

# **Properties and Behaviours of Fuzzy Cellular Automata**

Heather Betel

Thesis submitted to the  
Faculty of Graduate and Postdoctoral Studies  
in partial fulfilment of the requirements  
for the PhD degree in Electrical and Computer Engineering

School of Electrical Engineering and Computer Science  
Faculty of Engineering  
University of Ottawa

© Heather Betel, Ottawa, Canada, 2012

---

---

# Abstract

Cellular automata are systems of interconnected cells which are discrete in space, time and state. Cell states are updated synchronously according to a local rule which is dependent upon the current state of the given cell and those of its neighbours in a pre-defined neighbourhood. The local rule is common to all cells. Fuzzy cellular automata extend this notion to systems which are discrete in space and time but not state. In this thesis, we explore fuzzy cellular automata which are created from the extension of Boolean rules in disjunctive normal form to continuous functions. Motivated by recent results on the classification of these rules from empirical evidence, we set out first to show that fuzzy cellular automata can shed some light on classical cellular automata and then to prove that the observed results are mathematically correct.

The main results of this thesis can be divided into two categories. We first investigate the links between fuzzy cellular automata and their Boolean counter-parts. We prove that number conservation is preserved by this transformation. We further show that Boolean additive cellular automata have a definable property in their fuzzy form which we call self-oscillation. We then give a probabilistic interpretation of fuzzy cellular automata and show that homogeneous asymptotic states are equivalent to mean field approximations of Boolean cellular automata. We then turn our attention the asymptotic behaviour of fuzzy cellular automata.

In the second half of the thesis we investigate the observed behaviours of the fuzzy cellular automata derived from balanced Boolean rules. We show that the empirical results

of asymptotic behaviour are correct. In fuzzy form, the balanced rules can be categorized as one of three types: weighted average rules, self-averaging rules, and local majority rules. Each type is analyzed in a variety of ways using a range of tools to explain their behaviours.

---

---

# Acknowledgements

I would like to acknowledge the help of my committee in completing this thesis. I would especially like to thank my supervisors, Drs Flocchini and Karmouch for their support and encouragement. Extra special thanks are owed to Paola for the extra special attention she paid to this work.

I would also like to thank my family for their love and support throughout this process and beyond.

---

---

# Contents

<b>Abstract</b>	<b>ii</b>
<b>Acknowledgements</b>	<b>iv</b>
<b>List of Figures</b>	<b>viii</b>
<b>List of Tables</b>	<b>ix</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Discrete Cellular Automata . . . . .	2
1.2 Continuous Cellular Automata . . . . .	3
1.3 Thesis Objectives and Outline . . . . .	4
1.4 Published Research to Date . . . . .	6
<b>2 Discrete Cellular Automata</b>	<b>8</b>
2.1 Definitions . . . . .	8
2.2 Classification . . . . .	12
2.3 Injective, Surjective and Reversible Cellular Automata . . . . .	15
2.4 Number Conservation . . . . .	20
2.5 Additivity . . . . .	22
2.6 Summary . . . . .	24
<b>3 Fuzzy Cellular Automata</b>	<b>25</b>
3.1 Definitions . . . . .	25
3.2 Classification . . . . .	26
3.3 Evolution . . . . .	29
3.4 Applications . . . . .	30
3.5 CNF Fuzzification . . . . .	31
3.6 Summary . . . . .	34

<b>4</b>	<b>Preserved Properties</b>	<b>37</b>
4.1	Definitions and Basic Properties . . . . .	37
4.2	Number Conservation . . . . .	38
4.3	Spatial Number Conservation . . . . .	41
4.4	Conservation Revisited . . . . .	43
4.5	Probabilistic Interpretation of Fuzzification . . . . .	46
4.5.1	Probabilistic Properties of FCA . . . . .	47
4.5.2	Mean Field Approximation . . . . .	48
4.6	Summary . . . . .	50
<b>5</b>	<b>Additivity and Self-Oscillation</b>	<b>52</b>
5.1	Definitions and Basic Properties . . . . .	52
5.2	Equivalence between Self-Oscillation and Additivity . . . . .	55
5.3	Summary . . . . .	59
<b>6</b>	<b>Weighted Average Rules</b>	<b>60</b>
6.1	Definitions and Basic Properties . . . . .	61
6.1.1	Definitions for Periodic Behaviour . . . . .	61
6.1.2	Conjugate CA . . . . .	62
6.1.3	Definitions for Weighted Average Rules . . . . .	62
6.2	A General Convergence Theorem . . . . .	63
6.3	Convergence of Elementary Weighted Average Rules . . . . .	66
6.3.1	Rule 172 . . . . .	67
6.3.2	Rule 78 . . . . .	67
6.3.3	Rule 27 . . . . .	69
6.3.4	Rule 58 . . . . .	71
6.3.5	Rule 184 . . . . .	74
6.3.6	Rule 29 . . . . .	77
6.3.7	Rule 46 . . . . .	81
6.4	Creating Rules with a Given Periodicity . . . . .	84
6.5	Summary . . . . .	85
<b>7</b>	<b>Self-Averaging Rules</b>	<b>86</b>
7.1	Definitions and Basic Properties . . . . .	86
7.2	Convergence of Self-averaging Rules . . . . .	89
7.3	Behaviour around the point of convergence . . . . .	91
7.3.1	Self-Oscillation . . . . .	91
7.3.2	Other Oscillations . . . . .	97
7.3.3	Observations . . . . .	102
7.4	Error Propagation . . . . .	103
7.4.1	Additive Rules . . . . .	104
7.4.2	Centric Rules . . . . .	108

---

7.4.3	Non-Centric Rules . . . . .	115
7.4.4	Boolean Propagation . . . . .	123
7.5	Summary . . . . .	124
<b>8</b>	<b>Local Majority Rules</b>	<b>125</b>
8.1	Definitions and Basic Properties . . . . .	125
8.2	Rule 232 . . . . .	126
8.3	Rule 23 . . . . .	129
8.4	Rule 77 . . . . .	130
8.5	Rule 178 . . . . .	131
8.6	Summary . . . . .	133
<b>9</b>	<b>Conclusions</b>	<b>134</b>
9.1	Contributions of Thesis . . . . .	134
9.2	Future Work . . . . .	136
	<b>Bibliography</b>	<b>137</b>

---

---

# List of Figures

2.1	Sample Rules in Wolfram Classes. . . . .	13
2.2	de Bruijn graph for rule 184. . . . .	17
2.3	de Bruijn graph for rule 105. . . . .	18
2.4	Seirpinski triangles from rule 90 . . . . .	23
3.1	Evolution of Fuzzy Rules 105 and 12. . . . .	28
3.2	Evolution of Fuzzy Rule 29. . . . .	29
3.3	Evolution of Fuzzy Rule 46. . . . .	30
3.4	Evolution of Fuzzy Rule 138. . . . .	31
3.5	Sample Rules in Randomized CNF Classes . . . . .	36
6.1	Conjugate CA . . . . .	62
7.1	Evolution of fuzzy Rule 30 . . . . .	87
7.2	Additive rules: 60, 90, 150 and 105. . . . .	92
7.3	Centric-rules: 54, 57, 108, 156. . . . .	93
7.4	Non-centric rules: 30, 45,106,154. . . . .	94
8.1	Examples of Generalized Local Majority Rules . . . . .	127

---



---

# List of Tables

2.1 Example elementary rule . . . . .	11
2.2 Example of additivity in rule 150 . . . . .	23
3.1 Observed dynamics of Circular Elementary Cellular Automata. Rules equivalent under conjugation, reflection and both are not indicated. . . . .	35
5.1 Rule 90: fuzzy behaviour around $\frac{1}{2}$ (< indicates "< $\frac{1}{2}$ ", > indicates "> $\frac{1}{2}$ " ) (left); Boolean rule (right). . . . .	54
6.1 Weighted Average Rules. Rules equivalent under conjugation, reflection and both are indicated in parenthesis. . . . .	66
7.1 Self-averaging elementary fuzzy CA rules (the rules equivalent under conjugation, reflection, or both are not indicated). . . . .	88
7.2 Rule 60: fluctuations of the fuzzy rule around $\frac{1}{2}$ . . . . .	95
7.3 Rule 150: fluctuations of the fuzzy rule around $\frac{1}{2}$ (left). . . . .	96
7.4 Rules 30 and 45: fluctuations around $\frac{1}{2}$ . . . . .	99
7.5 Rules 106 and 154: fluctuations around $\frac{1}{2}$ . . . . .	100
7.6 Rules 108 and 156: fluctuations around $\frac{1}{2}$ . . . . .	101
7.7 Rules 54 and 57: fluctuations around $\frac{1}{2}$ . . . . .	101
7.8 The conditions for convergence to $\frac{1}{2}$ are given by indicating, for each rule, the subconfigurations that forbid convergence. . . . .	105

---

7.9 Description of convergent configurations $\alpha * \beta$ for rules $f_{108}$ and $f_{156}$ , where $\alpha$ is a left-infinite sequence, $\beta$ is a right-infinite sequence, and $*$ is the fuzzy value. . . . .	105
7.10 Rule 54 with fuzzy data . . . . .	108
7.11 Rule 57 with fuzzy data . . . . .	110
7.12 Rule 108 with fuzzy data . . . . .	111
7.13 Rule 156 with fuzzy data . . . . .	114
7.14 Rule 45 with fuzzy data . . . . .	117
7.15 Rule 30 with fuzzy data . . . . .	118
7.16 Rule 154 with fuzzy data . . . . .	120
7.17 Rule 106 with fuzzy data . . . . .	122
8.1 Majority rules. . . . .	126

---

---

# CHAPTER 1

---

---

## Introduction

The study of cellular automata dates back at least until the 1940s where these discrete systems were used to model dynamical biological systems. They gained popularity in the 70s with work by John Conway, the Game of Life, which was written up by Martin Gardner in Scientific American [38]. They moved further into the public consciousness with the enthusiastic work of Stephen Wolfram first in 1983 when he published a series of papers investigating the so-called elementary cellular automata and then in 2002 with the publication of his book *A New Kind of Science* which suggests that the universe is intrinsically discrete. Cellular automata have been used over the years to model many kinds of phenomena in a broad range of topics. The study of dynamical systems often benefitted from analysis as a cellular automata. With our increased reliance on the computer, an inherently discrete machine, the restrictions of the initial formalization of cellular automata were not considered too severe. However, much of the more recent study has concentrated on what happens when one or more of the requirements in the strict definition of cellular automata is relaxed. This thesis provides the first in depth study of cellular automata which are continuous in state, what we have called *fuzzy cellular automata*. We have looked at cellular automata which have a grid pattern, are discrete in time but can have states on the interval  $[0, 1]$ . We have compared these automata to the most studied classical cellular automata, those with binary states. In the rest of this chapter we give brief introductions to classical and continuous

cellular automata, then we outline the rest of the thesis. Finally, we give a list of publications which have resulted from this work so far.

## 1.1 Discrete Cellular Automata

Discrete dynamical systems known as cellular automata (CA) were first introduced by von Neumann as models of self-organizing/reproducing behaviours [74]. They consisted of a grid of cells whose states were synchronously updated following a rule applied to a pre-defined neighbourhood of each cell. As originally defined, cellular automata were discrete in space, time and state. It was quickly seen that even with simple rules and binary states, cellular automata could exhibit what appeared to be quite complex behaviours. Probably the best known example is the Game-of-Life which was invented by John Conway in 1970 [38]. Played on an infinite two-dimensional grid, the Game-of-Life started from an arbitrary Boolean configuration and updated according to the following rules:

1. A living cell (a Boolean one, usually depicted as a black cell) stayed alive if two or three of its neighbours in the eight surrounding cells on the grid were also alive. Fewer than two or more than three living neighbours caused a living cell to die at the next time step.
2. A non-living cell became alive at the next time step if it had exactly three living cells among its eight surrounding cells.

Even from these simple rules, amazing patterns appeared that seemed to mimic structures found in nature: gliders that floated across the screen, still life patterns that remained eternally fixed, oscillators that ran through a sequence of patterns before returning to their original form. It is not surprising then that cellular automata have been used in a wide array of disciplines to mimic and predict complex behaviours that eluded simple descriptions using other areas of mathematics such as differential equations.

Among the simplest cellular automata are the *elementary cellular automata* which are one dimensional, have two possible states, and a neighbourhood consisting of the cell itself and its left and right neighbours. Based on these simple parameters, there are 256 distinct possible local rules that exhibit a wide range of behaviours and properties. The richness and diversity of these simple rules has engendered an extensive amount of research since their introduction. They have been studied from many different perspectives: dynamic behaviour (e.g., see [22, 36]), algebraic properties (e.g., see [45, 47, 48]), topological properties (e.g., see [11, 12, 15]), and computational complexity (e.g., see [21, 23, 24]) are a few among many examples. We will look at some of those properties in the next chapter.

## 1.2 Continuous Cellular Automata

There have been several modifications to the original definition of cellular automata that have allowed continuity to be studied in otherwise classical cellular automata. One such modification of particular significance were the continuous coupled map lattices (a variant of cellular automata) introduced by Kaneko which were discrete in space and time, but continuous in state [43]. They were conceived of as simple models exhibiting spatio-temporal chaos, and now have applications in many different areas including fluid dynamics, biology, chemistry, etc. (e.g., [43, 49]) and in fields as divergent as ecology and theoretical computer science (e.g., see [7, 51, 81]).

Introduced in [13, 14] to study the impact that state-discretization has on the behaviour of these systems, fuzzy cellular automata (FCA), or disjunctive normal form (DNF)-continuous cellular automata, are a particular type of continuous cellular automata where the local transition rule is the “fuzzification” of the local rule of a corresponding Boolean cellular automaton in disjunctive normal form<sup>1</sup>. They have since gained currency as a modeling tool in pattern recognition (e.g., see [53–55]), and to mimic nature (e.g. [18, 68]), and

---

<sup>1</sup>These are not to be confused with a variant of cellular automata, also called fuzzy cellular automata, where the fuzziness refers to the local rule (e.g., see [1])

have been used to investigate the effect of perturbation (e.g. noisy source, computational error, mutation, etc.) on the evolution of Boolean cellular automata [34]. The asymptotic dynamics of elementary fuzzy cellular automata (i.e., with dimension and radius one) have been observed through simulations in [32] where an empirical classification was proposed. The asymptotic behaviour of some fuzzy cellular automata rules have only recently been studied analytically (e.g., see [33, 57, 59]); in particular it was shown in [57, 59] that none has chaotic dynamics, thus supporting the empirical evidence of [32]. Finally, methods for controlling the dynamics of fuzzy rule 90 have been investigated in [84].

Besides being interesting in their own right and providing a good model for certain applications, an important reason to study fuzzy cellular automata is to understand their relationship with Boolean cellular automata. In fact, the dynamics of fuzzy cellular automata might shed some light on their Boolean counter-parts, and properties of Boolean cellular automata could be interpreted differently in light of those of fuzzy cellular automata. If clear links between the two systems can be established, properties of Boolean cellular automata not previously observed might be revealed by their presence in fuzzy cellular automata. Unfortunately, until now, no such light had been shed and no such results existed.

To date, little is known about the dynamics of fuzzy cellular automata, except that, in quiescent backgrounds, none of them has chaotic dynamics [33, 57, 59]. The case of circular fuzzy cellular automata has been studied experimentally from random initial configurations; an empirical classification has been proposed based on these studies [32], suggesting that all elementary rules have, as expected, asymptotic periodic behaviour but, surprisingly, with periods of only certain lengths: 1, 2, 4, and  $n$  (where  $n$  is the size of the circular lattice). However, prior to this work, no analytical proofs existed.

### 1.3 Thesis Objectives and Outline

This thesis was initially motivated by the experimental work of Flocchini and Cezar, [32]. In this paper, all of the elementary DNF-fuzzy rules were implemented and their asymptotic

behaviours were observed and classified. The research which developed into this thesis began by providing analytical proofs for the observations made in that paper. As our understanding of the DNF-fuzzy rules developed, our study of them branched into two main directions. First, we explored the link between fuzzy cellular automata and their Boolean counterparts in order to show the usefulness of this approach to the larger community of researchers. We also continued to analyse the classification scheme for fuzzy cellular automata, described in detail in the following chapter, in order to prove that the behaviours observed for the various rules were mathematically true. To these ends we have found properties which are preserved by the fuzzification and explained the implications of some of the asymptotic behaviours to the Boolean rules. In some cases, the fuzzy behaviours have allowed us to look at the Boolean rules in a new light. We have also analytically proven the asymptotic behaviours of the fuzzifications of the balanced rules, those rules whose truth tables have exactly four transitions to zero and four transitions to one. We now give an outline of this thesis.

The thesis can be roughly divided into three parts. The next two chapters give a background into research to date on first Boolean and then fuzzy cellular automata. The new results of this thesis can then be divided into two main parts: the relationship between Boolean rules and their fuzzifications, and the asymptotic behaviour of fuzzy cellular automata.

In the first of these parts, chapters 4 through 6, we study the relationship between Boolean rules and their fuzzifications. The aim is to show that fuzzy cellular automata can be a useful tool to those studying more classical CA and, furthermore, that they are intrinsically linked. Despite the apparent differences in their asymptotic behaviours, the properties of interest can be defined in the fuzzy domain and are maintained. We begin in Chapter 4 by showing that number conservation properties are preserved through fuzzification. Further, we show that such properties are easily determined in the fuzzy case with mathematical tools available at the high school level. In Chapter 5, we show that additivity of Boolean rules produces certain easily observable behaviours on convergence of their fuzzy equivalents and we show why this property has this effect on convergence. Finally, in Chapter 6, we give a

probabilistic interpretation of our fuzzification which leads to a new understanding of the asymptotic results.

In the third and final part, we give analytical proofs of the results that had been observed for three different types of rules. Combined, these three types of rules include all of the balanced elementary rules - rules having an equal number of transitions to one as to zero. They are the weighted average rules, self-averaging (or permutative) rules, and generalized local majority rules.

## 1.4 Published Research to Date

The following is a list of publications that has resulted from the work outlined in this thesis.

- Journal Papers

1. H. Betel and P. Flocchini, (2011). On the asymptotic behaviour of circular fuzzy cellular automata. In *Journal of Cellular Automata*. 6: 25-52.
2. H. Betel and P. Flocchini, (2011). On the relationship between Boolean and fuzzy cellular automata. *Theoretical Computer Science*. 412: 703-713.

- Conference Papers

1. H. Betel, P. Flocchini, and A. Karmouch, (2010). Asymptotic behaviour of self-averaging continuous cellular automata. *Proceedings of the 16th International Workshop on Cellular Automata and Discrete Complex Systems*. 173-186. (Short paper).
2. H. Betel and P. Flocchini, (2009). On the asymptotic behaviour of circular fuzzy cellular automata. *Proc. of the 15th International Workshop on Cellular Automata and Discrete Complex Systems*. Brazil. *Electronic Notes in Theoretical Computer Science*. 252:23-40. (Full paper).

- 
3. H. Betel and P. Flocchini, (2009). On the relationship between Boolean and fuzzy cellular automata. Proc. of the 15th International Workshop on Cellular Automata and Discrete Complex Systems. Brazil. Electronic Notes in Theoretical Computer Science. 252: 5-21. (Full paper).
  4. H. Betel and P. Flocchini, (2009). Fluctuations of fuzzy cellular automata around their convergence point. 2009 International Symposium on Nonlinear Theory and its Applications NOLTA'09. 655-658.

---

---

## CHAPTER 2

---

---

# Discrete Cellular Automata

Cellular automata date back to the work of John von Neumann in the 1940s. In these early days of computing, he was interested in modelling how biological systems seemed to work: independent but interconnected cells, operating in their own environment, but according to a fixed set of rules. In this chapter we will review some of the work which has been done in the field of cellular automata. We will begin by examining classical cellular automata with an emphasis on Boolean CA, giving precise mathematical definitions.

### 2.1 Definitions

A  $d$ -dimensional infinite cellular automaton can be described by a quadruplet  $C = \langle \mathbb{Z}^d, S, N, g \rangle$  where:  $\mathbb{Z}^d$  represents the set of *cells*,  $S$  is a finite set of *states*,  $N$  is the *neighbourhood* of a cell and can be defined in different ways but usually contains the cell itself plus the neighbouring cells up to a certain radius, and  $g : S^{|N|} \rightarrow S$  is the *local transition rule* (or simply *local rule*) of the automaton. Among the possible states there is normally at least one state which is designated as *quiescent*, or in-active, and at least one active state. Given an initial configuration,  $C^0$ , that is a mapping  $C^0 : \mathbb{Z}^d \rightarrow S$ , cell states are synchronously updated at each time step by the local transition rule applied to their neighbourhoods. The set of all possible configurations is denoted by  $C$ . The global transition rule  $G : C^t \rightarrow C^{t+1}$  is

the mapping of the entire configuration. A configuration is the resulting map  $C^t : \mathbb{Z}^d \rightarrow S$  at any time  $t$ . In a Boolean cellular automata the set of possible states  $S$  is  $\{0, 1\}$ , with 0 being the quiescent state. Cellular automata as defined here are *infinite* and in fact, we will use the convention of choosing a point of origin in this case. A  $d$ -dimensional cellular automaton is said to have a *finite* configuration if it has a finite number of non-quiescent states in an infinite quiescent background. In the Boolean case, this would mean that  $C^t(\vec{z}) = 0$  for all but finitely many  $\vec{z} \in \mathbb{Z}^d$ . *Circular* cellular automata can be thought of as infinite CA with a periodic repeating pattern, or as a finite circular  $d$ -dimensional grid. When necessary, we will distinguish the restrictions of the global rule to finite and circular configurations from the infinite global rule by denoting them with subscripts,  $G_F$  and  $G_C$ , respectively. Similarly, we will denote the restricted configuration spaces as  $C_F$  and  $C_C$ . Because fuzzy cellular automata, the topic of this thesis, are formed from Boolean cellular automata, as we will see in the next chapter, we will restrict our detailed discussion here to the Boolean domain.

### 1-Dimensional Cellular Automata

In one dimension, infinite cellular automata are often regarded as having a point of origin, and radiating out in both directions, hence being *bi-infinite*. The cells of the configuration can then be referenced using subscripts rather than vector notation. In this thesis, we will assume bi-infinite CA whenever we are using infinite CA. Finite configurations can be redefined as those configurations where for some  $n$ ,  $x_i = 0$  for all  $|i| > n$ . In essence, a finite non-zero block bounded on both sides with infinite zero blocks. There are other definitions of finite configurations using other boundary conditions but they are not explored in this thesis. In the case of *one-dimensional circular* Boolean cellular automata, a configuration is a finite vector  $\mathbf{X}^t \in \{0, 1\}^n = (x_0^t, x_1^t, \dots, x_{n-1}^t)$  where cells are index modulo  $n$ , the length of the finite array. Alternatively, one can think of an infinite array containing a periodic configuration.

The neighbourhood of a cell in 1 dimension consists of the cell itself and its  $r$  left and right neighbours, thus the local transition rule has the form:  $g : \{0, 1\}^{2r+1} \rightarrow \{0, 1\}$ . The global dynamics of a one-dimensional cellular automaton composed of  $n$  cells is then defined

by the global transition rule:  $G : \{0, 1\}^n \rightarrow \{0, 1\}^n \quad s.t. \quad \forall \mathbf{X} \in \{0, 1\}^n, \forall i \in \{0, \dots, n-1\}$ , the  $i$ -th component  $G(X)_i$  of  $G(X)$  is  $G(X)_i = g(x_{i-r}, \dots, x_i, \dots, x_{i+r})$ , where all operations on indices are modulo  $n$ . Boolean cellular automata with dimension and radius one are called *elementary*. These CA were the subject of extensive study by Wolfram [78, 79, 81] and are the focus of much of this thesis.

The local transition rule  $g$  of a Boolean CA is typically given in tabular form by listing the  $2^{2r+1}$  binary tuples corresponding to the  $2^{2r+1}$  possible local configurations a cell can detect in its direct neighbourhood, and mapping each tuple to a Boolean value  $b_i$  ( $0 \leq i \leq 2^{2r+1} - 1$ ):  $(00 \dots 00, 00 \dots 01, \dots, 11 \dots 10, 11 \dots 11) \rightarrow (b_0, \dots, b_{2^{2r+1}})$ . In a naming system devised by Wolfram [78], the binary representation  $(b_0, \dots, b_{2^{2r+1}})$  is often converted into the decimal representation  $\sum_i 2^i b_i$ , and this value is typically used as the decimal code of the rule (or rule number).

Let us denote by  $d_i$  the tuple mapping to  $b_i$ , and by  $\mathcal{T}_1$  the set of tuples mapping to one. The local transition rule can also be canonically expressed in *disjunctive normal form* (DNF) as follows:

$$g(v_{-r}, \dots, v_r) = \bigvee_{i < 2^{2r+1}} b_i \bigwedge_{j=-r:r} v_j^{d_{i,(j+r)}}$$

where  $d_{ij}$  is the  $j$ -th digit, from left to right of  $d_i$  (counting from zero) and  $v_j^0$  (resp.  $v_j^1$ ) stands for  $\neg v_j$  (resp.  $v_j$ ) i.e.,  $\bigwedge_{j=-r:r} v_j^{d_{i,(j+r)}}$  will be equal to one precisely when  $v_{-r} \dots v_r$  viewed as a single binary number is equal to  $d_i$ .

**Example.** Consider the following truth table for an elementary rule: The rule code is given by reading the last column (the output) from bottom to top as a binary number. In this case, the rule has binary code  $00010010_{(2)}$  which is 18 decimal. Hence it will be referred to "elementary rule 18" or simply "rule 18". The values of  $b_i$  are obtained from reading down the column so that  $b_0 = 0, b_1 = 1$ , etc. The set  $\mathcal{T}_1$  is given by  $\{001, 100\}$ , while  $\mathcal{T}_0$  is its complement,  $\{000, 010, 011, 101, 110, 111\}$ . The local transition rule in DNF form is the following:

$$g(v_{-1}, v_0, v_1) = (\neg v_{-1} \wedge \neg v_0 \wedge v_1) \vee (v_{-1} \wedge \neg v_0 \wedge \neg v_1).$$

$v_{-1}$	$v_0$	$v_1$	$g(v_{-1}, v_0, v_1)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

Table 2.1: Example elementary rule

### Higher Dimensional Cellular Automata

Two dimensional cellular automata are perhaps the most interesting. The added dimensionality allows them to be used as models for a wider range of physical phenomena - for example forest fire spread [25], volcanic eruptions [72], and even coffee percolation [10] - while not overly increasing the complexity. The complexity, however, is increased. Consider the simple question of neighbourhood. On a 2-dimensional grid-like cellular automata there are two standard neighbourhood shapes: the Moore and von Neumann neighbourhoods. The Moore neighbourhood of radius 1 consists of the cell itself and any cell adjacent to it whether directly or on a diagonal. The von Neumann neighbourhood consists of the cell itself and cells which are adjacent either vertically or horizontally. So if cells are indexed by two subscript i.e.  $x_{i,j}$  then the radius 1 Moore neighbourhood is the set  $\{x_{k,l} : |k - i| \leq 1 \text{ and } |l - j| \leq 1\}$ . For radius  $n$ , we would allow  $|k - i|$  and  $|l - j|$  to be as large as  $n$ . By contrast in the von Neumann neighbourhood,  $k$  and  $l$  vary one at a time:  $\{x_{k,l} : k = i \text{ and } |l - j| \leq 1 \text{ or } |k - i| \leq 1 \text{ and } l = j\}$ . But in higher dimensions we are not restricted to square cells; our grid could be formed by hexagons or triangles, for example.

### Other Important Concepts

A configuration  $C$  is considered a *fixed point* for a cellular automata (or more precisely for the global function of a cellular automata) if  $G(C) = C$ . If for some  $k \geq 1$ ,  $G^k(C) = C$  then  $C$  is *temporally periodic* or simply *periodic* if no confusion arises. Configurations which lead to a periodic configuration are called *eventually periodic*. So if for some  $k$ ,  $G^k(C)$  is periodic, then  $C$  is eventually periodic. It is important to note here that cellular automata preserve spatial periodicity. This is what in essence makes the two characterisations of circular cellular automata equivalent. It also forces every circular cellular automata to be eventually periodic.

## 2.2 Classification

A fundamental problem in the study of cellular automata has always been their classification. The first attempt to classify cellular automata was done by Wolfram in [79] where they were grouped according to the observed behaviour of their space-time diagrams. They were divided into four classes based on the asymptotic behaviour of almost all initial configurations:

- (W1) homogeneous states,
- (W2) simple periodic structures,
- (W3) "chaotic" patterns,
- (W4) complex patterns with propagative localized structures .

Although not formally precise, this classification captured important distinctions among cellular automata. Figure 2.1 shows examples of each class. These classifications were made for both infinite and circular cellular automata and were later extended to include 2-dimensional cellular automata [66].

Since class membership was undecidable, as shown by Culik and Yu [20], observation of the evolution of a cellular automata starting from (possibly all) initial configurations

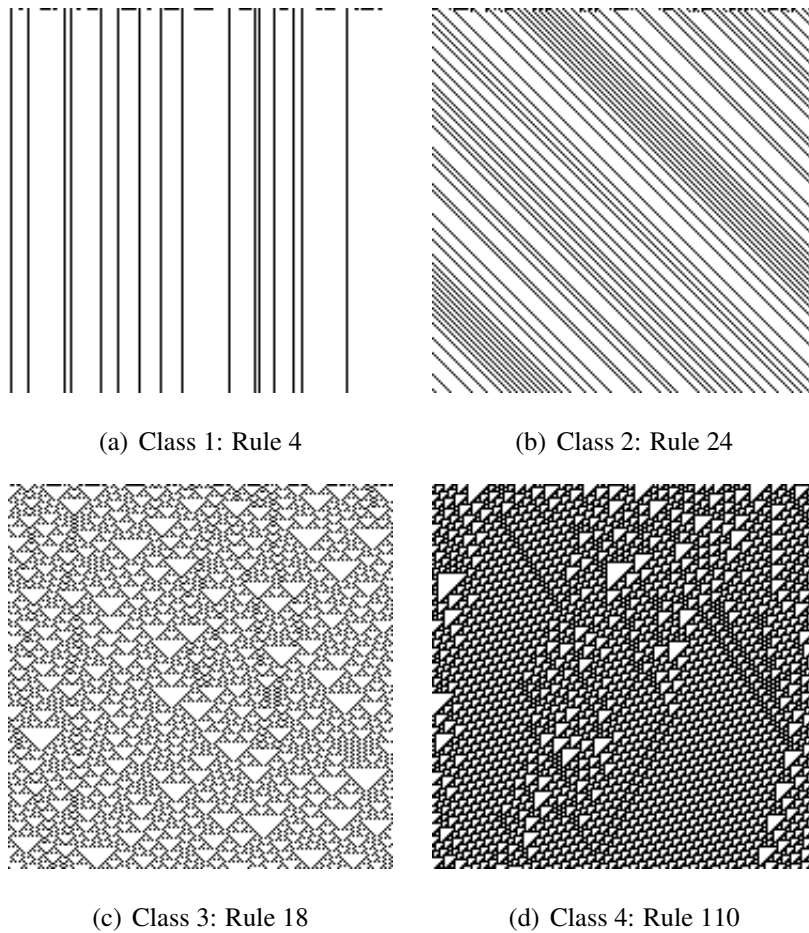


Figure 2.1: Sample Rules in Wolfram Classes.

became crucial to understanding its dynamics. In that same paper, Culik and Yu proposed their own classification which formalized Wolfram's and was based on finite configurations:

- (CY1) all  $s$ -finite configurations evolve to the homogeneous configuration  $s$ ;
- (CY2) all  $s$ -finite configurations are eventually periodic;
- (CY3) there exists a stable state  $s$  such that for any pair of  $s$ -finite configurations  $C_1$  and  $C_2$ , it is decidable if  $C_1$  evolves to  $C_2$ ;
- (CY4) all other rules .

Here,  $s$ -finite means that all but finitely many cells are in state  $s$ . Using this classification system, a rule belongs to the lowest class which applies. This scheme is also undecidable. The examples in Figure 2.1 would fall into the same categories in this classification scheme as well.

Several other criteria for grouping cellular automata have followed: some based on observable behaviours, some on intrinsic properties of cellular automata rules (e.g., see [19, 31, 40, 69]). For example, Eppstein developed a classification system for two-dimensional cellular automata based on their patterns of expansion and contraction [30]. He had three classes:

- (E1) patterns do not contract,
- (E2) patterns do not expand,
- (E3) patterns both contract and expand.

Eppstein believed that this classification scheme better captured some of the dynamics observed in two dimensions. The definition of "pattern" here comes from the study of Life-Like rules and is well defined.

A classification that has come out of viewing cellular automata as dynamical systems was that of Kurka [50] which put automata in four categories based on their sensitivity to initial conditions. This classification was made for infinite, 1-dimensional CA, but unlike the previous schemes described here, the alphabet could be any finite set, not just Boolean. In order to judge this sensitivity, we need first to be able to judge distance between configurations. To this end, we have the following metric. Given two configurations  $C_1$  and  $C_2$ , the distance between them is determined by the size of the region around the origin where the configurations are the same. The larger this region, the smaller the distance between the configurations. More precisely,

$$d(C_1, C_2) = 2^{-r}$$

where  $r = \inf\{i; C_1(i) \neq C_2(i) \text{ or } C_1(-i) \neq C_2(-i)\}$ . Now we can further define an *equicontinuous* point of an automaton  $G$  as a configuration that will remain close to any configuration that it is sufficiently close to begin with even as the automata evolves. So if  $E \in C$  is an equicontinuous point of  $G$ , then for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that

$$\forall C \in C : d(E, C) < \delta, d(G^t(E), G^t(C)) < \epsilon \text{ for all } t \geq 0.$$

If all possible configurations are equicontinuous points then  $G$  is called equicontinuous. The automaton is called *sensitive to initial conditions* if there are no equicontinuous points. Finally, it is called *positively expansive* if we can force a separation of at least some  $\delta$  no matter how close two configurations are to begin with. More precisely,  $G$  is positively expansive if there exist a  $\delta > 0$  such that for all  $C_1$  and  $C_2$ ,  $d(G^t(C_1), G^t(C_2)) > \delta$  for some  $t \geq 0$ . We can now define Kurka's four categories:

- (K1) equicontinuous CA,
- (K2) has some equicontinuous points,
- (K3) sensitive but not positively expansive,
- (K4) positively expansive.

Unfortunately, class membership for classes K1, K2, and K3 is undecidable, and K4 remains an open problem [29].

## 2.3 Injective, Surjective and Reversible Cellular Automata

A cellular automata is said to be injective or surjective if its global rule defined on the appropriate set of configurations is injective or surjective, respectively. More precisely, a CA is *injective* if every configuration maps to a unique output. That is, if the set of all possible configurations is  $C$ , then a CA with global rule  $G$  is injective if  $G(C_1) = G(C_2)$  implies  $C_1 = C_2$  for all  $C_1, C_2 \in C$ . A CA is *surjective* if every configuration has a pre-image

under the global function. Again more precisely, for every  $C_2 \in \mathcal{C}$  there exists an  $C_1 \in \mathcal{C}$  such that  $G(C_1) = C_2$ . We say that a CA is *reversible* if it is bijective and if its inverse function is also a CA. Because the determination of these three properties depends on the global rule and domain of configurations, we will often refer to injective or surjective rules where no ambiguity exists.

The distinction between reversibility and bijectivity exists because it is theoretically possible for a cellular automata to have an inverse function which is not a cellular automata, hence being bijective, but not reversible. In fact, a result of Hedlund [42] shows that the distinction between bijective and reversible CA is unnecessary since the inverse function will always be a CA itself. This result followed from the more general theorem that any continuous function which commutes with the shift functions is the global transition function of a CA.

**Theorem 1.** (Hedlund [42]) *A function  $G : S^{\mathbb{Z}^d} \rightarrow S^{\mathbb{Z}^d}$  is the global transition function of a CA if and only if  $G$  is continuous and it commutes with the shift functions on  $S^{\mathbb{Z}^d}$ .*

Since the inverse function of a bijective CA is easily shown to meet both conditions, we have the following corollary, also a result of Hedlund.

**Corollary 1.** *Bijective cellular automata are reversible.*

Surjectivity in infinite CA is somewhat daunting. How can one prove that an infinite sequence has or does not have a pre-image? A joint theorem of Moore [61] and Myhill [64] reduced this test to finding finite configurations, so called Garden-of-Eden configurations that had no pre-images (and therefore can only exist in initial configurations, hence the name). In a theorem commonly known as The Garden-of-Eden Theorem, they jointly showed that an infinite CA,  $G$  is surjective if and only if the global rule applied to finite configurations,  $G_F$  is injective. Since injectivity of the infinite CA implies injective of the finite CA, we have that injectivity, bijectivity and reversibility are all equivalent concepts.

**Theorem 2.** (Garden-of-Eden Theorem)  *$G_F$  is injective if and only if  $G$  is surjective.*

In one dimension, there exist quadratic algorithms for testing if a CA is surjective or injective. Amoroso and Patt [2] gave an elegant algorithm for injectivity while Sutner [70] tests for reversibility using ambiguity of de Bruijn graphs; the CA is surjective if and only if its de Bruijn graph is unambiguous. That is, there cannot exist two different paths starting and ending in the same nodes and having the same labels. The test for injectivity is slightly more complex and will be explained later on. We will now explain the construction and use of de Bruijn graphs.

Given a CA with neighbourhood size  $n$ , defined on an alphabet  $S$  of size  $m$ , the de Bruijn graph is a directional graph with of  $(n - 1)m$  nodes each with  $m$  outgoing edges. The nodes are labelled from the set  $S^{n-1}$ . There is an edge linking a node  $x_0, \dots, x_{n-2}$  to any node of the form  $x_1, \dots, x_{n-2}, x_{n-1}$  for any  $x_{n-1} \in S$ . We can think of this as a window of size  $n - 1$  sliding across our configuration. In the current window is  $x_0, \dots, x_{n-2}$ ; there will be an edge connecting this node to any node that could appear when we slide the window over one cell. The edge connecting these nodes is labelled with the result of  $f(x_0, \dots, x_{n-1})$ . Below is the de Bruijn graph for elementary rule 184. The edge connecting node "00" to node "01" is labeled "0" since  $f_{184}(001) = 0$ . Similarly, the edge connecting "01" to "10" is labeled "1" since  $f_{184}(011) = 1$ . There is no edge connecting "00" to "11" since the adjacent numbers are not the same.

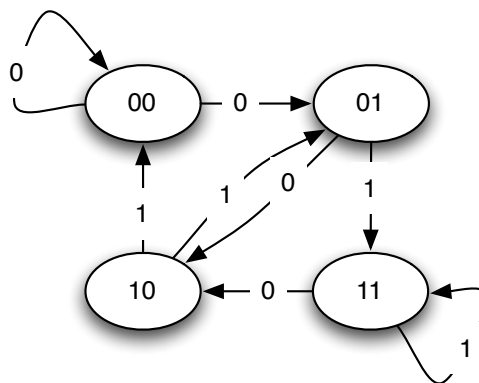


Figure 2.2: de Bruijn graph for rule 184.

