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ISBN 0-315-56366-4

Dynamic Simulation For a Robot With a Closed Kinematic Chain

by

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Thesis submitted to the School of Graduate Studies
as partial fulfillment of the requirements
for the degree of M.A.Sc. in
Mechanical Engineering

UNIVERSITY OF OTTAWA
OTTAWA, CANADA, 1989.

UNIVERSITÉ D'OTTAWA
UNIVERSITY OF OTTAWA

ABSTRACT

A computer program was developed in this thesis that simulates the dynamic behavior of a robot manipulator with a closed kinematic chain. This simulation embraces two major parts, the first part covers the rigid body analysis whereas the second part deals with the vibration analysis of the manipulator. The algorithms used in the first part are among the most efficient in their kind. In the second part a preliminary vibration analysis was conducted, using a general Finite Element Method on dynamical analysis, to determine the behavior of the manipulator. This vibration analysis proved to be very time consuming, and some improvements for this analysis are suggested.

ACKNOWLEDGEMENT

The author wishes to express his gratitude to Dr. D. Redekop for his continuous encouragement and invaluable suggestions throughout the research. I would like to thank my mother Zakia for her patience, encouragement and moral support.

Finally, I would like to express my sincere thanks to the Tunisian government for its scholarship program, without it this research would not have been available this early.

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Nomenclature

${}^{i}R_{j}$	Rotation matrix transforming vectors from frame j to frame i.
${}^{i}\omega_{j}$	Angular velocity of link j expressed in frame i.
${}^{i}\dot{\omega}_{j}$	Angular acceleration of link j expressed in frame i.
${}^{i}\mathbf{V}^{*}_{j}$	Linear velocity of center of mass of link j expressed in frame i.
${}^{i}\mathbf{V}_{j,k}$	Linear velocity of the joint between link j and link k expressed in frame i.
${}^{i}\mathbf{a}^{*}_{j}$	Linear acceleration of center of mass of link j expressed in frame i.
${}^{i}\mathbf{a}_{j,k}$	Linear acceleration of the joint between link j and link k expressed in frame i.
$\mathbf{K}^{*}_{i}$	Generalized inertia forces of link i
$\mathbf{K}_{i}$	Generalized active forces of link i
$\mathbf{T}^{i/j}$	Torque transmitted from link i to link j
$\mathbf{R}^{i/j}$	Force transmitted from link i to link j

Chapter 1

INTRODUCTION

The need for increased automation in industry has recently stimulated an intense research effort in the field of robotics. Robotic research requires major inputs from the fields of computer science and machine tool technology. Numerous fields are involvedsuch as computer programming, artificial intelligence, machine design, computer vision, and sensor technology. Important advances in design and development have been obtained using computer simulation. A major concern in simulation is the prediction of the dynamic behavior of the robot. This specific topic is addressed in the present thesis.

In dynamic simulation of robots the items of major concern include the location of the members of the robot in three dimensional space, and the set of torques that must be applied by the joints actuators. Also stresses and elastic displacement are of interest.

1.1 Kinematics of Manipulators

The kinematics problem in robotics, as in machinery, is the science of motion which treats motion without regard to the forces which cause it. This problem can be divided in two areas; the forward kinematics, which is the calculation of the position and orientation of the end effector given the joint values and rates, and the inverse kinematics, which is the computation of all possible joint angles which could be used to attain a given position and orientation.

1.2 Dynamics

Dynamics is a large discipline devoted to studying forces on bodies and the associated motion. The study of dynamics of robot manipulators is described in terms of the time rate of change of the arm configuration in relation to the joint torques exerted by the actuators. The problem of dynamics is complicated due to the high nonlinearity of the manipulator motion. The methods most frequently employed to date in robot dynamics are Lagrange formulation, Newton-Euler formulation and most recently Kane's dynamical equations.

The inverse dynamic problem, in which torques are found from specific link motion, is of interest to this study. The Lagrange method tends to lead to computational algorithms involving large numbers of unnecessary arithmetic operations, the Newton-Euler approach can force one to perform unnecessary calculations associated with the elimination of certain forces and torques of interaction between elements of a robot, particularly when such elements form closed loops. The Kane's equations are considered more efficient than the others. In this study they are further improved by computing the kinematic quantities using the recursive Newton-Euler algorithm, and producing a separate formulation for the inverse dynamics.

1.3 Analysis of High Speed Elastic Robots

The commonly used assumption in the force analysis, of the static or dynamic problem, is that the links forming the robot are rigid. The complexity of the mathematical analysis of mechanisms, in general, with elastic links has a deterrent against giving up the rigidity assumption. Omitting consideration of link deformation under dynamic conditions may contribute to a machine's failure to perform adequately at higher speeds. Hence it is clear that an accurate dynamic analysis of robotic manipulators which include the effects of link flexibility is an essential part of the design of these systems now and will become even more critical in the future in this rapidly growing industry.

1.4 Thesis Outline

The purpose of this thesis is to study the kinematics, dynamics and vibration characteristics of a robot with a kinematic closed chain having 6 degrees of freedom manipulator such as the ASEA IRB905/2. An efficient algorithm simulating the rigid body motion of such a robot is developed. A major aim in the algorithm is to make use of the most efficient techniques available to date. As well the vibration aspects of the robot are investigated by means of Finite Element Methods. The resulting computer program developed herein, is referred to as SIMASEA.

The text of this thesis is organized into five chapters. Chapter 2 provides an update of previous work on simulators of mechanisms and robot manipulators. Chapter 3 deals with the rigid body dynamics that has been used in SIMASEA, then it gives a summary on the VAL II language and discusses the structure of the SIMASEA program and its features. Chapter 4 presents some results of the program and some discussion. Finally conclusions are presented in Chapter 5.

Chapter 2

LITERATURE SURVEY

In this chapter a survey of the literature is presented on the development of motion simulators for the design and analysis of mechanisms and robot manipulators. Emphasis is placed on the development of dynamic models, suited for the simulation of high speed motion, and the use of these models as tools in computer-aided-design (CAD). This chapter is divided into three parts. The first part deals with models for the motion of mechanisms, with rigid and flexible links. The second part deals with models for the motion of conventional open-chain (serial) robots manipulators. Models permitting rigid and flexible links are considered. The final part deals with robot manipulators with closed kinematic chain mechanisms, i.e. combinations of mechanisms and conventional robots. Completed work for such configurations is discussed and further research needs are indicated.

2.1 Introduction

The continuing pressure for increased productivity in industry is leading to the development of machinery with ever higher performance capabilities. A direct result is the use of higher speeds in mechanisms and robot manipulators. Models for simulating the motion of such machinery must account for the increased capabilities. In particular models must account for flexibility of the components.

Simultaneous with the need for increased productivity in industry there is a need for increased efficiency in design. This requirement is being met by the introduction of computer-aided-design (CAD) methods for mechanisms and robots. Models for machinery motion form important analysis tools in CAD and must be designed to allow easy integration with other tools.

In this chapter a survey is presented of the literature on models of motion for mechanisms and robots. Such a parallel study is important due to the many similarities in the two fields and the opportunity to bring together the better parts of two specialities. Particular emphasis is placed on models for high speed motion and the integration of models into the CAD process. The survey is oriented towards an evolving new form of robot, a robot with closed-chain mechanisms.

2.2 Mechanisms

A classification of the various fields of study in mechanism theory was made by Erdman and Sandor [1] and is reproduced in Table 1. The four fields significant for the current study are

1. Kinematics.
2. Dynamic analysis.
3. Elastodynamic (ED) analysis.
4. Kineto-elastodynamic (KED) analysis.

The various aspects of mechanism motion considered in these fields are indicated in Table 1. The second of these fields is understood to represent the forward dynamic problem (input forces specified). The inverse problem (input kinematics specified) is considered a combination of a kinematic and static analysis when the d'Alembert Principle is applied.

Kinematic analysis concerns the examination of the displacements, velocities and accelerations of a mechanism with all its members regarded as rigid. Distinctions in problem complexity are made according to whether the mechanism is two or three dimensional [2-4], whether it is single or multi-loop and whether the constraints are holonomic or not. For two dimensional problems it is expedient to use a vector approach [4] while for three-dimensional problems a transformation matrix approach [2]. A separation of variables into dependent and independent variables is made. The displacement problem is non-linear and may be solved numerically using a Newton-Raphson method. Derivatives of the loop closure equations are required in this process and again later in the solution of the velocity and acceleration problem.

2.2.1 Dynamics of Mechanisms with Rigid Links (Dynamic Analysis)

Dynamic analysis concerns the determination of the displacements, velocities, and accelerations of a mechanism made up of rigid members for specified input forces. Solutions are obtained by integrating the non-linear differential equations of motion for time varying inputs. Paul [4] has reviewed five major methods of forming the equations of motion in the context of two-dimensional mechanisms. These methods form the bases of some of the major dynamic analysis computer packages [5-10].

Vectorial methods, based on Newton's Laws, are the most direct but possibly the least efficient. They serve as the basis of the MEDUSA [7] and VECNEC [9] programs. An approach based on Lagrange's form of the d'Alembert principle serves to increase the efficiency by reducing the number of independent variables. It is used as the basis of the DYMAC [8] program. A method based on Lagrange's equations is not as efficient but was used in early versions of the IMP [5] program. An attractive method of enforcing constraints is through the use of Lagrangian multipliers. This method has been employed in the two programs ADAMS and DRAM [6]. A method based on Hamilton's equation is not particularly advantageous for mechanisms but was used in another version of IMP [5].

Uicker [11] has presented a solution for the problem of a three-dimensional mechanism with multiple loops. The solution uses the transformation matrix approach and is based on the Lagrange equations of motion. This work is incorporated in the three-dimensional version of IMP.

2.2.2 Dynamics of Mechanisms with Flexible Links (ED and KED)

Lowen has presented two surveys [12-13] of the literature on high speed mechanisms with elastic components. In the second of these, which covers work up to 1983, he divides the studies into four categories

1. analytical methods.
2. finite element methods (FEM).
3. optimization.
4. general experimentation.

The second of these groups is of interest for the current study. The finite element method [14] appears most suited for the solution of general problems, and has been used in a number of studies [15-22].

In the early work a common assumption was that the small elastic motion within links may be uncoupled from the large rigid body motion. Many important studies, working within the limitations of this assumption, were carried out. Dubowsky and Gardner [15] considered impact and clearances at the joint as well as flexibility in links. This study, while not clearly FEM work, indicated the complexity of realistic models. Midha, Erdman and Frohrib [16] used a single beam element to model links of a four-bar mechanism. Nine uncoupled differential equations with time dependent coefficients were developed. Bagci [17] determined 'critical natural frequencies' of spatial mechanisms with elastic links. Specific mechanisms were treated as a series of instantaneous structures. Gamache and Thompson [18] compared FEM vibration responses of elastic bar linkages as determined by the Euler-Bernoulli and Timoshenko beam theories. Sunada and Dubowsky [19] used the NASTRAN FEM package to compute modal frequencies of members. A reduction in the size of the system matrix was achieved by the component mode synthesis (CMS) technique. Cleghorn, Fenton and Tabarrok [20] achieved some increased efficiency by removing the axial degrees of freedom of elastic links.

In more recent work [21-22] coupling of the elastic and the rigid body motion has been considered. In the study of Yoo and Haug [21] vibration and static correction modes are used to account for linear elastic deformation. Constraints between flexible bodies are enforced using Lagrangian multipliers, and coupled large displacement-small deformation equations of motion are derived. An application to a spatial four-bar linkage is discussed. In the study of Sandor and Zhuang [22] a linearization of the coupled equations of motion is considered. The theory is applied to the case of a planar four-bar linkage.

Finally it is to be noted that parallel problems exist in the study of space-craft dynamics. In both cases, however, the system problems do not permit the simple but comprehensive descriptions that are possible for three-dimensional flexible structures encountered in Civil

Engineering [23].

2.2.3 Software for Mechanisms

The computer-aided-design (CAD) approach promises to greatly increase the efficiency of mechanism design. Interactive environments are provided in which powerful synthesis and analysis tools are immediately available. Computer graphics facilities permit modelling and animated simulation of mechanisms. Solid modelling offers the possibility for automatic determination of member properties. A common data base together with plotting and drafting facilities minimizes the transfer-of-data overhead.

Erdman [24] has presented a summary of the capabilities of the major computer programs for mechanism synthesis and analysis available for the CAD process. He has also indicated specific areas in CAD requiring further study.

A number of CAD packages for mechanism design are under development including those of ref. [25-28]. In ref. [25] a package is described which links the IMP analysis program and the MECSYN synthesis program with a new program ANIMEC that provides for automatic geometric modelling, assembly and animation of mechanisms. The package of ref. [26] links the DRAM analysis program and the LINCAGES synthesis program with the ICEM design/drafting program. Both of these packages are restricted to two dimensional mechanisms. The package of ref.[27-28] are intended for three-dimensional work and the first of these is to make use of solid modelling.

2.3 Robot Manipulators

The main fields of study in robot motion theory were identified by Takano [29] and are reproduced in Table 2. The fields in the areas of kinematics and dynamics are considered of significance in the present study. It should be noted that simulation through theoretical models in the case of mechanisms was directed towards the objectives of mechanism synthesis and analysis while in the case of manipulators it is directed to robot design and task planning [29].

The kinematic problems for robots are mostly three-dimensional [30-32]. Many approaches are available [33] but those based on vectors and transformation matrices [34-35] are the most popular. The forward kinematic problem (joint kinematics specified) is readily solved by the latter approach through the formation of A (link) matrices, and their differentiation. The inverse kinematic problem (end-effector kinematics specified) corresponds to the synthesis problem of mechanisms, but must be solved for each path of motion rather than solely at the design stage. Furthermore the objective is to determine necessary joint kinematics rather than link dimensions. Solutions available for the inverse kinematics problem include ones based on geometric [34], matrix equations [30] and continuation [36] methods. Multiple solutions and singularities cause difficulties [31, p.45] as for mechanisms. Computer graphics [37-38] has played an important role in robot kinematics from the beginining and has proved useful in animation and task planning.

2.3.1 Dynamics of Robot Manipulators with Rigid Links

The inverse dynamic problem for robots with rigid links is important for real-time robot operation (control) while the forward problem is important for robot simulation (design). In the inverse problem where the end effector motion is specified, joint variables are found through an inverse kinematic solution, link kinematic quantities are found through forward kinematics relations and forces through a static analysis employing the d'Alembert principle. In the forward dynamic problem the non-linear equations of motion are integrated numerically. The equation complexity is severe, leading to the need for separate programs for the derivation of the basic equations [39].

Solutions to dynamic problems have been presented using Kane's equations [39-40], the Newton-Euler equations [41-43], and the Lagrange equations [44-46]. Efficient solutions to the inverse problem, which must be solved in real time, were given in ref. [41] and [44]. Silver [47] has indicated that computationally these are equivalent. Further efforts at increasing the efficiency are still continuing [40], [48]. Solutions to the forward problem have been presented in ref. [41],[43] and [46].

An approach [49] which assists in the understanding of the robot dynamics, as the arm configuration varies, is that of the generalized inertia ellipsoid (GIE). This approach can serve as the basis for the design of a robot with better dynamic characteristics [50]. Finally an approach using vectors rather than transformation matrices for the geometry has also proved useful [51].

2.3.2 Dynamics of Robot Manipulators with Flexible Links

The motivation for the study of robot manipulators with elastic components is given by Daniel and Davey [52]. The cross-sectional areas of robot links necessary to support the dynamic stresses can be considerably less than those required for it to behave as a 'stiff' structure. Accounting for the elasticity of components in the dynamics may permit a reduction in mass with all its' accompanying benefits.

Two main types of analyses are identified. In the first type, a small elastic motion is superimposed on the large rigid body motion of the links. Thus rigid body and elastic degrees-of-freedom are kept separate in this 'vibration' analysis. The driving forces in this analysis are the inertia effects of the rigid body motion. In the second type of analysis, rigid body and elastic degrees-of-freedom are coupled and must be determined simultaneously.

An analysis of the first type has been carried out by Sunada and Dubowsky [53], for the case of a robot with links of complex geometry. The basic rigid body motion is assumed known from an inverse dynamics solution (using the rigid link assumption). A finite element package is used to determine the mode shapes of the individual links. The component mode synthesis (CMS) method is then used to obtain the vibration characteristics of the robot as a whole. Results were calculated, and flexibility effects were found to be significant and to agree well with experimental data.

Analyses of the second type have been presented in ref. [54-57], using the Lagrangian method, and in ref. [58] using the Newton-Euler method. Book [54] used the transformation matrix approach and estimated that the number of calculation was 2.67 times that for a comparable rigid case. Singh and Likins [55] have used the vector dyadic approach but did not provide data about the number of calculations. Geradin, Robert and Bernardin [56] have also used a transformation matrix approach. A description of deflection, where each link was referred to the rigid configuration of the whole robot, proved preferable to one where each link was referred to its own rigid configuration. Finally Chedmail and Michel [57] and Mufti [58] have used the vector approach to consider, respectively, two and three dimensional geometries.

The distribution of the structural elasticity of a manipulator is an important issue. According to the experimental findings of Gradetskii, Gukasyan and Grudev [59] and Chernousko and Gradetsky [60] joint elasticity exceeds link elasticity for industrial robots. Further experimental work and theory concerning robot elasticity have been given for the static case by Dukovski and Leu [61] and Derby [62].

2.3.3 Software for Robot Manipulators

The development of CAD packages for robotics is less advanced than for mechanisms. CAD tools for robotics are used in the areas of robot design and robot applications. The necessary ingredients of CAD robotics packages are: an extensive data base, powerful graphics capability and a good dynamic simulator. A series of packages [63-73] have already been developed, with varying capabilities. Ardayfio [63] has provided a brief summary of recent developments.

Toyama and Takano [64] have developed a comprehensive package that contains a simulator which can model flexible links. The GRASP program developed by Derby [65] permits users to design or modify robots, and then evaluate them in their working environment. Liegeois et al [66] have developed the MIRE program that facilitates the automatic design of the geometry, the actuators and the control system of robots. Potkonjak and Vukobratovic [67] have developed a program that permits optimization of geometry of robots on the basis of velocity and energy consumption considerations. The CATIA package developed by Dassault Systèmes [68] is directed to robot applications. Wu [69] has developed a CAD design tool for the correction of kinematic errors of robots. A package developed by Anderson et al [70] employed ADAMS for analysis, and features a solid modelling capability. DACM developed by Palmquist and Duffy [71] provides tools for the design and application of cooperating manipulator systems. Finally the VAST package of Pfeifer and Neuman [72] and the SIR package of Stepanenko and Sankar [73] are oriented to the simulation of robot dynamics and control systems.

2.4 Robot Manipulators With Closed Chain Mechanisms

In this section robots with closed kinematic chain mechanisms are considered. Industrial robots with in-parallel-actuated arms as discussed by Hunt [74] may be analyzed using the theory for serial robots and are not considered here.

Examples of closed chain mechanisms are found in a number of electric-drive industrial robots, such as the ASEA IRB905/2 and Cincinatti T3. The use of such mechanisms permits the motors for the forearm and, possibly, for the 'pitch' of the wrist to be mounted on the first moving link. A beneficial effect is the greatly reduced mass and inertia of the arm. Unfortunately the motion of such robots can not be determined directly using the recursive dynamic analysis developed for serial robots [75]. The dynamic model must combine the theory for serial robots and multi-loop spatial mechanisms. Models developed for robot control must be in the efficient recursive form to permit real time calculations.

The cases for rigid [75-81] and flexible links [82-85] will be discussed separately. Not considered here are the closed kinematic chain mechanisms used in the associated 'finger' problem [31,p.71].

2.4.1 Case of Rigid Links

Luh and Zheng [75] have presented a detailed analysis of the inverse problem, that combines methods for closed-chain mechanisms [8], [11] with the recursive Newton-Euler method of serial robots [42]. Essentially the closed-chain mechanism is treated as if it were cut open at one of the joints, creating a tree structure. An inverse dynamic solution similar to that for serial robots is then performed yielding theoretical actuator forces at all joints. Using a Lagrange multiplier technique, which enforces the closed-chain constraints, transfer of the forces to the actual actuators is effected.

Megahed and Renaud [76-77] have set up the equations for the forward and inverse dynamic problems. The Lagrange equations are used together with Lagrangian multipliers for the closed-chain constraints. Vukobratovic and Potkonjak [78] consider the case when the closed-chain involves a kinematic parallelogram. A solution is formulated using Appel's equations. Based on the work of Wittenburg [3], Sol, Veldpaus and Janssen [79] have developed kinematic tools for multi-body systems. They give an example for a robot with three closed-chains. Kane and Faessler [85] applied the Kane's equations, which do not need Lagrangian multipliers. Finally Asada [80] has considered the design of a direct drive arm with a five-bar-link parallel drive mechanism. Through a 'balancing' procedure a time-invariant inertia is obtained for the first several links. By using specially designed motors an order of magnitude increase was achieved for the top speed and maximum acceleration.

2.4.2 Case of Flexible Links

Relatively little work has been done in this area so far. Two main types of analysis would appear appropriate, as for flexible serial robots. In the first, small elastic motions are superimposed on the large rigid body motion and in the second the elastic and rigid body motions are coupled. To the authors' knowledge no extensive studies on either type have so far been carried out.

Kiedrzynski and Becquet [82] have determined experimentally the stiffness matrix of a robot link consisting of three joints, such as would appear in a robot with a closed-chain mechanism. This experimental method of determining the properties of a link with complex geometry is to be contrasted with the finite element method used by Sunada and Dubowsky [53]. Thompson and Sung [83] have performed a tentative analysis of the first type mentioned in the preceding paragraph. The main objective was to demonstrate the advantage of composite materials versus steel for robot links. Composite material links were found much stiffer for comparable masses. Leu, Dukovski and Wang [84] used elementary statics to determine the forces in a robot with a closed-chain mechanism. The loading considered was a force on the final link. Deflection arising from 1) joint torsion, 2) structure bending, and 3) bearing deformation were calculated. Results indicating that joint torsion effects predominate were verified experimentally.

2.5 Conclusion

An extensive amount of work has been done to make the simulation of the motion of high speed machinery more realistic. Some of this work has been incorporated into the CAD process and is conveniently available to industry. Ample opportunities, however, still exist to improve the models and to incorporate the more advanced work into CAD. In this thesis such a package, SIMASEA, for a robot manipulator with a closed kinematic chain, is discussed in detail.

Table 1. Fields of Study in Mechanisms Theory [1, *p*.21]

		Aspects of Mechanisms in Motion							
Type of performance	Forces in members and joints	Stress and strains in members and joints	Rigid body position velocity ect.	Inertia forces due to rigid body motion	Elastic deformations	Inertia forces due to elastic deformation	Time as a reference variable	Torque or force balancing	Stability
1. Static analysis	*	*			*				
2. Kinematics analysis			*						
3. Dynamics analysis	*		*	*			*		
4. Elastic analysis	*	*			*				
5. Elasto-dynamic analysis	*	*	*	*	*		*		
6. Kineto-elasto-dynamic analysis	*	*	*	*	*	*	*		
7. Kinematic synthesis			*						
8. Kineto-elastostatic synthesis	*	*	*	*	*				
9. Dynamic synthesis	*		*	*			*		
10. Dynamic balancing	*		*	*			*	*	
11. Kineto-elasto-dynamic synthesis	*	*	*	*	*	*	*		
12. Kineto-elasto-dynamic balancing	*	*	*	*	*	*	*	*	
13. Kineto-elastodynamic stability of motion	*	*	*	*	*	*	*	*	*

Table 2.2: **Fields of Study in Robot Theory** [29, *p*.224]

1. Kinematics: to obtain displacement, velocity and acceleration from joint motion.
2. Inverse kinematics (synthesis): to obtain joint displacement from robot position and orientation.
3. Statics: to calculate joint torque (force) due to gravitational force and external force and torque.
4. Dynamics(I): to calculate joint torque due to dynamic force and moment.
5. Dynamics(II): to obtain the robot motion from joint torque (force) variation with time.
6. Deflection: to obtain the deflection due to static and dynamic force and moment.
7. Dynamics(III): to calculate the vibrations.
8. Control system analysis: to calculate the dynamic characteristics of control system with internal sensor feedback loop.
9. Path generation: to obtain the continous path data from discrete points series.
10. Compliance analysis: to determine compliance matrices from arm and joint dimensions by FEM.
11. Inertia data: to determine mass, gravity center, inertia matrix of arms and workpiece from their dimensions.

Chapter 3

KINEMATICS AND DYNAMICS

3.1 Introduction

The specification of the time history of the end effector is essential to a successful application of a robot to many tasks such as arc welding, spray painting and conveyor belt tracking. Also the transformation between the joint space and the Cartesian space is very important since robots are controlled in the joint space. The kinematics problem deals with the analytical study of the relation between these two spaces, whereas the dynamic problem deals with the relationship between the joint space coordinates and the set of the forces and torques needed to drive the members of the manipulator in such a way to achieve the desired end effector motions. To simplify the kinematic solution, manipulators are usually designed so that the neighboring axes are orthogonal or parallel to each other and the three last joints form a spherical wrist. The spherical wrist allow the decomposition of the six-degree-of-freedom kinematic problem into two three-degree-of-freedom kinematic subproblems. Some efficient algorithms have been developed for these transformations. Featherstone's algorithm [86] seems to be the most efficient for the inverse kinematic problem. Hollerbach and Sahar [87] have used the same technique to calculate inverse acceleration. In this study, these algorithms were used for the inverse kinematic problem.

As regards robot operations, dynamical equations are frequently used to compute the forces and torques to drive the members of the manipulator, a task that must be performed repeatedly and, in most cases rapidly. Consequently, it is important to make use of the most efficient possible computational algorithms. The dynamical analysis of this study is based on a new approach using Kane's dynamical equations [40],[85]. This new approach was developed by the author, and it is more efficient than the classical Kane's equations.

3.2 Robot Definition

The robot to be considered has nine revolute joints with a closed loop arranged as shown in Fig.1. The joint axes for the manipulator are derived from the Denavit-Härtenberg specifications [35], even though these specifications are not used directly. The rotation axis $\mathbf{z}_i$ corresponds to joint angle θ_{i+1} between link i and $i+1$. The internal coordinate system for link i is completed by defining $\mathbf{x}_i$ from the cross product $\mathbf{z}_{i-1} \times \mathbf{z}_i$, for $i = 1, 2, 4, 5$ and 6, and $\mathbf{x}_3 = -\mathbf{z}_2 \times \mathbf{z}_3$ because it is desired that the rotation axis $\mathbf{z}_3$ point at the wrist, and $\mathbf{y}_i = \mathbf{z}_i \times \mathbf{x}_i$. The reason behind using the Denavit-Hartenberg convention for the set up of the frames is that some of the kinematic parameters can be expressed easily in general form. For convenience the closed loop is disregarded initially in the kinematics study, see Fig. 2. Additional steps associated with it are included in the dynamic analysis. The rotation matrix ${}^{i-1}R_i$ that transforms points expressed in link i coordinates to link $i-1$ coordinates is

$$ {}^{i-1}R_i = \begin{bmatrix} c_i & -s_i c_{\alpha_i} & s_i s_{\alpha_i} \\ s_i & c_i c_{\alpha_i} & -c_i s_{\alpha_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} \end{bmatrix} $$

Here $s_i = \sin\theta_i$, $c_i = \cos\theta_i$, $s_{\alpha_i} = \sin\alpha_i$ and $c_{\alpha_i} = \cos\alpha_i$, and α_i is the angle between $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$. For convenience, the eight joint transformation matrices are listed below:

$$^{0}R_1 = \begin{bmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{bmatrix} \qquad ^{1}R_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{2}R_3 = \begin{bmatrix} c_3 & 0 & -s_3 \\ s_3 & 0 & c_3 \\ 0 & -1 & 0 \end{bmatrix} \qquad ^{3}R_4 = \begin{bmatrix} c_4 & 0 & s_4 \\ s_4 & 0 & -c_4 \\ 0 & 1 & 0 \end{bmatrix}$$

$$^{4}R_5 = \begin{bmatrix} c_5 & 0 & s_5 \\ s_5 & 0 & -c_5 \\ 0 & 1 & 0 \end{bmatrix} \qquad ^{5}R_6 = \begin{bmatrix} c_6 & -s_6 & 0 \\ s_6 & c_6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{1}R_7 = \begin{bmatrix} c_7 & -s_7 & 0 \\ s_7 & c_7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad ^{7}R_8 = \begin{bmatrix} c_8 & -s_8 & 0 \\ s_8 & c_8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The following identity is observed:

$$\left[^{i-1}R_i\right]^{-1} = {}^{i}R_{i-1}$$

as well as identities of the form:

$$^{i}R_{i+3} = {}^{i}R_{i+1}\,{}^{i+1}R_{i+2}\,{}^{i+2}R_{i+3}$$

Also a vector $^{i}\mathbf{V}$ is specified with a left superscript to denote that the vector $\mathbf{V}$ is referred to i^{th} coordinate system.

The position of the end effector can be defined uniquely in either joint space coordinates :

$$\theta_1, \theta_2, \ldots, \theta_6$$

or in cartesian space coordinates :

$$R_x, R_y, R_z, r_\phi, r_\theta, r_\psi$$

where $\mathbf{R} = (R_x, R_y, R_z)$ is the position vector of the origin of the end effector coordinates in base coordinates; and r_ϕ, r_θ, and r_ψ are the rotations about the **z**-axis, the new **y**-axis and the new z-axis that align the base coordinates with the end effector coordinates. These rotation parameters were chosen because they correspond with the arrangement of the joints at the wrist and hence simplify the equations later. The total rotation can be expressed, (Appendix B.1), as

$$E = Rot(\mathbf{z}, \phi) Rot(\mathbf{y}, \theta) Rot(\mathbf{z}, \psi)$$

or

$$E = {}^0R_1\,{}^1R_2\,{}^2R_3\,{}^3R_4\,{}^4R_5\,{}^5R_6$$

Finally the orientation matrix as a function of the orientation parameters is

$$\mathbf{E}(\phi, \theta, \psi) = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix} \quad (3.1)$$

3.3 Position Transformation

In applying Featherstone's method [86] the inverse kinematic positions are computed first, then the inverse kinematic velocities and finally the inverse kinematic accelerations.

3.3.1 Inverse Kinematic Positions

The first three joints of the robot serve to position the wrist and the remaining joints serve to orient the end effector. The position analysis is considered first. To find the wrist position we presume that the orientation matrix E of the gripper, (Appendix B.1), and the position of its tip P have been specified. Then the position of the wrist $\mathbf{W}_1$ in Fig. 3 is given by

$$\mathbf{W}_1 = \mathbf{R} - \mathbf{P}_6 \tag{3.2}$$

where $\mathbf{P}_6$ in base coordinates is

$$\mathbf{P}_6 = d_6(c_\phi s_\theta, s_\phi s_\theta, c_\theta) \tag{3.3}$$

Joint 1 is directly found from $\mathbf{W}_1$ since joint 2 and 3 act in a plane that does not alter the projection of the wrist, and the position of the wrist from joint two is

$$\mathbf{W}_2 = \mathbf{W}_1 - d_1 \mathbf{k}_0 \tag{3.4}$$

hence

$$\theta_1 = ATAN2(W_{1x}, W_{1y}) \tag{3.5}$$

This equation indicates that a degeneracy occurs when $W_{1x} = W_{1y} = 0$ that is when the wrist lies on the $\mathbf{z}_0$ axis. In this case θ_1 can take any value, Fig. 3. Application of the cosine rule to triangle $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{W}_2$ in Fig. 3 and 4, gives:

$$\cos\gamma = \frac{l_2^2 + d_4^2 - W_2^2}{2 l_2 d_4} \tag{3.6}$$

$$\theta_3 = \gamma + \frac{\pi}{2} \tag{3.7}$$

$$\theta_2 = ATAN2(\sqrt{W_{2x}^2 + W_{2y}^2}, W_{2z}) + ATAN2(d_4 \sin\gamma, l_2 + d_4 \cos\gamma) \tag{3.8}$$

This completes the position analysis. The analysis for the orientation will now be considered. The inverse orientation problem is most easily solved by using spherical trigonometry. Ref. [86] presents the formulas used here to solve the spherical triangle in Fig. 5. The desired orientation of the end effector is given by a rotation of r_ϕ about the z-axis, Fig. 5, which moves the x-z circle from A1 to AB, a rotation of r_θ about the y-axis, which moves the z-axis from A to B, and a rotation of r_ψ about the **z**-axis, which moves the x-z plane to BB1. The first three joint angles have caused a rotation θ_1 about the **z**-axis, moving the x-z circle from AA1 to AC, and a rotation of $\theta_2 + \theta_3$ about the **y**-axis, moving the **z**-axis point from A to C, Now θ_4 and θ_6 are rotation about the z-axis, and θ_5 is a rotation about the y-axis; thus θ_5, being the only rotation that moves the z-axis, must move it from C to B. In order to do this, θ_4 must move the x-z circle from AC to BC, since the **z**-axis points can only travel along this circle. Finally, θ_6 must move the y-z circle from CB to BB1, completing the transformation. The problem of finding θ_4, θ_5 and θ_6 is thus one of solving the spherical triangle ABC. The applications of the spherical cosine rule to ABC gives

$$\cos\theta_5 = \cos(\theta_2+\theta_3)\cos r_\theta + \sin(\theta_2+\theta_3)\sin r_\theta \cos(r_\phi - \theta_1) \tag{3.9}$$

The sign of θ_5 is arbitrary. A degeneracy take place when $\cos\theta_5 = 1$ and θ_4 and θ_6 become linearly dependant. Thus

$$\begin{aligned}
\sin\theta_4 &= \sin r_\theta \sin(r_\phi - \theta_1) \\
\cos\theta_4 &= \cos(\theta_2+\theta_3)\sin r\theta\cos(r_\phi - \theta_1) - \sin(\theta_2+\theta_3)\cos r\theta \\
\theta_4 &= ATAN2(\sin\theta_4, \cos\theta_4) && (3.10) \\
\sin\beta &= \sin(\theta_2+\theta_3)\sin(r_\phi - \theta_1) \\
\cos\beta &= \cos(\theta_2+\theta_3)\sin r_\theta - \cos(r_\phi - \theta_1)\sin(\theta_2+\theta_3)\cos r_\theta) \\
\beta &= ATAN2(\sin\beta, \cos\beta) && (3.11) \\
\theta_6 &= r_\psi - \beta && (3.12)
\end{aligned}$$

3.3.2 Forward Kinematic Position

The general method is to build up the transformation from base coordinates to the end effector by rotating successively about the joint and translating along the links until the end effector is reached. A faster technique is to take advantage of the spherical geometry,

[86], and apply the arguments of the previous section in reverse.

$$\cos r_\theta = \cos(\theta_2 + \theta_3)\cos\theta_5 - \sin(\theta_2 + \theta_3)\sin\theta_5\cos\theta_4 \tag{3.13}$$

A degeneracy occurs when $|\cos r_\phi| = 1$ and causes difficulty in the evaluation of r_θ and r_ϕ. In the case of the degeneracy one takes:

$$r_\phi = \theta_6 + ATAN2\,[\sin(\theta_2 + \theta_3)\sin\theta_4\sin\theta_5\cos(\theta_2 + \theta_3) + \cos\theta_5\sin(\theta_2 + \theta_3)\cos\theta_4] \tag{3.14}$$

$$r_\psi = \theta_6 + ATAN2\,[\sin(\theta_2 + \theta_3)\sin\theta_4, \sin\theta_5\cos(\theta_2 + \theta_3) + \cos\theta_5\sin(\theta_2 + \theta_3)\cos\theta_4)] \tag{3.15}$$

The position is calculated as:

$$W_{2xy} = l_2\sin\theta_2 + l_3\cos(\theta_2 + \theta_3) \tag{3.16}$$

$$W_{2z} = l_2\cos\theta_2 + l_3\sin(\theta_2 + \theta_3) \tag{3.17}$$

$$W_1 = (-\sin\theta_1 W_{2z}, \cos\theta_1 W_{2z}, l_1 + W_{2xy}) \tag{3.18}$$

Hence

$$\mathbf{R} = \mathbf{W}_1 + \mathbf{P}_6 \tag{3.19}$$

No ambiguities arise in the equations since there is only one solution.

