On the Conjugacy of Maximal Toral Subalgebras of Certain Infinite-Dimensional Lie Algebras

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Abstract

We will extend the conjugacy problem of maximal toral subalgebras for Lie algebras of the form $\mathfrak{g} \otimes_k R$ by considering $R = k[t, t^{-1}]$ and $R = k[t, t^{-1}, (t-1)^{-1}]$, where k is an algebraically closed field of characteristic zero and $\mathfrak{g}$ is a direct limit Lie algebra. In the process, we study properties of infinite matrices with entries in a Bézout domain and we also look at how our conjugacy results extend to universal central extensions of the suitable direct limit Lie algebras.

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Finally, I would like to thank my close friends and family, who have supported me in the pursuit of my goals.

Dedication

I dedicate this thesis to my mother Marina, without whom I would not have had the opportunity to pursue my education in Canada. Thank you for your continued support, guidance, and understanding.

Contents

Chapter 0

Introduction

The theory of Lie algebras is a vast subject and has been studied by the greatest mathematicians and physicists of our times. In the study and classification of Lie algebras, it is often convenient to break down the problem into smaller parts, by studying subalgebras and their properties. In the representation theory of Lie algebras one uses subalgebras, which have a simpler structure, to derive information about the original Lie algebra. From this point of view, a class of abelian subalgebras called Cartan subalgebras are a very important class. One example of a Cartan subalgebra is the subalgebra of $\mathfrak{gl}_n$ consisting of all diagonal $n \times n$ matrices. It is known that this subalgebra acts diagonally on finite-dimensional irreducible representations of $\mathfrak{gl}_n$. This leads to a highest weight vector theory, and helps to classify all finite-dimensional representations of semisimple Lie algebras over the complex field.

An important result in the theory of finite-dimensional semisimple Lie algebras is the conjugacy of Cartan subalgebras. The notion of a Cartan subalgebra can be defined for any Lie algebra due to the work of Bourbaki. If the field is algebraically closed of characteristic zero, then all Cartan subalgebras are conjugate under the action of automorphisms of the Lie algebra. In particular, all Cartan subalgebras are isomorphic.

The conjugacy of Cartan subalgebras of semisimple Lie algebras over an algebraically closed field of characteristic zero was proved more than one hundred years ago. Curiously, the infinite-dimensional case involving direct limit Lie algebras was not proved until the eighties and nineties.

Infinite-dimensional Lie algebras and their associated loop algebras appear in many fields including theoretical physics, string theory, and in number theory in relation to modular forms. In many areas of mathematics, the theory of infinite-dimensional Lie algebras and their representations is of great interest. A natural class of infinite-dimensional Lie algebras, namely the direct limit Lie algebras has recently received a lot of attention. The Lie algebras obtained as a direct limit construction from finite-dimensional simple or semisimple Lie algebras have been recently studied.

The techniques used by Stumme [21] to generalize the conjugacy of Cartan subalgebras of direct limit Lie algebras are more classical in nature and use original methods which lead to some challenges. The work of Salmasian [18] uses classical K-theory, and methods of Chernousov, Pianzola use scheme theory and algebraic geometry [3], in the language of Demazure, Grothendieck, and Gabriel [1].

The goal in this thesis is to introduce the subject of conjugacy of Cartan subalgebras of direct limit Lie algebras and generalize beyond the existing work. In the case of direct limit Lie algebras, the work of Salmasian [18] studies $\mathfrak{sl} \otimes_k R$, $\mathfrak{so} \otimes_k R$ and $\mathfrak{sp} \otimes_k R$ where $R = \mathbb{C}[t, t^{-1}]$ and k is an algebraically closed field of characteristic zero. In this thesis we prove that the maximal toral subalgebras are conjugate when $R = \mathbb{C}[t, t^{-1}, (t-1)^{-1}]$. Furthermore, we generalize this result to include universal central extensions of the Lie algebras. By proving this result for $R = \mathbb{C}[t, t^{-1}, (t-1)^{-1}]$, which is a Bézout domain, we strive to create a motivation to investigate these results for a broad class of rings, namely Bézout domains. For this reason, we study and classify Bézout domains to obtain many rich examples and understand their structure.

In the first Chapter of the thesis, we introduce the notion of a Bézout domain, which is a certain class of GCD domains, named after the French Mathematician

Étienne Bézout. With localization, we develop machinery that allows us to generate explicit examples of Bézout domains.

In the second Chapter, our goal is to introduce direct limits of semisimple Lie algebras. We start with an introduction to infinite matrices and their properties, which are an integral part of our work. We proceed by introducing the direct limit Lie algebras $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$ and $\mathfrak{sp}_\infty$ over an algebraically closed field k of characteristic zero. We conclude the chapter by defining these algebras over a ring R as a tensor product to get $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}_\infty(R)$, and $\mathfrak{so}_\infty(R)$.

The third Chapter introduces the notion of a central extension. We define central extensions and develop machinery for generating central extensions explicitly using 2-cocycles and other tools. We define the universal central extension and end the chapter with an explicit construction for an important example of Lie algebras, namely the N-point affine algebra.

In the fourth Chapter, we explore the root decompositions of the Lie algebras defined in Chapter two and define their maximal toral subalgebras. We define and construct root decompositions and present their corresponding Dynkin diagrams. We also include a discussion of weight spaces, which is an important part of the final theorem of our thesis.

In the fifth Chapter, we introduce the conjugacy problem and give a summary of the work that has been done up to this point with respect to this subject as seen in the work of Stumme [21] and Salmasian [18]. We proceed to prove the result for the Bézout domain $R = \mathbb{C}[t, t^{-1}, (t-1)^{-1}]$, and discuss the generalizations of these results for central extensions. The final result of this chapter is original.

In Chapter six we further develop the theory of Bézout domains by classifying finitely generated Bézout domains under certain assumptions. This further motivates the pursuit of the conjugacy problem over Bézout domains, by limiting the number of necessary examples that would need to be considered. The results of this chapter are also original, and we thank the valuable input of Dr. Daniel Daigle for providing us with help on this topic.

The main results of this thesis generalize the conjugacy result of direct limit Lie algebras over $R = \mathbb{C}[t, t^{-1}, (t-1)^{-1}]$. We have corrected some errors that were published in the literature and mention this as a remark whenever appropriate. The challenge in pursuit of this goal was to identify the direction in which we chose to proceed in this generalization. Through the study of the detailed proofs, we were able to observe that the weakest assumption necessary was the use of a Bézout identity and this led us to choose to generalize our rings to $R = \mathbb{C}[t, t^{-1}, (t-1)^{-1}]$. With some adjustments, this was a success.

Chapter 1

Localization and Bézout Domains

In this chapter we will explore the notion of a Bézout domain, which is a certain type of GCD domain. To generate examples of Bézout domains, we will discuss the concept of a localization of a ring. Indeed we will see that the localization of any given Bézout domain is a Bézout domain.

In later chapters we will look at infinite-dimensional matrices with entries in a Bézout domain. These matrices are the infinite-dimensional analogues of $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$ and $\mathfrak{so}_{2n+1}$ and we will look at the conjugacy properties of their maximal toral subalgebras. All rings will contain a unit element.

1.1 Localization

In this section we will explore the concept of localization of a ring which gives rise to many examples of Bézout rings. We will start with a definition and prove a standard theorem that can be found in [5, Section 15.4] to introduce the concept of localization.

Definition 1.1.1. *A subset S of a ring R is a* ***multiplicative set*** *if $1 \in S$ and $st \in S$, for all $s, t \in S$.*

Theorem 1.1.1. *Let R be a commutative ring. Let $S \subset R$ be a multiplicative set that does not contain zero or any zero divisors. Then there exists a commutative ring Q with 1 that satisfies the following properties:*

(1) Q contains (an isomorphic copy of) R as a subring.

(2) Every element of S is a unit in Q.

(3) Every element of Q is of the form rs^{-1} for $r \in R, s \in S$.

*(4) (**Universal Property**) Any ring T containing an isomorphic copy of R in which every element of S is a unit also contains an isomorphic copy of Q. More precisely, for any injective homomorphism $\phi : R \to T$ where $\phi(s)$ is a unit in T for all $s \in S$, there exists an injective homomorphism $\psi : Q \to T$ such that $\psi|_R = \phi$.*

Proof: Define the relation $\sim$ on $R \times S$ where $(r_1, s_1) \sim (r_2, s_2)$ if and only if $r_1 s_2 = r_2 s_1$. Let $(r_1, s_1), (r_2, s_2)$ and (r_3, s_3) be arbitrary elements in $R \times S$. Reflexivity of this relation is immediate since $(r_1, s_1) \sim (r_1, s_1)$, because $r_1 s_1 = r_1 s_1$. This relation is symmetric because if $(r_1, s_1) \sim (r_2, s_2)$, namely $r_1 s_2 = r_2 s_1$, we also have $r_2 s_1 = r_1 s_2$, hence $(r_2, s_2) \sim (r_1, s_1)$. Now assume $(r_1, s_1) \sim (r_2, s_2)$ and $(r_2, s_2) \sim (r_3, s_3)$. It follows that $r_1 s_2 = r_2 s_1$ and $r_2 s_3 = r_3 s_2$. Multiplying the first equation by s_3 and the second by s_1 gives $r_1 s_2 s_3 = r_2 s_1 s_3$ and $r_2 s_3 s_1 = r_3 s_2 s_1$. Subtracting the resulting equations gives us $r_1 s_2 s_3 - r_3 s_2 s_1 = s_2(r_1 s_3 - r_3 s_1) = 0$. Now since s_2 belongs to the set S which has no zero divisors and does not contain zero, it follows that $r_1 s_3 - r_3 s_1 = 0$ or $r_1 s_3 = r_3 s_1$, namely $(r_1, s_1) \sim (r_3, s_3)$, which gives us transitivity. Thus the relation $\sim$ is an equivalence relation.

Define Q to be the set of equivalence classes under $\sim$ and denote an equivalence class (r, s) by $\dfrac{r}{s}$.

We now define the operations of addition and multiplication on Q by

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + s_1 r_2}{s_1 s_2} \quad \text{and} \quad \frac{r_1}{s_1} \times \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}.$$

We first check that the operations are well defined. Assume that $\dfrac{r_1}{s_1} = \dfrac{r_1'}{s_1'}$ and

$\dfrac{r_2}{s_2} = \dfrac{r_2'}{s_2'}$, i.e. $r_1 s_1' = r_1' s_1$ and $r_2 s_2' = r_2' s_2$. Addition is well defined if and only if $\dfrac{r_1 s_2 + s_1 r_2}{s_1 s_2} = \dfrac{r_1' s_2' + s_1' r_2'}{s_1' s_2'}$ or equivalently $r_1 s_2 s_1' s_2' + s_1 r_2 s_1' s_2' = r_1' s_2' s_1 s_2 + s_1' r_2' s_1 s_2$. Substituting $r_1' s_1$ for $r_1 s_1'$, and $r_2' s_2$ for $r_2 s_2'$ gives us the right hand side. Thus the addition operation is well defined. Multiplication is well defined if and only if $\dfrac{r_1 r_2}{s_1 s_2} = \dfrac{r_1' r_2'}{s_1' s_2'}$. The same substitution into the left hand side of $r_1 r_2 s_1' s_2' = r_1' r_2' s_1 s_2$ gives us the right hand side, concluding that multiplication is well defined. Note also that S being a multiplicative set is needed in order for these operations to be well defined.

Next we check that Q is an abelian group under addition. Firstly, note that for any $\dfrac{r}{s} \in Q$ and $d \in S$ we have that

$$\frac{r}{s} + \frac{0}{d} = \frac{rd + s_1 0}{sd} = \frac{rd}{sd} = \frac{r}{s} \quad \text{and} \quad \frac{r}{s} + \frac{-r}{s} = \frac{rs - sr}{ss} = \frac{0}{ss}$$

making the equivalence class $\dfrac{0}{d}$ the additive identity and $\dfrac{-r}{s}$ the additive inverse for $\dfrac{r}{s}$. Note that in the previous calculation we used that $\dfrac{d}{d}$ is the multiplicative identity for any $d \in S$ since S is closed under multiplication.

Associativity follows from the following

$$\begin{aligned}\left(\frac{r_1}{s_1} + \frac{r_2}{s_2}\right) + \frac{r_3}{s_3} &= \frac{r_1 s_2 + s_1 r_2}{s_1 s_2} + \frac{r_3}{s_3} = \frac{r_1 s_2 s_3 + r_2 s_1 s_3 + s_1 s_2 r_3}{s_1 s_2 s_3} \\ &= \frac{r_1}{s_1} + \frac{r_2 s_3 + s_2 r_3}{s_2 s_3} = \frac{r_1}{s_1} + \left(\frac{r_2}{s_2} + \frac{r_3}{s_3}\right).\end{aligned}$$

The abelian property follows from the commutativity of the operation in R. Thus Q is an abelian group under addition.

Commutativity of multiplication and distributivity all follow in the same fashion. Thus we have that Q is a commutative ring with identity and we have left to check properties (1) to (4).

To show (1) consider the map $i : R \to Q$ given by $i(r) = \dfrac{rd}{d}$ where $d \in S$. Note that this map is well defined as it does not depend on the choice of d, since

$\frac{d}{d}$ is the multiplicative identity in Q for any $d \in S$. Letting $r_1, r_2 \in R$ we see that $i(r_1 + r_2) = \frac{(r_1 + r_2)d}{d} = \frac{r_1 d + r_2 d}{d} \times \frac{d}{d} = \frac{r_1 d}{d} + \frac{r_2 d}{d} = i(r_1) + i(r_2)$ and $i(r_1)i(r_2) = \frac{dr_1}{d}\frac{dr_2}{d} = \frac{ddr_1 r_2}{dd} = i(r_1 r_2)$, where $d^2 \in S$ as S is multiplicatively closed. So i is a ring homomorphism.

Now note that $i(r) = 0$ if and only if $\frac{rd}{d} = \frac{0}{d}$ or $rd^2 = 0$. Now since both d and d^2 are in S, and S has no zero divisors, we conclude that $r = 0$. Thus i is injective and we have shown that Q contains $i(R)$ as a subring, which is isomorphic to R. For the rest of the proof we refer to R as a subring of Q and S as a subset of Q by identifying R, S with $i(R), i(S)$, respectively.

To show (2), consider an element $s \in S$ which corresponds to the class $\frac{sd}{d}$ for $d \in S$. Since S is a multiplicative set we have that $sd \in S$, thus $\frac{d}{sd} \in Q$ and $\frac{sd}{d} \times \frac{d}{sd} = \frac{sd^2}{sd^2}$ which is the identity since $sd^2 \in S$, concluding that every element of s is a unit in Q.

To show (3) we take a class $\frac{r}{s} \in Q$, and note that $\frac{r}{s} = \frac{r}{1} \times \frac{1}{s}$.

To show (4) take any ring T containing R in which every element of S is a unit and let $\phi : R \to T$ be an injective homomorphism where $\phi(s)$ is a unit in T for all $s \in S$. Define $\psi : Q \to T$ by $\psi\left(\frac{r}{s}\right) = \phi(r)\phi(s)^{-1}$. Assume $\frac{r}{s} = \frac{r'}{s'}$. Then $rs' = sr'$ thus $\phi(r)\phi(s') = \phi(s)\phi(r')$. Since $\phi(s)$ and $\phi(s')$ are units in T we can multiply this equation by $\phi(s')^{-1}\phi(s)^{-1}$ to get $\phi(r)\phi(s)^{-1} = \phi(r')\phi(s')^{-1}$, i.e. $\psi\left(\frac{r}{s}\right) = \psi\left(\frac{r'}{s'}\right)$. Thus the map ψ is well defined.

To check that ψ is a ring homomorphism let $\frac{r}{s}, \frac{r'}{s'} \in Q$. Then

$$\psi\left(\frac{r}{s}\right)\psi\left(\frac{r'}{s'}\right) = \phi(r)\phi(s)^{-1}\phi(r')\phi(s')^{-1} = \phi(rr')\phi(ss')^{-1} = \psi\left(\frac{r}{s} \times \frac{r'}{s'}\right)$$

$$\begin{aligned}\psi\left(\frac{r}{s}\right) + \psi\left(\frac{r'}{s'}\right) &= \phi(r)\phi(s)^{-1} + \phi(r')\phi(s')^{-1} \\ &= \phi(rs' + sr')\phi(ss')^{-1} = \psi\left(\frac{rs' + sr'}{ss'}\right) = \psi\left(\frac{r}{s} + \frac{r'}{s'}\right)\end{aligned}$$

as desired.

Lastly, to check that ψ is injective, let $rs^{-1} \in Q$ such that $rs^{-1} \in \ker(\psi)$. Then $\phi(r)\phi(s)^{-1} = 0$ and we conclude that $\phi(r) = 0$, i.e. $r \in \ker(\phi)$. Since ϕ is injective we have that $r = 0$, hence $rs^{-1} = 0$. Thus ψ is injective. Lastly, let $r \in R$ and $\frac{rd}{d}$ be its equivalence class in Q. Then $\psi(\frac{rd}{d}) = \phi(rd)\phi(d)^{-1} = \phi(r)\phi(d)\phi(d)^{-1} = \phi(r)$. Thus $\psi|_R = \phi$. ∎

Example 1.1.1. If R is an integral domain and $S = R\backslash\{0\}$, then $S^{-1}R$ is the field of fractions of R.

The next result generalizes this construction of $S^{-1}R$ by allowing S to have zero divisors. Note that in this case R need not be embedded as a subring of $S^{-1}R$.

Theorem 1.1.2. *Let R be a commutative ring. Let S be a multiplicatively closed subset of R. Then there exists a commutative ring $S^{-1}R$ and a ring homomorphism $\pi : R \to S^{-1}R$ such that for any homomorphism $\phi : R \to T$ of commutative rings where $\phi(1) = 1$ and $\phi(s)$ is a unit in T for all $s \in S$, there exists a unique homomorphism $\psi : S^{-1}R \to T$ such that $\psi \circ \pi = \phi$.*

Proof: See [5, Section 15.4]. ∎

Definition 1.1.2. *The ring $S^{-1}R$ defined in Theorems 1.1.1 and 1.1.2 is called the* ***localization of*** *R* ***at*** *S.*

1.2 Bézout domains

Definition 1.2.1. *Recall that in a commutative ring, a* ***greatest common divisor*** *of a and b is a nonzero element d such that*

1. *d divides a and d divides b, and*

2. *if d' divides a and d' divides b, then d' divides d.*

Remark 1.2.1. Note that in an integral domain a greatest common divisor is unique up to a unit element.

Definition 1.2.2. *An integral domain R is called a* ***Bézout domain*** *or a* ***Bézout ring*** *if every pair of elements a, b of R has a greatest common divisor d in R that can be expressed as a linear combination of a and b, i.e. $d = ax + by$ for some x, y in R.*

The expression $d = ax + by$ of the greatest common divisor of two elements of a ring as a linear combination is called ***Bézout's identity****.*

Lemma 1.2.1. *An integral domain R is a Bézout domain if and only if every ideal (a, b) generated by two elements $a, b \in R$ is principal.*

Proof: Let R be a Bézout domain and take $a, b \in R$. Suppose that d is a greatest common divisor of a and b with Bézout identity $d = ax + by$ for some x, y in R. Thus, $d \in (a, b)$, hence $(d) \subset (a, b)$. The inclusion $(a, b) \subset (d)$ follows because $a \in (d)$ since d divides a, and $b \in (d)$ since d divides b. Thus every ideal (a, b) generated by two elements $a, b \in R$ is principal.

Conversely assume that every ideal (a, b) generated by two elements $a, b \in R$ is principal. Taking $a, b \in R$ we have $(a, b) = (d)$. From this equality we can see that there exist $x, y \in R$ such that $ax + by = d$ for some $d \in R$. The fact that $a \in (d)$ and $b \in (d)$ shows us that d is a common divisor of a and b. Lastly, assuming that there is another common divisor $d' \in R$ of a and b, we have that $a \in (d')$, $b \in (d')$, concluding that the ideal generated by a and b, namely $(d) \subset (d')$. Thus $d \in (d')$, meaning d' divides d. Thus d is a greatest common divisor of a and b, and together with the Bézout identity $d = ax + by$, we conclude that R is a Bézout domain. ∎

Example 1.2.1.

1. Every field, Euclidean domain and principal ideal domain is a Bézout domain.

2. If k is a field, then the ring of polynomials $k[t]$ is a Bézout domain as it is a Euclidean domain with Euclidean function given by $\deg(f(t)) =$ degree of $f(t)$.

The following theorem is given in [4, p. 252].

Theorem 1.2.2. *Let R be a Bézout domain whose quotient field is K. Then any ring T in between R and K is a localization of the ring R.*

Proof: Let T be a ring satisfying $R \subset T \subset K$. Consider the subset $S := \{x \in R : x^{-1} \in T\}$ of R. We can see that S is a multiplicative set since given $x, y \in S$ it follows that $x^{-1}, y^{-1} \in T$ hence $(xy)^{-1} \in T$, thus $xy \in S$.

We will show that $T = S^{-1}R$, that is, T is the localization of R at S. All elements of S are units in T and $S^{-1}R$ is the smallest ring with this property, so we conclude that $S^{-1}R \subset T$. Consider an element $a \in T$, then $a = rs^{-1}$ for some $r \in R, s \in R\backslash\{0\}$ such that the greatest common divisor of r and s is d. Since R is a Bézout domain we have the Bézout identity $d = rx + sy$ for some x, y in R. Since d is a divisor of r and s, there exist $s_1, r_1 \in R$ with $r = dr_1$ and $s = ds_1$ giving us $a = rs^{-1} = (dr_1)(ds_1)^{-1} = r_1 s_1^{-1}$. Substitution into the Bézout identity gives $d = dr_1 x + ds_1 y$ which implies $r_1 x + s_1 y = 1$ by the cancellation property of the integral domain R. Now multiplying the equation $r_1 x + s_1 y = 1$ by s_1^{-1} on both sides gives us $s_1^{-1} = ax + y$. Since a, x and y are all elements in T, it follows that $s_1^{-1} \in T$, hence $s_1 \in S$ implying that $a = r_1 s_1^{-1} \in S^{-1}R$. Thus we have $T = S^{-1}R$. ∎

The following theorem is found in [4].

Theorem 1.2.3. *Let R be a Bézout domain and S a multiplicative set in R. Then $S^{-1}R$ is a Bézout domain.*

Proof: Take $x, y \in S^{-1}R$. Then $x = s_1^{-1}\tilde{r}_1$ and $y = s_2^{-1}\tilde{r}_2$, where $\tilde{r}_1, \tilde{r}_2 \in R$ and $s_1, s_2 \in S$. Letting $s = s_1 s_2$, $r_1 = s_2\tilde{r}_1$ and $r_2 = s_1\tilde{r}_2$, we can write $x = s^{-1}r_1$ and $y = s^{-1}r_2$.

Since R is a Bézout domain, we can assume that $d \in R$ is a greatest common divisor of r_1 and r_2 with Bézout identity $d = r_1 a + r_2 b$ and $a, b \in R$. We claim that $ds^{-1} \in S^{-1}R$ is a greatest common divisor of x, y in $S^{-1}R$.

Since d is a common divisor of r_1 and r_2, we have that $r_1 = d\bar{r_1}$ and $r_2 = d\bar{r_2}$, for some $\bar{r_1}, \bar{r_2} \in R$. Multiplying these two equations by s^{-1} we get $x = (s^{-1}d)\bar{r_1}$ and $y = (s^{-1}d)\bar{r_2}$, namely, $ds^{-1} \in S^{-1}R$ is a common divisor of x, y.