Notice that the two paths:

$$00 \rightarrow 00 \rightarrow 01 \rightarrow 11 \rightarrow 10$$

and

$$00 \rightarrow 01 \rightarrow 10 \rightarrow 01 \rightarrow 10$$

are both labeled 0010. This means that the two finite sequences 000110 and 001010 both map to 0010. By the Garden of Eden Theorem, rule 184 is not surjective.

Now consider rule 105. Rule 105 is, in fact, reversible. In this case it is obvious from

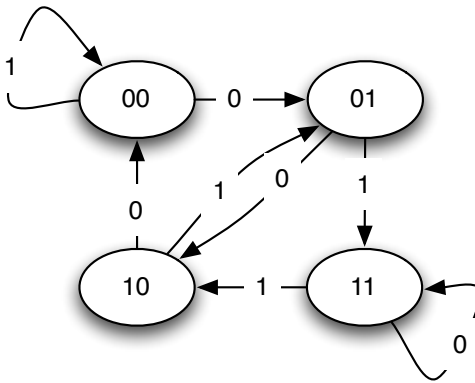


Figure 2.3: de Bruijn graph for rule 105.

the graph that no ambiguous paths exist since the outgoing edges at each node have unique labels. We will see later on that graphs having either unique outgoing edges or incoming edge labels fall into a category of rules which we have called self-oscillating and that their fuzzifications have particularly interesting behaviour.

Since the edge labels are the CA output, a CA is injective if and only if every two-way infinite path has unique labels. Luckily, it can be shown that injectivity of the global function on periodic configurations [?] implies infinite injectivity. (The converse is obviously true). So we need only show that no two periodic configurations, corresponding to loops in the graph have the same labels. The test for this is normally carried out on a reduced Cartesian product graph of the de Bruijn graph where the problem simplifies to finding cycles. In

fact, the test for surjectivity on this graph can also be reduced to finding cycles containing particular nodes.

In two dimensions and higher the questions of injectivity and surjectivity are at best semi-decidable. One can prove by contradiction that a CA is not surjective if an orphan configuration (a configuration without predecessor) can be found. Similarly, if an inverse CA can be found for a given CA then it must be reversible. The proof that no algorithm exists is profound, relying on an equation of cellular automata to Wang tiles [44].

In Kari's survey paper [46], a summary of the known relationships between injectivity and surjectivity in finite, circular, and infinite CA, denoted as  $G_F$ ,  $G_C$  and  $G$ , respectively, is given as follows:

**Theorem 3.** *The following implications are true in every dimension:*

- *If  $G$  is injective then  $G_C$  and  $G_F$  are injective.*
- *If  $G_C$  or  $G_F$  is surjective then  $G$  is surjective.*
- *If  $G_C$  is injective then  $G_C$  is surjective.*
- *If  $G$  is injective then  $G_F$  is surjective.*

*The following additional implications are true for one-dimensional CA:*

- *If  $G_C$  is injective then  $G$  is injective.*
- *If  $G$  is surjective then  $G_C$  is surjective.*

The proofs of this theorem can be found in [? ].

A paper by Kari summarizes the open problems in this area [46]:

**Open Problem 1.** *In two- and higher-dimensional cellular automata,*

- *Does injectivity of  $G_C$  imply surjectivity of  $G_F$ ?*
- *Does surjectivity of  $G_F$  imply surjectivity of  $G_C$ ?*
- *Does surjectivity of  $G$  imply surjectivity of  $G_C$ ?*

## 2.4 Number Conservation

Another question that has garnered considerable interest is the construction of cellular automata which possess or lack conservation properties. A cellular automata is said to be number conserving, or density conserving if the sum of the states is constant. Formally,

$\forall C \in C_F$

$$\sum C(i) = \sum F(C(i)).$$

A comparable definition holds for periodic CA where the sum is taken over a single period. In fact, a CA is number conserving on finite configurations if and only if it is number conserving on periodic configurations so the definitions are equivalent. Cellular automata having this property are of interest in engineering because of their applicability to classical problems such as traffic flow and random walks (e.g., see [35, 63]). Essentially, number conserving rules can be used to model systems which are or can be modeled as closed loop systems.

From the definition of number conservation, it is clear that for any state  $s$ ,  $f(s, \dots, s) = s$  if  $F$  is number conserving. Necessary and sufficient conditions for number conservation in one-dimensional cellular automata were given by Boccara and Fuks for binary and arbitrary finite states ([8, 9]).

**Theorem 4.** *A one-dimensional  $q$ -state,  $n$ -input CA rule  $f$  is number-conserving if and only if for all  $(x_0, \dots, x_{n-1}) \in S^n$*

$$f(x_0, \dots, x_{n-1}) = x_0 + \sum_{k=1}^{n-1} (f(0, 0, \dots, 0, x_1, x_2, \dots, x_{n-k}) - f(0, 0, \dots, 0, x_0, x_1, \dots, x_{n-1-k}))$$

We will see later on that equating Boolean cellular automata with their fuzzy counterparts greatly simplifies the identification of number conserving rules.

Durand, Formenti, et. al. [27, 28] produced a quasi-linear time algorithm to decide if a CA in any dimension and defined over any alphabet is number conserving thus proving that this question, at least, is decidable.

From a theoretical perspective, one-dimensional number conserving cellular automata have garnered a great deal of interest because they are universal. That is, any one-dimensional CA can be simulated by a number conserving CA of larger neighbourhood and with more states[62],

**Theorem 5.** *Let  $G$  be a CA with  $q$  states and neighbourhood size  $n$ . The  $G$  can be simulated by a number conserving CA with  $2q + 1$  states and neighbourhood size  $2n$ .*

The notion of number conservation was generalized by Hattori and Takesue to encompass any additive function on the configuration [41]. In their notation, given a configuration  $C$ , an additive quantity  $\Phi_C$  is defined as a pairing of a  $k$ -dimensional neighbourhood  $N = (\vec{x}_0, \vec{x}_1, \dots, \vec{x}_{k-1})$  and a density function  $\phi : S^k \rightarrow \mathbb{R}$  where  $S = \{0, 1\}$  in the Boolean case and  $\phi(0, \dots, 0) = 0$ . Then

$$\Phi_C(\vec{x}) = \phi(C(\vec{x} + \vec{x}_0), C(\vec{x} + \vec{x}_1), \dots, C(\vec{x} + \vec{x}_{k-1})).$$

They extend this definition to a function on finite configurations  $C$  (which includes all configurations in the periodic case) by

$$\hat{\phi}(C) = \sum_{\vec{x}} \Phi_C(\vec{x}).$$

A CA with global rule  $G$  is said to conserve the additive quantity  $\Phi_C$  if  $\hat{\phi}(G(\mathbf{C})) = \hat{\phi}(\mathbf{C})$  for all  $C \in \mathcal{C}$ . A CA is number conserving when it conserves the additive quantity  $\Phi_C$  which maps Boolean 0 to 0 and Boolean 1 to 1.

Knowing which quantities are preserved by a particular CA can be important for the use of CA in modelling. Hattori and Takesue have shown that it is straight forward to test if a given additive quantity is conserved by a given CA, and for a given CA and a fixed neighbourhood, preserved additive quantities can be found with reasonable efficiency [41]:

**Theorem 6.** *There is an algorithm to determine if a given additive quantity is conserved by a given CA. For a given CA and neighbourhood vector  $N$ , one can effectively find all conserved quantities of  $G$  that use neighbourhood  $N$ .*

They also give necessary and sufficient conditions for conservation laws. In [71], Takesue used these results to enumerate all the conserved laws in reversible elementary CA. Similarly, Pivato developed a method for finding all 1-dimensional cellular automata defined on a finite set that preserve a given additive quantity [67]. However, for arbitrary neighbourhoods  $N$ , the question of finding all preserved quantities in 1-dimension is open, and it is provably undecidable in higher dimensions [46].

## 2.5 Additivity

A subclass of cellular automata that have been studied extensively are the additive rules. These cellular automata are among the simplest algebraically but still create rich dynamics. A cellular automata is additive if its rule preserves addition of the states. That is, we can define a binary operation on the states such that  $G(A) + G(B) = G(A + B)$  where  $(A + B)_i = A_i + B_i$ . Usually, in Boolean cellular automata, the binary operator is the exclusive or but can also be defined as the XNOR or other operators [16]. An example of such a rule is elementary rule 90:

$$g_{90}(x, y, z) = x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z.$$

Since rule 90 can be written as an exclusive or  $g_{90}(x, y, z) = x \oplus z$ , we see immediately that  $G(A) \oplus G(B) = G(A \oplus B)$  since:

$$\begin{aligned} G(A)_i \oplus G(B)_i &= A_{i-1} \oplus A_{i+1} \oplus B_{i-1} \oplus B_{i+1} \\ &= A_{i-1} \oplus B_{i-1} \oplus A_{i+1} \oplus B_{i+1} \\ &= G(A \oplus B)_i \end{aligned}$$

Notice that this relationship continues to hold for subsequent iterations. So if we apply the global rule twice,  $G^2(A + B) = G(G(A) + G(B)) = G^2(A) + G^2(B)$ , hence in general,  $G^n(A+B) = G^n(A)+G^n(B)$ . Table 2.2, below, gives an example using rule 150 ( $x_{i-1} \oplus x_i \oplus x_{i+1}$ ). The  $z$  values can be obtained either directly by applying rule 150 to the previous values or by setting  $z_i = x_i \oplus y_i$ . This property can be used to efficiently calculate evolutions of the

rules from arbitrary starting points. For example, if we have calculated the evolutions for a single point on a quiescent background, we can derive the evolutions for any finite number of points by shifting and adding these evolutions to themselves. In this thesis, we used the extended definition of additive rules which includes XNOR operations, in which case  $G(A) \oplus G(B) = \overline{G(A \oplus B)}$ .

$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$		$y_0$	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$		$z_0$	$z_1$	$z_2$	$z_3$	$z_4$	$z_5$
0	0	0	1	1	1		0	1	0	1	1	0		0	1	0	0	0	1
1	0	1	0	1	0		1	1	0	0	0	1		0	1	1	0	1	1
1	0	1	0	1	0		1	0	1	0	1	0		0	0	0	0	0	0
1	0	1	0	1	0		1	0	1	0	1	0		0	0	0	0	0	0

Table 2.2: Example of additivity in rule 150

An interesting observation on the growth of infinite additive Boolean cellular automata is that they often evolve as fractal patterns. Take rule 90, for example. When evolved from a single 1 on a quiescent background, it creates the classic Sierpinski triangles, (Figure 2.4). These relationships, between fractal patterns and additive cellular automata, have been extensively studied [73, 75, 77].

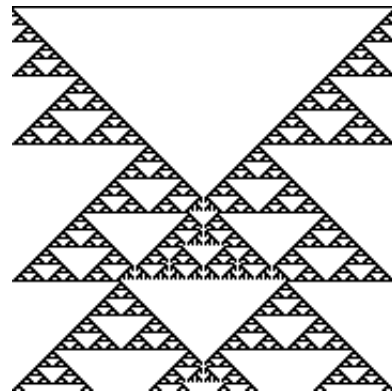


Figure 2.4: Sierpinski triangles from rule 90

The structure of these rules has led to their being used in a wide variety of applications. For example, Wolfram proposed using elementary additive cellular automata, often referred to linear cellular automata in the cryptographic community, as pseudo-random bit generators [80]. This led to applications as hash functions. When circular additive rules (without the XNOR) are allowed to evolve, they will eventually reach a stable periodic configuration. Each of these periodic configurations is called an attractor basins, or just attractors. Additive CA can be designed to have multiple attractors. (In the Table 2.2 above, we see two examples of attractors, the configurations 000000 and 101010.) Now, additive CA have two additional properties that make them useful as hash functions: each attractor has the same number of initial configurations that will eventually evolve to it and the maximum number of evolutions necessary to reach an attractor basin can be calculated. In order to use an additive CA as a hash function, each attractor basin is assigned a hash key. The data to be hashed is entered as the initial configuration of the CA which is then evolved the required number of times and a hash tag is assigned according to the attractor basin reached. The configurations are assigned uniformly to the hash tags with the probability of a collision being dependent on the number of possible initial configurations and the number of attractors. See, for example, [16, 56].

## 2.6 Summary

In this chapter we have reviewed some of the work to date on classical cellular automata with emphasis on 1-dimensional Boolean cellular automata. We have explored interesting and useful properties of classical CA: injectivity, surjectivity, and reversibility as well as additivity and number conservation. We have described several classification systems for classical CA. In the next chapter we will review some of the work to date on continuous extensions of Boolean cellular automata.

---

---

## CHAPTER 3

---

---

# Fuzzy Cellular Automata

While cellular automata have yielded many interesting results, the initial conditions of discrete time, space and state can be limiting especial when considering them as tools for modelling. Not surprisingly, many modifications have also arisen which affect one or more of these conditions. In this thesis, we are studying rules which are discrete in time and space but not state. We begin this chapter with the study of DNF fuzzification.

### 3.1 Definitions

A *fuzzy cellular automaton* (FCA) is a particular continuous cellular automaton where the local transition rule is obtained by *DNF-fuzzification* of the local transition rule of a classical Boolean CA in disjunctive normal form (DNF). The fuzzification consists of a fuzzy extension of the Boolean operators AND, OR, and NOT in the DNF expression of the Boolean rule. Depending on which fuzzy operators are used, different types of fuzzy cellular automata can be defined. Among the various possible choices, we consider the following:  $(a \vee b)$  is replaced by  $\min\{1, (a + b)\}$ ,  $(a \wedge b)$  by  $(ab)$ , and  $(\neg a)$  by  $(1 - a)$ . Note that, in the case of FCA,  $\min\{1, (a + b)\} = (a + b)$ . Whenever we talk about fuzzification, we are referring to the *DNF-fuzzification* defined above. The resulting local transition rule  $f : [0, 1]^{2r+1} \rightarrow [0, 1]$  becomes a real function that generalizes the canonical representation of the corresponding

Boolean CA:

$$f(v_{-r}, \dots, v_r) = \sum_{i < 2^{2r+1}} \hat{b}_i \prod_{j=-r:r} l(v_j, d_{i,j+r}) \quad (3.1.1)$$

where  $l(a, 0) = 1 - a$  and  $l(a, 1) = a$ , and  $\hat{b}_i = 0$  if  $b_i$  is false and  $\hat{b}_i = 1$  if  $b_i$  is true.

Note that the resulting function,  $f(v_{-r}, \dots, v_r)$ , is affine in each of its variables by which we mean that if one considers the restriction of  $f$  to  $v_i$ , the function has the form  $F(\dots, v_i, \dots) = a_i v_i + b_i$  where  $a_i$  and  $b_i$  are independent of  $v_i$ . Furthermore, it agrees with the Boolean function  $g(v_{-r}, \dots, v_r)$  at the  $2^n$  points in  $\{0, 1\}^n$ . It is therefore the only affine extension of  $g$ .

**Example.** Consider again elementary rule 18 whose local transition rule in DNF form is  $g(v_{-1}, v_0, v_1) = (\neg v_{-1} \wedge \neg v_0 \wedge v_1) \vee (v_{-1} \wedge \neg v_0 \wedge \neg v_1)$ , then the corresponding fuzzy local transition rule becomes:

$$f(v_{-1}, v_0, v_1) = (1 - v_{-1})(1 - v_0)v_1 + v_{-1}(1 - v_0)(1 - v_1).$$

Throughout this thesis, we will denote local rules of Boolean CA by  $g$  and their fuzzifications for the corresponding FCA by  $f$ . For ease of notation, we will denote  $g(y_{i-r}, \dots, y_i, \dots, y_{i+r})$  by  $g[y_i]$  and  $f(x_{i-r}, \dots, x_i, \dots, x_{i+r})$  by  $f[x_i]$ . The corresponding global rules are denoted by  $G$  and  $F$ .

## 3.2 Classification

Fuzzy cellular automata have been empirically classified according to various criteria. For example, fuzzy cellular automata in Boolean backgrounds have been grouped according to the level of ‘‘spread’’ of fuzziness in [13]. More recently, an empirical classification for elementary circular FCA was proposed based on experimental results from random initial configurations [32]. In this section, we will examine this classification in detail as it provides the basis for this thesis. In Chapters 7 through 9 of this thesis we will analytically prove the behaviours empirically observed for this classification for certain classes of fuzzy rules.

To obtain the classification, the evolution of the fuzzy rules was observed in two different ways. They were viewed using the same type of space-time images normally used for Boolean rules with the interval from zero to one discretized and colored but they were also viewed using a radial display. A unit circle was divided into  $n$  equal segments,  $n$  being the length of the circular automata. At each iteration, radii having length the current value of the cell were drawn. Effectively, if at some time  $t$  the  $i$ th cell had value  $x_i^t$ , then the radial view would display a dot at the point  $(x_i^t, \frac{2\pi}{n}i)$  in polar co-ordinates or a line to it from the centre. The rules evolved to one of four configurations: a dot, a circle, two concentric circles, or four concentric circles. The observed results suggested that all elementary rules have asymptotic periodic behaviour but with periods of only certain lengths: 1, 2, 4, and  $n$  (where  $n$  is the size of the circular lattice). The classifications were made according to the asymptotic behaviour of each cell and many of the classes can be subdivided.

In the first class, each cell appeared to converge toward a fixed point. These rules can be further divided into four groups: every cell tended toward zero (Quiescent) which appeared as a dot when viewed radially, every cell tended toward the same non-zero point (Homogeneous) which appeared as a circle, cells tended toward different points (Heterogeneous) which appeared as a fixed random pattern, cells tended toward at least 2 different values when  $n$  was even but were homogeneous when  $n$  was odd. (See examples in Figure 3.1. The square dot in this image was the last point plotted.)

In the second class, rules appeared to converge to period two, so at each cell, even time steps converged to one value while odd time steps converged to another. Some of these rules exhibited this behaviour for all values of  $n$ , while others only exhibited this behaviour when  $n$  was even. Furthermore, some rules converged homogeneously in space; that is, all cells values were tending toward the same two points one at even time steps, one at odd time steps. These rules showed one circle at a time but alternated between two different circles. Other rules converged to period two spatially as well as temporally. In these cases, two concentric circles appeared simultaneously but appeared to rotate one position with each time step. (See Example in Figure 3.2.)

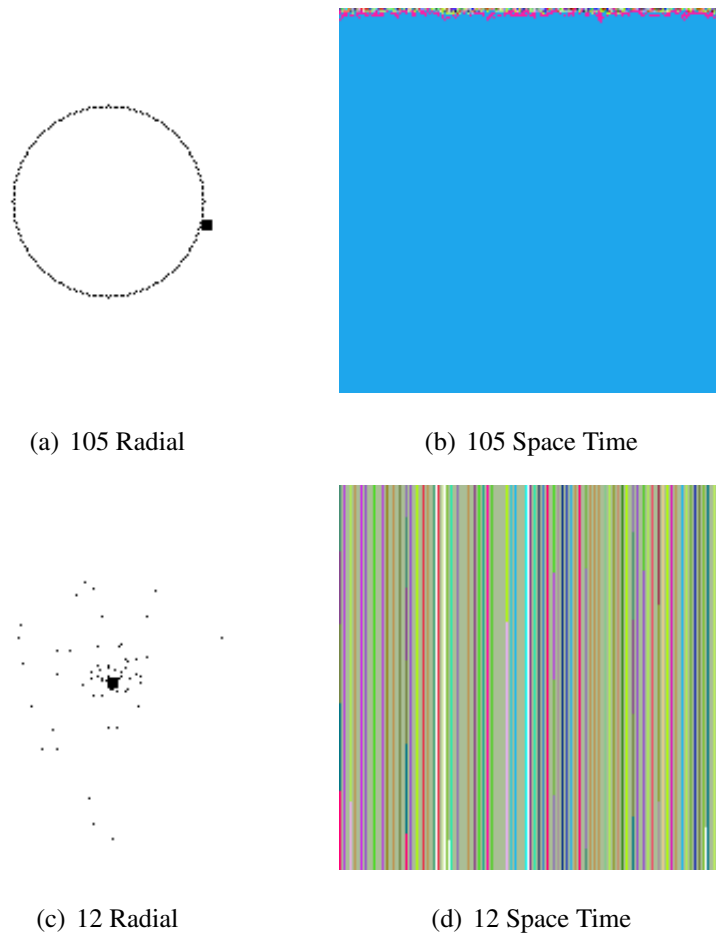


Figure 3.1: Evolution of Fuzzy Rules 105 and 12.

Only one rule, rule 46 converged to period four and only when the array size  $n$  was a multiple of four. These four circles appeared simultaneously and rotated at every time step suggesting that the rule was tending toward period four both temporally and spatially. (See Figure 3.3.)

In the last category were the period  $n$  rules. These can be thought of as rules that converged heterogeneously along diagonals or that tended to become simple shifts. (See example in Figure 3.4.)

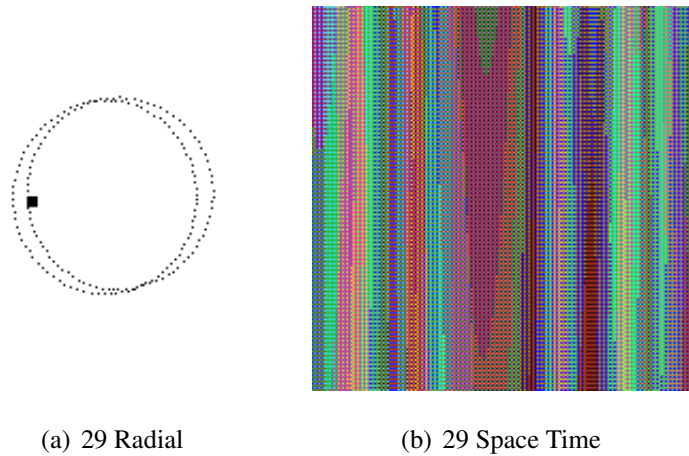


Figure 3.2: Evolution of Fuzzy Rule 29.

### 3.3 Evolution

Fuzzy cellular automata are a relatively new field of study so there is little work on them in the literature. However, A. Mingarelli, among others has studied their evolution from various backgrounds. In [33], Flocchini, Guerts, Mingarelli and Santoro studied rule 90, showing that it always converges to a fixed point but also showing that, if the level of discretization is high enough, fuzzy rule 90 displays the same behaviour as Boolean rule 90. This is an observation which we further explain and extend in our section on self-oscillating rules. Later, in [58], Mingarelli examined the evolution of rule 110 using this and other fuzzifications. Rule 110 is of particular interest in the Boolean version because it has been shown to be universal [17], and as a fuzzy rule because it converges to two values whose ratio is the golden number. Further, in [60], he showed that chaos was not possible for fuzzy rules of this type because sensitivity to initial conditions was not possible.

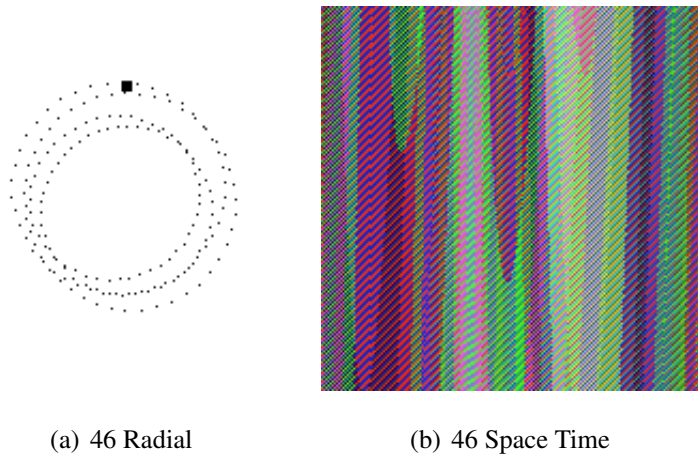


Figure 3.3: Evolution of Fuzzy Rule 46.

### 3.4 Applications

Fuzzy cellular automata, although only fairly recently defined, have been used in a variety of applications. We give two interesting examples here.

In [54], Maji and Chaudhuri use fuzzy CA for pattern classification of real-valued data. They used a genetic algorithm to develop fuzzy multiple-attractor cellular automata (FMACA) having two to ten attractor points. These FMACA are similar to the CA used for hash functions as discussed in the previous chapter, except that they are fuzzy. The resulting classifiers performed extremely well on classification accuracy, retrieval time and overhead size. They did, however take significantly longer to generate than the C4.5 algorithm against which they were tested. Further work was done on FMACA classifiers in [55] using hybrid fuzzy CA.

Fuzzy cellular automata have been used for modelling snow flakes. In [18], Coxe and Reiter used a hexagonal array with values in the initial configurations set between 0 and 0.5 except for a single point having a value of 1. A cell was "frozen", and therefore part of a snowflake, if it had a value greater than 0.5. The use of a hexagonal grid ensured that the resulting snowflake had the six-fold symmetry of real snowflakes (when ambient

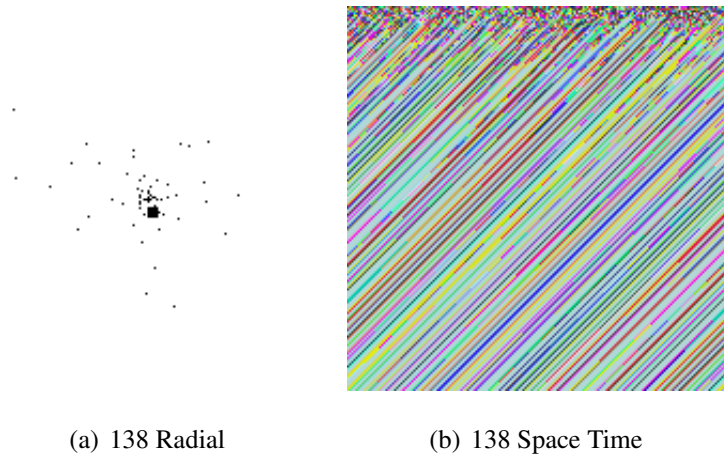


Figure 3.4: Evolution of Fuzzy Rule 138.

air temperature was constant) and the fuzzy rules created the diversity of snowflake design and sensitivity to initial conditions (which modeled air temperature) that were lacking in previous Boolean attempts to model snowflakes [83]. This model also allowed the physicists to simulate the effects of a heat source or sink near the snowflake.

### 3.5 CNF Fuzzification

While this thesis is examining rules fuzzified from the disjunctive normal form, it is certainly not the only way to extend rules on discrete space to continuous space. There are two things that must be considered when extending a rule in this way. First, all rules must be written in a standard form so that the binary representation of the rule is unique. In what was described above, the standard form chosen was the discrete normal form (or DNF). Then the binary operations must be translated to operations on the continuous space in such a way that the results of the binary operations are preserved. More formally, if  $g : \{0, 1\}^n \rightarrow \{0, 1\}$  is our Boolean function, and  $f : (0, 1)^n \rightarrow (0, 1)$  is its continuous extension, then, equating the Boolean 0, 1 with the real numbers 0, 1, the restriction of  $f$  to  $\{0, 1\}^n$  must be equal to  $g$ .

$$f|_{\{0,1\}} = g.$$

Another continuous extension which has been studied more recently is the Conjunctive Normal Form, or CNF. We can obtain this representation by considering the events which are mapped to 0 and applying De Morgan's law.

As an example, consider again rule 18. From the table, the following are all mapped to 0:  $\neg v_{-1} \wedge \neg v_0 \wedge \neg v_1$ ,  $\neg v_{-1} \wedge v_0 \wedge \neg v_1$ ,  $\neg v_{-1} \wedge v_0 \wedge v_1$ ,  $v_{-1} \wedge \neg v_0 \wedge v_1$ ,  $v_{-1} \wedge v_0 \wedge \neg v_1$ ,  $v_{-1} \wedge v_0 \wedge v_1$ .

Hence the rule can be written as:

$$\neg((\neg v_{-1} \wedge \neg v_0 \wedge \neg v_1) \vee (\neg v_{-1} \wedge v_0 \wedge \neg v_1) \vee (\neg v_{-1} \wedge v_0 \wedge v_1) \vee (v_{-1} \wedge \neg v_0 \wedge v_1) \vee (v_{-1} \wedge v_0 \wedge \neg v_1) \vee (v_{-1} \wedge v_0 \wedge v_1)).$$

Now, applying De Morgan's laws,  $\neg(a \vee b) = \neg a \wedge \neg b$  and  $\neg(a \wedge b) = \neg a \vee \neg b$ , we obtain:

$$\begin{aligned} & \neg(\neg v_{-1} \wedge \neg v_0 \wedge \neg v_1) \wedge \neg(\neg v_{-1} \wedge v_0 \wedge \neg v_1) \wedge \neg(\neg v_{-1} \wedge v_0 \wedge v_1) \\ & \wedge \neg(v_{-1} \wedge \neg v_0 \wedge v_1) \wedge \neg(v_{-1} \wedge v_0 \wedge \neg v_1) \wedge \neg(v_{-1} \wedge v_0 \wedge v_1) \\ = & (v_{-1} \vee v_0 \vee v_1) \wedge (v_{-1} \vee \neg v_0 \vee v_1) \wedge (v_{-1} \vee \neg v_0 \vee \neg v_1) \\ & \wedge \neg(v_{-1} \vee v_0 \vee \neg v_1) \wedge (\neg v_{-1} \vee \neg v_0 \vee v_1) \wedge (\neg v_{-1} \vee \neg v_0 \vee \neg v_1). \end{aligned}$$

So in *disjunctive* normal form, we obtain a disjunction (ORs) of terms containing only conjunctions (ANDs), whereas in *conjunctive* normal form, we obtain a conjunction of disjunctions, (we AND together terms containing only ORs). We can then translate the binary operators in the same way as before, AND becomes multiplication, OR addition, and NOT inversion ( $\neg x = (1 - x)$ ). However, in this case the stipulation that  $x \vee y = \min\{1, x + y\}$  becomes important since it is possible to obtain values greater than 1.

Continuous extensions of elementary cellular automata in CNF have recently been in [37]. In his thesis, Forrester develops software to simulate and display the elementary CNF rules. He then classifies these rules from observation and analysis. From homogenous initial configurations, 4 classes of rules were derived which are described by both the local function and the asymptotic behaviour.

- (CH1)  $f(0, 0, 0) = 1$  and  $f(1, 1, 1) = 1$ : Converge immediately to a fixed point.
- (CH2)  $f(0, 0, 0) = 0$  and  $f(1, 1, 1) = 1$ : Converge to 1 from any initial configuration except the 0 configuration.

- (CH3)  $f(0, 0, 0) = 1$  and  $f(1, 1, 1) = 0$ : Converge toward periodic behaviour, cycling between the 0 and 1 configurations.
- (CH4)  $f(0, 0, 0) = 0$  and  $f(1, 1, 1) = 0$ : May converge toward cyclic behaviour or may exhibit chaos.

An important observation which is proven here is that all rules in class 4 exhibit chaotic behaviour for some homogeneous initial configurations.

From random initial configurations, six distinct behaviours were observed and were dependent on key values in the local rule.

- (CR1)  $f(0, 0, 0) = 0$  and  $f(1, 1, 1) = 0$ : No obvious periodicity, behaviour is pseudo-random, possibly chaotic.
- (CR2)  $f(0, 0, 0) = 1$  and  $f(1, 1, 1) = 0$ : Different behaviours possible.
- (CR3)  $f(1, 1, 1) = 1$  and  $f(1, 1, 0) = 0$  and  $f(0, 1, 1) = 0$ : Produce Sierpinski triangles.
- (CR4)
  - a:  $f(1, 1, 1) = 1$  and  $f(1, 1, 0) = 0$  and  $f(0, 1, 1) = 1$ : Produce shifts.
  - b:  $f(1, 1, 1) = 1$  and  $f(1, 1, 0) = 1$  and  $f(0, 1, 1) = 1$ : Produce shifts.
- (CR5)  $f(1, 1, 1) = 1$  and  $f(1, 1, 0) = 1$  and  $f(1, 0, 1) = 0$  and  $f(0, 1, 1) = 1$ : Temporal fixed points.
- (CR6)  $f(1, 1, 1) = 1$  and  $f(1, 1, 0) = 1$  and  $f(1, 0, 1) = 1$  and  $f(0, 1, 1) = 1$ : All values converge to 1.

Examples of each of the random classes are given in Figure 3.5 below. Note that in the images below, 1 is represented by white and values approaching white are shown in magenta.

## 3.6 Summary

In this chapter we have reviewed the work to date on continuous cellular automata with emphasis on fuzzified DNF Boolean cellular automata. With respect to fuzzy CA, we have summarized the most significant results including the classification of these rules according to their asymptotic behaviour. We have also given a brief description of how these rules are used in other areas of study. We then described recent work done on fuzzified CNF rules and their classification. In the remainder of this thesis, we will be building on the results presented here and in the previous chapter.

Period 1 (Quiescent)	0, 8, 24, 32, 36, 40, 44, 56, 72, 74, 104, 128, 136, 152, 160, 164, 168, 200
Period 1 (Homogeneous)	6, 9, 22, 25, 26, 30, 33, 35, 37, 38, 41, 45, 54, 57, 60, 61, 62, 73, 90, 105, 106, 110, 122, 126, 134, 150, 154, 172.
Period 1 (Heterogeneous)	4, 12, 13, 28, 76, 77, 108, 132, 140, 156, 204, 232
Period 1 for $n$ even (Heterogeneous)	78, 94
Period 2 for all $n$	1, 5, 19, 23, 27, 50, 51, 178
Period 2 for $n$ even	18, 29, 58, 146, 184
Period 4 for $n$ multiple of 4	46
Period $n$ (Shifts)	2, 3, 7, 10, 11, 14, 15, 34, 42, 43, 130, 138, 142, 162, 170

Table 3.1: Observed dynamics of Circular Elementary Cellular Automata. Rules equivalent under conjugation, reflection and both are not indicated.

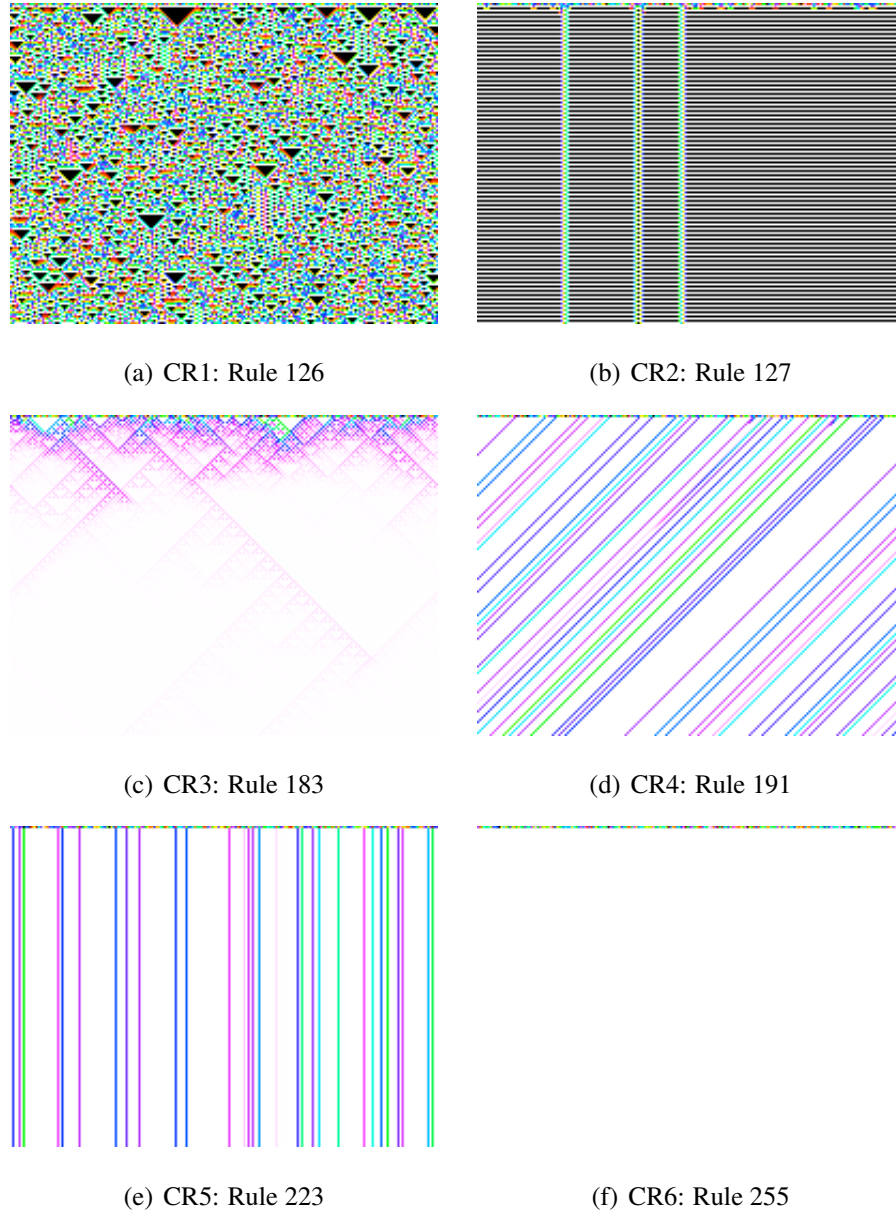


Figure 3.5: Sample Rules in Randomized CNF Classes

---

---

## CHAPTER 4

---

---

# Preserved Properties

In this chapter, we explore the link between Boolean and fuzzy CA proving that there are density conservation properties that are preserved through the fuzzification process. Since such properties are defined only for finite or circular CA, throughout this section we will be considering circular CA (the finite case is analogous).

### 4.1 Definitions and Basic Properties

Let  $C$  be the universe of all possible configurations for a CA (resp. FCA) of size  $n$ . As usual, let  $g$  be the local rule with corresponding global transition function  $G$ , ( $f$  and  $F$ , respectively for the fuzzy rule).

**Definition 1.** *We call a property  $\mathcal{P}$  of a CA (resp. FCA) a global property of the transition function if it holds for all configurations: i.e.,  $\mathcal{P}(G(\mathbf{Y}))$  is true for all  $\mathbf{Y} \in C$ .*

Note that to verify that a property is global for a CA (FCA) it suffices to prove it for all initial configurations  $\mathbf{Y} \in C^0$  (because  $C = C^0$ ). In this chapter we will start by considering the global property of number conservation where  $\Sigma G(Y) = \Sigma Y$ .

To begin with, we show an important property of DNF-fuzzification: that the function obtained through DNF-fuzzification is the only continuous extension of the Boolean function

from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  which is affine in every variable.