3.4 Inverse Kinematic Velocities

We present in this section the transformation of velocities from cartesian space to the joint space. It is assumed that the intermediate, and final results of the position transformation are available. The instantaneous angular velocities, (Appendix B.2), of the end effector coordinate frame about the principal axes of the reference frame are obtained as:

$$\omega_6 = \dot\phi\mathbf{z} + \dot\theta\mathbf{y} + \dot\psi\mathbf{z} \tag{3.20}$$

$$^0\omega_6 = \begin{pmatrix} -s_\phi\dot\theta + c_\phi s_\theta\dot\psi \\ c_\phi\dot\theta + s_\phi s_\theta\dot\psi \\ \dot\phi + c_\theta\dot\psi \end{pmatrix} \tag{3.21}$$

where ψ, θ and ϕ are Euler angles defined earlier. The problem is defined as follows; given the position and orientation of the end effector and its linear velocity $\dot{R}$ and the angular velocity ${}^0\omega_6$, find the vector of joint rates, $\dot{\theta}_1, \ldots, \dot{\theta}_6$. The first step is to calculate the linear velocity of the wrist, from which the values of the first three joints can be calculated. The position of the wrist is defined, Fig. 4, by the following equation:

$$\mathbf{W}_1 = d_1\mathbf{z}_0 + l_2\mathbf{x}_2 + d_4\mathbf{z}_3 \tag{3.22}$$

We differentiate W_1 , in eqn. (3.22), with respect to time and evaluate in link 2 coordinates:

$${}^2\dot{\mathbf{W}}_1 = {}^2\omega_2 \times l_2\,{}^2\mathbf{x}_2 + {}^2\omega_3 \times d_4\,{}^2\mathbf{z}_3 \tag{3.23}$$

Therefore

$${}^2\dot{\mathbf{W}}_1 = \begin{pmatrix} -d_4c_3(\dot{\theta}_2 + \dot{\theta}_3) \\ l_2\dot{\theta}_2 - d_4s_3(\dot{\theta}_2 + \dot{\theta}_3) \\ -{}^1W_{2x}\dot{\theta}_1 \end{pmatrix} \tag{3.24}$$

But the wrist's linear velocity as a function of the desired end-effector linear and angular velocities are obtained by deriving eqn. (3.2), thus:

$$\dot{\mathbf{W}}_1 = \dot{\mathbf{R}} - \omega_6 \times \mathbf{P}_6 \tag{3.25}$$

Solving for the first three joint rates, (Appendix B.3),

$$\dot{\theta}_1 = \frac{{}^2\dot{W}_{1z}}{-{}^1W_{2x}} \tag{3.26}$$

$$\dot{\theta}_2 = \frac{1}{l_2}\left({}^2\dot{W}_{1y} - \frac{s_3}{c_3}\,{}^1\dot{W}_{1x}\right) \tag{3.27}$$

$$\dot{\theta}_3 = -\frac{{}^2\dot{W}_{1x}}{d_4c_3} - \dot{\theta}_2 \tag{3.28}$$

The first three joint rates have determined the linear velocity of the wrist. The remaining rates have to determine the angular velocity of the end-effector. The first three joints have already imparted an angular velocity to the wrist, and hence to the end effector, so the remaining joint rates have to produce at the wrist the difference between the desired angular velocity of the end effector and the angular velocity produced by the first three joints. Now let $\omega_{6/3}$ be the angular velocity of the end effector with respect to frame 3, then $\omega_{6/3}$ is given by

$$\omega_{6/3} = \omega_6 - \omega_3 \tag{3.29}$$

Now $\omega_{6/3}$ is made up from $\dot{\theta}_4$, $\dot{\theta}_5$, and $\dot{\theta}_6$ as follows:

$$\omega_{6/3} = \dot{\theta}_4 \mathbf{z}_3 + \dot{\theta}_5 \mathbf{z}_4 + \dot{\theta}_6 \mathbf{z}_5 \tag{3.30}$$

Solving for the last three joint rates, (Appendix B.3),

$$\dot{\theta}_5 = {}^4\omega_{6/3z} \tag{3.31}$$

$$\dot{\theta}_6 = \frac{{}^4\omega_{6/3x}}{s_5} \tag{3.32}$$

$$\dot{\theta}_4 = {}^4\omega_{6/3y} + c_5\dot{\theta}_6 \tag{3.33}$$

The accuracy of these equations deteriorates as $\sin\theta_5$ tends to zero.

3.5 Acceleration Transformations

For the computation of acceleration, it is assumed that the previous results from positions and velocities transformations are available as well as the linear and angular accelerations of the end effector. The instantaneous angular acceleration of the wrist, (Appendix B.2), about the inertial frame are obtained by differentiating the angular velocities, eqn. (3.21);

$$^0\dot{\omega}_z = \begin{pmatrix} -s_\phi\ddot{\theta} + c_\phi s_\theta\ddot{\psi} - c_\phi\dot{\theta}\dot{\phi} + c_\phi c_\theta\dot{\theta}\dot{\psi} - s_\phi s_\theta\dot{\psi}\dot{\phi} \\ c_\phi\ddot{\theta} + s_\phi s_\theta\ddot{\psi} - s_\phi\dot{\theta}\dot{\phi} + s_\phi c_\theta\dot{\theta}\dot{\psi} + c_\phi s_\theta\dot{\psi}\dot{\phi} \\ \ddot{\phi} + c_\theta\ddot{\psi} - s_\theta\dot{\theta}\dot{\psi} \end{pmatrix} \tag{3.34}$$

The first step is to calculate the linear acceleration of the wrist, from which the acceleration of the three joints can be calculated. The linear acceleration of the end effector is obtained by deriving eqn. (3.25). One gets:

$$\ddot{\mathbf{W}}_1 = \ddot{\mathbf{R}} - \dot{\omega}_6 \times \mathbf{P}_6 - \omega_6 \times (\omega_6 \times \mathbf{P}_6) \tag{3.35}$$

The $\ddot{\mathbf{W}}_1$ can be readily found since the right hand side is known. The linear acceleration of the wrist $\ddot{\mathbf{W}}_1$ can also be calculated by differentiating the linear velocity eqn. (3.23)

$$\ddot{\mathbf{W}}_1 = \dot{\omega}_2 \times \mathbf{W}_2 + \omega_2 \times \dot{\mathbf{W}}_2 - d_4\ddot{\theta}_3\mathbf{x}_3 - d_4\dot{\theta}_3\omega_3 \times \mathbf{x}_3 \tag{3.36}$$

This equation is fully developed in appendix B.4, and the joint accelerations can be found

$$\ddot{\theta}_1 = -\frac{{}^2\ddot{U}_{4z}}{{}^1W_{2x}} \tag{3.37}$$

$$\ddot{\theta}_2 = \frac{1}{l_2}({}^2\ddot{U}_{4y} - \frac{s_3}{c_3}\,{}^2\ddot{U}_{4x}) \tag{3.38}$$

$$\ddot{\theta}_3 = -\frac{{}^2\ddot{U}_{4x}}{d_4c_3} - \ddot{\theta}_2 \tag{3.39}$$

The hand angular velocity relative to the wrist is found by differentiating eqn. (3.29).

$$\dot{\omega}_{6/3} = \dot{\omega}_6 - \dot{\omega}_3 - \omega_3 \times \omega_{6/3} \tag{3.40}$$

On the other hand $\dot{\omega}_{6/3}$ can be expressed in terms of the last three angular acceleration by deriving eqn. (3.30), that is

$$\dot{\omega}_{6/3} = \ddot{\theta}_4\mathbf{z}_3 + \ddot{\theta}_5\mathbf{z}_4 + \ddot{\theta}_6\mathbf{z}_5 + \dot{\theta}_4\mathbf{z}_3 \times \dot{\theta}_5\mathbf{z}_4 + (\dot{\theta}_4\mathbf{z}_3 + \dot{\theta}_5\mathbf{z}_4) \times \dot{\theta}_6\mathbf{z}_5 \tag{3.41}$$

The joint accelerations of the last three frames can be found as

$$\ddot{\theta}_6 = \frac{1}{s_5}({}^4\dot{\omega}_{6/3x} - \dot{\theta}_4\dot{\theta}_5 - c_5\theta_5\dot{\theta}_6) \tag{3.42}$$

$$\ddot{\theta}_5 = {}^4\dot{\omega}_{6/3z} + s_5\dot{\theta}_4\dot{\theta}_6 \tag{3.43}$$

$$\ddot{\theta}_4 = {}^4\dot{\omega}_{6/3y} + c_5\ddot{\theta}_6 - s_5\dot{\theta}_5\dot{\theta}_6 \tag{3.44}$$

These equations complete the accelerations analysis.

3.6 Dynamic Analysis

The dynamical equations developed herein are based on Kane's equations [40],[85] with some modifications. The closed loop present in the robot will now be considered. To characterize the configuration of the system, coordinates θ_i for $i = 1, \ldots, 8$ are introduced, where θ_i is the angle between x_{i-1} and x_i, Fig. 1. Since the system has only six degrees of freedom, it is possible to specify the configuration by assigning values of six of θ_i provided that they are independent, and express the remaining two in terms of these six. For the present robot one must take θ_i for $i = 1, \ldots, 6$ as independent, and expressing θ_7 and θ_8 as a function of the independent joints, Fig. 6. Considering $\dot{\theta}_i$ for $i = 1, 2, 4, 5, 6$ and 7 as our generalized speeds. Consideration of the closed loop, Fig. 1, leads to the following kinematical constraint equations:

$$O_1P_8 = L_7\mathbf{i}_7 + L_8\mathbf{i}_8 \tag{3.45}$$

$$O_1P_8 = L_2\mathbf{i}_2 - L_3\mathbf{k}_3 \tag{3.46}$$

$$V_{P8} = (\dot{\theta}_1\mathbf{k}_0 + \dot{\theta}_7\mathbf{k}_1) \times L_7\mathbf{i}_7 + (\dot{\theta}_1\mathbf{k}_0 + \dot{\theta}_7\mathbf{k}_1 + \dot{\theta}_8\mathbf{k}_7) \times L_8\mathbf{i}_8 \tag{3.47}$$

$$V_{P8} = (\dot{\theta}_1\mathbf{k}_0 + \dot{\theta}_2\mathbf{k}_1) \times L_2\mathbf{i}_2 - (\dot{\theta}_1\mathbf{k}_0 + \dot{\theta}_2\mathbf{k}_2 + \dot{\theta}_3\mathbf{k}_2) \times L_3\mathbf{k}_3 \tag{3.48}$$

or

$$V_{P8} = (\dot{\theta}_8\mathbf{k}_7 \times L_8\mathbf{i}_8) + (\dot{\theta}_1\mathbf{k}_0 + \dot{\theta}_7\mathbf{k}_1) \times (L_7\mathbf{i}_7 + L_8\mathbf{i}_8) \tag{3.49}$$

$$V_{P8} = -(\dot{\theta}_3\mathbf{k}_2 \times L_3\mathbf{k}_3) + (\dot{\theta}_1\mathbf{k}_0 + \dot{\theta}_2\mathbf{k}_1) \times (L_2\mathbf{i}_2 - L_3\mathbf{k}_3) \tag{3.50}$$

When expressing eqns. (3.47) and (3.48) in frame 1, one needs to express the unit vectors appearing in these equations in frame 1, then:

$${}^1\mathbf{k}_0 = {}^1R_0\,{}^0\mathbf{k}_0 = \begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{3.51}$$

$${}^1\mathbf{k}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.52}$$

$${}^1\mathbf{i}_2 = {}^1R_2\,{}^2\mathbf{i}_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_2 \\ s_2 \\ 0 \end{pmatrix} \tag{3.53}$$

$$
{}^1\mathbf{k}_2 = {}^1R_2\,{}^2\mathbf{k}_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.54}
$$

$$
\begin{aligned}
{}^1\mathbf{k}_3 &= {}^1R_2\,{}^2R_3\,{}^3\mathbf{k}_3 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & 0 & -s_3 \\ s_3 & 0 & c_3 \\ 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{bmatrix} c_{23} & 0 & -s_{23} \\ s_{23} & 0 & c_{23} \\ 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -s_{23} \\ c_{23} \\ 0 \end{pmatrix}
\end{aligned} \tag{3.55}
$$

$$
{}^1\mathbf{i}_7 = {}^1R_7\,{}^7\mathbf{i}_7 = \begin{bmatrix} c_7 & -s_7 & 0 \\ s_7 & c_7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_7 \\ s_7 \\ 0 \end{pmatrix} \tag{3.56}
$$

$$
{}^1\mathbf{k}_7 = {}^1R_7\,{}^7\mathbf{k}_7 = \begin{bmatrix} c_7 & -s_7 & 0 \\ s_7 & c_7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.57}
$$

$$
\begin{aligned}
{}^1\mathbf{i}_8 &= {}^1R_7\,{}^7R_8\,{}^8\mathbf{i}_8 = \begin{bmatrix} c_7 & -s_7 & 0 \\ s_7 & c_7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_8 & -s_8 & 0 \\ s_8 & c_8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{bmatrix} c_{78} & -s_{78} & 0 \\ s_{78} & c_{23} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_{78} \\ s_{78} \\ 0 \end{pmatrix}
\end{aligned} \tag{3.58}
$$

Hence eqn. (3.49) can be evaluated:

$$
{}^1\mathbf{V}_{P8} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_8 \end{pmatrix} \times \begin{pmatrix} L_8 c_{78} \\ L_8 s_{78} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{\theta}_1 \\ \dot{\theta}_7 \end{pmatrix} \times \left\{ \begin{pmatrix} L_7 c_7 \\ L_7 s_7 \\ 0 \end{pmatrix} + \begin{pmatrix} L_8 c_{78} \\ L_8 s_{78} \\ 0 \end{pmatrix} \right\}
$$

$$= \begin{pmatrix} 0 \\ 0 \\ \dot\theta_8 \end{pmatrix} \times \begin{pmatrix} L_8c_{78} \\ L_8s_{78} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot\theta_1 \\ \dot\theta_7 \end{pmatrix} \times \begin{pmatrix} L_7c_7 + L_8c_{78} \\ L_7s_7 + L_8s_{78} \\ 0 \end{pmatrix} \tag{3.59}$$

Now we introduce the process that is central to the method at hand, namely the process of defining quantities Z_n, which saves, much writing and assures the efficiency of the final computer code. The rule for introducing these quantities is that each time we encounter a mathematical expression involving previously defined quantities later not needed explicitly in this expression. For example eqn. (3.59) can be rewritten as

$${}^1\mathbf{V}_{P8} = \begin{pmatrix} 0 \\ 0 \\ \dot\theta_8 \end{pmatrix} \times \begin{pmatrix} Z_1 \\ Z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot\theta_1 \\ \dot\theta_7 \end{pmatrix} \times \begin{pmatrix} Z_3 \\ Z_4 \\ 0 \end{pmatrix} \tag{3.60}$$

where $Z_1, \ldots, Z_4$ are equal to the quantities that they replace. The Appendix C, we listed all Z_n developed in this dynamical analysis. Thus

$${}^1\mathbf{V}_{P8} = \begin{pmatrix} -Z_2\dot\theta_8 \\ Z_1\dot\theta_8 \\ 0 \end{pmatrix} + \begin{pmatrix} -Z_4\dot\theta_7 \\ Z_3\dot\theta_7 \\ -Z_3\dot\theta_1 \end{pmatrix} \tag{3.61}$$

Similarily for eqn. (3.50):

$$\begin{aligned} {}^1\mathbf{V}_{P8} &= -\begin{pmatrix} 0 \\ 0 \\ \dot\theta_3 \end{pmatrix} \times \begin{pmatrix} -L_3s_{23} \\ L_3c_{23} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot\theta_1 \\ \dot\theta_2 \end{pmatrix} \times \left\{ \begin{pmatrix} L_2c_2 \\ L_2s_2 \\ 0 \end{pmatrix} - \begin{pmatrix} -L_3s_{23} \\ L_3c_{23} \\ 0 \end{pmatrix} \right\} \\ &= -\begin{pmatrix} 0 \\ 0 \\ \dot\theta_3 \end{pmatrix} \times \begin{pmatrix} -L_3s_{23} \\ L_3c_{23} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot\theta_1 \\ \dot\theta_2 \end{pmatrix} \times \begin{pmatrix} L_2c_2 + L_3s_{23} \\ L_2s_2 - L_3c_{23} \\ 0 \end{pmatrix} \end{aligned} \tag{3.62}$$

$${}^1\mathbf{V}_{P8} = -\begin{pmatrix} 0 \\ 0 \\ \dot\theta_3 \end{pmatrix} \times \begin{pmatrix} Z_5 \\ Z_6 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot\theta_1 \\ \dot\theta_2 \end{pmatrix} \times \begin{pmatrix} Z_7 \\ Z_8 \\ 0 \end{pmatrix}$$

$$= -\begin{pmatrix} -Z_6\dot{\theta}_3 \\ Z_5\dot{\theta}_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -Z_8\dot{\theta}_2 \\ Z_7\dot{\theta}_2 \\ -Z_7\dot{\theta}_1 \end{pmatrix} \tag{3.63}$$

Equating the x and y components in eqn. (3.60) and (3.63);

$$-Z_2\dot{\theta}_8 - Z_4\dot{\theta}_7 = Z_6\dot{\theta}_3 - Z_8\dot{\theta}_2 \tag{3.64}$$

$$Z_1\dot{\theta}_8 + Z_3\dot{\theta}_7 = -Z_5\dot{\theta}_3 + Z_7\dot{\theta}_2 \tag{3.65}$$

Our goal is to express $\dot{\theta}_3$ and $\dot{\theta}_8$ as a function of $\dot{\theta}_2$ and $\dot{\theta}_7$.

$$Z_6\dot{\theta}_3 + Z_2\dot{\theta}_8 = Z_8\dot{\theta}_2 - Z_4\dot{\theta}_7 \tag{3.66}$$

$$Z_5\dot{\theta}_3 + Z_1\dot{\theta}_8 = Z_7\dot{\theta}_2 - Z_3\dot{\theta}_7 \tag{3.67}$$

let

$$Z_9 = Z_6Z_1 - Z_2Z_5 \tag{3.68}$$

Then

$$\dot{\theta}_3 = \left(\frac{Z_8Z_1 - Z_2Z_7}{Z_9}\right)\dot{\theta}_2 + \left(\frac{Z_2Z_3 - Z_1Z_4}{Z_9}\right)\dot{\theta}_7 \tag{3.69}$$

$$\dot{\theta}_8 = \left(\frac{Z_6Z_7 - Z_5Z_8}{Z_9}\right)\dot{\theta}_2 + \left(\frac{Z_4Z_5 - Z_3Z_6}{Z_9}\right)\dot{\theta}_7 \tag{3.70}$$

or

$$\dot{\theta}_3 = Z_{10}\dot{\theta}_2 + Z_{11}\dot{\theta}_7 \tag{3.71}$$

$$\dot{\theta}_8 = Z_{12}\dot{\theta}_2 + Z_{13}\dot{\theta}_7 \tag{3.72}$$

Next the dependant angular acceleration is to be evaluated, for that purpose we need to derive $\mathbf{V}_{P8}$ in both eqns. (3.49) and (3.50).

Taking the time derivative of eqn. (3.49) one obtains:

$$\begin{aligned}\dot{\mathbf{V}}_{P8} = {} & \left(\ddot{\theta}_1\mathbf{k}_0 + \ddot{\theta}_7\mathbf{k}_1 + \dot{\theta}_1\mathbf{k}_0 \times \dot{\theta}_7\mathbf{k}_1\right) \times (L_7\mathbf{i}_7 + L_8\mathbf{i}_8) + \left(\dot{\theta}_1\mathbf{k}_0 + \dot{\theta}_7\mathbf{k}_1\right) \times \mathbf{V}_{P8} + \\ & \left[\ddot{\theta}_8\mathbf{k}_7 + \left(\dot{\theta}_1\mathbf{k}_0 + \dot{\theta}_7\mathbf{k}_1\right) \times \dot{\theta}_8\mathbf{k}_7\right] \times L_8\mathbf{i}_8 + \dot{\theta}_8\mathbf{k}_7 \times \left[\left(\dot{\theta}_1\mathbf{k}_0 + \dot{\theta}_7\mathbf{k}_1 + \dot{\theta}_8\mathbf{k}_7\right) \times L_8\mathbf{i}_8\right]\end{aligned} \tag{3.73}$$

Note that $\frac{d}{dt}(L_7\mathbf{i}_7 + L_8\mathbf{i}_8) = \mathbf{V}_{P8}$. Then,

$$
\begin{aligned}
{}^1\dot{\mathbf{V}}_{P8} &= \left\{\begin{pmatrix}0\\ \ddot\theta_1\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 0\\ \ddot\theta_7\end{pmatrix} + \begin{pmatrix}0\\ \dot\theta_1\\ 0\end{pmatrix} \times \begin{pmatrix}0\\ 0\\ \dot\theta_7\end{pmatrix}\right\} \times \begin{pmatrix}Z_3\\ Z_4\\ 0\end{pmatrix} + \begin{pmatrix}0\\ \dot\theta_1\\ \dot\theta_7\end{pmatrix} \times \begin{pmatrix}{}^1V_{P8_x}\\ {}^1V_{P8_y}\\ {}^1V_{P8_z}\end{pmatrix} + \\
&\left\{\begin{pmatrix}0\\ 0\\ \ddot\theta_8\end{pmatrix} + \begin{pmatrix}0\\ \dot\theta_1\\ \dot\theta_7\end{pmatrix} \times \begin{pmatrix}0\\ 0\\ \dot\theta_8\end{pmatrix}\right\} \times \begin{pmatrix}Z_1\\ Z_2\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 0\\ \dot\theta_8\end{pmatrix} \times \left\{\begin{pmatrix}0\\ \dot\theta_1\\ \dot\theta_7+\dot\theta_8\end{pmatrix} \times \begin{pmatrix}Z_1\\ Z_2\\ 0\end{pmatrix}\right\} \\
&= \begin{pmatrix}\dot\theta_1\dot\theta_7\\ \ddot\theta_1\\ \ddot\theta_7\end{pmatrix} \times \begin{pmatrix}Z_3\\ Z_4\\ 0\end{pmatrix} + \begin{pmatrix}0\\ \dot\theta_1\\ \dot\theta_7\end{pmatrix} \times \begin{pmatrix}{}^1V_{P8_x}\\ {}^1V_{P8_y}\\ {}^1V_{P8_z}\end{pmatrix} + \begin{pmatrix}\dot\theta_1\dot\theta_8\\ 0\\ \ddot\theta_8\end{pmatrix} \times \begin{pmatrix}Z_1\\ Z_2\\ 0\end{pmatrix} + \\
&\begin{pmatrix}0\\ 0\\ \dot\theta_8\end{pmatrix} \times \begin{pmatrix}-Z_2\left(\dot\theta_7+\dot\theta_8\right)\\ Z_1\left(\dot\theta_7+\dot\theta_8\right)\\ -Z_1\dot\theta_1\end{pmatrix} \\
&= \begin{pmatrix}-Z_4\ddot\theta_7\\ Z_3\ddot\theta_7\\ Z_4\dot\theta_1\dot\theta_7 - Z_3\ddot\theta_1\end{pmatrix} + \begin{pmatrix}\dot\theta_1\,{}^1V_{P8_z} - \dot\theta_7\,{}^1V_{P8_y}\\ \dot\theta_7\,{}^1V_{P8_x}\\ -\dot\theta_1\,{}^1V_{P8_x}\end{pmatrix} + \begin{pmatrix}-Z_2\ddot\theta_8\\ Z_1\ddot\theta_8\\ Z_2\dot\theta_1\dot\theta_8\end{pmatrix} + \begin{pmatrix}-Z_1\left(\dot\theta_7+\dot\theta_8\right)\dot\theta_8\\ -Z_2\left(\dot\theta_7+\dot\theta_8\right)\dot\theta_8\\ 0\end{pmatrix} \\
&= \begin{pmatrix}-Z_4\ddot\theta_7 + \dot\theta_1\,{}^1V_{P8_z} - \dot\theta_7\,{}^1V_{P8_y} - Z_2\ddot\theta_8 - Z_1\left(\dot\theta_7+\dot\theta_8\right)\dot\theta_8\\ Z_3\ddot\theta_7 + \dot\theta_7\,{}^1V_{P8_x} + Z_1\ddot\theta_8 - Z_2\left(\dot\theta_7+\dot\theta_8\right)\dot\theta_8\\ Z_4\dot\theta_1\dot\theta_7 - Z_3\ddot\theta_1 - \dot\theta_1\,{}^1V_{P8_x} + Z_2\dot\theta_1\dot\theta_8\end{pmatrix} \qquad (3.74)
\end{aligned}
$$

Deriving eqn. (3.50) one gets:

$$
\begin{aligned}
\dot{\mathbf{V}}_{P8} &= \left(\ddot\theta_1\mathbf{k}_0 + \ddot\theta_2\mathbf{k}_1 + \dot\theta_1\mathbf{k}_0 \times \dot\theta_2\mathbf{k}_1\right) \times (L_2\mathbf{i}_2 - L_3\mathbf{k}_3) + \left(\dot\theta_1\mathbf{k}_0 + \dot\theta_2\mathbf{k}_1\right) \times \mathbf{V}_{P8} - \qquad (3.75)\\
&\left[\ddot\theta_3\mathbf{k}_2 + \left(\dot\theta_1\mathbf{k}_0 + \dot\theta_2\mathbf{k}_1\right) \times \dot\theta_3\mathbf{k}_2\right] \times L_3\mathbf{k}_3 - \dot\theta_3\mathbf{k}_2 \times \left[\left(\dot\theta_1\mathbf{k}_0 + \dot\theta_2\mathbf{k}_1 + \dot\theta_3\mathbf{k}_2\right) \times L_3\mathbf{k}_3\right] \\
{}^1\dot{\mathbf{V}}_{P8} &= \left\{\begin{pmatrix}0\\ \ddot\theta_1\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 0\\ \ddot\theta_2\end{pmatrix} + \begin{pmatrix}0\\ \dot\theta_1\\ 0\end{pmatrix} \times \begin{pmatrix}0\\ 0\\ \dot\theta_2\end{pmatrix}\right\} \times \begin{pmatrix}Z_7\\ Z_8\\ 0\end{pmatrix} + \begin{pmatrix}0\\ \dot\theta_1\\ \dot\theta_2\end{pmatrix} \times \begin{pmatrix}{}^1V_{P8_x}\\ {}^1V_{P8_y}\\ {}^1V_{P8_z}\end{pmatrix} -
\end{aligned}
$$

$$
\left\{\begin{pmatrix}0\\0\\\ddot\theta_3\end{pmatrix}+\begin{pmatrix}0\\\dot\theta_1\\\dot\theta_2\end{pmatrix}\times\begin{pmatrix}0\\0\\\dot\theta_3\end{pmatrix}\right\}\times\begin{pmatrix}Z_5\\Z_6\\0\end{pmatrix}-\begin{pmatrix}0\\0\\\dot\theta_3\end{pmatrix}\times\left\{\begin{pmatrix}0\\\dot\theta_1\\\dot\theta_2+\dot\theta_3\end{pmatrix}\times\begin{pmatrix}Z_5\\Z_6\\0\end{pmatrix}\right\}
$$

$$
=\begin{pmatrix}\dot\theta_1\dot\theta_2\\\ddot\theta_1\\\ddot\theta_2\end{pmatrix}\times\begin{pmatrix}Z_7\\Z_8\\0\end{pmatrix}+\begin{pmatrix}0\\\dot\theta_1\\\dot\theta_2\end{pmatrix}\times\begin{pmatrix}{}^1V_{P8_x}\\{}^1V_{P8_y}\\{}^1V_{P8_z}\end{pmatrix}-\begin{pmatrix}\dot\theta_1\dot\theta_3\\0\\\ddot\theta_3\end{pmatrix}\times\begin{pmatrix}Z_5\\Z_6\\0\end{pmatrix}-
$$

$$
\begin{pmatrix}0\\0\\\dot\theta_3\end{pmatrix}\times\begin{pmatrix}-Z_6\left(\dot\theta_2+\dot\theta_3\right)\\Z_5\left(\dot\theta_2+\dot\theta_3\right)\\-Z_5\dot\theta_1\end{pmatrix}
$$

$$
=\begin{pmatrix}-Z_8\ddot\theta_2\\Z_7\ddot\theta_2\\Z_8\dot\theta_1\dot\theta_2-Z_7\ddot\theta_1\end{pmatrix}+\begin{pmatrix}\dot\theta_1\,{}^1V_{P8_z}-\dot\theta_2\,{}^1V_{P8_y}\\\dot\theta_2\,{}^1V_{P8_x}\\-\dot\theta_1\,{}^1V_{P8_x}\end{pmatrix}-\begin{pmatrix}-Z_6\ddot\theta_3\\Z_5\ddot\theta_3\\Z_6\dot\theta_1\dot\theta_3\end{pmatrix}-\begin{pmatrix}-Z_5\left(\dot\theta_2+\dot\theta_3\right)\dot\theta_3\\-Z_6\left(\dot\theta_2+\dot\theta_3\right)\dot\theta_3\\0\end{pmatrix}
$$

$$
=\begin{pmatrix}-Z_8\ddot\theta_2+\dot\theta_1\,{}^1V_{P8_z}-\dot\theta_2\,{}^1V_{P8_y}+Z_6\ddot\theta_3+Z_5\left(\dot\theta_2+\dot\theta_3\right)\dot\theta_3\\Z_7\ddot\theta_2+\dot\theta_2\,{}^1V_{P8_x}-Z_5\ddot\theta_3+Z_6\left(\dot\theta_2+\dot\theta_3\right)\dot\theta_3\\Z_8\dot\theta_1\dot\theta_2-Z_7\ddot\theta_1-\dot\theta_1\,{}^1V_{P8_x}-Z_6\dot\theta_1\dot\theta_3\end{pmatrix}\tag{3.76}
$$

Equating eqns. (3.74) and (3.76) and arranging terms one obtains:

$$
-Z_4\ddot\theta_7-Z_2\ddot\theta_8 = C_1 \tag{3.77}
$$

$$
Z_3\ddot\theta_7+Z_1\ddot\theta_8 = C_2 \tag{3.78}
$$

where C_1 and C_2 are defined as follows:

$$
C_1 = -Z_8\ddot\theta_2+Z_6\ddot\theta_3+{}^1V_{P8y}\left(\dot\theta_7-\dot\theta_2\right)+Z_5\left(\dot\theta_2+\dot\theta_3\right)\dot\theta_3+Z_1\left(\dot\theta_7+\dot\theta_8\right)\dot\theta_8 \tag{3.79}
$$

$$
C_2 = Z_7\ddot\theta_2-Z_5\ddot\theta_3+{}^1V_{P8x}\left(\dot\theta_2-\dot\theta_7\right)+Z_6\left(\dot\theta_2+\dot\theta_3\right)\dot\theta_3+Z_2\left(\dot\theta_7+\dot\theta_8\right)\dot\theta_8 \tag{3.80}
$$

Let

$$
\Delta = Z_2Z_3-Z_1Z_4 \tag{3.81}
$$

Hence

$$
\ddot\theta_7 = \left(C_1Z_1+C_2Z_2\right)/\Delta \tag{3.82}
$$

$$\ddot{\theta}_8 = -(C_1 Z_3 + C_2 Z_4)/\Delta \tag{3.83}$$

The next task to be undertaken is that of expressing the angular velocity of each link in its frame in two forms, one involving the generalized speeds $\dot{\theta}_1, \ldots, \dot{\theta}_6$ implicitly, the other explicitly. Hence

$$^1\omega_1 = \dot{\theta}_1 \, ^1R_0 \, ^0k_0 = \begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{bmatrix}$$

$$^1\omega_1 = \begin{pmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{pmatrix} \tag{3.84}$$

$$^2\omega_2 = \, ^2R_1(^1\omega_1 + \dot{\theta}_2 \, ^1\mathbf{k}_1) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{pmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} \right\}$$

$$^2\omega_2 = \begin{pmatrix} s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} ^2\omega_{2x} \\ ^2\omega_{2y} \\ \dot{\theta}_2 \end{pmatrix} \tag{3.85}$$

$$^3\omega_3 = \, ^3R_2(^2\omega_2 + \dot{\theta}_3 \, ^2\mathbf{k}_2) = \begin{bmatrix} c_3 & s_3 & 0 \\ 0 & 0 & -1 \\ -s_3 & c_3 & 0 \end{bmatrix} \left\{ \begin{pmatrix} ^2\omega_{2x} \\ ^2\omega_{2y} \\ \dot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{pmatrix} \right\}$$

$$^3\omega_3 = \begin{pmatrix} ^3\omega_{3x} \\ ^3\omega_{3y} \\ ^3\omega_{3z} \end{pmatrix} = \begin{pmatrix} s_{23}\dot{\theta}_1 \\ Z_{14}\dot{\theta}_2 - Z_{11}\dot{\theta}_7 \\ c_{23}\dot{\theta}_1 \end{pmatrix} = \begin{pmatrix} c_3 \, ^2\omega_{2x} + s_3 \, ^2\omega_{2y} \\ -(\dot{\theta}_2 + \dot{\theta}_3) \\ -s_3 \, ^2\omega_{2x} + c_3 \, ^2\omega_{2y} \end{pmatrix} \tag{3.86}$$

$$^4\omega_4 = \, ^4R_3(^3\omega_3 + \dot{\theta}_4 \, ^3\mathbf{k}_3) = \begin{bmatrix} c_4 & s_4 & 0 \\ 0 & 0 & 1 \\ s_4 & -c_4 & 0 \end{bmatrix} \left\{ \begin{pmatrix} ^3\omega_{3x} \\ ^3\omega_{3y} \\ ^3\omega_{3z} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_4 \end{pmatrix} \right\}$$

$$
{}^4\omega_4 = \begin{pmatrix} {}^4\omega_{4x} \\ {}^4\omega_{4y} \\ {}^4\omega_{4z} \end{pmatrix} = \begin{pmatrix} Z_{15}\dot{\theta}_1 + Z_{16}\dot{\theta}_2 + Z_{17}\dot{\theta}_7 \\ c_{23}\dot{\theta}_1 + \dot{\theta}_4 \\ Z_{18}\dot{\theta}_1 + Z_{19}\dot{\theta}_2 + Z_{20}\dot{\theta}_7 \end{pmatrix} = \begin{pmatrix} c_4\,{}^3\omega_{3x} + s_4\,{}^3\omega_{3y} \\ {}^3\omega_{3z} + \dot{\theta}_4 \\ s_4\,{}^3\omega_{3x} - c_4\,{}^3\omega_{3y} \end{pmatrix} \quad (3.87)
$$

$$
{}^5\omega_5 = {}^5R_4({}^4\omega_4 + \dot{\theta}_5\,{}^4\mathbf{k}_4) = \begin{bmatrix} c_5 & s_5 & 0 \\ 0 & 0 & 1 \\ s_5 & -c_5 & 0 \end{bmatrix} \left\{ \begin{pmatrix} {}^4\omega_{4x} \\ {}^4\omega_{4y} \\ {}^4\omega_{4z} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_5 \end{pmatrix} \right\} \quad (3.88)
$$

$$
{}^5\omega_5 = \begin{pmatrix} {}^5\omega_{5x} \\ {}^5\omega_{5y} \\ {}^5\omega_{5z} \end{pmatrix} = \begin{pmatrix} Z_{21}\dot{\theta}_1 + Z_{22}\dot{\theta}_2 + s_5\dot{\theta}_4 + Z_{23}\dot{\theta}_7 \\ Z_{18}\dot{\theta}_1 + Z_{19}\dot{\theta}_2 + \dot{\theta}_5 + Z_{20}\dot{\theta}_7 \\ Z_{24}\dot{\theta}_1 + Z_{25}\dot{\theta}_2 - c_5\dot{\theta}_4 + Z_{26}\dot{\theta}_7 \end{pmatrix} = \begin{pmatrix} c_5\,{}^4\omega_{4x} + s_5\,{}^4\omega_{4y} \\ {}^4\omega_{4z} + \dot{\theta}_5 \\ s_5\,{}^4\omega_{4x} - c_5\,{}^4\omega_{4y} \end{pmatrix}
$$

$$
{}^6\omega_6 = {}^6R_5({}^5\omega_5 + \dot{\theta}_6\,{}^5\mathbf{k}_5) = \begin{bmatrix} c_6 & s_6 & 0 \\ -s_6 & c_6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{pmatrix} {}^5\omega_{5x} \\ {}^5\omega_{5y} \\ {}^5\omega_{5z} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_6 \end{pmatrix} \right\} \quad (3.89)
$$

$$
{}^6\omega_6 = \begin{pmatrix} {}^6\omega_{6x} \\ {}^6\omega_{6y} \\ {}^6\omega_{6z} \end{pmatrix} = \begin{pmatrix} Z_{27}\dot{\theta}_1 + Z_{28}\dot{\theta}_2 + Z_{29}\dot{\theta}_4 + s_6\dot{\theta}_5 + Z_{30}\dot{\theta}_7 \\ Z_{31}\dot{\theta}_1 + Z_{32}\dot{\theta}_2 + Z_{33}\dot{\theta}_4 + c_6\dot{\theta}_5 + Z_{34}\dot{\theta}_7 \\ Z_{24}\dot{\theta}_1 + Z_{25}\dot{\theta}_2 - c_5\dot{\theta}_4 + \dot{\theta}_6 + Z_{26}\dot{\theta}_7 \end{pmatrix} = \begin{pmatrix} c_6\,{}^5\omega_{5x} + s_6\,{}^5\omega_{5y} \\ -s_6\,{}^5\omega_{5x} + c_6\,{}^5\omega_{5y} \\ {}^5\omega_{5z} + \dot{\theta}_6 \end{pmatrix}
$$

$$
{}^7\omega_7 = {}^7R_1({}^1\omega_1 + \dot{\theta}_7\,{}^1\mathbf{k}_1) = \begin{bmatrix} c_7 & s_7 & 0 \\ -s_7 & c_7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{pmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_7 \end{pmatrix} \right\}
$$

$$
{}^7\omega_7 = \begin{pmatrix} {}^7\omega_{7x} \\ {}^7\omega_{7y} \\ \dot{\theta}_7 \end{pmatrix} = \begin{pmatrix} s_7\dot{\theta}_1 \\ c_7\dot{\theta}_1 \\ \dot{\theta}_7 \end{pmatrix} \quad (3.90)
$$

$$
{}^8\omega_8 = {}^8R_7({}^7\omega_7 + \dot{\theta}_8\,{}^7\mathbf{k}_7) = \begin{bmatrix} c_8 & s_8 & 0 \\ -s_8 & c_8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{pmatrix} {}^7\omega_{7x} \\ {}^7\omega_{7y} \\ \dot{\theta}_7 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_8 \end{pmatrix} \right\}.
$$

$$
{}^8\omega_8 = \begin{pmatrix} {}^8\omega_{8x} \\ {}^8\omega_{8y} \\ {}^8\omega_{8z} \end{pmatrix} = \begin{pmatrix} s_{78}\dot{\theta}_1 \\ c_{78}\dot{\theta}_1 \\ Z_{12}\dot{\theta}_2 + Z_{35}\dot{\theta}_7 \end{pmatrix} = \begin{pmatrix} c_8\,{}^7\omega_{7x} + s_8\,{}^7\omega_{7y} \\ -s_8\,{}^7\omega_{7x} + c_8\,{}^7\omega_{7y} \\ \dot{\theta}_7 + \dot{\theta}_8 \end{pmatrix} \tag{3.91}
$$

As has been noted the angular velocity of each link is expressed in the frame attached to it. This leads automatically to the expressions of partial angular velocities in their frames, and facilitates later work, where it shall be assumed that the central principal axes of inertia of each link are parallel to the frame vectors attached to it.