To show that $ds^{-1} \in S^{-1}R$ is a greatest common divisor, assume that pq^{-1} is a common divisor of x and y in $S^{-1}R$. From this we get two equations $x = pq^{-1}(u_1 v_1^{-1})$ and $y = pq^{-1}(u_2 v_2^{-1})$, for some $u_1, u_2 \in R$ and $v_1, v_2 \in S$. Now we multiply these two equations by a and b, respectively, to get $xa = pq^{-1}(u_1 v_1^{-1})a$ and $yb = pq^{-1}(u_2 v_2^{-1})b$. And now using the Bézout identity we get

$$
\begin{aligned}
ds^{-1} &= (r_1 a + r_2 b)s^{-1} = s^{-1}r_1 a + s^{-1}r_2 b \\
&= pq^{-1}(u_1 v_1^{-1})a + pq^{-1}(u_2 v_2^{-1})b \\
&= pq^{-1}((u_1 v_1^{-1})a + (u_2 v_2^{-1})b)
\end{aligned}
$$

In other words, pq^{-1} divides ds^{-1} in $S^{-1}R$, as desired.

Thus $ds^{-1} \in S^{-1}R$ is a greatest common divisor of x, y in $S^{-1}R$ with Bézout Identity $ds^{-1} = xa + yb$, concluding that $S^{-1}R$ is a Bézout domain. ∎

Example 1.2.2. If k is a field, then the ring of polynomials $k[t]$ is a Bézout domain by Example 1.2.1. Considering the intermediate rings between $k[t]$ and its field of fractions $k(t)$ provides a rich source of examples of Bézout domains.

In particular, if $a_1, \ldots, a_n \in k$ are distinct and nonzero, then

$$
k[t, (t - a_1)^{-1}, \ldots, (t - a_n)^{-1}]
$$

is a Bézout domain. This ring appears in algebraic geometry as the coordinate ring of the projective space $\mathbb{P}^1$ with $n+1$ points removed.

Example 1.2.3. Note that if k is a field, then $k[s,t]$ is not a Bézout domain. An example of a non-principal ideal in $k[s,t]$ that is generated by two elements is (s,t). Indeed if $(s,t) = \big(f(s,t)\big)$ for some $f(s,t) \in k[s,t]$, then s is a multiple of $f(s,t)$. Thus $f(s,t)$ does not contain the variable t. A symmetric argument shows that $f(s,t)$ does not contain the variable s. Thus we conclude that $f(s,t)$ must be a constant, contradicting the fact that the ideal (s,t) does not contain any constants.

Chapter 2

Classical Simple Direct Limit Lie Algebras

This chapter considers the infinite-dimensional analogues of the classical Lie algebras $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$ and $\mathfrak{so}_{2n+1}$ which are denoted by $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$. We will see that just like their finite-dimensional counterparts, they can be described concretely by $\infty \times \infty$ matrices. In order to define the Lie algebras $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$ in a precise way, we will identify the elements in these Lie algebras (which are endomorphisms) with infinite matrices. Throughout this chapter k will be an algebraically closed field of characteristic zero.

2.1 Infinite Matrices

Definition 2.1.1. *Let J be an infinite set. For simplicity we assume that J is countable, however this is not necessary. An **infinite matrix** or $J \times J$ **matrix** over k is a map $x : J \times J \to k$.*

We only consider a subset of all maps $x : J \times J \to k$, namely the maps that correspond to endomorphisms of the vector space $V := k^J$. As we will see in the next

Lemma, the (i, j)-entry of the product of two such matrices, given by the formula

$$(AB)_{i,j} = \sum_{r \in J} A_{i,r} B_{r,j},$$

is well defined. This is due to the fact that each column of such a matrix has only finitely many nonzero entries.

Consequently, the set of *all* $J \times J$ matrices is not the right object to consider. For example, the product of two such matrices need not exist since the result may involve infinite series that do not converge. Also, the multiplication need not be associative, see [21, Example 1.1].

This brings us to the first result.

Lemma 2.1.1. *An infinite matrix $x : J \times J \to k$ corresponds to an endomorphism of k^J if and only if x has finitely many entries in each column.*

Proof: The vector space $V = k^J$ has countable dimension. Let $\mathrm{End}_k(V)$ denote the set of all k-linear transformations from V to V. Fix an ordered k-basis $\mathcal{B} = \{e_1, e_2, \dots\}$ of V.

Then each linear transformation X in $\mathrm{End}_k(V)$ is determined by the set of vectors $X(e_i)$ for all positive integers i. Thus, under this basis, we can represent each X in $\mathrm{End}_k(V)$ as an infinite matrix $[X]_{\mathcal{B}}$ with countably many rows and countably many columns. Moreover, since each $X(e_i)$ in V is a finite linear combination of elements from $\mathcal{B}$, then the matrix $[X]_{\mathcal{B}}$ has only finitely many nonzero entries in each column.

For the converse take an infinite matrix $x : J \times J \to k$ such that x has finitely many entries in each column. That is, if we say that $x = (x_{ij})$ for $i, j \in J$, then j-th column $C_j = \{x_{ij} : i \in J\}$ contains only finitely many nonzero elements. If we define a map $X : V \to V$ given by $X(e_j) = \sum_{i \in J} x_{ij} e_i$, then we get an element $X \in \mathrm{End}_k(V)$. ∎

Remark 2.1.1. Given this convenient identification of elements of $\mathrm{End}_k(V)$ with infinite matrices, from now on, we will interchangeably refer to elements of $\mathrm{End}_k(V)$ as either matrices, endomorphisms, or maps, depending on the context.

Remark 2.1.2. Even though J is an arbitrary countable set in the context of an arbitrary infinite matrix $x : J \times J \to k$, one should note that in the context of a matrix $[X]_{\mathcal{B}}$ that corresponds to an endomorphism $X : V \to V$, J corresponds to the index set of the given basis $\mathcal{B}$. Since $\mathcal{B}$ was defined as $\{e_1, e_2, \dots\}$ in the previous proof we implicitly assumed that $J = \mathbb{N}$, but this need not be the case in all situations. However for our purposes, unless duly noted this will not make a difference.

Definition 2.1.2. *An element X in $\mathrm{End}_k(V)$ is called **$\mathcal{B}$-finitary** if $[X]_{\mathcal{B}}$ has only finitely many nonzero entries in each row. It is called **$\mathcal{B}$-finite** if $[X]_{\mathcal{B}}$ has only finitely many nonzero entries.*

From now on $\mathcal{B}$ will denote a fixed basis of V indexed by a countable set J.

Remark 2.1.3. It is worth noting that the above definition may differ depending on the author. For example, the definition of $\mathcal{B}$-finite given above is given in [21] as simply finitary. This should not be a point of confusion. The terms chosen here are consistent with the definitions in [18].

Definition 2.1.3. *Let X be a matrix in $\mathrm{End}_k(V)$ whose (i, j)-entry over a basis $\mathcal{B}$ is given by $[X_{i,j}]_{\mathcal{B}}$ and the sum $\sum_i X_{ii}$ has only finitely many nonzero terms. We define $\sum_i X_{ii}$ to be the **trace of** X.*

Lemma 2.1.2. *Suppose that $X, Y, Z \in \mathrm{End}_k(V)$ and that X is $\mathcal{B}$-finite and Y, Z are $\mathcal{B}$-finitary. Then*

(i) XY and YX are $\mathcal{B}$-finite.

(ii) YZ is $\mathcal{B}$-finitary.

(iii) $\mathrm{tr}(XY) = \mathrm{tr}(YX)$.

Proof: Let $\mathcal{B} = \{e_1, e_2, \dots\}$ and denote the ij-entries as $[X]_\mathcal{B} = (x_{ij})$, $[Y]_\mathcal{B} = (y_{ij})$, and $[Z]_\mathcal{B} = (z_{ij})$.

(i) To show that XY is $\mathcal{B}$-finite, we need to show that $[XY]_\mathcal{B}$ has only finitely many nonzero columns. It is not hard to see that $[XY]_\mathcal{B} = [X]_\mathcal{B}[Y]_\mathcal{B}$, thus every column of $[XY]_\mathcal{B}$ has only finitely many nonzero entries. Since X is $\mathcal{B}$-finite, there exists a positive integer n_0 such that $x_{ij} = 0$ for all $i, j > n_0$. Also since Y is $\mathcal{B}$-finitary, there is a positive integer n_1, such that $y_{ij} = 0$ for $1 \le i \le n_0$ and $j > n_1$. Let $n = \max(n_0, n_1)$.

The (i, j)-entry of $[X]_\mathcal{B}[Y]_\mathcal{B}$ is obtained by the dot-product

$$(x_{i1}, x_{i2}, \dots, x_{in}, 0, \dots) \cdot (y_{1j}, y_{2j}, \dots, y_{nj}, \dots).$$

If $j > n$ then $y_{1j} = y_{2j} = \cdots = y_{nj} = 0$, and the dot product is zero. Thus there are finitely many nonzero columns in the matrix $[X]_\mathcal{B}[Y]_\mathcal{B}$, so XY is $\mathcal{B}$-finite.

We can also observe that the (i, j)-entry of $[Y]_\mathcal{B}[X]_\mathcal{B}$ is obtained by the dot-product

$$(y_{i1}, y_{i2}, \dots, y_{in}, \dots) \cdot (x_{1j}, x_{2j}, \dots, x_{nj}, 0, \dots).$$

If $j > n$ then the dot product is zero since the j-th column of $[X]_\mathcal{B}$ is a zero column. Thus YX is $\mathcal{B}$-finite.

(ii) Fix a positive integer p. To show that YZ is $\mathcal{B}$-finitary, we need to show that $[YZ]_\mathcal{B}$ has only finitely many nonzero entries in the p-th row. Now since Y is $\mathcal{B}$-finitary, there exists a positive integer n such that $y_{pj} = 0$ for all $j > n$. The entries in the p-th row are given by $(YZ)_{pj} = (y_{p1}, y_{p2}, \dots, y_{pn}, 0, \dots)(z_{1j}, z_{2j}, \dots, z_{nj}, \dots) = y_{p1}z_{1j} + y_{p2}z_{2j} + \cdots + y_{pn}z_{nj}$ for $1 \le j < \infty$. Also since Z is $\mathcal{B}$-finitary, there is a positive integer m_0, such that $z_{ij} = 0$ for $1 \le i \le n$ and $j > m_0$. Thus $(YZ)_{pj} = 0$ for all $j > m_0$. Thus each row of $[YZ]_\mathcal{B}$ has only finitely many nonzero entries, hence YZ is $\mathcal{B}$-finitary.

(iii) Since both XY and YX are $\mathcal{B}$-finite, their trace is defined. Let $(XY)_{ii}$ and $(YX)_{ii}$ denote the diagonal entries of the matrices $[X]_\mathcal{B}[Y]_\mathcal{B}$ and $[Y]_\mathcal{B}[X]_\mathcal{B}$, respectively. This result follows by the following calculation

$$\mathrm{tr}(XY) = \sum_k (XY)_{kk} = \sum_i \sum_j x_{ij} y_{ji} = \sum_j \sum_i x_{ij} y_{ji} = \sum_k (YX)_{kk} = \mathrm{tr}(YX).$$

Note that the summations have only finitely many nonzero terms. ∎

Corollary 2.1.3. *Let* GL_∞ *be the set of all* X *in* $\mathrm{End}_k(V)$*, such that* X *is invertible and both* X *and* X^{-1} *are* $\mathcal{B}$*-finitary. Then* GL_∞ *is a group under composition.*

Proof: Every element in GL_∞ has an inverse. GL_∞ contains the identity element since the identity is $\mathcal{B}$-finitary. By Lemma 2.1.2 if X and Y are in GL_∞, then XY is $\mathcal{B}$-finitary. The inverse of XY is be given by $Y^{-1}X^{-1}$, which is again $\mathcal{B}$-finitary by Lemma 2.1.2. ∎

Definition 2.1.4. *We will denote by* E_{ij}*, the* $J \times J$ *matrix with a* 1 *in the* ij*-position and zeroes elsewhere. The matrices* $\mathbf{I}_n$ *and* $\mathbf{I}_\infty$ *are the* $n \times n$ *and* $J \times J$ *identity matrices, respectively.*

Let x *be an* $n \times n$ *matrix. The transpose is denoted by* x^t*. We denote by* $x^\dagger$ *the matrix given by* $\mathbf{J}_n x^t \mathbf{J}_n$ *where* $\mathbf{J}_n$ *is the* ***exchange matrix****, namely the* $n \times n$ *matrix with entries of* 1 *in the anti-diagonal and zeroes elsewhere. An* $n \times n$ *matrix* x *is called* ***persymmetric*** *if* $x^\dagger = x$ *and* ***skew persymmetric*** *if* $x^\dagger = -x$*.*

Example 2.1.1.

1. $\mathbf{I}_\infty = \sum_{j \in J} E_{jj}$

2. $\mathbf{J}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

3. If $x = \begin{pmatrix} 1 & 2 & 3 & 0 & 88 \\ 6 & 9 & 4 & 2 & 1 \\ 7 & 3 & 4 & 0 & 1 \\ 0 & 9 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ then $x^\dagger = \begin{pmatrix} 0 & 0 & 1 & 1 & 88 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 4 & 4 & 3 \\ 0 & 9 & 3 & 9 & 2 \\ 1 & 0 & 7 & 6 & 1 \end{pmatrix}$

Now we will define the Lie algebras $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$.

2.2 The Lie Algebra $\mathfrak{gl}_\infty$

Definition 2.2.1. *We define the* ***general linear Lie algebra*** $\mathfrak{gl}_\infty$ *as the subset of* $\mathrm{End}_k(V)$ *consisting of* $\mathcal{B}$*-finite elements. The corresponding space of* $J \times J$ *matrices is denoted by* $\mathfrak{gl}(J, k)$*. Likewise, the* $J \times J$ *matrices corresponding to the endomorphisms in* GL_∞*, defined in Corollary 2.1.3, are denoted by* $\mathrm{GL}(J, k)$*.*

Example 2.2.1. (1) Let $\mathcal{B} = \{e_i : i \in \mathbb{N}\}$ be a basis of V, then with respect to this basis, an element $x = (x_{ij})$ in $\mathfrak{gl}_\infty$, for $i, j \in \mathbb{N}$, whose nonzero part is contained in an $n \times n$ block for a positive integer n, can be pictured as

$$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} & 0 & \cdots \\ x_{21} & x_{22} & \cdots & x_{2n} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & 0 & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(2) Alternatively, if $\tilde{\mathcal{B}} = \{e_i : i \in \mathbb{Z}\backslash\{0\}\}$ is a basis of V, then with respect to this basis, an element $x = (x_{ij})$ in $\mathfrak{gl}_\infty$, for $i, j \in \mathbb{Z}\backslash\{0\}$, whose nonzero part is contained in a $2n \times 2n$ block for a positive integer n, can be pictured as

$$x = \left(\begin{array}{cccccc|cccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & x_{-n,-n} & \cdots & x_{-n,-2} & x_{-n,-1} & x_{-n,1} & x_{-n,2} & \cdots & x_{-n,n} & 0 & \cdots \\
\cdots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & 0 & \cdots \\
\cdots & 0 & x_{-2,-n} & \cdots & x_{-2,-2} & x_{-2,-1} & x_{-2,1} & x_{-2,2} & \cdots & x_{-2,n} & 0 & \cdots \\
\cdots & 0 & x_{-1,-n} & \cdots & x_{-1,-2} & x_{-1,-1} & x_{-1,1} & x_{-1,2} & \cdots & x_{-1,n} & 0 & \cdots \\
\hline
\cdots & 0 & x_{1,-n} & \cdots & x_{1,-2} & x_{1,-1} & x_{11} & x_{12} & \cdots & x_{1n} & 0 & \cdots \\
\cdots & 0 & x_{2,-n} & \cdots & x_{2,-2} & x_{2,-1} & x_{21} & x_{22} & \cdots & x_{2n} & 0 & \cdots \\
\cdots & 0 & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \cdots \\
\cdots & 0 & x_{n,-n} & \cdots & x_{n,-2} & x_{n,-1} & x_{n1} & x_{n2} & \cdots & x_{nn} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

(3) Alternatively, if $\tilde{\mathcal{B}} = \{e_i : i \in \mathbb{Z}\}$ is a basis of V, then with respect to this basis, an element $x = (x_{ij})$ in $\mathfrak{gl}_\infty$, for $i, j \in \mathbb{Z}$, whose nonzero part is contained in a $(2n+1) \times (2n+1)$ block for a positive integer n, can be pictured as

$$x = \left(\begin{array}{cccccc|c|cccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & x_{-n,-n} & \cdots & x_{-n,-2} & x_{-n,-1} & x_{-n,0} & x_{-n,1} & x_{-n,2} & \cdots & x_{-n,n} & 0 & \cdots \\
\cdots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & 0 & \cdots \\
\cdots & 0 & x_{-2,-n} & \cdots & x_{-2,-2} & x_{-2,-1} & x_{-2,0} & x_{-2,1} & x_{-2,2} & \cdots & x_{-2,n} & 0 & \cdots \\
\cdots & 0 & x_{-1,-n} & \cdots & x_{-1,-2} & x_{-1,-1} & x_{-1,0} & x_{-1,1} & x_{-1,2} & \cdots & x_{-1,n} & 0 & \cdots \\
\hline
\cdots & 0 & x_{0,-n} & \cdots & x_{0,-2} & x_{0,-1} & x_{0,0} & x_{0,1} & x_{0,2} & \cdots & x_{0,n} & 0 & \cdots \\
\hline
\cdots & 0 & x_{1,-n} & \cdots & x_{1,-2} & x_{1,-1} & x_{1,0} & x_{11} & x_{12} & \cdots & x_{1n} & 0 & \cdots \\
\cdots & 0 & x_{2,-n} & \cdots & x_{2,-2} & x_{2,-1} & x_{2,0} & x_{21} & x_{22} & \cdots & x_{2n} & 0 & \cdots \\
\cdots & 0 & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \cdots \\
\cdots & 0 & x_{n,-n} & \cdots & x_{n,-2} & x_{n,-1} & x_{n,0} & x_{n1} & x_{n2} & \cdots & x_{nn} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

Remark 2.2.1. The concepts of $\mathcal{B}$-finite and $\mathcal{B}$-finitary do not depend on the choice of J.

2.3 The Lie Algebra $\mathfrak{sl}_\infty$

Note that the trace is well defined for each matrix in $\mathfrak{gl}_\infty$, as each matrix is $\mathcal{B}$-finite.

Definition 2.3.1. ***The special linear Lie algebra*** $\mathfrak{sl}_\infty$ *is defined as the subset of* $\mathfrak{gl}_\infty$ *consisting of all matrices with trace zero. More precisely,* $\mathfrak{sl}_\infty$ *is the subset of* $\mathfrak{gl}_\infty$*, whose corresponding maps lie in the set*

$$\mathfrak{sl}(J,k) = \{x \in \mathfrak{gl}(J,k) : \mathrm{tr}(x) = 0\}.$$

2.4 The Lie Algebra $\mathfrak{sp}_\infty$

From now on $J^\pm := J \sqcup -J$ where $-J$ is a copy of J, whose elements are denoted $-j$, for $j \in J$. Similarly define $2J+1$ and $J_0^\pm$ as two equivalent notations for the set $J \sqcup \{0\} \sqcup -J$. Define the skew-symmetric $J^\pm \times J^\pm$ matrix $S = \sum_{j\in J}(E_{j,-j} - E_{-j,j})$.

Definition 2.4.1. *We define the* ***symplectic Lie algebra*** $\mathfrak{sp}_\infty \subset \mathfrak{gl}_\infty$ *as the set of matrices whose corresponding maps lie in the set*

$$\mathfrak{sp}(J,k) = \{x \in \mathfrak{gl}(J^\pm,k) : x^t S = -Sx\}.$$

Theorem 2.4.1. *Let* $x \in \mathfrak{sp}_\infty$ *and assume that* $\tilde{\mathcal{B}} = \{e_i : i \in \mathbb{Z}\backslash\{0\}\}$*. Then with respect to this basis, there exists a positive integer* n *such that the nonzero part of* x *is contained in a* $2n \times 2n$ *block which is denoted by* $\tilde{x}$*, given by*

$$\tilde{x} := \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right)$$

where X*,* Y*,* Z*, and* W *are* $n \times n$ *matrices. Then* $W^\dagger = -X$*,* $Y^\dagger = Y$*, and* $Z^\dagger = Z$*.*

Proof: Assume that x, $\tilde{x}$, $\mathcal{B}$ and S are defined as in the statement of the theorem.

Then with respect to the same basis $\mathcal{B}$, we have that,

$$x = \left(\begin{array}{ccc|ccc} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & X & Y & 0 & \cdots \\ \hline \cdots & 0 & Z & W & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \quad \text{and} \quad S = \left(\begin{array}{ccc|ccc} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 0 & 0 & 0 & -1 & \cdots \\ \cdots & 0 & 0 & -1 & 0 & \cdots \\ \hline \cdots & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Now to carry out the multiplication $x^t S = -Sx$ it is enough to consider

$$\tilde{x} := \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) \quad \text{and} \quad \tilde{S} := \left(\begin{array}{c|c} 0 & \mathbf{J}_n \\ \hline \mathbf{J}_n & 0 \end{array}\right).$$

After carrying out the multiplication $x^t S = -Sx$ by blocks, we get

$$\left(\begin{array}{c|c} W^\dagger & -Y^\dagger \\ \hline -Z^\dagger & X^\dagger \end{array}\right) = \left(\begin{array}{c|c} -X & -Y \\ \hline -Z & -W \end{array}\right)$$

as desired. ∎

Remark 2.4.1. Note that $\tilde{x}$ belongs to the Lie algebra $\mathfrak{sp}_{2n}$. This illustration gives insight into an alternate but equivalent construction of $\mathfrak{sp}_\infty$, namely that of the direct limit $\varinjlim \mathfrak{sp}_{2n}$ of the direct system of Lie algebras with monomorphisms $i_n : \mathfrak{sp}_{2n} \to \mathfrak{sp}_{2(n+1)}$ given by

$$X \mapsto \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & X & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad \text{for any } X \in \mathfrak{sp}_{2n}. \tag{2.4.1}$$

(See [18, Section 2.3]).

2.5 The Lie Algebra $\mathfrak{so}_\infty$

We will define the Lie algebra $\mathfrak{so}_\infty$ in three different ways, and then proceed to prove that the three definitions yield isomorphic Lie algebras.

Definition 2.5.1. *Define the three following subsets of* $\mathfrak{gl}_\infty$

$$\begin{aligned}
\mathfrak{so}(J,J,k) &= \{x \in \mathfrak{gl}(J^\pm,k) : x^tQ = -Qx\} \quad \text{where} \quad Q = \sum_{j\in J}(E_{j,j} - E_{-j,-j}) \\
\mathfrak{so}(2J+1,k) &= \{x \in \mathfrak{gl}(J_0^\pm,k) : x^tQ_o = -Q_ox\} \quad \text{where} \quad Q_o = -E_{0,0} + \sum_{j\in J}(E_{j,-j} + E_{-j,j}) \\
\mathfrak{so}(2J,k) &= \{x \in \mathfrak{gl}(J^\pm,k) : x^tQ_e = -Q_ex\} \quad \text{where} \quad Q_e = \sum_{j\in J}(E_{j,-j} + E_{-j,j}).
\end{aligned} \tag{2.5.1}$$

We define the ***orthogonal Lie algebra*** $\mathfrak{so}_\infty$ *as the subset of* $\mathfrak{gl}_\infty$ *whose corresponding maps are identified with either of the three sets* $\mathfrak{so}(2J+1,k)$*,* $\mathfrak{so}(2J,k)$ *or* $\mathfrak{so}(J,J,k)$*.*

We will now prove that the three definitions of $\mathfrak{so}_\infty$ indeed yield isomorphic Lie algebras.

