uOttawa

L'Université canadienne
Canada's university

FACULTÉ DES ÉTUDES SUPÉRIEURES ET POSTOCTORALES

uOttawa

L'Université canadienne
Canada's university

FACULTY OF GRADUATE AND POSDOCTORAL STUDIES

Trevor Wares

AUTEUR DE LA THÈSE / AUTHOR OF THESIS

M.Sc. (Mathematics)

GRADE / DEGREE

Department of Mathematics and Statistics

FACULTÉ, ÉCOLE, DÉPARTEMENT / FACULTY, SCHOOL, DEPARTMENT

Some Intuitionist Principles in the Free Topos

TITRE DE LA THÈSE / TITLE OF THESIS

Dr. P. Scott

DIRECTEUR (DIRECTRICE) DE LA THÈSE / THESIS SUPERVISOR

CO-DIRECTEUR (CO-DIRECTRICE) DE LA THÈSE / THESIS CO-SUPERVISOR

Dr. Pieter Hofstra

Dr. Paul-Eugène Parent

Dr. Jonathan Funk

Gary W. Slater

Le Doyen de la Faculté des études supérieures et postdoctorales / Dean of the Faculty of Graduate and Postdoctoral Studies

SOME INTUITIONIST PRINCIPLES IN THE FREE TOPOS

By
Trevor Wares, B.Sc.
June 2010

A Thesis
submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Master of Science in Mathematics[1]

[1]The M.Sc. Program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Your file Votre référence
ISBN: 978-0-494-69091-8
Our file Notre référence
ISBN: 978-0-494-69091-8

Abstract

Brouwer's principle/theorem states that all total functions $\mathbb{R} \longrightarrow \mathbb{R}$ are continuous. Obviously not classically true this result was a theorem of Brouwer in his intuitionistic setting. The formalization(s) of intuitionistic logic provides us with systems of logic in which to ask is this principle provable? In a higher order setting (e.g. higher order type theory) one has two ways of expressing this principle. First is simply the statement

$$\vdash \forall_f (f : \mathbb{R} \longrightarrow \mathbb{R} \Rightarrow \text{“}f \text{ is continuous”})$$

Second is as a meta theorem

$$\frac{\vdash f : \mathbb{R} \longrightarrow \mathbb{R}}{\vdash \text{“}f \text{ is continuous”}}$$

In this work we present a categorical proof that the second formulation holds in higher order Heyting arithmetic (HAH). Our proof is general enough however to obtain the same continuity principle with $\mathbb{R}$ replaced by a "sufficiently nice" space S, a notion which is made precise.

Acknowledgments

The topic of Brouwer's Principle in the free topos was presented to me by my thesis supervisor Philip Scott. Without Phil's insight, our mathematical discussions and his infinite patience this thesis could not have been completed.

Many thanks are due to Pieter Hofstra for our countless talks on topos theory. Both Michael Warren and Jonathon Funk have helpful in clarifying issues within the thesis. Thank you also to my thesis committee.

Finally I am happy to thank my funding sources: the University of Ottawa, it's Department of Mathematics and Statistics, the National Sciences and Engineering Research Council of Canada and Philip Scott.

Dedication

Dedicated to all my family; and for Mom and Dad.

Contents

Chapter 1

Introduction

This work is inspired by [5], some unpublished notes written by Fourman, and a personal communication with André Joyal.

Although we make some attempt at being self-contained, in that we discuss locale theory, higher order intuitionistic type theory, elementary toposes and their internal language, the reader is assumed to have a basic working knowledge of category theory.

Our goal is to give a categorical proof that Brouwer's principle, which states that all total functions from $\mathbb{R} \longrightarrow \mathbb{R}$ are continuous, holds in higher order Heyting arithmetic (HAH, also called pure type theory $\mathcal{L}_0$ below). Our proof is based on sketched notes by M. P. Fourman of some joint work with A. Joyal , in [3]. To the best of our knowledge, this work was never published. Part of the goal of this thesis is to expand on the ideas contained in Fourman's sketched proof.

In fact our proof is general enough to prove multiple continuity principles of the form

$$\frac{\vdash f : S \longrightarrow S}{\vdash \text{"f is continuous"}}$$

for spaces S which have a sufficiently nice definition in HAH (which is made precise below). This proceeds via a theory of formal spaces considered as locales, and sheaves thereon, which is developed in detail in the thesis.

Along the way we discuss other intuitionistic principles. Especially important is the principle of bar induction, which holds in HAH and which expands our notion of "sufficiently nice definition" to include Baire space $\mathbb{N}^{\mathbb{N}}$ and Cantor space $2^{\mathbb{N}}$. We believe our theory provides a method to prove that bar induction holds in HAH [3], however due to time limitations a detailed analysis will be left for future work.

1.1 Contributions

This work contains brief introductions to locale theory, higher order intuitionistic type theories, and elementary toposes. We develop some basic intuitionistic theory of Sheaves on a Grothendieck topology on a poset internal to a category. This allows us to internalize the notion of formal space for a geometric propositional theory to a base topos. We exploit this to give a categorical proof that Brouwer's principle holds in higher order Heyting arithmetic (HAH). Our proof is general enough to simultaneously provide similar continuity principles for a class of topological spaces. In addition these methods allow one to prove induction principles and choice principles in HAH, although this is not developed here [3]. For a general reference to formal versions of all these principles, see Troelstra's volume [13].

Chapter 2

Locales and frames

2.1 Basics

The idea of a locale can be reached by examining the opens of a topological space. The "localic" viewpoint discussed below was developed in detail in an influential monograph of Joyal and Tierney [2] and the book of Johnstone [7]. If X is a space, then its opens, denoted by $\mathcal{O}(X)$, form a complete lattice with the distributive law $U \cap \bigcup_i V_i = \bigcup_i (U \cap V_i)$. We call this the $\wedge, \bigvee$-Law. In addition, a continuous map $f : X \longrightarrow Y$ of spaces yields a lattice map $f^{-1} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ that preserves arbitrary unions. We will call the lattice $\mathcal{O}(X)$ the frame corresponding to X make the following generalization:

Definition 2.1.1. The category of *Frames*, denoted by **Frm**, is the subcategory of complete $\wedge, \bigvee$-lattices and lattice homomorphisms that preserve infinite joins (but not necessarily infinite meets).

Frm is, however, the wrong category if we wish to properly generalize spaces. Since continuous maps $X \longrightarrow Y$ correspond to frame maps $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ in the opposite direction we define the category of locales: $\mathbf{Loc} := \mathbf{Frm}^{op}$. Given a locale X we will write $\mathcal{O}(X)$ to denote the underlying frame of X and $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ for the frame map corresponding to the locale map $f : X \longrightarrow Y$.

There is the important notion of a *Heyting Algebra*, which is a lattice H with an operation $\rightarrow$, called implication, satisfying $x \wedge y \leq z$ if and only if $x \leq y \rightarrow z$. Regarding H as a category, meets are products, and so this simply says that H is cartesian closed. The above property of $\rightarrow$ actually is equivalent to saying $y \rightarrow x = \bigvee\{x : x \wedge y \leq z\}$, and it is a simple exercise to show that a complete lattice satisfies the $\wedge, \bigvee$-Law if and only if it is a Heyting algebra. It is important to note that frame maps need not preserve this implication.

Regarding a frame map $f^* : Y \longrightarrow X$ as a map of posets, we can always construct a right adjunction $f_* : X \longrightarrow Y$ to f^* in the following way: We need $f^*(x) \leq y \iff x \leq f_*(y)$, so define $f_*(y) = \bigvee\{x : f^*(x) \leq y\}$. It is routine to check that f_* is order preserving and right adjoint (as a poset map) to f^*. Since f_* is a right adjoint, it preserves all products, that is all meets. But in general f_* will not be frame map, it will not preserve joins.

So we may identify a locale map as actually an adjunction of poset maps $f^* \dashv f_*$ with f^* additionally preserving products. f^* is commonly called the inverse image part of the morphism.

The terminal object in **Loc** is given by the locale $\Omega := \mathcal{P}(1)$. The unique locale map $X \xrightarrow{\varphi_X} \Omega$ is given by $\varphi_X^*(U) = \bigvee\{X : * \in U\}$. We denote this locale by 1 so that $\mathcal{O}(1) = \Omega$.

2.2 Relationship between Locales and Spaces

We denote by **Space** the category of topological spaces and continuous maps. There is an obvious functor $\mathcal{O} : \mathbf{Space}^{op} \longrightarrow \mathbf{Frm}$ mapping a space to its frame of opens. We denote $\mathcal{L} = \mathcal{O}^{op} : \mathbf{Space} \longrightarrow \mathbf{Loc}$. In this section we will describe the right adjoint of $\mathcal{L}$.

2.2.1 Sober Spaces

In **Space**, and in any other category, a point of X is given by a continuous map from the terminal space 1 to X. The continuous map $1 \longrightarrow X$ sending $* \mapsto x \in X$ corresponds to the frame map $x^* : \mathcal{O}(X) \longrightarrow \Omega$ with $x^*(U) = \bigvee\{1 : x \in U\}$ and hence a locale map $x : 1 \longrightarrow \mathcal{L}(X)$. This gives a map $X \xrightarrow{\eta_X} \mathbf{Loc}(\Omega, \mathcal{O}(X))$. However this map need not be a bijection. Classically injectivity of η_X is equivalent to the separation property T_0, though we will not be assuming classical logic in this work.

Definition 2.2.1. *A space X is said to be* Sober *if η_X is bijective.*

An equivalent formulation of sobriety is given as follows: An order-preserving map $p : \mathcal{O}(X) \longrightarrow \Omega$ is equivalent to an upper set F_p of $\mathcal{O}(P)$ given by $F_p = \{U : p(U) = 1\}$. Now p preserving meets is equivalent to F_p being a prime filter, and p preserving arbitrary joins is equivalent to F_p being complete filter (i.e. closed under arbitrary joins). So a sober space is a space X in which the complete prime filters of $\mathcal{O}(X)$ are in bijection with the points of X via $x \mapsto \{U \in \mathcal{O}(X) : x \in U\}$. These ideas come from [11].

2.2.2 Points of a Locale

A point of a locale X will be a locale map $1 \longrightarrow X$. As above this is equivalent to a completely prime filter in X. The set of points of X will be denoted by $pt(X)$, and comes equipped with a canonical topology with opens given by $pt(U) = \{p \in pt(X) : p^*(U) = 1\}$. We will call this the *canonical or induced topology on $pt(X)$*. In fact pt determines a locale map $\epsilon_X : \mathcal{L}(pt(X)) \longrightarrow X$ with $\epsilon_X^*(U) = pt(U)$. To see this take $U, V \in X$; then

$$
\begin{aligned}
pt(U \wedge V) &= \{p \in pt(X) : p^*(U \wedge V) = 1\} \\
&= \{p \in pt(X) : p^*(U) \wedge p^*(V) = 1\} \\
&= \{p \in pt(X) : p^*(U) = 1 \text{ and } p^*(V) = 1\} \\
&= \{p \in pt(X) : p^*(U) = 1\} \cap \{p \in pt(X) : p^*(V) = 1\} \\
&= pt(U) \cap pt(V)
\end{aligned}
$$

And for $U_i \in X$ for $i \in I$:

$$\begin{aligned}
pt(\bigvee_{i\in I} U_i) &= \{p \in pt(X) : p^*(\bigvee_{i\in I} U_i) = 1\} \\
&= \{p \in pt(X) : \bigvee_{i\in I} p^*(U_i) = 1\} \\
&= \{p \in pt(X) : \exists i \in I . p^*(U_i) = 1\} \\
&= \{p \in pt(X) : \exists i \in I . p \in pt(U_i)\} \\
&= \bigcup_{i\in I} pt(U_i)
\end{aligned}$$

A locale X is said to be *spatial* if ϵ_X is an isomorphism. Note that it is always mono (since pt is always surjective). To be epi we require that pt be injective, which is to say given $U, V \in X$ there is exists a point $p : 1 \longrightarrow X$ for which $p^*(U) = 1$ and $p^*(V) < 1$ or vice versa. In this situation we say that the points of X separate the opens of X.

The pt construction extends to a functor, if $f : X \longrightarrow Y$ is a locale map, then $pt(f) : pt(X) \longrightarrow pt(Y)$ is defined by composition: $p \in pt(X) \mapsto fp$. We must check that this is indeed continuous: a typical open of $pt(Y)$ is of the form $pt(U)$ for $U \in Y$. Now calculate

$$\begin{aligned}
pt(f)^{-1}(pt(U)) &= \{p \in pt(X) : fp \in pt(U)\} \\
&= \{p \in pt(X) : (fp)^*(U) = 1\} \\
&= \{p \in pt(X) : p^*(f^{-1}(U)) = 1\} \\
&= pt(f^{-1}(U))
\end{aligned}$$

which is open in $pt(X)$.

Proposition 2.2.2. *pt : **Loc** $\longrightarrow$ **Space** is right adjoint to $\mathcal{L}$: **Space** $\longrightarrow$ **Loc**. The unit is ϵ_X and the counit is η_X. In addition this adjunction restricts to an equivalence between the full subcategories of Sober Spaces and Spatial Locales.*

This is a standard theorem, for example see [10] or [11].

2.3 Sublocales

This sections follows the treatment of sublocales in [11]. A sublocale is given by a monomorphism in **Loc**, that is an epimorphism (surjection) in **Frm**. Yet this does not help us to "see" what the sublocale is.

For example, in **Space**, an open subset u of a space X gives rise to an embedding $u \longrightarrow X$ via the inclusion. We can mimic this in locales: $\mathcal{O}(u)$ is all the opens contained in u, i.e. $\mathcal{O}(u) = \downarrow u$ in $\mathcal{O}(X)$. So for a locale X we define the open sublocale generated by $u \in X$ to be $\downarrow u$, which is a locale and comes with a canonical locale monomorphism $\downarrow u \stackrel{i}{\longrightarrow} X$ given by $i^*(v) = v \wedge u$. Additionally i_* is the inclusion of $\downarrow u$ into X.

An important aspect of the above example is that we can view the sublocale as a subposet and the inclusion of posets is the direct image part of the locale monomorphism. This happens in general as pointed out in the following theorem, from Mac Lane and Moerdijk ([11])

Theorem 2.3.1. *Let $f : X \longrightarrow Y$ be a map of locales. Then the following are equivalent:*

(a) f is a epimorphism of locales (i.e. f^* is injective)

(b) $f_*f^* = 1_Y$

(c) f_* is an epimorphism of posets

Also the following are equivalent:

(a') f is a monomorphism of locales (i.e. f^* is surjective)

(b') $f^*f_* = 1_X$

(c') f_* is an monomorphism of posets

The proof is just the application of the triangle equalities for the adjunction $f^* \dashv f_*$. It can be found in [11] (p.484).

There is another (equivalent) way of describing a sublocale, via closure operators:

Definition 2.3.2. A *nucleus* on a locale X is a map $j : \mathcal{O}(X) \longrightarrow \mathcal{O}(X)$ satisfying

(i) $u \leq ju$

(ii) $ju = j^2u$

(iii) $j(u \wedge v) = jv \wedge jv$

Theorem 2.3.3. *Let $j : X \longrightarrow X$ be a nucleus. Then the set of fixed points of j, denoted by X_j is a sublocale of X where $X_j \xrightarrow{i} X$ is given by $i^* = j$ and i_* is the poset inclusion of X_j into X. Moreover every sublocale $Y \xrightarrow{i} X$ of X is obtained this way by letting $j = f_*f^*$.*

This is also proved in [11] (p. 458).

2.4 Gluing Locales

Given a continuous map $f : X \longrightarrow Y$ of spaces, one may form the "gluing" of X and Y along f. There is an analogous construction for locales which we will use later.

Given a locale map $X \xrightarrow{p} Y$, we can think of p as a functor between categories $\mathcal{O}(Y) \xrightarrow{p^*} \mathcal{O}(X)$ and form the comma category $p^* \downarrow \mathcal{O}(X) = \{(U, V) : U \in \mathcal{O}(Y), V \in \mathcal{O}(X), p^*(U) \leq V\}$ which, under the order inherited from $\mathcal{O}(X)$ and $\mathcal{O}(Y)$, will also be a locale. This is the *Gluing of X and Y along p*. In general X is an open sublocale of $p^* \downarrow \mathcal{O}(X)$ and Y is a closed sublocale and retract of $p^* \downarrow \mathcal{O}(X)$. The inclusions are given by:

$i : X \longrightarrow p^* \downarrow X : i^*(U, V) = U$

$j : Y \longrightarrow p^* \downarrow X : j^*(U, V) = V$

The retraction is $\rho : p^* \downarrow \mathcal{O}(X) \longrightarrow \mathcal{O}(Y) : \rho^*(U) = (U, p^*(U))$ and further more $p = \rho i$.

Remark 2.4.1. For those already familiar with topos theory: this is a special case of a more general construction, that of gluing toposes along geometric morphisms. See [6] or [8] p.82 Vol I, for details. This construction also satisfies a universal mapping property, which we omit here (see the above references).

Chapter 3

Toposes

3.1 Algebraic View

One may define a topos as a cartesian closed category with a subobject classifier and a natural numbers object, or by using power objects. We do not wish our logical functors to necessarily preserve all exponentials on the nose so we choose the latter. This approach is taken from Lambek and Scott [10]. To be precise a topos will be a category with finite products, a subobject classifier Ω, a natural numbers object, and all exponentials of the form $\Omega^A := P(A)$. We let δ_A denote the characteristic map of the diagonal $A \xrightarrow{<1_A,1_A>} A \times A$ and ask that strict logical functors will preserve this structure on the nose (including δ_A). A consequence of this definition will be that strict logical functors will preserve all exponentials and kernels up to isomorphism and characteristic morphisms on the nose.

This weakens our choice of morphism from the "obvious" notion of logical functor. Following the treatment in [10] we also require our toposes to come equipped with a choice of canonical subobjects. That is a choice of representatives from the isomorphism classes of subobjects of all objets in the topos satisfying four conditions for all objects A, B, C:

C1) $1_A : A \longrightarrow A$ is a canonical subobject of A.

C2) If $f : B \longrightarrow A$ and $g : C \longrightarrow B$ are canonical subobjects then so is fg.

C3) If $f : B \longrightarrow A$ and $g : D \longrightarrow C$ are canonical subobjects then so is $f \times g$:

$B \times D \longrightarrow A \times C$.

C4) If $f : B \longrightarrow A$ is a canonical subobject then $\Omega^f : \Omega^B \longrightarrow \Omega^A$ is a canonical subobject.

For example *Set* has canonical subobjects (subsets) and likewise any presheaf (or sheaf) category will have canonical subobjects (subfunctors). Most importantly any topos is equivalent to one with canonical subobjects (see [10] for details). We now take $\mathbf{Top}_0$ to be the category of toposes with canonical subobjects and strict logical morphisms in the sense above.

The canonical subobjects are required to ensure we arrive at an adjunction; if we are willing to sacrifice the strictness of our logical functors in $\mathbf{Top}_0$ then we could avoid this.

3.1.1 Free Cartesian Closed Categories, Polynomial Arrows and Functional Completeness

In this section we assume familiarity with cartesian closed categories. For an introduction see [10].

Given a graph G we can always construct the free category on G by taking its vertices as objects and edges as morphisms, then adding in morphisms for identities and compositions, and finally identifying morphisms according to the theory of categories (i.e. making composition associative and unitial). This process extends to a functor which is left adjoint to the forgetful functor from categories to graphs.

Likewise we can construct the free cartesian closed category on a graph by further adding in new objects for products and exponentials and new morphisms for projections and evaluations, then proceeding to identify morphisms accordingly. This is formalized very nicely in [10] where cartesian closed categories are defined equationally.

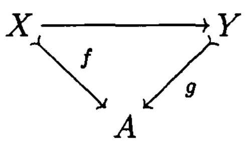
Suppose that $\mathbb{C}$ is a cartesian closed category. Given an object A we will wish to freely adjoin a new arrow $1 \longrightarrow A$. This can be done by adding an edge $x : 1 \longrightarrow A$ in the underlying graph of $\mathbb{C}$ and then taking the free cartesian closed category generated by that graph subject to the equations which hold in $\mathbb{C}$. We denote this new category by $\mathbb{C}[x]$ (actually we should denote it by $\mathbb{C}[x : 1 \longrightarrow A]$, or at least $\mathbb{C}[x^A]$, to include the codomain of x. But this will always be clear from the context). We refer to the arrows of $\mathbb{C}[x]$ as polynomial arrows of C in x. This process may be generalized in the obvious way to define categories $\mathbb{C}[x_1, \ldots, x_n]$, and we write a polynomial arrow $\varphi(x_1, \ldots, x_n)$ to mean that we are viewing it as a arrow in $\mathbb{C}[x_1, \ldots, x_n]$.

This construction comes with substitution functors $\mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathbb{C}$ which when given $a_i : 1 \longrightarrow A_i$ in $\mathbb{C}$ (assuming $x_i : 1 \longrightarrow A_i$) and $\varphi(x_1, \ldots, x_n) : 1 \longrightarrow B$ in $\mathbb{C}[x_1, \ldots, x_n]$ yields an arrow $\varphi(a_1, \ldots, a_n) : 1 \longrightarrow B$. Such a polynomial determines a unique arrow $\lambda\varphi : A_1 \times \cdots \times A_n \longrightarrow B$ such that substitution is given by composition: $\lambda\varphi\langle a_1, \ldots, a_n\rangle = \varphi(a_1, \ldots, a_n)$.

These facts are discussed in detail and formally proven in [10].

3.1.2 Internal Heyting Algebras Ω^A

In any category we can define the notion of *subobject* as an equivalence class of monos. To be precise we let $Mono(A)$ denote the set of monomorphisms with codomain A and define a preorder $\leq$ on $Mono(A)$ by $X \rightarrowtail^{f} A \leq Y \rightarrowtail^{g} A$ when f factors through g via some (necessarily unique) map $X \longrightarrow Y$:



(1)

The equivalence relation on $Mono(A)$ induced by this preorder is that the two monos f, g are equivalent if there is an isomorphism $X \longrightarrow Y$ making (1) commute. We call the equivalence class containing f the subobject determined by f, which will also be denoted by f. The context shall make it clear if we are referring to the map or the subobject. The collection of subobjects will be denoted by $Sub(A)$, which is a partial order with top element (given by 1_A).