When it comes to dealing with linear velocities of the center of mass of each link, which will be done next, it is not necessarily advantageous to express these velocities in their proper frame. Instead, it is best to use whatever vector basis permits one to write the simplest expression. However, it is necessary once again to construct for each velocity only an explicit expression as regards $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_4, \dot{\theta}_5, \dot{\theta}_6$ and $\dot{\theta}_7$, which is not the case in the classical Kane's equation. The velocity of the mass center of link i is defined as $\mathbf{V}_i^*$ and the velocity of the point of intersection of link i and link $i+1$ is defined as $\mathbf{V}_{i,i+1}$, hence the velocities of mass centers are:

$$
\mathbf{V}_1^* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.92}
$$

$$
{}^2\mathbf{V}_2^* = {}^2\omega_2 \times r_2\,{}^2\mathbf{i}_2 = \begin{pmatrix} s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} r_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ r_2\dot{\theta}_2 \\ Z_{36}\dot{\theta}_1 \end{pmatrix} \tag{3.93}
$$

$$
{}^2\mathbf{V}_{2,3} = {}^2\omega_2 \times l_2\,{}^2\mathbf{i}_2 = \begin{pmatrix} s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} l_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ l_2\dot{\theta}_2 \\ Z_{37}\dot{\theta}_1 \end{pmatrix} \tag{3.94}
$$

$$
{}^3\mathbf{V}_3^* = {}^3R_2\,{}^2\mathbf{V}_{2,3} + {}^3\omega_3 \times r_3\,{}^3\mathbf{k}_3
$$

$$
{}^{3}\mathbf{V}_3^* = \begin{pmatrix} (Z_{38} + r_3 Z_{14})\dot{\theta}_2 - r_3 Z_{11}\dot{\theta}_7 \\ -(Z_{37} + r_3 s_{23})\dot{\theta}_1 \\ Z_{39}\dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} Z_{40}\dot{\theta}_2 + Z_{41}\dot{\theta}_7 \\ Z_{42}\dot{\theta}_1 \\ Z_{39}\dot{\theta}_2 \end{pmatrix} \tag{3.95}
$$

in expressing $\mathbf{V}_4^*$ it is easier to express in frame 3, therefore it is just ${}^{3}\mathbf{V}_3^*$ where r_3 is replaced by r_4

$$
{}^{3}\mathbf{V}_4^* = \begin{pmatrix} Z_{43}\dot{\theta}_2 + Z_{44}\dot{\theta}_7 \\ Z_{45}\dot{\theta}_1 \\ Z_{39}\dot{\theta}_2 \end{pmatrix} \tag{3.96}
$$

The same for the point of intersect of link 4 and 5,

$$
{}^{3}\mathbf{V}_{4,5} = \begin{pmatrix} Z_{46}\dot{\theta}_2 + Z_{47}\dot{\theta}_7 \\ Z_{48}\dot{\theta}_1 \\ Z_{39}\dot{\theta}_2 \end{pmatrix} \tag{3.97}
$$

$$
{}^{5}\mathbf{V}_5^* = {}^{5}R_4\, {}^{4}R_3\, {}^{3}\mathbf{V}_{4,5} + {}^{5}\omega_5 \times r_5\, {}^{5}\mathbf{k}_5
$$

$$
{}^{5}\mathbf{V}_5^* = \begin{bmatrix} c_5 c_4 & c_5 s_4 & s_5 \\ s_4 & -c_4 & 0 \\ s_5 c_4 & s_5 s_4 & -c_5 \end{bmatrix} {}^{3}\mathbf{V}_{4,5} + {}^{5}\omega_5 \times r_5\, {}^{5}\mathbf{k}_5
$$

$$
{}^{5}\mathbf{V}_5^* = \begin{bmatrix} Z_{49} & Z_{50} & s_5 \\ s_4 & -c_4 & 0 \\ Z_{51} & Z_{52} & -c_5 \end{bmatrix} {}^{3}\mathbf{V}_{4,5} + {}^{5}\omega_5 \times r_5\, {}^{5}\mathbf{k}_5
$$

$$
{}^{5}\mathbf{V}_5^* = \begin{pmatrix} (Z_{50}Z_{48} + r_5 Z_{18})\dot{\theta}_1 + (Z_{49}Z_{46} + s_5 Z_{39} + r_5 Z_{19})\dot{\theta}_2 + r_5\dot{\theta}_5 + (Z_{49}Z_{47} + r_5 Z_{20})\dot{\theta}_7 \\ (-c_4 Z_{48} - r_5 Z_{21})\dot{\theta}_1 + (s_4 Z_{46} - r_5 Z_{22})\dot{\theta}_2 - s_5 r_5 \dot{\theta}_4 + (s_4 Z_{47} - r_5 Z_{23})\dot{\theta}_7 \\ Z_{52}Z_{48}\dot{\theta}_1 + (Z_{51}Z_{46} - c_5 Z_{39})\dot{\theta}_2 + Z_{51}Z_{47}\dot{\theta}_7 \end{pmatrix}
$$

$$
{}^{5}\mathbf{V}_5^* = \begin{pmatrix} (Z_{53} + r_5 Z_{18})\dot{\theta}_1 + (Z_{54} + r_5 Z_{19})\dot{\theta}_2 + r_5\dot{\theta}_5 + (Z_{55} + r_5 Z_{20})\dot{\theta}_7 \\ (Z_{56} - r_5 Z_{21})\dot{\theta}_1 + (Z_{57} - r_5 Z_{22})\dot{\theta}_2 - s_5 r_5 \dot{\theta}_4 + (Z_{58} - r_5 Z_{23})\dot{\theta}_7 \\ Z_{59}\dot{\theta}_1 + Z_{60}\dot{\theta}_2 + Z_{61}\dot{\theta}_7 \end{pmatrix} \tag{3.98}
$$

$$ {}^5\mathbf{V}_5^* = \begin{pmatrix} Z_{62}\dot\theta_1 + Z_{63}\dot\theta_2 + r_5\dot\theta_5 + Z_{64}\dot\theta_7 \\ Z_{65}\dot\theta_1 + Z_{66}\dot\theta_2 + Z_{67}\dot\theta_4 + Z_{68}\dot\theta_7 \\ Z_{59}\dot\theta_1 + Z_{60}\dot\theta_2 + Z_{61}\dot\theta_7 \end{pmatrix} \tag{3.99} $$

Expressing $\mathbf{V}_6^*$ in frame 5, this can be accomplished by replacing r_5 with r_6 in eqn. (3.98)

$$ {}^5\mathbf{V}_6^* = \begin{pmatrix} Z_{69}\dot\theta_1 + Z_{70}\dot\theta_2 + r_6\dot\theta_5 + Z_{71}\dot\theta_7 \\ Z_{72}\dot\theta_1 + Z_{73}\dot\theta_2 + Z_{74}\dot\theta_4 + Z_{75}\dot\theta_7 \\ Z_{59}\dot\theta_1 + Z_{60}\dot\theta_2 + Z_{61}\dot\theta_7 . \end{pmatrix} \tag{3.100} $$

$$ {}^7\mathbf{V}_7^* = {}^7\omega_7 \times r_7\, {}^7\mathbf{i}_7 = \begin{pmatrix} 0 \\ r_7\dot\theta_7 \\ Z_{76}\dot\theta_1 \end{pmatrix} \tag{3.101} $$

$$ {}^7\mathbf{V}_{7,8} = {}^7\omega_7 \times l_7\, {}^7\mathbf{i}_7 = \begin{pmatrix} 0 \\ l_7\dot\theta_7 \\ Z_{77}\dot\theta_1 \end{pmatrix} \tag{3.102} $$

$$ {}^8\mathbf{V}_8^* = {}^8R_7\, {}^7\mathbf{V}_{7,8} + {}^8\omega_8 \times r_8\, {}^8\mathbf{i}_8 $$

$$ {}^8\mathbf{V}_8^* = \begin{pmatrix} Z_{78}\dot\theta_7 \\ Z_{79}\dot\theta_2 + Z_{80}\dot\theta_7 \\ Z_{81}\dot\theta_1 \end{pmatrix} \tag{3.103} $$

In addition to partial angular velocities and partial linear velocities, we shall need the angular and linear accelerations of each link. To obtain the necessary expressions, (Appendix D.1), one can differentiate the available angular and linear velocities with respect to time, referring to eqns. (3.84)-(3.91);

$$ {}^1\dot\omega_1 = \begin{pmatrix} 0 \\ \ddot\theta_1 \\ 0 \end{pmatrix} \tag{3.104} $$

and for $i = 1, \ldots, 6$ and 8

$$ {}^{i+1}\dot\omega_{i+1} = {}^{i+1}R_i \left({}^i\dot\omega_i + \ddot\theta_{i+1}\, {}^i\mathbf{k}_i + {}^i\omega_i \times \dot\theta_{i+1}\, {}^i\mathbf{k}_i \right) \tag{3.105} $$

and

$$^7\dot{\omega}_7 = {}^7R_1\,({}^1\dot{\omega}_1 + \ddot{\theta}_7\,{}^1\mathbf{k}_1 + {}^1\omega_1 \times \dot{\theta}_7\,{}^1\mathbf{k}_1) \qquad (3.106)$$

and for the linear accelerations, (Appendix D.2), we differentiate eqns. (3.92)-(3.103):

$$^1\mathbf{a}_1^* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (3.107)$$

$$^2\mathbf{a}_2^* = {}^2\dot{\omega}_2 \times r_2\,{}^2\mathbf{i}_2 + {}^2\omega_2 \times ({}^2\omega_2 \times r_2\,{}^2\mathbf{i}_2) \qquad (3.108)$$

$$^2\mathbf{a}_{2,3} = {}^2\dot{\omega}_2 \times l_2\,{}^2\mathbf{i}_2 + {}^2\omega_2 \times ({}^2\omega_2 \times l_2\,{}^2\mathbf{i}_2) \qquad (3.109)$$

$$^3\mathbf{a}_{2,3} = {}^3R_2\,{}^2\mathbf{a}_{2,3} \qquad (3.110)$$

$$^3\mathbf{a}_3^* = {}^3\mathbf{a}_{2,3} + {}^3\dot{\omega}_3 \times r_3\,{}^3\mathbf{k}_3 + {}^3\omega_3 \times ({}^3\omega_3 \times r_3\,{}^3\mathbf{k}_3) \qquad (3.111)$$

$$^3\mathbf{a}_4^* = {}^3\mathbf{a}_{2,3} + {}^3\dot{\omega}_3 \times r_4\,{}^3\mathbf{k}_3 + {}^3\omega_3 \times ({}^3\omega_3 \times r_4\,{}^3\mathbf{k}_3) \qquad (3.112)$$

$$^3\mathbf{a}_{4,5} = {}^3\mathbf{a}_{2,3} + {}^3\dot{\omega}_3 \times d_4\,{}^3\mathbf{k}_3 + {}^3\omega_3 \times ({}^3\omega_3 \times d_4\,{}^3\mathbf{k}_3) \qquad (3.113)$$

$$^5\mathbf{a}_{4,5} = {}^5R_4\,{}^4R_3\,{}^3\mathbf{a}_{4,5} \qquad (3.114)$$

$$^5\mathbf{a}_5^* = {}^5\mathbf{a}_{4,5} + {}^5\dot{\omega}_5 \times r_5\,{}^5\mathbf{k}_5 + {}^5\omega_5 \times ({}^5\omega_5 \times r_5\,{}^5\mathbf{k}_5) \qquad (3.115)$$

$$^5\mathbf{a}_6^* = {}^5\mathbf{a}_{4,5} + {}^5\dot{\omega}_5 \times r_6\,{}^5\mathbf{k}_5 + {}^5\omega_5 \times ({}^5\omega_5 \times r_6\,{}^5\mathbf{k}_5) \qquad (3.116)$$

$$^7\mathbf{a}_7^* = {}^7\dot{\omega}_7 \times r_7\,{}^7\mathbf{i}_7 + {}^7\omega_7 \times ({}^7\omega_7 \times r_7\,{}^7\mathbf{i}_7) \qquad (3.117)$$

$$^7\mathbf{a}_{7,8} = {}^7\dot{\omega}_7 \times l_7\,{}^7\mathbf{i}_7 + {}^7\omega_7 \times ({}^7\omega_7 \times l_7\,{}^7\mathbf{i}_7) \qquad (3.118)$$

$$^8\mathbf{a}_{7,8} = {}^8R_7\,{}^7\mathbf{a}_{7,8} \qquad (3.119)$$

$$^8\mathbf{a}_8^* = {}^8\mathbf{a}_{7,8} + {}^8\dot{\omega}_8 \times r_8\,{}^8\mathbf{i}_8 + {}^8\omega_8 \times ({}^8\omega_8 \times r_8\,{}^8\mathbf{i}_8) \qquad (3.120)$$

The kinematical analysis performed so far provides the ingredients needed for the construction of expressions for the generalized inertia forces, which are:

$$\frac{\partial \mathrm{K}_i^*}{\partial \dot{\theta}_j} = \left(\frac{\partial \omega_i}{\partial \dot{\theta}_j}\right).\mathrm{T}_i^* + m_i \left(\frac{\partial \mathrm{V}_i}{\partial \dot{\theta}_j}\right).\mathbf{a}_i^* \qquad (3.121)$$

for $i = 1, \ldots, 8$ and $j = 1, 2, 4, 5, 6$ and 7

where

$$\mathbf{T}_i^* = -\alpha_i \mathbf{J}_i - \omega_i \times \mathbf{J}_i.\omega_i \qquad (3.122)$$

and

$$\mathbf{J}_i = \begin{pmatrix} J_{ix} \\ J_{iy} \\ J_{iz} \end{pmatrix} \tag{3.123}$$

is the moment of inertia along the principle axis that are parallel to the directions of frame i, (Appendix E). Eqn. (3.121) represents the contribution of link i to the generalized inertia $\frac{\partial \mathbf{K}_i^*}{\partial \dot\theta_j}$, (Appendix F.1). Therefore

$$\frac{\partial \mathbf{K}^*}{\partial \dot\theta_j} = \sum_{i=1}^{8} \frac{\partial \mathbf{K}_i^*}{\partial \dot\theta_j} = \sum_{i=1}^{8} \left(\frac{\partial \omega_i}{\partial \dot\theta_j}\right).\mathbf{T}_i^* + m_i \sum_{i=1}^{8} \left(\frac{\partial \mathbf{V}_i}{\partial \dot\theta_j}\right).\mathbf{a}_i^* \tag{3.124}$$

In addition to generalized inertia forces, we require the generalized active forces, that is

$$\frac{\partial \mathbf{K}_i}{\partial \dot\theta_j} = \left(\frac{\partial \omega_i}{\partial \dot\theta_j}\right).\mathbf{T}_i + \left(\frac{\partial \mathbf{V}_i}{\partial \dot\theta_j}\right).\mathbf{R}_i \tag{3.125}$$

and the generalized active forces of the whole system, (Appendix F.2), is

$$\frac{\partial \mathbf{K}}{\partial \dot\theta_j} = \sum_{i=1}^{8} \frac{\partial \mathbf{K}_i}{\partial \dot\theta_j} = \sum_{i=1}^{8} \left(\frac{\partial \omega_i}{\partial \dot\theta_j}\right).\mathbf{T}_i + \sum_{i=1}^{8} \left(\frac{\partial \mathbf{V}_i}{\partial \dot\theta_j}\right).\mathbf{R}_i \tag{3.126}$$

where $\mathbf{T}_i$ and $\mathbf{R}_i$ are the system of couple of torque and force equivalent to the set of contact and/or body forces acting on the i^{th} link of the manipulator applied at the mass center. In the case of the ASEA Robot arm there are two kind of forces that contribute to the generalized active forces $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_4, \mathbf{K}_5, \mathbf{K}_6$ and $\mathbf{K}_7$, namely, contact forces applied in order to drive the links, and the gravitational forces exerted on them by earth. Considering, first the contact forces, we replace the set of such forces transmitted from link $i-1$ to link i with a couple of torque $\mathbf{T}^{i-1/i}$ together with a force $\mathbf{R}^{i-1/i}$ applied at the mass center of link i. Similarly, the set of contact forces transmitted from link $i+1$ to link i is replaced with a couple of torque $\mathbf{T}^{i+1/i}$ together with a force $\mathbf{R}^{i+1/i}$ applied to i at a point fixed on i and coincides with the mass center of link $i+1$. The law of reaction requires from link i to apply a couple of torque $\mathbf{T}^{i/i+1}$ or $-\mathbf{T}^{i+1/i}$ together with the force $\mathbf{R}^{i/i+1}$ or $-\mathbf{R}^{i+1/i}$ applied to link $i+1$ at its mass center. As for gravitational forces exerted on link

i for $i = 1, \ldots, 8$ by the earth, these are denoted G_i and are expressed in the same frame as the linear velocities V_i^*, the reason for that is it facilitates the calculation of the dot multiplication with the partial velocities. Also g is the gravitational field. The starting value is

$$^1\mathbf{g} = \begin{pmatrix} 0 \\ -g \\ o \end{pmatrix} \tag{3.127}$$

where g is equal to $9.81 m/s^2$, hence

$$^1\mathbf{G}_1 = m_1 \, ^1\mathbf{g} = m_1^1 g(0, -1, 0) \, . \tag{3.128}$$

$$^2\mathbf{G}_2 = m_2 \, ^2R_1 \, ^1\mathbf{g} = m_2 g(-s_2, -c_2, 0) \tag{3.129}$$

$$^3\mathbf{G}_3 = m_3 \, ^3R_2 \, ^2\mathbf{g} = m_3 g(-s_{23}, 0, -c_{23}) \tag{3.130}$$

$$^3\mathbf{G}_4 = m_4 \, ^3R_2 \, ^2\mathbf{g} = m_4 g(-s_{23}, 0, -c_{23}) \tag{3.131}$$

$$^5\mathbf{G}_5 = m_5 \, ^5R_4 \, ^4R_3 \, ^3\mathbf{g} = m_5 g(-Z_{21}, -Z_{18}, -Z_{24}) \tag{3.132}$$

$$^5\mathbf{G}_6 = m_6 \, ^5R_4 \, ^4R_3 \, ^3\mathbf{g} = m_6 g(-Z_{21}, -Z_{18}, -Z_{24}) \tag{3.133}$$

$$^7\mathbf{G}_7 = m_7 \, ^7R_1 \, ^1\mathbf{g} = m_7 g(-s_7, -c_7, 0) \tag{3.134}$$

$$^8\mathbf{G}_8 = m_8 \, ^8R_7 \, ^7\mathbf{g} = m_8 g(-s_{78}, -c_{78}, 0) \tag{3.135}$$

Referring to Fig. 7, one can express $\frac{\partial \mathbf{K}_i}{\partial \theta_j}$, the contribution to the generalized active force $\frac{\partial \mathbf{K}}{\partial \theta_j}$ of all forces acting on particles of link i for $i = 1, \ldots, 8$. One must noted that $\mathbf{T}^{3/2} = \mathbf{T}^{8/3} = \mathbf{T}^{8/7} = 0$ and $\frac{\partial \mathbf{V}_i}{\partial \theta_j} = 0$ for $j = 1, 2, 4, 5, 6$ and 7, also knowing the following quantities:

$$\omega_1 - \omega_2 = -\dot{\theta}_2 \mathbf{k}_1 \tag{3.136}$$

$$\omega_1 - \omega_7 = -\dot{\theta}_7 \mathbf{k}_1 \tag{3.137}$$

$$\omega_2 - \omega_3 = -\dot{\theta}_3 \mathbf{k}_2 \tag{3.138}$$

$$\omega_3 - \omega_4 = -\dot{\theta}_4 \mathbf{k}_3 \tag{3.139}$$

$$\omega_4 - \omega_5 = -\dot{\theta}_5 \mathbf{k}_4 \tag{3.140}$$

$$\omega_5 - \omega_6 = -\dot{\theta}_6 \mathbf{k}_5 \tag{3.141}$$

One can set up the generalized active forces, (Appendix E.3), The results are:

$$\frac{\partial K}{\partial \dot{\theta}_1} = \tau_1 + Z_{83} \tag{3.142}$$

$$\frac{\partial K}{\partial \dot{\theta}_2} = -\tau_2 + Z_{86} \tag{3.143}$$

$$\frac{\partial K}{\partial \dot{\theta}_4} = -\tau_4 + Z_{87} \tag{3.144}$$

$$\frac{\partial K}{\partial \dot{\theta}_5} = -\tau_5 + Z_{88} \tag{3.145}$$

$$\frac{\partial K}{\partial \dot{\theta}_6} = -\tau_6 \tag{3.146}$$

$$\frac{\partial K}{\partial \dot{\theta}_7} = -\tau_7 + Z_{90} \tag{3.147}$$

Then equating the generalized active forces to the generalized inertia forces, the set of torques can be easily found.

3.7 SIMASEA Program Structure

The first problem in developing a simulator is the trajectory planning. For practical reasons the user of the robot system should not be concerned with solving complicated functions of space and time to specify the task. Instead he should specify the desired motion through single commands, with the system determining the details of the task. As a means of communication with the system, the user, should specify the task by programming the motion in a commercially available robot textual programming language.

In the same way, the user specifies the task to the SIMASEA program using a subset of the well known robot programming language VAL II [91], as if he or she were programming the task on a real robot. The main features of the VAL II are explained below followed by a brief description of the structure of the SIMASEA program. The chapter concludes with a discussion of the scheme used to interpolate between essential points of the trajectory.

3.7.1 VAL II Language and Features of SIMASEA

The commands from the VAL II language which are relevant to this simulation are summarized below;

1. PAUSE
2. SPEED
3. MOVE
4. MOVES
5. OPENI
6. CLOSEI
7. START
8. STOP

The basic instructions are the motion commands, namely MOVE and MOVES commands. They require the input of a destination point and the orientation of the gripper at that point. The former causes the robot to move by a joint-interpolated motion whereas the latter causes the end effector to move along a straight line. Both commands have an optional subcommand, namely VIA, which requests the specification of any VIA point before ending at the destination point. The speed setting can be changed anywhere within the program by the SPEED command. The command applies to all subsequent motions until the speed is again altered. This command also requires an argument (a real number between 0.0 and 100.0) that represents a percentage of the maximum speed at which the task can be completed. The default value of the speed is 100.0. The PAUSE command causes the robot to stay motionless for a certain amount of time equal to the argument of the command. The two commands OPENI and CLOSEI are used for gripper control. These commands cause the gripper to open or close immediately while the manipulator is motionless. These two commands are similar to the PAUSE command except that each has a time argument equal to the time required to open or to close the gripper. The START command defines the point in space where the gripper is initially located, and this must be the first *executable* statement. It can be issued only once during the editing. The STOP command is used to terminate the program.

At the beginning of editing of the VAL II program, the following commands can not be selected MOVE, MOVES and STOP commands. They can be used only after the START command has been issued.

Once the VAL II program is entirely typed in, SIMASEA displays the entire VAL II program, and then it computes the time history of position, velocity and acceleration of all members of the robot. It branches to the corresponding subprogram according to whether the command is a time delaying or a motion command. see Appendix A.2. In the motion commands, the joints kinematic quantities are provided by the trajectory planning subprogram for the MOVE command; whereas some further computations are needed in the case of MOVES command to evaluate the corresponding joints kinematic quantities. The execution then is transferred to the dynamics subprogram to compute the torques required to achieve the necessary task. Finally the vibration analysis is performed.
A sample VAL II program is given in Appendix A.1.

3.7.2 Motion Trajectory

It is desirable for the motion of the manipulator to be smooth, for that purpose continuity of position, velocity and acceleration are required. An approach developed by Paul [30] employing a quartic polynomial interpolation of motion achieves this objective. This approach of Paul is adopted in the present study. For a transition between different path segments a time of 2 t_{acc} is allowed to decelerate and accelerate from maximum negative velocity to maximum positive velocity. For continuity of motion a polynomial function of the fifth degree is required. There are six boundary conditions (position, velocity and acceleration at both ends of the transition). Due to symmetry a quartic polynomial is required;

$$q = a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \tag{3.148}$$

where q is the generalized joint position; linear, or angular. Deriving and applying the boundary conditions, one is able to obtain the functions specifying position q, velocity $\dot{q}$, and acceleration $\ddot{q}$ for the transition.

Chapter 4

RESULTS AND DISCUSSION

4.1 Result Validation

This analysis has required a good deal of very complex mathematical equations, that in the other hand would increase the probability of introducing faulty equations and some errors into SIMASEA. For that matter, an approach to assure an error free computer program would be very vital to this study.

For that consequence, three subprograms in the SIMASEA are mainly concerned, namely, these are the forward, the inverse kinematics and the dynamics subroutines. As a reminder, the dynamics study includes the forward velocity and acceleration kinematics, which gives an opportunity to cross examine these quantities with those computed earlier in the kinematics part. For the position study, the forward analysis has been conducted on a set of results computed by the inverse analysis on a given set of input data, has came back to that initial data set, as expected. The kinematic quantities of the closed loop were verified by considering point P8, Fig. 1, as a point on link 3, in one hand; and a point on link 8, in the other. The computation of dynamic quantities, present about one quarter of the whole dynamic analysis, offer no method to validate the result except probably some experimental procedures, or the use of an existing simulator package.

For the vibration analysis, the program was initially verified against some results given by the ADINA package, and were in good agreement. Then the vibration analysis program was added to the existing rigid body analysis part of the SIMASEA without any alteration.

4.2 Kinematic and Dynamics Results

The sample VAL II program in Appendix A.1 was used as a model input to the SIMASEA program since it contains a wide variety of VAL II commands. The relevant parameters to this study are listed in Appendix A.3. The results that are relevant to this section are the time history of the eight joint angles, the position and the orientation of the end effector, as well as their derivatives, and the torques that must applied to the joints to perform the desired trajectory.

The results were divided into two parts. The first part examines the study from the beginning of the VAL II program until the OPENI statement, which runs for 27 seconds. The second part runs from the OPENI statement until the end of the program, that is, from time 27 seconds until time 51.5 seconds:

As can be noticed, that the first part is made up of MOVE command as the only motion command, whereas in the second part, MOVES command is the sole motion command. It is expected that the joint angles $\theta_1, \theta_2, \ldots, \theta_6$ are interpolated using the algorithm developed by Paul [30], then the joint angles θ_7 and θ_8 are computed as dependent of the first six joint angles. Graph 1 represents the time history for the first three angles for the global task, graph 2 represents the position of the gripper for the whole task. Graphs 3 and 5 are identical to graphs 1 and 2 respectively, except that the curves are plotted for just the first part. Graph 4 is the plot of the second three joint angles for the first part. The shape of the trajectory in graphs 3 and 4, is as expected by the algorithm used. The via points are not on the trajectory, they are located at the intersection of the lines of constant velocities of the previous and the preceding portions to the point under consideration. The position and the orientation are plotted in graphs 5 and 6 respectively. Graph 7 shows the variations of θ_1, $\dot{\theta}_1$ and $\ddot{\theta}_1$ which are in good agreement with the algorithm for the trajectory planning. Graph 8 demonstrates the same principle of graph 7 for θ_4. Graphs 9 and 17 show the torque for both parts. It should be remarked that the torques at the PAUSE or the OPENI modes are the static forces to hold the members of the robot at the specified positions. Graphs 10 and 11 represent the position and the orientation of the gripper. Graphs 12 and 13 show the shape of the first and the second three joint angles respectively. Graph 14 and 15 demonstrate the second angle of the orientation and the z-component of the position vector respectively escorted by their derivatives. Graph 16 shows the variation of θ_1 and its derivatives when the motion is cartesian type.

4.3 Formulation of The Vibration Problem

The equation of equilibrium governing the linear dynamic response of a system of finite elements is:

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{C}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{R} \tag{4.1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the mass, damping, and stiffness matrices; $\mathbf{R}$ is the external load vector; and $\mathbf{U}, \dot{\mathbf{U}}$ and $\ddot{\mathbf{U}}$ are the displacement, the velocity, and the acceleration vectors of the finite element nodes. The coefficient matrices $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ of eqn. (4.1) are functions of the structure geometry and vary as the positions of the nodes are varied. This is consistent with the commonly made assumption in numerical methods [30] that the time dependent system parameters are held constant over each discretized time interval. The load vector is $-\mathbf{M}\ddot{U}_r$ where $\ddot{U}_r$ is the vector containing the rigid body accelerations at the nodes. The time step for the vibration analysis is very critical since this is directly related to the sampling time of the rigid body analysis, and to the natural frequencies of the system. The robot considered has been discretized into beam elements, Fig. 8. Node 1 is restricted to be motionless. All connections are assumed rigid at the instant of the analysis. This assumption is not very realistic since the three joints that are located at the closed loop except joint 1 are not rigid. If this fact has to be taken into account that would increase the global matrices, and eventually would extend the computation time.

4.4 Vibration Results

In this simulation analysis, the vibration analysis is optional to the user, because this part of the analysis consumes lots of computation time even though that the number of steps is 2 for the integration. The reason for this is that for each time step before the vibration analysis, the accelerations of the nodal points have to be transformed to the inertial frame, that are 83 accelerations to be transformed. There are 14 nodal points in the system, which is a bit too high for a faster computation. The major problem is most probably due to the subroutine to solve the set of the equations, since this one is a very general that does not take advantage of the zero elements of the global matrices.

4.5 Recommendations

This study can accommodate the computation of the stresses and the strains and the nodes easily, but again that would magnify the computation time. A future improvement would consider a better matrix solution subprogram to the FEM analysis that would examine a specific manipulator system with a fixed and fewer number of elements. Probably some experimental work would provide some information on the sensitivity of the result as some parameters are varied, such as the number of iterations in the vibration analysis, the way the manipulator is distretized into a number of elements, reduction of some degrees of freedom for some nodes.

Chapter 5

CONCLUSION

Manipulator dynamic simulation has long been one area of active development, more pronounced in the field of robotics. Considering the flexibility of the links is another complex and obscure problem that barely these two problems are examined in the same study. Moreover, much effort has been devoted to developing effective procedure to compute the inverse dynamics in real time.

In order to model the required simulation, for a robot with a closed loop, a computer program was developed to comply with the VAL II subset language and capable of compiling any VAL II program. In this study, almost all the areas in robot mechanics were studied in full depth, and were used in one single computer program. The use of efficient algorithms in the rigid body analysis makes it unique in its kind, with less than 700 statements for rigid body analysis and about 1100 statements for the whole analysis.

Vibration analysis to predict the behavior of the links of the robot was conducted, but enormous computation was involved that deterred the purpose of the dynamical deflection analysis, but some improvement can be brought to the analysis, as recommended in the previous chapter.

