A REFINED FINITE ELEMENT ANALYSIS OF STIFFENED AND UNSTIFFENED SKEW PLATES

By

M. S. Desikan

Thesis submitted to the Faculty of Science & Engineering through the Department of Civil Engineering in partial fulfillment of the requirements for the Degree of Master of Applied Science at the University of Ottawa.

Ottawa, Canada

September, 1970

ACKNOWLEDGEMENTS

The writer wishes to express his deep sense of gratitude to Dr. S. F. Ng, Department of Civil Engineering, University of Ottawa under whose supervision this investigation was carried out, for his guidance and constant encouragement throughout the work. He is very much indebted to Dr. M. D. Olson of the National Research Council of Canada with whom he had very useful discussions. The cooperation extended by the personnel of the Computer Centre, University of Ottawa, is highly appreciated. The financial grant No. 4357 of the National Research Council which made this study possible is gratefully acknowledged.

SYNOPSIS

Stiffened and unstiffened skew plates with varying skew angles and aspect ratios and with the following boundary conditions have been analysed using the finite element method:

(i) Clamped on all sides.

(ii) Simply supported on all sides.

(iii) Clamped on one side and free on the other three (cantilevered skew plates)

(iv) Two sides simply supported and two sides free.

(v) Two sides simply supported and two sides stiffened with edge beams.

A refined conforming triangular finite element based on a quintic polynomial is used. Results are compared wherever possible with existing analytical solutions obtained by other researchers.

It is found that in contrast to conventional analytical methods of analysis, the finite element method of solution gives consistent and converging results even for high angles of skew. In addition, the precise finite element adopted in this work yields accurate results even for coarse meshes as compared to other available finite elements.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

NOTATIONS

a, b, c	Element dimensions
a_i	Coefficients of the quintic polynomial for the triangular plate element
$\{A\}$	Column vector of the undetermined constants a_i for the plate element
$\{A\}_b$	Column vector of the undetermined constants for the beam element
b_i	Coefficients of the quintic polynomial for the beam element
$[C]$	Constraint equations matrix
$D,\ D_x,\ D_y$	Flexural rigidities of the plates
$E,\ E_x,\ E_y$	Moduli of elasticity of the plate material
$\{F\}$	Generalised force matrix
$\{\tilde{F}\}$	Generalised forces in the orthogonal system
$G(m, n)$	Modified Euler's beta function
I	Moment of inertia of the beam element
$[K]$	Stiffness matrix for the plate element with respect to the global coordinate system
$[K]_b$	Stiffness matrix for the beam element with respect to the global coordinate system
$[K_l]$	Stiffness matrix of the finite element with respect to the local coordinate system
$[\tilde{K}]$	Stiffness matrix in the orthogonal system
$[k]$	Stiffness matrix for the plate element relative to the vector $\{A\}$
$[k]_b$	Stiffness matrix for the beam relative relative to vector $\{A\}_b$
ℓ	length of the beam element
L	Length of the plate

m_i, n_i	Exponents of ξ, η in the i^{th} term of the polynomial expression
$\{P\}$	Column vector of consistent loads for the plate element
$\{P\}_b$	Column vector of consistent loads for the beam element
q	Intensity of uniformly distributed load for the plate
q_b	Intensity of uniformly distributed load for the beam
$[R]$	Rotation matrix..... (Eq. 3.32)
r	Length defined by Eq. (3.6)
S	Distance along the oblique boundary of the plate
$[T]$	Transformation matrix for the plate element (Eq. 3.13)
$[T]_b$	Transformation matrix for the beam element
$[T_2]$	Transformation matrix... (Eq. 3.15)
$[T_2]_b$	Transformation matrix... (Eq. 3.48)
t	Thickness of the plate
U	Strain energy of the plate element
U_b	Strain energy of the beam element
u, v	Non-dimensional co-ordinates within the triangular element
V	Virtual work of the applied loads for the plate
V_b	Virtual work of the applied loads for the beam
$\{W\}$, $\{W_1\}$	Column vector of generalized displacements for the plate element
$\{W\}_b$	Column vector of generalized displacements for the beam element

w	Displacement normal to the xy-plane
$w_x, w_y, w_{xx}, w_{xy}, w_{yy}$ -	Derivatives of w with respect to the global coordinate system
$w_\xi, w_\eta, w_{\xi\xi}, w_{\xi\eta}, w_{\eta\eta}$ -	Derivatives of w with respect to the local coordinate system
X, Y	Global coordinates
$\{X\}$	Generalised displacements
$\{\tilde{X}\}$	Generalised siaplacements in the orthogonal system
Z	Vertical axis
ξ, η	Local coordinates
$\bar{\Lambda}$	Potential energy
$[\lambda]$	Lagrangian multipliers
θ	Angle between the local and global coordinate systems
α	Acute angle that the skew side makes with the X-axis (Fig. 1)
β	Skew angle (Fig. 1)

CHAPTER 1

INTRODUCTION

1.1 General:

Skew plates are a class of structures which occur commonly as swept wings of aircrafts and as building and bridge slabs. The need for a rational analysis of skew bridge decks is particularly felt in Canada where nearly 36% of bridge decks are on skew alignments and 7 1/2 % are on heavy skews (16). Also, from the analyst's point of view, skewed plates offer a much more challenging problem than the rectangular ones due to the presence of unbounded stresses in corners. Even the experimental results are not considered very reliable. No explicit analytical solutions are available as yet. Several simplifying assumptions have been made in material properties, geometry and boundary conditions and the results obtained by different methods differ very much especially for higher skew angles. So it is desirable to have a method of analysis which involves fewer simplifying assumptions and giving consistent results. This work is concerned with the geometric idealisation of the actual structure using the finite element method.

1.2 Object and Scope:

The main objective of the thesis is to make a comprehensive study of skew plates with different boundary conditions and aspect ratios with or without edge beams and to develop a utility computer program. The scope of this work is limited to linear elastic structures subjected to uniformly distributed loads although point loads could be handled with very little modification of the program. It is not aimed at here to develop a general program which could handle arbitrary boundary conditions. However the changes to be incorporated in the program to handle different boundary conditions are explained in a note on computer program.

1.3 Outline of the Thesis:

In Chapter 2 the development of the finite element method and the analysis of skew plates by different methods are reviewed. In Chapter 3 the finite element used in this work is derived from plate theory and energy principles. Also the stiffness matrix of a beam element is derived. Two different methods of handling the boundary conditions are outlined. Chapter 4 is devoted to the discussion of results while comparison is made with existing analytical solutions.

In the final chapter the conclusions drawn are summarized and suggestions for future research are indicated.

A note on the computer program is included.

CHAPTER 2

REVIEW OF LITERATURE

2.1 THE FINITE ELEMENT METHOD OF ANALYSIS:

In problems of continuum mechanics, the actual prototype structures can be transformed into a mathematical model which is then reduced to a numerical procedure best suited to the computer. The finite element method falls under this class of problems. Two and three dimensional bodies such as plates, shells and dams can be considered as structures with infinite degrees of freedom. In using the finite element method, the structures having infinite degrees of freedom are approximated to ones having finite degrees of freedom. The stress and displacements fields can sometimes be assumed for certain classical problems and solutions could then be obtained from the governing differential equations. But for most of the problems encountered in practice, it is not easy to find a closed form solution due to complexity of geometry, loading, material properties and boundary conditions. The conventional method is to establish a simplified and idealized model of the real structure for which a closed form solution is available and apply the solution with proper interpretation or to resort to the method of finite difference. It is well known that in the method of finite differences the exact differential equation is solved at discrete points. However, the stress resultants at any point in between the grid points can only be approximated.

The concepts of stiffness and flexibility which are derived from load-displacement characteristics of structures using matrix formulation form the basis of the finite element method. The analysis of any structure by this method involves (i) Structural idealisation (ii) Element analysis (iii) Structural analysis. By structural idealisation it is meant to divide the actual structure conveniently by rectangular, triangular, quadrilateral or tetrahedral elements. The element analysis is the essence of the finite element method in which the force-displacement relationship within an element is established. Finally, as in conventional structural analysis, the usual requirements of equilibrium and compatability are met. There are two distinctly different approaches. One is based on an assumed stress distribution within an element and is known as the flexibility method and the other is based on an assumed displacement function and is called the stiffness method.

The stiffness method is more popular because of its easiness to matrix formulation. Besides, it gives a physical understanding of the problem. If the problem is formulated on the basis of variational principles the finite element method is equivalent to the Rayleigh-Ritz procedure in which the domain of a set of trial functions is restricted to the volume of an element and the same trial functions are used in the domain of each element. In other words, in the Rayleigh-Ritz procedure a displacement function is assumed for the whole structure satisfying the boundary conditions while in the displacement method the displacement function is chosen for an element in its own coordinate system and all such elements of which the structure is made of, are assembled in their common coordinate system and the boundary conditions are prescribed for those points lying on the boundary. The matrix manipulations involved in realising these are dealt with in greater detail in a later chapter.

Advantages of the Finite Element Method:

(i) Any irregular boundary can be approximated by choosing suitable sizes and shapes of elements. Triangular elements are the most favoured ones because of their adaptability to boundaries of most two or three dimensional structures.

(ii) Any type of loading can be considered.

(iii) Anisotropy and inhomogenity do not pose any problems.

(iv) By the method of incremental loads, problems in the non-linear range can be tackled.

(v) If the solution is based on variational principles, statements on the bounds of strain energy can be made.

Hence the finite element method finds its application in problems of structural mechanics, heat transfer, flow through porous media, flow of electric potential etc.

In order to understand and appreciate the use of the triangular element considered in this work it is necessary to trace the development of the finite element method to the present stage. The finite element method in its present form originated from a paper by Turner, Clough et. al. (24). Since then, a number of papers have been published on the suitability of different shapes and displacement functions for both plane stress and plate bending problems. The most satisfactory of the rectangular elements is by Bogner, Fox and Schmit (3). As pointed out earlier triangular elements have a greater adaptability to boundaries and hence only a review on plate bending triangular elements will be summarized here.

The first step in the formulation of any finite element is the choice of a displacement function which will nearly represent the deflected shape of the element under load. The various types of elements available differ only in the types of polynomial assumed. Here again it should be pointed out that polynomial functions have been assumed invariably as against trigonometric functions because of their simplicity for automatic computation and it is easier to show the convergence criteria based on calculus of variations. The accuracy of a particular finite element approximation depends on how closely the assumed displacement function can represent the actual displacements. The factors which affect the theoretical accuracy can be classified as follows:

(i) Element stiffness (the assumed displacement function and chosen nodal displacements).

(ii) Load representation.

(iii) Element mesh.

(iv) Boundary conditions.

Besides these discretization errors, there may be round off errors due to numerical calculations. These discretization errors occur as a consequence of approximating a continuum which has an infinite degrees of freedom with a model having finite degrees of freedom. But considerable progress has been made during the last decade to establish the minimum requirements for a set of functions for displacement model which will converge to the true solution as the sub division is refined. They are listed as follows:

(i) All rigid body states must be included.

(ii) Uniform strains states must be included.

(iii) Displacement function must depend on nodal displacements.

(iv) Internal and interface compatability i. e. displacement and slope must be continuous not only at the nodal points but across the interface also.

(v) A complete polynomial is required as it represents isotrophy.

The most obvious choice of terminal degrees of freedom is w and its first derivatives. Since for a triangle at least 9 independent displacement functions are required, the first attempts were made using a cubic function. Adini et. al. (1) suggested for a triangular plate bending element:

$$w(x, y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 + a_9 y^3$$

where w is the transverse deflection of the plate

It is evident that in the polynomial xy term is omitted from a full cubic function in order to maintain symmetry. This function although includes constant curvature condition yet the omission of the xy term makes it too stiff. Clough and Tocher (5) improved it by coupling the eighth and ninth term of the complete cubic polynomial and including the xy term. But this suffered from the disadvantage that for certain orientations the transformation matrix relating the nodal displacements to the undetermined constants was singular. Zienkiewicz, Cheung (28) obviated this trouble by resorting to area coordinates since they are intrinsically related to element geometry and free from external reference. However, this element did not possess the continuity requirements i.e. the displacementsare continuous at the nodes but not the normal slopes. Clough and Tocher (5) introduced a triangular element which was made up of 3 triangles. Although conformity was achieved by this yet normal slope varied only linearly between nodal points which made the element stiff (Note: A conforming element is one which satisfies all continuity requirements and hence leads to monotonic convergence of strain energy). Moreover, it was not good for automatic computation and it was also noted that some non-conforming elements having the same mesh size yielded better results. So it was found that in order to obtain a more flexible element it is necessary to introduce higher derivatives as degrees of freedom. A twist in arbitrary direction can be obtained only by second derivatives in x and y. So besides w, w_x and w_y, the second order derivatives namely w_{xx}, w_{xy}, and w_{yy} were introduced as degrees of freedom and hence higher degrees of polynomials (like quintic) were thought of. These higher derivatives have a further advantage in that moments could be obtained directly. In cases where only displacement and its first derivatives were considered as degrees of freedom, moments could be obtained only by differentiation, and hence some accuracy is lost. Also, the moments at the same point were not the same for two elements meeting at that point. Hence only average moment could be obtained.

Although it could be rigorously proved that conforming elements lead to monotonic convergence yet the problem of convergence in non conforming elements is still to be resolved. The element which has been considered for the present analysis of skew plates is based on a quintic polynomial and it will be discussed in greater detail in relation to convergence and conformity at a later chapter.

2. 2 Review on the Analysis of Skew Plates:

The various approaches attempted for the analysis of skew plates can be broadly classified under the following:

(1) By differential equation approach using series solutions.

(2) By energy principles.

(3) By mapping functions.

(4) By numerical approach:

(i) Finite difference methods, (ii) The finite element method.

(1) By differential equation approach:

Based on the usual assumptions of the small deflection of elastic thin plates the differential equation governing the lateral deflection w for an orthotropic plate can be written as:(22)

$$D_x w_{xxxx} + 2H\, w_{xxyy} + D_y w_{xxxy} = q(x,y)$$

where

$$D_x = \frac{E_x h^3}{12(1-\mu_x\mu_y)}, \quad D_y = \frac{E_y h^3}{12(1-\mu_x\mu_y)} \quad H = \left[\frac{\mu_y E_x}{1-\mu_x\mu_y} + 2G\right]\frac{h^3}{12}$$

where E_x and E_y are the Young's Moduli of the material in x and y directions; μ_x and μ_y are the Poisson's ratios of the material in x and y directions; G is the shear modulus; h is the thickness of the plate and w is the transverse deformation of the plate. The suffixes represent the order of differentiation in x and y.

For an isotropic plate this reduces to,

$$w_{xxxx} + 2\,w_{xxyy} + w_{yyyy} = \frac{q(xy)}{D} \quad \text{where} \quad D = \frac{E\,h^3}{12(1-\mu^2)}$$

In the case of skew plates the governing differential equation is usually transformed into skew coordinates. Then the equation above could be written in skew coordinates as:

$$w_{\eta\eta\eta\eta} + p^4\, w_{\xi\xi\xi\xi} + 2\,(1+2s^2)\,p^2\, w_{\xi\xi\eta\eta}$$

$$-\ 4s(p^3\, w_{\xi\xi\xi\eta} + p w_{\xi\eta\eta\eta}) = \frac{(bc)^4}{16D}\ q(\eta,\xi) \quad \text{where for this}$$

equation, ξ and η are the dimensionless oblique coordinates; p the aspect ratio of the plate (b/a); s = Sine β; c = Cosine β, where β is the skew angle. Then the analysis of skew plates is reduced to solving the above differential equation and a rigorous solution would involve adjusting certain constants to take care of the boundary conditions. Due to the absence of orthogonal relationship the analysis becomes more involved than the elegant Navier or Levy solution for rectangular plates.

Several attempts have been made to solve skew problems using series solutions. Both polynomial and trigonometric series were tried. It is reported (16) that Eharz solved the simply supported and clamped plates assuming a deflection function in polar coordinates as:

$$w = \frac{q r^4}{64D} + \sum_{n=0,1,2} (a_{2n}\, r^{2n} + C_{2n+2}\quad r^{2n+2})\ \text{Cos}\ 2n\,\theta$$

where r and θ are polar co-ordinates:

The arbitrary constants a_{2n} and C_{2n+2} were determined by satisfying the boundary conditions at 4 discrete points along one side. He also found solutions to the same problem using a polynomial power series representation of deflection function of the form:

$$w = \frac{q}{D}\left(y + \sqrt{3}x + \sqrt{\frac{3b}{2}}\right)\left(y - \sqrt{3}x - \sqrt{\frac{3b}{2}}\right)\left(y + \sqrt{3}x - \sqrt{\frac{3b}{2}}\right)$$

$$\left(y - \sqrt{3}x + \sqrt{\frac{3b}{2}}\right) \sum_{m=0,2,4\ldots} \quad \sum_{n=0,2,4\ldots} C_{mn} \left(\frac{x}{b}\right)^m \left(\frac{y}{b}\right)^n$$

where b is the oblique dimension of the plate.

Here also he could satisfy the deflection boundary condition along the simply supported edge. The other boundary condition and the governing differential equation were satisfied at discrete points along the boundary and in the plate region leading to enough equations to solve for the constants C_{mn}.

Lardy as reported by Kennedy & Tamberg (16) successfully solved the differential equation in oblique coordinates using a double infinite Fourier series. The solution of the governing equation in homogeneous form was obtained in terms of a combination of hyperbolic and trigonometric series. The residual deflection for the approximate solution for a simply supported plate did not exceed 2% of central deflection for two terms of the series. Results for moments were not given nor the convergence tested.

Quinlan (20) employed a double infinite series to represent the deflection of skew plates. Single plates with arbitrary boundary conditions can be analysed using his method. However, he could not establish convergence when the angle of skew exceeded 45°.

Kennedy and Huggins (14) obtained a solution in oblique dimensionless coordinates using a trigonometric series for the deflection function . The arbitrary constants were adjusted for various harmonics until they were convergent. Iyengar et. al. (12) obtained quite good results using a double Fourier series for clamped plates.

(2) By energy principles:

Rayleigh-Ritz procedure was used by Dorman (16) to obtain an approximate solution to uniformly loaded clamped plates. He assumed a deflection function of the form:

$$w = \frac{q}{D} \cos^2 \frac{\pi u}{b} \frac{\pi v}{\ell \phi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos \frac{n\pi u}{b} \cos \frac{m\pi v}{\ell \phi}$$

(where u and v are displacements parallel to x and y respectively and $\ell \phi$ and b are the oblique dimensions of the plate;) and minimized the total potential energy of the system to determine the unknown constants A_{mn}. As Kennedy observed the results presented by him are unreliable as the function satisfied not only polar symmetry but also quadrant symmetry which is non existent in parallelogrammic plates. Kennedy and Ng (15) used a simple polynomial of the form:

$$w = \left[\left(\frac{b}{2}\right)^2 - u^2 \right]^2 \left[(\ell \phi/2)^2 - v^2 \right]^2 (A_0 - A_1\, uv) \quad \text{where}$$

$\ell \phi$, u, v and b are the same as those defined in the deflection function that Dorman assumed.

The undetermined constants A_0 and A_1 were found by minimizing the potential energy. Recently Kennedy (16) used the following polynomial series for clamped plates:

$$w = (1-\eta^2)^2 (1-\xi^2)^2 (1-\eta\xi - K\eta^3\xi^3) \sum_{0,2,4} \sum_{0,2,4} A_{mn}\, \eta^m \xi^n$$

where, for this equation, η and ξ are the oblique dimensionless coordinates.

The constant K was adjusted and the undetermined constants A_{mn} was found using Galerkin's variational principles. He compared his results with experimental models made of aluminium for skew angles 30°, 50° and 55° and found that the maximum central deflection and central stresses were within 4% and 12% respectively.

(3) By mapping functions:

Aggarwala (2) analysed a simply supported rhombic plate using a mapping function. He transformed the parallellogram to a unit circle by a mapping function in complex form. Convergence was also studied. The results agreed within 1% of Morley's for skew angles less than 30°.

