The Eulerian Bratteli diagram and traces on its associated dimension group

Gustavo Felisberto Valente

A thesis submitted in partial fulfillment of the requirements for the
Doctorate in Philosophy degree in Mathematics[1]

Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

---

[1] The Ph.D. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

In this thesis we present two important closely related examples of Bratteli diagrams: the Pascal triangle and the Eulerian Bratteli diagram. The former is well-known and related to binomial coefficients. The latter, which is the main object of the thesis, is related to the Eulerian numbers. Bratteli diagrams were introduced in 1972 by Ola Bratteli in his study of approximately finite dimensional (AF) C*-algebras. In 1976, George Arthur Elliott associated to an AF C*-algebra or to a corresponding Bratteli diagram an ordered group, he called dimension group.

In the first part of the thesis we study the space of infinite paths of the Eulerian diagram, and we realize it as a projective limit of finite permutation groups. In the second part, we study the state space of the dimension group associated to the Eulerian Bratteli diagram. It is a compact convex set and we describe its extremal points. Finally, we use this description to give a necessary and sufficient condition for an element of this dimension group to be positive.

Dedications

À meu pai, Beto, que para sempre será uma fonte de inspiração.

Acknowledgement

Primeiramente gostaria de agradecer minha família, especialmente meu pai e minha mãe, Beto e Toda, por todo o apoio. Amo vocês!

Secondly, I would like to thank all the Professors that made this possible, namely: Thierry Giordano and David Handelman for the supervision and the assistance in my hardest challenges; Vladimir Pestov for supporting me in my career and to help me settle in Canada; and Charles Starling, James Mingo, Vadim Kaimanovich and Aaron Tikuisis for accepting the duty of being examiners of my thesis, for the careful reading, and to give relevant feedback to help me improve it.

Next I would like to thank all friends, colleagues and staff members that I met during the course of all these years. It's been a pleasure to share this part of my life with you all. Thank you very much!

Last but not least, I thank Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), for granting my studies with the scholarship Ciência sem Fronteiras.

Contents

List of Figures

Preface

In 1972, Ola Bratteli [5] introduced a convenient notation, using graphs, to describe inductive limits of ascending sequences of finite dimensional C*-algebras, the closure of these inductive limits are called AF-algebras and these graphs were later named Bratteli diagrams. In 1976, George Elliott [11] associated to a Bratteli diagram an inductive limit of simplicial groups, called the *dimension group*. These introductory concepts are defined on Chapter 1.

In this thesis we are going to focus on a single Bratteli diagram, called the *Eulerian diagram*, also known as the *Euler graph* [12] or the *Eulerian number triangle* [18]. This graph has many properties among different fields, e.g., it is related to the Eulerian numbers, which count the number of permutations of the set $\{1, \ldots, n\}$ with a certain number of rises. We are going to consider the dimension group associated to the Eulerian diagram and describe some of its properties.

In Chapter 2 we consider the *infinite path space* of the Eulerian diagram, i.e., the set of paths that can be formed by connecting edges in the Bratteli diagram across consecutive levels. We prove that this infinite path space is homeomorphic to the Cantor set (Section 2.1) and that there are bijections between the finite paths of length n in the Eulerian diagram and permutations of $\{1, \ldots, n\}$ (Section 2.3). One of the possible bijections is viewing each path arriving at a vertex as a permutation of $\{1, \ldots, n\}$ with a fixed number of rises. This representation of paths is useful for many combinatorial results in the Eulerian diagram and they are introduced in Section 2.5. Later on this chapter (Section 2.8) we define an action and a measure (called *symmetric*) on the path space and prove some topological properties (namely minimality and ergodicity).

In Chapter 3 we turn our attention to the elements of the dimension group arising from the Eulerian diagram. Dimension groups are inductive limits of simplicial groups. We refer to elements of a simplicial group as *distributions* and we see how these distribution behave under the connecting maps of the dimension group. The goal of this chapter is to prove that the Eulerian distributions (arising from the Eulerian Bratteli diagram) are *unimodal*, in other words, they have a “single peak”. We will also find the positions where the distributions attain their maximum value.

Chapter 4 started as a tentative proof of the approximate transitivity of the adic action in the path space of the Eulerian Bratteli diagram based on the proof of the same property for the Pascal triangle given by Thierry Giordano, Mitchell Baker and Radu Munteanu [3]. Although a proof could not be given, the process of adapting the proof gave rise to a combinatorial result that is interesting *per se*. It states that given two vertices v_1 and v_2 on a fixed level n (not necessarily adjacent), the difference between the their distributions on future levels is asymptotically less than twice the central Eulerian numbers.

Finally, in Chapter 5, we give a general criteria to identify an element of the positive cone in the dimension group arising from the Eulerian Bratteli diagram. In 1976, David Handelman and Kenneth R. Goodearl [19] proved that if a dimension group is simple then its positive elements are fully determined by its *state space* (or *traces*). Although the Eulerian Bratteli diagram does not give rise to a simple dimension group, its positive elements can also be fully described by its state space (Theorem 5.6.8).

Chapter 1

Introduction

In this chapter, we define Bratteli diagrams and discuss an application in the study of dimension groups. We aim to focus on the particular example of the Eulerian Bratteli diagram, which is the one we develop the results for.

1.1 Inductive limits and dimension groups

This section is based on [10]. Consider a sequence of sets and maps

$$X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \cdots .$$

If $n \leq m$ let $\varphi_{n,m} : X_n \to X_m$ be the function $\varphi_{n,m} := \varphi_{m-1} \circ \cdots \circ \varphi_n$ for $n < m$ and $\varphi_{n,n} = \mathrm{id}$. Let

$$\bigsqcup X_n := \{(x, n) \mid x \in X_n,\ n \in \mathbb{N}\}$$

and define R as the equivalence relation on $\sqcup X_n$ given by

$$(x, n)R(y, m) \Leftrightarrow \exists N :\ \varphi_{n,N}(x) = \varphi_{m,N}(y).$$

Let

$$X_\infty := \sqcup X_n / R$$

denoting by $[x, n]$ the equivalence class of (x, n). Also let

$$\begin{array}{rcl} \varphi_{n\infty} : X_n & \to & X_\infty \\ x & \mapsto & [x, n]. \end{array}$$

We call X_∞ the *direct limit* of $X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \cdots$ and denote by $\varinjlim(X_n, \varphi_n)$.

Definition 1.1.1. *An* ordered group *is a pair (G, G^+) where G is an abelian group and G^+ (named* positive cone*) satisfies the following conditions:*

- *$G^+ + G^+ \subseteq G^+$,*
- *$G^+ - G^+ = G$,*
- *$G^+ \cap (-G^+) = \{0\}$,*

We shall write $a \leq b$ (respectively, $a < b$) if $b - a \in G^+$ (respectively, $b - a \in G^+ \setminus \{0\}$).

Definition 1.1.2. *A* dimension group *is an ordered group G that is isomorphic to the direct limit given by sequences of the form*

$$\mathbb{Z}^{r(1)} \xrightarrow{\varphi_1} \mathbb{Z}^{r(2)} \xrightarrow{\varphi_2} \cdots, \qquad \varphi_n = [\kappa_{ij}(n)], \qquad \kappa_{ij}(n) \in \mathbb{Z}^+$$

where $\varphi_n : \mathbb{Z}^{r(n)} \to \mathbb{Z}^{r(n+1)}$ are homomorphisms given by ordinary matrix multiplication (these matrices are called transition matrices *or* adjacency matrices*).*

1.2 Bratteli diagrams

The following definition can be found in [15, Section 19.3.1].

Definition 1.2.1. *A* Bratteli diagram *is a pair (V, E) such that*

- *$V = \cup_{n\geq 1} V_n$ where $V_n \cap V_m = \emptyset$ if $n \neq m$;*
- *$E = \cup_{n\geq 1} E_n$ where $E_n \cap E_m = \emptyset$ if $n \neq m$;*
- *There exist maps $s : E_n \to V_n$ and $r : E_n \to V_{n+1}$ such that $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_1$.*

We refer to V as the set of vertices*, E as the set of* edges*, s as the* source *and r as the* range*.*

Bratteli diagrams were introduced by Ola Bratteli in [5, Page 13]. In that paper, he denotes a vertex in V_n as an ordered pair (n, k) where $k \in \{1, \ldots, \#V_n\}$. Moreover, he considers a sequence of relations $\langle \searrow^p \rangle_{p\geq 0}$ satisfying the following axioms:

- If $(n, k) \in V_n$, $(m, q) \in V_m$ and $m = n + 1$, there exists one and only one non negative integer p such that $(n, k) \searrow^p (m, q)$.
- If $m \neq n + 1$ no such integer exists.
- If $(n, k) \in V_n$, there exists $q \in \{1, \ldots, \#V_{n+1}\}$ and $p \geq 1$ such that $(n, k) \searrow^p (n + 1, q)$.

- If $(n,k) \in V_n$ and $n > 1$, there exists $q \in \{1, \dots, \#V_{n-1}\}$ and $p \geq 1$ such that $(n-1,q) \searrow^p (n,k)$.

The relation $(n,k) \searrow^p (n+1,q)$ means that there exist p edges $e_1, \dots, e_p \in E_n$ such that $s(e_i) = (n,k)$ and $r(e_i) = (n+1,q)$ for all $i \in \{1, \dots, p\}$.

Notice that these two definitions are equivalent. In fact, for $v = (n,k) \in V_n$, we have:

- $s^{-1}(v) \neq \emptyset$ if and only if there exist $q \in \{1, \dots, \#V_{n-1}\}$ and $p \geq 1$ such that $(n-1,q) \searrow^p (n,k)$.
- $r^{-1}(v) \neq \emptyset$ if and only if there exist $q \in \{1, \dots, \#V_{n+1}\}$ and $p \geq 1$ such that $(n,k) \searrow^p (n+1,q)$.

If $\mathcal{V}_n := \#V_n$, then the edge set E_n is described by a $(\mathcal{V}_{n+1} \times \mathcal{V}_n)$-matrix $\varphi_n = [\kappa_{ij}(n)]$ where $\kappa_{ij}(n) \in \mathbb{Z}^+$ is the number of edges connecting (n,j) with $(n+1,i)$. Conversely, given a dimension group

$$\mathbb{Z}^{\mathcal{V}_1} \xrightarrow{\varphi_1} \mathbb{Z}^{\mathcal{V}_2} \xrightarrow{\varphi_3} \cdots, \qquad \varphi_n = [\kappa_{ij}(n)], \qquad \kappa_{ij}(n) \in \mathbb{Z}^+$$

we construct a Bratteli diagram where each level n consists of $\mathcal{V}_n$ vertices and the number of edges connecting (n,j) to $(n+1,i)$ is $\kappa_{ij}(n)$.

Example: Below we present the Pascal triangle:

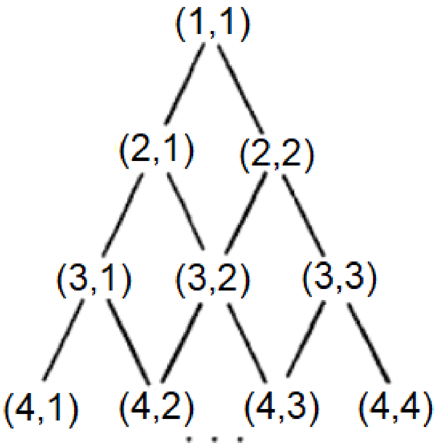

Figure 1.1: The Pascal triangle

- $(n,k) \searrow^1 (m,q) \Leftrightarrow (m = n+1) \wedge ((q = k) \vee (q = k+1))$;
- $(n,k) \searrow^0 (m,q)$ otherwise.

The dimension group associated to the Pascal triangle is

$$\mathbb{Z} \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}1&0\\1&1\\0&1\end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\\0&0&1\end{pmatrix}} \mathbb{Z}^4 \to \cdots \to \mathbb{Z}^n \xrightarrow{\varphi_n} \mathbb{Z}^{n+1} \to \cdots$$

where $\varphi_n = [\kappa_{ij}]_{n\times(n+1)}$ is the adjacency matrix given by

$$\varphi_n = \begin{pmatrix}1&0&0&\cdots&0&0\\1&1&0&\cdots&0&0\\0&1&1&\cdots&0&0\\\vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\0&0&\cdots&1&1&0\\0&0&\cdots&0&1&1\\0&0&\cdots&0&0&1\end{pmatrix} \qquad \kappa_{ij} = \begin{cases}1 & \text{if } i=j \text{ or } i=j+1\\0 & \text{otherwise}\end{cases}$$

Consider the element of the dimension group $[1,1] \in \varinjlim(\mathbb{Z}^n, \varphi_n)$ and observe below the image of $1 \in \mathbb{Z}$ under the maps φ_n:

$$1 \xrightarrow{\varphi_1} (1,1) \xrightarrow{\varphi_2} (1,2,1) \xrightarrow{\varphi_3} (1,3,3,1) \to \cdots \to \left(\binom{n}{0},\ldots,\binom{n}{n}\right) \to \cdots$$

These are the binomial numbers and they are often arranged in the following way:

$$\begin{array}{ccccccccccccc}
&&&&&&1&&&&&&\\
&&&&&1&&1&&&&&\\
&&&&1&&2&&1&&&&\\
&&&1&&3&&3&&1&&&\\
&&1&&4&&6&&4&&1&&\\
&1&&5&&10&&10&&5&&1&\\
1&&6&&15&&20&&15&&6&&1
\end{array}$$

Note that $\binom{n}{k}$ is the number of paths from the apex vertex to (n,k).

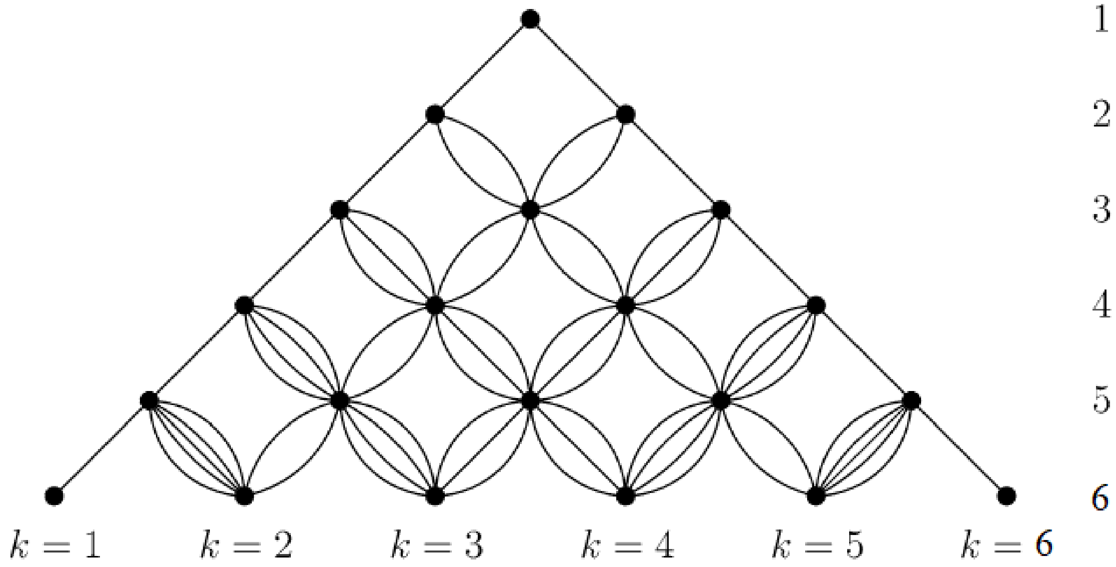

Figure 1.2: The Eulerian Bratteli diagram

1.2.1 The Eulerian Bratteli diagram

The *Eulerian Bratteli diagram*, also known as the *Euler graph* [12] or the *Eulerian number triangle* [18], is the graph with vertex sets given by

$$V = \{(n,k) \mid n \geq 1,\ k \in \{1,\dots,n\}\}$$

and the following rules for the edges:

- There are k edges from vertex (n,k) to vertex $(n+1,k)$;
- There are $n-k+1$ edges from (n,k) to $(n+1,k+1)$.

In terms of the original Bratelli diagram definition we have

- $(n,k) \searrow^{k} (m,q) \Leftrightarrow (m = n+1) \wedge (q = k)$;
- $(n,k) \searrow^{n-k+1} (m,q) \Leftrightarrow (m = n+1) \wedge (q = k+1)$;
- $(n,k) \searrow^{0} (m,q)$ otherwise.

The dimension group associated to the Eulerian Bratteli diagram is

$$\mathbb{Z} \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}1&0\\2&2\\0&1\end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix}1&0&0\\3&2&0\\0&2&3\\0&0&1\end{pmatrix}} \mathbb{Z}^4 \to \cdots \to \mathbb{Z}^n \xrightarrow{\varphi_n} \mathbb{Z}^{n+1} \to \cdots$$

where $\varphi_n = [\kappa_{ij}]$ is the adjacency matrix given by

$$\varphi_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ n & 2 & 0 & \cdots & 0 & 0 \\ 0 & n-1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 3 & n-1 & 0 \\ 0 & 0 & \cdots & 0 & 2 & n \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \qquad \kappa_{ij} = \begin{cases} j & \text{if } i = j \\ n-j+1 & \text{if } i = j+1 \\ 0 & \text{otherwise.} \end{cases}$$

If we consider the element $[1,1] \in \varinjlim(\mathbb{Z}^n, \varphi_n)$ and its image all the subsequent levels, we have:

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 4 \quad 1 \\ 1 \quad 11 \quad 11 \quad 1 \\ 1 \quad 26 \quad 66 \quad 26 \quad 1 \\ 1 \quad 57 \quad 302 \quad 302 \quad 57 \quad 1 \\ 1 \quad 120 \quad 1191 \quad 2416 \quad 1191 \quad 120 \quad 1 \end{array}$$

These are called the *Eulerian numbers* (thus the name of the diagram) and will be detailed in section 2.5.

1.2.2 Generalized Eulerian Bratteli diagram

The adjacency matrices for the Pascal triangle and the Eulerian diagram have only two nonzero diagonals. In fact, one can generalize these dimension groups to one of the form

$$\mathbb{Z} \xrightarrow{\varphi_1} \mathbb{Z}^2 \xrightarrow{\varphi_2} \mathbb{Z}^3 \to \cdots \to \mathbb{Z}^n \xrightarrow{\varphi_n} \mathbb{Z}^{n+1} \to \cdots$$

where φ_n is the *generalized transition matrix*:

$$\varphi_n = \begin{pmatrix} l(n,1) & 0 & 0 & \cdots & 0 & 0 \\ r(n,1) & l(n,2) & 0 & \cdots & 0 & 0 \\ 0 & r(n,2) & l(n,3) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r(n,n-2) & l(n,n-1) & 0 \\ 0 & 0 & \cdots & 0 & r(n,n-1) & l(n,n) \\ 0 & 0 & \cdots & 0 & 0 & r(n,n) \end{pmatrix}, \qquad l(n,k), r(n,k) \geq 1.$$

This means that every vertex (n,k) will have $l(n,k)$ edges connecting to $(n{+}1,k)$ (to the left) and $r(n,k)$ edges connecting to $(n{+}1,k{+}1)$ (to the right). The dimension group associated to the Pascal triangle is obtained when $l(n,k)=r(n,k)=1$ and the dimension group associated to the Eulerian diagram is obtained when $l(n,k)=k$ and $r(n,k)=n-k+1$. Whenever possible, we will prove results in this general form.

Lemma 1.2.2. *If φ_n is the generalized transition matrix and $x=(v_1,\ldots,v_n)\in\mathbb{Z}^n$ then $\varphi_n(x)=(w_1,\ldots,w_{n+1})$ where*

$$w_k = r(n,k-1)\cdot v_{k-1} + l(n,k)\cdot v_k.$$

Proof: Since $\varphi_n=[\kappa_{ij}]$ where $\kappa_{ij}=\begin{cases} l(n,j) & \text{if } i=j; \\ r(n,j) & \text{if } i=j+1; \\ 0 & \text{otherwise,}\end{cases}$ then

$$w_k=\sum_{j=1}^{n}\kappa_{kj}v_k=\kappa_{k,k-1}v_{k-1}+\kappa_{kk}v_k=r(n,k-1)v_{k-1}+l(n,k)v_k.$$

∎

Chapter 2

The path space and Eulerian numbers

In this chapter we will consider the space of infinite edge paths in the Eulerian diagram that we will describe in two different ways: one reading forwards (as an infinite set product) and another reading backwards (as an inverse limit of permutations). We refer to these as *path space* and we will then associate the path space in the Eulerian diagram with permutations with some properties (namely, the number of rises and falls).

2.1 The path space of a Bratteli Diagram

In this section, every Bratteli diagram $B = (V, E)$ will have a single vertex at level 1, i.e., V_1 is a singleton.