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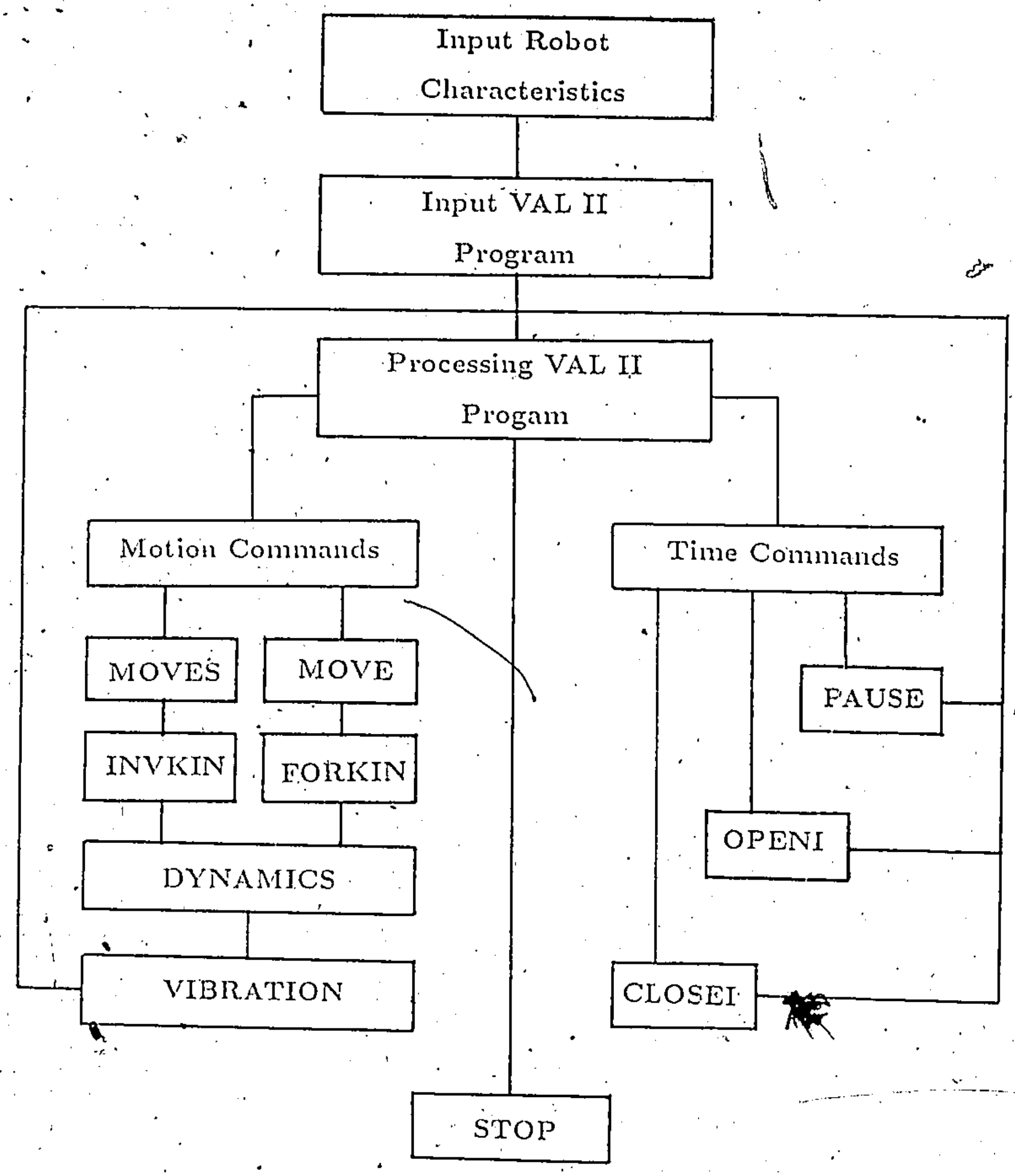
APPENDIX A

A.1 VAL II Sample Program

```
1. SPEED   100.
2. START   P1
3. MOVE    P2  VIA  P3  P4
4. MOVE    P6
5. OPENI
6. MOVES   P7
7. PAUSE   1.5
8. MOVES   P8  VIA  P9  P10
9. STOP
```

	The essential points					
	Position			*Orientation*		
P1	0.320	0.150	0.880	20.0	45.0	75.0
P2	0.270	0.280	0.820	37.0	15.0	23.0
P3	0.120	0.250	0.880	28.0	45.0	75.0
P4	0.190	0.220	0.940	68.0	18.0	42.0
P5	0.170	0.180	0.840	14.0	56.0	75.0
P6	0.260	0.180	0.740	35.0	70.0	43.0
P7	0.310	0.300	0.800	71.0	55.0	43.0
P8	0.170	0.240	0.880	14.0	56.0	75.0
P9	0.210	0.200	0.710	33.0	85.0	61.0
P10	0.204	0.185	0.720	23.0	19.0	33.0

A.2 Program Structure



A.3 Required Input Data

Parameter	Value	Description
t_{inc}	0.25	Sampling time
t_{acc}	1.50	Time to accelerate
t_{open}	1.00	Time to open gripper
t_{tclose}	1.00	Time to close gripper
$\dot{x}_m$, $\dot{y}_m$, $\dot{z}_m$	0.03	Maximun linear velocities of gripper
$\dot{\phi}_m$, $\dot{\theta}_m$, $\dot{\psi}_m$	50.0	Maximum rate of Euler's angle
$\dot{\theta}_{im}$	50.0	Maximum rate of joint angle for $i = 1, \ldots, 6$
D_1	0.55	Distance from the base to joint 2
L_2	0.60	Distance from joint 2 to joint 3
L_3	0.24	Distance from joint 2 to joint 8
D_4	0.54	Distance from joint 3 to joint 4
L_6	0.30	Distance from joint 5 to hand tip
L_7	0.25	Distance from joint 2 to joint 7
L_8	0.60	Distance from joint 7 to joint 8
E_x	2.0×10^{11}	Elastic modulus of element x
A_x	9.0×10^{-4}	Cross sectional area
ρ_x	7850.0	Density of element x
g	9.81	Gravitational field
ν	0.3	Poisson's ratio

	Range of Permissible angles					
Link	1	2	3	4	5	6
θ_{High}	180	90	270	180	180	180
θ_{Low}	-180	0	0	-180	-180	-180

	Distance from joint i to center of mass							
Link	1	2	3	4	5	6	7	8
r_i	0.40	0.30	0.20	0.30	0.10	0.20	0.12	0.30

	Mass of links							
Link	1	2	3	4	5	6	7	8
m_i	0.40	0.90	1.20	0.30	0.10	0.20	0.40	0.80

	Moment of inertia of links							
Link	1	2	3	4	5	6	7	8
J_{ix}	0.10	0.10	0.22	0.16	0.12	0.11	0.10	0.12
J_{iy}	0.10	0.24	0.25	0.14	0.10	0.09	0.18	0.20
J_{iz}	0.20	0.28	0.12	0.08	0.04	0.03	0.19	0.21

	Moment of inertia of elements													
Element	1	2	3	4	5	6	7	8	9	10	11	12	13	
J_x	0.20	0.10	0.10	0.10	0.12	0.1	0.12	0.12	0.12	0.12	0.14	0.04	0.03	0
J_y	0.04	0.07	0.07	0.08	0.08	0.08	0.08	0.07	0.07	0.07	0.05	0.03	0.04	0
J_z	0.03	0.06	0.06	0.09	0.09	0.09	0.09	0.08	0.08	0.08	0.05	0.02	0.05	0

	Element connectivity													
Element	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Left node	1	4	3	4	2	6	5	8	8	9	10	11	12	13
Right node	4	3	2	6	5	8	7	7	9	10	11	12	13	14

APPENDIX B

B.1 Orientation Matrix

$$
\begin{aligned}
\mathbf{E} &= {}^{a}R_{b}\,{}^{b}R_{c}\,{}^{c}R_{d} \\
&= Rot(\mathbf{z},\phi)Rot(\mathbf{y},\theta)Rot(\mathbf{z},\psi) \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathbf{E} &= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}
\end{aligned}
\tag{B.1}
$$

B.2 Rates of the Orientations

B.2.1 Angular velocity

$$
{}^{a}\omega_{b} = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix},
$$

$$
\begin{aligned}
{}^a\omega_c &= {}^a\omega_b + {}^a\omega_{c/b} \\
&= {}^a\omega_b + {}^aR_b\, {}^b\omega_{c/b} \\
&= {}^a\omega_b + {}^aR_b\, \dot\theta\, {}^by_b \\
&= \begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix} + \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix} + \begin{pmatrix} -s_\phi\dot\theta \\ c_\phi\dot\theta \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
{}^a\omega_c = \begin{pmatrix} -s_\phi\dot\theta \\ c_\phi\dot\theta \\ \dot\phi \end{pmatrix} \tag{B.2}
$$

$$
\begin{aligned}
{}^a\omega_d &= {}^a\omega_c + {}^a\omega_{d/c} \\
&= {}^a\omega_c + {}^aR_b\, {}^bR_c\, {}^c\omega_{d/c} \\
&= {}^a\omega_c + {}^aR_b\, {}^bR_c \dot\psi\, {}^cz_c \\
&= \begin{pmatrix} -s_\phi\dot\theta \\ c_\phi\dot\theta \\ \dot\phi \end{pmatrix} + \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix} \\
&= \begin{pmatrix} -s_\phi\dot\theta \\ c_\phi\dot\theta \\ \dot\phi \end{pmatrix} + \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} s_\theta\dot\psi \\ 0 \\ c_\theta\dot\psi \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^a\omega_d &= \begin{pmatrix} -s_\phi\dot\theta \\ c_\phi\dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} c_\phi s_\theta\dot\psi \\ s_\phi s_\theta\dot\psi \\ c_\theta\dot\psi \end{pmatrix} \\
&= \begin{pmatrix} -s_\phi\dot\theta + c_\phi s_\theta\dot\psi \\ c_\phi\dot\theta + s_\phi s_\theta\dot\psi \\ \dot\phi + c_\theta\dot\psi \end{pmatrix}
\end{aligned} \tag{B.3}
$$

B.2.2 Angular Acceleration

$$
\begin{aligned}
{}^a\dot{\omega}_b &= \begin{pmatrix} 0 \\ 0 \\ \ddot{\phi} \end{pmatrix} \\
{}^a\dot{\omega}_c &= {}^a\dot{\omega}_b + {}^a\dot{\omega}_{c/b} \\
&= \begin{pmatrix} 0 \\ 0 \\ \ddot{\phi} \end{pmatrix} + \begin{pmatrix} -s_\phi\ddot{\theta} \\ c_\phi\ddot{\theta} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} \times \begin{pmatrix} -s_\phi\dot{\theta} \\ c_\phi\dot{\theta} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -s_\phi\ddot{\theta} - c_\phi\dot{\theta}\dot{\phi} \\ c_\phi\ddot{\theta} - s_\phi\dot{\theta}\dot{\phi} \\ \ddot{\phi} \end{pmatrix}
\end{aligned} \tag{B.4}
$$

The same result would have been found if one derives ${}^a\omega_c$ in eqn. (B.2)

$$
\begin{aligned}
{}^a\dot{\omega}_d &= {}^a\dot{\omega}_c + {}^a\dot{\omega}_{d/c} + {}^a\omega_c \times {}^a\omega_{d/c} \\
&= \begin{pmatrix} -s_\phi\ddot{\theta} - c_\phi\dot{\theta}\dot{\phi} \\ c_\phi\ddot{\theta} - s_\phi\dot{\theta}\dot{\phi} \\ \ddot{\phi} \end{pmatrix} + {}^aR_b\,{}^bR_c \begin{pmatrix} 0 \\ 0 \\ \ddot{\psi} \end{pmatrix} + \begin{pmatrix} -s_\phi\dot{\theta} \\ c_\phi\dot{\theta} \\ \dot{\phi} \end{pmatrix} \times \begin{pmatrix} c_\phi s_\theta\dot{\psi} \\ s_\phi s_\theta\dot{\psi} \\ c_\theta\dot{\psi} \end{pmatrix} \\
&= \begin{pmatrix} -s_\phi\ddot{\theta} - c_\phi\dot{\theta}\dot{\phi} + c_\phi s_\theta\ddot{\psi} + c_\phi c_\theta\dot{\theta}\dot{\psi} - s_\phi s_\theta\dot{\phi}\dot{\psi} \\ c_\phi\ddot{\theta} - s_\phi\dot{\theta}\dot{\phi} + s_\phi s_\theta\ddot{\psi} + s_\phi c_\theta\dot{\theta}\dot{\psi} + c_\phi c_\theta\dot{\phi}\dot{\psi} \\ \ddot{\phi} + c_\theta\ddot{\psi} - s_\theta\dot{\theta}\dot{\psi} \end{pmatrix}
\end{aligned} \tag{B.5}
$$

This also could have been found by deriving eqn. (B.3) with respect to time.

B.3 Inverse Kinematic Velocity

$$
\mathbf{W}_1 = \mathbf{R} - \mathbf{P}_6 \tag{B.6}
$$

$$= \quad d_1\mathbf{k}_0 + l_2\mathbf{i}_2 + d_4\mathbf{k}_3 \tag{B.7}$$

deriving these eqns. (B.6) and (B.7), one obtains:

$$\dot{\mathbf{W}}_1 \quad = \quad \dot{\mathbf{R}} - \omega_6 \times {}^0\mathbf{P}_6 \tag{B.8}$$

$$= \quad \tilde{\omega}_2 \times l_2\mathbf{i}_2 + \omega_3 \times d_4\mathbf{k}_3 \tag{B.9}$$

Considering that:

$${}^1\omega_1 \quad = \quad \dot{\theta}_1\, {}^1R_0\, {}^0\mathbf{k}_0 = \begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{pmatrix} \tag{B.10}$$

$${}^2\omega_2 \quad = \quad {}^2R_1({}^1\omega_1 + \dot{\theta}_2\, {}^1\mathbf{k}_1) = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{pmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} {}^2\omega_{2x} \\ {}^2\omega_{2y} \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \tag{B.11}$$

$${}^2\omega_3 \quad = \quad ({}^2\omega_2 + \dot{\theta}_3\, {}^2\mathbf{k}_2) = \begin{pmatrix} s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix} \tag{B.12}$$

$${}^3\omega_3 \quad = \quad {}^3R_2\, {}^2\omega_3$$

$$= \begin{pmatrix} {}^3\omega_{3x} \\ {}^3\omega_{3y} \\ {}^3\omega_{3z} \end{pmatrix} = \begin{bmatrix} c_3 & s_3 & 0 \\ 0 & 0 & -1 \\ -s_3 & c_3 & 0 \end{bmatrix} \begin{pmatrix} {}^2\omega_{3x} \\ {}^2\omega_{3y} \\ {}^2\omega_{3z} \end{pmatrix} \tag{B.13}$$

$${}^2\dot{\mathbf{W}}_1 \quad = \quad {}^2\omega_2 \times l_2\, {}^2\mathbf{x}_2 + {}^2\omega_3 \times d_4\, {}^2R_3\, {}^3\mathbf{z}_3 \tag{B.14}$$

$$= \begin{pmatrix} s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} l_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix} \times \begin{bmatrix} c_3 & 0 & -s_3 \\ s_3 & 0 & c_3 \\ 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ d_4 \end{pmatrix}$$

$$= \begin{pmatrix} -d_4 c_3 \left(\dot\theta_2 + \dot\theta_3\right) \\ l_2\dot\theta_2 - d_4 s_3 \left(\dot\theta_2 + \dot\theta_3\right) \\ l_2 c_2 \dot\theta_1 + d_4 \left(s_2 c_3 + s_3 c_2\right) \dot\theta_1 \end{pmatrix} \tag{B.15}$$

$$\dot\theta_1 = \frac{{}^2\dot{W}_{1z}}{-l_2 c_2 + d_4 s_{23}} \tag{B.16}$$

$$\dot\theta_2 + \dot\theta_3 = \frac{{}^2\dot{W}_{1x}}{-d_4 c_3}$$

$$\dot\theta_2 = \frac{{}^2\dot{W}_{1y} + d_4 s_3 \left(\dot\theta_2 + \dot\theta_3\right)}{l_2}$$

$$\dot\theta_2 = \frac{{}^2\dot{W}_{1y} - \left(\frac{s_3}{c_3}\right) {}^2\dot{W}_{1x}}{l_2} \tag{B.17}$$

$$\dot\theta_3 = \frac{{}^2\dot{W}_{1x}}{-d_4 c_3} - \dot\theta_2 \tag{B.18}$$

The angular velocity of the hand is $\omega_{6/3}$, therefore:

$$\omega_{6/3} = \omega_6 - \omega_3 \tag{B.19}$$

$$^4\omega_{6/3} = {}^4\omega_6 - {}^4\omega_3 \tag{B.20}$$

$$^4\omega_{6/3} = {}^4R_3\,{}^3R_2 \left({}^2R_1\,{}^1R_0\,{}^0\omega_6 - {}^2\omega_3\right) \tag{B.21}$$

$$^4\omega_{6/3} = {}^4R_3\,{}^3R_2 \left\{ \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{bmatrix} \begin{pmatrix} {}^0\omega_{6x} \\ {}^0\omega_{6y} \\ {}^0\omega_{6z} \end{pmatrix} - \begin{pmatrix} {}^2\omega_{3x} \\ {}^2\omega_{3y} \\ {}^2\omega_{3z} \end{pmatrix} \right\}$$

$$^4\omega_{6/3} = \begin{bmatrix} c_4 & s_4 & 0 \\ 0 & 0 & 1 \\ s_4 & -c_4 & 0 \end{bmatrix} \begin{bmatrix} c_3 & s_3 & 0 \\ 0 & 0 & -1 \\ -s_3 & c_3 & 0 \end{bmatrix} \left\{ \begin{bmatrix} c_2 c_1 & c_2 s_1 & s_2 \\ -s_2 c_1 & -s_2 s_1 & c_2 \\ s_1 & -c_1 & 0 \end{bmatrix} \begin{pmatrix} {}^0\omega_{6x} \\ {}^0\omega_{6y} \\ {}^0\omega_{6z} \end{pmatrix} - \begin{pmatrix} {}^2\omega_{3x} \\ {}^2\omega_{3y} \\ {}^2\omega_{3z} \end{pmatrix} \right\}$$

$$^4\omega_{6/3} = \begin{bmatrix} c_4 c_3 & c_4 s_3 & s_4 \\ -s_3 & c_3 & 0 \\ s_4 c_3 & s_4 s_3 & -c_4 \end{bmatrix} \left\{ \begin{bmatrix} c_2 c_1 & c_2 s_1 & s_2 \\ -s_2 c_1 & -s_2 s_1 & c_2 \\ s_1 & -c_1 & 0 \end{bmatrix} \begin{pmatrix} {}^0\omega_{6x} \\ {}^0\omega_{6y} \\ {}^0\omega_{6z} \end{pmatrix} - \begin{pmatrix} {}^2\omega_{3x} \\ {}^2\omega_{3y} \\ {}^2\omega_{3z} \end{pmatrix} \right\} \tag{B.22}$$

In the other hand the augular velocity of the hand is expressed as a function of the hand joint variables;

$$\omega_{6/3} = \dot{\theta}_4 \mathbf{z}_3 + \dot{\theta}_5 \mathbf{z}_4 + \dot{\theta}_6 \mathbf{z}_5 \tag{B.23}$$

$$^4\omega_{6/3} = \dot{\theta}_4\, {}^4R_3\, {}^3\mathbf{z}_3 + \dot{\theta}_5\, {}^4\mathbf{z}_4 + \dot{\theta}_6\, {}^4R_5\, {}^5\mathbf{z}_5 \tag{B.24}$$

$$^4\omega_{6/3} = \begin{bmatrix} c_4 & s_4 & 0 \\ 0 & 0 & 1 \\ s_4 & -c_4 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_5 \end{pmatrix} + \begin{bmatrix} c_5 & 0 & s_5 \\ s_5 & 0 & -c_5 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_6 \end{pmatrix}$$

$$^4\omega_{6/3} = \begin{pmatrix} s_5\dot{\theta}_6 \\ \dot{\theta}_4 - c_5\dot{\theta}_6 \\ \dot{\theta}_5 \end{pmatrix} \tag{B.25}$$

Therefore equating these equations, one gets:

$$\dot{\theta}_5 = {}^4W_{6/3z} \tag{B.26}$$

$$\dot{\theta}_6 = \frac{{}^4W_{6/3x}}{s_5} \tag{B.27}$$

$$\dot{\theta}_4 = {}^4W_{6/3y} + c_5\ddot{\theta}_6 \tag{B.28}$$

B.4 Inverse Kinematic Accelerations

Find the wrist linear acceleration by deriving eqn. 8

$$\ddot{\mathbf{W}}_1 = \ddot{\mathbf{R}} - \dot{\omega}_6 \times \mathbf{P}_6 - \omega_6 \times (\omega_6 \times \mathbf{P}_6) \tag{B.29}$$

Differentiate eqn. 9

$$\ddot{\mathbf{W}}_1 = \dot{\omega}_2 \times \mathbf{W}_2 + \omega_2 \times \dot{\mathbf{W}}_2 - d_4\ddot{\theta}_3\, \mathbf{i}_3 - d_4\dot{\theta}_3\, \omega_3 \times \mathbf{i}_3 \tag{B.30}$$

Noting that $\dot{\mathbf{W}}_2 = \dot{\mathbf{W}}_1$, then we have

$$^2\ddot{\mathbf{W}}_1 = {}^2\dot{\omega}_2 \times {}^2\mathbf{W}_2 + {}^2\omega_2 \times {}^2\dot{\mathbf{W}}_1 - d_4\ddot{\theta}_3\, {}^2\mathbf{i}_3 - d_4\dot{\theta}_3\, {}^2\omega_3 \times {}^2\mathbf{i}_3 \tag{B.31}$$

Starting by evaluating the angular acceleration:

$$^{0}\dot{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{pmatrix} \tag{B.32}$$

$$^{1}\dot{\omega}_1 = {}^{1}R_0\, {}^{0}\dot{\omega}_1 = \begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{pmatrix}$$

$$^{1}\dot{\omega}_1 = \begin{pmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{pmatrix} \tag{B.33}$$

$$^{1}\dot{\omega}_2 = {}^{1}\dot{\omega}_1 + \ddot{\theta}_2\, {}^{1}\mathbf{k}_1 + {}^{1}\omega_1 \times \dot{\theta}_2\, {}^{1}\mathbf{k}_1$$

$$^{1}\dot{\omega}_2 = \begin{pmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix}$$

$$^{1}\dot{\omega}_2 = \begin{pmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} ^{1}\dot{\omega}_{2x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} \tag{B.34}$$

$$^{2}\dot{\omega}_2 = {}^{2}R_1\, {}^{1}\dot{\omega}_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} ^{1}\dot{\omega}_{2x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}$$

$$^{2}\dot{\omega}_2 = \begin{pmatrix} c_2\, {}^{1}\dot{\omega}_{2x} + s_2\ddot{\theta}_1 \\ -s_2\, {}^{1}\dot{\omega}_{2x} + c_2\ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} ^{2}\dot{\omega}_{2x} \\ ^{2}\dot{\omega}_{2y} \\ \ddot{\theta}_2 \end{pmatrix} \tag{B.35}$$

$$^{2}\dot{\omega}_3 = {}^{2}\dot{\omega}_2 + \ddot{\theta}_3\, {}^{2}\mathbf{k}_2 + {}^{2}\omega_2 \times \dot{\theta}_3\, {}^{2}\mathbf{k}_2$$

$$
{}^2\dot{\omega}_3 = \begin{pmatrix} {}^2\dot{\omega}_{2x} \\ {}^2\dot{\omega}_{2y} \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_3 \end{pmatrix} + \begin{pmatrix} {}^2\omega_{2x} \\ {}^2\omega_{2y} \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{pmatrix}
$$

$$
{}^2\dot{\omega}_3 = \begin{pmatrix} {}^2\dot{\omega}_{2x} + \dot{\theta}_3\, {}^2\omega_{2y} \\ {}^2\dot{\omega}_{2y} - \dot{\theta}_3\, {}^2\omega_{2x} \\ \ddot{\theta}_2 + \ddot{\theta}_3 \end{pmatrix} = \begin{pmatrix} {}^2\dot{\omega}_{3x} \\ {}^2\dot{\omega}_{3y} \\ {}^2\dot{\omega}_{3z} \end{pmatrix} \tag{B.36}
$$

$$
{}^3\dot{\omega}_3 = {}^3R_2\, {}^2\dot{\omega}_3 = \begin{bmatrix} c_3 & s_3 & 0 \\ 0 & 0 & -1 \\ -s_3 & c_3 & 0 \end{bmatrix} = \begin{pmatrix} {}^2\dot{\omega}_{3x} \\ {}^2\dot{\omega}_{3y} \\ {}^2\dot{\omega}_{3z} \end{pmatrix}
$$

$$
{}^3\dot{\omega}_3 = \begin{pmatrix} c_3\, {}^2\dot{\omega}_{3x} + s_3\, {}^2\dot{\omega}_{3y} \\ -\, {}^2\dot{\omega}_{3z} \\ -s_3\, {}^2\dot{\omega}_{3x} + c_3\, {}^2\dot{\omega}_{3y} \end{pmatrix} = \begin{pmatrix} {}^3\dot{\omega}_{3x} \\ -{}^2\dot{\omega}_{3z} \\ {}^3\dot{\omega}_{3z} \end{pmatrix} \tag{B.37}
$$

Also we know that;

$$
{}^1\mathbf{W}_2 = \begin{pmatrix} {}^1W_{2x} \\ {}^1W_{2y} \\ 0 \end{pmatrix} \tag{B.38}
$$

Then one can evaluate the following:

$$
{}^1\dot{\omega}_2 \times {}^1\mathbf{W}_2 = \begin{pmatrix} -\ddot{\theta}_2\, {}^1W_{2y} \\ \ddot{\theta}_2\, {}^1W_{2x} \\ \dot{\theta}_1\dot{\theta}_2\, {}^1W_{2y} - \ddot{\theta}_1\, {}^1W_{2x} \end{pmatrix} \tag{B.39}
$$

$$
\begin{aligned}
{}^2\dot{\omega}_2 \times {}^2\mathbf{W}_2 &= {}^2R_1\left({}^1\dot{\omega}_2 \times {}^1\mathbf{W}_2\right) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -\ddot{\theta}_2\, {}^1W_{2y} \\ \ddot{\theta}_2\, {}^1W_{2x} \\ \dot{\theta}_1\dot{\theta}_2\, {}^1W_{2y} - \ddot{\theta}_1\, {}^1W_{2x} \end{pmatrix} \\
&= \begin{pmatrix} \ddot{\theta}_2\left(-c_2\, {}^1W_{2y} + s_2\, {}^1W_{2x}\right) \\ \ddot{\theta}_2\left(s_2\, {}^1W_{2y} + c_2\, {}^1W_{2x}\right) \\ \dot{\theta}_1\dot{\theta}_2\, {}^1W_{2y} - \ddot{\theta}_1\, {}^1W_{2x} \end{pmatrix}
\end{aligned} \tag{B.40}
$$

This expression can be reduced further, noting:

$$-c_2\,{}^1W_{2y} + s_2\,{}^1W_{2x} = -d_4c_3 \tag{B.41}$$

$$s_2\,{}^1W_{2y} + c_2\,{}^1W_{2x} = l_2 - d_4s_3 \tag{B.42}$$

Therefore,

$${}^2\dot{\omega}_2 \times {}^2\mathbf{W}_2 = \begin{pmatrix} -\ddot{\theta}_2 d_4 c_3 \\ \ddot{\theta}_2\left(l_2 - d_4 s_3\right) \\ \dot{\theta}_1\dot{\theta}_2\,{}^1W_{2y} - \ddot{\theta}_1\,{}^1W_{2x} \end{pmatrix} \tag{B.43}$$

The last term is:

$${}^3\omega_3 \times {}^3\mathbf{i}_3 = \begin{pmatrix} {}^3\omega_{3x} \\ {}^3\omega_{3y} \\ {}^3\omega_{3z} \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ {}^3\omega_{3z} \\ -\,{}^3\omega_{3y} \end{pmatrix} \tag{B.44}$$

or

$$\begin{aligned} {}^2\omega_3 \times {}^2\mathbf{x}_3 &= {}^2R_3\left({}^3\omega_3 \times {}^3\mathbf{x}_3\right) = \begin{bmatrix} c_3 & 0 & -s_3 \\ s_3 & 0 & c_3 \\ 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ {}^3\omega_{3z} \\ -\,{}^3\omega_{3y} \end{pmatrix} \\ &= \begin{pmatrix} s_3\,{}^3\omega_{3y} \\ -c_3\,{}^3\omega_{3y} \\ -\,{}^3\omega_{3z} \end{pmatrix} \end{aligned} \tag{B.45}$$

The term

$$\begin{aligned} d_4\ddot{\theta}_3\,{}^2\mathbf{x}_3 &= d_4\ddot{\theta}_3\,{}^2R_3\,{}^3\mathbf{x}_3 = \begin{bmatrix} c_3 & 0 & -s_3 \\ s_3 & 0 & c_3 \\ 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= d_4\ddot{\theta}_3 \begin{pmatrix} c_3 \\ s_3 \\ 0 \end{pmatrix} \end{aligned} \tag{B.46}$$

Then eqn. (B.31) would be as follows:

$$^2\ddot{\mathbf{W}}_1 = \begin{pmatrix} -\ddot{\theta}_2 d_4 c_3 \\ \ddot{\theta}_2 (l_2 - d_4 s_3) \\ \dot{\theta}_1 \dot{\theta}_2 \, ^1W_{2y} - \ddot{\theta}_1 \, ^1W_{2x} \end{pmatrix} + \, ^2\omega_2 \times \, ^2\dot{\mathbf{W}}_1 - \begin{pmatrix} d_4 c_3 \ddot{\theta}_3 \\ d_4 s_3 \ddot{\theta}_3 \\ 0 \end{pmatrix} - d_4 \dot{\theta}_3 \begin{pmatrix} s_3 \, ^3\omega_{3y} \\ -c_3 \, ^3\omega_{3y} \\ - \, ^3\omega_{3z} \end{pmatrix}$$

Arranging terms,

$$\begin{pmatrix} -d_4 c_3 \left(\ddot{\theta}_2 + \ddot{\theta}_3\right) \\ \ddot{\theta}_2 (l_2 - d_4 s_3) - d_4 s_3 \ddot{\theta}_3 \\ -\ddot{\theta}_1 \, ^1W_{2x} \end{pmatrix} = \, ^2\ddot{\mathbf{W}}_1 - \, ^2\omega_2 \times \, ^2\dot{\mathbf{W}}_1 + \begin{pmatrix} d_4 s_3 \dot{\theta}_3 \, ^3\omega_{3y} \\ -d_4 c_3 \dot{\theta}_3 \, ^3\omega_{3y} \\ -\dot{\theta}_1 \dot{\theta}_2 \, ^1W_{2y} - d_4 \dot{\theta}_3 \, ^3\omega_{3z} \end{pmatrix}.$$

Defining the right side as $^2\ddot{\mathbf{U}}_4$, therefore we have:

$$\begin{pmatrix} -d_4 c_3 \left(\ddot{\theta}_2 + \ddot{\theta}_3\right) \\ \ddot{\theta}_2 l_2 - d_4 s_3 \left(\ddot{\theta}_2 + \ddot{\theta}_3\right) \\ -\ddot{\theta}_1 \, ^1W_{2x} \end{pmatrix} = \begin{pmatrix} ^2\ddot{U}_{4x} \\ ^2\ddot{U}_{4y} \\ ^2\ddot{U}_{4z} \end{pmatrix} \tag{B.47}$$

Now the joint accelerations can be found:

$$\ddot{\theta}_1 = -\frac{^2\ddot{U}_{4z}}{^1W_{2x}} \tag{B.48}$$

$$\left(\ddot{\theta}_2 + \ddot{\theta}_3\right) = -\frac{^2\ddot{U}_{4x}}{d_4 c_3}$$

$$\ddot{\theta}_2 = \left(^2\ddot{U}_{4y} - \frac{s_3}{c_3} \, ^2\ddot{U}_{4x}\right) \frac{1}{l_2} \tag{B.49}$$

$$\ddot{\theta}_3 = -\frac{^2\ddot{U}_{4x}}{d_4 c_3} - \ddot{\theta}_2 \tag{B.50}$$

The hand angular acceleration relative to the forearm is foud by differenciating eqn. (B.20) with respect to time, then one obtains:

$$^4\dot{\omega}_{6/3} = \, ^4R_3 \, ^3R_2 \left(^2R_1 \, ^1R_0 \, ^0\dot{\omega}_6 - \, ^2\dot{\omega}_3 - ^2\omega_3 \times \, ^2\omega_{6/3}\right). \tag{B.51}$$

The last three joint accelerations in terms of the last three angular acceleration, by deriving eqn. (B.24)

$$\dot{\omega}_{6/3} = \ddot{\theta}_4 \mathbf{z}_3 + \ddot{\theta}_5 \mathbf{z}_4 + \ddot{\theta}_6 \mathbf{z}_5 + \dot{\theta}_4 \mathbf{z}_3 \times \dot{\theta}_5 \mathbf{z}_4 + \left(\dot{\theta}_4 \mathbf{z}_3 + \dot{\theta}_5 \mathbf{z}_4\right) \times \dot{\theta}_6 \dot{\mathbf{z}}_5 \tag{B.52}$$

or

$$
\begin{aligned}
{}^4\dot{\omega}_{6/3} = \; & \ddot{\theta}_4\, {}^4R_3\, {}^3\mathbf{z}_3 + \ddot{\theta}_5\, {}^4\mathbf{z}_4 + \ddot{\theta}_6\, {}^4R_5\, {}^5\mathbf{z}_5 + \\
& \dot{\theta}_4\, {}^4R_3\, {}^3\mathbf{z}_3 \times \dot{\theta}_5\, {}^4\mathbf{z}_4 + \\
& \left(\dot{\theta}_4\, {}^4R_3\, {}^3\mathbf{z}_3 + \dot{\theta}_5\, {}^4\mathbf{z}_4\right) \times \dot{\theta}_6\, {}^4R_5\, {}^5\mathbf{z}_5
\end{aligned}
\tag{B.53}
$$

Hence

$$
{}^4\dot{\omega}_{6/3} = \begin{pmatrix} s_5\ddot{\theta}_6 \\ \ddot{\theta}_4 - c_5\ddot{\theta}_6 \\ \ddot{\theta}_5 \end{pmatrix} + \begin{pmatrix} \dot{\theta}_4\dot{\theta}_5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} c_5\dot{\theta}_5\dot{\theta}_6 \\ s_5\dot{\theta}_5\dot{\theta}_6 \\ -s_5\dot{\theta}_4\dot{\theta}_6 \end{pmatrix}
\tag{B.54}
$$

The joint acceleration can be found:

$$
\ddot{\theta}_6 = \frac{1}{s_5}\left({}^4\dot{W}_{6/3x} - \dot{\theta}_4\dot{\theta}_5 - c_5\dot{\theta}_5\dot{\theta}_6\right) \tag{B.55}
$$

$$
\ddot{\theta}_5 = {}^4\dot{W}_{6/3z} + s_5\dot{\theta}_4\dot{\theta}_6 \tag{B.56}
$$

$$
\ddot{\theta}_4 = {}^4\dot{W}_{6/3y} + c_5\ddot{\theta}_6 - s_5\dot{\theta}_5\dot{\theta}_6 \tag{B.57}
$$

APPENDIX C

$$
\begin{aligned}
Z_1 &= l_8 c_{78} \\
Z_2 &= l_8 s_{78} \\
Z_3 &= l_7 c_7 + Z_1 \\
Z_4 &= l_7 s_7 + Z_2 \\
Z_5 &= -l_3 s_{23} \\
Z_6 &= l_3 c_{23} \\
Z_7 &= l_2 c_2 - Z_5 \\
Z_8 &= l_2 s_2 - Z_6 \\
Z_9 &= Z_1 Z_6 - Z_2 Z_5 \\
Z_{10} &= (Z_1 Z_8 - Z_2 Z_7)/Z_9 \\
Z_{11} &= (Z_2 Z_3 - Z_1 Z_4)/Z_9 \\
Z_{12} &= (Z_5 Z_8 - Z_6 Z_7)/Z_9 \\
Z_{13} &= (Z_6 Z_3 - Z_4 Z_5)/Z_9 \\
Z_{14} &= -1 - Z_{10} \\
Z_{15} &= c_4 s_{23} \\
Z_{16} &= s_4 Z_{14}
\end{aligned}
$$

$$
\begin{aligned}
Z_{17} &= -s_4 Z_{11} \\
Z_{18} &= s_4 s_{23} \\
Z_{19} &= -c_4 Z_{14} \\
Z_{20} &= c_4 Z_{11} \\
Z_{21} &= c_5 Z_{15} + s_5 c_{23} \\
Z_{22} &= c_5 Z_{16} \\
Z_{23} &= c_5 Z_{17} \\
Z_{24} &= s_5 Z_{15} - c_5 c_{23} \\
Z_{25} &= s_5 Z_{16} \\
Z_{26} &= s_5 Z_{17} \\
Z_{27} &= c_6 Z_{21} + s_6 Z_{18} \\
Z_{28} &= c_6 Z_{22} + s_6 Z_{19} \\
Z_{29} &= c_6 s_5 \\
Z_{30} &= c_6 Z_{23} + s_6 Z_{20} \\
Z_{31} &= -s_6 Z_{21} + c_6 Z_{18} \\
Z_{32} &= -s_6 Z_{22} + c_6 Z_{19} \\
Z_{33} &= -s_6 s_5 \\
Z_{34} &= -s_6 Z_{23} + c_6 Z_{20} \\
Z_{35} &= 1 + Z_{13} \\
Z_{36} &= -r_2 c_2 \\
Z_{37} &= -l_2 c_2 \\
Z_{38} &= s_3 l_2 \\
Z_{39} &= c_3 l_2 \\
Z_{40} &= Z_{38} + r_3 Z_{14}
\end{aligned}
$$

$$
\begin{aligned}
Z_{41} &= -r_3 Z_{11} \\
Z_{42} &= -Z_{37} - r_3 s_{23} \\
Z_{43} &= Z_{38} + r_4 Z_{14} \\
Z_{44} &= -r_4 Z_{11} \\
Z_{45} &= -Z_{37} - r_4 s_{23} \\
Z_{46} &= Z_{38} + d_4 Z_{14} \\
Z_{47} &= -d_4 Z_{11} \\
Z_{48} &= -Z_{37} - d_4 s_{23} \\
Z_{49} &= c_5 c_4 \\
Z_{50} &= c_5 s_4 \\
Z_{51} &= s_5 c_4 \\
Z_{52} &= s_5 s_4 \\
Z_{53} &= Z_{50} Z_{48} \\
Z_{54} &= Z_{49} Z_{46} + s_5 Z_{39} \\
Z_{55} &= Z_{49} Z_{47} \\
Z_{56} &= -c_4 Z_{48} \\
Z_{57} &= s_4 Z_{46} \\
Z_{58} &= s_4 Z_{47} \\
Z_{59} &= Z_{52} Z_{48} \\
Z_{60} &= Z_{51} Z_{46} - c_5 Z_{39} \\
Z_{61} &= Z_{51} Z_{47} \\
Z_{62} &= Z_{53} + r_5 Z_{18} \\
Z_{63} &= Z_{54} + r_5 Z_{19} \\
Z_{64} &= Z_{55} + r_5 Z_{20}
\end{aligned}
$$

$$
\begin{aligned}
Z_{65} &= Z_{56} - r_5 Z_{21} \\
Z_{66} &= Z_{57} - r_5 Z_{22} \\
Z_{67} &= -r_5 s_5 \\
Z_{68} &= Z_{58} - r_5 Z_{23} \\
Z_{69} &= Z_{53} + r_6 Z_{18} \\
Z_{70} &= Z_{54} + r_6 Z_{19} \\
Z_{71} &= Z_{55} + r_6 Z_{20} \\
Z_{72} &= Z_{56} - r_6 Z_{21} \\
Z_{73} &= Z_{57} - r_6 Z_{22} \\
Z_{74} &= -r_6 s_5 \\
Z_{75} &= Z_{58} - r_6 Z_{23} \\
Z_{76} &= -r_7 c_7 \\
Z_{77} &= -l_7 r_7 \\
Z_{78} &= s_8 l_7 \\
Z_{79} &= r_8 Z_{12} \\
Z_{80} &= c_8 l_7 + r_8 Z_{35} \\
Z_{81} &= Z_{77} - r_8 c_{78} \\
Z_{82} &= Z_{59} Z_{24} \\
Z_{83} &= -g[m_5(Z_{62} Z_{21} + Z_{65} Z_{18} + Z_{82}) + m_6(Z_{69} Z_{21} + Z_{72} Z_{18} + Z_{82})] \\
Z_{84} &= c_{23} Z_{39} \\
Z_{85} &= Z_{60} Z_{24} \\
Z_{86} &= -g[m_2 r_2 c_2 + m_3(s_{23} Z_{40} + Z_{84}) + m_4(s_{23} Z_{43} + Z_{84}) + \\
&\quad m_5(Z_{63} Z_{21} + Z_{66} Z_{18} + Z_{85}) + m_6(Z_{70} Z_{21} + Z_{73} Z_{18} + Z_{85}) + m_8 c_{78} Z_{79}] \\
Z_{87} &= -g Z_{18}(m_5 Z_{67} + m_6 Z_{74})
\end{aligned}
$$

$$Z_{88} = -gZ_{21}(m_5r_5 + m_6r_6)$$

$$Z_{89} = Z_{61}Z_{24}$$

$$Z_{90} = -g[s_{23}(m_3Z_{41} + m_4Z_{44}) + m_5(Z_{64}Z_{21} + Z_{68}Z_{18} + Z_{89}) + m_6(Z_{71}Z_{21} + Z_{75}Z_{18} + Z_{89}) + m_7r_7c_7 + m_8(Z_{78}s_{78} + Z_{80}c_{78})]$$