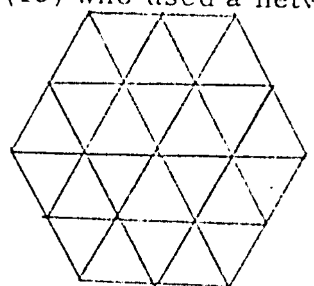
Tungl (16) analysed a simply supported skew plate with an arbitrary concentrated load. The particular solution was obtained by Green's function within the domain. Here again the solution to the homogeneous equation was obtained as a combination of hyperbolic and trigonometric functions.

(4) By numerical approach:

From the foregoing review on the analysis of skew plates by classical methods it is evident that the choice of the displacement function satisfying the skew boundary conditions is difficult and those applied are at best approximate. So finite difference methods were tried as an alternative.

(i) Finite difference methods:

It is well known that in the finite difference methods the actual structure is divided into a grid work and the governing differential equation is solved at discrete points of the grid using finite difference operators which relate the deflections, slopes, curvatures etc. of any point to its neighbouring points. This results in a set of linear equations which when solved simultaneously give an approximate solution to the problem. This technique has been applied for various combinations of boundary conditions. The most notable among them is that due to Jensen (13) who used a network as shown.



Morley (18) has presented results for boundary conditions considered in this thesis but the results were not found satisfactory for high skew angles. In using the finite difference method, since the moments and shears are based on the difference between adjacent points it is necessary that the deflections are estimated to a high degree of accuracy. Experiments have demonstrated that results obtained from the finite difference method usually have the following shortcomings:

1) A network which is not fine enough to represent steep changes in deflection will produce results which are far from actual values.

2) Application of boundary conditions are not satisfactory particularly for free edges.

3) Convergence deteriorates with increase in skew angles.

These shortcomings made researchers look for other numerical methods.

(ii) The finite element method:

Since the introduction of the finite element method into the area of continuum mechanics in 1956, many of the problems in structural engineering have been tried including skew plates. Several shapes of elements like triangular, rectangular and parallellogrammic were tried. Dawe (7) used a non-conforming parallellogrammic element for the solution of a skew cantilever plate. He used a fine grid for 60° skew angle to find stress singularities in the corners. The element used was a non-conforming one and hence convergence could not be established. Results varied by 20 to 30% of experimental values. The finite element method could not be attemped with confidence until conforming elements which have the inherent properties of convergence were introduced. Recently Monforton (17) used a rectangular element based on Hermitian polynomials and analysed clamped and cantilever skew plates. Cowper, Kosko, Lindberg and Olsen of the National Research Council of Canada (6) developed a fully conforming triangular element and suggested a method good for automatic computation. It was tried for rectangular plates with great success. For the same number of degrees of freedom it yielded better results compared to other finite elements available. So this element was chosen for the analysis of skew plates in this work to achieve the dual objectives of finding better solution to skew plate problems and to evaluate the element itself.

Stiffened Skew Plates:

Skew plates stiffened with edge beams or integral stiffeners occur very commonly in bridges. Comparatively less amount of literature is available on them. Basically three different approaches were tried. In one method the whole bridge slab with stiffeners is transformed into a frame work of intersecting beams whose stiffnesses are adjusted to conform to stiffnesses of slab and girders. This was the approach used by Hendry and Jaeger (9).

In the second approach the slab and girder is converted to an orthotropic plate of equivalent stiffness The problem in this method is to separate plate moment from girder moment and to find the torsional rigidity of an equivalent orthotropic plate. This is the method used by Chu and Krishnamoorthy (4). The third approach considers a plate supported on girders with interface shears and the flexural rigidity of girders is adjusted to compensate for the monolithic actions of slab and beam. This is found in the work of Newmark and Seiss (19). The above methods were tried for rectangular cases only.

Jensen (13) used finite difference method for solving skew plates stiffened with edge beams on two sides and simply supported on the other two. Quinlan (20) solved a similar problem using a double infinite series and established convergence for skew angles up to 45° only. Kennedy and Huggins (14) obtained the solution for such a problem using oblique dimensionless coordinates expressing deflection in a single Fourier series. The constants involved in the expression for w were adjusted to satisfy the boundary conditions. They observed that the percentage difference between solutions of 9 and 10 harmonics for skew angles up to 30° was less than 1%. However no convergence was established beyond 45° . Gustafson (8) used a rectangular finite element for such a problem. He used the non conforming element developed by Zienkiewickz and compared his results with those of Huggins and Kennedy. Gustafson's results for deflection differed from those obtained by Kennedy & Huggins by over 10% for skew angles over 45° . So this problem was also tried using the conforming triangular finite element.

CHAPTER 3

METHOD OF ANALYSIS

3.1 General:

In this chapter the refined triangular finite element used in this analysis is derived from plate theory. The required stiffness and consistent load matrices are derived from energy principles instead of the conventional method of arriving at stiffness matrices through the displacement-strain-stress procedure. A computational procedure very much suited for automatic computation was used in the derivation of stiffness matrices.

3.2 Assumptions:

In figure 3 is given a cross section of the plate element which is stiffened with a beam. The plate and the beams are assumed to be made up of elastic homogeneous and isotropic material. The stiffeners are assumed to be cast monolithically with the plate thus precluding any relative movement of the plate and beam at the time of deformation. In addition the usual assumptions of classical plate theory for small deflections hold good, namely (i) The deflections are small compared to the thickness of the plate (ii) Middle surface of the plate remains unstretched.

Finally the stiffeners are assumed to resist bending in vertical planes only and do not resist torsion.

The coordinate systems are shown in figure 4. The figure illustrates the positive directions of internal forces. The middle surface is taken as the reference surface and the arrows indicate the directions of positive displacements and loads. Since Z is considered positive downwards negative curvatures lead to positive moments.

3.3 Criteria for a conforming element:

The criteria required for a conforming element which leads to monotonic convergence of strain energy have been outlined in an earlier chapter. Here it is demonstrated that the element chosen for this analysis meets those requirements. The displacement function chosen is a quintic polynomial of the form (6)

$$w(\xi,\eta) = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2$$

$$+ a_{10}\eta^3 + a_{11}\xi^4 + a_{12}\xi^3\eta + a_{13}\xi^2\eta^2 + a_{14}\xi\eta^3 + a_{15}\eta^4 + a_{16}\xi^5 + a_{17}\xi^3\eta^2$$

$$+ a_{18}\xi^2\eta^3 + a_{19}\xi\eta^4 + a_{20}\eta^5 \ldots\ldots\ldots\ldots \qquad (3.1)$$

For the triangular plate bending element chosen, 6 degrees of freedom per node are allowed, namely the deflection and its first and second derivatives. This results in 18 degrees of freedom per element. A complete quintic polynomial has 21 constants. So it is required to furnish three additional equations. These conditions are easily furnished by making the normal slopes vary as a cubic between terminal vertices. One of the conditions is satisfied by the omission of the term $\xi^4\eta$ from the polynomial to make the normal slope $\frac{\partial\omega}{\partial\eta}$ along $\eta = 0$ a cubic in ξ. The other two conditions are met by finding expressions for normal slopes to the other two sides and equating the expressions having terms higher than cubic to zero. (These will be derived later). It is easily seen that the element includes all rigid body displacements (due to the inclusion of a_1, $a_2\xi$, $a_3\eta$) and uniform states of strain by the inclusion of $\xi\eta$ terms which provide for uniform twists. Continuity of both displacement and slope is maintained both at the nodal points and across the interface. Besides these essential conditions for conformity the variation of strain between adjacent points being cubic makes the element very flexible. It has been shown (6) that such an element leads to monotonic convergence of strain energy and the rate of convergence is of the order of the sixth power of the linear dimension of the element. An additional feature of the element is that the moments could be obtained directly by the inclusion of curvature as a degree of freedom and thus any loss of accuracy is avoided.

3.4 Stiffness matrix for triangular plate bending element:

3.4.1 Stiffness matrix in the local coordinate system:

The two reference systems used in this analysis are given in figure 4. The stiffness matrix is first derived in the local coordinate system and then transformed to the global coordinate system to facilitate assembly. The stiffness matrix in the local coordinate system will henceforth be referred to as the Element Stiffness Matrix and that in the global system as the Structure Stiffness Matrix. The eighteen generalized displacements are w_1, $w_{\xi 1}$, $w_{\eta 1}$, $w_{\xi\xi 1}$, $w_{\xi\eta}$, $w_{\eta\eta 1} \ldots w_{\eta\eta 3}$ at the three vertices. The subscripts 1, 2 and 3 denote the vertices p_1, p_2 and p_3 respectively of the element shown in figure 1. They can be put in a column vector designated by $\{W_1\}^{t}$ *.

---

* superscript t denotes transpose

Differentiating the expression w [Eq. (3. 1)] with respect to ξ, η and substituting the element dimensions, the 18 generalized coordinates can be related to the undetermined constants $a_1 \ldots a_{20}$ by a transformation matrix $[T_1]_{18 \times 20}$

Thus, $\{W_1\} = [T_1]\{A\}_{20 \times 1}$ .... (3. 2)

The element dimensions a, b, c as shown in figure 4 can be found by simple geometry and they are:

$$a = \{(x_2 - x_3)(x_2 - x_1) + (y_2 - y_3)(y_2 - y_1)\} / r \tag{3.3}$$

$$b = \{(x_3 - x_1)(x_2 - x_1) + (y_3 - y_1)(y_2 - y_1)\} / r \tag{3.4}$$

$$c = \{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\} / r \tag{3.5}$$

where $$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{3.6}$$

(For detailed derivation see appendix)

The transformation matrix $[T_1]_{18 \times 20}$ could be a augmented by the two constraint equations for normal slopes and a set of 20 equations can be obtained for the undetermined constants a_i, i = 1 to 20. The constraint equations are obtained by forcing the normal slopes for the sides p_1p_3 and p_2p_3 of the triangular element to vary as a cubic. The normal slope for O_1P_3 referring to figure 6 is given by $w_n = w_\xi \text{ Sin } \alpha - w_\eta \text{ Cos } \alpha$. assuming it makes an angle α with the ξ axis. It is necessary to consider only the terms having powers higher than four as the others automatically satisfy the condition.

Further, by making use of Eq. (3. 1)

$$w_n = 5a_{16}\xi^4 + 3a_{17}\xi^2\eta^2 + 2a_{18}\xi\eta^3 + a_{19}\eta^4) \text{ Sin } \alpha$$

$$-(2a_{17}\xi^3\eta + 3a_{18}\xi^2\eta^2 + 4a_{19}\xi\eta^3 + 5a_{20}\eta^4) \text{ Cos } \alpha$$

$$+ \text{ terms of third or lower degree in } \xi \text{ and } \eta \tag{3. 7}$$

$$\xi = S \text{ Cos } \alpha, \quad \eta = S \text{ Sin } \alpha \tag{3. 8}$$

where S is the distance along the line p_1p_3

Substituting Eqs. (3. 8) in (3. 7) gives,

$$w_n = S^4\{ 5a_{16} \text{Cos}^4 \alpha \text{ Sin } \alpha + a_{17}(3 \text{ Cos}^2 \alpha \text{ Sin}^3 \alpha - 2 \text{ Cos}^4 \alpha \text{ Sin } \alpha)$$

$$+ a_{18}(2 \text{ Cos } \alpha \text{ Sin}^4 \alpha - 3 \text{ Cos}^3 \alpha \text{ Sin}^2 \alpha)$$

$$+ a_{19}(\text{Sin}^5 \alpha - 4 \text{ Cos}^2 \alpha \text{ Sin}^3 \alpha) - 5a_{20} \text{ Cos } \alpha \text{ Sin}^4 \alpha\} + \ldots\ldots \tag{3. 9}$$

The condition that the normal slope varies as a cubic is realized only when the expressions inside the parentheses vanish. To apply this condition to edge p_1p_3 of the triangular element in figure 4 appropriate values of Cos α and Sin α must be inserted in Eq. (3. 8) namely,

$$\text{Cos } \alpha = b \Big/ \sqrt{b^2 + c^2} \quad \text{and} \quad \text{Sin } \alpha = c \Big/ \sqrt{b^2 + c^2} \tag{3. 10}$$

The condition for cubic variation of normal slope along p_1p_3 is then:

$$5b^4 c\, a_{16} + (3b^2c^3 - 2b^4c)\, a_{17} + (2bc^4 - 3b^3c^2)\, a_{18} + (c^5 - 4b^2c^3)\, a_{19} - 5bc^4\, a_{20} = 0 \qquad (3.11)$$

Similarly the condition for cubic variation of the normal slope along the edge p_2p_3 of the triangular element gives:

$$5a^4 c\, a_{16} + (3a^2c^3 - 2a^4c)\, a_{17} + (-2ac^4 + 3a^3c^2)\, a_{18} + (c^5 - 4a^2c^3)\, a_{19} + 5ac^4\, a_{20} = 0 \qquad (3.12)$$

When the eighteen equations of Eqn. (3.2) are augmented by the conditions posed by Eqns. (3.11) and (3.12) a set of twenty equations for the coefficients $a_1, a_2 \ldots a_{20}$ can be written as

$$\begin{Bmatrix} W_1 \\ 0 \\ 0 \end{Bmatrix} = [T]_{20 \times 20} \{A\}_{20 \times 1} \qquad (3.13)$$

where $[T]$ is the transformation matrix listed on page 78 . This transformation matrix is non-singular as its determinant is zero only when the area of the triangle vanishes.

From Eqn. (3.13)

$$\{A\}_{20 \times 1} = [T]^{-1}_{20 \times 20} \begin{Bmatrix} W_1 \\ 0 \\ 0 \end{Bmatrix}_{20 \times 1} \qquad (3.14)$$

or

$$\{A\} = [T_2]_{20 \times 18} \{W_1\}_{18 \times 1} \qquad (3.15)$$

where the 20×18 matrix $[T_2]$ consists of the first eighteen columns of $[T]^{-1}$

The stiffness matrix of the element in the local coordinate system could be obtained by a calculation of strain energy. The strain energy in bending for an isotropic plate element is

$$U = \frac{1}{2} D \int\int [w_{\xi\xi}^2 + w_{\eta\eta}^2 + 2\mu w_{\xi\xi} w_{\eta\eta} + 2(1-\mu) w_{\xi\eta}^2] d\xi d\eta \qquad (3.16)$$

where D is the flexural rigidity of the plate; μ is Poisson's ratio and the integration is carried out over the entire area of the element. By substituting Eq. (3.1) into Eq. (3.16) and carrying out the necessary integrations, the strain energy could also be expressed in terms of the undetermined constants as:

$$U = \frac{1}{2} D \{A\}^t [k] \{A\} \qquad (3.17)$$

Substituting Eqn. (3.15) in Eqn. (3.17),

$$U = \frac{1}{2} D \underset{18\times 1}{\{W_1\}^t} \underset{18\times 20}{[T_2]^t} \underset{20\times 20}{[k]} \underset{20\times 18}{[T_2]} \underset{18\times 1}{\{W_1\}} \qquad (3.18)$$

or

$$U = \frac{1}{2} D \underset{18\times 1}{\{W_1\}^t} \underset{18\times 18}{[K_1]} \underset{18\times 1}{\{W_1\}} \qquad (3.19)$$

where $\underset{18\times 18}{[K_1]} = [T_2]^t [k] [T_2]$ is the stiffness matrix in the local coordinate system

The $[k]$ matrix in Eqn. (3.18) can be found by numerical integration. The following expressions are developed to facilitate automatic computation. By using an abbreviated notation, Eq. (3.1) may be written as,

$$w = \sum_{i=1}^{20} a_i \xi^{m_i} \eta^{n_i} \qquad (3.20)$$

where m_i and n_i are the powers of ξ and η respectively of the ith term of the polynomial.

$$w_{\xi\xi} = \sum_i a_i m_i (m_i - 1) \xi^{m_i - 2} \eta^{n_i} \tag{3.21}$$

$$w_{\xi\xi}^2 = \sum_i \sum_j a_i a_j m_i m_j (m_i - 1)(m_j - 1) \xi^{m_i + m_j - 4} \eta^{n_i + n_j} \tag{3.22}$$

So that

$$\iint w_{\xi\xi}^2 \, d\xi \, d\eta = \sum_{i=1}^{20} \sum_{j=1}^{20} a_i a_j m_i m_j (m_i - 1)(m_j - 1) \iint \xi^{m_i + m_j - 4} \eta^{n_i + n_j} \, d\xi \, d\eta \tag{3.23}$$

To proceed further it is necessary to find a general formula for the ingeral $\iint \xi^m \eta^n \, d\xi \, d\eta$. Referring to figure 4 , the integral $\iint \xi^m \eta^n \, d\xi \, d\eta$ over the entire area of the triangle $p_1 p_2 p_3$ can be considered as that due to the triangles $p_1 Q p_3$ and $Q p_2 p_3$. Substituting $u = \xi / a$, $v = \eta / c$ the integral over the triangle $Q p_2 p_3$ becomes

$$\int_0^a \int_0^{c(1-\xi/a)} \xi^m \eta^n \, d\eta \, d\xi = a^{m+1} c^{n+1} \int_0^1 \int_0^{1-u} u^m v^n \, du \, dv$$

$$= \frac{a^{m+1} c^{n+1}}{n+1} \int_0^1 u^m (1-u)^{n+1} \, du \tag{3.24}$$

The last intregal $\int_0^1 u^m (1-u)^{n+1} \, du$ is the Euler's beta function and when integrated between these limits give $\frac{m!\,(n+1)!}{(m+n+2)!}$ where $m, n > 0$.

Similarly the integral over the area Qp_1p_2 can be found and is given by,

$$-\frac{m!\,n!}{(m+n+2)!}\,(-b)^{m+1}\,c^{n+1}$$

Thus

$$\iint \xi^m \eta^n \, d\xi \, d\eta = G(m,n)$$

on integrating over the entire area of the triangle QP_1P_2; where

$$G(m,n) = C^{n+1}\{a^{m+1} - (-b)^{m+1}\}\frac{m!\,n!}{(m+n+2)!}\ , \text{ for } m,n \geq 0 \tag{3.25}$$

Thus

$$\iint w_{\xi\xi}^2 = \Sigma\Sigma a_i a_j \{m_i m_j (m_{i-1})(m_{j-1})\ G(m_i + m_j - 4\ ,\ n_i + n_j) \tag{3.26}$$

Similarly,

$$\iint w_{\eta\eta}^2 = \Sigma\Sigma\ a_i a_j\ n_i n_j (n_{i-1})(n_{j-1})\ \ G(m_i + m_j,\ n_i + n_j - 4) \tag{3.27}$$

$$\iint w_{\xi\xi} w_{\eta\eta} = \Sigma\Sigma a_i a_j \{m_i (m_{i-1}) n_j (n_{j-1}) + m_j (m_{j-1}) n_i (n_{i-1})$$

$$G(m_i + m_j - 2\ ,\ n_i + n_j - 2)\} \tag{3.28}$$

$$\iint\ w_{\xi\eta} = a_i\ m_i\ n_i\ \xi^{m_i - 1}\ \ \eta^{n_i - 1} \tag{3.29}$$

$$\iint\ w_{\xi\eta}^2\, d\xi\, d\eta = a_i a_j m_i m_j n_i n_j \quad G(m_i + m_j - 2,\ n_i + n_j - 2) \tag{3.30}$$

Having obtained expressions for the numerical integration in a closed form for automatic computation the strain energy of the element can be written in the form,

$$U = \frac{D}{2} \{A\}^t [k]_{ij} \{A\} .$$

From this form and the requirement of symmetry the elements of the matrix $[k]_{ij}$ are given by,

$$k_{ij} = m_i m_j (m_i - 1)(m_j - 1)\, G(m_i + m_j - 4, n_i + n_j) + n_i n_j (n_i - 1)(n_j - 1)\, G(m_i + m_j, n_i + n_j - 4)$$

$$+ \{2(1-\mu)\, m_i m_j n_i n_j + \mu\, m_i (m_i - 1)\, n_j (n_j - 1) + \mu\, m_j n_i (m_j - 1)(n_i - 1)\} G(m_i + m_j - 2, n_i + n_j - 2) \quad (3.31)$$

Substituting Eq. (3.31) in Eq. (3.18) the stiffness matrix $[K_1]$ as given by Eq. (3.19) can be obtained.

