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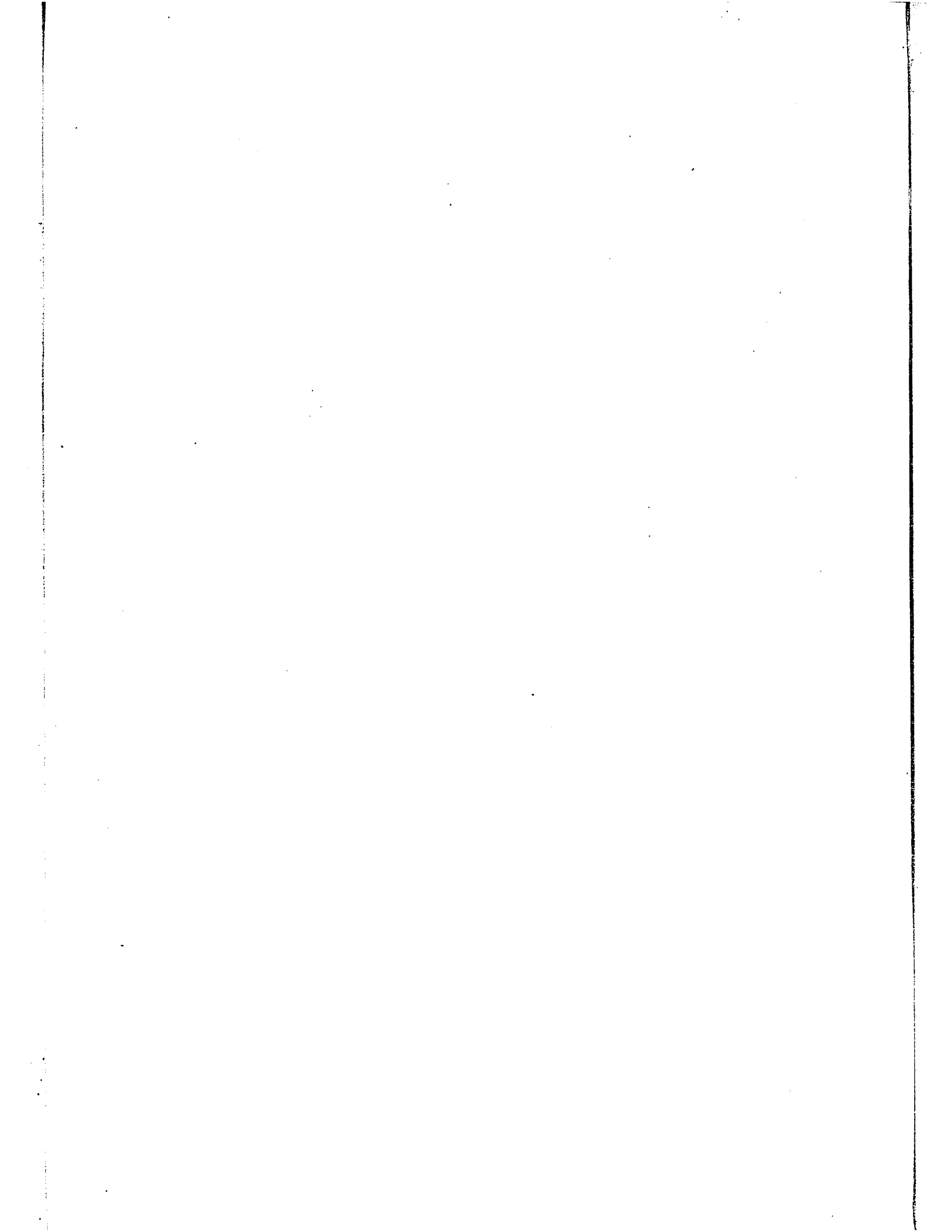
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DENSITY OF DIRECT FACTORS
IN NORMED ARITHMETICAL SEMI-GROUPS

A thesis submitted by John R. Thomas
to the School of Graduate Studies of the
University of Ottawa to complete fulfillment
of the requirements for the degree of Master
of Science in the subject of Mathematics.



UNIVERSITY OF OTTAWA
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ABSTRACT

In this thesis we study direct factors A and B of normed arithmetical semi-groups. Essentially, we prove that such direct factors have asymptotic densities. The generality of this result is restricted to unique factorization semi-groups whose counting functions satisfy some regularity conditions. The treatment is broad enough, however, to include, in addition to the semi-group \mathbb{N}^* of positive integers, analogous "normalized" semi-groups of Gaussian integers and other algebraic integers.

I. INTRODUCTION

Let N^* denote the set of all positive integers. Two subsets A and B of N^* are said to be *supplementary* if every integer $n \geq 1$ can be written uniquely in the form $n = ab$, where $a \in A$ and $b \in B$. It is easy to verify that if one of A and B is given then the other is determined uniquely. Such sets A and B are called *direct factors* of N^* and they occur in some divisor problems.

It may be of interest to ask whether such direct factors always have *asymptotic densities*, that is, whether or not the limits

$$(I.1) \quad d(A) = \lim_{x \rightarrow \infty} \frac{A(x)}{x} \quad \text{and} \quad d(B) = \lim_{x \rightarrow \infty} \frac{B(x)}{x}$$

exist, where $A(x)$ (respectively $B(x)$) is the number of elements of A (respectively B) not exceeding x . It is easy to prove (see §V) that A and B always have *logarithmic densities*

$$(I.2) \quad \delta(A) = \lim_{x \rightarrow \infty} \frac{1}{\ln x} \sum_{\substack{a \leq x \\ a \in A}} a^{-1} \quad \text{and} \quad \delta(B) = \lim_{x \rightarrow \infty} \frac{1}{\ln x} \sum_{\substack{b \leq x \\ b \in B}} b^{-1}$$

and that

$$(I.3) \quad \delta(A) = \left(\sum_{b \in B} \frac{1}{b} \right)^{-1} \quad \text{and} \quad \delta(B) = \left(\sum_{a \in A} \frac{1}{a} \right)^{-1} .$$

The existence of a logarithmic density for a set is weaker than that of an asymptotic density, in that it is known (see §V) that if a set $E \subseteq N^*$ has an asymptotic density, $d(E) = \lim_{x \rightarrow \infty} \frac{E(x)}{x}$, then it

also has a logarithmic density, $\delta(E) = \lim_{x \rightarrow \infty} \frac{1}{\ln x} \sum_{\substack{n \leq x \\ n \in E}} \frac{1}{n}$, and indeed $\delta(E) = d(E)$.

Therefore in view of (I.3), if the direct factors A and B have asymptotic densities, one necessarily has

$$(I.4) \quad d(A) = \left(\sum_{b \in B} \frac{1}{b} \right)^{-1} \quad \text{and} \quad d(B) = \left(\sum_{a \in A} \frac{1}{a} \right)^{-1}.$$

One will observe that at least one of $d(A)$ and $d(B)$ must be equal to zero -- it being immediate that $\max \left(\sum_{a \in A} a^{-1}, \sum_{b \in B} b^{-1} \right) = +\infty$.

In case $\min \left(\sum_{a \in A} a^{-1}, \sum_{b \in B} b^{-1} \right) < +\infty$, then exactly one of $d(A)$ and $d(B)$ is positive (if it exists).

Under this last assumption, B. Saffari [4] has proved the existence of the asymptotic densities $d(A)$ and $d(B)$. More precisely we have the following:

THEOREM A. If $\min \left(\sum_{a \in A} a^{-1}, \sum_{b \in B} b^{-1} \right) < +\infty$, then the asymptotic densities $d(A)$ and $d(B)$ exist and (I.4) is satisfied.

Recently Saffari, Erdős and Vaughan [5] have been able to settle the case for which $\min \left(\sum_{a \in A} a^{-1}, \sum_{b \in B} b^{-1} \right) = +\infty$. They have proven

THEOREM B. If $\min \left(\sum_{a \in A} a^{-1}, \sum_{b \in B} b^{-1} \right) = +\infty$ then the asymptotic densities $d(A)$ and $d(B)$ exist and $d(A) = d(B) = 0$.

The aim of this thesis is to extend Theorem A to a class of "normed arithmetical semi-groups" which contain, in addition to the

semi-group N^* of positive integers, analogous "normalized" semi-groups of gaussian integers and other algebraic integers. Since the methods used in the proof of Theorem B are entirely different from those used in Theorem A we will not attempt a generalization of Theorem B here.

To prepare for the statement of the theorem which generalizes Theorem A, the next section will be used to introduce some necessary definitions.

The proof of the Theorem of this thesis is the work of the author with outlines supplied by B. Saffari, who proved the special case of the Theorem in N^* .

II. DEFINITIONS

DEFINITION II.1 A commutative semi-group S , is called an *arithmetical semi-group* (abbreviation: ASG) if the following conditions are satisfied:

- (i) S has more than one element,
- (ii) the identity is the only invertible element in S ,
- (iii) cancellation is valid in S , i.e., the relation $ab = ac$ implies $b = c$,
- (iv) any element a in S has only finitely many divisors, i.e., the equation $a = bc$ is satisfied by only finitely many b and c in S .

DEFINITION II.2 An arithmetical semi-group S , is called a *normed arithmetical semi-group* (abbreviation: NASG) if one has defined a (multiplicative) norm on S , that is, a mapping $\rho : S \rightarrow R_+$ satisfying the following properties:

- (i) $\rho(ab) = \rho(a)\rho(b)$ for all a and b in S ,
- (ii) $\rho(a) > 1$ if $a \neq 1$,
- (iii) for each x in R_+ the inequality $\rho(a) \leq x$ is satisfied by only finitely many a in S .

In order to simplify our notation we shall write $\|a\|$ for $\rho(a)$.

We note that in order for the axioms of a (not necessarily normed) arithmetical semi-group to be consistent, it must a priori be an infinite set. Moreover, if S is an NASG then it is

denumerable since each of the sets $S_n = \{a \in S : \|a\| \leq n\}$ is finite and $S = \bigcup_{n=1}^{\infty} S_n$. We also note that the semi-group $\rho(S)$ of norm-values is, as a subset of R_+ , closed, discrete and unbounded.

Let S be an NASG, let $E \subseteq S$ and χ_E the characteristic function of E . For real $x \geq 1$ and positive σ set

$$(II.1) \quad L_E(\sigma, x) = \sum_{\substack{\|a\| \leq x \\ a \in E}} \|a\|^{-\sigma} = \sum_{\substack{\|s\| \leq x \\ s \in S}} \frac{\chi_E(s)}{\|s\|^\sigma},$$

and

$$(II.2) \quad L(\sigma, E) = \sum_{a \in E} \|a\|^{-\sigma}.$$

We also write $E(x) = \sum_{\substack{\|a\| \leq x \\ a \in E}} 1$, $L_E(x) = \sum_{\substack{\|a\| \leq x \\ a \in E}} \|a\|^{-1}$ and

$$L(E) = \lim_{x \rightarrow \infty} L_E(x) = \sum_{a \in E} \|a\|^{-1}.$$

Observe that the convergence abscissa, σ_a , of the (generalized) Dirichlet series in the right side of (II.2) is always non-negative. For technical reasons which will appear later, we confine ourselves to the case in which the conditions

$$(II.3) \quad 0 < \sigma_a < +\infty$$

and

$$(II.4) \quad L(\sigma_a, S) = \sum_{a \in S} \|a\|^{-\sigma_a} = +\infty$$

are satisfied. (Replacing, if necessary, the norm $\| \cdot \|$ by the modified norm $\| \cdot \|_a^{\sigma_a}$, these conditions are equivalent to assuming that $\sigma_a = 1$

and that the "harmonic sum" $L(S) = \sum_{s \in S} \|s\|^{-1}$ equals $+\infty$).

If f , g and h are positive real-valued functions respectively defined on $[1, \infty[$, $[1, \infty[\times \mathbb{R}_+^*$ and $[1, \infty[\times \mathbb{R}_+^*$ then by writing

$$g(x, \varepsilon) \ll_{\varepsilon} f(x) \ll_{\varepsilon} h(x, \varepsilon)$$

we shall mean that given any $\varepsilon > 0$, there exist positive "constants" η_{ε} and μ_{ε} such that whenever $x \geq 1$, then

$$\eta_{\varepsilon} g(x, \varepsilon) \leq f(x) \leq \mu_{\varepsilon} h(x, \varepsilon).$$

The assumption of the existence of a positive constant, σ_a , for which

$$(II.5) \quad x^{\sigma_a - \varepsilon} \ll_{\varepsilon} S(x) \ll_{\varepsilon} x^{\sigma_a + \varepsilon}$$

leads (by Proposition IV.1) to the conclusion that σ_a is the convergence abscissa of the "harmonic sum", $L(\sigma, S)$. On the other hand an assumption such as

$$(II.6) \quad S(\lambda x) \geq c \lambda^{\sigma} S(x), \quad \text{for } \lambda \geq \lambda_0 \text{ and } x \geq x_0,$$

where c is an absolute positive constant, is, in view of Corollary (IV.3.3) to be seen later, sufficient for Condition (II.4) to hold.

In this work we shall prove our theorem assuming conditions which we shall presently introduce. We shall show these conditions to be somewhat stronger than Conditions (II.5) and (II.6) -- assuming anything weaker appears to require the more powerful analytic techniques of Complex Function Theory.

Let f and ω denote positive-valued functions defined for $x \geq 1$.

DEFINITION II.3 (i) The function f is said to be of *regular growth* if for every fixed $\lambda \geq 1$

$$(II.7) \quad f(\lambda x) \sim \lambda^\alpha f(x) \quad (\text{as } x \rightarrow +\infty)$$

for some positive "constant" α (that is, α independent of λ).

(ii) The function ω is called *slowly varying* if for every fixed $\lambda \geq 1$

$$(II.8) \quad \omega(\lambda x) \sim \omega(x) \quad (\text{as } x \rightarrow +\infty) .$$

Observe that if $\alpha = 1$ in Condition (II.7) and $f(x) = x\omega(x)$, the function f is of regular growth if and only if ω is slowly varying.

DEFINITION II.4 We say that the function ω is *mildly decreasing* if and only if

$$(II.9) \quad \liminf_{\substack{y, x \rightarrow +\infty \\ y \geq x}} \frac{\omega(y)}{\omega(x)} > 0 .$$

As above if $f(x) = x\omega(x)$ then ω is mildly decreasing if and only if f satisfies the condition

$$(II.10) \quad \liminf_{\substack{y, x \rightarrow +\infty \\ y \geq x}} \frac{yf(y)}{yf(x)} > 0 .$$

The preceding comments are useful, for by employing the change of function

$$\omega(x) = x^{-1}S(x)$$

when dealing with the counting function $S(x)$ of an NASG we simplify the notation.

The notion of a mildly decreasing function has some connection with the (multiplicative) form of the notion of a "slowly decreasing" function as defined in Hardy [2], (Chap. VI, p. 124), although it is more general. To be precise, if $\theta(x)$ is "slowly decreasing" in the sense of Hardy [2], then $e^{\theta(x)}$ is mildly decreasing in our sense, but the converse is false (e.g., $\omega(x) = \exp(2 + \sin x)$ is mildly decreasing but $\ln \omega(x)$ is not slowly decreasing).

In §IV we shall examine in more detail the properties of functions of regular growth and functions which are mildly decreasing.

DEFINITION II.5 Let S be an NASG and let $\omega(x) = \frac{S(x)}{x}$.

- (i) S is called *regular* if $\omega(\lambda x) \sim \omega(x)$ for each fixed $\lambda \geq 1$, i.e., if ω is slowly varying.
- (ii) S is called *mild* if $\liminf_{\substack{y, x \rightarrow +\infty \\ y \geq x}} \frac{\omega(y)}{\omega(x)} > 0$.
- (iii) S is called a mild, regular NASG if both (i) and (ii) hold.

DEFINITION II.6 Let S be an NASG and let E be a subset of S .

The *upper and lower logarithmic densities* of E in S are defined as the numbers

$$(II.11) \quad \delta^*(E) = \limsup_{x \rightarrow +\infty} \frac{L_E(x)}{L_S(x)} \quad \text{and} \quad \delta_*(E) = \liminf_{x \rightarrow +\infty} \frac{L_E(x)}{L_S(x)},$$

respectively. If $\delta^*(E) = \delta_*(E)$, their common value is denoted by $\delta(E)$ and is called the *logarithmic density* of E . The upper and lower asymptotic densities of E in S are defined as the numbers

$$(II.12) \quad d^*(E) = \limsup_{x \rightarrow +\infty} \frac{E(x)}{S(x)} \quad \text{and} \quad d_*(E) = \liminf_{x \rightarrow +\infty} \frac{E(x)}{S(x)}$$

respectively. If $d^*(E) = d_*(E)$, the common value is denoted by $d(E)$ and is called the *asymptotic density* of E in S .