APPENDIX D

D.1 Angular Accelerations:

$$
\begin{aligned}
{}^{0}\dot{\omega}_1 &= \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{pmatrix} \\
{}^{1}\dot{\omega}_1 &= {}^{1}R_0\,{}^{0}\dot{\omega}_1 = \begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{pmatrix} \\
{}^{1}\dot{\omega}_1 &= \begin{pmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{pmatrix} \\
{}^{1}\dot{\omega}_2 &= {}^{1}\dot{\omega}_1 + \ddot{\theta}_2\,{}^{1}\mathbf{k}_1 + {}^{1}\omega_1 \times \dot{\theta}_2\,{}^{1}\mathbf{k}_1 \\
{}^{1}\dot{\omega}_2 &= \begin{pmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} \\
{}^{1}\dot{\omega}_2 &= \begin{pmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} {}^{1}\dot{\omega}_{2x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^2\dot{\omega}_2 &= {}^2R_1\,{}^1\dot{\omega}_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} {}^1\dot{\omega}_{2x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} \\
{}^2\dot{\omega}_2 &= \begin{pmatrix} c_2\,{}^1\dot{\omega}_{2x} + s_2\ddot{\theta}_1 \\ -s_2\,{}^1\dot{\omega}_{2x} + c_2\ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} {}^2\dot{\omega}_{2x} \\ {}^2\dot{\omega}_{2y} \\ \ddot{\theta}_2 \end{pmatrix} \\
{}^2\dot{\omega}_3 &= {}^2\dot{\omega}_2 + \ddot{\theta}_3\,{}^2\mathbf{k}_2 + {}^2\omega_2 \times \dot{\theta}_3\,{}^2\mathbf{k}_2 \\
{}^2\dot{\omega}_3 &= \begin{pmatrix} {}^2\dot{\omega}_{2x} \\ {}^2\dot{\omega}_{2y} \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_3 \end{pmatrix} + \begin{pmatrix} {}^2\omega_{2x} \\ {}^2\omega_{2y} \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{pmatrix} \\
{}^2\dot{\omega}_3 &= \begin{pmatrix} {}^2\dot{\omega}_{2x} + \dot{\theta}_3\,{}^2\omega_{2y} \\ {}^2\dot{\omega}_{2y} - \dot{\theta}_3\,{}^2\omega_{2x} \\ \ddot{\theta}_2 + \ddot{\theta}_3 \end{pmatrix} = \begin{pmatrix} {}^2\dot{\omega}_{3x} \\ {}^2\dot{\omega}_{3y} \\ {}^2\dot{\omega}_{3z} \end{pmatrix} \\
{}^3\dot{\omega}_3 &= {}^3R_2\,{}^2\dot{\omega}_3 = \begin{bmatrix} c_3 & s_3 & 0 \\ 0 & 0 & -1 \\ -s_3 & c_3 & 0 \end{bmatrix} \begin{pmatrix} {}^2\dot{\omega}_{3x} \\ {}^2\dot{\omega}_{3y} \\ {}^2\dot{\omega}_{3z} \end{pmatrix} \\
{}^3\dot{\omega}_3 &= \begin{pmatrix} c_3\,{}^2\dot{\omega}_{3x} + s_3\,{}^2\dot{\omega}_{3y} \\ -\,{}^2\dot{\omega}_{3z} \\ -s_3\,{}^2\dot{\omega}_{3x} + c_3\,{}^2\dot{\omega}_{3y} \end{pmatrix} = \begin{pmatrix} {}^3\dot{\omega}_{3x} \\ -{}^2\dot{\omega}_{3z} \\ {}^3\dot{\omega}_{3z} \end{pmatrix} \\
{}^3\dot{\omega}_4 &= {}^3\dot{\omega}_3 + \ddot{\theta}_4\,{}^3\mathbf{k}_3 + {}^3\omega_3 \times \dot{\theta}_4\,{}^3\mathbf{k}_3 \\
{}^3\dot{\omega}_4 &= \begin{pmatrix} {}^3\dot{\omega}_{3x} \\ -\,{}^2\dot{\omega}_{3z} \\ {}^3\dot{\omega}_{3z} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_4 \end{pmatrix} + \begin{pmatrix} {}^3\omega_{3x} \\ {}^3\omega_{3y} \\ {}^3\omega_{3z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_4 \end{pmatrix} \\
{}^3\dot{\omega}_4 &= \begin{pmatrix} {}^3\dot{\omega}_{3x} + \dot{\theta}_4\,{}^3\omega_{3y} \\ -\,{}^2\dot{\omega}_{3z} - \dot{\theta}_4\,{}^3\omega_{3x} \\ {}^3\dot{\omega}_{3z} + \ddot{\theta}_4 \end{pmatrix} = \begin{pmatrix} {}^3\dot{\omega}_{4x} \\ {}^3\dot{\omega}_{4y} \\ {}^3\dot{\omega}_{4z} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^4\dot{\omega}_4 &= {}^4R_3\,{}^3\dot{\omega}_4 = \begin{bmatrix} c_4 & s_4 & 0 \\ 0 & 0 & 1 \\ s_4 & -c_4 & 0 \end{bmatrix} \begin{pmatrix} {}^3\dot{\omega}_{4x} \\ {}^3\dot{\omega}_{4y} \\ {}^3\dot{\omega}_{4z} \end{pmatrix} \\
{}^4\dot{\omega}_4 &= \begin{pmatrix} c_4\,{}^3\dot{\omega}_{4x} + s_4\,{}^3\dot{\omega}_{4y} \\ {}^3\dot{\omega}_{4z} \\ s_4\,{}^3\dot{\omega}_{4x} - c_4\,{}^3\dot{\omega}_{4y} \end{pmatrix} = \begin{pmatrix} {}^4\dot{\omega}_{4x} \\ {}^3\dot{\omega}_{4z} \\ {}^4\dot{\omega}_{4z} \end{pmatrix} \\
{}^4\dot{\omega}_5 &= {}^4\dot{\omega}_4 + \ddot{\theta}_5\,{}^4\mathbf{k}_4 + {}^4\omega_4 \times \dot{\theta}_5\,{}^4\mathbf{k}_4 \\
{}^4\dot{\omega}_5 &= \begin{pmatrix} {}^4\dot{\omega}_{4x} \\ {}^3\dot{\omega}_{4z} \\ {}^4\dot{\omega}_{4z} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_5 \end{pmatrix} + \begin{pmatrix} {}^4\omega_{4x} \\ {}^4\omega_{4y} \\ {}^4\omega_{4z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_5 \end{pmatrix} \\
{}^4\dot{\omega}_5 &= \begin{pmatrix} {}^4\dot{\omega}_{4x} + \dot{\theta}_5\,{}^4\omega_{4y} \\ {}^3\dot{\omega}_{4z} - \dot{\theta}_5\,{}^4\omega_{4x} \\ {}^4\dot{\omega}_{4z} + \ddot{\theta}_5 \end{pmatrix} = \begin{pmatrix} {}^4\dot{\omega}_{5x} \\ {}^4\dot{\omega}_{5y} \\ {}^4\dot{\omega}_{5z} \end{pmatrix} \\
{}^5\dot{\omega}_5 &= {}^5R_4\,{}^4\dot{\omega}_5 = \begin{bmatrix} c_5 & s_5 & 0 \\ 0 & 0 & 1 \\ s_5 & -c_5 & 0 \end{bmatrix} \begin{pmatrix} {}^4\dot{\omega}_{5x} \\ {}^4\dot{\omega}_{5y} \\ {}^4\dot{\omega}_{5z} \end{pmatrix} \\
{}^5\dot{\omega}_5 &= \begin{pmatrix} c_5\,{}^4\dot{\omega}_{5x} + s_5\,{}^4\dot{\omega}_{5y} \\ {}^4\dot{\omega}_{5z} \\ s_5\,{}^4\dot{\omega}_{5x} - c_5\,{}^4\dot{\omega}_{5y} \end{pmatrix} = \begin{pmatrix} {}^5\dot{\omega}_{5x} \\ {}^4\dot{\omega}_{5z} \\ {}^5\dot{\omega}_{5z} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^5\dot{\omega}_6 &= {}^5\dot{\omega}_5 + \ddot{\theta}_6\,{}^5\mathbf{k}_5 + {}^5\omega_5 \times \dot{\theta}_6\,{}^5\mathbf{k}_5 \\
{}^5\dot{\omega}_6 &= \begin{pmatrix} {}^5\dot{\omega}_{5x} \\ {}^4\dot{\omega}_{5z} \\ {}^5\dot{\omega}_{5z} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_6 \end{pmatrix} + \begin{pmatrix} {}^5\omega_{5x} \\ {}^5\omega_{5y} \\ {}^5\omega_{5z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_6 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^5\dot{\omega}_6 &= \begin{pmatrix} {}^5\dot{\omega}_{5x} + \dot{\theta}_6\, {}^5\omega_{5y} \\ {}^4\dot{\omega}_{5z} - \dot{\theta}_6\, {}^5\omega_{5x} \\ {}^5\dot{\omega}_{5z} + \ddot{\theta}_6 \end{pmatrix} = \begin{pmatrix} {}^5\dot{\omega}_{6x} \\ {}^5\dot{\omega}_{6y} \\ {}^5\dot{\omega}_{6z} \end{pmatrix} \\
{}^6\dot{\omega}_6 &= {}^6R_5\, {}^5\dot{\omega}_6 = \begin{bmatrix} c_6 & s_6 & 0 \\ -s_6 & c_6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} {}^5\dot{\omega}_{6x} \\ {}^5\dot{\omega}_{6y} \\ {}^5\dot{\omega}_{6z} \end{pmatrix} \\
{}^6\dot{\omega}_6 &= \begin{pmatrix} c_6\, {}^5\dot{\omega}_{6x} + s_6\, {}^5\dot{\omega}_{6y} \\ -s_6\, {}^5\dot{\omega}_{6x} + c_6\, {}^5\dot{\omega}_{6y} \\ {}^5\dot{\omega}_{6z} \end{pmatrix} = \begin{pmatrix} {}^6\dot{\omega}_{6x} \\ {}^6\dot{\omega}_{6y} \\ {}^5\dot{\omega}_{6z} \end{pmatrix} \\
{}^1\dot{\omega}_7 &= {}^1\dot{\omega}_1 + \ddot{\theta}_7\, {}^1\mathbf{k}_1 + {}^1\omega_1 \times \dot{\theta}_7\, {}^1\mathbf{k}_1 \\
{}^1\dot{\omega}_7 &= \begin{pmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_7 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_7 \end{pmatrix} \\
{}^1\dot{\omega}_7 &= \begin{pmatrix} \dot{\theta}_1\dot{\theta}_7 \\ \ddot{\theta}_1 \\ \ddot{\theta}_7 \end{pmatrix} = \begin{pmatrix} {}^1\dot{\omega}_{7x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_7 \end{pmatrix} \\
{}^7\dot{\omega}_7 &= {}^7R_1\, {}^1\dot{\omega}_7 = \begin{bmatrix} c_7 & s_7 & 0 \\ -s_7 & c_7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} {}^1\dot{\omega}_{7x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_7 \end{pmatrix} \\
{}^7\dot{\omega}_7 &= \begin{pmatrix} c_7\, {}^1\dot{\omega}_{7x} + s_7\ddot{\theta}_1 \\ -s_7\, {}^1\dot{\omega}_{7x} + c_7\ddot{\theta}_1 \\ \ddot{\theta}_7 \end{pmatrix} = \begin{pmatrix} {}^7\dot{\omega}_{7x} \\ {}^7\dot{\omega}_{7y} \\ \ddot{\theta}_7 \end{pmatrix} \\
{}^7\dot{\omega}_8 &= {}^7\dot{\omega}_7 + \ddot{\theta}_8\, {}^7\mathbf{k}_7 + {}^7\omega_7 \times \dot{\theta}_8\, {}^7\mathbf{k}_7 \\
{}^7\dot{\omega}_8 &= \begin{pmatrix} {}^7\dot{\omega}_{7x} \\ {}^7\dot{\omega}_{7y} \\ \ddot{\theta}_7 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_8 \end{pmatrix} + \begin{pmatrix} {}^7\omega_{7x} \\ {}^7\omega_{7y} \\ \dot{\theta}_7 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_8 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{7}\dot{\omega}_{8} &= \begin{pmatrix} {}^{7}\dot{\omega}_{7x} + \dot{\theta}_{8}\,{}^{7}\omega_{7y} \\ {}^{7}\dot{\omega}_{7y} - \dot{\theta}_{8}\,{}^{7}\omega_{7x} \\ \ddot{\theta}_{7} + \ddot{\theta}_{8} \end{pmatrix} = \begin{pmatrix} {}^{7}\dot{\omega}_{8x} \\ {}^{7}\dot{\omega}_{8y} \\ {}^{7}\dot{\omega}_{8z} \end{pmatrix} \\
{}^{8}\dot{\omega}_{8} &= {}^{8}R_{7}\,{}^{7}\dot{\omega}_{8} = \begin{bmatrix} c_{8} & s_{8} & 0 \\ -s_{8} & c_{8} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} {}^{7}\dot{\omega}_{8x} \\ {}^{7}\dot{\omega}_{8y} \\ {}^{7}\dot{\omega}_{8z} \end{pmatrix} \\
{}^{8}\dot{\omega}_{8} &= \begin{pmatrix} c_{8}\,{}^{7}\dot{\omega}_{8x} + s_{8}\,{}^{7}\dot{\omega}_{8y} \\ -s_{8}\,{}^{7}\dot{\omega}_{8x} + c_{8}\,{}^{7}\dot{\omega}_{8y} \\ {}^{7}\dot{\omega}_{8z} \end{pmatrix} = \begin{pmatrix} {}^{8}\dot{\omega}_{8x} \\ {}^{8}\dot{\omega}_{8y} \\ {}^{7}\dot{\omega}_{8z} \end{pmatrix}
\end{aligned}
$$

D.2 Linear Accelerations:

$$
\begin{aligned}
{}^1\mathbf{a}_1^* &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
{}^2\mathbf{a}_2^* &= {}^2\dot{\omega}_2 \times r_2\,{}^2\mathbf{i}_2 + {}^2\omega_2 \times ({}^2\omega_2 \times r_2\,{}^2\mathbf{i}_2) \\
&= \begin{pmatrix} {}^2\dot{\omega}_{2x} \\ {}^2\dot{\omega}_{2y} \\ \ddot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} r_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} {}^2\omega_{2x} \\ {}^2\omega_{2y} \\ \dot{\theta}_2 \end{pmatrix} \times \left\{ \begin{pmatrix} {}^2\omega_{2x} \\ {}^2\omega_{2y} \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} r_2 \\ 0 \\ 0 \end{pmatrix} \right\} \\
&= \begin{pmatrix} {}^2\dot{\omega}_{2x} \\ {}^2\dot{\omega}_{2y} \\ \ddot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} r_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} {}^2\omega_{2x} \\ {}^2\omega_{2y} \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} 0 \\ r_2\dot{\theta}_2 \\ -r_2\,{}^2\omega_{2y} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ r_2\ddot{\theta}_2 \\ -r_2\,{}^2\dot{\omega}_{2y} \end{pmatrix} + \begin{pmatrix} -r_2({}^2\omega_{2y}^2 + \dot{\theta}_2^2) \\ r_2\,{}^2\omega_{2x}\,{}^2\omega_{2y} \\ r_2\,{}^2\omega_{2x}\dot{\theta}_2 \end{pmatrix} \\
&= \begin{pmatrix} -r_2({}^2\omega_{2y}^2 + \dot{\theta}_2^2) \\ r_2(\ddot{\theta}_2 + {}^2\omega_{2x}\,{}^2\omega_{2y}) \\ -r_2({}^2\dot{\omega}_{2y} - {}^2\omega_{2x}\dot{\theta}_2) \end{pmatrix} = \begin{pmatrix} {}^2a_{2x}^* \\ {}^2a_{2y}^* \\ {}^2a_{2z}^* \end{pmatrix} \\
{}^2\mathbf{a}_{2,3} &= {}^2\dot{\omega}_2 \times l_2\,{}^2\mathbf{i}_2 + {}^2\omega_2 \times ({}^2\omega_2 \times l_2\,{}^2\mathbf{i}_2) \\
&= \begin{pmatrix} -l_2({}^2\omega_{2y}^2 + \dot{\theta}_2^2) \\ l_2(\ddot{\theta}_2 + {}^2\omega_{2x}\,{}^2\omega_{2y}) \\ -l_2({}^2\dot{\omega}_{2y} - {}^2\omega_{2x}\dot{\theta}_2) \end{pmatrix} = \begin{pmatrix} {}^2a_{2,3x} \\ {}^2a_{2,3y} \\ {}^2a_{2,3z} \end{pmatrix} \\
{}^3\mathbf{a}_{2,3} &= {}^3R_2\,{}^2\mathbf{a}_{2,3} = \begin{bmatrix} c_3 & s_3 & 0 \\ 0 & 0 & -1 \\ -s_3 & c_3 & \end{bmatrix} \begin{pmatrix} {}^2a_{2,3x} \\ {}^2a_{2,3y} \\ {}^2a_{2,3z} \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} c_3\,{}^2a_{2,3x} + s_3\,{}^2a_{2,3y} \\ -\,{}^2a_{2,3z} \\ -s_3\,{}^2a_{2,3x} + c_3\,{}^2a_{2,3y} \end{pmatrix} = \begin{pmatrix} {}^3a_{2,3x} \\ -\,{}^2a_{2,3z} \\ {}^3a_{2,3z} \end{pmatrix}$$

$$\begin{aligned}
{}^3\mathbf{a}_3^* &= {}^3\mathbf{a}_{2,3} + {}^3\dot{\omega}_3 \times r_3\,{}^3\mathbf{k}_3 + {}^3\omega_3 \times (\,{}^3\omega_3 \times r_3\,{}^3\mathbf{k}_3) \\
&= \begin{pmatrix} {}^3a_{2,3x} \\ -\,{}^2a_{2,3z} \\ {}^3a_{2,3z} \end{pmatrix} + \begin{pmatrix} {}^3\dot{\omega}_{3x} \\ -\,{}^2\dot{\omega}_{3z} \\ {}^3\dot{\omega}_{3z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ r_3 \end{pmatrix} + \begin{pmatrix} {}^3\omega_{3x} \\ {}^3\omega_{3y} \\ {}^3\omega_{3z} \end{pmatrix} \times \left\{ \begin{pmatrix} {}^3\omega_{3x} \\ {}^3\omega_{3y} \\ {}^3\omega_{3z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ r_3 \end{pmatrix} \right\} \\
&= \begin{pmatrix} {}^3a_{2,3x} \\ -\,{}^2a_{2,3z} \\ {}^3a_{2,3z} \end{pmatrix} + \begin{pmatrix} {}^3\dot{\omega}_{3x} \\ -\,{}^2\dot{\omega}_{3z} \\ {}^3\dot{\omega}_{3z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ r_3 \end{pmatrix} + \begin{pmatrix} {}^3\omega_{3x} \\ {}^3\omega_{3y} \\ {}^3\omega_{3z} \end{pmatrix} \times \begin{pmatrix} r_3\,{}^3\omega_{3y} \\ -r_3\,{}^3\omega_{3x} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} {}^3a_{2,3x} \\ -\,{}^2a_{2,3z} \\ {}^3a_{2,3z} \end{pmatrix} + \begin{pmatrix} -r_3\,{}^2\dot{\omega}_{3z} \\ -r_3\,{}^3\dot{\omega}_{3x} \\ 0 \end{pmatrix} + \begin{pmatrix} r_3\,{}^3\omega_{3x}\,{}^3\omega_{3z} \\ r_3\,{}^3\omega_{3y}\,{}^3\omega_{3z} \\ -r_3(\,{}^3\omega_{3x}^2 + {}^3\omega_{3y}^2) \end{pmatrix} \\
&= \begin{pmatrix} {}^3a_{2,3x} - r_3(\,{}^2\dot{\omega}_{3z} - {}^3\omega_{3x}\,{}^3\omega_{3z}) \\ -\,{}^2a_{2,3z} - r_3(\,{}^3\dot{\omega}_{3x} - {}^3\omega_{3y}\,{}^3\omega_{3z}) \\ {}^3a_{2,3z} - r_3(\,{}^3\omega_{3x}^2 + {}^3\omega_{3y}^2) \end{pmatrix} = \begin{pmatrix} {}^3a_{3x}^* \\ {}^3a_{3y}^* \\ {}^3a_{3z}^* \end{pmatrix} \\
{}^3\mathbf{a}_4^* &= {}^3\mathbf{a}_{2,3} + {}^3\dot{\omega}_3 \times r_4\,{}^3\mathbf{k}_3 + {}^3\omega_3 \times (\,{}^3\omega_3 \times r_4\,{}^3\mathbf{k}_3) \\
&= \begin{pmatrix} {}^3a_{2,3x} - r_4(\,{}^2\dot{\omega}_{3z} - {}^3\omega_{3x}\,{}^3\omega_{3z}) \\ -\,{}^2a_{2,3z} - r_4(\,{}^3\dot{\omega}_{3x} - {}^3\omega_{3y}\,{}^3\omega_{3z}) \\ {}^3a_{2,3z} - r_4(\,{}^3\omega_{3x}^2 + {}^3\omega_{3y}^2) \end{pmatrix} = \begin{pmatrix} {}^3a_{4x}^* \\ {}^3a_{4y}^* \\ {}^3a_{4z}^* \end{pmatrix} \\
{}^3\mathbf{a}_{4,5} &= {}^3\mathbf{a}_{2,3} + {}^3\dot{\omega}_3 \times d_4\,{}^3\mathbf{k}_3 + {}^3\omega_3 \times (\,{}^3\omega_3 \times d_4\,{}^3\mathbf{k}_3) \\
&= \begin{pmatrix} {}^3a_{2,3x} - d_4(\,{}^2\dot{\omega}_{3z} - {}^3\omega_{3x}\,{}^3\omega_{3z}) \\ -\,{}^2a_{2,3z} - d_4(\,{}^3\dot{\omega}_{3x} - {}^3\omega_{3y}\,{}^3\omega_{3z}) \\ {}^3a_{2,3z} - d_4(\,{}^3\omega_{3x}^2 + {}^3\omega_{3y}^2) \end{pmatrix} = \begin{pmatrix} {}^3a_{4,5x} \\ {}^3a_{4,5y} \\ {}^3a_{4,5z} \end{pmatrix}
\end{aligned}$$

$$
{}^5\mathbf{a}_{4,5} = {}^5R_4\,{}^4R_3\,{}^3\mathbf{a}_{4,5} = \begin{bmatrix} Z_{49} & Z_{50} & s_5 \\ s_4 & -c_4 & 0 \\ Z_{51} & Z_{52} & -c_5 \end{bmatrix} \begin{pmatrix} {}^3a_{4,5x} \\ {}^3a_{4,5y} \\ {}^3a_{4,5z} \end{pmatrix}
$$

$$
= \begin{pmatrix} Z_{49}\,{}^3a_{4,5x} + Z_{50}\,{}^3a_{4,5y} + s_5\,{}^3a_{4,5z} \\ s_4\,{}^3a_{4,5x} - c_4\,{}^3a_{4,5y} \\ Z_{51}\,{}^3a_{4,5x} + Z_{52}\,{}^3a_{4,5y} - c_5\,{}^3a_{4,5z} \end{pmatrix} = \begin{pmatrix} {}^5a_{4,5x} \\ {}^5a_{4,5y} \\ {}^5a_{4,5z} \end{pmatrix}
$$

$$
{}^5\mathbf{a}_5^* = {}^5\mathbf{a}_{4,5} + {}^5\dot{\omega}_5 \times r_5\,{}^5\mathbf{k}_5 + {}^5\omega_5 \times (\,{}^5\omega_5 \times r_5\,{}^5\mathbf{k}_5)
$$

$$
= \begin{pmatrix} {}^5a_{4,5x} \\ {}^5a_{4,5y} \\ {}^5a_{4,5z} \end{pmatrix} + \begin{pmatrix} {}^5\dot{\omega}_{5x} \\ {}^4\dot{\omega}_{5z} \\ {}^5\dot{\omega}_{5z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ r_5 \end{pmatrix} + \begin{pmatrix} {}^5\omega_{5x} \\ {}^5\omega_{5y} \\ {}^5\omega_{5z} \end{pmatrix} \times \left\{ \begin{pmatrix} {}^5\omega_{5x} \\ {}^5\omega_{5y} \\ {}^5\omega_{5z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ r_5 \end{pmatrix} \right\}
$$

$$
= \begin{pmatrix} {}^5a_{4,5x} \\ {}^5a_{4,5y} \\ {}^5a_{4,5z} \end{pmatrix} + \begin{pmatrix} {}^5\dot{\omega}_{5x} \\ {}^4\dot{\omega}_{5z} \\ {}^5\dot{\omega}_{5z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ r_5 \end{pmatrix} + \begin{pmatrix} {}^5\omega_{5x} \\ {}^5\omega_{5y} \\ {}^5\omega_{5z} \end{pmatrix} \times \begin{pmatrix} r_5\,{}^5\omega_{5y} \\ -r_5\,{}^5\omega_{5x} \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} {}^5a_{4,5x} \\ {}^5a_{4,5y} \\ {}^5a_{4,5z} \end{pmatrix} + \begin{pmatrix} r_5\,{}^4\dot{\omega}_{5z} \\ -r_5\,{}^5\dot{\omega}_{5x} \\ 0 \end{pmatrix} + \begin{pmatrix} r_5\,{}^5\omega_{5x}\,{}^5\omega_{5z} \\ r_5\,{}^5\omega_{5y}\,{}^5\omega_{5z} \\ -r_5(\,{}^5\omega_{5x}^2 + {}^5\omega_{5y}^2) \end{pmatrix}
$$

$$
= \begin{pmatrix} {}^5a_{4,5x} + r_5(\,{}^4\dot{\omega}_{5z} + {}^5\omega_{5x}\,{}^5\omega_{5z}) \\ {}^5a_{4,5y} - r_5(\,{}^5\dot{\omega}_{5x} - {}^5\omega_{5y}\,{}^5\omega_{5z}) \\ {}^5a_{4,5z} - r_5(\,{}^5\omega_{5x}^2 + {}^5\omega_{5y}^2) \end{pmatrix} = \begin{pmatrix} {}^5a_{5x}^* \\ {}^5a_{5y}^* \\ {}^5a_{5z}^* \end{pmatrix}
$$

$$
{}^5\mathbf{a}_6^* = {}^5\mathbf{a}_{4,5} + {}^5\dot{\omega}_5 \times r_6\,{}^5\mathbf{k}_5 + {}^5\omega_5 \times (\,{}^5\omega_5 \times r_6\,{}^5\mathbf{k}_5)
$$

$$
= \begin{pmatrix} {}^5a_{4,5x} + r_6(\,{}^4\dot{\omega}_{5z} + {}^5\omega_{5x}\,{}^5\omega_{5z}) \\ {}^5a_{4,5y} - r_6(\,{}^5\dot{\omega}_{5x} - {}^5\omega_{5y}\,{}^5\omega_{5z}) \\ {}^5a_{4,5z} - r_6(\,{}^5\omega_{5x}^2 + {}^5\omega_{5y}^2) \end{pmatrix} = \begin{pmatrix} {}^5a_{6x}^* \\ {}^5a_{6y}^* \\ {}^5a_{6z}^* \end{pmatrix}
$$

$$
{}^7\mathbf{a}_7^* = {}^7\dot{\omega}_7 \times r_7\,{}^7\mathbf{i}_7 + {}^7\omega_7 \times (\,{}^7\omega_7 \times r_7\,{}^7\mathbf{i}_7)
$$

$$
\begin{aligned}
&= \begin{pmatrix} {}^7\dot{\omega}_{7x} \\ {}^7\dot{\omega}_{7y} \\ \ddot{\theta}_7 \end{pmatrix} \times \begin{pmatrix} r_7 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} {}^7\omega_{7x} \\ {}^7\omega_{7y} \\ \dot{\theta}_7 \end{pmatrix} \times \left\{ \begin{pmatrix} {}^7\omega_{7x} \\ {}^7\omega_{7y} \\ \dot{\theta}_7 \end{pmatrix} \times \begin{pmatrix} r_7 \\ 0 \\ 0 \end{pmatrix} \right\} \\
&= \begin{pmatrix} {}^7\dot{\omega}_{7x} \\ {}^7\dot{\omega}_{7y} \\ \ddot{\theta}_7 \end{pmatrix} \times \begin{pmatrix} r_7 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} {}^7\omega_{7x} \\ {}^7\omega_{7y} \\ \dot{\theta}_7 \end{pmatrix} \times \begin{pmatrix} 0 \\ r_7\dot{\theta}_7 \\ -r_7\,{}^7\omega_{7y} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ r_7\ddot{\theta}_7 \\ -r_7\,{}^7\dot{\omega}_{7y} \end{pmatrix} + \begin{pmatrix} -r_7({}^7\omega_{7y}^2 + \dot{\theta}_7^2) \\ r_7\,{}^7\omega_{7x}\,{}^7\omega_{7y} \\ r_7\,{}^7\omega_{7x}\dot{\theta}_7 \end{pmatrix} \\
&= \begin{pmatrix} -r_7({}^7\omega_{7y}^2 + \dot{\theta}_7^2) \\ r_7(\ddot{\theta}_7 + {}^7\omega_{7x}\,{}^7\omega_{7y}) \\ -r_7({}^7\dot{\omega}_{7y} - {}^7\omega_{7x}\dot{\theta}_7) \end{pmatrix} = \begin{pmatrix} {}^7a^*_{7x} \\ {}^7a^*_{7y} \\ {}^7a^*_{7z} \end{pmatrix} \\
{}^7\mathbf{a}_{7,8} &= {}^7\dot{\omega}_7 \times l_7\,{}^7\mathbf{i}_7 + {}^7\omega_7 \times ({}^7\omega_7 \times l_7\,{}^7\mathbf{i}_7) \\
&= \begin{pmatrix} -l_7({}^7\omega_{7y}^2 + \dot{\theta}_7^2) \\ l_7(\ddot{\theta}_7 + {}^7\omega_{7x}\,{}^7\omega_{7y}) \\ -l_7({}^7\dot{\omega}_{7y} - {}^7\omega_{7x}\dot{\theta}_7) \end{pmatrix} = \begin{pmatrix} {}^7a_{7,8x} \\ {}^7a_{7,8y} \\ {}^7a_{7,8z} \end{pmatrix} \\
{}^8\mathbf{a}_{7,8} &= {}^8R_7\,{}^7\mathbf{a}_{7,8} = \begin{bmatrix} c_8 & s_8 & 0 \\ -s_8 & c_8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} {}^7a_{7,8x} \\ {}^7a_{7,8y} \\ {}^7a_{7,8z} \end{pmatrix} \\
&= \begin{pmatrix} c_8\,{}^7a_{7,8x} + s_8\,{}^7a_{7,8y} \\ -s_8\,{}^7a_{7,8x} + c_8\,{}^7a_{7,8y} \\ {}^7a_{7,8z} \end{pmatrix} = \begin{pmatrix} {}^8a_{7,8x} \\ {}^8a_{7,8y} \\ {}^7a_{7,8z} \end{pmatrix} \\
{}^8\mathbf{a}^*_8 &= {}^8\mathbf{a}_{7,8} + {}^8\dot{\omega}_8 \times r_8\,{}^8\mathbf{i}_8 + {}^8\omega_8 \times ({}^8\omega_8 \times r_8\,{}^8\mathbf{i}_8) \\
&= \begin{pmatrix} {}^8a_{7,8x} \\ {}^8a_{7,8y} \\ {}^7a_{7,8z} \end{pmatrix} + \begin{pmatrix} {}^8\dot{\omega}_{8x} \\ {}^8\dot{\omega}_{8y} \\ {}^7\dot{\omega}_{8z} \end{pmatrix} \times \begin{pmatrix} r_8 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} {}^8\omega_{8x} \\ {}^8\omega_{8y} \\ {}^8\omega_{8z} \end{pmatrix} \times \left\{ \begin{pmatrix} {}^8\omega_{8x} \\ {}^8\omega_{8y} \\ {}^8\omega_{8z} \end{pmatrix} \times \begin{pmatrix} r_8 \\ 0 \\ 0 \end{pmatrix} \right\}
\end{aligned}
$$

$$
= \begin{pmatrix} {}^{8}a_{7,8x} \\ {}^{8}a_{7,8y} \\ {}^{7}a_{7,8z} \end{pmatrix} + \begin{pmatrix} {}^{8}\dot{\omega}_{8x} \\ {}^{8}\dot{\omega}_{8y} \\ {}^{7}\dot{\omega}_{8z} \end{pmatrix} \times \begin{pmatrix} r_8 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} {}^{8}\omega_{8x} \\ {}^{8}\omega_{8y} \\ {}^{8}\omega_{8z} \end{pmatrix} \times \begin{pmatrix} 0 \\ r_8\, {}^{8}\omega_{8z} \\ -r_8\, {}^{8}\omega_{8y} \end{pmatrix}
$$

$$
= \begin{pmatrix} {}^{8}a_{7,8x} \\ {}^{8}a_{7,8y} \\ {}^{7}a_{7,8z} \end{pmatrix} + \begin{pmatrix} 0 \\ r_8\, {}^{7}\dot{\omega}_{8z} \\ -r_8\, {}^{8}\dot{\omega}_{8y} \end{pmatrix} + \begin{pmatrix} -r_8({}^{8}\omega_{8y}^2 + {}^{8}\omega_{8z}^2) \\ r_8\, {}^{8}\omega_{8x}\, {}^{8}\omega_{8y} \\ r_8\, {}^{8}\omega_{8x}\, {}^{8}\omega_{8z} \end{pmatrix}
$$

$$
= \begin{pmatrix} {}^{8}a_{7,8x} - r_8({}^{8}\omega_{8y}^2 + {}^{8}\omega_{8z}^2) \\ {}^{8}a_{7,8y} + r_8({}^{7}\dot{\omega}_{8z} + {}^{8}\omega_{8x}\, {}^{8}\omega_{8y}) \\ {}^{7}a_{7,8z} - r_8({}^{8}\dot{\omega}_{8y} - {}^{8}\omega_{8x}\, {}^{8}\omega_{8z}) \end{pmatrix} = \begin{pmatrix} {}^{8}a_{8x}^{*} \\ {}^{8}a_{8y}^{*} \\ {}^{8}a_{8z}^{*} \end{pmatrix}
$$