3.4.2 Stiffeness matrix in the global coordinate system:

In order to assemble the stiffnesses of individual elements to get the stiffness of the whole structure it is necessary to transform the element into the global coordinate system. This is easily done by a rotation matrix $[R]$ relating vectors of generalized displacements in the local coordinate system to that of the global coordinate system. Let W be the vector of generalized displacements in the global coordinate system, its transpose being

$$\{W\}^t = (w_1, w_{x_1}, w_{y_1}, \ldots\ldots\ldots w_{yy3})$$

It can be related to the vector $\{W_1\}$ in the local coordinate system as follows:

$$\underset{18\times 1}{\{W_1\}} = [R] \underset{18\times 1}{\{W\}} \quad (3.32)$$

where R is the rotation matrix (see page 79)

Substituting in Eq. (3. 19)

$$U = \frac{D}{2} \{W\}^t [R]^t [K_1] [R] \{W\}$$

$$= \frac{D}{2} \{W\}^t [R]^t [T_2]^t [k][T_2][R] \{W\} \quad (3.33)$$

or

$$U = \frac{D}{2} \{W\}^t [K] \{W\} \quad (3.34)$$

where K is the stiffness matrix in the global coordinate system.

3. 5 Consistent load vector:

The consistent load vector for an element can be established by a calculation of the virtual work of the applied loads. If q is the intensity of the applied load then this work is,

$$V = \iint q\, w\, d\xi\, d\eta$$

$$= \Sigma a_i \iint q\, \xi^{m_i} \eta^{n_i}\, d\xi\, d\eta \quad (3.34)$$

The above can be put in matrix form as,

$$V = \{A\}^t \{p\} \quad (3.35)$$

where the elements in column vector $\{p\}$ are given by,

$$p_i = \iint q\, \xi^{m_i} \eta^{n_i}\, d\xi\, d\eta \quad (3.36)$$

Substituting Eqns. (3. 15) and (3. 32) into Eq. (3. 35)

$$V = \{W\}^t \{P\} \quad (3.37)$$

where $\{P\}$ is the required load vector for the element given by $\{R\}^t [T_2]^t \{p\}$.

If the load is uniformly distributed and of intensity q, the entries of $\{p\}$ are given by,

$$p_i = q \iint \xi^{m_i} \eta^{n_i} \, d\xi \, d\eta = q \; G(m_i, n_i) \tag{3.37}$$

a form suitable for automatic computation.

3.6 Stiffeners:

3.6.1 General:

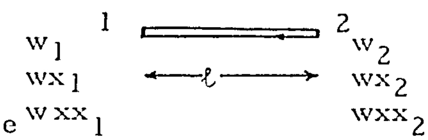
The advantage of the finite element method lies in its simplicity in the assembly of different elements like beams, plates, columns and even foundations at their common nodes. The only requirement is that the elements meeting at their common node must have the same number of degrees of freedom. In this analysis, a commonly occurring case of a bridge slab stiffened with edge beams on its free sides with the other two simply supported is considered. The beam is assumed to have no torsional rigidity and the slab and beam are assumed to be monolithic.

3.6.2 Stiffness of a beam element:

We have assumed a quintic polynomial for the deflection of the plate and complete monolithic action of slab and beam. So it becomes imperative that the deflection function for the beam lying parallel to the x-axis must be a quintic polynomial in x.

The assumed polynomial is of the form,

$$w(x) = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + b_5 x^4 + b_6 x^5 \tag{3.38}$$



The generalized coordinates are w and its first and second derivatives in x. A similar procedure as used for plate is used here to derive the stiffness of the beam element. The generalized displacements $w_1, w_{x1}, w_{xx1}, w_2, w_{x2}, w_{xx2}$ can be related to the undetermined constants by a transformation matrix as shown.

$$\begin{Bmatrix} w_1 \\ w_{x1} \\ w_{xx1} \\ w_2 \\ w_{x2} \\ w_{xx2} \end{Bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^3 & \ell^4 & \ell^5 \\ 0 & 1 & 2\ell & 3\ell^2 & 4\ell^3 & 5\ell^4 \\ 0 & 0 & 2 & 6\ell & 12\ell^2 & 20\ell^3 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{Bmatrix}$$

$$\{W\}_b = [T]_b \{A\}_b \qquad (3.39)$$

The strain energy of a beam element considering bending alone is given by,

$$U_b = EI \int_0^{\ell} \left(\frac{d^2 w}{dx^2} \right)^2 dx \qquad (3.40)$$

where ℓ is the length of the beam element

Eqn. (3.38) can be written in an abbreviated form as

$$w = \sum_{i=1}^{6} b_i x_i^{m_i} \qquad (3.41)$$

where $m_1 = 0$, $m_2 = 1$, $m_3 = 2$, $m_4 = 3$, $m_5 = 4$ and $m_6 = 5$

$$\left(\frac{dw}{dx} \right) = \sum_{i=1}^{6} b_i m_i x_i^{m_i - 1} \qquad (3.42)$$

$$\left(\frac{d^2 w}{dx^2}\right) = \sum_{i=1}^{6} b_i\, m_i\, (m_i - 1)\, x^{m_i - 2} \tag{3.43}$$

$$\left(\frac{d^2 w}{dx^2}\right)^2 = \sum_{i=1}^{6} \sum_{j=1}^{6} b_i\, b_j\, m_i\, m_j (m_i - 1)(m_j - 1)\, x^{m_i + m_j - 4} \tag{3.44}$$

Substituting Eq. (3.44) into Eq. (3.40)

$$U_b = \frac{EI}{2} \int_o^{\ell} \sum_{i=1}^{6} \sum_{j=1}^{6} b_i b_j m_i m_j (m_i - 1)(m_j - 1)\, x^{m_i + m_j - 4}$$

$$= \frac{EI}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} b_i\, b_j\, m_i\, m_j (m_i - 1)(m_j - 1) \frac{\ell^{m_i + m_j - 3}}{m_i + m_j - 3} \tag{3.45}$$

or

$$U_b = \frac{EI}{2} \Sigma\, \Sigma\, b_i\, b_j\, [k]_b$$

$$= \frac{EI}{2} \{A\}_b^t\, [k]_b\, \{A\}_b \tag{3.46}$$

From Eq. (3.39)

$$\{A\}_b = [T]_b^{-1}\, \{W\}_b \tag{3.47}$$

Substituting Eq. (3.47) into Eq. (3.46)

$$U_b = \frac{EI}{2} \{W\}_b^t\, [T_2]_b^t\, [k]_b\, [T_2]_b\, \{W\}_b \tag{3.48}$$

where $[T_2]_b = [T]_b^{-1}$

$$= \frac{EI}{2} \{W\}_b^t [K]_b \{W\}_b \tag{3.49}$$

where $[K]_b$ is the stiffness matrix of the beam element in the local coordinate system. Since in the problem considered in this thesis the local coordinate system for the beam elements coincides with that of the global coordinate system there is no need for a rotation matrix.

3.6.3 Consistent load vector for a beam element:

As in the case of a plate element the consistent load vector for a beam element can be established from a calculation of the virtual work of applied loads. If q_b is the intensity of the applied load, this work is

$$V_b = \int q_b \, w \, dx$$

$$= \Sigma \, b_i \int q_b \, x^{m_i} \, dx \tag{3.50}$$

In matrix from,

$$V_b = \{A\}_b^t \{p\}_b \tag{3.51}$$

where the entries in the column vector $\{p\}_b$ are given by,

$$p_{bi} = \int q_b \, x^{m_i} \, dx \tag{3.52}$$

From Eq. (3.39) $V_b = \{W\}_b^t [T_2]_b^t \{p\}_b$

or $$V_b = \{W\}_b^t \{P\}_b \tag{3.53}$$

where $\{P\}_b$ is the required consistent load vector for a beam element given by,

$$\{P\}_b = [T_2]_b^t \{p\}_b \tag{3.54}$$

where $[T_2]_b = [T]_b^{-1}$

If q_b is the intensity of uniformly distributed load, the terms in p_{bi} are given by,

$$p_{bi} = q_b \frac{\ell^{m_i+1}}{m+1} \qquad (3.55)$$

3.6.4 Assembly of plate and beam elements:

As mentioned earlier the condition for assembly of beam and plate elements is that they must have equal number of degrees of freedom at their common nodes. However, the beam element developed has only three degrees of freedom per node. So in order to make it compatible for assembly the 6 x 6 stiffness matrix for the beam element is expanded to a 12 x 12 stiffness matrix by including zeros in the corresponding rows and columns of the degrees of freedom not considered in the beam element. The assembly of these elements to plate can be considered in two different ways. Either the beam elements can be considered as part of the triangular plate elements to which they are attached so that their contribution can be considered in the stiffness of plate elements. In the other method which is considered here, the contributions of the beam elements are included in the structure stiffness matrix directly.

3.7 Boundary conditions:

The application of boundary conditions for skew plate considering it as a displacement model is slightly different from those derived for classical plate theory. Only natural or kinematic boundary conditions should be applied. This is by virtue of the principle of minimum potential energy. To cite an instance where in addition to natural boundary conditions, forced boundary conditions are applied, a reference is here made to the case of a simply supported edge. From classical plate theory since moment all along a simply supported edge is zero the corresponding displacement namely the curvature would have been considered as zero which would have added constraints to the system and result in a stiffer structure.

From variational principles $F \times \delta w = 0$ where F is the generalized force and δw is the corresponding displacement. Hence if at a particular point the force is known to be zero as in the case of a simply supported edge the corresponding displacement must be left as unknown as the product would any way result in zero.

This has been the guiding principle in the derivation of boundary conditions in this work. The boundary conditions could be applied in two different ways.

3.7.1 Rotation of coordinates:

In this case for those points lying on a skew boundary the stiffness matrix in global system is further transformed into an orthogonal system so that the generalized displacements become normal and tangential components to the skew boundary. The boundary conditions can then be applied directly by eliminating those displacements from the equations of the problem.

Mathematically this is done by premultiplying and post multiplying the structure stiffness matrix by the orthogonal transformation matrix. Incidentally this property of orthogonality has been judiciously made use of in storing it in an auxilliary storage which is explained in the section on the solution of equations. (Note: For the orthogonal matrix the transpose itself is the inverse).

The following is the derivation of orthogonal transformation.

$$\{F\} = [K] \quad \{X\}$$

where $\{F\}$ is the vector of generalized forces, $[K]$ is the stiffness matrix and $\{X\}$ is the vector of generalized displacements all in rectangular coordinate system.

But $$F = [T] \quad \{\tilde{F}\} \tag{3.57}$$

and $$X = [T] \{\tilde{X}\} \tag{3.58}$$

where $\tilde{F}$ and $\tilde{X}$ are the generalized forces and displacements respectively in an orthogonal coordinate system.

Substituting,

$$[T] \{\tilde{F}\} = [K] [T] \quad \{X\}$$

or $$\{\tilde{F}\} = [T]^{-1} [K] [T] \{\tilde{X}\}$$

$$= [\tilde{K}] \{\tilde{X}\} \tag{3.59}$$

where $[\tilde{K}]$ is the stiffness matrix in the orthogonal system

3.7.2 Use of Lagrangian multipliers:

An alternate procedure is to express the boundary conditions in terms of derivatives of the global coordinate system. To consider in more detail let us consider the case of a simple supported edge. The boundary conditions can be expressed as,

$$w = 0 \tag{3.60}$$

$$w_s = 0 = w_x \text{ Cos } \alpha + w_y \text{ Sin } \alpha \tag{3.61}$$

$$w_{ss} = 0 = w_{xx} \operatorname{Cos}^2 \alpha + 2\, w_{xy} \operatorname{Sin} \alpha \operatorname{Cos} \alpha + w_{yy} \operatorname{Sin}^2 \alpha \tag{3.62}$$

where the subscripts represent derivatives taken tangential to the plate boundary. One way is to express the one in terms of the other and eliminate the former from the equations of the problem. But the easier procedure would be to provide additional equations and to find the unknowns. This is done by the use of Lagrangian Multiplier, which are unknowns attached to the additional equations. The constraint equations for all points lying on the simply supported boundary are grouped together, given by $C = 0$ and the process is to minimize the potential energy $\overline{\Lambda} = \frac{1}{2} [X]^t [K] [X] - [X]^t [F]$ subject to the constraints. The coefficients of constraint equations are multiplied by the Lagrangian multipliers $\lambda_1, \lambda_2 \ldots$ and added to the original set of equations. The additional equations $[C]\{X\} = 0$ together with the above provide enough equations to solve for the unknowns $\{X\}$ and $(\lambda_1, \lambda_2 \ldots)$. They could be put in matrix from as,

$$\left[\begin{array}{c|c} K & C^t \\ \hline C & 0 \end{array}\right] \begin{Bmatrix} X \\ \lambda \end{Bmatrix} \quad \begin{Bmatrix} F \\ 0 \end{Bmatrix} \tag{3.63}$$

It is easy to set up the above equation. It should be noted that $[K]\{X\} + [C]^t [\lambda] = \{F\}$ takes up stationary value when no constraints are present and the Lagrangian multipliers automatically vanish.

3.8 Solution of equations:

One of the important aspects of the finite element method is the choice of a suitable method for solving the large number of equations resulting from it. Usually the number of equations are too large for an average size computer. The accuracy of the solution generally increases with the fineness of the grid system for which the number of equations proportionally increases. Also, while testing the convergence of a particular element the number of equations to handle is usually very large. This necessitates the economic use of available computer memory. Fortunately the matrices encountered in structural enginering problems have some special characteristics like bandedness and symmetry which would allow programming to effect an efficient and economic usage of storage.

What follows is a brief description of different methods in practice. The relative merits of them are discussed. The methods used in this analysis are in particular dealt with in detail. The most general method is to strike off the rows and columns corresponding to the known displacements and to move up and shift to the left the resulting matrix. This matrix is then inverted to find the unknowns. This way no saving of core is achieved.

In the second method (27) when a particular displacement is known the corresponding row and column is zeroed and unity is placed in the leading diagonal. The load vector is suitably modified to account for known displacements. Here also there is no saving of core and the time taken is perhaps more in the inversion. However, this has the advantage that the prescribed displacement need not necessarily be zero but can be any arbitrary value whereas in the previous one it is mandatory. This method has been used in the solution of equations for smaller sized matrices.

Apart from these mathematical manipulations there are yet other ways of saving computer memory. This is done by physically partitioning the structure and building up the skeletal matrices in auxilliary storages like disks or tapes. There are at least two ways of assembly of structure stiffness matrix from element stiffness. In the first method which is used in this work assembly is done by considering element by element. The element stiffness matrix in this case 18 x 18 is split up into 9, 6 x 6 submatrices as shown by the schematic diagram.

1, 1	1, 2	1, 3
2, 1	2, 2	2, 3
3, 1	3, 2	3, 3

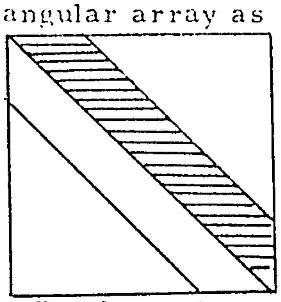
Element Stiffness matrix (18 x 18) is split up into submatrices (6 x 6)

Appropriate submatrix is added to the stiffness of the nodal point connected to an element and the process is repeated for all elements. This method is computationally simpler.

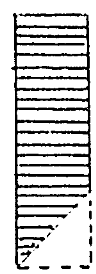
In the other method the assembly is done by considering nodes (27). This is a searching process in which at each node corresponding to each degree of freedom the contributions to that node from connected elements ia added up. This method has the advantage that equations could be generated one by one as in finite difference method and transferred to an auxilliary storage. These equations could be called in blocks at the time of solution.

In the analysis of clamped skew plates the boundary conditions have been applied after transforming the structure stiffness matrix in global system into an orthogonal system using rotation of coordinates. Here the structure stiffness matrix is premultiplied and post multiplied by the orthogonal transformation matrix. But no additional core space was required for storing the transformation matrix. In fact the transformation matrix is generated in the location for stiffness matrix and transferred to a disc. In its place later on the stiffness matrix is generated. The transformation matrix is called in one row or one column at a time for multiplication. Since it happens to be an orthogonal matrix (and hence its transpose itself is its inverse) it is called in a row at a time for premultiplication and a column at a time for post multiplication. The product of these three matrices are stored in one. Besides advantage could be taken of the banded nature of the stiffness matrix.

In the case of the cantilever plate the boundary conditions could be applied directly without resort to Lagrangian multipliers. So in this case the stiffness matrix is banded and symmetric. In this case only half the banded elements are generated and stored in a rectangular array as shown.

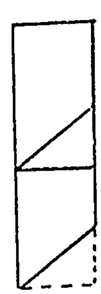


Band matrix



Storage of upper band as rectangular array

Here the application of boundary condition deserves some consideration. Since the rows have been shifted laterally the first column represents all diagonal elements and hence all column element have been shifted. This fact must be borne while zeroing the rows and columns, and it is shown in the schematic diagram.



The equations stored in this form could be solved by elimination, or by Cholesky's method of decomposition (25). This method has been used in solving large number of equations and is briefly described here.

Let us assume that we have to solve a set of equations given by,

$$[A] \cdot \{X\} = \{F\} \tag{3.64}$$

in which $\{X\}$ is the column vector of unknowns and $\{F\}$ that of load. The matrix $[A]$ which is symmetric can be decomposed.

Thus $$[A] = [U]^t \, [U] \tag{3.65}$$

It can be seen that the elements of the matrix A consists of inner products of the rows of $[U]^t$ and the columns $[U]$. Equivalently, the elements of $[A]$ may be calculated as inner products among the columns of $[U]$. Thus

$$A_{ii} = U_{1i}^2 + U_{2i}^2 + U_{3i}^2 + \dots\dots\dots + U_{ii}^2$$

or $$A_{ii} = \sum_{k=1}^{i} U_{ki}^2 \quad (i = j) \tag{3.66}$$

$$A_{ij} = U_{1i} U_{1j} + U_{2i} U_{2j} + \dots\dots\dots + U_{ii} U_{ij}$$

or $$A_{ij} = \sum_{k=1}^{i} U_{ki} U_{kj} \qquad (i < j) \tag{3.67}$$

The elements of $[U]$ may be determined by rearranging Eqns.(3.66) and 3.67 as follows:

$$U_{ii} = \sqrt{A_{ii} - \sum_{k=1}^{i-1} U_{ki}^{2}} \quad (1 < i = j) \tag{3.68}$$

$$U_{ij} = A_{ij} - \sum_{k=1}^{i-1} U_{ki} U_{kj} \quad (1 < i < j) \tag{3.69}$$

$$U_{ij} = 0 \qquad (i > j) \tag{3.70}$$

Substituting Eq. (3.65) into Eq. (3.64)

$$[U]^{t} [U] \{X\} = \{F\} \tag{3.71}$$

Defining the vector $\overline{X}$ to be: $UX = \overline{X}$ (3.72)

Substituting Eq. (3.72) into Eq. (3.71) yields:

$$[U]^{t} \{\overline{X}\} = \{F\} \tag{3.73}$$

The vector of unknowns $\{X\}$ may now be obtained in two steps using Eqs. (3.73) and (3.72). In the first step Eq. (3.73) is solved for vector $\{\overline{X}\}$ by a series of forward substitutions and the recurrence formula for elements $\{X\}$ is given by:

$$\overline{X}_i = \frac{B_i - \sum_{k=1}^{i-1} U_{ki} \overline{X}_k \; (1 < i)}{U_{ii}} \tag{3.74}$$

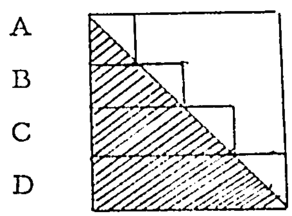
The second step consists of solving for the vector $\{X\}$ by a series of backward substitutions and the recurrence formula for this is given by:

$$X_i = \frac{\overline{X}_i - \sum_{k=i+1}^{n} U_{ik} X_k \; (i < n)}{U_{ii}} \tag{3.75}$$

This step completes the solution of original set of equations (3.64) for the unknowns.

