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Nilpotent derivations in exterior algebra

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The theta map

Consider the following map : for every element e_i in the standard basis of $\mathbb{R}^n$, define

$$\theta(e_i) = \begin{cases} e_{i-1} \text{ if } i \neq 1 \\ 0 \text{ otherwise} \end{cases}$$

Then, extend this map using the product rule as for a derivative for specific vectors in $\Lambda^k\mathbb{R}^n$:

$$\theta(e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}) = \sum_{j=1}^{k} e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_j - 1} \wedge ... \wedge e_{i_k}$$

For clarity, a vector of the form $e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}$ will be called a “weight vector”. Then, we can extend by linearity for an arbitrary element of $\Lambda^k\mathbb{R}^n$.

It is easy to see that this map is nilpotent. Indeed, define the weight of a vector in the standard basis of $\Lambda^k\mathbb{R}^n$ as the sum of its indices. Then, theta always decreases the weight by one, and so theta sends any vector of the lowest possible weight to 0.

Because theta is nilpotent, its Jordan canonical form will have 1s or 0s on the superdiagonal and 0s everywhere else. Part of the goal of this project is to find a Jordan basis, the number of Jordan blocks and the size of these Jordan blocks.

First result : $\mathbb{R}^4$

Let n = 4. Then k can take any value from 1 to 4. When k = 4, one can verify that the theta map is always 0, so its matrix will be [0].

When k = 1 or 3, one can pick the following vectors as basis :

k = 1

$$\begin{aligned} v_1 &= e_4 \\ v_2 &= e_3 \\ v_3 &= e_2 \\ v_4 &= e_1 \end{aligned} \quad \text{with matrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

k = 3

$$\begin{aligned} v_1 &= e_2 \wedge e_3 \wedge e_4 \\ v_2 &= e_1 \wedge e_3 \wedge e_4 \\ v_3 &= e_1 \wedge e_2 \wedge e_4 \\ v_4 &= e_1 \wedge e_2 \wedge e_3 \end{aligned} \quad \text{with matrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, when k = 2, things start to get more interesting. Indeed, there are two Jordan blocks this time. One can pick the following vectors as Jordan basis :

$$\begin{aligned} v_1 &= e_3 \wedge e_4 \\ v_2 &= e_2 \wedge e_4 \\ v_3 &= e_1 \wedge e_4 + e_2 \wedge e_3 \\ v_4 &= 2e_1 \wedge e_3 \\ v_5 &= 2e_1 \wedge e_2 \\ v_6 &= e_1 \wedge e_4 - e_2 \wedge e_3 \end{aligned}$$

The resulting matrix will be :

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This completes the full description of a low dimensional case. While it is possible to aready observe some pattern at this point, it does not give us much information about the general case.

An obvious relationship

In the last section, there was some symmetry in the results for k = 1 and k = 3. Let us consider what happens in $\Lambda^2\mathbb{R}^5$ and $\Lambda^3\mathbb{R}^5$. One can obtain the following Jordan basis :

	k = 2	k = 3
v_1	$= e_4 \wedge e_5$	$= e_3 \wedge e_4 \wedge e_5$
v_2	$= e_3 \wedge e_5$	$= e_2 \wedge e_4 \wedge e_5$
v_3	$= e_2 \wedge e_5 + e_3 \wedge e_4$	$= e_1 \wedge e_4 \wedge e_5 + e_2 \wedge e_3 \wedge e_5$
v_4	$= e_1 \wedge e_5 + 2e_1 \wedge e_4$	$= e_2 \wedge e_3 \wedge e_4 + 2e_1 \wedge e_3 \wedge e_5$
v_5	$= 3e_1 \wedge e_4 + 2e_2 \wedge e_3$	$= 3e_1 \wedge e_3 \wedge e_4 + 2e_1 \wedge e_2 \wedge e_5$
v_6	$= 5e_1 \wedge e_3$	$= 5e_1 \wedge e_2 \wedge e_4$
v_7	$= 5e_1 \wedge e_2$	$= 5e_1 \wedge e_2 \wedge e_3$
v_8	$= 2e_2 \wedge e_5 - 3e_3 \wedge e_4$	$= 2e_2 \wedge e_3 \wedge e_5 - 3e_1 \wedge e_4 \wedge e_5$
v_9	$= 2e_1 \wedge e_5 - e_2 \wedge e_4$	$= 2e_2 \wedge e_3 \wedge e_4 - e_1 \wedge e_3 \wedge e_5$
v_{10}	$= e_1 \wedge e_4 - e_2 \wedge e_3$	$= e_1 \wedge e_3 \wedge e_4 - e_1 \wedge e_2 \wedge e_5$

The matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As you can see, each vector is either mapped to the following vector or 0. This is exactly what we were looking for. However, can you observe some relationship between these two bases?

Indeed, it looks like the coefficients match! Not only do they share the same matrix, but there seems to be some hidden relationship between these two values of k. In fact, it turns out there is some relationship between n and n-k and that the bases we are looking for are related by a very interesting isomorphism.

The flip map

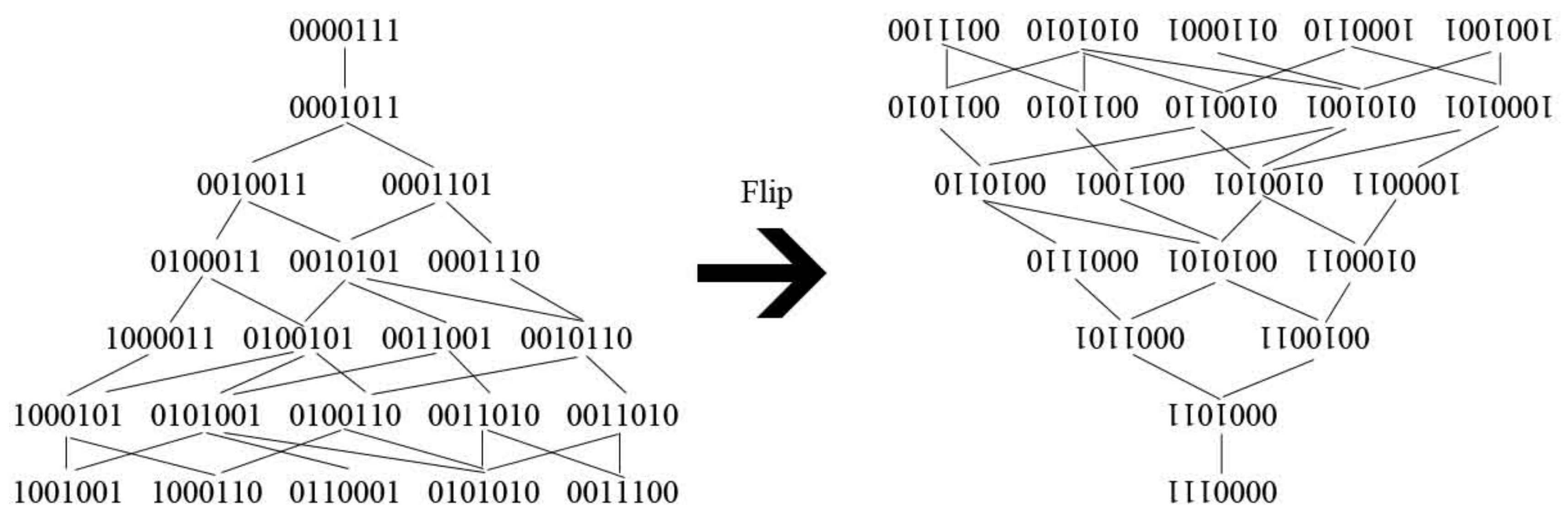
To find this relationship, two maps are required : the Hodge Star map and the flip map. It is easier to understand the effect of the flip maps with some examples. Take for example the vector $e_2 \wedge e_3 \wedge e_5 \wedge e_7$ in $\Lambda^4\mathbb{R}^7$. The flip map will return $e_1 \wedge e_3 \wedge e_5 \wedge e_6$ To understand this, write $e_2 \wedge e_3 \wedge e_5 \wedge e_7$ as “0110101” where a 0 in the ith position means that e_i is not in our vector, and a 1 in the ith position means that e_i is in it. To guarantee the bijection, we require that our indices are always increasing. Then, write this bit string backwards to obtain “1010110”. By converting back, $e_1 \wedge e_3 \wedge e_5 \wedge e_6$ is obtained. That is the effect of the flip map.