Lemma 2.5.1. *Let* Q_1*,* Q_2 *be two* $J \times J$ *matrices. For* $i = 1,2$ *define*

$$\mathfrak{g}_i = \{x \in \mathfrak{gl}(J,k) : x^tQ_i = -Q_ix\}.$$

If $Q_1 = D^tQ_2D$ *holds for a matrix* $D \in \mathrm{GL}(J,k)$*, then the conjugation map* $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$*, given by* $\phi(x) = DxD^{-1}$ *is an isomorphism of Lie algebras.*

Proof: Linearity of ϕ is clear, so we will check the Lie bracket condition. Let $x, y \in \mathfrak{g}_1$, then

$$\begin{aligned}
\phi([x,y]) = \phi(xy - yx) &= \phi(xy) - \phi(yx) \\
&= DxyD^{-1} - DyxD^{-1} \\
&= (DxD^{-1})(DyD^{-1}) - (DyD^{-1})(DxD^{-1}) \\
&= [\phi(x), \phi(y)]
\end{aligned}$$

Now suppose $Q_1 = D^t Q_2 D$. After a substitution into $x^t Q_1 = -Q_1 x$ and an application of part (i) of Lemma 2.1.2, we get that $(DxD^{-1})^t(Q_2) = -(Q_2)(DxD^{-1})$, showing that $DxD^{-1} \in \mathfrak{g}_2$.

Finally, we will show that ϕ is bijective. If $x \neq y$, then $DxD^{-1} \neq DyD^{-1}$ so ϕ is injective. And for all $x \in \mathfrak{g}_2$, a quick calculation shows that the pre-image is $D^{-1}xD \in \mathfrak{g}_1$, giving us that ϕ is surjective. ∎

Theorem 2.5.2. *The Lie algebras* $\mathfrak{so}(2J+1, k)$ *and* $\mathfrak{so}(2J, k)$ *are both isomorphic to* $\mathfrak{so}(J, J, k)$.

Proof: For arbitrary countable sets J_1 and J_2, define

$$Q_{J_1,J_2} = \sum_{j \in J_1} E_{j,j} - \sum_{j \in J_2} E_{j,j}$$

Define the Lie algebras

$$\mathfrak{so}(J_1, J_2, k) = \{x \in \mathfrak{gl}(J_1 \sqcup J_2, k) : x^t Q_{J_1,J_2} = -Q_{J_1,J_2} x\}.$$

First we observe that if J_1 and J_1' have the same cardinality, and J_2 and J_2' have the same cardinality, then the Lie algebras $\mathfrak{so}(J_1, J_2, k)$ and $\mathfrak{so}(J_1', J_2', k)$ are isomorphic.

To finish the proof, it is sufficient to show that the matrices Q_e and $Q_{J,\{0\}\cup -J}$ and the matrices Q_o and $Q_{J,-J}$ yield isomorphic Lie algebras in the sense of Lemma 2.5.1. Indeed, because of the cardinalities of these infinite sets, this will show that $\mathfrak{so}(2J+1, k)$ and $\mathfrak{so}(2J, k)$ are both isomorphic to $\mathfrak{so}(J, J, k)$.

Now we will find explicit matrices D_1 and D_2 such that $Q_{J,\{0\}\cup -J} = D_1 Q_e D_1$ and $Q_{J,-J} = D_2 Q_o D_2$. Indeed, the following matrices D_1 and D_2 give the result.

$$D_1 = \left(\frac{1}{2}\right) \sum_{j \in J} (E_{j,-j} + E_{-j,-j}) + \sum_{j \in J} (E_{j,j} + E_{-j,j})$$

$$D_2 = E_{0,0} + \left(\frac{1}{2}\right) \sum_{j \in J} (E_{j,-j} + E_{-j,-j}) + \sum_{j \in J} (E_{j,j} + E_{-j,j}).$$

∎

Remark 2.5.1. Note that $\frac{1}{2} \in k$ makes sense because k has characteristic zero.

Remark 2.5.2. In the proof [21, p.821, Lemma 1.8] I have found and corrected the following errors. My two equations above, namely $Q_{J,\{0\}\cup -J} = D_1 Q_e D_1$ and $Q_{J,-J} = D_2 Q_o D_2$, are the correct ones. In [2] the order of the Q's is reversed and the matrix D that is given is actually the transpose of the desired matrix.

Thus $\mathfrak{so}_\infty$ is the Lie subalgebra of $\mathfrak{gl}_\infty$ whose corresponding elements can be identified with either of the three sets $\mathfrak{so}(2J+1,k)$, $\mathfrak{so}(2J,k)$ or $\mathfrak{so}(J,J,k)$. In the case where J is a finite set, the Lie algebras $\mathfrak{so}(2J,k)$ and $\mathfrak{so}(J,J,k)$ are isomorphic. We have included all three types for completeness and it is occasionally more convenient to work with one specific realization over another.

Example 2.5.1. This example will illustrate an alternate way to obtain the three isomorphic versions of $\mathfrak{so}_\infty$ as direct limits.

Firstly, recall that $\mathfrak{so}_{2n} = \{x \in \mathfrak{gl}_{2n} : x^t E = -Ex\}$ where E is a symmetric $2n \times 2n$ matrix such that $(x,y) \mapsto xEy$ is a non-degenerate, symmetric, bilinear form.

1. (See [18, Section 2.3]) If we consider $\mathfrak{so}_\infty$ as the $J^\pm \times J^\pm$ matrices in $\mathfrak{so}(J,J,k)$ with respect to the basis $\mathcal{B} = \{e_i : i \in \mathbb{Z}\backslash\{0\}\}$ we observe that an arbitrary matrix $x \in \mathfrak{so}(J,J,k)$ has a nonzero part $\tilde{x} := \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right)$, where X, Y, Z and W are $n \times n$ matrices, and $X = -X^t$, $W = -W^t$ and $Z = Y^t$. (The proof of the previous statement is very similar to Theorem 2.4.1 so it is omitted.) Note that $\tilde{x} \in \mathfrak{so}_{2n} = \{x \in \mathfrak{gl}_{2n} : x^t \tilde{Q} = -\tilde{Q}x\}$, where $\tilde{Q} = \left(\begin{array}{c|c} -\mathbf{I}_n & 0 \\ \hline 0 & \mathbf{I}_n \end{array}\right)$.

Construction of $\mathfrak{so}(J,J,k)$ can be seen as the direct limit $\varinjlim \mathfrak{so}_{2n}$ of the direct system of Lie algebras with monomorphisms $i_n : \mathfrak{so}_{2n} \to \mathfrak{so}_{2(n+1)}$ given by

$$X \mapsto \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & X & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad \text{for any } X \in \mathfrak{so}_{2n}. \tag{2.5.2}$$

2. (See [18, Section 2.3]) If we consider $\mathfrak{so}_\infty$ as the $J_0^\pm \times J_0^\pm$ matrices in $\mathfrak{so}(2J+1,k)$ with respect to the basis $\mathcal{B} = \{e_i : i \in \mathbb{Z}\}$ we observe that an arbitrary matrix $x \in \mathfrak{so}(2J+1,k)$ has a nonzero part $\tilde{x} := \left(\begin{array}{c|c|c} X & s & Y \\ \hline u & a & t \\ \hline Z & v & W \end{array}\right)$, where X, Y, Z and W are $n \times n$ matrices and s, y are column vectors in k^n, and u, t are row vectors in k^n and $a \in k$. We then have $X = -W^\dagger$, $Y = -Y^\dagger$ and $Z = -Z^\dagger$, $v^t = u$, $s^t = t$ and $a = 0$. (The proof of these relations is very similar to Theorem 2.4.1 so it is omitted.) Note that $\tilde{x} \in \mathfrak{so}_{2n+1} = \{x \in \mathfrak{gl}_{2n+1} : x^t \tilde{Q}_o = -\tilde{Q}_o x\}$, where

$$\tilde{Q}_o = \left(\begin{array}{c|c|c} 0 & 0 & \mathbf{J}_n \\ \hline 0 & -1 & 0 \\ \hline \mathbf{J}_n & 0 & 0 \end{array}\right).$$

Construction of $\mathfrak{so}(J,J,k)$ can be seen as the direct limit $\varinjlim \mathfrak{so}_{2n}$ of the direct system of Lie algebras with monomorphisms $i_n : \mathfrak{so}_{2n+1} \to \mathfrak{so}_{2(n+1)+1}$ given by

$$X \mapsto \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & X & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad \text{for any } X \in \mathfrak{so}_{2n}. \tag{2.5.3}$$

3. (See [18, Section 2.3]) If we consider $\mathfrak{so}_\infty$ as the $J^\pm \times J^\pm$ matrices in $\mathfrak{so}(2J,k)$ with respect to the basis $\mathcal{B} = \{e_i : i \in \mathbb{Z}\backslash\{0\}\}$ we observe that an arbitrary matrix $x \in \mathfrak{so}(2J,k)$ has a nonzero part $\tilde{x} := \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right)$, where X, Y, Z and

W are $n \times n$ matrices, and $X = -W^\dagger$, $Y = -Y^\dagger$ and $Z = -Z^\dagger$. (The proof of the latter relations is very similar to Theorem 2.4.1 so it is omitted.) Note that $\tilde{x} \in \mathfrak{so}_{2n} = \{x \in \mathfrak{gl}_{2n} : x^t\tilde{Q}_e = -\tilde{Q}_e x\}$, where $\tilde{Q}_e = \left(\begin{array}{c|c} 0 & \mathbf{J}_n \\ \hline \mathbf{J}_n & 0 \end{array}\right)$.

Construction of $\mathfrak{so}(2J, k)$ can be seen as the direct limit $\varinjlim \mathfrak{so}_{2n}$ of the direct system of Lie algebras with monomorphisms $i_n : \mathfrak{so}_{2n} \to \mathfrak{so}_{2(n+1)}$ given by

$$X \mapsto \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & X & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad \text{for any } X \in \mathfrak{so}_{2n}. \tag{2.5.4}$$

2.6 Infinite Matrices over a Ring

So far the matrices we have considered have entries in the field k. We will now extend these definitions to allow the matrices to have entries in a commutative ring with unity.

Let V be as before and assume that R is a commutative ring containing the field k. Define $V_R := V \otimes_k R$. Then V_R is an R-module with a countable generating set. Denote by $\mathrm{End}_R(V_R)$ the set of all R-linear endomorphisms on V_R. Considering these endomorphisms as infinite matrices over R (as in Lemma 2.1), we may define $\mathfrak{gl}_\infty(R)$, or the corresponding space of matrices $\mathfrak{gl}(J, R)$, as the set of all $\mathcal{B}$-finite elements in $\mathrm{End}_R(V_R)$.

Definition 2.6.1. *We define $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}_\infty(R)$, and $\mathfrak{so}_\infty(R)$ as the following analogous* *subsets of* $\mathrm{End}_R(V_R)$*:*

$$\mathfrak{sl}(J,R) = \{x \in \mathfrak{gl}(J,R) : \mathrm{tr}(x) = 0\}$$

$$\mathfrak{sp}(J,R) = \{x \in \mathfrak{gl}(J^{\pm},R) : x^t S = -Sx\} \quad \textit{where} \quad Q = \sum_{j \in J}(E_{j,j} - E_{-j,-j})$$

$$\mathfrak{so}(J,J,R) = \{x \in \mathfrak{gl}(J^{\pm},R) : x^t Q = -Qx\} \quad \textit{where} \quad Q = \sum_{j \in J}(E_{j,j} - E_{-j,-j})$$

$$\mathfrak{so}(2J+1,R) = \{x \in \mathfrak{gl}(J_0^{\pm},R) : x^t Q_o = -Q_o x\} \quad \textit{where} \quad Q_o = -E_{0,0} + \sum_{j \in J}(E_{j,-j} + E_{-j,j})$$

$$\mathfrak{so}(2J,R) = \{x \in \mathfrak{gl}(J^{\pm},R) : x^t Q_e = -Q_e x\} \quad \textit{where} \quad Q_e = \sum_{j \in J}(E_{j,-j} + E_{-j,j}).$$

where the last three are indeed isomorphic. (The proof is similar to Remark 2.5.2)

Note that $\mathrm{End}_R(V_R) \cong \mathrm{End}_k(V) \otimes_k R$, and in particular $\mathfrak{gl}_\infty(R) \cong \mathfrak{gl}_\infty \otimes_k R$. Therefore we can see that a Lie bracket for $\mathfrak{gl}_\infty \otimes_k R$ can be defined by

$$[x \otimes t, y \otimes s] = [x,y] \otimes ts$$

for $x, y \in \mathfrak{gl}_\infty$ and $s, t \in R$.

Chapter 3

Central Extensions

In this chapter we will discuss the notion of a central extension of a Lie algebra. In later chapters we will prove a conjugacy theorem for maximal toral subalgebras of $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}_\infty(R)$, and $\mathfrak{so}_\infty(R)$, and we will see that this result extends to universal central extensions of these Lie algebras.

3.1 Definition and Examples

Definition 3.1.1. *Let $\mathfrak{g}$ be a Lie algebra. The **center** of $\mathfrak{g}$, namely the set*

$$\{x \in \mathfrak{g} : [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}$$

*is denoted by $\mathfrak{z}(\mathfrak{g})$. An **extension** of $\mathfrak{g}$ is a surjective homomorphism of Lie algebras $\pi : \mathfrak{h} \to \mathfrak{g}$. A **central extension** of $\mathfrak{g}$ is an extension $\pi : \mathfrak{h} \to \mathfrak{g}$ satisfying $\ker(\pi) \subset \mathfrak{z}(\mathfrak{h})$.*

Example 3.1.1.

1. Let $\pi : \mathfrak{h} \to \mathfrak{g}$ be a central extension. Then this gives rise to a short exact sequence

$$0 \to \ker \pi \xrightarrow{i} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0$$

where i is the inclusion map and we know that this implies $\mathfrak{h} \cong \mathfrak{g} \oplus \ker \pi$ as vector spaces. ([5, Section 10.5])

2. For any Lie algebra $\mathfrak{g}$ over k we can always construct a trivial central extension $\pi : \mathfrak{g} \oplus \mathfrak{c} \to \mathfrak{g}$ where π is the projection map onto the first summand, and $\mathfrak{c}$ is an abelian Lie algebra such that $\mathfrak{g} \oplus \mathfrak{c}$ has Lie bracket $[x + a, y + b]_{\mathfrak{g}\oplus\mathfrak{c}} = [x, y]_{\mathfrak{g}}$, $x, y \in \mathfrak{g}, a, b, \in \mathfrak{c}$.

3. For an arbitrary Lie algebra $\mathfrak{h}$ over k we may ask if $\mathfrak{h}$ itself is a central extension of some other Lie algebra $\mathfrak{g}$. Indeed we may always take the canonical projection

$$\pi : \mathfrak{h} \to \mathfrak{h}/\mathfrak{c}$$

where $\mathfrak{c}$ is any subalgebra of $\mathfrak{z}(\mathfrak{h})$.

4. Let $\mathfrak{h}_3$ denote the 3-dimensional Heisenberg Lie algebra which can be identified with strictly upper triangular 3×3 matrices with entries in $\mathbb{R}$. Then $\mathfrak{h}_3$ is a central extension of the commutative Lie algebra $\mathbb{R}^2$ with surjective homomorphism given by

$$\pi : \begin{pmatrix} 0 & p & c \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix} \mapsto (p, q).$$

5. The Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ is the $(2n + 1)$-dimensional real Lie algebra with basis $\{p_1, \dots, p_n, q_1, \dots, q_n, c\}$ and Lie bracket given by

$$[p_i, p_j] = [q_i, q_j] = [p_i, c] = [q_i, c] = [c, c] = 0 \quad \text{and} \quad [p_i, q_j] = c\delta_{ij}$$

where δ_{ij} is the Kronecker delta and $1 \leq i, j \leq n$.

To generalize the previous example, we show that the $(2n + 1)$-dimensional Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ is a central extension of the commutative Lie algebra $\mathbb{R}^{2n}$.

One can see that the elements in $\mathfrak{h}_{2n+1}$ can be identified with $(n+2) \times (n+2)$ matrices

$$\begin{pmatrix} 0 & p_1 & \cdots & p_n & c \\ 0 & 0 & \cdots & 0 & q_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and the surjective homomorphism $\pi : \mathfrak{h}_{2n+1} \to \mathbb{R}^{2n}$ given by

$$\begin{pmatrix} 0 & p_1 & \cdots & p_n & c \\ 0 & 0 & \cdots & 0 & q_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \mapsto (p_1, \ldots, p_n, q_1, \ldots, q_n)$$

shows that $\mathfrak{h}_{2n+1}$ is a central extension of $\mathbb{R}^{2n}$.

3.2 Constructing Central Extensions via 2-cocycles

In this section we will review basic results on the construction of a central extension of a given Lie algebra. For further study, see [13, Section 1.9].

Definition 3.2.1. *Let $\mathfrak{g}$ be a Lie algebra over k and let $\mathfrak{c}$ be a vector space over k. A bilinear function $\phi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{c}$ on a Lie algebra $\mathfrak{g}$ satisfying properties*

i. $\phi(x, y) = -\phi(y, x)$ *and*

ii. $\phi(x, [y, z]) + \phi(y, [z, x]) + \phi(z, [x, y]) = 0$ *for all* $x, y, z \in \mathfrak{g}$

is called a ***2-cocycle of*** $\mathfrak{g}$ ***with coefficients in*** $\mathfrak{c}$.

Theorem 3.2.1. *Let* $\mathfrak{g}$ *be a Lie algebra over* k *and let* $\phi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{c}$ *be a 2-cocycle on* $\mathfrak{g}$ *with coefficients in* $\mathfrak{c}$. *Viewing* $\mathfrak{g}$ *as a vector space we can extend it to the vector space* $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathfrak{c}$. *Define* $[\cdot,\cdot] : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ *by* $[x + a, y + b] := [x, y] + \phi(x, y)$. *Then*

1. $\tilde{\mathfrak{g}}$ *is a Lie algebra with* $[\cdot,\cdot]$ *as the Lie bracket,*

2. *The projection map on the first summand* $\psi : \mathfrak{g} \oplus \mathfrak{c} \to \mathfrak{g}$ *is a central extension of* $\mathfrak{g}$.

Proof: In order to prove (1) we must simply check that $[\cdot,\cdot]$ is a Lie bracket. Let $\alpha, \beta \in k$ and let $\tilde{x} = x + a$, $\tilde{y} = y + b$ and $\tilde{z} = z + c$ be in $\tilde{\mathfrak{g}}$. Bilinearity, i.e. $[\alpha\tilde{x} + \beta\tilde{y}, \tilde{z}] = \alpha[\tilde{x}, \tilde{z}] + \beta[\tilde{y}, \tilde{z}]$ follows from the bilinearity of ϕ. The properties $[\tilde{x}, \tilde{x}] = 0$ and the Jacobi identity follow from properties (i) and (ii) in the definition of a 2-cocycle. Thus $\tilde{\mathfrak{g}}$ is a Lie algebra with $[\cdot,\cdot]$ as the Lie bracket.

To prove (2) we note that the projection map on the first summand $\psi : \mathfrak{g} \oplus \mathfrak{c} \to \mathfrak{g}$ is a surjective map satisfying $\ker \pi = 0 \oplus \mathfrak{c}$ and $[0+a, x+b] := [0, x] + \phi(0, x) = 0$ since $\phi(0, x) = 0$ follows from the definition of 2-cocycle. Thus the kernel of π is contained in the center of $\tilde{\mathfrak{g}}$ making it a central extension. ∎

Thus for any Lie algebra $\mathfrak{g}$ that admits a 2-cocycle, we can construct a central extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ explicitly. The following example illustrates how an arbitrary central extension gives rise to a 2-cocycle.

Example 3.2.1. Let $\pi : \mathfrak{h} \to \mathfrak{g}$ be a central extension of $\mathfrak{g}$ and pick a subspace $\mathfrak{g}'$ of $\mathfrak{h}$ such that $\mathfrak{g}' \overset{\pi}{\cong} \mathfrak{g}$. Indeed, such a subspace can be constructed by taking a k-basis $\{x_j\}$ of $\mathfrak{g}$ and for each j choosing a fixed $x'_j \in \mathfrak{h}$ such that $\pi(x'_j) = x_j$ and then defining $\mathfrak{g}' = \mathrm{span}_k\{x'_j\}$. We let $s : \mathfrak{g} \to \mathfrak{g}'$ be defined by $s(x'_j) = x_j$ and extended linearly. Then we have

$$\pi(s([x, y]) - [s(x), s(y)]) = 0$$

and therefore we can define a bilinear map $\phi : \mathfrak{g} \times \mathfrak{g} \to \ker \pi$ given by

$$\phi(x, y) = s([x, y]) - [s(x), s(y)]$$

It is not hard to check that ϕ is a 2-cocycle of $\mathfrak{g}$ with coefficients in $\ker \pi$. Thus a central extension gives rise to a 2-cocycle and we will now show how this 2-cocycle can be used to reconstruct this central extension.

Using the 2-cocycle ϕ we define the central extension

$$\pi' : \mathfrak{g} \oplus \ker \pi \to \mathfrak{g}$$

where π' is a projection onto the first summand and the Lie bracket of $\mathfrak{g} \oplus \ker \pi$ is given by

$$[x + a, y + b]_* = [x, y]_{\mathfrak{g}} + \phi(x, y) \text{ for all } x, y \in \mathfrak{g} \text{ and } a, b \in \ker \pi.$$

Note that $\ker \pi' \cong \ker \pi$. As vector spaces it is clear that $\mathfrak{h} \cong \mathfrak{g} \oplus \ker \pi$.

To summarise, we have that a central extension $\pi : \mathfrak{h} \to \mathfrak{g}$ gives rise to a 2-cocycle ϕ, which in turn gives rise to another central extension $\pi' : \mathfrak{g} \oplus \ker \pi \to \mathfrak{g}$. Lastly, we will show that $\mathfrak{h}$ and $\mathfrak{g} \oplus \ker \pi$ are isomorphic *as Lie algebras*. The required isomorphism can be given by

$$\psi : \mathfrak{h} \to \mathfrak{g} \oplus \ker \pi, \quad \psi(h) = \pi(h) + (s(\pi(h)) - h), \text{ where } h \in \mathfrak{h}.$$

We can see that $\pi(h) \in \mathfrak{g}$, and $(h - s(\pi(h))) \in \ker \pi$ as $\pi(s(\pi(h)) - h) = \pi(s(\pi(h)))) - \pi(h) = \pi(h) - \pi(h) = 0$, so $\psi(h)$ is indeed an element of $\mathfrak{g} \oplus \ker \pi$. Also note that $\pi' \circ \psi = \pi$. Let $z, w \in \mathfrak{h}$, we have that $s(\pi(z)) - z$ and $s(\pi(w)) - w$ are in $\ker \pi$, therefore

$$[z, w]_{\mathfrak{h}} = [z, w]_{\mathfrak{h}} + [s(\pi(z)) - z, s(\pi(w))]_{\mathfrak{h}} + [z, s(\pi(w)) - w]_{\mathfrak{h}} = [s(\pi(z)), s(\pi(w))]_{\mathfrak{h}} \tag{3.2.1}$$

and thus we have

$$\begin{aligned}
\psi([z,w]_{\mathfrak{h}}) &= \pi([z,w]_{\mathfrak{h}}) + s(\pi([z,w]_{\mathfrak{h}})) - [z,w]_{\mathfrak{h}} \\
&= [\pi(z),\pi(w)]_{\mathfrak{g}} + s([\pi(z),\pi(w)]_{\mathfrak{g}} - [z,w]_{\mathfrak{h}} \\
&= [\pi(z),\pi(w)]_{\mathfrak{g}} + s([\pi(z),\pi(w)]_{\mathfrak{g}}) - [s(\pi(z)),s(\pi(w))]_{\mathfrak{h}} \text{ by Equation (3.2.1)} \\
&= [\pi(z),\pi(w)]_{\mathfrak{g}} + \phi\big(\pi(z),\pi(w)\big) \\
&= [\pi'(\psi(z)),\pi'\psi(w)]_{\mathfrak{g}} + \phi\big(\pi'(\psi(z)),\pi'(\psi(w))\big) \\
&= [\psi(z),\psi(w)]_*.
\end{aligned}$$

So it is indeed a Lie algebra isomorphism. ([13, Section 1.9])

Remark 3.2.1. The 2-cocycle ϕ constructed in the previous example depended on our choice of the subspace $\mathfrak{g}'$. In general, different subspaces $\mathfrak{g}'$ will lead to different 2-cocycles. As we have seen from the previous example, 2-cocycles constructed from different choices of $\mathfrak{g}'$ will lead to isomorphic central extensions of $\mathfrak{g}$. Thus it is possible for two different 2-cocycles to lead us to isomorphic central extensions. Such cocycles are called cohomologous.