APPENDIX E

E.1 Inertia Torques

$$
\begin{aligned}
\mathbf{T}_1^* &= \begin{pmatrix} 0 \\ -\ddot{\theta}_1 \, J_{1y} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ T_{1y}^* \\ 0 \end{pmatrix} \\
\mathbf{T}_2^* &= \begin{pmatrix} -\,{}^2\dot{\omega}_{2x} \, J_{2x} + {}^2\omega_{2y} \, \dot{\theta}_2 (J_{2y} - J_{2z}) \\ -\,{}^2\dot{\omega}_{2y} \, J_{2y} + \dot{\theta}_2 \, {}^2\omega_{2x} (J_{2z} - J_{2x}) \\ -\ddot{\theta}_2 \, J_{2z} + {}^2\omega_{2x} \, {}^2\omega_{2y} (J_{2x} - J_{2y}) \end{pmatrix} = \begin{pmatrix} T_{2x}^* \\ T_{2y}^* \\ T_{2z}^* \end{pmatrix} \\
\mathbf{T}_3^* &= \begin{pmatrix} -\,{}^3\dot{\omega}_{3x} \, J_{3x} + {}^3\omega_{3y} \, {}^3\omega_{3z} (J_{3y} - J_{3z}) \\ {}^2\dot{\omega}_{3z} \, J_{3y} + {}^3\omega_{3z} \, {}^3\omega_{3x} (J_{3z} - J_{3x}) \\ -\,{}^3\dot{\omega}_{3z} \, J_{3z} + {}^3\omega_{3x} \, {}^3\omega_{3y} (J_{3x} - J_{3y}) \end{pmatrix} = \begin{pmatrix} T_{3x}^* \\ T_{3y}^* \\ T_{3z}^* \end{pmatrix} \\
\mathbf{T}_4^* &= \begin{pmatrix} -\,{}^4\dot{\omega}_{4x} \, J_{4x} + {}^4\omega_{4y} \, {}^4\omega_{4z} (J_{4y} - J_{4z}) \\ -\,{}^3\dot{\omega}_{4z} \, J_{4y} + {}^4\omega_{4z} \, {}^4\omega_{4x} (J_{4z} - J_{4x}) \\ -\,{}^4\dot{\omega}_{4z} \, J_{4z} + {}^4\omega_{4x} \, {}^4\omega_{4y} (J_{4x} - J_{4y}) \end{pmatrix} = \begin{pmatrix} T_{4x}^* \\ T_{4y}^* \\ T_{4z}^* \end{pmatrix} \\
\mathbf{T}_5^* &= \begin{pmatrix} -\,{}^5\dot{\omega}_{5x} \, J_{5x} + {}^5\omega_{5y} \, {}^5\omega_{5z} (J_{5y} - J_{5z}) \\ -\,{}^4\dot{\omega}_{5z} \, J_{5y} + {}^5\omega_{5z} \, {}^5\omega_{5x} (J_{5z} - J_{5x}) \\ -\,{}^5\dot{\omega}_{5z} \, J_{5z} + {}^5\omega_{5x} \, {}^5\omega_{5y} (J_{5x} - J_{5y}) \end{pmatrix} = \begin{pmatrix} T_{5x}^* \\ T_{5y}^* \\ T_{5z}^* \end{pmatrix}
\end{aligned}
$$

$$\mathbf{T}_6^* = \begin{pmatrix} -\,{}^6\dot{\omega}_{6x}\, J_{6x} + {}^6\omega_{6y}\,{}^6\omega_{6z}(J_{6y} - J_{6z}) \\ -\,{}^6\dot{\omega}_{6y}\, J_{6y} + {}^6\omega_{6z}\,{}^6\omega_{6x}(J_{6z} - J_{6x}) \\ -\,{}^5\dot{\omega}_{6z}\, J_{6z} + {}^6\omega_{6x}\,{}^6\omega_{6y}(J_{6x} - J_{6y}) \end{pmatrix} = \begin{pmatrix} T_{6x}^* \\ T_{6y}^* \\ T_{6z}^* \end{pmatrix}$$

$$\mathbf{T}_7^* = \begin{pmatrix} -\,{}^7\dot{\omega}_{7x}\, J_{7x} + {}^7\omega_{7y}\,\dot{\theta}_7(J_{7y} - J_{7z}) \\ -\,{}^7\dot{\omega}_{7y}\, J_{7y} + \dot{\theta}_7\,{}^7\omega_{7x}(J_{7z} - J_{7x}) \\ -\ddot{\theta}_7\, J_{7z} + {}^7\omega_{7x}\,{}^7\omega_{7y}(J_{7x} - J_{7y}) \end{pmatrix} = \begin{pmatrix} T_{7x}^* \\ T_{7y}^* \\ T_{7z}^* \end{pmatrix}$$

$$\mathbf{T}_8^* = \begin{pmatrix} -\,{}^8\dot{\omega}_{8x}\, J_{8x} + {}^8\omega_{8y}\,{}^8\omega_{8z}(J_{8y} - J_{8z}) \\ -\,{}^8\dot{\omega}_{8y}\, J_{8y} + {}^8\omega_{8z}\,{}^8\omega_{8x}(J_{8z} - J_{8x}) \\ -\,{}^7\dot{\omega}_{8z}\, J_{8z} + {}^8\omega_{8x}\,{}^8\omega_{8y}(J_{8x} - J_{8y}) \end{pmatrix} = \begin{pmatrix} T_{8x}^* \\ T_{8y}^* \\ T_{8z}^* \end{pmatrix}$$

E.2 Generalized Inertia Forces

$$\frac{\partial K^*}{\partial \dot\theta_1} = T^*_{1y} + s_2 T^*_{2x} + c_2 T^*_{2y} + m_2 Z_{36}\,{}^2a_{2z} +$$
$$s_{23} T^*_{3x} + c_{23} T^*_{3z} + m_3 Z_{42}\,{}^3a_{3y} +$$
$$Z_{15} T^*_{4x} + c_{23} T^*_{4y} + Z_{18} T^*_{4z} + m_4 Z_{45}\,{}^3a_{4y} +$$
$$Z_{21} T^*_{5x} + Z_{18} T^*_{5y} + Z_{24} T^*_{5z} + m_5(Z_{62}\,{}^5a_{5x} + Z_{65}\,{}^5a_{5y} + Z_{59}\,{}^5a_{5z}) +$$
$$Z_{27} T^*_{6x} + Z_{31} T^*_{6y} + Z_{24} T^*_{6z} + m_6(Z_{69}\,{}^5a_{6x} + Z_{72}\,{}^5a_{6y} + Z_{59}\,{}^5a_{6z}) +$$
$$s_7 T^*_{7x} + c_7 T^*_{7y} + m_7 Z_{76}\,{}^7a_{7z} + s_{78} T^*_{8x} + c_{78} T^*_{8y} + m_8 Z_{81}\,{}^8a_{8z} \quad \text{(E.58)}$$

$$\frac{\partial K^*}{\partial \dot\theta_2} = T^*_{2z} + m_2 r_2\,{}^2a_{2y} + Z_{14} T^*_{3y} + m_3(Z_{40}\,{}^3a_{3x} + Z_{39}\,{}^3a_{3z}) +$$
$$Z_{16} T^*_{4x} + Z_{19} T^*_{4z} + m_4(Z_{43}\,{}^4a_{4x} + Z_{39}\,{}^4a_{4z}) +$$
$$Z_{22} T^*_{5x} + Z_{19} T^*_{5y} + Z_{25} T^*_{5z} + m_5(Z_{63}\,{}^5a_{5x} + Z_{66}\,{}^5a_{5y} + Z_{60}\,{}^5a_{5z}) +$$
$$Z_{28} T^*_{6x} + Z_{32} T^*_{6y} + Z_{25} T^*_{6z} + m_6(Z_{70}\,{}^5a_{6x} + Z_{73}\,{}^5a_{6y} + Z_{60}\,{}^5a_{6z}) +$$
$$Z_{12} T^*_{8z} + m_8 Z_{79}\,{}^8a_{8y} \quad \text{(E.59)}$$

$$\frac{\partial K^*}{\partial \dot\theta_4} = T^*_{4y} + s_5 T^*_{5x} - c_5 T^*_{5y} + m_5 Z_{67}\,{}^5a_{5y} +$$
$$Z_{29} T^*_{6x} + Z_{33} T^*_{6y} - c_5 T^*_{6z} + m_6 Z_{74}\,{}^5a_{6y} \quad \text{(E.60)}$$

$$\frac{\partial K^*}{\partial \dot\theta_5} = T^*_{5y} + m_5 r_5\,{}^5a_{5x} + s_6 T^*_{6x} + c_6 T^*_{6y} + m_6 r_6\,{}^5a_{6x} \quad \text{(E.61)}$$

$$\frac{\partial K^*}{\partial \dot\theta_6} = T^*_{6z}$$

$$\frac{\partial K^*}{\partial \dot\theta_7} = -Z_{11} T^*_{3y} + m_3 Z_{41}\,{}^3a_{3x} + Z_{17} T^*_{4x} + Z_{20} T^*_{4z} + m_4 Z_{44}\,{}^4a_{4x} +$$
$$Z_{23} T^*_{5x} + Z_{20} T^*_{5y} + Z_{26} T^*_{5z} + m_5(Z_{64}\,{}^5a_{5x} + Z_{68}\,{}^5a_{5y} + Z_{61}\,{}^5a_{5z}) +$$
$$Z_{30} T^*_{6x} + Z_{34} T^*_{6y} + Z_{26} T^*_{6z} + m_6(Z_{71}\,{}^5a_{6x} + Z_{75}\,{}^5a_{6y} + Z_{61}\,{}^5a_{6z}) +$$
$$T^*_{7z} + m_7 r_7\,{}^7a_{7y} + Z_{35} T^*_{8z} + m_8(Z_{78}\,{}^8a_{8x} + Z_{80}\,{}^8a_{8y}) \quad \text{(E.62)}$$

E.3 Generalized Active Forces

$$\frac{\partial \mathbf{K}}{\partial \dot{\theta}_1} = \left(\frac{\partial \omega_1}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{0/1}+\mathbf{T}^{2/1}+\mathbf{T}_{7/1}\right)+\left(\frac{\partial \mathbf{V}_1}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{0/1}+\mathbf{G}_1\right)+ \left(\frac{\partial \mathbf{V}_2}{\partial \dot{\theta}_j}\right).\mathbf{R}^{2/1}+\left(\frac{\partial \mathbf{V}_7}{\partial \dot{\theta}_j}\right).\mathbf{R}^{7/1}$$

$$\frac{\partial \mathbf{K}}{\partial \dot{\theta}_2} = \left(\frac{\partial \omega_2}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{1/2}+\mathbf{T}^{3/2}\right)+\left(\frac{\partial \mathbf{V}_2}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{1/2}+\mathbf{G}_2\right)+\left(\frac{\partial \mathbf{V}_3}{\partial \dot{\theta}_j}\right).\mathbf{R}^{3/2}$$

$$\frac{\partial \mathbf{K}}{\partial \dot{\theta}_3} = \left(\frac{\partial \omega_3}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{2/3}+\mathbf{T}^{8/3}+\mathbf{T}_{4/3}\right)+\left(\frac{\partial \mathbf{V}_3}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{2/3}+\mathbf{R}^{8/3}+\mathbf{G}_3\right)+ \left(\frac{\partial \mathbf{V}_4}{\partial \dot{\theta}_j}\right).\mathbf{R}^{4/3}$$

$$\frac{\partial \mathbf{K}}{\partial \dot{\theta}_4} = \left(\frac{\partial \omega_4}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{3/4}+\mathbf{T}^{5/4}\right)+\left(\frac{\partial \mathbf{V}_4}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{3/4}+\mathbf{G}_4\right)+\left(\frac{\partial \mathbf{V}_5}{\partial \dot{\theta}_j}\right).\mathbf{R}^{5/4}$$

$$\frac{\partial \mathbf{K}}{\partial \dot{\theta}_5} = \left(\frac{\partial \omega_5}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{4/5}+\mathbf{T}^{6/5}\right)+\left(\frac{\partial \mathbf{V}_5}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{4/5}+\mathbf{G}_5\right)+\left(\frac{\partial \mathbf{V}_6}{\partial \dot{\theta}_j}\right).\mathbf{R}^{6/5}$$

$$\frac{\partial \mathbf{K}}{\partial \dot{\theta}_6} = \left(\frac{\partial \omega_6}{\partial \dot{\theta}_j}\right).\mathbf{T}^{5/6}+\left(\frac{\partial \mathbf{V}_6}{\partial \dot{\theta}_\mathrm{j}}\right).\left(\mathbf{R}^{5/6}+\mathbf{G}_6\right)$$

$$\frac{\partial \mathbf{K}}{\partial \dot{\theta}_7} = \left(\frac{\partial \omega_7}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{1/7}+\mathbf{T}^{8/7}\right)+\left(\frac{\partial \mathbf{V}_7}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{1/7}+\mathbf{G}_7\right)+\left(\frac{\partial \mathbf{V}_8}{\partial \dot{\theta}_j}\right).\mathbf{R}^{8/7}$$

$$\frac{\partial \mathbf{K}}{\partial \dot{\theta}_8} = \left(\frac{\partial \omega_8}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{7/8}+\mathbf{T}^{3/8}\right)+\left(\frac{\partial \mathbf{V}_8}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{7/8}+\mathbf{G}_8\right)+\left(\frac{\partial \mathbf{V}_3}{\partial \dot{\theta}_j}\right).\mathbf{R}^{3/8}$$

The generalized active forces of the entire system is:

$$\frac{\partial K}{\partial \dot{\theta}_j} = \sum_{i=1}^{8} \frac{\partial K_i}{\partial \dot{\theta}_j} = \sum_{i=1}^{8} \left(\frac{\partial \omega_i}{\partial \dot{\theta}_j}\right).\mathbf{T}_i + \sum_{i=1}^{8} \left(\frac{\partial \mathbf{V}_i}{\partial \dot{\theta}_j}\right).\mathbf{R}_i$$

Then summing up the contribution of all the links, gives:

$$\begin{aligned}
\frac{\partial K}{\partial \dot{\theta}_j} = \; & \left(\frac{\partial \omega_1}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{0/1} + \mathbf{T}^{2/1} + \mathbf{T}_{7/1}\right) + \left(\frac{\partial \mathbf{V}_1}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{0/1} + \mathbf{G}_1\right) + \\
& \left(\frac{\partial \mathbf{V}_2}{\partial \dot{\theta}_j}\right).\mathbf{R}^{2/1} + \left(\frac{\partial \mathbf{V}_7}{\partial \dot{\theta}_j}\right).\mathbf{R}^{7/1} + \\
& \left(\frac{\partial \omega_2}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{1/2} + \mathbf{T}^{3/2}\right) + \left(\frac{\partial \mathbf{V}_2}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{1/2} + \mathbf{G}_2\right) + \left(\frac{\partial \mathbf{V}_3}{\partial \dot{\theta}_j}\right).\mathbf{R}^{3/2} + \\
& \left(\frac{\partial \omega_3}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{2/3} + \mathbf{T}^{8/3} + \mathbf{T}_{4/3}\right) + \left(\frac{\partial \mathbf{V}_3}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{2/3} + \mathbf{R}^{8/3} + \mathbf{G}_3\right) + \\
& \left(\frac{\partial \mathbf{V}_4}{\partial \dot{\theta}_j}\right).\mathbf{R}^{4/3} + \\
& \left(\frac{\partial \omega_4}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{3/4} + \mathbf{T}^{5/4}\right) + \left(\frac{\partial \mathbf{V}_4}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{3/4} + \mathbf{G}_4\right) + \left(\frac{\partial \mathbf{V}_5}{\partial \dot{\theta}_j}\right).\mathbf{R}^{5/4} + \\
& \left(\frac{\partial \omega_5}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{4/5} + \mathbf{T}^{6/5}\right) + \left(\frac{\partial \mathbf{V}_5}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{4/5} + \mathbf{G}_5\right) + \left(\frac{\partial \mathbf{V}_6}{\partial \dot{\theta}_j}\right).\mathbf{R}^{6/5} + \\
& \left(\frac{\partial \omega_6}{\partial \dot{\theta}_j}\right).\mathbf{T}^{5/6} + \left(\frac{\partial \mathbf{V}_6}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{5/6} + \mathbf{G}_6\right) + \\
& \left(\frac{\partial \omega_7}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{1/7} + \mathbf{T}^{8/7}\right) + \left(\frac{\partial \mathbf{V}_7}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{1/7} + \mathbf{G}_7\right) + \left(\frac{\partial \mathbf{V}_8}{\partial \dot{\theta}_j}\right).\mathbf{R}^{8/7} + \\
& \left(\frac{\partial \omega_8}{\partial \dot{\theta}_j}\right).\left(\mathbf{T}^{7/8} + \mathbf{T}^{3/8}\right) + \left(\frac{\partial \mathbf{V}_8}{\partial \dot{\theta}_j}\right).\left(\mathbf{R}^{7/8} + \mathbf{G}_8\right) + \left(\frac{\partial \mathbf{V}_3}{\partial \dot{\theta}_j}\right).\mathbf{R}^{3/8} \\
\frac{\partial K}{\partial \dot{\theta}_j} = \; & \left(\frac{\partial \omega_1}{\partial \dot{\theta}_j}\right).\mathbf{T}^{0/1} + \left(\frac{\partial \omega_1}{\partial \dot{\theta}_j} - \frac{\partial \omega_2}{\partial \dot{\theta}_j}\right).\mathbf{T}^{2/1} + \left(\frac{\partial \omega_1}{\partial \dot{\theta}_j} - \frac{\partial \omega_7}{\partial \dot{\theta}_j}\right).\mathbf{T}^{7/1} + \\
& \left(\frac{\partial \omega_3}{\partial \dot{\theta}_j} - \frac{\partial \omega_4}{\partial \dot{\theta}_j}\right).\mathbf{T}^{4/3} + \left(\frac{\partial \omega_4}{\partial \dot{\theta}_j} - \frac{\partial \omega_5}{\partial \dot{\theta}_j}\right).\mathbf{T}^{5/4} + \left(\frac{\partial \omega_5}{\partial \dot{\theta}_j} - \frac{\partial \omega_6}{\partial \dot{\theta}_j}\right).\mathbf{T}^{6/5} + \\
& \left(\frac{\partial \mathbf{V}_2}{\partial \dot{\theta}_j}\right).\mathbf{G}_2 + \left(\frac{\partial \mathbf{V}_3}{\partial \dot{\theta}_j}\right).\mathbf{G}_3 + \left(\frac{\partial \mathbf{V}_4}{\partial \dot{\theta}_j}\right).\mathbf{G}_4 + \left(\frac{\partial \mathbf{V}_5}{\partial \dot{\theta}_j}\right).\mathbf{G}_5 + \\
& \left(\frac{\partial \mathbf{V}_6}{\partial \dot{\theta}_j}\right).\mathbf{G}_6 + \left(\frac{\partial \mathbf{V}_7}{\partial \dot{\theta}_j}\right).\mathbf{G}_7 + \left(\frac{\partial \mathbf{V}_8}{\partial \dot{\theta}_j}\right).\mathbf{G}_8
\end{aligned}$$

Therefore the link contribution is:

$$\begin{aligned}
\frac{\partial K}{\partial \dot\theta_1} &= (\mathbf{j}_1.\mathbf{T}^{0/1}) - g[m_5(Z_{62}Z_{21} + Z_{65}Z_{18} + Z_{59}Z_{24}) + \\
&\quad m_6(Z_{69}Z_{21} + Z_{72}Z_{18} + Z_{59}Z_{24})] \\
&= (\mathbf{j}_1.\mathbf{T}^{0/1}) - g[m_5(Z_{62}Z_{21} + Z_{65}Z_{18} + Z_{82}) + \\
&\quad m_6(Z_{69}Z_{21} + Z_{72}Z_{18} + Z_{82})] \\
&= \tau_1 + Z_{83} \qquad \text{(E.63)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial K}{\partial \dot\theta_2} &= -(\mathbf{k}_1.\mathbf{T}^{2/1}) - g[m_2r_2c_2 + m_3(s_{23}Z_{40} + c_{23}Z_{39}) + m_4(s_{23}Z_{43} + c_{23}Z_{39}) + \\
&\quad m_5(Z_{63}Z_{21} + Z_{66}Z_{18} + Z_{60}Z_{24}) + m_6(Z_{70}Z_{21} + Z_{73}Z_{18} + Z_{60}Z_{24}) + m_8c_{78}Z_{79}] \\
&= -(\mathbf{k}_1.\mathbf{T}^{2/1}) - g[m_2r_2c_2 + m_3(s_{23}Z_{40} + Z_{84}) + m_4(s_{23}Z_{43} + Z_{84}) + \\
&\quad m_5(Z_{63}Z_{21} + Z_{66}Z_{18} + Z_{85}) + m_6(Z_{70}Z_{21} + Z_{73}Z_{18} + Z_{85}) + m_8c_{78}Z_{79}] \\
&= -\tau_2 + Z_{86} \qquad \text{(E.64)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial K}{\partial \dot\theta_4} &= -(\mathbf{k}_3.\mathbf{T}^{4/3}) - g(m_5Z_{67}Z_{18} + m_6Z_{74}Z_{18}) \\
&= -(\mathbf{k}_3.\mathbf{T}^{4/3}) - gZ_{18}(m_5Z_{67} + m_6Z_{74}) \\
&= -\tau_4 + Z_{87} \qquad \text{(E.65)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial K}{\partial \dot\theta_5} &= -(\mathbf{k}_4.\mathbf{T}^{5/4}) - g(m_5Z_{21}r_5 + m_6r_6Z_{21}) \\
&= -(\mathbf{k}_4.\mathbf{T}^{5/4}) - gZ_{21}(m_5r_5 + m_6r_6) \\
&= -\tau_5 + Z_{88} \qquad \text{(E.66)}
\end{aligned}$$

$$\frac{\partial K}{\partial \dot\theta_6} = -\mathbf{k}_5.\mathbf{T}^{6/5} = -\tau_6$$

$$\begin{aligned}
\frac{\partial K}{\partial \dot\theta_7} &= -(\mathbf{k}_1.\mathbf{T}^{7/1}) - g[m_3Z_{41}s_{23} + m_4Z_{44}s_{23} + m_5(Z_{64}Z_{21} + Z_{68}Z_{18} + Z_{61}Z_{24}) + \\
&\quad m_6(Z_{71}Z_{21} + Z_{75}Z_{18} + Z_{61}Z_{24}) + m_7r_7c_7 + m_8(Z_{78}s_{78} + Z_{80}c_{78})] \\
&= -(\mathbf{k}_1.\mathbf{T}^{7/1}) - g[s_{23}(m_3Z_{41} + m_4Z_{44}) + m_5(Z_{64}Z_{21} + Z_{68}Z_{18} + Z_{89}) + \\
&\quad m_6(Z_{71}Z_{21} + Z_{75}Z_{18} + Z_{89}) + m_7r_7c_7 + m_8(Z_{78}s_{78} + Z_{80}c_{78})] \\
&= -\tau_7 + Z_{90} \qquad \text{(E.67)}
\end{aligned}$$

APPENDIX F

F.1 The Mass Element Matrix

$$
\left[\mathrm{M}^{(e)}\right] = \rho A l \begin{bmatrix}
\frac{1}{3} & & & & & & & & & & & \\
0 & \frac{13}{35} & & & & & & & & & & \\
0 & 0 & \frac{13}{35} & & & & & & & & & \\
0 & 0 & 0 & \frac{J}{3A} & & & & & & & & \\
0 & 0 & \frac{-11}{210}l & 0 & \frac{l^2}{105} & & & & & & & \\
0 & \frac{11}{210}l & 0 & 0 & 0 & \frac{l^2}{105} & & & & & & \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & & & & & \\
0 & \frac{9}{70} & 0 & 0 & 0 & \frac{13}{420}l & 0 & \frac{13}{35} & & & & \\
0 & 0 & \frac{9}{70} & 0 & \frac{-13}{420}l & 0 & 0 & 0 & \frac{13}{35} & & & \\
0 & 0 & 0 & \frac{J}{6A} & 0 & 0 & 0 & 0 & 0 & \frac{J}{3A} & & \\
0 & 0 & \frac{13}{420}l & 0 & \frac{-l^2}{140} & 0 & 0 & 0 & \frac{11}{210}l & 0 & \frac{l^2}{105} & \\
0 & \frac{-13}{420}l & 0 & 0 & 0 & \frac{-l^2}{140} & 0 & \frac{-11}{210}l & 0 & 0 & 0 & \frac{l^2}{105}
\end{bmatrix}
$$

F.2 The Stiffness Element Matrix

$$
\left[\mathbf{K}^{(e)}\right] = E \begin{bmatrix}
\frac{A}{l} \\
0 & \frac{12I_z}{l^3} \\
0 & 0 & \frac{12I_y}{l^3} \\
0 & 0 & 0 & \frac{J}{2l(1+\nu)} \\
0 & 0 & \frac{-6I_y}{l^2} & 0 & \frac{4I_y}{l} \\
0 & \frac{6I_z}{l^2} & 0 & 0 & 0 & \frac{4I_z}{l} \\
\frac{-A}{l} & 0 & 0 & 0 & 0 & 0 & \frac{A}{l} \\
0 & \frac{-12I_z}{l^3} & 0 & 0 & 0 & \frac{-6I_z}{l^2} & 0 & \frac{12I_z}{l^3} \\
0 & 0 & \frac{-12I_y}{l^3} & 0 & \frac{6I_y}{l^2} & 0 & 0 & 0 & \frac{12I_y}{l^3} \\
0 & 0 & 0 & \frac{-J}{2l(1+\nu)} & 0 & 0 & 0 & 0 & 0 & \frac{J}{2l(1+\nu)} \\
0 & 0 & \frac{-6I_y}{l^2} & 0 & \frac{2I_y}{l} & 0 & 0 & 0 & \frac{6I_y}{l^2} & 0 & \frac{4I_y}{l} \\
0 & \frac{6I_z}{l^2} & 0 & 0 & 0 & \frac{2I_z}{l} & 0 & \frac{-6I_z}{l^2} & 0 & 0 & 0 & \frac{4I_z}{l}
\end{bmatrix}
$$

F.3 The Damping Element Matrix

$$
\left[\mathbf{C}^{(e)}\right] = \begin{bmatrix}
0 \\
\vdots \\
0 \ldots 0
\end{bmatrix}
$$