Definition 2.1.1. *Let $B = (V, E)$ be a Bratteli diagram The set*

$$X_B := \{(e_n)_{n\geq 1} \in \prod E_n \mid r(e_n) = s(e_{n+1}) \ \forall n \geq 1\}$$

is called the set of infinite edge paths *of the Bratteli diagram B.*

We consider on $\prod_{n\geq 1} E_n$ the product topology, and we endow $X_B \subset \prod_{n\geq 1} E_n$ with the relative topology. Being closed in $\prod_{n\geq 1} E_n$, X_B is compact. To any finite path $(e_1, e_2, \dots, e_n)$ starting at V_1, we associate the cylinder set

$$C(e_1 \dots e_n) := \{x \in X_B \mid x_i = e_i \ \forall i \in \{1, \dots, n\}\}.$$

Lemma 2.1.2. *With the above notation, the cylinder sets are clopen, and form a countale basis for the topology on X_B, that we denote by $\mathcal{T}$.*

Proof: First we prove that cylinder sets are open. Let $C := C(e_1, e_2, \ldots, e_n)$ be a cylinder set. We can write

$$C = \{e_1\} \times \{e_2\} \times \cdots \times \{e_n\} \times E_{n+1} \times E_{n+2} \times \cdots ,$$

which is an open set in the product topology of $\prod_{n\geq 1} E_n$. Since $C = C \cap X_B$, then C is also open in the relative topology of X_B.

Next, we prove that cylinder sets are closed. Let $C := C(e_1e_2 \cdots e_n)$ be a cylinder set. An element in C is a path that begins with $e_1e_2 \cdots e_n$. So every element in C does **not** belong to the cylinder set $C(e_1e_2 \cdots e)$ where $e \neq e_i$ for some $i \leq n$. Since there are only a finite number of such e, then $D := \bigcup_{e\neq e_i} C(e_1e_2 \cdots e)$ is a union of open sets (thus it is open). We now prove that $D^c = C$. By De Morgan's laws: $D^c = \bigcap_{e=e_i} C(e_1e_2 \cdots e)$. Since $C(e_1) \supset C(e_1e_2) \supset \cdots \supset C(e_1e_2 \cdots e_n)$ then $D^c = C(e_1e_2 \cdots e_n) = C$. So C is the complement of an open set, thus it is closed.

Now we prove that the collection of cylinder sets form a basis for the topology. Clearly, the collection of cylinder sets cover X_B. We just need to prove that if C_1 and C_2 are two cylinder sets, then for every $x \in I := C_1 \cap C_2$, there exists a cylinder set $C \subset I$ such that $x \in C$. Let $C_1 := C(e_1, \ldots, e_n)$ and $C_2 := C(f_1, \ldots, f_m)$ and suppose, without loss of generality, that $n \leq m$. If $e_i = f_i$ for every $i \in \{1, \ldots, n\}$ then $I = C_1$ and every $x \in I$ will have the property $x \in C_1 \subset I$. If $e_i \neq f_i$ for some $i \in \{1, \ldots, n\}$ then consider $k = \min\{i \mid e_i \neq f_i\}$. If $k = 1$ then $I = \emptyset$, so by vacuity every $x \in I$ have the desired property. If $k > 1$ then $I = C(e_1, \ldots, e_{k-1})$, so for $C = I$ we have $x \in C \subset I$. ∎

Recall that a Bratteli diagram $B = (V, E)$ is *simple* if there exists a telescoping (V', E') of (V, E) so that the transition matrices of (V', E') have only non-zero entries at each level.

Proposition 2.1.3. *Let $B = (V, E)$ be a Bratteli diagram and X_B be its path space. Then X_B is a compact, zero-dimensional metric space. Moreover, if $B = (V, E)$ is simple, then X_B is a Cantor set.*

Proof: Define the following metric in the path space: for $x = (e_1, \ldots, e_{n-1}, e_n, \ldots)$ and $y = (e_1, \ldots, e_{n-1}, f_n, \ldots)$ with $e_n \neq f_n$, set $d(x, y) := \frac{1}{n}$.

Let E_n be the set of edges that connect V_n to V_{n+1}. By definition of a Bratteli diagram, E_n is a finite set for every $n \geq 1$. Thus $\prod E_n$ is compact by Tychonoff's theorem. Since X_B is a closed subset of the compact set $\prod E_n$, X_B is compact.

By Lemma 2.1.2, the collection of cylinder sets form a countale basis for the topology on X_B consisting of clopen sets, thus X_B is zero-dimensional.

If a Bratteli diagrams is simple then its path space has no isolated points [16, Page 71], thus it is homeomorphic to the Cantor set. ∎

2.2 Tail-equivalence

Definition 2.2.1. *Let* B *be a Bratteli diagram and* X_B *be its path space. For each* $N \geq 1$*, let* R_N *be the finite equivalent relation on* X_B *given by*

$$R_N := \{(e, f) \in X_B \times X_B \mid \forall n \geq N,\ e_n = f_n\}.$$

Then $R := \cup_{n \geq 1} R_N$ *is the tail-equivalence on* X_B*, i.e.,*

$$R = \{(e, f) \in X_B \times X_B \mid \exists N \geq 1,\ \forall n > N,\ e_n = f_n\}.$$

We now consider partial orders on the edge set E of a Bratteli diagram.

Definition 2.2.2. *Let* (V, E) *be a Bratteli diagram. A* source-ordered Bratteli diagram $(V, E, \leq_s)$ *is a Bratteli diagram* (V, E) *together with a partial order* $\leq_s$ *on* E *so that two edges* $e, f \in E$ *are comparable if and only if* $s(e) = s(e')$*; in other words, we have a linear order on* $s^{-1}(v)$*, for all* $v \in V$*.*

Remark 2.2.3. Ordered Bratteli diagrams were introduced in 1992 by Herman, Putnam and Skau, and have been very important in Cantor dynamics [23] and in the study of the topological orbit equivalence of Cantor minimal systems (see [15] for a survey). In Cantor dynamics the studied ordered Bratteli diagrams are range-ordered, i.e., $e, f \in E$ are comparable if and only if $r(e) = r(f)$.

Bratteli diagram with a range and source-partial orders on E were introduced in [25, Definition 4.2].

Definition 2.2.4. *Let* $B = (V, E, \leq_s)$ *be a source-ordered Bratteli diagram. We denote, for each* $v \in V$*, the bijection*

$$\lambda_v : s^{-1}(v) \to \{1, \ldots, \#s^{-1}(v)\} \qquad (\textit{or } \{0, \ldots, \#s^{-1}(v) - 1\})$$

given by $\lambda_v(e) \leq \lambda_v(f)$ *if and only if* $e \leq_s f$*.*

For $e \in s^{-1}(v)$*, we call* $\lambda_v(e)$ *the label of the edge* e*.*

Example 2.2.5. On both the Pascal triangle and the Eulerian Bratteli diagram we consider the left to right (or lexicographic) source-partial order, and the corresponding *left-to-right* labellings. For any vertex $(n, k) \in V_n$ of the Pascal triangle, we set $\lambda_{(n,k)}(e) = 0$ if $r(e) = (n + 1, k)$ and $\lambda_{(n,k)}(f) = 1$ if $r(f) = (n + 1, k + 1)$. This labelling is illustrated in Figure 2.1.

In the Eulerian Bratteli diagram, $s^{-1}(n, k)$ have $n+1$ elements where k of them have range $(n+1, k)$, and $n-k+1$ have range $(n+1, k+1)$. We set $\lambda_{(n,k)}(e) = 1, \ldots, k$ for the k edges whose range is $(n+1, k)$, and $\lambda_{(n,k)}(e) = k+1, \ldots, n+1$ for the $n-k+1$ edges whose range is $(n + 1, k + 1)$. This labelling is illustrated in Figure 2.2.

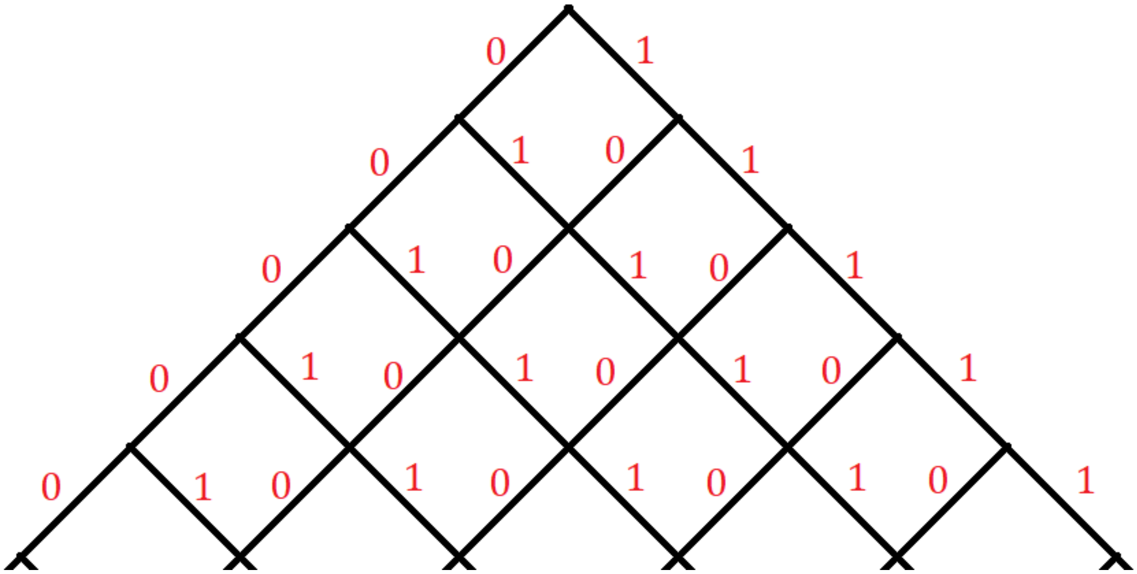

Figure 2.1: A path in the Pascal triangle is a string of 0's and 1's where 0 is a "left" turn and 1 is a "right" turn.

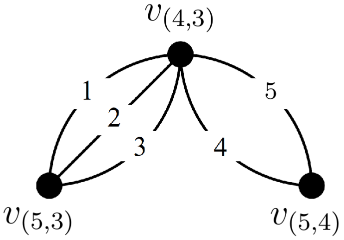

Figure 2.2: Labels in the Eulerian Bratteli diagram

Recall that the transition matrix φ_n of a Bratteli diagram (V, E) is the $(\#V_{n+1} \times \#V_n)$-matrix $\varphi_n = [\kappa_{ij}(n)]$ where $\kappa_{ij}(n)$ is the number of paths connecting vertex $(n, j) \in V_n$ to $(n+1, i) \in V_{n+1}$.

Definition 2.2.6. *A Bratteli diagram (V, E) has the* equal column sums *(ECS) property if there exists a sequence $(\mathcal{E}_n)_{n\geq 1}$ of positive integers such that for every $n \geq 1$ and $j \in \{1, \ldots, \#V_n\}$,*

$$\sum_{i=1}^{\#V_{n+1}} \kappa_{ij}(n) = \mathcal{E}_n.$$

In other words, the sum of the elements of each column of φ_n are equal.

Since the entries $\kappa_{ij}(n)$ of the transition matrix φ_n indicate the number of edges from the vertex (n,j) to the vertex $(n+1,i)$, the column sum $\mathcal{E}_n := \sum_i \kappa_{ij}(n)$ is the number of edges whose source is the vertex (n,j). Thus if (V,E) is an ECS Bratteli diagram, for each $n \geq 1$ and all $v \in V_n$, $\#s^{-1}(v) = \mathcal{E}_n$. Note that both the Pascal diagram (with $\mathcal{E}_n = 2$ for all $n \geq 1$) and the Eulerian Bratteli diagram (with $\mathcal{E}_n = n+1$ for all $n \geq 1$) are ECS Bratteli diagrams.

Definition 2.2.7. *Let $\mathcal{E} = (\mathcal{E}_n)_{n\geq 1}$ be a sequence of positive integers, $\mathcal{E}_n \geq 2$ for all $n \geq 1$. We denote by:*

- *UHF($\mathcal{E}$) the Bratteli diagram (V,E) such that $\#V_n = 1$ for all $n \geq 0$ and $\#E_n = \mathcal{E}_n$ for $n \geq 1$;*
- *$X_{\mathcal{E}}$ the path space of UHF($\mathcal{E}$).*

Remark 2.2.8. Let UFH($\mathcal{E}$) be a Bratteli diagram, then:

- The path space $X_{\mathcal{E}}$ is naturally homeomorphic to the product space $\prod_{n\geq 1}\{1,2,\dots,\mathcal{E}_n\}$, that we will also denote by $\overrightarrow{X}_{\mathcal{E}}$.
- Any linear order on $\mathcal{E}_n$ defines both a source and a range-order on $X_{\mathcal{E}}$.

Proposition 2.2.9. *Let $B = (V,E)$ be an ECS Bratteli diagram and $\mathcal{E} = (\mathcal{E}_n)_{n\geq 1}$ be the sequence of positive integers such that*

$$\forall n \geq 0,\ \forall v \in V_n, \qquad \#s^{-1}(v) = \mathcal{E}_n.$$

Let $\leq_s$ be a source-order on B, and let λ_v, $v \in V$, be its corresponding labelling functions. Then the map $\lambda : X_B \to X_{\mathcal{E}}$ given by

$$\lambda(e) = \lambda((e_n)_{n\geq 1}) = (\lambda_{s(e_n)}(e_n))_{n\geq 1}$$

is a homeomorphism.

Proof: Since X_B and $X_{\mathcal{E}}$ are compact, it is sufficient to prove that λ is a continuous bijection.

We start proving that it is surjective. let $x = (x_1, x_2, \dots) \in X_{\mathcal{E}}$, then $x_i \in \{1,\dots,\mathcal{E}_i\}$ for all $i \geq 1$. Since the map $\lambda_{(1,1)} : s^{-1}(1,1) \to \{1,\dots,\mathcal{E}_1\}$ is a bijection, there exists $e_1 \in s^{-1}(1,1) = E_1$ such that $\lambda_{(1,1)}(e_1) = x_1$.

Since $r(e_1) \in V_2$, there exists $1 \leq k \leq \#V_2$ such that $r(e_1) = (2,k)$. Since $\lambda_{(2,k)} : s^{-1}(2,k) \to \{1,\dots,\mathcal{E}_2\}$ is a bijection, there exists $e_2 \in s^{-1}(2,k) \subset E_2$ such that $\lambda_{(2,k)}(e_2) = x_2$. By construction, $r(e_1) = (2,k) = s(e_2)$.

By induction, for every $i \geq 1$, there exists $e_i \in E_i$ such that $r(e_{i-1}) = s(e_i)$ and $\lambda_{s(e_i)}(e_i) = x_i$. Let $(e_1, e_2, \dots)$ be the infinite edge path in X_B constructed from the edges $e_i \in E_i$ described above, then $\lambda(e_1, e_2, \dots) = (x_1, x_2, \dots)$. Therefore λ is surjective.

Next, we prove that λ is continuous. Given a cylinder set $C(x_1 \ldots x_n) \subset X_{\mathcal{E}}$, we can use the technique from the previous paragraphs to construct edges $e_i \in E_i$ such that $r(e_{i-1}) = s(e_i)$ and $\lambda_{s(e_i)}(e_i) = x_i$. The finite path $(e_1, \ldots, e_n)$ is such that $C(e_1 \ldots e_n) \subset C(x_1 \ldots x_n)$. Thus λ is continuous.

Finally, we prove that λ is injective. Let $e, f \in X_B$ such that $\lambda(e) = \lambda(f)$, then $\lambda_{s(e_1)}(e_1) = \lambda_{s(f_1)}(f_1)$. Since $e_1, f_1 \in E_1$, then $s(e_1) = s(f_1) = v_{(1,1)}$. So $\lambda_{(1,1)}(e_1) = \lambda_{(1,1)}(f_1)$. Since $\lambda_{(1,1)}$ is a bijection, we have $e_1 = f_1$.

As $e_1 = f_1$, their range are equal: $r(e_1) = r(f_1)$. Since $r(e_1) = s(e_2)$ and $r(f_1) = s(f_2)$, we have that $s(e_2) = s(f_2)$. Let $1 \leq k \leq \#V_2$ be such that $s(e_2) = s(f_2) = (2, k)$, then $\lambda_{s(e_2)}(e_2) = \lambda_{(2,k)}(e_2)$ and $\lambda_{s(f_2)}(f_2) = \lambda_{(2,k)}(f_2)$. Again, from the assumption that $\lambda(e) = \lambda(f)$, we have that $\lambda_{s(e_2)}(e_2) = \lambda_{s(f_2)}(f_2)$. It follows that $\lambda_{(2,k)}(e_2) = \lambda_{(2,k)}(f_2)$. Since $\lambda_{(2,k)}$ is bijective (thus injective), $e_2 = f_2$.

By induction, we have that $e_n = f_n$ for all $n \geq 1$. Therefore, $e = f$ and the function is injective. ∎

Lemma 2.2.10. *Let $(B, \leq_s)$ be a source-ordered ECS Bratteli diagram, UHF($\mathcal{E}$) be the associated UHF Bratteli diagram, and $\lambda : X_B \to X_{\mathbb{E}}$ be the homeomorphism defined in Proposition 2.2.9. If R_B (respectively $R_{\mathcal{E}}$) denote the tail-equivalence relation on X_B (respectively $X_{\mathcal{E}}$), then*

$$(\lambda \times \lambda)(R_B) \subset R_{\mathcal{E}}.$$

Proof: If $(e, f) \in R_B$ then there exists $N \in \mathbb{N}$, such that $\forall n > N$, $e_n = f_n$. Hence, for $n > N$, we have $r(e_n) = r(f_n)$. Since $r(e_n) = s(e_{n+1})$ and $r(f_n) = s(f_{n+1})$, then $s(e_{n+1}) = s(f_{n+1})$ for $n > N$. Thus $\lambda_{s(e_{n+1})}(e_{n+1}) = \lambda_{s(e_{n+1})}(f_{n+1})$ for all $n > N$. Therefore $(\lambda(e), \lambda(f)) \in R_{\mathcal{E}}$. ∎

We keep the notation introduced in Proposition 2.2.9 and Lemma 2.2.10. Let $P = P_\lambda$ be the equivalence relation on X_B given for $e, f \in X_B$ by

$$(e, f) \in P \Leftrightarrow (\lambda(e), \lambda(f)) \in R_{\mathcal{E}}. \tag{2.2.1}$$

By Lemma 2.2.10, R_B is a subequivalence relation of P_λ. In the case of the Pascal triangle Bratteli diagram, the following example shows that R_B is a strict subequivalence relation of P_λ.

Example 2.2.11. Consider the Pascal triangle as in Example 2.2.5. Let $e := (0, 0, 0, \ldots)$ and $f := (1, 0, 0, 0, \ldots)$ be two labelled paths in $X_{\mathcal{E}}$. Since $e_n = f_n = 0$ for all $n > 1$, then $(e, f) \in R_{\mathcal{E}}$. However, it is not true that $(e, f) \in R_B$ (as Figure 2.3 illustrates).

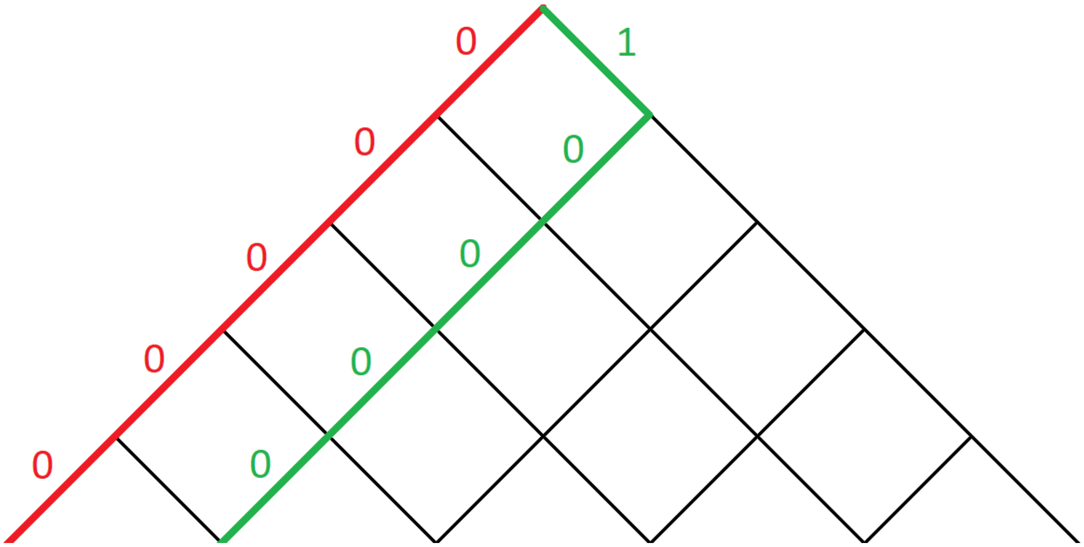

Figure 2.3: Paths $000\cdots$ and $1000\cdots$ in the Pascal triangle are parallel

In the rest of this section, $(B, \leq_s)$ denotes either the Pascal triangle or the Eulerian Bratteli diagram with the left-right source-ordering. We then construct a homomorphism $\varphi_\lambda : P_\lambda \to \mathbb{Z}$ such that the tail-equivalence relation R_B is equal to the kernel of φ_λ. To define the homomorphism φ_λ, we first need to introduce the following notation.