The next result shows us that 2-cocycles can actually be constructed if you have an invariant bilinear form and a derivation with suitable properties.

Theorem 3.2.2. *Let $\mathfrak{g}$ be a Lie algebra over k and assume that $(\cdot,\cdot)$ is a symmetric, invariant, bilinear form on $\mathfrak{g}$. Let d be a k-linear derivation of $\mathfrak{g}$ that satisfies $(d(x),y) = -(x,d(y))$. Then $\psi(x,y) = (d(x),y)$ defines a 2-cocycle on $\mathfrak{g}$ with coefficients in the vector space k.*

Proof: Let $x,y,z \in \mathfrak{g}$ and $a,b \in k$. Then

$$\begin{aligned}
&\psi(ax+by,z) = (d(ax+by),z) = -(ax+by,d(z)) = -a(x,d(z)) - b(y,d(z)) \\
&= a(d(x),z) + b(d(y),z) = a\psi(x,z) + b\psi(y,z).
\end{aligned}$$

So ψ is bilinear.

Also, $\psi(x,y) = (d(x),y) = -(x,d(y)) = -(d(y),x) = -\psi(y,x)$ and finally

$$\begin{aligned}
&\phi(x,[y,z]) + \phi(y,[z,x]) + \phi(z,[x,y]) = -(x,d([y,z])) - (y,d([z,x])) - (z,d([x,y])) \\
&= -(x,[y,d(z)] + [d(y),z]) - (y,[z,d(x)] + [d(z),x]) - (z,[x,d(y)] + [d(x),y]) \\
&= -(x,[y,d(z)]) - (x,[d(y),z]) - (y,[z,d(x)]) - (y,[d(z),x]) - (z,[x,d(y)]) - (z,[d(x),y]) \\
&= -(x,[y,d(z)]) + (x,[y,d(z)]) - (y,[z,d(x)]) + (y,[z,d(x)]) - (z,[x,d(y)]) + (z,[x,d(y)]) \\
&= 0 + 0 + 0 = 0.
\end{aligned}$$

∎

3.3 Universal Central Extensions

The material in this section is found in [13, Section 1.9].

Definition 3.3.1. *A* ***universal central extension (u.c.e.)*** *of $\mathfrak{g}$ is a central extension $\pi : \hat{\mathfrak{g}} \to \mathfrak{g}$ such that given any other central extension $\pi' : \mathfrak{h} \to \mathfrak{g}$, there exists a unique homomorphism $\psi : \hat{\mathfrak{g}} \to \mathfrak{h}$, such that $\pi' \circ \psi = \pi$, that is, the following diagram commutes:*

$$\begin{tikzcd}
\hat{\mathfrak{g}} \arrow[d, dashed, "\psi"'] \arrow[dr, "\pi"] & \\
\mathfrak{h} \arrow[r, "\pi'"'] & \mathfrak{g}
\end{tikzcd}$$

Theorem 3.3.1. *If a u.c.e exists, then it is unique up to isomorphism.*

Proof: Suppose that $\pi_1 : \hat{\mathfrak{g}}_1 \to \mathfrak{g}$ and $\pi_2 : \hat{\mathfrak{g}}_2 \to \mathfrak{g}$ are universal central extensions of $\mathfrak{g}$. Then by the definition of u.c.e there are unique homomorphisms $\psi : \hat{\mathfrak{g}}_1 \to \hat{\mathfrak{g}}_2$ and $\psi' : \hat{\mathfrak{g}}_2 \to \hat{\mathfrak{g}}_1$ such that $\pi_1 = \pi_2 \circ \psi$ and $\pi_2 = \pi_1 \circ \psi'$. This gives us $\pi_1 = \pi_1 \circ \psi' \circ \psi$ and $\pi_2 = \pi_2 \circ \psi \circ \psi'$ whence by uniqueness applied to $\pi_1 : \hat{\mathfrak{g}}_1 \to \mathfrak{g}$ and $\pi_2 : \hat{\mathfrak{g}}_2 \to \mathfrak{g}$,

$\psi' \circ \psi = 1_{\hat{\mathfrak{g}}_1}$ and $\psi \circ \psi' = 1_{\hat{\mathfrak{g}}_2}$. We conclude that ψ is an isomorphism and $\hat{\mathfrak{g}}_1 \cong \hat{\mathfrak{g}}_2$. ∎

Definition 3.3.2. *A Lie algebra $\mathfrak{g}$ is said to be **perfect** if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

Lemma 3.3.2. *Let $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ and $\pi' : \mathfrak{h} \to \mathfrak{g}$ be central extensions of $\mathfrak{g}$. If $\tilde{\mathfrak{g}}$ is perfect then there exists at most one homomorphism $\psi : \tilde{\mathfrak{g}} \to \mathfrak{h}$ such that $\pi' \circ \psi = \pi$.*

Proof: Let $\psi, \psi' : \tilde{\mathfrak{g}} \to \mathfrak{h}$ be homomorphisms such that $\pi' \circ \psi = \pi$ and $\pi' \circ \psi' = \pi$.

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[d, "\psi"] \arrow[d, bend right, "\psi'"'] \arrow[dr, "\pi"] & \\ \mathfrak{h} \arrow[r, "\pi'"'] & \mathfrak{g} \end{tikzcd}$$

Letting $z, w \in \tilde{\mathfrak{g}}$ we have that

$$\pi(z) = \pi'(\psi(z)) = \pi'(\psi'(z))$$
$$\pi(w) = \pi'(\psi(w)) = \pi'(\psi'(w))$$

and therefore elements $\psi'(z) - \psi(z)$ and $\psi(w) - \psi'(w)$ are in $\ker \pi' \subset \mathfrak{z}(\mathfrak{h})$. This gives us

$$\begin{aligned} \psi([z,w]_{\tilde{\mathfrak{g}}}) &= [\psi(z), \psi(w)]_{\mathfrak{h}} \\ &= [\psi(z), \psi'(w)]_{\mathfrak{h}} + [\psi(z), \psi(w) - \psi'(w)]_{\mathfrak{h}} \\ &= [\psi(z), \psi'(w)]_{\mathfrak{h}} \\ &= [\psi(z), \psi'(w)]_{\mathfrak{h}} + [\psi'(z) - \psi(z), \psi'(w)]_{\mathfrak{h}} \\ &= [\psi'(z), \psi'(w)]_{\mathfrak{h}} \\ &= \psi'([z,w]_{\tilde{\mathfrak{g}}}). \end{aligned}$$

Thus, ψ and ψ' are the same on $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$, hence on $\tilde{\mathfrak{g}}$. ∎

The following theorem and its proof is found in [13, Proposition 2, Section 1.9].

Theorem 3.3.3. *A u.c.e. of a Lie algebra* $\mathfrak{g}$ *exists if and only if* $\mathfrak{g}$ *is perfect.*

Proof: Assume that $\mathfrak{g}$ has a u.c.e. given by $\pi : \hat{\mathfrak{g}} \to \mathfrak{g}$ and set $\phi : \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to be the canonical projection map. Define the map $\pi \oplus \phi : \mathfrak{g} \to (\mathfrak{g} \oplus (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]))$ by $(\pi \oplus \phi)(x) = (\pi(x), \phi(x))$ and $\pi \oplus 0 : \mathfrak{g} \to (\mathfrak{g} \oplus (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]))$ by $(\pi \oplus 0)(x) = (\pi(x), 0)$. Let $p_1 : (\mathfrak{g} \oplus (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])) \to \mathfrak{g}$ be the projection onto the first summand.

$$\begin{tikzcd}
\hat{\mathfrak{g}} \arrow[d, bend right, "\pi \oplus \phi"'] \arrow[d, bend left, "\pi \oplus 0"] \arrow[dr, bend left, "\pi"] & \\
(\mathfrak{g} \oplus (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])) \arrow[r, "p_1"'] & \mathfrak{g}
\end{tikzcd}$$

Now notice that $\ker(p_1) = 0 \oplus (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]))$ and $\mathfrak{z}(\mathfrak{g} \oplus (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])) = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{z}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) = \mathfrak{z}(\mathfrak{g}) \oplus (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ and thus we have $\ker(p_1) \subset \mathfrak{z}(\mathfrak{g} \oplus (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]))$, making p_1 a central extension of $\mathfrak{g}$.

Since $p_1(\mathfrak{g} \oplus \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) = p_1(\mathfrak{g} \oplus 0)$, the above diagram commutes. By the uniqueness property in the definition of u.c.e. we conclude that $\pi \oplus \phi = \pi \oplus 0$, which gives us that $\phi = 0$, whence $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

For the converse, assume that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Consider the vector space given by the tensor product $T^2(\mathfrak{g}) := \mathfrak{g} \otimes_k \mathfrak{g}$. Letting J be the subspace of $T^2(\mathfrak{g})$ generated by elements of the form $x \otimes x$, define $\Lambda^2(\mathfrak{g}) := T^2(\mathfrak{g})/J$. If $x, y \in \mathfrak{g}$, we denote by $x \wedge y$ the equivalence class of $x \otimes y$ in $\Lambda^2(\mathfrak{g})$. Note that $x \wedge x = 0$ and $x \wedge y = -y \wedge x$. Further, we define $\mathfrak{l}$ to be the subspace of $\Lambda^2(\mathfrak{g})$ generated by elements of the form

$$x \wedge [y, z] + y \wedge [z, x] + z \wedge [x, y], \qquad x, y, z \in \mathfrak{g}$$

and define $\mathfrak{c} = \Lambda^2(\mathfrak{g})/\mathfrak{l}$ and let $x \wedge_{\mathfrak{c}} y$ denote the equivalence class of $x \wedge y$ in $\mathfrak{c}$.

Now define the map $\phi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{c}$ by $\phi(x, y) = x \wedge_{\mathfrak{c}} y$ and note that

$$\begin{aligned} &\phi(x, x) = \phi(0) = 0 \\ &\phi(x, [y, z]) + \phi(y, [z, x]) + \phi(z, [x, y]) = 0 \ \text{ for all } \ x, y, z \in \mathfrak{g}. \end{aligned} \tag{3.3.1}$$

Namely, ϕ is a 2-cocycle on $\mathfrak{g}$ with coefficients in $\mathfrak{c}$. Now by Theorem 3.2.1 we have a central extension $\pi : \hat{\mathfrak{g}} \to \mathfrak{g}$ where $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{c}$ and Lie bracket

$$[x + a, y + b]_{\hat{\mathfrak{g}}} := [x, y] + \phi(x, y).$$

Let $\pi' : \mathfrak{h} \to \mathfrak{g}$ be another central extension and let $\mathfrak{c}' = \ker \pi'$. As in Example 3.2.1, we write $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{c}'$ and associate with it a 2-cocycle $f : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{c}'$ and corresponding Lie bracket $[x + a, y + b]_{\mathfrak{h}} := [x, y] + f(x, y)$. Now because f is a 2-cocycle, we get an induced map $f : \Lambda^2(\mathfrak{g}) \to \mathfrak{c}'$ that vanishes on $\mathfrak{l}$. We define a linear map $\psi_0 : \mathfrak{c} \to \mathfrak{c}'$ by $\psi_0(\phi(x, y)) = f(x \wedge y)$. Note that the surjectivity of ϕ guarantees that the domain of ψ_0 is indeed $\mathfrak{c}$.

Now ψ_0 is well defined because if $\sum \phi(x_i, y_i) = \sum \phi(x_i', y_i')$, then

$$\begin{aligned}&\sum(x_i \wedge_{\mathfrak{c}} y_i) - \sum(x_i' \wedge_{\mathfrak{c}} y_i') = \\ &\sum \big((x_i'' \wedge_{\mathfrak{c}} [y_i'', z_i'']) + (y_i'' \wedge_{\mathfrak{c}} [z_i'', x_i'']) + (z_i'' \wedge_{\mathfrak{c}} [x_i'', y_i''])\big) + \sum(x_i''' \wedge_{\mathfrak{c}} x_i''').\end{aligned}$$

Then

$$\begin{aligned}&f\big(\sum(x_i \wedge y_i) - \sum(x_i' \wedge y_i')\big) = \\ &f\big(\sum \big(x_i'' \wedge [y_i'', z_i''] + y_i'' \wedge [z_i'', x_i''] + z_i'' \wedge [x_i'', y_i'']\big) + \sum x_i''' \wedge x_i'''\big) = 0\end{aligned}$$

since f vanishes on $\mathfrak{l}$. Thus $f\big(\sum(x_i \wedge y_i)\big) = f\big(\sum(x_i' \wedge y_i')\big)$ and ψ_0 is well defined.

We now prove that $\psi : \hat{\mathfrak{g}} \to \mathfrak{h}$ given by $\psi(x+c) = x+\psi_0(c)$ is a Lie homomorphism satisfying $\pi = \pi' \circ \psi$. First we have that $\pi(x + c) = x = \pi'(x + \psi_0(c)) = \pi'(\psi(x + c))$. Next we have that

$$\begin{aligned}\psi([x + c, y + d]_{\hat{\mathfrak{g}}}) &= \psi([x, y]_{\hat{\mathfrak{g}}}) \\ &= \psi([x, y]_{\mathfrak{g}} + \phi(x, y)) \\ &= [x, y]_{\mathfrak{g}} + \psi_0(\phi(x, y)) \\ &= [x, y]_{\mathfrak{g}} + f(x, y) \\ &= [x, y]_{\mathfrak{h}} \\ &= [x + \psi_0(c), y + \psi_0(d)]_{\mathfrak{h}} \\ &= [\psi(x + c), \psi(y + d)]_{\mathfrak{h}}\end{aligned}$$

and hence we have that ψ is a Lie algebra homomorphism.

Now $\hat{\mathfrak{g}}$ need not be perfect. We define $\check{\mathfrak{g}} = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]_{\hat{\mathfrak{g}}}$, the derived algebra of $\hat{\mathfrak{g}}$. Since $\mathfrak{g}$ is perfect, we have that $\pi(\check{\mathfrak{g}}) = \mathfrak{g}$ and therefore $\check{\mathfrak{g}} + \mathfrak{c} = \hat{\mathfrak{g}}$ and hence we have

$$\check{\mathfrak{g}} = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]_{\hat{\mathfrak{g}}} = [\check{\mathfrak{g}} + \mathfrak{c}, \check{\mathfrak{g}} + \mathfrak{c}]_{\hat{\mathfrak{g}}} = [\check{\mathfrak{g}}, \check{\mathfrak{g}}]_{\hat{\mathfrak{g}}}$$

and so $\check{\mathfrak{g}}$ is perfect. Now let $\check{\mathfrak{c}} = \check{\mathfrak{g}} \cap \mathfrak{c}$ and define the central extension

$$\check{\pi} : \check{\mathfrak{g}} \to \mathfrak{g}$$

where $\check{\pi} = \pi|_{\check{\mathfrak{g}}}$ and which has $\ker \check{\pi} = \check{\mathfrak{c}}$. We contend that this central extension is universal. Indeed, given any other central extension $\pi' : \mathfrak{h} \to \mathfrak{g}$ of $\mathfrak{g}$ we know there exists $\psi : \hat{\mathfrak{g}} \to \mathfrak{h}$ such that $\pi = \pi' \circ \psi$. Defining $\check{\psi}$ and $\check{\psi}_0$ to be the restrictions of ψ and ψ_0 to $\check{\mathfrak{g}}$ and $\check{\mathfrak{c}}$, respectively, we get a homomorphism from $\check{\mathfrak{g}}$ to $\mathfrak{h}$, and together with the fact that $\check{\mathfrak{g}}$ is perfect we apply Lemma 3.3.2 to conclude that this homomorphism is unique, whence $\check{\pi} : \check{\mathfrak{g}} \to \mathfrak{g}$ is universal. ∎

Corollary 3.3.4. *The universal central extension of a perfect Lie algebra is perfect.*

Definition 3.3.3. *A perfect Lie algebra $\mathfrak{g}$ is called **centrally closed** if the u.c.e. of $\mathfrak{g}$ is equal to $\mathfrak{g}$.*

Definition 3.3.4. *A nonabelian Lie algebra $\mathfrak{g}$ satisfying $[\mathfrak{g}, \mathfrak{g}] \neq 0$ is called **simple** if it has no ideals except $\{0\}$ and itself. A Lie algebra is **semisimple** if it is a direct sum of simple ideals.*

3.4 Examples

Example 3.4.1. The Lie algebra $\mathfrak{g} \otimes_k k[t, t^{-1}]$ is called a *loop algebra*. For finite-dimensional semisimple Lie algebras $\mathfrak{g}$, the universal central extension is given in [10].

In [18] the u.c.e for $\mathfrak{g} \otimes_k k[t, t^{-1}]$ is given where $\mathfrak{g}$ is assumed to be one of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, $\mathfrak{so}_\infty$ and $\mathfrak{gl}_\infty$. It is shown that the universal central extension of the direct limit is the direct limit of the universal central extension [15].

We now construct an explicit universal central extension. This topic is presented in [6, Section 9.4].

Example 3.4.2.

Let $R = k[t, t^{-1}]$. The loop algebra $\mathfrak{g}(R) := \mathfrak{g} \otimes_k R$ has Lie bracket

$$[x \otimes f(t), y \otimes g(t)] := [x, y] \otimes f(t)g(t).$$

Consider $\mathfrak{g}(R)$ where $\mathfrak{g}$ is semisimple. We aim to construct a central extension of $\mathfrak{g}(R)$ as in Theorem 3.2.1 using a 2-cocycle given by Theorem 3.2.2. For this we will need a symmetric, invariant bilinear form on $\mathfrak{g}(R)$ and a derivation on $\mathfrak{g}(R)$.

1. For this example, see [6, Section 9.4]. The Killing form on $\mathfrak{g}$ is given by

   $$\kappa(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to k, \quad \text{where } \kappa(x, y) := \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y).$$

   Now κ is a symmetric, invariant, bilinear form. Moreover, since we have that $\mathfrak{g}$ is *semisimple*, we may conclude that κ is non-degenerate. We may extend it to a non-degenerate, invariant, bilinear form κ_R on $\mathfrak{g}(R)$ by

   $$\kappa_R(x \otimes f(t), y \otimes g(t)) = \kappa(x, y)\mathrm{const}(f(t)g(t)),$$

   where $\mathrm{const}(f(t)g(t))$ denotes the constant term of the polynomial. Thus we have the desired form.

2. Now we will construct a derivation on $\mathfrak{g}(R)$. Take $t\frac{d}{dt}$ to be the derivation on R and define a bilinear form d on $\mathfrak{g}(R)$ by

   $$d(x \otimes f(t)) = x \otimes d(f(t)) = x \otimes t\frac{d}{dt}(f(t))$$

   It is not hard to check that d is a derivation on $\mathfrak{g}(R)$.

3. Now using Theorem 3.2.2 we combine κ_R and d to create a 2-cocycle on $\mathfrak{g}(R)$ with coefficients in k by

$$\phi(x \otimes f(t), y \otimes g(t)) = \kappa_R(d(x \otimes f(t)), y \otimes g(t))$$

and apply Theorem 3.2.1 to get a central extension of $\mathfrak{g}(R)$ given by $\widehat{\mathfrak{g}(R)} = \mathfrak{g}(R) \oplus kc$, (where kc denotes a 1-dimensional vector space over k spanned by c) with Lie bracket

$$[x \otimes f(t) + ac, y \otimes g(t) + bc] := [x, y] \otimes f(t)g(t) + \phi(x \otimes f(t), y \otimes g(t))c.$$

with $x, y \in \mathfrak{g}$, $f(t), g(t) \in R$ and $a, b \in k$.

4. Moreover, it is a well known result shown in [7] that $\pi : \widehat{\mathfrak{g}(R)} \to \mathfrak{g}(R)$ is indeed the *universal central extension* of the loop algebra $\mathfrak{g}(R)$.

The following section is optional. We demonstrate how one can construct an explicit u.c.e. of a specific Lie algebra. The Main Theorem 5.5.1 of this Thesis generalizes to u.c.e. and this section may be useful to those interested in understanding the explicit structure of a u.c.e.

3.5 N-point Affine Algebra

Let $\mathfrak{g}$ be a *finite-dimensional, semisimple* Lie algebra over an algebraically closed field k of characteristic zero. In this section we will describe the u.c.e of the Lie algebra $\mathfrak{g} \otimes_k R$, where $R = k[z, (z-a_1)^{-1}, \ldots, (z-a_n)^{-1}]$ for distinct nonzero $a_i \in k$. This ring may be regarded as the ring of regular functions on the projective line $\mathbb{P}^1(k)$ which have poles at $z \in \{\infty, a_1, \ldots, a_n\}$, and this gives a map algebra $\mathfrak{g} \otimes_k R$ sometimes known as the **N-point affine algebra**, where $N = n + 1$. The universal central extension $\widehat{\mathfrak{g} \otimes_k R}$ of $\mathfrak{g} \otimes_k R$ is considered in [2], and is described explicitly with Lie bracket relations and also by commutation relations. We first summarise the work of [11].

Definition 3.5.1. *Let R be any commutative algebra over k. Let $\{r_i\}$ be any basis for R over k, and let F be a free module over R with a basis $\{\tilde{d}r_i\}$ such that the set $\{\tilde{d}r_i\}$ is indexed by the same set of indices as the set $\{r_i\}$. Let $\tilde{d}: R \to F$ be the k-linear map that sends $\Sigma a_i r_i \mapsto \Sigma a_i \tilde{d}r_i$, for $a_i \in k$. We may consider F as a bi-module by setting $r\tilde{d}(s) = \sum a_i(r \cdot \tilde{d}r_i) = \tilde{d}(s)r$ for $r, s \in R$. Let K be the R-submodule of F generated by elements of the form*

$$\tilde{d}(rs) - \tilde{d}(r)s - r\tilde{d}(s), \qquad \text{where } r, s \in R.$$

*Define $\Omega_R := F/K$ and the **differential map** is defined as the canonical map $d: R \to \Omega_R$ by $d: r \to \tilde{d}r + K$. The pair (Ω_R, d) is known as the **module of differentials**.*

Theorem 3.5.1. *Up to isomorphism (Ω_R, d) is characterized by the property that for every R-module M and every k-linear derivation $D: R \to M$ there is a unique R-module map $f: \Omega_R \to M$ such that the diagram*

$$\begin{tikzcd} R \arrow[d, "d"'] \arrow[dr, "D"] & \\ \Omega_R \arrow[r, dashed, "f"'] & M \end{tikzcd}$$

commutes. In this way $\mathrm{Der}_k(R, M) \cong \mathrm{Hom}_R(\Omega_R, M)$.

