t,\lambda)]$ is sufficiently CS for $t \in [0,\infty)$, and $\lambda \geq 0$ small and $T(\lambda)$ for $\lambda \geq 0$ is sufficiently smooth , we get that $[X(\lambda)T(\lambda)]$ is sufficiently CS for $\lambda \geq 0$ small. The branch $X(\lambda)$ defined by (6.5.34) is such that $[X(\lambda)T(\lambda)] \in D$ for $\lambda \geq 0$ and it is the branch whose existence is proved in Theorem 6.2.1.

Therefore, the branch $X^f(\lambda)$ for $\lambda \geq 0$ is the branch constructed in Theorem 6.2.1. Thus, all the statements regarding $X(\lambda)$ and $X^f(\lambda)$ given in Theorem 6.2.1 are valid. □

Proof of Theorem 6.2.6. 1. Suppose that $|X_0| \neq k\omega_{bif}$ for any $k \in \mathbb{Z}$.

We have $B(t,\lambda) = e^{-X^f(\lambda)t}A(t,\lambda) = e^{-X^f(\lambda)t}e^{Z(t,\lambda)} = e^{BCH(-X^f(\lambda)t, Z(t,\lambda))}$ for $t \in [0,\infty)$ and $\lambda \geq 0$ small.

Since $X^f(\lambda)$, $[Z(t,\lambda)]$ and $A(t,\lambda)$ are sufficiently CS for $t \in [0,\infty)$ and $\lambda \geq 0$ small and the BCH formula in $so(3)$ is smooth, it follows that $[Per^f(t,\lambda)]$ and $B(t,\lambda)$ are sufficiently CS for $t \in [0,\infty)$ and $\lambda \geq 0$ small.

Since $|Per^f(t,\lambda)|$ is continuous for $t \in [0,\infty)$ and $\lambda \geq 0$ small, we get that $\lim_{\lambda\to 0} |Per^f(t,\lambda)| = |Per^f(t,0)| = |BCH(-X^f(0)t, Z(t,0))|$ uniformly for t in any compact interval in $[0,\infty)$.

Since $e^{-X^f(0)t}e^{Z(t,0)} = A(t)^{-1}A(t) = I_3 = e^{BCH(-X^f(0)t,Z(t,0))}$, it follows that $Per^f(t,0) = BCH(-X^f(0)t, Z(t,0)) = O_3$. Then, $\lim_{\lambda\to 0} |Per^f(t,\lambda)| = 0$ uniformly with respect to $t \in [0,M]$ for $\lambda \geq 0$ small, where $M > \sup_{\lambda\geq 0}(T(\lambda)+\rho) = \sup_{\lambda \geq 0}(T(0)+O(\lambda)+\rho)$ with $\rho > 0$ small, is chosen independent of λ for $\lambda \geq 0$ small.

It follows that for $\lambda \geq 0$ small and any $t \in [0,M]$ we have $|Per^f(t,\lambda)| < \pi$.

Then, $Per^f(t,\lambda)$ is sufficiently CS for $\lambda \geq 0$ small and $t \in [0, T(\lambda)+\rho]$.

The periodicity of $Per^f(.,\lambda)$ results from Remark 2.2.14.

Therefore, $Per^f(t,\lambda)$ is sufficiently CS for $\lambda \geq 0$ small and $t \in [0,\infty)$.

Writing the Taylor formula for $\epsilon \to Per^f(t,\epsilon^2)$ at $\epsilon = 0$ up to order of ϵ and taking $\epsilon = \lambda^{\frac{1}{2}}$, we get $Per^f(t,\lambda) = O_3 + \epsilon Y_1(t,\epsilon) = \lambda^{\frac{1}{2}} Y(t,\lambda))$ for $t \in [0,\infty)$ and $\lambda \geq 0$ small.

2. Suppose that $|X_0| = k\omega_{bif}$ for some $k \in \mathbb{Z}$, $k \neq 0$.
   We have $B(t,\lambda) = e^{-X(\lambda)t}A(t,\lambda) = e^{-X(\lambda)t}e^{Z(t,\lambda)} = e^{BCH(-X(\lambda)t,Z(t,\lambda))}$ for $t \in [0,\infty)$ and $\lambda \geq 0$ small. Since $X(\lambda)$, $A(t,\lambda)$ and $[Z(t,\lambda)]$ are sufficiently CS for $t \in [0,\infty)$ and $\lambda \geq 0$ small, and BCH formula in $so(3)$ is smooth, it is clear that $B(t,\lambda)$, $[Per(t,\lambda)]$ are sufficiently CS for $t \in [0,\infty)$ and $\lambda \geq 0$ small.

$\square$

Proof of Theorem 6.3.3. Let us define $Y(\alpha,\beta) = e^{\alpha X_1 + \beta X_2}$.

Let us denote $\lambda = \epsilon^2$, $\widetilde{X}(\epsilon,\mu) = X(\epsilon^2,\mu)$ and $\widetilde{T}(\epsilon,\mu) = T(\epsilon^2,\mu) = \frac{2\pi}{|\omega_{\epsilon^2,\mu}|}$. Let $\widetilde{T}(0,\mu) = \frac{2\pi}{|\omega_{0,\mu}|}$.

We have to prove that there exist sufficiently smooth curves

$$\mu = \mu(\epsilon), \alpha = \alpha(\epsilon) \text{ and } \beta = \beta(\epsilon) \text{ such that } \mu(0) = 0, \alpha(0) = 0 \text{ and } \beta(0) = 0$$

satisfying

$$Y(\alpha,\beta) = e^{\widetilde{X}(\epsilon,\mu)\widetilde{T}(\epsilon,\mu)} \text{ or } Y(\alpha,\beta)^{-1}e^{\widetilde{X}(\epsilon,\mu)\widetilde{T}(\epsilon,\mu)} = I_3. \tag{6.5.35}$$

Let us define the function $F\colon \mathbb{R}^4 \to SO(3)$ by

$$F(\alpha,\beta,\epsilon,\mu) = e^{-\alpha X_1 - \beta X_2}e^{\widetilde{X}(\epsilon,\mu)\widetilde{T}(\epsilon,\mu)}.$$

We will prove the existence of the smooth curves using the implicit function theorem for the function F at the point $(\alpha,\beta,\epsilon,\mu) = (0,0,0,0)$.
We now prove that:

1. $F(0,0,0,0) = I_3$
   and

2. $[(DF)_{(0,0,0,0)}]_{X_0^1, X_1, X_2}$ has rank 3.

We have that $F(0,0,0,0) = e^{-0\cdot X_1 - 0\cdot X_2}e^{\widetilde{X}(0,0)\widetilde{T}(0,0)} = e^{X(0,0)T(0,0)} = e^{X_0\frac{2\pi}{|\omega_{bif}|}} = I_3$ since $|X_0| = k\omega_{bif}$.
Recall that we have

$$X_G(q,\lambda,\mu) = x_0(q,\lambda,\mu)X_0^1 + x_1(q,\lambda,\mu)X_1 + x_2(q,\lambda,\mu)X_2.$$

Then $X_G(0,0,\mu) = a(\mu)X_0^1 + b(\mu)X_1 + c(\mu)X_2$, where

$$a(\mu) = x_0(0,0,\mu) = |X_0| + O(\mu), b(\mu) = x_1(0,0,\mu) = O(\mu) \text{ and } c(\mu) = x_2(0,0,\mu) = O(\mu).$$

Since $A(t,\lambda,\mu) = e^{X(\lambda,\mu)T(\lambda,\mu)}$, then $A(t,0,\mu) = e^{X(0,\mu)T(0,\mu)} = e^{\widetilde{X}(0,\mu)\widetilde{T}(0,\mu)}$. Also $A(t,0,\mu) = e^{X_G(0,0,\mu)T(0,\mu)}$. Using Theorem 2.2.13 (2), we get

$$\begin{aligned}\frac{\partial}{\partial\mu}e^{X(0,\mu)T(0,\mu)}|_{\mu=0} &= \frac{\partial}{\partial\mu}e^{X_G(0,0,\mu)T(0,\mu)}|_{\mu=0}\\ &= e^{X_G(0,0,0)T(0,0)}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(n+1)!}\left(ad(X_G(0,0)T(0,0))\right)^n\left(\left(X_G(0,0,\mu)T(0,\mu)\right)'|_{\mu=0}\right)\right].\end{aligned} \tag{6.5.36}$$

We have $T(0,0) = \frac{2\pi}{|\omega_{bif}|}$, $X_G(0,0,0) = X_0$ and

$$
\begin{aligned}
(X_G(0,0,\mu)T(0,\mu))'|_{\mu=0} &= (X_G(0,0,\mu))'|_{\mu=0}T(0,0) + X_G(0,0,0)(T(0,\mu))'|_{\mu=0} \\
&= \frac{2\pi}{|\omega_{bif}|}\left[(a'(0)X_0^1 + b'(0)X_1 + c'(0)X_2)\right] \\
&\quad + |X_0|\left[-2\pi \cdot sgn(\omega_{bif})\right]\frac{(\omega_{0,\mu})'|_{\mu=0}}{\omega_{bif}^2}X_0^1 \\
&= \frac{2\pi}{|\omega_{bif}|}\left[(a'(0) - k\cdot(sgn(\omega_{bif}))^2(\omega_{0,\mu})'|_{\mu=0})X_0^1 + b'(0)X_1 + c'(0)X_2\right],
\end{aligned}
\tag{6.5.37}
$$

where we have used $|X_0| = k\omega_{bif}$. Therefore, (6.5.36) becomes

$$
\begin{aligned}
& e^{\frac{2\pi|X_0|}{|\omega_{bif}|}X_0^1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\left(ad(\frac{2\pi\,|X_0|}{|\omega_{bif}|}X_0^1)\right)^n\left(\frac{2\pi}{|\omega_{bif}|}\left[(a'(0) - k(\omega_{0,\mu})'|_{\mu=0})X_0^1 + b'(0)X_1 + c'(0)X_2\right]\right) \\
&= \frac{2\pi}{|\omega_{bif}|}\left[(a'(0) - k(\omega_{0,\mu})'|_{\mu=0})X_0^1 + b'(0)X_1 + c'(0)X_2\right] \\
&+ \sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}\left(ad(2|k|\pi X_0^1)\right)^n\left(\frac{2\pi}{|\omega_{bif}|}\left[(a'(0) - k(\omega_{0,\mu})'|_{\mu=0})X_0^1 + b'(0)X_1 + c'(0)X_2\right]\right) \\
&= \frac{2\pi}{|\omega_{bif}|}\left[(a'(0) - k(\omega_{0,\mu})'|_{\mu=0})X_0^1 + (b'(0) + *)X_1 + (c'(0) + *)X_2\right],
\end{aligned}
\tag{6.5.38}
$$

where we have used $e^{\frac{2\pi|X_0|}{|\omega_{bif}|}X_0^1} = I_3$ and where $*$ denotes other terms. These term appear only in the coefficients of X_1 and X_2 since

$$
\begin{aligned}
& \left(ad(2|k|\pi X_0^1)\right)\left(\frac{2\pi}{|\omega_{bif}|}\left[(a'(0) - k(\omega_{0,\mu})'|_{\mu=0})X_0^1 + b'(0)X_1 + c'(0)X_2\right]\right) \\
&= \frac{2\pi}{|\omega_{bif}|}\left[2|k|\pi X_0^1, (a'(0) - k(\omega_{0,\mu})'|_{\mu=0})X_0^1 + b'(0)X_1 + c'(0)X_2\right] \\
&= \frac{4|k|\pi^2}{|\omega_{bif}|}(-c'(0)X_1 + b'(0)X_2),
\end{aligned}
\tag{6.5.39}
$$

and for any integer $n \geq 2$,

$$
\begin{aligned}
& \left(ad(2|k|\pi X_0^1)\right)^n\left(\frac{2\pi}{|\omega_{bif}|}\left[(a'(0) - k(\omega_{0,\mu})'|_{\mu=0})X_0^1 + b'(0)X_1 + c'(0)X_2\right]\right) \\
&= \left[2|k|\pi X_0^1, \left(ad(2|k|\pi X_0^1)\right)^{n-1}\left(\frac{2\pi}{|\omega_{bif}|}\left[(a'(0) - k(\omega_{0,\mu})'|_{\mu=0})X_0^1 + b'(0)X_1 + c'(0)X_2\right]\right)\right]
\end{aligned}
\tag{6.5.40}
$$

will contain only linear combinations of X_1 and X_2.
Then with respect to the basis $\{X_0^1, X_1, X_2\}$,

$$(DF)_{(0,0,0,0)} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ \frac{2\pi}{|\omega_{bif}|}\left(a^{'}(0) - k(\omega_{0,\mu})^{'}|_{\mu=0}\right) & \frac{2\pi}{|\omega_{bif}|}b^{'}(0) + * & \frac{2\pi}{|\omega_{bif}|}c^{'}(0) + * \end{pmatrix},$$

where $*$ denotes other terms (Below we show that in fact $\frac{2\pi}{|\omega_{bif}|}b^{'}(0) + * = 0$ and $\frac{2\pi}{|\omega_{bif}|}c^{'}(0) + * = 0$).
Let $U = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\left(ad(\frac{2\pi|X_0|}{|\omega_{bif}|}X_0^1)\right)^n \left(\frac{2\pi}{|\omega_{bif}|}\left[(a^{'}(0) - k(\omega_{0,\mu})^{'}|_{\mu=0})X_0^1 + b^{'}(0)X_1 + c^{'}(0)X_2\right]\right)$.
Using theorem 2.2.9 (1), we get

$$\begin{aligned} \overrightarrow{U} &= \overrightarrow{\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\left(ad(2\pi|k|X_0^1)\right)^n \left(\frac{2\pi}{|\omega_{bif}|}\left[(a^{'}(0) - k(\omega_{0,\mu})^{'}|_{\mu=0})X_0^1 + b^{'}(0)X_1 + c^{'}(0)X_2\right]\right)} \\ &= \frac{2\pi}{|\omega_{bif}|}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}(2\pi|k|X_0^1)^n \left[(a^{'}(0) - k(\omega_{0,\mu})^{'}|_{\mu=0})\overrightarrow{X}_0^1 + b^{'}(0)\overrightarrow{X}_1 + c^{'}(0)\overrightarrow{X}_2\right]. \end{aligned} \tag{6.5.41}$$

If we apply the relation (6.5.24) for $Z = 2\pi|k|X_0^1$, then taking into account that $|Z| = 2\pi|k|$, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}(2\pi|k|X_0^1)^n = I_3 + (X_0^1)^2. \tag{6.5.42}$$

If we substitute (6.5.42) into (6.5.41), it follows that

$$\begin{aligned} \overrightarrow{U} &= \frac{2\pi}{|\omega_{bif}|}[I_3 + (X_0^1)^2]\left[(a^{'}(0) - k(\omega_{0,\mu})^{'}|_{\mu=0})\overrightarrow{X}_0^1 + b^{'}(0)\overrightarrow{X}_1 + c^{'}(0)\overrightarrow{X}_2\right] \\ &= \frac{2\pi}{|\omega_{bif}|}\left[(a^{'}(0) - k(\omega_{0,\mu})^{'}|_{\mu=0})\overrightarrow{X}_0^1 + b^{'}(0)\overrightarrow{X}_1 + c^{'}(0)\overrightarrow{X}_2\right. \\ &\left. +(a^{'}(0) - k(\omega_{0,\mu})^{'}|_{\mu=0})(X_0^1)^2\overrightarrow{X}_0^1 + b^{'}(0)(X_0^1)^2\overrightarrow{X}_1 + c^{'}(0)(X_0^1)^2\overrightarrow{X}_2\right]. \end{aligned} \tag{6.5.43}$$

Since $(X_0^1)^2\overrightarrow{X}_0^1 = \overrightarrow{0}$, $(X_0^1)^2\overrightarrow{X}_1 = -\overrightarrow{X}_1$ and $(X_0^1)^2\overrightarrow{X}_2 = -\overrightarrow{X}_2$, we get $\overrightarrow{U} = \frac{2\pi}{|\omega_{bif}|}(a^{'}(0) - |k|(\omega_{0,\mu})^{'}|_{\mu=0})\overrightarrow{X}_0^1$.
In order to have the rank of $(DF)_{(0,0,0,0)}$ equal to 3, it is necessary and sufficient that

$$a^{'}(0) \neq k(\omega_{0,\mu})^{'}|_{\mu=0}.$$

We note that depending on the function $X^G(t,\lambda,\mu)$ we can establish if $\alpha(\epsilon) = 0$ and/or $\beta(\epsilon) = 0$.

Another proof of conclusion (1) of Theorem 6.3.3 We have that $A(T(\lambda,\mu),\lambda,\mu) = e^{X(\lambda,\mu)T(\lambda,\mu)}$, where $X(\lambda,\mu)$ is defined as in Theorem 6.2.5.

Let $X(\lambda,\mu) = a_1(\lambda,\mu)X_0^1 + b_1(\lambda,\mu)X_1 + c_1(\lambda,\mu)X_2$. Then, since $X(\lambda,\mu)$ is sufficiently CS, we get that a_1, b_1, c_1 are sufficiently CS and $b_1(0,\mu) = O(\mu), c_1(0,\mu) = O(\mu)$.

Since $e^{X(0,0)T(0,0)} = A(T(0,0),0,0) = I_3$, then $X(0,0) = O_3$, that is $a_1(0,0) = 0$.

We want to find a sufficiently CS branch $\mu = \mu(\lambda)$ such that $\mu(0) = 0$ and $a_1(\lambda,\mu(\lambda)) = 0$, for $\lambda \geq 0$ small. If $(a_1)_\mu(0,0) \neq 0$, then by applying the implicit function theorem we get the existence of the required sufficiently CS branch.

Now we show that $(a_1)_\mu(0,0) \neq 0$ is equivalent with $a'(0) \neq k(\omega_{0,\mu})'|_{\mu=0}$.

We have that $A(T(0,\mu),0,\mu) = e^{X(0,\mu)T(0,\mu)} = e^{X_G(0,0,\mu)T(0,\mu)}$ for $|\mu|$ small, then we get $X(0,\mu)T(\mu,0) = X_G(0,0,\mu)T(\mu,0) + \frac{2l(\mu)\pi}{|X_G(0,0,\mu)|}X_G(0,0,\mu)$ for some $l(\mu) \in \mathbf{Z}$ for $|\mu|$ small. Thus, $\left[a_1(0,\mu)X_0^1 + b_1(0,\mu)X_1 + c_1(0,\mu)X_2\right]T(0,\mu) = \left[a(\mu)X_0^1 + b(\mu)X_1 + c(\mu)X_2\right]T(0,\mu) + \frac{2l(\mu)\pi}{\sqrt{a(\mu)^2+O(\mu^2)}}\left[a(\mu)X_0^1 + b(\mu)X_1 + c(\mu)X_2\right]$ for $|\mu| \geq 0$ small.

This implies that

$$a_1(0,\mu) = a(\mu) + \frac{a(\mu)}{\sqrt{a(\mu)^2 + O(\mu^2)}}l(\mu)\omega_{0,\mu} \cdot sgn(\omega_{bif}), \tag{6.5.44}$$

which for $\mu = 0$ gives $l(0) = -|k|$, where we have used the fact that $|X_0| = k\omega_{bif} > 0$ implies $sgn(k) = sgn(\omega_{bif})$, and thus, $k \cdot sgn(\omega_{bif}) = |k|$. Since $l(\mu) \in \mathbb{Z}$ is continuous, we get $l(\mu) = -|k|$ for $|\mu| \geq 0$ small.

Then by taking into account that $sgn(\omega_{bif}) = sgn(k)$, (6.5.44) becomes

$$a_1(0,\mu) = a(\mu) - k\frac{a(\mu)}{\sqrt{a(\mu)^2 + O(\mu^2)}}\omega_{0,\mu}. \tag{6.5.45}$$

By differentiation of (6.5.45), it follows that $(a_1)_\mu(0,0) = a'(0) - k(\omega_{0,\mu})'|_{\mu=0} \neq 0$.

The other conclusions of Theorem 6.3.3 result in the same way as in the proof of Theorems 6.2.1, 6.2.5 and 6.2.6, except the scaling of the primary frequencies $|X(\lambda,\mu(\lambda))|$ that results from Remark 3.5 in [85]. $\square$

Chapter 7

Forced Symmetry-breaking for a Normally Hyperbolic $SO(3)$-Group Orbit of a Rotating Wave on $r\mathbf{S}^2$

Throughout this chapter, we will suppose that $\alpha \in (\frac{1}{2}, 1)$ is fixed.

7.1 Forced Symmetry-breaking

In mathematics and physics the phrase "symmetry-breaking" has distinct meanings. The first refers to the frequently observed phenomenon that a configuration of a physical system satisfying a law (a set of equations) which is invariant under a group of transformations, may itself be invariant under a subgroup of this group. This is referred to as *spontaneous symmetry-breaking.* The second meaning refers to the problem of explicitly adding symmetry-breaking terms to the equations which describe the system. This we call *induced* or *forced symmetry-breaking.* In another words, *forced symmetry-breaking* means the systematic study of an equivariant system of differential equations which is perturbed slightly so that is loses its symmetry properties partially or completely.

We consider a perturbed reaction-diffusion system of the form

$$\frac{\partial u}{\partial t}(t,x) = D\Delta u(t,x) + F(u(t,x),\lambda) + \epsilon G(u(t,x),x,\epsilon,\lambda) \text{ on } r\mathbf{S}^2, \tag{7.1.1}$$

where $r > 0$, $r\mathbf{S}^2$ is the sphere or radius r, $\mathbf{S}^2$ is the unit sphere in $\mathbb{R}^3$ and $u = (u_1,\ u_2,$

$\ldots, u_N)\colon r\mathbf{S^2} \to \mathbb{R}^N$ for $N \geq 1$,

$D = \begin{pmatrix} d_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & d_N \end{pmatrix}$ with $d_i > 0$ for $i = 1, 2, \ldots, N$ are the diffusion coefficients, $F = (F_1, F_2, \ldots, F_N)\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ and $G = (G_1, G_2, \ldots, G_N)\colon \mathbb{R}^N \times r\mathbf{S^2} \times [0,\infty) \times \mathbb{R} \to \mathbb{R}^N$ are sufficiently smooth functions for $|\lambda|$, $\epsilon \geq 0$ small parameters.

We remind that the superposition operator associated to F, denoted by $F\colon \mathbf{Y}^\alpha \times [0,\infty) \to \mathbf{Y}$, $F(u,\lambda)(x) = F(u(x),\lambda)$ is $SO(3)$-equivariant with respect to the action θ defined in (3.2.2), for any $\lambda \geq 0$.

We consider the group $SO(2)$ as being diffeomorphic with the subgroup of $SO(3)$ defined by $\{e^{Q\theta} \mid \theta \text{ is in } [0, 2\pi)\}$, Q being fixed in $so(3)$ such that $|Q| = 1$. Thus, we consider that the Lie algebra $so(2)$ of $SO(2)$ is isomorphic to $\{Qx \mid x \in \mathbb{R}\}$. We will study the reaction-diffusion system (7.1.1) on the function space $\mathbf{Y} = \mathbf{L^2}(r\mathbf{S^2}, \mathbb{R}^N)$.

Remark 7.1.1. *The reaction-diffusion system (7.1.1) can be studied on the function space* $\mathbf{Y} = \mathbf{H^2}(r\mathbf{S^2}, \mathbb{R}^N)$, *if there exists an* $i = 1, \ldots, N$ *such that* $d_i = 0$.

In the following sections we suppress the parameter λ.

Theorem 7.1.2. *Let* $F\colon \mathbb{R}^N \to \mathbb{R}^N$ *and* $G\colon \mathbb{R}^N \times r\mathbf{S^2} \times [0,\infty) \to \mathbb{R}^N$ *be sufficiently smooth functions such that* $F(0) = 0$ *and* $G(0, x, \epsilon) = 0$ *for* $\epsilon \geq 0$ *small. Also, suppose that the superposition operator associated to* G, *denoted by* $G\colon \mathbf{Y}^\alpha \times [0,\infty) \to \mathbf{Y}$, $G(u,\epsilon)(x) = G(u(x), x, \epsilon)$ *is* $SO(2)$*-equivariant with respect to the action* θ *defined in (3.2.2) restricted to* $SO(2)$, *but not* $SO(3)$*-equivariant with respect to the action* θ *for any* $\epsilon \geq 0$. *The reaction-diffusion system (7.1.1) defines a sufficiently smooth parameter-dependent local semiflow* $\mathbf{\Phi}$ *on the function space* $\mathbf{Y}^\alpha$. *Namely, for any* $u_0 \in \mathbf{Y}^\alpha$ *and* $\epsilon \in [0, \epsilon_0)$ *with* $\epsilon_0 > 0$ *small, let* $u(t, u_0, \epsilon)$ *be the sufficiently smooth solution of the initial value problem given by the reaction-diffusion system (7.1.1) and by the initial condition* $u(0) = u_0$, *defined on the maximal interval of existence* $I(u_0, \epsilon) = [0, t_0(u_0, \epsilon))$. *Let* $W = \{(t, u_0, \epsilon) \in [0,\infty) \times \mathbf{Y}^\alpha \times [0, \epsilon_0) \mid t \in I(u_0, \epsilon)\}$. *Then, the local semiflow* $\mathbf{\Phi}\colon W \subset [0,\infty) \times \mathbf{Y}^\alpha \times [0, \epsilon_0) \to \mathbf{Y}^\alpha$ *is defined by* $\mathbf{\Phi}(t, u_0, \epsilon) = u(t, u_0, \epsilon)$ *for any* $(t, u_0, \epsilon) \in W$.

Theorem 7.1.3. *The local semiflow* $\mathbf{\Phi}$ *defined by Theorem 7.1.2 is* $SO(3)$*-equivariant for* $\epsilon = 0$, *but it is only* $SO(2)$*-equivariant for* $\epsilon > 0$ *small.*

The definition of relative equilibrium, relative periodic orbits, rotating waves and modulated rotating waves for a $SO(2)$-equivariant reaction-diffusion system is the same as the definition of relative equilibrium, relative periodic orbits, rotating waves and modulated rotating waves for a $SO(3)$-equivariant reaction-diffusion system.