THE FIGURES

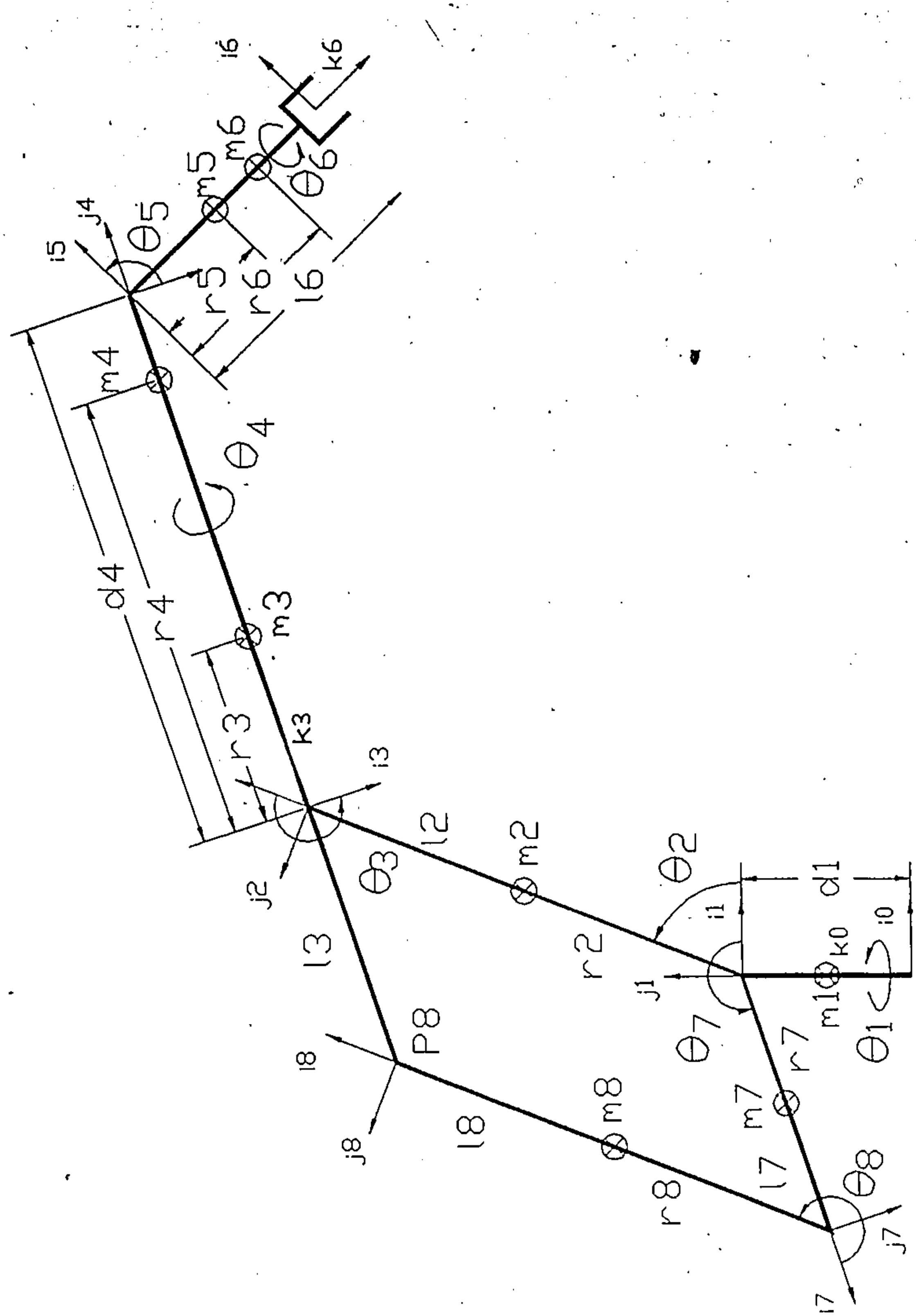


Figure 1: The robot definition

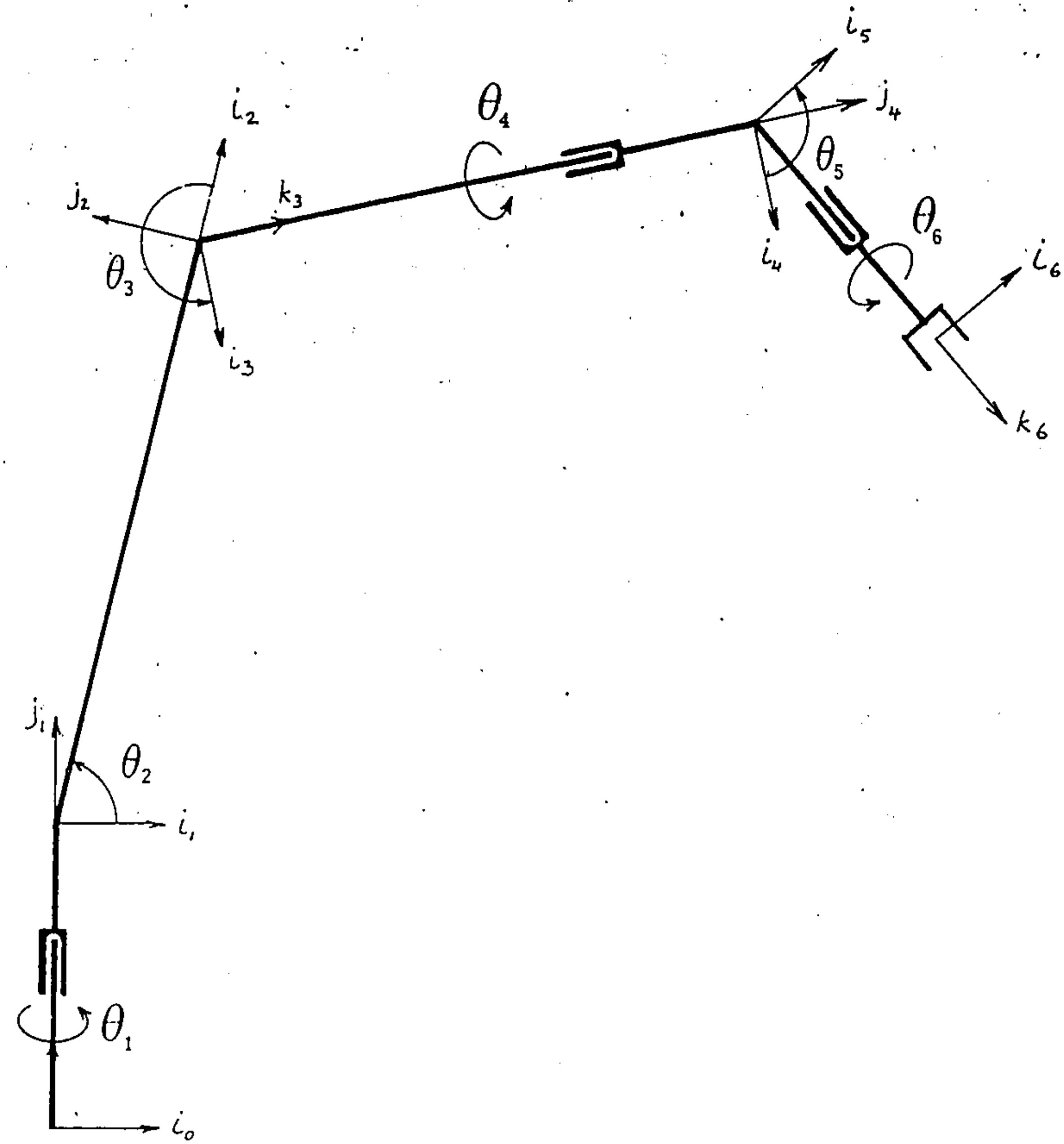


Figure 2: The corresponding robot without the loop

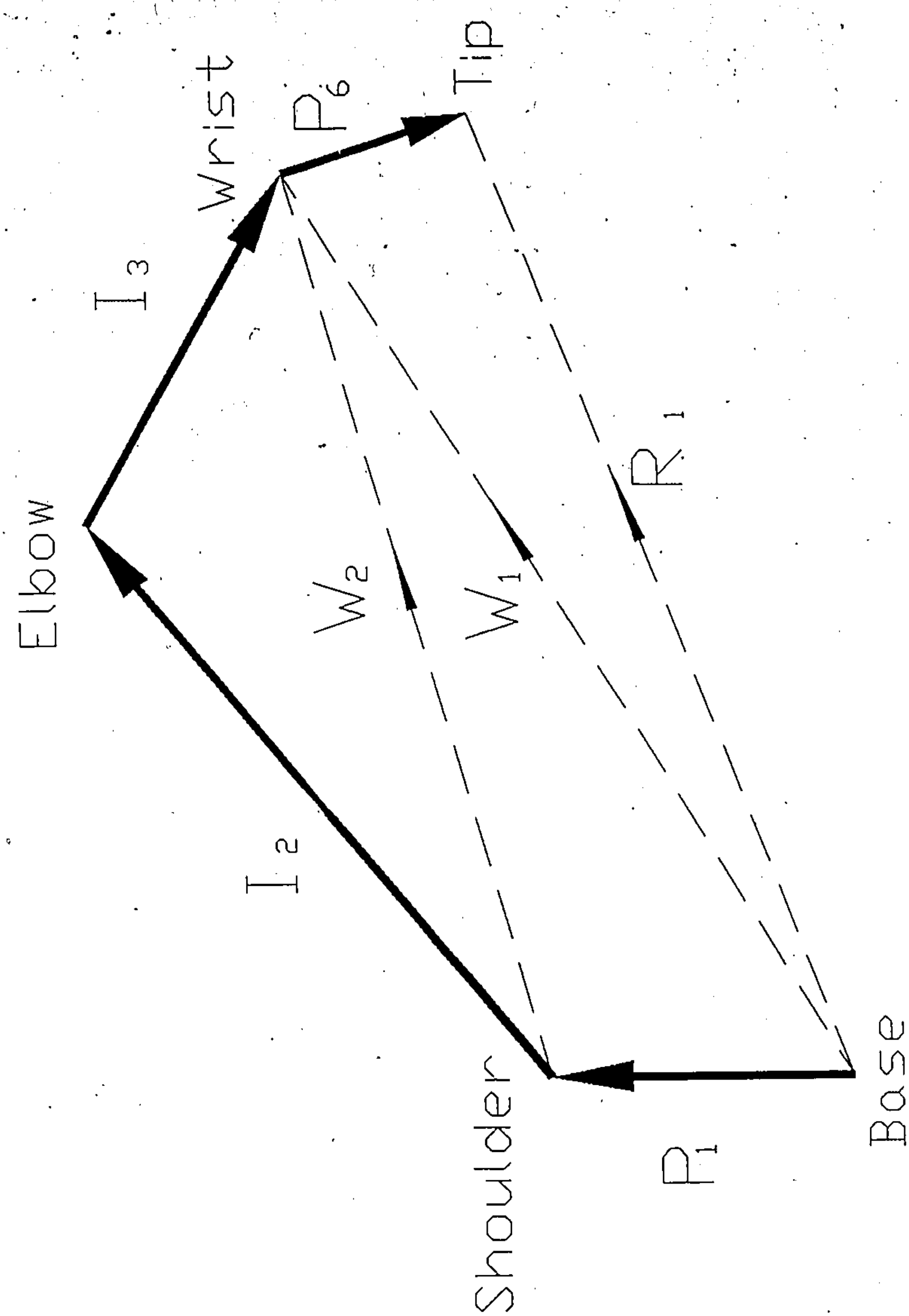


Figure 3: The vector representation of the robot

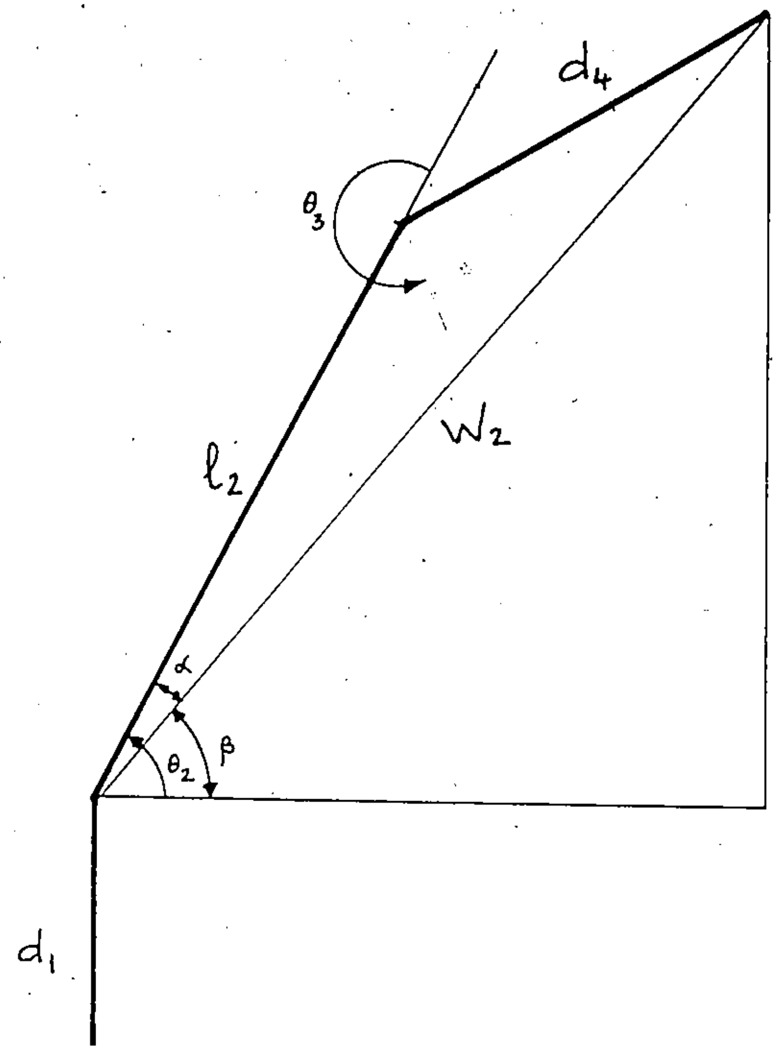


Figure 4: The shoulder-elbow angles

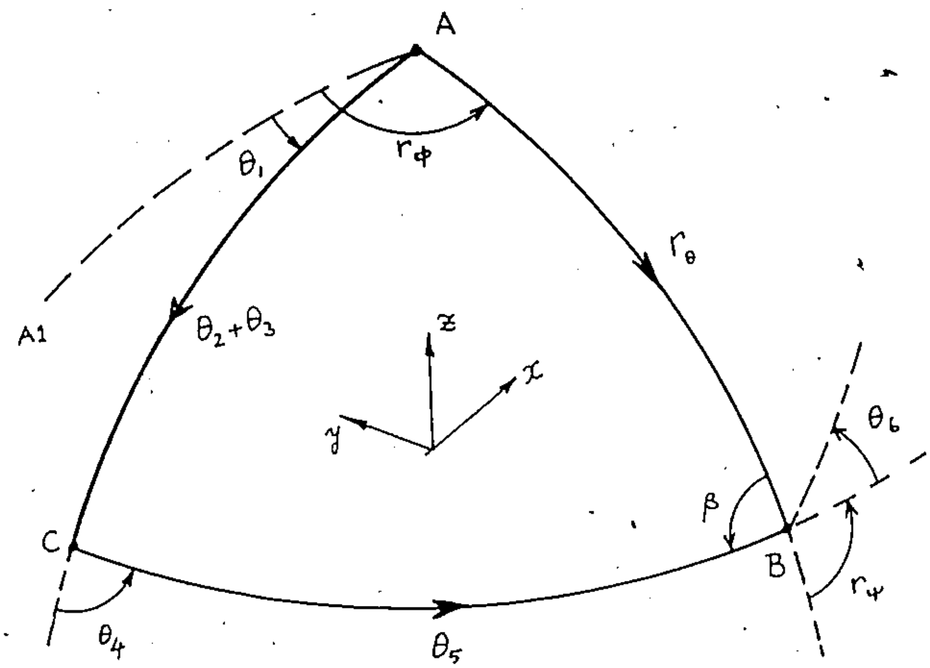


Figure 5: The spherical coordinate representation

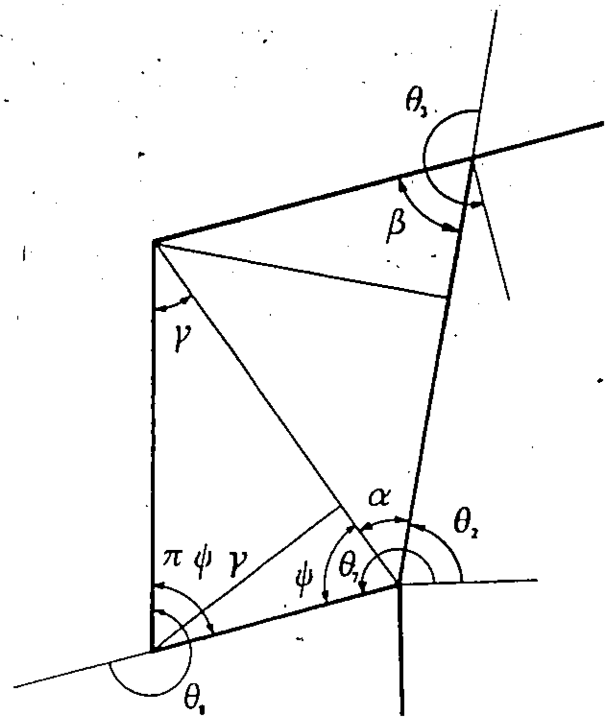


Figure 6: The loop dependant angles

$$
\begin{aligned}
\beta &= -\theta_3 - \frac{\pi}{2} \\
H^2 &= l_2^2 - l_3^2 + 2 l_2 l_3 \sin\theta_3 \\
\cos\psi &= \frac{H^2 + l_7^2 - l_8^2}{2 H l_7} \\
\sin\psi &= \sqrt{1 - \cos\psi^2} \\
\cos\gamma &= \frac{H - l_7 \cos\psi}{l_8} \\
\sin\gamma &= \frac{l_7 \sin\psi}{l_8} \\
\cos\alpha &= \frac{l_2 + l_3 \sin\theta_3}{H} \\
\sin\alpha &= \frac{-l_3 \cos\theta_3}{H} \\
\theta_8 &= -\gamma - \psi \\
\theta_7 &= \theta_2 + \alpha + \psi
\end{aligned}
$$

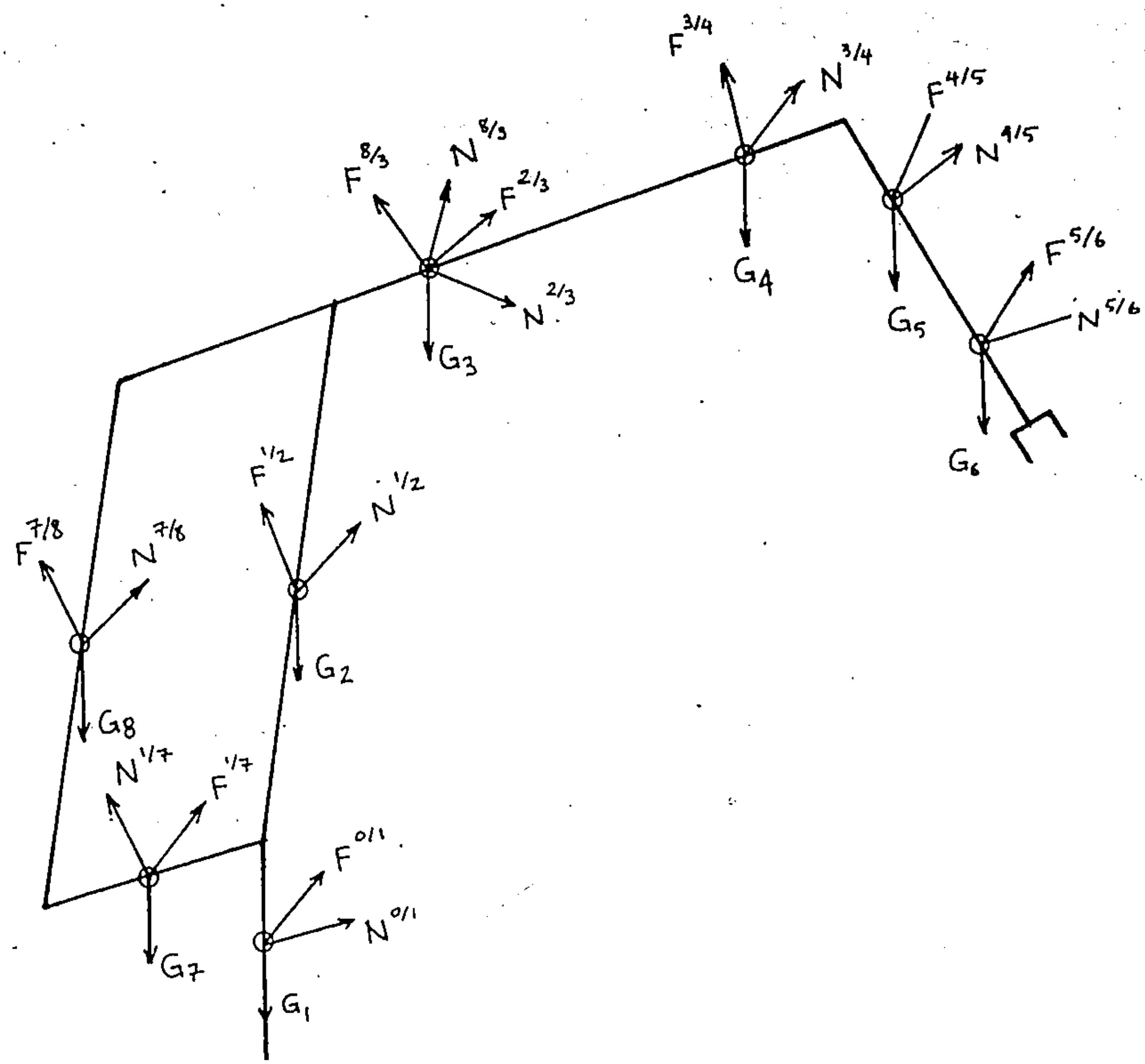


Figure 7: The generalized active forces

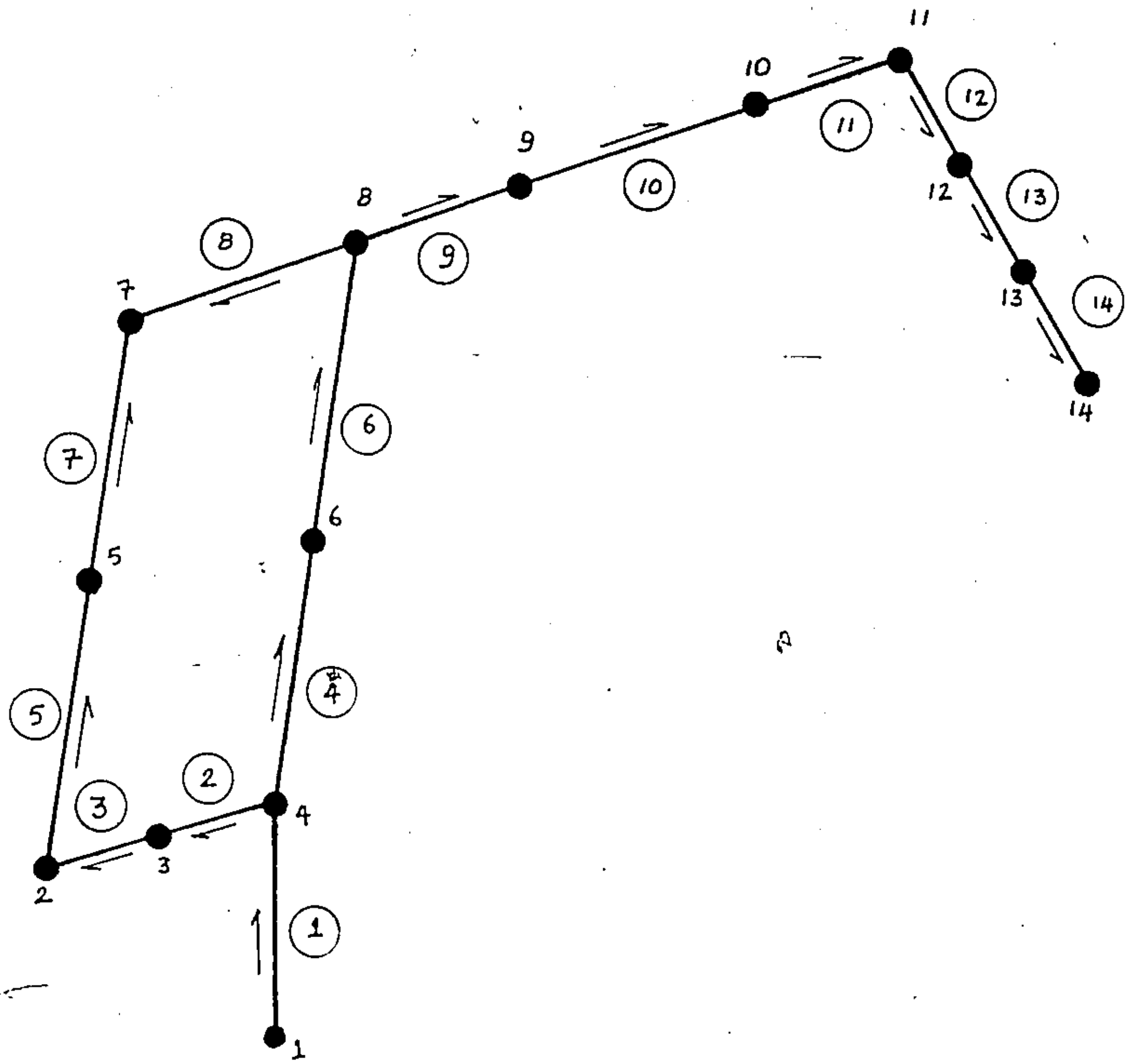
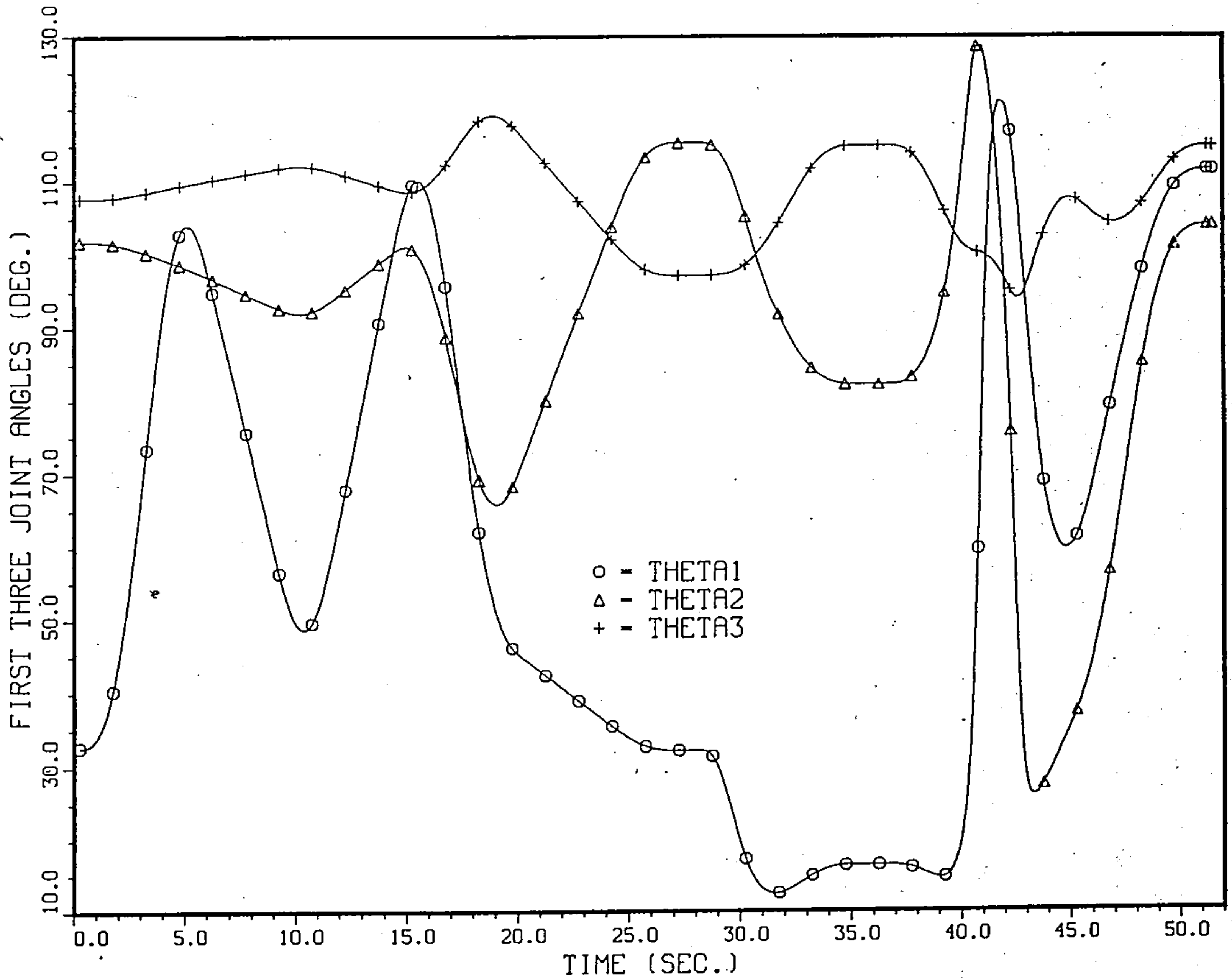