This method was found to be very fast and accurate.

Unfortunately the matrices resulting from the analysis of skew plates with the exception of cantilever plate is only symmetric and not banded. This is due to the addition of equations to the structure stiffness matrix by way of boundary constraints. However, symmetric nature of it could be made use of in effecting some saving of core. The symmetric matrix could be split up into as many parts as deemed necessary as shown:



During the assembly when the number of rows of structure stiffness matrix exceeds the size of one of the blocks it is automatically stored in the following one. Elements beyond the leading diagonal are not generated at all. The solution of these equations could be done by Cholesky's method.

CHAPTER 4

COMPARISON AND DISCUSSION OF RESULTS

4.1 Skew stiffened plate:

In order to assess the accuracy of this method, first results were obtained for rectangular plates with edge beams. Timoshenko has presented results for central deflection and moments for various ratios of flexural rigidity of beam and slab. Results given in Table 1 shows an excellent agreement between the two methods. As one would expect, results for $^{EI}/_{LD} = .0$ should concur with the results for two sides simply supported and two sides free case and those for higher rigidity ratios the structure should behave like one with all sides simply supported.

For skew plates comparison is made to that of Kennedy and Huggins (14) whose analysis is based on a single infinite Fourier series for the deflection of the plate and to that of Gustafson (8) who used a non-conforming rectangular element (Table 2).

The deflections and moments of Kennedy and Huggins show good agreement with the present method for skew angles 15 and 30° while they seem to differ by 15 to 20% for higher Skew angles. Kennedy and Huggins have stated that convergence was difficult to obtain and they did not consider convergence beyond 45°. In the finite element method considered here results for a 4 x 4 grid do not deviate appreciably from that of a 2 x 2 grid even for high angles of skew. Hence it is believed that the results presented herein are more reliable than those obtained by Kennedy and Huggins. Besides they compare reasonably with those presented by Gustafson (8) for all skew angles.

Effect of Rigidity Ratio on High Skew Stiffened Plate:

In order to determine the effect of rigidity ratios, different ratios ranging from 0. 5 to 50 were tried for a skew stiffened plate of skew angle 60°. The results for deflection and moments are presented in Table (3). It is observed that the maximum values for deflections shifted from the free edges towards centre. Both moments and deflections decreased considerably as the rigidity ratios are increased.

Effect of Poisson's Ratio:

Sawko and Cope (21) studied the effect of poisson's ratio on the stress distribution of a skew deck (two sides simply supported and the other two sides free). They found that the change of poisson's ratio from 0 to 0. 3 for 45° skew lead to an increase of 13 and 32 percent respectively for longitudinal and transverse moments at the centre.

The easiness with which such changes could be made was utilised to study this aspect. In Table (4) are given results for a 60° skew case for poisson's ratios from 0 to 0. 3. It is observed that there is an increase of over 70% in longitudinal moments and 23 percent in the transverse moments. It must be concluded that the effect of Poisson's ratio on the plate moments is very significant.

Effect of Aspect Ratio:

The results of central deflections, moments and strain energy are presented for aspect ratios 1, 1. 5 and 2. 0 in Tables 5, 6 and 7. It is found that they increase with increase in aspect ratio. The longitudinal moments seem to increase more rapidly than the transverse moments with increase in skew angle.

4.2 Skew cantilever plate:

The presence of infinite stresses local to the junction between fixed and trailing edges has been the subject of discussion since Williams (26) had shown that it was possible that unbounded stresses might occur at the obtuse corner of a cantilever plate when the angle exceeds 95°. This singularity according to him increases with increase in skew angle. So in this work an attempt was made to analyse a skew cantilever plate keeping this in view. A very fine mesh (figure 2) was used to find out the quantitative estimate of these stresses.

In Table (8) comparison is made to the works of Monforton (17) and Dawe (7) and the Experimental values of Williams (26) Monforton used a conforming rectangular element, and Dawe used a non-conforming parallelogrammic element. For skew angles of 20° and 40° there is excellent agreement between all the three analytical results. However all of the finite elements results are below that of the experimental values. The error however could be attributed to the experimental values in as far as skew angles of 20° and 40° are concerned. Williams himself has stated that deflection of clamped plates was found to be strongly affected by clamping pressures and it was difficult to match the theoretical boundary conditions that are imposed on the mid-plane of the plates with the experimental boundary conditions that are imposed on the surface. Dawe's values for deflection for 60° skew lay 20% below that of experimental value. However, there is excellent agreement between Monforton's and the present method. So it can be concluded that the actual values are somewhere between these analytical values and experimental values.

From the results obtained using a fine grid as shown in figure 2 it is observed that both the longitudinal moment M_x and the transverse moment M_y increase and the latter more rapidly near the obtuse corner. This, however should be verified by experiment. Monotonic convergence which is the inherent property of the element is established as shown in Table 9. The deflections and the moments also converge. The free corner deflections and the clamped end moments decrease with increase in aspect ratio. (Tables 9, 10 and 11). The rate of decrease for moments is about 50% for aspect ratios between 1 and 1.5 but it is only 15% for aspect ratios between 1.5 and 2.0.

4.3 Skew clamped plate:

A number of authors have analysed the bending of clamped skew plates under uniform load. Different variational methods like the Ritz and Galerkin variational techniques have been applied. Morley (18) used the finite difference method and conformal mapping was tried by Komatsu (16). But these methods gave adequate results only for relatively small skew angles. By far the best and dependable solution is that due to Iyengar et. al. (12) who used a double in finite series which satisfied the boundary conditions and coefficients were found so as to satisfy the governing differential equation.

Monforton (17) has compared his results with that of Iyengar et. al. and found good agreement except for the skew angle 75° and aspect ratio 1. It could be observed from the Table (12) that this method shows good agreement with that of Ref. (17) even for such a high skew angle as 75°. It is observed from Tables (13, 14 and 15) that the maximum deflection decreases with increase in aspect ratio. The maximum longitudinal moment decreases while the maximum transverse moment increases with increase in aspect ratio. This could be expected as the moments are transferred through the shorter sides.

4.4 Skew simply supported plate:

Results obtained for a skew simply supported plate have been compared to those of Morley (18) who used finite difference method. (Table 16) The variation between the results for the angle $\beta = 60^{o}$ is worth noting. For the same angle Jensen (13) using the finite differences technique but using a different grid obtained a maximum deflection of .000497 $qL^4/16D$ against .000384 $qL^4/16D$. For the same angle Favre (18) is reported to have obtained a value of .000309 $qL^4/16D$ which is closer to the present. So no definite conclusion could be drawn for this case especially for high skew angles. Referring to Tables 17, 18 and 19 it is observed that the maximum deflection increases with increase in aspect ratio but decreases with increase in skew angle. Maximum longitudinal moment decreases while the maximum transverse moment increases with increase in aspect ratio.

4.5 Skew plate simply supported on two sides and free on the other two:

Results for this case have been compared to those of Jensen as shown in Table 20. They show fairly good agreement for small skew angles. Jensen applied finite difference technique using only nine internal nodal points which in the writers opinion is not enough to obtain a reasonably accurate solution. It is observed that the maximum deflection shifts from centre of the free edges towards the centre of the plate as the angle ϕ decreases.

From the foregoing discussion on skew plates it may be observed that this finite element yields reasonably good results for all the cases considered. A 4x4 grid is enough to obtain reasonably accurate results. The numerical evidence of convergence of strain energy deflections and moments gives added confidence for the use of this element.

CHAPTER 5

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

5.1 Conclusions:

The following conclusions may be drawn from this study on the behaviour of stiffened and unstiffened skew plates.

(1) The finite element method is a useful topi in solving problems of this nature and this refined finite element in particular gives good results even for coarse meshes for moderate skew angles.

(2) In the case of skew plate and rectangular plate stiffened with edge beams, the maximum deflections shift from centre of the edge beams to the centre of the plate as the stiffness of the beam to plate increases.

(3) When the ratio $^{EI}/_{LD}$ is very large the plate tends to behave like one simply supported on all sides and when $^{EI}/_{LD}$ is small, the plate acts like one which is simply supported on two sides and free on the other two.

(4) The effect of Poisson's ratio influences the moments to a great extent especially for high skew angles and hence some of the methods which do not take the Poisson's ratio into account will underestimate the moments.

(5) In clamped skew plates the central deflection decreases as the skew angle increases which is due to the increased rigidity of the plate at obtuse corners.

(6) In the case of the plate simply supported on two sides and free on the other two the deflection shifts from centre of the free edge to the centre of plate with increase in skew angle.

(7) For cantilever plates with a relatively high angle of skew analytical methods generally give results lower than experimental ones. However the results obtained herein seem to be closer to experimental values than previous analytical techniques.

(8) From the fine grid pattern used for the cantilever plate it is observed that there is a definite tendency for the moments to increase rapidly near the obtuse corners.

(9) In general, the skewed plates behave like one way slabs when the aspect ratio is over 1.5.

5.2 Recommendations for further study:

This investigation on the stiffened skew plate can be extended for a grid work of longitudinal and transverse girders. These days different cross sections (16) have been used in longer girders. The most suitable section from the point of view of the stiffness may be found for a particular span. By repeated trials the optimum spacing of longitudinal and cross girders may also be found which will result in a most economic structure. The influence lines for track and wheel loads could be drawn for various boundary conditions which will be a useful contribution to the design engineers. Because of its great flexibility this method can be used to analyse continuous beams and non-linear problems in skew plates.

A NOTE ON THE COMPUTER PROGRAM

A computer program coded in Fortran IV and written for an IBM-360/65 model for the analysis of skew stiffened plate is given in the appendix. This program can handle skew stiffened plates with two sides simply supported and stiffened with edge beams on the other two, with any aspect ratio, skew angle and rigidity ratio. A sample output is also given. The same program can be used for analysing the unstiffened skew plates with any boundary condition by deleting that part of the program in which the stiffnesses of the beam elements are added to the stiffnesses of plate elements. As far as the boundary conditions are concerned that part of the program in which the constraint equations are added must be replaced by the corresponding ones for the appropriate boundary condition considered. In the case of the cantilever plate where no constraint equations are required two special routines have been written to make use of the banded nature of the stiffness matrix in storing and solving. They are also given in the appendix. It is only a matter of changing the load vector to take into account point loads.

REFERENCES

1. Adini, A. and Clough, R., "Analysis of Plate Bending by the Finite Element Method", Report submitted to the Nat. Sc. Found, Grant G7337, 1960.

2. Aggarwala, B. D., "Bending of Parallelogram Plates", Journal of Eng. Mech. Dov., A.S.C.E., Vol. 93, No. EM4, Aug., 1967.

3. Bogner, F. K., Fox, R. L., Schmit, L. A., Jr., "The Generation of Inter Element Compatible Stiffness and Mass Matrices by the Use of Interpolation Formulas", Conference on Matrix Methods in Structural Mechanics, Wright-Patterson AFB, Ohio, AFFDL-TR-66-80, 1966.

4. Chu, K., and Krishnamoorthy, G., "Use of Orthotropic Plate Theory in Bridge Design", Journal of the Structural Division, A.S.C.E., Vol. 88, No. ST3, Proc. paper 3157, 1962.

5. Clough, R. W., and Tocher, J. L., "Finite Element Stiffness Matrices for Analysis of Plate Bending", Conference on Matrix Methods in Structural Mechanics Wright-Patterson AFB, Ohio, AFFDL-TR-66-80, 1966.

6. Cowper, G. R., Kosko, E., Lindberg, G. M. and Olson, M. D., "A High Precision Triangular Plate Bending Element", Aeronautical Report LR-514, Dec., 1968.

7. Dawe, D. J., "Parallelogrammic Elements in the Solution of Rhombic Cantilever Plate Problems", Journal of Strain Analysis Vol. 1, No. 3, 1966.

8. Gustafson, W. C., "Analysis of Eccentrically Stiffened Skewed Plate Structures", Ph. D. Thesis, Univ. of Illinois, 1966.

9. Hendry, A. N., and Jaegar, A. N., "The Analysis of Grid Frame Works and Related Structures", Prentice Hall, New Jersey, 1959.

10. Hilderbrand, F. B., "Methods of Applied Mathematics", Prentice Hall, New Jersey, 1952.

11. Holand, I. and Bell, K., "Finite Element Methods in Stress Analysis".

12. Iyengar, K., Sunderra, R. and Srinivasan, R., "Clamped Skew Plate Under Uniform Normal Loading", Journal of the Roy. Aero. Soc., Vol. 71, Feb., 1967.

13. Jensen, V. P., "Analysis of Skew Slabs", Univ. of Illinois, Eng. Exp. Sta., Bul. 332.

14. Kennedy, J. B. and Huggins, M. W., "Series Solution of Skewed Stiffened Plates", Journal of Eng. Mech. Division, A. S. C. E., Vol. 90, No. EM1.

15. Kennedy, J. B. and Ng, S., "Analysis of Skewed Plate Structures With Clamped Edges", Trans. of the Eng. Inst. of Canada, Paper No. EIC-65-BR & STR9, Dec., 1965.

16. Kennedy, J. B. and Tamberg, K. G., "Problems of Skew in Concrete Bridge Design", Dept. of Highways, Ont., D. H. O. Report No. RR 144, March, 1969.

17. Monforton, G. R., "Finite Element Displacement and Vibration Analysis of Skew Plates", Case Western Reserve University, Report No. 18, Division of Solid Mechanics, Structure and Mechanical Design, March, 1968.

18. Morley, L. S. D., "Skew Plates and Structures", A Pergamon Press Book, 1963.

19. Newmark, N. M. and Seiss, C. P., "Moments in I-Beam Bridges", Univ. of Illinois, Eng. Exp. Sta., Bul. 336, 1942.

20. Quinlan, P. M., "The λ - Method for Skew Plates", Fourth U.S. Nat. Cong. of App. Mech., 1962.

21. Sawko, F. S. and Cope, R. J., "The Analysis of Skew Bridge Decks - a new finite element approach", Structural Engineer, June, 1969.

22. Timoshenko, S. and Woinowski Krieger, S., "Theory of Plates and Shells", McGraw-Hill, N. Y., 1959.

23. Tong, P. and Pian, T. H. H., "The Convergence of the Finite Element Method in Solving Linear Elastic Problems", Int. J. Solids and Structures, Vol. 3, 1967.

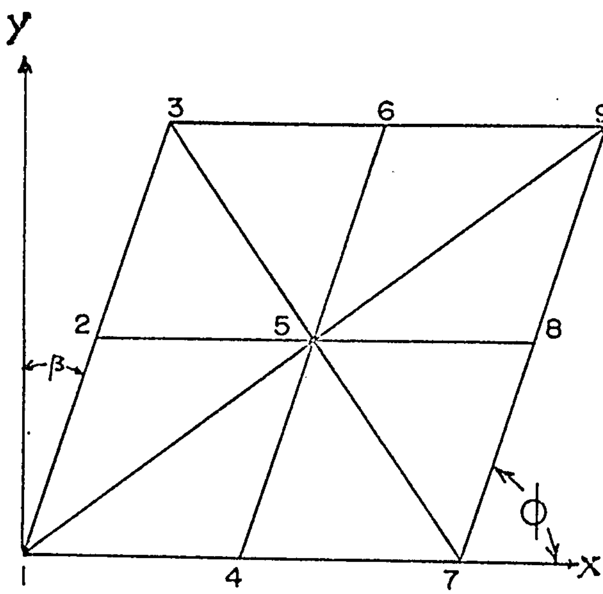
24. Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J. "Stiffness and Deflection Analysis of Complex Structures Journal of the Aero. Sci., Vol. 23, No. 9, Sept. 1956.

25. Weaver, Jr., W., "Computer Programs for Structural Analysis," D. Van Nostran Company Inc., 1967.

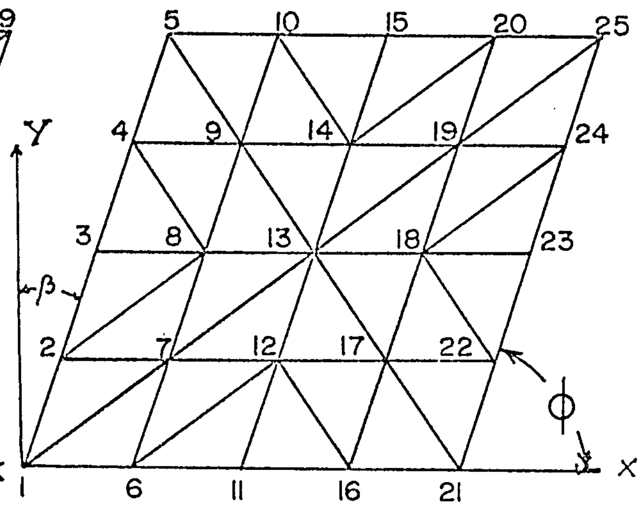
26. Williams, M. L., "A Review of Certain Analysis Methods for Swept-wing Structures", J. Aero. Science, 1952.

27. Wilson, E. L., "Finite Element Analysis of Two Dimensional Structures", Ph. D. Thesis, Univ. of California, 1963.

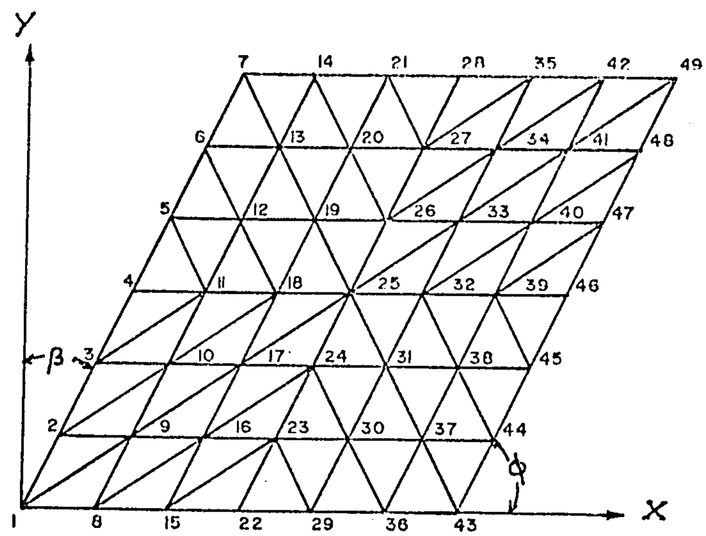
28. Zienkiewicz, O. C., Y. K. Cheung, "The Finite Element Method in Structural and Continuum Mechanics", Published by McGraw-Hill, 1963.