**Lemma 1.** *A local fuzzy rule  $f$  obtained from an Boolean rule is affine in each variable.*

**Proof:** This follows from the construction  $f$  as the sum of terms which are affine in each variable.

**Lemma 2.** *The local fuzzy rule  $f$  obtained from an elementary Boolean rule is the only affine extension of the Boolean rule.*

**Proof:** Consider the function on one variable,  $f(x) = ax + b$  with  $f(0)$  and  $f(1)$  fixed. Clearly,  $b = f(0)$  and  $a = f(1) - f(0)$  gives the unique affine extension. Now assume that such a unique extension exists for  $n - 1$  or fewer variables, and consider the function  $f(x_0, \dots, x_{n-1})$ . By induction, if we let  $x_{n-1} = 0$ , we obtain the unique affine extension  $f_0(x_0, \dots, x_{n-2})$ . Similarly, we obtain an affine extension  $f_1(x_0, \dots, x_{n-2})$  by letting  $x_1 = 1$ . We then obtain the unique affine extension as

$$f(x_0, \dots, x_{n-1}) = f_0(x_0, \dots, x_{n-2})(1 - x_{n-1}) + f_1(x_0, \dots, x_{n-2})x_{n-1}$$

.

The lemmas above imply that the global rule  $F$  obtained from such local rules are affine in each variable at each position.

## 4.2 Number Conservation

Number conservation is a global property that has been extensively investigated (e.g., see [8, 9, 27, 28, 35, 62, 63]) since its introduction in [65], a main focus being the study of linear time decision algorithms for the property of number conservation for finite or periodic configurations.

A Boolean CA is number conserving if the number of ones in the initial configuration is preserved at each subsequent iteration (we will also say that a rule is number preserving). Rules having this property are often used to model things such as traffic flow where we can assumed a constant number of cars for the duration of the simulation, (e.g., see [35, 63]).

**Example:** Rule 184 is an example of a number conserving rule. In DNF form, rule 184 is given by:  $f(x, y, z) = \bar{x}yz + x\bar{y}\bar{z} + x\bar{y}z + xyz$ . Here is an example of a small circular CA updated by rule 184. Notice that every row contains exactly six 1s: The analogous property in fuzzy

0	1	1	0	1	0	1	1	0	1
1	1	0	1	0	1	1	0	1	0
1	0	1	0	1	1	0	1	0	1
0	1	0	1	1	0	1	0	1	1
1	0	1	1	0	1	0	1	1	0

CA is that the sum of values of the initial configuration is preserved. These rules could be used to model fluid flow that cannot be measured discretely but where a total volume is conserved. Again we illustrate this property using fuzzy rule 184. Notice that every row now sums to 4. In this section, we wish to show that using DNF-fuzzification, a Boolean CA with

0.1	0.2	0.3	0.4	0	0.9	0.8	0.7	0.6	0
0.02	0.14	0.26	0.18	0.4	0.72	0.74	0.66	0.28	0.6
0.5908	0.0536	0.1504	0.2852	0.396	0.6448	0.6756	0.4364	0.6432	0.124

local rule  $g$  is number conserving if and only if the local rule  $f$  of the corresponding FCA is sum conserving. We will actually first prove a more general result.

We can extend the fuzzification process to any function  $g : \{0, 1\}^n \rightarrow \mathbb{R}^n$  by defining

$$C(g)(v_0, \dots, v_{n-1}) = \sum_{i < 2^{2r+1}} g(d_i) \prod_{j=-r:r} l(v_j, d_{i,j+r})$$

where  $d_i$  and  $l(v_j, d_{i,j+r})$  are defined as for the fuzzification of  $g$ . The function  $C(g) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and is once again affine in each variable.

**Lemma 3.** *Given any linear function  $E : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and any function  $g : \{0, 1\}^n \rightarrow \mathbb{R}^n$ , then  $C(E \circ g) = E \circ C(g)$*

**Proof:** Let  $E(x_0, \dots, x_{n-1}) = (\alpha_0 x_0, \dots, \alpha_{n-1} x_{n-1})$  and let  $g(d_i) = (\delta_{i0}, \dots, \delta_{in-1})$ . Then

$$\begin{aligned}
C(E \circ g)(v_0, \dots, v_{n-1}) &= \sum_{i < 2^{2r+1}} E \circ g(d_i) \prod_{j=-r:r} l(v_j, d_{i,j+r}) \\
&= \sum_{i < 2^{2r+1}} (\alpha_0 \delta_{i0}, \dots, \alpha_{n-1} \delta_{in-1}) \prod_{j=-r:r} l(v_j, d_{i,j+r}) \\
&= \sum_{i < 2^{2r+1}} (\alpha_0 \prod_{j=-r:r} l(v_j, d_{i,j+r}) \delta_{i0}, \dots, \alpha_{n-1} \prod_{j=-r:r} l(v_j, d_{i,j+r}) \delta_{in-1}) \\
&= E \left( \sum_{i < 2^{2r+1}} \left( \prod_{j=-r:r} l(v_j, d_{i,j+r}) \delta_{i0}, \dots, \prod_{j=-r:r} l(v_j, d_{i,j+r}) \delta_{in-1} \right) \right) \\
&= E \left( \sum_{i < 2^{2r+1}} (\delta_{i0}, \dots, \delta_{in-1}) \prod_{j=-r:r} l(v_j, d_{i,j+r}) \right) \\
&= E \circ C(g)(v_0, \dots, v_{n-1}).
\end{aligned}$$

**Theorem 7.** Given any two linear functions  $\Psi$ , and  $\Phi$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , and any function  $g : \{0, 1\}^n \rightarrow \mathbb{R}^n$

$$\Psi \circ C(g) = \Phi$$

if and only if

$$\Psi \circ g = \bar{\Phi}$$

where  $\bar{\Phi}$  is the restriction of  $\Phi$  to  $\{0, 1\}^n$ .

**Proof:** The proof is straightforward:

$$\begin{aligned}
\Psi \circ C(g) &= C(\Psi \circ g) \text{ by Lemma 3} \\
&= C(\bar{\Phi}) \\
&= \Phi \text{ by Lemma 2.}
\end{aligned}$$

**Theorem 8.** Let  $\Psi$  be a real linear function. Then:

$$\forall (y_0, \dots, y_{n-1}) \in \{0, 1\}^n \quad \Psi(g[y_0], \dots, g[y_{n-1}]) = \Psi(y_0, \dots, y_{n-1})$$

if and only if

$$\forall (x_0, \dots, x_{n-1}) \in [0, 1]^n \quad \Psi(f[x_0], \dots, f[x_{n-1}]) = \Psi(x_0, \dots, x_{n-1})$$

**Proof:** This follows from Theorem 7 by letting  $\Psi = \Phi$ .

Note that, when  $\Psi$  is the summation of all values, we have:  $\sum_{i=0}^{n-1} g[y_i] = \sum_{i=0}^{n-1} y_i$   $\forall (y_0, \dots, y_{n-1})$  if and only if  $\sum_{i=0}^{n-1} f[x_i] = \sum_{i=0}^{n-1} x_i$   $\forall (x_0, \dots, x_{n-1})$ , that is:

**Theorem 9.** *A Boolean CA is number conserving if and only if its corresponding FCA is sum conserving.*

**Theorem 10.** *Let  $f_{184}$  be fuzzy local rule 184. We have:*

$$\forall (x_0, \dots, x_n) \in [0, 1]^n \quad \sum_{i=0}^{n-1} f_{184}[x_i] = \sum_{i=0}^{n-1} x_i$$

**Proof:** Fuzzy rule 184 has the following form:  $x_i^{t+1} = x_{i-1}^t - x_{i-1}^t x_i^t + x_i^t x_{i+1}^t$ . Then we have:

$$\sum_{i=0}^{n-1} x_i^{t+1} = \sum_{i=0}^{n-1} x_{i-1}^t - \sum_{i=0}^{n-1} x_i^t x_{i-1}^t + \sum_{i=0}^{n-1} x_i^t x_{i+1}^t.$$

Since we are using a circular FCA,  $\sum_{i=0}^{n-1} x_{i-1}^t = \sum_{i=0}^{n-1} x_i^t$  and  $\sum_{i=0}^{n-1} x_i^t x_{i-1}^t = \sum_{i=0}^{n-1} x_i^t x_{i+1}^t$ , which implies:

$$\sum_{i=0}^{n-1} x_i^{t+1} = \sum_{i=0}^{n-1} x_i^t.$$

The result for the Boolean case (which is already known) follows as a corollary of Theorem 8.

**Corollary 2.** *Let  $g_{184}$  be elementary Boolean local rule 184. We have:*

$$\forall (y_0, \dots, y_n) \in \{0, 1\}^n \quad \sum_{i=0}^{n-1} g_{184}[y_i] = \sum_{i=0}^{n-1} y_i.$$

### 4.3 Spatial Number Conservation

We now describe another global property that is preserved by fuzzification. This property also deals with the density of configurations. Following an approach similar to the one of Theorem 8, we can show that in a CA, linear properties hold for the Boolean rule if and only if they hold for the corresponding fuzzy rule.

**Theorem 11.** Let  $g : \{0, 1\}^{2r+1} \rightarrow \{0, 1\}$  be the local rule of a Boolean CA and let  $f : [0, 1]^{2r+1} \rightarrow [0, 1]$  be its fuzzification. Let  $\Psi$  be a real linear function.

$$\forall (y_0, \dots, y_{n-1}) \in \{0, 1\}^n \quad \Psi(g[y_0], \dots, g[y_{n-1}]) = 0$$

if and only if

$$\forall (x_0, \dots, x_{n-1}) \in [0, 1]^n \quad \Psi(f[x_0], \dots, f[x_{n-1}]) = 0.$$

**Proof:** This follows from Theorem 7 by letting  $\Phi = 0$

Note that, when  $\Psi(x_0, \dots, x_{n-1}) = \sum_{i=0}^{n-1} (-1)^i x_i$  and  $n$  is even, we obtain the preservation through fuzzification of a spatial conservation property where the sum of the even numbered cells ( $x_{2i}$ ) is equal to the sum of the odd numbered cells ( $x_{2i+1}$ ) at any time after the initial configuration:

**Corollary 3.** Let  $n$  be even.  $\forall (y_0, \dots, y_{n-1}) \in \{0, 1\}^n \quad \sum_{i=0}^{n-1} (-1)^i g[y_i] = 0$

if and only if

$$\forall (x_0, \dots, x_{n-1}) \in [0, 1]^n \quad \sum_{i=0}^{n-1} (-1)^i f[x_i] = 0$$

**Example:** Rule 46 is an example of a spatially number conserving rule where the sum of the even numbered cells ( $x_{2i}$ ) is equal to the sum of the odd numbered cells ( $x_{2i+1}$ ) at any time after the initial configuration.

**Theorem 12.** Let  $f_{46}$  be fuzzy local rule 46 in a FCA of even size. We have:  $\forall (x_0, \dots, x_{n-1}) \in [0, 1]^n \quad \sum_{i=0}^{n-1} (-1)^i f_{46}[x_i] = 0$ .

**Proof:** Rule 46 is given by:  $x_i^{t+1} = x_i^t + x_{i+1}^t - x_{i-1}^t x_i^t - x_i^t x_{i+1}^t$ , so:

$$\sum_{i=0}^{n-1} (-1)^i x_i^{t+1} = \sum_{i=0}^{n-1} (-1)^i x_i^t + \sum_{i=0}^{n-1} (-1)^i x_{i+1}^t - \sum_{i=0}^{n-1} (-1)^i x_{i-1}^t x_i^t - \sum_{i=0}^{n-1} (-1)^i x_i^t x_{i+1}^t.$$

By a change of variables, due to circularity we have:  $\sum_{i=0}^{n-1} (-1)^i x_{i+1}^t = -(\sum_{i=0}^{n-1} (-1)^i x_i^t)$ , and  $\sum_{i=0}^{n-1} (-1)^i x_i^t x_{i+1}^t = -(\sum_{i=0}^{n-1} (-1)^i x_{i-1}^t x_i^t)$ . So we can conclude:

$$\sum_{i=0}^{n-1} (-1)^i x_i^{t+1} = \sum_{i=0}^{n-1} (-1)^i x_i^t - \sum_{i=0}^{n-1} (-1)^i x_i^t - \sum_{i=0}^{n-1} (-1)^i x_{i-1}^t x_i^t + \sum_{i=0}^{n-1} (-1)^i x_{i-1}^t x_i^t = 0.$$

The result for the Boolean case now follows as a corollary of Theorem 3.

**Corollary 4.** *Let  $g_{46}$  be elementary Boolean local rule 46. When  $n$  is even, we have:*

$$\forall (y_0, \dots, y_{n-1}) \in \{0, 1\}^n \quad \sum_{i=0}^{n-1} (-1)^i g_{46}[y_i] = 0.$$

## 4.4 Conservation Revisited

In this section, we give an alternate approach to the main theorems in Section 3.

In the notation of Hattori and Takesue, given a configuration  $c$ , we define an additive quantity  $\Phi_c$  as a pairing of a  $k$ -dimensional neighbourhood  $N = (\vec{x}_0, \vec{x}_1, \dots, \vec{x}_{k-1})$  and a density function  $\phi : S^k \rightarrow \mathbb{R}$  where  $S = \{0, 1\}$  in the Boolean case and  $S = [0, 1]$  in the fuzzy case and  $\phi(0, \dots, 0) = 0$  for both. Then

$$\Phi_c(\vec{x}) = \phi(c(\vec{x} + \vec{x}_0), c(\vec{x} + \vec{x}_1), \dots, c(\vec{x} + \vec{x}_{k-1}))$$

and we extend this definition to a function on finite configurations  $C$  (which includes all configurations in the periodic case) by

$$\hat{\phi}(c) = \sum_{\vec{x}} \Phi_c(\vec{x}).$$

A CA with global rule  $G$  is said to conserve the additive quantity  $\Phi_c$  if  $\hat{\phi}(G(\mathbf{c})) = \hat{\phi}(\mathbf{c})$  for all  $c \in C$ .

If the Boolean density function  $\phi$  is extended to  $[0, 1]$  as described earlier, then it is useful to understand when conservation by the Boolean rule implies conservation by the fuzzy rule. We will extend the definition of conservation to include any linear combination of the additive quantity and we show that when the additive quantity is affine in each variable, conservation is preserved through the process of fuzzification.

Given a Boolean density function  $\phi_B$  and additive quantity  $\Phi_{c_B}$ , then a Boolean rule  $G$  conserves  $\Phi_{c_B}$  if and only if its associated fuzzy rule  $F$  conserves  $\Phi_{c_f}$ . The extension of  $\phi_B$  can be obtained in a manor similar to the fuzzification as described in Section 3.

**Lemma 4.** *Let  $\Phi_{c_B}$  be an additive quantity with neighbourhood  $N$  of dimension  $k$  and density function  $\phi_B : \{0, 1\}^k \rightarrow \mathbb{R}$ , let  $\phi_f : [0, 1]^k \rightarrow \mathbb{R}$  be its affine extension and let  $\Phi_{c_f}$  be the associated additive quantity with neighbourhood  $N$ . Then a Boolean global rule  $G$  conserves  $\Phi_{c_B}$  if and only if its fuzzification  $F$  conserves  $\Phi_{c_f}$ .*

**Proof:** We denote by  $\hat{\phi}_B$  the extension of  $\Phi_{c_B}$  to  $C_B$  and by  $\hat{\phi}_f$  the extension of  $\Phi_{c_f}$  to  $C_f$ .  
 $\Leftarrow$ : Since the property applies to all values in  $[0, 1]$ , it must apply to  $\{0, 1\}$  as well and the implication follows from the construction of  $f$ .

$\Rightarrow$ : We will prove this by induction on the number of positions  $\vec{x}$  of a configuration  $c$  such that  $c(\vec{x}) \in (0, 1)$ .

When  $c(\vec{x}) \in \{0, 1\}$  for all  $\vec{x}$ ,  $\hat{\phi}_f(F(c)) = \hat{\phi}_B(G(c)) = \hat{\phi}_B(c) = \hat{\phi}_f(c)$ . The first and last equalities are by definition and the second is by assumption.

Now assume that whenever there are  $n$  or fewer positions such that  $c(\vec{x}) \in (0, 1)$ ,  $\hat{\phi}_f(F(c)) = \hat{\phi}_f(c)$ . Choose a configuration  $c_0$  with  $n + 1$  positions in  $(0, 1)$  and let  $\vec{y}$  be one such position. Now consider the subset  $C_0$  of  $C_f$  of configurations which agree with  $c_0$  at all but  $\vec{y}$ , i.e. for all  $c \in C_0$ , and for all  $\vec{x} \neq \vec{y}$ ,  $c(\vec{x}) = c_0(\vec{x})$ . If we restrict  $\hat{\phi}$  to  $C_0$ , we can see that it is an affine function in  $c(\vec{y})$ . We can write  $\hat{\phi}_f|_{C_0}(c) = ac(\vec{y}) + b$  for some  $a, b \in \mathbb{R}$ . Since  $\hat{\phi}_f(F(c)) = \hat{\phi}_f \circ F(c)$  is a composition of affine functions, it is also affine in  $c(\vec{y})$ . When  $c(\vec{y}) = 0$ ,  $c$  has  $n$  positions in  $(0, 1)$  so the inductive hypothesis applies and  $\hat{\phi}_f|_{C_0}(c) = \hat{\phi}_f(c) = \hat{\phi}_f(F(c))$ . Similarly, when  $c(\vec{y}) = 1$ ,  $\hat{\phi}_f|_{C_0}(c) = \hat{\phi}_f(c) = \hat{\phi}_f(F(c))$ . Since both  $\hat{\phi}_f$  and  $\hat{\phi}_f \circ F$  are affine in  $c(\vec{y})$ , and therefore completely defined by two points, we must have  $\hat{\phi}_f(F(c_0)) = \hat{\phi}_f(c_0)$ , completing the proof.

A more general result follows from the above. Let  $\Phi_c$  be an additive quantity with neighbourhood vector  $N$  and density function  $\phi$ , and let  $a$  be a weighting vector having the same dimensionality as elements in  $C$  and components in  $\mathbb{R}$ , a finite number of which are non-zero. We denote by  $a(\vec{x})$  the value of  $a$  at position  $\vec{x}$ . Then we can extend the function  $\Phi_c$  to a function  $\psi$  on  $C$  as follows:

$$\psi(c) = \sum_{\vec{x}} a(\vec{x})\Phi_c(\vec{x})$$

We say that the global rule  $G$  *conserves the  $\vec{a}$ -weighted additive quantity  $\Phi_c$*  if  $\psi(G(c)) = \psi(c)$  for all  $c \in C$ .

We are now ready to prove this general theorem.

**Theorem 13.** *Let  $\Phi_c$  be an additive quantity with neighbourhood  $N$ , and density function  $\phi_B$  which is affine in each variable. Let  $\psi_B$  be the extension of  $\Phi_{C_B}$  to  $C$  by weight vector  $\vec{a}$ ,  $\psi(c) = \sum_{\vec{x}} a(\vec{x})\Phi_c(\vec{x})$ . We can extend the definition of  $\psi_B$  to  $\psi_f$  on  $C_f$  and*

$$\psi_B(G(c)) = \psi_B(c) \quad \forall c \in C_B$$

*if and only if*

$$\psi_f(F(c)) = \psi_f(c) \quad \forall c \in C_f$$

**Proof:** It is sufficient to note that if  $\phi$  is affine in each of its components, then  $\Phi_c$ , as an affine function of affine functions, is also affine. Then the proof follows exactly as in Lemma 8.

Note that in the finite or periodic cases, when  $\phi_B$  is the local rule and  $\phi_f$  is its fuzzification the theorem implies that  $\sum_{i=0}^{n-1} g[y_i] = \sum_{i=0}^{n-1} y_i \quad \forall (y_0, \dots, y_{n-1})$  if and only if  $\sum_{i=0}^{n-1} f[x_i] = \sum_{i=0}^{n-1} x_i \quad \forall (x_0, \dots, x_{n-1})$ . That is, we have the following theorem.

**Theorem 14.** *A Boolean CA is number conserving if and only if the corresponding FCA is sum conserving.*

**Proof:** Let a Boolean CA have local rule  $g$ , global rule  $G$  and neighbourhood  $N$ . Let its fuzzification have local rule  $f$  and global rule  $F$ . Furthermore, let the additive quantity  $\Phi_{C_B}$  be defined by  $\phi_B = g$  and neighbourhood  $N$ , the neighbourhood used by the CA. Let  $\phi_f = f$  be the extension of  $\phi_B$  to  $[0, 1]^k$  defining  $\Phi_{C_f}$ . If  $\sum_{i=0}^{n-1} g[y_i] = \sum_{i=0}^{n-1} y_i \quad \forall (y_0, \dots, y_{n-1})$  then  $G$  conserves  $\Phi_{C_B}$ . In other words,  $G$  is number conserving if and only if  $G$  conserves  $\Phi_{C_B}$ . Similarly,  $F$  is sum conserving if and only if  $F$  conserves  $\Phi_{C_f}$ . Now from Lemma 1, we know that  $\phi_f = f$  is an affine function in each variable. So Theorem 4 holds, and  $G$  conserves  $\Phi_{C_B}$  if and only if  $F$  conserves  $\Phi_{C_f}$ .

We now describe another global property that is preserved by fuzzification. This property also deals with the density of configurations. Following an approach similar to the one of Theorem 4, we can show that in a CA, linear properties hold for the Boolean rule if and only if they hold for the corresponding fuzzy rule.

**Theorem 15.** *Let  $\Phi_c$  be an additive quantity with neighbourhood  $N$ , and density function  $\phi_B$ . Let  $\psi_B$  be the extension of  $\Phi_{c_B}$  to  $C_B$  by weight vector  $\vec{a}$ ,  $\psi(c) = \sum_{\vec{x}} a(\vec{x})\Phi_c(\vec{x})$ . Let  $\phi_f$  be the affine extension of  $\phi_B$ , we can extend the definition of  $\psi_B$  to  $\psi_f$  on  $C_f$  and*

$$\psi_B(G(c)) = 0 \quad \forall c \in C_B$$

*if and only if*

$$\psi_f(F(c)) = 0 \quad \forall c \in C_f$$

**Proof:** The proof of this theorem is almost identical to the proof of Theorem 4.

$\Rightarrow$ : We will prove this by induction on the number of position vectors  $\vec{x}$  such that  $c(\vec{x}) \in [0, 1]$ . By definition and hypothesis, given  $c$  such that all  $c(\vec{x}) \in \{0, 1\}$ ,  $\psi_f(F(c)) = \psi_B(G(c)) = 0$ . Now we assume, that for any configuration  $c$  with  $m$  or fewer values in  $[0, 1]$ ,  $\psi_f(F(c)) = 0$ . Fix  $c \in C_f$  with  $m + 1$  such values with one such value occurring at position  $\vec{y}$ . As before,  $\psi_f(F(c))$  is affine as a function in  $c(\vec{y})$ . Now by induction, if we fix the values at all other position ( $\vec{x} \neq \vec{y}$ ) when  $c(\vec{y}) = 0$ ,  $\psi(F(c))(0) = 0$  and at  $c(\vec{y}) = 1$ ,  $\psi(F(c))(1) = 0$ . Since the functions are affine they are completely described by two points, and therefore  $\psi(F(c)) = 0$  for all values of  $c(\vec{y})$ .

$\Leftarrow$ : Since the property applies to all values in  $[0, 1]$ , it must apply to  $\{0, 1\}$  as well and the implication follows from the construction of  $f$ .

## 4.5 Probabilistic Interpretation of Fuzzification

An interesting property of the DNF fuzzification is how it relates to the probability of a one occurring at a given time in a given cell. Since the fuzzy values are in the range  $[0, 1]$ , we

can interpret them as probabilities, i.e., we can let a fuzzy value  $x_i^t$  denote the probability that a cell  $y_i$  of a Boolean CA assumes value 1 at time  $t$ . Then, if the values were independent, the fuzzy rule applied to a neighbourhood would return the probability of having value 1 at the next time step:

$$f(x_{i-r}^t, \dots, x_i^t, \dots, x_{i+r}^t) = x_i^{t+1} = P(y_i^{t+1} = 1).$$

In this section we will establish some basic probabilistic results resulting from this interpretation.

### 4.5.1 Probabilistic Properties of FCA

We will begin here by introducing a property that will be needed later, relating the expectation of a Boolean local function to the fuzzy rule applied to expectations.

We will first need some notation. Given a random variable  $Z$ , let  $E(Z)$  denote its expected value. Note that when  $Z$  is a binary random variable, then  $E(Z)$  is the probability  $P(Z = 1)$ , the probability that  $Z$  is equal to 1. Random variables  $Z_1$  and  $Z_2$  are *independent* if  $E(Z_1|Z_2 = z) = E(Z_1)$  for any value of  $z$ . This implies that  $E(Z_1Z_2) = E(Z_1)E(Z_2)$ .

Essentially, we show that applying the fuzzification  $f$  of  $g$  to the expected values of a cell  $Y_i$  and its  $2r$  neighbouring cells, we obtain the expected value of  $g[Y_i]$ , the cell at the next time step.

**Theorem 16.** *Let  $(Y_0, \dots, Y_{n-1})$  be independent binary random variables. Then:*

$$\forall i = 0, \dots, n-1, \quad f[E(Y_i)] = E(g[Y_i]).$$

**Proof:** By definition,  $f[E(Y_i)] = \sum_{j=0}^{2^{2r+1}-1} b_j \prod_{k=-r}^r l(E(Y_{i+k}), d_{j,k+r})$ .

If  $d_{j,k+r} = 1$ , then

$$l(E(Y_{i+k}), d_{j,k+r}) = E(Y_{i+k}) = P(Y_{i+k} = d_{j,k+r}).$$

Similarly, if  $d_{j,k+r} = 0$ , then

$$l(E(Y_{i+k}), d_{j,k+r}) = 1 - E(Y_{i+k}) = 1 - P(Y_{i+k} = 1) = P(Y_{i+k} = 0) = P(Y_{i+k} = d_{j,k+r}).$$

So we have:

$$f[E(Y_i)] = \sum_{j=0}^{2^{2r+1}-1} b_j \prod_{k=-r}^{+r} P(Y_{i+k} = d_{j,k+r})$$

Since the variables are independent,

$$\prod_{k=-r}^{+r} P(Y_{i+k} = d_{j,k+r}) = P((Y_{i-r}, \dots, Y_{i+r}) = d_j)$$

thus:

$$f[E(Y_i)] = \sum_{j=0}^{2^{2r+1}-1} b_j \cdot P((Y_{i-r}, \dots, Y_{i+r}) = d_j).$$

Recall that  $b_j = 1$  if  $d_j \in \mathcal{T}_1$ , the set of Boolean tuples mapping to one, otherwise  $b_j = 0$ , thus:

$$\begin{aligned} f[E(Y_i)] &= \sum_{d_j \in \mathcal{T}_1} P((Y_{i-r}, \dots, Y_{i+r}) = d_j) \\ &= P((Y_{i-r}, \dots, Y_{i+r}) \in \mathcal{T}_1) \\ &= P(g[Y_i] = 1) \\ &= E(g[Y_i]). \end{aligned}$$

As a consequence of Theorem 16, we can intuitively see that the asymptotic behaviour of a FCA represents a rough approximation of the asymptotic density of the corresponding Boolean CA. In the next section, we show that such an intuition is in fact correct.

## 4.5.2 Mean Field Approximation

We will now show the connection between the asymptotic behaviour of fuzzy CA and of one descriptor of the asymptotic behaviour of Boolean CA.

The mean field approximation is an estimate of the asymptotic density of Boolean cellular automata when no spatial correlation among cells is taken into account. Thought of another way, it is again an estimate of the probability of a one occurring in a random place in a configuration once its density has stabilized [39, 82], not considering spatial correlations. Although in cellular automata spatial correlations play an important role and greatly

influence their dynamics, the mean field approximation can give a rough indication, although sometimes quite far from the exact value, of the asymptotic density. The approximation is derived by assuming that when the asymptotic probability is reached, then the likelihood of increasing in density is equal to the likelihood of decreasing in density. More formally, we assume that for all  $i$ ,  $P(y_i = 1) = p$  and that the  $y_i$  are independent. Then we can denote the probability of a transition from 0 to 1 as a function of  $p$  by  $P_{0 \rightarrow 1}(p)$ . This is equal to the probability that  $g[y_i] = 1$  given that  $y_i = 0$  or  $P(g[y_i] = 1 | y_i = 0)$ . Similarly, we denote the probability of a transition from 1 to 0 by  $P_{1 \rightarrow 0}(p)$ . A stable density of the mean field approximation is any  $p$  such that  $P_{0 \rightarrow 1}(p) - P_{1 \rightarrow 0}(p) = 0$ . We show in the following lemma that these probabilities can be evaluated as the sum of fuzzifications of the transitions from 0 to 1 evaluated at  $p$  which we denote by  $R_{0 \rightarrow 1}(p)$ , in the first instance, and as  $R_{1 \rightarrow 0}(p)$  the sum of fuzzifications of the transitions from 1 to 0 also evaluated at  $p$ , in the second.

**Lemma 5.**  $P_{0 \rightarrow 1}(p) = R_{0 \rightarrow 1}(p)$  and  $P_{1 \rightarrow 0}(p) = R_{1 \rightarrow 0}(p)$ .

**Proof:** We prove that  $P_{0 \rightarrow 1}(p) = R_{0 \rightarrow 1}(p)$ , the analogous proof holds for  $P_{1 \rightarrow 0}(p) = R_{1 \rightarrow 0}(p)$ . First note that since in the calculation of the mean field approximation we are assuming that the  $y_i$  are independent, the probability of any given neighbourhood combination  $[y_i]$  is the fuzzification of that neighbourhood evaluated at  $p$ . That is, let  $(v_{-r}, \dots, v_r)$  be a binary vector, then  $P((y_{i-r}, \dots, y_{i+r}) = (v_{-r}, \dots, v_r)) = \prod_{j=-r:r} l(p, v_j)$  where as before  $l(p, 1) = p$  and  $l(p, 0) = 1 - p$ . By definition,  $P_{0 \rightarrow 1}(p)$  is the probability that  $g[y_i] \in \tau_1$  given that  $y_i = 0$ , so it is equal to the sum of the fuzzifications of the transitions from 0 to 1, or  $R_{0 \rightarrow 1}(p)$ .

**Theorem 17.** *Given a global fuzzy rule  $F$ , if there exists an homogeneous configuration  $X = (p, \dots, p)$  such that  $F(X) = X$ , then  $p$  is a stable density of the mean field approximation of the Boolean rule  $G$  associated with  $F$ .*

**Proof:** Let  $f$  be the local rule associated with  $F$  and  $g$  its Boolean rule. Let  $R_{0 \rightarrow 1}(p)$  denote the sum of the fuzzifications of the transitions from 0 to 1 for  $g$ , evaluated at  $X = (p, \dots, p)$ .

Similarly,  $R_{0 \rightarrow 0}(p)$ ,  $R_{1 \rightarrow 0}(p)$ , and  $R_{1 \rightarrow 1}(p)$  denote sums of fuzzifications of transitions from 0 to 0, 1 to 0, and 1 to 1 evaluated at  $(p, \dots, p)$ , respectively. The sum of all these transition must be one. Since  $X$  is fixed by  $F$ , and since  $f(p, \dots, p) = R_{0 \rightarrow 1}(p) + R_{1 \rightarrow 1}(p)$  by definition, then  $R_{0 \rightarrow 1}(p) + R_{1 \rightarrow 1}(p) = p$ . Also,  $R_{1 \rightarrow 0}(p) + R_{1 \rightarrow 1}(p) = p$  since this is the sum of all terms in  $x_i$  (as opposed to terms in  $\bar{x}_i$ ), and the result is independent of  $f$ . Combining these two results, we have

$$\begin{aligned} (R_{1 \rightarrow 0}(p) + R_{1 \rightarrow 1}(p)) - (R_{0 \rightarrow 1}(p) + R_{1 \rightarrow 1}(p)) &= p - p \\ R_{1 \rightarrow 0}(p) - R_{0 \rightarrow 1}(p) &= 0. \end{aligned}$$

Thus at  $p$ ,  $P_{1 \rightarrow 0}(p) = P_{0 \rightarrow 1}(p)$  by Lemma 5 . Hence  $p$  is a stable density of the mean field approximation, as required.

Note that if  $p$  is not unique, then the mean field approximation has several stable densities.

It is easy to see that the reverse also holds.

**Theorem 18.** *If  $p$  is a stable density of the mean field approximation for a Boolean rule  $G$ , then the homogeneous configuration at that point is a fixed point for the fuzzification  $F$  of  $G$ .*

In the next section, we show further connections between the density of Boolean CA and their corresponding fuzzy CA.

## 4.6 Summary

In this chapter, we extend the definition of *number conserving* Boolean cellular automata to *sum preserving* fuzzy CA and we show that a Boolean CA is number conserving if and only if its fuzzy equivalent is sum preserving. In addition, we find an analogous spatial property which we call *spatial number conservation* and show that it too, along with any linear properties on the Boolean rule, will hold on the Boolean rule if and only if the fuzzy extensions hold on the fuzzy rule. These results hold for finite and circular CA having

---

any neighbourhood size. We also give a probabilistic interpretation of fuzzy CA whereby the fuzzy value can be regarded as the probability of a 1 occurring in a given cell in the initial configuration. Given this interpretation and an assumption of independence, after a first application of any rule, the values in each cell will again represent the probability of a 1. We have also shown, using this idea of probabilities, that when FCA converge to a fixed a point (spatially) that this point is the rough estimate of the asymptotic density of the corresponding Boolean CA given by the mean field approximation. This result suggests a further connection between the asymptotic behaviour of fuzzy and Boolean CA, in particular with respect to density. These results hold for circular CA of any neighbourhood size. The results in this chapter appeared in [5].

---

---

## CHAPTER 5

---

---

# Additivity and Self-Oscillation

In this chapter, we consider another property of Boolean cellular automata extensively studied in the literature: additivity (e.g., see [16, 52, 76]). Because these rules have a kind of semi-reversibility, they can be used in applications such as error correcting codes. We continue the investigation of the link between Boolean and fuzzy CA showing a connection between additivity and a new fuzzy property that we call *self-oscillation*. In doing so, we characterize the class of self-oscillating fuzzy CA. Although for simplicity we take our examples from one dimensional CA, the results of this section hold for any dimension.

### 5.1 Definitions and Basic Properties

A common definition of additivity is that a Boolean rule  $g$  is *additive* if  $g(y_0, \dots, y_{n-1}) \oplus g(z_0, \dots, z_{n-1}) = g(y_0 \oplus z_0, \dots, y_{n-1} \oplus z_{n-1})$ . These additive rules can be expressed as the XOR of some of their variables. An example is elementary rule 90, which can be expressed as:  $g_{90}(x, y, z) = (\bar{x} \wedge z) \vee (x \wedge \bar{z}) = x \oplus z$ . We will use a broader definition of additivity which includes rules that are additive by the definition above and their negations:

**Definition 2.** A Boolean rule  $g$  is additive if

$$g(y_0, \dots, y_{n-1}) \oplus g(z_0, \dots, z_{n-1}) = g(y_0 \oplus z_0, \dots, y_{n-1} \oplus z_{n-1}),$$

or

$$g(y_0, \dots, y_{n-1}) \oplus g(z_0, \dots, z_{n-1}) = \overline{g(y_0 \oplus z_0, \dots, y_{n-1} \oplus z_{n-1})}.$$

An example of a rule which is additive in this broader sense but not by the strict mathematical definition is rule  $g_{105}(x, y, z) = xy\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} = \overline{x \oplus y \oplus z}$ , which is equal to  $x \oplus y \oplus \bar{z} = x \oplus \bar{y} \oplus z = \bar{x} \oplus y \oplus z$ .

$$\begin{aligned} g_{105}(x_1, y_1, z_1) \oplus g_{105}(x_2, y_2, z_2) &= x_1 \oplus y_1 \oplus \bar{z}_1 \oplus x_2 \oplus y_2 \oplus \bar{z}_2 \\ &= x_1 \oplus x_2 \oplus y_1 \oplus y_2 \oplus z_1 \oplus z_2 \end{aligned}$$

while

$$\begin{aligned} g_{105}(x_1 \oplus x_2, y_1 \oplus y_2, z_1 \oplus z_2) &= x_1 \oplus x_2 \oplus y_1 \oplus y_2 \oplus \overline{z_1 \oplus z_2} \\ &= \overline{x_1 \oplus x_2 \oplus y_1 \oplus y_2 \oplus z_1 \oplus z_2}. \end{aligned}$$

In general, when  $g$  is additive,  $g(y_0, \dots, y_{n-1})$  can be expressed as the XOR of some of its variables  $y_i$  and at most one negation  $\bar{y}_i$ , which implies the following property:

**Property 1.** *An additive Boolean rule has the form:  $g(x_0, \dots, x_{n-1}) = \bigoplus_{i \in S} x_i$  or  $g(x_0, \dots, x_{n-1}) = \overline{\bigoplus_{i \in S} x_i}$ , where  $i$  ranges over  $S$ , a subset of the numbers from 0 to  $n - 1$ .*

We extend the definition of the XOR operator to fuzzy rules by defining  $x \oplus y = x\bar{y} + \bar{x}y = x(1 - y) + (1 - x)y$ . (In this section, to simplify notation we will often use  $\bar{x}$  to denote  $(1 - x)$  in a fuzzy rule.) If a Boolean rule is additive, its fuzzification is also additive and Property 1 holds for fuzzy rules as well.

A *fixed point*  $\mathbf{P}$  for a FCA with global transition rule  $F$  is a configuration  $\mathbf{P}$  such that  $F(\mathbf{P}) = \mathbf{P}$ . A configuration  $\mathbf{P} = (\dots, p_{i-1}, p_i, p_{i+1}, \dots)$  is *homogeneous* if  $p_i = p_j, \forall i, j$ ; in such a case, we obviously also have  $f(p, \dots, p) = p$ . A global rule is said to *converge* to an homogeneous configuration  $\mathbf{P} = (\dots, p, p, p, \dots)$  if, for all initial configurations  $\mathbf{X}^0 = (\dots, x_{i-1}^0, x_i^0, x_{i+1}^0, \dots)$  with  $x_i^0 \in (0, 1)$  for all  $i$ , then  $\forall \epsilon > 0, \exists T$  such that  $\forall t > T$  and  $\forall i, |x_i^t - p| < \epsilon$ . In this case, we will also say that the local rule  $f$  *converges to*  $p$ . Note that if a rule converges to a homogeneous configuration it must be a fixed point.

We can now introduce the notion of self-oscillation for fuzzy CA. Informally, a fuzzy rule  $f$  is self-oscillating if while converging towards an homogeneous fixed point, it behaves like the corresponding Boolean rule  $g$ ; in other words, when the dynamics of  $f$  around a fixed point coincides with the dynamics of  $g$ . In fact, the rule table of a fuzzy self-oscillating CA, written around its fixed point, coincides with the Boolean rule table. This is the case, for example, of elementary fuzzy rule 90 which has been shown in [33] to behave like its Boolean counter-part around  $\frac{1}{2}$ . (See Table 5.1 where  $>$  and  $<$  respectively indicate values greater than or smaller than  $\frac{1}{2}$ .)