Definition 2.2.12. *Let $B = (V, E)$ be a Bratteli diagram and X_B be its path space. For each $n \geq 1$, set $\mathcal{V}_n = \#V_n$. Ordering V_n from left to right, we have*

$$V_n = \{(n, 1), (n, 2), \ldots, (n, \mathcal{V}_n)\},$$

which we identify with $\{1, 2, \ldots, \mathcal{V}_n\}$. Then let $k : X_B \to \prod_{n\geq 1}\{1, 2, \ldots, \mathcal{V}_n\}$ be the map defined by $k(e) = (k_n(e))_{n\geq 1}$ where for $e = (e_n)_{n\geq 1}$, $k_n(e) = k$ if and only if $s(e_n) = (n, k)$.

Remark 2.2.13. Let $B = (V, E)$ be a Bratteli diagram and X_B its path space. If V_1 is a singleton, then $V_1 = \{(1, 1)\}$, so $k_1(e) = 1$ for all infinite edge paths $e \in X_B$. Thus, for $n \geq 1$, we have that $k_{n+1}(e) = k$ if and only if $s(e_{n+1}) = (n + 1, k) = r(e_n)$.

Example 2.2.14. Let B be the Pascal triangle from Example 2.2.11 and X_B its path space. If an infinite edge path $e = (e_i)_{i\geq 1} \in X_B$ is such that $s(e_n) = (n, k)$, then

$$k_{n+1}(e) = \begin{cases} k & \text{if } \lambda_{s(e_n)}(e_n) = 0; \\ k+1 & \text{if } \lambda_{s(e_n)}(e_n) = 1. \end{cases}$$

Thus we can write:

$$k_{n+1}(e) = \begin{cases} k_n(e) & \text{if } \lambda_{s(e_n)}(e_n) = 0; \\ 1 + k_n(e) & \text{if } \lambda_{s(e_n)}(e_n) = 1. \end{cases}$$

Note that $k_n(e) = 1 + \sum_{i<n} \lambda_{s(e_i)}(e_i)$ for all $n > 1$.

Example 2.2.15. Let X_B be the Eulerian Bratteli diagram from Example 2.2.5. If an infinite edge path $e = (e_i)_{i \geq 1} \in X_B$ is such that $s(e_n) = (n, k)$, then

$$k_{n+1}(e) = \begin{cases} k & \text{if } \lambda_{s(e_n)}(e_n) \leq k; \\ k + 1 & \text{if } \lambda_{s(e_n)}(e_n) > k. \end{cases}$$

Thus we can write:

$$k_{n+1}(e) = \begin{cases} k_n(e) & \text{if } \lambda_{s(e_n)}(e_n) \leq k_n(e); \\ 1 + k_n(e) & \text{if } \lambda_{s(e_n)}(e_n) > k_n(e). \end{cases} \tag{2.2.2}$$

Example 2.2.16. Consider a generalized Eulerian Bratteli diagram as described in Section 1.2.2. In these diagrams, each vertex (n, k) contains $l(n, k)$ edges connecting to $(n + 1, k)$, and $r(n, k)$ connecting to $(n + 1, k + 1)$. Thus we consider a labelling

$$\lambda_{(n,k)} : s^{-1}(n, k) \to \{1, \ldots, l(n, k) + r(n, k)\}$$

called *left to right*, where we assign, for $e_n \in s^{-1}(n, k)$, the following values:

$$\lambda_{(n,k)}(e_n) = \begin{cases} 1, \ldots, l(n, k) & \text{if } r(e) = (n + 1, k); \\ l(n, k) + 1, \ldots, l(n, k) + r(n, k) & \text{if } r(e) = (n + 1, k + 1). \end{cases}$$

With this labelling, for an infinite edge path $e = (e_i)_{i \geq 1} \in X_B$ with $s(e_n) = (n, k)$, we have that:

$$k_{n+1}(e) = \begin{cases} k & \text{if } \lambda_{s(e_n)}(e_n) \leq l(n, k), \\ k + 1 & \text{if } \lambda_{s(e_n)}(e_n) > l(n, k); \end{cases}$$

In other words, for $n \geq 1$,

$$k_{n+1}(e) = \begin{cases} k_n(e) & \text{if } \lambda_{s(e_n)}(e_n) \leq l(n, k_n(e)); \\ 1 + k_n(e) & \text{if } \lambda_{s(e_n)}(e_n) > l(n, k_n(e)). \end{cases}$$

Proposition 2.2.17. *Let X_B be the Eulerian Bratteli diagram path space and consider $e = (e_i)_{i \geq 1}, f = (f_i)_{i \geq 1} \in X_B$. If, for $n \geq 1$, $\lambda_{s(e_n)}(e_n) = \lambda_{s(f_n)}(f_n)$ then*

$$|k_{n+1}(e) - k_{n+1}(f)| \leq |k_n(e) - k_n(f)|.$$

Proof: Using formula (2.2.2), we have:

- If $\lambda_{s(e_n)}(e_n) = \lambda_{s(f_n)}(f_n) \leq \min\{k_n(e), k_n(f)\}$ then
$$|k_{n+1}(e) - k_{n+1}(f)| = |k_n(e) - k_n(f)|.$$
- If $\lambda_{s(e_n)}(e_n) = \lambda_{s(f_n)}(f_n) > \max\{k_n(e), k_n(f)\}$ then
$$|k_{n+1}(e) - k_{n+1}(f)| = |k_n(e) + 1 - (k_n(f) + 1)| = |k_n(e) - k_n(f)|.$$
- If $\min\{k_n(e), k_n(f)\} < \lambda_{s(e_n)}(e_n) = \lambda_{s(f_n)}(f_n) \leq \max\{k_n(e), k_n(f)\}$ then we consider two cases: if $k_n(e) < k_n(f)$ then
$$|k_{n+1}(e) - k_{n+1}(f)| = |k_n(e) + 1 - k_n(f)| \leq |k_n(e) - k_n(f)|,$$
while if $k_n(e) > k_n(f)$ we have
$$|k_{n+1}(e) - k_{n+1}(f)| = |k_n(e) - k_n(f) - 1| \leq |k_n(e) - k_n(f)|.$$

In all the possible cases we have the desired property ∎

Proposition 2.2.18. *Let X_B be the Eulerian Bratteli diagram path space and consider $e = (e_i)_{i\geq 1}, f = (f_i)_{i\geq 1} \in X_B$. If, for $n \geq 1$, $\lambda_{s(e_n)}(e_n) = \lambda_{s(f_n)}(f_n)$ then*
$$k_n(e) \leq k_n(f) \Rightarrow k_{n+1}(e) \leq k_{n+1}(f).$$

Proof: Using formula (2.2.2), we have:

- If $\lambda_{s(e_n)}(e_n) = \lambda_{s(f_n)}(f_n) \leq \min\{k_n(e), k_n(f)\}$ then
$$k_{n+1}(e) - k_{n+1}(f) = k_n(e) - k_n(f).$$
- If $\lambda_{s(e_n)}(e_n) = \lambda_{s(f_n)}(f_n) > \max\{k_n(e), k_n(f)\}$ then
$$k_{n+1}(e) - k_{n+1}(f) = k_n(e) + 1 - (k_n(f) + 1) = k_n(e) - k_n(f).$$
- If $\min\{k_n(e), k_n(f)\} < \lambda_{s(e_n)}(e_n) = \lambda_{s(f_n)}(f_n) \leq \max\{k_n(e), k_n(f)\}$ then we consider two cases: if $k_n(e) > k_n(f)$ we have
$$k_{n+1}(e) - k_{n+1}(f) = k_n(e) - k_n(f) - 1 \leq k_n(e) - k_n(f) \leq 0.$$
while if $k_n(e) < k_n(f)$ then $k_n(e) + 1 \leq k_n(f)$, or, $-k_n(f) \leq -(k_n(e) + 1)$. So
$$k_{n+1}(e) - k_{n+1}(f) = k_n(e) + 1 - k_n(f) \leq k_n(e) + 1 - (k_n(e) + 1) = 0,$$

In all the cases we have $k_{n+1}(e) - k_{n+1}(f) \leq 0$, equivalently, $k_{n+1}(e) \leq k_{n+1}(f)$. ∎

We proved that, given two infinite edge paths $e, f \in X_B$ that have cofinal labels, i.e., $\exists N$ such that $\lambda_{s(e_n)}(e_n) = \lambda_{s(f_n)}(f_n)\ \forall n > N$ (in other words, $(e, f) \in P$), then the sequence $(k_n(e) - k_n(f))_{n \geq N} \subset \mathbb{N}$ does not change sign. Combining with Proposition 2.2.17, we have that $(k_n(e) - k_n(f))$ is monotone, bounded by zero, and assumes values in a discrete set. Thus the limit $\lim_{n\to\infty}(k_n(e) - k_n(f))$ exists for $(e, f) \in P$.

Proposition 2.2.19. *Let B be an ordered ECS Bratteli diagram, X_B its path space and consider the equivalence relation P defined in (2.2.1). Then the map $\varphi : P \to \mathbb{Z}$ defined by*

$$\varphi(e, f) := \lim_{n\to\infty}(k_n(e) - k_n(f))$$

is a groupoid homomorphism, i.e., for every $g \in X_B$ such that $(e, g), (g, f) \in P$, we have $\varphi(e, f) = \varphi(e, g) + \varphi(g, f)$.

Proof: Since $k_n(e) - k_n(f) = k_n(e) - k_n(g) + k_n(g) - k_n(f)$ and

$$\begin{aligned} \varphi(e, f) &= \lim_{n\to\infty}(k_n(e) - k_n(f)), \\ \varphi(e, g) &= \lim_{n\to\infty}(k_n(e) - k_n(g)), \\ \varphi(g, f) &= \lim_{n\to\infty}(k_n(g) - k_n(f)); \end{aligned}$$

we have that $\varphi(e, f) = \varphi(e, g) + \varphi(g, f)$. ∎

What follows is the main theorem of this section. It states that two paths e and f are tail-equivalent (i.e. $(e, f) \in R$) if and only if e and f are cofinal in their labels (i.e. $(e, f) \in P$) and $\varphi(e, f) = 0$. More precisely:

Theorem 2.2.20. *Let B be the Eulerian Bratteli diagram and X_B be its infinite edge path space. If $\varphi : P \to \mathbb{Z}$ is the map from Proposition 2.2.19, and R is tail-equivalence on X_B; then $\ker \varphi = R$.*

Proof: Let $(e, f) \in \ker \varphi$, then $(e, f) \in P$ and $\varphi(e, f) = 0$. From $(e, f) \in P$, we have that $\exists N_1$ such that $\lambda_n(e) = \lambda_n(f)$ for all $n > N_1$. This means that e and f have the same label at levels $n > N_1$. From $\varphi(e, f) = 0$, we have that $\exists N_2$ such that $k_n(e) = k_n(f)$ for all $n > N_2$, i.e., $r(e_n) = r(f_n)$. Since $r(e_n) = s(e_{n+1})$ and $r(f_n) = s(f_{n+1})$, then $s(e_{n+1}) = s(f_{n+1})$ for all $n > N_2$.

We have that for every $n > \max\{N_1, N_2\}$, the edges e_n and f_n share the same source and have the same label. So $e_n = f_n$ for every $n > \max\{N_1, N_2\}$. Therefore, $(e, f) \in R$. ∎

For the particular case of the Pascal triangle Bratteli diagram (V, E) with the canonical $\leq_s$ order relation on E (see Example 2.2.5), we have $\lambda \times \lambda(R_B) = S \subset R_{\mathcal{E}}$ where S is the equivalence relation induced by the action of the group of finite permutations of $\mathbb{N}^*$ (see next section its precise definition) on $X_{\mathcal{E}}$ given by:

$$\sigma(x_1, x_2, x_3, \dots) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \dots), \qquad \sigma \in S_\infty, \quad x = (x_n)_{n\geq 1} \in X_{\mathcal{E}}.$$

2.3 Permutations in the Eulerian diagram

Let S_n be the group of bijections of $\{1, \dots, n\}$, i.e.

$$S_n := \{x : \mathbb{N} \to \mathbb{N} \mid x \text{ is bijective and } \forall i > n,\ x_i := x(i) = i\}.$$

Denote by S_∞ the group of finite permutations of $\mathbb{N}^*$, i.e.,

$$S_\infty := \{x : \mathbb{N} \to \mathbb{N} \mid x \text{ is bijective and } \exists n \in \mathbb{N},\ \forall i > n,\ x_i = i\} = \cup S_n.$$

We can write $x \in S_n$ as $x = \begin{pmatrix} 1 & 2 & \cdots & n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$. The number of permutations in S_n is $n!$. In this section, we will prove that the number of paths in the Eulerian diagram arriving at level n is also $n!$, so there exists a bijection between *paths arriving at level* n and S_n.

Proposition 2.3.1. *The number of paths in the Eulerian Bratteli diagram arriving at level n is $n!$.*

Proof: By induction on n. For $n = 2$ we have only two paths arriving, one connecting $(1, 1)$ to $(2, 1)$ and another connecting $(1, 1)$ to $(2, 2)$. Assume that the number of paths arriving at level n is $n!$ and let us prove that the number of paths arriving at level $n+1$ is $(n+1)!$. Since each vertex (n, k) has $n+1$ edges emanating from it, the total number of paths arriving at level $n+1$ is $n! \cdot (n+1) = (n+1)!$. ∎

Since $n!$ is also the number of permutations of the set $\{1, \dots, n\}$ we can associate each path arriving at level n to a permutation of $\{1, \dots, n\}$. For example, we can associate the edge connecting $(1, 1)$ to $(2, 1)$ with the permutation 21 and the edge connecting $(1, 1)$ to $(2, 2)$ with the permutation 12. Extending a permutation in S_n to a permutation in S_{n+1} corresponds to extending a finite path arriving at level n to a path arriving at level $n+1$ (and vice-versa). This extension is described in the next paragraph.

When a finite path C of length n, corresponding to a permutation $\pi(C) \in S_n$, is extended by an edge from level n to level $n+1$, we extend $\pi(C)$ in a unique way to a permutation in $n+1$ as follows: let e be the edge on level $n+1$ that connects $v \in V_n$ to $w \in V_{n+1}$ and order the edges whose source is v. Since there are $n+1$ of them, we have $s^{-1}(v) = \{e_1, \ldots, e_{n+1}\}$. If $e = e_k$ then insert $n+1$ into $\pi(C)$ in the k-th position to create a permutation in S_{n+1}. See Figure 2.4 below for an illustration of the method.

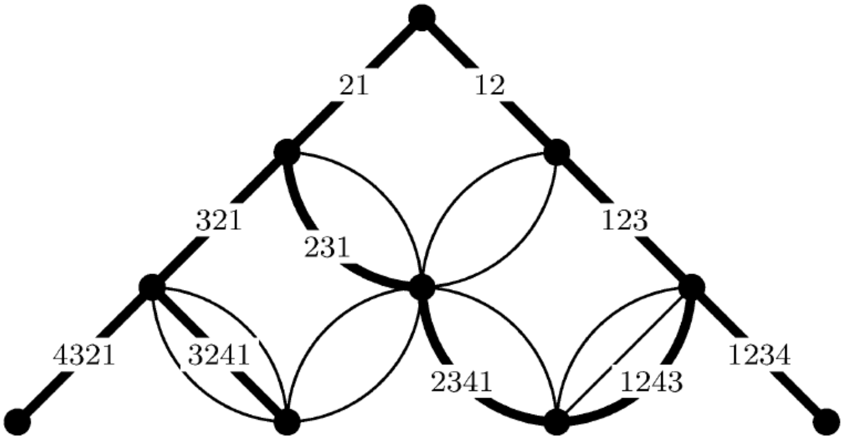

Figure 2.4: Consistency among the path.

For example, continuing from the edge connecting $(1,1)$ to $(2,1)$, represented by the permutation $\pi(C) = 21 \in S_2$, we have 3 possibilities of permutation in S_3: they are 321, 231 and 213. The permutations 321, 231 and 213 differ in the position where we place the digit 3. If we insert the digit 3 on the kth position of the permutation _2_1_, we write $\iota_3^k(21)$, i.e., $321 = \iota_3^1(21)$, $231 = \iota_3^2(21)$ and $213 = \iota_3^3(21)$. In general, if $x = x_1 \ldots x_{n-1} \in S_{n-1}$ then we set $\iota_n^k(x) := x_1 \ldots x_{k-1}, n, x_k \ldots x_{n-1} \in S_n$. Formally, we write, for $k \in \{1, \ldots, n\}$, a map $\iota_n^k : S_{n-1} \to S_n$ where $(\iota_n^k(x))_i = \begin{cases} x_i & i < k; \\ n & i = k; \\ x_{i-1} & i > k. \end{cases}$

Definition 2.3.2. *We refer to ι_n^k as* insert digit n at position k*.*

The maps ι_n^k admit a left inverse, called the *removal of digit* n, defined as follows. Let $p_n : S_n \to S_{n-1}$ and $x = x_1 \cdots x_n \in S_n$ with $x_k = n$ (or $k = x^{-1}(n)$); then $p_n(x) = x_1 \cdots x_{k-1} x_{k+1} \cdots x_n$. More precisely,

$$(p_n(x))_i := \begin{cases} x_i & \text{if } i < x^{-1}(n), \\ x_{i+1} & \text{otherwise.} \end{cases}$$

Since $p_n \circ \iota_n^k = \mathrm{id}$, the maps p_n are surjective and the ι_n^k are injective.

Definition 2.3.3. *Let* $\overleftarrow{X} := \varprojlim(S_n, p_n)$ *be the inverse limit of the finite permutation groups* S_n *with respect to* $p_n : S_n \to S_{n-1}$*. We call this space the* backward path space of the Eulerian Bratteli diagram *or* backward representation*. An element of* $\overleftarrow{X}$ *will be denoted* $(x^{(2)}, x^{(3)}, \dots)$*, where each* $x^{(i)} \in S_i$*.*

Example: In Figure 2.4 we highlight the paths $(21, 321, 4321, \dots)$, $(21, 321, 3241, \dots)$, $(21, 231, 2341, \dots)$, $(12, 123, 1243, \dots)$ and $(12, 123, 1234, \dots)$.

2.4 Duality between the backward and the forward representations

Let $B = (V, E)$ be the Eulerian Bratteli diagram and recall that B is an ECS Bratteli diagram with $\mathcal{E}_n = n + 1$. By Proposition 2.2.9, there is a homeomorphism between X_B an the path space $X_{\mathcal{E}}$ of $\mathrm{UHF}(\mathcal{E}_n)$. On Remark 2.2.8, we noted that $X_{\mathcal{E}}$ is homeomorphic to $\prod_{n\geq 1}\{1, \dots, \mathcal{E}_n\}$. Therefore, the path space of the Eulerian Bratteli diagram is homeomorphic to $\prod_{n\geq 1}\{1, \dots, n+1\}$. We denote by $\overrightarrow{X}$ the set $\prod_{n\geq 1}\{1, \dots, n+1\}$ and call it the *forward representation* of the path space of the Eulerian Bratteli diagram.

There is a natural bijection between $\overrightarrow{X}$ (the forward representation) and $\overleftarrow{X}$ (the backward representation). Given $\overrightarrow{x} = (x_1, x_2, \dots) \in \overrightarrow{X}$, define $f(\overrightarrow{x}) := \overleftarrow{x}$ where $\overleftarrow{x} = (x^{(2)}, x^{(3)}, \dots)$ and

- $x^{(2)} = 21$ if $x_1 = 1$;
- $x^{(2)} = 12$ if $x_1 = 2$;
- $x^{(n)} = \iota_n^k(x^{(n-1)})$ if $x_{n-1} = k$ (recall that ι_n^k is the insertion of digit n at position k).

The inverse of f is the function $f^{-1} : \overleftarrow{X} \to \overrightarrow{X}$ defined as follows. Given an element $\overleftarrow{x} = (x^{(2)}, x^{(3)}, \dots) \in \overleftarrow{X}$, set $f(\overrightarrow{X}) = (x_1, x_2, \dots)$ where $x_n = k$ and k is the position of digit $n+1$ in the permutation $x^{(n+1)} \in S_{n+1}$. In other words, $x_n = (x^{(n+1)})^{-1}(n+1)$.