Example 7.1.4. *For any $x \in r\mathbf{S}^2$, we denote by $r(x) = d(x,l)^2$, where l is the line containing the vector $\overrightarrow{Q}$ and d is the Euclidean distance in $\mathbb{R}^3$. Let us define $G(u,x) = \widetilde{G}(u, r(x))$, where $\widetilde{G}\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is a sufficiently smooth function. Then the superposition operator associated to G is $SO(2)$-equivariant, but is not $SO(3)$-equivariant in general.*

7.2 The Center Manifold Reduction

Throughout this chapter we suppose that for $\epsilon = 0$ we have a normally hyperbolic (see Definition 2.5.5) relative equilibrium $u_0 \in \mathbf{Y}^\alpha$ such that $\Sigma_{u_0} = I_3$. Let $\mathbf{\Phi}(t, u_0, 0) = e^{X_0 t} u_0$.

Theorem 7.2.1 ([6], [23], [40]). *There exists a sufficiently smooth parameter-dependent $SO(2)$-invariant normally hyperbolic manifold $M(\epsilon)$ for the reaction-diffusion system (7.1.1) such that $M(0) = SO(3)u_0$, and, for $\epsilon > 0$ small, $M(\epsilon)$ is diffeomorphic to $SO(3)u_0$ and this diffeomorphism is $SO(2)$-equivariant.*

Since $\Sigma_{u_0} = I_3$, we have that $SO(3)u_0$ is diffeomorphic to $SO(3)$, so $M(\epsilon)$ is $SO(2)$-equivariant diffeomorphic to $SO(3)$.

Theorem 7.2.2. *Let L be the linearization of the right-hand side of (7.1.1) with respect to the rotating wave $\mathbf{\Phi}(t, u_0, 0) = e^{X_0 t} u_0$ in the co-rotating frame, that is*

$$L = D\Delta + D_u F(u_0) - X_0.$$

Suppose that:

1. *$\sigma(L) \cap \{z \in \mathbb{C} \mid Re(z) \geq 0\}$ is a spectral set with spectral projection P_*, and $dim(R(P_*)) < \infty$;*
2. *the semigroup e^{Lt} satisfies $\left|e^{Lt}|_{R(1-P_*)}\right| \leq Ce^{-\beta_0 t}$ for some $\beta_0 > 0$ and $C > 0$.*

Let V_* be the orthogonal complement of $T_{u_0}(SO(3)u_0)$ in $E^{cu} = R(P_*)$.
Then there exists a sufficiently smooth parameter-dependent center manifold $M^{cu}(\epsilon)$ of the normally hyperbolic relative equilibrium $SO(3)u_0$ such that $M(\epsilon) \subset M^{cu}(\epsilon)$ and $M^{cu}(\epsilon)$ is diffeomorphic to $SO(3) \times V_*$ for $\epsilon \geq 0$ small. For $\epsilon = 0$, $M^{cu}(\epsilon)$ is $SO(3)$-invariant, and for $\epsilon > 0$ small, $M^{cu}(\epsilon)$ is $SO(2)$-invariant. Furthermore, there exist sufficiently smooth

functions $f_1\colon V_* \to so(3)$, $f_2\colon V_* \to V_*$, $G^2_{pert}\colon SO(3) \times V_* \times [0,\infty) \to V_*$ *and a sufficiently smooth vector field* $G^1_{pert}\colon SO(3) \times V_* \times [0,\infty) \to T(SO(3))$ *with* $G^1_{pert}(.,q,\epsilon)$ *being* $SO(2)$*-equivariant,* $G^2_{pert}(.,q,\epsilon)$ *being* $SO(2)$*-invariant such that any solution of*

$$\begin{aligned} \dot{A} &= Af_1(q) + \epsilon G^1_{pert}(A,q,\epsilon) \\ \dot{q} &= f_2(q) + \epsilon G^2_{pert}(A,q,\epsilon), \end{aligned} \tag{7.2.1}$$

on $SO(3) \times V_*$ *corresponds to a solution of the reaction-diffusion system (7.1.1) on* $M^{cu}(\epsilon)$ *under the diffeomorphic identification for* $\epsilon \geq 0$ *small. Also,* $f_1(0) = X_0$, $f_2(0) = 0$ *and* $\sigma(D_q f_2(0)) = \sigma(Q_* L|_{V_*})$*, where* Q_* *is the projection onto* V_* *along* $T_{u_0}(SO(3)u_0)$*. Moreover,* 0 *is a hyperbolic equilibrium in* V_* *for the second differential equation in (7.2.1) for* $\epsilon = 0$*.*

Remark 7.2.3. *Henceforth, we will identify* $M(\epsilon)$ *with* $SO(3)$ *and* $M^{cu}(\epsilon)$ *with* $SO(3) \times V_*$ *for* $\epsilon \geq 0$ *small. Therefore, we will talk about the semiflow* $\boldsymbol{\Phi}$ *on* $SO(3)$ *and on* $SO(3) \times V_*$ *for* $\epsilon \geq 0$ *small.*

7.3 The General Form of the $SO(2)$-Equivariant Perturbation on the Center Manifold

Theorem 7.3.1. *The general form of the* $SO(2)$*-equivariant perturbation on* $M^{cu}(\epsilon)$ *given by Theorem 7.2.2 is*

$$\begin{aligned} G^1_{pert}(A,q,\epsilon) &= Ak(A,q,\epsilon), \\ G^2_{pert}(A,q,\epsilon) &= h(A,q,\epsilon), \end{aligned} \tag{7.3.1}$$

where $A \in SO(3)$, $q \in V_*$, $\epsilon \geq 0$ *is small, and* $k\colon SO(3) \times V_* \times [0,\epsilon_0) \to so(3)$ *and* $h\colon SO(3) \times V_* \times [0,\epsilon_0) \to so(3)$ *are sufficiently smooth* $SO(2)$*-invariant functions for* $\epsilon \in [0,\epsilon_0)$ *and* $q \in V_*$ *with* $\epsilon_0 > 0$ *small.*

Generically, $M(\epsilon)$ corresponds to a hyperbolic equilibrium $q_\epsilon \in V_*$ near 0 for $\epsilon \geq 0$ small. Therefore, we can restrict the study to the $SO(2)$-equivariant dynamics on $SO(3)$, by substituting $q = q_\epsilon$ in the differential equations (7.2.1).

Theorem 7.3.2. *The perturbed differential equations on* $M(\epsilon)$ *given by Theorem 7.2.2 are*

$$\dot{A} = A\,[X_0 + \epsilon g(A,\epsilon)] \tag{7.3.2}$$

where $A \in SO(3)$, $\epsilon \geq 0$ *is small and, up to a constant matrix in* $so(3)$ *that depends sufficiently smoothly on* ϵ, $g(A,\epsilon) = k(A,q_\epsilon,\epsilon)$ *for* k *defined in Theorem 7.3.1. Thus* $g(.,\epsilon)$ *is* $SO(2)$*-invariant for* $\epsilon \geq 0$ *small.*

The definition of the tip position function for a $SO(2)$-equivariant reaction-diffusion system is similar to the definition of the previously defined tip position function for a $SO(3)$-equivariant reaction-diffusion system. We define the following tip motion on $M(\epsilon)$ that we use throughout this chapter:

$$x_{tip}(\Phi(t, A_0 q_\epsilon, \epsilon)) = A(t, \epsilon)x_0,$$

where $\epsilon \geq 0$ is small, $A_0 \in SO(3)$, $A(t,\epsilon)$ is the solution of the initial value problem given by the differential equation (7.3.2) and $A(0) = A_0$, and $x_0 \in r\mathbf{S}^2$ is fixed.

7.4 The Analysis of the Projected Dynamics on the Orbit Space $SO(3)/SO(2)$

Let x_1^0 and x_2^0 be the intersections of the line having as direction the vector $\overrightarrow{X}_0$ with the unit sphere $\mathbf{S}^2$; $x_1^0 = \frac{1}{|X_0|}\overrightarrow{X}_0$ and $x_2^0 = -\frac{1}{|X_0|}\overrightarrow{X}_0$.
Let x_1^Q and x_2^Q be the intersections of the line having as direction the vector $\overrightarrow{Q}$ with the unit sphere $\mathbf{S}^2$; $x_1^Q = \overrightarrow{Q}$ and $x_2^Q = -\overrightarrow{Q}$.
We consider the left action $\widetilde{\theta}$: $SO(3) \times SO(3) \to SO(3)$ given by

$$\widetilde{\theta}(A, C) = CA^{-1}. \tag{7.4.1}$$

We substitute $C = A^{-1}$ in (7.3.2).
Since $O_3 = \dot{I}_3 = \dot{\widehat{AC}} = \dot{A}C + A\dot{C}$, it follows that $\dot{C} = -C\dot{A}C$. Therefore, the differential equations (7.3.2) are equivalent to the following differential equations

$$\dot{C} = -\left[X_0 + \epsilon\widetilde{g}(C, \epsilon)\right]C, \tag{7.4.2}$$

where $C \in SO(3)$, $\epsilon \geq 0$ small and $\widetilde{g}$ is a sufficiently smooth function such that, for $\epsilon \geq 0$ small, $\widetilde{g}(C, \epsilon) = g(C^{-1}, \epsilon)$ is invariant under the restriction of the action $\widetilde{\theta}$ to the subgroup $SO(2)$ of $SO(3)$, with g defined in Theorem 7.3.2.
Let us denote the flow associated to the differential equations (7.4.2) by Ψ. It is clear that $\Psi(t, C, \epsilon) = [\Phi(t, C^{-1}, \epsilon)]^{-1}$. The differential equations (7.4.2) have the following property:

- for $\epsilon = 0$, they are $SO(3)$-equivariant under the action $\widetilde{\theta}$;
- for $\epsilon > 0$ small, they are $SO(2)$-equivariant under the restriction of the action $\widetilde{\theta}$ to the subgroup $SO(2)$ of $SO(3)$.

Definition 7.4.1. *The quotient of $SO(3)$ under the equivalence relation $A \sim B$ iff A and B are on the same group orbit under the action $\widetilde{\theta}$ defined in (7.4.1) and restricted to $SO(2)$ is called the* orbit space *of $SO(3)$ under the action $\widetilde{\theta}$ and it is denoted by $SO(3)/SO(2)$. It is the left coset of $SO(3)$ modulo $SO(2)$ under the action (7.4.1).*

Theorem 7.4.2 ([11], [45], [79]). *There is a unique C^∞ manifold structure for $SO(3)/SO(2)$ such that the projection $\pi\colon SO(3) \to SO(3)/SO(2)$ defined by $\pi(C) = C \cdot SO(2)$ is C^∞ and such that there exist local smooth sections of $SO(3)/SO(2)$ in $SO(3)$; that is, for any $A \cdot SO(2) \in SO(3)/SO(2)$ there exists a local smooth function π_s from a neighborhood U of $A \cdot SO(2)$ into $SO(3)$ such that $\pi \circ \pi_s = id$ on U.*

Theorem 7.4.3 ([11], [45], [79]). *The orbit space of $SO(3)$ under the action $\widetilde{\theta}$ is diffeomorphic as a manifold to the unit sphere $\mathbf{S}^2$. The diffeomorphism can be chosen such that*

$$\beta\colon SO(3)/SO(2) \to \mathbf{S}^2, \beta(A \cdot SO(2)) = Ax_1^Q \text{ for any } A \in SO(3). \tag{7.4.3}$$

Definition 7.4.4. *The* projected flow *on $SO(3)/SO(2)$ associated to the flow $\mathbf{\Psi}$ is defined by*

$$\widetilde{\mathbf{\Psi}}(t, \pi(C), \epsilon) = \pi(\mathbf{\Psi}(t, C, \epsilon)) \text{ for any } \epsilon \geq 0 \text{ small }, C \in SO(3) \text{ and } t \geq 0. \tag{7.4.4}$$

Remark 7.4.5. *The projected flow on $SO(3)/SO(2)$, $\widetilde{\mathbf{\Psi}}$, is well-defined and sufficiently smooth.*

Remark 7.4.6. *From [45] and [67], we know that every sufficiently smooth vector field on $SO(3)/SO(2)$ lifts to a sufficiently smooth $SO(2)$-equivariant vector field on $SO(3)$, unique up to vector fields tangent to the $SO(2)$-orbits.*

Definition 7.4.7. *Using the diffeomorphism β defined in (7.4.3), we define the following flow on $\mathbf{S}^2$:*

$$\widetilde{\mathbf{\Psi}_1}(t, Cx_1^Q, \epsilon) = \beta(\widetilde{\mathbf{\Psi}}(t, \pi(C), \epsilon)) \text{ for any } \epsilon \geq 0 \text{ small }, C \in SO(3) \text{ and } t \geq 0. \tag{7.4.5}$$

The flows $\widetilde{\mathbf{\Psi}}$ and $\widetilde{\mathbf{\Psi}_1}$ are topologically equivalent.

Theorem 7.4.8. *The differential equations given by (7.4.2) projected on the orbit space $SO(3)/SO(2)$ have the following form (after taking into account Theorem 7.4.3 and Definition 7.4.7):*

$$\dot{x} = -\left[X_0 + \epsilon g^S(x, \epsilon)\right] x, \tag{7.4.6}$$

where $x \in \mathbf{S}^2$ and $\epsilon \geq 0$ is small; that is, the vector field given by (7.4.6) is the vector field associated to the flow $\widetilde{\Psi_1}(t, Cx_1^Q, \epsilon)$. The function $g^S(x, \epsilon)$ is given by

$$g^S(x, \epsilon) = \widetilde{g}(C, \epsilon), \text{ where } x = Cx_1^Q \in \mathbf{S}^2, \epsilon \geq 0 \text{ is small and } \widetilde{g} \text{ is defined in (7.4.2).} \quad (7.4.7)$$

Remark 7.4.9. *1. For $\epsilon = 0$, the differential equations (7.4.6) have reflectional symmetry with respect to the equatorial plane orthogonal to (the line containing) the vector $\overrightarrow{X}_0$.*

2. For $\epsilon > 0$, generically, the differential equations (7.4.6) have no reflectional symmetry with respect to the equatorial plane orthogonal to (the line containing) the vector $\overrightarrow{X}_0$.

Theorem 7.4.10 ($\epsilon = 0$). *1. For $\epsilon = 0$ the solutions of the differential equations (7.3.2) are of the form*

$$A(t) = A_0 e^{X_0 t}, \text{ where } A_0 \in SO(3). \quad (7.4.8)$$

2. For $\epsilon = 0$ the solutions of the differential equations (7.4.6) are the equilibrium points x_1^0 and x_2^0, and the $\frac{2\pi}{|X_0|}$-periodic solutions of the form $x(t, x_0) = e^{-X_0 t} x_0$, whenever $x_0 \neq x_1^0$ and $x_0 \neq x_2^0$.

3. The correspondence between the solutions of the differential equations (7.3.2) and the differential equations (7.4.6) is as follows:

(a) The solution $A(t)$ of (7.3.2) corresponding to A_0 such that $x_1^Q = A_0 x_1^0$ projects onto the equilibrium x_1^0.

(b) The solution $A(t)$ of (7.3.2) corresponding to A_0 such that $x_1^Q = A_0 x_2^0$ projects onto the equilibrium x_2^0.

(c) The other solutions of (7.3.2) with the initial condition A_0 projects onto the periodic solutions of (7.4.6) having the initial condition x_0 such that $x_1^Q = A_0 x_0$.

Remark 7.4.11. *For $\epsilon = 0$, the flow given by the differential equations (7.4.2) is the only $SO(3)$-equivariant flow that can be obtained when we lift from the differential equations (7.4.6) on $SO(3)/SO(2)$ to $SO(3)$.*

Proof. The differential equations (7.4.6) for $\epsilon = 0$ are $\dot{x} = -X_0 x$ and its flow is given by $\widetilde{\Psi}(t, C \cdot SO(2), 0) = e^{-X_0 t} C \cdot SO(2)$. We get $\pi(\Psi(t, C, 0)) = \widetilde{\Psi}(t, C \cdot SO(2), 0) = e^{-X_0 t} C \cdot SO(2)$. It follows that $\Psi(t, C, 0) = e^{-X_0 t} C e^{\tau(t,C) Q}$. This is $SO(3)$-equivariant under the action $\widetilde{\theta}$ if and only if $\tau(t,C)=0$ for $t\geq 0$ and $C\in SO(3)$; that is, $\Psi(t,C,0)$ is the $SO(3)$-equivariant flow given by differential equations (7.4.2) for $\epsilon=0$. $\square$

Theorem 7.4.12 **($\epsilon = 0$).** *We will consider two distinct unit spheres S_1 and S_2. Let the points x_1^Q and x_2^Q be on sphere S_2 as they were defined at the beginning of this section. Also, let the points x_1^0 and x_2^0 be on sphere S_1 as they were defined at the beginning of this section. For $\epsilon=0$, the semiflow Φ on $SO(3)u_0$ contains only rotating waves. We have:*

1. *The rotating waves for the reaction-diffusion system (7.1.1) on $SO(3)u_0$ with the unit frequency vectors $\overrightarrow{X}$ on a circle around $\overrightarrow{Q}$ on the unit sphere S_2 project onto the periodic solution of (7.4.6), that has as an orbit the circle around $\overrightarrow{X}_0$ on the unit sphere S_1 passing through the point $x_0\in S_1$ such that $\cos(\angle(\vec{X},\vec{Q}))=\cos(\angle(\vec{X}_0,\vec{x}_0))$.*

2. *The rotating waves for the reaction-diffusion system (7.1.1) on $SO(3)u_0$ with the frequency vector $|X_0|\,\overrightarrow{Q}$ project onto x_1^0. The rotating waves for the reaction-diffusion system (7.1.1) on $SO(3)u_0$ with the frequency vector $-|X_0|\,\overrightarrow{Q}$ project onto x_2^0.*

3. *Let C_1 and C_2 be two distinct circles on the unit sphere S_2 with the centers on the line having the direction vector $\overrightarrow{Q}$ and such that $d(0,C_1)=d(0,C_2)$. Let us denote by C_1' the circle on S_1 onto which project the rotating waves on $SO(3)u_0$ with the unit frequency vectors on C_1. Let us denote by C_2' the circle on S_1 onto which project the rotating waves on $SO(3)u_0$ with the unit frequency vectors on C_2. Then $d(0,C_1')=d(0,C_2')$ and $C_1'\neq C_2'$, where d is the Euclidean distance in $\mathbb{R}^3$.*

Remark 7.4.13. *The following two figures illustrate Theorem 7.4.12: The sphere S_2 (Figure (7.4.1)) and The sphere S_1 (Figure (7.4.13)).*

Let $B\in SO(3)$ such that $Bx_1^0=(0,0,1)^T$. Let $g^{SS}(s,\epsilon)=Bg^S(B^{-1}s,\epsilon)B^{-1}$, where g^S is defined in (7.4.7) and for any $s=(x_1,x_2,x_3)\in\mathbf{S}^2$ and $\epsilon\geq 0$ small,

$$g^{SS}(s,\epsilon)=g^{SS}(x_1,x_2,x_3,\epsilon)=\begin{pmatrix} 0 & -G_1(x_1,x_2,x_3,\epsilon) & G_2(x_1,x_2,x_3,\epsilon)\\ G_1(x_1,x_2,x_3,\epsilon) & 0 & -G_3(x_1,x_2,x_3,\epsilon)\\ -G_2(x_1,x_2,x_3,\epsilon) & G_3(x_1,x_2,x_3,\epsilon) & 0\end{pmatrix}.$$

Let

$$a(\phi,\theta,\epsilon)=G_1(\sin\phi\cos\theta,\sin\phi\sin\theta,\cos\phi,\epsilon),$$

$$b(\phi,\theta,\epsilon)=G_2(\sin\phi\cos\theta,\sin\phi\sin\theta,\cos\phi,\epsilon),$$

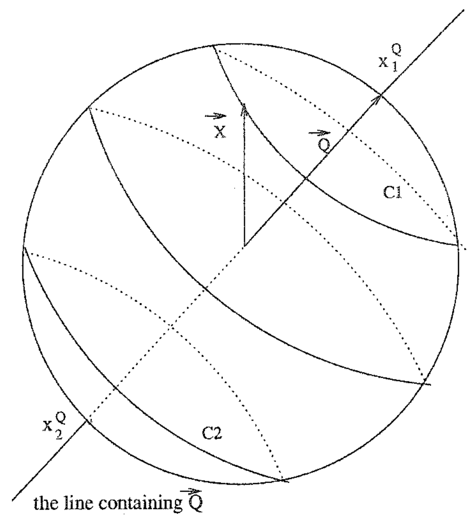

Figure 7.4.1: The Sphere S_2

and

$$c(\phi, \theta, \epsilon) = G_3(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi, \epsilon),$$

where $\theta \in [0, 2\pi)$, $\phi \in [0, \pi]$, $\epsilon \geq 0$ small. The analysis of the perturbed differential equations (7.4.6) on the orbit space $SO(3)/SO(2)$ for $\epsilon > 0$ small gives us :

Theorem 7.4.14 ($\epsilon > 0$). *1. There exist two sufficiently smooth branches of equilibria for (7.4.6) defined for $\epsilon \geq 0$ small, one branch $x^1(\epsilon)$, near x_1^0 and the other one $x^2(\epsilon)$, near x_2^0, such that $x^1(0) = x_1^0$ and $x^2(0) = x_2^0$.*
Generically, we have $\frac{\partial G_3}{\partial x_2}(0, 0, 1, 0) - \frac{\partial G_2}{\partial x_1}(0, 0, 1, 0) \neq 0$. The stability of the generically hyperbolic equilibria $x^1(\epsilon)$ is as follows:

(a) if $\frac{\partial G_3}{\partial x_2}(0, 0, 1, 0) - \frac{\partial G_2}{\partial x_1}(0, 0, 1, 0) < 0$, then $x^1(\epsilon)$ is locally asymptotically stable for the differential equations (7.4.6);

(b) if $\frac{\partial G_3}{\partial x_2}(0, 0, 1, 0) - \frac{\partial G_2}{\partial x_1}(0, 0, 1, 0) > 0$, then $x^1(\epsilon)$ is unstable for the differential equations (7.4.6).

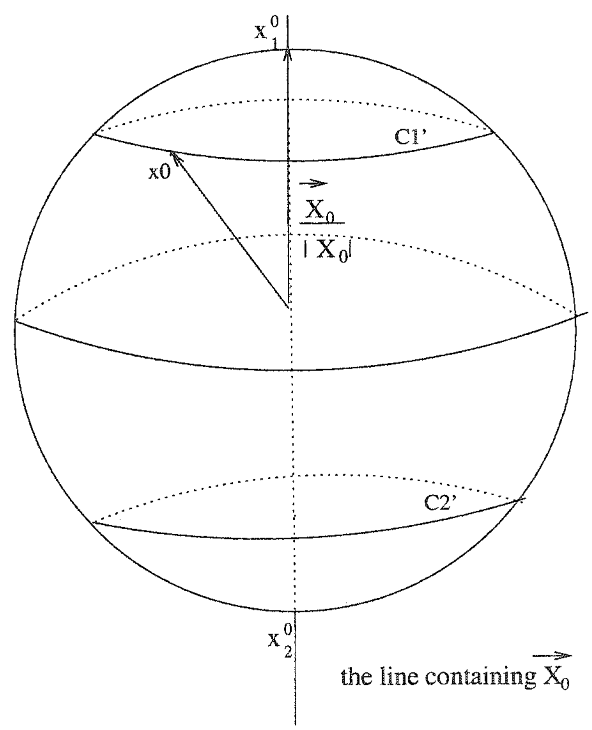

Figure 7.4.2: The Sphere S_1

Generically, we have $\frac{\partial G_3}{\partial x_2}(0,0,-1,0) - \frac{\partial G_2}{\partial x_1}(0,0,-1,0) \neq 0$. *The stability of the generically hyperbolic equilibria* $x^2(\epsilon)$ *is as follows:*

(a) *if* $\frac{\partial G_3}{\partial x_2}(0,0,-1,0) - \frac{\partial G_2}{\partial x_1}(0,0,-1,0) < 0$, *then* $x^2(\epsilon)$ *is locally asymptotically stable for the differential equations (7.4.6);*

(b) *if* $\frac{\partial G_3}{\partial x_2}(0,0,-1,0) - \frac{\partial G_2}{\partial x_1}(0,0,-1,0) > 0$, *then* $x^2(\epsilon)$ *is unstable for the differential equations (7.4.6).*

2. *Let*

$$I(\phi) = \int_0^{2\pi} [b(\phi, t, 0)\cos t - c(\phi, t, 0)\sin t] \; dt$$

for $\phi \in (0, \pi)$, *and consider*

$$x_0 = B^{-1}\begin{pmatrix} \sin\phi_0 \cos\theta_0 \\ \sin\phi_0 \sin\theta_0 \\ \cos\phi_0 \end{pmatrix}$$ *with* $\phi_0 \in (0,\pi)$, $\theta_0 \in [0, 2\pi)$. *For small* $\epsilon > 0$, *the periodic solution* $x(t, x_0) = e^{-X_0 t}x_0$ *of the differential equations (7.4.6) obtained for* $\epsilon = 0$ *is continuously deformed when* ϵ *increases to another periodic solution* $x^\epsilon(t)$ *with period* $T(\epsilon) = \frac{2\pi}{|X_0|} + O(\epsilon)$ *if*

$$I(\phi_0) = 0 \text{ and } I'(\phi_0) \neq 0.$$

Also $x^0(t) = x(t, x_0)$.

The stability of the periodic solution $x^\epsilon(t)$ *is as follows:*

(a) *if* $I'(\phi_0) < 0$, *then* $x^\epsilon(t)$ *is locally asymptotically stable for the differential equations (7.4.6);*

(b) *if* $I'(\phi_0) > 0$, *then* $x^\epsilon(t)$ *is unstable for the differential equations (7.4.6).*

Using the Poincare-Bendixson theorem on a sphere (see Theorem 2.5.12), we have a complete picture of the phase portrait for the differential equations (7.4.6).