2 x 2 Grid



4 x 4 Grid



6 x 6 Grid

FIG. I FINITE ELEMENT LAYOUTS

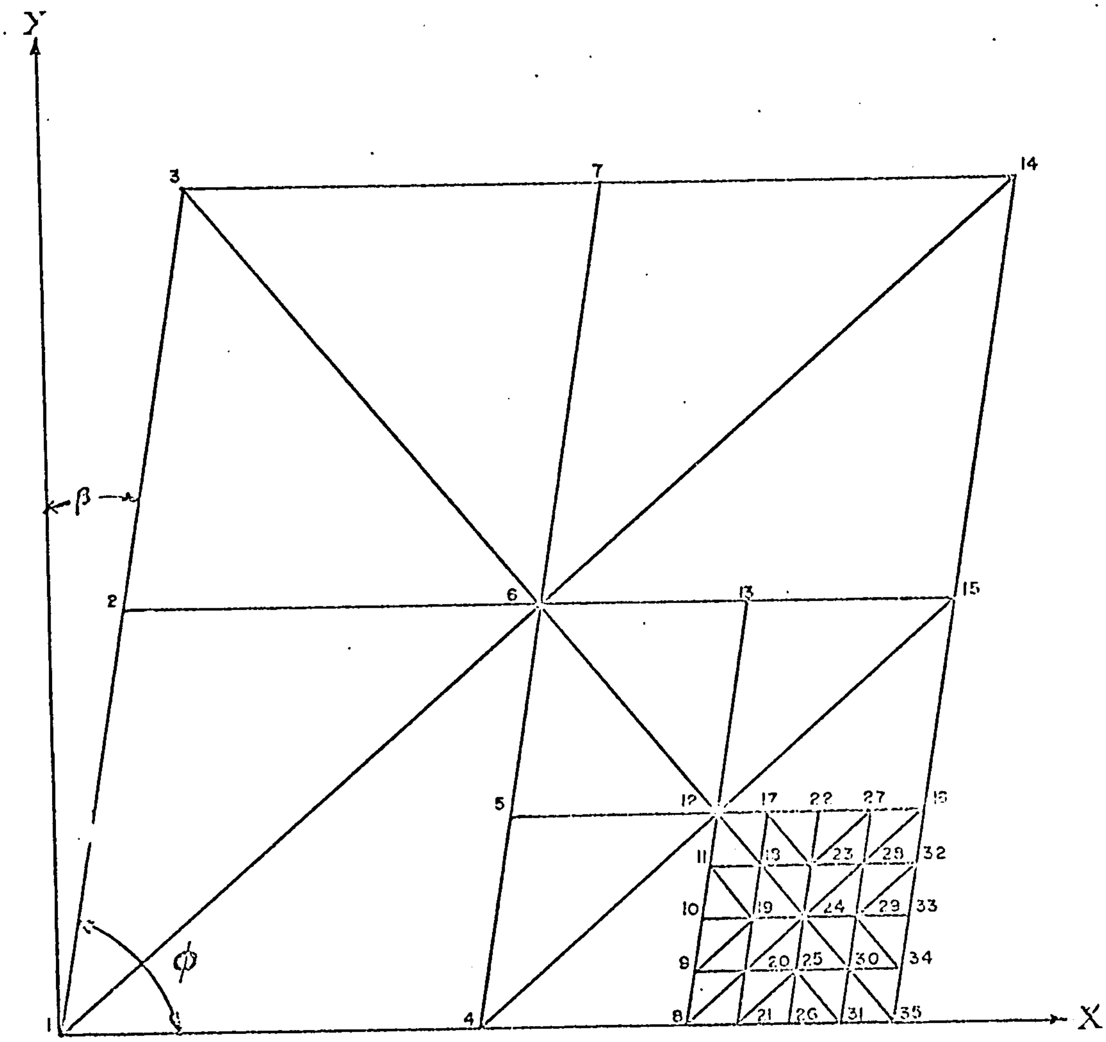


FIG.2 FINE MESH ARRANGEMENT FOR CANTILEVER PLATE

NOTE: Compatability is maintained between the grid points 4 and 6 all along the line.

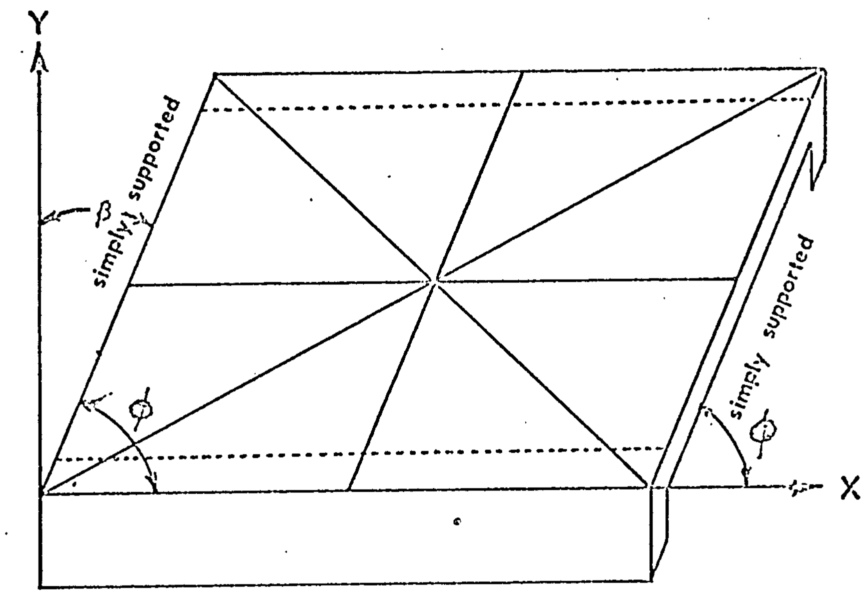


FIG. 3 SKEW PLATE STIFFENED WITH EDGE BEAMS

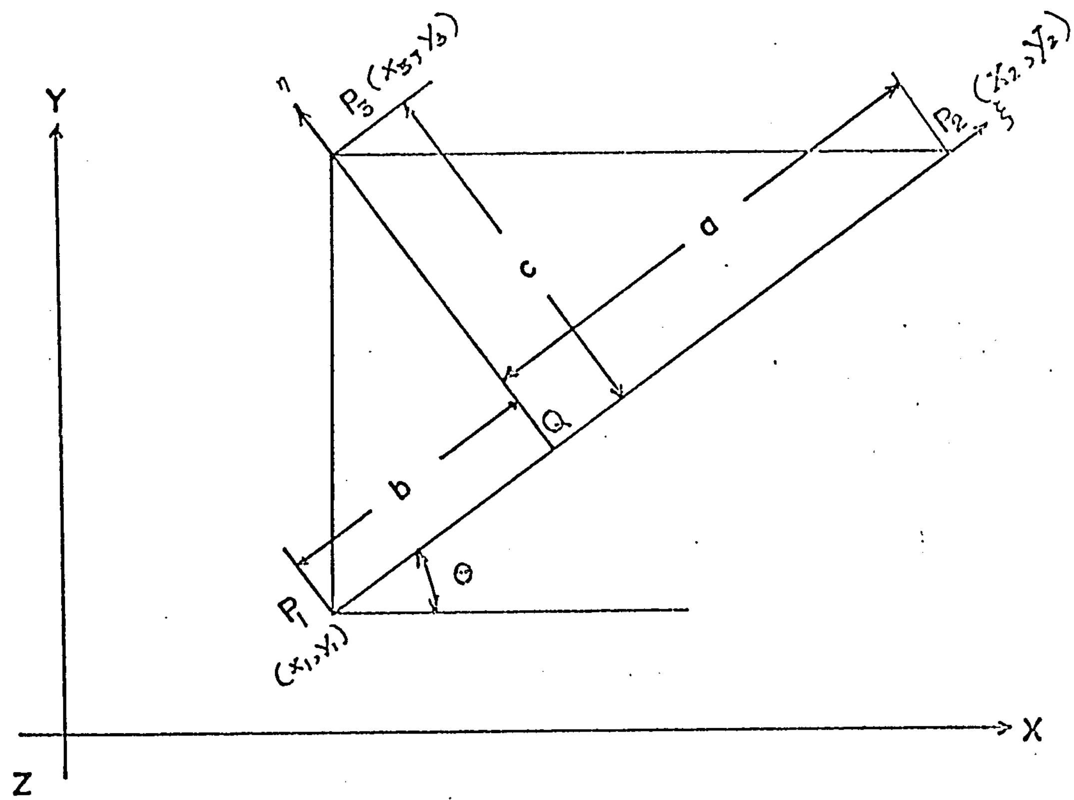


FIG. 4 GLOBAL AND LOCAL COORDINATE SYSTEMS

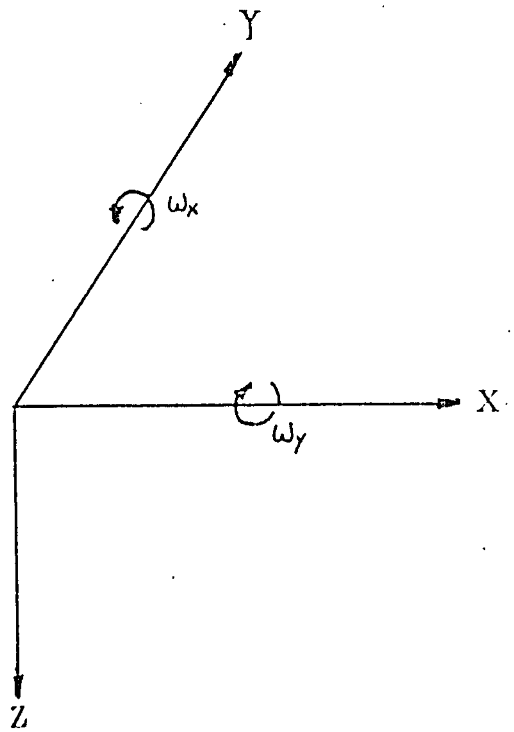


FIG. 5 SIGN CONVENTION

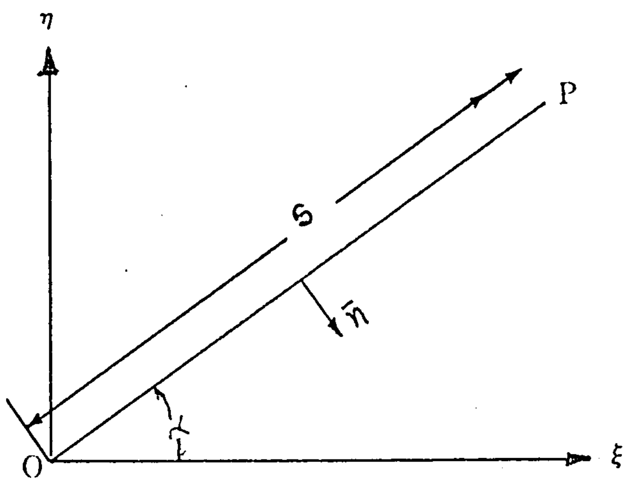


FIG. 6 OBLIQUE BOUNDARY

TABLE 1

EFFECT OF FLEXURAL RIGIDITY RATIO ON STIFFENED PLATE (Square)

Aspect Ratio : 1 Poisson's Ratio : 0.3

$\frac{EI}{LD}$	Central Deflection / $\frac{D}{qL^4}$		Central Moment M_x / qL^2		Central Moment M_y / qL^2	
	Timoshenko	Present	Timoshenko	Present	Timoshenko	Present
0	.01309	.01309370	.1225	.12242134	.0271	.02730093
1	.00624	.00624038	.0643	.06582018	.0376	.04284531
2	.00529	.00530067	.0571	.05805731	.0419	.04499319
4	.00472	.00472708	.0528	.05331833	.0447	.04630500
10	.00434	.00434052	.0500	.05012445	.0500	.04718944
30	.00409	.00415704	.0486	.04860838	.0456	.04760928

TABLE 2

COMPARISON OF CENTRAL DEFLECTIONS AND MOMENTS FOR THE SKEW STIFFENED PLATE

Aspect Ratio = 1.0 $\qquad \frac{EI}{LD} = 0.26 \qquad \mu = 0.2$

Skew angle β	Central Deflection/ $\frac{16D}{qL^4}$			M_x (central)/ $\frac{qL^2}{4}$			M_y (central)/ $\frac{qL^2}{4}$			M_{xy} (central)/ $\frac{qL^2}{4}$		
	Kennedy & Huggins	Gustafson	Present	Kennedy & Huggins	Gustafson	Present	Kennedy & Huggins	Gustafson	Present	Kennedy & Huggins	Gustafson	Present
0°		.14672	.14639		.3609	.3372		.1178	.1184		.00002	.00000
15°	.13293	.13120	.1308	.33346	.3285	.3204	.09927	.1212	.1212	.06315	.05964	.0592
30°	.08926	.09248	.09184	.23628	.2432	.2392	.11310	.1248	.1260	.08926	.0960	.0944
45°	.04183	.04832	.04752	.1208	.1352	.1372	.12225	.1164	,1220	.07618	.0968	.09248
60°	.01279	.01584	.01468	.04024	.04068	.0464	.11732	.0700	.0968	.03091	.0660	.0569

TABLE 3

EFFECT OF FLEXURAL RIGIDITY RATIO ON SKEW STIFFENED PLATE

Aspect Ratio = 2.0, $\beta = 60°$, Grid = 4x4, $\mu = 0.2$

$EI/_{LD}$	Deflection (central) $/\frac{D}{qL^4}$	Deflection (max)/ $\frac{D}{qL^4}$	Moment M_x (central)/ qL^2	Moment M_x (max)/ qL^2	Moment M_y (central)/ qL^2	Moment M_y (max)/ qL^2
0.5	.02086869	.02535305 (* point 11)	.05777863	.06032061 (* point 12)	.0302174	.04802041 (* point 9)
1	.01609998	.01951753 (* point 11)	.04297681	.04622100 (* point 12)	.03316001	.04572183 (* point 9)
2	.01135802	.01349234 (* point 11)	.03084690	.03302911 (* point 12)	.03372718	.03989392 (* point 9)
5	.00634658	.00707231 (* point 11)	.01949096	.0199772 (* point 12)	.032159	same as central
10	.00389790	.00395883 (* point 11)	.01420879	same as central	.03064422	same as central
50	.00144870	same as central	.00899436	same as central	.02865281	same as central

* Refer. fig. (2) for the location of the point

TABLE 4

EFFECT OF POISSON'S RATIO ON SKEW STIFFENED PLATES

$\beta = 60^{\circ}$; Aspect Ratio : 1.0

Poisson's Ratio	M_x / qL^2 (central)	M_y / qL^2 (central)
0.0	.00877246	.02006921
0.1	.01086439	.02091213
0.2	.01296527	.02178545
0.3	.01508003	.02269488

TABLE 5

CENTRAL DEFLECTIONS, CENTRAL MOMENTS, MAXIMUM DEFLECTIONS, MAXIMUM MOMENTS AND STRAIN ENERGY FOR THE SKEW STIFFENED PLATES

Aspect Ratio : 1 Poisson's Ratio : 0.2 $\frac{EI}{LD} = 0.26$

		Skew angle					
	Grid	0°	15°	30°	45°	60°	75°
Deflection / $\frac{qL^4}{D}$	2x2	.00914896	.00817690	.00574494	.00297536	.00091860	.00007312
	4x4	.00914987	.00817970	.00574735	.00297933	.00094439	.00009086
		(.00914987)	(.00817970)	(.00574735)	(.00353607)[11]	(.00160742)[11]	(.00043346)[11]
Strain/ Energy $\frac{D}{qL^6}$	2x2	.00279558	.00242805	.00156426	.00069693	.00017688	.00001481
	4x4	.00279573	.00242866	.00156645	.00070401	.00018566	.00001872
			(.00242866)				
M_x/qL^2	2x2	.08813375	.08017783	.05980100	.03433492	.01162322	.00297010
	4x4	.08930319	.08124059	.06023389	.03431293	.01292104	.00099118
		(.08930319)	(.08124059)	(.06023389)	(.03449004)[12]	(.01562215)[12]	(.00478493)[15]
M_y/qL^2	2x2	.02958859	.02955766	.02877474	.04062121	.04023524	.01387191
	4x4	.02960218	.03033054	.03152643	.03054397	.02422398	.01308708
		(.02960218)	(.03033054)	(.03152643)	(.03054397)	(.02422398)	(.01313723)[10]
M_{xy}/qL^2	2x2	.00000	.01449356	.02263816	.02192629	.01398100	.00432872
	4x4	.00000	.01493405	.02369016	.02406122	.01413819	.00391752

The values inside the brackets show the maximum values and the superscripts denote their locations in the 4x4 grid as shown in Fig. 1.

TABLE 6

CENTRAL DEFLECTIONS, CENTRAL MOMENTS, MAXIMUM DEFLECTIONS, MAXIMUM MOMENTS AND STRAIN ENERGY FOR THE SKEW STIFFENED PLATES

Aspect Ratio : 1.5 Poisson's Ratio : 0.2 $\frac{EI}{LD} = 0.26$

		Skew angle					
	Grid	0°	15°	30°	45°	60°	75°
Deflection/ $\frac{qL^4}{D}$	2x2	.04378928	.03962936	.02901498	.01629992	.00603746	.00007312
	4x4	.04379273	.03964038	.02907276	.01647191	.00634554	.00122750
		(.04379273)	(.03964038)	$(.03081154)^{11}$	$(.01926735)^{11}$	$(.00899712)^{11}$	$(.00243101)^{11}$
Strain Energy $/ \frac{D}{qL^6}$	2x2	.02090875	.01833927	.01217949	.00574535	.00164352	.00001481
	4x4	.02091033	.01834665	.01222121	.00584088	.00173580	.0002136
M_x / qL^2	2x2	.18509788	.16968613	.13300704	.08548541	.03170213	.00081833
	4x4	.18719510	.17219090	.13258689	.07927120	.02873539	.00172483
		(.18719510)	(.17219090)	(.13258689)	(.08172983)	$(.03774491)^{11}$	$(.00968959)^{12}$
M_y / qL^2	2x2	.05013034	.04921363	.04993252	.05209495	.04826503	.01387191
	4x4	.04783471	.04847427	.04853573	.04453826	.03282161	.01674362
		(.04783471)	(.04847427)	(.04853573)	(.04453826)	$(.04376857)^{9}$	$(.02468527)^{12}$
M_{xy} / qL^2	2x2	0.00000000	.03245863	.05503692	.06198604	.05230252	.00290744
	4x4	0.00000000	.03215327	.05427860	.06039106	.05147784	.0332454

The values inside the brackets show the maximum values and the superscripts denote their locations in the 4x4 grid as shown in Fig. 1.

TABLE 7

CENTRAL DEFLECTIONS, CENTRAL MOMENTS, MAXIMUM DEFLECTIONS, MAXIMUM MOMENTS AND STRAIN ENERGY FOR THE SKEW STIFFENED PLATES

Aspect Ratio : 2 Poisson's Ratio : 0.2 $\frac{EI}{LD} = 0.26$

	Grid	Skew angle 0°	15°	30°	45°	60°	75°
Deflection/ $\frac{qL^4}{D}$	2x2	.13735424	.12533420	.09421987	.05588396	.02349567	.00500460
	4x4	.13736457	.12539769	.09460662	.05686674	.02479963	.00606253
		(.13948594)^{11}	(.12852325)^{11}	(.09974312)^{11}	(.05686674)	(.0298264)^{11}	(.00788959)^{11}
Strain Energy / $\frac{D}{qL^6}$	2x2	.08828563	.07790133	.05271928	.02578762	.00789584	.00094344
	4x4	.08828920	.07793544	.05293070	.02625592	.00830036	.00107683
M_x / qL^2	2x2	.32409262	.29813615	.23620364	.15943678	.07330777	.012151103
	4x4	.32644626	.30252048	.23967869	.15544863	.07322430	.01830151
		(.32644626)	(.30252048)	(.23967869)	(.15544863)	(.07322430)	(.01976039)^{10}
M_y / qL^2	2x2	.06106755	.05916063	.05672025	.05103450	.04066128	.02684532
	4x4	.05861284	.05889124	.05672661	.04735804	.02560416	.01785575
		(.058611284)	(.05889124)	(.05672661	(.26458556)	(.2295203)^{5}	(.02689449)^{8}
M_{xy} / qL^2	2x2	0.00000000	.05738850	.10017849	.11752601	.10767729	.06706620
	4x4	0.00000000	.05569277	.09571176	.11122610	.10101307	.06525648

The values inside the brackets show the maximum values and the superscripts denote their locations in the 4x4 grid as shown in Fig. 1.