Proof: Let M be an R-module and suppose that $D: R \to M$ is a k-linear derivation. Let $\tilde{d}$ be as before and define $D': F \to M$ to be an R-linear map such that $D'(\tilde{d}r_i) = D(r_i)$. Note that D' is well defined and makes the following diagram commute:

$$\begin{tikzcd} R \arrow[d, "\tilde{d}"'] \arrow[dr, "D"] & \\ F \arrow[r, dashed, "D'"'] & M \end{tikzcd}$$

Now suppose that $r \in R$, such that $\tilde{d}(r) \in K$. Then $\tilde{d}(r) = \sum_i(\tilde{d}(r_i s_i) - \tilde{d}(r_i)s_i - r_i\tilde{d}(s_i))$ and thus $D(r) = D'(\tilde{d}(r)) = \sum_i(D(r_i s_i) - D(r_i)s_i - r_i D(s_i)) = 0$ since D is a derivation. Thus we have shown that the map D' factors through K and thus inducing the desired map $f : \Omega_R \to M$ by $f\big(\sum a_i(dr_i)\big) = \sum a_i(D(r_i))$. ∎

Let dR denote the image of the map $d : R \to \Omega_R$, and consider $\overline{d(r)}$ to be the equivalence class of $d(r)$ in Ω_R/dR.

Consider the vector space

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes_k R \oplus \Omega_R/dR$$

and define a bilinear operation

$$[x \otimes r, y \otimes s]' = [x, y] \otimes rs + (\overline{(dr)s})\kappa(x, y)$$

where κ is the Killing form on $\mathfrak{g}$.

This makes $\hat{\mathfrak{g}}$ into a Lie algebra.

Theorem 3.5.2. *$\hat{\mathfrak{g}}$ is the u.c.e of $\mathfrak{g}$.*

Proof: See [14] and [11]. ∎

Now that we have summarized the work of [11], we can apply it to the specific case of the N-point affine algebra. We will start with a basis of R and construct Ω_R and Ω_R/dR explicitly.

Lemma 3.5.3. *Let $R = k[z, (z - a_1)^{-1}, \dots, (z - a_n)^{-1}]$ where $a_1, \dots, a_n$ are distinct. Then a k-basis of R consists of the elements*

$$\{1\} \cup \{z^k, (z - a_i)^{-k} : k \in \mathbb{Z}^+, 1 \leq i \leq n\}$$

.

Proof: This basis spans R by the theory of partial fractions. That is, every element $f(z) \in R$ can be expressed as

$$\frac{h(z)}{(z-a_1)^{m_1}\dots(z-a_1)^{m_n}} = p(z) + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\frac{b_{ij}}{(z-a_i)^j}.$$

This shows that the given set spans R. Thus we only need to show that the elements in the given set are linearly independent. Suppose that for $m_0, \dots, m_n \in \mathbb{Z}^+ \cup \{0\}$ and $b_i^j \in k$ we have

$$g(z) := b_0^0 + b_1^0 z + b_2^0 z^2 + b_3^0 z^3 + \cdots + b_{m_0}^0 z^{m_0} + \sum_{j=1}^{n}(b_1^j(z-a_j)^{-1} + \cdots + b_{m_j}^j(z-a_j)^{-m_j}) = 0.$$

First we take the case where $m_1 = m_2 = \cdots = m_n = 0$. Then $g(z)$ is a polynomial, so all its coefficients $b_0^0, b_1^0, b_2^0, \cdots, b_{m_0}^0$ must be zero. Otherwise assume that $m_j \geq 1$ and $b_{m_j}^j \neq 0$ for some j. In that case we consider the new polynomial $g(z)(z-a_j)^{m_j} = 0$ and notice that in this polynomial the term $(z-a_j)$ never occurs with a negative exponent and every other term whose coefficient is not $b_{m_j}^j$ has a factor of $(z-a_j)$. By evaluating the equation $g(z)(z-a_j)^{m_j} = 0$ at $z = a_j$ the result is $b_{m_j}^j = 0$, a contradiction. Thus all of the coefficients of $g(z)$ are zero and we have shown that the given set is indeed a basis. ∎

Let $B = \{f_i\}$ denote the k-basis of R in the preceding Lemma, where we index the elements of the basis with some indexing set $\mathcal{I}$, and as before we consider a free R-module with basis $\tilde{d}B = \{\tilde{d}f_i\}$. Define $\tilde{d} : R \to F$ and K and Ω_R as before.

Proposition 3.5.1. In Ω_R, the following elements are zero:

$$\tilde{d}(1), \tilde{d}(z^k) - kz^{k-1}\tilde{d}(z), \tilde{d}(z-a_i)^{-k} + k(z-a_i)^{-k-1}\tilde{d}(z-a_i),$$

where $k \geq 1, 1 \leq i \leq n$.

Remark 3.5.1. Note that there is no proof in [2]. We give a proof.

Proof: Firstly, we know that elements of the form $\tilde{d}(rs)-\tilde{d}(r)s-r\tilde{d}(s)$, where $r, s \in R$ are zero in Ω_R. Then letting $r = s = 1$, we get that $\tilde{d}(1\cdot 1)-\tilde{d}(1)1-1\tilde{d}(1) = -\tilde{d}(1)$ is zero.

To show that $\tilde{d}(z^k) - kz^{k-1}\tilde{d}z$ is zero for $k \geq 1$ we proceed by induction. For the base case $k = 1$ we let $r = z$ and $s = z^0 = 1$ to give us

$$\tilde{d}(zz^0) - \tilde{d}(z)z^0 - z\tilde{d}(z^0) = \tilde{d}(z) - \tilde{d}(z)\cdot 1 - z\cdot\tilde{d}(1) = \tilde{d}(z) - \tilde{d}(z)$$

so the result holds.

Now assume that the result holds for $k-1$, and let $r = z^{k-1}$, and $s = z$, then

$$\tilde{d}(z^{k-1}z)-\tilde{d}(z^{k-1})z-z^{k-1}\tilde{d}(z) = \tilde{d}(z^k)-(k-1)z^{k-1}\tilde{d}(z)-z^{k-1}\tilde{d}(z) = \tilde{d}(z^k)-kz^{k-1}\tilde{d}z,$$

as desired.

Fix i to satisfy $1 \leq i \leq n$. For $k = 1$ we let $r = (z-a_i)^{-1}$ and $s = (z-a_i)$ to give us

$$\begin{aligned}
&\tilde{d}((z-a_i)^{-1}(z-a_i)) - \tilde{d}((z-a_i)^{-1})(z-a_i) - (z-a_i)^{-1}\tilde{d}((z-a_i)) \\
&= \tilde{d}(1) - \tilde{d}((z-a_i)^{-1})(z-a_i) - (z-a_i)^{-1}\tilde{d}((z-a_i)) \\
&= -\tilde{d}((z-a_i)^{-1})(z-a_i) - (z-a_i)^{-1}\tilde{d}((z-a_i))
\end{aligned}$$

which gives us that $-\tilde{d}((z-a_i)^{-1})(z-a_i)-(z-a_i)^{-1}\tilde{d}((z-a_i))$ is zero, and multiplying this expression by $-(z-a_i)^{-1}$ gives us that $\tilde{d}((z-a_i)^{-1}) + (z-a_i)^{-2}\tilde{d}((z-a_i))$ is zero as desired.

Now assume that the result holds for $k-1$, and let $r = (z-a_i)^{-k+1}$ and $s = (z-a_i)^{-1}$, then

$$\begin{aligned}
&\tilde{d}((z-a_i)^{-k+1}(z-a_i)^{-1}) - \tilde{d}((z-a_i)^{-k+1})(z-a_i)^{-1} - (z-a_i)^{-k+1}\tilde{d}((z-a_i)^{-1}) \\
&= \tilde{d}((z-a_i)^{-k}) + (k-1)(z-a_i)^{-k-1}\tilde{d}(z-a_i) + (z-a_i)^{-k-1}\tilde{d}(z-a_i) \\
&= \tilde{d}((z-a_i)^{-k}) + k(z-a_i)^{-k-1}\tilde{d}(z-a_i)
\end{aligned}$$

Thus we have shown that $\tilde{d}(z-a_i)^{-k} + k(z-a_i)^{-k-1}\tilde{d}(z-a_i)$ are zero for $k \geq 1, 1 \leq i \leq n$. ∎

Now we can define the differential map as the canonical map $d : R \to \Omega_R$ by $d : r \to \tilde{d}r + K$. Take the basis B of R, and the basis $\tilde{d}B$ of F as before. The equivalence classes df_i for each $f_i \in B$ can always be written as a linear combination of the elements $z^k dz$ where $k \geq 0$, and $(z-a_i)^{-k}dz$ where $k \geq 1, 1 \leq i \leq n$. Thus we have shown the following proposition:

Proposition 3.5.2. Let $R = k[z,(z-a_1)^{-1},\ldots,(z-a_n)^{-1}]$. Then a k-basis of Ω_R consists of the elements

$$z^k dz \text{ where } k \geq 0, \quad (z-a_i)^{-k}dz \text{ where } k \geq 1, 1 \leq i \leq n.$$

Now note that of the elements in the basis of Ω_R the elements $z^k dz$ can be expressed as $d(\frac{1}{k+1}z^{k+1})$ where $k \geq 0$ and the elements $(z-a_i)^{-k}dz$ for $1 \leq i \leq n$ can be expressed as $d(\frac{1}{-k+1}(z-a_i)^{-k+1})$, $k \neq 1$. Thus we have shown that all elements of the form $d(\frac{1}{k+1}z^{k+1})$ for $k \geq 0$ are in the same coset (in fact, these elements are in the zero coset dR) of Ω_R/dR, and the elements $(z-a_i)^{-k}dz$, $k \neq 1$, for fixed i are in the same coset (i.e. the zero coset) of Ω_R/dR. To summarise, we have shown:

Theorem 3.5.4. *The k-basis of Ω_R/dR consists of the cosets of the elements*

$$(z-a_i)^{-1}dz$$

for $1 \leq i \leq n$. In particular the dimension is n.

Now using Theorem 3.5.2 we may conclude that the u.c.e. of $\mathfrak{g} \otimes_k R$ is

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes_k R \oplus \Omega_R/dR,$$

where Ω_R/dR is n-dimensional, and the Lie bracket of $\hat{\mathfrak{g}}$ can be given by

$$[x \otimes r, y \otimes s]' = [x,y] \otimes rs + (\overline{(dr)s})\kappa(x,y)$$

where κ is the Killing form on $\mathfrak{g}$.

Chapter 4

Root Decompositions and Maximal Toral Subalgebras

In Chapter 2 we defined the Lie algebras $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$ over a field k of characteristic zero. This chapter will explore the root decompositions of these Lie algebras and will define their *maximal toral subalgebras*. A conjugacy theorem about the maximal toral subalgebras of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$ is given in [21], and similar conjugacy results for $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}(R)$, and $\mathfrak{so}_\infty(R)$ where $R = k[t, t^{-1}]$ are given in [18]. The results in this chapter will be used in Chapter 5 to generalize these conjugacy theorems for $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}(R)$, and $\mathfrak{so}_\infty(R)$ for a more general ring R. Assume that all Lie algebras are vector spaces over k. We start with some definitions. Further study of some topics in this chapter may be found in [21].

4.1 Definitions

Definition 4.1.1. *Let $\mathfrak{h}$ be an abelian subalgebra of a Lie algebra $\mathfrak{g}$ over k.*

1. *The subalgebra $\mathfrak{h}$ is called **maximal abelian** if whenever $\mathfrak{h} \subset \mathfrak{h}'$ for another abelian subalgebra $\mathfrak{h}'$, then $\mathfrak{h}' = \mathfrak{h}$.*

2. *A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a* ***splitting Cartan subalgebra*** *if it is maximal abelian and there exists a basis $\mathcal{C}$ of $\mathfrak{g}$ such that $[\mathrm{ad}_h]_{\mathcal{C}}$ is diagonal for all $h \in \mathfrak{h}$. Namely, all non-diagonal entries of $[\mathrm{ad}_h]_{\mathcal{C}}$ are zero.*

3. *A nonzero subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a* ***maximal toral subalgebra*** *if the following two conditions hold:*

   (a) There exists a basis $\mathcal{C}$ of $\mathfrak{g}$ such that $[\mathrm{ad}_h]_{\mathcal{C}}$ is diagonal for all $h \in \mathfrak{h}$

   (b) If $\mathfrak{h} \subset \mathfrak{h}'$ for another subalgebra $\mathfrak{h}'$ that satisfies property (a), then $\mathfrak{h}' = \mathfrak{h}$.

The following Lemma is standard:

Lemma 4.1.1. *Every maximal toral subalgebra is abelian.*

Proof: Suppose that $\mathfrak{h}$ is a maximal toral subalgebra of a Lie algebra $\mathfrak{g}$ over k. Let $x, y \in \mathfrak{h}$, such that $[x, y] \neq 0$. Write $y = y_1 + \cdots + y_k$ such that each y_i is an ad_x-eigenvector with eigenvalue λ_i, and the λ_i's are mutually distinct. Now note that each y_i belongs to $\mathfrak{h}$ because

$$\begin{aligned} y &= y_1 + \cdots + y_k \in \mathfrak{h} \\ \mathrm{ad}_x(y) &= \lambda_1 y_1 + \cdots + \lambda_k y_k \in \mathfrak{h} \\ &\vdots \\ \mathrm{ad}_x^{k-1}(y) &= \lambda_1^{k-1} y_1 + \cdots + \lambda_k^{k-1} y_k \in \mathfrak{h} \end{aligned}$$

Thus if we define the $k \times k$ matrix $A = (a_{ij}) = (\lambda_j^{i-1})$, sometimes called the *Vandermonde matrix*, we can conclude that it has a nonzero determinant

$$\det A = \prod_{1 \leq i < j \leq k} (\lambda_j - \lambda_i)$$

so it is invertible. If we let $\overline{y}$ be the column vector $(y_1, \ldots, y_k)^t$ we get that $A\overline{y} := \overline{z}$ is a column vector such that every entry is in $\mathfrak{h}$. We conclude that $\overline{y} = A^{-1}\overline{z}$ is a column vector where every entry y_i can be expressed as a linear combination of elements of $\mathfrak{h}$, so each $y_i \in \mathfrak{h}$.

Next we choose $i \in \{1, \dots, k\}$ such that $\lambda_i \neq 0$. Such an i exists because $[x, y] \neq 0$. Thus $[x, y_i] = \lambda_i y_i$. On the other hand $[\mathrm{ad}_{y_i}]_{\mathcal{C}}$ is diagonal too. Hence we may write $x = \sum_j x_j$ with linearly independent eigenvectors x_j of ad_{y_i}, say $[y_i, x_j] = a_j x_j$ for a suitable $a_j \in k$. But then $-\lambda_i y_i = [y_i, x] = \sum_j a_j x_j$, and applying ad_{y_i} once more gives us $0 = \sum_j a_j^2 x_j$. This implies $a_j = 0$ for all j, hence $\lambda_i = 0$, which is a contradiction. ∎

Note that an alternate proof of the above Lemma can be found in [9, Lemma 8.1].

Remark 4.1.1. We have just shown that any maximal toral subalgebra is abelian. A priori it is not immediately obvious that a maximal toral subalgebra is also maximal abelian. By an argument similar to the proof of the main theorem of this thesis, namely Theorem 5.5.1, every maximal toral subalgebra of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$ is also maximal abelian. In this thesis, we will focus on maximal toral subalgebras.

Definition 4.1.2. *Let $\mathfrak{g}$ be a Lie algebra over k. We define $\mathfrak{g}^* := \hom_k(\mathfrak{g}, k)$.*

Definition 4.1.3. *Suppose that $\mathfrak{g}$ contains a maximal toral subalgebra $\mathfrak{h}$. Then the pair $(\mathfrak{g}, \mathfrak{h})$, or simply $\mathfrak{g}$, is called a* ***split Lie algebra****.*

Let $(\mathfrak{g}, \mathfrak{h})$ be a split Lie algebra and let $\alpha \in \mathfrak{h}^*$, define

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x, \ \forall\ h \in \mathfrak{h}\}$$

The set of all $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}^\alpha \neq \{0\}$ is called the ***root system*** *corresponding to $\mathfrak{h}$ and is denoted by $\Delta(\mathfrak{g}, \mathfrak{h})$. Elements of $\Delta(\mathfrak{g}, \mathfrak{h})$ are called* ***roots****. Any nonzero $\mathfrak{g}^\alpha$ is called a* ***root space****, and any nonzero element $x \in \mathfrak{g}^\alpha$ is called a* ***root vector****.*

Remark 4.1.2. The definition of maximal toral subalgebra says that if $(\mathfrak{g}, \mathfrak{h})$ is a split Lie algebra, then we can can take a basis $\mathcal{C} = \{e_1, e_2, \dots\}$ of $\mathfrak{g}$ such that $[\mathrm{ad}_h]_{\mathcal{C}}$ is diagonal for each $h \in \mathfrak{h}$. Thus we induce functionals $\alpha_i \in \mathfrak{h}^*$ such that

$\mathrm{ad}_h(e_i) = \alpha_i(h)e_i$ for each $h \in \mathfrak{h}$. Take $x \in \mathfrak{g}$. If $x = c_1 e_{p1} + \quad + c_l e_{pl}$ for some $c_1, \ldots, c_l \in k$ then

$$[h, x] = \mathrm{ad}_h(x) = c_1\alpha_{p1}(h)e_{p1} + \cdots + c_l\alpha_{pl}(h)e_{pl}.$$

Noting that $\mathfrak{h} = \mathrm{span}\{v \in \mathcal{C} : \mathrm{ad}_h(v) = 0 \text{ for all } h \in \mathfrak{h}\}$, giving us

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha_i \in \Delta(\mathfrak{g},\mathfrak{h})} \mathfrak{g}^{\alpha_i}.$$

Now we will give explicit examples of maximal toral subalgebras for $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$.

Example 4.1.1. Let $\mathfrak{g} = \mathfrak{sl}_\infty$. Define

$$\mathfrak{h}_d = \mathrm{span}\{E_{j,j} - E_{k,k}, j \neq k\}$$

where E_{ij} denotes the matrix with an entry of 1 in the ij-position and zero entries elsewhere. We claim that $\mathfrak{h}_d$ is a maximal toral subalgebra of $\mathfrak{g}$. Indeed, it consists of diagonal matrices so condition (a) is immediate, and condition (b) is true because if $y \notin \mathfrak{h}_d$ then y is not diagonal, so in particular it will not commute with one of the elements of the form $E_{j,j} - E_{k,k}$ for appropriate j and k. This violates Lemma 4.1.1.

Let $\mathfrak{g}$ be one of $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$. The Lie subalgebra $\mathfrak{h}_d = \mathrm{span}\{E_{j,j} - E_{-j,-j}, j \in J\}$ of diagonal matrices is a maximal toral subalgebra of $\mathfrak{g}$. An analogous argument shows that this is a maximal toral subalgebra.

We will now look at the structure theory of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$ which helps us to understand the conjugacy results of their maximal toral subalgebras.

4.2 The Root System A_J of $\mathfrak{sl}_\infty$

As we have seen in Example 4.1.1 the subalgebra $\mathfrak{h}_d = \mathrm{span}\{E_{j,j} - E_{k,k}, j \neq k\}$ is a maximal toral subalgebra of $\mathfrak{sl}_\infty$. In this section we will now look at the root system

$A_J := \Delta(\mathfrak{sl}_\infty, \mathfrak{h}_d)$. We will see that

$$A_J = \{\varepsilon_j - \varepsilon_k : j \neq k\}$$

where $\varepsilon_j : \mathfrak{h}_d \to k$ is given by $\varepsilon_j(\sum x_j E_{jj}) = x_j$. Indeed the root space corresponding to any $\varepsilon_j - \varepsilon_k$ for $j \neq k$ contains a nonzero element E_{jk} since

$$\begin{aligned} \mathrm{ad}_h(E_{jk}) &= [h, E_{jk}] = hE_{jk} - E_{jk}h \\ &= (h_j - h_k)(E_{jk}) \\ &= (\varepsilon_j - \varepsilon_k)(h)(E_{jk}) \end{aligned}$$

for any $h = (h_1, h_2, \dots) \in \mathfrak{h}_d$. Moreover this root space is one dimensional. In fact, if x belongs to this root space then $[h, x] = (\varepsilon_j - \varepsilon_k)(h)x$ for all $h \in \mathfrak{h}_d$ and we can conclude that $x \in \mathrm{span}_k\{E_{jk}\}$.

Since we know that $\mathfrak{sl}_\infty$ is spanned by elements of the form

$$\{E_{jj} - E_{kk}, E_{jk} : j \neq k\},$$

and $\mathfrak{sl}_\infty = \mathfrak{h}_d \oplus \sum_{\alpha_i \in \Delta(\mathfrak{g},\mathfrak{h})} {\mathfrak{sl}_\infty}^{\alpha_i}$, then we can conclude that there are no other root spaces because

$$\mathfrak{h}_d \oplus \sum_{\varepsilon_j - \varepsilon_k, j \neq k} {\mathfrak{sl}_\infty}^{\varepsilon_j - \varepsilon_k} = \mathrm{span}\{E_{jj} - E_{kk} : j \neq k\} \oplus \mathrm{span}\{E_{jk} : j \neq k\} = \mathfrak{sl}_\infty\,.$$

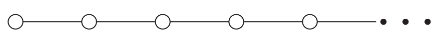

Figure 4.1: Dynkin Diagram for A_J

4.3 The Root System C_J of $\mathfrak{sp}_\infty$

We have seen in Example 4.1.1 that $\mathfrak{h}_d = \mathrm{span}\{E_{j,j} - E_{-j,-j}, j \in J\}$ is a maximal toral subalgebra of $\mathfrak{sp}_\infty$. In this section we will look at the root system $C_J := \Delta(\mathfrak{sp}_\infty, \mathfrak{h}_d)$.

We claim that

$$C_J = \{\pm 2\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j \neq k\}$$

where $\varepsilon_j(E_{kk} - E_{-k,-k}) = \delta_{ij}$, the Kronecker delta.

It can be shown by a similar argument as in the case for A_J that the root space corresponding to $2\varepsilon_j$ is one-dimensional and is equal to $\operatorname{span}\{E_{j,-j}\}$ and the root space corresponding to $-2\varepsilon_j$ is equal to $\operatorname{span}\{E_{-j,j}\}$. For $\varepsilon_j + \varepsilon_k$ it can be shown that the corresponding root space is equal to $\operatorname{span}\{E_{j,-k} - E_{k,-j}\}$, and the root space corresponding to $-\varepsilon_j - \varepsilon_k$ is given by $\operatorname{span}\{E_{-j,k} - E_{-k,j}\}$. Likewise for $\varepsilon_j - \varepsilon_k$ it can be shown that the corresponding root space is $\operatorname{span}\{E_{j,k} - E_{-k,-j}\}$.

Similarly as before we can conclude that there are no other nonzero root spaces since the union of all of the spanning vectors of the root spaces, together with $\mathfrak{h}_d$ spans all of $\mathfrak{sp}_\infty$.

Furthermore, because all of the root spaces are one-dimensional, for each root $\alpha \in C_J$ we can find spanning vectors $x \in \mathfrak{sp}_\infty^\alpha$ and $y \in \mathfrak{sp}_\infty^{-\alpha}$ such that $\tilde{\alpha} := [x, y]$ is the **coroot** of α. Namely, $\tilde{\alpha}$ satisfies $\alpha(\tilde{\alpha}) = 2$, so that $\operatorname{span}\{x, y, \tilde{\alpha}\}$ is a Lie subalgebra isomorphic to $\mathfrak{sl}_2$.

In fact, the spanning vectors given above for root spaces are precisely the vectors that will give us the coroots.