If our category has pullbacks then $Sub(A)$ will become a meet-semi lattice with top element. The meet of two subobjects is given by their pullback. In **Set**, $Sub(A)$ is a complete boolean algebra, under union intersection and complement.

In a topos however $Sub(A)$ will always be a Heyting algebra. Moreover, for each arrow $f : A \longrightarrow B$, pullback along f yields a map $f^* : Sub(B) \longrightarrow Sub(A)$, and this map will have left and right adjoints $\exists_f \dashv f^* \dashv \forall_f$. This construction is also functorial in that $\exists_g \exists_f = \exists_{gf}$ and similarly for $\forall$.

This structure is also reflected "internally" in that each object Ω^A is an internal complete Heyting algebra object, i.e. has maps $\Omega^A \times \Omega^A \xrightarrow{\wedge} \Omega^A$ and $\Omega^{\Omega^A} \xrightarrow{\bigvee} \Omega^A$ satisfying the appropriate diagrams. Also there are arrows $\Omega^B \xrightarrow{\exists_f, \forall_f} \Omega^A$ for each map $f : A \longrightarrow B$ which are internal adjoints to the internal pullback functor $f^* : \Omega^B \longrightarrow \Omega^A$.

These facts, and more, are discussed in detail throughout the literature, see for example [1].

Additionally as a (non-trivial) consequence of the definition of topos, all toposes have all finite colimits (see [1],[8]).

3.1.3 Cartesian Functors and Comparison Maps

Given a topos $\mathcal{E}$ and an object A in $\mathcal{E}$, the evaluation $ev_A : \Omega^A \times A \longrightarrow \Omega$ gives rise to a subobject $\in_A$ which we call the *Element Relation on A*

$$\begin{array}{ccc} \in_A & \longrightarrow & 1 \\ \downarrow & & \downarrow \top \\ \Omega^A \times A & \xrightarrow{ev_A} & \Omega \end{array}$$

Suppose that $F : \mathcal{E} \longrightarrow \mathcal{S}$ is a cartesian functor (i.e. preserves finite limits) between toposes. Following [8] (p.69) for each object A of $\mathcal{E}$ we define a map $\varphi_A : F(\Omega^A) \longrightarrow \Omega^{F(A)}$ as follows; Apply F to the subobject $\in_A$ to obtain a new subobject $F(\in_A)$:

$$\begin{array}{ccccc} F(\in_A) & \longrightarrow & F(1) & \xrightarrow{\cong} & 1 \\ \downarrow & & \downarrow F(\top) & & \downarrow \top \\ F(\Omega^A) \times F(A) & \xrightarrow{F(ev_A)} & F(\Omega) & \xrightarrow{\alpha} & \Omega \end{array}$$

We define φ_A to be the transpose of $(\alpha F(ev_A))$. The functor F is logical if and only if each comparison map φ_A is an isomorphism.

3.2 Logical View: Higher-Order Logic

The category $\mathcal{L}ang$ consists of Intuitionistic type theories and translations between them. Following [10] we present two equivalent formulations of type theory, one based on equality and one based on logic.

3.2.1 Type Theory based on Logic.

A type theory $\mathcal{L}$ has types generated from a given list of basic types (which will always include $1, N, \Omega$) and given type-forming operations (which will always include a binary operation $\times$ and a unary operation $P(-)$ (power set)). In general there may be other types, basic types or type-forming operations and non-trivial identifications between them.

The terms of $\mathcal{L}$ are formed from basic terms and term forming operations where the following are basic terms:

$$0 : N \quad * : 1 \quad \top : \Omega \quad \bot : \Omega \quad x_i^A : A$$

for each $i \in \mathbb{N}$ and A any type, along with the term forming operations summarized by:

$$\frac{n : N}{Sn : N} \qquad \frac{a : A \quad b : B}{\langle a, b\rangle : A \times B}$$

$$\frac{p : \Omega \quad q : \Omega}{p \wedge q,\ p \vee q,\ p \Rightarrow q\ : \Omega} \qquad \frac{a : A \quad \alpha : P(A)}{a \in \alpha : \Omega}$$

$$\frac{\varphi(x) : \Omega}{\{x \in A | \varphi(x)\} : P(A)} \qquad \frac{\varphi(x) : \Omega}{\forall_{x \in A}\varphi(x),\ \exists_{x \in A}\varphi(x) : \Omega}$$

We define the usual short hands $\neg p \equiv p \Rightarrow \bot$, $p \iff q \equiv p \Rightarrow q \wedge q \Rightarrow p$ and equality is defined following Leibniz $a = a' \equiv \forall_{u \in P(A)}(a \in u \iff a' \in u)$.

Free variables of terms are defined in the usual way, and formulae are defined to be terms of type Ω. For each context (a set of variables) X, $\mathcal{L}$ is equipped with an entailment relation $\vdash_X$ between formulas with free variables included in X. For short hand we write $\vdash$ for $\vdash_{\emptyset}$, and $\vdash_X p$ for $\top \vdash_X p$. This relation must satisfy three groups of axioms:

Structural Rules:

$$\frac{}{p \vdash_X p}$$

$$\frac{p \vdash_X q \quad q \vdash_X r}{p \vdash_X r}$$

$$\frac{p \vdash_X q}{p \vdash_{X \cup \{y\}} q}$$

$$\frac{\varphi(x^B) \vdash_{X \cup \{x^B\}} \psi(x^B)}{\varphi(b) \vdash_X \psi(b)}$$

Where $b : B$ has its free variables contained in X and no free variable of b becomes bound in $\varphi(b)$ or in $\psi(b)$.

Logical Rules:

$$p \vdash_X \top$$

$$\bot \vdash_X p$$

$$r \vdash_X p \wedge q \quad \text{if and only if} \quad r \vdash_X p \text{ and } r \vdash_X q$$

$$p \vee q \vdash_X r \quad \text{if and only if} \quad p \vdash_X r \text{ and } p \vdash_X q$$

$$p \vdash_X q \Rightarrow r \quad \text{if and only if} \quad p \wedge q \vdash_X r$$

$$p \vdash_X \forall_{y \in B} \psi(y) \quad \text{if and only if} \quad p \vdash_{X \cup \{y\}} \psi(y)$$

$$\exists_{y \in B} \psi(y) \vdash_X p \quad \text{if and only if} \quad \psi(y) \vdash_{X \cup \{y\}} p$$

Extralogical Axioms:

Comprehension

$$\vdash_X \forall_{x \in A}(x \in \{x \in A | \psi(x)\} \iff \psi(x))$$

Extensionality

$$\vdash \forall_{u \in P(A)} \forall_{v \in P(A)} (x \in u \iff x \in v) \Rightarrow u = v$$

$$\vdash \forall_{s \in \Omega} \forall_{t \in \Omega} (s \iff t) \Rightarrow s = t$$

Products

$$\vdash \forall_{z \in 1} z = *$$

$$\vdash \forall_{z \in A \times B} \exists_{x \in A} \exists_{y \in B} z = \langle x, y \rangle$$

$$\vdash \forall_{x \in A} \forall_{x' \in A} \forall_{y \in B} \forall_{y' \in B} (\langle x, y \rangle = \langle x', y' \rangle \Rightarrow (x = x' \wedge y = y'))$$

Peano Axioms

$$\vdash \forall_{x \in N} \neg (Sx = 0)$$

$$\vdash \forall_{x \in N} \forall_{y \in N} (Sx = Sy \Rightarrow x = y)$$

$$\vdash \forall_{u \in P(N)} (0 \in u \wedge \forall_{x \in N} (x \in u \Rightarrow Sx \in u) \Rightarrow \forall_{y \in N} y \in u)$$

Classical type theory would also include one of the two equivalent axioms: $\vdash \forall_{t \in \Omega}(t \vee \neg t)$ or $\vdash \forall_{t \in \Omega}(\neg\neg t \Rightarrow t)$, but we do not impose these here.

Pure Type theory $\mathcal{L}_0$ is the type theory obtained adding no types, identifications between types, or terms other than what is required above, and taking $\vdash_X$ to be the least such relation satisfying the above axioms.

3.2.2 Type Theory based on Equality.

A type theory $\mathcal{L}$ based on equality has types and terms subject to the same conditions as a type theory based on logic except that we no longer require the basic terms $\top, \bot : \Omega$ and change the term forming operations to include:

$$\frac{n : N}{Sn : N} \qquad \frac{a : A \quad b : B}{\langle a, b\rangle : A \times B} \qquad \frac{\varphi(x) : \Omega}{\{x \in A | \varphi(x)\} : P(A)}$$

$$\frac{a : A \quad a' : A}{a = a' : \Omega} \qquad \frac{a : A \quad \alpha : P(A)}{a \in \alpha : \Omega}$$

The difference is that equality of formulae is now a basic concept with which we must define the logical symbols, which we do as follows:

$$\begin{aligned}
\top &\equiv * = * \\
p \wedge q &\equiv \langle p, q\rangle = \langle \top, \top\rangle \\
p \Rightarrow q &\equiv \langle p, q\rangle = p \\
\forall_{x \in A}\varphi(x) &\equiv \{x \in A : \varphi(x)\} = \{x \in A : \top\} \\
\bot &\equiv \forall_{t \in \Omega} t \\
p \vee q &\equiv \forall_{t \in \Omega}(((p \Rightarrow t) \wedge (q \Rightarrow t)) \Rightarrow t) \\
\exists_{x \in A}\varphi(x) &\equiv \forall_{t \in \Omega}(\forall_{x \in A}(\varphi(x) \Rightarrow t) \Rightarrow t)
\end{aligned}$$

For convenience we describe the entailment relation here as a sequent between a finite set of formulae and a formula $\Gamma \vdash_X q$ subject to the following axioms:

Structural Rules

$$p \vdash_X p$$

$$\frac{\Gamma \vdash_X p \quad \Gamma \cup \{p\} \vdash_X q}{\Gamma \vdash_X q}$$

$$\frac{\Gamma \vdash_X q}{\Gamma \cup \{p\} \vdash_X q}$$

$$\frac{\Gamma \vdash_X q}{\Gamma \vdash_{X\cup\{y\}} q}$$

$$\frac{\Gamma(y) \vdash_{X\cup\{y\}} \varphi(y)}{\Gamma(b) \vdash_X \varphi(b)}$$

where it is assumed that b is substitutable for y in $\varphi(y)$ and in each element of $\Gamma(y) = \{\psi(y) : \psi \in \Gamma\}$.

Pure Equality Rules

$$\vdash_X a = a$$

$$\{a = b, \varphi(a)\} \vdash_X \varphi(b)$$

where a and b are substitutable for x in $\varphi(x)$.

$$\frac{\Gamma \cup \{p\} \vdash_X q \quad \Gamma \cup \{q\} \vdash_X p}{\Gamma \vdash_X p = q}$$

Products and Terminal Axioms

$$\langle a, b\rangle = \langle c, d\rangle \vdash_X a = c$$

$$\langle a, b\rangle = \langle c, d\rangle \vdash_X b = d$$

$$\vdash_x x = *$$

where x is of type 1.

$$\frac{\Gamma \cup \{z = \langle x, y\rangle\} \vdash_{X\cup\{x,y,z\}} \varphi(z)}{\Gamma_{X\cup\{x\}} \varphi(x)}$$

where x and y are not free in Γ or $\varphi(z)$.

Comprehension Axioms

$$\vdash_X \varphi(x) = (x \in \{x \in A : \varphi(x)\})$$

where $x \in X$.

$$\frac{\Gamma \vdash_{X \cup \{x\}} \varphi(x) = (x \in \alpha)}{\Gamma \vdash_X \alpha = \{x \in A : \varphi(x)\}}$$

where x is not free in Γ.

Peano Axioms

$$Sx = 0 \vdash_{X \cup \{x\}} p$$

$$Sx = Sy \vdash_{\{x,y\}} x = y$$

$$\frac{\Gamma \vdash_X \varphi(0) \quad \Gamma \cup \{\varphi(x)\} \vdash_{X \cup \{x\}} \varphi(Sx)}{\Gamma \vdash_{X \cup \{x\}} \varphi(x)}$$

where x is not free in Γ.

Discussed in [10] is the fact that it does not matter whether we base a type theory on logic or on equality, since each one proves the necessary theorems and derived rules of inference to qualify being a member of both formulations. For example Leibniz's rule for equality is a theorem in any type theory based on equality, and the definitions of the logical connectives in a type theory based on equality become theorems in a type theory based on logic.

3.2.3 Translations

A translation from one language $\mathcal{L}_1$ to another $\mathcal{L}_2$ is a function F mapping the types of $\mathcal{L}_1$ to $\mathcal{L}_2$ and the terms of $\mathcal{L}_1$ to $\mathcal{L}_2$ that respects the special types $1, N, \Omega$ and terms $0 : N, \top : \Omega, * : 1$, the type and term forming operations, any identifications between types or terms, and of course terms of type A are mapped to terms of type $F(A)$ and x_i^A is mapped to $x_i^{F(A)}$. Two translations F, G are said to be equal if they are provably equal, that is $F(A) = G(A)$ for all types A and $\vdash_{F(X)} F(t(X)) = G(t(X))$ for all terms $t(X)$.

A consequence of this definition is that translations preserve theorems. Indeed p is a theorem if and only if $\vdash p = \top$ which is preserved by translations.

Type theories and translations between then form a category which we will denote by **Lang**. Pure type theory $\mathcal{L}_0$ is the initial object of **Lang** with the obvious canonical translation of $\mathcal{L}_0$ into any other language.

3.3 Internal Language of a Topos

To each topos $\mathcal{E}$ we associate a type theory (based on equality) denoted by $L(\mathcal{E})$ which will have as types the objets of $\mathcal{E}$ with the obvious identification that the object 1 is the type 1 and similar identifications for N and Ω. The terms of type A are polynomial arrows $\varphi(x_1, \ldots, x_n) : 1 \longrightarrow A$ where we interpret:

$*$	$1 \longrightarrow 1$
0	$0 : 1 \longrightarrow N$
Sn	$1 \xrightarrow{n} N \xrightarrow{S} N$
$\langle a, b \rangle$	$1 \xrightarrow{\langle a,b \rangle} A \times B$
$a = a'$	$1 \xrightarrow{\langle a,a' \rangle} A \times A \xrightarrow{\delta_A} \Omega$
$a \in \alpha$	$1 \xrightarrow{\langle \alpha,a \rangle} \Omega^A \times A \xrightarrow{ev_A} \Omega$
$\{x \in A \mid \varphi(x)\}$	$(\lambda\varphi)^* : 1 \longrightarrow \Omega^A$

where a, a', b, α are all of the appropriate type.

To be clear we will reserve "=" for the formal symbol in the internal language of a topos and use $\cdot = \cdot$ for equality of arrows.

For $X = \{x_1^{A_1}, \ldots, x_m^{A_m}\}$ we put $\{\varphi_1(X), \ldots, \varphi_n(X)\} \vdash_X \varphi_{n+1}(X)$ to mean that for all $h : C \longrightarrow A_1 \times \cdots \times A_n$ if for $i = 1, \ldots, n$ $\lambda(\varphi_i(X))h \cdot = \cdot \top 0_C$ then also $\lambda(\varphi_{n+1}(X))h \cdot = \cdot \top 0_C$, where 0_C is the unique map $C \longrightarrow 1$.

Since each $\lambda\varphi_i(X) : A_1 \times \cdots \times A_m \longrightarrow \Omega$ we can rephrase this in terms of subobjects: Each $\varphi(X)$ determines a subobject of $A_1 \times \cdots \times A_n$. To say $\{\varphi_1(X), \ldots, \varphi_n(X)\} \vdash_X \varphi_{n+1}(X)$ is equivalent that the subobject determined by φ_{n+1} is greater than (in the subobject lattice $Sub(A_1 \times \cdots \times A_n)$) the meet of the subobjects determined by φ_i, $i = 1, \ldots, n$.

It is routine calculation to show that $\mathcal{L}(\mathcal{E})$ satisfies all the axioms except for Peano's. The proof that $\mathcal{L}(\mathcal{E})$ satisfies Peano's axioms can be found in [10].

To each logical functor $F : \mathcal{E} \longrightarrow \mathcal{S}$ arises a translation $\mathcal{L}(F) : \mathcal{L}(\mathcal{E}) \longrightarrow \mathcal{L}(\mathcal{S})$ in a natural way, which makes $\mathcal{L}$ into a functor $Top_0 \longrightarrow$ **Lang**.

3.3.1 Some Applications of the Internal Language

In [10] the following useful facts (and more) are proved:

Theorem 3.3.1. *Let $\mathcal{E}$ be a topos.*

(i) *For arrows $f, g : A \longrightarrow B$ in $\mathcal{E}$ one has $f\cdot = \cdot g \iff \vdash \forall_{x \in A}. fx = gx$.*

(ii) *If $h : A \times B \longrightarrow \Omega$ and $h^* : A \longrightarrow \mathcal{P}B$ is the transpose then $h^*x\cdot = \cdot\{y \in B : h\langle x, y\rangle\}$.*

(iii) *Provably functional relations in $\mathcal{L}(\mathcal{E})$ determine arrows in $\mathcal{E}$. That is if $\vdash \forall_{x\in A}\exists!_{y\in B}\varphi(x, y)$ then there is a unique arrow $g : A \longrightarrow B$ (with $y = gx\cdot = \cdot\varphi(x, y)$) such that $\vdash \forall_{x\in A}\varphi(x, gx)$.*

(iv) *$f : A \longrightarrow B$ is a mono if and only if $\vdash \forall_{x\in A}\forall_{y\in A}(fx = fy \Rightarrow x = y)$.*

(v) *$f : A \longrightarrow B$ is epi if and only if $\vdash \forall_{y\in B}\exists_{x\in A}.fx = y$.*

(vi) *$f : A \longrightarrow B$ is an iso if and only if $\vdash \forall_{y\in B}\exists!_{x\in A}.fx = y$.*

Number (iii) will particularly useful later on, where we will rely heavily on the internal language to define arrows in a topos.

3.4 Topos Generated by Type Theory

The construction of a topos $T(\mathcal{L})$ from a type theory $\mathcal{L}$ is a formalization of the proof that the category of sets and functions forms a topos with a few minor adjustments. $T(\mathcal{L})$ will have as objects closed terms of type $P(A)$ for some type A, modulo provable equality, and a map $f : \alpha \longrightarrow \beta$ where α and β are closed terms of type $P(A)$ and $P(B)$ respectively will be a "provably functional relation" from α to β (modulo provable equality); that is a closed term $f : P(A \times B)$ satisfying $\vdash f : \alpha \longrightarrow \beta$ where for notational convenience we define $f : \alpha \longrightarrow \beta \equiv (f \subset \alpha \times \beta) \wedge \forall_{x \in A}(x \in \alpha \Rightarrow \exists!_{y \in B}\langle x, y\rangle \in f)$ (with the usual definitions for $\subset$, $\alpha \times \beta$, and $\exists!$).

To each type A there is the closed term $\boldsymbol{A} := \{x \in A : \top\}$. Identities, composition, products, and pullbacks are defined in the obvious way as in set theory. The natural numbers object is given by $\boldsymbol{N}$ with the obvious maps for zero and successor. Peano's axioms guarantee that this gives a natural numbers object.

The subobject classifier is given by $\{\langle *, \top\rangle\} : \mathbf{1} \longrightarrow \boldsymbol{\Omega}$ and if $f : \alpha \longrightarrow \beta$ is a mono then f has the characteristic map given by

$$\{\langle y, t\rangle \in B \times \Omega | y \in \beta \wedge t = \exists_{x \in A}\langle x, y\rangle \in f\} : \beta \longrightarrow \boldsymbol{\Omega}$$

.

A note of caution, we can define function spaces $\beta^\alpha = \{f \in P(\alpha \times \beta) : \vdash f : \alpha \longrightarrow \beta\}$, but when it comes to power objects we have two candidates: $\boldsymbol{\Omega}^\alpha$ and $P\alpha := \{w \in PA : w \subseteq \alpha\}$. Although these are not equal, they are isomorphic.

The cartesian closed structure is given as follows: the evaluation $ev_\alpha : P\alpha \times \alpha \longrightarrow \boldsymbol{\Omega}$ (for $\alpha : A$) is the term

$$\{\langle\langle u, x\rangle, t\rangle \in (PA \times A) \times \Omega : x \in \alpha \wedge u \subseteq \alpha \wedge t = (x \in u)\}$$

For $h : \beta \times \alpha \longrightarrow \Omega$ (i.e. $h : A \times B \times \Omega$) we define the transpose $h^* : \beta \longrightarrow P\alpha$ by

$$\{\langle y, u\rangle \in B \times PA : u = \{x \in A : \langle\langle y, x\rangle, \top\rangle \in h\}\}$$

Each translation $F : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ preserves closed terms and theorems, hence also preserves provably functional relations between closed terms. Thus we associate a logical functor $T(F) : LT(\mathcal{L}_1) \longrightarrow T(\mathcal{L}_2)$ in the obvious way, applying F. From this arises a functor $T : \mathbf{Lang} \longrightarrow Topo_0$

In [10] it is proved that for any type theory $\mathcal{L}$, $LT(\mathcal{L})$ is a conservative extension of $\mathcal{L}$, for any topos $\mathcal{E}$, $TL(\mathcal{E})$ is equivalent to $\mathcal{E}$, and $T \dashv L$. In particular T preserves the initial object so we have an initial topos called the *free topos* $\mathcal{F} := T(\mathcal{L}_0)$. These results are used in [10] to reduce proof theoretic properties of $\mathcal{L}_0$ to algebraic properties of $\mathcal{F}$ in order to establish intuitionistic principles in $\mathcal{L}_0$.