Also required for the statement of the theorem is the following definition.

DEFINITION II.7 Let S be an NASG. Two subsets A and B of S are called *supplementary direct factors* of S if every $s \in S$ can be written uniquely in the form $s = ab$ where $a \in A$ and $b \in B$.

III. STATEMENT OF THE MAIN THEOREM

As in the classical case of N^* it is not difficult to show that direct factors A and B of a regular NASG in which Equation (II.4) holds always have logarithmic densities given by

$$(III.1) \quad \delta(A) = \left(\sum_{b \in B} \|b\|^{-1} \right)^{-1} \quad \text{and} \quad \delta(B) = \left(\sum_{a \in A} \|a\|^{-1} \right)^{-1},$$

(see §V). The main result is the following theorem:

THEOREM. *Let S be a mild, regular, normed arithmetical semi-group with unique factorization, and let A and B be supplementary direct factors of S . If $\min \left(\sum_{a \in A} \|a\|^{-1}, \sum_{b \in B} \|b\|^{-1} \right) < +\infty$, then the asymptotic densities $d(A)$ and $d(B)$ exist, with*

$$(III.2) \quad d(A) = \left(\sum_{b \in B} \|b\|^{-1} \right)^{-1} \quad \text{and} \quad d(B) = \left(\sum_{a \in A} \|a\|^{-1} \right)^{-1}.$$

(Here we are assuming that $\sigma_a = 1$, without loss of generality as observed earlier.)

IV. FUNCTIONS OF REGULAR GROWTH, FUNCTIONS OF SLOW VARIATION,
AND MILDLY DECREASING FUNCTIONS

In what is to follow f will denote (unless otherwise specified) a positive-valued, non-decreasing, right-continuous function defined for $x \geq 1$, (for $x < 1$ we shall assume $f(x) = 0$).

PROPOSITION IV.1 If α is a positive constant for which the condition

$$(IV.1) \quad x^{\alpha-\varepsilon} \underset{\varepsilon}{\ll} f(x) \underset{\varepsilon}{\ll} x^{\alpha+\varepsilon}$$

holds, then α is the convergence abscissa of the integral

$$(IV.2) \quad \int_1^{\infty} t^{-\sigma} df(t) .$$

PROOF. For fixed $\sigma > 0$, consider the function (of $x \geq 1$)

$$F(\sigma, x) = \int_1^x t^{-\sigma} df(t) .$$

Since f is right-continuous, this integral is always defined and we can integrate it by parts to get

$$(IV.3) \quad F(\sigma, x) = \frac{f(x)}{x^{\sigma}} + \sigma \int_1^x t^{-\sigma-1} f(t) dt .$$

Now let $\varepsilon > \varepsilon_1 > 0$ and choose $\sigma = \alpha + \varepsilon$. In view of Relation (IV.1) there exists positive constants μ_{ε} and μ_{ε_1} such that

$$f(x) \leq \mu_{\varepsilon} x^{\alpha+\varepsilon} \quad \text{and} \quad f(x) \leq \mu_{\varepsilon_1} x^{\alpha+\varepsilon_1} .$$

From the second inequality it follows that

$$\begin{aligned}
\int_1^x t^{-\sigma-1} f(t) dt &= \int_1^x t^{-(\alpha+\varepsilon_1+\varepsilon-\varepsilon_1)-1} f(t) dt \\
&\leq \mu_{\varepsilon_1} \int_1^x t^{-(\varepsilon-\varepsilon_1)-1} dt \\
&< \mu_{\varepsilon_1} \int_1^{+\infty} t^{-(\varepsilon-\varepsilon_1)-1} dt \\
&= \mu_{\varepsilon_1} \frac{1}{\varepsilon - \varepsilon_1} < +\infty .
\end{aligned}$$

Thus both terms on the right hand side of Relation (IV.3) are bounded.

Therefore $F(\sigma, x)$ is bounded, and since it increases with x ,

$$\lim_{x \rightarrow +\infty} F(\sigma, x) < +\infty .$$

Now set $\sigma = \alpha - \varepsilon$. Again from Relation (IV.1)

$$\int_1^x t^{-(\alpha-\varepsilon)-1} f(t) dt \geq \eta_{\varepsilon} \ln x$$

for some positive constant η_{ε} . Therefore, in view of Relation

$$(IV.3), \lim_{x \rightarrow +\infty} F(\sigma, x) = +\infty . \quad \text{Q.E.D.}$$

If f is of regular growth with exponent of growth equal to α , then

$$\left(\frac{x}{y}\right)^{\alpha-\varepsilon} \ll_{\varepsilon} \frac{f(x)}{f(y)} \ll_{\varepsilon} \left(\frac{x}{y}\right)^{\alpha+\varepsilon}, \quad 1 \leq y \leq x,$$

by Lemma (4A) of [1]. In particular with $y = 1$

$$(IV.4) \quad x^{\alpha-\varepsilon} \ll_{\varepsilon} f(x) \ll_{\varepsilon} x^{\alpha+\varepsilon} .$$

Interpreting the function f as the counting function $S(x)$ of an NASG, we obtain the following Corollary to Proposition (IV.1):

COROLLARY IV.1.1 If S is a regular NASG, the convergence abscissa of the sum $L(\sigma, S)$ equals 1.

On the other hand, regular growth is not sufficient to determine the behaviour of the integral (IV.2) at $\sigma = \sigma_a$. For example both of the functions

$$f_1(x) = \begin{cases} 0, & x < 1 \\ x, & x \geq 1 \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 0, & x \leq e^2 \\ x \ln^{-2}x, & x \geq e^2 \end{cases}$$

are of regular growth with abscissa of convergence in both cases equal to 1. However, the integral $F(\sigma, x)$ diverges if $f = f_1$, while it converges if $f = f_2$.

To determine a sufficient condition for the divergence to $+\infty$ of the integral $F(\sigma, x)$ as $x \rightarrow +\infty$ at $\sigma = \sigma_a$, we will make use of the following results.

PROPOSITION IV.2 Let f denote a positive-valued, measurable function defined for $x \geq 1$ (for $x < 1$ we assume $f(x) = 0$). Let $\sigma > 0$.

Then

$$(IV.5) \quad \int_1^{+\infty} \frac{f(x)}{x^{1+\sigma}} dx = +\infty,$$

under each of the following assumptions:

(A) there is a "constant" $x_0 \geq 1$ (x_0 possibly depending on f and σ) such that

$$(IV.6) \quad \limsup_{\lambda \rightarrow +\infty} \left(\inf_{x \geq x_0} \frac{f(\lambda x)}{\lambda^\sigma f(x)} \right) > 0;$$

(B) there are "constants" $x_0 \geq 1$ and $\lambda_0 > 1$ (x_0 and λ_0 possibly depending on f and σ) such that

$$(IV.7) \quad \sup_{\lambda \geq \lambda_0 > 1} \left[\inf_{x \geq x_0} \frac{f(\lambda x)}{\lambda^\sigma f(x)} \right] \geq 1 .$$

PROOF. (A) There is a sequence $(\lambda_k)_{k \geq 1}$ tending to $+\infty$ such that

$$\lim_{k \rightarrow +\infty} \left[\inf_{x \geq x_0} \frac{f(\lambda_k x)}{\lambda_k^\sigma f(x)} \right] = \Gamma > 0 .$$

(We may of course assume a priori that $\lambda_k > 1$ for all $k \geq 1$.)

Let ϵ be chosen so that $0 < \epsilon < \min(1, \Gamma)$. Then for

$k \geq k_0(\epsilon)$,

$$\inf_{x \geq x_0} \frac{f(\lambda_k x)}{\lambda_k^\sigma f(x)} \geq \Gamma - \epsilon ,$$

that is,

$$f(\lambda_k x) \geq (\Gamma - \epsilon) \lambda_k^\sigma f(x)$$

for $k \geq k_0(\epsilon)$ and $x \geq x_0$. Setting $\Delta = \min(1, \Gamma)$ we get

$$f(\lambda_k x) \geq (\Delta - \epsilon) \lambda_k^\sigma f(x)$$

for $k \geq k_0$ and $x \geq x_0$. Now with $k \geq k_0$, use the substitution $x = \lambda_k u$ and the preceding inequality to obtain

$$\int_{\lambda_k x_0}^{\lambda_k^2 x_0} \frac{f(x)}{x^{1+\sigma}} dx = \int_{x_0}^{\lambda_k x_0} \frac{f(\lambda_k u)}{\lambda_k^\sigma u^{1+\sigma}} du \geq (\Delta - \epsilon) \int_{x_0}^{\lambda_k x_0} \frac{f(u)}{u^{1+\sigma}} du .$$

Similarly, for all integers $r \geq 1$,

$$\int_{\lambda_k^r x_0}^{\lambda_k^{r+1} x_0} \frac{f(x)}{x^{1+\sigma}} dx \geq (\Delta - \epsilon)^r \int_{x_0}^{\lambda_k x_0} \frac{f(u)}{u^{1+\sigma}} du .$$

Summing from $r = 0$ (in which case the preceding inequality holds trivially) to $r = R$ we get

$$\int_{x_0}^{\lambda_k^R x_0} \frac{f(x)}{x^{1+\sigma}} dx \geq \left(\sum_{r=0}^R (\Delta - \epsilon)^r \right) \int_{x_0}^{\lambda_k x_0} \frac{f(u)}{u^{1+\sigma}} du .$$

Now let $R \rightarrow +\infty$. Since $\sum_{r=0}^{\infty} (\Delta - \epsilon)^r = (1 - \Delta + \epsilon)^{-1}$, then

$$(IV.8) \quad \int_{x_0}^{+\infty} \frac{f(x)}{x^{1+\sigma}} dx \geq \frac{1}{1 - \Delta + \epsilon} \int_{x_0}^{\lambda_k x_0} \frac{f(x)}{x^{1+\sigma}} dx .$$

The integral on the right hand side of (IV.8) is positive (because the function f is positive valued) and $(1 - \Delta + \epsilon)^{-1} > 1$, (because $0 < \epsilon < \Delta < 1$).

We conclude that

$$(IV.9) \quad \int_{x_0}^{+\infty} \frac{f(x)}{x^{1+\sigma}} dx = +\infty ,$$

for otherwise if we let $\lambda_k \rightarrow +\infty$ in Inequality (IV.8) we would obtain a contradiction in view of the preceding remarks. In turn Equation (IV.9) implies that

$$\int_1^{+\infty} \frac{f(x)}{x^{1+\sigma}} dx = +\infty ,$$

as required.

(B) There is a λ (fixed) > 1 such that for all $x \geq x_0$

$$f(\lambda x) \geq \lambda^\sigma f(x)(1 - \epsilon) .$$

Exactly in the same manner as in the proof of (A) (λ replacing λ_k and $1 - \epsilon$ replacing $\Delta - \epsilon$) we get

$$\int_{\lambda^r x_0}^{\lambda^{r+1} x_0} \frac{f(x)}{x^{1+\sigma}} dx \geq (1 - \epsilon)^r \int_{x_0}^{\lambda x_0} \frac{f(u)}{u^{1+\sigma}} du$$

for all integers $r \geq 0$. Summing as in (A) from $r = 0$ to $r = +\infty$ we obtain

$$(IV.10) \quad \int_{x_0}^{+\infty} \frac{f(x)}{x^{1+\sigma}} dx \geq \frac{1}{\epsilon} \int_{x_0}^{\lambda x_0} \frac{f(u)}{u^{1+\sigma}} du .$$

Since Inequality (IV.10) holds for every $\epsilon > 0$ (with $\int_{x_0}^{\lambda x_0} f(u)u^{-1-\sigma} du$ a fixed positive constant independent of ϵ), letting $\epsilon \rightarrow 0$ implies

$$\int_{x_0}^{+\infty} f(x)x^{-1-\sigma} dx = +\infty ,$$

which again leads to Equation (IV.5).

Q.E.D.

COROLLARY IV.2.1 Let f be as in Proposition (IV.2) and let

$f(x) = x^\sigma \omega(x)$, where $\sigma > 0$. Then

$$(IV.11) \quad \int_1^{+\infty} \frac{\omega(x)}{x} dx = +\infty$$

under each of the following assumptions:

(A') there is a "constant" $x_0 \geq 1$ (x_0 possibly depending on f and σ) such that

$$(IV.12) \quad \limsup_{\lambda \rightarrow +\infty} \left(\inf_{x \geq x_0} \frac{\omega(\lambda x)}{\omega(x)} \right) = \Gamma > 0 ;$$

(B') there are "constants" $x_0 \geq 1$ and $\lambda_0 \geq 1$ (x_0 and λ_0 possibly depending on f and σ) such that

$$(IV.13) \quad \sup_{\lambda \geq \lambda_0} \inf_{x \geq x_0} \frac{\omega(\lambda x)}{\omega(x)} \geq 1 .$$

COROLLARY IV.2.2 Let f be as in Proposition (IV.2) with the added stipulation that f be right continuous. Then

$$(IV.14) \quad \int_1^{+\infty} t^{-\sigma} df(t) = +\infty ,$$

providing one of the Conditions (IV.6) or (IV.7) holds.

PROOF. Fix $x \geq 1$. Since f is right-continuous we can write

(see Verley [7], p. 138)

$$\int_1^x t^{-\sigma} df(t) = x^{-\sigma} f(x) + \int_1^x t^{-\sigma-1} f(t) dt .$$

Letting $x \rightarrow +\infty$, we obtain

$$\int_1^{+\infty} t^{-\sigma} df(t) = +\infty .$$

Q.E.D.

It is our intention to apply Proposition (IV.2) and its corollaries to mild normed arithmetical semi-groups in which $\sigma_a = 1$. We begin by examining the elementary properties of mildly decreasing functions in general.

PROPOSITION IV.3 Let ω be a mildly decreasing function. Then

- (i) either ω is bounded or $\lim_{x \rightarrow +\infty} \omega(x) = +\infty$;
 (ii) $\liminf_{x \rightarrow +\infty} \omega(x) > 0$; i.e., ω is bounded away from 0 for sufficiently large x .

PROOF. (i) Suppose ω is unbounded and has no limit in the extended real line $\bar{\mathbb{R}}$ as $x \rightarrow +\infty$. Then there is a constant $c > 0$ such that $\omega(y_n) \leq c$ for a sequence of numbers $(y_n)_{n \geq 1}$ tending to $+\infty$. There is also a sequence $(x_n)_{n \geq 1}$ tending to $+\infty$ such that $\omega(x_n) \rightarrow +\infty$ as $n \rightarrow +\infty$. Now for each x_n we can find some y_m with $y_m > x_n$ with the property that

$$\frac{\omega(y_m)}{\omega(x_n)} \leq \frac{c}{\omega(x_n)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty ,$$

contradicting our assumption that ω is mildly decreasing.

(ii) By assumption $\liminf_{x \rightarrow +\infty} \frac{\omega(y)}{\omega(x)} = \alpha > 0$, so there is an $x_0 \geq 1$ such that

$$\inf_{\substack{x \geq x_0 \\ y \geq x}} \frac{\omega(y)}{\omega(x)} \geq \frac{\alpha}{2} > 0 ;$$

and so $\omega(x) \geq \frac{\alpha}{2} \omega(x_0) > 0$ for $x \geq x_0$. Therefore, $\inf_{x \geq x_0} \omega(x) > 0$.

In other words, $\liminf_{x \rightarrow +\infty} \omega(x) > 0$, which is Condition (ii) of the

proposition.

Q.E.D.

REMARK IV.1 Observe that if ω is a bounded function then the condition $\liminf_{x \rightarrow +\infty} \omega(x) > 0$ is also a *sufficient* condition for ω to be mildly decreasing. Note also that a bounded mildly decreasing function,

contrary to an unbounded one, does not necessarily have a limit as $x \rightarrow +\infty$ even if it is slowly varying. An example of such a function is

$$f(x) = a + \sin(\ln \ln x), \quad a > 1.$$

On the other hand, a slowly varying function which tends to $+\infty$ need not be mildly decreasing. Consider for example

$$f(x) = (1 + (\ln x)\sin^2(\ln \ln x))\ln \ln x.$$

COROLLARY IV.3.1 Let S be a mild NASG (with $\sigma_a = 1$). Also, let $S(x) = x\omega(x)$. Then

- (i) ω is bounded on every compact interval;
- (ii) $\inf_{x \geq 1} \omega(x) > 0$;
- (iii) $\inf_{y \geq x \geq 1} \frac{\omega(y)}{\omega(x)} > 0$.