Remark 7.4.15. *The matrix* $B \in SO(3)$ *such that* $Bx_1^0 = (0,0,1)^T$ *is not unique, but Theorem 7.4.14 does not depend on the choice of the matrix* B. *Therefore, in the statement and proof of Theorem 7.4.14, without loss of generality we may assume that* $x_1^0 = (0,0,1)^T$.

Remark 7.4.16. *Let* $\epsilon > 0$ *be fixed and* $\phi_0 \in (0,\pi)$ *such that it corresponds to the* $\frac{2\pi}{|X_0|}$-*periodic solution* $x(t, x_0) = e^{-X_0 t}x_0$ *of the differential equations (7.4.6) for* $\epsilon = 0$. *Then, the condition* $I(\phi_0) = 0$ *is necessary for the persistence of* $\frac{2\pi}{|X_0|}$-*periodic solution* $x(t, x_0)$ *to* $T(\epsilon)$-*periodic solution* $x^\epsilon(t)$.

7.5 The Analysis of the $SO(2)$-Equivariant Dynamics on the Phase Space $SO(3)$

Theorem 7.5.1 ($\epsilon > 0$). *For $\epsilon > 0$ small we have:*

1. *The equilibrium $x^1(\epsilon)$ of the differential equations (7.4.6) gives the $SO(2)$-orbit of a solution of the differential equations (7.3.2) having the form $A(t,\epsilon) = e^{\alpha_1(\epsilon)Qt}A_\epsilon$, where $x_1^Q = A_\epsilon x^1(\epsilon)$ and $\alpha_1(\epsilon) = |X_0| + O(\epsilon)$. The equilibrium $x^1(\epsilon)$ is generically hyperbolic, therefore the corresponding $SO(2)$-orbit is generically normally hyperbolic (see [11], [22]);*

2. *The equilibrium $x^2(\epsilon)$ of the differential equations (7.4.6) gives the $SO(2)$-orbit of a solution of the differential equations (7.3.2) having the form $A(t,\epsilon) = e^{\alpha_2(\epsilon)Qt}A_\epsilon$, where $x_1^Q = A_\epsilon x^2(\epsilon)$ and $\alpha_2(\epsilon) = -|X_0| + O(\epsilon)$. The equilibrium $x^2(\epsilon)$ is generically hyperbolic, therefore the corresponding $SO(2)$-orbit is generically normally hyperbolic (see [11], [22]);*

3. *If $I(\phi_0) = 0$ and $I'(\phi_0) \neq 0$, where I and ϕ_0 are defined in Theorem 7.4.14, then any $T(\epsilon)$-periodic solution $x^\epsilon(t)$ of (7.4.6) gives the $SO(2)$-orbit of a solution of the differential equations (7.3.2) having the form $A(t,\epsilon) = e^{\beta(\epsilon)Qt}B(t,\epsilon)$, where $\beta(\epsilon) = O(\epsilon)$ and $B(t,\epsilon)$ is a $T(\epsilon) = \frac{2\pi}{|X_0|} + O(\epsilon)$-periodic function such that $x_1^Q = B(0,\epsilon)x^\epsilon(0)$.*

The orbital stability of the above solutions is the same as the stability of the solutions of (7.4.6) from which they were obtained (see [11]).
Using the Poincare-Bendixson theorem on a sphere (see Theorem 2.5.12) and [11], we have a complete picture of the phase portrait for the differential equations (7.3.2).

The definition of the concepts of orbital stability for rotating waves and modulated rotating waves for a $SO(2)$-equivariant reaction-diffusion system is the same as those for a $SO(3)$-equivariant reaction-diffusion system.

Theorem 7.5.2 (The Effects of Forced Symmetry-Breaking). *For small $\epsilon > 0$ we have:*

1. *The $SO(2)$-orbit of the rotating wave $\Phi(t, A_0u_0, 0) = e^{Q|X_0|t}A_0u_0$ is deformed as ϵ increases to the $SO(2)$-orbit of a rotating wave having as the frequency vector $(|X_0| + O(\epsilon))\ \vec{Q}$*

2. *The $SO(2)$-orbit of the rotating wave $\Phi(t, A_0u_0, 0) = e^{-Q|X_0|t}A_0u_0$ is deformed as ϵ increases to a rotating wave having as the frequency vector $(-|X_0| + O(\epsilon))\overrightarrow{Q}$;*

3. *The $SO(2)$-orbit of the rotating waves $\Phi(t, A_0u_0, 0) = e^{Xt}A_0u_0$ having the frequency vectors $\overrightarrow{X} = \overrightarrow{A_0X_0A_0^{-1}}$ different from $|X_0|\overrightarrow{Q}$ and $-|X_0|\overrightarrow{Q}$, and satisfying $I(\phi_0) = 0$ and $I'(\phi_0) \neq 0$, where I and ϕ_0 are defined in Theorem 7.4.14, transform as ϵ increases into $SO(2)$-orbits of modulated rotating waves having primary frequency vectors of the form $O(\epsilon)\overrightarrow{Q}$ and the second period of the form $T(\epsilon) = \frac{2\pi}{|X_0|} + O(\epsilon)$.*

The orbital stability of the above solutions of the reaction-diffusion system (7.1.1) is related to the orbital stability of the rotating wave $\Phi(t, u_0, 0)$ and to the stability of the solutions of (7.4.6) from which they were obtained as follows:

- *If the rotating wave $\Phi(t, u_0, 0)$ is orbitally unstable, then the above solutions of the reaction-diffusion system (7.1.1) are orbitally unstable;*

- *If the rotating wave $\Phi(t, u_0, 0)$ is orbitally stable (that is $E^{cu} = T_{u_0}(SO(3)u_0)$), then the above solutions of the reaction-diffusion system (7.1.1) have the same orbital stability as the stability of the solutions of (7.4.6) from which they were obtained.*

The results obtained by forced symmetry-breaking may explain the anchoring of a spiral wave near a localized inhomogeneity, the repelling of a spiral wave away from the localized inhomogeneity, as well as the drifting of a spiral wave tip around the equator, the parallel of a sphere or around the boundary of an abnormal cardiac tissue.

7.6 Examples

We can extend the definition of $I(\phi_1)$ to $\phi_1 = 0$ and $\phi_1 = \pi$ as follows:
$I(0) = \int_0^{2\pi} [b(0,t,0)\cos t - c(0,t,0)\sin t]\, dt = 0$ and
$I(\pi) = \int_0^{2\pi} [b(\pi,t,0)\cos t - c(\pi,t,0)\sin t]\, dt = 0$. Also, we get that
$I'(0) = 2\pi[\frac{\partial G_2}{\partial x_1}(0,0,0,1) - \frac{\partial G_3}{\partial x_2}(0,0,0,1)]$ and $I'(\pi) = 2\pi[\frac{\partial G_2}{\partial x_1}(0,0,0,-1) - \frac{\partial G_3}{\partial x_2}(0,0,0,-1)]$.
In the proof of Theorem 7.4.14, the Poincaré section is $\theta = 0, \phi \in (0, \pi)$, therefore the Poincaré map and the function I are constructed independent of the periodic solution of the differential equations (7.4.6) for $\epsilon = 0$, whose persistence is discussed, that is independent of $\theta_0 \in (0, \pi)$. Therefore, the zeros of I in $(0, \pi)$ that satisfy the transversality condition will persist to periodic solutions. If the $SO(2)$-equivariant perturbation is chosen such that

$b(\phi,\theta,\epsilon) = h(\phi)\cos\theta$, $c(\phi,\theta,\epsilon) = -h(\phi)\sin\theta$, where $h(0) = h(\pi) = 0$ and $h^{'}(0) \neq 0$, $h^{'}(\pi) \neq 0$, then we get $I(\phi) = 2\pi h(\phi)$.

Example 7.6.1 (Anchoring at the inhomogeneity). *If we choose h such that $h^{'}(0) < 0$, we have that the rotating wave corresponding to $x^1(\epsilon)$ is orbitally stable.*

Example 7.6.2 (Repelling from the inhomogeneity vs. Boundary drifting). *If we choose h such that $h^{'}(0) > 0$, and there exists a unique $\phi_0 \in (0,\pi)$ such that $h(\phi_0) = 0$ and $h^{'}(\phi_0) < 0$, then there exists an orbitally stable modulated rotating wave, and the rotating wave corresponding to $x^1(\epsilon)$ is orbitally unstable.*

Example 7.6.3 (Anchoring at the inhomogeneity vs. Boundary drifting). *If we choose h such that $h^{'}(0) < 0$, and there exists only ϕ_1, $\phi_2 \in (0,\pi)$ such that $\phi_1 < \phi_2$, $h(\phi_1) = h(\phi_2) = 0$ and $h^{'}(\phi_1) > 0$, $h^{'}(\phi_2) < 0$, then there exist an orbitally unstable modulated rotating wave corresponding to ϕ_1 and an orbitally stable modulated rotating wave corresponding to ϕ_2. Moreover, the rotating wave corresponding to $x^1(\epsilon)$ is orbitally stable.*

Example 7.6.4 (Repelling from the inhomogeneity vs. Anchoring at the opposite site). *If we choose h such that $h^{'}(0) > 0$, $h^{'}(\pi) < 0$ and h has no other zeros in $(0,\pi)$, we have that the rotating wave corresponding to $x^1(\epsilon)$ is orbitally unstable, and rotating wave corresponding to $x^2(\epsilon)$ is orbitally stable.*

7.7 Proofs of some Theorems of Chapter 7

Proof of Theorem 7.1.2. The proof that the superposition operator G is a sufficiently smooth operator defined from $\mathbf{Y}^\alpha \times [0,\infty)$ into $\mathbf{Y}$ is done in the same way as the proof that the superposition operator F is a sufficiently smooth operator $\mathbf{Y}^\alpha \times [0,\infty)$ into $\mathbf{Y}$ (see proof of Theorem 3.1.16), because $r\mathbf{S}^2$ is compact and connected. □

Proof of Theorem 7.1.3. Since the operator $D\Delta$ (see Theorems 3.2.4 and 3.3.1) and the superposition operator F (see Theorem 3.3.1) are $SO(3)$-equivariant, and the superposition operator G is $SO(2)$-equivariant, the fact that $\mathbf{Y}^\alpha$ is $SO(3)$-invariant, the conclusion of the theorem is obvious. □

Proof of Theorem 7.2.2. We find an open set $U \subset \mathbf{Y}^\alpha$ such that there exists a constant $M_2 > 0$ satisfying $|G(u,\epsilon)|_{\mathbf{Y}} < M_2$, for any $u \in U$ and $\epsilon \geq 0$ small, where G represents the superposition operator associated to $G(u(x), x, \epsilon)$. Then the proof follows from [20], [48] and [70]. Let U be an open set included in a compact tubular neighborhood of $SO(3)u_0 \subset \mathbf{Y}^\alpha$. Then, by the Sobolev embedding Theorem 2.5.16 and 3.1.5, since $r\mathbf{S}^2$ is compact and connected, there exists a constant $M_1 > 0$ such that for any $u \in U$ and any $x \in r\mathbf{S}^2$, we have that $|u(x)| < M_1$. Since G is sufficiently smooth and $r\mathbf{S}^2$ is compact, there exists a constant $M > 0$ such that $|G(u(x), x, \epsilon)| < M$ for any $u \in U$, $x \in r\mathbf{S}^2$ and $\epsilon \geq 0$ small, where M is independent of x, ϵ and u. Since $r\mathbf{S}^2$ is compact, it follows that $|G(u, \epsilon)|_{\mathbf{Y}} < M_2$ for any $u \in U$ and $\epsilon \geq 0$ small.

Since $SO(3)u_0$ is normally hyperbolic (see [22]), then the equilibrium $0 \in V_*$ of the second differential equation in (7.2.1) is hyperbolic for $\epsilon = 0$ and it is continuously deformed under the small perturbation $\epsilon G^2_{pert}(A, q, \epsilon)$ to another hyperbolic equilibrium $q_\epsilon = O(\epsilon)$. $M(\epsilon)$ corresponds to q_ϵ. □

Proof of Theorem 7.3.1. There is nothing to prove for $G^2_{pert}(A, q, \epsilon)$.

Let $k(A, q, \epsilon) = A^{-1}G^1_{pert}(A, q, \epsilon)$ for $A \in SO(3)$, $q \in V_*$ and $\epsilon \geq 0$ small.

Since $G^1_{pert}(A, q, \epsilon) \in T_A(SO(3)) = A \cdot so(3)$, then $k(A, q, \epsilon) \in so(3)$ for any $A \in SO(3), q \in V_*$ and $\epsilon \geq 0$ small.

Using the $SO(2)$-equivariance of $G^1_{pert}(., q, \epsilon)$, we get that the function $k(., q, \epsilon)$ is $SO(2)$-invariant for $\epsilon \geq 0$ small and $q \in V_*$, that is

$k(e^{Q\theta}A, q, \epsilon) = (e^{Q\theta}A)^{-1}G^1_{pert}(e^{Q\theta}A, q, \epsilon) = A^{-1}e^{-Q\theta}e^{Q\theta}G^1_{pert}(A, q, \epsilon) = A^{-1}G^1_{pert}(A, q, \epsilon) = k(A, q, \epsilon)$ for any $A \in SO(3)$, $q \in V_*$, $\theta \in [0, 2\pi)$ and $\epsilon \geq 0$ small. □

Proof of Theorem 7.3.2. For $\epsilon > 0$ small, $M(\epsilon)$ corresponds to the hyperbolic equilibrium $q_\epsilon \in V_*$ of the second differential equation of (7.2.1). Therefore, we only consider the $SO(2)$-equivariant dynamics on $SO(3)$ obtained by substituting $q = q_\epsilon$ in the differential equations (7.2.1). Using theorem 7.3.1, we get

$$\dot{A} = A\left[f_1(q_\epsilon) + \epsilon k(A, q_\epsilon, \epsilon)\right]. \tag{7.7.1}$$

Since $q_\epsilon = O(\epsilon)$ and f_1 is sufficiently smooth, we get $f_1(q_\epsilon) = f_1(0) + \epsilon H(\epsilon) = X_0 + \epsilon H(\epsilon)$ with $H(\epsilon) \in so(3)$. The differential equations (7.7.1) become

$$\dot{A} = A\left[X_0 + \epsilon H(\epsilon) + \epsilon k(A, q_\epsilon, \epsilon)\right]. \tag{7.7.2}$$

Let $g(A, \epsilon) = H(\epsilon) + k(A, q_\epsilon, \epsilon)$. We get the differential equations (7.3.2), where $g(., \epsilon)$ is $SO(2)$-invariant, since $k(., q_\epsilon, \epsilon)$ is $SO(2)$-invariant. □

Proof of Theorem 7.4.8. It is enough to prove that there exists a sufficiently smooth function $g^S\colon \mathbf{S^2} \times [0, \epsilon_0) \to so(3)$ for ϵ_0 small such that

$$\widetilde{g}(C,\epsilon) = g^S(Cx_Q^1,\epsilon) \text{ for any } C \in SO(3) \text{ and } \epsilon \geq 0 \text{ small .} \tag{7.7.3}$$

Since $\mathbf{S^2}$ is compact, it is enough to prove (7.7.3) locally on $\mathbf{S^2}$. We may then use the partition of unity to get a sufficiently smooth function g^S globally defined on $\mathbf{S^2}$.
Let $C_0 \in SO(3)$ be fixed, but arbitrary. There exists a smooth local section π_s of $SO(3)/SO(2)$ in $SO(3)$ near $C_0 \cdot SO(2)$ (see Theorem 7.4.2).
We have:

$$\begin{aligned}\pi \circ \pi_s &= id \\ &\Rightarrow \pi(\pi_s(C \cdot SO(2))) = C \cdot SO(2) \\ &\Rightarrow \pi_s(C \cdot SO(2))SO(2) = C \cdot SO(2) \\ &\Rightarrow C = \pi_s(C \cdot SO(2)) \cdot C^*(C) \text{ for } C^*(C) \in SO(2) \text{ near } C_0 \cdot SO(2).\end{aligned} \tag{7.7.4}$$

Let us define the function

$$g^S = \widetilde{g} \circ \pi_s \circ \beta^{-1} \text{ near } C_0 x_Q^1. \tag{7.7.5}$$

Then g^S is a sufficiently smooth function defined locally near $C_0x_Q^1$ on $\mathbf{S^2}$.
Also,

$$g^S(Cx_Q^1,\epsilon) = \widetilde{g}(\pi_s(C \cdot SO(2)),\epsilon) = \widetilde{g}(\pi_s(C \cdot SO(2)) \cdot C^*(C),\epsilon) = \widetilde{g}(C,\epsilon) \tag{7.7.6}$$

for C near C_0 and $\epsilon \geq 0$ small, where for the second equality we use the invariance of $\widetilde{g}(.,\epsilon)$ under the restriction of the action $\widetilde{\theta}$ to $SO(2)$ and for the third equality we use (7.7.4).
The function g^S is well-defined since the function $\widetilde{g}$ is $SO(2)$-invariant under the action $\widetilde{\theta}$ restricted to $SO(2)$; that is, for any $\theta \in [0, 2\pi)$, $C \in SO(3)$ and $\epsilon \geq 0$ small, we have $g^S(Ce^{Q\theta}x_Q^1,\epsilon) = \widetilde{g}(Ce^{Q\theta},\epsilon) = \widetilde{g}(C,\epsilon) = g^S(Cx_Q^1,\epsilon)$.
Also, taking into account that the action θ^R is transitive, we can look at the function g^S as being defined on $\mathbf{S^2}$.
We check that the vector field given by the differential equations (7.4.6) is associated to the semiflow $\widetilde{\Psi_1}$.

We have $\widetilde{\Psi_1}(t, Cx_Q^1, \epsilon) = (\beta \circ \pi)(\Psi(t, C, \epsilon)) = \Psi(t, C, \epsilon)x_Q^1$. Therefore,

$$\begin{aligned} \frac{\partial \widetilde{\Psi_1}}{\partial t}(t, Cx_Q^1, \epsilon) &= \frac{\partial}{\partial t}(\Psi(t, C, \epsilon)x_Q^1) = \frac{\partial \Psi}{\partial t}(t, C, \epsilon)x_Q^1 \\ &= -\left[X_0 + \epsilon\widetilde{g}(\Psi(t, C, \epsilon), \epsilon)\right]\Psi(t, C, \epsilon)x_Q^1 \\ &= -\left[X_0 + \epsilon g^S(\Psi(t, C, \epsilon)x_Q^1, \epsilon)\right]\Psi(t, C, \epsilon)x_Q^1 \\ &= -\left[X_0 + \epsilon g^S(\widetilde{\Psi_1}(t, Cx_Q^1, \epsilon), \epsilon)\right]\widetilde{\Psi_1}(t, Cx_Q^1, \epsilon) \end{aligned} \tag{7.7.7}$$

for any $C \in SO(3)$, $t \in [0, \infty)$ and $\epsilon \geq 0$ small. □

Proof of Theorem 7.4.10. Conclusions (1) and (2) are obvious.
The third conclusion results from the definition of the flows Ψ and $\widetilde{\Psi_1}$, as well as $\Psi(t, C, \epsilon) = [\Phi(t, C^{-1}, \epsilon)]^{-1}$. □

Proof of Theorem 7.4.12. Conclusions (1) and (2) result from Theorems 7.2.2 and 7.4.10.
Using Theorem 7.2.2, we get that the rotating waves for the reaction-diffusion system (7.1.1) for $\epsilon = 0$ are of the form $\Phi(t, A_0u_0, 0) = A_0e^{X_0t}u_0$ for $A_0 \in SO(3)$ and $t \in [0, \infty)$.
Therefore, the primary frequency vector of $\Phi(t, A_0u_0, 0)$ is $\overrightarrow{A_0X_0A_0^{-1}}$. By Theorem 7.4.10, the rotating wave $\Phi(t, A_0u_0, 0)$ projects onto $x_0 \in \mathbf{S}^2$ such that $x_1^Q = A_0x_0$.
Let $X = \frac{1}{|X_0|}A_0X_0A_0^{-1}$. Using Theorem 2.2.9 (1) and $x_1^0 = \frac{1}{|X_0|}\overrightarrow{X}_0$, we get $A_0x_1^0 = \overrightarrow{X}$.
Also, $(x_1^Q)^T = x_0^TA_0^T$.
Therefore, $\cos(\angle(\vec{X}, \vec{Q})) = (x_1^Q)^T\overrightarrow{X} = x_0^TA_0^TA_0x_1^0 = x_0^Tx_1^0 = \cos(\angle(\vec{x_0}, \vec{X_0}))$.
The third conclusion results from the first conclusion. □

Proof of Theorem 7.4.14. For $\epsilon = 0$, the differential equation (7.4.6) has two equilibria, x_1^0 and x_2^0. Using the implicit function theorem (see Theorem 2.1.1), we will prove the persistence of these two equilibria for small $\epsilon > 0$.
We have defined $B \in SO(3)$ by $Bx_1^0 = (0, 0, 1)^T$. Then $Bx_2^0 = (0, 0, -1)^T$. Let $s = Bx$. Then,

$$\dot{s} = B\dot{x} = -\left[BX_0B^{-1} + \epsilon Bg^S(B^{-1}s, \epsilon)B^{-1}\right]s, \tag{7.7.8}$$

and, if we write $X_{00} = BX_0B^{-1}$ and $g^{SS}(s, \epsilon) = Bg^S(B^{-1}s, \epsilon)B^{-1}$, then we get

$$\dot{s} = -\left[X_{00} + \epsilon g^{SS}(s, \epsilon)\right]s, \tag{7.7.9}$$

Since $B\overrightarrow{X_0} = |X_0|\,(0,0,1)^T$, using Theorem 2.2.9 (2) we have

$$X_{00} = BX_0B^{-1} = |X_0| \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us define the function

$$F(s,\epsilon) = -\left[X_{00} + \epsilon g^{SS}(s,\epsilon)\right] s, \text{ where } F\colon \mathbf{S^2} \times [0,\epsilon_0) \to T\mathbf{S^2} \text{ for } \epsilon_0 > 0 \text{ small}. \quad (7.7.10)$$

It is clear that $F((0,0,1),0) = 0$. We will compute $(D_sF)_{((0,0,1),0)}\colon T_{(0,0,1)}\mathbf{S^2} \to T\mathbf{S^2} \simeq \mathbb{R}^2$. Let $s = (x_1, x_2, x_3) \in \mathbf{S^2}$ and recall that

$$g^{SS}(s,\epsilon) = g^{SS}(x_1,x_2,x_3,\epsilon) = \begin{pmatrix} 0 & -G_1(x_1,x_2,x_3,\epsilon) & G_2(x_1,x_2,x_3,\epsilon) \\ G_1(x_1,x_2,x_3,\epsilon) & 0 & -G_3(x_1,x_2,x_3,\epsilon) \\ -G_2(x_1,x_2,x_3,\epsilon) & G_3(x_1,x_2,x_3,\epsilon) & 0 \end{pmatrix}.$$

For $(x_1,x_2,x_3) \in \mathbf{S^2}$ near $(0,0,1)$, one has $x_3 = \sqrt{1-x_1^2-x_2^2} > 0$ for $\|(x_1,x_2)\| \geq 0$ small enough. If we substitute this expression of x_3 into the differential equations (7.7.9), we have:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \frac{d}{dt}\sqrt{1-x_1^2-x_2^2} \end{pmatrix} = -\left[\begin{pmatrix} 0 & -|X_0| & 0 \\ |X_0| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right.$$
$$\left. +\epsilon \begin{pmatrix} 0 & -G_1(x_1,x_2,\sqrt{1-x_1^2-x_2^2},\epsilon) & G_2(x_1,x_2,\sqrt{1-x_1^2-x_2^2},\epsilon) \\ G_1(x_1,x_2,\sqrt{1-x_1^2-x_2^2},\epsilon) & 0 & -G_3(x_1,x_2,\sqrt{1-x_1^2-x_2^2},\epsilon) \\ -G_2(x_1,x_2,\sqrt{1-x_1^2-x_2^2},\epsilon) & G_3(x_1,x_2,\sqrt{1-x_1^2-x_2^2},\epsilon) & 0 \end{pmatrix}\right]$$
$$\cdot \begin{pmatrix} x_1 \\ x_2 \\ \sqrt{1-x_1^2-x_2^2} \end{pmatrix} \quad (7.7.11)$$

or

$$\begin{aligned} \dot{x_1} &= |X_0|\,x_2 + \epsilon G_1(x,\epsilon)x_2 - \epsilon G_2(x,\epsilon)\sqrt{1-x_1^2-x_2^2}, \\ \dot{x_2} &= -|X_0|\,x_1 - \epsilon G_1(x,\epsilon)x_1 + \epsilon G_3(x,\epsilon)\sqrt{1-x_1^2-x_2^2}, \;, \\ \tfrac{d}{dt}\sqrt{1-x_1^2-x_2^2} &= \epsilon G_2(x,\epsilon)x_1 - \epsilon G_3(x,\epsilon)x_2, \end{aligned} \quad (7.7.12)$$

where $x = (x_1, x_2, \sqrt{1-x_1^2-x_2^2})$.

If the first two equations in the system (7.7.12) are satisfied, then the third one is immediately satisfied.

Therefore, we get the system

$$\begin{aligned} \dot{x_1} &= H_1(x_1, x_2, \epsilon), \\ \dot{x_2} &= H_2(x_1, x_2, \epsilon), \end{aligned} \tag{7.7.13}$$

where

$$\begin{aligned} H_1(x_1, x_2, \epsilon) = |X_0|\, x_2 + \epsilon \Big[& G_1(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}, \epsilon) x_2 \\ & - G_2(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}, \epsilon) \sqrt{1 - x_1^2 - x_2^2} \Big], \end{aligned} \tag{7.7.14}$$

and

$$\begin{aligned} H_2(x_1, x_2, \epsilon) = -\,|X_0|\, x_1 - \epsilon \Big[& G_1(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}, \epsilon) x_1 \\ & - G_3(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}, \epsilon) \sqrt{1 - x_1^2 - x_2^2} \Big]. \end{aligned} \tag{7.7.15}$$

Let $H(x_1, x_2, \epsilon) = (H_1(x_1, x_2, \epsilon), H_2(x_1, x_2, \epsilon))$. We have $H(0, 0, 0) = (0, 0)$ and the linearization of $H(x_1, x_2, \epsilon)$ about $(0, 0)$ at $\epsilon = 0$ is

$$D_{(x_1, x_2)} H(0, 0, 0) = \begin{pmatrix} 0 & |X_0| \\ -|X_0| & 0 \end{pmatrix},$$

which is an invertible matrix. Using the implicit function theorem there exists a sufficiently smooth branch $(x_1^1(\epsilon), x_2^1(\epsilon))$ near $(0, 0)$ such that $(x_1^1(0), x_2^1(0)) = (0, 0)$, and $H_1(x_1^1(\epsilon), x_2^1(\epsilon), \epsilon) = 0$ and $H_2(x_1^1(\epsilon), x_2^1(\epsilon), \epsilon) = 0$ for $\epsilon \geq 0$ small.