Example: If $x^{(2)} = 21$ then the position of digit 2 is $(x^{(2)})^{-1} = 1$, thus $x_1 = 1$. Similarly, if $x^{(2)} = 12$ then the position of digit 2 is $(x^{(2)})^{-1} = 2$, thus $x_1 = 2$. Below we write the bijection of some paths highlighted in Figure 2.5:

$$\overrightarrow{X} \;\leftrightarrow\; \overleftarrow{X}$$

$$\begin{aligned}
111\ldots &\leftrightarrow (21, 321, 4321, \ldots) \\
113\ldots &\leftrightarrow (21, 321, 3241, \ldots) \\
123\ldots &\leftrightarrow (21, 231, 2341, \ldots) \\
233\ldots &\leftrightarrow (12, 123, 1243, \ldots) \\
234\ldots &\leftrightarrow (12, 123, 1234, \ldots)
\end{aligned}$$

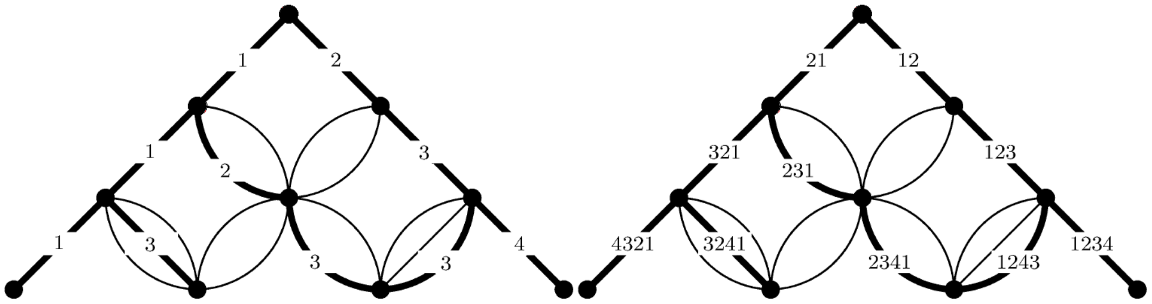

Figure 2.5: Bijection between representations.

We prove that this bijection is a homeomorphism using the following well known result.

Theorem 2.4.1. *If* X *is a compact topological space and* Y *is a Hausdorff topological space then every continuous bijective function* $f : X \to Y$ *is a homeomorphism.*

Since $f : \overrightarrow{X} \to \overleftarrow{X}$ and $\overleftarrow{X}$ is compact, the only thing we need is to define a topology for $\overrightarrow{X}$ that makes it Hausdorff and the function continuous. We endow $\overleftarrow{X} = \underleftarrow{\lim}(S_n, p_n)$ with a topology generated by sets indexed by $\tau \in S_n$ of the form $C(\tau) := \{x \in \overleftarrow{X} \mid x^{(n)} = \tau\}$.

Lemma 2.4.2. $\overleftarrow{X}$ *is Hausdorff.*

Proof: Let $x, y \in \overleftarrow{X}$ such that $x \neq y$, then there exists $n \in \mathbb{N}$ such that $x^{(n)} \neq y^{(n)}$. Consider $U = C(x^{(n)})$ and $V = C(y^{(n)})$. Since U and V are sets of the form $C(\tau)$, $\tau \in S_n$, they are open sets, and by construction, $x \in U \cap V^c$ and $y \in V \cap U^c$. ∎

Lemma 2.4.3. *The function* $f : \overrightarrow{X} \to \overleftarrow{X}$ *defined by* $f(x_1, x_2, \ldots) = (x^{(2)}, x^{(3)}, \ldots)$, *where*

- $x^{(2)} = 21$ *if* $x_1 = 1$,

- $x^{(2)} = 12$ *if* $x_1 = 2$,
- $x^{(n)} = \iota_n^k(x^{(n-1)})$ *if* $x_n = k$;

is continuous.

Proof: We will prove that for any basic open set $U \subset \overleftarrow{X}$, there exists a cylinder set $V \subset \overrightarrow{X}$ such that $f^{-1}(U) = V$. Since the basic sets for the topology of $\overleftarrow{X}$ are of the form $C(\tau)$, $\tau \in S_n$, and the cylinder sets of $\overrightarrow{X}$ are of the form $C(x_1 x_2 \cdots x_n)$ where $x_i \in \{1, \ldots, i+1\}$; then we must prove that for any $\tau \in S_n$, there exists a finite path $(x_1, x_2, \ldots, x_n) \in \prod_i \{1, \ldots, i+1\}$, such that $f^{-1}(C(x_1 x_2 \cdots x_n)) = C(\tau)$.

Let $\tau \in S_n$. Set $x^{(n)} := \tau$ and, for $i = 2, \ldots, n-1$, define $x^{(i)} = p_{i+1}(x^{(i+1)})$. Consider a finite path $(x_1, x_2, \ldots, x_{n-1}) \in \prod_i \{1, \ldots, i+1\}$ given by $x_i = (x^{(i+1)})^{-1}(i+1)$ for $i = 1, \ldots, n-1$. By construction, $f(C(x_1 x_2 \cdots x_{n-1})) = C(\tau)$. ∎

2.5 The rises and falls representation

In this section we introduce a representation for the Eulerian Bratteli diagram that takes into consideration its relation to Eulerian numbers (first mentioned in Section 1.2.1). This section is based on [20, Section 6.2].

Definition 2.5.1. *Let* $x_1 x_2 \ldots x_n \in S_n$. *We say that position* i *is a* rise *or an* ascent *if* $x_i > x_{i-1}$. *We say that position* i *is a* fall *or a* descent *if* $x_i < x_{i-1}$.

Example: The permutation 5472163 has two rises and four falls since

$$5 > 4 < 7 > 2 > 1 < 6 > 3.$$

Every $<$ symbol corresponds to a rise and every $>$ symbol to a fall.

Remark 2.5.2. If a permutation $x_1 \cdots x_n \in S_n$ has k rises, then it necessarily has $n-k-1$ falls. And vice versa: if it has k falls then it has $n-k-1$ rises.

Definition 2.5.3. *The* Eulerian number $A(n,k)$ *(*$n, k \geq 1$*) is the number of permutations of* $\{1, 2, \ldots, n\}$ *with exactly* $k-1$ *rises.*

Let us write the first few Eulerian numbers:

$$\begin{array}{cccccccccc}
 & & & & 1 & & & & & n=1 \\
 & & & 1 & & 1 & & & & n=2 \\
 & & 1 & & 4 & & 1 & & & n=3 \\
 & 1 & & 11 & & 11 & & 1 & & n=4 \\
1 & & 26 & & 66 & & 26 & & 1 & n=5 \\
k=1 & & k=2 & & k=3 & & k=4 & & k=5 &
\end{array}$$

Indeed, there is a general formula for the Eulerian numbers [20, Equation (6.38)]:

$$A(n,k) = \sum_{j=0}^{k-1} (-1)^j \binom{n+1}{j} (k-j)^n,$$

and also a recursive formula ([20, Equation (6.35)] and [29]): if $n, k \in \mathbb{Z}_+^*$ then

$$\begin{cases} A(n,k) = 1 & \text{for } k = 1, n; \\ A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k) & \text{for } 1 < k < n; \\ A(n,k) = 0 & \text{for } k > n. \end{cases}$$

In order to compute $A(n,k)$, we need k copies of $A(n-1,k)$ and $(n-k+1)$ copies of $A(n-1,k-1)$. This is similar to the edge connections in the Eulerian Bratteli diagram (see Figure 2.6).

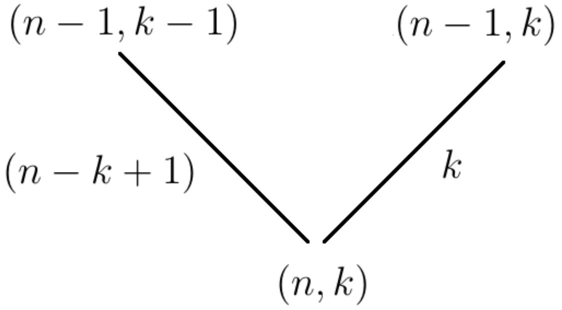

Figure 2.6: Edge connections.

This means that we can associate Eulerian numbers with the Eulerian diagram. The Eulerian number $A(n,k)$ is exactly the number of paths from the vertex $(1,1)$ (also called the *root* or *apex*) to (n,k). With respect to the Bratteli Diagram, there is a bijective correspondence between paths of length n starting at the root and terminating at (n,k), and permutations of $\{1,2,\ldots,n\}$ with $k-1$ rises. For more details refer to [18, Lemma 5] or [12].

The paths up to level 3 will be labeled as in Section 2.3 (see Figure 2.7). The differences are in level 4. In the previous representation, we ordered the edges emanating from vertex (n,k) from left to right, this implies that labels 1 through k will represent a left turn and labels $k+1$ to $n+1$ will represent a right turn (see Figure 2.8, top). In the rises and falls representation, if the placement of digit 4 creates a new fall then we represent it by a left turn; respectively, if the placement of digit 4 creates a new rise then we represent it by a rise turn (Figure 2.8, bottom).

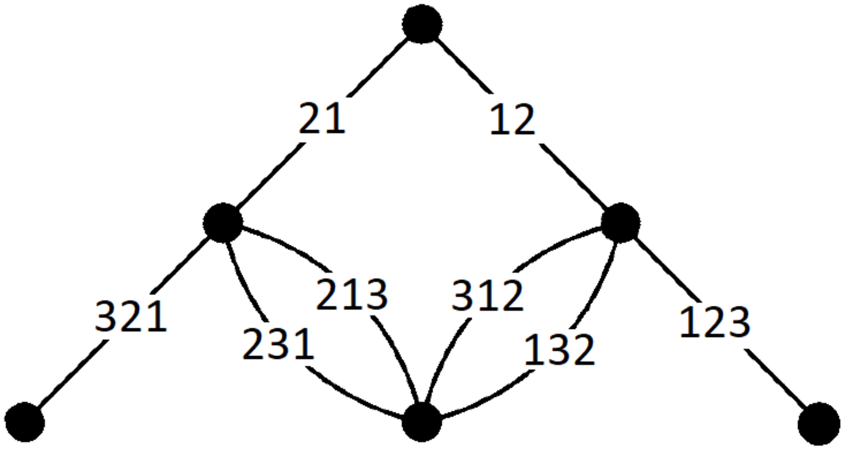

Figure 2.7: Paths up to level 3.

Example: In Figure 2.8 we have 4 possibilities of placement of digit 4 into the permutation 213.

1. If we insert 4 in the first position, then we create a new fall:

$$2 > 1 < 3 \qquad \Longrightarrow \qquad 4 > 2 > 1 < 3 \quad \text{(extra fall ``>'')}$$

2. If we insert 4 in the second position, then we create a new rise:

$$2 > 1 < 3 \qquad \Longrightarrow \qquad 2 < 4 > 1 < 3 \quad \text{(extra rise ``<'')}$$

3. If we insert 4 in the third position, then we create a new fall:

$$2 > 1 < 3 \qquad \Longrightarrow \qquad 2 > 1 < 4 > 3 \quad \text{(extra fall ``>'')}$$

4. If we insert 4 in the fourth position, then we create a new rise:

$$2 > 1 < 3 \qquad \Longrightarrow \qquad 2 > 1 < 3 < 4 \quad \text{(extra rise ``<'')}$$

Items 1 (first position) and 4 (last position) are true for any starting permutation (see Proposition below). In item 2 (second position), since at that position we had a fall ($2 > 1$), then the new permutation will have an additional rise at that place. Lastly, in item 3 (third position), since there was a rise ($1 < 3$), the new permutation will have an additional fall at that position. This is also proven on the next proposition.

Note that, what was a right turn in the left-to-right labelling (e.g. 2143) will not always be a left turn in the rises and falls representation (compare top and bottom of

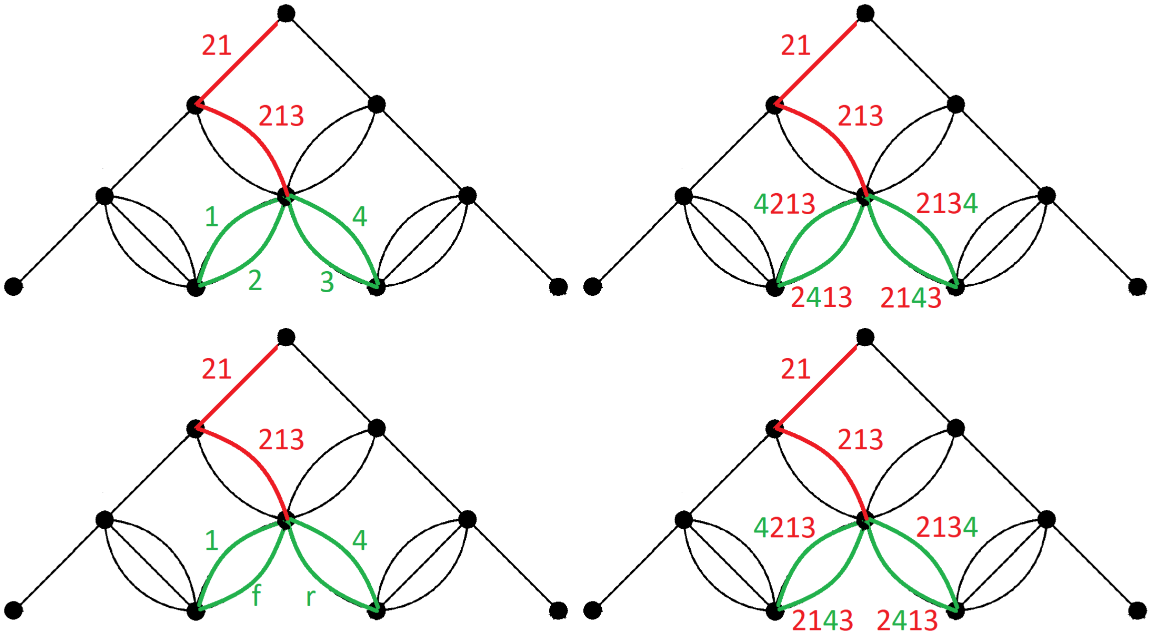

Figure 2.8: Top: left-to-right labelling. Bottom: rises and falls labelling.

Figure 2.8). The following Proposition will show what we described in this example: if we insert the "next" digit in the first position *or* between a rise, then the new permutation will have one more fall compared to the original; if we insert in the last position *or* between a fall then we create a new rise. Recall that we defined an *insert* function on Definition 2.3.2; more precisely, the function ι_{n+1}^{k} reads as *insert digit* $n+1$ *at position* k.

Proposition 2.5.4. *Let* $x = x_1 \cdots x_n \in S_n$ *and consider* $R_x := \{i \mid x_{i-1} < x_i\}$ *and* $F_x := \{i \mid x_{i-1} > x_i\}$ *to be, respectively, the set of rises and falls of the permutation* x*. Denote by* r_x *the number of rises of* x *(i.e.,* $r_x = \#R_x$*) and* f_x *the number of falls of* x *(i.e.,* $f_x = \#F_x$*). Then:*

- *If* $k = 1$ *then the number of falls of* $\iota_{n+1}^{k}(x)$ *is* $f_x + 1$*;*
- *If* $k \in R_x$ *then the number of falls of* $\iota_{n+1}^{k}(x)$ *is* $f_x + 1$*;*
- *If* $k \in F_x$ *then the number of rises of* $\iota_{n+1}^{k}(x)$ *is* $r_x + 1$*;*
- *If* $k = n+1$ *then the number of rises of* $\iota_{n+1}^{k}(x)$ *is* $r_x + 1$*;*

Proof: Indeed, if $x = x_1 \cdots x_n \in S_n$ then $\iota_{n+1}^{1}(x) = n+1, x_1 \cdots x_n \in S_{n+1}$. Since $x_i \in \{1, \ldots, n\}$ for all i then $n+1 > x_1$. So $\iota_{n+1}^{1}(x)$ has $f_x + 1$ falls. On the other hand, $\iota_{n+1}^{n+1}(x) = x_1 \cdots x_n, n+1$, so since $x_n < n+1$ then $\iota_{n+1}^{n+1}(x)$ has $r_x + 1$ rises.

Now if $k \neq 1, n+1$ then $\iota^k_{n+1}(x) = x_1 \cdots x_{k-1}, n+1, x_k \cdots x_n$. If $k \in R_x$ then $x_{k-1} < x_k$ becomes $x_{k-1} < n+1$ (rise) and $n+1 > x_k$ (fall), meaning that we create a new fall. If $k \in F_x$ then $x_{k-1} > x_k$ becomes $x_{k-1} < n+1$ (rise) and $n+1 > x_k$ (fall), meaning that we create a new rise. ∎

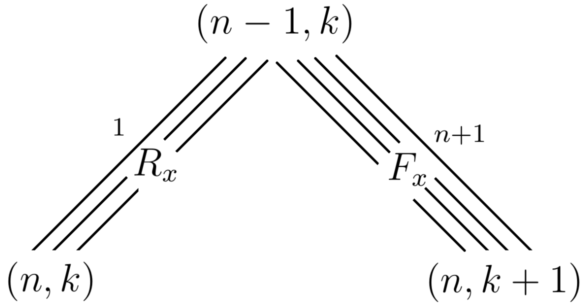

Figure 2.9: Labelling in the rises and falls representation.

2.6 A group-action on the backward path space

In this section, we will define an action of S_∞ on the backward path space. Consider an element $\sigma \in S_\infty$ and a path $x \in \varprojlim(S_n, p_n)$. Since $S_\infty = \cup S_n$, there exists $n \in \mathbb{N}$ such that $\sigma \in S_n$, i.e., σ is a permutation on the set $\{1, 2, \ldots, n\}$. We can extend this permutation to a new one, called $\overline{\sigma}$, that behaves identically to σ in $\{1, 2, \ldots, n\}$ and fixes the digits $i > n$. More precisely, $\overline{\sigma}(i) = \begin{cases} \sigma(i) & i \leq n; \\ i & \text{otherwise.} \end{cases}$

Since a path is an element in the projective limit $\varprojlim(S_i, p_i)$, x is a sequence $(x^{(2)}, x^{(3)}, \ldots)$ where $x^{(i)} \in S_i$ and $p_i(x^{(i)}) = x^{(i-1)}$. The idea of the action is to apply the permutation σ to each level of the path, e.g., if

$$x = (21, 231, 2431, 24531, 246531, 6245731, \ldots)$$

and σ is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$, then we want $\sigma \cdot x$ to be the path given by replacing 12345 by 23541 at each level, i.e.:

$$\sigma \cdot x = (??, ???, ????, 34152, 346152, 6341752, \ldots).$$

The first three entries of the path do not have all the digits 12345, thus we cannot apply σ directly. But since we want $\sigma \cdot x$ to be a path in $\varprojlim(S_n, p_n)$ we will set

???? = 3412 (removal of digit 5), ??? = 312 (removal of digit 4) and so on. Thus $\sigma \cdot x = (12, 312, 3412, 34152, 346152, 6341752, \dots)$. Thus,

$$\begin{aligned} x &= (x^{(2)}, x^{(3)}, \dots, x^{(n-1)}, x^{(n)}, x^{(n+1)}, \dots) \\ \sigma \cdot x &= (y^{(2)}, y^{(3)}, \dots, y^{(n-1)}, \sigma \cdot x^{(n)}, \overline{\sigma} \cdot x^{(n+1)}, \dots), \end{aligned}$$

where $y^{(n-1)} = p_n(\sigma \cdot x^{(n)})$ and $y^{(i-1)} = p_i(y^{(i)})$ for $i < n$.

Lemma 2.6.1. *If $\sigma \in S_n$ and $x \in S_{n+1}$, then $\sigma \cdot p_{n+1}(x) = p_{n+1}(\overline{\sigma} \cdot x)$. In other words, the following diagram commutes:*

$$\begin{tikzcd} S_n \arrow[d, "\sigma"] & S_{n+1} \arrow[l, "p_{n+1}"'] \arrow[d, "\overline{\sigma}"] \\ S_n & S_{n+1} \arrow[l, "p_{n+1}"'] \end{tikzcd}$$

Proof: On one hand, we have $p_{n+1}(\overline{\sigma} \cdot x)(i) = \begin{cases} \overline{\sigma} \cdot x(i) & i < (\overline{\sigma} \cdot x)^{-1}(n+1); \\ \overline{\sigma} \cdot x(i+1) & \text{otherwise.} \end{cases}$
Since $\overline{\sigma}(i) = \sigma(i)$ for $i \leq n$ and the image of p_{n+1} is a permutation in S_n, we have

$$p_{n+1}(\overline{\sigma} \cdot x)(i) = \begin{cases} \sigma \cdot x(i) & i < (\overline{\sigma} \cdot x)^{-1}(n+1); \\ \sigma \cdot x(i+1) & \text{otherwise.} \end{cases}$$

Since $\overline{\sigma}(n+1) = n+1$ then $\overline{\sigma}^{-1}(n+1) = (n+1)$, so $(\overline{\sigma} \cdot x)^{-1}(n+1) = x^{-1}(n+1)$. Thus we can rewrite $p_{n+1}(\overline{\sigma} \cdot x)$ as

$$p_{n+1}(\overline{\sigma} \cdot x)(i) = \begin{cases} \sigma \cdot x(i) & i < x^{-1}(n+1); \\ \sigma \cdot x(i+1) & \text{otherwise.} \end{cases}$$

On the other hand, $p_{n+1}(x)(i) = \begin{cases} x(i) & i < x^{-1}(n+1); \\ x(i+1) & \text{otherwise.} \end{cases}$ Thus

$$\sigma \cdot p_{n+1}(x)(i) = \begin{cases} \sigma \cdot x(i) & i < x^{-1}(n+1); \\ \sigma \cdot x(i+1) & \text{otherwise.} \end{cases}$$

Since $\sigma \cdot p_{n+1}(x)(i) = p_{n+1}(\overline{\sigma} \cdot x)(i)$ for every i then $\sigma \cdot p_{n+1}(x) = p_{n+1}(\overline{\sigma} \cdot x)$. ∎

To simplify the notation, we set, for $i > n$, $\sigma \cdot x^{(i)} := \overline{\sigma} \cdot x^{(i)}$. We can thus write $\sigma \cdot p_i(x^{(i)}) = p_i(\sigma \cdot x^{(i)})$ for all $i > n$.