However, the flip map is merely an isomorphism from $\Lambda^k\mathbb{R}^n$ to itself. To find the required relationship, it is necessary to compose it with the Hodge Star map. In this case however, we ignore the sign of the permutation given from the Hodge Star map. Then, not only do we find an isomorphism between $\Lambda^k\mathbb{R}^n$ and $\Lambda^k\mathbb{R}^n$, but it commutes with the theta map.
Sadly, the proof of this claim is too long to fit in this small poster, but here is a motivating example :

$$* \circ f(3e_1 \wedge e_4 + 2e_2 \wedge e_3) = * \circ f(3(10010)+2(01100)) = *(3(01001)+2(00110)) = 3(10110) + 2(11001) = 3e_1 \wedge e_3 \wedge e_4 + 2e_1 \wedge e_2 \wedge e_5$$

We find that the resulting term is exactly the corresponding one from the basis in the last section!

More can be done with the flip map. Let us define a relation between our weight vectors defined earlier. Two weight vectors are related if one appears in the sum of weight vectors given by the theta map applied to the other. For example, we could have $\theta(e_2 \wedge e_4 \wedge e_5) = e_1 \wedge e_4 \wedge e_5 + e_2 \wedge e_3 \wedge e_5$, so $e_2 \wedge e_4 \wedge e_5$ is related to both $e_1 \wedge e_4 \wedge e_5$ and $e_2 \wedge e_3 \wedge e_5$ .Then, a very amusing phenomenon occurs : by writing about half of the graph of this relation in the following way, one can obtain the other half by flipping the graph horizontally and vertically. Here is an example for n = 7, k = 3.



The use of Mathematica and LateX

In higher dimensions, it becomes very difficult to compute all those matrices and to solve complicated linear systems. Mathematica allowed me to make these calculations more easily. Furthermore, I have learned how to use LateX for typesetting. Learning how to use and apply these two programs can only be useful in further projects.

The conjecture

The project resulted in a conjecture and an algorithm that seems to meet our goal. Here is the statement of the conjecture :

Let r_i be the number of positive integer solutions to the system $\sum_{j=1}^{k} x_j = i$ such that $1 \leq x_1 < x_2 < ... < x_k \leq n$. Then the number of blocks is given by $\max\{r_i\}$ and the size of the jth block is given by $| \{r_i ; r_i \geq j\} |$

It is also possible to develop an algorithm from this conjecture to obtain the basis for we were seeking. It is the following :