We have

$$x_1^1(\epsilon) = x_{11}\epsilon + O(\epsilon^2) \text{ and } x_2^1(\epsilon) = x_{22}\epsilon + O(\epsilon^2), \quad \text{for small } \epsilon \geq 0. \tag{7.7.16}$$

By using implicit differentiation we get

$$\begin{aligned} \frac{\partial H_1}{\partial x_1}\frac{dx_1}{d\epsilon} + \frac{\partial H_1}{\partial x_2}\frac{dx_2}{d\epsilon} &= -\frac{\partial H_1}{\partial \epsilon}, \\ \frac{\partial H_2}{\partial x_1}\frac{dx_1}{d\epsilon} + \frac{\partial H_2}{\partial x_2}\frac{dx_2}{d\epsilon} &= -\frac{\partial H_2}{\partial \epsilon}. \end{aligned} \tag{7.7.17}$$

Generically, we have

$$G_2(0, 0, 1, 0) \neq 0 \text{ and } G_3(0, 0, 1, 0) \neq 0. \tag{7.7.18}$$

From (7.7.17) it follows that at $\epsilon = 0$ we have

$$\frac{dx_1}{d\epsilon} = -\frac{\begin{vmatrix} \frac{\partial H_1}{\partial \epsilon} & \frac{\partial H_1}{\partial x_2} \\ \frac{\partial H_2}{\partial \epsilon} & \frac{\partial H_2}{\partial x_2} \end{vmatrix}}{\begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} \end{vmatrix}} \tag{7.7.19}$$

and

$$\frac{dx_2}{d\epsilon} = -\frac{\begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial \epsilon} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial \epsilon} \end{vmatrix}}{\begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} \end{vmatrix}}. \tag{7.7.20}$$

Since $\frac{\partial H_1}{\partial x_1}(0,0,0) = \frac{\partial H_2}{\partial x_2}(0,0,0) = 0$, $\frac{\partial H_1}{\partial x_2}(0,0,0) = -\frac{\partial H_2}{\partial x_1}(0,0,0) = |X_0|$, $\frac{\partial H_1}{\partial \epsilon}(0,0,0) = -G_2(0,0,1,0)$ and $\frac{\partial H_2}{\partial \epsilon}(0,0,0) = G_3(0,0,1,0)$, it follows that

$$\begin{aligned} x_{11} &= \tfrac{1}{|X_0|} G_3(0,0,1,0), \\ x_{22} &= \tfrac{1}{|X_0|} G_2(0,0,1,0). \end{aligned} \tag{7.7.21}$$

So, for $\epsilon \geq 0$ small, we get

$$x_1^1(\epsilon) = \frac{1}{|X_0|} G_3(0,0,1,0)\epsilon + O(\epsilon^2), x_2^1(\epsilon) = \frac{1}{|X_0|} G_2(0,0,1,0)\epsilon + O(\epsilon^2). \tag{7.7.22}$$

Therefore, there exists a sufficiently smooth branch $s^1(\epsilon) = (x_1^1(\epsilon), x_2^1(\epsilon), x_3^1(\epsilon))$ near $(0,0,1)$ of equilibria of the differential equations (7.7.9), such that
$s^1(0) = (x_1^1(0), x_2^1(0), x_3^1(0)) = (0,0,1)$ and $F(s^1(\epsilon), \epsilon) = (0,0,0)$ for $\epsilon \geq 0$ small.
For $\epsilon \geq 0$ small, there is a sufficiently smooth branch $x^1(\epsilon) \in \mathbf{S^2}$ of equilibria for the differential equations (7.4.6) such that $x^1(0) = x_1^0$.
In the same way, for $\epsilon \geq 0$ small there exists a sufficiently smooth branch $x^2(\epsilon)$ of equilibria for the differential equations (7.4.6) such that $x^2(0) = x_2^0$.
The stability of the equilibrium $x^1(\epsilon)$ is the same as the stability of the equilibrium $(x_1^1(\epsilon), x_2^1(\epsilon))$ for $\epsilon \geq 0$ small, and the stability of the equilibrium $x^2(\epsilon)$ is the same as the stability of the equilibrium $(x_1^2(\epsilon), x_2^2(\epsilon))$ for $\epsilon \geq 0$ small.
Let us now obtain the stability of the equilibrium $(x_1^1(\epsilon), x_2^1(\epsilon))$.
For $\epsilon \geq 0$ small, the linearization of $H(x_1, x_2, \epsilon)$ around $(x_1^1(\epsilon), x_2^1(\epsilon))$ is given by

$$D_{(x_1,x_2)} H(x_1^1(\epsilon), x_2^1(\epsilon), \epsilon) = \begin{pmatrix} a_{11}(\epsilon) & |X_0| + O(\epsilon) \\ -|X_0| + O(\epsilon) & a_{22}(\epsilon) \end{pmatrix}, \tag{7.7.23}$$

where

$$a_{11}(\epsilon) = \frac{\partial H_1}{\partial x_1}(x_1^1(\epsilon), x_2^1(\epsilon), \epsilon) = a_{11}\epsilon + O(\epsilon^2) \text{ since } a_{11}(0) = 0 \tag{7.7.24}$$

and

$$a_{22}(\epsilon) = \frac{\partial H_2}{\partial x_2}(x_1^1(\epsilon), x_2^1(\epsilon), \epsilon) = a_{22}\epsilon + O(\epsilon^2) \text{ since } a_{22}(0) = 0. \tag{7.7.25}$$

The eigenvalues of the linearization of $H(x_1, x_2, \epsilon)$ around $(x_1^1(\epsilon), x_2^1(\epsilon))$ satisfies

$$\begin{vmatrix} a_{11}(\epsilon) - \lambda & |X_0| + O(\epsilon) \\ -|X_0| + O(\epsilon) & a_{22}(\epsilon) - \lambda \end{vmatrix} = 0. \tag{7.7.26}$$

Then, it follows that $(a_{11}(\epsilon) - \lambda)(a_{22}(\epsilon) - \lambda) - (|X_0| + O(\epsilon))(-|X_0| + O(\epsilon) = 0$ or $\lambda^2 - (a_{11}(\epsilon) + a_{22}(\epsilon))\lambda + |X_0|^2 + O(\epsilon) = 0$.
We get the eigenvalues

$$\lambda_{1,2}(\epsilon) = \frac{a_{11}(\epsilon) + a_{22}(\epsilon)}{2} \pm i(|X_0| + O(\epsilon)) = \left[\frac{a_{11} + a_{22}}{2}\epsilon + O(\epsilon^2)\right] \pm i(|X_0| + O(\epsilon)). \tag{7.7.27}$$

In the case $a_{11}(\epsilon) + a_{22}(\epsilon) = 0$, the following theorem holds:

Theorem 7.7.1 ([83]). *The equilibrium $(x_1^1(\epsilon), x_2^1(\epsilon))$ is either a center, a center-focus or a (stable or unstable) focus for the differential equations (7.7.13). In an analytic system a center-focus cannot occur.*

Generically, $a_{11}(\epsilon) + a_{22}(\epsilon) \neq 0$, therefore the stability of $(x_1^1(\epsilon), x_2^1(\epsilon))$ is given by the sign of $a_{11}(\epsilon) + a_{22}(\epsilon)$. In fact, $(x_1^1(\epsilon), x_2^1(\epsilon))$ is generically a hyperbolic equilibrium for the differential equations (7.4.6).
Generically, $a_{11} + a_{22} \neq 0$.
If $a_{11} + a_{22} < 0$, then $(x_1^1(\epsilon), x_2^1(\epsilon))$ is an asymptotically stable equilibrium for the differential equations (7.4.6), and if $a_{11} + a_{22} > 0$, then $(x_1^1(\epsilon), x_2^1(\epsilon))$ is an unstable equilibrium for the differential equations (7.4.6).
We express a_{11} and a_{22} in terms of $G_1((0,0,1),0)$, $G_2((0,0,1),0)$ and $G_3((0,0,1),0)$ and their partial derivatives at $(0,0,1)$.
The Taylor expansion of $H_1(x_1, x_2, \epsilon)$ about $(0,0,0)$ up to terms of order 2 (h.o.t. contains all the terms of order ≥ 3 in x_1, x_2, ϵ) is:

$$\begin{aligned} H_1(x_1, x_2, \epsilon) = {} & a_0^1 + a_1^1 x_1 + a_2^1 x_2 + a_3^1 x_1^2 + a_4^1 x_2^2 + a_5^1 x_1 x_2 \\ & + a_6^1 \epsilon + a_7^1 x_1 \epsilon + a_8^1 x_2 \epsilon + a_9^1 \epsilon^2 + h.o.t.. \end{aligned} \tag{7.7.28}$$

Since $a_0^1 = 0$, $a_1^1 = 0$, $a_3^1 = 0$, $a_4^1 = 0$ and $a_5^1 = 0$, we get

$$H_1(x_1, x_2, \epsilon) = a_2^1 x_2 + a_6^1 \epsilon + a_7^1 x_1 \epsilon + a_8^1 x_2 \epsilon + a_9^1 \epsilon^2 + h.o.t.. \tag{7.7.29}$$

The Taylor expansion of $H_2(x_1, x_2, \epsilon)$ about $(0,0,0)$ up to terms of order 2 is:

$$\begin{aligned} H_2(x_1, x_2, \epsilon) = {} & a_0^2 + a_1^2 x_1 + a_2^2 x_2 + a_3^2 x_1^2 + a_4^2 x_2^2 + a_5^2 x_1 x_2 \\ & + a_6^2 \epsilon + a_7^2 x_1 \epsilon + a_8^2 x_2 \epsilon + a_9^2 \epsilon^2 + h.o.t.. \end{aligned} \tag{7.7.30}$$

Since $a_0^2 = 0$, $a_2^2 = 0$, $a_3^2 = 0$, $a_4^2 = 0$ and $a_5^2 = 0$, we get

$$H_2(x_1, x_2, \epsilon) = a_1^2 x_1 + a_6^2 \epsilon + a_7^2 x_1 \epsilon + a_8^2 x_2 \epsilon + a_9^2 \epsilon^2 + h.o.t.. \tag{7.7.31}$$

We have

$$\frac{\partial H_1}{\partial x_1}(x_1^1(\epsilon), x_2^1(\epsilon), \epsilon) = a_7^1 \epsilon + o(\epsilon^2) \tag{7.7.32}$$

and

$$\frac{\partial H_2}{\partial x_2}(x_1^1(\epsilon), x_2^1(\epsilon), \epsilon) = a_8^2 \epsilon + o(\epsilon^2). \tag{7.7.33}$$

We have that

$$a_{11} = a_7^1 = \frac{\partial^2 H_1}{\partial x_1 \partial \epsilon}(0,0,0) = -\frac{\partial G_2}{\partial x_1}(0,0,1,0), \tag{7.7.34}$$

by the definition of H_1 and

$$a_{22} = a_8^2 = \frac{\partial^2 H_2}{\partial x_2 \partial \epsilon}(0,0,0) = \frac{\partial G_3}{\partial x_2}(0,0,1,0), \tag{7.7.35}$$

by the definition of H_2.

Generically, we have $a_7^1 + a_8^2 \neq 0$.

If $a_7^1 + a_8^2 < 0$, then $(x_1^1(\epsilon), x_2^1(\epsilon))$ is an asymptotically stable equilibrium for the differential equations (7.4.6), and if $a_7^1 + a_8^2 > 0$, then $(x_1^1(\epsilon), x_2^1(\epsilon))$ is an unstable equilibrium for the differential equations (7.4.6).

If $a_7^1 + a_8^2 = 0$, then we compute the terms of order ϵ^2 in $a_{11}(\epsilon) + a_{22}(\epsilon)$.

In the same way, we can establish the stability of the equilibrium $x^2(\epsilon)$.

This ends the proof of the first conclusion of Theorem 7.4.14.

For $\epsilon = 0$ the differential equation (7.4.6) has $\frac{2\pi}{|X_0|}$-periodic solutions of the form $x(t, x_0) = e^{-X_0 t} x_0$, where $x_0 \neq x_1^0$ and $x_0 \neq x_2^0$.

Let us fix a point x_0 on the unit sphere such that $x_0 \neq x_1^0$ and $x_0 \neq x_2^0$.

We will prove the persistence of the periodic solution $x(t, x_0)$ if some conditions are satisfied. Making the same change of variable as before, we get the system (7.7.9) and the periodic solution $x(t, x_0)$ becomes $s(t, s_0) = e^{-X_{00} t} s_0$, where $s_0 = B x_0$ and $s_0 \neq (0,0,1)$, $s_0 \neq (0,0,-1)$.

For $s \neq (0,0,1)$ and $s \neq (0,0,-1)$ we use the spherical polar coordinates (ϕ, θ) on $\mathbf{S}^2$.

Let $s = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$, where $\phi \in (0, \pi)$ and $\theta \in [0, 2\pi)$.

Recall that $s_0 = \begin{pmatrix} \sin\phi_0\cos\theta_0 \\ \sin\phi_0\sin\theta_0 \\ \cos\phi_0 \end{pmatrix}$

and

$$a(\phi,\theta,\epsilon) = G_1(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi, \epsilon),$$

$$b(\phi,\theta,\epsilon) = G_2(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi, \epsilon),$$

$$c(\phi,\theta,\epsilon) = G_3(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi, \epsilon)$$

for $\theta \in [0, 2\pi)$, $\phi \in [0,\pi]$, $\epsilon \geq 0$ small and such that

$$a(0,\theta,\epsilon) = G_1(0,0,1,\epsilon), a(\pi,\theta,\epsilon) = G_1(0,0,-1,\epsilon)\text{for any } \theta \in [0,2\pi),$$

$$b(0,\theta,\epsilon) = G_2(0,0,1,\epsilon), b(\pi,\theta,\epsilon) = G_2(0,0,-1,\epsilon)\text{for any } \theta \in [0,2\pi),$$

$$c(0,\theta,\epsilon) = G_3(0,0,1,\epsilon), c(\pi,\theta,\epsilon) = G_{13}(0,0,-1,\epsilon)\text{for any } \theta \in [0,2\pi).$$

The system (7.7.9) in (ϕ,θ) coordinates yields:

$$\begin{pmatrix} \frac{d}{dt}(\sin\phi\cos\theta) \\ \frac{d}{dt}(\sin\phi\sin\theta) \\ \frac{d}{dt}(\cos\phi) \end{pmatrix} = -\left[\begin{pmatrix} 0 & -|X_0| & 0 \\ |X_0| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & -a(\phi,\theta,\epsilon) & b(\phi,\theta,\epsilon) \\ a(\phi,\theta,\epsilon) & 0 & -c(\phi,\theta,\epsilon) \\ -b(\phi,\theta,\epsilon) & c(\phi,\theta,\epsilon) & 0 \end{pmatrix}\right] \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$$

$\Longrightarrow$

$$\begin{pmatrix} \dot\phi\cos\phi\cos\theta - \dot\theta\sin\phi\sin\theta \\ \dot\phi\cos\phi\sin\theta + \dot\theta\sin\phi\cos\theta \\ -\dot\phi\cos\phi \end{pmatrix} = -\begin{pmatrix} 0 & -|X_0| - \epsilon a(\phi,\theta,\epsilon) & \epsilon b(\phi,\theta,\epsilon) \\ |X_0| + \epsilon a(\phi,\theta,\epsilon) & 0 & -\epsilon c(\phi,\theta,\epsilon) \\ -\epsilon b(\phi,\theta,\epsilon) & \epsilon c(\phi,\theta,\epsilon) & 0 \end{pmatrix} \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$$

$\Longrightarrow$

$$\begin{aligned} \dot\phi\cos\phi\cos\theta - \dot\theta\sin\phi\sin\theta &= [|X_0| + \epsilon a(\phi,\theta,\epsilon)]\sin\phi\sin\theta - \epsilon b(\phi,\theta,\epsilon)\cos\phi, \\ \dot\phi\cos\phi\sin\theta + \dot\theta\sin\phi\cos\theta &= -[|X_0| + \epsilon a(\phi,\theta,\epsilon)]\sin\phi\cos\theta + \epsilon c(\phi,\theta,\epsilon)\cos\phi, \\ -\dot\phi\sin\phi &= \epsilon b(\phi,\theta,\epsilon)\sin\phi\cos\theta - \epsilon c(\phi,\theta,\epsilon)\sin\phi\sin\theta. \end{aligned} \tag{7.7.36}$$

We divide the third equation in (7.7.36) by $\sin\phi \neq 0$, where $\phi \in (0, \pi)$ to get

$$\dot{\phi} = -\epsilon b(\phi,\theta,\epsilon)\cos\theta + \epsilon c(\phi,\theta,\epsilon)\sin\theta. \tag{7.7.37}$$

We substitute $\dot{\phi}$ given by (7.7.37) into the first two equations of (7.7.36) to get

$$\begin{aligned}
\dot{\theta}\sin\phi\sin\theta = -\left[|X_0| + \epsilon a(\phi,\theta,\epsilon)\right]\sin\theta\sin\phi &+ \epsilon c(\phi,\theta,\epsilon)\cos\theta\sin\theta\cos\phi \\
+\epsilon b(\phi,\theta,\epsilon)\cos\phi\sin^2\theta, \\
\dot{\theta}\sin\phi\cos\theta = -\left[|X_0| + \epsilon a(\phi,\theta,\epsilon)\right]\cos\theta\sin\phi &+ \epsilon c(\phi,\theta,\epsilon)\cos\phi\cos^2\theta \\
+\epsilon b(\phi,\theta,\epsilon)\sin\theta\cos\theta\cos\phi.
\end{aligned} \tag{7.7.38}$$

If $\sin\theta \neq 0$, we divide the first equation in (7.7.38) by $\sin\phi\sin\theta$, to get

$$\dot{\theta} = -\left[|X_0| + \epsilon a(\phi,\theta,\epsilon)\right] + \epsilon b(\phi,\theta,\epsilon)\sin\theta\cot\phi + \epsilon c(\phi,\theta,\epsilon)\cos\theta\cot\phi. \tag{7.7.39}$$

Thus, the second differential equation of (7.7.38) is satisfied.
If $\sin\theta = 0$, then $\cos\theta \neq 0$ and we divide the second equation in (7.7.38) by $\sin\phi\cos\theta$ to get

$$\dot{\theta} = -\left[|X_0| + \epsilon a(\phi,\theta,\epsilon)\right] + \epsilon b(\phi,\theta,\epsilon)\sin\theta\cot\phi + \epsilon c(\phi,\theta,\epsilon)\cos\theta\cot\phi. \tag{7.7.40}$$

Thus, the first of the differential equations (7.7.38) is satisfied.
Therefore, we obtain the system

$$\begin{aligned}
\dot{\phi} &= -\epsilon b(\phi,\theta,\epsilon)\cos\theta + \epsilon c(\phi,\theta,\epsilon)\sin\theta, \\
\dot{\theta} &= -\left[|X_0| + \epsilon a(\phi,\theta,\epsilon)\right] + \epsilon b(\phi,\theta,\epsilon)\sin\theta\cot\phi + \epsilon c(\phi,\theta,\epsilon)\cos\theta\cot\phi.
\end{aligned} \tag{7.7.41}$$

or

$$\begin{aligned}
\dot{\phi} &= \epsilon\left[-b(\phi,\theta,\epsilon)\cos\theta + c(\phi,\theta,\epsilon)\sin\theta\right], \\
\dot{\theta} &= -|X_0| + \epsilon\{a(\phi,\theta,\epsilon) + [b(\phi,\theta,\epsilon)\sin\theta + c(\phi,\theta,\epsilon)\cos\theta]\cot\phi\}.
\end{aligned} \tag{7.7.42}$$

For $\phi \in (0,\pi)$, $\theta \in [0, 2\pi)$ and for $\epsilon > 0$ small enough, we can have that

$$-|X_0| + \epsilon\{a(\phi,\theta,\epsilon) + [b(\phi,\theta,\epsilon)\sin\theta + c(\phi,\theta,\epsilon)\cos\theta]\cot\phi\} > 0$$

and we can rescale time, choosing $\tau = \tau(t)$ such that

$$\frac{d\tau}{dt} = -|X_0| + \epsilon\left\{a(\phi,\theta,\epsilon) + [b(\phi,\theta,\epsilon)\sin\theta + c(\phi,\theta,\epsilon)\cos\theta]\cot\phi\right\}. \tag{7.7.43}$$

We get

$$
\begin{aligned}
\frac{d\phi}{d\tau} &= \frac{d\phi}{dt}\frac{dt}{d\tau} = \frac{\epsilon[-b(\phi,\theta,\epsilon)\cos\theta+c(\phi,\theta,\epsilon)\sin\theta]}{-|X_0|+\epsilon\{a(\phi,\theta,\epsilon)+[b(\phi,\theta,\epsilon)\sin\theta+c(\phi,\theta,\epsilon)\cos\theta]\cot\phi\}}, \\
\frac{d\theta}{d\tau} &= \frac{d\theta}{dt}\frac{dt}{d\tau} = 1.
\end{aligned}
\tag{7.7.44}
$$

Since

$$b(\phi, \theta, \epsilon) = b(\phi, \theta, 0) + \epsilon h_1(\phi, \theta, \epsilon)$$

and

$$c(\phi, \theta, \epsilon) = c(\phi, \theta, 0) + \epsilon h_2(\phi, \theta, \epsilon),$$

it results that

$$
\begin{aligned}
\frac{d\phi}{d\tau} &= \frac{\epsilon}{|X_0|}\left[b(\phi, \theta, 0)\cos\theta - c(\phi, \theta, 0)\sin\theta\right] + \epsilon^2 h(\phi, \theta, \epsilon), \\
\frac{d\theta}{d\tau} &= 1.
\end{aligned}
\tag{7.7.45}
$$

Let us construct the Poincaré map (that is the time 2π map) for the flow given by the differential equations (7.7.45).

The Poincaré section associated to the flow given by the differential equations (7.7.45) is given by $\theta = 0$, $\phi \in (0, \pi)$, with ϕ chosen independent of $\theta_0 \in (0, \pi)$. The Poincaré map is given by

$$P(\phi_1, \epsilon) = \phi(2\pi, \phi_1, \epsilon),$$

where $\phi(t, \phi_1, \epsilon)$ is the solution of the following initial problem after we relabel $\tau = t$

$$
\begin{aligned}
\frac{d\phi}{dt} &= \frac{\epsilon}{|X_0|}\left[b(\phi, t, 0)\cos t - c(\phi, t, 0)\sin t\right] + \epsilon^2 h(\phi, t, \epsilon), \\
\phi(0) &= \phi_1.
\end{aligned}
\tag{7.7.46}
$$

Therefore, we get by using (7.7.46)

$$
\begin{aligned}
P(\phi_1, \epsilon) &= \phi(0, \phi_1, \epsilon) + \int_0^{2\pi} \frac{d\phi}{dt}(t, \phi_1, \epsilon)\, dt \\
&= \phi_1 + \int_0^{2\pi} \left\{ \frac{\epsilon}{|X_0|}\left[b(\phi(t, \phi_1, \epsilon), t, 0)\cos t - c(\phi(t, \phi_1, \epsilon), t, 0)\sin t\right] \right. \\
&\left. +\epsilon^2 h(\phi(t, \phi_1, \epsilon), t, \epsilon) \right\} dt
\end{aligned}
$$

or

$$P(\phi_1, \epsilon) = \phi_1 + \int_0^{2\pi} \frac{\epsilon}{|X_0|}\left[b(t, \phi_1, 0)\cos t - c(t, \phi_1, 0)\sin t\right] dt + \epsilon^2 h_*(\phi_1, \epsilon).$$

We have that

$$P(\phi_1, \epsilon) = \phi_1 + \frac{\epsilon}{|X_0|} I(\phi_1) + \epsilon^2 h_*(\phi_1, \epsilon),$$

where we define

$$I(\phi_1) = \int_0^{2\pi} [b(\phi_1, t, 0)\cos t - c(\phi_1, t, 0)\sin t]\; dt.$$

Then,

$$P(\phi_0, 0) - \phi_0 = 0,$$

since $x(t, x_0) = e^{-X_0 t}x_0$ is a periodic solution of the differential equations (7.4.6) for $\epsilon = 0$, that is $s(t, s_0) = e^{-X_{00}t}s_0$ is a periodic solution of the differential equations (7.7.9) for $\epsilon = 0$ and $s_0 = \begin{pmatrix} \sin\phi_0\cos\theta_0 \\ \sin\phi_0\sin\theta_0 \\ \cos\phi_0 \end{pmatrix}$. Let us consider the equation

$$P(\phi_1, \epsilon) - \phi_1 = \frac{\epsilon}{|X_0|}\,[I(\phi_1) + \epsilon h_*(\phi_1, \epsilon)] = 0,$$

that is

$$I(\phi_1) + \epsilon h_*(\phi_1, \epsilon) = 0.$$

If

$$I(\phi_0) = 0 \text{ and } I^{'}(\phi_0) \neq 0,$$

then, using the implicit function theorem, we find for $\epsilon \geq 0$ small a sufficiently smooth branch $\phi(\epsilon)$ of fixed points of P such that $\phi(0) = \phi_0$.