TABLE 8

COMPARISON OF FREE CORNER DEFLECTIONS OF RHOMBIC CANTILEVER PLATE

		$w_3^* / \frac{D}{qL^4}$				$w_{14}^* / \frac{D}{qL^4}$			
Skew angle β		30°	50°	70°	90°	30°	50°	70°	90°
	2 x 2	.01308	.0493	.09438	.1262	.06654	.10437	.1297	.1262
Present	4 x 4	.01378	.04942	.0948	.12691	.07305	.1067	.1300	.12691
	6 x 6	.0139494	.0494614	.0949399	.12708489	.07487046	.1071632	.13008489	.12708489
Ref (17) Monforton		.0140	.0496	.0950		.0746	.1071	.1300	
Ref (7) Dawe		.0143	.0496	.0952		.0674	.1035	.1292	
Ref (26) Williams		.0159	.0529	.0994		.0855	.1132	.1367	

(*For locations of point 3 and 14 refer Fig. 2)

TABLE 9

FREE CORNER DEFLECTIONS, MAXIMUM MOMENTS AND STRAIN ENERGY OF A SKEW CANTILEVER PLATE

Aspect Ratio : 1 Poisson's Ratio : 0.3

Skew angle	Grid	Deflections / $\frac{D}{qL^4}$ w_3	w_{14}	M_x / qL^2 max	M_y / qL^2 max	Strain Energy $\frac{D}{q^2 L^6}$
60°	2x2	.01308251	.06654037	-.20971058	-.69903528	.00297313
	4x4	.01378606	.07305490	-.30920012	-1.03066706	.00328949
	6x6	.01394940	.07487046	-.37857549	-1.26191833	.00337850
50°	2x2	.02871013	.08562107	-.23576873	-.78589577	.00617636
	4x4	.02915520	.08995530	-.32558112	-1.0852704	.00645699
	6x6	.02925113	.09104805	-.38246764	-1.27489214	.00652764
40°	2x2	.04929773	.10437029	-.2442764	-.81475878	.01067407
	4x4	.04942908	.10673060	-.31455970	-1.04865967	.01085965
	6x6	.04961400	.10716320	-.35452055	-1.18173517	.01890020
30°	2x2	.07214692	.11997099	-.23748336	-.79161120	.01583869
	4x4	.07231345	.12082452	-.28426942	-.94756474	.01594222
	6x6	.07235920	.12097608	-.30599823	-1.01999409	.01594524
20°	2x2	.09437740	.12626436	-.21591319	-.71971064	.02066752
	4x4	.09483229	.12691240	-.23807978	-.79359927	.02076054
	6x6	.09493990	.12708489	-.24226165	-.80753884	.02079158
0°	2x2	.12621434	.12971291	-.13969961	-.52256713	.02534679
	4x4	.12691237	.13002447	-.1586765	-.52892251	.02549046
	6x6	.12708489	.13008489	-.1590862	-.53028730	.02553000

TABLE 10

FREE CORNER DEFLECTIONS, MAXIMUM MOMENTS AND STRAIN ENERGY OF A SKEW CANTILEVER PLATE

Aspect Ratio : 1.5 Poisson's Ratio : 0.3

Skew angle	Grid	Deflections / $\frac{D}{qL^4}$ w_3	Deflections / $\frac{D}{qL^4}$ w_{14}	$M_{x\,max} / qL^2$	$M_{y\,max} / qL^2$	Strain Energy $\frac{D}{q^2 L^6}$
60°	2x2	.00699896	.05841601	-.13678911	-.45596370	.00304318
	4x4	.0071088	.06823570	-.20970604	-.69902014	.00350433
	6x6					
50°	2x2	.01920651	.07931494	-.18141146	-.60470486	.00730343
	4x4	.01928420	.08594939	-.25307405	-.84358016	.0077
	6x6					
40°	2x2	.03838055	.10002398	-.20489280	-.68129759	.01387291
	4x4	.03855409	.10389579	-.26671994	-.88906648	.01420733
	6x6					
30°	2x2	.06217707	.1175967	-.20993175	-.69977249	.02195078
	4x4	.06252451	.11942464	-.25559741	-.85199135	.02217421
	6x6					
20°	2x2	.08702102	.12887205	-.20033838	-.66779459	.02987679
	4x4	.08753900	.12957319	-.22518356	-.75061186	.03003996
	6x6					
0°	2x2	.12461165	.12461165	-.15474554	-.51581848	.03782234
	4x4	.12503392	.12503392	-.15571130	-.51843115	.03797929
	6x6					

TABLE 11

FREE CORNER DEFLECTIONS, MAXIMUM MOMENTS AND STRAIN ENERGY OF A SKEW CANTILEVER PLATE

Aspect Ratio : 2 Poisson's Ratio : 0.3

Skew angle	Grid	Deflections / $\frac{D}{qL^4}$ w_3	w_{14}	M_x / qL^2 max	M_y / qL^2 max	Strain Energy $\frac{D}{q^2 L^6}$
60°	2x2	.006065	.054022	-.11334751	-.37782503	.003252
	4x4	.006158	.065890	-.17890992	-.59636641	.003798
	6x6					
50°	2x2	.016803	.075156	-.15613599	-.52045331	.008461
	4x4	.017044	.084175	-.22074915	-.73583050	.009030
	6x6					
40°	2x2	.034574	.097237	-.18417348	-.61391158	.017016
	4x4	.035040	.107893	-.24094032	-.80313490	.017481
	6x6					
30°	2x2	.057946	.116785	-.19563740	-.65212468	.027966
	4x4	.058615	.119105	-.23763103	-.79210343	.028288
	6x6					
20°	2x2	.083529	.128503	-.19130922	-.63769739	.039005
	4x4	.084199	.129604	-.21545979	-.71819930	.039227
	6x6					
0°	2x2	.123784	.123784	-.15316106	-.51053686	.050297
	4x4	.124061	.124068	-.15435544	-.51451814	.050458
	6x6					

TABLE 12

COMPARISON OF CENTRAL DEFLECTIONS FOR THE SKEW CLAMPED PLATE

Aspect Ratio = 1 $\mu = 0.3$

Skew angle	Present* $w_c / \frac{D}{qL^4}$	Monforton $w_c / \frac{D}{qL^4}$	Iyengar et. al. $w_c / \frac{D}{qL^4}$
75°	.0000070 .00000908	.00000847	.00000902
60°	.0000965 .0001073	.0001073	.0001061
45°	.0003709 .0003762	.0003753	.0003761
30°	.0007685 .0007683	.0007683	.0007687
15°	.00111113 .0011211	.001123	.001123
0°	.0012612 .0012640		

* Results for 2 x 2 and 4 x 4 grids respectively

TABLE 13

CENTRAL DEFLECTIONS, CENTRAL MOMENTS, MAXIMUM DEFLECTIONS, MAXIMUM MOMENTS AND STRAIN ENERGY FOR THE SKEW CLAMPED PLATES

Aspect Ratio : 1 Poisson's Ratio : 0. 3

		Skew angle					
	Grid	0°	15°	30°	45°	60°	75°
Deflection/ $\frac{qL^4}{D}$	2 x 2	.00126130	.00111128	.00076847	.00037091	.00009655	.00000694
	4 x 4	.00126432	.00112114	.00076783	.00037624	.00010739	.00000909
		(.00126432)	(.00112114)	(.00076783)	(.00037624)	(.00010739)	(.00000909)
Strain/ $\frac{D}{}$ Energy q^2L^6	2 x 2	.00019244	.00016137	.00009505	.00003528	.00000624	.00000023
	4 x 4	.00019436	.00016474	.00009780	.00003678	.00000673	.00000026
M_x/qL^2	2 x 2	.02838855	.02464027	.01888231	.01104799	.00381464	.00058115
	4 x 4	.02218570	.02035712	.01611193	.01075651	.00543134	.00127059
		$(-.05118702)^{3}$	$(-.04537095)^{3}$	$(-.03837479)^{4}$	$(-.02832507)^{4}$	(-.01439423)	$(-.00336193)^{4}$
M_y/qL^2	2 x 2	.02838855	.02376507	.01678070	.01056836	.00518968	.00137443
	4 x 4	.02218570	.021066021	.01832118	.01382506	.00803326	.00260767
		$(-.05118702)^{15}$	$(-.04537095)^{15}$	$(-.03837479)^{10}$		$(-.01439425)^{16}$	$(-.00348441)^{16}$
M_{xy}/qL^2	2 x 2	0.00000000	-.00163314	-.00182005	-.00023981	.00039636	.00010628
	4 x 4	.00000000	.00132286	.00191327	.00153478	.00075115	.00017915

The values inside the brackets show the maximum values and the superscripts denote their locations in the 4 x 4 grid as shown in Fig. 1.

TABLE 14

CENTRAL DEFLECTIONS, CENTRAL MOMENTS, MAXIMUM DEFLECTIONS, MAXIMUM MOMENTS AND STRAIN ENERGY FOR THE SKEW CANTILEVER PLATES

Aspect Ratio : 1.5 Poisson's Ratio : 0.3

		Skew angle					
	Grid	0°	15°	30°	45°	60°	75°
Deflection/ $\frac{qL^4}{D}$	2x2	.00219432	.00191200	.00127652	.00059603	.00015268	.00001090
	4x4	.00219329	.00193289	.00129498	.00060664	.00016066	.00001262
			(.00193289)	(.00129498)	(.00060664)	(.00016066)	(.00001262)
Strain Energy / $\frac{D}{q^2L^6}$	2x2	.00051629	.00042595	.00023928	.00008404	.00001409	.00000049
	4x4	.0005258	.00044440	.00026111	.00009625	.00001710	.00000064
M_x / qL^2	2x2	.02254047	.02151918	.01964463	.01376479	.00635008	.00151931
	4x4	.01917648	.01720338	.01353125	.00880754	.00427245	.00116965
		$(-.05331475)^3$	$(-.04740944)^3$	$(-.04008695)^4$	$(-.02737711)^4$	$(-.01276919)^4$	$(-.00167277)^{10}$
M_y / qL^2	2x2	.05049629	.04496105	.03513086	.02473477	.01426526	.00449568
	4x4	.0362173	.03426305	.02929403	.02113923	.01146115	.00344567
		$(-.07558345)^{15}$	$(-.06867044)^{15}$	$(-.05586072)^{16}$	$(-.03974832)^{16}$	$(-.02027404)^{16}$	$(-.00557591)^{16}$
M_{xy} / qL^2	2x2	.00000000	.00144564	.00110807	.0005824	.00097557	.00024220
	4x4	.00000000		.00136460	.00029742	.00024593	.00003065

The values inside the brackets show the maximum values and the superscripts denote their locations in the 4x4 grid as shown in Fig. 1.

TABLE 15

CENTRAL DEFLECTIONS, CENTRAL MOMENTS, MAXIMUM DEFLECTIONS, MAXIMUM MOMENTS AND STRAIN ENERGY FOR THE SKEW CLAMPED PLATES

Aspect Ratio : 2 Poisson's Ratio : 0. 3

		Skew angle					
	Grid	0°	15°	30°	45°	60°	75°
Deflection/ $\frac{qL^4}{D}$	2 x 2	.00257546	.00225004	.00149200	.00068913	.00017755	.00001290
	4 x 4	.00253153			.00065355	.00016520	.00001213
		(.00253153)			(.00065355)	(.00016520)	(.00001213)
Strain/ Energy $\frac{D}{q^2L^6}$	2 x 2	.0083397	.00068444	.00037956	.00013217	.00002253	.00000081
	4 x 4	.00087130			.00015572	.00002745	.00000103
M_x/qL^2	2 x 2	.0904234	.01958423	.01876802	.01331559	.00622462	.00148351
	4 x 4	.01497556			.00680612	.00347355	.00103221
		(-.04708589)^{3}			(-.02362066)^{4}	(-.01104221)^{4}	(-.00277300)^{4}
M_y/qL^2	2 x 2	.06414782	.05852976	.04643786	.0314612	.01638430	.00438367
	4 x 4	.04105325			.02220340	.01191375	.00347854
		(-.08267541)^{15}			(-.04153625)^{16}	(-.02067590)^{16}	(.00551660)^{16}
M_{xy}/qL^2	2 x 2	.00000000	.00053706	.000124	.00001	.00021	.00004
	4 x 4	.00000000			.000029	.000019	.00004452

The values inside the brackets show the maximum values and the superscripts denote their locations in the 4x4 grid as shown in Fig. 1.

TABLE 16

COMPARISON OF CENTRAL DEFLECTIONS FOR THE SKEW SIMPLY SUPPORTED PLATE

Aspect Ratio = 1 $\mu = 0.3$

Skew angle	Present $w_c / \frac{D}{qL^4}$	Ref. (18) Morley $w_c / \frac{D}{qL^4}$
60°	.000278 .000304	.000384
50°	.000761 .00081894	.000912
40°	.00152 .001595	.00167
30°	.00244 .002492	.00254
0°	.004061	.00406

TABLE 17

CENTRAL DEFLECTIONS, MOMENTS AND STRAIN ENERGY FOR THE SKEW SIMPLY SUPPORTED PLATE

Aspect Ratio : 1   Poisson's Ratio : 0.3

		Skew angle δ					
	Grid	0°	15°	30°	40°	50°	60°
Deflection / $\frac{qL^4}{D}$	2 x 2	.00406094	.00328368	.00243746	.00152265	.00076100	.00027753
	4 x 4	.00406235	.00330765	.00249204	.00159492	.00081894	.00030436
Strain Energy / $\frac{D}{q^2L^6}$	2 x 2	.00085100	.000633115	.00042270	.00022620	.00009133	.00002495
	4 x 4	.00085124	.00063986	.00043426	.00023819	.00009894	.00002776
M_x / qL^2	2 x 2	.04702609	.04364310	.03807196	.02951925	.01935335	.011001186
	4 x 4	.04782595	.04124866	.03383845	.02488204	.01548496	.00733624
M_y / qL^2	2 x 2	.04702609	.04754005	.04538925	.03941246	.02990275	.01885210
	4 x 4	.04782595	.04350213	.03841466	.03183735	.02402261	.01542832
M_{xy} / qL^2	2 x 2	0.00000000	.00535359	.00633696	.00589513	.00442600	.00255196
	4 x 4	0.00000000	.00314301	.00404822	.00424937	.00366263	.00237416

TABLE 18

CENTRAL DEFLECTIONS, MOMENTS AND STRAIN ENERGY FOR THE SKEW SIMPLY SUPPORTED PLATE

Aspect Ratio : 1.5 Poisson's Ratio : 0.3

		Skew angle β					
	Grid	0°	20°	30°	40°	50°	60°
Deflection / $\frac{qL^4}{D}$	2 x 2	.00771843	.00615166	.00452348	.00281125	.00140162	.00051016
	4 x 4	.00772300	.00623688	.00465431	.00295265	.00151932	.00058365
Strain Energy / $\frac{D}{q^2 L^6}$	2 x 2	.00244971	.00179769	.00118980	.00063452	.00025662	.00007040
	4 x 4	.00245568	.00183473	.00123617	.00067757	.00049015	.00007810
M_x / qL^2	2 x 2	.04896559	.04252430	.03557537	.02693816	.01741991	.00889142
	4 x 4	.04931759	.04230943	.03460617	.02634412	.01864231	.01157186
M_y / qL^2	2 x 2	.08978400	.07992425	.06786083	.05211084	.03502653	.01968663
	4 x 4	.08160543	.07330164	.06380156	.05174159	.03793624	.02373114
M_{xy} / qL^2	2 x 2	0.00000000	.00144	.001663841	.00123791	.00053902	.00005901
	4 x 4	.00000000	.00050932	.00083152	.000989220	.00087527	.00062853

TABLE 19

CENTRAL DEFLECTIONS, MOMENTS ANS STRAIN ENERGY FOR THE SKEW SIMPLY SUPPORTED PLATE.

Aspect Ratio : 2.0 Poisson's Ratio : 0.3

		Skew angle β					
	Grid	0°	20°	30°	40°	50°	60°
Deflection / $\frac{qL^4}{D}$	2x2	.01011607	.00802417	.00596874	.00370915	.00188499	.00070355
	4x4	.01012863	.00808759	.00597323	.00375909	.00193039	.00074458
Strain Energy / $\frac{D}{q^2L^6}$	2x2	.00438572	.00318609	.00210128	.00172658	.00046258	.00012966
	4x4	.00440317	.00326588	.00218720	.00118615	.00049131	.00013920
M_x / qL^2	2x2	.04774496	.04177325	.03537896	.02751322	.01874690	.01042582
	4x4	.04576953	.03844480	.03109630	.02345662	.01649393	.01038168
M_y / qL^2	2x2	.11590087	.10138131	.08535804	.06590047	.04551175	.02687144
	4x4	.10250393	.09063856	.07802742	.0625280	.0454936	.02857807
M_{xy} / qL^2	2x2	.00000000	.00492669	.00537167	.00430446	.00250340	.00099217
	4x4	.00000000	.00043166	.00039068	.00043662	.00060	.00062840

TABLE 20

COMPARISON OF DEFLECTIONS FOR A SKEW PLATE SIMPLY SUPPORTED ON TWO SIDES AND FREE ON THE OTHER TWO

Aspect Ratio : 1 Poisson's Ratio : 0.2

		Deflection $w_c^* / \frac{D}{qL^4}$			Deflection $w_e^* / \frac{D}{qL^4}$		
		30°	45°	90°	30°	45°	90°
Present	2x2	.01603	.0669	.01309	.0351	.08279	.01502
	4x4	.01819	.0729	.01309	.0433	.09360	.01501
Jensen		.0185	.0707		.0393	.0841	
Timoshenko				.01309			.01509

w_c^* - central deflection

w_e^* - deflection at the centre of the free side

TRANSFORMATION MATRIX [T]

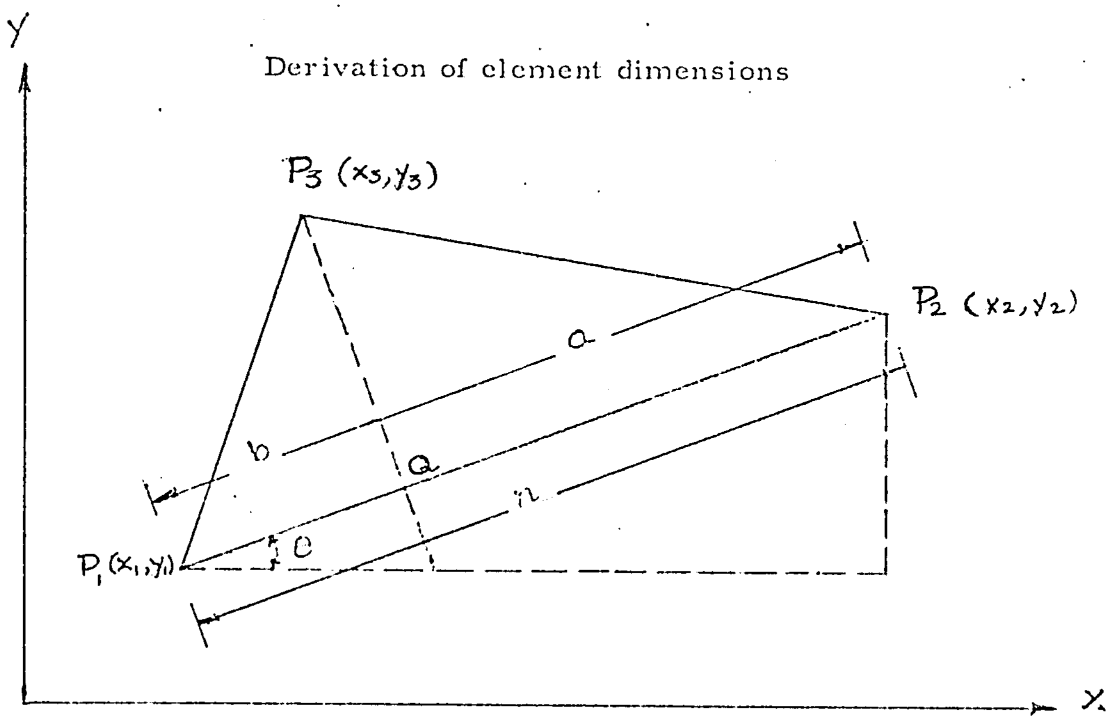
1	$-b$	0	b^2	0	0	$-b^3$	0	0	0	b^4	0	0	0	0	$-b^5$	0	0	0	0
0	1	0	$-2b$	0	0	$3b^2$	0	0	0	$-4b^3$	0	0	0	0	$5b^4$	0	0	0	0
0	0	1	0	$-b$	0	0	b^2	0	0	0	$-b^3$	0	0	0	0	0	0	0	0
0	0	0	2	0	0	$-6b$	0	0	0	$12b^2$	0	0	0	0	$-20b^3$	0	0	0	0
0	0	0	0	1	0	0	$-2b$	0	0	0	$3b^2$	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	$-2b$	0	0	0	$2b^2$	0	0	0	$-2b^3$	0	0	0
1	a	0	a^2	0	0	a^3	0	0	0	a^4	0	0	0	0	a^5	0	0	0	0
0	1	0	$2a$	0	0	$3a^2$	0	0	0	$4a^3$	0	0	0	0	$5a^4$	0	0	0	0
0	0	1	0	a	0	0	a^2	0	0	0	a^3	0	0	0	0	0	0	0	0
0	0	0	2	0	0	$6a$	0	0	0	$12a^2$	0	0	0	0	$20a^3$	0	0	0	0
0	0	0	0	1	0	0	$2a$	0	0	0	$3a^2$	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	$2a$	0	0	0	$2a^2$	0	0	0	$2a^3$	0	0	0
1	0	c	0	0	c^2	0	0	0	c^3	0	0	0	0	c^4	0	0	0	0	c^5
0	1	0	0	c	0	0	0	c^2	0	0	0	0	c^3	0	0	0	0	c^4	0
0	0	1	0	0	$2c$	0	0	0	$3c^2$	0	0	0	0	$4c^3$	0	0	0	0	$5c^4$
0	0	0	2	0	0	0	$2c$	0	0	0	0	$2c^2$	0	0	0	0	$2c^3$	0	0
0	0	0	0	1	0	0	0	$2c$	0	0	0	0	$3c^2$	0	0	0	0	$4c^3$	0
0	0	0	0	0	2	0	0	0	$6c$	0	0	0	0	$12c^2$	0	0	0	0	$20c^3$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$5a^4c$	$3a^2c^3-2a^4c$	$-2ac^4+3a^3c^2$	$c^5-4a^2c^3$	$5ac^4$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$5b^4c$	$3b^2c^3-2b^4c$	$2bc^4-3b^3c^2$	$c^5-4b^2c^3$	$-5bc^4$