3.5 Topos Semantics

Thinking of an arrow $C \xrightarrow{a} A$ as a "generalized element of A at stage C" , and given a term $\varphi(x)$ of type A, we write $C \Vdash \varphi(a)$ for $\varphi(a)\cdot = \cdot\top 0_C$ (read "C forces $\varphi(a)$" or "$\varphi(a)$ is true at stage C"). This says a factors through the subobject of A determined by φ via $C \xrightarrow{\lambda(\varphi)} \Omega$.

Some simple facts follow this definition:

$$C \Vdash \varphi(a) \text{ and } h : D \longrightarrow C \text{ implies } D \Vdash \varphi(ah) \tag{2}$$

$$L(\mathcal{E}) \vdash_{x^A} \varphi(x^A) \iff \text{for all } C \xrightarrow{a} A \quad C \Vdash \varphi(a) \tag{3}$$

$$C \Vdash \varphi(a) \iff \vdash \forall_{z \in C} \varphi(az) \tag{4}$$

$$D \xrightarrow{h} C \text{ epi and } D \Vdash \varphi(ah) \text{ implies } C \Vdash \varphi(a) \tag{5}$$

The big theorem is that this notion of satisfiability can be inductively defined.

Theorem 3.5.1 (Beth-Kripke-Joyal Semantics). *Given $C \xrightarrow{a} A$ and two formulae φ, ψ:*

0) If $A = \Omega$ then $C \Vdash a \iff a\cdot = \cdot\top 0_C$.

1)For $b(x^A)$ a term of type B and $\beta(x^A)$ a term of type $P(B)$, $C \Vdash b(a) \in \beta(a) \iff ev_B\langle\beta(a), b(a)\rangle\cdot = \cdot\top 0_C$.

2) $C \Vdash \top$ always.

3) $C \Vdash \bot \iff C \cong 0$.

4) $C \Vdash \varphi(a) \wedge \psi(a) \iff C \Vdash \varphi(a)$ and $C \Vdash \psi(a)$.

5) $C \Vdash \varphi(a) \vee \psi(a) \iff$ *there exists an epi* $[k, l] : D + E \longrightarrow C$ *where* $D \Vdash \varphi(ak)$ *and* $E \Vdash \psi(al)$.

6) $C \Vdash \varphi(a) \Rightarrow \psi(a) \iff$ *for all* $D \xrightarrow{h} C$, $D \Vdash \varphi(ah)$ *implies* $C \Vdash \psi(ah)$.

7) $C \Vdash \forall_{y \in B} \psi(y, a) \iff$ *for all* $D \xrightarrow{h} C$ *and* $D \xrightarrow{b} B$, $D \Vdash \psi(b, ah)$.

8) $C \Vdash \exists_{y \in B} \psi(y, a) \iff$ *there exists an epi* $D \xrightarrow{h} C$ *and* $D \xrightarrow{b} B$ *where* $D \Vdash \psi(n, ah)$.

9) For terms $b(x^A)$ *and* $b'(x^A)$ *of type* B, $C \Vdash b(a) = b'(a) \iff b(a) \cdot = \cdot b'(a)$.

A discussion of these ideas and a detailed proof can be found in [11] (p. 302).

In light of (3) we may reduce the provability internal statements to meta-statements about arrows in the topos.

Chapter 4

Intuitionistic Principles

A key idea of constructive mathematics is that a proof should be a construction (or a method to do a construction, e.g. an algorithm). In the study of constructive mathematics this leads to several metamathematical principles that have become basic to the subject. Although these principles may not be universally acknowledged as required for a constructive foundation, they serve as basic notions in intuitionistic mathematics.

For example, for an intuitionist to prove an existential statement, the proof should actually construct an explicit witness. Or a proof of a disjunction should be a construction of a proof of one of the disjuncts.

Although Brouwer himself was not interested in formalizing the logic of intuitionism, his student Heyting (in the 1930's) and numerous logicians since then, have spent considerable time formalizing large parts of constructive mathematics and the logical principles underlying the many different versions of constructivism. We should mention in particular the work of Heyting, Kleene, Vesley, Markov, Troelstra, van Dalen, E. Bishop, D. Bridges, Brouwer. For a brief introduction to the history of constructivism see [14].

More recently, with the advent of categorical logic and topos theory, the formal connections of intuitionist logics to categorical constructions has opened up new approaches to the formalization and the semantics of constructive mathematics.

In what follows, we present a few familiar intuitionistic proof principles, along the lines of [10], with some of their categorical meanings.

4.1 Proof Theoretic Principles

Let us consider some proof-theoretic principles that hold in pure type theory and their translations into algebraic statements in the free topos $\mathcal{F}$ (for proofs see [10]).

Disjunction Property (DP): if $\vdash p \vee q$ then $\vdash p$ or $\vdash q$.

DP may be formulated as saying the terminal object 1 in the free topos $\mathcal{F}$ is indecomposable. That is, whenever $[k, l] : A + B \longrightarrow 1$ is an epimorphism then one of k or l is an epimorphism.

Existence Property (EP): if $\vdash \exists_{x \in A} \varphi(x)$ then there is a (closed) term $a : A$ for which $\vdash \varphi(a)$.

The existence property EP essentially says that 1 is projective, but not exactly. It is in general a little weaker. The premise of EP says there is a surjection $\{x \in A \mid \varphi(x)\} \longrightarrow 1$. The conclusion of projectivity of 1 is that there is an arrow $1 \longrightarrow \{x \in A \mid \varphi(x)\}$. By definition of the free topos, this arrow is a term of type $P(A \times 1)$ satisfying a unique existence condition (since it is a provably functional relation). Hence, the projectivity of 1 witnesses existential statements by points which are provably functional relations. The EP wants something stronger: it needs closed terms as witnesses, not points. Hence the projectivity of 1 reduces the existence property to the Unique Existence Property ([10], Lemma 20.3, p. 229), namely

Unique Existence Property (E!P): if $\vdash \exists!_{x \in A} \psi(x)$ then there is a (closed) term $a : A$ for which $\vdash \psi(a)$.

This latter is proved by structural induction on the syntax of pure intuitionistic type theory. So we indeed can conclude (EP) by showing the projectivity of 1 in $\mathcal{F}$, together with (E!P).

A consequence of this is that *numerals are standard*, that is, the only arrows $1 \longrightarrow N$ in $\mathcal{F}$ are standard numerals $S^n 0$, for $n \in \mathbb{N}$.

Troelstra's Uniformity Rule (UR): if $\vdash \forall_{x \in PC} \exists_{y \in N} \varphi(x, y)$ then $\vdash \exists_{y \in N} \forall_{x \in PC} \varphi(x, y)$.

Equivalently, by (EP), the conclusion of (UR) says: $\vdash \forall_{x \in PC}\varphi(x, \bar{n})$, for some numeral $\bar{n}$. Intuitively this, this says the only arrows into N from a (pure) powerset are constant. In $\mathcal{F}$, this means that any arrow $P(C) \longrightarrow N$ factors through a standard numeral.

Markov's Rule: if $\vdash \forall_{x \in A}(\varphi(x) \vee \neg\varphi(x))$ and $\vdash \neg\forall_{x \in A}\neg\varphi(x)$ then $\vdash \exists_{x \in A}\varphi(x)$.

This says: for decidable predicates, classical reasoning is valid. More specifically, if $\varphi(x)$ is decidable and if $\vdash \neg\neg\exists_{x \in A}\varphi(x)$, then there is some closed term of type A which must provably witness φ.

Independence of premisses (IP): if $\vdash \neg q \Rightarrow \exists_{x \in A}\varphi(x)$ then $\vdash \exists_{x \in A}(\neg q \Rightarrow \varphi(x))$, where x is not in the free variables of q.

Another interpretation would be: if $\neg q \vdash \exists_{x \in A}\varphi(x)$ then $\neg q \vdash \varphi(a)$, for some closed term of type A, which we might call **Existence Property mod** $\neg q$. This can be interpreted in the free topos as saying $\neg q$ (more precisely, $\neg \boldsymbol{q} = \{x \in 1 : \neg q\}$) is a projective subobject of 1.

4.2 Bar Induction

Bar induction is a principle for reasoning about predicates (subsets) on certain types of trees. We consider the concrete case of sequences of natural numbers. We start with stating the principle for the tree $\mathbb{N}^* = \mathbb{N}^{<\mathbb{N}} =$ all finite sequences with values in $\mathbb{N}$ (where $a \leq b$, for finite sequences a and b, if b is an initial segment of a).

Before we give the principle of Bar Induction we need some notation. Given $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$ we define $\overline{\alpha}(n)$ to be the sequence $\langle \alpha(0), \ldots, \alpha(n-1) \rangle$. Finite sequences will be written with lower case letters a, b, c... and concatenation will be denoted by $a * b$. For $n \in \mathbb{N}$ we write $a * n$ as shorthand for $a * \langle n \rangle$.

Given $R \subseteq \mathbb{N}^*$ define $\overline{R} = \{a \in \mathbb{N}^* : \forall_{\alpha \in \mathbb{N}^{\mathbb{N}}}\exists_{x \in \mathbb{N}}.R(a * \overline{\alpha}(x))\}$ i.e. those sequences for which every extension to an infinite sequence must "pass though" R. For $a \in \overline{R}$ we say "a is bared by R", or "R is a bar for a"

It is easy to see that $\overline{R}$ possesses two properties:

(i) $R \subseteq \overline{R}$

(ii) $(\forall_{n\in\mathbb{N}}.a * n \in \overline{R}) \Rightarrow a \in \overline{R}$

Any subset of $\mathbb{N}^*$ having property (ii) is called *inductive.* Bar induction says that if R is decidable then $\overline{R}$ is the smallest inductive set containing R, i.e. the smallest subset of $\mathbb{N}^*$ satisfying (i) and (ii). This statement is known as *Decidable Bar Induction* and is formalized as follows: for $R, A \subseteq \mathbb{N}^*$ and $b \in \mathbb{N}^*$ we have

$$(R\vee\neg R)\wedge(R \subseteq A)\wedge(\forall_{\alpha\in\mathbb{N}^{\mathbb{N}}}\exists_{x\in\mathbb{N}}.b*\overline{\alpha}(x) \in R)\wedge(\forall_{a\in\mathbb{N}}.(\forall_{n\in\mathbb{N}}.a*n \in A) \Rightarrow a \in A) \Rightarrow b \in A$$

Classically this statement is true but intuitionistically it is a nontrivial statement, known as the Brouwer Bar Theorem [9]. The methods we employ in this paper allow one to prove that Bar Induction holds for N^* internal to the free topos, however due to time restrictions this will be done in a sequel.

There are equivalent alternate forms of Bar induction that we shall not discuss here (see [9] p. 54-56). There is, however a stronger version called *Monotonic Bar Induction* where the requirement that R be decidable is replaced with R being "monotonic", i.e. a down set in the tree $\mathbb{N}^*$.

Both Monotonic and Decidable Bar Induction have, what appears to be, a weakening by taking $A = R$, but it turns out that this weaker version is equivalent to the full bar induction [14]. For clarity we state the "weak" version of Monotonic Bar Induction, where $R \subseteq \mathbb{N}^*$ and $b \in \mathbb{N}^*$:

Monotonic Bar Induction

$$(\forall_{a\in\mathbb{N}^*}(a \in R \iff \forall_{n\in\mathbb{N}} a * n \in R)) \wedge (\forall_{\alpha\in\mathbb{N}^{\mathbb{N}}}\exists_{x\in\mathbb{N}}.b * \overline{\alpha}(x) \in R) \Rightarrow b \in R$$

The first conjunct says that R is "monotonic inductive" (i.e. down-closed and inductive). This is the form we are interested in formalizing in the internal logic of the free topos.

Below we will give an equivalent formalization of Monotonic Bar Induction as the statement that a certain locale is spatial and isomorphic to the locale of opens of $\mathbb{N}^{\mathbb{N}}$ (with the usual product topology).

We state a general bar induction principle for sub-trees of N^*. Given a sub-tree P of N^* we define the following notation:

1. 1_P is the top element of P
2. Given $p \in P$ we denote by V_p the set of infinite paths in P beginning at p.
3. For a monotonic $R \subseteq P$ and $\alpha \in V_p$ we say that "α goes through R" if there exists some node of α in R.
4. We say that "R bars $p \in P$" if every path in V_p goes through R.

Monotonic Bar Induction for P can now be stated as follows: A monotonic inductive subset of P contains p if and only if it bars p.

4.3 Continuity Principles

The continuity principle for a space S is the statement that all functions $S \longrightarrow S$ are continuous. In a type theory we regard this as derived rule of inference:

$$\frac{\vdash f : S \longrightarrow S}{\vdash \text{“}f \text{ is continuous”}}$$

The goal of this work is to demonstrate that the continuity principle holds in the free topos for the spaces $R, 2^N, N^N$, and any space which can be suitably constructed from N (the meaning of which will be made precise below). In addition the principle for R and N^N has been demonstrated in other toposes (see [11] p. 324).

Chapter 5

Topologies and Formal Spaces

For this chapter (and the next) we take special care to restrict ourselves to the logic of toposes. In particular the powerset of 1 will be denoted as Ω and be thought of as a frame (not a boolean algebra). We will be arguing informally in this chapter, but all the results may be restated formally within the internal logic of a topos (relativization). In the next chapter we will make this transition into the internal logic of a topos more explicit.

5.1 Topologies on a Poset

Let P be a poset and $D(P)$ be the object of all down subsets of P. The poset $D(P)$ is closed under arbitrary intersections and so forms a complete lattice. In fact it is also a Heyting algebra, and hence we may think of it as a locale.

Remark 5.1.1. For those familiar with the terminology: $D(P)$ is in bijection with order-preserving maps $P^{op} \longrightarrow \Omega$; these maps $P^{op} \longrightarrow \Omega$ are the same as subterminal presheaves on P (definition below). This gives an alternate description $D(P) = \mathrm{Sub}_{Pr(P)}(1)$.

We have a canonical order-preserving map $P \longrightarrow D(P)$ sending x to its principal down set $\downarrow x$. For notational convenience we use the following shorthand notation:

- $D(x)$ for the set of down sets of $\downarrow x$

- $x \downarrow y = \downarrow x \cap \downarrow y$
- For $R \in D(P)$ put $R\restriction_x = R \cap \downarrow x$

Definition 5.1.2. A *Topology* on a poset P is given by a function J assigning to each $x \in P$ a subset $J(x) \subseteq D(x)$ satisfying the following:

1. $\downarrow x \in J(x)$.
2. If $R \in J(x)$ and $y \leq x$ then $R\restriction_y \in J(y)$.

We call J a Grothendieck topology when J also satisfies

(L) If $R \in J(x)$, $S \in D(x)$, and for each $y \in R$, $S\restriction_y \in J(y)$ then $S \in J(x)$.

A category equipped with a *Grothendieck topology* is called a *site.*

Remark 5.1.3. This is a special case of the more general notion of site on an arbitrary category (for e.g. see [8] chapter C2).

For us the term "topology" will always mean one that is not necessarily Grothendieck. When we wish to assume a topology is Grothendieck or stress that a particular topology is Grothendieck then we will use the term "Grothendieck topology".

Property (L) is called the *locale character condition.* Note that 1. 2. and (L) are all preserved under intersections of topologies. In particular, given a topology J one may form the Grothendieck completion $\overline{J}$, which is the intersection of all Grothendieck topologies containing J. $\overline{J}$ is the smallest Grothendieck topology containing J.

Alternatively one may try to form $\overline{J}$ by inductively adding all covers which are required by (L). In general there is no guarantee this process may stop after even a transfinite induction ([8] p. 541, when P is a "large" category). However the posets and topologies we consider will be small (in fact countable) and so this method will always work for us.

An immediate consequence of (L) is that the set of covers of $p \in P$ is closed upwards and under intersections, that is the covers of p form a filter in $D(p)$.

Given a join semi-lattice (complete or otherwise) L, we may form the *canonical* Grothendieck topology $J(x) = \{A \subseteq \downarrow x : \bigvee A \text{ exists and equals } x\}$ and the *coherent* Grothendieck topology $J(x) = \{A \subseteq \downarrow x : A \text{ is finite and } \bigvee A = x\}$.

Given a topology J we may ask for down sets which are suitably closed under covers, meaning down sets A such that if $A\restriction_x \in J(x)$ then $x \in A$. Such down sets will be called J-ideals, and the collection of them will be denoted by Ideal(P, J), which is ordered by inclusion. J-ideals are closed under arbitrary intersection and so form a complete sub-lattice of $D(P)$ (and hence also form a locale).

Remark 5.1.4. In terms of presheaves, if J is Grothendieck then J-ideals correspond to J-sheaves (definition below), and so Ideal$(P, J) = \text{Sub}_{Sh(P,J)}(1)$ is also a locale, under inclusion.

Note that J-ideals are not ordinary poset ideals in that they need not be directed upwards. But if P is a join-semi-lattice, and J is the coherent topology, then J-ideals correspond to ordinary ideals. Moreover if P is a complete join-semilattice and J is the canonical topology then J-ideals correspond exactly to the principal ideals of P.

Given a down-set A we may form the closure of A with respect to J: $cl_J(A) = \{x \in P : A\restriction_x \in J(x)\}$, which will be written as $\overline{A}$ when the topology J is understood (one can also define $cl_J(A)$ to be the intersection of all J-ideals containing A).

Theorem 5.1.5. *The operator cl_J regarded as a map $D(P) \longrightarrow D(P)$ is a nucleus of locales. The sublocale corresponding to cl_J is precisely Ideal(P, J) and there is a canonical order map $P \longrightarrow$ Ideal$(P, J))$ sending x to $\overline{\downarrow x}$.*

Proof. The fact that cl_J is a nucleus is clear. By (2.3.3) the corresponding sublocale of $D(P)$ is given by the fixed points of cl_J, i.e. the J Ideals of P. The canonical order-preserving map is just the composite $P \longrightarrow D(P) \xrightarrow{cl_J} \text{Ideal}(P, J)$. □

This allows us to think of topologies on a poset as a kind of presentation of a locale. Conversely, each locale X can be presented as a Grothendieck topology in a canonical way, as the canonical topology on its underlying poset. This works since the J Ideals will then be principle ideals of X (and every lattice is isomorphic to its lattice of principle ideals).

5.2 Filters and J-Prime maps

The following discussion is inspired from [5]. A filter in a poset is an up-set which is directed downwards, meaning that any two elements with an upper bound have a lower bound. In the case of a meet-semilattice with a top element this reduces to the usual definition of filter. Since our posets will always have a top element a filter for us will be an up-set in which any two elements have a lower bound.

In this section we give a generalization of the notion of filter in a poset (which is essentially a model of a geometric theory in a frame, and is rephrasing of the notion of model in [5]). The generalization is a natural abstraction of the properties of a filter in a meet-semilattice.

Definition 5.2.1. Given a locale H, an *H-valued filter* (or H-filter) of P is an order-preserving map $P \xrightarrow{\varphi} H$ satisfying $\varphi(x) \wedge \varphi(y) \leq \bigvee\{\varphi(z) : z \in x \downarrow y\}$. In addition if J is a topology on P then φ is said to be *J-prime* if φ maps covers into joins, i.e. if $R \in J(x)$ then $\varphi(x) = \bigvee\{\varphi(y) : y \in R\}$. The collection of J-prime maps $(P, J) \longrightarrow H$ will be denoted by $\mathrm{Prime}(P, J; H)$.

Remark 5.2.2. Definition 5.2.1 is related to the notion of a continuous flat morphisms functors and the general notion of morphisms of sites (see [8] for details).

The reason for calling the above an H-filter is because Ω-filters are the characteristic maps of the standard filters in a poset.

A J-prime map φ is automatically a filter (see below), and is named as prime because if P taken to be a join semi-lattice and J to be the coherent topology, then J-prime Ω-valued maps correspond exactly to prime filters of P (and if J is the canonical topology then J-Prime Ω-valued maps correspond to complectly prime filters). Also when discussing the actual subset of P characterized by a J-prime map, we will use the term *J-prime filter.*

Proposition 5.2.3. *A map $P \longrightarrow H$ is an H-filter if and only if it is J-prime for J the minimal topology on P. Also J and J' are topologies on P with $J \subseteq J'$ then J'-prime maps are automatically J-prime. Consequently all J-prime maps are filters.*

Proof. If $\varphi : P \longrightarrow H$ is an H-filter then $\varphi(x) = \varphi(x) \wedge \varphi(x) \leq \bigvee\{\varphi(z) : z \in x \downarrow x = \downarrow x\}$ which shows that φ is J-prime for the minimal topology J. For the converse, given any topology J and φ a J-prime map:

$$
\begin{aligned}
\varphi(x) \wedge \varphi(y) &= \bigvee\{\varphi(z) : z \leq x\} \wedge \bigvee\{\varphi(w) : w \leq y\} \\
&= \bigvee\{\varphi(z) \wedge \bigvee\{\varphi(w) : w \leq y\} : z \leq x\} \\
&= \bigvee\{\bigvee\{\varphi(z) \wedge \varphi(w) : w \leq y\} : z \leq x\} \\
&= \bigvee\{\varphi(z) \wedge \varphi(w) : z \leq x \text{ and } w \leq y\} \\
&\leq \bigvee\{\varphi(z) \wedge \varphi(z) : z \leq x \text{ and } z \leq y\} \\
&= \bigvee\{\varphi(z) : z \in x \downarrow y\}
\end{aligned}
$$

□

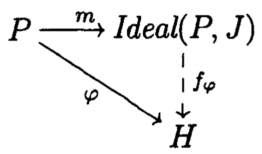
Lemma 1. *Given a J-prime map $\varphi : (P, J) \longrightarrow H$ and $A \in D(P)$ we have*

$$\bigvee\{\varphi(x) : x \in A\} = \bigvee\{\varphi(x) : x \in \overline{A}\} \tag{6}$$

Proof. Since $A \subseteq \overline{A}$ we clearly have the "$\leq$" direction. For the converse, if $x \in \overline{A}$ then $A{\restriction_x} \in J(x)$. In which case $\varphi(x) = \bigvee\{\varphi(y) : y \in A{\restriction_x}\} \leq \bigvee\{\varphi(y) : y \in A\}$. □

The next theorem is fundamental.