PROOF. (i) This follows from the fact that the counting function $S(x)$ is a step function.

(ii) ω is mildly decreasing. Therefore, by Condition (ii) of Proposition (IV.3), $\inf_{x \geq x_0} \omega(x) > 0$ for some $x_0 > 1$. On the other hand we also have

$$0 < \frac{1}{x_0} \leq \frac{S(x)}{x} = \omega(x) \quad \text{for } 1 \leq x \leq x_0;$$

hence, $\inf_{x \geq 1} \omega(x) > 0$.

- (iii) Since ω is slowly decreasing, $\inf_{y \geq x \geq x_0} \frac{\omega(y)}{\omega(x)} > 0$ for

some $x_0 \geq 1$. On the other hand

$$0 < \alpha = \sup_{1 \leq x \leq x_0} \omega(x) < +\infty,$$

since $x^{-1}S(x) = \omega(x)$ is bounded on every interval $[1, a]$ where $a > 0$.

Consequently $\frac{x}{S(x)} \geq \frac{1}{\alpha} > 0$ whenever $1 \leq x \leq x_0$. Therefore

$$\inf_{\substack{y \geq x \\ 1 \leq x \leq x_0}} \frac{\omega(y)}{\omega(x)} \geq \frac{1}{\alpha} \inf_{y \geq 1} \omega(y) > 0,$$

so Condition (iii) holds.

Q.E.D.

COROLLARY IV.3.3 If S is a mild NASG ($\sigma_a = 1$), then the "harmonic sum"

$$L(S) = \sum_{a \in S} a^{-1}$$

is divergent.

PROOF. Fix $x \geq 1$ and write

$$(IV.15) \quad L_S(x) = \int_1^x t^{-1} dS(t) = \omega(x) + \int_1^x t^{-1} \omega(t) dt.$$

By Condition (ii) of Proposition (IV.3), $\omega(x)$ is either bounded or $\lim_{x \rightarrow +\infty} \omega(x) = +\infty$. In the latter case the result follows by letting

$x \rightarrow +\infty$ in Equation (IV.15) above. On the other hand, if ω is bounded we apply Corollary (IV.3.1), i.e., the fact that there is a constant $c > 0$ such that $c < \omega(x)$ for $x \geq 1$. This assures the divergence of the second term on the right hand side of Equation (IV.15) to $+\infty$ as $x \rightarrow +\infty$. Therefore, in either case the result follows.

Q.E.D.

V. ELEMENTARY RESULTS

LEMMA V.1 Let S be an NASG for which $L(S) = +\infty$ (here again we assume $\sigma_a = 1$). Then for any subset $E \subseteq S$

$$(V.1) \quad 0 \leq d_*(E) \leq \delta_*(E) \leq \delta^*(E) \leq d^*(E) \leq 1 .$$

PROOF. Clearly the first, third and fifth inequalities in (V.1) are valid. To show the others let $\varepsilon > 0$. By Definition (II.7) there is an $x_0 = x_0(\varepsilon)$ such that whenever $x \geq x_0$

$$(V.2) \quad \frac{E(x)}{S(x)} \geq d_*(E) - \varepsilon .$$

We apply this inequality as follows: first we write

$$L_E(x) = \int_1^x t^{-1} dE(t) ;$$

then integrate by parts to get

$$(V.3) \quad L_E(x) = \frac{E(x)}{x} + \int_1^x t^{-2} E(t) dt .$$

Letting $x \geq x_0$ in the above expression and rewriting it in the form

$$L_E(x) = \frac{E(x)}{S(x)} \frac{S(x)}{x} + \int_1^{x_0} t^{-2} E(t) dt + \int_{x_0}^x t^{-2} S(t) \frac{E(t)}{S(t)} dt ,$$

we obtain the following inequality (using (V.2)):

$$\begin{aligned}
L_E(x) &\geq (d_*(E) - \epsilon) \frac{S(x)}{x} + \int_1^{x_0} t^{-2}S(t) dt \\
&\quad + (d_*(E) - \epsilon) \int_1^x t^{-2}S(t) dt \\
&= (d_*(E) - \epsilon) \left[\frac{S(x)}{x} + \int_1^x t^{-2}S(t) dt \right] \\
&\quad + \int_1^{x_0} t^{-2}S(t) dt - (d_*(E) - \epsilon) \int_1^{x_0} t^{-2}S(t) dt .
\end{aligned}$$

Noting that, as in Equation (V.3),

$$L_S(x) = \frac{S(x)}{x} + \int_1^x t^{-2}S(t) dt ,$$

we arrive at the inequality

$$L_E(x) \geq (d_*(E) - \epsilon)L_S(x) + C(\epsilon)$$

or

$$(V.4) \quad \frac{L_E(x)}{L_S(x)} \geq d_*(E) - \epsilon + \frac{C(\epsilon)}{L_S(x)} , \quad (x \geq x_0) ,$$

where we have set

$$C(\epsilon) = (1 - d_*(E) + \epsilon) \int_1^{x_0} t^{-2}S(t) dt .$$

Letting $x \rightarrow +\infty$ in Inequality (V.4) we obtain

$$\delta_*(E) \geq d_*(E) - \epsilon ,$$

since $C(\epsilon)$ is independent of x and $L_S(x) \rightarrow +\infty$ as $x \rightarrow +\infty$.

Because this inequality is valid for every $\epsilon > 0$, the second inequality in (V.1), $\delta_*(E) \geq \delta^*(E)$, is obtained.

Replacing E by $E^c = S \setminus E$ in the inequality $\delta_*(E) \geq d_*(E)$ and applying the identities

$$\delta_*(E^c) = 1 - \delta^*(E) , \quad d_*(E^c) = 1 - d^*(E)$$

we get

$$1 - \delta^*(E) \geq 1 - d^*(E)$$

from which we obtain the fourth inequality, $\delta^*(E) \leq d^*(E)$, in (V.1). Q.E.D.

LEMMA V.2 *Let S be a regular NASG and let A and B be direct factors of S . Then A and B have logarithmic densities given by*

$$(V.5) \quad \delta(A) = \frac{1}{L(B)} \quad \text{and} \quad \delta(B) = \frac{1}{L(A)} .$$

PROOF. Since every s in S can be written uniquely in the form $s = ab$ where $a \in A$ and $b \in B$ then

$$(V.6) \quad L_S(x) = \sum_{\|s\| \leq x} \|s\|^{-1} = \sum_{\substack{\|ab\| \leq x \\ a \in A, b \in B}} \|a\|^{-1} \|b\|^{-1} .$$

From this it follows that $L_A(x)L_B(x) \geq L_S(x)$, or $\frac{L_A(x)}{L_S(x)} \geq \frac{1}{L_B(x)}$.

Taking the limit as $x \rightarrow +\infty$ on both sides of this inequality gives

$$(V.7) \quad \delta_*(A) \geq \lim_{x \rightarrow +\infty} \frac{1}{L_B(x)} .$$

On the other hand, let $\lambda > 1$ be given. By Equation (V.6), $L_A(x)L_B(\lambda) \leq L_S(\lambda x)$; so

$$(V.8) \quad \frac{L_A(x)}{L_S(x)} \leq \frac{L_S(\lambda x)}{L_S(x)} \frac{1}{E_B(\lambda)} .$$

Writing $L_S(\lambda x)$ as a Stieltjes integral and integrating by parts we have

$$L_S(\lambda x) = \frac{S(\lambda x)}{\lambda x} + \int_1^{\lambda x} t^{-2} S(t) dt .$$

Using the substitution $t = \lambda u$, the preceding reduces to

$$L_S(\lambda x) = \frac{S(\lambda x)}{\lambda x} + \int_{1/\lambda}^x \frac{S(\lambda u)}{\lambda u^2} du .$$

Now let $\varepsilon > 0$ be given. Since $S(x)$ is of regular growth there is an $x_0 = x_0(\varepsilon) > 1$ such that for $x \geq x_0$, $S(\lambda x) < (\lambda + \varepsilon)S(x)$. Therefore, for $x \geq x_0$ we have

$$(V.9) \quad L_S(\lambda x) < \left(1 + \frac{\varepsilon}{\lambda}\right) \frac{S(x)}{x} + \int_{1/\lambda}^{x_0} \frac{S(\lambda u)}{\lambda u^2} du + \left(1 + \frac{\varepsilon}{\lambda}\right) \int_{x_0}^x \frac{S(u)}{u^2} du .$$

Now if we set

$$C(\varepsilon) = \int_{1/\lambda}^{x_0} \frac{S(\lambda u)}{\lambda u^2} du - \left(1 + \frac{\varepsilon}{\lambda}\right) \int_1^{x_0} \frac{S(u)}{u^2} du ,$$

we can then write Inequality (V.9) in the form

$$L_S(\lambda x) < \left(1 + \frac{\varepsilon}{\lambda}\right) L_S(x) + C(\varepsilon) .$$

Dividing both sides of the preceding inequality by $L_S(x)$ and then letting $x \rightarrow +\infty$ gives

$$(V.10) \quad \limsup_{x \rightarrow +\infty} \frac{L_S(\lambda x)}{L_S(x)} \leq 1 + \frac{\varepsilon}{\lambda}.$$

Since (V.10) holds for every $\varepsilon > 0$ and since $\frac{L_S(\lambda x)}{L_S(x)} \geq 1$ we have the following:

$$(V.11) \quad \lim_{x \rightarrow +\infty} \frac{L_S(\lambda x)}{L_S(x)} = 1.$$

Now by Equation (V.11) above, and Inequality (V.8), we observe that

$$(V.12) \quad \limsup_{x \rightarrow +\infty} \frac{L_A(x)}{L_S(x)} \leq \frac{1}{L_B(\lambda)}.$$

Since the left hand side of Inequality (V.11) does not depend on λ , then

$$(V.13) \quad \delta^*(A) \leq \lim_{\lambda \rightarrow +\infty} \frac{1}{L_B(\lambda)}.$$

Inequalities (V.13) and (V.7) then imply that

$$(V.14) \quad \delta_*(A) = \frac{1}{L(B)} = \delta^*(A)$$

as required.

Q.E.D.

PROPOSITION V.1 *Let S be a mild NASG and let $E \subseteq S$ with*

$\sum_{v \in E} \frac{1}{\|v\|} < +\infty$. Then E has an asymptotic density equal to zero.

PROOF. Write $E(x)$ as a Stieltjes integral:

$$E(x) = \int_1^x t \, d \left(\sum_{\|v\| \leq t} \frac{1}{\|v\|} \chi_E(v) \right) = \int_1^x t \, dL_E(t).$$

Since the functions $f(x) = x$ and $L_E(x)$ are continuous and right continuous respectively we can integrate by parts to obtain

$$(V.15) \quad E(x) = xL_E(x) - \int_1^x L_E(t) dt .$$

Now let $\epsilon > 0$ be given. By assumption $L(E) < +\infty$, so there is an $x_0(\epsilon)$ such that for $x \geq x_0(\epsilon)$

$$L(E) - \epsilon \leq L_E(x) ,$$

which in turn implies that

$$\begin{aligned} \int_1^x L_E(t) dt &= \int_1^{x_0} L_E(t) dt + \int_{x_0}^x L_E(t) dt \\ &\geq x(L(E) - \epsilon) + O(1) . \end{aligned}$$

Therefore, in view of Equation (V.15), for $x \geq x_0(\epsilon)$

$$\begin{aligned} \frac{E(x)}{S(x)} &\leq \frac{x}{S(x)} L_E(x) - \frac{x}{S(x)} (L_E(x) - \epsilon) + O\left(\frac{1}{S(x)}\right) \\ &= \epsilon \frac{x}{S(x)} + O\left(\frac{1}{S(x)}\right) . \end{aligned}$$

Since S is mild by assumption, then by Corollary (IV.3.1) there is a positive constant θ such that $\frac{S(x)}{x} \geq \theta > 0$ for $x \geq 1$. Thus taking $x > \max(x_0, 1)$ we have

$$\frac{E(x)}{S(x)} \leq \frac{\epsilon}{\theta} + O\left(\frac{1}{S(x)}\right) .$$

Letting $x \rightarrow +\infty$ in the preceding inequality and then letting $\epsilon \rightarrow 0$ implies that E has an asymptotic density equal to zero.

Q.E.D.

REMARK V.1 The theorem assumes $\min(L(A), L(B)) < +\infty$. Henceforth we shall suppose that $L(B) < +\infty$. This implies that $d(B) = 0$,

(Proposition V.1), and also that $L(A) = +\infty$. Thus the equality $d(B) = 1/L(A)$ holds true, and the theorem essentially reduces to the relation

$$(V.16) \quad d(A) = \frac{1}{L(B)},$$

to the proof of which the rest of this thesis is devoted.

By Lemmas (V.1) and (V.2),

$$(V.17) \quad d_*(A) \leq \frac{1}{L(B)} \leq d^*(A).$$

The proof of the theorem is based on the somewhat surprising fact, which we have adopted from Lemma (2) in [4], that if any of the two inequalities in (V.17) is an equality, then the other is also an equality:

LEMMA V.3 *The equalities*

$$(V.18) \quad d_*(A) = \frac{1}{L(B)}$$

and

$$(V.19) \quad d^*(A) = \frac{1}{L(B)}$$

are equivalent.

PROOF. Since A and B are direct factors of S ,

$$(V.20) \quad \begin{aligned} S(x) &= \sum_{\|s\| \leq x} \chi_S(s) = \sum_{\|ab\| \leq x} \chi_A(a) \chi_B(b) \\ &= \sum_{\substack{\|b\| \leq x \\ b \in B}} A\left(\frac{x}{\|b\|}\right). \end{aligned}$$

We may assume that B is not reduced to the identity 1 , otherwise $A = S$ and (V.18) and (V.19) both hold trivially. Let b run

over the set of those elements of B whose norm-values are > 1 , i.e., those which are $\neq 1$, and let b_2 denote an element of B of least norm-value > 1 . By (V.20)

$$(V.21) \quad S(x) - A(x) = \sum_{\|b\| \geq \|b_2\|} A\left(\frac{x}{\|b\|}\right).$$

Let $\varepsilon > 0$ and let $x_0 = x_0(\varepsilon)$ be such that $x \geq x_0$ implies

$$(d_*(A) - \varepsilon)S(x) \leq A(x) \leq (d^*(A) + \varepsilon)S(x).$$

Then if $T \geq \|b_2\|$ and $x \geq Tx_0$,

$$(V.22) \quad (d_*(A) - \varepsilon) \sum_{\|b_2\| \leq \|b\| \leq T} S\left(\frac{x}{\|b\|}\right) \leq \sum_{\|b_2\| \leq \|b\| \leq T} A\left(\frac{x}{\|b\|}\right) \\ \leq (d^*(A) + \varepsilon) \sum_{\|b_2\| \leq \|b\| \leq T} S\left(\frac{x}{\|b\|}\right).$$

By (V.21) and the left inequality (V.22), if $T \geq \|b_2\|$ and $x \geq Tx_0$, then

$$(V.23) \quad (d_*(A) - \varepsilon) \sum_{\|b_2\| \leq \|b\| \leq T} S\left(\frac{x}{\|b\|}\right) \leq S(x) - A(x).$$

By assumption $S\left(\frac{x}{\lambda}\right)/S(x) \rightarrow 1/\lambda$ as $x \rightarrow +\infty$, for each fixed $\lambda > 1$. Therefore dividing both sides of Inequality (V.23) by $S(x)$ and then letting $x \rightarrow +\infty$, $T \rightarrow +\infty$ and $\varepsilon \rightarrow 0$, in that order, we get

$$(V.24) \quad d_*(A)(L(B) - 1) \leq 1 - d^*(A).$$

On the other hand since we are assuming that $\omega(x) = \frac{S(x)}{x}$ is mildly decreasing then by Corollary (IV.3.1)

$$\inf_{x \geq 1} \frac{S(\lambda x)}{\lambda S(x)} = \theta > 0$$

for all $\lambda \geq 1$. Replacing x by $\frac{x}{\|b\|}$ and λ by $\|b\|$ in the preceding inequality so that λx is replaced by x , we have

$$(V.25) \quad S\left(\frac{x}{\|b\|}\right) \leq \frac{1}{\theta} \frac{S(x)}{\|b\|}$$

when $1 \leq \|b\| \leq x$. Since $A(x) \leq S(x)$ for all $x \geq 1$, then from Inequality (V.25),

$$(V.26) \quad \sum_{\|b\| > T} A\left(\frac{x}{\|b\|}\right) \leq \frac{S(x)}{\theta} \sum_{\|b\| > T} \frac{1}{\|b\|}.$$

From Equality (V.21), Inequality (V.26) and the right inequality (V.22), if $T \geq \|b_2\|$ and $x \geq Tx_0$, then

$$(V.27) \quad S(x) - A(x) \leq \frac{S(x)}{\theta} \sum_{\|b\| > T} \frac{1}{\|b\|} + (d^*(A) + \epsilon) \sum_{\|b_2\| \leq \|b\| \leq T} S\left(\frac{x}{\|b\|}\right).$$

Dividing both sides of Inequality (V.27) by $S(x)$, then letting $x \rightarrow +\infty$ and using the fact that $S(x)$ is of regular growth we get

$$(V.28) \quad 1 - d_*(A) \leq \frac{1}{\theta} \sum_{\|b\| > T} \frac{1}{\|b\|} + (d^*(A) + \epsilon) \sum_{\|b_2\| \leq \|b\| \leq T} \frac{1}{\|b\|}.$$

On letting $T \rightarrow +\infty$ and $\epsilon \rightarrow 0$, Inequality (V.28) yields

$$(V.29) \quad 1 - d_*(A) \leq d^*(A)(L(B) - 1).$$

Now, if Equation (V.18) holds, then

$$d_*(A)(L(B) - 1) = 1 - \frac{1}{L(B)},$$

hence by Inequality (V.24),

$$d^*(A) \leq \frac{1}{L(B)},$$

which implies Inequality (V.19) in view of the right inequality (V.17).