Define $S_\infty \times X \ni (\sigma, x) \mapsto \sigma \cdot x := (y^{(2)}, y^{(3)}, \dots) \in X$, where

$$y^{(i)} := \begin{cases} \sigma \cdot x^{(i)} & \text{if } \sigma \in S_i; \\ p_{i+1} y^{(i+1)} & \text{otherwise.} \end{cases}$$

Proposition 2.6.2. *The map* $(\sigma, x) \mapsto \sigma \cdot x$ *is well-defined, i.e.,* $\sigma \cdot x \in \varprojlim(S_n, p_n)$.

Proof: Write $\sigma \cdot x := (y^{(2)}, y^{(3)}, \dots)$. We must prove for every i, that $y^{(i)} \in S_i$ and $y^{(i)} = p_{i+1}(y^{(i+1)})$. It is clear that $y^{(i)} \in S_i$ because, if $\sigma \in S_i$, then $\sigma \cdot x^{(i)}$ is a product in S_i; if $\sigma \notin S_i$ then $p_{i+1}(y^{(i+1)}) \in S_i$ by the definition of $p_{i+1} : S_{i+1} \to S_i$. The only thing left to prove is that $y^{(i)} = p_{i+1}(y^{(i+1)})$ for every i. It is already true that $y^{(i)} = p_{i+1}(y^{(i+1)})$ if $\sigma \notin S_i$ by the definition of $y^{(i)}$; so we just need to prove that $y^{(i)} = p_{i+1}(y^{(i+1)})$ when $\sigma \in S_i$. If $\sigma \in S_i$ then, by Lemma 2.6.1,

$$p_{i+1}(y^{(i+1)}) = p_{i+1}(\sigma \cdot x^{(i+1)}) = \sigma \cdot p_{i+1}(x^{(i+1)}) = \sigma \cdot x^{(i)} = y^{(i)}.$$

∎

In order to prove that the map $S_\infty \times X \to X$ defined above is a group action, we must prove that:

- $\mathrm{id} \cdot x = x$ for all $x \in X$;
- $\sigma \cdot (\tau \cdot x) = (\sigma\tau) \cdot x$ for all $\sigma, \tau \in S_\infty$ and all $x \in X$.

The first property is trivial.

Proposition 2.6.3. *If* $\sigma, \tau \in S_\infty$ *and* $x \in X$ *then* $\sigma \cdot (\tau \cdot x) = (\sigma\tau) \cdot x$.

Proof: We must prove that $(\sigma \cdot (\tau \cdot x))^{(i)} = ((\sigma\tau) \cdot x)^{(i)}$ for all i. Let $\sigma, \tau \in S_\infty$; then there exist n, m such that $\sigma \in S_n$ and $\tau \in S_m$.

Case $i \geq \max\{n, m\}$. In this case, $(\sigma \cdot (\tau \cdot x))^{(i)} = \sigma \cdot (\tau \cdot x)^{(i)} = \sigma \cdot (\tau \cdot x^{(i)})$ from the definition of the action. Since $\sigma, \tau, x^{(i)} \in S_i$, the product is associative, thus $\sigma \cdot (\tau \cdot x^{(i)}) = (\sigma\tau) \cdot x^{(i)}$. And since $\sigma\tau \in S_{\max\{n,m\}}$ then again, by definition, $(\sigma\tau) \cdot x^{(i)} = ((\sigma\tau) \cdot x)^{(i)}$.

Case $i < \max\{n, m\}$. We will use backwards induction.

For $i = \max\{n, m\} - 1$. We have that $(\sigma \cdot (\tau \cdot x))^{(i)} = p_{i+1}((\sigma(\tau \cdot x))^{(i+1)})$. Since $i + 1 = \max\{n, m\}$, the previous case applies: $(\sigma \cdot (\tau \cdot x))^{(i+1)} = ((\sigma\tau) \cdot x)^{(i+1)}$. It follows that $p_{i+1}((\sigma(\tau \cdot x))^{(i+1)}) = p_{i+1}(((\sigma\tau) \cdot x)^{(i+1)}) = ((\sigma\tau) \cdot x)^{(i)}$.

Induction hypothesis: Assume $(\sigma \cdot (\tau \cdot x))^{(i)} = ((\sigma\tau) \cdot x)^{(i)}$ for a fixed index $i < \max\{n, m\}$.

Proof of $i-1$: Indeed,
$(\sigma \cdot (\tau \cdot x))^{(i-1)} = p_i((\sigma \cdot (\tau \cdot x))^{(i)}) = p_i(((\sigma\tau) \cdot x)^{(i)}) = ((\sigma\tau) \cdot x)^{(i-1)}$. $\blacksquare$

2.6.1 Example

Let $x \in X$ be the path $(21, 231, 2431, 24531, 245316, \dots)$ and consider $\sigma \in S_\infty$ given by $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$. We evaluate $\sigma \cdot x$.

Note that σ replaces 1234 by 4312, thus:

$$\begin{array}{rclllllllll} x & = & 9 & 2 & 4 & 5 & 8 & 3 & 1 & 7 & 6 & \cdots \\ & & & \downarrow & \downarrow & & & \downarrow & \downarrow & & & \\ \sigma \cdot x & = & 9 & 3 & 2 & 5 & 8 & 1 & 4 & 7 & 6 & \cdots \end{array}$$

Then $\sigma \cdot x = (21, 321, 3214, 32514, 325146, 3251476, 32581476, 932581476, \dots)$. More explicitly:

$$\begin{tikzcd} x \arrow[d] & = & 21 & 231 \arrow[l] & 2431 \arrow[l] \arrow[d] & 24531 \arrow[l] \arrow[d] & 245316 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ \sigma \cdot x & = & 21 & 321 \arrow[l] & 3214 \arrow[l] & 32514 \arrow[l] & 325146 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

2.7 The symmetric probability measure

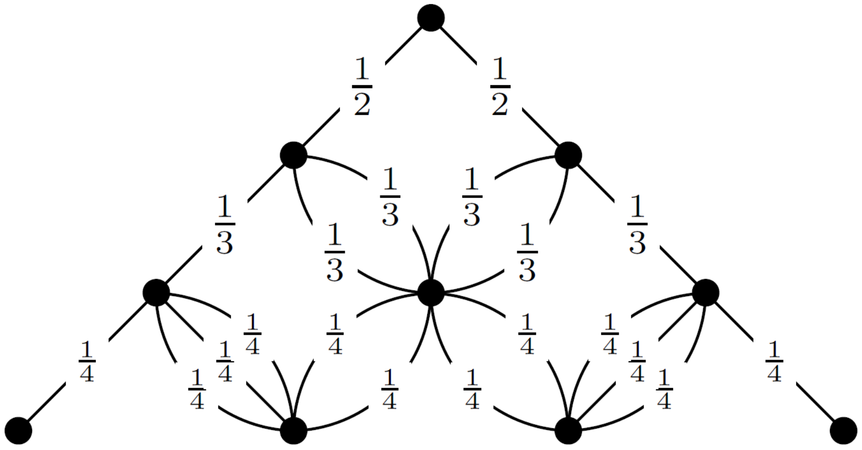

Figure 2.10: The symmetric measure.

Recall that a cylinder set $C(c_1c_2\cdots c_n)$ is the set of infinite paths $x \in \overrightarrow{X}$ such that $x_i = c_i$ for all $i \in \{1, \dots, n\}$. We consider the Borel σ-algebra $\mathcal{T}$ generated by the cylinder sets and define $\nu : \mathcal{T} \to \mathbb{R}$ as being $\nu(C(c_1, \ldots, c_n)) = \frac{1}{n!}$. Since the cylinder sets are the base for the topology of $\overrightarrow{X}$ then ν is a Borel measure.

Equivalently, as discussed in Section 2.4, when working with the backward path space $\overleftarrow{X} = \varprojlim(S_n, p_n)$, the cylinder sets are of the form $C(\tau) := \{x \in \overleftarrow{X} \mid x^{(n)} = \tau\}$. Thus, we have $\nu(C(\tau)) = \frac{1}{n!}$ for any $\tau \in S_n$.

Remark 2.7.1. If $x \in C(\tau)$ then $x^{(n)} = \tau$. Recall that there are $n+1$ possible elements $x^{(n+1)} \in S_{n+1}$ such that $p_{n+1}(x^{(n+1)}) = x^{(n)}$, and they are $\iota^k_{n+1}(x^{(n)})$ for $k \in \{1, \ldots, n+1\}$ (as explained in Section 2.3). Thus $C(\tau) = \bigsqcup_{k=1}^{n+1} C(\iota^k_{n+1}(\tau))$.

Lemma 2.7.2. *If $\sigma, \tau \in S_n$ then $\sigma \cdot C(\tau) = C(\sigma\tau)$.*

Proof: Let $x \in \sigma \cdot C(\tau)$, then $x = \sigma \cdot y$ for $y \in C(\tau)$, i.e., $y^{(n)} = \tau$. So $x^{(n)} = (\sigma \cdot y)^{(n)} = \sigma \cdot y^{(n)} = \sigma\tau$. Thus $x \in C(\sigma\tau)$.

On the other hand, if $x \in C(\sigma\tau)$ then $x^{(n)} = \sigma\tau$. Picking any path y such that $y^{(n)} = \tau$ we have $x^{(n)} = \sigma \cdot y^{(n)} = (\sigma \cdot y)^{(n)}$. So $x \in \sigma \cdot C(\tau)$. ∎

Corollary 2.7.3. *If $\sigma \in S_n$ and $\tau \in S_m$, $n < m$, then $\sigma \cdot C(\tau) = C(\sigma\tau)$.*

Proof: Extend $\sigma \in S_n$ to $\overline{\sigma} \in S_m$ and apply Lemma 2.7.2. ∎

The following lemma proves that the S_∞-action on the path space $\varprojlim(S_n, p_n)$ is measure-preserving. We will start with the case given by Lemma 2.7.2 and Corollary 2.7.3 above ($\sigma \in S_n$, $\tau \in S_m$, $n \leq m$).

Lemma 2.7.4. *If $\sigma \in S_n$ and $\tau \in S_m$ with $n \leq m$, then $\nu(\sigma \cdot C(\tau)) = \nu(C(\tau))$.*

Proof: If $\tau \in S_m$, then $\nu(C(\tau)) = \frac{1}{m!}$ and the same for $C(\sigma\tau)$ (as $\sigma\tau = \overline{\sigma}\tau \in S_m$). By the previous lemma and corollary, $\nu(\sigma \cdot C(\tau)) = \nu(C(\sigma\tau)) = \frac{1}{m!}$. ∎

The following results cover the cases where $\sigma \in S_n$ and $\tau \in S_m$ for $n > m$.

Lemma 2.7.5. *If $\tau \in S_n$ and $\sigma \in S_{n+1}$, then $\nu(\sigma \cdot C(\tau)) = \nu(C(\tau))$.*

Proof: If $\tau \in S_n$, then $\iota^k_{n+1}(\tau) \in S_{n+1}$ for all $k \in \{1, \ldots, n+1\}$. Note that

$$\sigma \cdot C(\tau) = \sigma \cdot \bigsqcup_{k=1}^{n+1} C(\iota^k_{n+1}(\tau)) = \bigsqcup_{k=1}^{n+1} \sigma \cdot C(\iota^k_{n+1}(\tau)).$$

Since both σ and $\iota_{n+1}^k(\tau)$ belong to S_{n+1}, we can apply Lemma 2.7.2:

$$\nu(\sigma \cdot C(\iota_{n+1}^k(\tau))) = \nu(C(\iota_{n+1}^k(\tau))) = \frac{1}{(n+1)!}$$

Therefore,

$$\nu(\sigma \cdot C(\tau)) = \sum_{k=1}^{n+1} \nu(C(\iota_{n+1}^k(\tau))) = \sum_{k=1}^{n+1} \frac{1}{(n+1)!} = \frac{n+1}{(n+1)!} = \frac{1}{n!} = \nu(C(\tau)).$$

∎

Corollary 2.7.6. *If $\tau \in S_n$ and $\sigma \in S_m$ with $n < m$, then $\nu(\sigma \cdot C(\tau)) = \nu(C(\tau))$.*

Proof: Induction on m using Lemma 2.7.5. ∎

2.8 Minimality and ergodicity of the action

We consider the topological space $(\overleftarrow{X}, \mathcal{T})$ where $\overleftarrow{X} = \varprojlim(S_n, p_n)$ and $\mathcal{T}$ is the topology generated by having the cylinder sets as a base. Let $\cdot : S_\infty \times \overleftarrow{X} \to \overleftarrow{X}$ the action by S_∞ defined on Section 2.6.

Proposition 2.8.1. *The action $\cdot : S_\infty \times \overleftarrow{X} \to \overleftarrow{X}$ is continuous.*

Proof: Let $\sigma \in S_\infty$ and consider the map $\sigma(\cdot) \in \mathrm{Aut}(\overleftarrow{X}, \mathcal{T})$ given by $x \mapsto \sigma(x)$. We prove that $\sigma(\cdot)$ is continuous with respect to the topology $\mathcal{T}$. Since $\sigma \in S_\infty = \cup S_i$, there exists $n \in \mathbb{N}$ such that $\sigma \in S_n$.

Let $x \in \overleftarrow{X}$ and consider a neighbourhood $\mathcal{V}$ of $\sigma(x) = (y^{(2)}, y^{(3)}, \dots)$, so $y^{(i)} = \sigma \cdot x^{(i)}$ for all $i > n$. Since cylinder sets are a base for the topology, there exist a cylinder set $C(\tau)$, with $\tau \in S_m$, such that $\sigma(x) \in C(\tau) \subset \mathcal{V}$. So $\tau = y^{(m)}$. We let $N > \max\{n, m\}$, then $C(y^{(N)}) \subset C(y^{(m)}) \subset \mathcal{V}$.

Let $\mathcal{U} = C(x^{(N)})$ be a neighbourhood of x. By Corollary 2.7.3,

$$\sigma(\mathcal{U}) = \sigma(C(x^{(N)})) = C(\sigma \cdot x^{(N)}) = C(y^{(N)}) \subset \mathcal{V}.$$

∎

Proposition 2.8.2. *Every orbit for the S_∞ action is dense.*

Proof: Let $x \in \overleftarrow{X}$ and consider its orbit $S_\infty \cdot x$. In order to show that the orbit is dense, it is sufficient to prove that for an arbitrary cylinder set $C(y)$, $y \in S_n$, there exist $\sigma \in S_\infty$ such that $\sigma \cdot x \in C(y)$. Since $x^{(n)} \in S_n$, we can use $\sigma := y \cdot (x^{(n)})^{-1} \in S_n$ and see that $y = \sigma \cdot x^{(n)} = (\sigma \cdot x)^{(n)}$. ∎

The following result is stated in [2, Proposition 1] and [12, Lemma 1]. In the former, they adapted the proof in [21, Lemma 17.1]. This is also analogous to a similar result for the Pascal Bratteli diagram as seen in [22], [26, Lemma 3.4] and [27, Proposition 2.4].

Lemma 2.8.3 (Vershik-Kerov [32, 33])**.** *If A is a measurable subset of $(\overleftarrow{X}, \mathcal{T}, \nu)$ then for almost every $x \in \overleftarrow{X}$ we have:*

$$\lim_{n\to\infty} \frac{\nu(A \cap C(x^{(n)}))}{\nu(C(x^{(n)}))} = \chi_A(x)$$

Theorem 2.8.4. *The permutation action is ergodic with respect to the symmetric measure.*

Proof: Let A be a measurable subset of X such that A is S_∞-invariant and assume, by contradiction, that $0 < \nu(A) < 1$. By Lemma 2.8.3, for almost every $x \in A$ and $y \in X \setminus A$ we have:

$$\lim_{n\to\infty} \frac{\nu(A \cap C(x^{(n)}))}{\nu(C(x^{(n)}))} = 1 \qquad \lim_{n\to\infty} \frac{\nu((X \setminus A) \cap C(y^{(n)}))}{\nu(C(y^{(n)}))} = 1$$

Hence for almost every $x \in A$ and $y \in X \setminus A$, there exist N such that for all $n > N$,

$$\frac{\nu(A \cap C(x^{(n)}))}{\nu(C(x^{(n)}))} > \frac{1}{2} \qquad \frac{\nu((X \setminus A) \cap C(y^{(n)}))}{\nu(C(y^{(n)}))} > \frac{1}{2} \tag{2.8.1}$$

Note that we can map $x^{(n)}$ into $y^{(n)}$ via permutation, this means that there exist $\sigma \in S_\infty$ such that $\sigma \cdot C(x^{(n)}) = C(y^{(n)})$. Since A is S_∞-invariant, this contradicts (2.8.1). Thus $\nu(A) = 0$ or $\nu(A) = 1$. Therefore the action by S_∞ is ergodic with respect to the symmetric measure. ∎

Chapter 3

Distributions on the Eulerian Bratteli diagram

Recall from Section 1.1 that an element of the Eulerian Bratteli diagram dimension group $G = \varinjlim(\mathbb{Z}^n, \varphi_n)$ is a pair $[x, n]$ where $x \in \mathbb{Z}^n$. We represent elements of $\mathbb{Z}^n$ as row vectors and we refer to $\varphi_{nm}(x) \in \mathbb{Z}^m$ as *distribution of x at level m.*

Example 3.0.1. In the dimension group associated to the Eulerian Bratteli diagram (Section 1.2.1), if we set $x = (1, 0) \in \mathbb{Z}^2$, then $\varphi_2(x) = (1, 2, 0)$ and $\varphi_{2,4}(x) = \varphi_3 \circ \varphi_2(x) = (1, 7, 4, 0)$. So we say that $(1, 7, 4, 0)$ is the distribution of $(1, 0)$ at level 4.

The goal of this chapter is to prove that any distribution arising from the Eulerian Bratteli diagram (we call them *Eulerian distributions*) are *unimodal*, in other words, they have a "single peak". We will also find the positions where the distributions attain their maximum value. In order to do so, we structure this chapter into three sections:

- The Eulerian distributions are *log concave* (Section 3.1). This implies unimodality [31].
- Polynomials of the form $\sum A(n, k)x^k$ have only real roots. This means that the Eulerian numbers (the coefficients of the polynomials) form *Pólya frequency* sequences (defined and detailed in Section 3.2).
- In Section 3.3 we define the *centre* of an Eulerian distribution and prove that the *mean* of the Eulerian distributions converge to it. This implies that the Eulerian distributions have their peak at the centre of the diagram (a single centre on odd levels and two centres on even levels in case of symmetric distributions).

3.1 Log concavity

This section is based in multiple references on combinatorics, namely, [7, Section 7.1], [31], [4] and [6].

Definition 3.1.1. *Let I be a finite subset of $\mathbb{Z}$. An element a_k of a sequence $(a_i)_{i\in I}$ of nonnegative real numbers is* log concave *if $a_k^2 \geq a_{k-1}a_{k+1}$.*

Remark 3.1.2. Log concavity is not defined for $a_{\min I}$ and $a_{\max I}$ because $\min I - 1, \max I + 1 \notin I$. If all a_i, except for $a_{\min I}$ and $a_{\max I}$, is log concave, and moreover $a_{\min I}, a_{\max I} > 0$, then $a_i > 0$ for all $i \in I$.

Example 3.1.3. The sequence of consecutive numbers $(1, 2, 3, \dots, n)$ is log concave since for every $k \neq 1, n$ we have $k^2 \geq k^2 - 1 = (k-1)(k+1)$.

Example 3.1.4. The sequence of binomial numbers $(\binom{n}{k})_k$ arising from the Pascal triangle at level n is log concave. Indeed, at level 3 we have $\binom{2}{1}^2 = 2^2 = 4$ and $\binom{2}{0} \cdot \binom{2}{2} = 1 \cdot 1 = 1 \leq 4$. For later levels, it will be a consequence of Theorem 3.1.7 below.

Example 3.1.5. The sequence of Eulerian numbers $(A(n,k))_k$ arising from the Eulerian diagram at level n is log concave. Indeed, at level 3 we have $A(3,2)^2 = 4^2 = 16$ and $A(3,1) \cdot A(3,3) = 1 \cdot 1 = 1 \leq 16$. For later levels, this will be a consequence of Theorem 3.1.7 below.