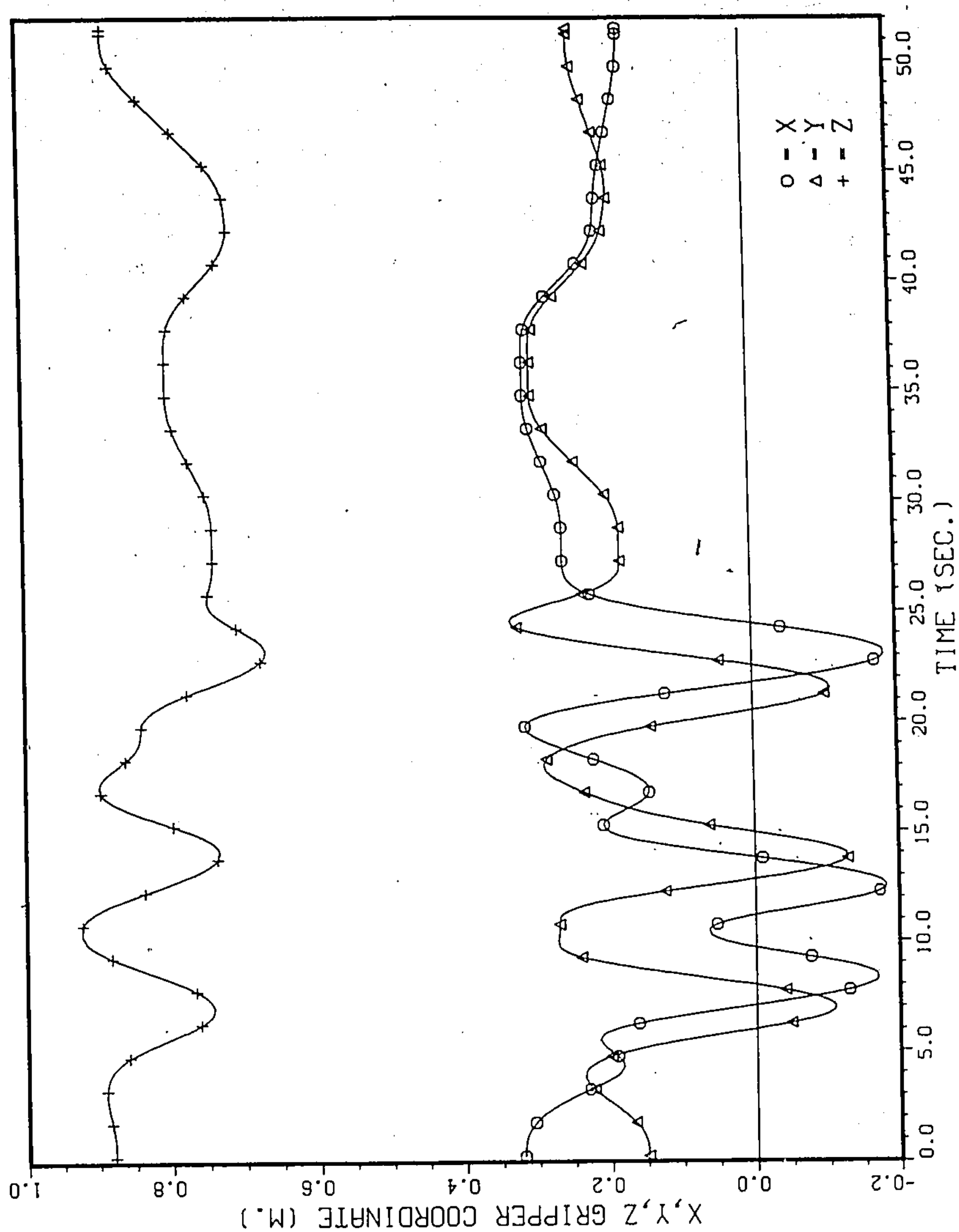
Figure 8: The finite element Mesh

THE GRAPHS

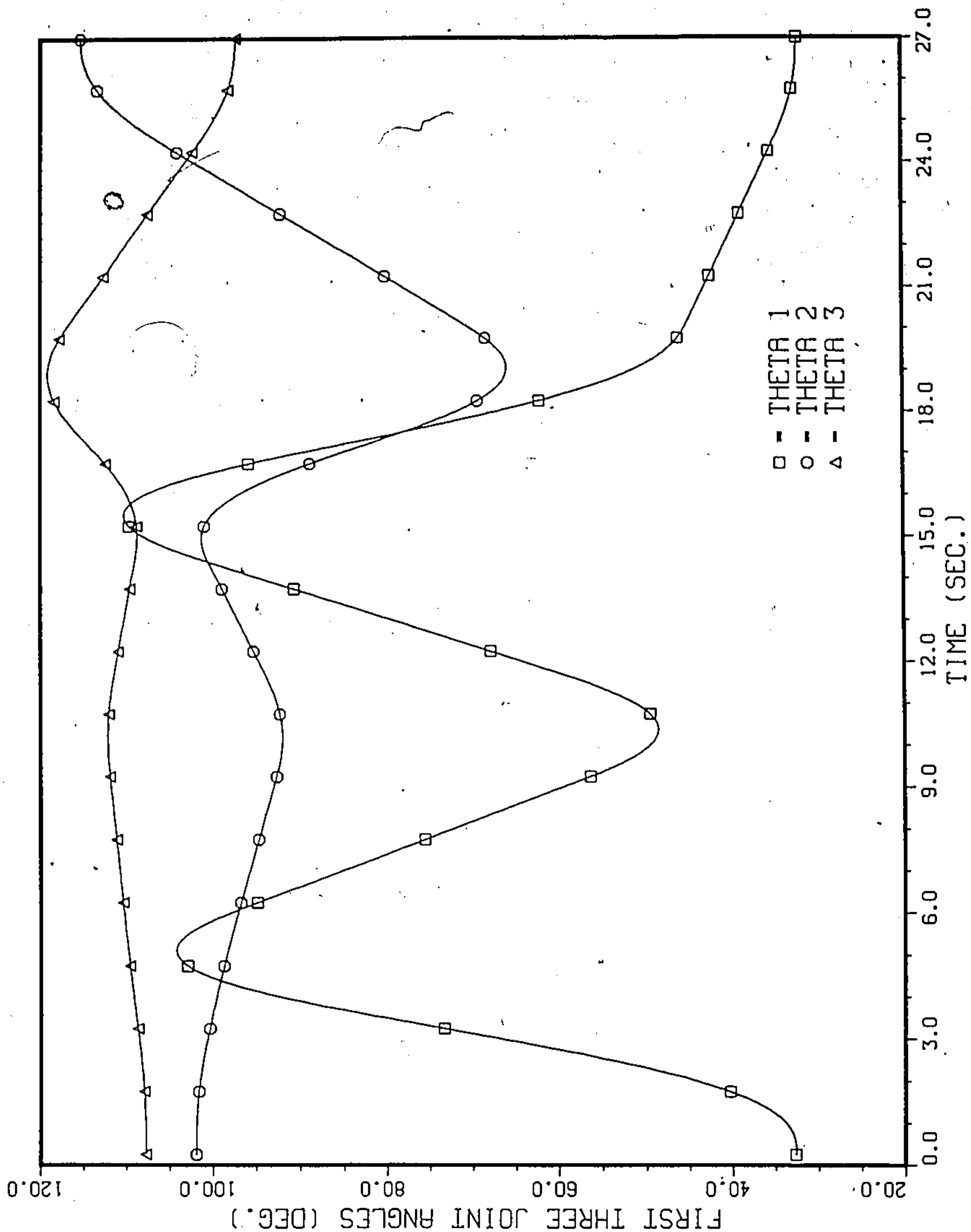
GRAPH 1 : GLOBAL JOINT MOTION



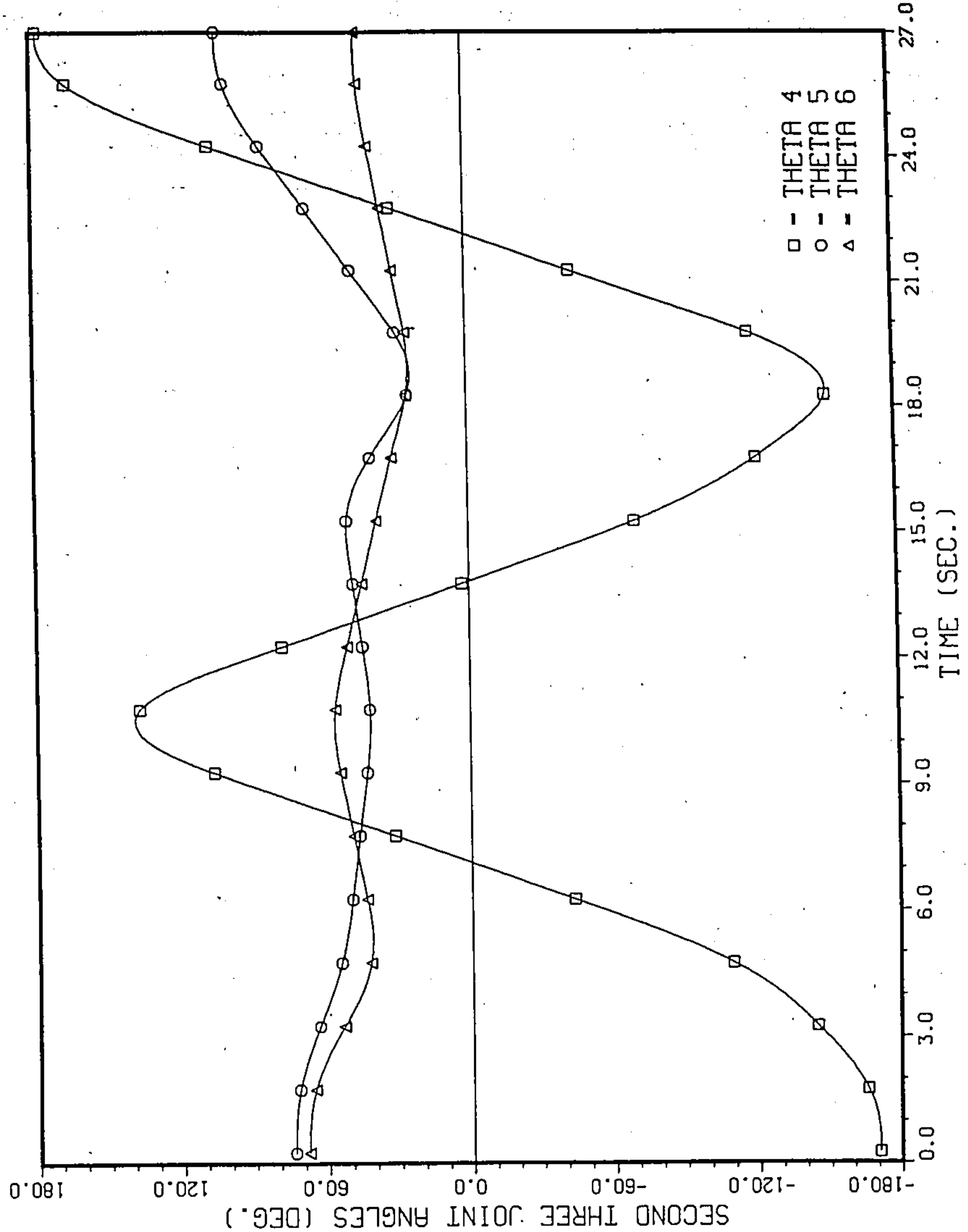
GRAPH 2 : GLOBAL GRIPPER POSITION



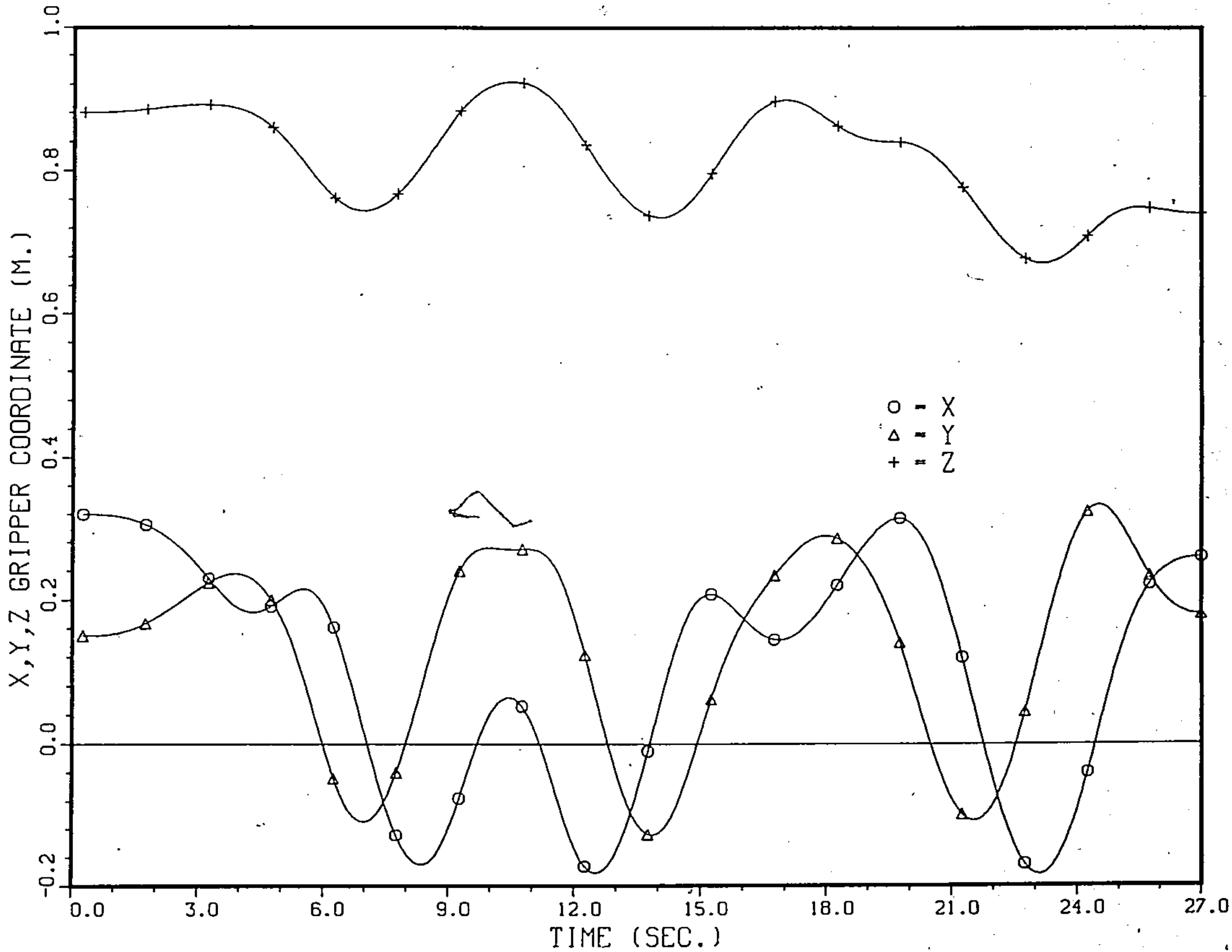
GRAPH 3 : JOINT MOTION I



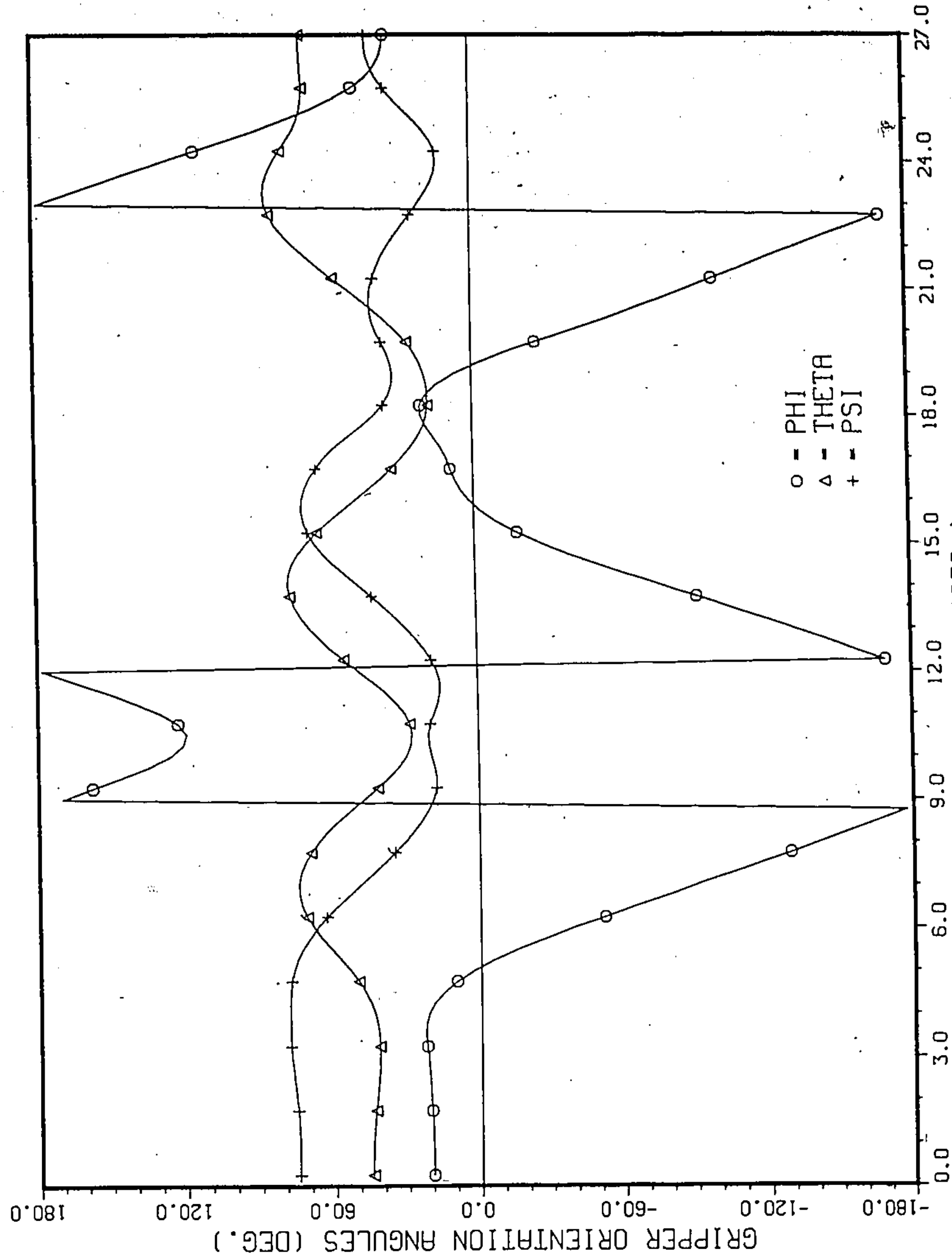
GRAPH 4 : JOINT MOTION II



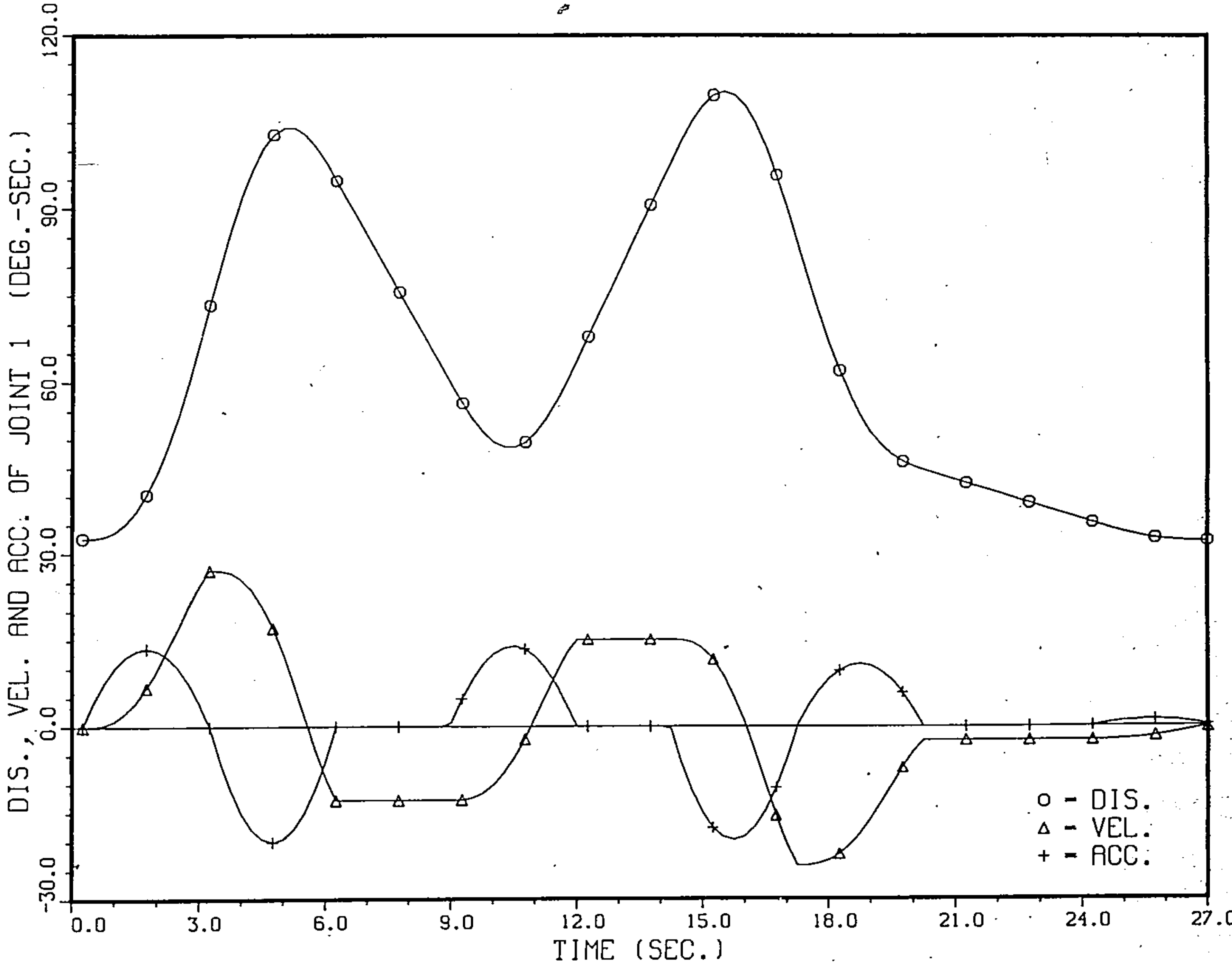
GRAPH 5 : GRIPPER POSITION



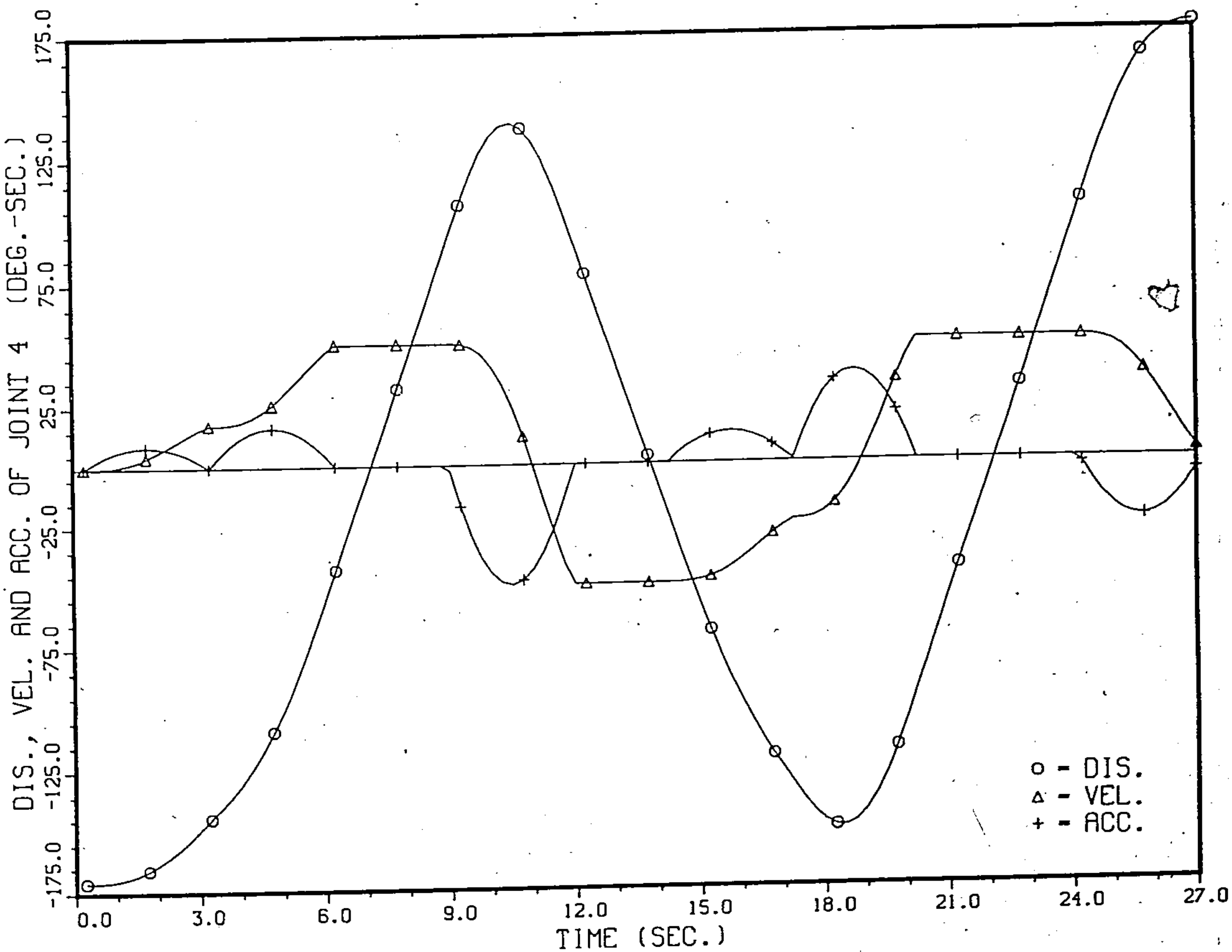
GRAPH 6 : GRIPPER ORIENTATION



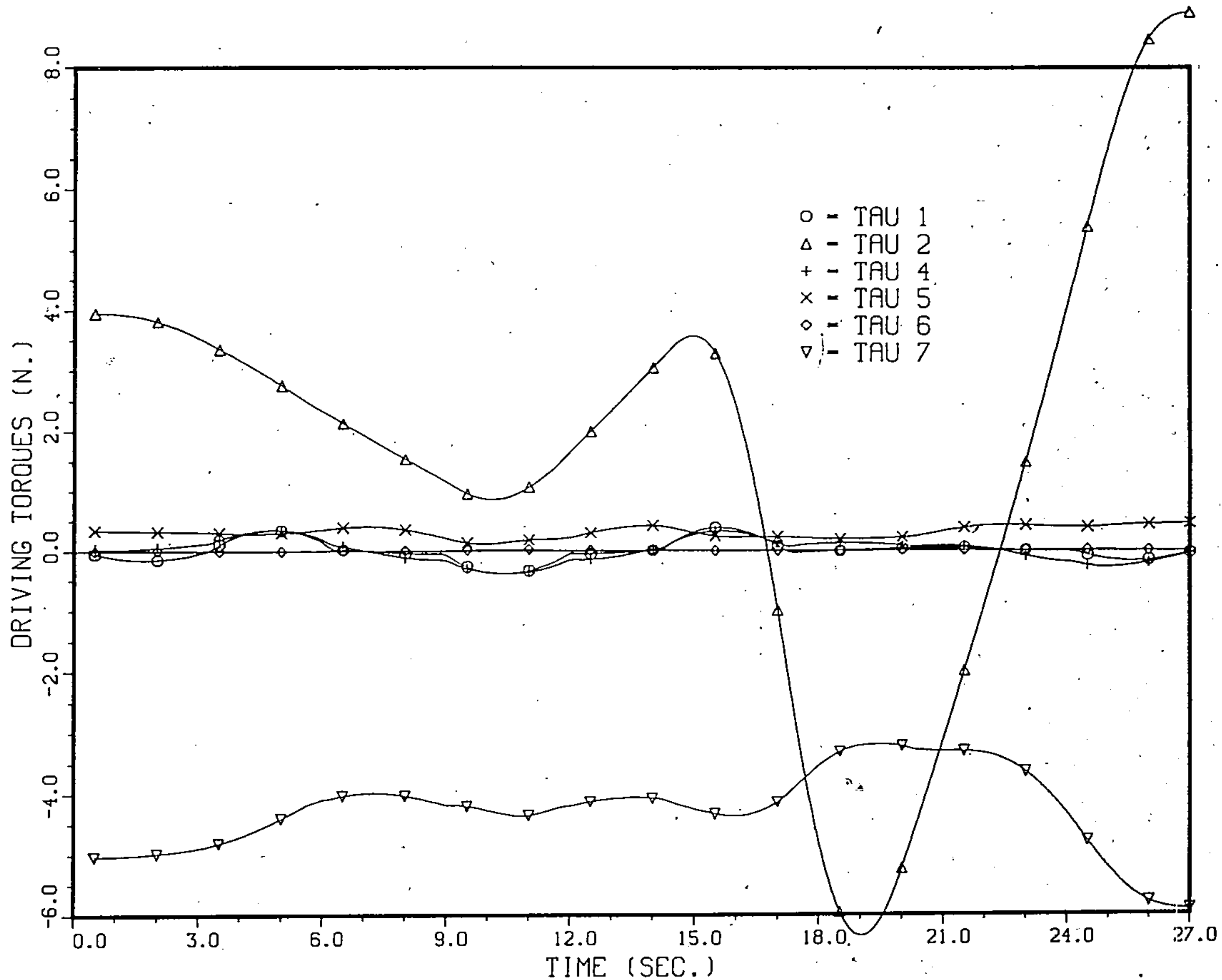
GRAPH 7 : KINEMATICS OF JOINT 1



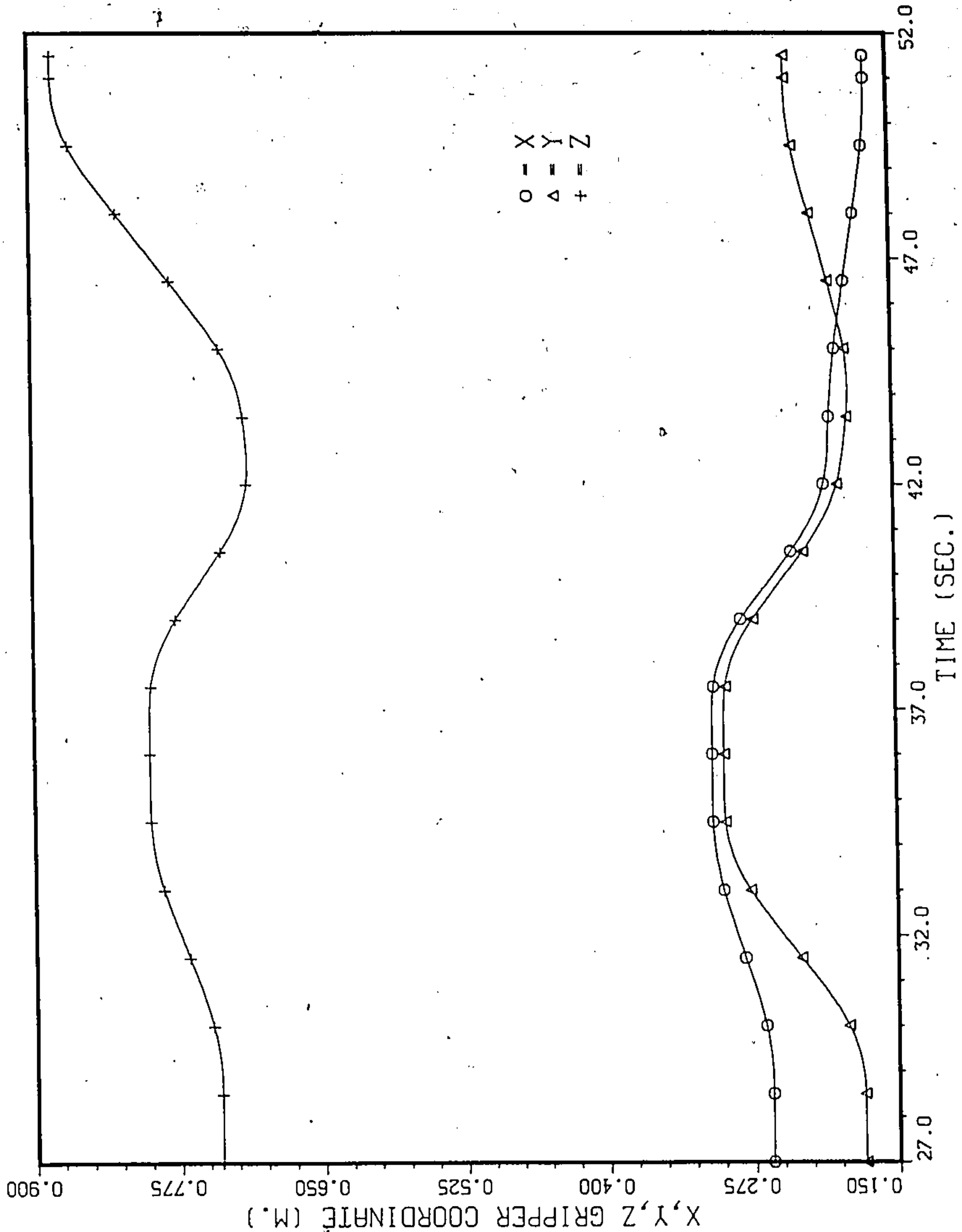
GRAPH 8 : KINEMATICS OF JOINT 4



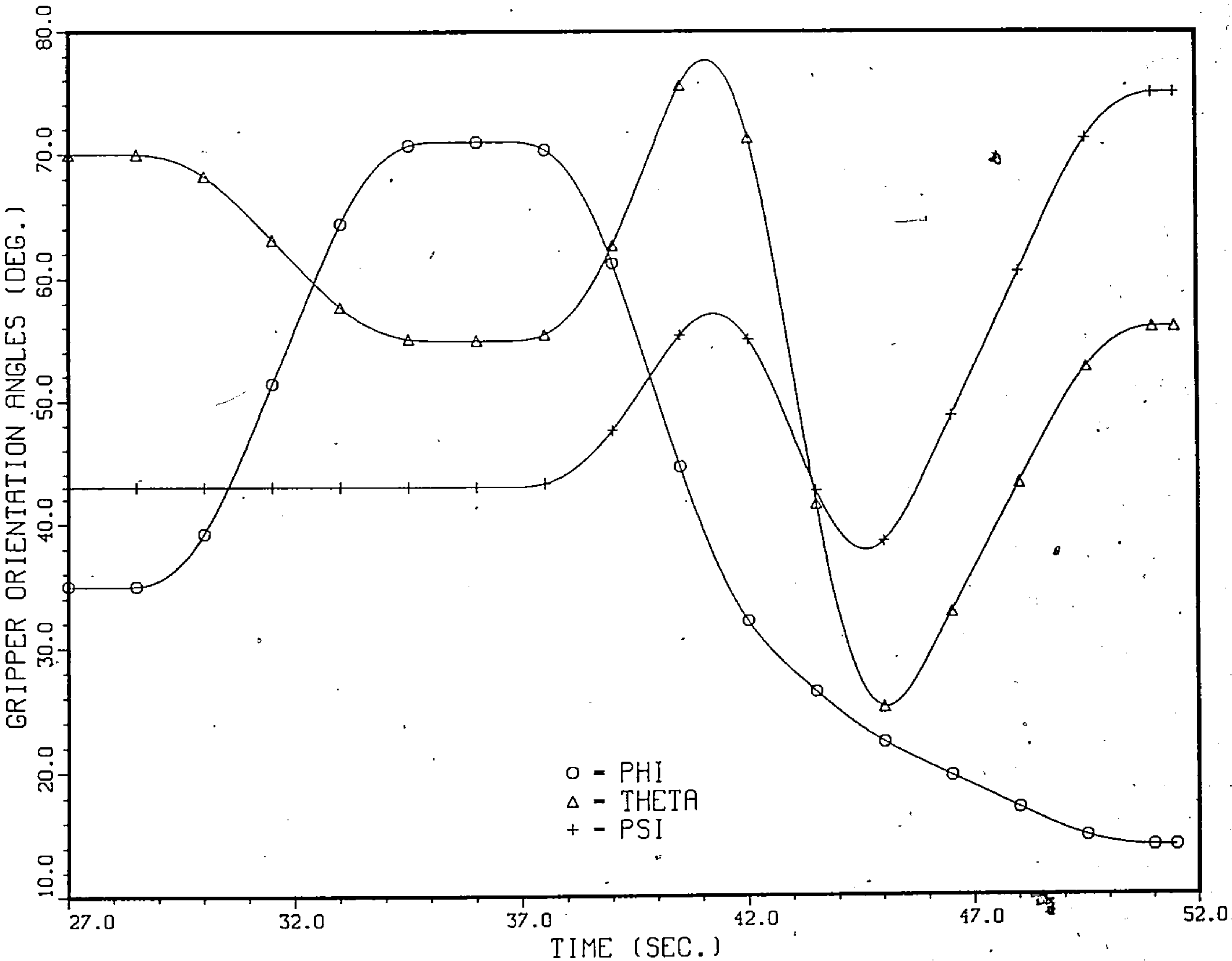
GRAPH 9 : TORQUES



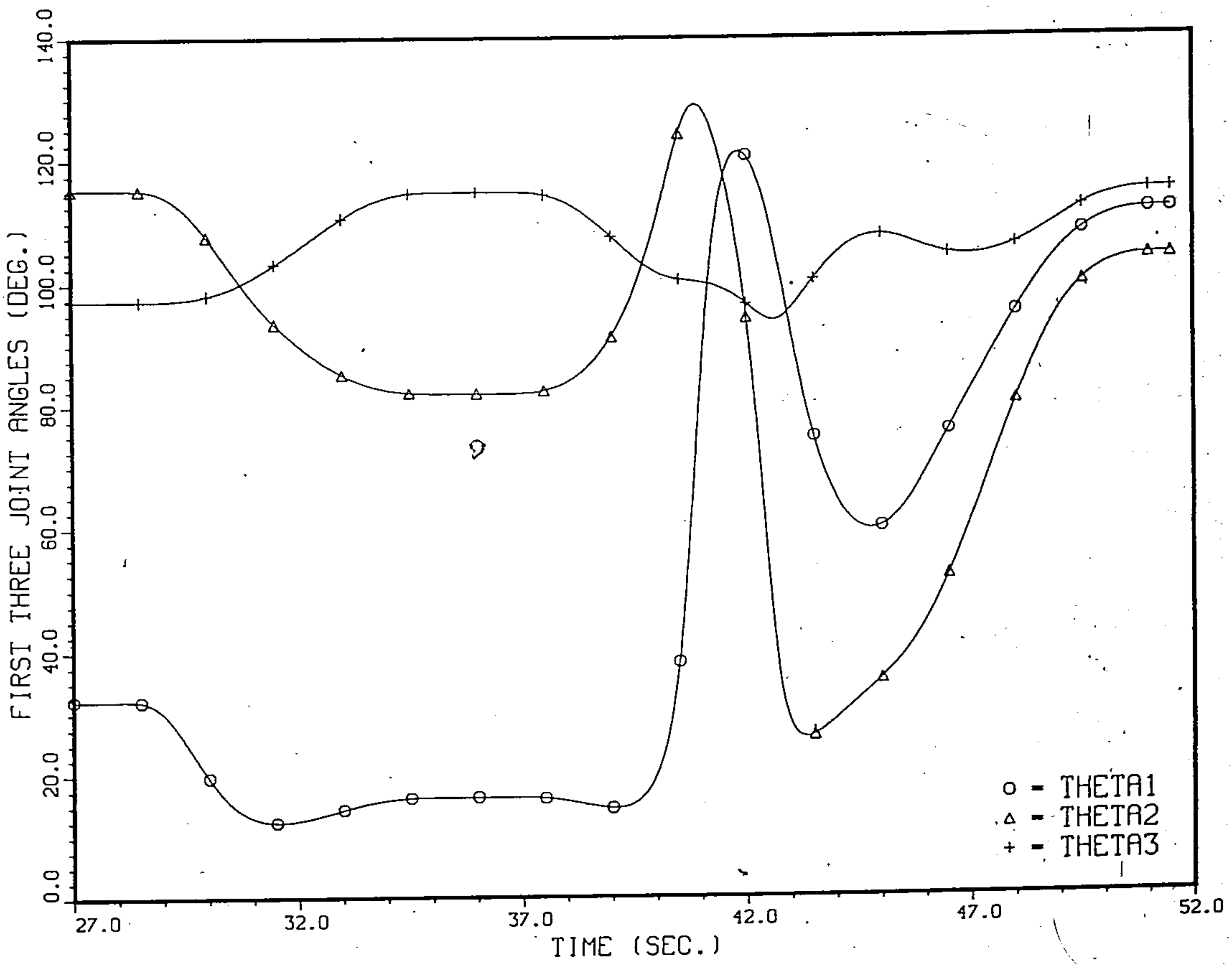
GRAPH 10 : GRIPPER POSITION



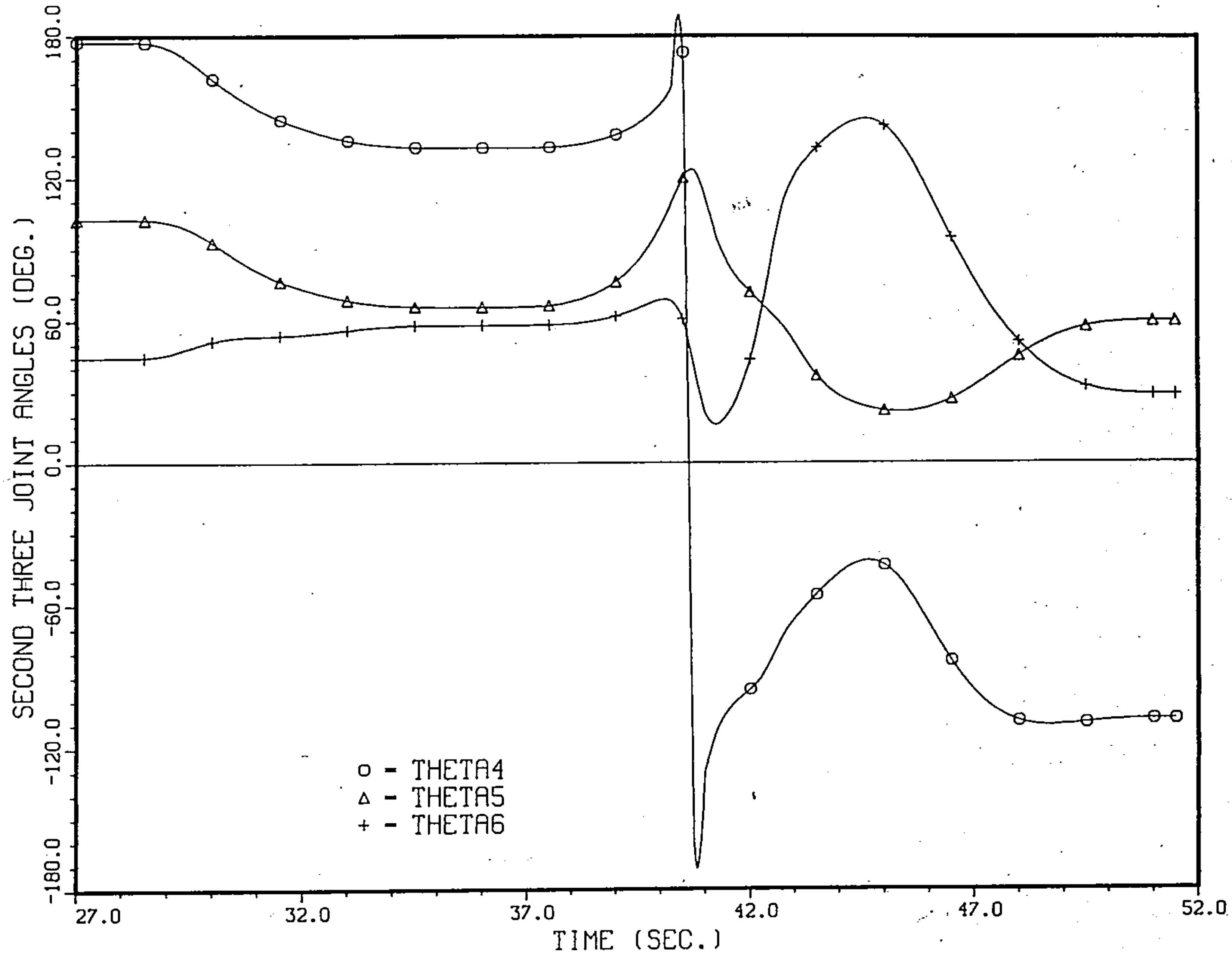
GRAPH 11 : GRIPPER ORIENTATION



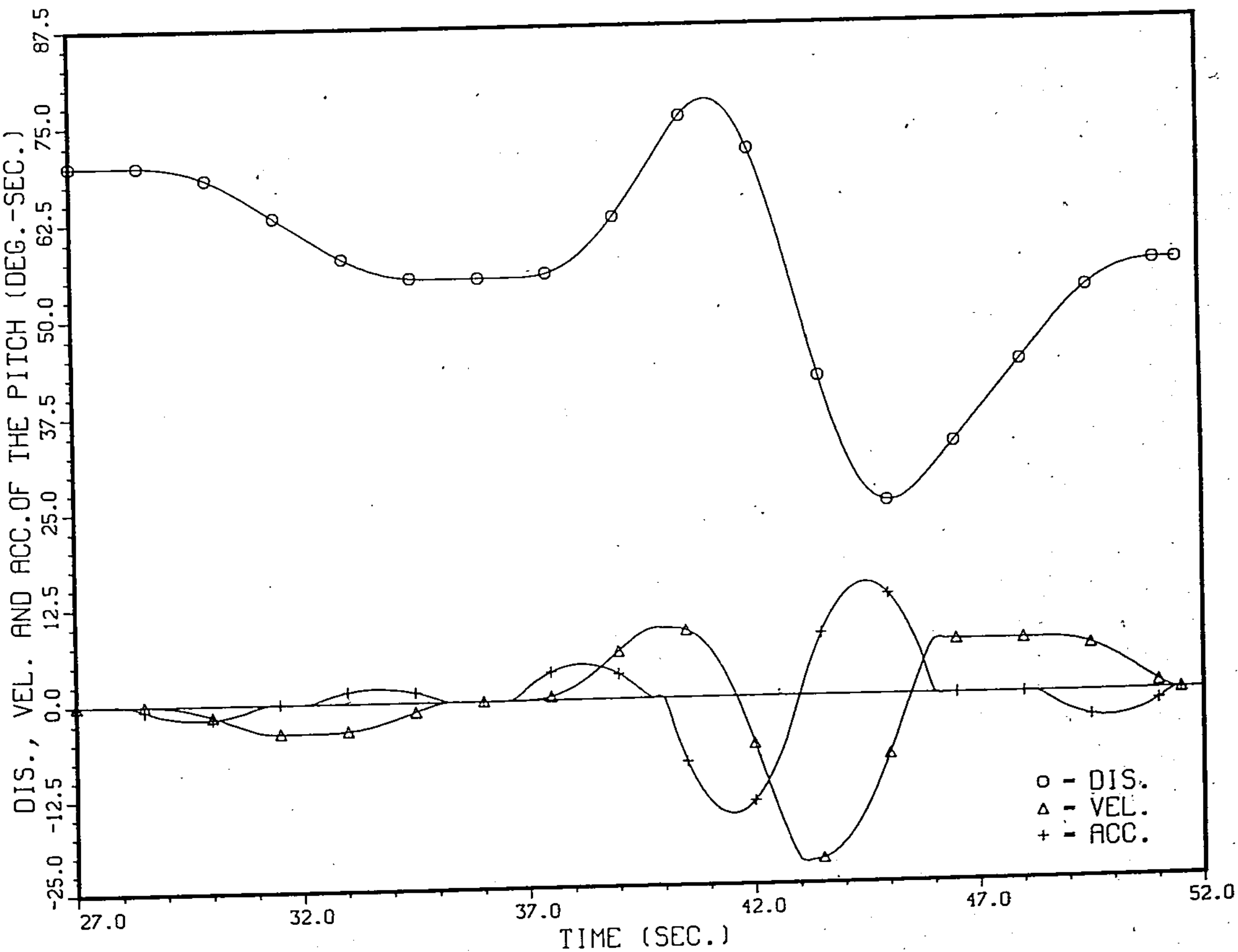
GRAPH 12 : JOINT MOTION I



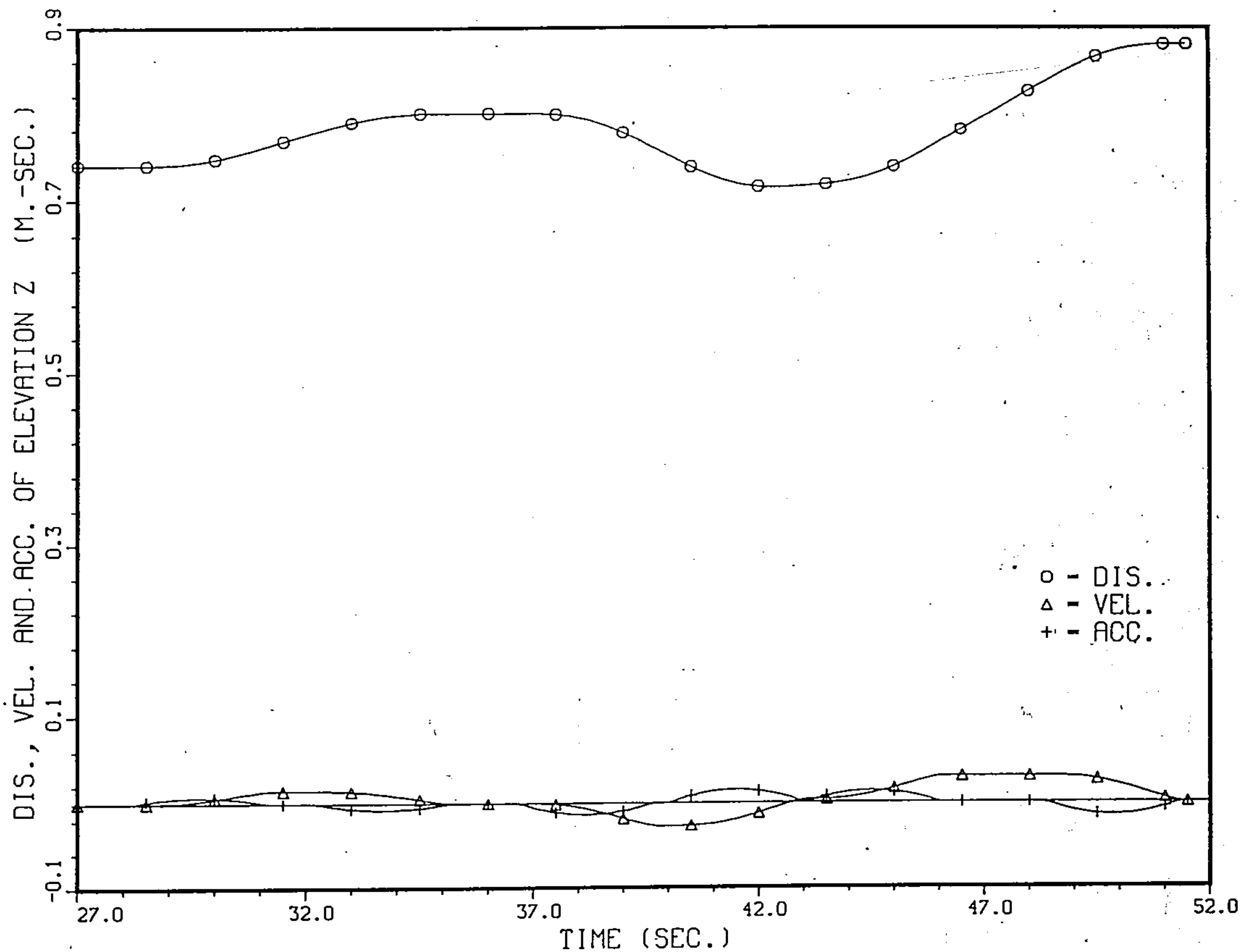
GRAPH 13 : JOINT MOTION II



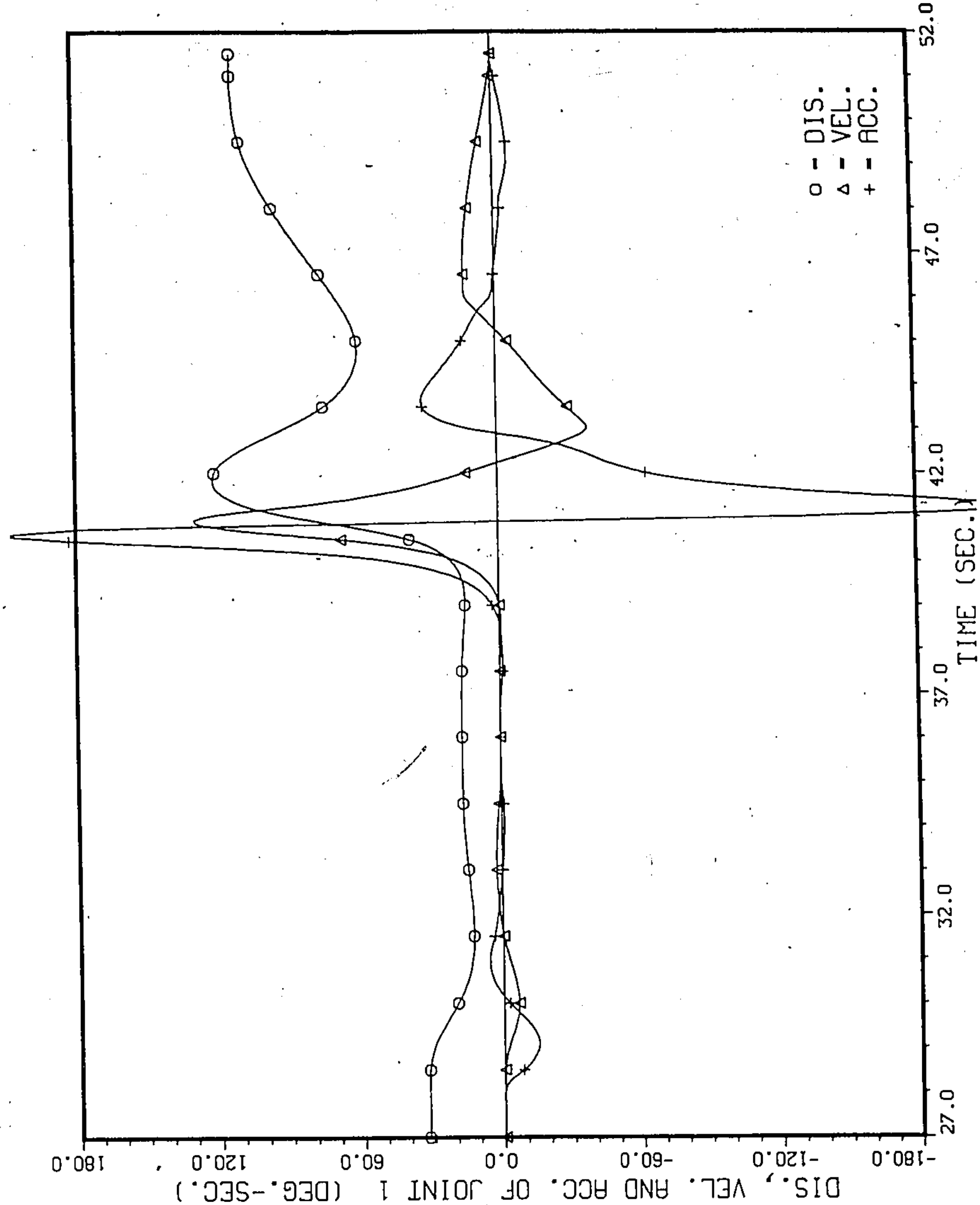
GRAPH 14 : KINEMATICS OF THE PITCH



GRAPH 15 : KINEMATICS OF THE ELEVATION



GRAPH 16 : KINEMATICS OF JOINT 1



GRAPH 17 : TORQUES