Similarly, assuming Equation (V.19) holds, then

$$d^*(A)(L(B) - 1) = 1 - \frac{1}{L(B)},$$

hence by Inequality (V.29)

$$d_*(A) \geq \frac{1}{L(B)},$$

which implies Equation (V.18) in view of the left inequality (V.17).

This completes the proof of Lemma (V.3).

Q.E.D.

VI. DIRICHLET CONVOLUTION

In what follows S will denote an ASG, R a ring with identity 1 , and $\mathcal{F}(S, R)$ the set of functions $S \rightarrow R$.

DEFINITION VI.1 Let $f, g : S \rightarrow R$. The *Dirichlet convolution*, denoted by $f * g$, of f and g is the function $S \rightarrow R$ defined on points $s \in S$ by

$$(f * g)(s) = \sum_{xy=s} f(x)g(y) .$$

LEMMA VI.1 With addition defined pointwise and Dirichlet convolution taken as the product, the set $\mathcal{F}(S, R)$ is a ring with identity e_0 , where $e_0(s) = 1$ if $s = 1$ and $e_0(s) = 0$ if $s \neq 1$.

PROOF. With addition defined pointwise $\mathcal{F}(S, R)$ is clearly a commutative group. Multiplication is associative since for any f, g and h in $\mathcal{F}(S, R)$ and $s \in S$ we have

$$\begin{aligned} ((f * g) * h)(s) &= \sum_{xy=s} (f * g)(x)h(y) \\ &= \sum_{xy=s} \left(\sum_{uv=x} f(u)g(v) \right) h(y) = \sum_{uvy=s} (f(u)g(v))h(y) \\ &= \sum_{uvy=s} f(u)(g(v)h(y)) = \sum_{uz=s} f(u) \left(\sum_{vy=z} g(v)h(y) \right) \\ &= \sum_{uz=s} f(u)(g * h)(z) = (f * (g * h))(s) . \end{aligned}$$

Multiplication is distributive over addition since

$$\begin{aligned}
(f * (g + h))(s) &= \sum_{xy=s} f(x)(g(y) + h(y)) \\
&= \sum_{xy=s} f(x)g(y) + \sum_{xy=s} f(x)h(y) \\
&= (f * g)(s) + (f * h)(s) .
\end{aligned}$$

(A similar calculation yields right distributivity.) Clearly the function $e_0 : S \rightarrow R$ defined by $e_0(1) = 1$ and $e_0(s) = 0$ if $s \neq 1$ is an identity for $\mathcal{F}(S, R)$.

REMARK VI.1 If one of A and B is given, where A and B are *supplementary direct factors* of S then the other is uniquely determined.

PROOF. Indeed, the definition of A and B is equivalent to the condition that $\chi_A * \chi_B = Z$, where $Z(s) = 1$ for all $s \in Z$. (To see this note that

$$(\chi_A * \chi_B)(s) = \sum_{xy=s} \chi_A(x)\chi_B(y) = \sum_{\substack{s=ab \\ a \in A, b \in B}} 1 .$$

Thus $(\chi_A * \chi_B)(s)$ is equal to the number of ways of writing $s = ab$, $a \in A$, $b \in B$. This by definition is exactly 1.) Now χ_A and χ_B are both invertible. Let μ_A and μ_B denote their convolution inverses. Then $\chi_A = Z * \mu_B$ and $\chi_B = Z * \mu_A$, so if A is given, B is uniquely defined and conversely.

We shall make use of the following elementary facts, the proofs of which are adopted from those given for the case $S = N^*$ found in Shockley [6]. We also give an alternate proof to part (ii) of Lemma (VI.3) in the Appendix, which has been adopted from one due to Kuipers [3].

LEMMA VI.2 Let S be an NASG, R a commutative ring with identity 1 and $f \in \mathcal{F}(S, R)$. Then f is invertible if and only if $f(1)$ is a unit in R .

PROOF. Necessity is evident. Suppose $f(1)$ is a unit in R . Define the function g inductively by

$$g(1) = \frac{1}{f(1)}$$

and

$$g(s) = -g(1) \sum_{\substack{xy=s \\ x \neq s}} g(x)f(y) \quad \text{if } \|s\| > 1.$$

We will prove that $g = f^{-*}$ (the convolution inverse of f). If $s = 1$, then

$$(g * f)(1) = g(1)f(1) = 1 = e_0(1),$$

and if $\|s\| > 1$, then

$$\begin{aligned} (g * f)(s) &= \sum_{xy=s} g(x)f(y) = g(s)f(1) + \sum_{\substack{xy=s \\ x \neq s}} g(x)f(y) \\ &= 0 = e_0(s). \end{aligned}$$

Thus $g * f = e_0$ and so the lemma follows.

Q.E.D.

If there is an element q in S such that $x = qy$, we say y divides x . We denote this fact by writing " $y|x$ ". If y^k is the highest integral power of y that divides x we will denote this fact by writing " $y^k \parallel x$ ".

Let S denote an ASG. We shall say that elements x and y in S are *coprime* in S and write $(x;y) = 1$ if their only common divisor is the identity 1 . Clearly if x and y are coprime in S then for any divisor d of x and any divisor h of y , $(d;h) = 1$.

We shall call a function $f : S \rightarrow R$ *multiplicative* if $f(1) = 1$ and $f(xy) = f(x)f(y)$ whenever $(x;y) = 1$; and *completely multiplicative* if $f(xy) = f(x)f(y)$ for all $x, y \in S$.

LEMMA VI.3 Let R be a commutative ring with identity, S an NASG and $f, g \in \mathcal{F}(S, R)$.

(i) If f and g are multiplicative, then $f * g$ is multiplicative;

(ii) If f is multiplicative then so is its convolution inverse.

PROOF. (i) Suppose $(x;y) = 1$. If d is a divisor of xy , d can be written as $d = uv$ where $u|x$ and $v|y$. Since

$$(u;v) = \left(\frac{x}{u}; \frac{y}{v} \right) = 1,$$

then

$$\begin{aligned} (f * g)(xy) &= \sum_{d|xy} f(d)g\left(\frac{xy}{d}\right) = \sum_{u|x, v|y} f(uv)g\left(\frac{x}{u}\frac{y}{v}\right) \\ &= \sum_{u|x, v|y} f(u)g\left(\frac{x}{u}\right)f(v)g\left(\frac{y}{v}\right) \\ &= \left(\sum_{u|x} f(u)g\left(\frac{x}{u}\right) \right) \left(\sum_{v|y} f(v)g\left(\frac{y}{v}\right) \right) \\ &= ((f * g)(x))((f * g)(y)). \end{aligned}$$

Thus $f * g$ is multiplicative.

(ii) Since f is multiplicative, then $f(1) = 1$. Thus from Lemma (VI.2) it follows that the convolution inverse of f , say g , exists, that

$$g(1) = 1$$

and

$$g(\bar{s}) = - \sum_{\substack{xy=s \\ x \neq s}} g(x)f(y) \quad \text{if } \|s\| > 1.$$

We must prove that if x and y are coprime in S , $g(xy) = g(x)g(y)$. Suppose there is a pair of coprime elements x and y in S such that $g(xy) \neq g(x)g(y)$. From the well-ordering principle we know that there exists at least one pair of coprime elements of S with this property such that the norm of their product is the smallest in the set of norm-values of all such products. Let u and v denote such a pair. Then $(u;v) = 1$, $g(uv) \neq g(u)g(v)$ and if c and d are coprime such that $\|cd\| < \|uv\|$, then $g(cd) = g(c)g(d)$. It is clear that neither u nor v is equal to the identity 1 in S . Consider the quantity $g(u)g(v) - g(uv)$. We have

$$g(u)g(v) - g(uv) = g(u)g(v) + \sum_{\substack{r|uv \\ \|r\| < \|uv\|}} g(r)f\left(\frac{uv}{r}\right).$$

If $r|uv$, we can write $r = mn$ where $m|u$ and n divides v . We can rewrite the above equation as

$$g(u)g(v) - g(uv) = g(u)g(v) + \sum_{\substack{m|u, n|v \\ \|mn\| < \|uv\|}} g(mn)f\left(\frac{u}{m} \frac{v}{n}\right).$$

Because $\|mn\| < \|uv\|$, then $g(mn) = g(m)g(n)$. Using the fact that $f(1) = 1$ we obtain

$$\begin{aligned}
 g(u)g(v) - g(uv) &= g(u)g(v) + \sum_{\substack{m|u, n|v \\ \|mn\| < \|uv\|}} g(m)g(n)f\left(\frac{u}{m}\right)f\left(\frac{v}{n}\right) \\
 &= g(u)g(1)g(v)f(1) + \sum_{\substack{m|u, n|v \\ \|mn\| < \|uv\|}} g(m)f\left(\frac{u}{m}\right)g(n)f\left(\frac{v}{n}\right) \\
 &= \sum_{m|u, n|v} g(m)f\left(\frac{u}{m}\right)g(n)f\left(\frac{v}{n}\right) \\
 &= \left(\sum_{m|u} g(m)f\left(\frac{u}{m}\right)\right)\left(\sum_{n|v} g(n)f\left(\frac{v}{n}\right)\right) \\
 &= (g * f)(u)(g * f)(v) \\
 &= e_0(u)e(v) = 0,
 \end{aligned}$$

since u and v are not both equal to 1. Thus $g(uv) = g(u)g(v)$ which contradicts the choice of u and v . Therefore $g(xy) = g(x)g(y)$ for any pair of coprime elements x and y in S ; that is, g is multiplicative. Q.E.D.

DEFINITION VI.2 Let S be an NASG and let D be a subset of S . D is said to be *divisor closed* if the divisors of each element of D are also in D .

For a commutative ring R with identity we shall denote by $\mathcal{F}_D(S, R)$ the set of maps $f \in \mathcal{F}(S, R)$ satisfying $f(u) = 0$ for $u \notin D$.

Consider the product defined on elements f and g in $\mathcal{F}_D(S, R)$ by

$$(f *_D g)(x) = \chi_D(x)(f * g)(x), \quad x \in S.$$

Since D is divisor closed this product is associative. In effect, if $x \notin D$,

$$\begin{aligned} ((f *_D g) *_D h)(x) &= \chi_D(x)((f *_D g) * h)(x) \\ &= 0 \\ &= \chi_D(x)(f * (g *_D h))(x) \\ &= (f *_D (g *_D h))(x), \end{aligned}$$

and if $x \in D$,

$$\begin{aligned} ((f *_D g) *_D h)(x) &= ((f *_D g) * h)(x) \\ &= \sum_{uv=x} (f *_D g)(u)h(v); \end{aligned}$$

since each divisor of u is a divisor of x and therefore in D the last sum equals

$$(VI.1) \quad \sum_{(zw)v=x} (f(z)g(w))h(v).$$

Similarly,

$$(f *_D (g *_D h))(x) = \sum_{uv=x} f(u)(g *_D h)(v).$$

Since each divisor of v , being a divisor of x , is in D the left hand sum in the above equation equals

$$(VI.2) \quad \sum_{u(rs)=x} f(u)(g(r)h(s)) .$$

Since (VI.1) and (VI.2) are clearly equivalent

$$((f *_D g) *_D h)(x) = (f *_D (g *_D h))(x) .$$

Thus with addition of elements in $\mathcal{F}_D(S, R)$ defined pointwise and with the product taken as the convolution $*_D$, the set $\mathcal{F}_D(S, R)$ is a ring with identity e_0 where $e_0(x) = 0$ if $x \neq 1$ and $e_0(1) = 1$ if $x = 1$.

LEMMA VI.4 Let S be an NASG, let D be a divisor closed subset of S and let R be a commutative ring with identity. If $f \in \mathcal{F}_D(S, R)$ is $*_D$ -invertible then its inverse g is given by

$$(VI.3) \quad g(x) = x_D(x) \left\{ \sum_{k \geq 1} (-1)^k K^{-(k+1)} \sum_{\substack{x=x_1 x_2 \dots x_k \\ \|x_j\| > 1}} f(x_1) f(x_2) \dots f(x_k) \right\}$$

when $x \neq 1$ and $g(1) = K^{-1} = \frac{1}{f(1)}$.

REMARK VI.2 Note that the above sum is always finite with the number of terms depending only on x .

PROOF. Suppose f is $*_D$ -invertible with inverse g ; then
 $(f *_D g)(x) = e_0(x)$ for all $x \in S$. If $x \notin D$ then $g(x) = 0$
and the result is clearly valid. Hence assume $x \in D$. Then

$$(f *_D g)(x) = (f * g)(x) = \sum_{x=uv} f(u)g(v) = e_0(x) .$$

Setting $K = f(1)$ we can then write

$$Kg(x) + \sum_{\substack{x_1|x \\ \|x_1\|>1}} f(x_1)g\left(\frac{x}{x_1}\right) = e_0(x) .$$

Thus for any $x \in D$ we have

$$(VI.4) \quad g(x) = K^{-1}e_0(x) - K^{-1} \sum_{\substack{x_1|x \\ \|x_1\|>1}} f(x_1)g\left(\frac{x}{x_1}\right) .$$

In particular since D is divisor closed

$$g\left(\frac{x}{x_1}\right) = K^{-1}e_0\left(\frac{x}{x_1}\right) - K^{-1} \sum_{\substack{x_1x_2|x \\ \|x_1\|, \|x_2\|>1}} f(x_2)g\left(\frac{x}{x_1x_2}\right)$$

for each divisor x_1 of x . Substituting this into Equation (VI.4),
we obtain

$$\begin{aligned} g(x) &= K^{-1}e_0(x) - K^{-2} \sum_{\substack{x_1|x \\ \|x_1\|>1}} f(x_1)e_0\left(\frac{x}{x_1}\right) \\ &\quad + K^{-3} \sum_{\substack{x_1x_2|x \\ \|x_1\|, \|x_2\|>1}} f(x_1)f(x_2)g\left(\frac{x}{x_1x_2}\right) . \end{aligned}$$

Continuing this process we obtain after the $(k + 1)$ -th step that

$$\begin{aligned}
 \text{(VI.5)} \quad g(x) &= K^{-1}e_0(x) - K^{-2} \sum_{\substack{x_1 | x \\ \|x_1\| > 1}} f(x_1)g\left(\frac{x}{x_1}\right) \\
 &+ K^{-3} \sum_{\substack{x_1 x_2 | x \\ \|x_1\|, \|x_2\| > 1}} f(x_1)f(x_2)e_0\left(\frac{x}{x_1 x_2}\right) - \dots \\
 &+ (-1)^k K^{-k-1} \sum_{\substack{x_1 x_2 \dots x_k | x \\ \|x_1\|, \|x_2\|, \dots, \|x_k\| > 1}} f(x_1)f(x_2)\dots f(x_k)e_0\left(\frac{x}{x_1 x_2 \dots x_k}\right) \\
 &+ (-1)^{k+1} K^{-k-2} \sum_{\substack{x_1 x_2 \dots x_{k+1} | x \\ \|x_1\|, \|x_2\|, \dots, \|x_{k+1}\| > 1}} f(x_1)f(x_2)\dots f(x_{k+1})g\left(\frac{x}{x_1 x_2 \dots x_{k+1}}\right).
 \end{aligned}$$

This procedure cannot continue indefinitely since

$$1 < \|x_1 x_2 \dots x_k\| \leq \|x\|.$$

Therefore for some finite k_0 ,

$$g\left(\frac{x}{x_1 x_2 \dots x_{k_0}}\right) = K^{-1}e_0\left(\frac{x}{x_1 x_2 \dots x_{k_0}}\right) = K^{-1};$$

and so the last sum on the right hand side of Equation (VI.5) is empty for $k = k_0$. Moreover for any value of $k \leq k_0$ we have

$$\begin{aligned}
 \text{(VI.6)} \quad &\sum_{\substack{x_1 x_2 \dots x_k | x \\ \|x_1\|, \|x_2\|, \dots, \|x_k\| > 1}} f(x_1)f(x_2)\dots f(x_k)e_0\left(\frac{x}{x_1 x_2 \dots x_k}\right) \\
 &= \sum_{\substack{x = x_1 x_2 \dots x_k \\ \|x_1\|, \|x_2\|, \dots, \|x_k\| > 1}} f(x_1)f(x_2)\dots f(x_k)
 \end{aligned}$$

Thus from Equations (VI.5) and (VI.6)

$$g(x) = \sum_{k \geq 1} (-1)^k K^{-(k+1)} \sum_{\substack{x=x_1 x_2 \dots x_k \\ \|x_1\|, \|x_2\|, \dots, \|x_k\| > 1}} f(x_1) f(x_2) \dots f(x_k),$$

which yields (VI.3) as required.