Theorem 5.2.4 (Formal Space). *The canonical map $m : P \longrightarrow Ideal(P, J)$ is J-prime and any J-prime map $P \xrightarrow{\varphi} H$ uniquely factors through the canonical one via the frame morphism $Ideal(P, J) \xrightarrow{f_\varphi} H$ with $f_\varphi(K) = \bigvee\{\varphi(x) : x \in K\}$.*



This gives an isomorphism $Prime(P, J; H) \cong Frame(Ideal(P, J), H) \cong Loc(H, Ideal(P, J))$.

Proof. The canonical map $m : P \longrightarrow \text{Ideal}(P, J)$ is $p \mapsto \overline{\downarrow p}$. First we check that m is a J-prime map. To this end take $R \in J(x)$ and note that then $\overline{R} = \overline{\downarrow x}$ ($R \subseteq \downarrow x$ gives one direction, and $x \in \overline{R}$ gives the other).

$$
\begin{aligned}
m(x) &= \overline{\downarrow x} \\
&= \overline{R} \\
&= \overline{\bigcup\{\downarrow y : y \in R\}} \\
&= \bigvee\{\overline{\downarrow y} : y \in R\} \\
&= \bigvee\{m(y) : y \in R\}
\end{aligned}
$$

So m is a J-Prime map. Next we must check that f_φ is indeed a frame morphism:

$$
\begin{aligned}
f_\varphi(R_1) \wedge f_\varphi(R_2) &= \bigvee\{\varphi(x) : x \in R_1\} \wedge \bigvee\{\varphi(y) : y \in R_2\} \\
&= \bigvee\{\varphi(x) \wedge \varphi(y) : x \in R_1 \text{ and } y \in R_2\} \\
&= \bigvee\{\bigvee\{\varphi(z) : z \in x \downarrow y\} : x \in R_1 \text{ and } y \in R_2\} \\
&= \bigvee\{\varphi(z) : z \in R_1 \text{ and } z \in R_2\} \\
&= f_\varphi(R_1 \cap R_2)
\end{aligned}
$$

$$
\begin{aligned}
f_\varphi(\bigvee_i K_i) &= \bigvee\{\varphi(x) : x \in \bigvee_i K_i\} \\
&= \bigvee\{\varphi(x) : x \in \overline{\bigcup_i K_i}\} \\
&= \bigvee\{\varphi(x) : x \in \bigcup_i K_i\} \qquad \text{by Lemma 1} \\
&= \bigvee_i\{\varphi(x) : x \in K_i\} \\
&= \bigvee_i f_\varphi(K_i)
\end{aligned}
$$

Finally we check that $\varphi = f_\varphi m$ (this also shows uniqueness):

$$\begin{aligned} f_\varphi(m(x)) &= \bigvee\{\varphi(y) : y \in \overline{\downarrow x}\} \\ &= \bigvee\{\varphi(y) : y \in \downarrow x\} \\ &= \varphi(x) \end{aligned}$$

The only thing left is to note that the composition of a J-prime map with a frame map is again J-prime (since frame maps preserve all joins and hence all covers), this establishes the bijection. □

The Grothendieck topology J naturally induces a classical topology τ_J on the set Prime$(P, J; H)$ given by the basic opens $U_x = \{\varphi \in Prime(P, J; H) : \varphi(x) = \top\}$, where $x \in P$.

Remark 5.2.5. In light of 5.2.4, when $H = \Omega$ J-prime maps are the points of the locale Ideal(P, J) and the spaces pt(Ideal(P, J)) and Prime$(P, J; \Omega)$ coincide.

We call the locale Ideal(P, J) the *Formal Space* of the topology (P, J).

5.3 Examples of Topologies

Example 5.3.1 (Monotonic Inductive Topology). Consider the case that P is sub-tree of N^*. We describe the *Monotonic Inductive* topology on P: for $p \in P$ let p^+ denote the set of immediate successors of p. The only covers of $p \in P$ are $\downarrow p$ and $\downarrow p^+$. We will denote this topology by MI_P.

Now a MI_P-ideal is a down set of P (i.e. is monotonic), say R, which satisfies the following: $p^+ \subseteq R \Rightarrow p \in R$ (i.e. R is inductive). Thus MI_P-ideals are monotonic inductive subsets of P (in the sense of section 4.2).

A MI_P-Prime (Ω-valued) filter is first and foremost a filter in P, i.e. a path starting at the top of P (since P is a tree a filter must be a path) which satisfies $p \in R \Rightarrow \exists_{q \in p^+}.q \in R$. Thus MI_P-prime filters are infinite paths through P (beginning at the top of P).

Example 5.3.2 (Bar-Closed (Grothendieck) Topology). The Bar-Closed Topology, denoted by Bar_P on a sub-tree P of N^* has as covers of $p \in P$ the down-sets of P which bar p. These trivially include the covers in the monotonic inductive topology, and so $MI_P \subseteq Bar_P$.

A Bar_P-ideal is now an monotonic subset, R, of P for which R bars $p \Rightarrow p \in R$. We will call these subsets "bar closed". Note that the principle of monotonic bar induction for P can now be restated as saying that a MI_P-ideal is also a Bar_P ideal.

The Bar_P-filters are also infinite paths in P.

We consider two specific cases of the monotonic inductive topology which we find interesting:

Example 5.3.3. Take $P = 2^*$ the tree of finite binary sequences. A monotonic inductive prime filter is an infinite binary sequence (i.e. element of $2^{\mathbb{N}}$). And so we view the locale Ideal$(2^*, MI_{2^*})$ as the "formal Cantor space".

Example 5.3.4. Similarly if we take $P = \mathbb{N}^*$ then a $MI_{\mathbb{N}^*}$-prime filter is an infinite sequence (element of $\mathbb{N}^{\mathbb{N}}$). And so we view the locale Ideal$(\mathbb{N}^*, MI_{\mathbb{N}^*})$ as the "formal Baire space".

Since the locales Ideal$(\mathbb{N}^*, MI_{\mathbb{N}^*})$ and Ideal$(\mathbb{N}^*, Bar_{\mathbb{N}^*})$ have the same points it is natural to ask why we choose the former as the "formal space" rather than the latter. The answer is that it is the former for which one can prove intuitionistic versions of certain classical results (Such as every compact, stably locally compact separable locale X is the image (under a locale map) of the formal cantor space [5]).

Theorem 5.3.5. *Given a sub-tree P of N^* the locale Ideal(P, Bar_P) is spatial. If the order relation on P is decidable and every $p \in P$ has at least two distinct successors then Ideal(P, Bar_P) has a basis given by $\{\downarrow p : p \in P\}$. In particular the space pt(Ideal(P, Bar_P) has basis consisting of the sets $V_p = pt(\downarrow p)$.*

Moreover Ideal(P, MI_P) is spatial if and only if bar induction for P holds (i.e. Ideal$(P, MI_P) = Ideal(P, Bar_P)$.

Proof. To see that Ideal(P, Bar_P) is spatial, it suffices to show that the frame map $pt^* : \text{Ideal}(P, Bar_P) \longrightarrow pt(P, Bar_P)$ is injective. So suppose that $R, S \in \text{Ideal}(P, Bar_P)$ are such that $pt(R) = pt(S)$.

Note that a prime map α in P (infinite path in P starting at 1_P) is a point of R if and only if α goes through R. Now suppose that $p \in R$. Then every $\alpha \in V_p$ is a point of R, hence a point of S an so goes through S. This means that S bars p and so $p \in S$. Therefore $R \subseteq S$. By symmetry R and S must be equal and so Ideal(P, Bar_P) is spatial.

Since every $R \in \text{Ideal}(P, Bar_P)$ can be written as $R = \bigcup_{p \in P} cl_{Bar_P}(\downarrow p) = \bigvee_{p \in P} cl_{Bar_P}(\downarrow p)$, to show that the $\downarrow p$'s form a basis it suffices to show that each $\downarrow p$ is in fact a Bar_P-ideal (i.e. bar closed). This follows from the decidability of the order on P: Suppose that $\downarrow p$ bars q, then in particular p and q have a lower bound and we conclude that $q \leq p \vee p \leq q$. Suppose that $\neg(q \leq p)$, then we have $p \leq q$. Since q has at least two distinct successors, one of these successors, say q' will not be above p (otherwise they couldn't both be successors of q). Now pick some $\alpha \in V_{q'}$. This α is in V_q as well and cannot go through $\downarrow p$, contradicting the assumption that $\downarrow p$ bars q. Therefore we have $\neg\neg(q \leq p)$ which implies $q \leq p$. It follows that $\downarrow p$ is bar closed.

Now clearly if bar induction for P holds then Ideal(P, MI_P) is spatial. For the converse, suppose that Ideal(P, MI_P) is spatial, and that R is a monotone inductive subset of P. Recall that $\overline{R}$ is the set of all elements of P for which R is a bar. Since $pt(R)$ is the set of sequences which go through R we have $pt(R) = pt(\overline{R})$, and hence the assumption of spatiality implies that $R = \overline{R}$. This proves monotone bar induction holds in P. □

In the case that P is $\mathbb{N}^*$ (or 2^*) the opens V_p, for some finite sequence p, are just the usual basis elements of all sequences with p as an initial segment. Therefore Ideal(P, Bar_P) is the usual topological Baire Space $\mathbb{N}^{\mathbb{N}}$ (or Cantor space $2^{\mathbb{N}}$).

Corollary 5.3.6. *Monotonic Bar Induction for $\mathbb{N}^*$ is equivalent to the formal Baire Space being spatial and isomorphic to the locale of opens of $\mathbb{N}^{\mathbb{N}}$ (with the product topology), and similarly for Cantor space.*

Example 5.3.7. Take $P =$ the poset of (non-trivial) open intervals in $\mathbb{Q}$ (including infinite intervals) ordered by inclusion (so that $(-\infty, \infty)$ is the top of P). Given an interval (p, q) define the down-set of *proper subintervals* of (a, b) to be $SI(p, q) := \{(a, b) \in P : p < a < b < q\}$.

The Dedekind topology denoted by J_D will be as follows: each interval (p, q) (possibly infinite) will have covers of the form: $SI(p, b) \cup SI(a, q)$, where $p \leq a < b \leq q$. Note that this includes as a special case $SI(p, q)$.

Next we show J_D-prime filters correspond to Dedekind cuts in the following sense (see [5]):

Definition 5.3.8. A *Dedekind Cut* is a pair (L, U) of disjoint subsets of $\mathbb{Q}$ satisfying

(i) L and U are "close together": $p < q \Rightarrow p \in L \vee q \in U$

(ii) L and U are open and inhabited

To this end let F be a J_D-filter and define $L, U \subseteq \mathbb{Q}$ by $r \in L \iff (r, \infty) \in L$ and $r \in U \iff (-\infty, r) \in F$. We show (L, U) is a Dedekind cut.

First suppose that $p < q$. Since $(-\infty, \infty) \in F$ and $(-\infty, \infty)$ is covered by $SI(-\infty, q) \cup SI(p, \infty)$ there exists $(a, b) \in SI(-\infty, q) \cup SI(p, \infty)$ with $(a, b) \in F$. But since F is closed upwards this implies $(-\infty, q) \in F \vee (p, \infty) \in F$, in which case $p \in L \vee q \in U$. This establishes (i).

L and U are disjoint since if $p \in L \cap U$ then we'd have $(-\infty, p), (p, \infty) \in F$, which would contradict the fact that F is a filter (since these intervals are disjoint and hence have no lower bound). Note that this automatically implies that L is closed downwards and U is closed upwards, for if $p < q \in L$ then $p \in L \vee q \in U$. But $\neg(q \in U)$ and so we conclude $p \in L$, hence L is closed downwards (similarly U is closed upwards).

Since L is closed downwards, to show openness it suffices to show that if $p \in L$ then there exists some $q \in \mathbb{Q}$ with $q > p$ and $q \in L$. So suppose that $p \in L$, then $(p, \infty) \in F$ and since (p, ∞) is covered by $SI(p, \infty)$ there exists some $(a, b) \in SI(p, \infty)$ in F. Again this implies $(a, \infty) \in F$, i.e. $a \in L$. Further $a > p$ and so L is open. Similarly one shows that U is open as well.

To see that L and U are inhabited note that since $(-\infty, \infty)$ is covered by $SI(-\infty, \infty)$ there exists $(p, q) \in SI(-\infty, \infty)$ with $(p, q) \in F$, in which case both $(-\infty, q), (p, \infty) \in F$; i.e. $p \in L$ and $q \in U$.

This shows us that every J_D-prime filter corresponds to some Dedekind cut. For the converse (that all cuts are realized) let (L, U) be a Dedekind cut and define $F \subseteq P$ as follows:

- For $q \in \mathbb{Q}$, $(-\infty, q) \in F \iff q \in U$
- For $p \in \mathbb{Q}$, $(p, \infty) \in F \iff p \in L$
- For $p, q \in \mathbb{Q}$, $p < q$, $(p, q) \in F \iff p \in L \wedge q \in U$

It is clear that if F is indeed a J_D-prime filter then it corresponds to the Dedekind cut (L, U). It is straightforward to show that F is a filter in P, we show here that it is J-prime. To this end suppose that $(p, q) \in F$ and that $p \leq a < b \leq q$ so that (p, q) is covered by $SI(p, b) \cup SI(a, q)$ In particular $a < b$ and so $a \in L \vee b \in U$. If $a \in L$ then $(a, q) \in F$, but this is not enough since $(a, q) \notin SI(a, q)$. But since L and U are open, we can find $a' > a$ and $q' < q$ with $a' \in L$ and $q' \in U$, in which case $(a', q') \in SI(a, q)$ and $(a', q') \in F$. Similarly if $b \in U$ one can find $(p', b') \in SI(p, b) \cap F$.

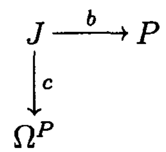
This establishes the fact that the J_D-prime filters are exactly the Dedekind cuts. Regarding the J_D-prime filters as the *points* of the locale Ideal(P, J_D) we've not only constructed the reals, but also endowed them with a topology. This topology arising on the reals is the usual topology. To see why this is so, the locale of J_D ideals has a basis consisting of principle ideals $\downarrow(p, q)$. The points of such a locale consist of the Dedekind cuts representing a real between p and q (i.e. the interval (p, q) of reals). These opens of reals form a basis for the usual topology on R.

5.4 Formalization

As stated at the beginning of this chapter, the above definitions and results may be formalized in a topos. Here we sketch a few highlights on how this is done.

A poset P is taken to be an internal poset object. The "down sets" of P are a class of subobjects of P and the collection of all down sets in P, denoted by $D_{\mathcal{E}}(P)$, is an actual subobject of Ω^P.

To formalize a topology on P, we may take J not to be a function as above, but instead the graph of a function, i.e. a span:



which satisfies appropriate diagrams corresponding to the definition of a topology (see [8] C 2.4 p.578 for details). Also one may define internal Grothendieck topologies by asking that the appropriate diagram corresponding to the condition (L) commute. There is one extra condition we ask of internal topologies which is that the maps b and c be jointly monic. This is just to say that J contains no extra elements other than what would be in the graph of an (external) topology.

The J-ideals form a subobject of $D_{\mathcal{E}}(P)$, denoted by $\mathrm{Ideal}_{\mathcal{E}}(P, J)$, are definable from the internal logic of $\mathcal{E}$. J-prime maps are easy to define, an H valued J-prime map $\varphi : P \longrightarrow H$ (for H an internal locale), is simply a order-preserving map for which figure 1 commutes.

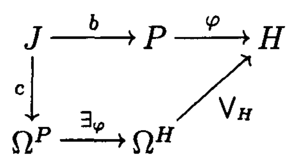


Figure 1: Condition for φ to be J-prime

The "collection" of H-valued J-prime maps forms a subobject $\mathrm{Prime}_{\mathcal{E}}(P, J; H)$ of H^P.

Chapter 6

Presheaf and Sheaf Toposes on Preorders

6.1 Using the Internal Language

From here on, until otherwise specified we will be assuming some base topos $\mathcal{E}$ in which we work using the internal logic (or direct categorical construction). Often we will argue or define notions informally and when it is unclear we will describe how our methods can be formalized.

We will use the internal logic to describe objects of $\mathcal{E}$. For example, given an internal poset P in $\mathcal{E}$, we can define a formula $\varphi(x)$ (of $L(\mathcal{E})$), that says "x is a down set of P", and thus the "set of down sets of P" is an actual object in $\mathcal{E}$ given by the term $\{x \in \Omega^P : \varphi(x)\}$ (i.e. a subobject of Ω^P). We may at times just refer to the "object of down sets of P".

Equally important for reasoning about any topos $\mathcal{E}$ is the use of the internal logic to describe arrows of $\mathcal{E}$ (see 3.3.1 (iii)). An arrow between objects $A \xrightarrow{f} B$ can be given via the internal language as a provably functional relation (term) $f : \boldsymbol{A} \longrightarrow \boldsymbol{B}$. At times we will describe this term using set-theoretic notation, such as $f(x) = t(x)$ or $x \mapsto t(x)$ (where $t(x)$ is some term depending on x), to mean we are considering $f = \{\langle x, y\rangle \in \boldsymbol{A} \times \boldsymbol{B} : y = t(x)\}$.

6.2 Internally indexed Coproducts in a topos.

Not all toposes are cocomplete, but all are so-called "internally cocomplete". We will only have need to discuss here internal coproducts. Given objects X_i as i varies over another object I in a topos $\mathcal{E}$, how shall we define the "internally indexed coproduct" $\coprod_{i\in I} X_i$? The X_i's should be uniformly defined in terms of I, in that we have map $\iota : I \longrightarrow \Omega^X$ for some object X in which each X_i embeds, satisfying $\iota(i) = X_i$. In this case we can take the transpose, $\iota^{\wr}$, of ι and consider the kernel, $\ker(\iota^{\wr}) = \{\langle i, x\rangle \in I \times X : \iota(i)(x) = \top\} = \{\langle i, x\rangle \in I \times X : x \in X_i\}$. This is exactly what the coproduct "should" be and we define

Definition 6.2.1. For ι as above, the *internal coproduct* $\coprod_{i\in} X_i := ker(\iota^*)$.

The injections $\kappa_i : X_i \longrightarrow \coprod_{i\in I} X_i$ are defined just as in set theory: $\kappa_i(x) = \langle i, x\rangle$.

Note that this serves only as a shorthand notation to facilitate the use of "set-theoretic like arguments".

6.3 Presheaves on P

Definition 6.3.1. A *Presheaf* on P is a pair of arrows $f : F \longrightarrow P$ and $\upharpoonright : Q_f \longrightarrow F$ where Q_f is defined by the pull back:

$$\begin{array}{ccc} Q_f & \longrightarrow & \leq_P \\ \downarrow & & \downarrow \\ P \times F & \xrightarrow{1_P \times f} & P \times P \end{array}$$

Note: Q_f is the object given by the closed term of the internal language of $\mathcal{E}$: $\{\langle p, x\rangle \in P \times F : p \leq f(x)\}$. Alternatively we can describe Q_f as the coproduct $\coprod_{p\in P} f^*(\downarrow p)$, where $f^* : \Omega^P \longrightarrow \Omega^F$ and $\downarrow p$ is the obvious subobject of P.

We write $x\upharpoonright_p$ as shorthand for $\upharpoonright\langle p, x\rangle$ (when we write $x\upharpoonright_p$ it will be implicitly assumed that $p \leq f(x)$) and require f and $\upharpoonright$ satisfy the equations

1. $f(x\upharpoonright_p) = p$

2. $x = x\upharpoonright_{f(x)}$

3. *For* $q \leq p$, $(x\restriction_p)\restriction_q = x\restriction_q$

For each $p \in P$ put $F(p) := \{x \in F : f(x) = p\}$ (a subobject of F). A morphism of presheaves $F \longrightarrow G$ is given by a map $F \xrightarrow{\alpha} G$ in $\mathcal{E}/P$ which satisfies $\alpha(x\restriction_p) = \alpha(x)\restriction_p$.

For example $P \xrightarrow{1_P} P$ is a presheaf with $p\restriction_q = q$. This is called the *terminal presheaf* on P. More generally given an object A the *constant presheaf* on P with value A is given by the projection $A \times P \longrightarrow P$ with $\langle a, p\rangle\restriction_q = \langle a, q\rangle$.

Given any presheaf $F \xrightarrow{f} P$ and $p \in P$ we define the presheaf $F\restriction_p \xrightarrow{f\restriction_p} \downarrow p$ by putting $F\restriction_p = \{x \in F : f(x) \leq p\}$ and taking $f\restriction_p$ to be the restriction of f to this subobject of F, i.e. $f\restriction_p$ is the pullback

$$\begin{array}{ccc} F\restriction_p & \rightarrowtail & F \\ {\scriptstyle f\restriction_p}\downarrow & & \downarrow{\scriptstyle f} \\ \downarrow p & \rightarrowtail & P \end{array}$$

The collection of presheaves on P and their morphisms forms a category which we will denote by $Pr_{\mathcal{E}}(P)$. In this category monos, epi's, and iso's are all given by underlying monos, epi's and iso's (respectively) in $\mathcal{E}$.

Subpresheaves are given by monics, and conversely any monic from an object G to a presheaf F, say $G \xrightarrow{i} F$, induces a presheaf structure on G via $G \xrightarrow{i} F \xrightarrow{f} P$. The presheaves $F\restriction_p$ are always subpresheaves of F. The "collection" of subpresheaves of a presheaf F forms a subobject of Ω^F denoted by $Sub_{Pr_{\mathcal{E}}(P)}(F)$. More generally the "collection" of presheaf maps $G \longrightarrow F$ forms a subobject of F^G denoted by $Pr_{\mathcal{E}}(P)(G, F)$. One can construct this subobject using the internal langauge as the term $\{x \in F^G : g = fx\}$.

The following theorem can be found in [1] (with general sites) for the case $\mathcal{E} = \mathbf{Set}$, but the logic used is intuitionistically valid and hence formalizes to any topos.