We have

$$b(\phi_1, t, 0) = G_2(\sin\phi_1\cos t, \sin\phi_1\sin t, \cos\phi_1, 0)$$

and

$$c(\phi_1, t, 0) = G_3(\sin\phi_1\cos t, \sin\phi_1\sin t, \cos\phi_1, 0).$$

Therefore,

$$\begin{aligned}\frac{\partial b}{\partial\phi_1}(\phi_1, t, 0) = {} & \frac{\partial G_2}{\partial x_1}(\sin\phi_1\cos t, \sin\phi_1\sin t, \cos\phi_1, 0)\cos\phi_1\cos t \\ & + \frac{\partial G_2}{\partial x_2}(\sin\phi_1\cos t, \sin\phi_1\sin t, \cos\phi_1, 0)\cos\phi_1\sin t \\ & - \frac{\partial G_2}{\partial x_3}(\sin\phi_1\cos t, \sin\phi_1\sin t, \cos\phi_1, 0)\sin\phi_1\end{aligned}$$

and

$$\begin{aligned}\frac{\partial c}{\partial \phi_1}(\phi_1, t, 0) = &\frac{\partial G_3}{\partial x_1}(\sin\phi_1\cos t, \sin\phi_1\sin t, \cos\phi_1, 0)\cos\phi_1\cos t\\ &+\frac{\partial G_3}{\partial x_2}(\sin\phi_1\cos t, \sin\phi_1\sin t, \cos\phi_1, 0)\cos\phi_1\sin t\\ &-\frac{\partial G_3}{\partial x_3}(\sin\phi_1\cos t, \sin\phi_1\sin t, \cos\phi_1, 0)\sin\phi_1.\end{aligned}$$

We have

$$\begin{aligned}I(\phi_0) = \int_0^{2\pi} &[G_2(\sin\phi_0\cos t, \sin\phi_0\sin t, \cos\phi_0, 0)\cos t\\ &-G_3(\sin\phi_0\cos t, \sin\phi_0\sin t, \cos\phi_0, 0)\sin t]\ dt = 0\end{aligned} \tag{7.7.47}$$

and

$$\begin{aligned}I'(\phi_0) = \int_0^{2\pi} \Bigg\{ &\Bigg[\frac{\partial G_2}{\partial x_1}(\sin\phi_0\cos t, \sin\phi_0\sin t, \cos\phi_0, 0)\cos\phi_0\cos t\\ &+\frac{\partial G_2}{\partial x_2}(\sin\phi_0\cos t, \sin\phi_0\sin t, \cos\phi_0, 0)\cos\phi_0\sin t\\ &-\frac{\partial G_2}{\partial x_3}(\sin\phi_0\cos t, \sin\phi_0\sin t, \cos\phi_0, 0)\sin\phi_0\Bigg]\cos t\\ &-\Bigg[\frac{\partial G_3}{\partial x_1}(\sin\phi_0\cos t, \sin\phi_0\sin t, \cos\phi_0, 0)\cos\phi_0\cos t\\ &+\frac{\partial G_3}{\partial x_2}(\sin\phi_0\cos t, \sin\phi_0\sin t, \cos\phi_0, 0)\cos\phi_0\sin t\\ &-\frac{\partial G_3}{\partial x_3}(\sin\phi_0\cos t, \sin\phi_{10}\sin t, \cos\phi_0, 0)\sin\phi_0\Bigg]\sin t\Bigg\}\ dt \neq 0.\end{aligned} \tag{7.7.48}$$

For $\epsilon \geq 0$ small, we get a sufficiently smooth branch $s^\epsilon(t)$ of periodic solutions for the differential equations (7.7.9) such that $s^0(t) = e^{-X_{00}t}s_0$.
Thus, for $\epsilon \geq 0$ small, we get a sufficiently smooth branch $x^\epsilon(t)$ of periodic solutions for the differential equations (7.4.6) such that $x^0(t) = e^{-X_0 t}x_0$.
If $I'(\phi_0) < 0$, the fixed point $\phi(\epsilon)$ is locally asymptotically stable for P, and if $I'(\phi_0) > 0$, the fixed point $\phi(\epsilon)$ is unstable for P.
The stability of the periodic solutions $x^\epsilon(t)$ which persist for $\epsilon > 0$ is the same as the stability of the fixed points $\phi(\epsilon)$ of the Poincaré map P from which the periodic solutions $x^\epsilon(t)$ are obtained.
This ends the proof of the second conclusion of Theorem 7.4.14. □

Proof of Theorem 7.5.1. We prove the first conclusion for the equilibria $x^1(\epsilon)$. A similar proof can be done for the equilibria $x^2(\epsilon)$. Let $A_\epsilon \in SO(3)$ be such that $x_1^Q = A_\epsilon x^1(\epsilon)$. We can choose a sufficiently smooth branch A_ϵ, by taking $C_\epsilon = (\pi_s \circ \beta^{-1})(x^1(\epsilon))$ and then $A_\epsilon = C_\epsilon^{-1}$. We have $C_\epsilon x_1^Q = x^1(\epsilon)$.
Taking into account that $\Psi(t, C, \epsilon) = [\Phi(t, A, \epsilon)]^{-1}$, with $A = C^{-1}$, we check that the differential equations (7.4.2) have a solution of the form

$$C^{RW}(t,\epsilon) \stackrel{\text{def}}{=} C_\epsilon e^{-\alpha_1(\epsilon)Qt} \text{ with } \alpha_1(\epsilon) = |X_0| + O(\epsilon).$$

By the definitions of the projected flows $\widetilde{\Psi_1}$ and $\widetilde{\Psi}$, we have $\widetilde{\Psi_1}(t, C_\epsilon x_1^Q, \epsilon) = C_\epsilon x_1^Q = x^1(\epsilon)$, then

$$\widetilde{\Psi}(t, C_\epsilon \cdot SO(2), \epsilon) = \pi(\Psi(t, C_\epsilon, \epsilon)) = C_\epsilon \cdot SO(2),$$

which implies

$$C^{RW}(t,\epsilon) = \Psi(t, C_\epsilon, \epsilon) = C_\epsilon e^{\tau(t,\epsilon)Q}, \tag{7.7.49}$$

where $\tau(0,\epsilon) = 0$. Without loss of generality, we assume that $\tau(t,\epsilon)$ sufficiently smooth with respect to t.
It follows that $C^{RW}(t,\epsilon)$ verifies the differential equations (7.4.2).
If we substitute (7.7.49) into the differential equations (7.4.2), we get

$$C_\epsilon \frac{\partial \tau}{\partial t}(t,\epsilon) Q e^{\tau(t,\epsilon)Q} = -\left[X_0 + \epsilon \widetilde{g}(C_\epsilon e^{\tau(t,\epsilon)Q}, \epsilon)\right] C_\epsilon e^{\tau(t,\epsilon)Q}$$

or using the $SO(2)$-invariance of $\widetilde{g}(.,\epsilon)$, we get

$$\begin{aligned} C_\epsilon \frac{\partial \tau}{\partial t}(t,\epsilon) Q &= -\left[X_0 + \epsilon \widetilde{g}(C_\epsilon, \epsilon)\right] C_\epsilon \\ \Rightarrow \frac{\partial \tau}{\partial t}(t,\epsilon) Q &= -C_\epsilon^{-1}\left[X_0 + \epsilon \widetilde{g}(C_\epsilon, \epsilon)\right] C_\epsilon \\ \Rightarrow \left|\frac{\partial \tau}{\partial t}(t,\epsilon) Q\right| &= \left|-C_\epsilon^{-1}\left[X_0 + \epsilon \widetilde{g}(C_\epsilon, \epsilon)\right] C_\epsilon\right| \end{aligned}$$

or, using $|Q| = 1$ and Theorem 2.2.9, we get

$$\begin{aligned} \left|\frac{\partial \tau}{\partial t}(t,\epsilon)\right| &= |X_0 + \epsilon \widetilde{g}(C_\epsilon, \epsilon)| \\ \Rightarrow \frac{\partial \tau}{\partial t}(t,\epsilon) &= \text{ constant } \stackrel{\text{def}}{=} \alpha_1(\epsilon). \end{aligned} \tag{7.7.50}$$

Therefore,

$$\tau(t,\epsilon) = -\alpha_1(\epsilon)t + \tau(0,\epsilon) \Rightarrow \tau(t,\epsilon) = -\alpha_1(\epsilon)t.$$

Also, we get $\alpha_1(\epsilon)Q = C_\epsilon^{-1}\left[X_0 + \epsilon\widetilde{g}(C_\epsilon,\epsilon)C_\epsilon\right]$ and $\alpha_1(\epsilon)$ is sufficiently smooth. Then,

$$\alpha_1(\epsilon) = |X_0| + O(\epsilon),$$

since $\alpha_1(0)Q = C_0^{-1}X_0C_0$ implies $\alpha_1(0)\overrightarrow{Q} = |X_0|\,\overrightarrow{Q}$, namely $\alpha_1(0) = |X_0|$ because $x_1^Q = C_0^{-1}x^1(0)$, $x^1(0) = x_1^0$ and Theorem 2.2.9, (2).
The stability issue for $C^{RW}(t,\epsilon)$ is proved in [11].
It is now easy to get the results for the differential equations (7.3.2) using the fact that $\Psi(t,C,\epsilon) = [\Phi(t,A,\epsilon)]^{-1}$, where $A = C^{-1}$.
This proves the first conclusion. Similarly, we can prove second conclusion of Theorem 7.5.1.
We now prove the conclusion (3) of Theorem 7.5.1.
Let $D(t,\epsilon) \in SO(3)$ be such that $D(t,\epsilon)x_1^Q = x^\epsilon(t)$.
Let $T(\epsilon) = \frac{2\pi}{|X_0|} + O(\epsilon)$ be the period of the function $x^\epsilon(t)$. We can choose a sufficiently smooth branch $D(t,\epsilon)$, by taking $D(t,\epsilon) = (\pi_s \circ \beta^{-1})(x^\epsilon(t))$.
Since $x^\epsilon(t)$ is $T(\epsilon)$-periodic, we have $D(t+T(\epsilon),\epsilon)x_1^Q = D(t,\epsilon)x_1^Q$. Therefore, there exists a sufficiently smooth function $g(t,\epsilon)$ such that

$$D(t+T(\epsilon),\epsilon) = D(t,\epsilon)e^{g(t,\epsilon)Q}. \tag{7.7.51}$$

Without loss of generality, we assume that $g(0,\epsilon) = 0$, otherwise we take $D_1(t,\epsilon) = D(t,\epsilon)e^{-g(0,\epsilon)Q}$.
Taking into the account that $\Psi(t,C,\epsilon) = [\Phi(t,A,\epsilon)]^{-1}$, where $A = C^{-1}$, we check that the differential equations (7.4.2) have a solution of the form $C^{MRW}(t,\epsilon) = B^*(t,\epsilon)e^{-\beta(\epsilon)Qt}$ with $\beta(\epsilon) = O(\epsilon)$, $B^*(0,\epsilon) = D(0,\epsilon)$ and $B^*(t,\epsilon)$ is $T(\epsilon)$-periodic.
Let $C_\epsilon^1 = D(0,\epsilon)$. We construct $B^*(t,\epsilon)$ and $\beta(\epsilon)$.
By the definitions of the projected flow $\widetilde{\Psi_1}$ and $\widetilde{\Psi}$, we have $\widetilde{\Psi_1}(t, C_\epsilon^1 x_1^Q,\epsilon) = x^\epsilon(t) = D(t,\epsilon)x_1^Q$ or $\Psi(t, C_\epsilon^1 \cdot SO(2),\epsilon) = \pi(\Psi(t,C_\epsilon^1,\epsilon)) = D(t,\epsilon)\cdot SO(2)$, which implies

$$\Psi(t,C_\epsilon^1,\epsilon) = D(t,\epsilon)e^{-\tau(t,\epsilon)Q} \text{ with } \tau(0,\epsilon) = 0. \tag{7.7.52}$$

Without loss of generality, we assume that $\tau(t,\epsilon)$ is a sufficiently smooth function with respect to t.
If we substitute (7.7.52) into the differential equations (7.4.2), we get

$$\dot{D} = -\left[X_0 + \epsilon\widetilde{g}(D,\epsilon)\right]D + \dot{\tau}DQ \Rightarrow \dot{\tau}DQ = D^{-1}\dot{D} + D^{-1}\left[X_0 + \epsilon\widetilde{g}(D,\epsilon)\right]D.$$

Therefore, the function $\tau(t,\epsilon)$ is sufficiently smooth in ϵ and in t. We define

$$C^{MRW}(t,\epsilon) \stackrel{\text{def}}{=} D(t,\epsilon)e^{-\tau(t,\epsilon)Q}e^{\beta(\epsilon)Qt}e^{-\beta(\epsilon)Qt}, \text{ where } \beta(\epsilon) = \frac{\tau(T(\epsilon),\epsilon)}{T(\epsilon)}.$$

Let us define $B^*(t,\epsilon) = D(t,\epsilon)e^{-\tau(t,\epsilon)Q}e^{\beta(\epsilon)Qt}$.
Then, $C^{MRW}(t,\epsilon) = B^*(t,\epsilon)e^{-\beta(\epsilon)Qt}$.
We have that $B^*(T(\epsilon),\epsilon) = D(T(\epsilon),\epsilon)e^{-\tau(T(\epsilon),\epsilon)Q}e^{\beta(\epsilon)T(\epsilon)Q} = D(0,\epsilon)e^{g(0,\epsilon)Q} = D(0,\epsilon) = B^*(0,\epsilon)$ by the definition of $\beta(\epsilon)$, relation (7.7.51) and the fact that $g(0,\epsilon) = 0$.
Since τ is sufficiently smooth in t and ϵ and $T(\epsilon)$ is sufficiently smooth, it follows from the definition of $\beta(\epsilon)$ that $\beta(\epsilon)$ is a sufficiently smooth function.
Thus, $\beta(\epsilon) = \frac{\tau(T(0),0)}{T(0)} + O(\epsilon)$.

We check that $B^*(t,\epsilon)$ is $T(\epsilon)$-periodic.
If we substitute $C = Be^{-\beta(\epsilon)Qt}$ into the differential equations (7.4.2), we get

$$\dot{B}e^{-\beta(\epsilon)Qt} + B(-\beta(\epsilon)Q)e^{-\beta(\epsilon)Qt} = -\left[X_0 + \epsilon\tilde{g}(B,\epsilon)\right]Be^{-\beta(\epsilon)Qt}$$

or

$$\dot{B} = \beta(\epsilon)BQ - \left[X_0 + \epsilon\tilde{g}(B,\epsilon)\right]B. \tag{7.7.53}$$

Since $C^{MRW}(t,\epsilon) = B^*(t,\epsilon)e^{-\beta(\epsilon)Qt}$ and $C^{MRW}(t,\epsilon)$ is a solution of the differential equations (7.4.2), we get that $B^*(t,\epsilon)$ is a solution of the differential equations (7.7.53). Since $B^*(T(\epsilon),\epsilon) = B^*(0,\epsilon)$, we get that $C^*(t,\epsilon) = B^*(t+T(\epsilon),\epsilon)$ is also a solution of the differential equations (7.7.53) such that $C^*(0,\epsilon) = B^*(0,\epsilon)$. Therefore, the function $B^*(t,\epsilon)$ is $T(\epsilon)$-periodic.

We check that $\beta(\epsilon) = O(\epsilon)$ up to $k\frac{2\pi}{T(\epsilon)}$, for some $k \in \mathbb{Z}$. Therefore, we show that $\tau(T(0),0) = g_1(T(0)) = 2k\pi$ for some $k \in \mathbb{Z}$.
By the definition of D we get that $D(t,0)x_1^Q = x^0(t) = x(t,x_0)e^{-X_0t}x_0$ or

$$D(t,0) = e^{-X_0t}D(0,0)e^{g_1(t)Q}, \tag{7.7.54}$$

with g_1 sufficiently smooth such that $g_1(0) = 0$.
Using (7.7.51), it follows that $D(t+T(0),0) = D(t,0)e^{g(t,0)Q}$. Since $g(0,0) = 0$, we get $D(T(0),0) = D(0,0) = e^{-X_0T(0)}D(0,0)e^{g_1(T(0))Q}$. Taking into account that $|X(0)|\,T(0) = 2\pi$, we get $e^{g_1(T(0))Q} = I_3$. It follows that $g_1(T(0)) = 2k\pi$ for some $k \in \mathbb{Z}$. Also,

$\Psi(t, C_0^1, 0) = D(t,0)e^{-\tau(t,0)Q} = e^{-X_0 t}D(0,0)e^{g_1(t)Q}e^{-\tau(t,0)Q}$ by (7.7.54), thus $\Psi(t, C_0^1, 0) = e^{-X_0 t}D(0,0)$. Then, we get $\tau(t,0) = g_1(t) + 2l(t)\pi$ with $l(t) \in \mathbb{Z}$. Since g_1 and τ are sufficiently smooth, we get that $l(t)$ is sufficiently smooth and, since $l(t) \in \mathbb{Z}$, we get $l(t)$ =constant. Since $\tau(0,0) = g_1(0) = 0$, we get $l = 0$.
Thus $\tau(t,0) = g_1(t)$ implies $\tau(T(0),0) = g_1(T(0)) = 2k\pi$.

Then, $\beta(0) = \frac{\tau(T(0),0)}{T(0)} = \frac{g_1(T(0))}{T(0)} = \frac{2k\pi}{T(0)}$.
Therefore, $\beta(\epsilon) = \frac{2k\pi}{T(0)} + O(\epsilon) = k\frac{2\pi}{T(\epsilon)} + \beta_1(\epsilon)$, where $\beta_1(\epsilon) = O(\epsilon)$.
We have $C^{MRW}(t,\epsilon) = B^*(t,\epsilon)e^{-\beta(\epsilon)Qt} = B^*(t,\epsilon)e^{-\frac{2\pi}{T(\epsilon)}kQt}e^{-\beta_1(\epsilon)Qt}$.
Let $B_1^*(t,\epsilon) = B^*(t,\epsilon)e^{-\frac{2\pi}{T(\epsilon)}kQt}$, $B_1^*(t,\epsilon)$ is $T(\epsilon)$-periodic and $B_1^*(0,\epsilon) = B^*(0,\epsilon)$.
We may drop the subscript 1 in B_1^* and $\beta_1(\epsilon)$.
The stability issue for $C^{MRW}(t,\epsilon)$ is proved in [11].
It is now easy to get the results for the differential equations (7.3.2) using the fact that $\Psi(C,t) = [\Phi(A,t)]^{-1}$, where $A = C^{-1}$.
This ends the proof of the third conclusion of Theorem 7.5.1 and its proof. □

Proof of Theorem 7.5.2. It is a consequence of Theorems 7.2.2, 7.5.1 and of the $SO(2)$-equivariance of the flow Φ.

□

Chapter 8

Numerical Analysis of the Spiral Waves Dynamics on $r\mathbb{S}^2$

8.1 Results of Numerical Analysis for the Transition from Rotating Waves to Modulated Rotating Waves

Let $L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$R_x(\theta) = e^{L_x\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$, $R_y(\theta) = e^{L_y\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$,

$R_z(\theta) = e^{L_z\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

In Chapter 5, we have seen that the study of the reaction-diffusion system (3.4.1) on $\mathbf{Y}^\alpha$ reduces to the study of the finite-dimensional system (5.3.1) on the center manifold $M^{cu}_{u0}(\lambda)$ of the relative equilibrium $SO(3)u_0$, with $\mathbf{\Phi}(t, u_0, 0) = e^{X_0 t}u_0$.

Let us denote $X_G(q, \lambda) = F^x(q, \lambda)L_x + F^y(q, \lambda)L_y + F^z(q, \lambda)L_z$ for any $q \in V_*$ and $|\lambda|$ small. If we parameterize $SO(3)$ by Euler angles, that is $A = R_z(\psi)R_x(\theta)R_z(\phi)$, where $\phi \in [0, 2\pi)$,

$\theta \in [0,\pi]$, $\psi \in [0,2\pi)$, the finite dimensional system (5.3.1) becomes

$$\begin{array}{rcl} \dot\phi &=& F^z(q,\lambda) - \cot\theta \left[F^y(q,\lambda)\cos\phi + F^x(q,\lambda)\sin\phi\right], \\ \dot\theta &=& -F^y(q,\lambda)\sin\phi + F^x(q,\lambda)\cos\phi, \\ \dot\psi &=& \frac{1}{\sin\theta}\left[F^y(q,\lambda)\cos\phi + F^x(q,\lambda)\sin\phi\right], \\ \dot q &=& X_N(q,\lambda). \end{array} \qquad (8.1.1)$$

We present the proof of (8.1.1):

If we parameterize $SO(3)$ by Euler angles, that is $A = R_z(\psi)R_x(\theta)R_z(\phi)$, where $\phi \in [0,2\pi)$, $\theta \in [0,\pi]$, $\psi \in [0,2\pi)$, and we substitute this into the equation $\dot A = AX_G(q,\lambda)$, we get

$$\dot\psi\left(\frac{dR_z}{d\psi}(\psi)\right)R_x(\theta)R_z(\phi) + \dot\theta R_z(\psi)\left(\frac{dR_x}{d\theta}(\theta)\right)R_z(\phi) + \dot\phi R_z(\psi)R_x(\theta)\left(\frac{dR_z}{d\phi}(\phi)\right)$$
$$= R_z(\psi)R_x(\theta)R_z(\phi)\left[F^x(q,\lambda)L_x + F^y(q,\lambda)L_y + F^z(q,\lambda)L_z\right] \quad (8.1.2)$$

or

$$\begin{array}{l} \dot\psi R_z(-\phi)R_x(-\theta)R_z(-\psi)\left(\frac{dR_z}{d\psi}(\psi)\right)R_z(\phi)R_x(\theta)R_z(\phi) \\ +\,\dot\theta R_z(-\phi)R_x(-\theta)\left(\frac{dR_x}{d\theta}(\theta)\right)R_z(\phi) + \dot\phi R_z(-\phi)\left(\frac{dR_z}{d\phi}(\phi)\right) \\ = F^x(q,\lambda)L_x + F^y(q,\lambda)L_y + F^z(q,\lambda)L_z. \end{array} \qquad (8.1.3)$$

By computation, it follows that

$$\begin{array}{rcl} R_z(-\phi)\left(\frac{dR_z}{d\phi}(\phi)\right) &=& L_z, \\ R_z(-\phi)R_x(-\theta)\left(\frac{dR_x}{d\theta}(\theta)\right)R_z(\phi) &=& (-\sin\phi)L_y + \cos\phi L_x, \\ R_z(-\phi)R_x(-\theta)R_z(-\psi)\left(\frac{dR_z}{d\psi}(\psi)\right)R_z(\phi)R_x(\theta)R_z(\phi) &=& \cos\theta L_z + \cos\phi\sin\theta L_y + \sin\phi\sin\theta L_x. \end{array} \qquad (8.1.4)$$

If we substitute (8.1.4) into (8.1.3), we get

$$\dot\psi(\cos\theta L_z + \cos\phi\sin\theta L_y + \sin\phi\sin\theta L_x) + \dot\theta((-\sin\phi)L_y + \cos\phi L_x) + \dot\phi L_z$$
$$= F^x(q,\lambda)L_x + F^y(q,\lambda)L_y + F^z(q,\lambda)L_z. \quad (8.1.5)$$

If we identify the corresponding coefficients of L_x, L_y and L_z from both sides of the equation (8.1.5), we get the system

$$\begin{array}{rcl} \dot\psi\cos\theta + \dot\phi &=& F^z(q,\lambda), \\ \dot\psi\cos\phi\sin\theta - \dot\theta\sin\phi &=& F^y(q,\lambda), \\ \dot\psi\sin\phi\sin\theta + \dot\theta\cos\phi &=& F^x(q,\lambda). \end{array} \qquad (8.1.6)$$

Then, by multiplying the second equation in (8.1.6) by $-\sin\phi$ and the third equation in (8.1.6) by $\cos\phi$ and adding them, we get $\dot{\theta} = -F^y(q,\lambda)\sin\phi + F^x(q,\lambda)\cos\phi$. Similarly, by multiplying the second equation in (8.1.6) by $\cos\phi$ and the third equation in (8.1.6) by $\sin\phi$ and adding them, we get $\dot{\psi} = \frac{1}{\sin\theta}\left[F^y(q,\lambda)\cos\phi + F^x(q,\lambda)\sin\phi\right]$. If we substitute $\dot{\psi}$ given by the above relation into the first equation of the system (8.1.6), we get $\dot{\phi} = F^z(q,\lambda) - \cot\theta\left[F^y(q,\lambda)\cos\phi + F^x(q,\lambda)\sin\phi\right]$.

We consider the following initial value problem associated with the system (8.1.1):

$$\begin{array}{rcl} \dot{\phi} &=& F^z(q,\lambda) - \cot\theta[F^y(q,\lambda)\cos\phi + F^x(q,\lambda)\sin\phi], \\ \dot{\theta} &=& -F^y(q,\lambda)\sin\phi + F^x(q,\lambda)\cos\phi, \\ \dot{\psi} &=& \frac{1}{\sin\theta}[F^y(q,\lambda)\cos\phi + F^x(q,\lambda)\sin\phi], \\ \phi(0) &=& 0, \\ \theta(0) &\neq& 0, \\ \psi(0) &=& 0, \\ \dot{q} &=& X_N(q,\lambda). \end{array} \tag{8.1.7}$$

We choose $\theta(0) \neq 0$ near 0 in (8.1.7).

We consider $V_* \simeq \mathbb{C}$. A supercritical Hopf bifurcation takes place in $\dot{q} = X_N(q,\lambda)$ at $q = 0$ for $\lambda = 0$ with eigenvalues $\pm i\omega_{bif}$. Let $q(t,\lambda)$ be the periodic solution of the second differential equation in (5.3.1), that appears by a supercritical Hopf bifurcation for $\lambda > 0$ small. Let $T(\lambda) = \frac{2\pi}{|\omega_\lambda|}$ be its period, where $\omega_\lambda = \omega_{bif} + O(\lambda)$ for $\lambda \geq 0$ small.