Q.E.D.

We now consider the problem of finding a necessary and sufficient condition for the convolution $*_D$ to have the following property:

Wiener's Property: Let C denote the set of complex numbers and let S be an NASG. If D is a divisor closed subset of S then the convolution $*_D$, is said to have Wiener's Property if whenever $f \in \mathcal{F}_D(S, C)$ is $*_D$ -invertible and satisfies the condition

$$\sum_{x \in D} |f(x)| < +\infty, \text{ then its } *_D\text{-inverse } g \text{ also satisfies}$$

$$\sum_{x \in D} |g(x)| < +\infty.$$

LEMMA VI.5 Let S be an NASG and let D denote a divisor closed subset of S . The convolution product $*_D$ of $\mathcal{F}_D(S, C)$ has Wiener's Property if and only if for each non-unit $x \in D$ there exists an integer $N(x)$ such that $x^n \notin D$ whenever $n > N(x)$.

PROOF OF "IF". Suppose $f \in \mathcal{F}_D(S, C)$ is $*_D$ -invertible and satisfies the condition $\sum_{x \in S} |f(x)| < +\infty$. Let g be the $*_D$ -inverse of f . Then, by Lemma (VI.3), whenever $\|x\| > 1$,

$$(VI.7) \quad g(x) = x_D(x) \sum_{k \geq 1} (-1)^k K^{-(k+1)} \sum_{\substack{x=x_1 x_2 \dots x_k \\ \|x_1\|, \|x_2\|, \dots, \|x_k\| > 1}} f(x_1) f(x_2) \dots f(x_k),$$

where $K = f(1)$.

We have to prove that

$$(VI.8) \quad \sum_{x \in S} |g(x)| < +\infty .$$

Observing that the $*_D$ -inverse of $\tilde{f}(x) = K^{-1}f(x)$ is given by $\tilde{g}(x) = Kg(x)$, and that $\tilde{f}(1) = \tilde{g}(1) = 1$, it follows that without loss of generality we may assume $K = 1$. Hence by Equation (VI.7) to prove Inequality (VI.8) it suffices to show that

$$(VI.9) \quad \sum_{k \geq 1} F(k) < +\infty ,$$

where

$$F(k) = \sum_{\substack{x_1, x_2, \dots, x_k \in S \\ \|x_1\|, \|x_2\|, \dots, \|x_k\| > 1}} |f(x_1)f(x_2)\dots f(x_k)| .$$

Let

$$(VI.10) \quad 0 < \rho < \min(1, \|f\|_\infty) ,$$

let y denote the element of S of minimum norm-value > 1 , and let $T = T(\rho) \geq \|y\|$ be such that

$$(VI.11) \quad \sum_{\|x\| > T} |f(x)| < \rho .$$

The hypothesis of the lemma implies that given any $x \in S$, there exists a positive integer λ_x so that $x^m \notin D$ whenever $m > \lambda_x$. Let $R = \sum_{\|y\| \leq \|x\| \leq T} \lambda_x$. Then if $k > R$, given any k non-unit elements x_1, x_2, \dots, x_k such that $x_1 x_2 \dots x_k \in D$, at most R of them have norm $\leq T$ (since, given any $x \in S$ such that $\|y\| \leq \|x\| \leq T$, at most λ_x of them can take the value x). Hence if $k > R$, then

$$(VI.12) \quad F(k) \leq \sum_{j=0}^R \binom{k}{j} \sum_{x_1, x_2, \dots, x_j} |f(x_1)f(x_2)\dots f(x_j)| \times \\ \sum_{x_{j+1}, x_{j+2}, \dots, x_k > T} |f(x_{j+1})f(x_{j+2})\dots f(x_k)|$$

where $\binom{k}{j} = k!/j!(k-j)!$. By Inequalities (VI.10), (VI.11) and (VI.12),

$$F(k) < \sum_{j=0}^R \binom{k}{j} \|f\|_1^j \rho^{k-j} < (R+1)\rho^R \|f\|_1^R \rho^{k-R},$$

and with Inequality (VI.10) this implies Inequality (VI.9) and therefore (VI.8) is satisfied.

PROOF OF "ONLY IF". If for some non-unit x , D contains an infinity of powers of x (and therefore every power of x , since D is divisor closed), let

$$f(s) = \begin{cases} 1 & \text{if } s = 1 \\ -1 & \text{if } s = x \\ 0 & \text{otherwise} \end{cases}.$$

Then by Lemma (VI.2) f is invertible with respect to the convolution $*_D$. Using Equation (VI.3) of Lemma (VI.4) we find the $*_D$ -inverse g of f to be given by $g(s) = 1$ if $s = x^n$ for some integer $n \geq 0$, $g(s) = 0$ otherwise. Thus $f \in \mathcal{L}^1$ and $g \notin \mathcal{L}^1$, which proves the "only if".

Q.E.D.

LEMMA VI.6 Let $D \subseteq S$ be divisor closed and let A and B be direct factors of S . If $A_D = A \cap D$ and $B_D = B \cap D$ then

$$(VI.13) \quad \chi_{A_D} *_D \chi_{B_D} = \chi_D.$$

PROOF. We have to show that for all $s \in S$

$$\chi_D(s) \left(\sum_{\substack{s=ab \\ a \in A_D, b \in B_D}} 1 \right) = \chi_D(s) .$$

If $s \notin D$ this is obvious. If $s \in D$ Relation (VI.13) reduces to showing that

$$(VI.14) \quad \sum_{\substack{s=ab \\ a \in A_D, b \in B_D}} 1 = 1 .$$

Since D is divisor closed and $s \in D$, the conditions $a \in A_D$, $b \in B_D$ reduce to $a \in A$, $b \in B$. Therefore Relation (VI.14) follows from the definition of A and B . Q.E.D.

VII. THE GENERALIZATION OF SEVERAL CLASSICAL LEMMAS TO MILD REGULAR
NORMED ARITHMETICAL SEMI-GROUPS WITH UNIQUE FACTORIZATION

In what is now to follow S will denote a mild regular NASG with *unique* factorization.

An element $s \in S$ will be called k -free where k is a positive integer if s is not divisible by the k -th power of any prime $p \in S$. We denote the set of k -free elements of S by Q_k . We note that for $k \geq 2$, $*_{Q_k}$ (which we shall henceforth denote simply as $*_k$) is divisor closed and possesses Wiener's Property. We will apply the results of Lemmas (VI.5) and (VI.6) to Q_k .

REMARK VII.1 If A and B are direct factors of S and if $D = Q_k$ then, by Lemma (VI.6),

$$(VII.1) \quad \chi_{A_k} *_k \chi_{B_k} = \chi_k,$$

where χ_{A_k} and χ_{B_k} denote the characteristic functions of $A_k = A \cap Q_k$ and $B_k = B \cap Q_k$, respectively. Letting μ_k denote the $*_k$ -inverse of χ_{B_k} , Identity (VII.1) yields

$$(VII.2) \quad \chi_{A_k} = \mu_k *_k \chi_k.$$

We wish to prove that A_k has an asymptotic density. One observes that by Identity (VII.2)

$$\begin{aligned} A_k(x) &= \sum_{\|a\| \leq x} \chi_{A_k}(a) = \sum_{\|a\| \leq x} \chi_k(a) \sum_{a=uv} \mu_k(u) \chi_k(v) \\ &= \sum_{\|uv\| \leq x} \chi_k(uv) \mu_k(u) \chi_k(v). \end{aligned}$$

Therefore, because of the relation $\chi_k(uv) = \chi_k(u)\chi_k(v)$,

$$(VII.3) \quad A_k(x) = \sum_{\|u\| \leq x} \mu_k(u) \left(\sum_{\substack{\|v\| \leq \frac{x}{\|u\|}} \chi_k(uv) \right) .$$

In order to evaluate $A_k(x)$ by Equation (VII.3) we will make use of the following lemmas.

LEMMA VII.1 Let $f : S \rightarrow C$ be a multiplicative function satisfying the condition

$$(VII.4) \quad \sum_p \sum_{n=1}^{\infty} |f(p^n)| < +\infty ,$$

where p runs over the primes in S . Then

$$(VII.5) \quad \sum_{s \in S} |f(s)| = \prod_p (1 + |f(p)| + |f(p^2)| + \dots) < +\infty .$$

PROOF. Let $u_p = \sum_{n=1}^{\infty} |f(p^n)|$ for each prime $p \in S$. By Condition (VII.4), $\sum_p u_p < +\infty$; hence the right hand side of Relation (VII.5), formally equal to $\prod_p (1 + u_p)$, converges to a finite positive number. Equation (VII.5) then follows from the assumption that f is multiplicative since we can formally expand the right hand side of Relation (VII.5). Q.E.D.

We will apply this result in Lemma (VII.2).

REMARK VII.2 Let $Z : S \rightarrow C$ be the function defined on points $s \in S$ by $Z(s) = 1$ and let $\mu : S \rightarrow C$ denote the convolution inverse of Z . Clearly $\mu(1) = 1$ and

$$\mu(s) = -\sum_{\substack{d|s \\ \|d\| < \|s\|}} \mu(d) Z\left(\frac{s}{d}\right) = -\sum_{\substack{d|s \\ \|d\| < \|s\|}} \mu(d) .$$

Thus if p is a prime in S ,

$$\mu(p) = -\sum_{\substack{d|p \\ \|d\| < \|p\|}} \mu(d) = -\mu(1) = -1 .$$

Because μ is multiplicative (it is the convolution inverse of a multiplicative function), then if p_1, p_2, \dots, p_k are distinct primes,

$$\mu(p_1 p_2 \dots p_k) = \mu(p_1) \mu(p_2) \dots \mu(p_k) = (-1)^k .$$

It follows that

$$\mu(p^2) = -(\mu(1) + \mu(p)) = 0$$

$$\mu(p^3) = -(\mu(1) + \mu(p) + \mu(p^2)) = 0 .$$

If we continue by induction on n , we see that $\mu(p^n) = 0$ for $n \geq 2$. Since μ is multiplicative, it follows that if s is divisible by a square then $\mu(s) = 0$. Because of this analogy with the classical case we call μ the *Möbius function for S* . We have introduced it here since it plays a role in what is to follow.

LEMMA VII.2 Let $E \subseteq S$ and suppose χ_E is multiplicative. Then E has an asymptotic density. Moreover,

$$(VII.6) \quad \sum_{p \notin E} \|p\|^{-1} < +\infty \quad \text{implies} \quad d(E) > 0 ;$$

$$(VII.7) \quad \sum_{p \notin E} \|p\|^{-1} = +\infty \quad \text{implies} \quad d(E) = 0 .$$

PROOF. Set $g = \mu * \chi_E$ so that $\chi_E = Z * g$. Then

$$\begin{aligned} E(x) &= \sum_{\|s\| \leq x} \chi_E(s) = \sum_{\|s\| \leq x} \left(\sum_{s=uv} Z(u)g(v) \right) \\ &= \sum_{\|v\| \leq x} g(v) \left(\sum_{\|u\| \leq \frac{x}{\|v\|}} 1 \right); \end{aligned}$$

so

$$(VII.8) \quad E(x) = \sum_{\|v\| \leq x} g(v) S\left(\frac{x}{\|v\|}\right).$$

Now for fixed v , $S\left(\frac{x}{\|v\|}\right) \sim \frac{S(x)}{\|v\|}$ as $x \rightarrow +\infty$ by assumption. Thus to each $\varepsilon > 0$ there corresponds an $x_0 = x_0(\varepsilon, v)$ such that for $x \geq x_0$

$$(VII.9) \quad \left| S\left(\frac{x}{\|v\|}\right) - \frac{S(x)}{\|v\|} \right| < \varepsilon \frac{S(x)}{\|v\|}.$$

On the other hand, since we are also assuming that $\omega(x) = \frac{S(x)}{x}$ is mildly decreasing then

$$\inf_{x \geq 1} \frac{S(\lambda x)}{\lambda S(x)} = \theta > 0, \quad \text{for all } \lambda \geq 1;$$

hence

$$S(x) \leq \frac{1}{\theta} \frac{S(\lambda x)}{\lambda}, \quad \text{for all } x \geq 1, \lambda \geq 1.$$

Replacing x by $\frac{x}{\|v\|}$ and λ by $\|v\|$ in the preceding inequality so that λx is replaced by x , we get

$$(VII.10) \quad S\left(\frac{x}{\|v\|}\right) \leq \frac{1}{\theta} \frac{S(x)}{\|v\|}, \quad \text{for all } x \geq 1, v \in S.$$

Now let $\varepsilon > 0$ be given and choose Y ($Y \geq 1$) fixed. Set

$$z_0(\varepsilon, Y) = \max_{\substack{v \\ \|v\| \leq Y}} x_0(\varepsilon, v) \quad \text{and} \quad W_0(\varepsilon, Y) = \max(Y, z_0(\varepsilon, v)), \text{ then}$$

for $x \geq W_0$,

$$\begin{aligned} \text{(VII.11)} \quad & \left| \sum_{\|v\| \leq x} g(v) S\left(\frac{x}{\|v\|}\right) - \sum_{\|v\| \leq x} g(v) \frac{S(x)}{\|v\|} \right| \\ & \leq \sum_{\|v\| \leq Y} |g(v)| \left| S\left(\frac{x}{\|v\|}\right) - \frac{S(x)}{\|v\|} \right| + \sum_{Y < \|v\| \leq x} |g(v)| \left| S\left(\frac{x}{\|v\|}\right) - \frac{S(x)}{\|v\|} \right| \\ & \leq \sum_{\|v\| \leq Y} |g(v)| \left| S\left(\frac{x}{\|v\|}\right) - \frac{S(x)}{\|v\|} \right| + \sum_{Y < \|v\| \leq x} |g(v)| S\left(\frac{x}{\|v\|}\right) \\ & \qquad \qquad \qquad + S(x) \sum_{Y < \|v\| \leq x} \frac{|g(v)|}{\|v\|}. \end{aligned}$$

For $\|v\| \leq Y$ and $x \geq x_0$ Relation (VII.9) is valid; so we apply it to the first sum on the right hand side of Inequality (VII.11). To the second sum we apply Inequality (VII.10). In view of Equation (VII.8), this yields

$$\text{(VII.12)} \quad \left| \frac{E(x)}{S(x)} - \sum_{\|v\| \leq x} \frac{g(v)}{\|v\|} \right| \leq \varepsilon \left(\sum_{\|v\| \leq Y} \frac{|g(v)|}{\|v\|} \right) + \left(\frac{1+\theta}{\theta} \right) \sum_{Y < \|v\|} \frac{|g(v)|}{\|v\|}.$$

If we can show that $\sum_v \frac{|g(v)|}{\|v\|} < +\infty$, we will be able to use the preceding inequality to show that $d(E) = \sum_v \frac{g(v)}{\|v\|}$. We start by developing an expression for $g(v)$ assuming (VII.6).