Lemma 3.1.6. *Let (v_i) be a log concave sequence and $k \neq \min I, \max I$. Assume that $v_k, v_{k-1} \neq 0$. Then $v_{k-1}v_k \geq v_{k-2}v_{k+1}$.*

Proof: If (v_k) is log concave then $v_k^2 \geq v_{k-1}v_{k+1}$, this means that $v_k \geq \frac{v_{k-1}v_{k+1}}{v_k}$, in particular, $v_{k-1} \geq \frac{v_{k-2}v_k}{v_{k-1}}$. Multiplying these inequalities we have $v_{k-1}v_k \geq v_{k-2}v_{k+1}$. ∎

Theorem 3.1.7. *Let $\varphi_n \in \mathbb{M}_{n+1,n}(\mathbb{R})$ be the following matrix*

$$\begin{pmatrix} l_1 & 0 & 0 & \cdots & 0 & 0 \\ r_1 & l_2 & 0 & \cdots & 0 & 0 \\ 0 & r_2 & l_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{n-2} & l_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & r_{n-1} & l_n \\ 0 & 0 & \cdots & 0 & 0 & r_n \end{pmatrix} \tag{3.1.1}$$

and suppose that (l_k), (r_k) *and* $x = (v_1, \dots, v_n) \in \mathbb{R}^n$ *are log concave. Write, for* $1 < k < n$, $L_k := l_k^2 - l_{k-1}l_{k+1}$, $R_k := r_k^2 - r_{k-1}r_{k+1}$, *and*

$$\Phi_k := 2r_{k-1}l_k - r_{k-2}l_{k+1} - l_{k-1}r_k \qquad (r_0 := 0)$$

Then $\varphi_n(x)$ *is log concave if and only if* $2\sqrt{R_{k-1}L_k} + \Phi_k \geq 0 \; \forall 1 < k < n$.

Remark 3.1.8. If (l_k) and (r_k) are log concave then $L_k, R_k \geq 0$, thus $\sqrt{R_{k-1}L_k}$ is well-defined.

Remark 3.1.9. The matrix above is the generalized transition matrix defined in Section 1.2.2 but omitting the index n (n is fixed in this theorem). Thus the matrix above describes the connection between levels illustrated in the figure below.

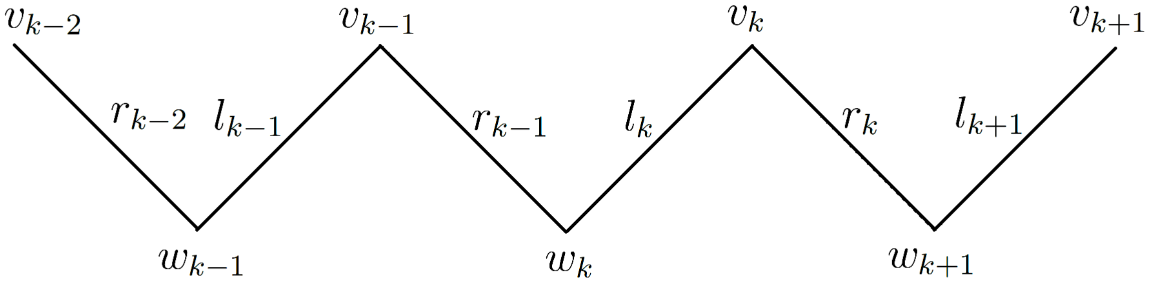

Figure 3.1: Generalized edge connections

Proof: Set $\varphi_n(x) := (w_1, \dots, w_{n+1})_{n+1} \in \mathbb{R}^{n+1}$. We must prove $w_k^2 \geq w_{k-1}w_{k+1}$. By Lemma 1.2.2, $w_k = r_{k-1} \cdot v_{k-1} + l_k \cdot v_k$, thus

$$w_k^2 = (r_{k-1} \cdot v_{k-1} + l_k \cdot v_k)^2 = r_{k-1}^2 \cdot v_{k-1}^2 + l_k^2 \cdot v_k^2 + 2r_{k-1}l_k \cdot v_{k-1}v_k.$$

Using the log concavity of (v_k) and the Lemma 3.1.6 we have that $w_{k-1}w_{k+1} =$

$$\begin{aligned}
&= (r_{k-2} \cdot v_{k-2} + l_{k-1} \cdot v_{k-1})(r_k \cdot v_k + l_{k+1} \cdot v_{k+1}) \\
&= r_{k-2}r_k \cdot v_{k-2}v_k + l_{k-1}l_{k+1} \cdot v_{k-1}v_{k+1} + r_{k-2}l_{k+1} \cdot v_{k-2}v_{k+1} + l_{k-1}r_k \cdot v_{k-1}v_k \\
&\leq r_{k-2}r_k \cdot v_{k-1}^2 + l_{k-1}l_{k+1} \cdot v_k^2 + r_{k-2}l_{k+1} \cdot v_{k-1}v_k + l_{k-1}r_k \cdot v_{k-1}v_k \\
&= r_{k-2}r_k \cdot v_{k-1}^2 + l_{k-1}l_{k+1} \cdot v_k^2 + (r_{k-2}l_{k+1} + l_{k-1}r_k) \cdot v_{k-1}v_k.
\end{aligned}$$

Now we evaluate: $w_k^2 - w_{k-1}w_{k+1} \geq$

$$\begin{aligned}
\geq\; & r_{k-1}^2 \cdot v_{k-1}^2 + l_k^2 \cdot v_k^2 + 2r_{k-1}l_k \cdot v_{k-1}v_k \\
& -[r_{k-2}r_k \cdot v_{k-1}^2 + l_{k-1}l_{k+1} \cdot v_k^2 + (r_{k-2}l_{k+1} + l_{k-1}r_k) \cdot v_{k-1}v_k] \\
=\; & (r_{k-1}^2 - r_{k-2}r_k) \cdot v_{k-1}^2 + (l_k^2 - l_{k-1}l_{k+1}) \cdot v_k^2 + (2r_{k-1}l_k - r_{k-2}l_{k+1} - l_{k-1}r_k) \cdot v_{k-1}v_k
\end{aligned}$$

$$= \quad R_{k-1} \cdot v_{k-1}^2 + L_k \cdot v_k^2 + \Phi_k \cdot v_{k-1}v_k$$

Since $(\sqrt{R_{k-1}} \cdot v_{k-1} - \sqrt{L_k} \cdot v_k)^2 = R_{k-1} \cdot v_{k-1}^2 + L_k \cdot v_k^2 - 2\sqrt{R_{k-1}L_k} \cdot v_{k-1}v_k$ then $R_{k-1} \cdot v_{k-1}^2 + L_k \cdot v_k^2 = (\sqrt{R_{k-1}} \cdot v_{k-1} - \sqrt{L_k} \cdot v_k)^2 + 2\sqrt{R_{k-1}L_k} \cdot v_{k-1}v_k$, thus

$$\begin{aligned} w_k^2 - w_{k-1}w_{k+1} &\geq R_{k-1} \cdot v_{k-1}^2 + L_k \cdot v_k^2 + \Phi_k \cdot v_{k-1}v_k \\ &= (\sqrt{R_{k-1}} \cdot v_{k-1} - \sqrt{L_k} \cdot v_k)^2 + 2\sqrt{R_{k-1}L_k} \cdot v_{k-1}v_k + \Phi_k \cdot v_{k-1}v_k \\ &= (\sqrt{R_{k-1}} \cdot v_{k-1} - \sqrt{L_k} \cdot v_k)^2 + (2\sqrt{R_{k-1}L_k} + \Phi_k) \cdot v_{k-1}v_k \end{aligned}$$

Since $(\sqrt{R_{k-1}} \cdot v_{k-1} - \sqrt{L_k} \cdot v_k)^2 > 0$ then $w_k^2 \geq w_{k-1}w_{k+1}$ if and only if $2\sqrt{R_{k-1}L_k} + \Phi_k \geq 0$. ∎

The hypothesis of Theorem 3.1.7 are satisfied by both the Pascal and the Eulerian diagram. For Pascal, this is trivial since $l_k, r_k = 1$ for all k thus $L_k, R_k, \Phi_k = 0$. As for the Eulerian diagram, since $l_k = k$ and $r_k = n-k+1$, then (l_k) and (r_k) are sequences of consecutive numbers, thus by Example 3.1.3 they are log concave sequences. It follows that, for all $1 < k < n$, $L_k = R_k = 1$ and

$$\Phi_k = \begin{cases} 3n+1 & \text{if } k = 2; \\ -2 & \text{otherwise.} \end{cases}$$

Therefore, $2\sqrt{R_{k-1}L_k} + \Phi_k \geq 0$. By Theorem 3.1.7, $\varphi_n(x)$ is log concave provided x is. By Example 3.1.5, $(A(3,k))_{k=1}^3$ is log concave, thus so is $(A(n,k))_{k=1}^n$ for every $n > 3$.

3.2 Pólya frequency and Darroch's rule

In this section we set $(0,0)$ to be the apex of the Eulerian Bratteli diagram, so $A(0,0) = 1$ and $A(n,k) = (n-k+1)A(n-1,k-1) + (k+1)A(n-1,k)$ for $n > 0$ and $0 \leq k \leq n$. We study *Eulerian polynomials*, which are polynomials of the form $\sum_{k=0}^n A(n,k)x^k$ (see Figure 3.2 for a few examples). It will be proved that Eulerian polynomials have only real roots and, therefore, the coefficients form a *Pólya frequency sequence* (or *PF sequence*, defined below). These sequences are the hypothesis of *Darroch's rule* (Theorem 3.3.2), an easy method to identify the position where the maximum (or maxima) of the sequence is located. We will use this rule to prove that the Eulerian distributions have their maxima at the centre of the diagram. This section is based on [8] and [30].

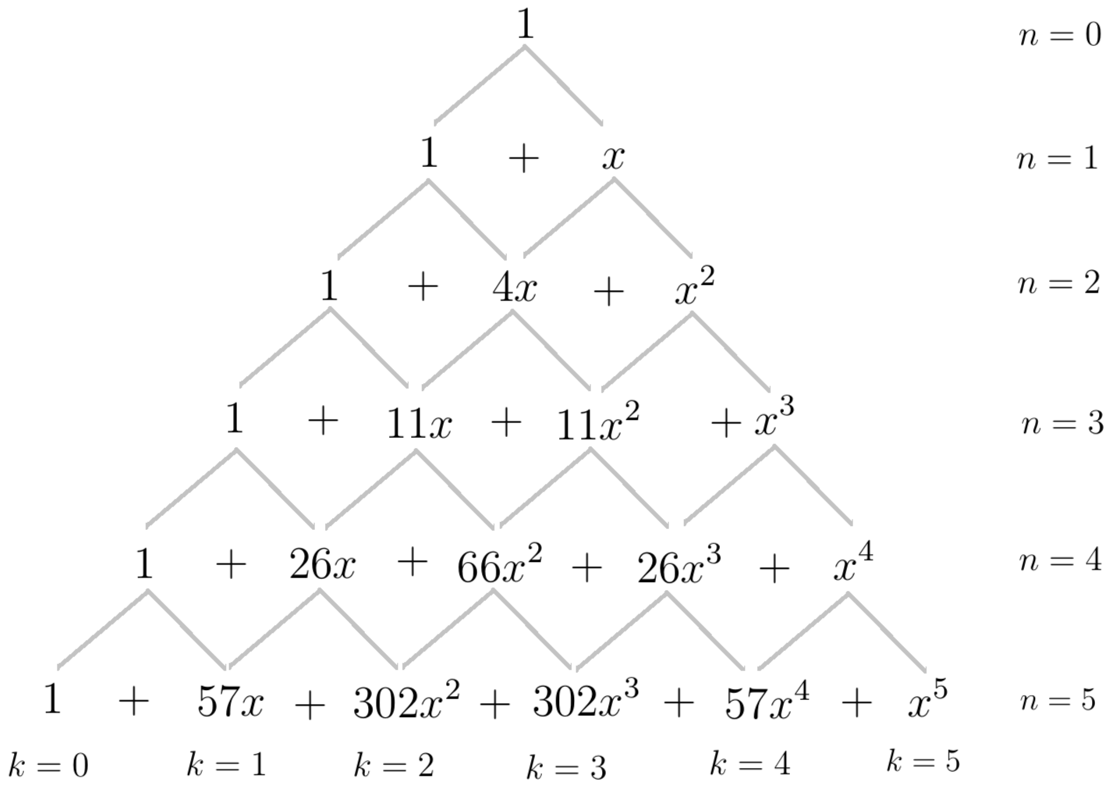

Figure 3.2: Eulerian polynomials

Definition 3.2.1. *A finite sequence of real numbers* $(a_k)_{k\in K}$ *is a* Pólya frequency sequence *if the polynomial* $\sum_{k\in K} a_k x^k$ *is either constant or has only real roots.*

We refer to Pólya frequency sequences as *PF sequences.* We will prove in Proposition 3.2.6 that the Eulerian polynomial $\sum_k A(n,k)x^k$ has only real roots, so the Eulerian numbers form a PF sequence. The next three lemmas will cover technical properties of polynomials that will be needed in order to prove the main result.

Lemma 3.2.2. *Let* α *be a root of a polynomial* $p \in \mathbb{C}[x]$*. Then* α *has multiplicity 1 if and only if* $p'(\alpha) \neq 0$*.*

Proof: If α is a root of a polynomial $p \in \mathbb{C}[x]$ then write $p(x) = (x-\alpha)^n r(x)$ where $r(x) \in \mathbb{C}[x]$, $r(\alpha) \neq 0$ and $n \geq 1$ is the multiplicity of α. Then

$$p'(x) = n(x-\alpha)^{n-1}r(x) + (x-\alpha)^n r(x) = (x-\alpha)^{n-1}[nr(x) + (x-\alpha)r'(x)].$$

As $r(\alpha) \neq 0$, then $p'(\alpha) \neq 0$ if and only if $n = 1$. ∎

Lemma 3.2.3. *Let* $p \in \mathbb{R}[x]$ *be a polynomial and* $\alpha < \beta$ *be real roots of* p *such that* $p(x) \neq 0$ *for all* $\alpha < x < \beta$*. If* α, β *are simple roots (i.e., have multiplicity 1), then* $p'(\alpha)p'(\beta) < 0$ *(i.e., have opposite signs).*

Proof: If α, β are simple roots of p, then $p(x) = (x-\alpha)(x-\beta)s(x)$ where $s \in \mathbb{R}[x]$ and $s(\alpha), s(\beta) \neq 0$. Thus $p'(x) = [(x-\alpha)+(x-\beta)]s(x) + (x-\alpha)(x-\beta)s'(x)$. By Lemma 3.2.2, $p'(\alpha) = (\alpha-\beta)s(\alpha) \neq 0$ and $p'(\beta) = (\beta-\alpha)s(\beta) \neq 0$. It follows that

$$p'(\alpha)p'(\beta) = -(\alpha-\beta)^2 s(\alpha)s(\beta).$$

Since, by hypothesis, $p(x) \neq 0$ for all $\alpha < x < \beta$, then $s(x) \neq 0$ for all $\alpha < x < \beta$. Thus s does not change sign in the interval $[\alpha, \beta]$, in other words, $s(\alpha)s(\beta) > 0$. Therefore $p'(\alpha)p'(\beta) < 0$ ∎

Lemma 3.2.4. *Let* $p \in \mathbb{R}[x]$*. If* $p(x_0) \neq 0$ *and* $p(y_0) \neq 0$ *have opposite signs, then there exists one root between* x_0 *and* y_0 *with odd multiplicity.*

Proof: Assume, without loss of generality, that $x_0 < y_0$, and let $A = \{x \in (x_0, y_0) \mid p(x) = 0\}$. Since $p(x_0)$ and $p(y_0)$ have opposite signs, by the intermediate value theorem, $A \neq \emptyset$, and p factorizes as

$$p(x) = \prod_{\alpha \in A} (x-\alpha)^{n_\alpha} r(x) \quad \text{and} \quad r(x) \neq 0 \ \forall x \in [x_0, y_0].$$

Suppose, by contradiction, that $\sum_{\alpha \in A} n_\alpha$ is even, then $\prod_{\alpha \in A}(x_0-\alpha)^{n_\alpha} > 0$ and $\prod_{\alpha \in A}(y_0-\alpha)^{n_\alpha} > 0$. Thus,

$$p(x_0)r(x_0) = \prod_{\alpha \in A}(x_0-\alpha)^{n_\alpha} r(x_0)^2 > 0,$$

$$p(y_0)r(y_0) = \prod_{\alpha \in A}(y_0-\alpha)^{n_\alpha} r(y_0)^2 > 0.$$

This means that $p(x_0)$ and $r(x_0)$ have the same sign, as well as $p(y_0)$ and $r(y_0)$. Since $p(x_0)$ and $p(y_0)$ have opposite signs, $r(x_0)$ and $r(y_0)$ would also have opposite signs. Contradiction with the fact that $r(x) \neq 0$ for all $x \in [x_0, y_0]$. Therefore, $\sum_{\alpha \in A} n_\alpha$ is odd, i.e., there exists one root $\alpha \in A$ with odd multiplicity. ∎

Proposition 3.2.5. *Suppose that*

- $p \in \mathbb{R}[x]$ *is a polynomial of degree* n *with* n *distinct real roots*

$$\alpha_n < \cdots < \alpha_2 < \alpha_1 < 0;$$

- $q \in \mathbb{R}[x]$ *is a polynomial of degree* $n+1$*;*
- $p(0), q(0) > 0$*;*
- $p'(\alpha_i)$ *and* $q(\alpha_i)$ *have opposite signs (or* $p'(\alpha_i)q(\alpha_i) < 0 \ \forall i \in \{1, \ldots, n\}$*).*

Then q *has* $n+1$ *distinct roots.*

Proof: By hypothesis, $p(x) = c(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ where $c \in \mathbb{R}$ and $\alpha_n < \cdots < \alpha_2 < \alpha_1 < 0$. Since for all $i \in \{1, \ldots, n\}$, α_i is a simple root of p; then, by Lemma 3.2.2, $p'(\alpha_i) \neq 0$. As $p(0) > 0$, $p'(\alpha_1) > 0$. By Lemma 3.2.3, $p'(\alpha_i)p'(\alpha_{i+1}) < 0$ for all $i \in \{1, \ldots, n-1\}$, so

$$\begin{cases} p'(\alpha_i) > 0 & \text{if } i \text{ is odd;} \\ p'(\alpha_i) < 0 & \text{if } i \text{ is even.} \end{cases}$$

By hypothesis, $p'(\alpha_i)q(\alpha_i) < 0$ for all $i \in \{1, \ldots, n-1\}$, so

$$\begin{cases} q(\alpha_i) < 0 & \text{if } i \text{ is odd;} \\ q(\alpha_i) > 0 & \text{if } i \text{ is even.} \end{cases} \tag{3.2.1}$$

By the intermediate value theorem, for all $1 \leq i \leq n$, there exists a real root β_i in the interval (α_i, α_{i-1}) (with $\alpha_0 := 0$), such that $q(\beta_i) = 0$. So we can factorize q as follows:

$$q(x) = (x-\beta_1)\cdots(x-\beta_n)r(x).$$

Since q has degree $n+1$ then r has degree 1. Thus we can rewrite q as follows:

$$q(x) = (x-\beta_1)\cdots(x-\beta_n)(d(x-\beta)) \qquad (d, \beta \in \mathbb{R}).$$

We prove that $\beta \neq \beta_i$ for every $i \in \{1, \ldots, n\}$. Suppose, by contradiction, that there exists $i \in \{1, \ldots, n\}$ such that $q(x) = (x-\beta_i)^2 s(x)$ with $s(x) \neq 0$ for all $\alpha_i \leq x \leq \alpha_{i-1}$. By (3.2.1) and Lemma 3.2.4, q has a root of odd multiplicity in (α_i, α_{i-1}). Contradiction with the fact that the degree of q is $n+1$. Hence q has $n+1$ distinct real roots $\{\beta, \beta_1, \ldots, \beta_n\}$. Moreover, $\beta \notin [\alpha_n, 0]$. ∎

Proposition 3.2.6. *Let* $f_n(x) := \sum_{k=0}^{n} A(n,k)x^k$ *be the Eulerian polynomial at level* n*. Then, for* $n \geq 1$*,* f_n *has only real roots.*

Proof: As $x = -1$ is the root of $f_1(x) = 1 + x$, we prove the proposition by induction. Assume, for $n > 1$, that f_n has n distinct real roots and note that

$$f_{n+1}(x) = \sum_{k=0}^{n+1} A(n+1,k)x^k = \sum_{k=0}^{n+1} [(n-k+2)A(n,k-1) + (k+1)A(n,k)]x^k$$

$$= (n+2)\sum_{k=0}^{n+1} A(n,k-1)x^k + \sum_{k=0}^{n+1} A(n,k)x^k + \sum_{k=0}^{n+1}[A(n,k)-A(n,k-1)]kx^k.$$

Now let us expand each summand individually. Since $A(n,-1)=0$ then

$$\sum_{k=0}^{n+1} A(n,k-1)x^k = x\sum_{k=0}^{n+1} A(n,k-1)x^{k-1} = x\sum_{k=0}^{n} A(n,k)x^k = xf_n(x). \tag{3.2.2}$$

Since $A(n,n+1)=0$ then

$$\sum_{k=0}^{n+1} A(n,k)x^k = \sum_{k=0}^{n} A(n,k)x^k = f_n(x). \tag{3.2.3}$$

Now we separate $\sum_{k=0}^{n+1}[A(n,k)-A(n,k-1)]kx^k$ in two sums:

$$\sum_{k=0}^{n+1} A(n,k)kx^k = \sum_{k=1}^{n} A(n,k)kx^k = x\sum_{k=1}^{n} A(n,k)kx^{k-1} = xf'_n(x). \tag{3.2.4}$$

$$\sum_{k=0}^{n+1} A(n,k-1)kx^k = \sum_{k=1}^{n+1} A(n,k-1)kx^k = \sum_{k=0}^{n} A(n,k)(k+1)x^{k+1}.$$

This last sum can be separated again into two:

$$\begin{aligned}\sum_{k=0}^{n} A(n,k)(k+1)x^{k+1} &= \sum_{k=1}^{n} A(n,k)kx^{k+1} + \sum_{k=0}^{n} A(n,k)x^{k+1} \\ &= x^2\sum_{k=1}^{n} A(n,k)kx^{k-1} + x\sum_{k=0}^{n} A(n,k)x^k \\ &= x^2 f'_n(x) + xf_n(x). \end{aligned} \tag{3.2.5}$$

If we input (3.2.2), (3.2.3), (3.2.4), (3.2.5) into $f_{n+1}(x)$, we have

$$\begin{aligned} f_{n+1}(x) &= (n+2)xf_n(x) + f_n(x) + xf'_n(x) - x^2f'_n(x) - xf_n(x) \\ &= (x-x^2)f'_n(x) + (1+x+nx)f_n(x). \end{aligned}$$

Note that f_n and f_{n+1} satisfy the hypothesis of Proposition 3.2.5 because:

- f_n is a polynomial of degree n with n distinct real roots $\alpha_n < \cdots < \alpha_2 < \alpha_1 < 0$ (the roots are negative because the coefficients are positive);

- f_{n+1} is a polynomial of degree $n+1$;
- $f_n(0) = f_{n+1}(0) = 1 > 0$;
- $f'_n(\alpha_i)$ and $f_{n+1}(\alpha_i)$ have opposite signs because $f_{n+1}(\alpha_i) = (\alpha_i - \alpha_i^2)f'_n(\alpha_i)$.