The normal form for $\dot{q} = X_N(q,\lambda)$ is $\dot{q} = [\alpha(\lambda) + i(\omega_{bif} + \beta(\lambda))]q + [c(\lambda) + id(\lambda)]q\,|q|^2$, where $\alpha(0) = \beta(0) = 0$, $c(0) < 0$, $\alpha'(0) > 0$. Since $q(t,\lambda) = \sqrt{-\frac{\alpha(\lambda)}{c(\lambda)}} = \left(\sqrt{\lambda}\sqrt{\frac{-\alpha'(0)}{c(0)}} + O(\lambda)\right)e^{i\omega_\lambda t}$ for $t \in \mathbb{R}$ and $\lambda > 0$ small, where $\omega_\lambda = \omega_{bif} + \beta(\lambda) - \frac{\lambda d(\lambda)\alpha(\lambda)}{c(\lambda)}$. The simplest case is when $\alpha(\lambda) = \lambda, \beta(\lambda) = 0$, $c(\lambda) = -1$ and $d(\lambda) = d \in \mathbb{R}$. Thus, $\omega_\lambda = \omega_{bif} + \lambda d$ and $q(t,\lambda) = \sqrt{\lambda}e^{i(\omega_{bif}+\lambda d)t}$.

Let $q(t,0) = 0$ for any $t \in \mathbb{R}$. If we substitute $q(t,\lambda)$ in $X_G(q,\lambda)$ and in the first three equations of the system (8.1.7), we get $X_G(q(t,\lambda),\lambda) = F^x(q(t,\lambda),\lambda)L_x + F^y(q(t,\lambda),\lambda)L_y + F^z(q(t,\lambda),\lambda)L_z = |X_0|\,X_0^1 + \lambda^{\frac{1}{2}}H(t,\lambda)$ for $\lambda \geq 0$ small and $t \in \mathbb{R}$. The following initial value problem is obtained

$$\begin{array}{lcl}\dot{\phi} & = & F^z(q(t,\lambda),\lambda) - \cot\theta[F^y(q(t,\lambda),\lambda)\cos\phi + F^x(q(t,\lambda),\lambda)\sin\phi], \\ \dot{\theta} & = & -F^y(q(t,\lambda),\lambda)\sin\phi + F^x(q(t,\lambda),\lambda)\cos\phi, \\ \dot{\psi} & = & \frac{1}{\sin\theta}[F^y(q(t,\lambda),\lambda)\cos\phi + F^x(q(t,\lambda),\lambda)\sin\phi], \\ \phi(0) & = & 0, \\ \theta(0) & \neq & 0, \\ \psi(0) & = & 0.\end{array} \tag{8.1.8}$$

We choose $\theta(0) \neq 0$ near 0 in (8.1.8).
We recall that $X^G(t,\lambda) = X_G(q(t,\lambda),\lambda)$ and, if we write $F^{xx}(t,\lambda) = F^x(q(t,\lambda),\lambda)$, $F^{yy}(t,\lambda) = F^y(q(t,\lambda),\lambda)$ and $F^{zz}(t,\lambda) = F^z(q(t,\lambda),\lambda)$, we get that $X^G(t,\lambda) = F^{xx}(t,\lambda)L_x + F^{yy}(t,\lambda)L_y + F^{zz}(t,\lambda)L_z$ is $\frac{2\pi}{|\omega_\lambda|}$-periodic.
We use a Maple program that integrates numerically the system (8.1.8) and finds numerically a primary frequency vector (if its norm is not 0 or π) of the modulated rotating wave that appears by a supercritical Hopf bifurcation, as discussed in Chapter 6. Then, for a choice of the point $x_0 \in r\mathbf{S}^2$ near $\frac{r}{|X_0|}\overrightarrow{X}_0$, we represent the tip motion

$$x_{tip}(\mathbf{\Phi}(t,u_\lambda,\lambda)) = A(0,\lambda)^{-1}A(t,\lambda)x_0 \text{ on } r\mathbf{S}^2 \tag{8.1.9}$$

for $\lambda \geq 0$ small (see Chapter 4, Section 4.3). Also, we have $\mathbf{\Phi}(t,u_\lambda,\lambda) = A(0,\lambda)^{-1}A(t,\lambda)\Psi(q(t,\lambda))$.
We consider the following cases:

Case 1. $X(t,\lambda)$ is given in Example (6.4.1) from Section 6.4; $X_0^1 = L_z, X_1 = L_x, X_2 = L_y$;

$$\omega_{bif} = 20,\ |X_0| = 2,\ g(t,\lambda) = sin((\omega_{bif}+\lambda)t),\ r = 3,\ \theta(0) = 0.01,\ x_0 = \begin{pmatrix} 0 \\ 0.92 \\ 2.85 \end{pmatrix}.$$

We get Figures 8.1.1 to 8.1.3 for $\lambda = 0.01,\ 0.05$.
Since $|X_0| \neq k\omega_{bif}$ for any $k \in \mathbb{Z}$, we have the nonresonant case. On Figures 8.1.1 and 8.1.3, it is visualized the tip motion given by (8.1.9) of the modulated rotating waves $\mathbf{\Phi}(t,u_\lambda,\lambda)$ obtained by a supercritical Hopf bifurcation that takes place in $\dot{q} = X_N(q,\lambda)$ at $q = 0$ for $\lambda = 0$. On Figure 8.1.1 we consider $\lambda = 0.01$. On Figure 8.1.3 we consider $\lambda = 0.05$.
On Figure 8.1.2 we plot the points $x_{tip}(\mathbf{\Phi}(i\frac{2\pi}{\omega_\lambda},u_\lambda,\lambda))$ for some $i = 0, 1, 2, \ldots$ and we can see that they are points of a circle on the sphere $r\mathbf{S}^2$ with the center on the line having the direction of the primary frequency vector of $\mathbf{\Phi}(t,u_\lambda,\lambda)$, where $\lambda = 0.01$.

The grey line is the line containing the frequency of the rotating wave undergoing the Hopf bifurcation, $\overrightarrow{X_0}$. The black lines are the lines corresponding to the primary frequency vector associated to the modulated rotating wave. One of them is computed numerically and the other is the exact one. We can see that they are very close.

Case 2. $X(t,\lambda)$ is given in Example (6.4.2) from Section 6.4; $X_0^1 = L_z, X_1 = L_x, X_2 = L_y$;

$$\omega_{bif} = |X_0| = 20,\ g(t,\lambda) = sin((\omega_{bif}+\lambda)t),\ r = 3,\ \theta(0) = 0.02,\ x_0 = \begin{pmatrix} 0 \\ 0.92 \\ 2.85 \end{pmatrix}.$$

We get Figures 8.1.4 to 8.1.6 for $\lambda = 0.05, 0.1$.

Since $|X_0| = k\omega_{bif}$ for $k = 1 \in \mathbb{Z}$, we have the resonant case. On Figures 8.1.4 and 8.1.6, it is visualized the tip motion given by (8.1.9) of the modulated rotating waves $\Phi(t, u_\lambda, \lambda)$ obtained by a supercritical Hopf bifurcation that takes place in $\dot{q} = X_N(q,\lambda)$ at $q = 0$ for $\lambda = 0$. On Figure 8.1.4 we consider $\lambda = 0.05$. On Figure 8.1.6 we consider $\lambda = 0.1$. We can see that the primary frequency vectors associated to the modulated rotating waves are orthogonal to $\overrightarrow{X_0}$. We call this phenomenon resonant drift.

On Figure 8.1.5 we plot the points $x_{tip}(\Phi(i\frac{2\pi}{\omega_\lambda}, u_\lambda, \lambda))$ for some $i = 0, 1, 2, \ldots$ and we can see that they are points of a circle on the sphere $r\mathbf{S}^2$ with the center on the line having the direction of the primary frequency vector of $\Phi(t, u_\lambda, \lambda)$, where $\lambda = 0.05$.

The grey line is the line containing the frequency of the rotating wave undergoing the Hopf bifurcation, $\overrightarrow{X_0}$. The black line is the line corresponding to the primary frequency vector associated to the modulated rotating wave.

Case 3. $X(t,\lambda)$ is given in Example (6.4.3) from Section 6.4; $X_0^1 = L_z, X_1 = L_x, X_2 = L_y$;

$$\omega_{bif} = |X_0| = 20,\ g(t,\lambda) = sin((\omega_{bif}+\lambda)t),\ r = 3,\ \theta(0) = 0.5,\ x_0 = \begin{pmatrix} 0.44 \\ 0.14 \\ 2.96 \end{pmatrix}.$$

We get Figures 8.1.7 to 8.1.9 for $\lambda = 0.05, 0.25$.

Since $|X_0| = k\omega_{bif}$ for $k = 1 \in \mathbb{Z}$, we have the resonant case. On Figures 8.1.7 and 8.1.9, it is visualized the tip motion given by (8.1.9) of the modulated rotating waves $\Phi(t, u_\lambda, \lambda)$ obtained by a supercritical Hopf bifurcation that takes place in $\dot{q} = X_N(q,\lambda)$ at $q = 0$ for $\lambda = 0$. On Figure 8.1.7 we consider $\lambda = 0.05$. On Figure 8.1.9 we consider $\lambda = 0.25$. We can see that the primary frequency vectors associated to the modulated rotating waves are not orthogonal to $\overrightarrow{X_0}$. This can happen in the

resonant case if we consider only one parameter λ.
On Figure 8.1.8 we plot the points $x_{tip}(\Phi(i\frac{2\pi}{\omega_\lambda}, u_\lambda, \lambda))$ for some $i = 0, 1, 2, \ldots$ and we can see that they are points of a circle on the sphere $r\mathbf{S}^2$ with the center on the line having the direction of the primary frequency vector of $\Phi(t, u_\lambda, \lambda)$, where $\lambda = 0.05$. The grey line is the line containing the frequency of the rotating wave undergoing the Hopf bifurcation, $\overrightarrow{X_0}$. The black line is the line corresponding to the primary frequency vector associated to the modulated rotating wave.

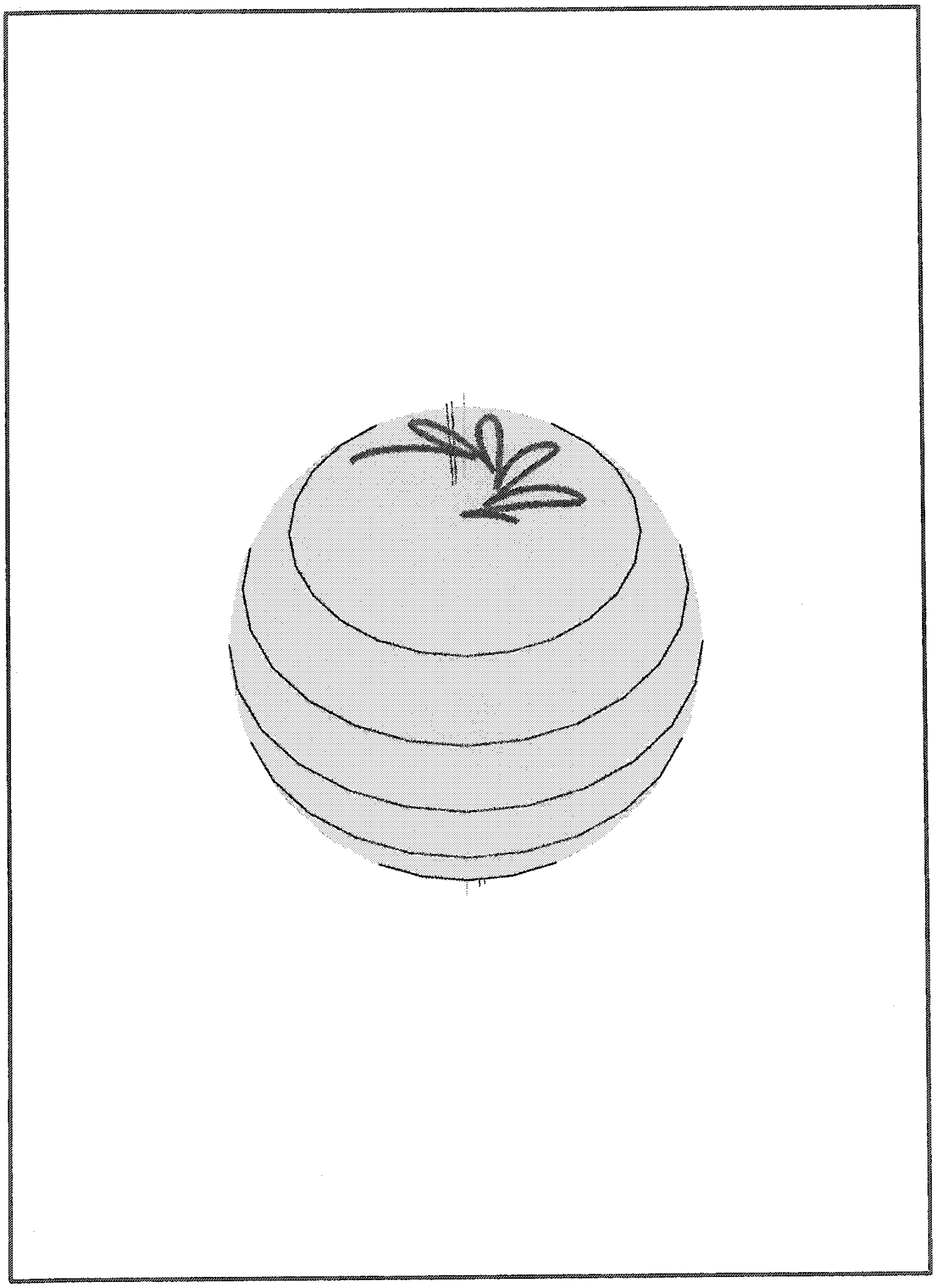

Figure 8.1.1: Case 1, $\lambda = 0.01$, $x_{tip}(\Phi(t, u_{0.01}, 0.01))$

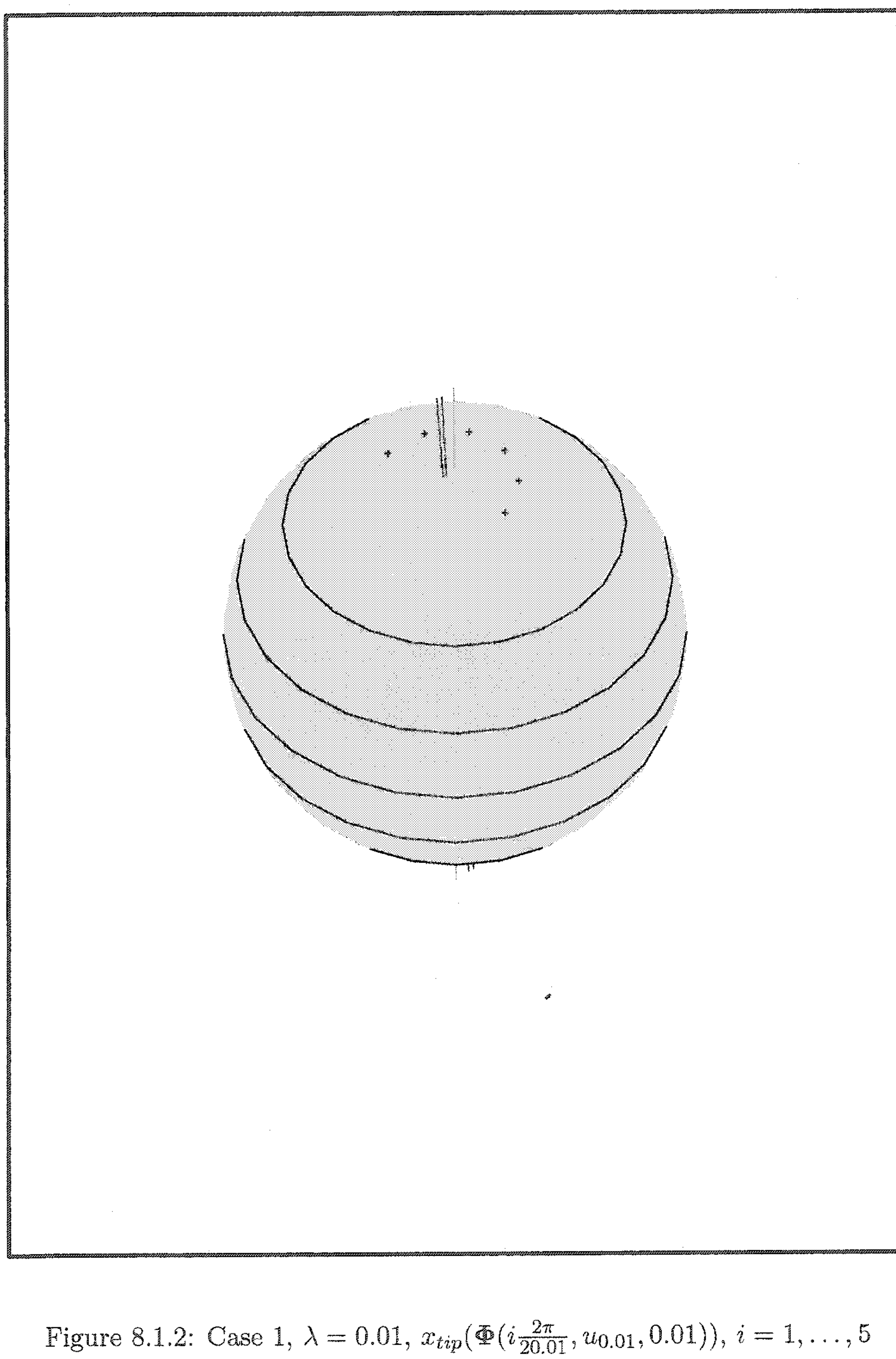

Figure 8.1.2: Case 1, $\lambda = 0.01$, $x_{tip}(\Phi(i\frac{2\pi}{20.01}, u_{0.01}, 0.01))$, $i = 1, \ldots, 5$

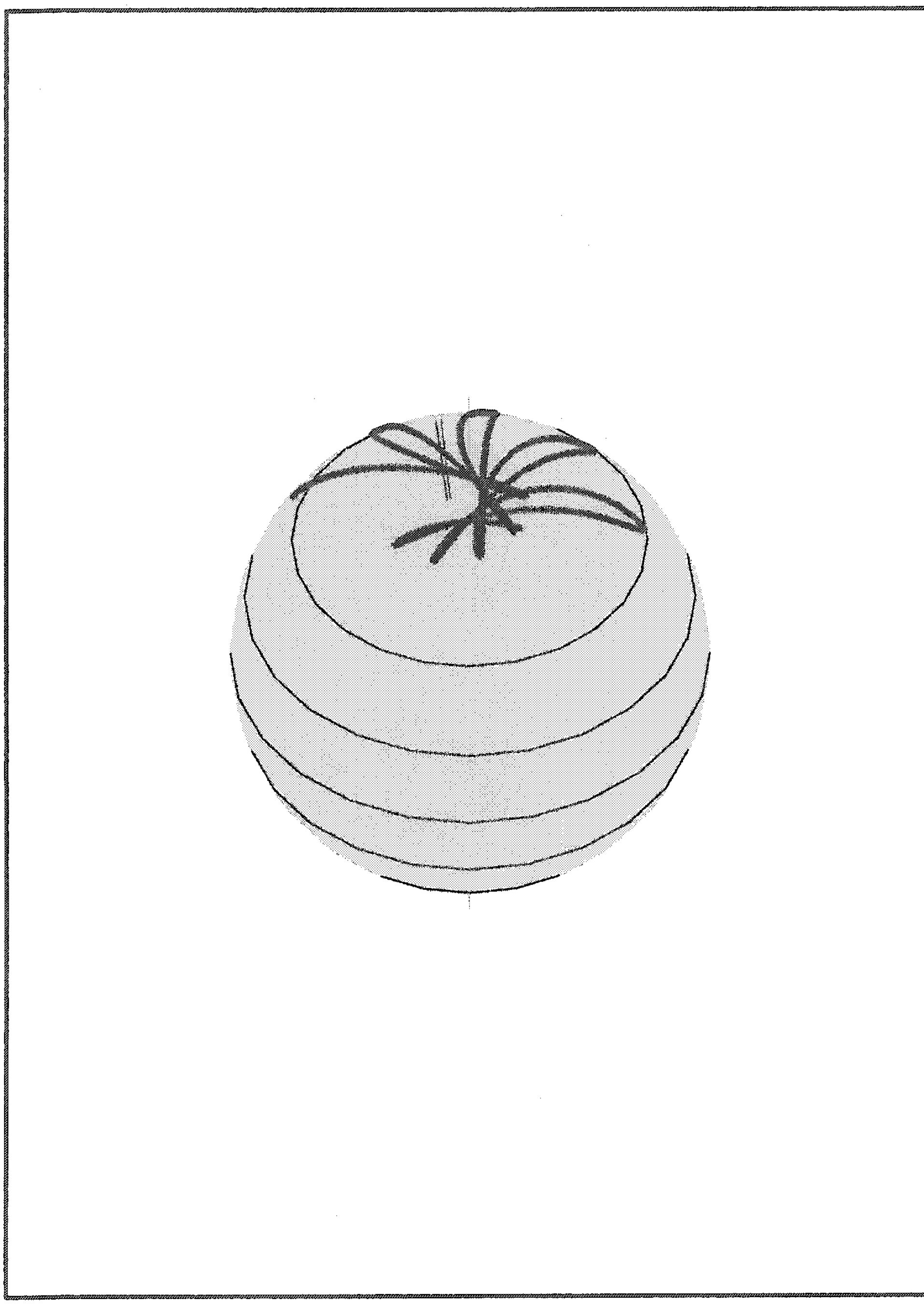

Figure 8.1.3: Case 1, $\lambda = 0.05$, $x_{tip}(\Phi(t, u_{0.05}, 0.05))$

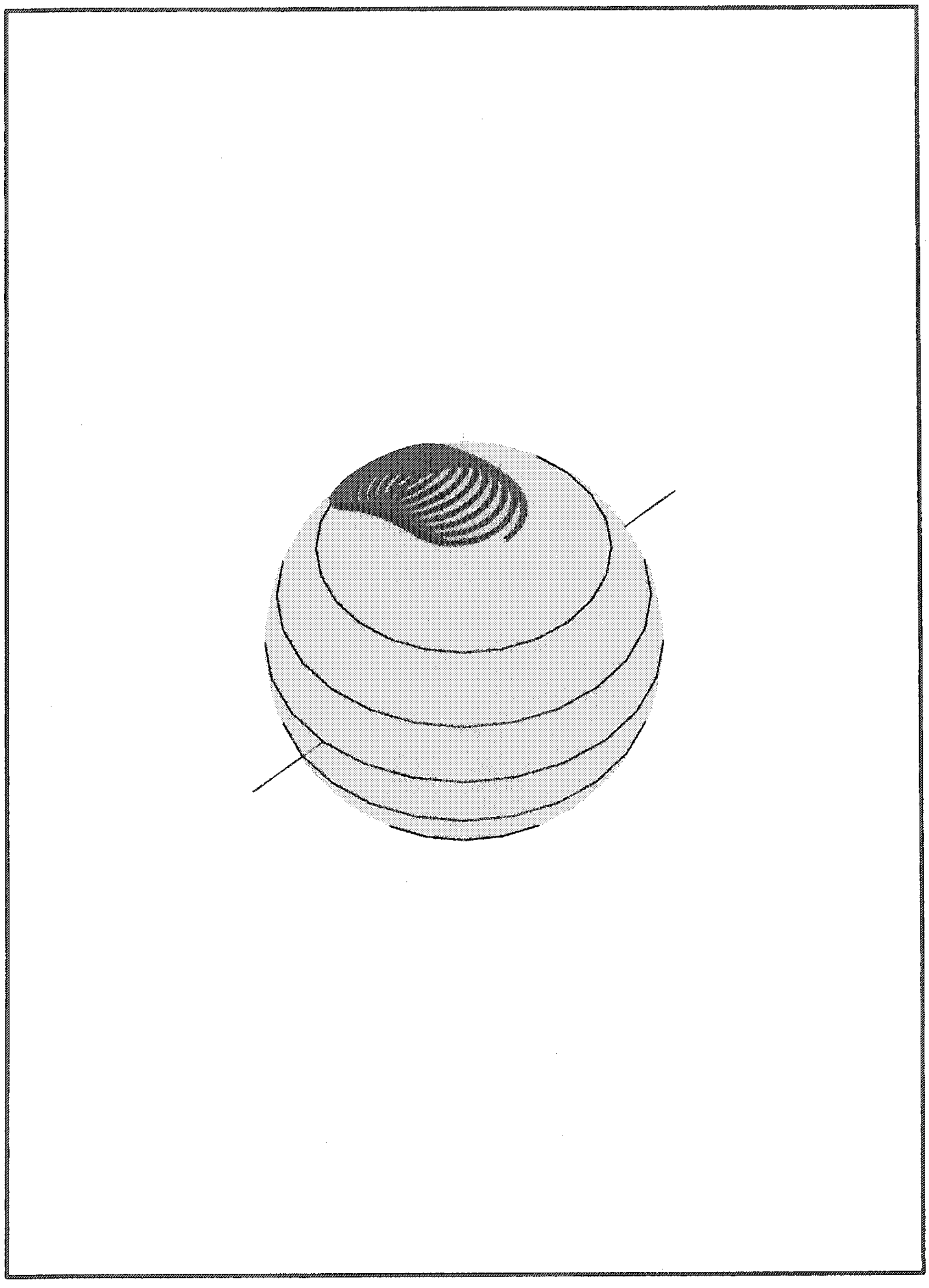

Figure 8.1.4: Case 2, $\lambda = 0.05$, $x_{tip}(\Phi(t, u_{0.05}, 0.05))$

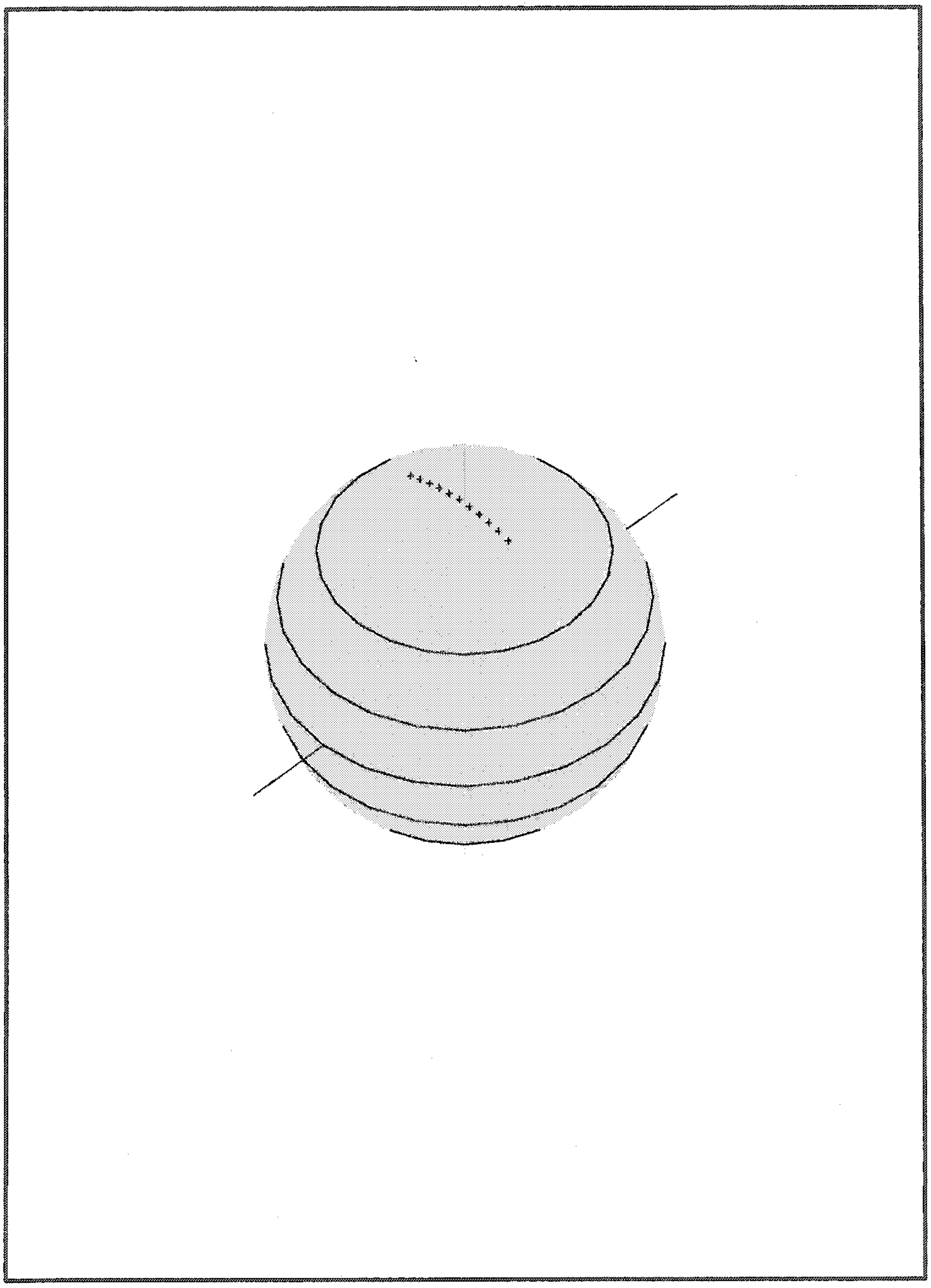

Figure 8.1.5: Case 2, $\lambda = 0.05$, $x_{tip}(\Phi(i\frac{2\pi}{20.05}, u_{0.05}, 0.05))$, $i = 1, \ldots, 10$