The function $g = \mu * \chi_E$ is multiplicative since μ and χ_E are. Thus to determine g let us compute $g(p^\alpha)$ for p prime and α an integer. If $\alpha = 1$, then

$$g(p) = \mu(1)\chi_E(p) + \mu(p)\chi_E(1) = \chi_E(p) - 1,$$

and if $\alpha \geq 2$, then

$$g(p^\alpha) = \mu(1)\chi_E(p^\alpha) + \mu(p)\chi_E(p^{\alpha-1}) + \sum_{k=2}^{\alpha} \mu(p^k)\chi_E(p^{\alpha-k});$$

hence

$$g(p^\alpha) = \chi_E(p^\alpha) - \chi_E(p^{\alpha-1}),$$

since $\mu(p^k) = 0$ for $k \geq 2$. Therefore, $g(v)$ is given by

$$g(v) = \begin{cases} 1 & \text{if } v = 1 \\ \prod_{p^\alpha \| v} (\chi_E(p^\alpha) - \chi_E(p^{\alpha-1})) & \text{if } v \neq 1 \end{cases}.$$

Therefore the multiplicative function $g(v)/\|v\|$ is given by

$$(VII.13) \quad \frac{g(v)}{\|v\|} = \begin{cases} 1 & \text{if } v = 1 \\ \prod_{p^\alpha \| v} \frac{\chi_E(p^\alpha) - \chi_E(p^{\alpha-1})}{\|p^\alpha\|} & \text{if } v \neq 1 \end{cases}.$$

We now apply Lemma (VII.1) to the multiplicative function $f(v) = g(v)/\|v\|$. By Equation (VII.13),

$$\frac{g(p^n)}{\|p^n\|} \leq \|p\|^{-n},$$

for p a prime and n an integer, $n \geq 2$. Therefore, summing over the positive integers ≥ 2 and then over the primes $p \in S$, we get

$$(VII.14) \quad \sum_p \sum_{n \geq 2} \frac{|g(p^n)|}{\|p\|^n} \leq \sum_p \sum_{n \geq 2} \|p\|^{-n} = \sum_p \frac{1}{\|p\|(\|p\|-1)}.$$

If s_0 is an element of S of minimum norm-value > 1 and $K = \left(1 - \frac{1}{\|s_0\|}\right)^{-1}$, we have $\sum_p \frac{1}{\|p\|(\|p\|-1)} < +\infty$, since the sum $\sum_{v \in S} \|v\|^{-\sigma}$ has exponent of convergence $\sigma = 1$ and

$$(VII.15) \quad \sum_p \frac{1}{\|p\|(\|p\|-1)} < K \sum_{\|v\|>1} \|v\|^{-2}.$$

From Inequalities (VII.14) and (VII.15) we conclude that

$$(VII.16) \quad \sum_p \sum_{n \geq 2} \frac{|g(p^n)|}{\|p\|} < +\infty.$$

On the other hand we have $\frac{g(p)}{\|p\|} = \frac{\chi_E(p) - 1}{\|p\|}$, so

$$(VII.17) \quad \sum_p \frac{|g(p)|}{\|p\|} = \sum_{p \notin E} \frac{1}{\|p\|} < +\infty \quad (\text{by assumption}).$$

From Inequalities (VII.16) and (VII.17), Condition (VII.4) is satisfied; therefore, by Lemma (VII.1)

$$(VII.18) \quad \sum_v \frac{|g(v)|}{\|v\|} < +\infty.$$

Recall that for $x \geq W_0(\epsilon, Y)$ we obtained

$$(VII.12) \quad \left| \frac{E(x)}{S(x)} - \sum_{\|v\| \leq x} \frac{g(v)}{\|v\|} \right| \leq \epsilon \sum_{\|v\| \leq Y} \frac{|g(v)|}{\|v\|} + \frac{1+\theta}{\theta} \sum_{Y < \|v\|} \frac{|g(v)|}{\|v\|}.$$

By Inequality (VII.18) above, we can choose $Y = Y_0(\epsilon)$ so that the last term on the right hand side is $\leq \epsilon$. Then, for all $x \geq W_0(\epsilon, Y_0(\epsilon))$ we have

$$-\left[\sum_{\|v\| \geq 1} \frac{|g(v)|}{\|v\|} + 1 \right] \epsilon \leq \frac{E(x)}{S(x)} - \sum_{\|v\| \leq x} \frac{g(v)}{\|v\|} \leq \left[\sum_{\|v\| \geq 1} \frac{|g(v)|}{\|v\|} + 1 \right] \epsilon$$

Letting $x \rightarrow +\infty$ and then letting $\epsilon \rightarrow 0$ we get

$$d(E) = \sum_v \frac{g(v)}{\|v\|}.$$

This completes the proof of Case (a) of Lemma (VII.2). The proof of Case (b) will require the following preparatory material.

DEFINITION VII.1 Let p be a prime. An element $s \in S$ is called *square-full* if $p^2|s$ whenever $p|s$ and *square-free* if $p \nmid s$ whenever $p|s$.

LEMMA VII.3 Let S denote an NASG with unique factorization and let F denote the set of all square-full elements of S . Then

$$(VII.19) \quad F(x) = O(x^{1/2+\epsilon})$$

for every $\epsilon > 0$ if S is regular and

$$(VII.20) \quad F(x) = O(S(\sqrt{x}))$$

if S is mild.

PROOF. Let s be a non-unit in S . Then s can be written uniquely in the form $s = d^2q$ where q is square-free. Now s is square-full if and only if $q|d$. Write $d = qr$ for some r . Thus s is square-full if and only if it can be written in the form $s = r^2q^3$ with q square free. Therefore,

$$(VII.21) \quad F(x) = \sum_{\substack{s=r^2q^3 \\ \|s\| \leq x}} 1 = \sum_{\|q\| \leq x^{1/3}} \left(\sum_{\substack{\|r\| \leq \frac{x^{1/2}}{\|q\|^{3/2}}} 1 \right) = \sum_{\|q\| \leq x^{1/3}} S\left(\frac{x^{1/2}}{\|q\|^{3/2}}\right).$$

Now suppose S is regular. By Lemma (4A) of [1], for each $\epsilon > 0$ there is a constant $D = D(\epsilon)$ such that

$$(VII.22) \quad S\left(\frac{x^{1/2}}{\|q\|^{3/2}}\right) \leq D\left(\frac{x^{1/2}}{\|q\|^{3/2}}\right)^{1+2\epsilon}.$$

From the preceding inequality and the fact that $\sum_{\|q\| \geq 1} \|q\|^{-(3/2+3\epsilon)} < +\infty$,

$$F(x) \leq Dx^{1/2+\epsilon} \left(\sum_{\|q\| \leq x^{1/3}} \|q\|^{-3/2-3\epsilon} \right) \leq Dx^{1/2+\epsilon} \left(\sum_{\|q\| \geq 1} \|q\|^{-3/2-3\epsilon} \right),$$

so $F(x) = O(x^{1/2+\epsilon})$ for every $\epsilon > 0$.

On the other hand, if S is mild, then by comparison with Inequality (VII.10),

$$S\left(\frac{x^{1/2}}{\|q\|^{3/2}}\right) \leq \frac{S(x^{1/2})}{\theta \|q\|^{3/2}}, \quad \theta \text{ a positive constant.}$$

Applying this inequality to Equation (VII.21) as was done with (VII.22) we get $F(x) = O(S(\sqrt{x}))$, since $\sum_{\|q\| \geq 1} \|q\|^{-3/2} < +\infty$ and

$$F(x) \leq \frac{S(\sqrt{x})}{\theta} \sum_{\|q\| \geq 1} \|q\|^{-3/2}. \quad \text{Q.E.D.}$$

Although we have shown Relation (VII.20) to hold, Relation (VII.19) will be sufficient for our purposes. We apply the latter in the next lemma where again we assume S to be a mild, regular NASG with unique factorization.

LEMMA VII.4 Let Q denote the set of square-free elements of S . Let E be a subset of S whose characteristic function is multiplicative. If $E \cap Q$ has an asymptotic density equal to zero, so does E .

PROOF. Let F_1 denote the set of those $s \in E$ such that $s = 1$ or s is square-full; i.e., $F_1 = E \cap F$. Let $E_1 = E \cap Q$. Then every $s \in E$ can be written uniquely as $s = uv$, $u \in E$, $v \in F_1$; consequently, since χ_E is multiplicative

$$\begin{aligned} \text{(VII.23)} \quad E(x) &= \sum_{\|s\| \leq x} \chi_E(s) = \sum_{\|uv\| \leq x} \chi_{E_1}(u) \chi_{F_1}(v) \\ &= \sum_{\|v\| \leq x} \chi_{F_1}(v) \left(\sum_{\|u\| \leq \frac{x}{\|v\|}} \chi_{E_1}(u) \right) = \sum_{\|v\| \leq x} \chi_{F_1}(v) E_1\left(\frac{x}{\|v\|}\right). \end{aligned}$$

Since E_1 has asymptotic density equal to zero by assumption, then given $\epsilon > 0$, there exists an $x_0 = x_0(\epsilon) \geq 1$ such that for $\frac{x}{\|v\|} \geq x_0$, $E_1\left(\frac{x}{\|v\|}\right) \leq \epsilon S\left(\frac{x}{\|v\|}\right)$. Since we are also assuming that $\omega(x) = \frac{S(x)}{x}$ is mildly decreasing, there is (as was argued in the proof of Lemma VII.2) a positive constant θ such that $S\left(\frac{x}{\|v\|}\right) \leq \frac{1}{\theta} \frac{S(x)}{\|v\|}$ for all $x \geq 1$, $v \in S$. Therefore, if we take $\|v\| \leq \frac{x}{x_0}$, we have

$$\text{(VII.24)} \quad \sum_{\|u\| \leq \frac{x}{x_0}} \chi_{F_1}(v) E_1\left(\frac{x}{\|v\|}\right) \leq \epsilon S(x) \sum_{\|v\| \leq \frac{x}{x_0}} \frac{\chi_{F_1}(v)}{\|v\|},$$

and if we take $\frac{x}{x_0} < \|v\| \leq x$, we have

$$\text{(VII.25)} \quad \sum_{\frac{x}{x_0} < \|v\| \leq x} \chi_{F_1}(v) E_1\left(\frac{x}{\|v\|}\right) \leq \frac{S(x)}{\theta} \sum_{\frac{x}{x_0} < \|v\| \leq x} \frac{\chi_{F_1}(v)}{\|v\|}.$$

On the other hand from Equation (VII.23) we have

$$E(x) = \sum_{\|v\| \leq \frac{x}{x_0}} \chi_{F_1}(v) E_1\left(\frac{x}{\|v\|}\right) + \frac{x}{x_0} \sum_{\frac{x}{x_0} < \|v\| \leq x} \chi_{F_1}(v) E_1\left(\frac{x}{\|v\|}\right),$$

so that, from Inequalities (VII.24) and (VII.25) we get

$$\frac{E(x)}{S(x)} \leq \varepsilon \sum_{\|v\| \leq \frac{x}{x_0}} \frac{\chi_{F_1}(v)}{\|v\|} + \frac{1}{\theta} \frac{x}{x_0} \sum_{\frac{x}{x_0} < \|v\| \leq x} \frac{\chi_{F_1}(v)}{\|v\|}.$$

By (VII.19), the sum $\sum_{v \in F_1} \|v\|^{-1}$ (and therefore also the sum $\sum_{v \in F_1} \|v\|^{-1}$)

converges. Hence, by letting successively $x \rightarrow \infty$ and $\varepsilon \rightarrow 0$, we obtain

$$\lim_{x \rightarrow \infty} \frac{E(x)}{S(x)} = 0,$$

as required.

We now prove case (b) of Lemma (VII.2).

LEMMA VII.2, case (b). Let E be a subset of S whose characteristic function is multiplicative. If $\sum_{\substack{p \notin E \\ p \text{ prime}}} \|p\|^{-1} = +\infty$, then E has

an asymptotic density equal to 0.

PROOF. Recall that Q_k^* for $k \in \mathbb{N}^*$, ($k \geq 2$) denotes the divisor closed set consisting of those elements of S which are k -free, and that $*_k \equiv *_{Q_k}$ has Wiener's Property.

By Lemma (VII.4) it suffices to show that $d(E \cap Q_2) = 0$.

Let $E_1 = E \cap Q_2$. Since $\sum_{p \notin E} \|p\|^{-1} = +\infty$, then

$\prod_{p \notin E} (1 - \|p\|^{-1}) = 0$. Therefore, for any $\epsilon > 0$ we can find primes

p_1, p_2, \dots, p_r none of which are in E and such that

$$(1 - \|p_1\|^{-1})(1 - \|p_2\|^{-1}) \dots (1 - \|p_r\|^{-1}) < \epsilon.$$

Now none of the primes p_1, p_2, \dots, p_r can divide any element $s \in E_1$;

for, if $p_j | s$ for some j ($1 \leq j \leq r$) then $s = p_j q$ with $(p_j, q) = 1$

(since s is square-free). Since χ_E is multiplicative this would

imply that $\chi_E(s) = \chi_E(p_j)\chi_E(q) = 0$, a contradiction. Thus, E_1 is

a subset of the set E_2 consisting of all those elements of S which

are not divisible by any of the primes: p_1, p_2, \dots, p_r . The characteristic

function of E_2 is clearly multiplicative and $\sum_{p \notin E_2} \|p\|^{-1} = \sum_{j=1}^r \|p_j\|^{-1}$ is

$< +\infty$. Therefore, by case (a) of Lemma (VII.2), E_2 has an asymptotic

density given by

$$d(E_2) = \sum_{v \in S} \frac{g(v)}{\|v\|} = \prod_{\substack{p \in S \\ p \text{ prime}}} \left(1 + \sum_{n=1}^{\infty} \frac{\chi_{E_2}(p^n) - \chi_{E_2}(p^{n-1})}{\|p\|^n} \right)$$

$$= \left(1 - \frac{1}{\|p_1\|}\right) \left(1 - \frac{1}{\|p_2\|}\right) \dots \left(1 - \frac{1}{\|p_r\|}\right) < \epsilon, \quad (\epsilon > 0),$$

i.e., $d(E_2) = 0$. Thus E_1 and hence E has an asymptotic density equal to zero.

This completes the proof of Lemma (VII.2).

Q.E.D.

VIII. DIRICHLET SERIES

Later on we will require some elementary properties concerning Dirichlet series of the form:

$$\sum_{v \in S} \frac{f(v)}{\|v\|^s}, \quad s = \sigma + it, \quad f \in \mathcal{F}(S, \mathbb{C}).$$

We begin with the following lemma:

LEMMA VIII.1 Let $f, g \in \mathcal{F}(S, \mathbb{C})$. If the series

$$\sum_{v \in S} \frac{f(v)}{\|v\|^s} \quad \text{and} \quad \sum_{v \in S} \frac{g(v)}{\|v\|^s}$$

are both absolutely convergent for $\sigma > \sigma_0$ then so is the series

$$\sum_{s \in S} \frac{(f * g)(s)}{\|s\|^s}, \quad \text{and moreover,}$$

$$(VIII.1) \quad \sum_{v \in S} \frac{(f * g)(v)}{\|v\|^s} = \left(\sum_{v \in S} \frac{f(v)}{\|v\|^s} \right) \left(\sum_{v \in S} \frac{g(v)}{\|v\|^s} \right)$$

for $\sigma > \sigma_0$.

PROOF. Fix $\sigma > \sigma_0$. Let $F(\sigma) = \sum_{v \in S} \frac{|f(v)|}{\|v\|^\sigma}$ and $G(\sigma) = \sum_{v \in S} \frac{|g(v)|}{\|v\|^\sigma}$.