Thus f_{n+1} has $n+1$ distinct roots and the induction is complete. ▮

A distribution of the form $[(0,\ldots,0,1,0,\ldots,0),n]$ where 1 is at position k is called the *point-mass distribution at* (n,k) and is denoted by δ_{nk}. We will prove that the distributions of δ_{nk} at future levels $p > n$ are also PF sequences. To do this, set $a(0,j) := \delta_{0j}$ and, for $m \geq 1$ and $j \in \mathbb{Z}$,

$$a(m,j) = (n+m-k-j+1)a(m-1,j-1) + (k+j+1)a(m-1,j). \qquad (3.2.6)$$

Note that $a(m,j)$ is the number of paths from the vertex (n,k) to the vertex $(n+m,k+j)$. We also set $a(m,j)=0$ if $m,j<0$.

Corollary 3.2.7. *The distributions of δ_{nk} at levels $p \geq n$ are PF sequences.*

Proof: At level $p = n$ we have $g_n(x) = x^k$ (because the only nonzero coefficient of δ_{nk} is 1 at position k), which has only real roots ($x=0$ is the only root). At level $p = n+1$ we have

$$g_{n+1}(x) = (k+1)x^k + (n-k+1)x^{k+1} = x^k[(k+1)+(n-k+1)x].$$

This polynomial has only real roots since it is a product of x^k and $h_1(x) = (k+1) + (n-k+1)x$, which has only one real root, namely, $x = -\frac{k+1}{n-k+1}$.

In general, write $p = n+m$ (for $m \geq 0$), then $g_p = g_{n+m}(x) = x^k \cdot h_m(x)$ where $h_m(x) = \sum_{j=0}^{m} a(m,j)x^j$ and $a(m,j)$ is given by equation (3.2.6). In the previous paragraph we proved that $h_m(x)$ has only one real roots for $m=1$. Now we assume the induction hypothesis that for $m>1$, $h_m(x)$ has m distinct real roots. Note that

$$\begin{aligned}
h_{m+1}(x) &= \sum_{j=0}^{m+1} a(m+1,j)x^j \\
&= \sum_{j=0}^{m+1} [(n+m-k-j+2)a(m,j-1) + (k+j+1)a(m,j)]x^j \\
&= \sum_{j=0}^{m+1} [(p-k-j+2)a(m,j-1) + (k+j+1)a(m,j)]x^j \\
&= \sum_{j=0}^{m+1} (p-k-j+2)a(m,j-1)x^j + \sum_{j=0}^{m+1} (k+j+1)a(m,j)x^j
\end{aligned}$$

$$\begin{aligned} &= \sum_{j=0}^{m+1}(p-k+2)a(m,j-1)x^j - \sum_{j=0}^{m+1} ja(m,j-1)x^j \\ &\quad + \sum_{j=0}^{m+1}(k+1)a(m,j)x^j + \sum_{j=0}^{m+1} ja(m,j)x^j. \end{aligned}$$

Now let us expand each sum individually. Since $a(m,-1)=0$ then

$$\sum_{j=0}^{m+1} a(m,j-1)x^j = x\sum_{j=0}^{m+1} a(m,j-1)x^{j-1} = x\sum_{j=0}^{m} a(m,j)x^j = xh_m(x). \tag{3.2.7}$$

Thus the first sum is equal to $(p-k+2)xh_m(x)$. Since $a(m,m+1)=0$ then

$$\sum_{j=0}^{m+1} a(m,j)x^j = \sum_{j=0}^{m} a(m,j)x^j = h_m(x). \tag{3.2.8}$$

Thus the third sum is equal to $(k+1)h_m(x)$. As for the fourth sum, we have

$$\sum_{j=0}^{m+1} ja(m,j)x^j = \sum_{j=1}^{m} ja(m,j)x^j = x\sum_{j=1}^{m} ja(m,j)x^{j-1} = xh'_m(x). \tag{3.2.9}$$

Lastly, let us simplify the second sum:

$$\begin{aligned} \sum_{j=0}^{m+1} ja(m,j-1)x^j &= \sum_{j=1}^{m+1} ja(m,j-1)x^j = \sum_{j=0}^{m}(j+1)a(m,j)x^{j+1} \\ &= \sum_{j=1}^{m} ja(m,j)x^{j+1} + \sum_{j=0}^{m} a(m,j)x^{j+1} \\ &= x^2\sum_{j=1}^{m} ja(m,j)x^{j-1} + x\sum_{j=0}^{m} a(m,j)x^j \\ &= x^2h'_m(x) + xh_m(x). \end{aligned} \tag{3.2.10}$$

If we input (3.2.7), (3.2.8), (3.2.9), (3.2.10) into $h_{m+1}(x)$, we have

$$\begin{aligned} h_{m+1}(x) &= (p-k+2)xh_m(x) + (k+1)h_m(x) + xh'_m(x) - x^2h'_m(x) - xh_m(x) \\ &= (x-x^2)h'_m(x) + [k+1+(p-k+1)x]h_m(x). \end{aligned}$$

Again, notice that h_m and h_{m+1} satisfy the hypothesis of Proposition 3.2.5 because:

- h_m is a polynomial of degree m with m distinct real roots $\alpha_m < \cdots < \alpha_2 < \alpha_1 < 0$ (the roots are negative because the coefficients are positive);
- h_{m+1} is a polynomial of degree $m+1$;
- $h_m(0) = h_{m+1}(0) = 1 > 0$;
- $h'_m(\alpha_i)$ and $h_{m+1}(\alpha_i)$ have opposite signs because $h_{m+1}(\alpha_i) = (\alpha_i - \alpha_i^2)h'_n(\alpha_i)$.

Thus h_{m+1} has $m+1$ distinct roots and the induction is complete. ∎

3.3 Means of Eulerian distributions

Definition 3.3.1. *Let $a = (a_k)_{k\in K}$ be a sequence of real numbers indexed by a finite subset $K \subset \mathbb{Z}$. The* mean *of the sequence a is the number*

$$\mu(a) := \frac{1}{\sum_{k\in K} a_k} \sum_{k\in K} k \cdot a_k.$$

Write $\mu = \lfloor \mu \rfloor + \delta(\mu)$ where $\lfloor \mu \rfloor$ is the integral part of μ.

Theorem 3.3.2 (Darroch's rule). *Let $(a_k)_{k\in K}$ be a PF sequence indexed by the set $K = \{1, \ldots, n\}$ and μ its mean. Then this PF sequence has either a unique index m or two consecutive indices m such that $a_m = \max_k a_k$. More precisely:*

- *$m = \lfloor \mu \rfloor$ if $0 \le \delta(\mu) < \frac{1}{\lfloor \mu \rfloor + 2}$;*
- *$m = \lfloor \mu \rfloor$ or $m = \lfloor \mu \rfloor + 1$, or both if $\frac{1}{\lfloor \mu \rfloor + 2} \le \delta(\mu) \le \frac{n - \lfloor \mu \rfloor}{n - \lfloor \mu \rfloor + 1}$;*
- *$m = \lfloor \mu \rfloor + 1$ if $\frac{n - \lfloor \mu \rfloor}{n - \lfloor \mu \rfloor + 1} < \delta(\mu) \le 1$.*

Also, for all $k \in K$,

$$\begin{aligned} \frac{a_k}{a_{k+1}} &\ge 1 \quad \text{if} \quad \mu \le k + \frac{1}{k+2}; \\ \frac{a_k}{a_{k+1}} &\le 1 \quad \text{if} \quad \mu \le k + \frac{n-k}{n-k+1}. \end{aligned}$$

The theorem above is proved in [8, Theorem 4]. We will use Darroch's rule on Eulerian distributions to prove that they attain their maxima in the "centre" of the distribution. In order to use Darroch's rule on Eulerian distributions, we introduce a notation for the Eulerian Bratteli diagram that highlights the position of the vertices within the centre of the diagram. We will denote this by $[n, k]$ (see Figure 3.3).

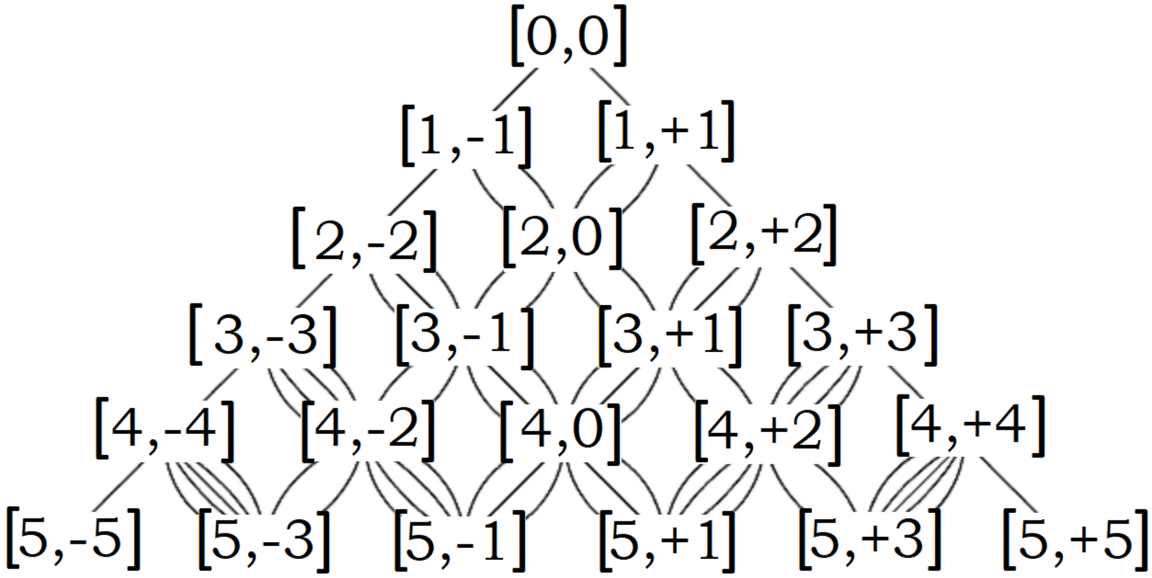

Figure 3.3: The Symmetric Notation

With this notation, a vertex is a pair of the form $[n, n-2j]$ (if it's on the right side of the diagram) or $[n, -n+2j]$ (if it's on the left side) where $j \in J_n = \{0, \dots, \lfloor \frac{n}{2} \rfloor\}$. Note that $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ if n is even, and $[n, n-2j] = [n, -n+2j] = [n, 0]$. Vertices of the form $[n, 0]$ for n even and $[n, -1]$, $[n, +1]$ for n odd are named the *centre* of the diagram. The edge connection is the following:

- There are $j+1$ edges from $[n, n-2j]$ to $[n+1, n-2j+1]$;
- There are $n-j+1$ edges from $[n, n-2j]$ to $[n+1, n-2j-1]$;
- There are $j+1$ edges from $[n, -n+2j]$ to $[n+1, -n+2j-1]$;
- There are $n-j+1$ edges from $[n, -n+2j]$ to $[n+1, -n+2j+1]$.

Proposition 3.3.3. *Let δ_{nk} be the point-mass distribution on the vertex $[n, k]$, i.e., $\delta_{nk} = (0, \dots, 0, 1, 0, \dots, 0)$ where 1 is at position k. Consider the map $\varphi_n : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$ given by the matrix*

$$\varphi_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ n & 2 & 0 & \cdots & 0 & 0 \\ 0 & n-1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 3 & n-1 & 0 \\ 0 & 0 & \cdots & 0 & 2 & n \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Then $\mu(\varphi_n(\delta_{nk})) = \frac{n+1}{n+2}k$.

Proof: There are two cases to consider.

If $k = -n + 2j$**.** In this case, $\varphi_n(\delta_{nk}) = (0, \ldots, j+1, n-j+1, 0, \ldots, 0)$ where $j+1$ is at position $-n+2j-1$ and $n-j+1$ is at position $-n+2j+1$. Therefore:

$$\mu(\varphi_n(\delta_{n,-n+2j})) = \frac{1}{n+2}[(j+1)(-n+2j-1)+(n-j+1)(n-2j+1)] = \frac{n+1}{n+2}(-n+2j).$$

If $k = n - 2j$**.** In this case, $\varphi_n(\delta_{nk}) = (0, \ldots, n-j+1, j+1, 0, \ldots, 0)$ where $n-j+1$ is at position $n-2j-1$ and $j+1$ is at position $n-2j+1$. Therefore:

$$\mu(\varphi_n(\delta_{n,n-2j})) = \frac{1}{n+2}[(n-j+1)(-n+2j-1)+(j+1)(n-2j+1)] = \frac{n+1}{n+2}(n-2j).$$

∎

The following proposition will show how the mean of an arbitrary distribution at level n behaves at future levels $p > n$. Recall that $\varphi_{n,p} = \varphi_{p-1} \circ \cdots \circ \varphi_{n+1} \circ \varphi_n$.

Proposition 3.3.4. *If* a *is an Eulerian distribution at level* n *then for all* $p > n$*:*

$$\mu(\varphi_{n,p}(a)) = \frac{n+1}{p+1}\mu(a) \tag{3.3.1}$$

Proof: This proof is by induction on p. For $p = n+1$, we must prove that

$$\mu(\varphi_n(a)) = \frac{n+1}{n+2}\mu(a) \qquad \text{(recall that } \varphi_n := \varphi_{n,n+1}\text{)}.$$

For $a = (v_{-n}, v_{-n+2}, \ldots, v_{+n-2}, v_{+n})$ we have

$$a = \sum_{j \in J_n} v_{-n+2j}\delta_{n,-n+2j} + v_{n-2j}\delta_{n,n-2j} - v_0\delta_{n,0}.$$

It follows that, using linearity of the mean:

$$\begin{aligned}\mu(\varphi_n(a)) \quad = \quad & \frac{1}{n+2}\Bigg[\sum_{j \in J_n} v_{-n+2j}[(j+1)(-n+2j-1) + (n-j+1)(-n+2j+1)] \\ & + \sum_{j \in J_n} v_{n-2j}[(n-j+1)(n-2j-1) + (j+1)(n-2j+1)]\Bigg]\end{aligned}$$

Note that

$$\begin{aligned}(j+1)(-n+2j-1)+(n-j+1)(-n+2j+1) &= (n+1)(-n+2j)\\ (n-j+1)(n-2j-1)+(j+1)(n-2j+1) &= (n+1)(n-2j)\end{aligned}$$

So

$$\mu(\varphi_n(a)) = \frac{n+1}{n+2}\left[\sum_{j\in J_n} v_{-n+2j}(-n+2j) + \sum_{j\in J_n} v_{n-2j}(n-2j)\right] = \frac{n+1}{n+2}\mu(a)$$

We use the same strategy for the general case $p > n$. Assuming the equation (3.3.1) is valid for a fixed $p > n$, we will prove that

$$\mu(\varphi_{n,p+1}(a)) = \frac{n+1}{p+2}\mu(a).$$

Set

$$\varphi_{n,p}(a) = \sum_{j\in J_p} v_{-p+2j}\delta_{p,-p+2j} + v_{p-2j}\delta_{p,p-2j} - v_0\delta_{p,0}$$

and use the linearity of the mean function to write:

$$\begin{aligned}\mu(\varphi_{n,p+1}(a)) \;=\; & \frac{1}{p+2}\Bigg[\sum_{j\in J_p} v_{-p+2j}[(j+1)(-p+2j-1)+(p-j+1)(-p+2j+1)]\\ & + \sum_{j\in J_p} v_{p-2j}[(p-j+1)(p-2j-1)+(j+1)(p-2j+1)]\Bigg]\end{aligned}$$

Note that $(j+1)(-p+2j-1)+(p-j+1)(-p+2j+1) = (p+1)(-p+2j)$ and $(p-j+1)(p-2j-1)+(j+1)(p-2j+1) = (p+1)(p-2j)$. So

$$\mu(\varphi_{n,p+1}(a)) = \frac{p+1}{p+2}\left[\sum_{j\in J_p} v_{-p+2j}(-p+2j) + \sum_{j\in J_p} v_{p-2j}(p-2j)\right] = \frac{p+1}{p+2}\mu(\varphi_{n,p}(a)).$$

Now, using the induction hypothesis (equation 3.3.1), we conclude:

$$\mu(\varphi_{n,p+1}(a)) = \frac{p+1}{p+2}\mu(\varphi_{n,p}(a)) = \frac{(p+1)}{(p+2)}\frac{(n+1)}{(p+1)}\mu(a) = \frac{n+1}{p+2}\mu(a).$$

∎

It follows that if $p \to \infty$ then $\mu(\varphi_{n,p}(a)) \to 0$, therefore the maximum values of the Eulerian distributions $\varphi_{n,p}(a)$ are, for a large p, at position $k = 0$, i.e., at the centre of the Bratteli diagram.

3.4 Further developments on Eulerian distributions

In the previous section we proved that the Eulerian distributions $(A(n,k))_{k=1}^{n}$ are asymptotically normally distributed with the mean asymptotic to $\frac{1}{2}n$. This was proven using the particular nature of the Eulerian numbers that they satisfy the recursion $A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k)$ with $A(1,1) = 1$. One can generalize these results by considering different recursion formulas to generate distinct sequences of numbers.

In Proposition 3.2.6, we found a recursion formula for the Eulerian polynomials $f_n(x) := \sum_{k=0}^{n} A(n,k)x^k$:

$$f_{n+1}(x) = (x-x^2)f_n'(x) + (1+x+nx)f_n(x) \qquad (n \geq 0).$$

In [24], Hsien-Kuei Hwang, Hua-Huai Chern and Guan-Huei Duh study linear recurrences of the form

$$P_n(v) = (\alpha(v)n + \gamma(v))P_{n-1}(v) + \beta(v)(1-v)P_{n-1}'(v) \qquad (n \geq 1).$$

Note that the former is a particular case of the latter when $\alpha(v) = \beta(v) = v$ and $\gamma(v) = 1+v$. In that same paper, they state the following theorem [24, Theorem 1]:

Theorem 3.4.1 (Asymptotic normality of X_n). *Assume that the sequence of functions $P_n(v)$ is defined recursively by*

$$\begin{cases} P_0(v) \text{ given} \\ P_n(v) = a_n(v)P_{n-1}(v) + b_n(v)(1-v)P_{n-1}'(v) \quad (n \geq 1) \end{cases}$$

satisfying:

- *The Taylor coefficients $[v^k]P_n(v)$ are nonnegative for $k, n \geq 0$,*
- *$P_n(v) \not\equiv 0$ for $k, n \geq 0$, and*
- *$P_0(v)$, $\alpha(v)$, $\beta(v)$ and $\gamma(v)$ analytic in $|v| \leq 1$.*

If, furthermore,

$$\alpha(1) + 2\beta(1) > 0 \qquad \text{and} \qquad \sigma^2 > 0,$$

where

$$\mu := \frac{\alpha'(1)}{\alpha(1)+\beta(1)} \qquad \text{and} \qquad \sigma^2 := \mu + \frac{\alpha''(1) - 2\mu\beta'(1) - \alpha(1)\mu^2}{\alpha(1)+2\beta(1)},$$

then the sequence of random variables X_n, defined by

$$\mathbb{P}(X_n = k) = \frac{[v^k]P_n(v)}{P_n(1)} \qquad (k, n \geq 0),$$

satisfies $X_n \sim \mathcal{N}(\mu n, \sigma^2 n)$, namely, X_n is asymptotically normally distributed with the mean and the variance asymptotic to μn and $\sigma^2 n$, respectively.