For the case of $\alpha = 2\varepsilon_j$, the coroot of α can be calculated by carrying out the Lie bracket $[E_{j,-j}, E_{-j,j}]$ to give us $E_{j,j} - E_{-j,-j}$ which can be seen to satisfy

$$2\varepsilon_j(E_{j,j} - E_{-j,-j}) = 2.$$

For $\alpha = \varepsilon_j + \varepsilon_k$, the coroot of α can be calculated by carrying out the Lie bracket $[E_{j,-k} - E_{k,-j}, E_{-j,k} - E_{-k,j}]$ to give us $E_{j,j} - E_{-j,-j} + E_{k,k} - E_{-k,-k}$ which can be seen to satisfy

$$(\varepsilon_j + \varepsilon_k)(E_{j,j} - E_{-j,-j} + E_{k,k} - E_{-k,-k}) = 1 + 1 = 2.$$

Finally, for $\alpha = \varepsilon_j - \varepsilon_k$, the coroot of α can be calculated by carrying out the Lie bracket $[E_{j,k} - E_{-k,-j}, E_{k,j} - E_{-j,-k}]$ to give us $(E_{j,j} - E_{-j,-j}) - (E_{k,k} - E_{-k,-k})$ which can be seen to satisfy

$$(\varepsilon_j - \varepsilon_k)((E_{j,j} - E_{-j,-j}) - (E_{k,k} - E_{-k,-k})) = 1 - (-1) = 2.$$

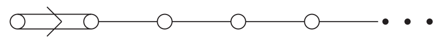

Figure 4.2: Dynkin Diagram for C_J

4.4 The Root System B_J of $\mathfrak{so}(2J+1, k)$

In Example 4.1.1 we have shown that $\mathfrak{h}_d = \text{span}\{E_{j,j} - E_{-j,-j}, j \in J\}$ is a maximal toral subalgebra of $\mathfrak{so}(2J+1, k)$. In this section we will look at the root system $B_J := \Delta(\mathfrak{so}(2J+1, k), \mathfrak{h}_d)$.

We claim that

$$B_J = \{\pm\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j \neq k\}$$

where $\varepsilon_j(E_{kk} - E_{-k,-k}) = \delta_{jk}$, the Kronecker delta.

It can be shown by a similar argument as in the case for A_J that the root space corresponding to $\alpha = \varepsilon_j$ is one-dimensional and is equal to $\text{span}\{2(E_{j,0} + E_{0,-j})\}$ and the root space corresponding to $-\alpha = -\varepsilon_j$ is equal to $\text{span}\{E_{-j,0} + E_{0,j}\}$. The coroot of α can be calculated by carrying out the Lie bracket $[E_{j,-j}, E_{-j,j}]$, which can be demonstrated by the case $[2(E_{2,0} + E_{0,-2}), (E_{-2,0} + E_{0,2})]$, where $2J+1 = \mathbb{Z}$ and $j = 2$, as follows

$$\left[\begin{pmatrix} \begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array} \end{pmatrix}, \begin{pmatrix} \begin{array}{cc|c|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \end{pmatrix}\right] = \begin{pmatrix} \begin{array}{cc|c|cc} -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array} \end{pmatrix}$$

to give us $E_{j,j} - E_{-j,-j}$ which can be seen to satisfy

$$\varepsilon_j(2(E_{j,j} - E_{-j,-j})) = 2.$$

For $\alpha = \varepsilon_j + \varepsilon_k$ it can be shown that the corresponding root space is equal to $\operatorname{span}\{E_{j,-k} - E_{k,-j}\}$, and the root space corresponding to $-\alpha = -\varepsilon_j - \varepsilon_k$ is given by $\operatorname{span}\{E_{-k,j} - E_{-j,k}\}$ with coroot equal to $(E_{k,k} - E_{-k,-k}) - (E_{j,j} - E_{-j,-j})$.

Finally, for $\alpha = \varepsilon_j - \varepsilon_k$ it can be shown that the corresponding root space is equal to $\operatorname{span}\{E_{j,k} - E_{-k,-j}\}$, and the root space corresponding to $-\alpha = \varepsilon_k - \varepsilon_j$ is given by $\operatorname{span}\{E_{k,j} - E_{-j,-k}\}$ with coroot equal to $(E_{j,j} - E_{-j,-j}) - (E_{k,k} - E_{-k,-k})$.

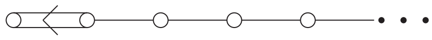

Figure 4.3: Dynkin Diagram for B_J

4.5 The Root System D_J of $\mathfrak{so}(2J, k)$

It was seen in Example 4.1.1 that $\mathfrak{h}_d = \operatorname{span}\{E_{j,j} - E_{-j,-j}, j \in J\}$ is a maximal toral subalgebra of $\mathfrak{so}(2J, k)$. In this section we will look at the root system $D_J := \Delta(\mathfrak{so}(2J, k), \mathfrak{h}_d)$.

We claim that

$$B_J = \{\pm\varepsilon_j \pm \varepsilon_k : j \neq k\}$$

where $\varepsilon_j(E_{kk} - E_{-k,-k}) = \delta_{jk}$, the Kronecker delta. For $\alpha = \varepsilon_j + \varepsilon_k$ it can be shown that the corresponding root space is equal to $\operatorname{span}\{E_{j,-k} - E_{k,-j}\}$, and the root space corresponding to $-\alpha = -\varepsilon_j - \varepsilon_k$ is given by $\mathrm{span}\{E_{-k,j} - E_{-j,k}\}$ with coroot equal to $(E_{k,k} - E_{-k,-k}) - (E_{j,j} - E_{-j,-j})$. The following calculation demonstrates the coroot calculation where we assume that $2J = \mathbb{Z}\backslash\{0\}$ and $j = 1$ and $k = 2$:

$$\left[\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\right] = \left(\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Finally, for $\alpha = \varepsilon_j - \varepsilon_k$ it can be shown that the corresponding root space is equal to $\mathrm{span}\{E_{j,k} - E_{-k,-j}\}$, and the root space corresponding to $-\alpha = \varepsilon_k - \varepsilon_j$ is given by $\mathrm{span}\{E_{k,j} - E_{-j,-k}\}$ with coroot equal to $(E_{j,j} - E_{-j,-j}) - (E_{k,k} - E_{-k,-k})$. In the following $j = 1$ and $k = 2$:

$$\left[\left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)\right] = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

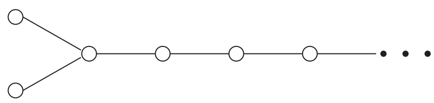

Figure 4.4: Dynkin Diagram for D_J

4.6 Local Finiteness

Definition 4.6.1. *A Lie algebra $\mathfrak{g}$ is called **locally finite** if every finite subset generates a finite-dimensional subalgebra of $\mathfrak{g}$.*

Example 4.6.1. If $\mathfrak{g}$ is one $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$ then $\mathfrak{g}$ is locally finite. Indeed, for any finite subset A of $\mathfrak{g}$, we have that all matrices in this subset are $\mathcal{B}$-finite, so we can find a positive integer n such that for all $x \in A$, we have $x_{ij} = 0$ if one of i, j is greater than n. Thus the dimension of the subalgebra generated by A is at most the dimension of $\mathfrak{gl}_n(k)$.

Lemma 4.6.1. *If $\mathfrak{g}$ is one $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$, then $\mathfrak{g}$ is a directed union of a set of finite-dimensional simple subalgebras $\mathcal{F}_{(\mathfrak{g},\mathfrak{h}_d)}$ with the property that for each $\mathfrak{f} \in \mathcal{F}_{(\mathfrak{g},\mathfrak{h}_d)}$ the intersection $\mathfrak{f} \cap \mathfrak{h}_d$ is a maximal toral subalgebra of $\mathfrak{f}$.*

Proof: Let $\mathfrak{g}$ be $\mathfrak{sl}(J, K)$ and take a finite subset M of J and consider the Lie subalgebra $\mathfrak{sl}(M, K) \subset \mathfrak{sl}(J, K)$. Since M is a finite set, then

$$\mathfrak{sl}(M, K) = \operatorname{span}\{E_{jj} - E_{kk}, E_{jk} : j, k \in M\}$$

is a finite-dimensional subalgebra and it is simple since it is isomorphic to $\mathfrak{sl}(|M|, k)$, where $|M|$ is the cardinality of M.

Moreover $\mathfrak{sl}(M, K) \cap \mathfrak{h}_d = \operatorname{span}\{E_{jj} - E_{kk} : j, k \in M\}$ is a maximal toral subalgebra of $\mathfrak{sl}(M, K)$. If $M_1 \subset M_2 \subset \ldots$ is a sequence of proper subsets of J, such that $M_1 \cup M_2 \cup \cdots = J$, then it is evident that

$$\mathfrak{sl}(J, K) = \bigcup_{i \geq 1} \mathfrak{sl}(M_i, K).$$

The proofs for $\mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$ are analogous. ∎

4.7 Weight Spaces

In this section we will look at the structure of the natural (or standard) module V of $\mathfrak{g}$. Namely, for $\mathfrak{gl}(J, k)$ it is the representation obtained by matrix multiplication on $V = k^J$, and for other Lie algebras it is obtained by restriction from $\mathfrak{gl}(J,k)$. We will see that if $(\mathfrak{g},\mathfrak{h})$ is a split Lie algebra with $\mathfrak{g}$ being one of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$ then there is a weight space decomposition $V = \bigoplus_{\lambda\in\mathcal{P}} V^\lambda$ such that each weight space V^λ corresponding to a nonzero λ is finite-dimensional, and that the dimension of V^0 is 0 or 1.

Definition 4.7.1. *Let $\mathfrak{g}$ be one of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$. Consider the natural module V of $\mathfrak{g}$. Take $\lambda \in \mathfrak{h}^*$ and define $V^\lambda = \{v \in V : h.v = \lambda(h)v, \forall h \in \mathfrak{h}\}$. A functional λ is called a* ***weight*** *for $\mathfrak{h}$ if $V^\lambda \neq \{0\}$, in which case V^λ is said to be the* ***weight space associated to*** *λ. The set $\mathcal{P} = \{\lambda \in \mathfrak{h}^* \backslash \{0\} : V^\lambda \neq \{0\}\}$ is called the corresponding* ***weight system****. A vector $v \in V$ is called* ***$\mathfrak{h}$-finite****, if the $\mathfrak{h}$-submodule generated by v is finite-dimensional.*

Remark 4.7.1. It is easy to show that the natural modules for $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$ are simple modules [18].

The following is a technical Lemma which will be used in the proof of Lemma 4.7.3.

Lemma 4.7.1. *Let $\mathfrak{g}$ be one of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$. Choose $h \in \mathfrak{g}$ such that ad_h is diagonalizable. Then we can find nilpotent elements $x_1, x_2, \ldots, x_n \in \mathfrak{g}$ such that $\pi := \exp(\mathrm{ad}_{x_1}) \cdots \exp(\mathrm{ad}_{x_n})$ is an automorphism of $\mathfrak{g}$ with $\pi(h) \in \mathfrak{h}_d$.*

Proof: See [21, Lemma 3.5]. ∎

The elements of $\mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$ can be realized in two equivalent ways. Firstly as an endomorphism of $V = k^J$ and secondly as a map $x : J \times J \to k$ with finitely many nonzero entries. Also, the maps corresponding to the endomorphisms in $\mathfrak{sp}_\infty$ are in

$$\mathfrak{sp}(J,k) = \{x \in \mathfrak{gl}(J^\pm, k) : x^t S = -Sx\}$$

The maps corresponding to the endomorphisms in $\mathfrak{so}_\infty$ are in either of the two following sets, which we proved to yield isomorphic Lie algebras,

$$\mathfrak{so}(2J+1,k) = \{x \in \mathfrak{gl}(J_0^\pm, k) : x^t Q_o = -Q_o x\}$$

$$\mathfrak{so}(2J,k) = \{x \in \mathfrak{gl}(J^\pm, k) : x^t Q_e = -Q_e x\}.$$

Letting B be one of the matrices S, Q_o or Q_e, define $\beta : V \times V \to k$ to be the corresponding skew-symmetric, resp. symmetric bilinear form, i.e. $\beta(v,w) = v^t B w$.

Lemma 4.7.2. *Let $\mathfrak{g}$ be one of $\mathfrak{sp}_\infty$ or $\mathfrak{so}_\infty$ and let $x \in \mathrm{End}_k(V)$. Then $x \in \mathfrak{g}$ if and only if x has finite rank and $\beta(x.v,w) + \beta(v,x.w) = 0$ for all $v,w \in V$.*

Proof: Let $\mathcal{B}$ be the canonical basis of V. Let $x \in \mathfrak{g}$, then x is $\mathcal{B}$-finite, thus it has finite rank. Also $x^t tB = -Bx$, so if $v,w \in V$, then $\beta(x.v,w) + \beta(v,x.w) = (xv)^t Bw + v^t B(xw) = (v)^t (x)^t Bw + v^t(-x^t B)w = 0$.

Conversely, take x to have finite rank. Since $[x]_\mathcal{B}$ has finitely many entries in each column and the number of linearly independent columns is finite (i.e. finite rank), then there exists an integer n_0 such that the first n_0 columns contain all of the linearly independent columns. Since each of the first n_0 columns have finitely many nonzero entries, we can take m_0 such that $x_{ij} = 0$ for $1 \le j \le n_0$ and $i > m_0$. Now if $j \ge n_0$, then the j-th column is a linear combination of the first n_0 columns, hence it has no nonzero entries x_{ij} when $i > m_0$. Thus $[x]_\mathcal{B}$ has finitely many nonzero rows.

If moreover $\beta(x.v,w) + \beta(v,x.w) = 0$ for all $v,w \in V$, then $-B^{-1}x^t B = x$. From this equation it follows that x has finitely many nonzero columns. Thus $x \in \mathfrak{g}$. ∎

Lemma 4.7.3. *Let $\mathfrak{g}$ be one of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$. Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. If $h \in \mathfrak{h}$, then h is diagonalizable. That is, there exists $D \in \mathrm{GL}(V)$ such that $DhD^{-1} \in \mathfrak{h}_d$.*

Proof: Let $h \in \mathfrak{h}$. We know ad_h is diagonalizable so by Lemma 4.7.1 we can find nilpotent elements $x_1, x_2, \ldots, x_n \in \mathfrak{g}$ such that $\pi := \exp(\mathrm{ad}_{x_1}) \cdots \exp(\mathrm{ad}_{x_n})$ is an automorphism of $\mathfrak{g}$ with $\pi(h) \in \mathfrak{h}_d$.

Let $y \in \mathfrak{g}$, then $\exp(\mathrm{ad}_{x_i})(y) = \exp(x_i)(y)\exp(-x_i)$ (By [9, Section 2.3]) Note that $\exp(\mathrm{ad}_{x_i})$ is defined because x_i is nilpotent. Thus,

$$\pi(h) = \exp(x_1)\cdots\exp(x_n)y\exp(-x_1)\cdots\exp(-x_n) = DhD^{-1}$$

is in $\mathfrak{h}_d$, where $D = \exp(x_1)\cdots\exp(x_n) \in \mathrm{GL}(V)$. Thus, h is diagonalizable. ∎

Lemma 4.7.4. *Let $\mathfrak{g}$ be one of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$. Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$ and let V be the natural $\mathfrak{g}$-module. Then every vector in V is $\mathfrak{h}$-finite.*

Proof: Let h be a nonzero endomorphism in $\mathfrak{h}$. By Lemma 4.7.3, we know that h is diagonalizable. Let μ be a nonzero eigenvalue of h and denote by

$$W = \{v \in V : hv = \mu v\}$$

the associated eigenspace. Observe that W is finite-dimensional since the dimension of W equals the multiplicity of the eigenvalue μ and h has finitely many nonzero eigenvalues, as it is $\mathcal{B}$-finite.

By Lemma 4.7.3, we know that the commutative vector space endomorphisms $h' : V \to V \in \mathfrak{h}$ are diagonalizable. So in particular, they are simultaneously diagonalizable on the finite-dimensional space W. Thus we can find a nonzero simultaneous eigenvector $w \in W$ for all the endomorphisms $h' \in \mathfrak{h}$. Now observe that the $\mathfrak{h}$-submodule of V generated by w (i.e. $\mathrm{span}\{h_1 \ldots h_k w : h_i \in \mathfrak{h}\} = \mathrm{span}\{w\}$) is one-dimensional. That is, w is $\mathfrak{h}$-finite.

Consider the subcollection V_{fin} of V defined by $V_{\mathrm{fin}} = \{v \in V : v \text{ is } \mathfrak{h}\text{-finite}\}$. We will show that V_{fin} is a submodule of V. First, note that V_{fin} is nonempty, as it contains w and note that V_{fin} is a subspace of V. Let $v \in V_{\text{fin}}$ and let $x \in \mathfrak{g}$ be a $\mathcal{B}$-finite matrix such that all of its nonzero entries can be contained in an $m \times m$ matrix. Observe that the $\mathfrak{h}$-submodule of V generated by $x.v$ equals

$$\mathcal{U}(\mathfrak{h}) \cdot (x \cdot v) \subset (\text{ad}(\mathcal{U}(\mathfrak{h}) \cdot x) \cdot v + x \cdot (\mathcal{U}(\mathfrak{h}) \cdot v)$$

Since x is a $\mathcal{B}$-finite matrix such that all of its nonzero entries can be contained in an $m \times m$ matrix, the vector space $\text{ad}(\mathcal{U}(\mathfrak{h})) \cdot x$ contains only matrices with nonzero part of size $m \times m$ or smaller, so it is finite-dimensional. Also $\mathcal{U}(\mathfrak{h}).v$ is finite-dimensional since v is an $\mathfrak{h}$-finite vector.

Thus V_{fin} is a nonzero $\mathfrak{g}$-submodule of V and since V is a simple $\mathfrak{g}$-module, it follows that $V_{\text{fin}} = V$. That is, every vector in V is $\mathfrak{h}$-finite. ∎

Theorem 4.7.5. *Let $\mathfrak{g}$ be one of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$. Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then there is a weight space decomposition $V = \bigoplus_{\lambda \in \mathcal{P}} V^\lambda$ such that each weight space V^λ corresponding to a nonzero λ is finite-dimensional.*

Proof: Let $v \in V$. By Lemma 4.7.4 we have that v is $\mathfrak{h}$-finite, thus v belongs to a finite-dimensional $\mathfrak{h}$-submodule U in V. By Lemma 4.7.3, we have that the vector space endomorphisms $h' : V \to V \in \mathfrak{h}$ are diagonalizable. Thus, in particular, they are simultaneously diagonalizable when restricted to U. Pick a basis $B = \{v_1, \dots, v_n\}$ of U, under which the elements of $\mathfrak{h}$ are diagonal. Then v can be expressed as a linear combination of the v_i's and each v_i satisfies $hv = \lambda_i(h)v$ for all $h \in \mathfrak{h}$ and $\lambda_i \in \mathfrak{h}$. That is $v_i \in V^{\lambda_i}$.

Thus there is a weight space decomposition $V = \bigoplus_{\lambda \in \mathcal{P}} V^\lambda$. Also, if $\lambda \neq 0$, then there exists nonzero h such that $\lambda(h) \neq 0$ and if $v \in V^\lambda$ then $hv = \lambda(h)v$, thus v belongs to the eigenspace of h with nonzero eigenvalue $\lambda(h)$. We have seen in the proof of Lemma 4.7.4 that such eigenspaces are finite-dimensional, thus each weight space V^λ corresponding to a nonzero λ is finite-dimensional. ∎

Remark 4.7.2. The only result left to show is that the dimension of V^0 is either 0 or 1. This was shown in Stumme [21, Lemma 4.3]. The proof, as presented in the paper by Stumme, uses a fact about coroots that does not follow immediately and would require a development of a theory of coroots. For that reason, this version of the proof will not be used. We will see a different proof of this fact in the next chapter.

Chapter 5

Structure of Classical Locally Finite Lie Algebras Over a Ring

The conjugacy of the maximal toral subalgebras for $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$ over an algebraically closed field k of characteristic zero were discussed by Stumme in [21]. This result was further generalized by Salmasian in [18] by considering $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}_\infty(R)$, and $\mathfrak{so}_\infty(R)$ for $R = k[t, t^{-1}]$. In this chapter we will discuss how these results could be generalized by weakening the assumptions about the underlying field or ring. The final result of this chapter is original.

5.1 The Action of Conjugation on Maximal Toral Subalgebras

We will start with some results that will be required in order to define the conjugation action on a maximal toral subalgebra.

Lemma 5.1.1. *[21, Lemma 2.1]*

(a) Let A and B be $J \times J$ matrices and $j, k \in J$. Then $AE_{jk}B$ equals the product

of the j-th column of A with the k-th row of B. In particular $AE_{jk}B$ is $\mathcal{B}$-finite if and only if the j-th column of A and the k-th row of B are finitary.

(b) *Let $\mathfrak{g}$ denote one of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$. Let A be an invertible $J' \times J'$ matrix where J' is one of J, $J^\pm$, or $J_0^\pm$. If for all $x \in \mathfrak{g}$ the matrix AxA^{-1} is $\mathcal{B}$-finite, then the rows of A^{-1} have finitely many nonzero entries. In particular, if for all $x \in \mathfrak{g}$ both AxA^{-1} and $A^{-1}xA$ are $\mathcal{B}$-finite, then $A \in \mathrm{GL}(J', k)$.*

Proof: (a) Easy calculation. (b) The second statement follows from the first, so it is sufficient to prove only the first statement. Suppose AxA^{-1} is $\mathcal{B}$-finite for all $x \in \mathfrak{sl}_\infty$. Then, in particular, $AE_{k,k+1}A^{-1}$ is $\mathcal{B}$-finite for all $k \geq 1$, giving us that the rows of A^{-1} are finitary. Suppose AxA^{-1} is $\mathcal{B}$-finite for all $x \in \mathfrak{sp}_\infty$, we have $AE_{j,-j}A^{-1}$ and $AE_{-j,j}A^{-1}$ is $\mathcal{B}$-finite, so the rows of A^{-1} are finitary. Suppose AxA^{-1} is $\mathcal{B}$-finite for all $x \in \mathfrak{so}_\infty$. Identifying $x \in \mathfrak{so}_\infty$ with $\mathfrak{so}(J, J, k)$, we have that A$(E_{j,k} - E_{k,j})A^{-1}$ for $j \neq k$ is $\mathcal{B}$-finite, so $(E_{j,k} - E_{k,j})A^{-1} = A^{-1}y$ for some $y \in \mathfrak{gl}(J^\pm, k)$. An easy calculation shows that the j-th row of $(E_{j,k} - E_{k,j})A^{-1}$ is the negative of the j-th row of A^{-1} and that the k-th row of $(E_{j,k} - E_{k,j})A^{-1}$ equals the k-th row of A^{-1}, and all other entries of $(E_{j,k} - E_{k,j})A^{-1}$ are zero. Now y is $\mathcal{B}$-finite, hence yA is $\mathcal{B}$-finite by Lemma 2.1.2, and this implies that the j-th row of A^{-1} is finitary. Since we also have that $A(E_{-j,-k} - E_{-k,-j})A^{-1}$ for $j \neq k$ is $\mathcal{B}$-finite, the same calculation shows that the $(-j)$-th row of A^{-1} is finitary. Thus all the rows of A^{-1} are finitary. ∎

The next Lemma shows us which matrices A result in an automorphism $\pi_A : \mathfrak{gl}_\infty \to \mathfrak{gl}_\infty$ given by $x \mapsto AxA^{-1}$.

Lemma 5.1.2. *[21, Lemma 2.2]*

(a) *The conjugation*

$$\pi_A : \mathfrak{gl}_\infty \to \mathfrak{gl}_\infty, x \mapsto AxA^{-1}$$

defines an injective endomorphism of the Lie algebra $\mathfrak{gl}_\infty$ if and only if every row of A^{-1} has finitely many nonzero entries.