ROTATION MATRIX [R]

$$[R] = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & R_1 \end{bmatrix}$$

$$[R_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos^2\theta & 2\sin\theta\cos\theta & \sin^2\theta \\ 0 & 0 & 0 & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta & \sin\theta\cos\theta \\ 0 & 0 & 0 & \sin^2\theta & -2\sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

Derivation of element dimensions



$$b = P_1O = P_1R \ \mathrm{Cos}\ \theta \ (P_1Q + QR) \ \mathrm{Cos}\ \theta$$

$$= \left[(x_3 - x_1) + P_3O \ \tan\theta \right] \mathrm{Cos}\ \theta$$

$$= \left[(x_3 - x_1) \ \mathrm{Cos}\ \theta + P_3O \ \mathrm{Sin}\ \theta \right.$$

$$= (x_3 - x_1)(x_2 - x_1)/_r + (y_3 - y_1)(y_2 - y_1)/_r$$

$$= \frac{1}{r}\left[(x_3 - x_1)(x_2 - x_1) + (y_3 - y_1)(y_2 - y_1) \right]$$

$$a = r - b = r - \left[(x_3 - x_1)(x_2 - x_1) + (y_3 - y_1)(y_2 - y_1) \right]/_r$$

$$= \frac{r^2 - \left[(x_3 - x_1)(x_2 - x_1) + (y_3 - y_1)(y_2 - y_1) \right]}{r}$$

$$= \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 - (x_3 - x_1)(x_2 - x_1) - (y_3 - y_1)(y_2 - y_1)}{r}$$

$$= \frac{(x_2 - x_1)[(x_2 - x_1) - (x_3 - x_1)] + (y_2 - y_1)[(y_2 - y_1) - (y_3 - y_1)]}{r}$$

$$a = [(x_2 - x_1)(x_2 - x_3) + (y_2 - y_1)(y_2 - y_3)]/_r$$

$$c = P_3R - OR$$

$$= \frac{y_3 - y_1}{\text{Cos } \theta} - b \tan \theta$$

$$= \frac{(y_3 - y_1)}{(x_2 - x_1)} r - \frac{(y_2 - y_1)}{(x_2 - x_1)} [\{(x_3 - x_1)(x_2 - x_1)\} + \{(y_3 - y_1)(y_2 - y_1)\}]/_r$$

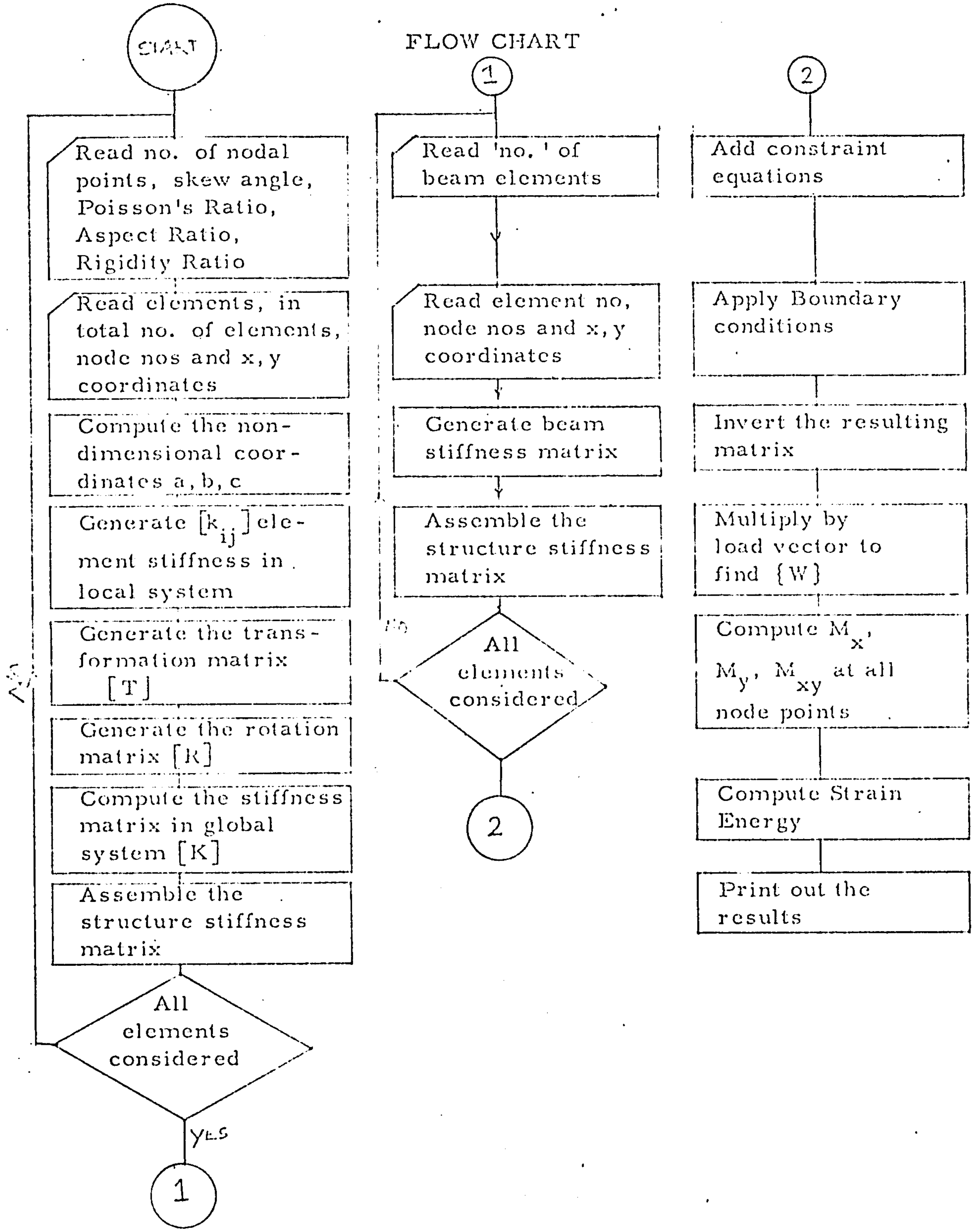
$$= \frac{(y_3 - y_1) r^2 - (y_2 - y_1)[(x_3 - x_1)(x_2 - x_1) + (y_3 - y_1)(y_2 - y_1)]}{(x_2 - x_1) r}$$

$$= \frac{(y_3 - y_1)[(x_2 - x_1)^2 + (y_2 - y_1)^2] - (y_2 - y_1)[(x_3 - x_1)(x_2 - x_1) + (y_3 - y_1)(y_2 - y_1)]}{(x_2 - x_1) r}$$

$$= \frac{(y_3 - y_1)(x_2 - x_1)[(x_2 - x_1)] + (y_3 - y_1)(y_2 - y_1)^2 - (y_3 - y_1)(y_2 - y_1)^2 - (y_2 - y_1)(x_3 - x_1)(x_2 - x_1)}{(x_2 - x_1) r}$$