$x$	$y$	$z$	$f_{90}(x, y, z)$	$x$	$y$	$z$	$g_{90}(x, y, z)$
$<$	$<$	$<$	$<$	0	0	0	0
$<$	$<$	$>$	$>$	0	0	1	1
$<$	$>$	$<$	$<$	0	1	0	0
$<$	$>$	$>$	$>$	0	1	1	1
$>$	$<$	$<$	$>$	1	0	0	1
$>$	$<$	$>$	$<$	1	0	1	0
$>$	$>$	$<$	$>$	1	1	0	1
$>$	$>$	$>$	$<$	1	1	1	0

Table 5.1: Rule 90: fuzzy behaviour around  $\frac{1}{2}$  ( $<$  indicates " $<\frac{1}{2}$ ",  $>$  indicates " $>\frac{1}{2}$ " ) (left); Boolean rule (right).

We now introduce the formal definition of self-oscillation. Let  $p$  be a fixed point for  $f$ . Let  $(x_1, \dots, x_{n-1})$  be an arbitrary fuzzy configuration, let  $x_n = f(x_0, \dots, x_{n-1})$ , and let us define  $y_i$ , for  $i = 0, \dots, n$ , as follows:

$$y_i = \begin{cases} 0 & \text{if } x_i < p \\ 1 & \text{if } x_i > p \end{cases}$$

**Definition 3.** *Rule  $f$  is self-oscillating around  $p$  if it converges to  $p$  and if  $f(x_0, \dots, x_{n-1}) = x_n$  implies that  $g(y_0, \dots, y_{n-1}) = y_n$ .*

Elementary rule 90 has been shown to have this type of behaviour in [33]. The other self-oscillating elementary rules will be identified using a case by case analysis in Chapter 8. However, the general implications of this behaviour were left unexplained. What was clear was that self-oscillation did not occur for all fuzzy rules with an homogeneous fixed point, but a characterization of the class of rules displaying self-oscillation was lacking until now.

## 5.2 Equivalence between Self-Oscillation and Additivity

In this section, we characterize the class of self-oscillating FCA proving the following result: *a non-trivial fuzzy CA rule is self-oscillating if and only if the corresponding Boolean CA rule is additive.*

We begin with some lemmas. We first describe the behaviour of the fuzzification of the XOR operator ( $x \oplus y = x\bar{y} + \bar{x}y$ ) around  $\frac{1}{2}$ , and then prove that convergence to  $\frac{1}{2}$  is necessary for self-oscillation.

**Lemma 6.**  *$xy + \bar{x}\bar{y}$  is greater than  $\frac{1}{2}$  if and only if both  $x$  and  $y$  are greater than  $\frac{1}{2}$  or both are smaller.*

**Lemma 7.** *A necessary condition for a convergent non-trivial rule to be self-oscillating is for it to converge to one half.*

### Proof:

To begin we note that functions converging to either zero or one can never be self-oscillating since values are, respectively, either always greater than or always less than the point of convergence. We will now prove this lemma by induction on  $n$ , the number of variables in  $f$ , i.e., on the size of the neighbourhood.

It is easy (but tedious) to show that when  $f$  is a non-trivial function on two variables only the following converge to homogeneous fixed points on  $(0, 1)$ :  $f_1(x_0, x_1) = x_0\bar{x}_1 + \bar{x}_0x_1$

and  $f_2(x_0, x_1) = x_0x_1 + \bar{x}_0\bar{x}_1$  which converge to  $\frac{1}{2}$  and are self-oscillating, and  $f_3(x_0, x_1) = \bar{x}_0\bar{x}_1$  which converges to  $p = \frac{3-\sqrt{5}}{2}$  and is not self-oscillating. For  $f_3$  to be self-oscillating, it would have to be greater than  $p$  only when both  $x_0$  and  $x_1$  were less than  $p$ . A counterexample occurs when  $x_0 = 0$  and  $x_1 = \frac{1}{2}$ , then  $f_3(x_0, x_1) = \frac{1}{2} > p$ .

Now assume that the lemma holds for all functions in  $n$  or fewer variables and consider the function  $f$  with global rule  $F$  which converges to a fixed point  $p$ . We re-write it as:  $f_+(x_0, \dots, x_{n-1})x_n + f_-(x_0, \dots, x_{n-1})\bar{x}_n$ . We wish to show that if  $f$  is convergent and non-trivial, then at least one of  $f_+$  and  $f_-$  must take on values greater than and less than  $p$ . If both  $f_+$  and  $f_-$  are always greater than  $p$ , then  $f > px_n + p(1 - x_n) = p$ . Self-oscillation implies that  $f = 1$ . Similarly, if  $f_+$  and  $f_-$  are both less than  $p$ , then  $f$  must be trivially 0. Now consider  $f_+$  always greater than  $p$  and  $f_-$  always less than  $p$ . When  $x_n = 1$ ,  $f(x_0, \dots, x_{n-1}, 1) = f_+(x_0, \dots, x_{n-1}) > p$ . Self-oscillation implies that  $f(x_0, \dots, x_n) > p$  whenever  $x_n > p$ . When  $x_n = 0$ ,  $f(x_0, \dots, x_{n-1}, 0) = f_-(x_0, \dots, x_{n-1}) < p$ . Again, self-oscillation implies  $f < p$  whenever  $x_n < p$ . Taking the two together, we must have  $f(x_0, \dots, x_n) = x_n$  which is not a convergent function. Similarly, if  $f_+ < p$  and  $f_- > p$ , we obtain  $f = \bar{x}_n$ . We conclude that at least one of  $f_+$  and  $f_-$  must have some values greater than  $p$  and some smaller. Assume, without loss of generality since the proofs are analogous, that  $f_+$  is sometimes greater than  $p$  and sometimes smaller, and again consider  $x_n = 1$  so that  $f(x_0, \dots, x_{n-1}, 1) = f_+(x_0, \dots, x_{n-1})$ . The function  $f_+$  is completely determined by  $f$  and so must be self-oscillating around  $p$ . By the inductive hypothesis,  $p = \frac{1}{2}$ .

As we know, given a Boolean rule  $g$ , we can derive its fuzzification  $f$  as the sum of the fuzzifications of each of its transitions to 1. In the following, we refer to each of the products in this sum as a *term of  $f$* .

**Lemma 8.** *If  $f(x_0, \dots, x_{n-1})$  converges to  $\frac{1}{2}$ ,  $f$  is the sum of  $2^{n-1}$  terms.*

**Proof:** The terms of any function evaluated at  $(\frac{1}{2}, \dots, \frac{1}{2})$  are all equal to  $(\frac{1}{2})^n$ . For  $f(\frac{1}{2}, \dots, \frac{1}{2}) = \frac{1}{2}$ , we must have  $2^{n-1}$  such terms summed together.

We now prove that for a fuzzy rule on  $n$  variables to be self-oscillating, it must be

balanced in  $x_i$  and  $\bar{x}_i$ . That is, it must be the sum of the same number of terms in  $x_i$  as in  $\bar{x}_i$  for all  $i$ .

**Lemma 9.** *Let  $f(x_0, \dots, x_{n-1})$  be self-oscillating. Then for all  $i$ , there are as many terms in the sum of  $f$  in  $x_i$  as there are terms in  $\bar{x}_i$ .*

**Proof:** We will show by contradiction that there are as many terms in the sum of  $f$  in  $x_i$  as there are terms in  $\bar{x}_i$ . We begin by writing  $f$  as:

$$f(x_0, \dots, x_{n-1}) = f_{i+}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1})x_i + f_{i-}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1})\bar{x}_i$$

Assume without loss of generality that more than half the terms are in  $f_+$ . Let there be  $m > 2^{n-2}$  terms in  $f_{i+}$ . Then, by Lemma 8 there must be  $2^{n-1} - m$  terms in  $f_{i-}$ . Then as  $x_j \rightarrow \frac{1}{2}$  for all  $j \neq i$ , each term of  $f_{i+}$  tends to  $\frac{1}{2}^{n-1}$  and thus  $f_{i+} \rightarrow \frac{m}{2^{n-1}}$ , which is  $> \frac{1}{2}$  because we assumed  $m > 2^{n-2}$ . Moreover,  $f_{i-} \rightarrow \frac{2^{n-1}-m}{2^{n-1}} < \frac{1}{2}$ . Note that this convergence happens as the  $x_j$  approach  $\frac{1}{2}$  from both directions. Choosing  $x_j$  close enough to  $\frac{1}{2}$ , we can assume that  $f_{i+} > \frac{1}{2}$  and  $f_{i-} < \frac{1}{2}$ . Now:  $f(x_0, \dots, x_{n-1}) = f_{i+}x_i + f_{i-}(1 - x_i) = (f_{i+} - f_{i-})x_i + f_{i-}$ . At  $x_i = 1$ ,  $f(x_0, \dots, x_{n-1}) = f_{i+} > \frac{1}{2}$ . That is for all values of  $x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1}$  close enough to  $\frac{1}{2}$ , whether greater than or less than  $\frac{1}{2}$ ,  $f(x_0, \dots, x_{n-1}) = f_{i+} > \frac{1}{2}$ . Similarly, when  $x_i = 0$ ,  $f(x_0, \dots, x_{n-1}) = f_{i-} < \frac{1}{2}$ . Self-oscillation then implies that  $f(x_0, \dots, x_{n-1}) = x_i$  which is not a convergent function.

We are finally able to characterize the form of a self-oscillating rule. We will see that these rules are fuzzifications of Boolean rules which are the XOR of single variables or their negations.

**Theorem 19.** *A rule  $f(x_0, \dots, x_{n-1})$  is self-oscillating if and only if its corresponding Boolean rule is additive.*

**Proof:**  $\Rightarrow$ :

We will prove that if a self-oscillating rule is additive,  $f(x_0, \dots, x_{n-1}) = \bigoplus_{i \in S} x_i$  or  $f(x_0, \dots, x_{n-1}) = \bigoplus_{i \in S} \bar{x}_i$ , (and thus the corresponding Boolean rule is additive) by induction on  $n$ .

For  $n = 2$ , from Lemma 9, we must have one term in  $x_i$  and one term in  $\bar{x}_i$  for  $i \in \{0, 1\}$  giving us only two possibilities:  $f(x_0, x_1) = x_0\bar{x}_1 + \bar{x}_0x_1 = x_0 \oplus x_1$  or  $f(x_0, x_1) = \bar{x}_0\bar{x}_1 + x_0x_1 = \bar{x}_0 \oplus x_1$  as required.

Now assume the hypothesis for all self-oscillating rules in less than or equal to  $n$  variables. Given a self-oscillating rule  $f(x_0, \dots, x_n)$ , if  $f$  is not dependent on all  $n + 1$  variables, then it can be rewritten as a self-oscillating rule on  $n$  or fewer variables and the inductive hypothesis holds. So we may continue on the assumption that  $f$  is dependent on all  $n + 1$  variables. We can write:

$$\begin{aligned} f(x_0, \dots, x_n) &= [f_{1-}(x_0, \dots, x_{n-2})\bar{x}_{n-1} + f_{1+}(x_0, \dots, x_{n-2})x_{n-1}]\bar{x}_n \\ &\quad + [f_{2-}(x_0, \dots, x_{n-2})\bar{x}_{n-1} + f_{2+}(x_0, \dots, x_{n-2})x_{n-1}]x_n \end{aligned}$$

Letting  $x_n = 0$ ,  $f(x_0, \dots, x_{n-1}, 0)$  is a self-oscillating rule on  $n$  variables so the inductive hypothesis applies and

$$f_{1-}(x_0, \dots, x_{n-2})\bar{x}_{n-1} + f_{1+}(x_0, \dots, x_{n-2})x_{n-1} = x_0 \oplus \dots \oplus x_{n-1}$$

or

$$f_{1-}(x_0, \dots, x_{n-2})\bar{x}_{n-1} + f_{1+}(x_0, \dots, x_{n-2})x_{n-1} = \bar{x}_0 \oplus x_1 \oplus \dots \oplus x_{n-1}.$$

Specifically, we must have  $f_{1-}(x_0, \dots, x_{n-2}) = x_0 \oplus x_1 \oplus \dots \oplus x_{n-2}$ ,  $f_{1+}(x_0, \dots, x_{n-2}) = \bar{x}_0 \oplus x_1 \oplus \dots \oplus x_{n-2}$  or the opposite. Setting  $x_n$  to 1, we can say the same thing about  $f_{2-}$  and  $f_{2+}$ .

Using the same argument, if we let  $x_{n-1} = 0$ , we see that  $f_{2-} = \bar{f}_{1-}$ . Thus we have only two possibilities for  $f$ :

$$\begin{aligned} f(x_0, \dots, x_n) &= [(x_0 \oplus \dots \oplus x_{n-2})\bar{x}_{n-1} + (\bar{x}_0 \oplus \dots \oplus x_{n-2})x_{n-1}]\bar{x}_n \\ &\quad + [(\bar{x}_0 \oplus \dots \oplus x_{n-2})\bar{x}_{n-1} + (x_0 \oplus \dots \oplus x_{n-2})x_{n-1}]x_n \\ &= x_0 \oplus \dots \oplus x_n \end{aligned}$$

or

$$f(x_0, \dots, x_n) = [(\bar{x}_0 \oplus \dots \oplus x_{n-2})\bar{x}_{n-1} + (x_0 \oplus \dots \oplus x_{n-2})x_{n-1}]\bar{x}_n$$

$$\begin{aligned}
& + [(x_0 \oplus \cdots \oplus x_{n-2})\bar{x}_{n-1} + (\bar{x}_0 \oplus \cdots \oplus x_{n-2})x_{n-1}]x_n \\
& = \bar{x}_0 \oplus x_1 \oplus \cdots \oplus x_n.
\end{aligned}$$

$\Leftarrow$ : We will assume that the Boolean rule corresponding to  $f$  is additive (and thus  $f(x_0, \dots, x_{n-1})$  is also additive) and proceed by induction on  $n$  to show that it is self-oscillating. When  $n = 2$ ,  $f(x_0, x_1)$  is equal to  $x_0 \oplus x_1$  or  $x_0 \oplus \bar{x}_1$ . In either case, by Lemma 15  $f$  is self-oscillating.

Now assume that for  $n$  or fewer variables, additivity implies self-oscillation and consider  $f(x_0, \dots, x_n)$ . Without loss of generality, assume that  $f$  is not independent of  $x_n$ , then we can write it as  $f(x_0, \dots, x_n) = f_1(x_0, \dots, x_{n-1}) \oplus x_n$  for an additive rule  $f_1$  which is self-oscillating by the induction hypothesis. Again applying Lemma 15,  $f$  must be self-oscillating.

If we restrict our examination to elementary rules, we obtain the following corollary.

**Corollary 5.** *Up to equivalence, all and only rules  $f_{60}$ ,  $f_{90}$ ,  $f_{105}$ , and  $f_{150}$  are elementary self-oscillating rules.*

### 5.3 Summary

In this chapter, we formally define *self-oscillation* and we show that a Boolean rule is additive if and only if its fuzzy equivalent exhibits self-oscillation. That is, the fuzzy equivalents of additive Boolean rules will converge to one half but will fluctuate around this point of convergence in a sense continuing to obey the Boolean rule. The results in this chapter hold for all Boolean CA, whether finite, circular or infinite and regardless of neighbourhood size.

The results in this chapter appeared in [3] and [5].

---

---

## CHAPTER 6

---

---

# Weighted Average Rules

In this chapter, we begin the study of the asymptotic behaviour of fuzzy cellular automata with a class of FCA, Weighted Average rules, which was of particular interest because it contained rules displaying most of the observed dynamics: fixed points, periods of length 2, and periods of length 4. In addition, it includes both the number conserving rule, 184 and the spatial number conserving rule, 46. Also, from an applications perspective, these rules, especially the generalized versions developed in this section, can be used to model such things as electoral influence in voting. We analytically study the asymptotic behaviour of all the elementary rules belonging to this class and we prove that they all exhibit spatial and temporal periodic behaviour from arbitrary initial configurations. The emphasis in this and the remaining chapters is on formal proof. The behaviours described here have been observed and classified, but they have not been proven to occur as a result of the structure of the rule and under any conditions on initial configurations. At the end of this chapter, we will show how weighted average rules of varying neighbourhood size can be constructed so that the resulting fuzzy rule has a desired periodicity.

## 6.1 Definitions and Basic Properties

### 6.1.1 Definitions for Periodic Behaviour

In this section we define periodicity and asymptotic periodicity, concepts which will be used to describe the convergent behaviours of several categories of rules in this and the following chapters.

A rule is said to be *Temporally Periodic* with period  $\tau$  if  $\exists T$  such that  $\forall t > T: F(\mathbf{X}^t) = F(\mathbf{X}^{t+\tau})$ . Similarly, a rule is *Spatially Periodic* with period  $\omega$  if  $\exists T$  such that  $\forall t > T, \forall i: x_i^t = x_{i+\omega}^t$

**Definition 4.** A rule is *Asymptotically Periodic in Time* (or *Asymptotically Temporally Periodic*) with period  $\tau$  if  $\forall \epsilon > 0 \exists T$  such that  $\forall t > T$  and  $\forall i: |x_i^t - x_i^{t+\tau}| < \epsilon$

**Definition 5.** A rule is *Asymptotically Periodic in Space* (or *Asymptotically Spatially Periodic*) with period  $\omega$  if  $\forall \epsilon > 0 \exists T$  such that  $\forall t > T$  and  $\forall i: |x_i^t - x_{i+\omega}^t| < \epsilon$

A rule that is asymptotically periodic in space with period 1 will be called *asymptotically homogeneous*. A rule which is asymptotically periodic in time with period 1 will be said to be *convergent to a fixed point*.

A *fixed point*  $\mathbf{P}$  for a FCA with global transition rule  $F$  is a configuration  $\mathbf{P}$  such that  $F(\mathbf{P}) = \mathbf{P}$ . A configuration  $\mathbf{P} = (\dots, p_{i-1}, p_i, p_{i+1}, \dots)$  is *homogeneous* if  $p_i = p_j, \forall i, j$ ; in such a case, we obviously also have  $f(p, \dots, p) = p$ .

**Definition 6.** A global rule is said to converge to an homogeneous configuration  $\mathbf{P} = (\dots p, p, p, \dots)$  if, for all initial configurations  $\mathbf{X}^0 = (\dots, x_{i-1}^0, x_i^0, x_{i+1}^0, \dots)$  with  $x_i^0 \in (0, 1)$  for all  $i$ , then  $\forall \epsilon > 0, \exists T$  such that  $\forall t > T$  and  $\forall i, |x_i^t - p| < \epsilon$ . In this case, we will also say that the local rule  $f$  converges to  $p$ .

Note that if a rule converges to a homogeneous configuration, it must be a fixed point.

### 6.1.2 Conjugate CA

A tool that we will be using in the proofs of convergence in this and subsequent chapters is to relate the CA to conjugate systems which are easily shown to converge. Hence, we need to define conjugate CA. Two cellular automata  $G_1$  and  $G_2$  having the same neighbourhood size and acting on the same space of configurations are said to be *conjugate* if there exists a homeomorphism,  $h$  on that configuration space such that the following diagram commutes. That is, for all configurations  $X \in C$ , the following holds:

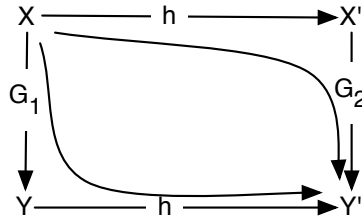


Figure 6.1: Conjugate CA

$$h \circ G_1(X) = G_2 \circ h(X).$$

Generally speaking and throughout this thesis, when discussing the elementary Boolean rules, we assume that results hold for all conjugates under conjugation ( $h(\cdots, x_{-1}, x_0, x_1, \cdots) = (\cdots, \bar{x}_{-1}, \bar{x}_0, \bar{x}_1, \cdots)$ ), reflection ( $h(\cdots, x_{-1}, x_0, x_1, \cdots) = (\cdots, x_1, x_0, x_{-1}, \cdots)$ ), or both combined.

### 6.1.3 Definitions for Weighted Average Rules

A weighted average of  $\alpha$  and  $\beta$  is a quantity of the form  $\mu = \gamma\alpha + (1 - \gamma)\beta$  where  $\gamma \in [0, 1]$ . The usual mean occurs when  $\gamma$  equals one half. Weighted averages have the following useful properties.

**Property 2.** If  $\mu = (1 - \gamma)\alpha + \gamma\beta$  then  $(1 - \mu) = (1 - \gamma)(1 - \alpha) + \gamma(1 - \beta)$

**Property 3.** If  $\gamma_1 < \gamma_2$  and  $\alpha < \beta$  then  $(1 - \gamma_1)\alpha + \gamma_1\beta < (1 - \gamma_2)\alpha + \gamma_2\beta$

A *generalized fuzzy CA* is a lattice where every cell updates its state according to a different local rule. In this paper, we will consider generalized fuzzy CA with neighbourhood 1 ( $q = 1$ ) and with rules of the type defined below.

**Definition 7.** *GWCA*

A generalized fuzzy cellular automata with weighted average rules (*GWCA*) is a generalized fuzzy CA where the local rule has the following form:

$$x_i^{t+1} = \gamma_i^t x_i^t + (1 - \gamma_i^t) x_{i+1}^t \quad (6.1.1)$$

$$(or \ x_i^{t+1} = \gamma_i^t x_i^t + (1 - \gamma_i^t) x_{i-1}^t) \quad (6.1.2)$$

with bounded weights. That is, there exists  $0 < \gamma < \frac{1}{2}$  such that  $\gamma_i^t \in (\gamma, 1 - \gamma)$  for all  $i$  and for all  $t$ .

In other words, in a *GWCA* the state of a cell  $i$  at time  $t + 1$  takes the average of the state of the cell itself and of one of its neighbours at time  $t$  weighted by a value in  $(0, 1)$  that varies from cell to cell.

## 6.2 A General Convergence Theorem

Recall that a *GWCA* is a generalized fuzzy CA where the local rule at every cell may vary but will always have the form of a weighted average as in 6.1.2. We will now prove a convergence result for *GWCA* when the initial configurations are in the open interval  $(0, 1)^n$ . We wish to show that repeated weighted averaging of a circular array results in the values in the array converging to a fixed point, as stated in the general theorem below.

**Theorem 20. General Theorem**

Consider a *GWCA* starting from configuration  $\mathbf{X}^0 = (x_0^0, \dots, x_{n-1}^0)$ . Then for some  $p \in [0, 1]$ ,  $x_i^t \rightarrow p$  for all  $i$  as  $t \rightarrow \infty$ .

We will prove the theorem by a sequence of lemmas.

**Lemma 10.** *Consider the sequence  $\min_i\{x_i^0\}, \min_i\{x_i^1\}, \dots, \min_i\{x_i^t\}, \dots$ . Such a sequence converges to some value  $l_m$ .*

**Proof:** Since each value is averaged with its neighbour, the result is always between these two values. Assume that  $x_i^t \leq x_{i+1}^t$ . Then,

$$x_i^{t+1} = \gamma_i x_i^t + (1 - \gamma_i) x_{i+1}^t \geq \gamma_i x_i^t + (1 - \gamma_i) x_i^t \geq x_i^t$$

with equality if and only if  $x_i^t = x_{i+1}^t$ . In particular, the previous holds when  $x_i^t$  is the minimum value at time  $t$ . Since all values at time  $t + 1$  are the weighted averages of values greater than or equal to the minimum value at time  $t$ , all values at time  $t + 1$  must be greater than or equal to the minimum at time  $t$ . Thus the sequence is increasing and bounded above by 1 and therefore convergent.

Analogously we have:

**Lemma 11.** *Consider the sequence  $\max_i\{x_i^0\}, \max_i\{x_i^1\}, \dots, \max_i\{x_i^t\}, \dots$ . Such a sequence converges to some value  $l_M$ .*

We wish to show that  $l_m = l_M$ . We will proceed by showing that if there is a difference in the maximum and minimum values in the configuration at some time  $t$ , then in  $n - 1$  iterations (i.e., at time  $t + n - 1$ ), the minimum must increase by at least an amount proportional to this difference.

**Lemma 12.** *Given a configuration of length  $n$ , if at any time  $t$ ,  $\max_i\{x_i^t\} - \min_i\{x_i^t\} \geq \delta$  then  $\min_i\{x_i^{t+n-1}\} - \min_i\{x_i^t\} \geq \gamma^{n-1} \delta$  where  $\gamma < \gamma_i < 1 - \gamma$ .*

**Proof:** Let  $m = \min_i\{x_i^t\}$  and  $M = \max_i\{x_i^t\}$  so that  $M - m \geq \delta$ . Renumbering if necessary, we may assume that the maximum occurs at  $x_{n-1}^t$ , so that:

$$x_i^t \geq \begin{cases} m & \text{for } 0 \leq i < n - 1 \\ M \geq m + \gamma^0 \delta & \text{for } i = n - 1 \end{cases} \quad (6.2.1)$$

with  $x_i^t = m$  for at least one  $i$  and  $x_{n-1}^t = M$ .

We wish to show by induction that for  $0 \leq s \leq n-1$ ,

$$x_i^{t+s} \geq \begin{cases} m & \text{for } 0 \leq i < n-s-1 \\ m + \gamma^s \delta & \text{for } n-s-1 \leq i < n \end{cases}$$

This is true when  $s = 0$  by equation (6.2.1). Now assume it is true for  $x_i^{t+s}$  with  $s \leq n-2$ , and consider  $x_i^{t+s+1}$ .

We prove this separately for  $i = n-s-2$ , for  $i = n-1$ , and for  $0 \leq n-s-1 < i < n-1$ . In each case, we obtain a new lower bound by averaging the current lower bounds, giving the highest weight,  $(1-\gamma)$ , to the lower of the two values being averaged, as in Property 3. For  $i = n-s-2$ , we have that:

$$\begin{aligned} x_{n-s-2}^{t+s+1} &= (1 - \gamma_{n-s-2}^{t+s})x_{n-s-2}^{t+s} + \gamma_{n-s-2}^{t+s}x_{n-s-1}^{t+s} \text{ by definition} \\ &\geq (1 - \gamma_{n-s-2}^{t+s})m + \gamma_{n-s-2}^{t+s}(m + \gamma^s \delta) \text{ by the induction hypothesis} \\ &\geq (1 - \gamma)m + \gamma(m + \gamma^s \delta) \text{ by Property 3 and by the fact that } \forall i \forall t, \gamma < \gamma_i^t < 1 - \gamma \\ &\geq m + \gamma(m + \gamma^s \delta - m) \\ &\geq m + \gamma^{s+1} \delta \end{aligned}$$

In the case of  $i = n-1$ , we are averaging  $m$  and  $m + \gamma^s \delta$ , hence  $x_{n-1}^{t+s+1} \geq (1-\gamma)m + \gamma \gamma^s \delta \geq m + \gamma^{s+1} \delta$ . For  $0 < i < n-s-2$ , we are averaging values greater than or equal to  $m$ , and thus the result will be greater than or equal to  $m$ . Finally, for  $n-s-1 < i < n-1$ , we have weighted averages of values greater than  $m + \gamma^s \delta$  and therefore,  $x_i^{t+s+1} \geq m + \gamma^s \delta > m + \gamma^{s+1} \delta$ .

As a consequence, when  $s = n-1$ , that is at time  $t+n-1$ , we have that  $x_i^{t+n-1} \geq m + \gamma^{n-1} \delta$  for all  $i$ . Hence  $\min_i \{x_i^{n-1}\} - \min_i \{x_i^0\} \geq (m + \gamma^{n-1} \delta) - m \geq \gamma^{n-1} \delta$ .

We are now ready to prove the General Theorem.

**Proof:** (of the General Theorem) We will show by contradiction that  $l_m = l_M$ . Let  $l_M - l_m = \delta > 0$ . Fix  $\epsilon$  such that  $0 < \epsilon < \gamma^{n-1} \delta$  and let  $T$  be such that for all  $t > T$ , if  $m = \min \{x_1^t, \dots, x_n^t\}$  and  $M = \max \{x_1^t, \dots, x_n^t\}$  then  $l_m - m < \epsilon$  and  $M - l_M < \epsilon$ . Note that  $M - m > \delta$ .

Let  $m^t = \min_i \{x_i^t\}$ . Fix  $t > T$ . By Lemma 12,

$$m^{t+n-1} > m^t + \gamma^{n-1} \delta > m + \epsilon > l_m$$

which is a contradiction. Hence  $l_M - l_m = 0$ . Let us call this limit  $p$ .

For all  $x_i^t$ , we have  $m^t \leq x_i^t \leq M^t$ . Since as  $t \rightarrow \infty$ ,  $m^t \rightarrow p$  and  $M^t \rightarrow p$ ,  $x_i^t \rightarrow p$  also.

### 6.3 Convergence of Elementary Weighted Average Rules

Elementary FCA rules 184 and 46, as well as some others, can be viewed as weighted averages of two values in a neighbourhood (or their negations) weighted by the third value (or its negation), closely resembling *GWCA*. Table 7.1 lists the elementary FCA with this feature. Note that we will use  $(1 - x)$  and  $\bar{x}$  interchangeably.

Rule	Equation	Averaged	Weight
<b>R<sub>184</sub></b> (= $R_{226}$ )	$(1 - x_i)x_{i-1} + x_i x_{i+1}$	$x_{i-1}, x_{i+1}$	$x_i$
<b>R<sub>46</sub></b> (= $R_{116}, R_{139}, R_{209}$ )	$(1 - x_i)x_{i+1} + x_i(1 - x_{i-1})$	$x_{i+1}, \bar{x}_{i-1}$	$x_i$
$R_{27}$ (= $R_{39}, R_{53}, R_{83}$ )	$(1 - x_{i+1})(1 - x_i) + x_{i+1}(1 - x_{i-1})$	$\bar{x}_i, \bar{x}_{i-1}$	$x_{i+1}$
$R_{29}$ (= $R_{71}$ )	$(1 - x_i)(1 - x_{i+1}) + x_i(1 - x_{i-1})$	$\bar{x}_{i+1}, \bar{x}_{i-1}$	$x_i$
$R_{58}$ (= $R_{114}, R_{163}, R_{177}$ )	$(1 - x_{i-1})x_{i+1} + x_{i-1}(1 - x_i)$	$x_{i+1}, \bar{x}_i$	$x_{i-1}$
$R_{78}$ (= $R_{92}, R_{141}, R_{197}$ )	$(1 - x_{i+1})x_i + x_{i+1}(1 - x_{i-1})$	$x_i, \bar{x}_{i-1}$	$x_{i+1}$
$R_{172}$ (= $R_{212}, R_{202}, R_{228}$ )	$(1 - x_{i-1})x_i + x_{i-1}x_{i+1}$	$x_i, x_{i+1}$	$x_{i-1}$

Table 6.1: Weighted Average Rules. Rules equivalent under conjugation, reflection and both are indicated in parenthesis.

Although Theorem 20 does not apply directly to these rules (except for the case of rule  $R_{172}$ ), we can determine their asymptotic behavior by constructing, for each of them, a topologically conjugate system for which the theorem does apply. In this section we assume

that the initial configurations are in the open interval  $(0, 1)^n$ . We show convergence beginning with the simplest and progressing to the most complex.

### 6.3.1 Rule 172

The simplest rule to analyze is rule 172:

$$x_i^{t+1} = (1 - x_{i-1}^t)x_i^t + x_{i-1}^t x_{i+1}^t$$

**Theorem 21.** *Rule 172 converges spatially and temporally to a homogeneous configuration.*

**Proof:** Rule 172 is the weighted average of  $x_i^t$  and  $x_{i+1}^t$ , so, to apply Theorem 20 we only need to show that there exists a value  $0 < \gamma < \frac{1}{2}$  such that the weights are bounded by  $\gamma$  and  $(1 - \gamma)$ . Let  $\gamma = \min \{x_0^0, \dots, x_{n-1}^0, 1 - x_0^0, \dots, 1 - x_{n-1}^0\}$  then  $\forall t$ ,

$$\begin{aligned} \gamma &\leq \min \{x_0^t, \dots, x_{n-1}^t, 1 - x_0^t, \dots, 1 - x_{n-1}^t\} \\ 1 - \gamma &\geq \max \{x_0^t, \dots, x_{n-1}^t, 1 - x_0^t, \dots, 1 - x_{n-1}^t\} \end{aligned}$$

Since the weights used by rule 172 are taken from these sets, it follows that  $\gamma \leq \gamma_i^t \leq 1 - \gamma$  and the conditions of Theorem 20 hold. Hence, there exists a value  $p$  such that  $x_i^t \rightarrow p$  for all  $i$  as  $t \rightarrow \infty$ .

In all subsequent rules, the lowest weight occurring in any equation at the first iteration provides a lower bound  $\gamma$  on the weights.

### 6.3.2 Rule 78

Consider rule  $R_{78}$  as a weighted average:

$$x_i^{t+1} = (1 - x_{i+1}^t)x_i^t + x_{i+1}^t(1 - x_{i-1}^t)$$

**Theorem 22.** *When  $n$  is even, rule 78 converges temporally to a homogeneous configuration. Furthermore, if  $x_0^t$  converges to a value  $p$ , then for all  $i$ ,  $x_{2i}^t$  converges to  $p$ , and  $x_{2i+1}^t$  converges to  $(1 - p)$ . When  $n$  is odd, rule 78 converges to the homogeneous configuration  $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ .*

**Proof:**

Let  $A$  be a fuzzy cellular automaton following rule 78 and let  $f : (0, 1)^3 \rightarrow (0, 1)$  be its local rule and  $F$  its global rule.

Assume that  $n$  is even. Consider a  $GWCA$   $A'$  with global rule  $F'$  given by two local rules  $f'_1, f'_2 : (0, 1)^3 \rightarrow (0, 1)$  defined as follows:

$$f'_1(x, y, z) = zy + (1 - z)x$$

$$f'_2(x, y, z) = (1 - z)y + zx$$

which are applied in alternation to a configuration  $(x_0, \dots, x_{n-1})$  as follows:

$$F'(x_0, \dots, x_{n-1}) = (f'_1(x_{n-1}, x_0, x_1), f'_2(x_0, x_1, x_2), \dots, f'_2(x_{n-2}, x_{n-1}, x_0))$$

Let  $h : (0, 1)^n \rightarrow (0, 1)^n$  be a homeomorphism defined by:

$$h(x_0, x_1, x_2, \dots, x_{n-1}) = (x_0, 1 - x_1, x_2, \dots, 1 - x_{n-1})$$

We want to show that  $F$  and  $F'$  are conjugates, that is that  $h \circ F = F' \circ h$ . Let  $\mathbf{X} = (x_0, x_1, \dots, x_{n-1})$  be an arbitrary configuration for  $A$ .

$$\begin{aligned} h \circ F(x_0, x_1, \dots, x_{n-1}) &= h((1 - x_1)x_0 + x_1(1 - x_{n-1}), (1 - x_2)x_1 + x_2(1 - x_0), \dots, (1 - x_0)x_{n-1} + x_0(1 - x_{n-2})) \\ &= ((1 - x_1)x_0 + x_1(1 - x_{n-1}), \overline{(1 - x_2)x_1 + x_2(1 - x_0)}, \dots, \overline{(1 - x_0)x_{n-1} + x_0(1 - x_{n-2})}) \\ &= ((1 - x_1)x_0 + x_1(1 - x_{n-1}), (1 - x_2)(1 - x_1) + x_2x_0, \dots, (1 - x_0)(1 - x_{n-1}) + x_0x_{n-2}) \\ F' \circ h(x_0, x_1, \dots, x_{n-1}) &= F'(x_0, (1 - x_1), \dots, (1 - x_{n-1})) \\ &= ((1 - x_1)x_0 + x_1(1 - x_{n-1}), (1 - x_2)(1 - x_1) + x_2x_0, \dots, (1 - x_0)(1 - x_{n-1}) + x_0x_{n-2}) \end{aligned}$$

By Theorem 20, the  $GWCA$   $A'$  converges to some point  $(p, p, \dots, p)$ . Then  $h^{-1}(p, p, \dots, p) = (p, 1 - p, p, \dots, 1 - p)$  is the point of convergence  $A$ .

$$\mathbf{X}^t \rightarrow (p, 1 - p, \dots, p, 1 - p)$$

When  $n$  is odd, consider a GWCA  $B$  of length  $2n$  with global rule  $F'$  as above, and a homeomorphism  $h : A \rightarrow B$  defined as:

$$h(x_0, x_1, \dots, x_{n-1}) = (x_0, \bar{x}_1, \dots, x_{n-1}, \bar{x}_0, x_1, \dots, \bar{x}_{n-1}).$$

Functions  $F$  and  $F'$  are conjugate since  $h \circ F = F' \circ h$ . In fact:

$$\begin{aligned} F' \circ h(x_0, x_1, \dots, x_{n-1}) &= F'(x_0, \bar{x}_1, \dots, x_{n-1}, \bar{x}_0, x_1, \dots, \bar{x}_{n-1}) \\ &= (\bar{x}_1 x_0 + x_1 \bar{x}_{n-1}, \bar{x}_2 \bar{x}_1 + x_2 x_0, \dots, \bar{x}_0 x_{n-1} + x_0 \bar{x}_{n-2}, \bar{x}_1 \bar{x}_0 + x_1 x_{n-1}, \dots, \bar{x}_0 \bar{x}_{n-1} + x_0 x_{n-2}), \end{aligned}$$

while

$$\begin{aligned} h \circ F(x_0, x_1, \dots, x_{n-1}) &= h(\bar{x}_1 x_0 + x_1 \bar{x}_{n-1}, \bar{x}_2 x_1 + x_2 \bar{x}_0, \dots, \bar{x}_0 x_{n-1} + x_0 \bar{x}_{n-2}) \\ &= (\bar{x}_1 x_0 + x_1 \bar{x}_{n-1}, \bar{x}_2 \bar{x}_1 + x_2 x_0, \dots, \bar{x}_0 x_{n-1} + x_0 \bar{x}_{n-2}, \bar{x}_1 \bar{x}_0 + x_1 x_{n-1}, \dots, \bar{x}_0 \bar{x}_{n-1} + x_0 x_{n-2}). \end{aligned}$$

Since  $f'_1$  and  $f'_2$  are weighted sums of neighbours,  $F'$  must converge to a single value,  $p$ . But we also notice that for all  $i$ ,  $x_i^t = (1 - x_i^t)$  as  $t \rightarrow \infty$ , since  $x_{i+n}^t = (1 - x_i^t) \rightarrow p$  also. So we must have that  $p = \frac{1}{2}$  and  $h^{-1}(\frac{1}{2}, \dots, \frac{1}{2}) = (\frac{1}{2}, \dots, \frac{1}{2})$  is the point of convergence of  $A$ .