Theorem 6.3.2. *The category* $Pr_{\mathcal{E}}(P)$ *is a topos with canonical subobjects, and the topos structure is given as:*

1. *The terminal presheaf is given by* $P \xrightarrow{1_P} P$

2. *The product of* $F \xrightarrow{f} P$ *and* $G \xrightarrow{g} P$ *is given by the pullback:*

$$\begin{array}{ccc} F \times_P G & \longrightarrow & F \\ \big\downarrow & & \big\downarrow f \\ G & \xrightarrow{\;g\;} & P \end{array}$$

where $\langle x, y\rangle\restriction_p = \langle x\restriction_p, y\restriction_p\rangle$.

3. *The pullback of* $F \xrightarrow{\alpha} H$ *and* $G \xrightarrow{\beta} H$ *is given by the pullback in* $\mathcal{E}$*:*

$$\begin{array}{ccccc} F \times_H G & \longrightarrow & F & & \\ \big\downarrow & & \big\downarrow \alpha & \searrow f & \\ G & \xrightarrow{\;\beta\;} & H & & \\ & \searrow g & & \searrow h & \\ & & & & P \end{array}$$

where $\langle x, y\rangle\restriction_p = \langle x\restriction_p, y\restriction_p\rangle$.

4. *The subobject classifier* $\underline{\Omega}$ *(We use the underline to distinguish* $\underline{\Omega}$ *from the subobject classifier* Ω *in* $\mathcal{E}$*) is given by*

$$\begin{array}{ccc} \coprod_{p\in P} D_{\mathcal{E}}(p) & \qquad & \langle p, K\rangle \\ \big\downarrow & & \big\downarrow \\ P & & p \end{array}$$

with $\langle p, K\rangle\restriction_q = \langle q, K\restriction_q\rangle$ *and* $1 \xrightarrow{\top} \underline{\Omega}$ *is given by* $\downarrow(\;) : P \longrightarrow \underline{\Omega}$.

Note: to define this coproduct we use $\iota : P \longrightarrow \Omega^P$ *with* $\iota(p) = D_{\mathcal{E}}(p)$*. Also* $D_{\mathcal{E}}(p)$ *is the object of down subsets of* $\downarrow p$.

5. *The power presheaf,* $\mathcal{P}(F)$*, of a presheaf* F *is given by*

$$\begin{array}{ccc} \coprod_{p\in P} Sub_{Pr_{\mathcal{E}}(P)}(F\restriction_p) & \qquad & \langle p, G\rangle \\ \big\downarrow & & \big\downarrow \\ P & & p \end{array}$$

with $\langle p, G\rangle \restriction_q = \langle q, G\restriction_q\rangle$ *and evaluation map* $ev_F : \mathcal{P}(F) \times_P F \longrightarrow \underline{\Omega}$ *is* $ev_F\langle\langle p, G\rangle, x\rangle = \langle p, \{q \leq p : x\restriction_q \in G\restriction_q\}\rangle$.

Note: to define this coproduct we use $\iota : P \longrightarrow \Omega^{\Omega^F}$ *where*

$\iota(p) = Sub_{Pr_{\mathcal{E}}(P)}(F\restriction_p) \subset \Omega^F$.

6. *The Exponent* G^F *of is given by*

$$\begin{array}{ccc} \coprod_{p\in P} Pr_{\mathcal{E}}(F\restriction_p, G) & & \langle p, \varphi\rangle \\ \downarrow & & \downarrow \\ P & & p \end{array}$$

with $\langle p, \varphi\rangle\restriction_q = \langle q, \varphi\restriction_q\rangle$.

Note: to define this coproduct we use $\iota : P \longrightarrow \Omega^{F\times G}$ *where*

$\iota(p) = \{x \in F \times G : x : F\restriction_p \longrightarrow G\}$ *(i.e.* ι *sends* p *to the object of graphs of maps* $F\restriction_p$ *to* G*, which is isomorphic to* $Pr_{\mathcal{E}}(F\restriction_p, G)$*).*

7. *If* $\alpha : F \longrightarrow \underline{\Omega}$ *then the canonical kernel of* α *is given by*

$ker(\alpha) = \{x \in F : \alpha(x) = \langle\downarrow f(x), f(x)\rangle\}$

8. *If* $G \overset{\alpha}{\rightarrowtail} F$ *then* $char(\alpha) : F \longrightarrow \underline{\Omega}$ *is given by*

$char(\alpha)(x) = \langle f(x), \{p \in\downarrow f(x) : x\restriction_p \in G\restriction_p\}\rangle$.

9. *The canonical subobjects are given by the canonical kernels above.*

The examples of constant presheaves above gives rise to a functor $\Delta : \mathcal{E} \longrightarrow Pr_{\mathcal{E}}(P)$ where $\Delta(A) = A \times P \xrightarrow{\pi_2} P$ and for $f : A \longrightarrow B$ we put $\Delta(f) = f \times 1_P$. This functor has a right adjoint called the *Global Sections* functor Γ, defined by $\Gamma(F) = Pr_{\mathcal{E}}(P)(1, F)$ (note that in $\mathcal{E}$, these are maps $\alpha : P \longrightarrow F$). Given a presheaf morphism $\beta : F \longrightarrow G$ put $\Gamma(\beta)(\alpha) = \beta\alpha$. If P has a top element, 1_P, we can identify the global sections of F with $F(1_P) = \{x \in F : f(x) = 1_P\}$ via an isomorphism $F(1_P) \longrightarrow \Gamma(F)$ which is defined using the internal logic as follows: $x \in F(1_P)$ corresponds to the presheaf map $P \longrightarrow F$ with $q \mapsto x\restriction_q$. Since any posets P we will consider will have a top element, we will assume from now on, that P has a top element 1_P.

Theorem 6.3.3. *With* Δ *and* Γ *as above, we have an adjointness* $\Delta \dashv \Gamma$ *with unit* $\eta_A : A \longrightarrow \Gamma\Delta(A) = 1_A$ *and counit* $\epsilon_F : \Delta\Gamma F \longrightarrow F$ *mapping* $\langle p, x\rangle \mapsto x\restriction_p$.

The standard way of defining presheaves are as functors $F : P^{op} \longrightarrow Set$, however since we insist on working internal to a topos $\mathcal{E}$ this definition does not work[1]. However we can still describe a presheaf using the data one would use to define such a functor F:

- Objects $F(p)$ for each $p \in P$.
- For each $q \leq p \in P$, restriction maps $\restriction^p_q : F(p) \longrightarrow F(q)$.
- If $r \leq q \leq p$ then $\restriction^q_r \circ \restriction^p_q = \restriction^p_q$.

We require that indexing the $F(p)$ for $p \in P$ must be internal to $\mathcal{E}$. For our purposes it will suffice that we have a map $P \longrightarrow \Omega^A$, for some object A, sending p to $F(P)$ as a subobject of A and another map $\{\langle q, p\rangle \in P \times P : q \leq p\} \longrightarrow \Omega^{A\times A}$ sending a pair $\langle q, p\rangle$ to the subobject of $A \times A$ which is the graph of $\restriction^p_q$. In this case we can construct our presheaf as the coproduct

$$\begin{array}{ccc} \coprod_{p\in P} F(p) & \qquad & \langle p, x\rangle \\ \downarrow & & \big\downarrow \\ P & & p \end{array}$$

with restrictions $\langle p, x\rangle\restriction_q = \langle q, x\restriction^p_q\rangle$. Similarly one may describe presheaf morphisms in this manner, although we will not have such need.

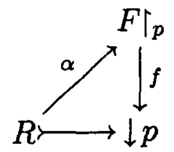
6.4 Sheaves on P

In this section we assume we have a site $\langle P, J\rangle$ internal to $\mathcal{E}$. Note this means we assume that J is Grothendieck.

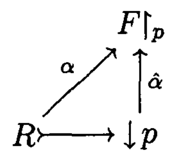
[1] This is not accurate. There is a theory of internal diagram categories in a fibration which generalizes the functor definition of a presheaf to internal categories. See the conclusion for ideas for further research utilizing this theory.

Definition 6.4.1. Given a presheaf F on P a *Compatible Family* in F is a triple $\langle p, R, \alpha\rangle$ (or more formally a pair $\langle\langle p, R\rangle, \alpha\rangle$) where $\langle R, p\rangle \in J$ and $\alpha : R \longrightarrow F{\restriction_p}$ is a presheaf morphism (when R is regarded as a subpresheaf of $\downarrow p$).

The requirement that α be a presheaf morphism implies that the following commutes:



This is all done internally: for each $\langle R, p\rangle \in J$ there is an object $Compat(R, p) = \{\alpha \in (F{\restriction_p})^R : \forall_{x\in R}.f(\alpha(x)) = x\}$. This itself defines a map[2] $J \longrightarrow \Omega^{\Omega^{F\times P}}$ which allows us to define the object of all compatible families in F: $Compat(F) := \coprod_{\langle R,p\rangle\in J} Compat(R, p)$.
If there exists a presheaf morphism $\hat{\alpha} :\downarrow p \longrightarrow F{\restriction_p}$ making



Then we say that $\hat{\alpha}$ is an *Amalgamation* of α. Just as with global sections above, the amalgamation is determined by its value at $p \in\downarrow p$.

Definition 6.4.2. A presheaf F is said to be *J-Separated* if every compatible family in F has at most one amalgamation. F is said to be a *J-Sheaf* if every compatible family has a unique amalgamation.

The collections of J-separated presheaves and of J-sheaves form full subcategories of $Pr_{\mathcal{E}}(P)$ denoted by $Sep_{\mathcal{E}}(P, J)$ and $Sh_{\mathcal{E}}(P, J)$ respectively.

In the case that F is a sheaf, the condition that $Compat(F)$ have unique amalgamations can be written in the internal language as a formula

$$\forall_{x\in Compat(F)}\exists!_{y\in F}.\text{“}y \text{ is the amalgamation of } x\text{”}$$

[2]Here one must use the correspondence of maps $R \longrightarrow F{\restriction_p}$ and their graphs as subobjects of $R \times F$.

This is a provably functional relation and so determines a unique map $Amalg : Compat(F) \longrightarrow F$.

It is convenient to write a compatible family as $\langle p, R; (x_r)_{r\in R}\rangle$, where $(x_r)_{r\in R} : R \longrightarrow F\restriction_p$ with $r \mapsto x_r$. The condition that this be a presheaf morphism is equivalent to $f(x_r) = r$ and $x_r\restriction_s = x_s$. An amalgamation of $\langle p, R, (x_r)_{r\in R}\rangle$ can be identified as some $x \in F(p)$ with the property that $x\restriction_r = x_r$ for all $r \in R$.

Note that $Compat(F)$ has a canonical map $Compat(F) \longrightarrow P$ sending a compatible family $\langle p, R; (x_r)_{r\in R}\rangle$ to p. Define restrictions on $Compat(F)$ by $\langle p, R; (x_r)_{r\in R}\rangle\restriction_q = \langle q, R\restriction_q; (x_r)_{r\in R\restriction_q}\rangle$. This makes $Compat(F)$ into a presheaf on P and $Amalg : Compat(F) \longrightarrow F$ a presheaf morphism.

In $Sh_{\mathcal{E}}(P, J)$ monos and epi's are given by underlying monos and epi's in $\mathcal{E}$. Just as with presheaves we can define the object of subsheaves $Sub_{Sh_{\mathcal{E}}(P,J)}(F)$ and the object of sheaf maps $Sh_{\mathcal{E}}(P, J)(G, F)$ (these are objects in $\mathcal{E}$).

Our main example is the sheaf of sections of a locale map; Let $Y \xrightarrow{f} X$ be a locale map internal to $\mathcal{E}$. We regard X as an internal site with the canonical Grothendieck topology. The sheaf of sections of f will be denoted by $\Sigma(f) \xrightarrow{\sigma} X$ and is given as follows: For each map $u : 1 \longrightarrow X$ we can internally construct the open sublocale $\downarrow u$ of X in the usual way. Then we consider the object $\Sigma(f)_u$, of internal locale maps $s : \downarrow u \longrightarrow Y$ making the following commute

$$\begin{array}{ccc} & & Y \\ & \overset{s}{\nearrow} & \downarrow f \\ \downarrow u & \rightarrowtail & X \end{array}$$

These are called the local sections of f with support u. Taking the coproduct[3] of all these local sections gives us $\Sigma(f)$:

$$\begin{array}{ccc} \coprod_{u\in X} \Sigma(f)_u X & \qquad & \langle s, u\rangle \\ \downarrow \sigma & & \downarrow \\ X & & u \end{array}$$

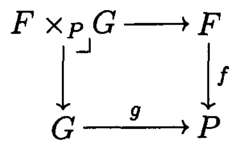
[3]The map used to define this coproduct is given from the internal language of $\mathcal{E}$ using the termconvenient $y = \Sigma(f)_x$ to define a provably functional relation $X \longrightarrow \Omega^{\Omega^{X\times Y}}$.

The following theorem is also in [1] for the case $\mathcal{E} = \mathbf{Set}$ with an intuitionistically valid proof.

Theorem 6.4.3. *The category $Sh_{\mathcal{E}}(P, J)$ is a topos with the following data:*

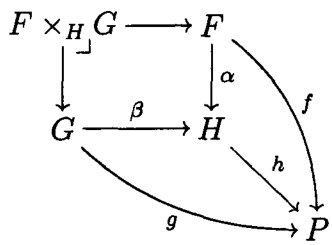
1. *The terminal presheaf is given by $P \xrightarrow{1_P} P$*

2. *The product of $F \xrightarrow{f} P$ and $G \xrightarrow{g} P$ is given by the pullback:*

   where $\langle x, y\rangle\restriction_p = \langle x\restriction_p, y\restriction_p\rangle$.

3. *The pullback of $F \xrightarrow{\alpha} H$ and $G \xrightarrow{\beta} H$ is given by the pullback in $\mathcal{E}$:*

   where $\langle x, y\rangle\restriction_p = \langle x\restriction_p, y\restriction_p\rangle$.

4. *The subobject classifier $\underline{\Omega}$ (we use the same underline notation as for presheaves to distinguish Ω in $\mathcal{E}$ from $\underline{\Omega}$) is given by*

$$\begin{array}{ccc} \coprod_{p\in P} Ideal_{\mathcal{E}}(\downarrow p, J) & \qquad & \langle p, K\rangle \\ \downarrow & & \downarrow \\ P & & p \end{array}$$

   with $\langle p, K\rangle\restriction_q = \langle q, K\restriction_q\rangle$ and $1 \xrightarrow{\top} \underline{\Omega}$ is given by $\downarrow(\ ) : P \longrightarrow \underline{\Omega}$.

   Note: to define this coproduct we use $\iota : P \longrightarrow \Omega^P$ with $\iota(p) = Ideal_{\mathcal{E}}\langle\downarrow p, J\rangle$.

5. *The power sheaf,* $\mathcal{P}(F)$*, of a sheaf* F *is given by*

$$\begin{array}{ccc} \coprod_{p\in P} Sub_{Sh_{\mathcal{E}}(P,J)}(F\restriction_p) & & \langle p, G\rangle \\ \downarrow & & \downarrow \\ P & & p \end{array}$$

with $\langle p, G\rangle \restriction_q = (\langle q, G\restriction_q\rangle$ *and evaluation map* $ev_F : \mathcal{P}(F) \times_P F \longrightarrow \underline{\Omega}$ *is* $ev_F\langle\langle p, G\rangle, x\rangle = \langle p, \{q \leq p : x\restriction_q \in G\restriction_q\}\rangle$.

Note: to define this coproduct we use

$\iota : P \longrightarrow \Omega^{\Omega^F}$ *where* $\iota(p) = Sub_{Pr_{\mathcal{E}}(P)}(F\restriction_p) \subset \Omega^F$.

6. *The exponent* G^F *of sheaves is given by:*

$$\begin{array}{ccc} \coprod_{p\in P} Sh_{\mathcal{E}}(P,J)(F\restriction_p, G) & & \langle p, \varphi\rangle \\ \downarrow & & \downarrow \\ P & & p \end{array}$$

with $\langle p, \varphi\rangle \restriction_q = (\langle q, \varphi\restriction_q\rangle$ *and evaluation map* $ev_{G,F} : G^F \times F \longrightarrow G$ *is* $ev_{G,F}\langle\langle p, \varphi\rangle, x\rangle = \langle p, \varphi(x)\rangle$.

Note: to define this coproduct we use $\iota : P \longrightarrow \Omega^{G\times F}$ *where*

$\iota(p) = \{x : G \times F : x : G\restriction_p \longrightarrow F\}$.

7. *If* $\alpha : F \longrightarrow \underline{\Omega}$ *then the canonical kernel of* α *is given by*

$ker(\alpha) = \{x \in F : \alpha(x) = \langle\downarrow f(x), f(x)\rangle\}$

8. *If* $G \overset{\alpha}{\rightarrowtail} F$ *then* $char(\alpha) : F \longrightarrow \underline{\Omega}$ *is given by*

$char(\alpha)(x) = \langle f(x), \{p \in\downarrow f(x) : x\restriction_p \in G\restriction_p\}\rangle$.

9. *The canonical subobjects are given by the canonical kernels above.*

6.5 The Plus Construction and the Associated Sheaf Functor

Next we describe an endofunctor called the plus construction $(\ )^+ : Pr_{\mathcal{E}}(P) \longrightarrow Pr_{\mathcal{E}}(P)$ which will allow us to construct a left adjoint to the forgetful functor $Sh_{\mathcal{E}}(P, J) \longrightarrow Pr_{\mathcal{E}}(P)$.

First define an (internal) equivalence relation on compatible families of F: $\langle p, R, (x_q)_{q\in R}\rangle \sim \langle p', S, (y_q)_{q\in S}\rangle$ if and only if $p = p'$ and for some cover $T \subseteq R \cap S$ of p we have, for all $q \in T$, $x_q = y_q$. This is exactly the condition for two compatible families to have the same amalgamation if F were a sheaf. The equivalence class containing $\langle p, R, (x_q)_{q\in R}\rangle$ will be denoted $[p, R, (x_q)_{q\in R}]$. [4]

Now given F we define $(F)^+$ as the object of equivalence classes of elements in F. For $r \leq p$ we define the restriction maps $\restriction^p_r$: $(F)^+(p) \longrightarrow (F)^+(r)$ by $[p, R, (x_q)_{q\in R}] \mapsto [r, R\restriction_r, (x_q)_{q\in R\restriction r}]$. Routine calculation shows that this is indeed well defined.

Since the amalgamation of any two equivalent compatible families are equal, in the case that F is a sheaf, $Amalg : Compat(F) \longrightarrow F$ factors through F^+. By abuse of notation we write $Amalg : F^+ \longrightarrow F$ for the resulting map.

Given a morphism of presheaves $\theta : F \longrightarrow G$ define $(\theta)^+[p, R, (x_q)_{q\in R}] = [p, R, (\theta(x_q))_{q\in R}]$.

We have a canonical presheaf morphism $\eta_F : F \longrightarrow (F)^+$ given by $(\eta_F)(x) = [\downarrow f(x), f(x), (x)_{q\leq f(x)}]$.

The following lemmas are all standard and have intuitionistically valid proofs which may be found in [12].

Lemma 6.5.1.

- *F is separated if and only if η_F is mono.*
- *F is a sheaf if and only if η_F is an isomorphism.*

Lemma 6.5.2. *For all presheaves F, $(F)^+$ is separated.*

[4]The equivalence relation $\sim \rightarrowtail Compat(F) \times Compat(F)$ is defined using the internal logic via the obvious formula described above. The object of equivalence classes is formed as the coequalizer of the two composites $\sim \rightarrowtail Compat(F) \times Compat(F) \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} Compat(F)$

Lemma 6.5.3. *If F is separated then $(F)^+$ is a sheaf.*

In light of these lemmas we define the *Associated Sheaf* functor $a : Pr_{\mathcal{E}}(P) \longrightarrow Sh_{\mathcal{E}}(P, J)$ as $a(F) = F^{++}$. The elements of $a(F)$ are compatible families of compatible families, we will denote them as $[p, R, (S_r)_{r\in R}; (x_{r,s})_{s\in S_q}]$. Compatibility requires that $S_r \in Cov(r)$ and so there is no ambiguity.

Theorem 6.5.4. *Let $U : Sh_{\mathcal{E}}(P, J) \longrightarrow Pr_{\mathcal{E}}(P)$ denote the forgetful functor. Then $a \dashv U$ with unit $\eta_F : F \longrightarrow a(F)$ given by*

$$x \mapsto [f(x), \downarrow f(x), (\downarrow q)_{q\leq f(x)}; (x\restriction_r)_{r\leq q}]$$

and counit $\epsilon_F : a(F) \longrightarrow F$ given by

$$[p, R, (S_r)_{r\in R}; (x_{r,s})_{s\in S_r}] \mapsto Amalg[p, R; (Amalg[r, S_r; (x_{r,s})_{s\in S_r}])_{r\in R}]$$

Proof. It is clear that η_F and ϵ_F are presheaf morphisms and form natural transformations. We check that they satisfy the triangle equalities. Suppressing the forgetful functor U we must show that for presheaves F and sheaves G

$$\begin{array}{ccc} aaF & \xrightarrow{\epsilon_{aF}} & aF \\ & \searrow^{1_{aaF}} & \downarrow{\scriptstyle a\eta_F} \\ & & aaF \end{array} \qquad \begin{array}{ccc} aG & \xrightarrow{\epsilon_G} & G \\ & \searrow^{1_{aG}} & \downarrow{\scriptstyle \eta_G} \\ & & aG \end{array}$$

First suppose that $[p, R, (S_r)_{r\in R}; (x_{r,s})_{s\in S_r}] \in aG$.

$$\begin{aligned} \eta_G\epsilon_G[p, R, (S_r)_{r\in R}; (x_{r,s})_{s\in S_r}] &= \eta_G\text{Amalg}[p, R; (\text{Amalg}[r, S_r; (x_{r,s})_{s\in S_r}])_{r\in R}] \\ &= [p, \downarrow p, (\downarrow q)_{q\leq p}; (y\restriction_r)_{r\leq q}] \\ &= [p, R, (S_r)_{r\in R}; (y\restriction_s)_{s\in S_r}] \qquad (7) \end{aligned}$$

where $y = \text{Amalg}[p, R; (\text{Amalg}[r, S_r; (x_{r,s})_{s\in S_r}])_{r\in R}]$. For $r \in R$ and $s \in S_r$ we have

$$\begin{aligned} y\restriction_s &= (y\restriction_r)\restriction_s \\ &= \text{Amalg}[r, S_r; (x_{r,s})_{s\in S_r}]\restriction_s \\ &= x_{r,s} \end{aligned}$$

Thus (7) is equal to $[p, R, (S_r)_{r\in R}; (x_{r,s})_{s\in S_r}]$ as desired.