(b) The conjugation π_A defines an automorphism of $\mathfrak{gl}_\infty$ if and only if $A \in \mathrm{GL}_\infty$.

Proof: (a) By the previous Lemma, the image of π_A lies in $\mathfrak{gl}_\infty$ if and only if A^{-1} has finitary rows. It is easy to see that π_A is linear and injective.

(b) $A \in \mathrm{GL}_\infty$ if and only if both A^{-1} and A have finitary rows. So the result follows from part (a). ∎

We will now show for which matrices A the map $\pi_A : \mathfrak{g} \to \mathfrak{g}$ such that $x \mapsto AxA^{-1}$ is an automorphism where $\mathfrak{g}$ is one of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$.

Recall the matrices S, Q_o and Q_e given by

$$S = \sum_{j \in J} (E_{j,-j} - E_{-j,j}), \quad j \in J^\pm$$

$$Q_o = -E_{0,0} + \sum_{j \in J} (E_{j,-j} + E_{-j,j}), \quad j \in J_0^\pm$$

$$Q_e = \sum_{j \in J} (E_{j,-j} + E_{-j,j}), \quad j \in J^\pm.$$

Definition 5.1.1. *The **symplectic group** SP_∞ is the subset of GL_∞ whose elements correspond to matrices in*

$$\mathrm{SP}(J,k) = \{A \in \mathrm{GL}(J^\pm, k) : A^t S A = S\}$$

*The **conformal symplectic group** GSP_∞ is the subset of GL_∞ whose elements correspond to matrices in*

$$\mathrm{GSP}(J,k) = \{A \in \mathrm{GL}(J^\pm, k) : A^t S A = hS, h \in k\}.$$

*Likewise the **orthogonal group** SO_∞ is the subset of GL_∞ whose elements corre-*

spond to matrices in either of the following three sets

$$\mathrm{SO}(2J,k) = \{A \in \mathrm{GL}(J^{\pm},k) : A^tQ_eA = Q_e\}$$
$$\mathrm{SO}(2J+1,k) = \{A \in \mathrm{GL}(J_0^{\pm},k) : A^tQ_oA = Q_o\}$$
$$\mathrm{SO}(J,J,k) = \{A \in \mathrm{GL}(J^{\pm},k) : A^tQA = Q\}.$$

The ***conformal orthogonal group*** GSO_∞ *is the subset of* GL_∞ *whose elements correspond to matrices in either of the following three sets*

$$\mathrm{GSO}(2J,k) = \{A \in \mathrm{GL}(J^{\pm},k) : A^tQ_eA = hQ_e, h \in k\}$$
$$\mathrm{GSO}(2J+1,k) = \{A \in \mathrm{GL}(J_0^{\pm},k) : A^tQ_oA = hQ_o, h \in k\}$$
$$\mathrm{GSO}(J,J,k) = \{A \in \mathrm{GL}(J^{\pm},k) : A^tQA = hQ, h \in k\}.$$

Theorem 5.1.3. *[21, Proposition 2.4]*

(a) *The conjugation π_A defines an automorphism of $\mathfrak{sl}_\infty$ if and only if $A \in \mathrm{GL}_\infty$.*

(b) *The conjugation π_A defines an automorphism of $\mathfrak{sp}_\infty$ if and only if $A \in \mathrm{GSP}_\infty$.*

(c) *The conjugation π_A defines an automorphism of $\mathfrak{so}(2J+1,k)$, $\mathfrak{so}(2J,k)$ and $\mathfrak{so}(J,J,k)$ if and only if $A \in \mathrm{GSO}(2J+1,k), \mathrm{GSO}(2J,k), \mathrm{GSO}(J,J,k)$ respectively.*

Proof: Proof presented in [21]. ∎

Thus we have shown the necessary conditions for a conjugation action on the Lie algebras $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}(R)$, and $\mathfrak{so}_\infty(R)$ for $R = k$. Replacing k with a general commutative k-algebra R in the definitions of $\mathrm{GSP}_\infty, \mathrm{GL}_\infty$ and GSO_∞ will generalize this notion of conjugation to a general ring R.

The maps in the previous theorem are a type of isomorphism and we will explore the effect of this isomorphism on the maximal toral subalgebras of $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}(R)$, and

$\mathfrak{so}_\infty(R)$. The case for $R = k[t, t^{-1}]$ was discussed in [18] in which it was shown that for $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}(R)$ every maximal toral subalgebra is conjugate, and that for $\mathfrak{so}_\infty(R)$ there are at most 5 conjugacy classes. The case for $R = k$ was discussed in [21] and we will include it in the next section for completeness.

5.2 Conjugacy of Maximal Toral Subalgebras of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$ over k

Lemma 5.2.1. *[21, Lemma 4.4] Let $\mathfrak{g}$ be one of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, or $\mathfrak{so}_\infty$. Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Let $\mathcal{C} = \{f_j : j \in J\}$ be a basis of the natrual module V consisting of $\mathfrak{h}$-weight vectors and let $\mathcal{E} = \{e_j : j \in J\}$ be the standard basis of V. Consider the endomorphism of V which maps f_j to e_j for all $j \in J$. Then the matrix A of this endomorphism belongs to GL_∞.*

Proof: Let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha_i \in \Delta} \mathfrak{g}^{\alpha_i}$ be the root decomposition. Let x_α be a root vector in $\mathfrak{g}^\alpha$ and let λ be a weight in $\mathcal{P}$. Let v be a nonzero vector in V^λ. Then $hv = \lambda(h)v$ and $[h, x_\alpha] = \alpha(h)x_\alpha$ for all $h \in \mathfrak{h}$. Then $x_\alpha v \in V^{\alpha+\lambda}$ since

$$\begin{aligned}
h(x_\alpha v) &= hx_\alpha v - \lambda(h)x_\alpha v + \lambda(h)x_\alpha v \\
&= hx_\alpha v - x_\alpha hv + \lambda(h)x_\alpha v \\
&= (hx_\alpha - x_\alpha h)v + \lambda(h)x_\alpha v \\
&= [h, x_\alpha]v + \lambda(h)x_\alpha v \\
&= (\alpha(h) + \lambda(h))x_\alpha v
\end{aligned}$$

By Theorem 4.7.5, we have that V^λ is finite-dimensional. Therefore the matrix $[x_\alpha]_\mathcal{C}$ has block form, whose blocks are of finite size. Moreover, since this matrix has finite rank, it has nonzero entries in a finite number of rows, so $[x_\alpha]_\mathcal{C}$ is $\mathcal{C}$-finite. We also know that for any matrix $h \in \mathfrak{h}$, $[h]_\mathcal{C}$ is $\mathcal{C}$-finite because it has finite rank and it is diagonal, since $\mathcal{C}$ consists of $\mathfrak{h}$-weight vectors. Thus we have shown that every matrix x in $\mathfrak{g}$ is $\mathcal{C}$-finite.

The matrix A represents an endomorphism of the vector space V, so $A[x]_{\mathcal{E}}A^{-1} = [x]_{\mathcal{C}}$ for all $x \in \mathfrak{g}$. By Lemma 5.1.1 (b) we have that the rows of A^{-1} are finitary.

Also we have that $Af_i = e_i$, thus we have that $e_i = \sum_{k\in J} A_{ik} f_k$ showing that A has only finitely many nonzero entries in each row. ∎

Theorem 5.2.2. *[21, Proposition 4.5] All maximal toral subalgebras of $\mathfrak{sl}_\infty$ are conjugate under the action by the group* GL_∞.

Proof: Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{sl}_\infty$. Letting $A \in \mathrm{GL}_\infty$ be as in Lemma 5.2.1, the conjugation $\pi_A : \mathfrak{sl}_\infty \to \mathfrak{sl}_\infty, x \mapsto AxA^{-1}$ is an automorphism of $\mathfrak{sl}_\infty$ by Theorem 5.1.3. Moreover $(AhA^{-1})e_j \in \mathrm{span}\{e_j\}$ for all $h \in \mathfrak{h}$, thus $A\mathfrak{h}A^{-1} \subset \mathfrak{h}_d$. Equality follows since they are both maximal abelian.

Thus any maximal toral subalgebra of $\mathfrak{sl}_\infty$ is conjugate to $\mathfrak{h}_d$. Therefore all maximal toral subalgebras of $\mathfrak{sl}_\infty$ are conjugate under the conjugation action by the group GL_∞. ∎

Theorem 5.2.3. *[21, Proposition 4.6] All maximal toral subalgebras of $\mathfrak{sp}_\infty$ are conjugate.*

Proof: Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{sp}_\infty$. By Theorem 5.1.3, it suffices to find a matrix $A \in \mathrm{GSP}_\infty$, such that

$$\pi_A : \mathfrak{sp}_\infty \to \mathfrak{sp}_\infty, x \mapsto AxA^{-1}$$

satisfies $\pi_A(\mathfrak{h}) \subset \mathfrak{h}_d$.

Let β be as in Lemma 4.7.2, and take distinct elements $\lambda, \mu \in \mathcal{P}$, such that $\lambda \neq -\mu$. Then for $v_\lambda \in V^\lambda, v_\mu \in V^\mu$ and $h \in \mathfrak{h}$, we have the equation

$$\lambda(h)\beta(v_\lambda, v_\mu) = \beta(h.v_\lambda, v_\mu) = -\beta(v_\mu, h.v_\lambda) = -\mu(h)\beta(v_\lambda, v_\mu)$$

which implies that $\beta(V^\lambda, V^\mu) = \{0\}$. Hence β is non-degenerate on the finite-dimensional subspace $V^\lambda + V^{-\lambda}$ for $\lambda \in \mathcal{P}$. Thus for each $\lambda \in \mathcal{P}$ we can choose a finite basis of $V^\lambda + V^{-\lambda}$ consisting of finitely pairs of elements f_j, f_{-j}, such that $\beta(f_j, f_{-j}) = 1$ and $\beta(f_k, f_j) = 0$ for $j \neq k$. Taking the union of all the bases, over all $\lambda \in \mathcal{P}$, we get a basis of V with $\beta(f_j, f_{-j}) = 1$ and $\beta(f_k, f_j) = 0$ for $j \in J, k \in J^\pm \backslash \{-j\}$. Consider the endomorphism of V that maps f_j to the canonical basis vector e_j for $j \in J^\pm$ and let A be the matrix of this endomorphism relative to the canonical basis. Then we have $A \in \mathrm{GL}_\infty$ by Lemma 5.2.1. Moreover, we have $\beta(f_j, f_k) = \beta(e_j, e_k) = \beta(A.f_j, A.f_k)$, so $A \in \mathrm{SP}_\infty$.

Since all elements of the basis $\{f_j\}$ are weight vectors we have that $\pi_A(\mathfrak{h}) \subset \mathfrak{h}_d$. Equality follows since both $\mathfrak{h}$ and $\mathfrak{h}_d$ are maximal abelian. ∎

Theorem 5.2.4. *[21, Proposition 4.7]*

(a)All maximal toral subalgebras $\mathfrak{h}$ of $\mathfrak{so}_\infty$, that have the property that $V^0 = \{0\}$ are conjugate.

(b)Likewise all maximal toral subalgebras $\mathfrak{h}$ of $\mathfrak{so}_\infty$, that have the property that $\dim V^0 = 1$ are conjugate.

Proof: We proceed as in the previous proof and get a set of linearly independent vectors f_j of V with $\beta(f_j, f_{-j}) = 1$ and $\beta(f_k, f_j) = 0$ for $j \in J, k \in J^\pm \backslash \{-j\}$. For part (a) we consider $\mathfrak{so}_\infty$ to correspond with $\mathfrak{so}(2J, k)$ so these vectors form a basis of V.

For part (b) we consider $\mathfrak{so}_\infty$ to correspond with $\mathfrak{so}(2J+1, k)$ so these vectors form a basis of V, together with a vector f_0 such that $\beta(f_0, f_0) = -1$.

Finally we get that for part (a) $A \in \mathrm{SO}(2J, k)$ and for part (b) $A \in \mathrm{SO}(2J+1, k)$. As in the proof of Theorem 5.2.3, we apply Theorem 5.1.3, and see that $\pi_A(\mathfrak{h}) \subset \mathfrak{h}_d$. ∎

5.3 Representations as Weight Modules of V_R, where R is a Bézout Domain

So far we have discussed the conjugacy results of the maximal toral subalgebras for $R = k$. In the generalization of this result from k to $k[t, t^{-1}]$ in [18], some general results about principal ideal domains were used. Rather than focus on other specific rings, we have decided to weaken the general assumptions about R and investigate which conjugacy results still hold for these weaker assumptions.

We will from now on assume (unless otherwise stated) that R is a Bézout Domain. Thus $\mathfrak{h}$ represents an arbitrary maximal toral subalgebra of either $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}_\infty(R)$ or $\mathfrak{so}_\infty(R)$, depending on the context. In order to discuss this more general case of R we begin with some definitions.

Definition 5.3.1. *We define V_R to be the standard representation of $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}_\infty(R)$, or $\mathfrak{so}_\infty(R)$. That is, V_R consists of infinite columns $[a_1, a_2, \dots]^t$ where $a_i \in R$ and all but finitely many of the a_i's are zero. The Lie algebras $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}_\infty(R)$, or $\mathfrak{so}_\infty(R)$ act on V_R by left matrix multiplication.*

As in section 4.7, the notions of **weight, weight space associated to** λ for $\lambda \in \mathfrak{h}^*$ are defined in the same way.

We will now give three general lemmas with references that will be used to generalize to the case of Bézout Domains. The following Lemma is found in [18, Proposition 2.1.1] and will be restated here.

Lemma 5.3.1. *[13, Proposition 2.1.1] Let $\mathfrak{a}$ be any Lie algebra over k, and let W be a vector space over k which is also an $\mathfrak{a}$-module such that*

$$W = \bigoplus_{\lambda \in \mathfrak{a}^*} W_\lambda$$

and

$$W_\lambda = \{w \in W : \forall x \in \mathfrak{a}, x \cdot w = \lambda(x)w\}.$$

Suppose $W' \subset W$ is a $\mathfrak{a}$-submodule of W. Then $W' = \bigoplus_{\lambda \in \mathfrak{a}^*} W'_\lambda$ where $W'_\lambda = W_\lambda \cap W'$.

Proof: See [13, Proposition 2.1.1] . ∎

Lemma 5.3.2. *[18, Lemma 3.2] Let W be an arbitrary vector space over k. Suppose $T \in \mathrm{End}_k(W)$ and $\mathcal{A} \subset \mathrm{End}_k(W)$ are such that*

(a) There exists a k-basis $\mathcal{C}_1$ of W consisting of eigenvectors of T with eigenvalues in k.

(b) There exists a k-basis $\mathcal{C}_2$ of W consisting of common eigenvectors of elements of $\mathcal{A}$ with eigenvalues in k.

(c) T commutes with elements of $\mathcal{A}$.

Then there exists a basis $\mathcal{C}$ of W consisting of common eigenvectors of elements of $\{T\} \cup \mathcal{A}$ with eigenvalues in k.

Proof: See [18, Lemma 3.2] . ∎

Lemma 5.3.3. *Let $R = k[t, t^{-1}]$ or $R = k[t, t^{-1}, (t-1)^{-1}]$. Consider $\mathfrak{g}_m(R)$ to be one of $\mathfrak{sl}_m(R)$, $\mathfrak{sp}_m(R)$, or $\mathfrak{so}_m(R)$. Let $\mathfrak{h}_d$ be the standard (i.e. diagonal) Cartan* *subalgebra of* $\mathfrak{g}_m(R)$ *(with entries in* k*). If* $\mathfrak{h}_m$ *is a maximal Lie subalgebra of* $\mathfrak{g}_m(R)$ *such that there exists a basis* $\mathcal{C}$ *of* $\mathfrak{g}_m(R)$ *consisting of common eigenvectors of* ad_h *for all* $h \in \mathfrak{h}_m$*, then* $\mathfrak{h}_m$ *and* $\mathfrak{h}_d \otimes_k 1$ *are conjugate by an element of the group* $\mathrm{GL}_m(R)$*.*

Proof: The case $R = k[t, t^{-1}]$ is proved in [16, Theorem 1]. The case $R = k[t, t^{-1}, (t-1)^{-1}]$ will appear in a subsequent paper by Chernousov, Gille, and Pianzola. ∎

Lemma 5.3.4. *[18, Lemma 3.4] Let* $\mathfrak{g}(R)$ *be one of* $\mathfrak{sl}_\infty(R)$*,* $\mathfrak{sp}_\infty(R)$*, or* $\mathfrak{so}_\infty(R)$ *and let* S *be a finite subset of* $\mathfrak{g}(R)$ *and let* $\mathfrak{h}$ *be a maximal toral subalgebra of* $\mathfrak{g}(R)$*. Then* S *lies inside a finite-dimensional* k*-subspace* W_S *of* $\mathfrak{g}(R)$ *such that* $[\mathfrak{h}, W_S] \subset W_S$*.*

Proof: See [18, Lemma 3.4]. ∎

Lemma 5.3.5. *[18, Lemma 3.5] Consider the action of* h *on* V_R *for some* $h \in \mathfrak{h}$*, where* $R = k[t, t^{-1}]$ *or* $R = k[t, t^{-1}, (t-1)^{-1}]$*. Then we have*

$$V_R = \bigoplus_{s \in k} V_{R,s} \text{ where } V_{R,s} = \{v \in V_R : h.v = sv\}.$$

Each $V_{R,s}$ *is a free* R*-module. If* $s \in k - \{0\}$*, then* $\mathrm{rank}_R(V_{R,s}) < \infty$*.*

Proof: Let $h \in \mathfrak{h}$. Then h is a $\mathcal{B}$-finite matrix so that all of its nonzero entries can be contained in an $m \times m$ matrix, for some positive integer m. Let $\mathfrak{g}_m(R)$ denote the Lie subalgebra of $\mathfrak{g}(R)$, whose elements consist of matrices with nonzero entries contained in an $m \times m$ matrix. That is, $h \in \mathfrak{g}_m(R)$. It follows that $[h, \mathfrak{g}_m(R)] = \mathfrak{g}_m(R)$. By applying Lemma 5.3.1 with the one dimensional space spanned by h playing the role of $\mathfrak{a}$ and $\mathfrak{g}(R)$ playing the role of W, we conclude that there is a k-basis of $\mathfrak{g}_m(R)$ which consists of eigenvectors of ad_h with eigenvalues in k. By Lemma 5.3.3 we have that the one dimensional space spanned by h is conjugate to a subalgebra of $\mathfrak{h}_d \otimes_k 1$. Thus there exists a $D \in \mathrm{GL}_m$ such that $DhD^{-1} \in \mathfrak{h}_d \otimes_k 1 \subset \mathfrak{g}_m(R)$. The embedding map $\mathrm{GL}_m \to \mathrm{GL}_\infty$ maps D to an element of GL_∞ which we will also denote by D. The action of DhD^{-1} on V_R is diagonal with respect to the standard R-basis, with eigenvalues in k. Thus we can decompose into a direct of of R-modules

$$V_R = \bigoplus_{s \in k} V_{R,D,s} \text{ where } V_{R,D,s} = \{v \in V_R : DhD^{-1}.v = sv\}.$$

There are finitely many nonzero summands, and each nonzero summand is a free R-module, spanned by a subset of the standard basis. Also since DhD^{-1} is in $\mathfrak{g}_m(R)$ we have that if $s \in k - \{0\}$, then $rank_R(V_{R,D,s}) < \infty$.

To complete the proof, we define a map $\sigma_D : V_R \to V_R$ by $\sigma_D(v) = Dv$ and set $V_{R,s} = \sigma^{-1}(V_{R,D,s})$. ∎

Lemma 5.3.6. *[18, Lemma 3.6] Let the notation be as in Lemma 5.3.5. There exists a nonzero element $v \in V_R$ and a $\lambda \in \mathfrak{h}^*$ such that for any $h \in \mathfrak{h}$, we have $h.v = \lambda(h)v$.*

Proof: Choose $h \in \mathfrak{h}$ and take $s \in k - \{0\}$ such that $V_{R,s} \neq 0$. Now $\mathfrak{h}$ is a commutative Lie algebra, so $\mathfrak{h} \cdot V_{R,s} \subset V_{R,s}$, that is, if $x, y \in \mathfrak{h}$ and $v \in V_{R,s}$ we conclude that $yv \in V_{R,s}$ since $xyv = yxv = ysv = s(yv)$. Thus for each element $x \in \mathfrak{h}$ we have an R-linear map $T_x : V_{R,s} \to V_{R,s}$. If we define $\mathcal{L} := \{T_x : x \in \mathfrak{h}\}$, then it forms a commutative Lie subalgebra of $End_R(V_{R,s})$. Since $V_{R,s}$ is a free R-module of finite rank it follows that $End_R(V_{R,s})$ is isomorphic to $M_{d\times d}(R)$, that is matrices of size $d \times d$ for some positive integer d. From Lemma 5.3.1 we have that for any $T_x \in \mathcal{L}$, there is a k-basis of $V_{R,s}$ which consists of eigenvectors of T_x with eigenvalues in k.

We extend each $T_x \in \mathcal{L}$ to a linear map $\hat{T}_x : V_{R,s} \otimes_R K \to V_{R,s} \otimes_R K$ using the standard embedding $End_R(V_{R,s}) \to End_K(V_{R,s} \otimes_R K)$, where K is the quotient field of R.

It follows that there exists a K-basis of $V_{R,s} \otimes_R K$ consisting of eigenvectors of T_x, with eigenvalues in k. Therefore $\mathcal{L}$ is a commuting and diagonalizable set of elements in $End_K(V_{R,s} \otimes_R K)$. Since $End_K(V_{R,s} \otimes_R K)$ is isomorphic to $M_{d\times d}(K)$, we can diagonalize elements of $\mathcal{L}$ simultaneously. So we can find common eigenvector of element of $\mathcal{L}$ in $V_{R,s} \otimes_R K$ with eigenvalues in k. To finish the proof we rescale, and get a common eigenvector in $V_{R,s}$. ∎

Lemma 5.3.7. *[18, Lemma 3.7] Let the notation be as in Lemma 5.3.5. For every $v \in V_R$, we have $\dim_k(\mathcal{U}(\mathfrak{h}) \cdot v) < \infty$.*

Proof: (The proof is taken from [18]) Define the set $W = \{v \in V_R | \dim_k(\mathcal{U}(h)\cdot v) < \infty\}$. From Lemma 5.3.6 we have that $W \neq \{0\}$. We want to show that W is invariant under the action of $\mathfrak{g}(R)$. Take $w \in W$, so that for any $x \in \mathfrak{g}(R)$ we have

$$\mathcal{U}(\mathfrak{h}) \cdot (x \cdot v) \subset (\mathrm{ad}(\mathcal{U}(\mathfrak{h}) \cdot x) \cdot v + x \cdot (\mathcal{U}(\mathfrak{h}) \cdot v)$$

from the proof of Lemma 4.7.4. From Lemma 5.3.4 we know that $\dim_k([\mathcal{U}(\mathfrak{h}), X]) < \infty$, so we conclude that $x \cdot w \in W$. Thus $V_R = W$ because of simplicity. ∎

Lemma 5.3.8. *[18, Lemma 3.8] Let the notation be as in Lemma 5.3.5. Let $\mathfrak{g}(R)$ be one of $\mathfrak{sl}_\infty(R)$, $\mathfrak{sp}_\infty(R)$, or $\mathfrak{so}_\infty(R)$ and $\mathfrak{h}$ be a maximal toral subalgebra. The module V_R can be decomposed as*

$$V_R = \bigoplus_{\alpha\in\mathfrak{h}^*} V_{R,\alpha}$$

such that $V_{R,\alpha} = \{w \in V_R : \forall h \in \mathfrak{h}, h.w = \alpha(h)w\}$.
Moreover, if $\alpha \neq 0$, then $V_{R,\alpha}$ is a free R-module of finite rank.