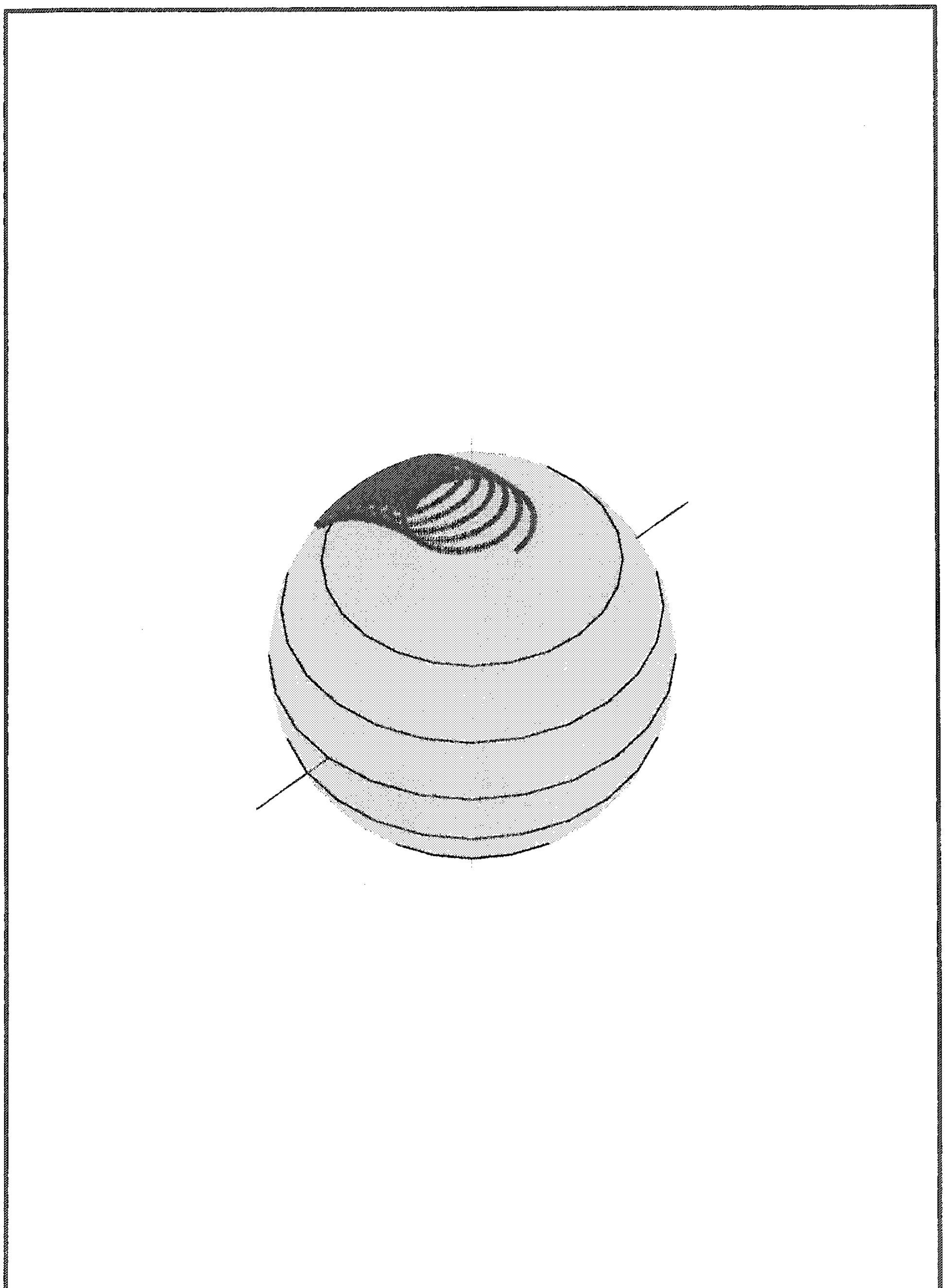

Figure 8.1.6: Case 2, $\lambda = 0.1$, $x_{tip}(\Phi(t, u_{0.1}, 0.1))$

Figure 8.1.7: Case 3, $\lambda = 0.05$, $x_{tip}(\Phi(t, u_{0.05}, 0.05))$

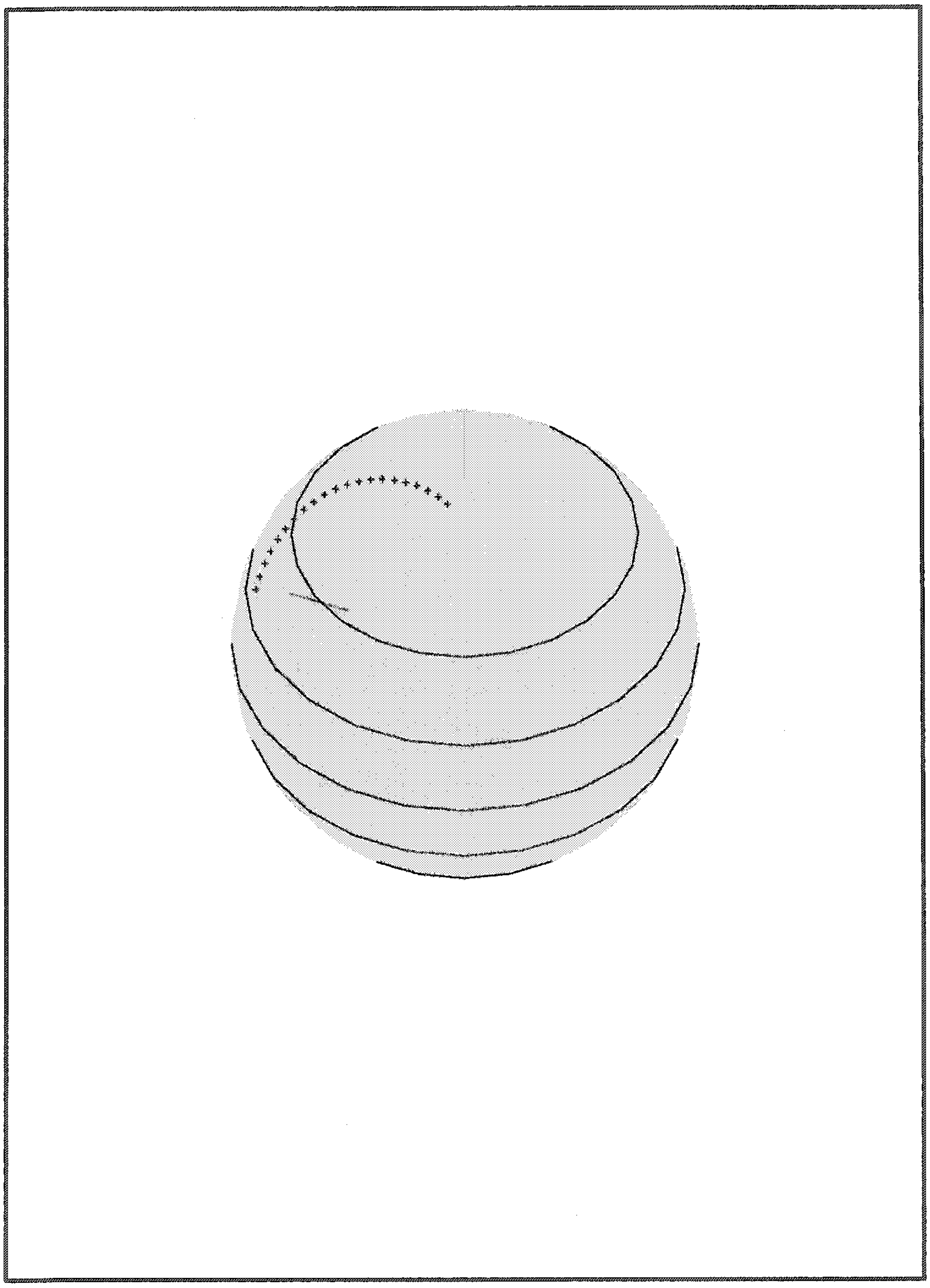

Figure 8.1.8: Case 3, $\lambda = 0.05$, $x_{tip}(\Phi(i\frac{2\pi}{20.05}, u_{0.05}, 0.05))$, $i = 1, \ldots, 20$

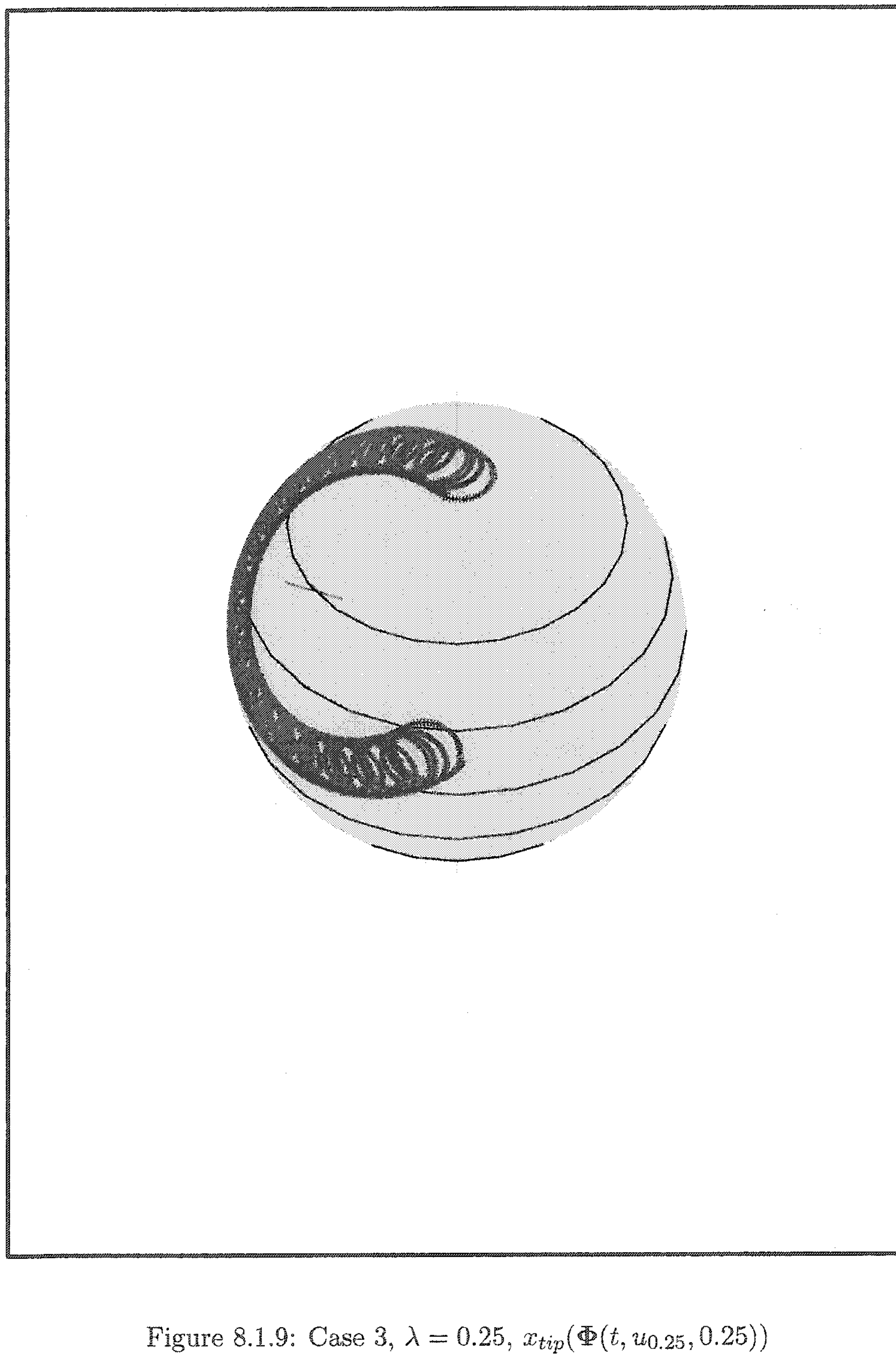

Figure 8.1.9: Case 3, $\lambda = 0.25$, $x_{tip}(\Phi(t, u_{0.25}, 0.25))$

Chapter 9

Conclusions

In this thesis a dynamical systems analysis of the dynamics and bifurcations of spiral waves in excitable media with spherical or approximately spherical geometry was undertaken. First we consider a parameter-dependent system of reaction-diffusion partial differential equations on a sphere that is equivariant under the group $SO(3)$ of all rigid rotations on a sphere. Two main types of spatial-temporal patterns that can be defined in such systems are rotating waves (stationary in a co-rotating frame) and modulated rotating waves (periodic in a co-rotating frame). These give us relative equilibria and relative periodic orbits. We have studied the transition from rotating waves to modulated rotating waves on spherical domains via Hopf bifurcation and studied the case of resonance between the critical eigenvalue leading to Hopf bifurcation and the primary frequency vector of the rotating wave undergoing the bifurcation. In the second part of the thesis, in order to characterize the effects of localized inhomogeneities (e.g. small blood vessels) inherent in cardiac tissue and the fact that the shape of the heart is not exactly a sphere, we have studied the effects of forced symmetry-breaking from $SO(3)$ to $SO(2)$ for a normally hyperbolic $SO(3)$ group orbit of a rotating wave on a sphere. This was done by introducing a small $SO(2)$-equivariant perturbation in the $SO(3)$-equivariant reaction-diffusion system considered above and studying the $SO(2)$-equivariant perturbed dynamics on the $SO(2)$-invariant normally hyperbolic manifold that persists under this perturbation. The results of this work may give a mathematical explanation of some of the phenomena observed in experiments (numerical or physical) on spiral waves on spherical or approximately spherical domains.

The study of dynamics and bifurcations of spiral waves on spherical or approximately spherical domains is important, because it may provide a clue to the cause of cardiac arrhythmia that can lead to ventricular fibrillation (the surface of the heart can be approximated by a sphere, that is a non-planar surface).

The tools that we have used to investigate these phenomena are highly interdisciplinary, drawing from dynamical systems theory, bifurcation theory, group theory, functional analysis, semilinear parabolic differential equations theory to the theory of Lie groups and Lie algebras, representation theory, equivariant dynamical systems, equivariant bifurcation theory, forced symmetry-breaking, orbit space reduction.

Some further problems related to the subject of this thesis are:

- forced symmetry-breaking from $SO(3)$ to $SO(2)$ for modulated rotating waves on spherical domains as LeBlanc and Wulff did for the planar modulated rotating waves (see [48]). The tools to solve this problem will be similar to those used in my doctoral thesis. In addition, the averaging methods may be necessary. The study of modulated rotating waves on a sphere will be complicated by the extra frequency that appears.

- the forced symmetry-breaking from $SO(3)$ taking into account two or more localized inhomogeneities for both rotating and modulated rotating waves on spherical domains.

- it is important to try to extend the study done in my doctoral thesis for spiral waves on a sphere to the case of a "breathing" sphere that has the radius periodically changing with time. The case of a breathing sphere is important because of the contractible function of the cardiac muscle. As Abramychev, Davydov and Zykov pointed out in [1], this leads to the drift of the spiral wave on the sphere, which is similar to the resonance of a spiral wave on the plane.

- Scroll waves are another type of spatio-temporal patterns which can appear in cardiac tissue, leading to cardiac arrhythmia and ventricular fibrillation. Because the ventricular muscle of the heart is thick, more realistic anatomical models of the heart are proposed. Scroll waves are the three-dimensional analog of spiral waves of electrical activity in thick cardiac muscle. Researchers such as Barkley, Biktashev, Chavez, Davidsen, Keener, Kapral, Glass, Winfree (see [7], [10], [44], [52], [82]) and many others have studied scroll waves by the means of numerical simulations, physical experiments and kinematical theory . Winfree introduced the concept of scroll waves to describe the three-dimensional rotating waves that, in cross-section, appear as a spiral. Some mathematical studies of scroll waves have been done by Fiedler, Mantel, Wulff and al. (see [19], [20], [70]). Therefore, the study of the possible types of scroll waves, their organization and their dynamics as a function of the geometry of the excitable medium in which waves propagate, namely in the spherical annulus and in the three dimensional space, is an important future undertaking.

Appendix A

Baker-Campbell-Hausdorff Formula in $so(3)$

Proof of Theorem 2.3.4. We treat the case where $X \neq O_3$, $Y \neq O_3$.
The formulae for the case $(e^X e^Y)^2 \neq I_3$ follow from [17] and the fact that $a = a_1 e$, $b = b_1 e$, $c = c_1 e$, $d = d_1 e$, taking into account that $d = d_1 e = \left|\frac{e^X e^Y - e^{-Y} e^{-X}}{2}\right| \neq 0$.
Here we prove the BCH formula in $so(3)$ in the case $(e^X e^Y)^2 = I_3$, $e^X e^Y \neq I_3$.
Let us denote $A = e^X e^Y = e^B$. Since $A \neq I_3$ and $A^2 = I_3$, then $|B| = (2k+1)\pi$, for some $k \in \mathbb{Z}$, $k \geq 0$. Using the Rodrigues' formula, we get $B^2 = [(2k+1)\pi]^2 \frac{A - I_3}{2}$. Then $BCH(X,Y) = [B]$ with $k = 0$. We have to prove that $B = \alpha X + \beta Y + \gamma[X,Y]$ for α, β, γ given in Theorem 2.3.4 for the case $(e^X e^Y)^2 = I_3$, $e^X e^Y \neq I_3$ and show that BCH formula in $so(3)$ is smooth.
Then $A = \frac{2}{\pi^2}(\alpha X + \beta Y + \gamma[X,Y])^2 + I_3$, that is

$$\begin{aligned} A = I_3 + \frac{2}{\pi^2}\left[\alpha^2 X^2 + \beta^2 Y^2 + \gamma^2 [X,Y]^2 + \alpha\beta(XY + YX)\right. \\ \left. + \alpha\gamma(X[X,Y] + [X,Y]X) + \gamma\beta(Y[X,Y] + [X,Y]Y)\right]. \end{aligned} \tag{A.0.10}$$

We have

$$X[X,Y] + [X,Y]X = X^2 Y - Y X^2, \tag{A.0.11}$$

$$Y[X,Y] + [X,Y]Y = -Y^2 X + X Y^2, \tag{A.0.12}$$

$$([X,Y])^2 = XYXY + YXYX - XY^2X - YX^2Y, \tag{A.0.13}$$

$$\begin{aligned} XY^2X + YX^2Y &= X(Y^2X + XY^2) + Y(X^2Y + YX^2) \\ &- X^2Y^2 - Y^2X^2. \end{aligned} \tag{A.0.14}$$

Also (see [17]), we have

$$\begin{aligned} (XY)^2 &= -(\vec{X}^T\vec{Y})XY, && \text{(A.0.15)}\\ (YX)^2 &= -(\vec{Y}^T\vec{X})YX, && \text{(A.0.16)}\\ X^2Y+YX^2 &= -|X|^2Y-(\vec{X}^T\vec{Y})X, && \text{(A.0.17)}\\ Y^2X+XY^2 &= -|Y|^2X-(\vec{Y}^T\vec{X})Y. && \text{(A.0.18)} \end{aligned}$$

Therefore, if we substitute (A.0.11)-(A.0.14) and (A.0.15)-(A.0.18) into (A.0.10), we get

$$\begin{aligned} A = \frac{2}{\pi^2}\Big[&(\alpha^2+|Y|^2\gamma^2)X^2+(\beta^2+|X|^2\gamma^2)Y^2\\ &+\alpha\beta(XY+YX)+\alpha\gamma(X^2Y-YX^2)\\ &+\gamma\beta(-Y^2X+XY^2)+\gamma^2(X^2Y^2+Y^2X^2)\Big]+I_3. \end{aligned} \tag{A.0.19}$$

Also, $A = e^Xe^Y = A^T = e^{-Y}e^{-X}$ and, using the Rodrigues' formulae for e^X, e^Y, e^{-X}, e^{-Y}, we get

$$\begin{aligned} A = I_3 &+ \frac{\sin^2\frac{|X|}{2}}{|X|^2}X^2+\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}Y^2+\frac{\sin|X|}{|X|}X+\frac{\sin|Y|}{|Y|}Y\\ &+\frac{\sin|X|}{|X|}\frac{\sin|Y|}{|Y|}XY+4\frac{\sin^2\frac{|X|}{2}}{|X|^2}\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}X^2Y^2\\ &+2\frac{\sin|X|}{|X|}\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}XY^2+2\frac{\sin|Y|}{|Y|}\frac{\sin^2\frac{|X|}{2}}{|X|^2}X^2Y \end{aligned} \tag{A.0.20}$$

and

$$\begin{aligned} A = I_3 &+ \frac{\sin^2\frac{|Y|}{2}}{|Y|^2}Y^2+\frac{\sin^2\frac{|X|}{2}}{|X|^2}X^2-\frac{\sin|Y|}{|Y|}Y-\frac{\sin|X|}{|X|}X\\ &+\frac{\sin|Y|}{|Y|}\frac{\sin|X|}{|X|}YX+4\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}\frac{\sin^2\frac{|X|}{2}}{|X|^2}Y^2X^2\\ &-2\frac{\sin|Y|}{|Y|}\frac{\sin^2\frac{|X|}{2}}{|X|^2}YX^2-2\frac{\sin|X|}{|X|}\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}Y^2X. \end{aligned} \tag{A.0.21}$$

If we add the relations (A.0.20) and (A.0.21), then we get

$$\begin{aligned} A = I_3 &+ 2\frac{\sin^2\frac{|X|}{2}}{|X|^2}X^2+2\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}Y^2\\ &+\frac{1}{2}\frac{\sin|X|}{|X|}\frac{\sin|Y|}{|Y|}(XY+YX)+2\frac{\sin^2\frac{|X|}{2}}{|X|^2}\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}(X^2Y^2+Y^2X^2)\\ &+\frac{\sin|X|}{|X|}\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}(XY^2-Y^2X)+\frac{\sin|Y|}{|Y|}\frac{\sin^2\frac{|X|}{2}}{|X|^2}(X^2Y-YX^2). \end{aligned} \tag{A.0.22}$$

Comparing (A.0.19) and (A.0.22) we get

$$\begin{aligned} \frac{2}{\pi^2}\gamma^2 &= 2\frac{\sin^2\frac{|X|}{2}}{|X|^2}\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}, && \text{(A.0.23)}\\ \frac{2}{\pi^2}(\alpha^2+|Y|^2\gamma^2) &= 2\frac{\sin^2\frac{|X|}{2}}{|X|^2}, && \text{(A.0.24)}\\ \frac{2}{\pi^2}(\beta^2+|X|^2\gamma^2) &= 2\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}, && \text{(A.0.25)}\\ \frac{2}{\pi^2}\alpha\beta &= \frac{1}{2}\frac{\sin|X|}{|X|}\frac{\sin|Y|}{|Y|}, && \text{(A.0.26)}\\ \frac{2}{\pi^2}\beta\gamma &= \frac{\sin|X|}{|X|}\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}, && \text{(A.0.27)}\\ \frac{2}{\pi^2}\alpha\gamma &= \frac{\sin|Y|}{|Y|}\frac{\sin^2\frac{|X|}{2}}{|X|^2}. && \text{(A.0.28)} \end{aligned}$$