Now for the other equality suppose that we have an element of aaF say

$$[p, R, (S_r)_{r\in R}, (T_{r,s})_{s\in S_r}, (U_{r,s,t})_{t\in T_{r,s}}; (x_{r,s,t,u})_{u\in U_{r,s,t}}] \tag{8}$$

We apply ϵ_{aF} to get:

$$\begin{aligned}\text{Amalg}[p, R; (\text{Amalg}[r, (S_r)_{r\in R}, (T_{r,s})_{s\in S_r}, (U_{r,s,t})_{t\in T_{r,s}}; (x_{r,s,t,u})_{u\in U_{r,s,t}}])_{r\in R}] &= \\ = \text{Amalg}[p, R, (\bigcup_{s\in S_r} T_{r,s})_{r\in R}, (U_{r,s,t})_{t\in\bigcup_{s\in S_r} T_{r,s}}; (x_{r,s,t,u})_{u\in U_{r,s,t}}] \\ = [p, \bigcup_{r\in R, s\in S_r} T_{r,s}, (U_{r,s,t})_{t\in\bigcup_{r\in R, s\in S_r} T_{r,s}}; (x_{r,s,t,u})_{u\in U_{r,s,t}}] \end{aligned}\tag{9}$$

For simplicity let $R' = \bigcup_{r\in R, s\in S_r} T_{r,s}$, for $r' \in T_{r,s} \subseteq R'$ let $S'_{r'} = U_{r,s,r'}$ and for $s' \in S'_{r'}$ let $x'_{r',s'} = x_{r',s',r',s'}$. Note that $R' \subseteq R$ and $S_{r'} \subseteq S_{r'}$. Now (9) is equal to $[p, R', (S_{r'})_{r'\in R'}; (x'_{r',s'})_{s'\in S'_{r'}}]$

Now we apply $a\eta_F$ to get:

$$[p, R', (S'_{r'})_{r'\in R'}, (\downarrow s')_{s'\in S'_{r'}}, (\downarrow t')_{t'\leq s'}; (x'_{r',s'}\restriction_{u'})_{u'\leq t'}] \tag{10}$$

Since $R' \subseteq R$ and $S'_{r'} \subseteq S_{r'}$ and further $\downarrow s' \subseteq T_{r',s'}$ and $\downarrow t' \subseteq U_{r',s',t'}$ and even further $x'_{r',s'}\restriction_{u'} = x_{r',s',t',u'}$ we have that (10) is equal to (8).

□

We can now define the notion of constant sheaf. The *Constant Sheaf Functor* $\Delta_{Sh} : \mathcal{E} \longrightarrow Sh_{\mathcal{E}}(P, J)$ is given as $a\Delta_{Pr} : \mathcal{E} \longrightarrow Pr_{\mathcal{E}}(P) \longrightarrow Sh_{\mathcal{E}}(P, J)$. The elements (i.e. global sections) of a constant sheaf $\Delta_{Sh}(A)$ look like $[p, R, (S_r)_{r\in R}; (x_{r,s})_{s\in S_q}]$ where $[r, S_r; (x_{r,s})_{s\in S_q}]$ is a compatible family in $\Delta_{Pr}(A)$. But a compatible family in a constant presheaf is constant, i.e. $x_{r,s_0} = x_{r,s_1}$ for each $s_0, s_1 \in S_r$, and so we can write

$$\Delta_{Sh}(A) = \{[p, R, (S_r)_{r\in R}; (x_r)_{s\in S_q}] : (R, p) \in J \text{ and } \forall_{r\in R} x_r \in A\}$$

Composing the two adjunctions in 6.3.3 and 6.5.4 we obtain

Corollary 6.5.5. *Another adjunction* $\mathcal{E} \underset{\Gamma}{\overset{\Delta_{Sh}}{\rightleftarrows}} Sh_{\mathcal{E}}(P, J)$ ($\Delta_{Sh} \dashv \Gamma$) *with unit* $\eta_A : A \longrightarrow \Gamma\Delta_{Sh}(A)$ *given by:*

$$\eta_A(a) = [1_P, \downarrow 1_P, (\downarrow r)_{r\in P}; (a)_{s\leq r}]$$

and counit $\epsilon_F : \Delta_{Sh}\Gamma F \longrightarrow F$ *given by:*

$$\begin{aligned}[p, R, (S_r)_{r\in R}; (x_r)_{s\in S_r}] &\mapsto Amalg[p, R; (Amalg[r, S_r, (x_r\restriction_s)_{s\in S_r}])_{r\in R}] \\ &= Amalg[p, R; (x_r\restriction_r)_{r\in R}]\end{aligned}$$

6.5.1 A Logical Functor

Lemma 6.5.6. *Given an object* A *in* $\mathcal{E}$ *there is an equivalence of categories* $\Theta : Sh_{\mathcal{E}}(\Omega^A) \longrightarrow \mathcal{E}/A$*, where* Ω^A *is regarded as a discrete locale given by the composition of the forgetful functor and the pullback functor* $Sh_{\mathcal{E}}(\Omega^A) \longrightarrow \mathcal{E}/\Omega^A \xrightarrow{\{\,\}^*} \mathcal{E}/A$*. In particular* Θ *is logical. Alternatively this may calculated*

$$\begin{array}{ccc} F & & \coprod_{a\in A} F(\{a\}) \\ {\scriptstyle f}\downarrow & \stackrel{\Theta}{\longmapsto} & \downarrow{\scriptstyle projection} \\ \Omega^A & & A \end{array}$$

$$(a, x) \longmapsto (a, \alpha(x))$$

$$\begin{array}{ccc} F \xrightarrow{\ \alpha\ } G & & \coprod_{a\in A} F(\{a\}) \xrightarrow{\ \Theta(\alpha)\ } \coprod_{a\in A} G(\{a\}) \\ {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \stackrel{\Theta}{\longmapsto} & \searrow \quad \swarrow \\ \Omega^A & & A \end{array}$$

Proof. Define $\Xi : \mathcal{E}/A \longrightarrow Sh_{\mathcal{E}}(\Omega^A)$ as follows: given $F \xrightarrow{f} A$ in $\mathcal{E}/A$ and a subobject $U \rightarrowtail A$ let F_U be the subobject of F^U given by the closed term $\{x \in F^U : \forall_{y \in U} f x^{\wr} y = f y\}$ (where the notation $(\ )^{\wr}$ is as follows: If t is a term of type A^B, that is a map with codomain A^B, then $t^{\wr}$ is the transpose of t with codomain B). Intuitively the formula in this term says that $x^{\wr} : U \longrightarrow F$ sends y to some element in the fiber $f^{-1}(y)$, i.e. in **Set** this is the product $\prod_{y \in U} f^{-1}(y)$.

For $u, v \in \Omega^A$ we let U, V stand for $Ker(u^{\wr})$ and $Ker(v^{\wr})$ respectively. Let $\Xi(f)$ be the sheaf

$$\begin{array}{c} \coprod_{u \in \Omega^A} F^U \\ \Big\downarrow \text{Projection} \\ \Omega^A \end{array}$$

with restrictions $\langle u, x\rangle \restriction_v = \langle v, x \restriction_v\rangle$ where $x \restriction_v$ is given by the composite $V \rightarrowtail U \xrightarrow{x} F$. These restrictions correspond to the projections $\prod_{y \in U} f^{-1}(y) \longrightarrow \prod_{y \in V} f^{-1}(y)$ in **Set**.

It is straight forward to check that $\Theta\Xi \cong 1_{\mathcal{E}/A}$ and $\Xi\Theta \cong 1_{Sh_{\mathcal{E}}(\Omega^A)}$, and hence form an equivalence. □

6.5.2 Sheaves on a Locale

If we restrict ourselves to sites that are locales along with the canonical topology some of the previous results simplify. Let X denote an internal locale, regarded as a site with the canonical topology.

A J-Ideal becomes a principal ideal, which is identified by its generator. Thus $\underline{\Omega} = \coprod_{u \in X} \downarrow u$ is just pairs $\langle u, v\rangle$ with $v \leq u$. More conveniently we may write $\underline{\Omega} = \{\langle u, v\rangle \in X \times X : v \leq u\}$, with the first projection $\underline{\Omega} \longrightarrow X$. Also important will be the observation: given $u : 1 \longrightarrow X$ we have $\underline{\Omega}\restriction_u = \{\langle u', v\rangle \in X \times X : v \leq u'\}$. We also calculate the global sections $\Gamma(\underline{\Omega}\restriction_u) = \downarrow u$.

The kernel of some $\alpha : F \longrightarrow \Omega$ is $ker(\alpha) = \{x \in F : \alpha(x) = \langle f(x), f(x)\rangle\}$. In particular $ker(\alpha)(u) = \{x \in F(u) : \alpha(x) = \langle u, u\rangle\}$.

The characteristic map of $G \stackrel{\alpha}{\rightarrowtail} F$ is given by $char(\alpha)(x) = \langle f(x), \bigvee\{v \leq f(x) : x\restriction_v \in G\restriction_v\}\rangle$.

There is a well-known equivalence of categories([8, 10, 1]): for X a topological space (or even locale), the category of Sheaves on X is equivalent to the category of etale maps over X. Due to time restraints we cannot prove this here, nor will we define an etale map of locales. We point out that in [4] this theory is developed in an intuitionistic setting, which allows us to conclude that $Sh_{\mathcal{E}}(X)$ is equivalent to the category of internal etale maps over X. What will be important for us is the following consequence of this theory:

Lemma 6.5.7. *Internal to $\mathcal{E}$ every sheaf $F \stackrel{f}{\longrightarrow} X$ is a sheaf of sections of some locale map $Y_F \longrightarrow X$. Moreover there is natural bijection between the sheaf maps:*

$$\begin{array}{ccc} F & \longrightarrow & G \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & X & \end{array}$$

and between locale maps over X:

$$\begin{array}{ccc} Y_F & \longrightarrow & Y_G \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

It is also important to point out that projections are etale, and etale maps are stable under pullback.

If one has a locale $f : X \longrightarrow Y$ then pullback along f yields a map $f^* : \mathrm{Loc}_{\mathcal{E}}/Y \longrightarrow \mathrm{Loc}_{\mathcal{E}}/X$ which will preserve etale maps and hence give us a functor $Sh_{\mathcal{E}}(Y) \longrightarrow Sh_{\mathcal{E}}(X)$. Most important is the observation that if f is the inclusion of an open sublocale then this map will be logical (see [8] Vol II p. 609). We calculate what this functors does on arrows for a special case:

Suppose that $i : X \longrightarrow Y$ is the inclusion of an open sublocale and consider the etale map over Y given by a projection of locales $L \times Y \stackrel{\pi}{\longrightarrow} Y$. The sheaf of sections $\Sigma(\pi)$ has endomorphisms given by maps making the following commute

$$\begin{array}{ccc} L\times Y & \xrightarrow{\qquad\qquad} & L\times Y \\ & {\scriptstyle\pi}\searrow \qquad \swarrow{\scriptstyle\pi} & \\ & Y & \end{array}$$

These correspond to locale maps $L \longrightarrow L$.

The functor i^* maps $L \times Y \xrightarrow{\pi} Y$ to $L \times X \xrightarrow{\pi} X$. Hence endomorphisms of $i^*(\Sigma(\pi))$ are also given by maps $L \longrightarrow L$.

Lemma 6.5.8. *With i as above and $g : L \longrightarrow L$ regarded as a sheaf morphism $\Sigma(\pi) \longrightarrow \Sigma(\pi)$, one has $i^*(g) = g$.*

Proof. One obtains $i^*(g)$ by pulling back the commutative triangle where g sits by i:

$$\begin{array}{ccccc} X\times L & & \xrightarrow{\quad 1_X\times g \quad} & & X\times L \\ & {\scriptstyle i\times 1_L}\searrow & & \swarrow{\scriptstyle i\times 1_L} & \\ {\scriptstyle\pi}\downarrow & Y\times L & \xrightarrow{1_Y\times g} & Y\times L & \downarrow{\scriptstyle\pi} \\ & {\scriptstyle\pi}\searrow & & \swarrow{\scriptstyle\pi} & \\ X & & & & X \\ & {\scriptstyle i}\searrow & & \swarrow{\scriptstyle i} & \\ & & Y & & \end{array}$$

□

Chapter 7

Interpretation in a Sheaf Topos

Given a site $\langle P, J\rangle$ internal to a topos $\mathcal{E}$ consider a functor $F : \mathcal{E} \longrightarrow \mathcal{S}$. Assuming that F preserves pullbacks $F(P)$ is also an internal poset in $\mathcal{S}$. We ask what is the induced topology on $F(P)$? We take it to be the span $F(P) \xleftarrow{F(b)} F(J) \xrightarrow{F(c)} F(\Omega^P) \xrightarrow{\varphi_P} \Omega^{F(P)}$, where φ_P is the comparison map for F (see Section 3.1.3). We denote this new span by $F(P, J)$.

Lemma 2 ([8] p. 584 Lemma 2.4.4). *If $\langle P, J\rangle$ an internal topology then so is $F(P, J)$.*

Note that we cannot conclude that $F(P, J)$ is Grothendieck (i.e. satisfying the locale character condition (L)), but fortunately this will not be of consequence to us.

7.1 Constant Sheaves and Prime Maps

The goal of this section is to establish the following lemma which is fundamental to our results:

Lemma 7.1.1. *Given a locale X, $u : 1 \longrightarrow X$ and a site $\langle P, J\rangle$ internal to a topos $\mathcal{E}$ the adjunction $\mathcal{E} \underset{\Gamma}{\overset{\Delta_{Sh}}{\rightleftarrows}} Sh_{\mathcal{E}}(X)$ (with $\Delta_{Sh} \dashv \Gamma$) induces an isomorphism $\mathcal{E}(P, \downarrow u) \cong Sh_{\mathcal{E}}(X)(\Delta_{Sh}(P), \underline{\Omega}\restriction_u)$. If (P, J) has no non-empty covers then this isomorphism restricts to J prime maps: $Prime_{\mathcal{E}}(P, J; \downarrow u) \cong Prime_{Sh_{\mathcal{E}}(X)}(\Delta(P, J); \underline{\Omega}\restriction_u)$.*

Note that this uses the fact that $\Gamma(\Omega\restriction_u) \cong \downarrow u$. To prove this we will need some lemmas:

Lemma 7.1.2. *If $m : P \longrightarrow X$ is a J-prime map and $\langle P, J\rangle$ has no empty covers, all internal to $\mathcal{E}$ and $F : \mathcal{E} \longrightarrow \mathcal{S}$ preserves pullbacks then the following commutes:*

$$\begin{array}{ccc} F(P) & & \\ \downarrow{\scriptstyle F(c)} & & \\ F(\Omega^P) & \xrightarrow{F(\exists_m)} & F(\Omega^X) \\ \downarrow{\scriptstyle \varphi_P^F} & & \downarrow{\scriptstyle \varphi_X^F} \\ \underline{\Omega}^{F(P)} & \xrightarrow{\exists_{F(m)}} & \underline{\Omega}^{F(X)} \end{array}$$

Proof. Consider the following diagram:

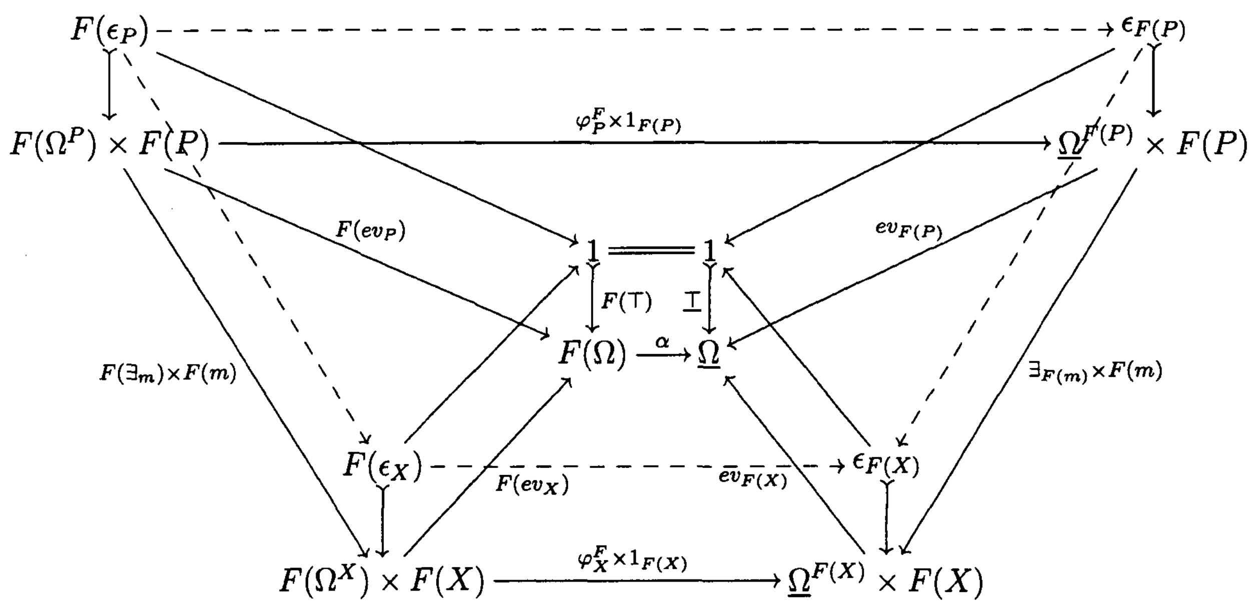


Figure 2: Big Diagram

The four solid squares involving $F(\epsilon_P), \epsilon_{F(P)}, F(\epsilon_X)$, and $\epsilon_{F(X)}$ as well as the middle square are all pullbacks. The definitions of $\exists_m$ and $\exists_{F(m)}$ ensure that the two bottom triangles commute. The dotted arrows are induced via the pullbacks making all the squares and triangles commute, with the exception of the bottom outer square (the top outer square does commute though!). We can now conclude that the following commutes:

$$\begin{array}{ccc}
F(\epsilon_P) & & \\
\downarrow & & \\
F(\Omega^P) \times F(P) & \xrightarrow{\varphi_P \times 1_{F(P)}} & \underline{\Omega}^{F(P)} \times F(P) \\
{\scriptstyle F(\exists_m) \times F(m)} \downarrow & & \downarrow {\scriptstyle \exists_{F(m)} \times F(m)} \\
\underline{F}(\Omega^X) \times F(X) & \xrightarrow[\varphi_X \times 1_{F(m)}]{} & \underline{\Omega}^{F(X)} \times F(X)
\end{array}$$

Taking the first projection gives

$$\begin{array}{ccc}
F(\epsilon_P) & & \\
\downarrow & & \\
F(\Omega^P) & \xrightarrow{\varphi_P} & \underline{\Omega}^{F(P)} \\
{\scriptstyle F(\exists_m)} \downarrow & & \downarrow {\scriptstyle \exists_{F(m)}} \\
\underline{F}(\Omega^X) & \xrightarrow[\varphi_X]{} & \underline{\Omega}^{F(X)}
\end{array}$$

Combining this with the fact that since J has no non-empty covers $J \xrightarrow{c} \Omega^P$ factors through $\epsilon_P \longrightarrow \Omega^P$ the result follows. □

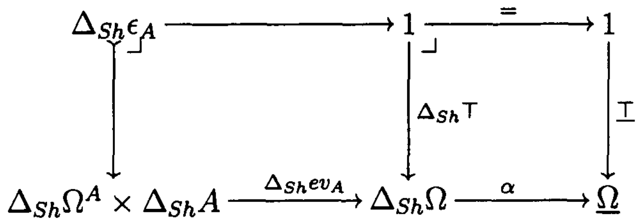
The next lemma is a formula for how to calculate the comparison map for Δ_{Sh}.

Lemma 7.1.3. *If $\varphi^{\Delta_{Sh}}$ is the comparison map for the Δ_{Sh} then one may calculate:*

$$\varphi_A^{\Delta_{Sh}} : \Delta_{Sh}\Omega^A \longrightarrow \underline{\Omega}^{\Delta_{Sh}A}$$

$$[u, R, (S_r)_{r\in R}; (A_{r,s})_{s\in S_r}] \mapsto \{[u', R', (S'_r)_{r\in R'}; (x_{r,s})_{s\in S'_r}] : u' \leq u, R' \subseteq R, \forall_{r\in R'}.S'_r \subseteq S_r, \forall_{s\in S'_r}.x_{r,s} \in A_{r,s}\}$$

Proof. $\varphi_A^{\Delta_{Sh}}$ can be identified as the transpose of the composite $\alpha\Delta ev_A$:



Here α can be computed by (see 6.5.2):

$$[u, R, (S_r)_{r\in R}; (t_{r,s})_{s\in S_r}] \mapsto \langle u, \bigvee\{v \leq u : [u, R, (S_r)_{r\in R}; (t_{r,s})_{s\in S_r}]\restriction_v = \Delta_{Sh}\top(*)\restriction_v\}\rangle$$
$$= \langle u, \bigvee\{v \leq u : [v, R\restriction_v, (S_r)_{r\in R\restriction_v}; (t_{r,s})_{s\in S_r}] = [v, \downarrow v, (\downarrow w)_{w\leq v}; \top]\}\rangle$$

And so $\Delta_{Sh}ev_A\alpha$ is given by:

$$[u, R, (S_r)_{r\in R}; (x_{r,s}, A_{r,s})_{s\in S_r}] \mapsto \langle u, \bigvee\{v \leq u : [v, R\restriction_v, (S_r)_{r\in R\restriction_v}; (ev_A(x_{r,s}, A_{r,s}))_{s\in S_r}] = [v, \downarrow v, (\downarrow w)_{w\leq v}; \top]\}\rangle$$

In which case we can write (see 6.5.2):

$$\varphi_A^{\Delta_{Sh}}[u, R, (S_r)_{r\in R}; (A_{r,s})_{s\in S_r}] = \{[u', R', (S'_r)_{r\in R'}; (x_{r,s})_{s\in S'_r}] \in \Delta_{Sh}A : u' \leq u, R' \subseteq R, \forall_{r\in R'}.S'_r \subseteq S_r,$$
$$u' = \bigvee\{v \leq u' : [v, R'\restriction_v, (S'_r)_{r\in R'\restriction_v}; (ev_A(x_{r,s}, A_{r,s}))_{s\in S'_r}] = [v, \downarrow v, (\downarrow w)_{w\leq v}; \top]\}\}$$

Now we claim that $[u', R', (S'_r)_{r\in R'}; (x_{r,s})_{s\in S'_r}] \in \varphi_A^{\Delta_{Sh}}[u, R, (S_r)_{r\in R}; (A_{r,s})_{s\in S_r}]$ if and only if there are subcovers T of R and Q_r of S_r forming a compatible family so that $[u', R', (S'_r)_{r\in R'}; (x_{r,s})_{s\in S'_r}] = [u', T', (Q'_r)_{r\in R'}; (x_{r,s})_{s\in Q'_r}]$ where for all $r \in T'$ and $s \in Q'_r$ we have $x_{r,s} \in A_{r,s}$.