### 6.3.3 Rule 27

We restate  $R_{27}$  as a weighted average:

$$x_i^{t+1} = (1 - x_{i+1}^t)(1 - x_i^t) + x_{i+1}^t(1 - x_{i-1}^t)$$

**Theorem 23.** *Rule 27 converges spatially to a single value and temporally it has asymptotic periodicity with period 2. Furthermore, if for all  $i$ , as  $t \rightarrow \infty$   $x_i^{2t} \rightarrow p$  then  $x_i^{2t+1} \rightarrow 1 - p$ .*

**Proof:** Let  $A$  be a fuzzy cellular automaton following rule 27 and let  $f : (0, 1)^3 \rightarrow (0, 1)$  be its local rule and  $F$  its global rule.

Let  $B$  be a fuzzy cellular automaton with global rule  $F \circ F$ . Consider a  $GWCA$   $B'$  with global rule  $F'_2 \circ F'_1$  where  $F'_1$  is defined by local rule  $f'_1$  and  $F'_2$  by  $f'_2$ . Local rules  $f'_1, f'_2 : (0, 1)^3 \rightarrow (0, 1)$  are defined as follows:

$$f'_1(x, y, z) = (1 - z)y + zx$$

$$f'_2(x, y, z) = zy + (1 - z)x$$

We want to show that  $F \circ F = F'_2 \circ F'_1$ . Let  $id$  be the identity homeomorphism and let  $inv$  be the negation homeomorphism:  $inv(x_0, x_1, \dots, x_{n-1}) = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{n-1})$ .

We will prove the equality of  $F \circ F$  and  $F'_2 \circ F'_1$  by showing that the following diagram commutes.

$$\begin{array}{ccc}
 & id & \\
 A & \longrightarrow & B \\
 \\
 F \downarrow & & \downarrow F'_1 \\
 & inv & \\
 A & \longrightarrow & B \\
 \\
 F \downarrow & & \downarrow F'_2 \\
 & id & \\
 A & \longrightarrow & B
 \end{array}$$

That is, we will show that  $F'_1 = inv \circ F$  and  $F = F'_2 \circ inv$

$$\begin{aligned}
 inv \circ F(\dots, x_{i-1}, x_i, x_{i+1}, \dots) & \\
 = inv(\dots, \bar{x}_{i+1}\bar{x}_i + x_{i+1}\bar{x}_{i-1}, \dots) & \\
 = (\dots, \bar{x}_{i+1}x_i + x_{i+1}x_{i-1}, \dots) & \\
 = F'_1(\dots, x_{i-1}, x_i, x_{i+1}, \dots) &
 \end{aligned}$$

$$\begin{aligned}
F'_2 \circ \text{inv}(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots) & \\
= F'_2(\cdots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \cdots) & \\
= (\cdots, \bar{x}_{i+1}\bar{x}_i + x_{i+1}\bar{x}_{i-1}, \cdots) & \\
= F(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots) &
\end{aligned}$$

Finally,

$$\begin{aligned}
F \circ F &= F'_2 \circ \text{inv} \circ F \\
&= F'_2 \circ F'_1
\end{aligned}$$

Thus  $F \circ F$  and  $F'_2 \circ F'_1$  are equal. Since both  $F'_1$  and  $F'_2$  are weighted averages of neighbours with bounded weights, Theorem 20 applies, thus  $B'$  is converging to a homogeneous configuration. Moreover, every other iteration of  $A$  must also be converging.

If we let the limit of the even iterations be  $p$ , then

$$\begin{aligned}
\lim_{t \rightarrow \infty} (x_0^{2t+1}, x_1^{2t+1}, \cdots, x_{n-1}^{2t+1}) &= \lim_{t \rightarrow \infty} F(x_0^{2t}, x_1^{2t}, \cdots, x_{n-1}^{2t}) \\
&= F(\lim_{t \rightarrow \infty} x_0^{2t}, \lim_{t \rightarrow \infty} x_1^{2t}, \cdots, \lim_{t \rightarrow \infty} x_{n-1}^{2t}) \\
&= F(p, p, \cdots, p) \\
&= (1-p, 1-p, \cdots, 1-p)
\end{aligned}$$

So

$$\begin{aligned}
\mathbf{X}^{2t} &\rightarrow (p, \cdots, p) \\
\mathbf{X}^{2t+1} &\rightarrow (1-p, \cdots, 1-p)
\end{aligned}$$

### 6.3.4 Rule 58

To begin with, we recall the rule:

$$x_i^{t+1} = x_{i-1}^t \bar{x}_i^t + (1 - x_{i-1}^t) x_{i+1}^t$$

which can also be written:

$$\bar{x}_i^{t+1} = x_{i-1}^t x_i^t + (1 - x_{i-1}^t) \bar{x}_{i+1}^t$$

**Theorem 24.** *When  $n$  is even, rule 58 is spatially and temporally asymptotic with period 2. Furthermore, if  $x_i^{2t} \rightarrow p$  then  $x_i^{2t+1} \rightarrow 1 - p$ ,  $x_{i+1}^{2t} \rightarrow 1 - p$  and  $x_{i+1}^{2t+1} \rightarrow p$ . When  $n$  is odd, rule 58 converges to the homogeneous configuration  $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ .*

**Proof:** Let  $A$  be a fuzzy cellular automaton following rule 58 and let  $f : (0, 1)^3 \rightarrow (0, 1)$  be its local rule and  $F$  its global rule.

Assume that  $n$  is even. Let  $B$  be a fuzzy cellular automaton with global rule  $F \circ F$ . Consider a GWCA  $B'$  with global rule  $F'_2 \circ F'_1$  where  $F'_1$  and  $F'_2$  are given by two local rules  $f'_1, f'_2 : (0, 1)^3 \rightarrow (0, 1)$  defined as follows:

$$f'_1(x, y, z) = (1 - x)y + xz$$

$$f'_2(x, y, z) = xy + (1 - x)z$$

and where  $F'_1$  and  $F'_2$  alternate  $f'_1$  and  $f'_2$  as shown:

$$F'_1(x_0, x_1, \dots, x_{n-1}) = (f'_1(x_{n-1}, x_0, x_1), f'_2(x_0, x_1, x_2), \dots, f'_2(x_{n-2}, x_{n-1}, x_0))$$

$$F'_2(x_0, x_1, \dots, x_{n-1}) = (f'_2(x_{n-1}, x_0, x_1), f'_1(x_0, x_1, x_2), \dots, f'_1(x_{n-2}, x_{n-1}, x_0))$$

We want to show that  $F \circ F$  and  $F'_2 \circ F'_1$  are conjugate. Let  $h$  be the homeomorphism previously defined for rule 78.

$$h(x_0, x_1, x_2, \dots, x_{n-1}) = (x_0, 1 - x_1, x_2, \dots, 1 - x_{n-1})$$

Again we need to show that  $h \circ F \circ F = F'_2 \circ F'_1 \circ h$ . We will first show that  $F'_1 \circ h = h' \circ F$ , with  $h' = h \circ inv$ . Without loss of generality, assume that  $i$  is even.

$$\begin{aligned} F'_1 \circ h(\dots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \dots) \\ &= F'_1(\dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, x_{i+2}, \dots) \\ &(\dots, x_{i-1}x_i + \bar{x}_{i-1}\bar{x}_{i+1}, x_i\bar{x}_{i+1} + \bar{x}_i x_{i+2}, \dots) \end{aligned}$$

$$\begin{aligned}
h' \circ F(\cdots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \cdots) \\
&= h'(\cdots, x_{i-1}\bar{x}_i + \bar{x}_{i-1}x_{i+1}, x_i\bar{x}_{i+1} + \bar{x}_ix_{i+2}, \cdots) \\
&= (\cdots, x_{i-1}x_i + \bar{x}_{i-1}\bar{x}_{i+1}, x_i\bar{x}_{i+1} + \bar{x}_ix_{i+2}, \cdots)
\end{aligned}$$

Similarly,  $h \circ F = F'_2 \circ h'$ .

Then

$$\begin{aligned}
h \circ F \circ F &= F'_2 \circ h' \circ F \\
&= F'_2 \circ F'_1 \circ h
\end{aligned}$$

Thus  $h \circ F \circ F$  and  $F'_2 \circ F'_1 \circ h$  are conjugate. By Theorem 20,  $B'$  is converging to a homogeneous configuration of the form  $(p, p, \cdots, p)$ . Every other iteration of  $B$  must also be converging to a configuration of the form  $h^{-1}(p, p, \cdots, p) = (p, 1-p, p, \cdots, 1-p)$ .

Now for odd time steps we have:

$$\begin{aligned}
\lim_{t \rightarrow \infty} (x_0^{2t+1}, x_1^{2t+1}, \cdots, x_{n-1}^{2t+1}) &= \lim_{t \rightarrow \infty} F(x_0^{2t}, x_1^{2t}, \cdots, x_{n-1}^{2t}) \\
&= F(\lim_{t \rightarrow \infty} x_0^{2t}, \lim_{t \rightarrow \infty} x_1^{2t}, \cdots, \lim_{t \rightarrow \infty} x_{n-1}^{2t}) \\
&= F(p, 1-p, p, \cdots, 1-p) \\
&= (1-p, p, 1-p, \cdots, p)
\end{aligned}$$

Summarizing,

$$\begin{aligned}
\mathbf{X}^{2t} &\rightarrow (p, 1-p, \cdots, p, 1-p) \\
\mathbf{X}^{2t+1} &\rightarrow (1-p, p, \cdots, 1-p, p)
\end{aligned}$$

We now turn our attention to the case where  $n$  is odd. We consider the FCA  $B$  described in Theorem 22 and we will show that  $A$  with global rule  $F \circ F$  is conjugate to  $B$  with global rule  $F'_2 \circ F'_1$  and homeomorphism  $h$  as in the proof of Theorem 22 when  $n$  is odd.

If we let  $h' = h \circ inv$  as before, we again have that  $F'_1 \circ h = h' \circ F$  and  $h \circ F = F'_2 \circ h'$ . We give the details for the second equality.

$$h \circ F(x_0, x_1, \cdots, x_{n-1})$$

$$\begin{aligned}
&= h(x_{n-1}\bar{x}_0 + \bar{x}_{n-1}x_1, x_0\bar{x}_1 + \bar{x}_0x_2, \dots, x_{n-2}\bar{x}_{n-1} + \bar{x}_{n-2}x_0) \\
&= (x_{n-1}\bar{x}_0 + \bar{x}_{n-1}x_1, x_0x_1 + \bar{x}_0\bar{x}_2, \dots, x_{n-2}\bar{x}_{n-1} + \bar{x}_{n-2}x_0, x_{n-1}x_0 + \bar{x}_{n-1}\bar{x}_1, x_0\bar{x}_1 + \bar{x}_0x_2, \dots, \\
&\quad \dots x_{n-2}x_{n-1} + \bar{x}_{n-2}\bar{x}_0)
\end{aligned}$$

$$\begin{aligned}
F'_2 &\circ h'(x_0, x_1, \dots, x_{n-1}) \\
&= F'_2(\bar{x}_0, x_1, \dots, \bar{x}_{n-1}, x_0, \bar{x}_1, \dots, x_{n-1}) \\
&= (x_{n-1}\bar{x}_0 + \bar{x}_{n-1}x_1, x_0x_1 + \bar{x}_0\bar{x}_2, \dots, x_{n-2}\bar{x}_{n-1} + \bar{x}_{n-2}x_0, x_{n-1}x_0 + \bar{x}_{n-1}\bar{x}_1, x_0\bar{x}_1 + \bar{x}_0x_2, \dots, \\
&\quad \dots x_{n-2}x_{n-1} + \bar{x}_{n-2}\bar{x}_0)
\end{aligned}$$

As in the case of rule 78 with  $n$  odd,  $B$  must converge to  $\frac{1}{2}$  so even time steps of  $A$  must also converge to  $\frac{1}{2}$ . Odd time steps will converge to  $\frac{1}{2}$  since

$$F\left(\frac{1}{2}, \dots, \frac{1}{2}\right) = \left(\frac{1}{2}, \dots, \frac{1}{2}\right).$$

### 6.3.5 Rule 184

Rules 184, 46, and 29 use  $x_i^t$  itself as the weighting factor in the average of its two neighbours. When  $n$  is even, we effectively have two separate weighted averages, one of the even indices, the other of the odd. The weight factors in each case come from the other set of values. We will exploit this structure to determine topologically conjugate FCA where we can apply Theorem 20.

We recall rule 184:

$$x_i^{t+1} = (1 - x_i^t)x_{i-1}^t + x_i^t x_{i+1}^t.$$

**Theorem 25.** *When  $n$  is even, rule 184 is asymptotically periodic with period 2 both spatially and temporally. Furthermore, if  $\forall i$  as  $t \rightarrow \infty$ ,  $x_{2i}^{2t} \rightarrow p$  and  $x_{2i+1}^{2t} \rightarrow q$ , then  $x_{2i+1}^{2t+1} \rightarrow p$ , and  $x_{2i}^{2t+1} \rightarrow q$  with  $p+q = \frac{2}{n} \sum_{i=0}^{n-1} x_i^0$ . When  $n$  is odd, rule 184 converges spatially and temporally to a homogeneous configuration. Moreover, if  $\forall i$  as  $t \rightarrow \infty$ ,  $x_i^t \rightarrow p$ , then  $p = \frac{1}{n} \sum_{i=0}^{n-1} x_i^0$ .*

**Proof:** Let  $A$  be a fuzzy cellular automaton following rule 184 and let  $f : (0, 1)^3 \rightarrow (0, 1)$  by  $f(x, y, z) = \bar{y}x + yz$  be its local rule and  $F$  its global rule.

When  $n$  is even, let  $B$  be the fuzzy cellular automaton where the global rule  $G$  is the left shift of  $F$ :

$$G(x_0, \dots, x_{n-1}) = (f(x_0, x_1, x_2), f(x_1, x_2, x_3), \dots, f(x_{n-1}, x_0, x_1))$$

Now let  $S$  be the shift self-homeomorphism of  $B$ :

$$S(x_0, \dots, x_{n-1}) = (x_{n-1}, x_0, \dots, x_{n-2})$$

Notice that  $S \circ G = G \circ S$ , also that  $F = S \circ G$ . Now consider  $F^n: F^n = (S \circ G)^n = S^n G^n = G^n$ .

Thus every  $n$ -th iteration of  $A$  is equal to every  $n$ -th iteration of  $B$ . Furthermore, in-between steps can be determined by the appropriate number of shifts of  $B$ . We will now determine the asymptotic behaviour of  $B$ .

Let  $B'$  be the cross product of two  $GWCA$  of length  $m = n/2$  with global rule  $G'$  given by:

$$\begin{aligned} & G'((u_0, \dots, u_{m-1}) \times (v_0, \dots, v_{m-1})) \\ &= (f(u_0, v_0, u_1), f(u_1, v_1, u_2), \dots, f(u_{m-1}, v_{m-1}, u_0)) \\ &\times (f(v_0, u_1, v_1), f(v_1, u_2, v_2), \dots, f(v_{m-1}, u_0, v_0)) \end{aligned}$$

with  $\gamma$  as in the proof of Theorem 21.

The topological space  $B$  is conjugate to  $B'$  under the homeomorphism  $h$  defined as follows:

$$h(x_0, \dots, x_{n-1}) = (x_0, x_2, \dots, x_{n-2}) \times (x_1, x_3, \dots, x_{n-1})$$

To prove this we need to show that  $h \circ G = G' \circ h$ . The  $i$ -th position of  $G(x_0, \dots, x_{n-2})$  is  $(1 - x_{i+1})x_i + x_{i+1}x_{i+2}$  so

$$\begin{aligned} h \circ G(x_0, \dots, x_{n-1}) \\ &= h((1 - x_1)x_0 + x_1x_2, (1 - x_2)x_1 + x_2x_3, \dots, (1 - x_0)x_{n-1} + x_0x_1) \end{aligned}$$

$$\begin{aligned}
&= ((1 - x_1)x_0 + x_1x_2, \dots, (1 - x_{n-1})x_{n-2} + x_{n-1}x_0) \times ((1 - x_2)x_1 + x_2x_3, \dots, (1 - x_0)x_{n-1} + x_0x_1) \\
&= G'((x_0, x_2, \dots, x_{n-2}) \times (x_1, x_3, \dots, x_{n-1})) \\
&= G' \circ h(x_0, \dots, x_{n-1})
\end{aligned}$$

Both  $GWCA$  will converge to fixed points by Theorem 20. If  $(p, \dots, p) \times (q, \dots, q)$  is the point of convergence of  $B'$ , then  $h^{-1}((p, \dots, p) \times (q, \dots, q)) = (p, q, p, \dots, q)$  is the point of convergence of  $B$ .

Since, by definition,  $A$ 's global rule is given by the right shift of  $B$ 's global rule, the theorem follows. In summary we have,

$$\begin{aligned}
\mathbf{X}^{2t} &\rightarrow (p, q, \dots, p, q) \\
\mathbf{X}^{2t+1} &\rightarrow (q, p, \dots, q, p)
\end{aligned}$$

Now the sum of all values at the point of convergence is  $\frac{n}{2}(p + q)$ . From Theorem 10, this must be equal to the sum of values in the initial configuration. Hence,  $p + q = \frac{2}{n} \sum_{i=0}^{n-1} x_i^0$ .

With  $n$  odd, we consider the system  $B$  described as above. However, this time we will consider a homeomorphism  $h$  given by:

$$h(x_0, x_1, \dots, x_{n-1}) = (x_0, x_2, \dots, x_{n-1}, x_1, x_3, \dots, x_{n-2})$$

and a  $GWCA$   $B'$  of size  $2n$  with a new global rule  $G'$ . Let  $m = \frac{n+1}{2}$ , then

$$G'(x_0, x_1, \dots, x_{n-1}) = (f(x_0, x_m, x_1), f(x_1, x_{m+1}, x_2), \dots, f(x_i, x_{m+i}, f_{i+1}), \dots)$$

where  $f$  is local rule 184, and  $\gamma$  as in the proof of Theorem 21.

The system  $B'$  is conjugate to  $B$  using  $h$ . To prove this we show that  $h \circ G = G' \circ h$

$$\begin{aligned}
h \circ G(x_0, x_1, \dots, x_{n-1}) &= h((1 - x_1)x_0 + x_1x_2, (1 - x_2)x_1 + x_2x_3, \dots, (1 - x_0)x_{n-1} + x_0x_1) \\
&= (\bar{x}_1x_0 + x_1x_2, \bar{x}_3x_2 + x_3x_4, \dots, \bar{x}_0x_{n-1} + x_0x_1, \bar{x}_2x_1 + x_2x_3, \dots, \bar{x}_{n-1}x_{n-2} + x_{n-1}x_0)
\end{aligned}$$

$$\begin{aligned}
G' &\circ h(x_0, x_1, \dots, x_{n-1}) \\
&= G'(x_0, x_2, \dots, x_{n-1}, x_1, x_3, \dots, x_{n-2}) \\
&= (\bar{x}_1 x_0 + x_1 x_2, \bar{x}_3 x_2 + x_3 x_4, \dots, \bar{x}_0 x_{n-1} + x_0 x_1, \bar{x}_2 x_1 + x_2 x_3, \dots, \bar{x}_{n-1} x_{n-2} + x_{n-1} x_0)
\end{aligned}$$

Now  $G'$  must converge to a single value and since  $h$  merely reorders those values,  $G$  must also converge to a single value. Finally, since the shift is the identity of the point of convergence,  $A$  must converge to a single value both spatially and temporally. Moreover, if  $\forall i$  as  $t \rightarrow \infty$ ,  $x_i^t \rightarrow p$ , then  $p = \frac{1}{n} \sum_{i=0}^{n-1} x_i^0$  by Theorem 10.

### 6.3.6 Rule 29

We now examine rule 29:

$$x_i^{t+1} = x_i^t(1 - x_{i-1}^t) + (1 - x_i^t)(1 - x_{i+1}^t)$$

**Theorem 26.** *When  $n$  is even, rule 29 is asymptotically periodic with period 2 both spatially and temporally. Furthermore, if  $\forall i$  as  $t \rightarrow \infty$ ,  $x_{2i}^{2t} \rightarrow p$ , and  $x_{2i+1}^{2t} \rightarrow q$  then  $x_{2i}^{2t+1} \rightarrow 1 - q$ , and  $x_{2i+1}^{2t+1} \rightarrow 1 - p$ . When  $n$  is odd, rule 29 converges spatially to a homogeneous configuration and is asymptotically periodic with period 2 temporally.*

**Proof:** We proceed as for rule 184. Let  $A$  be a fuzzy cellular automaton following rule 29 and let  $f : (0, 1)^3 \rightarrow (0, 1)$  be its local rule and  $F$  its global rule.

Let  $B$  be a fuzzy cellular automaton with global rule  $G$  given by the shift of  $F$ :

$$G(x_0, \dots, x_{n-1}) = (f(x_0, x_1, x_2), f(x_1, x_2, x_3), \dots, f(x_{n-1}, x_0, x_1))$$

When  $n$  is even, we let  $B'$  be the cross product of two  $GWCA$  of length  $m = n/2$  with global rules  $G'_1$  and  $G'_2$  given by local rules  $f_1$  and  $f_2$  as follows:

$$f_1(x, y, z) = yx + (1 - y)z$$

$$f_2(x, y, z) = (1 - y)x + yz$$

$$\begin{aligned}
& G'_i((u_0, \dots, u_{m-1}) \times (u_0, \dots, u_{m-1})) \\
&= (f_i(u_0, v_0, u_1), f_i(u_1, v_1, u_2), \dots, f_i(u_{m-1}, v_{m-1}, u_0)) \\
&\times (f_i(v_0, u_1, v_1), f_i(v_1, u_2, v_2), \dots, f_i(v_{m-1}, u_0, v_0))
\end{aligned}$$

for  $i = 1, 2$ .

The topological space  $B$  using rule  $G \circ G$  is conjugate to  $B'$  using rule  $G'_2 \circ G'_1$  under the homeomorphism  $h$  defined as follows:

$$h(x_0, \dots, x_{n-1}) = (x_0, x_2, \dots, x_{n-2}) \times (x_1, x_3, \dots, x_{n-1})$$

To show this, we consider a second homeomorphism  $h' = h \circ inv$  which maps  $B$  to  $B'$  by

$$h'(x_0, \dots, x_{n-1}) = (\bar{x}_0, \bar{x}_2, \dots, \bar{x}_{n-2}) \times (\bar{x}_1, \bar{x}_3, \dots, \bar{x}_{n-1})$$

We first show that  $G'_2 \circ h' = h \circ G$

$$\begin{aligned}
G'_2 \circ h'(x_0, x_1, x_2, x_3, \dots) &= G'_2((\bar{x}_0, \bar{x}_2, \dots, \bar{x}_{n-2}) \times (\bar{x}_1, \bar{x}_3, \dots, \bar{x}_{n-1})) \\
&= (f_2(\bar{x}_0, \bar{x}_1, \bar{x}_2), f_2(\bar{x}_2, \bar{x}_3, \bar{x}_4), \dots) \\
&\times (f_2(\bar{x}_1, \bar{x}_2, \bar{x}_3), f_2(\bar{x}_3, \bar{x}_4, \bar{x}_5), \dots) \\
&= ((x_1 \bar{x}_0 + \bar{x}_1 \bar{x}_2), (x_3 \bar{x}_2 + \bar{x}_3 \bar{x}_4), \dots) \\
&\times ((x_2 \bar{x}_1 + \bar{x}_2 \bar{x}_3), (x_4 \bar{x}_3 + \bar{x}_4 \bar{x}_5), \dots) \\
h \circ G(x_0, x_1, x_2, x_3, \dots) &= h((x_1 \bar{x}_0 + \bar{x}_1 \bar{x}_2, x_2 \bar{x}_1 + \bar{x}_2 \bar{x}_3, x_3 \bar{x}_2 + \bar{x}_3 \bar{x}_4, \dots)) \\
&= ((x_1 \bar{x}_0 + \bar{x}_1 \bar{x}_2), (x_3 \bar{x}_2 + \bar{x}_3 \bar{x}_4), \dots) \\
&\times ((x_2 \bar{x}_1 + \bar{x}_2 \bar{x}_3), (x_4 \bar{x}_3 + \bar{x}_4 \bar{x}_5), \dots)
\end{aligned}$$

Furthermore,  $G'_1 \circ h = h' \circ G$ :

$$G'_1 \circ h(x_0, x_1, x_2, x_3, \dots)$$

$$\begin{aligned}
&= G'_1((x_0, x_2, \dots, x_{n-2}) \times (x_1, x_3, \dots, x_{n-1})) \\
&= (x_1 x_0 + \bar{x}_1 x_2, x_3 x_2 + \bar{x}_3 x_4, \dots) \times (x_2 x_1 + \bar{x}_2 x_3, x_4 x_3 + \bar{x}_4 x_5, \dots) \\
&= h'((x_1 \bar{x}_0 + \bar{x}_1 \bar{x}_2, x_2 \bar{x}_1 + \bar{x}_2 \bar{x}_3, x_3 \bar{x}_2 + \bar{x}_3 \bar{x}_4, x_4 \bar{x}_3 + \bar{x}_4 \bar{x}_5, \dots)) \\
&= h' \circ G(x_0, x_1, x_2, x_3, \dots)
\end{aligned}$$

Finally we have,

$$\begin{aligned}
G'_2 \circ (G'_1 \circ h) &= (G'_2 \circ h') \circ G \\
&= h \circ G \circ G
\end{aligned}$$

as required.

By Theorem 20, both  $GWCA$  in  $B'$  converge to homogeneous configurations. If  $(p, \dots, p) \times (q, \dots, q)$  is the point of convergence of  $B'$ , then  $h^{-1}((p, \dots, p) \times (q, \dots, q)) = (p, q, p, \dots, q)$  is the point of convergence of  $B$ .

Since, by definition,  $A$ 's global rule is given by the right shift of  $B$ 's global rule, the theorem follows and we have,

$$\mathbf{X}^{2t} \rightarrow (p, q, \dots, p, q)$$

and

$$\begin{aligned}
\mathbf{X}^{2t+1} &\rightarrow F(p, q, \dots, p, q) \\
&\rightarrow (1 - q, 1 - p, \dots, 1 - q, 1 - p)
\end{aligned}$$

When  $n$  is odd, let  $B'$  and  $h$  be as in the proof of Theorem 25 the case of rule 184 when  $n$  is odd. But let  $B'$  transform under  $G'_2 \circ G'_1$  where

$$G'_j(x_0, x_1, \dots, x_{2n-1}) = (f_j(x_0, x_m, x_1), f_j(x_1, x_{m+1}, x_2), \dots, f_j(x_i, x_{i+m}, x_{i+1}), \dots)$$

for  $j = 1, 2$  and  $f_j$  defined as for  $n$  even above.

Let  $h'$  be the negation of  $h$ , so that  $h' = h \circ inv$

$$h'(x_0, x_1, \dots, x_{n-1}) = (\bar{x}_0, \bar{x}_2, \dots, \bar{x}_{n-1}, \bar{x}_1, \dots, \bar{x}_{n-2})$$

then  $h \circ G \circ G = G'_2 \circ G'_1 \circ h$ . Again we prove this by showing that  $h \circ G = G'_2 \circ h'$  and  $h' \circ G = G'_1 \circ h$

$$\begin{aligned} h \circ G(x_0, x_1, \dots, x_{n-1}) &= h(x_1 \bar{x}_0 + \bar{x}_1 \bar{x}_2, x_2 \bar{x}_1 + \bar{x}_2 \bar{x}_3, \dots, x_0 \bar{x}_{n-1} + \bar{x}_0 \bar{x}_1) \\ &= (x_1 \bar{x}_0 + \bar{x}_1 \bar{x}_2, x_3 \bar{x}_2 + \bar{x}_3 \bar{x}_4, \dots, x_0 \bar{x}_{n-1} + \bar{x}_0 \bar{x}_1, x_2 \bar{x}_1 + \bar{x}_2 \bar{x}_3, \dots, x_{n-1} \bar{x}_{n-2} + \bar{x}_{n-1} \bar{x}_0) \end{aligned}$$

$$\begin{aligned} G'_2 \circ h'(x_0, x_1, \dots, x_{n-1}) &= G'_2(\bar{x}_0, \bar{x}_2, \dots, \bar{x}_{n-1}, \bar{x}_1, \dots, \bar{x}_{n-2}) \\ &= (x_1 \bar{x}_0 + \bar{x}_1 \bar{x}_2, x_3 \bar{x}_2 + \bar{x}_3 \bar{x}_4, \dots, x_0 \bar{x}_{n-1} + \bar{x}_0 \bar{x}_1, x_2 \bar{x}_1 + \bar{x}_2 \bar{x}_3, \dots, x_{n-1} \bar{x}_{n-2} + \bar{x}_{n-1} \bar{x}_0) \\ &= h \circ G(x_0, x_1, \dots, x_{n-1}) \end{aligned}$$

$$\begin{aligned} h' \circ G(x_0, x_1, \dots, x_{n-1}) &= h(x_1 \bar{x}_0 + \bar{x}_1 \bar{x}_2, x_2 \bar{x}_1 + \bar{x}_2 \bar{x}_3, \dots, x_0 \bar{x}_{n-1} + \bar{x}_0 \bar{x}_1) \\ &= (x_1 x_0 + \bar{x}_1 x_2, x_3 x_2 + \bar{x}_3 x_4, \dots, x_0 x_{n-1} + \bar{x}_0 x_1, x_2 x_1 + \bar{x}_2 x_3, \dots, x_{n-1} x_{n-2} + \bar{x}_{n-1} x_0) \\ &= G'_1(x_0, x_2, \dots, x_{n-1}, x_1, \dots, x_{n-2}) \\ &= G'_1 \circ h(x_0, x_1, \dots, x_{n-1}) \end{aligned}$$

Since  $B'$  satisfies Theorem 20, it converges to a homogeneous configuration  $(p, \dots, p)$  and even time steps of  $B$  must also converge to a  $(p, \dots, p)$ . At odd time interval,

$$\begin{aligned} \lim_{t \rightarrow \infty} x_i^{2t+1} &= \lim_{t \rightarrow \infty} f(x_{i-1}^{2t}, x_i^{2t}, x_{i+1}^{2t}) \\ &= f(\lim_{t \rightarrow \infty} x_{i-1}^{2t}, \lim_{t \rightarrow \infty} x_i^{2t}, \lim_{t \rightarrow \infty} x_{i+1}^{2t}) \\ &= f(p, p, p) \\ &= p(1-p) + (1-p)(1-p) \\ &= 1-p \end{aligned}$$

In other words,

$$\begin{aligned}\mathbf{X}^{2t} &\rightarrow (p, \dots, p) \\ \mathbf{X}^{2t+1} &\rightarrow (1-p, \dots, 1-p).\end{aligned}$$

### 6.3.7 Rule 46

Rule 46 yields the most interesting results of all FCA in this class and it actually appears to be unique among all elementary FCA. In fact, from the experimental observations of [32] it is the only rule with periodic behavior of length 4. It once again uses the central value as the weight factor, but the resulting rule individuates sub-automata where the rule uses one value directly while the other is inverted resulting in temporally periodic behaviour in each of the GWCA in the cross product.

We restate the rule:

$$x_i^{t+1} = (1 - x_i^t)x_{i+1}^t + x_i^t\bar{x}_{i-1}^t$$

We have two cases: when  $n$  is a multiple of 4, and when it is not.

**Theorem 27.** *When  $n$  modulo 4 is equal to 0, rule 46 is asymptotically periodic with period 4 both spatially and temporally. Furthermore, if  $\forall i$  as  $t \rightarrow \infty$ ,  $x_{4i}^{At} \rightarrow p$ , and  $x_{4i+1}^{At} \rightarrow q$  then  $x_{4i+2}^{At} \rightarrow 1-p$ , and  $x_{4i+3}^{At} \rightarrow 1-q$ . Also,  $x_{4i}^{At+1} \rightarrow q$ ,  $x_{4i}^{At+2} \rightarrow 1-p$ , and  $x_{4i}^{At+3} \rightarrow 1-q$ . When  $n$  modulo 4 is not equal to 0, rule 46 converges to the homogeneous configuration  $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ .*

**Proof:** Let  $A$  be a fuzzy cellular automaton following rule 46 and let  $f : (0, 1)^3 \rightarrow (0, 1)$  be its local rule and  $F$  its global rule.

Assume to begin with, that  $n$  modulo 4 is 0. Let  $B$  be a fuzzy CA with global rule  $G$  given by the left shift of  $F$ :

$$G(x_0, \dots, x_{n-1}) = (f(x_0, x_1, x_2), f(x_1, x_2, x_3), \dots, f(x_{n-1}, x_0, x_1))$$

Let  $B'$  be the usual cross product with global rule  $G'$  given by local rules  $f_1$  and  $f_2$  as follows:

$$f_1(x, y, z) = yx + (1-y)z$$

$$f_2(x, y, z) = (1 - y)x + yz$$

$$\begin{aligned} & G'((u_0, \dots, u_{m-1}) \times (u_0, \dots, u_{m-1})) \\ &= (f_1(u_0, v_0, u_1), f_2(u_1, v_1, u_2), f_1(u_2, v_2, u_3), \dots, f_2(u_{m-1}, v_{m-1}, u_0)) \\ &\times (f_2(v_0, u_1, v_1), f_1(v_1, u_2, v_2), f_2(v_2, u_3, v_3) \dots, f_1(v_{m-1}, u_0, v_0)). \end{aligned}$$

The topological space  $B$  using the rule  $G \circ G$  is conjugate to  $B'$  using rule  $G' \circ G'$  under the homeomorphism  $h$ :

$$h(x_0, \dots, x_{n-1}) = (x_0, \bar{x}_2, x_4, \dots, \bar{x}_{n-2}) \times (x_1, \bar{x}_3, x_5, \dots, \bar{x}_{n-1})$$

To show this we consider a second homeomorphism  $h' = h \circ inv$  which maps  $B$  to  $B'$  by

$$h'(x_0, \dots, x_{n-1}) = (\bar{x}_0, x_2, \bar{x}_4, \dots, x_{n-2}) \times (\bar{x}_1, x_3, \bar{x}_5, \dots, x_{n-1})$$

We first show that  $G' \circ h = h' \circ G$

$$\begin{aligned} & G' \circ h(x_0, x_1, x_2, x_3, \dots) \\ &= G'(x_0, \bar{x}_2, x_4, \dots, \bar{x}_{n-2}) \times (x_1, \bar{x}_3, x_5, \dots, \bar{x}_{n-1}) \\ &= (f_1(x_0, x_1, \bar{x}_2), f_2(\bar{x}_2, \bar{x}_3, x_4), \dots) \\ &\times (f_2(x_1, \bar{x}_2, \bar{x}_3), f_1(\bar{x}_3, x_4, x_5), \dots) \\ &= ((x_1 x_0 + \bar{x}_1 \bar{x}_2), (x_3 \bar{x}_2 + \bar{x}_3 x_4), \dots) \\ &\times ((x_2 x_1 + \bar{x}_2 \bar{x}_3), (x_4 \bar{x}_3 + \bar{x}_4 x_5), \dots) \\ & h' \circ G(x_0, x_1, x_2, x_3, \dots) \\ &= h'((x_1 \bar{x}_0 + \bar{x}_1 x_2, x_2 \bar{x}_1 + \bar{x}_2 x_3, x_3 \bar{x}_2 + \bar{x}_3 x_4, x_4 \bar{x}_3 + \bar{x}_4 x_5, \dots)) \\ &= ((x_1 x_0 + \bar{x}_1 \bar{x}_2), (x_3 \bar{x}_2 + \bar{x}_3 x_4), \dots) \\ &\times ((x_2 x_1 + \bar{x}_2 \bar{x}_3), (x_4 \bar{x}_3 + \bar{x}_4 x_5), \dots) \end{aligned}$$

Similarly,  $h \circ G = G' \circ h'$ .

Finally we have,

$$\begin{aligned} G' \circ (G' \circ h) &= (G' \circ h') \circ G \\ &= h \circ G \circ G \end{aligned}$$

as required.

Since both systems in the cross product of  $B'$  are GWCA with  $\gamma$  as in the proof of Theorem 21, they will both converge to fixed points by Theorem 20. If  $(p, \dots, p) \times (q, \dots, q)$  is the point of convergence of  $B'$ , then  $h^{-1}((p, \dots, p) \times (q, \dots, q)) = (p, q, 1-p, 1-q, p, q, 1-p, 1-q, \dots, p, q, 1-p, 1-q)$  is the point of convergence of  $B$ .

Since  $A$  is the right shift of  $B$ .

$$\begin{aligned} \mathbf{X}^{4t} &\rightarrow (p, q, 1-p, 1-q, p, q, \dots, 1-p, 1-q) \\ \mathbf{X}^{4t+1} &\rightarrow (q, 1-p, 1-q, p, \dots, q, 1-p, 1-q, p) \\ \mathbf{X}^{4t+2} &\rightarrow (1-p, 1-q, p, q, \dots, 1-p, 1-q, p, q) \\ \mathbf{X}^{4t+3} &\rightarrow (1-q, p, q, 1-p, \dots, 1-q, p, q, 1-p) \end{aligned}$$

When  $n$  is even but not divisible by 4. Let  $B'$  be a cross product of 2 GWCA's of length  $n$ , and let  $h$  map  $B$  to  $B'$  as follows:

$$\begin{aligned} h(x_0, x_1, \dots, x_{n-1}) &= (x_0, \bar{x}_2, x_4, \dots, x_{n-2}, \bar{x}_0, x_2, \dots, \bar{x}_{n-2}) \\ &\times (x_1, \bar{x}_3, \dots, x_{n-1}, \bar{x}_1, x_3, \dots, \bar{x}_{n-1}) \end{aligned}$$

Let  $h' = h \circ inv$ . Let  $G'$  be as in the proof of the previous theorem.

As before, we can show that  $h \circ G \circ G = G' \circ G' \circ h$  by showing the intermediary steps using  $h'$ . This time, however, we see that if  $x_i \rightarrow p$  then  $1 - x_i \rightarrow p$  also and so  $p = \frac{1}{2}$ .

When  $n$  is odd, we consider one GWCA of length  $2n$ , which alternates original values and negations. So for  $n$  modulo 4 equal to 1, we will have:

$$(x_0, \bar{x}_2, x_4, \dots, x_{n-1}, \bar{x}_1, x_3, \dots, x_{n-2}, \bar{x}_0, x_2, \dots, \bar{x}_{n-1}, x_1, \bar{x}_3, \dots, \bar{x}_{n-2}),$$

and for  $n$  modulo 4 equal to 3 we will have:

$$(x_0, \bar{x}_2, x_4, \dots, \bar{x}_{n-1}, x_1, \bar{x}_3, \dots, x_{n-2}, \bar{x}_0, x_2, \dots, x_{n-1}, \bar{x}_1, x_3, \dots, \bar{x}_{n-2}).$$

The rule  $G'$  uses appropriate weights to average neighbours in this array. Clearly, due to its structure,  $B'$  must converge to  $\frac{1}{2}$ .