Then for $x \geq 1$,

$$\begin{aligned} \sum_{\|v\| \leq x} \frac{|(f * g)(v)|}{\|v\|^\sigma} &= \sum_{\|v\| \leq x} \left(\left| \sum_{v=uv} \frac{f(u)g(w)}{\|u\|^\sigma \|w\|^\sigma} \right| \right) \\ &\leq \sum_{\|v\| \leq x} \left(\sum_{v=uv} \frac{|f(u)g(w)|}{\|uv\|^\sigma} \right) \\ &\leq \left(\sum_{\|u\| \leq x} \frac{|f(u)|}{\|u\|^\sigma} \right) \left(\sum_{\|w\| \leq x} \frac{|g(w)|}{\|w\|^\sigma} \right) \\ &\leq F(\sigma)G(\sigma) < +\infty. \end{aligned}$$

This shows that the series $\sum_{v \in S} \frac{(f * g)(v)}{\|v\|^s}$ converges absolutely for $\sigma \geq \sigma_0$. Equation (VIII.1) is (for $\sigma \geq \sigma_0$) then obtained by taking the product on the right hand side of (VIII.1) and rearranging the terms.

Q.E.D.

Although irrelevant to our purpose we have included the following lemma due to Delange which has an interest of its own.

LEMMA VIII.2 Let $f \in \tilde{\mathcal{F}}(S, \mathbb{C})$ be invertible with convolution inverse g . If the series $\sum_{v \in S} \frac{f(v)}{\|v\|^s}$ has a finite abscissa of absolute convergence then so does the series $\sum_{v \in S} \frac{g(v)}{\|v\|^s}$.

PROOF. Since the series $\sum_{v \in S} \frac{f(v)}{\|v\|^s}$ has a finite abscissa of absolute convergence, then

$$\lim_{\sigma \rightarrow \infty} \sum_{v \in S} \frac{f(v)}{\|v\|^\sigma} = f(1).$$

As was noted in the proof of Lemma (VI.5) we may assume without loss of generality that $f(1) = 1$, so that there are values of σ for which

$$F(\sigma) = \sum_{v \in S} \frac{|f(v)|}{\|v\|^s} < 2.$$

Let ν denote the infimum of these values and take $\sigma > \nu$. By Lemma (VI.4),

$$g(v) = e_0(v) + \sum_{k \geq 1} (-1)^k \sum_{\substack{\|v_j\| > 1, j=1,2,\dots,k \\ v^j = v_1 v_2 \dots v_k}} f(v_1) f(v_2) \dots f(v_k);$$

therefore,

$$\frac{|g(v)|}{\|v\|^\sigma} \leq e_0(v) + \sum_{k \geq 1} \sum_{\substack{\|v_j\| > 1, j=1,2,\dots,k \\ \|v\|=v_1 v_2 \dots v_k}} \frac{|f(v_1)|}{\|v_1\|^\sigma} \frac{|f(v_2)|}{\|v_2\|^\sigma} \dots \frac{|f(v_k)|}{\|v_k\|^\sigma},$$

and so,

$$\begin{aligned} \sum_{v \in S} \frac{|g(v)|}{\|v\|^\sigma} &\leq 1 + \sum_{k \geq 1} \left(\sum_{v \in S} \sum_{\substack{\|v_j\| > 1, j=1,2,\dots,k \\ v=v_1 v_2 \dots v_k}} \frac{f(v_1)}{\|v_1\|^\sigma} \frac{f(v_2)}{\|v_2\|^\sigma} \dots \frac{f(v_k)}{\|v_k\|^\sigma} \right) \\ &= \frac{1}{1 - (F(\sigma) - 1)} < +\infty, \quad (\text{since } F(\sigma) < 2). \end{aligned}$$

Thus the abscissa of absolute convergence of g is finite and less than or equal to ν . Q.E.D.

ZETA FUNCTION OF A NORMED SEMI-GROUP

Consider the Dirichlet sum: $\sum_{v \in S} \frac{1}{\|v\|^s}$, $s = \sigma + it$. If S is regular then by Corollary (IV.1.1), $\sum_{v \in S} \frac{1}{\|v\|^s}$ has a finite abscissa of absolute convergence equal to 1. In that case the sum function so defined will be denoted by $\zeta_S(s)$ and will be called the *Zeta Function* for S .

Recall that in Remark (VII.2) we denoted by μ the convolution inverse of the arithmetical function $Z \in \mathcal{F}(S, \mathbb{C})$ defined by $Z(v) = 1$ for all $v \in S$, and as in the classical case we called μ the *Mobius Function* for S . Since $|\mu(v)| \leq 1$, the series $\sum_{v \in S} \frac{\mu(v)}{\|v\|^s}$ has an abscissa of absolute convergence less than or equal to one. Since μ and Z are convolution inverses of each other,

$$\sum_{v \in S} \frac{\mu(v)}{\|v\|^s} = \frac{1}{\zeta_S(s)} ; \quad \text{for } \operatorname{Re} s > 1 ,$$

by Lemma (VIII.1).

IX. PROOF OF THE MAIN THEOREM

Recall that Q_k denotes the set of k -free elements of S . Let $A_k = A \cap Q_k$ and $B_k = B \cap Q_k$ where A and B are direct factors of S . We obtained

$$(VII.3) \quad A_k(x) = \sum_{\|r\| \leq x} \mu_k(r) \left(\sum_{\|v\| \leq \frac{x}{\|r\|}} \chi_k(rv) \right),$$

where $\chi_k = \chi_{Q_k}$ and μ_k is the convolution inverse of χ_{B_k} in $\mathcal{F}_{Q_k}(S, C)$. In the next lemma we evaluate $A_k(x)$ from Equation (VII.3) by applying Lemma (VII.2) to the set $E_{k,r}$ consisting of all those k -free elements v in S such that rv is also k -free. Since the characteristic function $\chi_{E_{k,r}}$ is easily seen to be multiplicative, $E_{k,r}$ has an asymptotic density by Lemma (VII.2).

LEMMA IX.1 *With the preceding notation, A_k has an asymptotic density.*

PROOF. Let $\sigma(k,r) = d(E_{k,r})$ and write

$$(IX.1) \quad \Delta(x, k, r) = \frac{E_{k,r}(x)}{S(x)} - \sigma(k, r).$$

Let $T > 0$ and use (IX.1) to write Equation (VII.3) as

$$(IX.2) \quad \frac{A_k(x)}{S(x)} = J_1 + J_2 + J_3,$$

where

$$(IX.3) \quad J_1 = \frac{1}{S(x)} \sum_{\|r\| \leq T} \mu_k(r) \sigma(k, r) S\left(\frac{x}{\|r\|}\right),$$

$$(IX.4) \quad J_2 = \frac{1}{S(x)} \sum_{T < \|r\| \leq x} \mu_k(r) E_{k,r} \left(\frac{x}{\|r\|} \right),$$

and

$$(IX.5) \quad J_3 = \frac{1}{S(x)} \sum_{\|r\| \leq T} \mu_k(r) \Delta \left(\frac{x}{\|r\|}, k, r \right) S \left(\frac{x}{\|r\|} \right).$$

Now the $*_k$ -inverse of the function $\frac{1}{\|r\|} \mu_k(r)$ is $\frac{1}{\|r\|} \chi_{B_k}$, and since $L(B_k) \leq L(B) < +\infty$, Lemma (VI.5) implies

$$(IX.6) \quad \sum_{r \in S} \frac{1}{\|r\|} \left| \mu_k(r) \right| < +\infty,$$

which in turn implies that the sum

$$(IX.7) \quad \sum_{r \in S} \frac{1}{\|r\|} \mu_k(r) \sigma(k, r)$$

converges absolutely. We will now show that the value of the sum in (IX.7) is the asymptotic density of A_k .

Let $\epsilon > 0$ be given. Since $S(x)$ is of regular growth there is an $x_0(\epsilon, r)$ such that for $x \geq x_0(\epsilon, r)$,

$$\frac{1}{S(x)} \left| S \left(\frac{x}{\|r\|} \right) - \frac{1}{\|r\|} S(x) \right| < \epsilon \frac{1}{\|r\|}.$$

Also $\Delta \left(\frac{x}{\|r\|}, k, r \right) \rightarrow 0$ as $x \rightarrow \infty$ for fixed k and r , so there is an $x_1(\epsilon, k, r)$ such that for $x \geq x_1(\epsilon, k, r)$,

$$\left| \Delta \left(\frac{x}{\|r\|}, k, r \right) \right| < \epsilon.$$

Thus for $x \geq \max \left(\max_{\|r\| \leq T} x_0(\epsilon, r), \max_{\|r\| \leq T} x_1(\epsilon, k, r) \right)$,

$$\begin{aligned}
\text{(IX.8)} \quad & \left| J_1 - \sum_{r \in S} \frac{1}{\|r\|} \mu_k(r) \sigma(k, r) \right| \\
& \leq \frac{1}{S(x)} \sum_{\|r\| \leq T} \left| \mu_k(r) \right| \left| \sigma(k, r) \right| \left| S\left(\frac{x}{\|r\|}\right) - \frac{1}{\|r\|} S(x) \right| \\
& < \epsilon \sum_{\|r\| \leq T} \frac{1}{\|r\|} \left| \mu_k(r) \right| \left| \sigma(k, r) \right| \\
& \leq \epsilon \sum_{r \in S} \frac{1}{\|r\|} \left| \mu_k(r) \right| ,
\end{aligned}$$

and

$$\begin{aligned}
\text{(IX.9)} \quad & \left| J_3 - \sum_{\|r\| \leq T} \frac{1}{\|r\|} \mu_k(r) \Delta\left(\frac{x}{\|r\|}, k, r\right) \right| \\
& \leq \frac{1}{S(x)} \sum_{\|r\| \leq T} \left| \mu_k(r) \right| \left| \Delta\left(\frac{x}{\|r\|}, k, r\right) \right| \left| S\left(\frac{x}{\|r\|}\right) - \frac{1}{\|r\|} S(x) \right| \\
& < \epsilon^2 \sum_{\|r\| \leq T} \frac{1}{\|r\|} \left| \mu_k(r) \right| \\
& \leq \epsilon^2 \sum_{r \in S} \frac{1}{\|r\|} \left| \mu_k(r) \right| .
\end{aligned}$$

On the other hand since S is mild then (as was argued in the proof of Lemma (VII.2)) there is a positive constant θ such that

$$S\left(\frac{x}{\|r\|}\right) \leq \frac{1}{\theta} \frac{1}{\|r\|} S(x) , \quad \text{for all } x \geq 1 \text{ and } r \in S .$$

Therefore,

$$\begin{aligned}
\text{(IX.10)} \quad & |J_2| \leq \frac{1}{S(x)} \sum_{T \leq \|r\| \leq x} \left| \mu_k(r) \right| S\left(\frac{x}{\|r\|}\right) \\
& \leq \frac{1}{\theta} \sum_{T \leq \|r\|} \frac{1}{\|r\|} \left| \mu_k(r) \right| .
\end{aligned}$$

Now we let $x \rightarrow +\infty$ and then $T \rightarrow +\infty$. Noting that (IX.9) reduces to

$$(IX.11) \quad |J_3| < \varepsilon^2 \sum_{r \in S} \frac{1}{\|r\|} \left| \mu_k(r) \right| ,$$

and that $|J_2| \rightarrow 0$, we then obtain from the preceding results combined that

$$(IX.12) \quad \begin{aligned} -(\varepsilon + \varepsilon^2) \sum_{r \in S} \frac{1}{\|r\|} \left| \mu_k(r) \right| &\leq d_*(A_k) - \sum_{r \in S} \frac{1}{\|r\|} \mu_k(r) \sigma(k, r) \\ &\leq d^*(A_k) - \sum_{r \in S} \frac{1}{\|r\|} \mu_k(r) \sigma(k, r) \\ &\leq (\varepsilon + \varepsilon^2) \sum_{r \in S} \frac{1}{\|r\|} \left| \mu_k(r) \right| . \end{aligned}$$

Finally letting $\varepsilon \rightarrow 0$ in (IX.12) we conclude that

$$(IX.13) \quad d(A_k) = \sum_{r \in S} \frac{1}{\|r\|} \mu_k(r) \sigma(k, r) .$$

Q.E.D.

In the next lemma we obtain a useful lower bound for $d(A_k)$. Equation (IX.13) will not be applicable in obtaining that bound since we have little information on the value of $\mu_k(r)$.

LEMMA IX.2 *With the above notation, for $k \geq 2$,*

$$(IX.14) \quad d(A_k) \geq \frac{1}{\zeta_S(k)} \frac{1}{L(B)} .$$

PROOF. We know from Lemma (V.1) that $d(A_k)$ is also the logarithmic density of A_k . Therefore it suffices to show that

$$\lim_{x \rightarrow +\infty} \frac{L_A(x)}{L_S(x)} \geq \frac{1}{\zeta_S(k)} \frac{1}{L(B)} .$$

By Lemma (VI.6), for any $v \in S$

$$\begin{aligned} \chi_k(v) &= (\chi_{A_k} * \chi_{B_k})(v) = \chi_k(v) \left(\sum_{\substack{v=ab \\ a \in A_k, b \in B_k}} 1 \right) \\ &\leq \sum_{a \in A_k, b \in B_k} 1 . \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{\|v\| \leq x} \frac{1}{\|v\|} \chi_k(v) &\leq \sum_{\|v\| \leq x} \frac{1}{\|v\|} \left(\sum_{\substack{v=ab \\ a \in A_k, b \in B_k}} 1 \right) = \sum_{\substack{\|ab\| \leq x \\ a \in A_k, b \in B_k}} \frac{1}{\|ab\|} \\ &\leq \left(\sum_{\substack{\|a\| \leq x \\ a \in A_k}} \frac{1}{\|a\|} \right) \left(\sum_{\substack{\|b\| \leq x \\ b \in B_k}} \frac{1}{\|b\|} \right) \\ &\leq \left(\sum_{\substack{\|a\| \leq x \\ a \in A_k}} \frac{1}{\|a\|} \right) L(B) ; \end{aligned}$$

hence $L_{A_k}(x) \geq L_{Q_k}(x) \frac{1}{L(B)}$. Therefore,

$$(IX.15) \quad d(A_k) = \lim_{x \rightarrow +\infty} \frac{L_{A_k}(x)}{L_S(x)} \geq \left(\limsup_{x \rightarrow +\infty} \frac{L_{Q_k}(x)}{L_S(x)} \right) \frac{1}{L(B)} .$$

Applying what was seen in the proof of Lemma (VII.2) to the set Q_k

and the expression for μ , one can see that Q_k has an asymptotic density given by

$$d(Q_k) = \prod_{p \in S} \left(1 - \frac{1}{\|p\|^k} \right) = \frac{1}{\zeta_S(k)},$$

where the product runs over all primes $p \in S$. Thus by Lemma (VI.1)

$$\limsup_{x \rightarrow +\infty} \frac{L_{Q_k}(x)}{L_S(x)} = \frac{1}{\zeta_S(k)}$$

and therefore by (IX.15) we obtain

$$d(A_k) \geq \frac{1}{\zeta_S(k)} \frac{1}{L(B)}.$$

Q.E.D.

We now prove the main theorem (stated in §III).

PROOF OF THE THEOREM. For $k \geq 2$, $A_k \subseteq A$; therefore

$d_*(A) \geq d_*(A_k) = d(A_k)$ (by Lemma IX.1). However, by Lemma (IX.2), assuming $L(B) < +\infty$, $d(A_k) \geq \frac{1}{\zeta_S(k)} \frac{1}{L(B)}$, so $d_*(A) \geq \frac{1}{\zeta_S(k)} \frac{1}{L(B)}$.

Since the latter inequality is valid for $k \geq 2$, we have on letting $k \rightarrow \infty$ that $d_*(A) \geq \frac{1}{L(B)}$. Therefore, by Lemma (V.3), A has an asymptotic density given by $d(A) = \frac{1}{L(B)}$.

The fact that B has an asymptotic density equal to zero is a special case of Proposition (V.1).

X. APPENDIX1. An Alternate Proof of Part (ii) of Lemma (VI.3)

Let R be a commutative ring with identity, S an NASG and $f, g \in \tilde{\mathcal{F}}(S, R)$.

LEMMA VI.3 (ii) If f is multiplicative, then so is its convolution inverse.

The proof presented here is adopted from an original one for the case $S = N^*$ due to Kuipers [3].