The sufficiency direction is straight forward, for if so, then $\{v \leq u' : [v, R'\restriction_v, (S'_r)_{r\in R'\restriction_v}; (ev_A(x_{r,s}, A_{r,s}))_{s\in S'_r}] = [v, \downarrow v, (\downarrow w)_{w\leq v}; \top]\} \supseteq R' \in Cov(u')$

For the converse, let $T = \{v \leq u' : [v, R'\restriction_v, (S'_r)_{r\in R'\restriction_v}; (ev_A(x_{r,s}, A_{r,s}))_{s\in S'_r}] = [v, \downarrow v, (\downarrow w)_{w\leq v}; \top]\}$ and suppose that $\bigvee T = u'$. Define $T' = \{s \in S'_r : r \in R' \cap T, x_{r,s} \in A_{r,s}\}$. Then the locale character property of J implies T' covers u'; for given $r \in R' \cap T$, $T' \cap \downarrow r = \{s \in S'_r : x_{r,s} \in A_{r,s}\}$, which covers r since $r \in T$. Now we can take $Q'_r = T'\restriction_r$ and the claim follows.

□

Now we are in a position to prove 7.1.1:

Proof. Suppose that $m : P \longrightarrow X$ is a J-prime map. Consider the diagram:

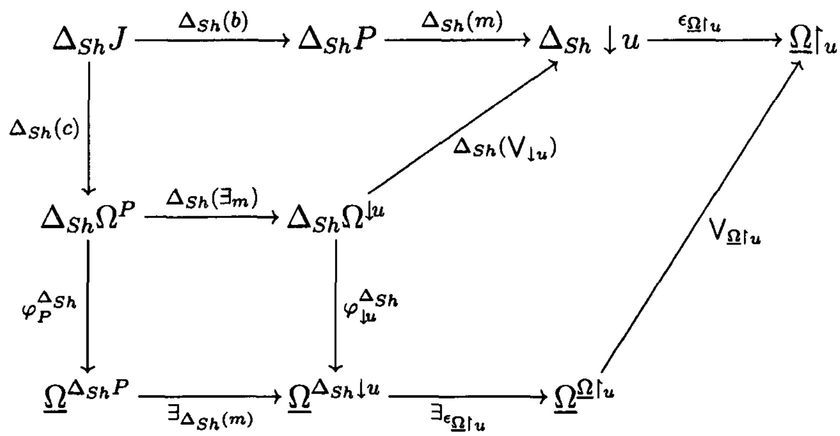


Figure 3:

We need to show that the outside of figure 3 commutes. We know the left top section commutes by assumption that m is J-prime. Also 7.1.2 gives us that the bottom left square will commute upon precomposition by $\Delta_{Sh}(c)$. Hence all that remains to check is that the right sections commutes. We do this directly; let $[u', R, (S_r)_{r\in R}; (A_r)_{s\in S_r}] \in \Delta_{Sh}\Omega^{\downarrow u}$:

$$\begin{aligned}
\epsilon_{\underline{\Omega}\upharpoonright_u}\Delta_{Sh}(\bigvee\nolimits_{\downarrow u})[u', R, (S_r)_{r\in R}; (A_r)_{s\in S_r}] &= \epsilon_{\underline{\Omega}\upharpoonright_u}([u', R, (S_r)_{r\in R}; (\bigvee A_r)_{s\in S_r}]) \\
&= \mathrm{Amalg}[u', R; (r \wedge \bigvee A_r)_{r\in R}] \\
&= \bigvee_{r\in R}(r \wedge \bigvee A_r)
\end{aligned}$$

See 6.5.2 for the formula for the above amalgamation.

$$\bigvee_{\underline{\Omega}\restriction_u} \exists_{\epsilon_{\underline{\Omega}\restriction u}} \varphi_{\downarrow u}^{\Delta_{Sh}}[u', R, (S_r)_{r\in R}; (A_r)_{s\in S_r}] =$$

$$\bigvee_{\underline{\Omega}\restriction_u} \exists_{\epsilon_{\underline{\Omega}\restriction u}} \{[u'', R', (S'_r)_{r\in R'}; (x_r)_{s\in S'_r}] \in \Delta_{Sh} \downarrow u : u'' \leq u', R' \subseteq R\restriction_{u''} \text{ and } \forall r \in R'.S'_r \subseteq S_r, x_r \in A_r\} =$$

$$\bigvee\{\bigvee_{r\in R'} (r\wedge x_r) : [u'', R', (S'_r)_{r\in R'}; (x_r)_{s\in S'_r}] \in \Delta_{Sh} \downarrow u, u'' \leq u', R' \subseteq R\restriction_{u''} \text{ and } \forall r \in R'.S'_r \subseteq S_r, x_r \in A_r\} \tag{11}$$

The last line is a join of elements of X of the form $r \wedge a$ for some $a \in A_r$ and $r \in R$. For each such pair $\langle a, r_0\rangle$ with $r_0 \in R$ and $a \in A_{r_0}$ there is $[r_0, \downarrow r_0, (S_r)_{r\leq r_0}; (a)_{s\in S_r}] \in \Delta_{Sh} \downarrow u$ which witnesses that $r_0 \wedge a$ is in the above join. Thus the above join is really

$$\bigvee\{r \wedge a : r \in R \text{ and } a \in A_r\} = \bigvee_{r\in R}(r \wedge \bigvee A_r)$$

This establishes one direction. For the converse, suppose that $m : \Delta_{Sh}P \longrightarrow \underline{\Omega}\restriction_u$ is a $\Delta_{Sh}J$-prime map and consider figure 4.
Squares 1 and 2 are naturality squares for η and so commute. 3 commutes by the assumption that m is a ΔJ-prime map. An easy calculation shows that 5 and 6 commute as well. But 4 does not commute. So we do the calculation showing the outside commutes directly. First we need to know what it means for m to be $\Delta_{Sh}J$-prime. Chasing $[u', R, (S_r)_{r\in R}; (A_r, p_r)_{s\in S_r}] \in \Delta_{Sh}J$ through figure 5 yields that:

$$m[u', R, (S_r)_{r\in R}; (p_r)_{s\in S_r}] = \bigvee\{m[u'', R', (S'_r)_{r\in R'}; (x_r)_{s\in S'_r}] : u'' \leq u', R' \subseteq R\restriction_{u''} \text{ and } \forall r \in R'.S'_r \subseteq S_r, x_r \in A_r\} \tag{12}$$

Now take $(A, p) \in J$ and chase through 4:

Top: $m[1_X, X, (\downarrow r)_{r\in X}; (p)_{s\leq r}]$

Bottom: $\bigvee\{m[1_X, \downarrow X, (\downarrow r)_{r\in X}; (a)_{s\leq r}] : a \in A\}$

To see that these are equal, substitute $[\top_X, \downarrow X, (\downarrow r)_{r\in X}; (A, p)_{s\leq r}]$ into 12 to obtain:

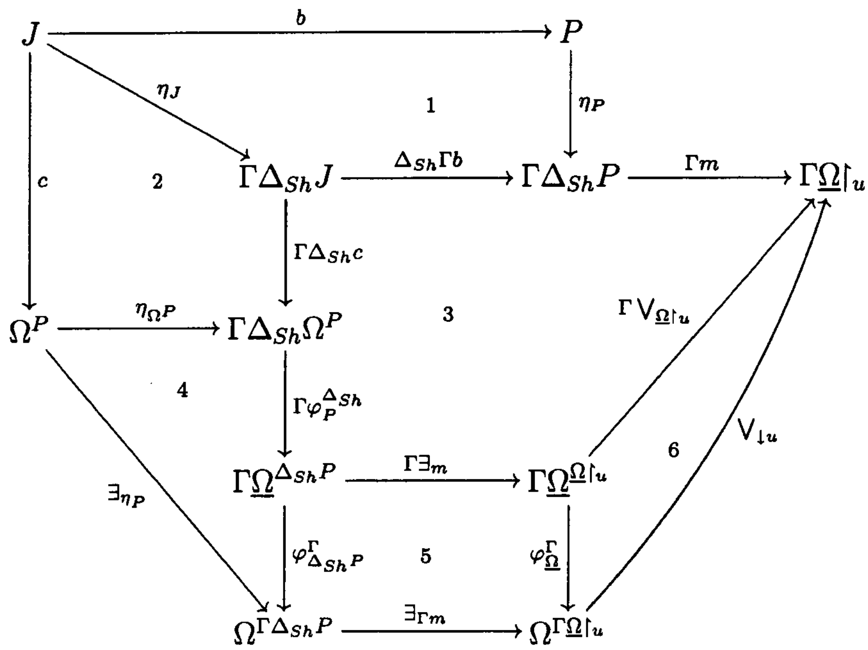


Figure 4:

$$
\begin{aligned}
m([\top_X, \downarrow X, (\downarrow r)_{r\in X}; (p)_{s\le r}] &= \bigvee\{m[u', R, (S_r)_{r\in R}; (x_r)_{s\in S_r}] : \\
&\qquad u' \le u, R \subseteq \downarrow u' \text{ and } \forall_{r\in R}.S_r \subseteq \downarrow r, x_r \in A\} \\
&= \bigvee\{m[u', R, (S_r)_{r\in R}; (x_r)_{s\in S_r}] : \forall_{r\in R}.x_r \in A\} \qquad (13)
\end{aligned}
$$

For each $m[u', R, (S_r)_{r\in R}; (x_r)_{s\in S_r}]$, consider it's restriction by some $w \in \bigcup_{r\in R} S_r$:

$$
\begin{aligned}
m[u', R, (S_r)_{r\in R}; (x_r)_{s\in S_r}]\restriction_w &= m[w, \downarrow w, (S_r)_{r\le w}; (x_r)_{s\in S_r}] \\
&= m[w, \downarrow w, (\downarrow r)_{r\le w}; (x_r)_{w\le r}] \\
&= m[w, \downarrow w, (\downarrow r)_{r\le w}; (x_w)_{s\le r}]
\end{aligned}
$$

The first line follows from $w \in R$. For the second line, where S_r is replaced by $\downarrow r$, proceed as follows: suppose $w \in S_{r_0}$. Then $r \in S_{r_0}$ and so $\downarrow r \subset S_{r_0}$. Now

$$\begin{array}{ccccc}
\Delta_{Sh}J & \xrightarrow{\Delta_{Sh}b} & \Delta_{Sh}P & \xrightarrow{m} & \underline{\Omega}\restriction_u \\
\downarrow{\scriptstyle \Delta_{Sh}c} & & & \nearrow{\scriptstyle \bigvee_{\Omega_u}} & \\
\Delta_{Sh}\Omega^P & & & & \\
\downarrow{\scriptstyle \varphi_P} & & & & \\
\underline{\Omega}^{\Delta_{Sh}P} & \xrightarrow{\exists m} & \underline{\Omega}^{\Omega_u} & &
\end{array}$$

Figure 5:

$S_r = S_{r_0}\restriction_r = \downarrow r$. In the third line x_r is independent of r and so replaced by x_w. To see this let $r \leq w$, then $[r, \downarrow r; (x_r)_{s \leq r}] = [w, \downarrow w; (x_w)_{s \leq w}]\restriction_r = [r, \downarrow r, (x_w)_{s \leq r}]$ which implies $x_r = x_w$.

It follows that

$$\begin{aligned}
m[u', R, (S_r)_{r \in R}; (x_r)_{s \in S_r}] &= \text{Amalg}[u', \bigcup_{r \in R} S_r; (m[w, \downarrow w, (\downarrow v)_{v \leq w}; (x_w)_{s \leq v}])_{w \in \bigcup_{r \in R} S_r}] \\
&= \bigvee_{w \in \bigcup_{r \in R} S_r} w \wedge m[w, \downarrow w, (\downarrow v)_{v \leq w}; (x_w)_{s \leq v}]
\end{aligned}$$

Now continuing 13

$$
\begin{aligned}
m([\top_X, \downarrow X, (\downarrow r)_{r\in X}; (p)_{s\leq r}] &= \bigvee\{m[u', R, (S_r)_{r\in R}; (x_r)_{s\in S_r}] : \\
&\qquad u' \leq u, R \subseteq\downarrow u' \text{ and } \forall r \in R.S_r \subseteq\downarrow r, x_r \in A\} \\
&= \bigvee\{m[u', R, (S_r)_{r\in R}; (x_r)_{s\in S_r}] : \forall r \in R.x_r \in A\} \\
= \bigvee\{ \bigvee_{w\in\bigcup_{r\in R} S_r} & w \wedge m[w, \downarrow w, (\downarrow v)_{v\leq w}; (x_w)_{s\leq v}] : \forall r \in R.x_r \in A\} \\
= \bigvee\{ \bigvee_{w\in X} & w \wedge m[w, \downarrow w, (\downarrow v)_{v\leq w}; (a)_{s\leq v}] : a \in A\} \\
= \bigvee\{&\text{Amalg}[\top_X, X; (m[w, \downarrow w, (\downarrow v)_{v\leq w}; (a)_{s\leq v}])_{w\in X}] : a \in A\} \\
= \bigvee\{&m(\text{Amalg}[\top_X, X; ([w, \downarrow w, (\downarrow v)_{v\leq w}; (a)_{s\leq v}])_{w\in X}]) : a \in A\} \\
= \bigvee\{&m([\top_X, X; (\downarrow v)_{v\leq w}; (a)_{s\leq v}] : a \in A\}
\end{aligned}
$$

and this is what we needed to show. □

7.2 Prime Maps in a Sheaf Topos

Now we switch from an arbitrary ambient topos $\mathcal{E}$ to the free topos $\mathcal{F}$. Given an internal locale X we let $\Psi_X : \mathcal{F} \longrightarrow Sh_{\mathcal{F}}(X)$ be the unique logical functor. When X is clear from context we may drop the subscript on Ψ_X.

Given locales L, L' internal to $Sh_{\mathcal{F}}(X)$ we have a potential ambiguity of notation. Inside $\mathcal{F}$ we may form the subobject of $Sh_{\mathcal{F}}(X)(L, L')$ consisting of those sheaf maps which are also locale maps. This object will be denoted by $\text{Loc}_{Sh_{\mathcal{F}}(X)}(L, L')$. Alternatively we form in $Sh_{\mathcal{F}}(X)$ the subobject of L'^L consisting of those arrows which (internal to $Sh_{\mathcal{F}}(X)$) are locale maps. This latter object will be denoted by $\underline{\text{Loc}_{Sh_{\mathcal{F}}(X)}(L, L')}$. Similarly we can extend this to other notations where there is possible ambiguity.

We wish to identify the sheaf which represents

$$\Psi_X(\text{Prime}_{\mathcal{F}}(P, J; \Omega)) \cong \underline{\text{Prime}_{Sh_{\mathcal{F}}(X)}(\Psi_X(P, J); \underline{\Omega})}$$

For this purpose we make the following definition.

Definition 7.2.1. Define an internal topology $\langle P, J\rangle$ in $\mathcal{F}$ to be *compatible for* X if $\Delta(P,J) \cong \Psi_X(P,J)$.

In the case $\langle P, J\rangle$ is compatible for X we have

$$\begin{aligned}\Psi(\text{Prime}_{\mathcal{F}}(P,J;\Omega)) &\cong \underline{\text{Prime}_{Sh_{\mathcal{F}}(X)}(\Psi(P,J);\underline{\Omega})}\\ &\cong \underline{\text{Prime}_{Sh_{\mathcal{F}}(X)}(\Delta(P,J);\underline{\Omega})}\end{aligned}$$

and so the problem of identifying $\Psi(\text{Prime}_{Sh_{\mathcal{F}}(X)}(P,J;\Omega))$ is reduced to identifying $\underline{\text{Prime}_{Sh_{\mathcal{F}}(X)}(\Delta(P,J);\underline{\Omega})}$. To this end we have a lemma, but first recall the following notation: For a sheaf $F \xrightarrow{f} X$ we write $F(u) = \{x \in F : f(x) = u\}$ (an object in $\mathcal{F}$ for each $u : 1 \longrightarrow X$), and that F is the coproduct of these $F(u)$'s.

Lemma 7.2.2. *Given a locale* L *internal to* $Sh_{\mathcal{F}}(X)$

$$\underline{Loc_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}, L)}(u) = Loc_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}\restriction_u, L).$$

i.e. $\underline{Loc_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}, L)}$ *is the sheaf of locale maps* $\underline{\Omega} \longrightarrow L$.

This lemma says that the sheaf that represents locale maps $\underline{\Omega} \longrightarrow L$ internally in $Sh_{\mathcal{F}}(X)$ is given by the sheaf of locale maps $\underline{\Omega} \longrightarrow L$. The proof requires sheaf semantics and is relegated to the Appendix.

Theorem 7.2.3. $\underline{Prime_{Sh_{\mathcal{F}}(X)}(\Delta(P,J);\underline{\Omega})}$ *is given by the sheaf of locale maps* $X \longrightarrow I_{\mathcal{F}}(P,J)$.

Proof.

$$\begin{aligned}\underline{\text{Prime}_{Sh_{\mathcal{F}}(X)}(\Delta(P,J);\underline{\Omega})}(u) &\cong \underline{\text{Loc}_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}, I_{Sh_{\mathcal{F}}(X)}(\Delta(P,J)))}(u)\\ &\cong \text{Loc}_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}\restriction_u, I_{Sh_{\mathcal{F}}(X)}(\Delta(P,J)))\\ &\cong \text{Prime}_{Sh_{\mathcal{F}}(X)}(\Delta(P,J);\underline{\Omega}\restriction_u)\\ &\cong \text{Prime}_{\mathcal{F}}(P,J;\downarrow u)\\ &\cong \text{Loc}_{\mathcal{F}}(\downarrow u, I_{\mathcal{F}}(P,J))\end{aligned}$$

□

Corollary 7.2.4. *If (P, J) is compatible for X then $\underline{Prime_{Sh_{\mathcal{F}}(X)}(\Psi(P, J); \Omega)}$ is the sheaf of locale maps $X \longrightarrow I_{\mathcal{F}}(P, J)$.*

Thus we have identified the sheaf $\underline{\mathrm{Prime}_{Sh_{\mathcal{F}}(X)}(\Psi(P, J); \Omega)}$k, further on we will need to know what the endo-maps of this sheaf look like.

Corollary 7.2.5. *In $Sh_{\mathcal{F}}(X)$ global sections of $\underline{Prime_{Sh_{\mathcal{F}}(X)}(\Delta(P, J); \Omega)}$ are given by locale maps $X \longrightarrow Ideal_{\mathcal{F}}(P, J)$ in $\mathcal{F}$. Moreover arrows $\underline{Prime_{Sh_{\mathcal{F}}(X)}(\Delta(P, J); \Omega)} \longrightarrow \underline{Prime_{Sh_{\mathcal{F}}(X)}(\Delta(P, J); \Omega)}$ in $Sh_{\mathcal{F}}(X)$ are determined by maps $Ideal_{\mathcal{F}}(P, J) \longrightarrow Ideal_{\mathcal{F}}(P, J)$.*

Proof. The global section part is obvious. Since $\underline{\mathrm{Prime}_{Sh_{\mathcal{F}}(X)}(\Delta(P, J); \Omega)}$ is the sheaf of locale maps $X \longrightarrow \mathrm{Ideal}_{\mathcal{F}}(P, J)$ we can view this as the sheaf of sections of the following locale over X:

$$\begin{array}{c} X \times \mathrm{Ideal}_{\mathcal{F}}(P, J) \\ \big\downarrow \pi_1 \\ X \end{array}$$

In view of 6.5.7 we have a correspondence with sheaf maps $\underline{\mathrm{Prime}_{Sh_{\mathcal{F}}(X)}(\Delta(P, J); \Omega)} \longrightarrow \underline{\mathrm{Prime}_{Sh_{\mathcal{F}}(X)}(\Delta(P, J); \Omega)}$ and with Loc$/X$ maps:

$$\begin{array}{ccc} X \times \mathrm{Ideal}_{\mathcal{F}}(P, J) & \longrightarrow & X \times \mathrm{Ideal}_{\mathcal{F}}(P, J) \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & X & \end{array}$$

These are exactly the locale maps $\mathrm{Ideal}_{\mathcal{F}}(P, J) \longrightarrow \mathrm{Ideal}_{\mathcal{F}}(P, J)$. □

Remark 7.2.6. In order to apply theorem 7.2.3 we need a result regarding when a topology $\langle P, J \rangle$ is compatible for a locale. Each site $\langle P, J \rangle$ in which P is countable corresponds to a propositional geometric theory. In [5] Fourman states (in different language) that if the theory corresponding to a site is categorical (in the model-theoretic sense; up to isomorphism has a unique model) then indeed $\Delta(P, J)$ and $\Psi(P, J)$ are both models of this theory and hence are isomorphic. In addition Fourman states that sufficient for categoricity is that P be countable and P,J are given "arithmetically". In particular the theories associated with examples 5.3.1 and 5.3.7 are categorical.