Also, it is true that (see [17])

$$\begin{aligned} \vec{X}^T\vec{Y} &= \vec{Y}^T\vec{X} = |X|\,|Y|\cos(\angle(\vec{X},\vec{Y})), && \text{(A.0.29)}\\ -Y^2X^2+X^2Y^2 &= -(\vec{X}^T\vec{Y})[X,Y]. && \text{(A.0.30)} \end{aligned}$$

If we substract the relation (A.0.21) from the relation (A.0.20), using the relations (A.0.17), (A.0.18), (A.0.29) and (A.0.30), we also get

$$\begin{aligned} &2\frac{\sin|X|}{|X|}X+2\frac{\sin|Y|}{|Y|}Y+\frac{\sin|X|}{|X|}\frac{\sin|Y|}{|Y|}[X,Y]\\ &-4\frac{\sin^2\frac{|X|}{2}}{|X|^2}\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}(\vec{X}^T\vec{Y})[X,Y]+2\frac{\sin|X|}{|X|}\frac{\sin^2\frac{|Y|}{2}}{|Y|^2}\left[-|Y|^2X-(\vec{Y}^T\vec{X})Y\right] \qquad \text{(A.0.31)}\\ &+2\frac{\sin|Y|}{|Y|}\frac{\sin^2\frac{|X|}{2}}{|X|^2}\left[-|X|^2Y-(\vec{X}^T\vec{Y})X\right]=0 \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} &\left[2\frac{\sin|X|}{|X|}-2\sin|X|\frac{\sin^2\frac{|Y|}{2}}{|X|}-2\sin|Y|\frac{\sin^2\frac{|X|}{2}}{|X|}\cos(\angle(\vec{X},\vec{Y}))\right]X\\ &+\left[2\frac{\sin|Y|}{|Y|}-2\sin|Y|\frac{\sin^2\frac{|X|}{2}}{|Y|}-2\sin|X|\frac{\sin^2\frac{|Y|}{2}}{|Y|}\cos(\angle(\vec{X},\vec{Y}))\right]Y \qquad \text{(A.0.32)}\\ &+\left[\frac{\sin|X|}{|X|}\frac{\sin|Y|}{|Y|}-4\frac{\sin^2\frac{|X|}{2}}{|X|}\frac{\sin^2\frac{|Y|}{2}}{|Y|}\cos(\angle(\vec{X},\vec{Y})\right][X,Y]=0 \end{aligned}$$

$\Longrightarrow$

$$2\left(\cos\frac{|X|}{2}\cos\frac{|Y|}{2}-\sin\frac{|X|}{2}\sin\frac{|Y|}{2}\cos(\angle(\vec{X},\vec{Y}))\right)$$
$$\cdot\left[\frac{1}{|X|}\sin\frac{|X|}{2}\cos\frac{|Y|}{2}X+\frac{1}{|Y|}\sin\frac{|Y|}{2}\cos\frac{|X|}{2}Y+2\frac{1}{|X||Y|}\sin\frac{|X|}{2}\sin\frac{|Y|}{2}[X,Y]\right]=0 \tag{A.0.33}$$

From this, we get $\cos\frac{|X|}{2}\cos\frac{|Y|}{2}=\sin\frac{|X|}{2}\sin\frac{|Y|}{2}\cos(\angle(\vec{X},\vec{Y}))$, because we can not have $\sin\frac{|X|}{2}=0$, $\sin\frac{|Y|}{2}=0$ (that would imply $A=I_3$) if $\vec{X}$, $\vec{Y}$ are linearly independent or $\sin\frac{|X|(1+a_2)}{2}=0$ if $\vec{Y}=a_2\vec{X}$ with $a_2\in\mathbb{R}$ (that would imply $A=I_3$).
We also have that $\cos\frac{|X|}{2}\cos\frac{|Y|}{2}\neq\sin\frac{|X|}{2}\sin\frac{|Y|}{2}\cos(\angle(\vec{X},\vec{Y}))$ for any $(X,Y)\in so(3)\times so(3)$ such that $(e^Xe^Y)^2\neq I_3$ or $e^Xe^Y=I_3$.
We choose the formulae for α,β,γ from the relations (A.0.23)-(A.0.28) independent of the sign of $\cos\frac{|X|}{2}\cos\frac{|Y|}{2}-\sin\frac{|X|}{2}\sin\frac{|Y|}{2}\cos(\angle(\vec{X},\vec{Y}))$ for $(X,Y)\in so(3)\times so(3)$ such that $(e^Xe^Y)^2\neq I_3$; we take $\alpha=\pi\sin\frac{|X|}{2}\cos\frac{|Y|}{2}$, $\beta=\pi\sin\frac{|Y|}{2}\cos\frac{|X|}{2}$, $\gamma=\pi\sin\frac{|X|}{2}\sin\frac{|Y|}{2}$.
The formulae for the case $e^Xe^Y=I_3$ result by computation. Also, in this case we have $d=0$, $d_1=0$ and $e\neq 0$.
The formulae when at least one of X or Y is O_3 are obtained by taking the limit to O_3 in the formulae for the case $X\neq O_3$, $Y\neq O_3$.

Remark A.0.1. *There exist $(X_1,Y_1)\in so(3)\times so(3)$ such that $(e^{X_1}e^{Y_1})^2\neq I_3$, $\cos\frac{|X_1|}{2}\cos\frac{|Y_1|}{2}<\sin\frac{|X_1|}{2}\sin\frac{|Y_1|}{2}\cos(\angle(\vec{X}_1,\vec{Y}_1))$ and $(X_2,Y_2)\in so(3)\times so(3)$ such that $(e^{X_2}e^{Y_2})^2\neq I_3$ and $\cos\frac{|X_2|}{2}\cos\frac{|Y_2|}{2}>\sin\frac{|X_2|}{2}\sin\frac{|Y_2|}{2}\cos(\angle(\vec{X}_2,\vec{Y}_2))$.*
Example:
Let $X_1=Y_1$ be such that $0<\cos|X_1|<\frac{1}{\sqrt{2}}$ and $X_2=Y_2$ be such that $0>\cos|X_2|>-\frac{1}{\sqrt{2}}$.

Proof of the smoothness of BCH formula in $so(3)$
We have $BCH(X,Y)=[\alpha(X,Y)X+\beta(X,Y)Y+\gamma(X,Y)[X,Y]]$. We prove that $|\alpha|$ is continuous and α^2 is smooth on $so(3)\times so(3)$. The same can be done for β and γ.
Let us define the following two smooth functions:

$$f(y)=\begin{cases}\frac{\arccos(y)}{\sin(\arccos(y))}=\frac{\arccos(y)}{\sqrt{1-y^2}} & \text{if } y\in(-1,1);\\ 1 & \text{if } y=1,\end{cases}$$

$$g_\alpha(X,Y)=\begin{cases}\frac{\alpha}{|X|} & \text{if } X\neq O_3;\\ \cos^2\frac{|Y|}{2} & \text{if } X=O_3.\end{cases}$$

Recall that

$e = \cos\frac{|X|}{2}\cos\frac{|Y|}{2} - \sin\frac{|X|}{2}\sin\frac{|Y|}{2}\cos(\angle(\vec{X},\vec{Y}))$,
$a_1 = \sin\frac{|X|}{2}\cos\frac{|Y|}{2}$, $a = a_1 e$,
$b_1 = \sin\frac{|Y|}{2}\cos\frac{|X|}{2}$, $b = b_1 e$,
$c_1 = \sin\frac{|X|}{2}\sin\frac{|Y|}{2}$, $c = c_1 e$,
$d_1 = \sqrt{a_1^2 + b_1^2 + 2a_1 b_1\cos(\angle(\vec{X},\vec{Y})) + c_1^2(\sin(\angle(\vec{X},\vec{Y}))^2}$,
$d = d_1\,|e|$,

$$h_a(X,Y) = \begin{cases} \frac{a_1}{|X|} & \text{if } X \neq O_3; \\ \cos\frac{|Y|}{2} & \text{if } X = O_3, \end{cases}$$

$$k(X,Y) = \begin{cases} s\frac{\arcsin(d)}{d_1} & \text{if } e^X e^Y)^2 \neq I_3,\ e^X e^Y \text{ has eigenvalues with positive real parts ;} \\ s\frac{\pi - \arcsin(d)}{d_1} & \text{if } (e^X e^Y)^2 \neq I_3,\ e^X e^Y \text{ has two eigenvalues with} \\ & \quad \text{negative or zero real parts ;} \\ \pi & \text{if } (e^X e^Y)^2 = I_3,\ e^X e^Y \neq I_3 \text{ ;} \\ s & \text{if } e^X e^Y = I_3, \end{cases}$$

where

$s = sgn(e) = \begin{cases} 1 & \text{if } e > 0; \\ -1 & \text{if } e < 0, \end{cases}$ for any $(X,Y) \in so(3) \times so(3)$ such that $(e^X e^Y)^2 \neq I_3$ or $e^X e^Y = I_3$.

We have that

$\alpha(X,Y) = f(\frac{Tr(e^X e^Y)-1}{2})g_a(X,Y)$ on $so(3) \times so(3) \backslash \{(X,Y) \in so(3) \times so(3) \mid (e^X e^Y)^2 = I_3, e^X e^Y \neq I_3\}$.

Also, the set $\{(X,Y) \in so(3) \times so(3) \mid (e^X e^Y)^2 = I_3, e^X e^Y \neq I_3\} \cup \{(X,Y) \in so(3) \times so(3) \mid (e^X e^Y)^2 \neq I_3, e^X e^Y$ has two eigenvalues with negative real parts$\} = \{(X,Y) \in so(3) \times so(3) \mid e^X e^Y$ has two eigenvalues with negative real parts$\}$ and on this set we have

$\alpha(X,Y) = k(X,Y)h_a(X,Y)$, where

$$k(X,Y) = \begin{cases} s\frac{\arccos\frac{Tr(e^X e^Y)-1}{2}}{d_1} & \text{if } e^X e^Y)^2 \neq I_3\ e^X e^Y \text{ has two eigenvalues} \\ & \quad \text{with negative real parts;} \\ \pi & \text{if } (e^X e^Y)^2 = I_3,\ e^X e^Y \neq I_3. \end{cases}$$

Since $d_1 = 1$, $e = 0$ for any $(X,Y) \in so(3) \times so(3)$ such that $(e^X e^Y)^2 = I_3$, $e^X e^Y \neq I_3$, we get $|k(X,Y)| = \frac{|d^*(e^X e^Y)|}{d_1}$ for any $(X,Y) \in so(3) \times so(3)$ such that $e^X e^Y$ has two eigenvalues with negative real parts, where d^* is the smooth function defined in Section 2.3 by (2.3.4). Also, $k(X,Y)^2 = \frac{|d^*(e^X e^Y)|^2}{d_1^2}$ for any $(X,Y) \in so(3) \times so(3)$ such that $e^X e^Y$ has

two eigenvalues with negative real parts.
Since d^*, $d_1 \neq 0$ and exp are smooth , it follows that $|k|$ is continuous and k^2 is smooth. Then, since $|k|$ is continuous and k^2, h_α, g_α, f, exp are smooth, we get that $|\alpha|$ is continuous and α^2 is smooth.
The fact that BCH is smooth from $so(3) \times so(3)$ into D follows from the closed formulae for α, β and γ, as well as the smoothness properties for α, β and γ. $\square$

Bibliography

[1] Abramychev, A. Yu., Davydov, V.A., Zykov, V.S. [1990], Drift of Spiral Waves on Nonuniformly Curved Surfaces, Sov. Phys.-JTEP 70, 666-671.

[2] Ashwin, P., Melbourne, I. [1997], Noncompact Drift for Relative Equilibria and Relative Periodic Orbits, Nonlinearity, 10, 595-616.

[3] Ashwin, P., Melbourne, I., Nicol, M. [1999], Drift Bifurcations of Relative Equilibria and Transitions of Spiral Waves, Nonlinearity, 12, 741-755.

[4] Barkley, D. [1992], Linear Stability Analysis of Rotating Spiral Waves in Excitable Media, Phys. Rev. Lett., vol. 68, no. 13, 2090-2093.

[5] Barkley, D. [1994], Euclidean Symmetry and the Dynamics of Rotating Waves, Phys. Rev. Lett., 72, 164-167.

[6] Bates, P.W., Lu, K., Zeng, C. [1998], Persistence of Invariant Manifolds, Memoirs of the American Mathematical Society, 645.

[7] Biktashev, V.N., Holden, A.V., Zhang, H. [1994], Tension of organizing filaments of scroll waves, Philos. Trans. Roy. Soc. London, Ser. A, 347, no. 1685, 611-630.

[8] Bröcker, T., tom Dieck, T. [1985], Representations of Compact Lie Groups, Springer-Verlag, New York.

[9] Brazhnik, P.K., Davydov, V.A., Mikailov, A.S. [1988], Kinematic Approach to the Description of Autowave Processes in Active Media, Theor. Math. Phys., 74, 300-306.

[10] Chavez, F., Kapral, R., Rousseau, G., Glass, L. [2001], Scroll Waves in Spherical Shell Geometries, Chaos, vol.11, no. 4, 757-765.

[11] Chossat, P. [2002], The Reduction of Equivariant Dynamics to the Orbit Space for Compact Group Actions, Acta Applicandae Mathematicae, 70, 71-94.

[12] Chossat, P., Iooss, G. [1994], The Couette-Taylor Problem, Appl. Math. Sciences, no.102, Springer-Verlag, New York.

[13] Davidsen, J., Glass, L., Kapral, R. [2004], Topological Constraints on Spiral Wave Dynamics in Spherical Shells with Inhomogeneous Excitability, preprint, to appear.

[14] Davidenko, J.M., Pertsov A.V., Salomonsz, R., Baxter, W., Jalife, J. [1992], Stationary and Drifting Spiral Waves of Excitation in Isolated Cardiac Muscle, Nature, 355, 349-351.

[15] Dym, H., McKean, H. P. [1972], Fourier Series and Integrals, Academic Press, New York/London.

[16] Davydov, V.A., Zykov, V.S. [1991], Kinematics of Spiral Waves on Nonuniformly Curved Surfaces, Physica D, 49, 71-74.

[17] Engo, K. [2001], On the BCH-Formula in $so(3)$, BIT Numerical Mathematics, 41, 629-632.

[18] Fiedler, B. [1988], Global Bifurcation of Periodic Solutions with Symmetry, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York.

[19] Fiedler, B., Mantel, R.M. [2000], Crossover Collision of Scroll Wave Filaments, Doc. Math., 5, 695-731.

[20] Fiedler, B., Sandstede, B., Scheel, A., Wulff, C. [1996], Bifurcation from Relative Equilibria of Noncompact Group Actions: Skew Products, Meanders, and Drifts, Doc. Math., 1, 479-505.

[21] Fiedler, B., Turaev, D. [1998], Normal forms, Resonances, and Meandering Tip Positions near Relative Equilibria of Euclidean Group Actions, Arch. Rational Mech. Analysis, 145, 129-159.

[22] Field, M. [1991], Local Structure of Equivariant Dynamics, Singularity Theory and Its Applications Warwick Part 1989 II, Lecture Notes in Mathematics, 1463, Springer-Verlag, Heidelberg, 143-166.

[23] Field, M. [1996], Symmetry Breaking for Compact Lie Groups, Memoirs of the American Mathematical Society, 574.

[24] Friedman, A. [1969], Partial Differential Equations, Holt, Rinehart and Winston, Inc.

[25] Gerisch, G. [1983], Naturwissenschaften, 58,420.

[26] Golubitsky, M., LeBlanc, V.G. and Melbourne, I. [1997], Meandering of the Spiral Tip: An Alternative Approach, J. Nonlinear Sci., 7, 557-586.

[27] Golubitsky, M., LeBlanc, V.G. and Melbourne, I. [2000], Hopf Bifurcation from Rotating Waves and Patterns in Physical Space, J. Nonlinear Sci., 10, 69-101.

[28] Golubitsky, M., Schaeffer, D.G. [1985], Singularities and Groups in Bifurcation Theory, vol. I, Springer-Verlag, Berlin.

[29] Golubitsky, M., Stewart, I., Schaeffer, D.G. [1988], Singularities and Groups in Bifurcation Theory, vol. II, Springer-Verlag, Berlin.

[30] Gomatam, J., Amdjadi, F. [1997], Reaction-diffusion on a Sphere: Meandering of Spiral Waves, Physical Review E, vol. 56, no. 4, 3913-3919.

[31] Gomatam, J., Hodson, D.A. [1991], The Eikonal Equation: Stability of Reaction-Diffusion Waves on a Sphere, Physica D, 49, 82-89.

[32] Grindrod, P., Gomatam, J. [1987], The Geometry and Motion of Reaction-Diffusion waves on Closed Two-Dimensional Manifolds, J. Math. Biol., 25, 597-610.

[33] Guyard, F., Lauterbach, R. [1997], Forced Symmetry Breaking Perturbations for Periodic Solutions, Nonlinearity, 10, 291-310.

[34] Guyard, F., Lauterbach, R. [1999], Forced Symmetry Breaking: Theory and Applications, Pattern formation in continuous and coupled systems (Minneapolis,MN, 1998), IMA Vol. Math. Appl., Springer, New York, 115, 121-135.

[35] Guyard, F., Lauterbach, R. [2001], Forced Symmetry Breaking and Relative Periodic Orbits, Ergodic Theory, Analysis, and Efficient Simulation of Dynamical Systems, Springer, Berlin, 453-468.

[36] Grill S., Zykov, V.S., Muller S.C. [1996], Spiral Wave Dynamics under Pulsatory Modulation of Excitability, J. Phys. Chem., 100, 19082-19088.

[37] Hale, J.K. [1969], Ordinary Differential Equations, Wiley, New York.

[38] Hebey, E. [1996], Sobolev Spaces on Riemannian Manifolds, Lectures Notes in Mathematics, Springer-Verlag, Berlin, 1635.

[39] Henry, D. [1981], Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, 840, Springer-Verlag.

[40] Hirsch, M.W., Pugh, C.C., Shub, M. [1977], Invariant Manifolds, Lecture Notes in Mathematics, 583, Springer-Verlag, New York/Berlin.

[41] Hirsch, M.W., Smale, S. [1974], Differential Equations, Dynamical Systems, and Linear Algebra, Acamedic Press, San Diego.

[42] Kapral, R., Showalter, K. [1995], Chemical Waves and Patterns, Kluwer Academic Publishers, Netherlands.

[43] Keener, J., Sneyd, J. [1998], Mathematical Physiology, Interdisciplinary Applied Mathematics, 8, Springer-Verlag, New York.

[44] Keener, J. [1998], The Dynamics of Three-Dimensional Scroll Waves in Excitable Media, Phys. D., 31, no.2, 269-276.

[45] Køenig, M.[1997], Linearization of Vector Fields on the Orbit Space of the Action of a Compact Lie group, Math. Proc. Camb. Phil. Soc., 121, 401-424.

[46] Krupa, M. [1990], Bifurcations of Relative Equilibria, SIAM J. Math. Anal., vol. 21, no. 6, 1453-1486.

[47] Lamb, J. W., Melbourne, I., Wulff, C. [2001], Bifurcation from Relative Periodic Solutions, Ergodic Theory Dynam. Systems, vol. 21, no. 2, 605-635.

[48] LeBlanc, V.G., Wulff, C. [2000], Translational Symmetry-Breaking for Spiral Waves, J. Nonlinear Sci., 10, 569-601.

[49] LeBlanc, V. G. [2002], Rotational Symmetry-Breaking for Spiral Waves, Nonlinearity, vol. 15, 1179-1203.

[50] Li, G., Ouyang, Q., Petrov, V., Swinney, H.L. [1996], Transition from Simple Rotating Chemical Spirals to Meandering and Traveling Spirals, Phys. Rev. Lett., 77, 2105-2108.

[51] Lions, J.L., Magenes, E. [1972], Non-homogeneous Boundary Value Problems and Applications, Springer-Verlag, Berlin, New York.

[52] Margerit, D., Barkley, D. [2002], Cookbook Asymptotics for Spiral and Scroll Waves in Excitable Media, Chaos, 12, no.3, 636-649.

[53] Marsden, J.E., Ratiu, T. S. [1999], Introduction to Mechanics and Symmetry, Springer-Verlag, New York.

[54] Maselko, J. [1998], Symmetrical Double Rotor Spiral Waves on Spherical Surfaces, J. Chem. Soc., Faraday Trans., vol. 94, no. 16, 2343-2345.

[55] Maselko, J., Showalter, K. [1989], Chemical Waves on Spherical Surfaces, Letters to Nature, 339, 609-611.

[56] McQuillan, P., Gomatam, J. [1996], Rotating Chemical Waves on the Sphere, J. Physical Chemistry, vol. 100, no. 13.

[57] Munuzuri, A.P., Perez-Munuzuri V., Perez-Villar, V. [1998], Attraction and Repulsion of Spiral Waves by Localized Inhomogeneities in Excitable Media, Phys. Rev. E, 58, R2689-R2692.

[58] Nam, K., Ott, E., Gabbay, M., Guzdar, P.N. [1997], Spiral wave dynamics in the complex Ginzburg-Landau equation with broken chiral symmetry, Physica D,118, 68-83.

[59] Onischchik, A. L. [1993], Lie Groups and Lie Algebras, vol.I, Springer-Verlag, Berlin/Heidelberg.

[60] Palis, J., Jr., de Melo, W. [1982], Geometric Theory of Dynamical Systems, An Introduction, Springer-Verlag, New York.

[61] Pazy, A. [1983], Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York.

[62] Rand, D. [1982], Dynamics and Symmetry. Predictions for Modulated Waves in Rotating Fluids, Arch. Ratl. Mech. and Anal., 79, 1-38.

[63] Renardy, M. [1982], Bifurcation from Rotating Waves, Arch. Ratl. Mech. and Anal., 79, 49-84.

[64] Rossmann, W. [2002], Lie Groups , An Introduction through Linear Groups, Oxford University Press, Oxford, New York.

[65] Rudin, W. [1987], Real and Complex Analysis, McGraw-Hill, New York, Montreal.

[66] Scheel, A. [1998], Bifurcation to Spiral Waves in Reaction-Diffusion Systems, Siam J. Math. Anal., vol. 29, no. 6, 1399-1415.

[67] Schwarz, G. [1975], Smooth functions invariant under the action of a compact Lie Group, Topology, 14, 63-68.

[68] Selig, J.M. [1996], Geometrical Methods in Robotics, Springer-Verlag, New York.

[69] Sandstede, B., Fiedler, B. [1992], Dynamics of Periodically Forced Parabolic Equations on the Circle, Ergod. Th. and Dynam. Sys., 12, 559-572.

[70] Sandstede, B., Scheel, A., Wulff, C. [1997], Dynamics of Spiral Waves on Unbounded Domains Using Center-Manifold reductions, Journal of Differential Equations, 141, 122-149.

[71] Sandstede, B., Scheel, A., Wulff, C. [1999], Bifurcations and Dynamics of Spiral Waves, J. Nonlinear Sci., vol. 9, no. 4, 439-478.

[72] Sugiura, M. [1990], Unitary Representations and Harmonic Analysis, an Introduction, North-Holland, Amsterdam.

[73] Taylor, M. [1996], Partial Differential Equations, vol. I-III, Springer-Verlag, New York.

[74] Terras, A. [1985], Harmonic Analysis on Symmetric Spaces and Applications I, Springer-Verlag, New York.

[75] Trayanova, N.A., Roth, B.J., Malden, L.J. [1993], The Response of a Spherical Heart to a Uniform Electric Field: a Bidomain Analysis of Cardiac Stimulation, IEEE Transactions on Biomedical Engineering, vol. 40, no. 9, 899-908.

[76] Turing, A.M. [1952], The Chemical Basis of Morphogenesis, Phil. Trans. R. Soc. Lond., B, 237, 37-72.

[77] Varadarajan, V.S. [1974], Lie Groups, Lie Algebras, And Their Representations, Prentice-Hall.

[78] Varadarajan, V.S. [1989], An Introduction to Harmonic Analysis on Semi-simple Lie Groups, Cambridge Studies in Advanced Mathematics, vol. 16, Cambridge University Press, Cambridge.

[79] Warner, F. [1971], Foundations of Differentiable Manifolds and Lie Groups, Scott, Foresman and Company, Illinois.

[80] Wiener, N., Rosenblueth, A. [1946], The Mathematical Formulation of the Problem of Conduction of Impulses in a Network of Connected Excitable Elements, Specifically in Cardiac Muscle, Arch. Inst. Cardiol. Mex., 16, 205-265.

[81] Winfree, A.T. [1972], Spiral Waves of Chemical Activity, Science, 175, 634-636.

[82] Winfree, A.T., Strogatz, S.H. [1984], Organizing Centres for Three-Dimensional Chemical Waves, Nature, 311, 611-615.

[83] Wiggings, S. [1990], Introduction to Applied Non-Linear Dynamical Systems and Chaos, Springer-Verlag, New York.

[84] Wulff, C. [1996], Theory of Meandering and Drifting Spiral Waves in Reaction-Diffusion Systems, Freie Universität, Berlin, Ph.D. thesis.

[85] Wulff, C. [2000], Transitions from Relative Equilibria to Relative Periodic Orbits, Documeta Math., 5, 227-274.

[86] Yermakova, Y.A., Pertsov, A.M. [1986], Interaction of Rotation Spiral Waves with a Boundary, Biophys., 31, 932-940.

[87] Yagisita, H., Mimura, M., Yamada, M. [1998], Spiral Wave Behaviors in an Excitable Reaction-Diffusion System on a Sphere, Physica D, 124, 126-136.

[88] Zaikin, A.N., Zhabotinsky, A.M. [1970], Concentration Wave Propagation in Two-Dimensional Liquid-Phase Self-Oscillating System, Nature, 225, 535-537.

[89] Zykov, V.S., Müller, S.C. [1996], Spiral Waves on Circular and Spherical Domains of Excitable Medium, Physica D, 97, 322-332.