## 6.4 Creating Rules with a Given Periodicity

As mentioned in the introduction to this section, the weighted average rules are drawn from almost every classification of elementary rules. From our knowledge of how they converge, we can use them as templates to construct rules which converge to a given periodicity for larger neighbourhoods. We can show that for circular CA with neighbourhood size  $m$ , the following periods are possible  $\{i, 2i : 1 \leq i < m\}$ .

**Theorem 28.** *Given an arbitrary integer  $k$ , a circular CA of length  $n \equiv 0 \pmod{k}$ , neighbourhood size  $m > k$  and local rule  $f(x_0, \dots, x_{m-1}) = x_j x_0 + (1 - x_j) x_k$  for some  $j \neq 0$ ,  $k$  will converge with spatial period  $k$ . Furthermore, if  $k \neq m/2$ , this CA will also have temporal period  $k$ .*

**Proof:** As in the proof of convergence for rule 184, the Ca is effectively divided into  $k$  arrays, each of length  $n/k$ , each one representing a generalized weighted average CA. Each of these  $k$  CAs will converge to some value which is independent of the values of the other GWCA. If  $k$  happens to be the middle value in the neighbourhood then no shifting occurs and we have spatial period  $k$ , temporal period 1. However, if  $k$  is not the middle value we will have shifting and hence temporal period  $k$  as well. Note that with neighbourhood size 3, the former cannot occur since  $k = 1$  would require that it be averaged with an immediate neighbour resulting in period 1.

Similarly, we can create CA with period  $2k$  using local rule  $f(x_0, \dots, x_{m-1}) = x_j x_0 + (1 - x_j)(1 - x_k)$  and array length  $m \equiv 0 \pmod{2j}$ .

## 6.5 Summary

In this chapter, we mathematically proved the periodic behaviours previously observed for weighted average rules, rules 26, 27, 46, 58, 78, 172, 184, a class exhibiting almost all the behaviours observed for elementary FCA. In order to do so, we generalized the concept of FCA to GWCA where each cell can be updated by a different weighted average rule and then we showed how each of these rules is topologically conjugate to a GWCA or a cross product of GWCA with known behaviour. Finally, we show how this style of rule can be used to create FCA which will converge to a given period and we begin the study of which periods can be expected from FCA with larger neighbourhoods.

The results in this chapter appeared in [4].

---

---

## CHAPTER 7

---

---

### Self-Averaging Rules

In the chapter we show that a class of rules which we call the *self-averaging rules* all converge to one half. As in the previous chapter, the emphasis here is on formal proof to substantiate what has been observed in simulations. Furthermore, we show that as these rules converge, their behaviour around one half fluctuates according to an elementary Boolean rule, either the corresponding rule or a simpler one. The results of this asymptotic behaviour led us to reconsider the Boolean rules and their asymptotic behaviour with interesting results. We then looked at error propagation in these rules, necessary and sufficient conditions for fuzzy data to contaminate an otherwise Boolean CA.

#### 7.1 Definitions and Basic Properties

A rule is said to be *self-averaging* if it can be written in the form of a weighted average of one of its variables,  $x_i$  and its negation,  $(1-x_i)$ :  $f(x_0, \dots, x_{n-1}) = f_i(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1})x_i + (1 - f_i(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1}))(1 - x_i)$ . With neighbourhood of size 3, this takes the following form  $f(x, y, z) = \gamma x + (1 - \gamma)(1 - x)$  (analogously for variables  $y$  and  $z$ ). For example, rule 30 can be written as:  $(1 - y)(1 - z)x + ((1 - y)z + y(1 - z) + yz)(1 - x)$  and it is easy to see that, in this case,  $\gamma = (1 - y)(1 - z)$  so that we can write rule 30 as  $\gamma x + (1 - \gamma)(1 - x)$ . Since the new value is always between  $x$  and  $1 - x$ , it will always be closer to one half than

either, generating a rule evolution characteristic of the rules in this class. Note that those values which have not yet converged to one half propagate along diagonals, in keeping with this being a self-average of  $x$ .

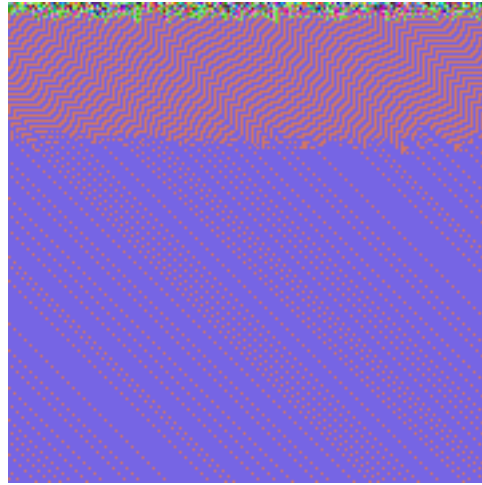


Figure 7.1: Evolution of fuzzy Rule 30

Boolean rules having this property are said to be *permutive* and are often described by the following definition which holds for their fuzzifications as well. A Boolean function is *permutive* in  $x_i$  if  $f(x_0, \dots, x_i, \dots, x_{n-1}) = 1 - f(x_0, \dots, 1 - x_i, \dots, x_{n-1})$ . In our example, rule 30 is permutive in  $x$  since

$$\begin{aligned}
 f_{30}(1 - x, y, z) &= \gamma(1 - x) + (1 - \gamma)x \\
 &= 1 - \gamma x - (1 - x - \gamma + \gamma x) \\
 &= 1 - \gamma x - (1 - \gamma)(1 - x) \\
 &= 1 - f_{30}(x, y, z).
 \end{aligned}$$

Rules such as rule 30 which are permutive in the left most variable in their neighbourhoods (no matter what the size of the neighbourhood) have de Bruijn diagrams where each node has incoming edges with different labels. These are easily proven to be injective. Similarly, if the rule is permutive in the last cell in the neighbourhood, nodes in the de Bruijn diagram

have distinct outgoing edges ensuring no ambiguity and hence injective. In the circular case they are therefore surjective.

Table 7.1 contains all the elementary self-averaging rules where  $\bar{x}$  indicates the value  $(1 - x)$ .

Rule	Equation	Boolean Equation for Behavior around $\frac{1}{2}$
$f_{60}$	$(\bar{x})y + (x)\bar{y}$	$g_{60}$
$f_{90}$	$(\bar{x})z + (x)\bar{z}$	$g_{90}$
$f_{105}$	$(\bar{x}y + x\bar{y})z + (\bar{x}\bar{y} + xy)\bar{z}$	$g_{105}$
$f_{150}$	$(\bar{x}\bar{z} + xz)y + (\bar{x}z + x\bar{z})\bar{y}$	$g_{150}$
$f_{30}$	$(\bar{y}\bar{z})x + (\bar{y}z + y\bar{z} + yz)\bar{x}$	$g_{15} = \bar{x}$
$f_{45}$	$(\bar{y}z)x + (\bar{y}\bar{z} + y\bar{z} + yz)\bar{x}$	$g_{15} = \bar{x}$
$f_{106}$	$(\bar{x}\bar{y} + \bar{x}y + x\bar{y})z + (xy)\bar{z}$	$g_{170} = z$
$f_{154}$	$(\bar{x}y + \bar{x}\bar{y} + xy)z + (x\bar{y})\bar{z}$	$g_{170} = z$
$f_{108}$	$(\bar{x}\bar{z} + \bar{x}z + x\bar{z})y + (xz)\bar{y}$	$g_{204} = y$
$f_{156}$	$(\bar{x}\bar{z} + \bar{x}z + xz)y + (x\bar{z})\bar{y}$	$g_{204} = y$
$f_{54}$	$(\bar{x}\bar{z})y + (\bar{x}z + x\bar{z} + xz)\bar{y}$	$g_{51} = \bar{y}$
$f_{57}$	$(\bar{x}z)y + (\bar{x}\bar{z} + x\bar{z} + xz)\bar{y}$	$g_{51} = \bar{y}$

Table 7.1: Self-averaging elementary fuzzy CA rules (the rules equivalent under conjugation, reflection, or both are not indicated).

The definitions of convergence and periodicity are the same as in the previous chapter therefore we do not repeat them here. However, we will use two additional terms.

**Definition 8.** A rule will be said to converge vertically to  $p$  if, for all  $i$ , the sequence  $x_i^0, x_i^1, x_i^2, \dots$  converges to  $p$ .

Similarly,

**Definition 9.** A rule will be said to converge diagonally to  $p$  if, for all  $i$ , the sequence  $x_i^0, x_{i+r}^1, x_{i+2r}^2, \dots$  converges to  $p$ .

Normally,  $r$  will be equal to 1 or  $-1$ .

## 7.2 Convergence of Self-averaging Rules

In this section we prove that the self-averaging rules converge to one half from any strictly fuzzy initial configuration, and are the only elementary rules to do so.

We begin with two lemmas about self-averaging rules; the first establishes that fuzzy data cannot become binary and the second that fuzzy weights will cause values to converge to one half point-wise. We will then proceed to examine each of the self-averaging rules showing that they must converge to one half from strictly fuzzy configurations and studying how they behave as they approach this point of convergence.

**Lemma 13.** Given  $x \in (0, 1)$  then  $\gamma x + (1 - \gamma)(1 - x) \in (0, 1)$  for all  $\gamma \in [0, 1]$ .

**Lemma 14.** Given  $\gamma \in (0, 1)$  and  $x \in [0, 1]$  then  $|(\gamma x + (1 - \gamma)(1 - x)) - \frac{1}{2}| = |1 - 2\gamma||x - \frac{1}{2}| \leq |x - \frac{1}{2}|$  with equality if and only if  $x = \frac{1}{2}$ .

**Proof:** Without loss of generality, assume  $x < 1 - x$  and let  $x = \frac{1}{2} - \epsilon$  for some  $\epsilon \in [0, \frac{1}{2}]$ . Then  $1 - x = \frac{1}{2} + \epsilon$ . So  $|x - \frac{1}{2}| = \epsilon$  and we need to show that  $|(\gamma x + (1 - \gamma)(1 - x)) - \frac{1}{2}| < \epsilon$ .

$$\begin{aligned}
 |(\gamma x + (1 - \gamma)(1 - x)) - \frac{1}{2}| &= |\gamma(\frac{1}{2} - \epsilon) + (1 - \gamma)(\frac{1}{2} + \epsilon) - \frac{1}{2}| \\
 &= |\frac{1}{2} + \epsilon - 2\gamma\epsilon - \frac{1}{2}| \\
 &= |\epsilon - 2\gamma\epsilon| \\
 &= |1 - 2\gamma|\epsilon \\
 &\leq \epsilon
 \end{aligned}$$

with equality if and only if  $\epsilon = \frac{1}{2}$

We can now easily show that, *when none of the initial values is Boolean*, all self-averaging rules have diagonals which converge to  $\frac{1}{2}$ .

**Theorem 29.** *Given initial configurations in  $(0, 1)$ , all non-trivial self-averaging rules converge to  $\frac{1}{2}$  along diagonals.*

**Proof:** Considering any of the non-trivial self-averaging rules, we can see that for the weights to be precisely equal to 0 or 1, we must have values equal to 0 and 1 in the calculation. With initial configuration in  $(0, 1)$ , this will never happen by Lemma 13. Thus by Lemma 14, all values must approach one half. Since the weighting factors  $\gamma$  is a sum of products of values which are approaching one half,  $|1 - 2\gamma|$  is bounded away from zero and one, hence from the proof of Lemma 2 we can say that after  $n$  iterations  $|x_i^{t+n} - \frac{1}{2}| < \gamma_0 |x_i^t - \frac{1}{2}|$  (in the centric case), thus the sequences  $x_i^t, x_i^{t+1}, \dots, x_i^{t+n}, \dots$  converges to one half. Similarly one can see the sequences  $x_i^t, x_{i-1}^{t+1}, \dots, x_{i-n}^{t+n}, \dots$  or  $x_i^t, x_{i+1}^{t+1}, \dots, x_{i+n}^{t+n}, \dots$  in the non-centric cases must also converge to one half.

In the case of circular cellular automata, this is enough to force convergence everywhere.

**Corollary 6.** *All non-trivial circular self-averaging rules converge to one half.*

**Proof:** Given  $\epsilon > 0$ , we can choose  $T_i$  such that the diagonal beginning at  $x_i^0$  is within  $\epsilon$  of one half after  $T_i$  iteration. Then since there is a finite number of diagonals, we can let  $T = \min\{T_i\}_{i=0..n}$  so that after  $T$  iterations every diagonal, and hence every cell, is within  $\epsilon$  of one half.

By the same token, it can be shown that any finite subsequence of an infinite non-trivial centric self-averaging rule can be shown to converge to one half.

We can also show the converse of Theorem 1, that rules which always converge point-wise to  $\frac{1}{2}$  along a diagonal when none of the initial values is Boolean, must be self-averaging.

**Theorem 30.** *Any fuzzy rule which converges point-wise to  $\frac{1}{2}$  along a diagonal from any initial configuration in  $(0, 1)$  must be a self-averaging rule.*

**Proof:** Without loss of generality assume that  $|f(x_0, \dots, x_{n-1}) - \frac{1}{2}| \leq |x_i - \frac{1}{2}|$ . Then we can write  $f$  as  $\alpha x_i + \beta \bar{x}_i$  where  $\alpha$  and  $\beta$  are functions in  $x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1}$ .

Consider the case where  $x_i = \frac{1}{2}$ . Then

$$\begin{aligned} |f(x_0, \dots, x_{n-1}) - \frac{1}{2}| &\leq |x_i - \frac{1}{2}| = 0 \\ \alpha x_i + \beta \bar{x}_i - \frac{1}{2} &= 0 \\ \frac{1}{2}\alpha + \frac{1}{2}\beta - \frac{1}{2} &= 0 \\ \frac{1}{2}(\alpha + \beta) - \frac{1}{2} &= 0 \\ \alpha + \beta &= 1 \end{aligned}$$

Clearly, if either  $\alpha$  or  $\beta$  are identically 0 or 1, then we do not have convergence when  $x_i$  is not equal to  $\frac{1}{2}$  so  $f$  must have the form of a self-averaging rule.

## 7.3 Behaviour around the point of convergence

Motivated by the observations made of the convergence behaviour of rule 90 and all the other additive rules that mimic, in some sense, the corresponding Boolean rule itself, we now study the convergence behaviours of all the elementary self-averaging rules.

### 7.3.1 Self-Oscillation

We now study the self-averaging fuzzy rules whose behaviour around  $\frac{1}{2}$  obeys the corresponding Boolean rule. This is the type of dynamics already observed for rule 90 in [33] and for all the additive rules in Chapter 5. We will show that, for these rules, we can determine whether  $f(x, y, z)$  is greater than or less than one half simply by knowing the relationships of each  $x$ ,  $y$ , and  $z$  relative to one half and without knowing their actual values. This is true for all  $x, y, z$  in  $[0, 1]$ , including binary values. Hence tables of the relationships to one half will resemble the corresponding binary rule tables with  $<$  replacing 0 and  $>$  replacing 1. Before starting the analysis, we introduce a technical lemma.

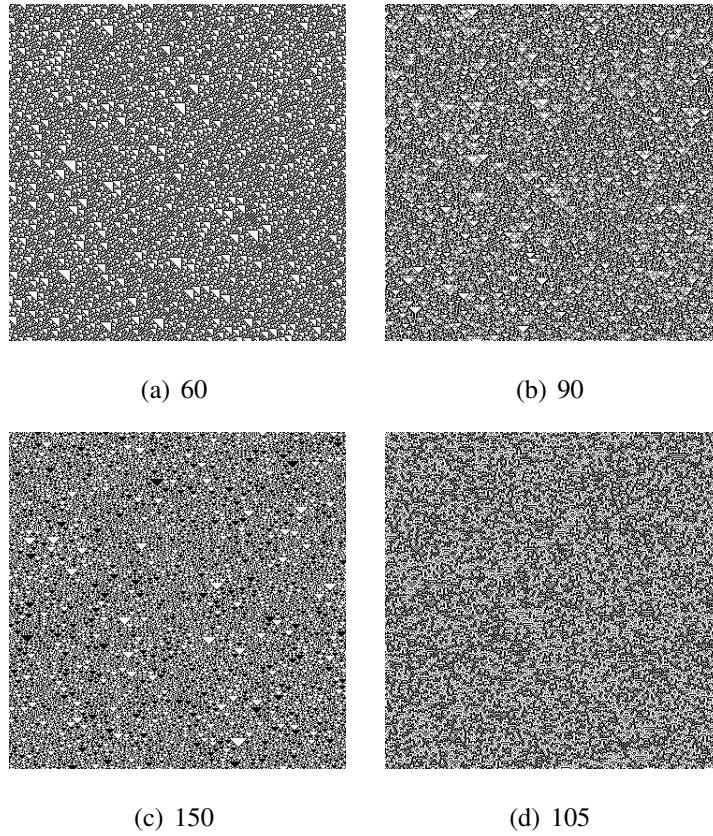


Figure 7.2: Additive rules: 60, 90, 150 and 105.

**Lemma 15.**  $\alpha\beta + \bar{\alpha}\bar{\beta}$  is greater than  $\frac{1}{2}$  if and only if both  $\beta$  and  $\alpha$  are greater than  $\frac{1}{2}$  or both are smaller.

**Proof:** Assume  $\alpha\beta + (1 - \alpha)(1 - \beta) > \frac{1}{2}$ . Rearranging we obtain:  $(2\alpha - 1)\beta > \frac{1}{2}(2\alpha - 1)$ . If  $\alpha > \frac{1}{2}$  then  $(2\alpha - 1) > 0$ , and  $\beta > \frac{1}{2}$ . Otherwise, if  $\alpha < \frac{1}{2}$  then  $\beta < \frac{1}{2}$ .

As a consequence of this lemma, if we have a rule of the form  $f(x, y, z) = h(x, y)z + (1 - h(x, y))\bar{z}$  (for example) and if we can say that  $h(x, y) > \frac{1}{2}$  under certain restrictions on  $x$  and  $y$ , then under those conditions,  $f(x, y, z) > \frac{1}{2}$  if and only if  $z > \frac{1}{2}$ . So we can begin to construct a truth table describing the behaviour of the rule around  $\frac{1}{2}$ . We will, in fact, show that the conditions required on  $x$  and  $y$  in the case of self-averaging rules is that they be near enough to one half and that they be either greater than or less than one half, depending on

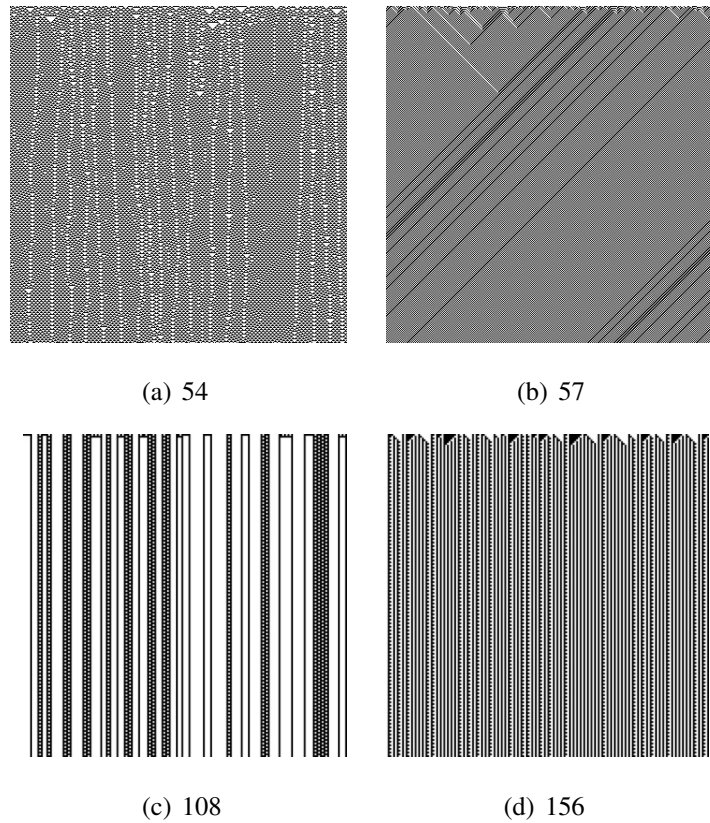


Figure 7.3: Centric-rules: 54, 57, 108, 156.

the precise rule.

We are now ready to examine the fluctuating behaviour of the self-averaging rules. The results of the first theorem, Theorem 31, have already been proven in Chapter 5. We have restated it here because the proofs are completely different, the proof given here exploring the mechanism of convergence.

**Theorem 31.** *The fluctuations of fuzzy rules 60, 90, 105, and 150 around their point of convergence of  $\frac{1}{2}$  obey their corresponding Boolean rule.*

**Proof:**

By Theorem 29 we know that all these rules converge to  $\frac{1}{2}$  along vertical or diagonal lines, we now derive their dynamics around it. Rule 60 has the following analytical form:

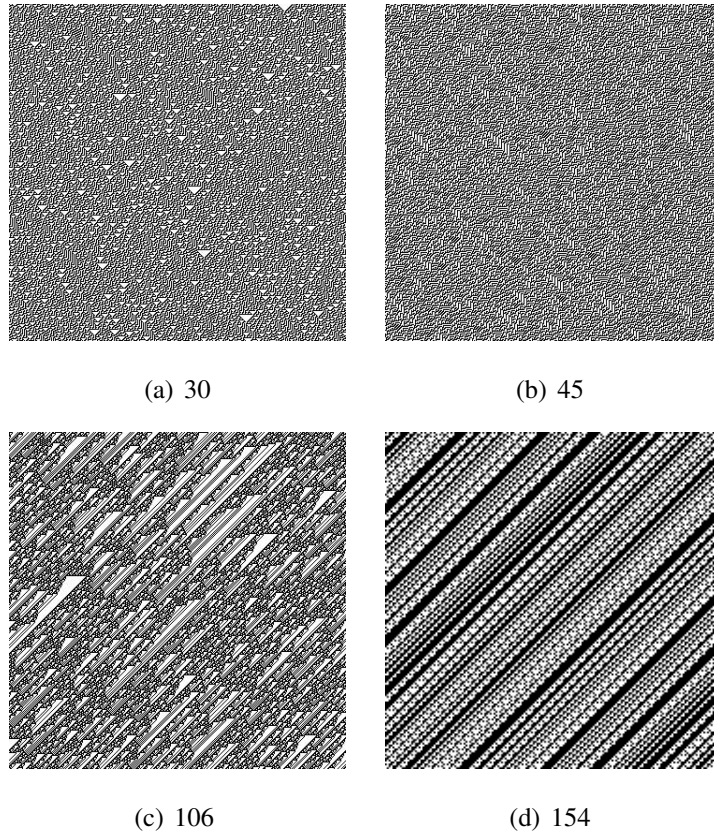


Figure 7.4: Non-centric rules: 30, 45, 106, 154.

$f_{60}(x, y, z) = (\bar{x})y + (x)\bar{y}$ . By Lemma 15 letting  $\alpha = x$  and  $\beta = y$ , we have:

$$f_{60}(x, y, z) > \frac{1}{2} \quad \text{if} \quad \begin{cases} x < \frac{1}{2} \text{ and } y > \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y < \frac{1}{2} \end{cases}$$

$$f_{60}(x, y, z) < \frac{1}{2} \quad \text{if} \quad \begin{cases} x < \frac{1}{2} \text{ and } y < \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y > \frac{1}{2} \end{cases}$$

which can be written around  $\frac{1}{2}$  as in Table 7.2 (left), and which coincides with Boolean rule 60 (right) where 0 corresponds to  $<$  and 1 to  $>$ . The proof for rule 90 is identical, letting  $\alpha = x$  and  $\beta = z$ .

Consider now rule 150:  $f_{150}(x, y, z) = (\bar{x}\bar{z} + xz)y + (\bar{x}z + x\bar{z})\bar{y}$ . We apply Lemma 15 to

$x$	$y$	$z$	$f_{60}(x, y, z)$	$x$	$y$	$z$	$g_{60}(x, y, z)$
<	<	<	<	0	0	0	0
<	<	>	<	0	0	1	0
<	>	<	>	0	1	0	1
<	>	>	>	0	1	1	1
>	<	<	>	1	0	0	1
>	<	>	>	1	0	1	1
>	>	<	<	1	1	0	0
>	>	>	<	1	1	1	0

Table 7.2: Rule 60: fluctuations of the fuzzy rule around  $\frac{1}{2}$ .

this rule letting  $\alpha = (\bar{x}z + x\bar{z})$  and  $\beta = y$  so that

$$f_{150}(x, y, z) > \frac{1}{2} \quad \text{if} \quad \begin{cases} \alpha < \frac{1}{2} \text{ and } y > \frac{1}{2} \\ \alpha > \frac{1}{2} \text{ and } y < \frac{1}{2} \end{cases}$$

$$f_{150}(x, y, z) < \frac{1}{2} \quad \text{if} \quad \begin{cases} \alpha < \frac{1}{2} \text{ and } y < \frac{1}{2} \\ \alpha > \frac{1}{2} \text{ and } y > \frac{1}{2} \end{cases} .$$

Then we apply the lemma directly to  $\alpha$ :

$$\alpha > \frac{1}{2} \quad \text{if} \quad \begin{cases} x < \frac{1}{2} \text{ and } z > \frac{1}{2} \\ x > \frac{1}{2} \text{ and } z < \frac{1}{2} \end{cases}$$

$$\alpha < \frac{1}{2} \quad \text{if} \quad \begin{cases} x < \frac{1}{2} \text{ and } z < \frac{1}{2} \\ x > \frac{1}{2} \text{ and } z > \frac{1}{2} \end{cases} .$$

$x$	$y$	$z$	$f_{150}(x, y, z)$	$x$	$y$	$z$	$g_{150}(x, y, z)$
<	<	<	<	0	0	0	0
<	<	>	>	0	0	1	1
<	>	<	>	0	1	0	1
<	>	>	<	0	1	1	0
>	<	<	>	1	0	0	1
>	<	>	<	1	0	1	0
>	>	<	<	1	1	0	0
>	>	>	>	1	1	1	1

Table 7.3: Rule 150: fluctuations of the fuzzy rule around  $\frac{1}{2}$  (left).

Combining these results, we obtain:

$$\begin{aligned}
 f_{150}(x, y, z) > \frac{1}{2} & \text{ if } \left\{ \begin{array}{l} x < \frac{1}{2} \text{ and } y < \frac{1}{2} \text{ and } z > \frac{1}{2} \\ x < \frac{1}{2} \text{ and } y > \frac{1}{2} \text{ and } z < \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y < \frac{1}{2} \text{ and } z < \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y > \frac{1}{2} \text{ and } z > \frac{1}{2} \end{array} \right. \\
 f_{150}(x, y, z) < \frac{1}{2} & \text{ if } \left\{ \begin{array}{l} x < \frac{1}{2} \text{ and } y < \frac{1}{2} \text{ and } z < \frac{1}{2} \\ x < \frac{1}{2} \text{ and } y > \frac{1}{2} \text{ and } z > \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y < \frac{1}{2} \text{ and } z > \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y > \frac{1}{2} \text{ and } z < \frac{1}{2} \end{array} \right.
 \end{aligned}$$

which again describes a fluctuation table around  $\frac{1}{2}$  that coincides with the corresponding Boolean rule table, Table 7.3, of rule 150. The proof for rule 105 follows in the same way letting  $\alpha = (\bar{x}\bar{y} + xy)$  and  $\beta = z$  when applying Lemma 15.

### 7.3.2 Other Oscillations

Analysis of the remaining self-averaging rules is more complex. But looking at the analytical form of the rules, we can see that all the self-oscillating rules in the previous section have weights which will converge to one half, while the remaining rules have weights which will converge to one quarter and three quarters hence one of the terms is consistently weighted more heavily than the other. If we consider rule 45, for example, as  $y$  and  $z$  converge towards one half, rule 45 weights  $\bar{x}$  more heavily than  $x$ . Consequently, when we observe the fluctuations around the point of convergence, rule 45 begins to behave like the fuzzification of Boolean rule 15,  $g_{15}(x, y, z) = \bar{x}$ .

From the table, near its point of convergence each of the remaining rules behaves like one of the four following rules :  $g_{15}(x, y, z) = \bar{x}$ ,  $g_{51}(x, y, z) = \bar{y}$ ,  $g_{170}(x, y, z) = z$ ,  $g_{204}(x, y, z) = y$ . What we expect to observe then, is point-wise convergence along the diagonal for rules which tend towards  $g_{15}$  or  $g_{170}$  and vertical convergence for the other two rules. In the case of  $g_{15}$ , values will converge point-wise to one half along the south-east diagonal while alternating between greater than and less than one half. For rules behaving like  $g_{170}$ , south-west diagonals will stabilize as either greater than or less than one half and will converge point-wise. Rules which mimic  $g_{51}$  will have values which oscillate around one half as they converge in columns and finally, rules which obey  $g_{204}$  will have values that stabilize as either greater than or less than one half and then converge vertically.

We begin the proofs of the descriptions above by establishing a range for the variables used in the calculation of the weights where we can be certain that the weight containing more terms will be greater than the weight with only one term.

**Lemma 16.** *For  $x$  and  $y$  in  $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ , all the following products are less than one half:  $xy$ ,  $x\bar{y}$ ,  $\bar{x}y$ ,  $\bar{x}\bar{y}$ .*

**Proof:** Note that  $1 - x$  and  $1 - y$  are also on the interval  $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ . Hence each of these products is less than  $(\frac{\sqrt{2}}{2})^2 = \frac{1}{2}$ .

**Theorem 32.** *Rule  $f_{30}$  converges to  $\frac{1}{2}$  along diagonals, its fluctuations around  $\frac{1}{2}$  obeying Boolean rule  $g_{15}$ . Furthermore, for all  $i$  there exist a time  $T$  such that for all  $t > T$ ,  $|x_{i+(t+1-T)}^{t+1} - \frac{1}{2}| < |x_{i+(t-T)}^t - \frac{1}{2}|$ , and  $x_{i+(t+1-T)}^{t+1} > \frac{1}{2}$  if and only if  $x_{i+t-T}^t < \frac{1}{2}$ .*

**Proof:** By Theorem 29, we know that rule 30 converges to  $\frac{1}{2}$  along diagonals. By Lemma 16, when  $y$  and  $z$  are in the open interval  $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ ,  $f_{30}(x, y, z) = \alpha x + (1 - \alpha)(1 - x)$  for some  $\alpha < \frac{1}{2}$ . Then by Lemma 15,  $f_{30}(x, y, z) > \frac{1}{2}$  precisely when  $x < \frac{1}{2}$  and it is greater than  $\frac{1}{2}$  when  $x$  is greater than  $\frac{1}{2}$ .

To prove point-wise convergence of the diagonals, we first note that for all  $i$ , Theorem 29 guarantees the existence of a  $T$  such that for all  $t > T$ ,  $x_{i+t-T}^t$ ,  $x_{i+t-T+1}^t$ , and  $x_{i+t-T-1}^t$  are in  $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ . Then  $x_{i+t-T}^{t+1} = \alpha x_{i+t-T-1}^t + (1 - \alpha)(1 - x_{i+t-T-1}^t)$  for some  $\alpha < \frac{1}{2}$ . By Lemma 14,  $|x_{i+(t+1-T)}^{t+1} - \frac{1}{2}| < |x_{i+(t-T)}^t - \frac{1}{2}|$ , and by Lemma 15,  $x_{i+(t+1-T)}^{t+1} > \frac{1}{2}$  if and only if  $x_{i+t-T}^t < \frac{1}{2}$ .

In other words, fuzzy rule 30 is converging along the south-easterly diagonals, oscillating around the point of convergence, one half. If we consider only the behaviour of the weights, rule 30 is converging to a rule of the form:  $\frac{1}{4}x + \frac{3}{4}\bar{x}$ . Although this rule is not identical to fuzzy rule 15, its behaviour around the point of convergence can be predicted by rule 15, as in the following table:

**Theorem 33.** *Rule  $f_{45}$  converges to  $\frac{1}{2}$  along diagonals, its fluctuations around  $\frac{1}{2}$  obeying Boolean rule  $g_{15}$ .*

**Proof:** By Theorem 29, we know that rule 45 converges to  $\frac{1}{2}$  along diagonals. Using the same argument as for rule 30, when  $y$  and  $z$  are in  $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ ,  $f(x, y, z) \geq \frac{1}{2}$  if and only if  $x \geq \frac{1}{2}$  with equality when  $x = \frac{1}{2}$  and  $|f(x, y, z) - \frac{1}{2}| < |x - \frac{1}{2}|$ .

**Theorem 34.** *Rule  $f_{106}$  converges to  $\frac{1}{2}$  along diagonals, its fluctuations around  $\frac{1}{2}$  obeying Boolean rule  $g_{170}$ .*

**Proof:** By Theorem 29, we know that rule  $f_{106}$  converges to  $\frac{1}{2}$  along diagonals. By Lemmas 16 and 15, when  $x$  and  $y$  are close enough to one half, that is, when they are in  $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ ,  $f_{106}(x, y, z)$  is greater than one half if and only if  $z$  is greater than one half. Hence,

$x$		$z$	$f_{30 45}(x, y, z)$	$x$	$y$	$z$	$g_{30 45}(x, y, z)$
<	<	<	>	0	0	0	1
<	<	>	>	0	0	1	1
<	>	<	>	0	1	0	1
<	>	>	>	0	1	1	1
>	<	<	<	1	0	0	0
>	<	>	<	1	0	1	0
>	>	<	<	1	1	0	0
>	>	>	<	1	1	1	0

Table 7.4: Rules 30 and 45: fluctuations around  $\frac{1}{2}$ 

on this interval,  $f_{106}$  obeys Boolean rule  $g_{170}(x, y, z) = z$ . Furthermore,  $|f_{106}(x, y, z) - \frac{1}{2}| < |z - \frac{1}{2}|$

Rule 154 is again the same as rule 30 except that the roles of  $\bar{x}$  and  $z$  are exchanged. Since rule 30 begins to approximate  $\bar{x}$ , we can see that rule 154 will approximate  $z$  as it nears its point of convergence.

**Theorem 35.** *Rule  $f_{154}$  converges to  $\frac{1}{2}$  along diagonals, its fluctuations around  $\frac{1}{2}$  obeying Boolean rule  $g_{170}$ .*

**Proof:** As in the proof of Theorem 32, for  $x$  and  $y$  close enough to  $\frac{1}{2}$ , precisely,  $1 - \frac{\sqrt{2}}{2} < x, y < \frac{\sqrt{2}}{2}$ ,  $f_{154}(x, y, z) > \frac{1}{2}$  if and only if  $z > \frac{1}{2}$ . Thus, on this interval, rule 154 obeys Boolean rule 170,  $g_{170}(x, y, z) = z$ .

**Theorem 36.** *Rule  $f_{108}$  converges vertically to  $\frac{1}{2}$ , its fluctuations around  $\frac{1}{2}$  obeying Boolean rule  $g_{204}$ .*

**Proof:** We can obtain the proof for rule 108 by noting that it is equivalent to rule 106

$x$	$y$	$z$	$f_{106 154}(x, y, z)$	$x$	$y$	$z$	$g_{106 154}(x, y, z)$
<	<	<	<	0	0	0	0
<	<	>	>	0	0	1	1
<	>	<	<	0	1	0	0
<	>	>	>	0	1	1	1
>	<	<	<	1	0	0	0
>	<	>	>	1	0	1	1
>	>	<	<	1	1	0	0
>	>	>	>	1	1	1	1

Table 7.5: Rules 106 and 154: fluctuations around  $\frac{1}{2}$ 

except that the roles of  $y$  and  $z$  are exchanged. Hence for  $x$  and  $z$  in  $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ ,  $f_{108}(x, y, z) > \frac{1}{2}$  when  $y > \frac{1}{2}$  and  $f_{108}(x, y, z) < \frac{1}{2}$  when  $y < \frac{1}{2}$ .

**Theorem 37.** Rule  $f_{156}$  converges vertically to  $\frac{1}{2}$ , its fluctuations around  $\frac{1}{2}$  obeying Boolean rule  $g_{204}$ .

**Proof:** We obtain the proof for rule 156 by noting that it is equivalent to rule 154 except that the roles of  $y$  and  $z$  are exchanged. Hence for  $x$  and  $z$  in  $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ ,  $f_{156}(x, y, z) > \frac{1}{2}$  when  $y > \frac{1}{2}$  and  $f_{156}(x, y, z) < \frac{1}{2}$  when  $y < \frac{1}{2}$ .

Rule 54 is the same as rule 30 except that the roles of  $x$  and  $y$  are exchanged. As a result, we would expect rule 54 to look like  $\bar{y}$  as it converges.

**Theorem 38.** Rule  $f_{54}$  converges vertically to  $\frac{1}{2}$ , its fluctuations around  $\frac{1}{2}$  obeying Boolean rule  $g_{51}$ .

**Proof:** The proof is the same as in the previous theorem, reversing the roles of  $x$  and  $y$ . Hence, for  $x$  and  $z$  in  $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ ,  $f_{54}(x, y, z)$  is greater than one half if and only if  $y$  is less than one half. Thus rule 54 obeys Boolean rule 51,  $g_{51}(x, y, z) = \bar{y}$ .

$x$	$y$	$z$	$f_{108 156}(x, y, z)$	$x$	$y$	$z$	$g_{108 156}(x, y, z)$
<	<	<	<	0	0	0	0
<	<	>	<	0	0	1	0
<	>	<	>	0	1	0	1
<	>	>	>	0	1	1	1
>	<	<	<	1	0	0	0
>	<	>	<	1	0	1	0
>	>	<	>	1	1	0	1
>	>	>	>	1	1	1	1

Table 7.6: Rules 108 and 156: fluctuations around  $\frac{1}{2}$ 

$x$	$y$	$z$	$f_{30 54}(x, y, z)$	$x$	$y$	$z$	$g_{30 54}(x, y, z)$
<	<	<	>	0	0	0	1
<	<	>	>	0	0	1	1
<	>	<	<	0	1	0	0
<	>	>	<	0	1	1	0
>	<	<	>	1	0	0	1
>	<	>	>	1	0	1	1
>	>	<	<	1	1	0	0
>	>	>	<	1	1	1	0

Table 7.7: Rules 54 and 57: fluctuations around  $\frac{1}{2}$

**Theorem 39.** *Rule  $f_{57}$  converges vertically to  $\frac{1}{2}$ , its fluctuations around  $\frac{1}{2}$  obeying Boolean rule  $g_{51}$ .*

**Proof:** Rule 57 can be obtained from rule 45 by exchanging the roles of  $x$  and  $y$ . For  $x$  and  $z$  in  $(\frac{1}{2} - \frac{(\sqrt{2}-1)}{2}, \frac{1}{2} + \frac{(\sqrt{2}-1)}{2})$ ,  $f_{57}(x, y, z)$  is greater than one half if and only if  $y$  is less than one half. We therefore obtain the following table. Rule 54 can similarly be viewed as a grid with anomalies which cause errors in the general pattern of  $\bar{y}$ . However these anomalies first cause larger disruptions in the grid pattern before gradually being resolved.