$$c = [\{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\}]/_r$$

FLOW CHART


START
Read no. of nodal points, skew angle, Poisson's Ratio, Aspect Ratio, Rigidity Ratio
Read elements, in total no. of elements, node nos and x, y coordinates
Compute the non-dimensional coordinates a, b, c
Generate [k_ij] element stiffness in local system
Generate the transformation matrix [T]
Generate the rotation matrix [R]
Compute the stiffness matrix in global system [K]
Assemble the structure stiffness matrix
All elements considered
YES
1
1
Read 'no.' of beam elements
Read element no, node nos and x, y coordinates
Generate beam stiffness matrix
Assemble the structure stiffness matrix
All elements considered
2
2
Add constraint equations
Apply Boundary conditions
Invert the resulting matrix
Multiply by load vector to find {W}
Compute M_x, M_y, M_xy at all node points
Compute Strain Energy
Print out the results

```
C     -------------------------------------------------------------------
C
C
C     A REFINED FINITE ELEMENT ANALYSIS OF STIFFENED SKEW PLATE-BY M.S.DESIKAN
C     -------------------------------------------------------------------
C
C
C     CASE-TWO SIDES SIMPLY SUPPORTED AND TWO SIDES STIFFENED WITH EDGE BEAMS
C     GRID-4 BY 4
C     AR-ASPECT RATIO
C     PR-POISSON RATIO
C     FRR-FLEXURAL RIGIDITY RATIO
C     ALFA-SKEW ANGLE
C     NUMNP--NUMBER OF NODAL POINTS
C     MLAST-NUMBER OF ELEMENTS
C     NBL-NUMBER OF BEAM ELEMENTS
C     NBP-NUMBER OF BOUNDARY POINTS ON S.S.EDGES
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION M(20),N(20),S(20,20),ES(18,18),T(20,20),RM(18,18)
      DIMENSION RMT(18,18),TT(18,20),W(18,20),WS (18,20),WST (18,18)
      DIMENSION SS(170,170),V(170),RS(170),KODE(25,6),PT(20),PL(18)
      DIMENSION BSM(12,12)
      COMMON AR,FRR
C
C     CLEAR ALL ARRAYS
C
      DO 87 I=1,170
87    V(I)=0.0
      DO 222 I=1,170
      DO 222 J=1,170
222   SS(I,J)=0.0
      DO 5000 I=1,18
      DO 5000 J=1,20
      TT(I,J)=0.0
      W(I,J)=0.0
5000  WS(I,J)=0.0
      DO 5001 I=1,18
```

```
FORTRAN IV G LEVEL  18                 MAIN               DATE = 70245          22/10/15          PA

 0018                DO 5001 J=1,18
 0019                ES(I,J)=0.0
 0020                RMT(I,J)=0.0
 0021         5001   WST(I,J)=0.0
 0022                DO 5003 I=1,20
 0023                DO 5003 J=1,20
 0024         5003   S(I,J)=0.0
 0025                DATA M/0,1,0,2,1,0,3,2,1,0,4,3,2,1,0,5,3,2,1,0/,N/0,0,1,0,1,2,0,1,
                    &2,3,0,1,2,3,4,0,2,3,4,5/
              C
              C      READ AND WRITE INPUT DATA
              C
 0026                READ(1,939)NUMNP,ALFA,PR,AR,FRR
 0027         939    FORMAT(I10,4F10.5)
 0028                WRITE(3,7777)ALFA,AR,FRR,PR
 0029         7777   FORMAT(1H0,'SKEW ANGLE=',F5.2 ,10X,'ASPECT RATIO='F6.4,10X,'RIGIDI
                    *TY RATIO=',F5.2,'      POISSON RATIO=',F5.2)
 0030         126    FORMAT(I3)
 0031                PI=3.14159265
 0032                ALFA=(PI*ALFA)/180.0
 0033                NEQ=NUMNP*6
 0034         555    READ(1,100)NEL,NLAST,IN,JN,KN,X1,Y1,X2,Y2,X3,Y3
 0035                X1=X1*AR
 0036                X2=X2*AR
 0037                X3=X3*AR
 0038                WRITE(3,100)NEL,NLAST,IN,JN-KN,X1,Y1,X2,Y2,X3,Y3
 0039         100    FORMAT(5I5,6F5.3)
              C
              C      CALCULATE THE NON-DIMENSIONAL PARAMETERS A , B , C
              C
 0040                X1=X1+Y1*DCOS(ALFA)
 0041                Y1=Y1*DSIN(ALFA)
 0042                X2=X2+Y2*DCOS(ALFA)
 0043                Y2=Y2*DSIN(ALFA)
 0044                X3=X3+Y3*DCOS(ALFA)
 0045                Y3=Y3*DSIN(ALFA)
 0046         31     XU=((X2-X1)**2)
 0047                XV=((Y2-Y1)**2)
```

```
FORTRAN IV G LEVEL  18                MAIN              DATE = 70245           22/10/15            D

                XT=XU+XV
                RAD=DSQRT(XT)
                COST=(X2-X1)/RAD
                SINT=(Y2-Y1)/RAD
                A=((X2-X3)*(X2-X1)+(Y2-Y3)*(Y2-Y1))/RAD
                B=((X3-X1)*(X2-X1)+(Y3-Y1)*(Y2-Y1))/RAD
                C=((X2-X1)*(Y3-Y1)-(X3-X1)*(Y2-Y1))/RAD
          C
          C     GENERATE ELEMENT STIFFNESS MATRIX IN LOCAL SYSTEM
          C
          32    DO 33 I=1,20
                DO 33 J=1,20
                PP=M(I)*M(J)*(M(I)-1)*(M(J)-1)
                IF(PP)41,42,41
          41    ICOM=M(I)+M(J)-4
                IF(ICOM) 42,151,151
          151   JCOM=N(I)+N(J)
                IF(JCOM) 42,152,152
          152   CALL FACTOR(A,B,C,ICOM,JCOM,FUNC1)
                P=PP*FUNC1
                GO TO 300
          42    P=0.
          300   QQ=N(I)*N(J)*(N(I)-1)*(N(J)-1)
                IF(QQ)43,44,43
          43    ITOM=M(I)+M(J)
                IF(ITOM)44,153,153
          153   JTOM=N(I)+N(J)-4
                IF(JTOM)44,154,154
          154   CALL FACTOR(A,B,C,ITOM,JTOM,FUNC2)
                Q=QQ*FUNC2
                GO TO301
          44    Q=0.
          301   PR1=2.*(1.-PR)
                PR2=M(I)*M(J)*N(I)*N(J)
                R1=PR1*PR2
                PR3=M(I)*N(J)*(M(I)-1)*(N(J)-1)
                R2=PR*PR3
                PR4=M(J)*N(I)*(M(J)-1)*(N(I)-1)
```

```
0083              R3=PR*PR4
0084              BRAC=R1+R2+R3
0085              IRAM=M(I)+M(J)-2
0086              IF(IRAM)157,158,158
0087        158   JRAM=N(I)+N(J)-2
0088              IF(JRAM)157,159,159
0089        159   CALL FACTOR(A,B,C,IRAM,JRAM,FUNC3)
0090              R=BRAC*FUNC3
0091              GO TO 33
0092        157   R=0.
0093        33    S(I,J)=P+Q+R
0094              DO 786 I=1,20
0095              IVEC=M(I)
0096              JVEC=N(I)
0097              CALL FACTOR (A,B,C,IVEC,JVEC,FUNC4)
0098        786   PT(I)=FUNC4
            C
            C     GENERATE TRANSFORMATION MATRIX
            C
0099              DO 35 I=1,20
0100              DO 35 J=1,20
0101        35    T(I,J)=0.
0102              I=1
0103              X=-B
0104        36    T(I,1)=1.
0105              T(I,2)=X
0106              T(I,4)=X**2
0107              T(I,7)=X**3
0108              T(I,11)=X**4
0109              T(I,16)=X**5
0110              I=I+1
0111              T(I,2)=1.
0112              T(I,4)=2.*X
0113              T(I,7)=3. *(X**2)
0114              T(I,11)=4.*(X**3)
0115              T(I,16)=5.*(X**4)
0116              I=I+1
0117              T(I,3)=1.
```

```
      T(I,5)=X
      T(I,8)=X**2
      T(I,12)=X**3
      I=I+1
      T(I,4)=2.
      T(I,7)=6.*X
      T(I,11)=12.*(X**2)
      T(I,16)=20.*(X**3)
      I=I+1
      T(I,5)=1.
      T(I,8)=2.*X
      T(I,12)=3.*(X**2)
      I=I+1
      T(I,6)=2.
      T(I,9)= (2.*X)
      T(I,13)=2.*(X**2)
      T(I,17)= (2.*X**3)
      I=I+1
      X=A
      IF(I.EQ.7) GO TO 36
      T(13,1)=1.0
      T(13,3)=C
      T(13,6)=C**2
      T(13,10)=C**3
      T(13,15)=C**4
      T(13,20)=C**5
      T(14,2)=1.
      T(14,5)=C
      T(14,9)=C**2
      T(14,14)=C**3
      T(14,19)=C**4
      T(15,3)=1.
      T(15,6)=2.*C
      T(15,10)=3.*C**2
      T(15,15)=4.*C**3
      T(15,20)=5.*C**4
      T(16,4)=2.
      T(16,8)=2.*C
```

```
 0156               T(16,13)=2.*C**2
 0157               T(16,18)=2.*C**3
 0158               T(17,5)=1.
 0159               T(17,9)=2.*C
 0160               T(17,14)=3.*C**2
 0161               T(17,19)=4.*C**3
 0162               T(18,6)=2.
 0163               T(18,10)=6.*C
 0164               T(18,15)=12.*C**2
 0165               T(18,20)=20.*C**3
 0166               T(19,16)=5.*(A**4)*C
 0167               T(19,17)=3.*(A**2)*(C**3)-2.*(A**4)*C
 0168               T(19,18)=-2.*A*(C**4)+3.*(A**3)*(C**2)
 0169               T(19,19)=(C**5)-4.*(A**2)*(C**3)
 0170               T(19,20)=5.*A*(C**4)
 0171               T(20,16)=5.*(B**4)*C
 0172               T(20,17)=3.*(B**2)*(C**3)-2.*(B**4)*C
 0173               T(20,18)=(2.*B*C**4)-3.*(B**3)*C**2
 0174               T(20,19)=C**5-4.*(B**2)*(C**3)
 0175               T(20,20)=-5.*B*C**4
            C
            C       GENERATE ROTATION MATRIX
            C
 0176               DO142 I=1,18
 0177               DO142 J=1,18
 0178         142   RM(I,J)=0.
 0179               RM(1,1)=1.
 0180               RM(2,2)=COST
 0181               RM(2,3)=SINT
 0182               RM(3,2)=-SINT
 0183               RM(3,3)=COST
 0184               RM(4,4)=COST*COST
 0185               RM(4,5)=2*SINT*COST
 0186               RM(4,6)=SINT*SINT
 0187               RM(5,4)=-SINT*COST
 0188               RM(5,5)=COST*COST-SINT*SINT
 0189               RM(5,6)=SINT*COST
 0190               RM(6,4)=SINT*SINT
```

```
          RM(6,5)=-2.*SINT*COST
          RM(6,6)=COST*COST
          DO 45 I=1,6
          II=I+6
          IJ=I+12
          DO 45 J=1,6
          JJ=J+6
          JK=J+12
          RM(II,JJ)=RM(I,J)
   45     RM(IJ,JK)=RM(I,J)
          CALL MATINV (T,20)
          DO 47 I=1,18
          DO 47 J=1,18
   47     RMT(I,J)=RM(J,I)
   C
   C      COMPUTE ELEMENT STIFFNESS MATRIX IN GLOBAL SYSTEM  AND ASSEMBLE
   C
          DO 48 I=1,20
          DO 48 J=1,18
   48     TT(J,I)= T(I,J)
          CALL MATMUL (RMT,TT,W,18,18,20)
          CALL MATMUL(W,PT,PL,18,20,1)
          CALL MATMUL(W,S,WS,18,20,20)
          CALL MATMUL(WS,T,WST,18,20,18)
          CALL MATMUL (WST,RM,ES,18,18,18)
          CALL BUILD(IN,JN,KN,SS,170,170,ES,PL,V)
          IF(NEL-NLAST)70,71,71
   70     GO TO 555
   C
   C      READ ELEMENT COORDINATES AND NODES FOR BEAM ELEMENTS
   C
   71     READ(1,4824)NBEL
 4824     FORMAT(I3)
          DO 40 I=1,NBEL
          READ(1,10)NE,NC,IT,JT,T1,T2
   10     FORMAT(4I3,2F10.6)
   C
   C      GENERATE ELEMENT STIFFNESS MATRICES FOR BEAM ELEMENTS
```

```
FORTRAN IV G LEVEL  18                    MAIN                DATE = 70245          22/10/15            PA
                C
 0221                 CALL BEAM(NC,T1,T2,BSM,12,12)
                C
                C     ASSEMBLY IN GLOBAL SYSTEM
                C
 0222                 CALL SETUP(IT,JT,SS,170,170,BSM)
 0223           40    CONTINUE
                C
                C     ADD CONSTRAINT EQUATIONS
                C
 0224                 SINA=DSIN(ALFA)
 0225                 COSA=DCOS(ALFA)
 0226                 DO 55 KK=1,10
 0227                 READ(1,126)NBP
 0228                 K=149+2*KK
 0229                 IA=NBP*6-6+2
 0230                 IB=NBP*6-6+3
 0231                 IC=NBP*6-6+4
 0232                 ID=NBP*6-6+5
 0233                 IE=NBP*6-6+6
 0234                 SS(K,IA)=COSA
 0235                 SS(K,IB)=SINA
 0236                 K2=K+1
 0237                 SS(K2,IC)=COSA*COSA
 0238                 SS(K2,ID)=2.0*SINA*COSA
 0239                 SS(K2,IE)=SINA*SINA
 0240           55    CONTINUE
 0241                 DO 60 K=151,170
 0242                 DO 60 J=1,170
 0243           60    SS(J,K)=SS(K,J)
 0244                 WRITE(3,792)
 0245          792    FORMAT(1H1,45X,'LOAD VECTOR')
 0246                 WRITE(3,794)(V(I),I=1,170)
 0247          794    FORMAT(1H0,30X,6F12.6)
                C
                C     APPLY BOUNDARY CONDITIONS
                C
 0248                 DO 500 I=1,NUMNP
```

```
          READ(1,127)NP,(KODE(NP,K),K=1,6)
          WRITE(3,127)NP,(KODE(NP,K),K=1,6)
   127    FORMAT(7I10)
          DO 500 J=1,6
          NODE=KODE(I,J)
          IF (NODE)500,250,500
   250    NN=(NP-1)*6+J
          DO 600 L=1,170
          SS (L,NN)=0.0
          SS (NN,L)=0.0
   600    V(NN)=0.0
          SS (NN,NN)=1.0
   500    CONTINUE
          CALL MATINV(SS ,170)
C
C         CALCULATE DEFLECTIONS ,MOMENTS AND STRAIN ENERGY
C
          CALL MATMUL(SS,V,RS,170,170,1)
          WRITE(3,5200)
   5200   FORMAT(1H1,10X,'NODAL POINT',20X,'DEFLECTION',15X,'MX',20X,'MY',20
         *X,'MXY')
          DO 5201 I=1,NUMNP
          II=I*6-2
          IJ=I*6-1
          IK=I*6
          IL=I*6-5
          EMX=RS(II)+(PR*RS(IK))
          EMY=RS(IK)+(PR*RS(II))
          EMXY=(1.-PR)*RS(IJ)
   5201   WRITE(3,5203)I, RS(IL),EMX,EMY,EMXY
          CALL MATMUL(V,RS,SE,1,170,1)
          SE=SE*0.5
   5206   FORMAT(////,10X,'STRAIN ENERGY=',F12.8)
          WRITE(3,5206)SE
   5203   FORMAT(18X,I3,18X,F12.8,10X,F12.8,10X,F12.8,11X,F12.8/)
          RETURN
          END
```

```
FORTRAN IV G LEVEL  18                MAIN                DATE = 70245          22/10/15          P

      C
      C      SUBROUTINE TO FIND THE FACTORS G(M,N)
      C
             SUBROUTINE FACTOR(A,B,C,M,N,FUNC)
             IMPLICIT REAL*8(A-H,O-Z)
             NN=N+1
             MN=M+1
             IFACT=1
             IF(M.EQ.0) GO TO 24
             DO 20 I=1,M
      20     IFACT=IFACT*I
      24     JFACT=1
             IF(N.EQ.0) GO TO 25
             DO 21 J=1,N
      21     JFACT=JFACT*J
      25     IDIV=M+N+2
             IF(IDIV)30,31,32
      32     IF3=1
             DO 22 I=1,IDIV
      22     IF3=IF3*I
             GO TO 52
      31     IF3=1
      52     AFACT=IFACT
             BFACT=JFACT
             AF3=IF3
             ANR=(C** NN)*((A** MN)-(-B)** MN)
             FUNC=(ANR*AFACT*BFACT)/AF3
             GO TO 50
      30     FUNC=0.
      50     RETURN
             END
```

```
      C
      C       SUBROUTINE FOR MATRIX INVERSION
      C
              SUBROUTINE MATINV(A,N)
              IMPLICIT REAL*8(A-H,O-Z)
              DIMENSION INDEX(170,2),A(N,N)
              DO 108 I=1,N
        108   INDEX(I,1)=0
              II=0
        109   AMAX=-1.
              DO 110 I=1,N
              IF(INDEX(I,1))110,111,110
        111   DO 112 J=1,N
              IF(INDEX(J,1))112,113,112
        113   TEMP=DABS(A(I,J))
              IF(TEMP-AMAX)112,112,114
        114   IROW=I
              ICOL=J
              AMAX=TEMP
        112   CONTINUE
        110    CONTINUE
              IF(AMAX)225,115,116
        116   INDEX(ICOL,1)=IROW
              IF(IROW-ICOL)119,118,119
        119   DO 120 J=1,N
              TEMP=A(IROW,J)
              A(IROW,J)=A(ICOL,J)
        120   A(ICOL,J)=TEMP
              II=II+1
              INDEX(II,2)=ICOL
        118   PIVOT=A(ICOL,ICOL)
              A(ICOL,ICOL)=1.
              PIVOT=1./PIVOT
              DO 121 J=1,N
              IF(DABS(A(ICOL,J)).LT.0.0000001) GO TO 200
              A(ICOL,J)=A(ICOL,J)*PIVOT
              GO TO 201
        200   A(ICOL,J)=0.
```

```
 201    CONTINUE
 121    CONTINUE
        DO 122 I=1,N
        IF(I-ICOL)123,122,123
 123    TEMP=A(I,ICOL)
        A(I,ICOL)=0.
        DO 124 J=1,N
        IF(DABS(A(ICOL,J)).LT.0.0000001) GO TO 124
        A(I,J)=A(I,J)-A(ICOL,J)*TEMP
 124    CONTINUE
 122    CONTINUE
        GO TO 100
 125    ICOL=INDEX(II,2)
        IROW=INDEX(ICOL,1)
        DO 126 I=1,N
        TEMP=A(I,IROW)
        A(I,IROW)=A(I,ICOL)
 126    A(I,ICOL)=TEMP
        II=II-1
 225    IF(II)125,127,125
 115    WRITE(6,1001)
 1001   FORMAT(1H0,2X,11H ZERO PIVOT)
 127    CONTINUE
        RETURN
        END
```

```
FORTRAN IV G LEVEL  18                 MAIN               DATE = 70245          22/10/15          P
      C
      C     SUBROUTINE FOR MATRIX MULTIPLICATION
      C
            SUBROUTINE MATMUL(A,B,C,M,N,K)
            IMPLICIT REAL*8(A-H,O-Z)
            DIMENSION A(M,N),B(N,K),C(M,K)
            DO 21 I=1,M
            DO 21 J=1,K
            C(I,J)=0.0
            DO 20 L=1,N
      20    C(I,J)=C(I,J)+A(I,L)*B(L,J)
      21    CONTINUE
            RETURN
            END
```

```
FORTRAN IV G LEVEL  18                MAIN                DATE = 70245          22/10/15

      C
      C      SUBROUTINE TO ASEMBLE PLATE ELEMETS IN GLOBAL SYSTEM
      C
             SUBROUTINE BUILD (I,J,K,A,M1,N1,D,PL,V)
             IMPLICIT REAL*8(A-H,O-Z)
      C      I,J,K-THE THREE NODES OF AN ELEMENT
      C      A-STIFFNESS MATRIX IN GLOBAL SYSTEM
      C      MI-NUMBER OF ROWS OF THE MATRIX
      C      NI-NUMBER OF COLUMNS OF THE MATRIX
      C      D-ELEMENT STIFFNESS MATRIX
      C      PL-LOAD VECTOR OF AN ELEMENT
      C      V-LOAD VECTOR OF THE STRUCTURE
      C
             DIMENSION A(170,170),D(18,18),PT(20),PL(18),V(170)
             DO 1 M=1,6
             I1=6*I-6+M
             J1=6*J-6+M
             K1=6*K-6+M
             V(I1)=V(I1)+PL(M)
             V(J1)=V(J1)+PL(M+6)
             V(K1)=V(K1)+PL(M+12)
             DO 1 N=1,6
             I2=6*I-6+N
             J2=6*J-6+N
             K2=6*K-6+N
             A(I1,I2)=A(I1,I2)+D(M,N)
             A(I1,J2)=A(I1,J2)+D(M,N+6)
             A(I1,K2)=A(I1,K2)+D(M,N+12)
             A(J1,I2)=A(J1,I2)+D(M+6,N)
             A(J1,J2)=A(J1,J2)+D(M+6,N+6)
             A(J1,K2)=A(J1,K2)+D(M+6,N+12)
             A(K1,I2)=A(K1,I2)+D(M+12,N)
             A(K1,J2)=A(K1,J2)+D(M+12,N+6)
           1 A(K1,K2)=A(K1,K2)+D(M+12,N+12)
             RETURN
             END
```

```
FORTRAN IV G LEVEL  18                MAIN                DATE = 70245          22/10/15
      C
      C      SUBROUTINE TO GENERATE ELEMENT STIFFNESS MATRICES FOR BEAM ELEMENTS
      C
             SUBROUTINE BEAM(NC,X1,X2,BSM,L1,L2)
             IMPLICIT REAL*8(A-H,O-Z)
             DIMENSION T(6,6),TT(6,6),BM(6,6),TP(6,6),SM(6,6),M(6)
             DIMENSION BSM(12,12)
             COMMON AR,FRR
             DO 70 I=1,6
             DO 70 J=1,6
             T(I,J)=0.0
             BM(I,J)=0.0
             TT(I,J)=0.0
             TP(I,J)=0.0
        70   SM(I,J)=0.0
             M(1)=0
             M(2)=1
             M(3)=2
             M(4)=3
             M(5)=4
             M(6)=5
             BL=DABS(X2-X1)*AR
             T(1,1)=1.0
             T(2,2)=1.0
             T(3,3)=2.0
             T(4,1)=1.0
             T(4,2)=BL
             T(4,3)=BL**2
             T(4,4)=BL**3
             T(4,5)=BL**4
             T(4,6)=BL**5
             T(5,2)=1.0
             T(5,3)=2.0*BL
             T(5,4)=3.0*(BL**2)
             T(5,5)=4.0*(BL**3)
             T(5,6)=5.0*(BL**4)
             T(6,3)=2.0
             T(6,4)=6.0*BL
```

```
      T(6,5)=12.0*(BL**2)
      T(6,6)=20.0*(BL**3)
      DO 41 I=1,6
      IF(I.EQ.1)GO TO 41
      DO 41 J=1,6
      IF(J.EQ.1) GO TO 41
      MT=M(I)*M(J)*M(I-1)*M(J-1)
      IF(MT)41,41,30
   30 ML=M(I)+M(J)-3
      IF(ML)41,41,33
   33 BM(I,J)=MT*((BL**ML)/ML)
   41 CONTINUE
      CALL MATINV(T,6)
      DO 46 I=1,6
      DO 46 J=1,6
   46 TT(I,J)=T(J,I)
      CALL MATMUL(BM,T,TR,6,6,6)
      CALL MATMUL(TT,TR,SM,6,6,6)
      DO 6759 I=1,6
      DO 6759 J=1,6
 6759 SM(I,J)=SM(I,J)*ERR
      DO 6500 I=1,12
      DO 6500 J=1,12
 6500 BSM(I,J)=0.0
      IF(NC.EQ.2)GO TO 6502
      DO 6509 I=1,6
      II=I
      IF(I.EQ.3) II=I+1
      IF(I.EQ.4.OR.I.EQ.5)II=I+3
      IF(I.EQ.6) II=I+4
      DO 6501 J=1,6
      JJ=J
      IF(J.EQ.3) JJ=J+1
      IF(J.EQ.4.OR.J.EQ.5)JJ=J+3
      IF(J.EQ.6) JJ=J+4
 6501 BSM(II,JJ)=SM(I,J)
 6509 CONTINUE
      GO TO 6520
```

FORTRAN IV G LEVEL 18 BEAM DATE = 70245 22/10/15 PA

```
6502  DO 6504 I=1,6
      II=I
      IF(I.EQ.2) II=I+1
      IF(I.EQ.3.OR.I.EQ.4) II=I+3
      IF(I.EQ.5) II=I+4
      IF (I.EQ.6) II=I+6
      DO 6503 J=1,6
      JJ=J
      IF(J.EQ.2) JJ=J+1
      IF(J.EQ.3.OR.J.EQ.4) JJ=J+3
      IF(J.EQ.5) JJ=J+4
      IF(J.EQ.6) JJ=J+6
6503  BSM(II,JJ)=SM(I,J)
6504  CONTINUE
6520  CONTINUE
      RETURN
      END
```

```
C
C       SUBROUTINE TO ASSEMBLE BEAM ELEMENTS IN GLOBAL SYSTEM
C
        SUBROUTINE SETUP (I,J,  A,MI,NI,D)
        IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION A(170,170),D(12,12)
        DO 1 M=1,6
        I1=6*I-6+M
        J1=6*J-6+M
        DO 1 N=1,6
        I2=6*I-6+N
        J2=6*J-6+N
        A(I1,I2)=A(I1,I2)+D(M,N)
        A(I1,J2)=A(I1,J2)+D(M,N+6)
        A(J1,I2)=A(J1,I2)+D(M+6,N)
  1     A(J1,J2)=A(J1,J2)+D(M+6,N+6)
        RETURN
        END
```

```
FORTRAN IV G LEVEL  18                  MAIN                DATE = 70245          22/10/15
      C
      C      SUBROUTINE TO SOLVE A BANDED SYMMETRIC MATRIX USING CHOLESKI'S METHOD
      C
      C      NUBW-HALF BAND WIDTH
      C      N-SIZE OF MATRIX
      C      A-MATRIX TO BE SOLVED
      C      B-LOAD VECTOR
      C      C-DISPLACEMENTS VECTOR
      C
             SUBROUTINE BNDSOL(NUBW,A,N,B,C)
             DIMENSION A(294,54 ),C(294),B(294)
      C
      C      DECOMPOSE THE ELEMENTS
      C
             DO 140 I=1,N
             N1=N-I+1
             IF(NUBW.LT.N1) N1=NUBW
             DO 140 J=1,N1
             N2=NUBW-J
             IF(N2.GT.(I-1)) N2=I-1
             SUM=A(I,J)
             IF(N2.EQ.0) GO TO 110
             DO 100 K=1,N2
             SUM=SUM-A(I-K,1+K)*A(I-K,J+K)
      100    CONTINUE
      110    IF(J.EQ.1) GO TO 120
             A(I,J)=SUM*TEMP
             GO TO 140
      120    IF(SUM.GT.0.0) GO TO 130
             WRITE(3,1) I,J,SUM
      1      FORMAT(1H0,2I4,2E20.14,'DIA. ELE IS ZEROOR -VE')
             GO TO 140
      130    SUM=SUM*100.0
             TEMP=10.0/ SQRT(SUM)
             A(I,J)=TEMP
      140    CONTINUE
      C
      C      THE SOLUTION PART
```

```
C
      DO 150 I=1,N
      J=I-NUBW+1
      IF((I+1).LE.NUBW) J=1
      SUM=B(I)
      IP1=I-1
      IF(IP1.EQ.0) GO TO 146
      DO 145 K=J,IP1
      SUM=SUM-A(K,I-K+1)*C(K)
145   CONTINUE
146   C(I)=SUM*A(I,1)
150   CONTINUE
      DO 160 I1=1,N
      I=N+1-I1
      J=I+NUBW-1
      IF(J.GT.N) J=N
      SUM=C(I)
      IP1=I+1
      IF(IP1.GT.N) GO TO 158
      DO 155 K=IP1,J
      SUM=SUM-A(I,K-I+1)*C(K)
155   CONTINUE
158   C(I)=SUM*A(I,1)
160   CONTINUE
      RETURN
      END
```

```
FORTRAN IV G LEVEL  18                 MAIN                DATE = 70245          22/10/15          P

C
C       THIS SUBROUTINE ASSEMBLES STIFFNESS MATRIX STORING ELEMENTS IN
C       HALF BAND WIDTH ONLY
C       I,J,K-THE THREE NODES OF AN ELEMENT
C       A-STIFFNESS MATRIX IN GLOBAL SYSTEM
C       M1-NUMBER OF ROWS OF THE MATRIX
C       N1-HALF BAND WIDTH
C       B-ELEMENT STIFFNESS MATRIX
C       PL-LOAD VECTOR OF AN ELEMENT
C       V-LOAD VECTOR OF THE STRUCTURE
C
        SUBROUTINE SWING (I,J,K,A,M1,N1,B,PL,V)
        DIMENSIONA(294,54 ),B(18,18),PT(20),PL(18),V(294)
        DO 20 M=1,6
        I1=I*6-6+M
        J1=J*6-6+M
        K1=K*6-6+M
        V(I1)=V(I1)+PL(M)
        V(J1)=V(J1)+PL(M+6)
        V(K1)=V(K1)+PL(M+12)
        DO 20 N=1,6
        I2=I*6-6+N
        J2=J*6-6+N
        K2=K*6-6+N
        IF(I2.LT.I1) GO TO 10
        A(I1,I2-I1+1)=A(I1,I2-I1+1)+B(M,N)
   10   IF(J2.LT.I1) GO TO 11
        A(I1,J2-I1+1)=A(I1,J2-I1+1)+B(M,N+6)
   11   IF(K2.LT.I1) GO TO 12
        A(I1,K2-I1+1)=A(I1,K2-I1+1)+B(M,N+12)
   12   IF(I2.LT.J1) GO TO 16
        A(J1,I2-J1+1)=A(J1,I2-J1+1)+B(M+6,N)
   16   IF(J2.LT.J1) GO TO 17
        A(J1,J2-J1+1)=A(J1,J2-J1+1)+B(M+6,N+6)
   17   IF(K2.LT.J1) GO TO 18
        A(J1,K2-J1+1)=A(J1,K2-J1+1)+B(M+6,N+12)
   18   IF(I2.LT.K1) GO TO 19
        A(K1,I2-K1+1)=A(K1,I2-K1+1)+B(M+12,N)
```

```
19    IF(J2.LT.K1) GO TO 21
      A(K1,J2-K1+1)=A(K1,J2-K1+1)+B(M+12,N+6)
21    IF(K2.LT.K1) GO TO 20
      A(K1,K2-K1+1)=A(K1,K2-K1+1)+B(M+12,N+12)
20    CONTINUE
      RETURN
      END
```