We believe this might mean the following: that these theories are in some sense categorical in **Set** and hence by Barr's Theorem must be categorical in every topos (see [11] p. 515 for this theorem). This would give us the result that $\langle P, J \rangle$ is compatible for every locale.

We have not yet formalized these remarks but believe that one can. A sketch is given below.

Proof Sketch:
Supposing that P is countable we might as well take $P = N$. In which case we ask that the order on P be given by a recursive set (We believe that this requirement may be relaxed to arithmetical set). For N^* this is the standard encoding of sequences. For covers we also restrict ourselves to recursive sets which we encode into N as well. Then we take $J = N$ where each $n \in J$ encodes a pair $\langle a, b \rangle$ with b an element of our order P and a encoding a recursive set. We ask that J be recursive as well. Thus our whole topology is reduced to a single recursive set.

Next we prove that Δ preserves recursive functions and sets, from which it will follow that Δ agrees with Ψ on the topology $\langle P, J \rangle$.

Chapter 8

Proof of Brouwer's Continuity Rule

In this chapter we assume that we have a topology $\langle P, J\rangle$ compatible for a locale X internal to $\mathcal{F}$. Let Pt_X denote the object (in $Sh_{\mathcal{F}}(X)$) of points of $\text{Ideal}_{Sh_{\mathcal{F}}(X)}(\Psi_X(P, J))$. Theorem 7.2.3 identifies this as the sheaf of locale maps $X \longrightarrow \text{Ideal}_{\mathcal{F}}(P, J)$. In addition let Pt denote the points of $\text{Ideal}_{\mathcal{F}}(P, J)$. Note that these notations hide the dependence on $\langle P, J\rangle$, this is needed for notational convenience.

Note that $\Psi_X(Pt) = Pt_X$. Therefore we think of Pt_X as the *set of points of the formal space of* $\langle P, J\rangle$, internal to $Sh_{\mathcal{F}}(X)$.

As an example, if we take $\langle P, J\rangle$ to be N^* with the monotonic inductive topology, then $Pt = N^N$ and $Pt_X = N^N$ (in $Sh_{\mathcal{F}}(X)$).

In section 2.2.2 we showed there is a canonical locale map $pt : \Omega^{Pt} \longrightarrow \text{Ideal}_{\mathcal{F}}(P, J)$ with $pt^*(R) = pt(R)$. Let $M = pt^* \downarrow \Omega^{Pt}$ the gluing of $\text{Ideal}_{\mathcal{F}}(P, J)$ (see section 2.4) with the open inclusion $i : \Omega^{Pt} \longrightarrow M$ as defined in section 2.4. Consider the following diagram of *logical* functors[1] (where Ω^{Pt} is regarded as a discrete locale):

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\Psi} & Sh_{\mathcal{F}}(M) \\ \downarrow & & \downarrow{\scriptstyle i^*} \\ \mathcal{F}/Pt & \xleftarrow[\Theta]{} & Sh_{\mathcal{F}}(\Omega^{Pt}) \end{array}$$

where $\mathcal{F} \longrightarrow \mathcal{F}/Pt$ is the slicing map and Θ is defined in 6.5.1. This square commutes

[1] Regarding the logical functor i^* see 6.5.7 and the discussion afterwards.

since there is a unique logical functor $\mathcal{F} \longrightarrow \mathcal{F}/Pt$.

We should note: $Pt_{\Omega^{Pt}}(\{\alpha\}) \cong \mathrm{Loc}_{\mathcal{F}}(\Omega^{\{\alpha\}}, \mathrm{Ideal}_{\mathcal{F}}(P, J)) \cong Pt$, and so

$$\Theta(Pt_{\Omega^{Pt}}) \cong \coprod_{\alpha \in Pt} \mathrm{Loc}_{\mathcal{F}}(\Omega^{\{\alpha\}}, \mathrm{Ideal}_{\mathcal{F}}(P, J)) \cong Pt \times Pt$$

Lemma 8.0.7. *A morphism $Pt_{\Omega^{Pt}} \longrightarrow Pt_{\Omega^{Pt}}$ in $Sh_{\mathcal{F}}(\Omega^{Pt})$ given by $g : Ideal_{\mathcal{F}}(P, J) \longrightarrow Ideal_{\mathcal{F}}(P, J)$ in $\mathcal{F}$ is mapped via Θ to $1_{Pt} \times pt(g)$:*

$$\begin{array}{ccc} Pt \times Pt & \xrightarrow{1_{Pt} \times pt(g)} & Pt \times Pt \\ & {}_{\pi}\searrow \quad \swarrow_{\pi} & \\ & Pt & \end{array}$$

Proof. Let $\gamma : \Omega^{\{\alpha\}} \cong \Omega$ be the canonical iso.

$$Pt \times Pt \xrightarrow{\cong} \textstyle\coprod_{\alpha \in Pt} \mathrm{Loc}_{\mathcal{F}}(\Omega^{\{\alpha\}}, \mathrm{Ideal}_{\mathcal{F}}(P, J)) \xrightarrow{\Theta(g)} \coprod_{\alpha \in Pt} \mathrm{Loc}_{\mathcal{F}}(\Omega^{\{\alpha\}}, \mathrm{Ideal}_{\mathcal{F}}(P, J)) \xrightarrow{\cong} Pt \times Pt$$

$$(\alpha, \beta) \quad \mapsto \quad \beta\gamma \quad \mapsto \quad g\beta\gamma \quad \mapsto \quad (\alpha, g\beta)$$

The result follows (recall that $pt(g) : \beta \mapsto g\beta$). □

We now give our main theorem:

Theorem 8.0.8. *Every map $f : Pt \longrightarrow Pt$ in $\mathcal{F}$ is of the form $pt(g)$ for some locale map $g : Ideal_{\mathcal{F}}(P, J) \longrightarrow Ideal_{\mathcal{F}}(P, J)$. Hence every map is continuous in the induced topology on pt.*

Proof. Let $f : Pt \longrightarrow Pt$ be given. Then $\Psi(f) : Pt_M \longrightarrow Pt_M$ is given by some $\mathrm{Ideal}_{\mathcal{F}}(P, J) \xrightarrow{g} \mathrm{Ideal}_{\mathcal{F}}(P, J)$. Then $i^*\Psi(f) : Pt_{\Omega^{Pt}} \longrightarrow Pt_{\Omega^{Pt}}$ is also given by g (see 6.5.8). Now slicing by Pt maps f to $1_{Pt} \times f$, but this equals $\Theta i^*\Psi(f) = 1_{Pt} \times pt(g)$ and so $f = pt(g)$, from which it follows that f is continuous. □

By instantiating[2] this theorem with example 5.3.7 we obtain

Corollary 8.0.9 (Brouwer's Principle). *In the free topos all arrows $R \longrightarrow R$ are provably continuous.*

Examples 5.3.3 and 5.3.4 also give

Corollary 8.0.10. *In the free topos all arrows $N^N \longrightarrow N^N$ (or $2^N \longrightarrow 2^N$) are provably continuous in the topology provided by viewing them as the points of the formal Baire Space (formal Cantor Space).*

Remark 8.0.11. By theorem 5.3.5 in order to complete the proof of the continuity principle for N^N (or 2^N) we need to show that monotonic bar induction for N^* holds in $\mathcal{F}$. *As mentioned above this will be left for future work.*

---

[2]This relies on knowing that the topology in example 5.3.7 is compatible for the appropriate locale. See remark 7.2.6 and the sketch afterwards for a reason why this is true.

Chapter 9

Conclusions

In this thesis we have obtained the following:

1. Brouwer's principle for the free topos (HAH).

2. Obtained the continuity rules for spaces which can be defined as the space of points of the formal space of some suitable topology (see Remark 7.2.6)

3. Theorem 7.2.3 yields a general result about representing spaces in a sheaf topos (internal to the free topos).

We end with a few directions we hope to pursue as follow-ups to this work.

First, as mentioned earlier, we believe these techniques can be used to prove bar induction for suitable trees (including N^* and 2^*) which will give the appropriate continuity rules.

Second, perhaps the most important direction is to incorporate the theory of fibrations, diagram categories, etc. alongside the internal language, to this construction. This will allow us to use well developed techniques to avoid the use of external reasoning about internal structures and then appealing to relativization. This may also help to weed out the correct categorical setting in which this argument may be applied, thus allowing for maximal generalization. Perhaps there are applications to other free toposes or other continuity rules (between different spaces?). Indeed, we believe that using sheaf semantics, as sketched earlier for Bar Induction, will also prove various uniformity rules, local choice rules, etc.

Finally, we find very interesting the need for the logical functor Ψ to match the inverse image part of the geometric morphism $Sh_{\mathcal{F}}(X) \longrightarrow \mathcal{F}$. What is really going on here? Is there some kind of general phenomenon that occurs when logic and geometry line up in this way? Perhaps some of the techniques mentioned for further work may shed some light on these questions.

Chapter 10

Appendix

10.1 Sheaf-Topos Semantics

We now specialize the semantics presented in 3.5.1 to the case that our topos is $Sh_{\mathcal{E}}(X)$, for some locale, internal to $\mathcal{E}$. We write $u \Vdash \varphi$ as a short hand for $\downarrow u \Vdash \varphi$. A major simplification is the reduction from arbitrary forcing $C \Vdash \varphi$ to considering only the subterminal sheaves $\downarrow u$. This is possible because the subterminal sheaves form a *generating set*. We do not go into these details.

Theorem 10.1.1 (Beth-Kripke-Joyal Semantics for Sheaf-Toposes). *Given $u \in X$, $\downarrow u \xrightarrow{a} F$ and two formulae φ, ψ:*

0) If $F = \underline{\Omega}$ then $u \Vdash a \iff a\cdot = \cdot \downarrow u \longrightarrow \underline{\Omega}$ (i.e. $a(v) = \langle u, v\rangle$ for all $v \leq u$).

1)For $b(x^F)$ a term of type G and $\beta(x^F)$ a term of type $\mathcal{P}(G)$, $u \Vdash b(a) \in \beta(a) \iff b(a)(u) \in \beta(a)(u)$.

2) $u \Vdash \top$ always.

3) $u \Vdash \bot \iff u = 0_X$ (the bottom of X).

4) $u \Vdash \varphi(a) \wedge \psi(a) \iff u \Vdash \varphi(a)$ and $u \Vdash \psi(a)$.

5) $u \Vdash \varphi(a) \vee \psi(a) \iff$ for some $v, w \in X$, $u = v \vee w$ and $v \Vdash \varphi(a{\restriction_v})$ and $w \Vdash \psi(a{\restriction_w})$.

6) $u \Vdash \varphi(a) \Rightarrow \psi(a) \iff$ for all $v \leq u$, $v \Vdash \varphi(a{\restriction_v})$ implies $u \Vdash \psi(a{\restriction_v})$.

7) $u \Vdash \forall_{y \in G}\psi(y, a) \iff$ for all $v \leq u$ and $\downarrow v \xrightarrow{b} G$, $\quad v \Vdash \psi(b, a{\restriction_v})$.

8) $u \Vdash \exists_{y \in G} \psi(y, a) \iff$ *there exists a cover* R *of* u *(i.e. a subobject of* $\downarrow u$ *with* $\bigvee_{\downarrow u} R = u$*) with a family* [1] *of arrows* $b_r : \downarrow r \longrightarrow G$ *such that for all* $r : 1 \longrightarrow R$ *we have* $r \Vdash \psi(b_r, a\restriction_r)$.

$= \bigvee R$ *of* u *and arrows* $b_r : \downarrow r \longrightarrow G$ *(for* $r : 1 \longrightarrow R$*) such that each* $b_r \Vdash \psi(b_r, a\restriction_r$ *).*

9) For terms $b(x^F)$ *and* $b'(x^F)$ *of type* G, $u \Vdash b(a) = b'(a) \iff b(a)\cdot = \cdot b'(a)$.

10.2 Proof of 7.2.2

Recall that Lemma 7.2.2 is the following:

Given a locale L internal to $Sh_{\mathcal{F}}(X)$

$$\underline{\mathrm{Loc}_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}, L)}(u) = \mathrm{Loc}_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}\restriction_u, L).$$

i.e. $\underline{\mathrm{Loc}_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}, L)}$ is the sheaf of locale maps $\underline{\Omega} \longrightarrow L$.

Proof.

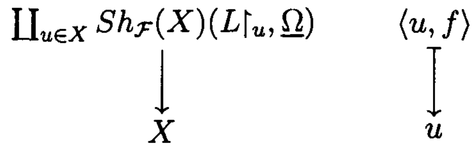
$$\begin{aligned} \underline{\mathrm{Loc}_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}, L)} &= \underline{\mathrm{Frame}_{Sh_{\mathcal{F}}(X)}(L, \underline{\Omega})} \\ &= \underline{\{x \in \underline{\Omega}^L : \varphi(x)\}} \end{aligned}$$

The formula $\varphi(x)$ is some formula to be defined below ($\varphi(f)$ will say "f is a frame map"). Here we identify $\underline{\Omega}^L$ as exponent

[1] *This family of arrows is internal to* $\mathcal{E}$. *So what we really mean is a map over* R: $\coprod_{r\in R} \downarrow r \xrightarrow{\;b\;} G \times R$ *where a given* b_{r_0} *is the composite*

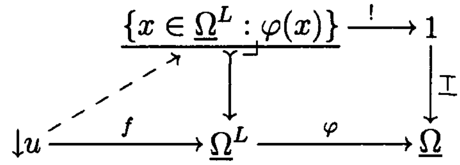
$$\begin{array}{ccc} \coprod_{r\in R} \downarrow r & \xrightarrow{\quad b \quad} & G \times R \\ & \searrow \quad \swarrow_{\pi_2} & \\ & R & \end{array}$$

$$\downarrow r \longrightarrow \coprod_{r\in R} \downarrow r \xrightarrow{b} G \times R \longrightarrow G$$



So that we may identify $\underline{\Omega}^L(u)$ as $Sh_{\mathcal{F}}(X)(L{\restriction_u}, \underline{\Omega})$. Therefore $\underline{\mathrm{Loc}_{Sh_{\mathcal{F}}(X)}(\Omega, L)}(u)$ will be a subobject of $Sh_{\mathcal{F}}(X)(L{\restriction_u}, \underline{\Omega})$. We must determine which subobject it is.

Given a sheaf morphism $f : L{\restriction_u} \longrightarrow \underline{\Omega}$ regarded as an arrow (compatible family) $\downarrow u \xrightarrow{f} \underline{\Omega}^L$, f will "be in" the object $\underline{\{x \in \underline{\Omega}^L : \varphi(x)\}}$ when we have a factorization:

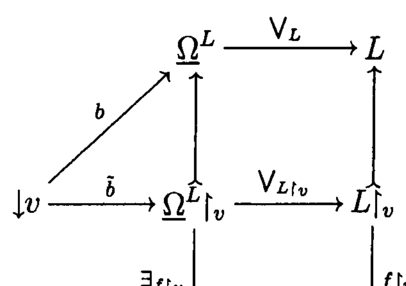


This happens exactly when $\varphi(f) = \underline{\top}!$, i.e. when $u \Vdash \varphi(f)$.

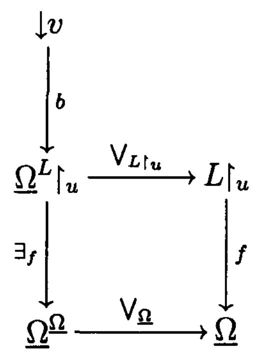
We take $\varphi(x) \equiv \forall_{y \in \underline{\Omega}^L}(x^l \bigvee_L y = \bigvee_{\underline{\Omega}} \exists_{x^l} y) \wedge \forall_{y_1 \in L} \forall_{y_2 \in L}(x^l(y_1 \wedge_L y_2) = x^l y_1 \wedge_L x^l y_2)$ (i.e. x^l preserves arbitrary joins and binary meets).

Now we apply the sheaf semantics to determine when $u \Vdash \varphi(f)$. For simplicity we take $\varphi(x) = \varphi_1(x) \wedge \varphi_2(x)$ where $\varphi_1(x)$ is the first conjunct and $\varphi_2(x)$ is the second. Thus $u \Vdash \varphi(f)$ if and only if $u \Vdash \varphi_1(f)$ and $u \Vdash \varphi_2(f)$.

First lets examine $u \Vdash \varphi_1(f)$. This happens if and only if for every $v \leq u$ and $\downarrow v \xrightarrow{b} \underline{\Omega}^L$ we have $v \Vdash f{\restriction_v} \bigvee_L b = \bigvee_{\underline{\Omega}} \exists_{f{\restriction_v}} b$. This is an equality of arrows giving a commutative diagram.

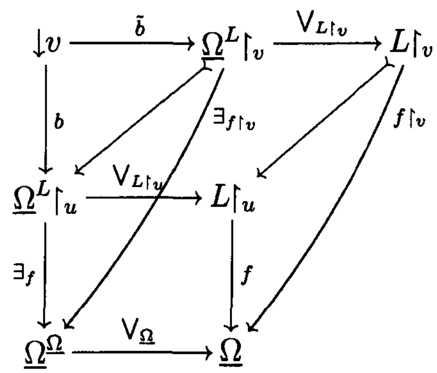


We claim that his implies that f is join-preserving, i.e. $f \bigvee_{L{\restriction_u}} = \bigvee_{\underline{\Omega}} \exists_f$. For this it suffices to show that for any $\downarrow v \xrightarrow{b} \underline{\Omega}^L{\restriction_u}$ that the following commutes (note that the existence of such a b implies that $v \leq u$):



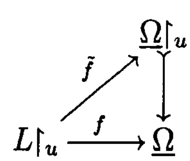
(14)

To see this we draw in the restrictions to v:



Chasing though this diagram now shows that (14) commutes. Therefore f is join-preserving. Clearly the converse holds, that if f is join-preserving then $u \Vdash \varphi_1(f)$. Similarly one can show that $u \Vdash \varphi_2(f)$ is equivalent to the condition that f preserve binary meets. Therefore we have $u \Vdash \varphi(f)$ if and only if $f : L\restriction_u \longrightarrow \underline{\Omega}$ is an (internal) locale map.

So far we've proven that $\underline{\mathrm{loc}_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}, L)}(u) \cong \mathrm{Frm}_{Sh_{\mathcal{F}}(X)}(L\restriction_u, \underline{\Omega})$. Now we just observe that a frame map $L\restriction_u \longrightarrow \underline{\Omega}$ factors uniquely through $\underline{\Omega}\restriction_u$:



Thus we have

$$\begin{aligned} \underline{\mathrm{loc}_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}, L)}(u) &\cong \mathrm{Frm}_{Sh_{\mathcal{F}}(X)}(L\restriction_u, \underline{\Omega}) \\ &\cong \mathrm{Frm}_{Sh_{\mathcal{F}}(X)}(L\restriction_u, \underline{\Omega}\restriction_u) \\ &\cong \mathrm{Loc}_{Sh_{\mathcal{F}}(X)}(\underline{\Omega}\restriction_u, L\restriction_u) \end{aligned}$$

A locale map $\underline{\Omega}\restriction_u \longrightarrow L\restriction_u$ may be composed the locale map $L\restriction_u \longrightarrow L$ to yield a map $\underline{\Omega}\restriction_u \longrightarrow L$. Conversely all such maps $\underline{\Omega}\restriction_u \longrightarrow L$ arise in this way (just pullback along $\downarrow u$. □

Bibliography

[1] Francis Borceux, (1994), *Handbook of Categorical Algebra 3*,Cambridge University Press, Cambridge

[2] Joyal, Tierney, (1984), *An extension of the Galois theory of Grothendieck*, Mem. Amer. Math. Soc. 51 (1984), no. 309, vii+71 pp.

[3] M. P. Fourman: *Proofs of some derived rules for HAH*, Handwritten notes of some joint work with A. Joyal, June 8, 1981.

[4] M.P. Fourman, D.S. Scott, (1979), *Sheaves and Logic*, Lecture Notes in Mathematics Vol 753: Applications of Sheaves, Springer-Verlag, Germany.

[5] M.P. Fourman, R.J. Grayson, (1982), *Formal Spaces*, Studies in Logic and the Foundations of Mathematics Volume 110 : The L.E.J. Brouwer Centenary Symposium, North-Holland Publishing Company, Amsterdam

[6] Gavin Wraith, (1974), *Artin Glueing*, Journal of Pure and Applied Algebra Volume 4 p.345 - 348

[7] Peter T. Johnstone, *Stone Spaces*, Cambridge Univ. Press, 1986, 396 pp.

[8] Peter T. Johnstone, (2002), *Sketches of an Elephant A topos Theory Compendium (Volumes I and II)*, Oxford University Press, New York

[9] Keene, Vesley, (1965), *The Foundations of Intuitionistic Mathematics*, North-Holland Publishing Company, Amsterdam

[10] J. Lambek and P.J. Scott, (1986), *Introduction to Higher Order Categorical Logic*, Cambridge University Press, Cambridge, UK.

[11] Mac Lane, Moerdijk (1992), *Sheaves in Geometry and Logic*, Springer-Verlag, New York.

[12] I. Moerdijk and J. van Oosten, *Topos Theory*, Online notes available at: http://www.math.uu.nl/people/jvoosten/syllabi/toposmoeder.pdf

[13] A. S. Troelstra, ed. *Metamathematical Investigations of Intuitionistic Arithmetic and Analysis*, Springer LNM 344, Springer, 1973.

[14] A.S. Troelstra, D. van Dalen, (1988), *Studies in Logic and the Foundations of Mathematics Volume 121 : Constructivism in Mathematics. An Introduction*, Elsevier Science Publishers B.V., Netherlands.