We may generalize these results to rules of higher dimensions.

**Theorem 40.** *Given a Boolean self-averaging rule  $g(y_0, \dots, y_{n-1}) = \alpha_g(y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_{n-1})y_i + \overline{\alpha_g(y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_{n-1})}\bar{y}_i$  with weight function  $\alpha_g$  such that there is only one element  $y \in \{0, 1\}^{n-1}$  such that  $\alpha_g(y) = 0$ , then the fuzzification  $f$  of  $g$  converges to a shift.*

**Proof:** Let  $\gamma = (\frac{1}{2})^{\frac{1}{n-1}}$  and consider the interval  $(1 - \gamma, \gamma)$ . If all variables  $x_i$  are on this interval then so are  $\bar{x}_i$ . Furthermore, the product of up to  $n - 1$  such variables is less than  $[(\frac{1}{2})^{\frac{1}{n-1}}]^{n-1} = \frac{1}{2}$ . Thus, when the entire configuration is on this interval,  $f$  is greater than  $\frac{1}{2}$  precisely when  $x_i$  is. Hence it fluctuates as the shift  $g(y_0, \dots, y_{n-1}) = y_i$ .

Note that as the neighbourhood size grows, the interval on which the function behaves as a shift also grows.

### 7.3.3 Observations

Let us now reconsider the Boolean rules in light of the asymptotic behaviour of their fuzzy equivalents. We have seen that the self-oscillating rules continue to exhibit the same behaviour as their binary equivalents even as they converge. But what about the other rules? Can their convergent behaviour be seen in anyway in their corresponding Boolean rules? In fact, testing has shown that the same convergent behaviour as we have described here does occur. Line by line comparisons of Boolean rules 30, 45, 106, 154, 108, 156, 54, and 57

with the Boolean equivalent of the asymptotic functions of their fuzzy rules (that is, rules  $\bar{x}$ ,  $\bar{x}$ ,  $z$ ,  $z$ ,  $y$ ,  $y$ ,  $\bar{y}$ ,  $\bar{y}$ , respectively) show that they agree with these rules over 84% of the time, starting from random initial configurations. From a random configuration, and, indeed, at the beginning of a simulation, we would expect agreement only 75% of the time. In fact, in most cases, we can observe this type of behaviour in the patterns formed.

Consider rules 106 and 154 illustrated in Figure 7.4. There is clearly, in both of them, a strong component of the rule  $z$  to which  $f_{106}$  and  $f_{154}$  converge. Rule 57 in Figure 7.3 appears to converge to  $z$  as well. However, this rule will sometimes have strong  $z$  components and sometimes strong  $x$  components. In fact, what it appears to be converging towards is a grid pattern ( $\dots 10101010\dots$ ) with 'deviations'. The behaviours of such a pattern under rules  $x$   $\bar{y}$  and  $z$  are identical. Anomalies in the grid of the form 010010 are mapped to 101001 which appears to be obeying  $f(x, y, z) = x$  but can equally be seen as an deviation or exception to the rule  $f(x, y, z) = \bar{y}$  while anomalies of the form 101101 map to 011010 which can again be either  $z$  or  $\bar{y}$  with deviations. Although rules 30 and 45 appear to be almost completely random, from the testing described above we know that they obey  $\bar{x}$  more than the 84.80 and 84.92 percent of the time, respectively, which is considerable. Rules 108 and 156, by contrast, can easily be seen to converge quickly to  $y$  with deviations.

## 7.4 Error Propagation

In this section, we consider the elementary self-averaging rules and examine exactly how much fuzzy data is actually required to force convergence to one half. In particular, we derive necessary and sufficient conditions for a single fuzzy value to "propagate", that is to spread everywhere, forcing convergence to one half. If we think of fuzziness as error or contamination, we see that understanding the degree and extend of spread has interesting possible applications in simulations. In this section when we refer to fuzzy values, we will mean values on the open interval  $(0, 1)$  and we will be distinguishing them from Boolean values which we have equated to the numbers  $\{0, 1\}$ .

When no fuzzy values are present all rules behave like Boolean rules and fuzziness never appears, so the presence of a single fuzzy value is always a necessary condition for convergence. Moreover, by Lemma 13, fuzziness never disappears since the fuzzy data will always appear as the value being averaged in some local neighbourhood, so we can expect four types of behaviours: fuzziness stays contained in a constant size area, fuzziness grows a finite amount and then stops, it propagates infinitely in one direction, or it propagates infinitely in both directions. Only in this last case will these rules converge to one half.

We have divided this section into three subsections according to the analytical form of the rule. In the first section we examine the additive rules (60, 90, 105, and 150) whose Boolean evolution is shown in Figure 7.2. While the fuzzy data always propagates infinitely, convergence to one half only happens for a single value for rules 105 and 150. Next we look at the centric, non-additive rules (54, 57, 108 and 156), rules in which  $y$  is self-averaged (the Boolean evolution of these rules is shown in Figure 7.3). Rules 54 and 57, which tend towards  $\bar{y}$  as shown in Section 7.3, will converge to one half except under certain specific conditions. Conversely, rules 108 and 156, which have been shown to tend towards  $y$  in Section 7.3, will not converge except under certain specific circumstances. Finally, we look at the non-centric rules (30, 45, 106, and 154 in Figure 7.4). All of these rules will tend to converge to one half except under specific conditions on initial configuration. We give a full description of these configurations for every rule except 106 for which only a partial description is given. Tables 7.8 and 7.9 summarize our results: Table 7.8 describes the rules that propagate except in certain cases as described, while Table 7.9 contains the ones that do not generally propagate except in the cases described.

### 7.4.1 Additive Rules

Our first theorem examines rules 105 and 150 but holds equally for self-averaging rules of larger neighbourhoods which are exclusive ors of all their variables. These rules can be described as a self-average of any of their variables.

Generally propagating rules.

Rule	Left Forbidden	Right Forbidden
$f_{54}$	none	none
$f_{57}$	none	none
$f_{45}$	$(001)^\infty$	none
$f_{30}$	$(01)^\infty$	none
$f_{154}$	$\alpha 10^m$ where $m$ is odd	$1, 1, 1, \dots$
$f_{106}$	?	$1, 1, 1, \dots$ , and $0, 0, 0, \dots$

Table 7.8: The conditions for convergence to  $\frac{1}{2}$  are given by indicating, for each rule, the subconfigurations that forbid convergence.

Generally non-propagating rules.

Rule	Configuration	Condition for convergence
$f_{60}$	$\alpha * \beta$	none
$f_{90}$	$\alpha * \beta$	none
$f_{108}$	$\alpha * \beta$	$\beta = (1110)^\infty, (1101)^\infty, (1011)^\infty, (1111)^\infty$ or $0(1011)^\infty, \alpha = \beta$
$f_{156}$	$\alpha * \beta$	$\alpha = (1)^\infty$ or $0(1)^\infty$ and $\beta = (0)^\infty$ or $1(0)^\infty$

Table 7.9: Description of convergent configurations  $\alpha * \beta$  for rules  $f_{108}$  and  $f_{156}$ , where  $\alpha$  is a left-infinite sequence,  $\beta$  is a right-infinite sequence, and  $*$  is the fuzzy value.

**Theorem 41.** *A single non-binary value in the initial configuration is necessary sufficient to force rules 105 and 150 to converge to  $\frac{1}{2}$ .*

**Proof:** We assume that we have one non-binary value in our initial configuration at  $x_0^0$ . First note that, for rules 105 and 150, we can write these equations as self-averaging rules of any of their three variables. So the equations for  $x_{-1}^0$ ,  $x_0^0$ , and  $x_1^0$  can all be written in the  $\alpha x + (1 - \alpha)(1 - x)$  where  $\alpha \in \{0, 1\}$  and  $x \in (0, 1)$ . Then by Lemma 13 all three of these values must be fuzzy. Thus for any given cell  $i$ ,  $x_i^t \in (0, 1)$  for all  $t \geq |i|$  and the convergence follows from Theorem 29.

For rules 60 and 90 a single value will never force convergence. In these cases we establish conditions for convergence with minimal fuzzy data. First consider rule 90; it is easy to see that a single fuzzy value does not force the entire CA to converge to  $\frac{1}{2}$ , but it makes half of the cells converge. We can show that for total convergence we need two strategically placed fuzzy values.

**Theorem 42.** *Two fuzzy values  $x_i^0$  and  $x_j^0$  in the initial configuration where  $i$  is odd and  $j$  is even (or vice-versa) is a necessary and sufficient condition for rule 90 to converge to one half.*

**Proof:** Assume that  $x_i^0, x_j^0 \in (0, 1)$  with  $i = 0$  and  $j$  odd. Then as in the proof above, at time  $t = 1$ ,  $x_{-1}^1$  and  $x_1^1$  will both be in  $(0, 1)$ . At  $t = 2$ ,  $x_{-2}^2$ ,  $x_0^2$  and  $x_2^2$  will all have fuzzy values. Continuing on in this way, we see that for any  $i$   $x_i^t$  will have a fuzzy value for all  $t \geq i$  such that the parity of  $t$  and  $i$  are the same. Similarly, if we also have a fuzzy value with odd index, for all  $i$  there exists a  $T$  such that  $x_i^t$  has a fuzzy value whenever  $t > T$  and  $i$  and  $t$  have opposite parity. Hence all values must converge to one half. If both fuzzy values were to be even (or odd) then at most every other value would be fuzzy at any given time.

Note that for circular CA, rule 90 will converge to one half with a single fuzzy value if the array has odd length (and therefore a single value can be considered at once even and odd).

For rule 60, the results are a little different. With only a finite number of non-binary values, we can only show that rule 60 converges to one half to the right of any such values.

**Theorem 43.** *Given fuzzy rule 60 and at least one non-binary value in the initial configuration at  $x_i^0$ , then for all  $j > i$ ,  $x_j^t$  will converge to  $\frac{1}{2}$  as  $t \rightarrow \infty$ .*

**Proof:** By Lemma 13,  $x_i^t$  will remain in  $(0, 1)$  for all  $t$ . Now since  $x_i^t$  is the weighting factor in the calculation of  $x_{i+1}^{t+1}$ , for all  $t > 0$   $x_{i+1}^t$  will be a fuzzy value. In fact, for any  $j > i$ , for any  $t \geq j - i$ ,  $x_j^t$  will be a fuzzy value and by Lemma 14, they will all be converging to one half.

In other words, a single fuzzy value is never sufficient to force total convergence. For this to happen we need the condition of the following corollary.

**Corollary 7.** *Fuzzy rule 60 converges to one half if and only if the initial configuration is such that for all  $i$  there exists a  $j \leq i$  such that  $x_j^0$  is fuzzy.*

Note that, as a consequence, given a single non-binary value in a circular CA with rule  $f_{60}$ , all values will converge to one half.

We can generalize these results to additive rules in higher dimensions. One way for a single fuzzy value to infect an entire system is if it is used non-trivially in the calculations of all the values in its von Neumann neighbourhood of range 1. This would imply that all of the values in its von Neumann neighbourhood are in the neighbourhood  $N$  of the cellular automaton and that they are not dummy variables.

**Theorem 44.** *Assume an additive self-averaging rule has a single non-binary value in its initial configuration. Further assume that the neighbourhood of a cell includes the von Neumann neighbourhood and that the function is non-trivial in the von Neumann neighbourhood. Then the entire configuration will converge to  $\frac{1}{2}$ .*

**PROOF** Assume, without loss of generality, that  $x_{\vec{0}}$  is in  $(0, 1)$ . Then at time  $t = 1$ , every cell in the von Neumann neighbourhood of range 1 of  $x_{\vec{0}}$  will be in  $(0, 1)$  by Lemma 13 since

the local rule is self-averaging in  $x_{\vec{0}}$ . Inductively, at time  $t$ , the von Neumann neighbourhood of range  $t$  will be fuzzy. Given any  $x_{\vec{i}}$ , it is in the von Neumann neighbourhood of range  $T$  of  $x_{\vec{0}}$  for some  $T$ . So at time  $T + 1$  its von Neumann neighbourhood of range 1 will consist entirely of fuzzy value. Hence by Lemma 14,  $x_{\vec{i}}^t$  is converging to  $\frac{1}{2}$  for  $t > T + 1$ . ■

We would expect one-sided convergence if only half of the range 1 neighbourhood was used non-trivially and a multi-dimensional rule 60 effect, that is, a fixed pattern of every other cell converging to  $\frac{1}{2}$ , if the local rule used the cell itself and the cells in its range 2 neighbourhood that are not in the range 1 neighbourhood. We obtain rule 90 from the cells in range 1 except for the centre cell. In multiple dimensions, this causes every other cell to converge to  $\frac{1}{2}$  along “diagonals”, or, thought of another way, at every other time step.

### 7.4.2 Centric Rules

We now consider the centric rules (that is, rules that are a weighted average of  $y$  and  $\bar{y}$ ) which are not additive: rules 54, 57, 108, and 156. We will see that rules that tend towards  $\bar{y}$  will always converge while those that tend towards  $y$  will only rarely converge. We start the analysis by considering rule 54.

x	y	z	=	$f_{54}$	x	y	z	=	$f_{54}$
*	0	0	=	*	0	0	*	=	*
*	0	1	=	1	0	1	*	=	$\bar{*}$
*	1	0	=	$\bar{*}$	1	0	*	=	1
*	1	1	=	0	1	1	*	=	0

Table 7.10: Rule 54 with fuzzy data

**Theorem 45.** *A single non-binary value in the initial configuration is necessary and sufficient to force rule  $f_{54}$  to converge to  $\frac{1}{2}$ .*

**Proof:** We first note that if at some time  $T$ ,  $x_i^T \in (0, 1)$  then for all  $t > T$ ,  $x_i^t \in (0, 1)$  since when  $x$  and  $z$  are Boolean,  $f(x, y, z)$  is equal to either  $y$  or  $\bar{y}$  and therefore fuzzy. Moreover, since rule 54 is symmetric in  $x$  and  $z$ , it is sufficient to show that any fuzzy data will propagate to one side (the right, for example) in a finite number of time steps. So assume, without loss of generality that  $x_0^0$  is fuzzy and that for all  $i > 0$   $x_i^0 \in \{0, 1\}$ . We want to show that in at most three steps both  $x_0$  and  $x_1$  will become fuzzy. If the initial fuzzy data is followed by either 00 or 10, then by Table ??  $x_1^1$  will be fuzzy. If, however,  $x_2^0 = 1$  then  $x_1^1 = \bar{x}_1^0$ , and will therefore still be Boolean after a single time step. We then consider what happens when  $x_2^0 = 1$ . If  $x_1^0 = 1$ , so that the fuzzy data is followed by 11 then  $x_1^1 = x_2^1 = 0$ , and  $x_2^2$  is fuzzy. In the final case, where the fuzzy data is followed by 01,  $x_1^0 = 0$  and  $x_2^0 = 1$  so  $x_1^1 = 1$  and therefore at time 1, the fuzzy data must be followed by either 10 or 11 which will propagate in at most 2 steps as before. We illustrate the maximum number of time steps before fuzzy data spreads to the right (i.e., three steps):

$$\begin{array}{cccc}
 * & 0 & 1 & 0 \\
 * & 1 & 1 & ? \\
 * & 0 & 0 & ? \\
 * & * & ? & ?
 \end{array}$$

Here, \* represents arbitrary fuzzy data and ? is arbitrary Boolean data.

We have shown, then, whenever there is a fuzzy value, it will propagate at least every three steps and every value will eventually be fuzzy, hence by Theorem 29 they will all converge to one half.

**Theorem 46.** *A single non-binary value in the initial configuration is necessary and sufficient to force rule  $f_{57}$  to converge to  $\frac{1}{2}$ .*

**Proof:** Again it suffices to show that every value will eventually be fuzzy. We first note that if at some time  $T$ ,  $x_i^T \in (0, 1)$  then for all  $t > T$ ,  $x_i^t \in (0, 1)$  since when  $x$  and  $z$  are Boolean,  $f(x, y, z)$  is equal to either  $y$  or  $\bar{y}$  which are fuzzy. We begin by showing that fuzzy

x	y	z	=	$f_{57}$	x	y	z	=	$f_{57}$
*	0	0	=	1	0	0	*	=	$\bar{*}$
*	0	1	=	*	0	1	*	=	*
*	1	0	=	0	1	0	*	=	1
*	1	1	=	$\bar{*}$	1	1	*	=	0

Table 7.11: Rule 57 with fuzzy data

data will propagate to the right in a finite number of time steps. Assume, without loss of generality that  $x_0^0$  is fuzzy and that for all  $i > 0$   $x_i^0 \in \{0, 1\}$ . Now if  $x_2^0 = 1$  then by Table ??,  $x_1^1$  will be fuzzy. If, however,  $x_2^0 = 0$  then  $x_1^1 = \bar{x}_1^0$ , and will therefore still be Boolean after a single time step. In this case, there are several possibilities and it is easy to see that they all result in the fuzziness propagating in three steps or less. We will illustrate below the case leading to the largest number of steps before propagation which corresponds to  $x_3^0 = 1$  and thus  $x_2^1 = 0$ . In this case we have  $x_1^1 = 1$ , which implies  $x_2^2 = 1$  and thus  $x_1^3$  is fuzzy. This case is illustrated below:

$$\begin{array}{cccc}
 * & 0 & 0 & 1 \\
 * & 1 & 0 & ? \\
 * & 0 & 1 & ? \\
 * & * & ? & ?
 \end{array}$$

Again, \* represents arbitrary fuzzy data and ? is arbitrary Boolean data.

Similarly, on the left hand side,  $x_{-1}^1$  is fuzzy when  $x_{-2}^0$  is equal to 0 and is equal to  $\bar{x}_{-1}^0$  otherwise (see Table ??). By the same reasoning as before, the following sequence represents

the maximum number of time steps for  $x_{-1}^t$  to become fuzzy:

$$\begin{array}{cccc} 0 & 1 & 1 & * \\ ? & 1 & 0 & * \\ ? & 0 & 1 & * \\ ? & ? & * & * \end{array}$$

Rules 108 and 156 do not necessarily converge. Both of these rules have simple stable configurations, patterns of cells that once established do not move or change regardless of the states of the cells surrounding them. These stable configurations create what we call forbidden blocks. That is, blocks that are stable under the Boolean rule and that create boundaries beyond which fuzzy data cannot spread. If a forbidden block is present in the initial configuration or if they are created in subsequent iterations, the fuzzy data cannot spread to the whole configuration. We will also refer to blocks which always evolve into forbidden blocks as forbidden.

$x$	$y$	$z$	$=$	$f_{108}$	$x$	$y$	$z$	$=$	$f_{108}$
*	0	0	=	0	0	0	*	=	0
*	0	1	=	*	0	1	*	=	1
*	1	0	=	1	1	0	*	=	*
*	1	1	=	$\bar{*}$	1	1	*	=	$\bar{*}$

Table 7.12: Rule 108 with fuzzy data

**Lemma 17.** *Any stable configuration is a forbidden block.*

**Proof:** If a configuration is stable, then it remains the same no matter which value precedes it. In other words, the result of the fuzzy function will be the same at the two points, where the stable block is preceded by either 0 or 1. Since the fuzzy rules are affine

in each variable it means that the fuzzy function on this block is the line at 0 or 1. Similarly the fuzzy function must be defined as a line to the right of the stable configuration.

Note that the lemma above is as a result of this fuzzification being affine in each variable and thus holds for all fuzzy rules obtained in this way and for larger neighbourhoods as well. In addition, infinite stable sequences, by the same argument, will also limit fuzzy growth. We now turn our attention to rule 108.

**Lemma 18.** *The block (00) is a forbidden block for fuzzy rule  $f_{108}$ .*

**Proof:** First note that rule 108 preserves Since  $f(?, 0, 0) = f(0, 0, ?) = 0$ , 00 is a stable configuration and hence a forbidden block by Lemma 7.4.2.

The consecutive zeros need not be present in the initial configuration, but once created they will not disappear. Hence propagation in rule 108 will tend to stop quickly. So the above lemma begs the question, under what initial conditions will rule 108 never produce two consecutive zeros? The answer turns out to be quite easily described.

**Theorem 47.** *Given a single fuzzy value in its initial configuration, rule 108 will converge to one half if and only if the binary data to the right of the fuzzy value has a repeating pattern of the form  $(1110)^\infty$ ,  $(1101)^\infty$ ,  $(1011)^\infty$ ,  $(0111)^\infty$ , or  $0(1011)^\infty$ . To the left of the fuzzy data, the Boolean data must be the mirror image of one of these configurations.*

**Proof:** Since rule 108 is symmetric in  $x$  and  $z$ , it suffices to prove the theorem for either the left or the right hand side. We prove the theorem by constructing a propagating configuration. In view of the previous lemma, we are trying to avoid double 0s occurring at any time in the configuration. The following are some of the blocks to be avoided for this rule because in at most two steps they will generate the forbidden block (00): 0000, 01010, 10101, 1111, 101101.

Consider the following two-bit words  $a = 01$ ,  $b = 10$ ,  $c = 11$  and  $d = 00$ . In any configuration, we are trying to avoid the appearance of  $d$  or  $ba$  because they are forbidden. Notice that, if we rewrite our initial configuration using these words, neither  $b$  nor  $c$  can be

repeated, since  $bba$ ,  $bbb$ ,  $bbc$  and  $cc$  are forbidden blocks. So  $b$  can only be followed by  $c$ . Now we note that, since both blocks  $1111$  and  $00$  are forbidden, we must have a  $b$  block occurring and therefore the sequence  $bc$  appearing somewhere to the right of the fuzzy value. Since  $bca = 101101$  is also forbidden,  $bc$  must be followed by another  $b$  in order for fuzzy data to propagate to the right, so following an initial  $10$  we must have the repeating sequence  $(cb) = 1110$ .

At this point we must establish what can occur before the first appearance of  $b$  to the right of the fuzzy value. There are several possibilities. If the sequence begins with a  $1$ , we can have at most 3 consecutive  $1$ s before the first zero, creating the following the patterns:  $1011101110 \dots = (1011)^\infty$ , or  $11011101110 \dots = (1101)^\infty$  or  $111011101110 \dots = (1110)^\infty$ . If the sequence begins with a leading  $0$ , it must be followed immediately by either one or three consecutive ones  $1$ , and then the infinite repeating pattern  $0111$ . If exactly two  $1$ s were to occur, we would have  $*01101 \rightarrow **1111 \rightarrow **00$  and propagation ends.

To summarize, we will have one of the following five sequences to the right of the fuzzy data, and a mirror image of any one of them to its left:

011101110111 ...

101110111011 ...

110111011101 ...

111011101110 ...

0101110111011 ...

By symmetry, the mirror image of one of these sequences must appear to the left of the fuzzy data in the initial configuration for propagation to occur.

Next we consider rule 156.

**Lemma 19.** *The combination (01) is a forbidden block for rule 156*

x	y	z	=	$f_{156}$	x	y	z	=	$f_{156}$
*	0	0	=	*	0	0	*	=	0
*	0	1	=	0	0	1	*	=	1
*	1	0	=	$\bar{*}$	1	0	*	=	$\bar{*}$
*	1	1	=	1	1	1	*	=	*

Table 7.13: Rule 156 with fuzzy data

**Proof:** The block 01 is a stable configuration under rule 156 since, from Table ??  $f(*, 0, 1) = 0$  and  $f(0, 1, *) = 1$ .

**Lemma 20.** *The right infinite sequence  $1, 1, 1, \dots$  is a forbidden block for rule 156.*

**Proof:** Since  $f(*, 1, 1) = 1$  from the table, this is a stable configuration. Notice that since  $f(1, 1, *) = *$  the left infinite sequence is not stable and therefore not forbidden.

**Lemma 21.** *The left infinite sequence  $\dots, 0, 0, 0$  is a forbidden block for rule 156.*

**Proof:** Since  $f(0, 0, *) = 0$ , this is a stable configuration. Again note that the right infinite sequence is not stable.

Again, we would like to know under which conditions on initial configuration will any of these forbidden blocks occur. Since the left and right hand sides of this rules are very different, we investigate them separately.

**Lemma 22.** *A single fuzzy value in the initial configuration of rule 156 will propagate infinitely to the right if and only if it is followed, on the right, by infinite 0s or a single 1 and then infinite 0s.*

**Proof:** If we have a 0 anywhere after the fuzzy value, it must be followed by infinite 0s otherwise we have the forbidden block 01. Since infinite 1s are also forbidden, we can only have finite 1s followed by infinite 0s. However, we cannot have more than one initial 1 since

we would eventually have the sequence 1100 which maps to the forbidden block01 and will therefore terminate the propagation. So the only two possibilities are infinite 0s or one 1 followed by infinite 0s.

Since rule 156 is symmetric in  $x$  and  $\bar{z}$  we would expect the mirrored inverse of the previous lemma.

**Lemma 23.** *A single fuzzy value in the initial configuration of rule 156 will propagate infinitely to the left if and only if it is followed, on the left, by infinite 1s or a single 0 and then infinite 1s.*

**Proof:** The proof follows from the previous lemma and the fact that rule 156 is symmetric in  $x$  and  $\bar{z}$ .

Finally, we can fully describe all initial configuration which allow an initial fuzzy value to propagate and therefore force rule 156 to converge towards one half.

**Theorem 48.** *Given a single fuzzy value in the initial configuration of rule 156, all values will eventually converge toward one half if and only if the initial configuration has one of the four following forms:*

$$\dots 111111111 * 000000000 \dots$$

$$\dots 111111111 * 100000000 \dots$$

$$\dots 111111110 * 000000000 \dots$$

$$\dots 111111110 * 100000000 \dots$$

### 7.4.3 Non-Centric Rules

We now turn our attention to the rules which are not centric. For these rules, the conditions for propagation to one side (to the left for averages of  $z$  and to the right for averages of  $x$ ) depend on having at least two consecutive fuzzy values at any point in time and arise as a result of the following lemma.

**Lemma 24.** *If any two values in the evaluation of an elementary self-averaging rule are strictly fuzzy, the result will be strictly fuzzy.*

**Proof:** If one of the two fuzzy values is the variable being self-averaged then the result will be fuzzy even with only a single fuzzy value, so consider the case where the variable being self-averaged is Boolean. In this situation it selects one of the two weights which will both be fuzzy.

**Lemma 25.** *If a non-centric rule has a configuration with two consecutive fuzzy values, fuzzy data will propagate to one side infinitely.*

**Proof:** Consider a self-averaging rule in  $x$  and assume without loss of generality that  $x_0^0$  and  $x_{-1}^0$  are both fuzzy. We will show by induction on  $i$  that  $x_i^t$  will be fuzzy for all  $t \geq i$ . By Lemma 7.4.3, for all  $t \geq 0$ ,  $x_0^t$  and  $x_{-1}^t$  will be fuzzy. Now if we assume that for some  $i$  and for all  $j \leq i$   $x_j^t$  will be fuzzy for all  $t \geq j$ , then for  $t \geq i + 1$ ,  $x_{i+1}^t$  will be fuzzy since it is a self-average of  $x_i^t$  which will be fuzzy by hypothesis. So we see that the fuzzy data will propagate right at every step. Rules that are self-averages of  $z$  can similarly be shown to propagate fuzziness left with two consecutive fuzzy values.

So propagation in one direction depends only on generating a single appropriately placed extra fuzzy value. In the other direction, however, we will need specific conditions to propagate. By their vary nature, these rules cannot have finite forbidden blocks since changing the value to the left or the right, (in the case of self-averages of  $x$  or  $z$ , respectively) will always change the outcome. However, they can have infinite one sided forbidden blocks inhibiting propagation on the other side (to the right for averages of  $z$  and to the left for averages of  $x$ ). The first two rules that we will discuss, 45 and 30, will propagate if and only if these infinite forbidden blocks do not appear in the initial configuration.

**Lemma 26.** *The left infinite sequence  $\dots, 0, 0, 1, 0, 0, 1$  is a forbidden block for rule 45.*

**Proof:** This follows from the Boolean truth table for rule 45 and from Table ??.

x	y	z	=	$f_{45}$	x	y	z	=	$f_{45}$
0	*	0	=	1	0	0	*	=	$\bar{*}$
0	*	1	=	*	0	1	*	=	1
1	*	0	=	0	1	0	*	=	*
1	*	1	=	$\bar{*}$	1	1	*	=	0

Table 7.14: Rule 45 with fuzzy data

**Theorem 49.** *Rule 45 will eventually converge to one half if and only if the data to the left of the initial fuzzy value does not contain an infinite sequence of  $(001)^\infty$ .*

**Proof:**

We will show the contrapositive, that is: rule 45 will not converge to one half if and only if it contains the infinite sequence described in its initial configuration.

$\Leftarrow$ : Since this is a forbidden block, it will limit spread to the left.

$\Rightarrow$ : To prove this, we must show that the fuzzy data will always spread to the right so that the only way to prevent convergence is to limit spread to the left. Now if the fuzzy data is ever followed by a 1, since by Table ?? there will be two consecutive fuzzy values at the next iteration, the fuzzy data will propagate infinitely to the right by Lemma 7.4.3.

$$\begin{array}{cccc} ? & * & 1 & \\ ? & * & * & \\ ? & * & * & * \end{array} .$$

But the fuzzy data can never be followed by 0s in perpetuity. In fact, assume that it is initially followed by a zero, if it is to be followed by a zero after the first iteration, it must actually be

followed by 001 in the initial configuration so we have:

$$\begin{array}{cccccc}
 ? & * & 0 & 0 & 1 & ? \\
 ? & ? & * & 0 & 1 & ? \\
 ? & ? & ? & * & 1 & ? \\
 ? & ? & ? & * & * & ?
 \end{array}$$

We see that the fuzzy data will always spread to the right and the only way to stop it from spreading everywhere is to stop the spread on the left.

Now by Table ??, the data spreads to the left every time it is preceded by a 0. In order to stop spread to the left we must have a stable (possibly left infinite) block which ends in a 1. Working backwards we can see that the only possible sequence is  $\dots, 0, 0, 1, 0, 0, 1$ . Since any predecessor of this sequence must contain this sequence, we are done.

The proof for rule 30 follows a similar pattern.

x	y	z	=	$f_{30}$	x	y	z	=	$f_{30}$
0	*	0	=	*	0	0	*	=	*
0	*	1	=	1	0	1	*	=	1
1	*	0	=	$\bar{*}$	1	0	*	=	$\bar{*}$
1	*	1	=	0	1	1	*	=	0

Table 7.15: Rule 30 with fuzzy data

**Lemma 27.** *The left infinite sequence  $\dots, 0, 1, 0, 1, 0, 1$  is a forbidden block for rule 30.*

**Proof:** This follows from the Boolean truth table for rule 30 and from Table ??.

**Theorem 50.** *Rule 30 will converge to one half with a single fuzzy value in its initial configuration if and only if the data to the left of the fuzzy value does not contain the left infinite sequence  $\dots, 0, 1, 0, 1, 0, 1$ .*

**Proof:**

$\Rightarrow$ : This follows immediately from Lemma 7.4.3.  $\Leftarrow$ : We first show that rule 30 will always propagate right from a single value. Since the fuzzy data is moving right at each iteration, all we require is two consecutive fuzzy values in order to have right propagation by Lemma 7.4.3. Now, if the single fuzzy value is ever followed by a 0 from Table ?? we see that there will be two consecutive fuzzy values at the next iteration. So let us assume that it is always followed by a 1. Then we have two possible cases, it is either preceded by a 0 in the initial configuration or by a 1. In either case, the data will begin to propagate right:

$$\begin{aligned} & \dots?????????0 * 1000 \\ & \dots????????? * 1 * 1000 \\ & \dots????????? * 0 * 1000 \\ & \dots????????? * 1 * 1000 \end{aligned}$$

or similarly:

$$\begin{aligned} & \dots?????????1 * 1000 \\ & \dots?????????0 * 1000 \\ & \dots????????? * 1 * 1000 \\ & \dots????????? * 0 * 1000 \\ & \dots????????? * 1 * 1000 \end{aligned}$$

So the only way to prevent convergence to one half is by preventing propagation on the left.

Rule 30 does not propagate to the left whenever the fuzzy data is preceded by a 1. For this to happen in perpetuity, it must at some point be preceded by the forbidden sequence  $\dots, 0, 1, 0, 1, 0, 1$ , otherwise it will continue to propagate left and will be tending toward one

half everywhere. Since the predecessors of this sequence must contain this sequence, the data will not propagate infinitely to the left if and only if  $\dots, 0, 1, 0, 1, 0, 1$  appears in the initial configuration.

x	y	z	=	$f_{154}$	x	y	z	=	$f_{154}$
0	*	0	=	0	*	0	0	=	*
0	*	1	=	1	*	0	1	=	*
1	*	0	=	*	*	1	0	=	0
1	*	1	=	*	*	1	1	=	1

Table 7.16: Rule 154 with fuzzy data

**Lemma 28.** *The right infinite sequence  $1, 1, 1, \dots$  is a forbidden block for rule 154.*

**Proof:** This follows from the Boolean truth table for rule 154 and from Table ??.

**Theorem 51.** *Rule 154 will converge to one half if and only if there is an even number of 0s before the first 1 to the left of the fuzzy data and the data to the right of the fuzzy data does not contain an infinite sequence of 1s.*

**Proof:**

$\Leftarrow$  Let  $(\dots x_{-2}^0, x_{-1}^0, x_0^0, x_1^0, x_2^0 \dots)$  be the initial configuration with  $x_0$  being the fuzzy value. By Table ?? and Lemma 7.4.3, as soon as the fuzzy data is preceded by a 1 on the left it will begin propagating left so if  $x_{-1}^0 = 1$  the fuzzy data will propagate left infinitely. If it is not 1, let the number of zeros before the first 1 to the left of the fuzzy value be even. That is, let  $x_{-2m}^0 \dots x_{-1}^0 = 0$  with  $m \geq 0$ , and let  $x_{-2m-1}^0 = 1$ . Since  $f(x, 1, 0) = 0$ ,  $f(1, 0, 0) = 1$ ,  $f(0, 0, 0) = 0$  and  $f(0, 0, *) = *$ , at the next step the number of zeros between the fuzzy value and the first one to the left has decreased by 2. In other words, if  $m > 1$  we would have:  $x_{-2m}^1 = 1$ ,  $x_{-2m+1}^1, x_{-2m+2}^1, \dots, x_{-2}^1 = 0$ , and  $x_{-1}^1 = *$ , and if  $m = 1$   $x_{-2}^1 = 1$ , and  $x_{-1}^1 = *$ . After  $t = \frac{m}{2}$  steps  $x_{-2}^t = 1$ , and  $x_{-1}^t = *$ . At this point fuzziness propagates to the left.

Now consider the right side and assume that the data has already propagated left so that, without loss of generality,  $x_t^i$  is fuzzy for all  $t$ . Now fuzziness propagates right whenever the current right most fuzzy data is followed by a 0. Furthermore, since  $f(*, 1, 0) = 0$ , it will propagate one step right in two iterations if it is followed by 10. So in order for the fuzziness to not propagate, the fuzzy data must be followed by 11. If it is to never propagate again, it must be followed by 11 in perpetuity. Since  $f(1, 1, 0) = 0$  while  $f(1, 1, 1) = 1$ , the only way for the fuzzy data to always be followed by 11 is for it to be followed by an infinite sequence of 1s. Since any predecessor of an infinite sequence of 1s contains an infinite sequence of 1s, if this is not present in the initial configuration, the fuzziness will (perhaps slowly) propagate right infinitely. When this configuration is not present, coupled with the left propagation, the configuration is becoming completely fuzzy and converging to one half by Theorem 29.

$\Rightarrow$  In this direction we will prove the contrapositive; we will show that if either of the two conditions on the initial configuration is not met, then the configuration will not converge to one half. We begin by assuming that there is an odd number of 0s before the first one on the left of the fuzzy value. Then by the same argument as above we would eventually arrive at a configuration where 10 precedes the fuzzy data. We need to show that from this configuration we can never arrive at a situation where the fuzzy data is preceded by a 1. First consider the two possible cases for data preceding the 10, that is, either we have 110\* or 010\*. If we have the former case, at the next iteration the data preceding the fuzzy data is again 10:

$$? \ 1 \ 1 \ 0 \ *$$

$$? \ 1 \ 0 \ *$$

So we only need to investigate the case 010\*. We claim that at the next iteration, we have either returned to the case where there is an odd number of 0s before the first 1, or there is an infinite number of 0 to the left of the fuzzy data. In fact, 0010\* will take us right back to

01\*, so we can consider only 1010\*. Consider these two examples:

$$\begin{array}{cccccccc}
 1 & 1 & 0 & 1 & 0 & 1 & 0 & * \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
 \\ 
 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 * \\
 ? & 1 & 0 & 0 & 0 & 0 & 0 & *
 \end{array}$$

We see that if we have an infinite sequence of 10s, then at the next iteration we have infinite 0s and the fuzzy data will never propagate. However, if break this pattern by either 11 or 00 (the only two possibilities), we are back to having an odd number of 0s before the first 1 to the left of the fuzzy data. Notice in the examples above that if we added another 10 pair before breaking the pattern that it would have the effect of adding two more 0s, and so would keep the number of 0s odd. In other words, we will never have a zero immediately to the left of the fuzzy data and the fuzziness will never propagate left.

A right infinite sequence of 1s is a forbidden block for rule 154 by Lemma 7.4.3 and its presence will therefore end propagation.

We now turn our attention to rule 106. Unfortunately, we cannot completely characterize under what conditions on initial configuration this rule will or will not propagate. However, we can describe what is ultimately required for propagation.

x	y	z	=	$f_{106}$	x	y	z	=	$f_{106}$
0	*	0	=	0	*	0	0	=	0
0	*	1	=	1	*	0	1	=	1
1	*	0	=	*	*	1	0	=	*
1	*	1	=	$\bar{*}$	*	1	1	=	$\bar{*}$

Table 7.17: Rule 106 with fuzzy data

**Lemma 29.** *The right infinite sequence  $0, 0, 0, \dots$  is a forbidden block for rule 106.*



**Theorem 53.** *If a self-averaging rule in  $n = 3$  or more variables acts non-trivial on all of its variables, then Boolean data spaced no closer than  $n$  will disappear.*

**Proof:** At most one non-fuzzy data bit will occur in any neighbourhood. If the variable being averaged is fuzzy, then the result is clearly fuzzy. If it is Boolean, then it essentially serves to select one of the two weights, which, as sums of products of fuzzy values, will be fuzzy.