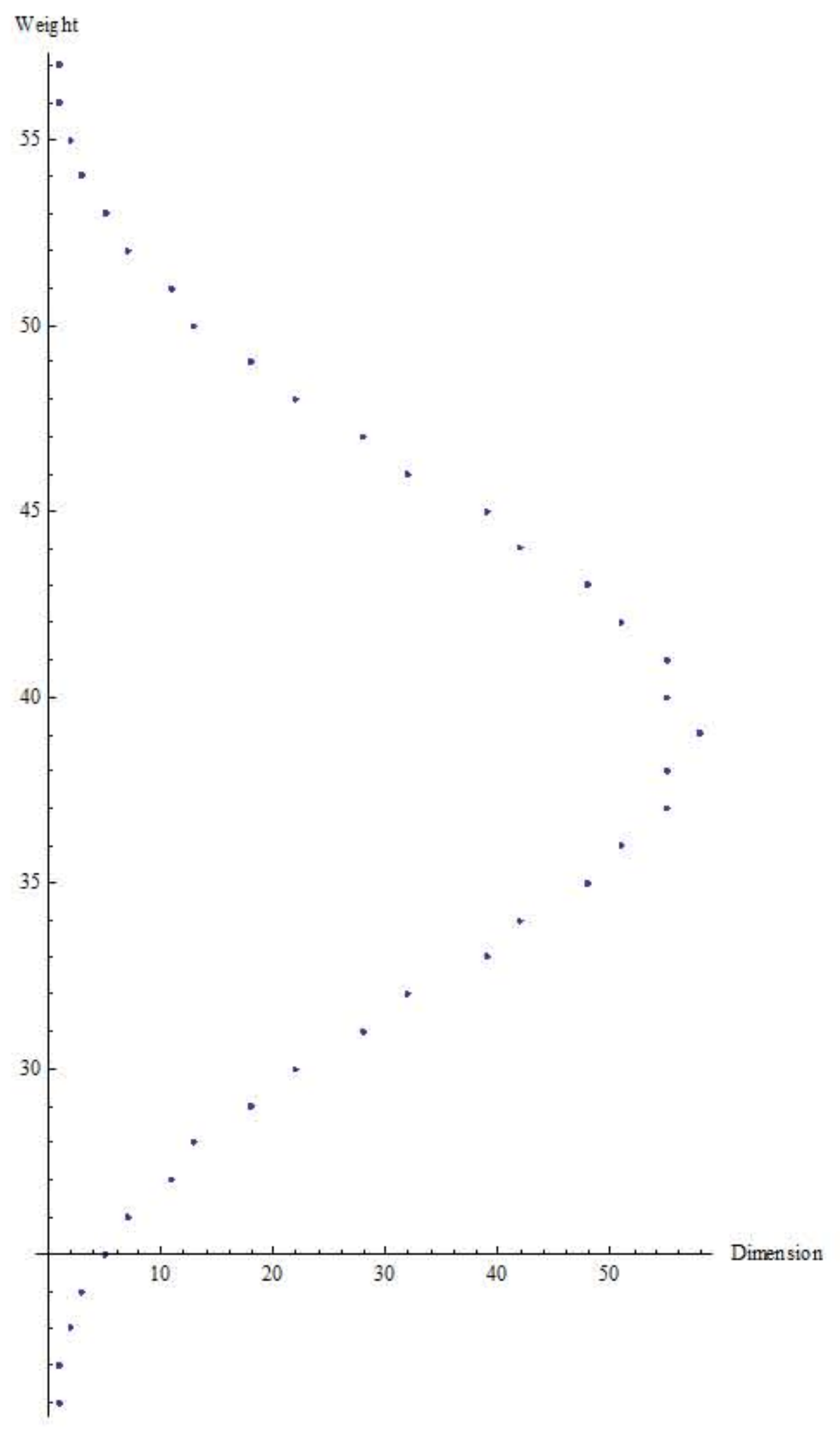
For every weight space W_s, let $r = dimW_s - dimW_{s-1}$. If $r \geq 1$, then Ker θ over W_s is r-dimensional. Compute the kernel of W_s. Then, consider the highest weight q with weight space W_q of the same dimension as W_s. Compute $\theta^{q-s}(W_q)$ and solve for vectors in a chosen basis of Ker W_s. Exactly r vectors in W_q should be obtained, and from there it is possible to get the full Jordan block. Repeat for every s.

Sadly, there is no proof for this conjecture yet, even if the algorithm seems to work. However, there are many motivating results that suggest it is true in general. The first thing to prove would be that for every weight q > s such that $dimW_q = dimW_s$ we have $\theta^{q-s}(W_q) = W_s$.
So far, our efforts to prove this result have been unsuccessful.

However, it has been possible to verify this conjecture for many different values of k and n. It has been verified for all k up to n = 8, and has been verified for k = 6, n = 12, which is 924-dimensional, so it seems like this conjecture may hold.

Many smaller results would also be necessary. For example, it is required to have the dimension of our weight spaces follow some kind of “Gaussian” distribution. The symmetry can already be proven from the flip map, but the increasing nature of the dimensions on the first half does not look like an easy thing to prove. This project seems to need ideas from combinatorics as well as from multilinear algebra.

Here is a graph of the “Gaussian” nature of these weight spaces in k = 6, n = 12.



Conclusion

To conclude, the conjecture remains open. I want to thank Professor Barry Jessup for his help and, especially, his patience throughout. I have learned a lot from this project. I also want to thank the Undergraduate Research Opportunity program to have offered me such a great opportunity.