Proof: (The proof is taken from [18]) Choose $v \in V_R$ and define $W_v = \mathcal{U}(\mathfrak{h}) \cdot v$. By Lemma 5.3.7 we have that $\dim_k(W_v) < \infty$. From Lemma 5.3.1, it follows that for every element $h \in \mathfrak{h}$, there exists a k-basis of W_v consisting of eigenvectors of h, with eigenvalues in k. Each element of $\mathfrak{h}$ gives rise to an element of $End_k(W_v)$. Since these elements commute and are diagonalizable, it follows that they are simultaneously diagonalizable. Thus every element of W_v is a sum of $\mathfrak{h}$-weight vectors to give us

$$V_R = \bigoplus_{\alpha \in \mathfrak{h}^*} V_{R,\alpha}$$

such that $V_{R,\alpha} = \{w \in V_R : \forall h \in \mathfrak{h}, h.w = \alpha(h)w\}$. Every $V_{R,\alpha}$ is an R-module.

If $\alpha \neq 0$, then by definition $\alpha(h) \neq 0$ for some $h \in \mathfrak{h}$. If we consider the direct sum decomposition of V_R given by Lemma 5.3.1 with respect to h, then we have $V_{R,\alpha} \subset V_{R,s}$ for some $s \neq 0$. Since we know by Lemma 5.3.5 that $V_{R,s}$ is a free R-module of finite rank, then there exists a positive integer d such that $End_R(V_{R,s})$ is isomorphic to $M_{d\times d}(R)$. Similarly as in the proof of Lemma 5.3.6, the elements of $\mathfrak{h}$ give rise to a commuting set $\mathcal{L}$ of elements of $\mathrm{End}_R(V_{R,s})$. The elements of $\mathcal{L}$ satisfy the assumptions of [18, Lemma 3.3]. Using this lemma we conclude that $\mathcal{L}$ is simultaneously diagonalizable by an element of $SL_d(R)$. Thus there exists an R-basis of $V_{R,s}$ with elements that are $\mathfrak{h}$-weight vectors. For all $\alpha \in \mathfrak{h}^*$ such that $\alpha(h) = s$, $V_{R,\alpha}$ is an R-submodule of $V_{R,s}$ generated by a subset of this R-basis. Consequently, $V_{R,\alpha}$ is a free R-module of finite rank. ∎

5.4 The Trivial $\mathfrak{h}$-submodule in V_R

Let $R = k[t, t^{-1}]$ or $R = k[t, t^{-1}, (t-1)^{-1}]$. Now that we have expressed V_R as a direct sum in Lemma 5.3.8, we consider $V_{R,0}$ corresponding to the zero functional $0 \in \mathfrak{h}^*$. We represent by V_R^* the $\mathfrak{sl}_\infty(R)$-module given by $\mathfrak{sl}_\infty(R)$ acting on V_R on the right by

$$v \cdot x = x^t \cdot v \text{ for any } v \in V_R, x \in \mathfrak{sl}_\infty .$$

We identify V_R^* with the free R-module of infinite rows $[a_1, a_2, \dots]$ where $a_i \in R$ and all but finitely many a_i's are zero. If we denote the standard basis of V_R by $\mathcal{B} = \{e_1, e_2, \dots\}$, then we denote the standard basis of V_R^* by $\mathcal{B}^* = \{e_1^*, e_2^*, \dots\}$. There is an R-linear pairing $\langle \cdot, \cdot \rangle : V_R \times V_R^* \to R$ given by $\langle x, y \rangle = yx$. Thus $V_R \otimes_R V_R^*$ is an associative algebra with multiplication $x \otimes x' \cdot y \otimes y' = \langle y, x' \rangle x \otimes y'$. So we get that $V_R \otimes_R V_R^*$ is a Lie algebra with the Lie bracket given by $[a, b] = a \cdot b - b \cdot a$. This yields an R-linear isomorphism of Lie algebras

$$\iota : V_R \otimes_R V_R^* \to \mathfrak{gl}(V_R)$$

where $v \otimes w \mapsto T_{v,w}$, where $T_{v,w}(x) = vwx$.

From now on, when $\mathfrak{g} \neq \mathfrak{sl}_\infty$, we set $V_R^* = V_R$ and $V_{R,\alpha}^* = V_{R,\alpha}$. We now define an element $v \star w \in V_R \otimes_R V_R^*$, given $v \in V_R$ and $w \in V_R^*$, by

$$v \star w = \begin{cases} v \otimes w & \text{if } \mathfrak{g} = \mathfrak{sl}_\infty \\ v \otimes w + w \otimes v & \text{if } \mathfrak{g} = \mathfrak{sp}_\infty \\ v \otimes w - w \otimes v & \text{if } \mathfrak{g} = \mathfrak{so}_\infty \end{cases}$$

Note that for each pair $v \in V_R$ and $w \in V_R^*$, for each Lie algebra $\mathfrak{g}(R)$ the element $v \star w$ in the definition above lies in the corresponding algebra.

Lemma 5.4.1. *[18, Lemma 4.1] Let $x \in V_{R,\alpha}$ and $y \in V_{R,\beta}^*$. For each $h \in \mathfrak{h}$ we have*

$$[h, \iota(xy)] = (\alpha(h) + \beta(h))\iota(xy).$$

Proof: See [18, Lemma 4.1]. ∎

Lemma 5.4.2. *[18, Lemma 4.2] If $x \in V_{R,\alpha}$ and $y \in V_{R,\beta}^*$, and $\alpha + \beta \neq 0$ then $\langle x, y \rangle = 0$.*

Proof: First, we take the case where $\mathfrak{g} = \mathfrak{sl}_\infty$. Let $h \in \mathfrak{h}$, then we have

$$\alpha(h)\langle x, y\rangle = \langle h \cdot x, y\rangle = \langle x, y \cdot h\rangle = -\beta(h)\langle x, y\rangle$$

which gives us $(\alpha(h) + \beta(h))\langle x, y\rangle = 0$ hence $\langle x, y\rangle = 0$.

For the case where $\mathfrak{g} \neq \mathfrak{sl}_\infty$, we have

$$\alpha(h)\langle x, y\rangle = \langle h \cdot x, y\rangle = -\langle x, h \cdot y\rangle = -\beta(h)\langle x, y\rangle$$

which gives us $(\alpha(h) + \beta(h))\langle x, y\rangle = 0$ hence $\langle x, y\rangle = 0$. ∎

Lemma 5.4.3. *[18, Lemma 4.4] If $\alpha \in \mathfrak{h}^*$ satisfies $V_{R,\alpha} \neq 0$, then there exist $x \in V_{R,\alpha}$ and $y \in V^*_{R,-\alpha}$ such that $\langle x, y\rangle = 1$.*

Proof: We prove the case for $\mathfrak{g} = \mathfrak{sl}_\infty$ and the proof for the remaining cases is the same. From Lemma 5.4.2 it follows that $\langle V_{R,\alpha}, V^*_{R,-\alpha}\rangle \neq 0$. Choose $v \in V_{R,\alpha}$ such that $\langle v, V^*_{R,-\alpha}\rangle \neq 0$. Since $v \in V_R$ we express it as a linear combination of the elements of the standard basis $\mathcal{B}$ to get

$$v = c_1 e_{p_1} + \cdots + c_l e_{p_l} \text{ where } c_1, \ldots, c_l \in R$$

Since R is a Bézout domain, we can let $d \in R$ be a greatest common divisor of $c_1, \ldots, c_l \in R$, and we can find a Bézout identity

$$c_1 c'_1 + \cdots + c_l c'_l = d \text{ where } c'_1, \ldots, c'_l \in R.$$

If we set $x := d^{-1}v$ and $w := c'_1 e^*_{p_1} + \cdots + c'_l e^*_{p_l}$, this gives us $\langle x, w\rangle = 1$. Now by Lemma 5.3.8 we can express $w = \sum_{\beta \in \mathfrak{h}^*} w_\beta$, where $w_\beta \in V^*_{R,\beta}$. By applying Lemma 5.4.2 we get that

$$\langle x, w\rangle = \langle x, w_{-\alpha}\rangle = 1.$$

∎

Remark 5.4.1. Lemma 5.4.3 uses the fact that R is a Bézout domain.

Lemma 5.4.4. *[18, Proposition 4.1]*

When $\mathfrak{g} = \mathfrak{sl}_\infty$ or $\mathfrak{sp}_\infty$, we have $V_{R,0} = \{0\}$. If $\mathfrak{g} = \mathfrak{so}_\infty$, then it is impossible to find vectors $v, w \in V_{R,0}$ satisfying

$$\langle v, v\rangle = \langle w, w\rangle = 0 \text{ and } \langle v, w\rangle = 1.$$

Proof: We only prove the case for $\mathfrak{g} = \mathfrak{sl}_\infty$ since the case for $\mathfrak{g} = \mathfrak{sp}_\infty$ is similar. We proceed by contradiction.

Suppose $V_{R,0} \neq \{0\}$. Since $V_{R,0} \neq V_R$, there exists $\alpha \neq 0$ such that $V_{R,\alpha} \neq \{0\}$. If we apply Lemma 5.4.3 we can get elements $x \in V_{R,0}$, $y \in V^*_{R,0}$ and $x_1 \in V_{R,\alpha}$, $y_1 \in V^*_{R,-\alpha}$ such that $\langle x, y\rangle = \langle x_1, y_1\rangle = 1$. Likewise by Lemma 5.4.2 we get that $\langle x_1, y\rangle = \langle x, y_1\rangle = 0$. Since the elements xy and x_1y_1 belong to $\mathfrak{gl}(V_R)$, we get that

$$xyxy = \langle x, y\rangle xy = xy \text{ and } x_1y_1x_1y_1 = \langle x_1, y_1\rangle x_1y_1 = x_1y_1.$$

Thus we get $xyx_1y_1 = x_1y_1xy = 0$. Namely, xy and x_1y_1 are commuting idempotents of $\mathfrak{gl}(V_R)$. Also $xy - x_1y_1 \in \mathfrak{sl}_\infty(R)$ and by [18, Lemma 4.3] $xy - x_1y_1 \in \mathfrak{z}(\mathfrak{h})$. Furthermore a proof similar to [18, Lemma 4.4] shows that $xy - x_1y_1 \in \mathfrak{h}$.

Next we note that

$$(xy - x_1y_1)x = xyx - x_1y_1x = \langle x, y\rangle x - \langle x, y_1\rangle x_1 = x - 0 = x$$

and similarly since $x \in V_{R,0}$ and $xy - x_1y_1 \in \mathfrak{h}$ it follows that

$$(xy - x_1y_1)x = 0.$$

This is a contradiction since $x \neq 0$.

This completes our proof of the Lemma for $\mathfrak{g} = \mathfrak{sl}_\infty$. The proof of the first statement of the Lemma can be adapted to a proof of the second statement. ∎

Lemma 5.4.5. *[18, Lemma 4.6]*

Let $\mathfrak{g} = \mathfrak{sp}_\infty$ or $\mathfrak{so}_\infty$. If $\lambda \in \mathfrak{h}^*$ is nonzero then $rank_R(V_{R,\lambda}) = rank_R(V_{R,-\lambda})$ and there exist R-bases $\{a_1, \ldots, a_r\}$ for $V_{R,\lambda}$ and $\{b_1, \ldots, b_r\}$ for $V_{R,-\lambda}$ such that

$$\langle a_i, a_j \rangle = \langle b_i, b_j \rangle = 0 \text{ and } \langle a_i, b_j \rangle = \delta_{ij} \text{ for every } 1 \leq i, j \leq r.$$

Proof: See [18, Lemma 4.6]. ∎

5.5 Main Theorem

Let $R = k[t, t^{-1}]$ or $R = k[t, t^{-1}, (t-1)^{-1}]$.

Theorem 5.5.1. *[Main Theorem] When $\mathfrak{g} = \mathfrak{sl}_\infty$ or $\mathfrak{sp}_\infty$, any two maximal toral subalgebras of $\mathfrak{g}(R)$ are conjugate under the action of* GL_∞.

Proof: We include the case $\mathfrak{g} = \mathfrak{sl}_\infty$. Firstly, from Lemma 5.4.4 it follows that

$$V_R = \bigoplus_{\lambda \in \mathfrak{h}^* - \{0\}} V_{R,\lambda}$$

where each $V_{R,\lambda}$ is a free R-module of finite rank. Choose an R-basis $\mathcal{B}' = \{f_1', f_2', f_3', \ldots\}$ of V_R that is a union of R-bases of $V_{R,\lambda}$'s. Let $\mathcal{B} = \{e_1, e_2, e_3, \ldots\}$ be the standard basis of V_R and define

$$T : V_R \to V_R, \text{ such that } T(e_i) = f_i'.$$

The matrix of T in the R-basis $\mathcal{B}$ has only finitely many nonzero entries in each column. Furthermore we have that the set $T^{-1}\mathfrak{h}T \subset \mathrm{End}_R(V_R)$ such that $\mathcal{B}$ gets mapped to an R-basis consisting of common eigenvectors, with eigenvalues in k. Now we will show that T and T^{-1} are both $\mathcal{B}$-finitary, so that any element of $T^{-1}\mathfrak{g}(R)T$ is $\mathcal{B}$-finite.

Suppose $e_i = \sum a_{i,j} f'_j$. We fix $j = 1$ and without loss of generality prove that there are only finitely many nonzero $a_{i,1}$'s. Suppose $f'_1 \in V_{R,\lambda}$. It follows that there is an $h \in \mathfrak{h}$ such that $\lambda(h) \neq 0$. Therefore if $a_{i,1} \neq 0$ then $h \cdot e_i \neq 0$. But since h is $\mathcal{B}$-finite, then $h \cdot e_i \neq 0$ can only be true for finitely many e_i's. Thus T^{-1} is $\mathcal{B}$-finitary.

Now to show that T is $\mathcal{B}$-finitary suppose $f'_i = \sum b_{i,j} e_j$. We fix $j = 1$ and without loss of generality prove that there are only finitely many nonzero $b_{i,1}$'s. Set $x = E_{1,1} - E_{2,2}$, then $x \cdot f'_i = b_{i,1}e_1 - b_{i,2}e_2$. We also know that

$$x = x_0 + \sum_{\alpha \in \Delta(\mathfrak{g}(R),\mathfrak{h})} x_\alpha \text{ where } x_0 \in \mathfrak{z}(\mathfrak{h}) \text{ and } x_\alpha \in \mathfrak{g}(R)_\alpha.$$

It is left to show that if $y = x_\alpha$ for some α, then $y \cdot f'_i \neq 0$ happens for only finitely many f'_is.

The first case is that $y \in \mathfrak{z}(\mathfrak{h})$. Then $y \cdot f'_i \neq 0$ happens for only finitely many f'_is, since y is $\mathcal{B}$-finite thus $\mathrm{Im}(y)$ lies in the R-submodule of V_R generated by $e_1, \dots, e_{n_0}$, for some $n_0 < \infty$. Now $y \cdot V_{R,\lambda} \subset V_{R,\lambda}$, since y commutes with $\mathfrak{h}$. Since for every λ we have $rank_R(V_{R,\lambda}) < \infty$, it suffices to show that the restriction of y to $V_{R,\lambda}$ can be nonzero only for finitely many λ's. Suppose, by contradiction that there exists an infinite set $\Lambda = \{\lambda_1, \lambda_2, \dots\} \in \mathfrak{h}^*$, such that for every $\lambda \in \Lambda$ we can find $v_\lambda \in V_{R,\lambda}$ such that $y \cdot v_\lambda \neq 0$. The vectors $y \cdot v_{\lambda_p}$ belong to $\mathrm{Im}(y)$ and are linearly independent over R, because they belong to distinct weight spaces of $\mathfrak{h}$. This contradicts the fact that $rank_R(\mathrm{Im}(y)) < \infty$.

Lastly, assume $y \in \mathfrak{g}(R)_\alpha$, then $y \cdot f'_i \neq 0$ happens for only finitely many i. Note that $y \cdot V_{R,\lambda} \subset V_{R,\lambda+\alpha}$. If there are infinitely many λ's such that $y \cdot V_{R,\lambda} \neq \{0\}$, then linear independence over R of $V_{R,\lambda+\alpha}$'s contradicts the fact that $rank_R(\mathrm{Im}(y)) < \infty$. ∎

Now that we have proved the main theorem, we now state that it can be generalized to universal central extensions of the corresponding algebras.

Theorem 5.5.2. *Let $R = k[t, t^{-1}]$ or $R = k[t, t^{-1}, (t-1)^{-1}]$. Let $\mathfrak{g}$ be one of $\mathfrak{sl}_\infty$ or*

$\mathfrak{sp}_\infty$. *Let* $\widehat{\mathfrak{g}(R)}$ *denote the universal central extension of* $\mathfrak{g}(R)$. *Then any two maximal toral subalgebras of* $\widehat{\mathfrak{g}(R)}$ *are conjugate.*

Proof: In [18, Section 6] it is shown that a direct system

$$\mathfrak{a}_1 \xrightarrow{i_1} \mathfrak{a}_2 \xrightarrow{i_2} \mathfrak{a}_3 \xrightarrow{i_3} \cdots$$

of perfect Lie algebras such that all the i_r's are monomorphisms induces a direct system

$$\hat{\mathfrak{a}_1} \xrightarrow{\hat{i_1}} \hat{\mathfrak{a}_2} \xrightarrow{\hat{i_2}} \hat{\mathfrak{a}_3} \xrightarrow{\hat{i_3}} \cdots$$

where $\hat{\mathfrak{a}_1}$ is the u.c.e. of $\mathfrak{a}_1$. It is found that if $\mathfrak{a}$ is the direct limit of the $\mathfrak{a}_n$'s and $\hat{\mathfrak{a}}$ is the direct limit of the $\hat{\mathfrak{a}_n}$'s, then $\hat{\mathfrak{a}}$ is the u.c.e of $\mathfrak{a}$. Therefore one can adopt the argument of [18, Cor 5.1] to the case under consideration. ∎

Chapter 6

Finitely Generated Bézout Domains

In this chapter we classify finitely generated (as a k-algebra) Bézout domains $k \subset R \subset k(x_1, \ldots, x_n)$. This will show that to solve the conjugacy problem over such rings, we should only consider a limited number of examples.

Let R be a commutative ring with unity. Recall that a prime ideal P of R is a proper ideal of R with the property that if a and b are two elements of R such that their product ab is an element of P, then a is in P or b is in P. This brings us to our first definition.

6.1 Definitions and Results

Definition 6.1.1. *The* ***Krull dimension*** *of a ring R is defined by*

$$\dim_{\mathrm{krull}}(R) := \sup\{d : \textit{There is a chain of prime ideals } P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d\}$$

Example 6.1.1. 1. Let $R = k$, then $\dim_{\mathrm{krull}}(R) = 0$.

2. Let R be a ring such that every prime ideal is maximal, then $\dim_{\mathrm{krull}}(R) = 1$.

3. A theorem in commutative algebra says that $\dim_{\text{krull}}(k[x_1,\ldots,x_n]) = n$, (See [17]).

Definition 6.1.2. *A ring R is called **Noetherian** if every ascending chain of ideals $I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq$ is finite. Namely, we have $I_m = I_{m+1} = I_{m+2} = \cdots$ for some finite number m.*

Remark 6.1.1. It is known by Hilbert's theorem that $k[x_1, x_2, \ldots x_n]$ is Noetherian, (See [17]).

Definition 6.1.3. *Let R be a Bézout domain that contains k. Then R is called **finitely generated** if there exist elements $r_1, \ldots r_n$ such that every element of R can be written as a polynomial in variables $r_1, \ldots r_n$ with coefficients in k.*

Theorem 6.1.1. *If R is a finitely generated Bézout domain satisfying $k \subset R \subset k(x_1, \ldots, x_n)$, then $\dim_{\text{krull}}(R) = 1$.*

Proof: Since R is a finitely generated Bézout domain, it is Noetherian. We have that every ideal of R is finitely generated. But in a Bézout domain every finitely generated ideal is principal by Lemma 1.2.1, hence R is a PID. Therefore every prime ideal of R is maximal. Thus $\dim_{\text{krull}}(R) = 1$. ∎

Definition 6.1.4. *Let $k \subset K$ be fields. Then elements $a_1, \ldots, a_m \in K$ are called **algebraically independent elements of** K **over** k if for every nonzero polynomial $p \in k[x_1, \ldots, x_m]$ we have $p(a_1, \ldots, a_m) \neq 0$.*

Definition 6.1.5. *Let $k \subset K$ be fields. We define the **transcendence degree of** K **over** k, denoted $\operatorname{tr}\deg(K/k)$, as the maximum number of algebraically independent elements of K over k.*

Theorem 6.1.2. *Let R be a finitely generated k-algebra which is also an integral domain. Let K be the field of fractions of R. Then $\dim_{\text{krull}}(R) = \operatorname{tr}\deg(K/k)$.*

Proof: See [12, Chapter 5]. ∎

Theorem 6.1.3. *Let $k \subset R \subset k(x_1, \ldots, x_n)$ be a finitely generated Bézout domain. Let K be the field of fractions of R. Then there exists an element $y \in k(x_1, \ldots, x_n)$ such that $K = k(y) := \{\frac{p(y)}{q(y)} : p, q$ are 1-variable polynomials with coefficients in k, $q \neq 0\}$*

Proof: By Theorem 6.1.1 we have that $\dim_{\text{krull}}(R) = \operatorname{tr}\deg(K/k) = 1$. The proposition follows from the general Lüroth theorem [19]. ∎

Remark 6.1.2. From Theorem 6.1.3 it follows that if $k \subset R \subset k(x_1, \ldots, x_n)$ is a finitely generated Bézout domain, then R is isomorphic (as a k-algebra) to a k-subalgebra of the field $k(x)$.

Definition 6.1.6. *Let R be a finitely generated k-algebra which is also an integral domain. Then R is called a **nonsingular affine curve** if $\dim_{\text{krull}}(R) = 1$ and for every prime ideal $P \subset R$ we have $\dim\left(PR_P/(PR_P)^2\right) = 1$, where R_P is the localization of R with respect to the multiplicative set $R - P$ and $PR_P = \{\frac{a}{b} : a \in P, b \in R - P\}$ is the unique maximal ideal of R_P. Note that $\left(PR_P/(PR_P)^2\right)$ is a vector space over k, since $k \subset R$. (Thus $\dim\left(PR_P/(PR_P)^2\right)$ is the dimension as a vector space and not krull dimension.)*

Theorem 6.1.4. *Every nonsingular affine curve R whose fraction field k is isomorphic to $k(x)$, is isomorphic to a localization of $k[x]$ of the form $k[x, \frac{1}{p(x)}]$.*

Proof: This is a consequence of [8, Chapter 1, Exercise 6.1]. ∎

Theorem 6.1.5. *Let $k \subset R \subset k(x_1, \ldots, x_n)$ be a finitely generated Bézout domain. Then $R = k[x, \frac{1}{p(x)}]$ is a k-algebra, where $p(x) \in k[x]$.*

Proof: By Theorem 6.1.4 it is enough to show R is a nonsingular affine curve with fraction field isomorphic to $k(x)$. The statement about fraction fields follows from Theorem 6.1.3. The statement $\dim_{\mathrm{krull}}(R) = 1$ follows from Theorem 6.1.1. Next we prove the nonsingularity condition.

Since it follows from the proof of Theorem 6.1.1 that R is a PID, it is also a UFD, thus R is integrally closed [20, Section 2, 5.1]. Hence R is normal, in the sense of [20, Section 2, 5.1]. But if R is normal and $\dim_{\mathrm{krull}}(R) = 1$, then R is nonsingular [20, Corollary 2, 5.1]. ∎

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