## 7.5 Summary

In this chapter, we have mathematically proven the observed behaviours of a class of rules which we have called the self-averaging rules, (rules 30,45, 54, 57, 60, 90, 105, 106, 108, 150, 154, 156). While the behaviours in this class are at first unremarkable - all rules converge to one half - the class itself is very interesting. This class contains all of the additive and surjective (hence reversible) rules. It also contains all of the rules which will converge to one half from any given initial configuration on the open interval  $(0, 1)$ . (This result holds for self-averaging rules of any neighbourhood size.) We know that the additive rules in this class exhibit self-oscillation and in this chapter we have studied how the other self-averaging rules converge showing that they all obey a simple Boolean rule: a shift, an inversion, or a combination of both. Finally, we consider self-averaging rules on  $[0, 1]$  and we establish necessary and sufficient conditions in almost all cases for convergence to one half. In the case of additive rules defined on the whole neighbourhood, a single fuzzy value will eventually force convergence to one half. (Again, this result holds for all neighbourhood sizes.) The other rules could all be shown to have very specific configuration either preventing or enabling convergence to one half from a single fuzzy value.

The results in this chapter appeared in [6].

---

---

## CHAPTER 8

---

---

# Local Majority Rules

In this chapter we investigate the last of the balanced rules. Unlike the weighted average rules or the self-averaging rules, these rules do not have easily described convergent behaviour, resulting in only a few values in the asymptotic configuration. They do have tendencies, however and converge towards stable patterns. These rules are pull towards extremities, evolving into binary or mostly binary configurations. We will see, in this section, how a form of voting in the Boolean rule tends the fuzzy rule to move towards extremes.

### 8.1 Definitions and Basic Properties

Boolean *local majority rules* simply return a majority vote of the local neighbourhood. The rules in the table below are *generalized local majority rules* in that the Boolean result of  $g(x, y, z)$  is the majority of the quantities  $g_1(x), g_2(y), g_3(z)$  where each function  $g_i(x)$  may return the variable  $x$  itself or its negation  $\bar{x}$ . We have used the notation  $M\{f_1(x), f_2(y), f_3(z)\}$  to emphasize how each variable is "voting". While these rules are drawn from every group in the classification of fuzzy rules, they have in common that asymptotically most of their values tend towards the binary as illustrated in the Figure 8.1.

The fuzzification of these rules have a well defined property: If a fuzzy rule  $f$  is a *generalized local majority rule*, then it has the following property:  $f(x, y, z) > \min\{f_1(x), f_2(y), f_3(z)\}$

Rule	Equation	Majority
$R_{23}$	$(\bar{x} \oplus \bar{y})\bar{z} + \bar{x}\bar{y}$	$M\{\bar{x}, \bar{y}, \bar{z}\}$
$R_{43}$	$(\bar{x} \oplus \bar{y})z + \bar{x}\bar{y}$	$M\{\bar{x}, \bar{y}, z\}$
$R_{77}$	$(\bar{x} \oplus y)\bar{z} + \bar{x}y$	$M\{\bar{x}, y, \bar{z}\}$
$R_{142}$	$(\bar{x} \oplus y)z + \bar{x}y$	$M\{\bar{x}, y, z\}$
$R_{178}$	$(x \oplus \bar{y})z + x\bar{y}$	$M\{x, \bar{y}, z\}$
$R_{232}$	$(x \oplus y)z + xy$	$M\{x, y, z\}$

Table 8.1: Majority rules.

if the majority of  $\{f_1(x), f_2(y), f_3(z)\}$  are greater than one half and  $f(x, y, z) < \max\{f_1(x), f_2(y), f_3(z)\}$  if the majority of  $\{f_1(x), f_2(y), f_3(z)\}$  are less than one half for some functions  $f_i$ , where  $f_i(x) = x$  or  $(1 - x)$ . The usual majority rule, rule 232, is a majority rule under this definition, as are all the rules listed above.

For each of these rules, we will first prove that it is a generalized local majority rule and then describe its asymptotic behaviour.

## 8.2 Rule 232

We will now prove that rule 232 is a majority rule according to our definition.

**Theorem 54.** *Rule 232 is a majority rule. That is, if  $f$  is the fuzzy local rule 232, then  $f(x, y, z) > \min\{x, y, z\}$  if at least two of  $\{x, y, z\}$  are greater than one half and  $f(x, y, z) < \max\{x, y, z\}$  if at least two of  $\{x, y, z\}$  are less than one half.*

**Proof:** Recall that rule 232 can be written as:

$$\begin{aligned} f(x, y, z) &= \bar{x}yz + x\bar{y}z + xy\bar{z} + xyz \\ &= (x \oplus y)z + xy \end{aligned}$$

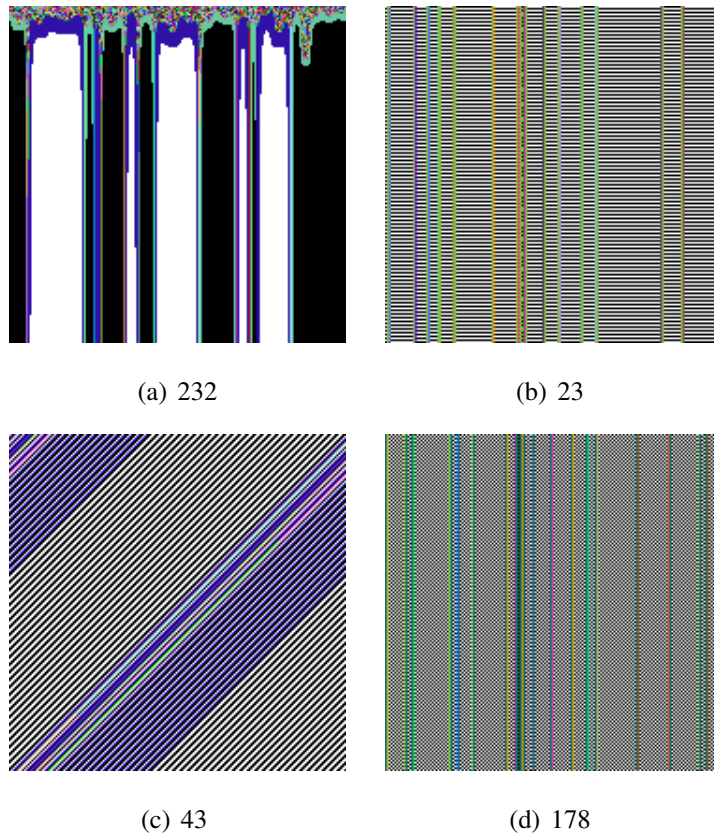


Figure 8.1: Examples of Generalized Local Majority Rules

Since rule 232 is symmetric in  $x$ ,  $y$ , and  $z$  we can assume without loss of generality that  $x, y > \frac{1}{2}$  and  $z < x, y$ , then,

$$\begin{aligned}
 f(x, y, z) &= (x \oplus y)z + xy \\
 &> (x \oplus y)z + xz \\
 &= (2x + y - 2xy)z
 \end{aligned}$$

Now,  $(2x + y - 2xy) < 1$  implies that  $x < \frac{1}{2}$ :

$$\begin{aligned}
 2x + y - 2xy &< 1 \\
 2x - 2xy &< 1 - y \\
 2x(1 - y) &< 1 - y
 \end{aligned}$$

$$\begin{aligned} 2x &< 1 \\ x &< \frac{1}{2} \end{aligned}$$

So  $2x + y - 2xy > 1$  by assumption and  $f(x, y, z) > z$ . Since  $z = \min\{x, y, z\}$ , we have our assertion. In fact, we can see that if  $x > y > z > \frac{1}{2}$ ,  $f(x, y, z) > y$  also:

$$\begin{aligned} f(x, y, z) &= (x \oplus z)y + xz \\ &> (x + z - 2xz)y + zy \\ &= (x + 2z - 2xz)y \\ &> y \end{aligned}$$

Finally, to see that  $f(x, y, z) < \max\{x, y, z\}$  whenever the majority of  $\{x, y, z\}$  are less than one half, assume that  $x$  and  $y$  are less than one half and that  $z > x, y$

$$\begin{aligned} f(x, y, z) &= (x \oplus y)z + xy \\ &< (x + y - 2xy)z + xz \\ &= (2x + y - 2xy)z \\ &< z, \end{aligned}$$

since  $x$  is less than  $\frac{1}{2}$ . As before, when all three values are less than one half, the function will be smaller than the second smallest.

When all values,  $f_1(x)$ ,  $f_2(y)$ , and  $f_3(z)$ , are less than one half, majority rules will tend towards 0; while when all values are greater than one half they will tend towards one.

**Theorem 55.** *Given an initial configuration  $X = (x_0^0, \dots, x_{n-1}^0)$  such that  $x_i^0 < \frac{1}{2}$  for all  $i$ , then  $F_{232}(X) \rightarrow (0, \dots, 0)$ .*

**Proof:** Clearly from the above, the maximum value at any time  $t$  are decreasing and bounded below and therefore convergent. Assume that the maximum converges to some values  $M \neq 0$ .

As a consequence of this behaviour, blocks of fuzzy data which are greater than one half, or having only isolated values which are less than one half will tend towards 1, while blocks of values less than one half or having only isolated values greater than one half will tend towards 0. Interestingly, fuzzy data at the boundaries of such blocks will tend to be preserved since

$$\begin{aligned} f(0, y, 1) &= y \\ f(1, y, 0) &= y. \end{aligned}$$

### 8.3 Rule 23

We will analyse rule 23 by showing that  $F_{23} \circ F_{23} = F_{232} \circ F_{232}$ .

First we will need the following Lemma.

**Lemma 30.** *Given any majority rule,  $M\{x, y, z\}$ ,  $\overline{M\{x, y, z\}} = M\{\bar{x}, \bar{y}, \bar{z}\}$ .*

**Proof:**

$$\begin{aligned} \overline{M\{x, y, z\}} &= \overline{\bar{x}yz + x\bar{y}z + xy\bar{z} + xyz} \\ &= \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y\bar{z} + x\bar{y}\bar{z} \\ &= M\{\bar{x}, \bar{y}, \bar{z}\} \end{aligned}$$

**Theorem 56.** *Let  $F_{23}$  and  $F_{232}$  be the global rules associated with local rules 23 and 232, respectively. Then  $F_{23} \circ F_{23} = F_{232} \circ F_{232}$*

**Proof:** First note that  $f_{23}(x, y, z) = M\{\bar{x}, \bar{y}, \bar{z}\} = \overline{f_{232}(x, y, z)}$  and  $f_{23}(\bar{x}, \bar{y}, \bar{z}) = M\{x, y, z\} = f_{232}(x, y, z)$ . For a given array  $(x_0^0, \dots, x_{n-1}^0)$ , let  $F_{232}(x_0^0, \dots, x_{n-1}^0) = (x_0^1, \dots, x_{n-1}^1)$  and  $F_{232}(x_0^1, \dots, x_{n-1}^1) = (x_0^2, \dots, x_{n-1}^2)$ . Then

$$\begin{aligned} F_{23} \circ F_{23}(x_0^0, \dots, x_{n-1}^0) &= F_{23}(\bar{x}_0^1, \dots, \bar{x}_{n-1}^1) \\ &= (x_0^2, \dots, x_{n-1}^2). \end{aligned}$$

In other words, convergent blocks under rule 232 will alternate between 0 and 1 while border fuzzy values will remain fixed.

## 8.4 Rule 77

= Similar to rule 23, we will analyse rule 77 by showing it to be conjugate to rule 232.

**Theorem 57.** *For  $n$  even, rule 77 is conjugate to rule 232 under the homeomorphism  $h$  :*

$$(x_0, \dots, x_{n-1}) = (\bar{x}_0, x_1, \bar{x}_2, \dots, x_{n-1})$$

**Proof:** Let  $F_{77}$  be global rule 77 and  $f_{77}$  its local rule and let  $F_{232}$  be global rule 232 and  $f_{232}$  its local rule. Now, since both local rules are majority rules, we can write them as  $M\{f_1(x), f_2(y), f_3(z)\} = (f_1(x) \oplus f_2(y))f_3(z) + f_1(x)f_2(y)$ . In particular,  $f_{232}(x, y, z) = M\{x, y, z\}$  while  $f_{77}(x, y, z) = M\{\bar{x}, y, \bar{z}\}$ . We need to show that  $h \circ F_{77} = F_{232} \circ h$ .

$$\begin{aligned} F_{232} \circ h(x_0, \dots, x_{n-1}) &= F_{232}(\bar{x}_0, x_1, \dots, x_{n-1}) \\ &= (M\{x_{n-1}, \bar{x}_0, x_1\}, M\{\bar{x}_0, x_1, \bar{x}_2\}, M\{x_1, \bar{x}_2, x_3\}, \dots, M\{\bar{x}_{n-2}, x_{n-1}, \bar{x}_0\}). \end{aligned}$$

While

$$\begin{aligned} h \circ F_{77}(x_0, \dots, x_{n-1}) &= h(M\{\bar{x}_{n-1}, x_0, \bar{x}_1\}, M\{\bar{x}_0, x_1, \bar{x}_2\}, M\{\bar{x}_1, x_2, \bar{x}_3\}, \dots, M\{\bar{x}_{n-2}, x_{n-1}, \bar{x}_0\}) \\ &= (\overline{M\{\bar{x}_{n-1}, x_0, \bar{x}_1\}}, \overline{M\{\bar{x}_0, x_1, \bar{x}_2\}}, \overline{M\{\bar{x}_1, x_2, \bar{x}_3\}}, \dots, \overline{M\{\bar{x}_{n-2}, x_{n-1}, \bar{x}_0\}}) \\ &= (M\{x_{n-1}, \bar{x}_0, x_1\}, M\{\bar{x}_0, x_1, \bar{x}_2\}, M\{x_1, \bar{x}_2, x_3\}, \dots, M\{\bar{x}_{n-2}, x_{n-1}, \bar{x}_0\}). \end{aligned}$$

From the above, we would expect that, asymptotically, rule 77 would produce blocks of alternating zeros and ones separated by fuzzy values which are maintained because  $f(0, y, 0) = y$  and  $f(1, y, 1) = y$ .

**Theorem 58.** *For  $n$  odd, rule 77 is conjugate to rule 232 under the homeomorphism  $h$  :*

$$(x_0, \dots, x_{n-1}) = (\bar{x}_0, x_1, \bar{x}_2, \dots, \bar{x}_{n-1}, x_0, \bar{x}_1, \dots, x_{n-1})$$

**Proof:**

$$F_{232} \circ h(x_0, \dots, x_{n-1}) = F_{232}(\bar{x}_0, x_1, \dots, \bar{x}_{n-1}, x_0, \bar{x}_1, \dots, x_{n-1})$$

$$= (M\{x_{n-1}, \bar{x}_0, x_1\}, M\{\bar{x}_0, x_1, \bar{x}_2\}, \dots, M\{x_{n-2}, \bar{x}_{n-1}, x_0\}, \\ M\{\bar{x}_{n-1}, x_0, \bar{x}_1\}, M\{x_0, \bar{x}_1, x_2\}, \dots, M\{\bar{x}_{n-2}, x_{n-1}, \bar{x}_0\}).$$

While

$$\begin{aligned} h \circ F_{77}(x_0, \dots, x_{n-1}) &= h(M\{\bar{x}_{n-1}, x_0, \bar{x}_1\}, M\{\bar{x}_0, x_1, \bar{x}_2\}, M\{\bar{x}_1, x_2, \bar{x}_3\}, \dots, M\{\bar{x}_{n-2}, x_{n-1}, \bar{x}_0\}) \\ &= (\overline{M\{\bar{x}_{n-1}, x_0, \bar{x}_1\}}, \overline{M\{\bar{x}_0, x_1, \bar{x}_2\}}, \dots, \overline{M\{\bar{x}_{n-2}, x_{n-1}, \bar{x}_0\}}), \\ &\quad M\{\bar{x}_{n-1}, x_0, \bar{x}_1\}, \overline{M\{\bar{x}_0, x_1, \bar{x}_2\}}, \dots, M\{\bar{x}_{n-2}, x_{n-1}, \bar{x}_0\}) \\ &= (M\{x_{n-1}, \bar{x}_0, x_1\}, M\{\bar{x}_0, x_1, \bar{x}_2\}, \dots, M\{x_{n-2}, \bar{x}_{n-1}, x_0\}, \\ &\quad M\{\bar{x}_{n-1}, x_0, \bar{x}_1\}, M\{x_0, \bar{x}_1, x_2\}, \dots, M\{\bar{x}_{n-2}, x_{n-1}, \bar{x}_0\}). \end{aligned}$$

From the above, we would expect that, asymptotically, rule 77 would produce blocks of alternating zeros and ones separated by fuzzy values which are maintained because  $f(0, y, 1) = y$  and  $f(1, y, 0) = y$ .

## 8.5 Rule 178

We will analyse rule 178 in the same way as rule 77, by showing it to be conjugate to rule 232 in 2 steps.

**Theorem 59.** *For  $n$  even,  $F_{178} \circ F_{178}$  is conjugate to rule  $F_{232} \circ F_{232}$  under the homeomorphism  $h : (x_0, \dots, x_{n-1}) = (x_0, \bar{x}_1, x_2, \dots, \bar{x}_{n-1})$*

**Proof:** Let  $F_{178}$  be global rule 178 and  $f_{178}$  its local rule and let  $F_{232}$  be global rule 232 and  $f_{232}$  its local rule. Now  $f_{232}(x, y, z) = M\{x, y, z\}$  while  $f_{178}(x, y, z) = M\{x, \bar{y}, z\}$ . We need to show that  $h \circ F_{178} \circ F_{178} = F_{232} \circ F_{232} \circ h$ . We will do so by showing that the following diagram commutes.

$$\begin{array}{c} h \\ \longrightarrow \end{array}$$

$$\begin{array}{ccc}
F_{178} \downarrow & & \downarrow F_{232} \\
& h_1 & \\
& \longrightarrow & \\
F_{178} \downarrow & & \downarrow F_{232} \\
& h & \\
& \longrightarrow &
\end{array}$$

where  $h_1 = inv \circ h$ . We will proceed by showing that  $F_{232} \circ h = h_1 \circ F_{178}$  and  $F_{232} \circ h_1 = h \circ F_{178}$ .

$$\begin{aligned}
F_{232} \circ h(x_0, \dots, x_{n-1}) &= F_{232}(x_0, \bar{x}_1, x_2, \dots, \bar{x}_{n-1}) \\
&= (M\{\bar{x}_{n-1}, x_0, \bar{x}_1\}, M\{x_0, \bar{x}_1, x_2\}, M\{\bar{x}_1, x_2, \bar{x}_3\}, \dots, M\{x_{n-2}, \bar{x}_{n-1}, x_0\}).
\end{aligned}$$

While

$$\begin{aligned}
h_1 \circ F_{178}(x_0, \dots, x_{n-1}) &= h_1(M\{x_{n-1}, \bar{x}_0, x_1\}, M\{x_0, \bar{x}_1, x_2\}, M\{x_1, \bar{x}_2, x_3\}, \dots, M\{x_{n-2}, \bar{x}_{n-1}, x_0\}) \\
&= (\overline{M\{x_{n-1}, \bar{x}_0, x_1\}}, \overline{M\{x_0, \bar{x}_1, x_2\}}, \overline{M\{x_1, \bar{x}_2, x_3\}}, \dots, \overline{M\{x_{n-2}, \bar{x}_{n-1}, x_0\}}) \\
&= (M\{\bar{x}_{n-1}, x_0, \bar{x}_1\}, M\{x_0, \bar{x}_1, x_2\}, M\{\bar{x}_1, x_2, \bar{x}_3\}, \dots, M\{x_{n-2}, \bar{x}_{n-1}, x_0\}).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
h \circ F_{178}(x_0, \dots, x_{n-1}) &= h(M\{x_{n-1}, \bar{x}_0, x_1\}, M\{x_0, \bar{x}_1, x_2\}, M\{x_1, \bar{x}_2, x_3\}, \dots, M\{x_{n-2}, \bar{x}_{n-1}, x_0\}) \\
&= (M\{x_{n-1}, \bar{x}_0, x_1\}, \overline{M\{x_0, \bar{x}_1, x_2\}}, M\{x_1, \bar{x}_2, x_3\}, \dots, \overline{M\{x_{n-2}, \bar{x}_{n-1}, x_0\}}) \\
&= (M\{x_{n-1}, \bar{x}_0, x_1\}, M\{\bar{x}_0, x_1, \bar{x}_2\}, M\{x_1, \bar{x}_2, x_3\}, \dots, M\{\bar{x}_{n-2}, x_{n-1}, \bar{x}_0\}) \\
&= F_{232} \circ h_1(x_0, \dots, x_{n-1}).
\end{aligned}$$

Now

$$\begin{aligned}
h \circ F_{178} \circ F_{178} &= F_{232} \circ h_1 \circ F_{178} \\
&= F_{232} \circ F_{232} \circ h.
\end{aligned}$$

As with rule 77, rule 178 will produce blocks of alternating zeros and ones separated by fuzzy values. However, in this case the fuzzy values will be inverted at each step since  $f(0, y, 0) = \bar{y}$  and  $f(1, y, 1) = \bar{y}$ .

## 8.6 Summary

In this chapter, we describe the asymptotic tendencies of a class of rules which we have called the local majority rules and which includes rules 23, 43, 77, 142, 178, and 232. We show, using some of the techniques first explored for weighted average rules, why these rules tend towards binary results from fuzzy initial configurations. With the analysis of this class of rules, we have looked at all of the "balanced" elementary rules - those rules having exactly four 1s in their Boolean truth tables.

---

---

## CHAPTER 9

---

---

# Conclusions

### 9.1 Contributions of Thesis

The results obtained so far can be divided into two categories: links between fuzzy and Boolean cellular automata, and proofs of the asymptotic behaviour of fuzzy cellular automata. In chapters 4 through 6 we established three strong links between our fuzzified rules and their Boolean equivalents providing a new way to study Boolean CA. In chapters 7 through 9 we provided formal proofs of phenomena previously known only through empirical observation.

We first showed that number conservation, an important property in the study of Boolean cellular automata, is preserved through fuzzification. In fact, number conservation is so easily detected in fuzzy cellular automata, thanks to the wider variety of tools at our disposal, that we described another form of conservation, spatial number conservation, not previously known.

Next we proved results that were anticipated in the chapter on self-averaging rules, proving that additive rules will have a certain asymptotic behaviour as they fluctuate around their point of convergence. This behaviour had previously been observed and studied for rule 90; in this thesis it was thoroughly explained and all rules exhibiting such behaviour were

found and described.

Finally, we turned our attention to a probabilistic interpretation of fuzzy cellular automata and showed that when fuzzy cellular automata converge homogeneously, the point of convergence is a stable value of the mean field approximation for the Boolean cellular automata. This points to the use of fuzzy CA for measuring the dynamic behaviour of Boolean CA.

Taken together, these three chapters show the importance of fuzzy cellular automata as a tool for exploring Boolean cellular automata and establish them as a significant field of research.

In chapters 7 through 9, we have analysed the asymptotic behaviour of all the balanced rules, those rules having as many transitions to one as to zero, providing formal proofs that the observed behaviours were, indeed, mathematically correct. To do so, the rules were divided into three types: weighted average rules discussed in Chapter 7; self-averaging, or permutive, rules in Chapter 8; and generalized local majority rules in Chapter 9.

Weighted average rules had a variety of asymptotic periodic behaviours, depending on which variable was being used as the weighting factor and the presence of inversions in the variables being averaged. Equivalences in a generalization to fuzzy cellular automata were used in the proofs. In this chapter, we developed new tools for analyzing and creating FCA and generalized FCA that could be useful in further study or applications of fuzzy or discrete cellular automata.

The definition of self-averaging rules was extended to rules of higher neighbourhoods and dimensions. In one dimension, whether finite, circular, or infinite, they were shown to converge to one half from purely fuzzy initial conditions for any neighbourhood size and in 2 dimensions criteria were established for neighbourhood shape which would force such a convergence. In addition, the affect of a single fuzzy value was shown to be infectious for a subset of these rules. In this section, we also analysed the way in which these rules converge showing that they begin to mimic other, simpler rules. Based on these results, we re-examined the Boolean rules, showing that they also tended towards these simpler rules.

The techniques developed for weighted average rules were used to describe and explain the asymptotic behaviours of generalized local majority rules, although exact outcomes were difficult to predict.

## 9.2 Future Work

There are several ways in which the work begun here could be continued.

The classification of fuzzy CA studied in this thesis showed that every rule converged, asymptotically to one of the following periods: 1, 2, 4, and  $n$ , the length of the circular automata. It would be interesting to show that these are the only periods possible and, further, to do this in such a way that the possible periods for larger neighbourhood sizes can be determined and rules having particular desired periods constructed.

The study of the relationship between fuzzy cellular automata and Boolean cellular automata could also be extended. In particular, we believe that FCA could prove to be useful in the study of surjective Boolean CA, possibly providing a necessary condition for surjectivity. The result on self-oscillation and additivity tend to point in this direction.

The techniques developed here could be used to study other continuous extensions of Boolean CA, including, for example, CNF-fuzzification. In fact, one could study fuzzification from the DNF form using different fuzzy operators. Asynchronous FCA could also be investigated. Some as yet unpublished work has begun in this latter direction.

Finally, one could further investigate the applications of FCA, particularly in the area of cryptography. An obvious place to begin is in the field of hash functions where Boolean CA have already been used and where techniques already developed to extend Boolean CA to fuzzy for pattern recognition may be applicable. Similar techniques could be used in data mining.

---

---

## Bibliography

- [1] A. I. Adamatzky. Hierarchy of fuzzy cellular automata. *Fuzzy Sets and Systems*, (62):167–174, 1994.
- [2] S. Amoroso and Y. Patt. Decision procedures for surjectivity and injectivity of parallel maps for tessellation structures. *J. of Computer and System Sciences*, 6(5):448–464, 1972.
- [3] H. Betel and P. Flocchini. Fluctuations of fuzzy cellular automata around their convergence point. In *Proc. of the International Symposium on Nonlinear Theory and its Applications, Japan*, 2009.
- [4] H. Betel and P. Flocchini. On the asymptotic behavior of circular fuzzy cellular automata. *Journal of Cellular Automata*, 6:25–52, 2011.
- [5] H. Betel and P. Flocchini. On the relationship between boolean and fuzzy cellular automata. *Theoretical Computer Science*, 412:703–713, 2011.
- [6] H. Betel, P. Flocchini, and A. Karmouch. Asymptotic behavior of self-averaging continuous cellular automata. In *Proc. of the 16th International Workshop on Cellular Automata and Discrete Complex Systems, France*, pages 173–186, 2010.
- [7] N. Boccara and K. Cheong. Automata network epidemic models. In *Cellular Automata and Cooperative Systems*, volume 396, pages 29–44. Kluwer, 1993.

- [8] N. Boccara and H. Fuks. Cellular automaton rules conserving the number of active sites. *J. of Physics A: math. gen.*, 31:6007–6018, 1998.
- [9] N. Boccara and H. Fuks. Number-conserving cellular automaton rules. *Fundamenta Informaticae*, 52:1–13, 2001.
- [10] R. Cappuccio, G. Cattaneo, G. Erbacci, and U. Jocher. Aparallel implementation of a cellular automata based model for coffee percolation. *Parallel Computing*, 27(5):719–735, 2001.
- [11] G. Cattaneo, A. Dennunzio, and L. Margara. Solution of some conjectures about topological properties of linear cellular automata. *Theor. Comput. Sci.*, 325(2):249–271, 2004.
- [12] G. Cattaneo, M. Finelli, and L. Margara. Investigating topological chaos by elementary cellular automata dynamics. *Theor. Comput. Sci.*, 244(1-2):219–241, 2000.
- [13] G. Cattaneo, P. Flocchini, G. Mauri, C. Quaranta-Vogliotti, and N. Santoro. Cellular automata in fuzzy backgrounds. *Physica D*, 105:105–120, 1997.
- [14] G. Cattaneo, P. Flocchini, G. Mauri, and N. Santoro. Fuzzy cellular automata and their chaotic behavior. In *Proc. International Symposium on Nonlinear Theory and its Applications (NOLTA)*, volume 4, pages 1285–1289. IEICE, 1993.
- [15] G. Cattaneo, E. Formenti, G. Manzini, and L. Margara. Ergodicity, transitivity, and regularity for linear cellular automata over  $z_m$ . *Theor. Comput. Sci.*, 233(1-2):147–164, 2000.
- [16] P. Pal Chaudhuri, D. Roy Chowdhury, S. Nandi, and S. Chattopadhyay. *Theory and Applications in Additive Cellular Automata*. IEEE Press, 1997.
- [17] M. Cook. Universality in cellular automata. *Complex Systems*, 15:1–40, 2004.

- [18] A.M. Coxe and C.A. Reiter. Fuzzy hexagonal automata and snowflakes. *Computers and Graphics*, 27:447–454, 2003.
- [19] K. Culik II, L. P. Hurd, and S. Yu. On the limit sets of cellular automatas. *SIAM Journal on Computing*, 18:831–842, 1989.
- [20] K. Culik II and S. Yu. Undecidability of CA classification schemes. *Complex Systems*, 2:177–190, 1988.
- [21] M. D’amico, G. Manzini, and L. Margara. On computing the entropy of cellular automata. *Theoretical Computer Science*, 290(3):1629–1646, 2003.
- [22] A. Dennunzio, P. di Lena, E. Formenti, and L. Margara. On the directional dynamics of additive cellular automata. *Theoretical Computer Science*, 410(47-49):4823–4833, 2009.
- [23] P. di Lena and L. Margara. Computational complexity of dynamical systems: The case of cellular automata. *Inf. Comput.*, 206(9-10):1104–1116, 2008.
- [24] P. di Lena and L. Margara. On the undecidability of the limit behavior of cellular automata. *Theoretical Computer Science*, 411(7-9):1075–1084, 2010.
- [25] B. Drossel and F. Schwabl. Self-organized critical forest-fire models. *Phys. Lett. A*, 69:1629–1632, 1992.
- [26] B. Durand. Global properties of 2d cellular automata. In E. Goles and S. Martinez, editors, *Cellular Automata and Complex Systems*. Kluwer, 1998.
- [27] B. Durand, E. Formenti, A. Grange, and Z. Ròka. Number conserving cellular automata: new results on decidability and dynamics. *Discrete Mathematics and Theoretical Computer Science*, pages 129–140, 2003.
- [28] B. Durand, E. Formenti, and Z. Ròka. Number-conserving cellular automata I: decidability. *Theoretical Computer Science*, 299(1-3):523–535, 2003.

- [29] B. Durand, E. Formenti, and G. Varouchas. On undecidability of equicontinuity classification for cellular automata. In M. Morvand and E. Rémila, editors, *Discrete Models for Complex Systems, DMCS'03*, volume AB of *DMTCS Proceedings*, pages 117–128. Discrete Mathematics and Theoretical Computer Science Proceedings, 2003.
- [30] D. Eppstein. Growth and decay in life-like cellular automata. In A. Adamatzky, editor, *Game of Life Cellular Automata*, chapter 6, pages 71–97. Springer-Verlag, 2010.
- [31] N. Fatès. Experimental study of elementary cellular automata dynamics using the density parameter. In *Discrete Models for Complex Systems*, pages 155–166, 2003.
- [32] P. Flocchini and V. Cezar. Radial view of continuous cellular automata. *Fundamenta Informaticae*, 87(3):165–183, 2008.
- [33] P. Flocchini, F. Geurts, A. Mingarelli, and N. Santoro. Convergence and aperiodicity in fuzzy cellular automata: revisiting rule 90. *Physica D*, 42:20–28, 2000.
- [34] P. Flocchini and N. Santoro. The chaotic evolution of information in the interaction between knowledge and uncertainty. In R. J. Stonier and X. H. Yu, editors, *Complex Systems, Mechanism of Adaptation*, pages 337–343. IOS Press, 1994.
- [35] E. Formenti and A. Grange. Number conserving cellular automata II: dynamics. *Theoretical Computer Science*, 304(1-3):269–290, 2003.
- [36] E. Formenti and P. Kurka. Dynamics of cellular automata in non-compact spaces. In *Encyclopedia of Complexity and Systems Science*, pages 2232–2242, 2009.
- [37] D. M. Forrester. Fuzzy cellular automata in conjunctive normal form. Master’s thesis, University of Ottawa, 2011.
- [38] M. Gardner. Mathematical games - the fantastic combinations of jon conway’s new solitaire game ‘life’. *Scientific American*, 223:120–123, 1970.

- [39] Peter Grassberger. New mechanism for deterministic diffusion. *Phys. Rev. A*, 28:3666–3667, 1983.
- [40] H. A. Gutowitz. A hierarchical classification of cellular automata. *Physica D*, 45:136–156, 1990.
- [41] T. Hattori and S. Takesue. Additive conserved quantities in discrete-time lattice dynamical systems. *Physica D*, 49:295–322, 1991.
- [42] G. A. Hedlund. Endomorphisms and automorphisms of the shift dynamical system. *Mathematical Systems Theory*, 3:320–375, 1969.
- [43] K. Kaneko. *Theory and Application of Coupled Map Lattices*. John Wiley & Sons Ltd, 1993.
- [44] J. Kari. Reversibility of 2d cellular automata is undecidable. *Physica D*, 45:379–385, 1990.
- [45] J. Kari. Reversible cellular automata. *Developments in Language Theory*, pages 57–68, 2005.
- [46] J. Kari. Theory of cellular automata: A survey. *Theoretical Computer Science*, 334:3–33, 2005.
- [47] J. Kari. Undecidable properties on the dynamics of reversible one-dimensional cellular automata. *Jour. of Cellular Automata*, pages 3–14, 2008.
- [48] J. Kari, P. Vanier, and T. Zeume. Bounds on non-surjective cellular automata. In *Proceedings of the Mathematical Foundations of Computer Science*, pages 439–450, 2009.
- [49] G. Keller, M. Kunze, and T. Nowiki. Some phase transitions in coupled map lattices. *Physica D*, 59:39–51, 1992.

- [50] P. Kurka. Languages, equicontinuity and attractors in cellular automata. *Ergodic Theory and Dynamical Systems*, 17(2):417–433, 1997.
- [51] C. G. Langton. Studying artificial life with cellular automata. In *Evolution, Games, and Learning*. North Holland, 1986.
- [52] B. Litow and P. Dumas. Additive cellular automata and algebraic series. *Theoretical Computer Science*, 119:345–354, 1993.
- [53] P. Maji. On characterization of attractor basins of fuzzy multiple attractor cellular automata. *Fundamenta Informaticae*, 86(1-2):143–168, 2008.
- [54] P. Maji and P. P. Chaudhuri. Fuzzy cellular automata for modeling pattern classifier. *IEICE Transactions on Information and Systems*, 88(4):691–702, 2005.
- [55] P. Maji and P. P. Chaudhuri. RBFFCA: A hybrid pattern classifier using radial basis function and fuzzy cellular automata. *Fundamenta Informaticae*, 78(3):369–396, 2007.
- [56] M. Mihaljevic, Y. Zheng, and H. Imai. A family of fast dedicated one-way hash functions based on programmable linear cellular automata over  $GF(q)$  and time variant table. *IEICE Transactions on Fundamentals*, E82-A:32–39, 1999.
- [57] A. Mingarelli. The global evolution of general fuzzy automata. *J. of Cellular Automata*, 1(2):141–164, 2006.
- [58] A. Mingarelli. The global evolution of general fuzzy cellular automata. *Journal of Cellular Automata*, 1(2):141–164, 2006.
- [59] A. Mingarelli. A study of fuzzy and many-valued logics in cellular automata. *J. of Cellular Automata*, 1(3):233–252, 2006.
- [60] A. Mingarelli. A study of fuzzy and many-valued logics in cellular automata. *Journal of Cellular Automata*, 1(3):233–252, 2006.

- [61] E. F. Moore. Machine models of self-reproduction. *Proc. of the Symp. in Applied Mathematics*, 14:17–33, 1962.
- [62] A. Moreira. Universality and decidability of number-conserving cellular automata. *Theoretical Computer Science*, 292(3):711–721, 2003.
- [63] A. Moreira, N. Boccara, and E. Goles. On conservative and monotone one-dimensional cellular automata and their particle representation. *Theory of Computer Science*, 325(2):285–316, 2004.
- [64] J. Myhill. The converse to moore’s garden-of-eden theorem. *Proc. of the American Mathematical Society*, 14:685–686, 1963.
- [65] K. Nagel and M. Schreckenberg. A cellular automaton for freeway traffic. *J. of Physics I*, 2:2221–2229, 1992.
- [66] N. H. Packard and S. Wolfram. Two-dimensional cellular automata. *Journal of Statistical Physics*, 38(5-6):901–946, 1985.
- [67] M. Pivato. Conservation laws in cellular automata. *Nonlinearity*, 15(6):1781–1793, 2002.
- [68] C. A. Reiter. Fuzzy automata and life. *Complexity*, 7(3):19–29, 2002.
- [69] K. Sutner. Classifying circular cellular automata. *Physica D*, 45:386–395, 1990.
- [70] K. Sutner. De bruijn graphs and linear cellular automata. *Complex Systems*, 5:19–30, 1991.
- [71] S. Takesue. Ergodic properties and thermodynamic behaviour of elementary reversible cellular automata. *J. Stat. Phys.*, 56(3-4):371–402, 1989.
- [72] A. Vicari, A. Herault, C. Del Negro, M. Coltelli, M. Marsella, and C. Proietti. Modeling of the 2001 lava flow at etna volcano by a cellular automata approach. *Environmental Modeling and Software*, 22:1465–1471, 2007.

- [73] F. von Haeseler, H-O. Peitgen, and G. Skordev. Self-similar structures of rescaled evolution sets of cellular automata ii. *Int Journal of Bifurcation Chaos*, 11(4):927–941, 2001.
- [74] J. von Neumann. *Theory of Self-Reproducing Automata*. University of Illinois Press, Urbana, 1966.
- [75] B. Voorhees. Cellular automata, pascal’s triangle, and generation of order. *Physica D*, 31:135–140, 1988.
- [76] B. Voorhees. *Computational Analysis of One-Dimensional Cellular Automata*. World Scientific, 1996.
- [77] S. Wilson. Cellular automata can generate fractals. *Discrete Applied Mathematics*, 8:91–99, 1984.
- [78] S. Wolfram. Statistical mechanics of cellular automata. *Reviews of Modern Physics*, 55(3):601–644, 1983.
- [79] S. Wolfram. Universality and complexity in cellular automata. *Phy. D*, 10:1–35, 1984.
- [80] S. Wolfram. Random sequence generation by cellular automata. *Advances in Applied Mathematics*, 7:123–169, 1986.
- [81] S. Wolfram. *Theory and Applications of Cellular Automata*. World Scientific, 1986.
- [82] S. Wolfram. *Cellular Automata and Complexity: Collected Papers*. Addison-Wesley, 1994.
- [83] S. Wolfram. *A New Kind of Science*. Wolfram Media, 2002.
- [84] S. El Yacoubi and A. Mingarelli. Controlling the dynamics of the fuzzy cellular automaton rule 90. In *Proc. of 8th International Conference on Cellular Automata for Research and Industry (ACRI)*, pages 174–183, 2008.