PROOF. Let g denote the convolution inverse of f . In order to show that g is multiplicative, or that $g(xy) = g(x)g(y)$ if $(x;y) = 1$, we apply induction with respect to the total number of divisors of x and y . If $x = y = 1$ then clearly $g(xy) = g(x)g(y)$. Now assume that $\|x\| > 1$ or $\|y\| > 1$. To simplify notation let d be a divisor of x , let e be a divisor of y and r a divisor of xy . Since $(x;y) = 1$ we have $(d;e) = \left(\frac{x}{d}; \frac{y}{e}\right) = 1$, and so

$$\begin{aligned} g(xy) &= - \sum_{\substack{r|xy \\ \|r\| \leq \|xy\|}} g(r)f\left(\frac{xy}{r}\right) \\ &= - \sum_{\substack{d|x, e|y \\ \|de\| < \|xy\|}} g(de)f\left(\frac{x}{d} \frac{y}{e}\right) \\ &= - \sum_{\substack{d|x, e|y \\ \|de\| < \|xy\|}} g(d)g(e)f\left(\frac{x}{d}\right)f\left(\frac{y}{e}\right). \end{aligned}$$

The last summation is extended over all pairs d, e described above, excluding $d = x, e = y$. The induction hypothesis, $g(de) = g(d)g(e)$, applies here since the total number of divisors of any pair d, e ($\|de\| < \|xy\|$) is less than the number of divisors of the pair x and y . By adding one term to the preceding sum we obtain

$$\begin{aligned}
 g(xy) &= - \sum_{d|x, e|y} g(d)g(e)f\left(\frac{x}{d}\right)f\left(\frac{y}{e}\right) + g(x)g(y) \\
 &= - \left[\sum_{d|x} g(d)f\left(\frac{x}{d}\right) \right] \left[\sum_{e|y} g(e)f\left(\frac{y}{e}\right) \right] + g(x)g(y) \\
 &= -((g * f)(x))((g * f)(y)) + g(x)g(y) \\
 &= -e_0(x)e_0(y) + g(x)g(y) \\
 &= g(x)g(y),
 \end{aligned}$$

proving the lemma.

Q.E.D.

2. A Modified Proof of the Theorem

In the proof of Theorem (A) for the classical case of N^* , H. Daboussi has observed in [4] that one can deduce the existence of $d(A)$ from that of $d(A_k)$ without having to appeal to Lemma (2) of that paper. An exactly analogous argument applies here, namely that one can deduce the existence of $d(A)$ from that of $d(A_k)$ without having to appeal to Lemma (V.3). In effect, let C_k denote the complement of Q_k in S . Since $A_k = A \cap Q_k$ and

$$A_k \subset A \subset A_k \cup C_k,$$

then

$$A_k(x) \leq A(x) \leq A_k(x) + C_k(x)$$

or

$$A_k(x) \leq A(x) \leq A_k(x) + S(x) - Q_k(x) .$$

Thus since A_k and Q_k have asymptotic densities with $d(Q_k) = 1/\zeta_S(k)$

we get

$$(X.1) \quad d(A_k) \leq d_*(A) \leq d^*(A) \leq d(A_k) + 1 - 1/\zeta_S(k) .$$

This implies

$$d^*(A) - d_*(A) \leq 1 - 1/\zeta_S(k) ,$$

whence $d^*(A) = d_*(A)$ on letting $k \rightarrow +\infty$. Alternatively, we can directly observe that $\lim_{k \rightarrow +\infty} d(A_k)$ exists, since $d(A_k)$ is a bounded non-decreasing function of k . Then (X.1) implies that

$$d_*(A) = d^*(A) = \lim_{k \rightarrow +\infty} d(A_k) .$$

XI. MISSING DIGITS

NOTATION

Let k and q be integers with $q \geq 2$ and $0 \leq k \leq q - 1$. For any subset D , consisting of k elements of the set of q -adic digits $\{0, 1, \dots, q-1\}$, let A denote the set of non-negative integers whose q -adic representations do not involve any of the digits of D . Let χ_A denote the characteristic function of A , that is, $\chi_A(n) = 1$ if $n \in A$ and $\chi_A(n) = 0$ otherwise. For x a non-negative real number, let $A(x)$ be defined as follows:

$$(XI.1) \quad A(x) = \sum_{0 \leq n < x} \chi_A(n) .$$

AN EXPRESSION FOR $A(x)$

First we observe that for each n the number of q -adic representations,

$$a_{n-1}q^{n-1} + a_{n-2}q^{n-2} + \dots + a_1q + a_0, a_j \in \{0, 1, \dots, q-1\} \setminus D,$$

is $(q-k)^n$ since there are exactly $q-k$ choices for each a_j in each of the above n terms. Thus

$$(XI.2) \quad A(q^n) = (q-k)^n, \quad n = 0, 1, 2, \dots .$$

Next observe that for $0 \leq d \leq q-1$, d an integer,

$$(XI.3) \quad A(dq^n) = \sum_{j=1}^d (d-j)q^n \sum_{(d-j)q^n \leq m < (d-j+1)q^n} \chi_A(m) .$$

If for each m where $(d-j)q^n \leq m < (d-j+1)q^n$, we write $m = (d-j)q^n + n$, then $0 \leq n < q^n$ and $\chi_A(m) = \chi_A(d-j) \chi_A(n)$. Substituting for each inner sum on the R.H.S. of Equation (XI.3) we obtain,

$$(XI.4) \quad A(dq^n) = \sum_{j=1}^d \chi_A(d-j) \sum_{0 \leq n < q^n} \chi_A(n) ,$$

or, since each inner sum on the R.H.S. of (XI.4) is, by (XI.2), equal to $(q-k)^n$ we have

$$(XI.5) \quad A(dq^n) = d^*(q-k)^n ,$$

where we have set $d^* = \sum_{j=1}^d \chi_A(d-j) = d - |D \cap \{0, 1, \dots, d-1\}|$.

Now let x be a non-negative integer with the following q -adic representation:

$$x = d_n q^n + d_{n-1} q^{n-1} + \dots + d_1 q + d_0 ,$$

and let $\lambda = \max \{j: d_j \in D\}$. Noting that $\chi_A(m) = 0$ for those integers m such that $d_n q^n + d_{n-1} q^{n-1} + \dots + d_\lambda q^\lambda \leq m < x$, we can write

$$(XI.6) \quad A(x) = \sum_{0 \leq m < d_n q^n} \chi_A(m) + \sum_{d_n q^n \leq m < d_n q^n + d_{n-1} q^{n-1}} \chi_A(m) + \dots + \sum_{d_n q^n + d_{n-1} q^{n-1} + \dots + d_{\lambda+1} q^{\lambda+1} \leq m < d_n q^n + d_{n-1} q^{n-1} + \dots + d_{\lambda+1} q^{\lambda+1} + d_\lambda q^\lambda} \chi_A(m)$$

Since $d_j \notin D$ for $j > \lambda$, each inner sum on the R.H.S. of (XI.6) equals

$\sum_{0 \leq m < d_n q^n} \chi_A(m)$, and so the R.H.S. of (XI.6) turns equals

$$\sum_{0 \leq m < d} \chi_A^{(m)} + \sum_{0 \leq m < d} \chi_A^{(m)} + \dots + \sum_{0 \leq m < d} \chi_A^{(m)} .$$

Therefore, in view of (XI.5) we arrive at the following expression:

$$(XI.7) \quad A(x) = d_n^* (q-k)^n + d_{n-1}^* (q-k)^{n-1} + \dots + d_\lambda^* (q-k)^\lambda ,$$

where for $\lambda \leq j \leq n$, $d_j^* = d_j - |D \cap \{0, 1, \dots, d_j - 1\}|$.

THE CASE OF THE LAST DIGIT MISSING BASE q .

For each positive integer n consider

$$x_n = (q-2)q^n + (q-2)q^{n-1} + \dots + (q-2) \text{ and } y_n = q^n .$$

Because of the dominant gaps in the sequence of positive integers that occur from x_n to y_n for each n , when all the integers containing the digit $q-1$ are removed, one expects that

$$\liminf_{x \rightarrow +\infty} \frac{A(x)}{x^\alpha} = \lim_{n \rightarrow +\infty} \frac{A(y_n)}{y_n^\alpha} = 1 ,$$

and

$$\limsup_{x \rightarrow +\infty} \frac{A(x)}{x^\alpha} = \lim_{n \rightarrow +\infty} \frac{A(x_n)}{x_n^\alpha} = \frac{(q-1)^\alpha}{(q-2)^\alpha} ,$$

where $\alpha = \frac{\ln(q-1)}{\ln q}$. In attempting to obtain these results we will make use of the following inequality.

Let σ , u , v and M be positive numbers with $0 \leq \sigma \leq 1$ and $u > v$. Then

$$(XI.8) \quad \frac{v+M}{(u+M)^\alpha} \geq \frac{v}{u^\alpha} .$$

PROOF Since $u > v > 0$ and $M > 0$,

$$(XI.9) \quad 1 + \frac{M}{v} \geq 1 + \frac{M}{u} .$$

Now since $0 \geq \sigma \leq 1$, and $1 + \frac{M}{u} > 1$, we get from (XI.9) that

$$1 + \frac{M}{v} \geq \left(1 + \frac{M}{u}\right)^\alpha,$$

which is just another form of (XI.8).

Q.E.D.

LOWER ESTIMATES FOR $A(x) / x^\alpha$

We assume that D consists of the last digit base q . Suppose

$x = \sum_{j=0}^n d_j q^j$ and $\lambda = \max \{j: d_j \in D\}$. If $\lambda = n$ then by (XI.7), with $\alpha = \ln(q-1) / \ln q$,

$$A(x) = d_n^* (q-1)^n = (q-1)^{n+1};$$

so

$$\frac{A(x)}{x^\alpha} = \frac{(q-1)^{n+1}}{\left(\sum_{j=0}^n d_j q^j\right)^\alpha} \geq \frac{(q-1)^{n+1}}{(q^{n+1})^\alpha} = 1.$$

Thus we may assume that $0 \leq \lambda < n$. Again by (XI.7),

$$\begin{aligned} A(x) &= \sum_{j=\lambda+1}^n d_j^* (q-1)^j + d_\lambda (q-1)^\lambda \\ &= \sum_{j=\lambda+1}^n d_j (q-1)^j + (q-1)^{\lambda+1}; \end{aligned}$$

so

$$\frac{A(x)}{x^\alpha} = \frac{\sum_{j=\lambda+1}^n d_j (q-1)^j + (q-1)^{\lambda+1}}{\left(\sum_{j=0}^n d_j q^j\right)^\alpha}.$$

Using the fact that $q^{\lambda+1} > \sum_{j=0}^{\lambda} d_j q^j$ we then find that

$$\frac{A(x)}{x^\alpha} > \frac{\sum_{j=\lambda+1}^n d_j (q-1)^j + (q-1)^{\lambda+1}}{\left(\sum_{j=\lambda+1}^n d_j q^j\right) + q^{\lambda+1} \alpha}.$$

Next we observe that factoring $q^{(\lambda+1)\alpha} = (q-1)^{\lambda+1}$ from the numerator and denominator on the right hand side of the preceding inequality gives

$$\frac{A(x)}{x^\alpha} > \frac{\sum_{j=\lambda+1}^n d_j (q-1)^{j-\lambda-1} + 1}{\left(\sum_{j=\lambda+1}^n d_j q^{j-\lambda-1} + 1 \right)^\alpha} .$$

To this last inequality we apply (XI.8) with

$$v = \sum_{j=\lambda+1}^n d_j (q-1)^{j-\lambda-1}, \quad u = \sum_{j=\lambda+1}^n d_j q^j, \quad M = 1$$

and $\alpha = \ln(q-1) / \ln q$ and we conclude that

$$\frac{A(x)}{x^\alpha} > \frac{\sum_{j=\lambda+1}^n d_j (q-1)^{j-\lambda-1}}{\left(\sum_{j=\lambda+1}^n d_j q^{j-\lambda-1} \right)^\alpha} .$$

Repeating this process of factoring a power of q^α and then erasing the residual digit, we see that ultimately we will arrive at the inequality

$$\frac{A(x)}{x^\alpha} > \frac{dn}{d_n^\alpha} \geq 1 .$$

We conclude that for x an integer,

$$(10) \quad \min_{x>0} \frac{A(x)}{x^\alpha} = 1 .$$

Since the function $f(x) = \frac{x^\alpha + 1}{(x+1)^\alpha}$, $x > 1$,

has a negative first derivative, then the terms of the sequence

$$y_n = \frac{A(q^{n+1})}{(q^{n+1})^\alpha} = \frac{q^{n\alpha} + 1}{(q^n + 1)^\alpha}, \quad n \geq 0,$$

are pairwise

distinct. Moreover, since $y_n \rightarrow 1$ as $n \rightarrow +\infty$, then in view of (XI.10), $y = 1$ is the smallest accumulation point of the set $\{A(x)/x^\alpha : x > 0\}$; i.e.,

$$(XI.11) \quad \liminf_{x \rightarrow +\infty} \frac{A(x)}{x^\alpha} = 1, \quad (x \text{ an integer}).$$

Now let x be any positive real number. Because of the relations

$$\frac{A(x)}{x^\alpha} = \frac{A([x])}{[x]^\alpha} \frac{1}{\left(1 + \frac{\{x\}}{[x]}\right)^\alpha} \sim \frac{A([x])}{[x]^\alpha},$$

where $\{x\}$ and $[x]$ denote the fractional and integral part of x respectively, Equation (XI.11) is still valid.

UPPER ESTIMATES FOR $A(x)/x^\alpha$

As in the preceding we take $D = \{q-1\}$ and $\alpha = \ln(q-1) / \ln q$. Combining the results of theorems (1) and (4) of [8], we obtain the formula

$$\sup_{0 \leq y < x} \frac{A(x) - A(y)}{(x-y)^\alpha} = \left(\frac{q-1}{q-2}\right)^\alpha.$$

Since

$$\sup_{x > 0} \frac{A(x)}{x^\alpha} \leq \sup_{0 \leq y < x} \frac{A(x) - A(y)}{(x-y)^\alpha},$$

we have the following inequality:

$$(XI.12) \quad \sup_{0 < x} \frac{A(x)}{x^\alpha} \leq \left(\frac{q-1}{q-2}\right)^\alpha.$$

Since the function $g(x) = (x^\alpha - 1) / (x-1)^\alpha$ has a positive first derivative, then each term of the sequence

$$y_n = \frac{A \left(\sum_{j=0}^n (q-2) q^j \right)}{\left(\sum_{j=0}^n (q-2) q^j \right)^\alpha} = \left(\frac{q-1}{q-2} \right)^\alpha \frac{q^{n\alpha+1} - 1}{(q^{n+1} - 1)^\alpha}$$

is distinct. Moreover since $y_n \rightarrow \left(\frac{q-1}{q-2} \right)^\alpha$ as $n \rightarrow +\infty$ then $(q-1)^\alpha / (q-2)^\alpha$ is in view of (XI.12) the largest accumulation point, i.e.,

$$\limsup_{x \rightarrow +\infty} \frac{A(x)}{x^\alpha} = \left(\frac{q-1}{q-2} \right)^\alpha .$$

Remark :

More generally, if D is of the form $\{0, 1, \dots, k_1 - 1, q - k_2, q - k_2 + 1, \dots, q - 1\}$ where the integers k_1 and k_2 satisfy $0 \leq k_1 + k_2 \leq q$, methods similar to those developed by Wegmann in [8] have been used to obtain the following result, which we state without proof :

$$\limsup_{x \rightarrow \infty} \frac{A(x)}{x^\alpha} = \left(1 - \frac{k_2}{q-1} \right)^{-\alpha} .$$

BIBLIOGRAPHY

- [1] P.T. Bateman & H.G. Diamond (1969), "Asymtotic distribution of Beurling's generalized prime numbers," M.A.A. Studies in Number Theory, Vol. 6, Prentice-Hall, Inc., Englewood Cliffs, N.J., 166-168. MR 39 # 4105.
- [2] G.H. Hardy (1949), *Divergent Series*, Oxford at the Clarendon Press. MR 11-25.
- [3] L. Kuipers, Unpublished manuscript.
- [4] B. Saffari (1976), "On the asymptotic density of sets of integers: I," *J. London Math. Soc.* (2) 13, 475-485.
- [5] P. Erdős, B. Saffari & R.C. Vaughan, "On the asymptotic density of sets of integers: II," *Proc. London Math. Soc.* (to appear).
- [6] J.E. Shockley (1967), *Introduction to Number Theory*, Holt, Rinehart and Winston, New York and London, 102-115. MR 35 # 1535.
- [7] J.L. Verley (1968), *Théorie Élémentaire de l'Intégration* (Deuxième édition), "Les Cours de Sorbonne," Centre de Documentation Universitaire, Paris. MR 41 # 7052.
- [8] H. Wegmann (1971), "Das Hausdorff-Mass von Cantormengen," *Math. Ann.* 193, 7-20. MR 45 # 2122.