This result is consistent with Proposition 3.3.4. In fact, for the Eulerian recurrence, we have $\alpha(v) = \beta(v) = v$, thus

$$\mu = \frac{\alpha'(1)}{\alpha(1) + \beta(1)} = \frac{1}{1+1} = \frac{1}{2}$$

and

$$\sigma^2 := \mu + \frac{\alpha''(1) - 2\mu\beta'(1) - \alpha\mu^2}{\alpha(1) + 2\beta(1)} = \frac{1}{2} + \frac{0 - 1 - \frac{1}{4}}{1+2} = \frac{1}{12}$$

Therefore, X_n is asymptotically normally distributed with the mean and the variance asymptotic to $\frac{1}{2}n$ and $\frac{1}{12}n$, respectively. This implies that the peak of the Eulerian distributions are located at the centre of the finite sequences $A(n,k)_{k=1}^{n}$ for large values of n.

Chapter 4

Total variation distance of distributions in the Eulerian Bratteli diagram

Let δ_{nk} and $\delta_{n,k+1}$ be two adjacent point-mass distributions at level n and consider the distribution $\delta_{nk} - \delta_{n,k+1} = (0, \dots, 0, 1, -1, 0, \dots, 0)$ where 1 is at position k. If $n < p$ and $\varphi_{n,p} : \mathbb{Z}^n \to \mathbb{Z}^p$ are the transition matrices for the Eulerian diagram, we refer to $\varphi_{n,p}(\delta_{nk} - \delta_{n,k+1})$ as *difference distributions* (examples are given in the next subsections). In this chapter we are going to count paths between vertices in the Eulerian diagram. Denote by $\dim(v, w)$ the number of paths connecting vertices v and w. Since we have multiple notations for a vertex (e.g., (n, k) in Chapter 1 and $[n, k]$ in Section 3.3), we may use $\dim(v, w)$ replacing v and w by whatever notation is convenient for the occasion.

The goal of this chapter is to prove a combinatorial property of the difference distributions in the Eulerian diagram and use it to prove that these differences converge to zero in the symmetric measure. More precisely, the first result is the following (Theorem 4.4.1): if $n < p$ and (n, k) and (p, j) are two vertices in the Eulerian diagram, then

$$\dim((n,k),(p,j)) = \sum_{i \le j} \dim((n+1,k),(p+1,i)) - \dim((n+1,k+1),(p+1,i)),$$

$$\dim((n,k),(p,j)) = \sum_{i > j} \dim((n+1,k+1),(p+1,i)) - \dim((n+1,k),(p+1,i)).$$

Later, we prove in Theorem 4.5.5 that there exists an index $j_0 \in \{1, \dots, p+1\}$ such that

$$\dim((n+1,k),(p+1,i)) \ge \dim((n+1,k+1),(p+1,i)) \qquad \text{for } i \le j_0,$$

$$\dim((n+1,k),(p+1,i)) < \dim((n+1,k+1),(p+1,i)) \qquad \text{for } i > j_0.$$

This allows us to conclude that there is a bound for the sum of absolute value of the differences (Corollary 4.5.6):

$$\sum_{i=1}^{p+1} |\dim((n+1,k),(p+1,i)) - \dim((n+1,k+1),(p+1,i))| \leq 2A(p,c)$$

where $A(p,c)$ is the *central Eulerian number* at level p. The centre c of level p is $c=\frac{p}{2}$ is p is even and $c=\frac{p+1}{2}$ if p is odd.

Finally, in Theorem 4.6.3, we prove that given two finite paths x and y and their respective cylinder sets $C(x)$ and $C(y)$; if we set $C(p,j)$ to be the union of cylinder sets $C(z)$ where z are all the paths arriving at the vertex (p,j) then

$$\lim_{p\to\infty} \sum_{j=1}^{p} |\mu(C(p,j)\cap C(x)) - \mu(C(p,j)\cap C(y))| = 0.$$

We illustrate these results with a couple of examples.

4.1 Example: difference distributions from vertices $(2,1)$ and $(2,2)$

This example is illustrated in Figure 4.1.

If $\delta_{21}=(1,0)$ and $\delta_{22}=(0,1)$, then

$$\begin{aligned}
\delta_{21}-\delta_{22} &= (1,-1);\\
\varphi_{2,3}(\delta_{21}-\delta_{22}) &= (1,0,-1);\\
\varphi_{2,4}(\delta_{21}-\delta_{22}) &= (1,3,-3,-1);\\
\varphi_{2,5}(\delta_{21}-\delta_{22}) &= (1,10,0,-10,-1);\\
\varphi_{2,6}(\delta_{21}-\delta_{22}) &= (1,25,40,-40,-25,-1).
\end{aligned}$$

Notice that the difference distributions are antisymmetrical in the sense that the first value is equal in absolute value to the last one; the second is equal in absolute value to the second-last; and so on. This happens because the starting vertices $(2,1)$ and $(2,2)$ are in opposite sides of the diagram (this will be studied more precisely in Section 4.4).

We will prove later the following combinatorial result. Consider the sequence of sums at each side of the diagram (highlighted in Figure 4.1, top) and notice that not only they are equal in absolute value (as expected due to the symmetry of the

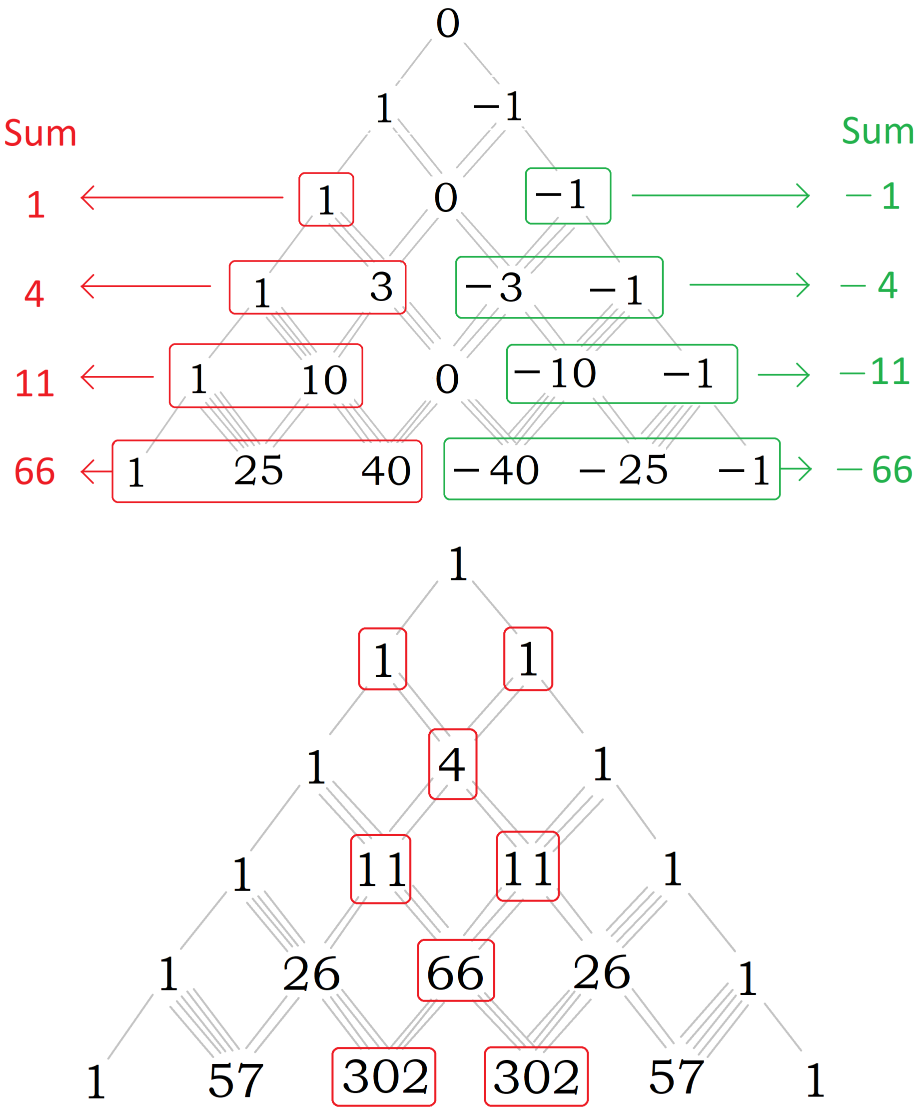

Figure 4.1: Difference distributions and the central Eulerian numbers

Eulerian Bratteli diagram) but they also satisfy the following equation:

$$\sum_{j=1}^{p+1} |\operatorname{dim}((2,1),(p+1,j)) - \operatorname{dim}((2,2),(p+1,j))| = 2A(p,c)$$

where $A(p,c)$ is the *central Eulerian numbers* at level p (highlighted in Figure 4.1, bottom).

In the next example we will illustrate that even when we start from two nonsymmetric adjacent vertices (n, k) and $(n, k+1)$, we still have

$$\sum_{j=1}^{p+1} |\dim((n,k),(p+1,j)) - \dim((n,k+1),(p+1,j))| \leq 2A(p,c).$$

The proof of this statement will be given in Section 4.4.

4.2 Example: difference distributions from vertices $(4,1)$ and $(4,2)$

This example is illustrated in Figure 4.2. The difference distributions are written below:

$$\begin{aligned}
\delta_{41} - \delta_{42} &= (1,-1,0,0); \\
\varphi_{4,5}(\delta_{41} - \delta_{42}) &= (1,2,-3,0,0); \\
\varphi_{4,6}(\delta_{41} - \delta_{42}) &= (1,9,-1,-9,0,0); \\
\varphi_{4,7}(\delta_{41} - \delta_{42}) &= (1,24,42,-40,-27,0,0); \\
\varphi_{4,8}(\delta_{41} - \delta_{42}) &= (1,55,270,50,-295,-81,0,0).
\end{aligned}$$

If we choose, at a future level $p > 4$, a certain position $j_0 \in \{1, \ldots, p\}$, and sum the coefficients of the difference distribution $\varphi_{4,p}(\delta_{41} - \delta_{42})$ from 1 to j_0, the outcome will be equal in absolute value to the sum of the coefficients of the difference distribution from $j_0 + 1$ to p. For example, if $p = 8$, then $\varphi_{4,8}(\delta_{41} - \delta_{42}) = (1,55,270,50,-295,-81,0,0)$. If we choose $j_0 = 4$ then

$$\begin{aligned}
\sum_{j \leq j_0} \varphi_{4,8}(\delta_{41} - \delta_{42})|_j &= \sum_{j=1}^{4} \varphi_{4,8}(\delta_{41} - \delta_{42})|_j = 1 + 55 + 270 + 50 = 376; \\
\sum_{j > j_0} \varphi_{4,8}(\delta_{41} - \delta_{42})|_j &= \sum_{j=5}^{8} \varphi_{4,8}(\delta_{41} - \delta_{42})|_j = -295 - 81 = -376.
\end{aligned}$$

The number 376 is exactly the number of paths from the vertex right above $(4,1)$ and $(4,2)$ (which is the vertex $(3,1)$ highlighted with a circle in Figure 4.2) arriving to the vertex $(7,4) = (p-1, j_0)$. Recall that the j-th coefficient of $\varphi_{4,p}(\delta_{41} - \delta_{42})$ is the difference between $\dim((4,1),(p,j))$ and $\dim((4,2),(p,j))$. Therefore we can rewrite the above result as

$$\sum_{j \leq j_0} \dim((4,1),(p,j)) - \dim((4,2),(p,j)) = \dim((3,1),(p-1,j_0));$$

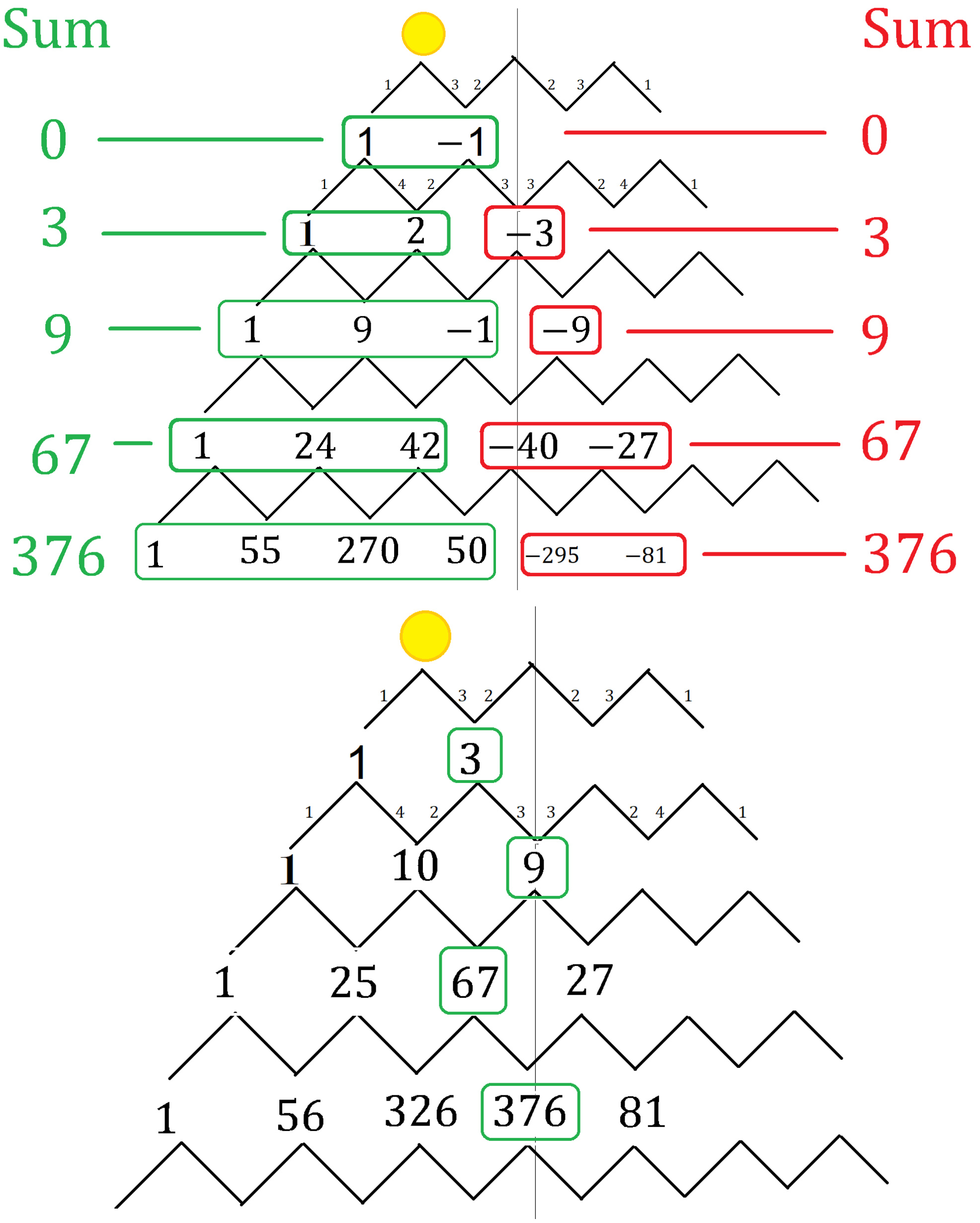

Figure 4.2: Another difference distribution

$$\sum_{j>j_0} \dim((4,1),(p,j)) - \dim((4,2),(p,j)) \quad = \quad -\dim((3,1),(p-1,j_0)).$$

or summarise the sums above as:

$$\sum_{j=1}^{p} |\dim((4,1),(p,j)) - \dim((4,2),(p,j))| = 2\dim((3,1),(p-1,j_0))$$

for some $j_0 \in \{1,\ldots,p\}$.

Since the Eulerian distributions have their maxima around the centre (as proved in Section 3.3) then $\dim((3,1),(p-1,j_0)) \leq \dim((3,1),(p-1,c))$ where c is the centre of the diagram. Furthermore, $\dim((3,1),(p-1,c)) \leq A(p-1,c)$ since the central Eulerian numbers count the number of paths from the apex instead of starting from an arbitrary point $(3,1)$. Thus we can conclude that for all $p > 4$ there exists $j_0 \in \{1,\ldots,p-1\}$ such that

$$\sum_{j=1}^{p} |\dim((4,1),(p,j)) - \dim((4,2),(p,j))| \leq 2A(p-1,j_0).$$

4.3 Left-right notation

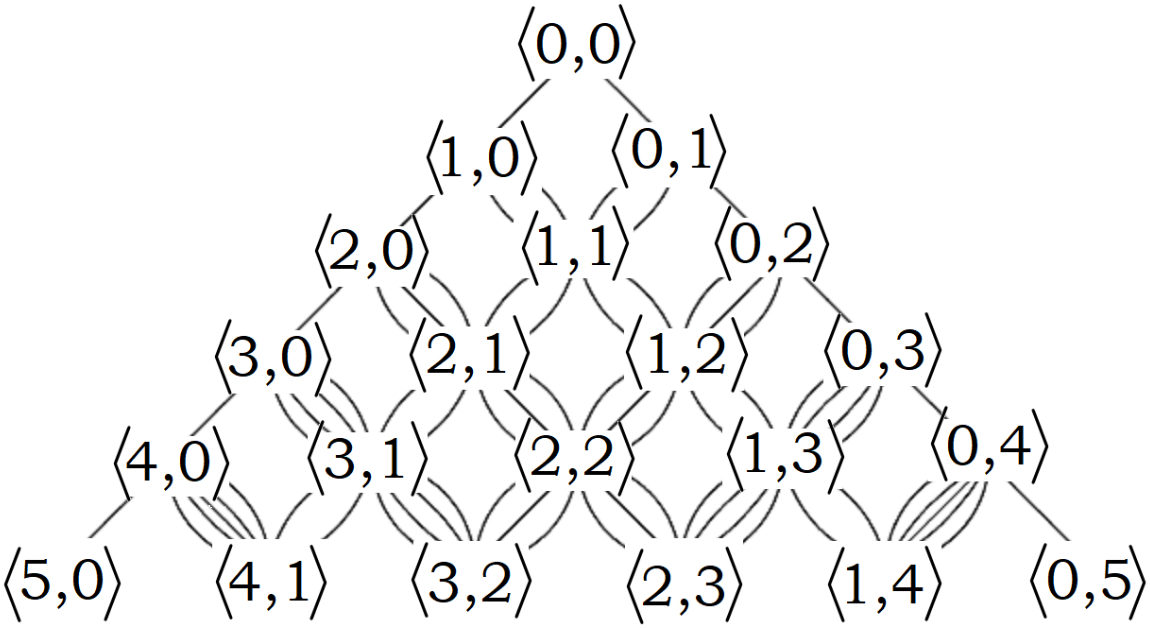

Figure 4.3: The Left-right notation

In order to prove the aforementioned result for a general (n,k), let us describe the left-right notation introduced by Karl Petersen and Alexander Varchenko [28].

This notation will greatly help to count the number of paths in the Eulerian Bratteli diagram.

In this notation, a vertex is represented by a pair $\langle p, q\rangle$ where p is the number of "left turns" from the root and q is the number of "right turns" from the root (see Figure 4.3). Starting from $\langle p, q\rangle$, there are $q+1$ edges connecting it to $\langle p+1, q\rangle$ and $p+1$ edges to $\langle p, q+1\rangle$. Notice that, within a level, the sum of the two coordinates is fixed, thus we say say that the level of vertex $\langle p, q\rangle$ is $p+q$. We say that a vertex $\langle p, q\rangle$ is at the *left side* (respectively *right side*) of the diagram if $p > q$ (respectively $p < q$). Observe that within a level, say, n, the vertices are of the form $\langle n, 0\rangle$, $\langle n-1, 1\rangle$,..., $\langle n-k, k\rangle$,..., $\langle 1, n-1\rangle$, $\langle 0, n\rangle$.

By [28, Theorem 3.1], the number of paths connecting the vertices $\langle p, q\rangle$ and $\langle p+i, q+j\rangle$ is given by

$$A_{p,q}\langle i, j\rangle = \sum_{t=0}^{i}(-1)^{i-t}\binom{p+q+t+1}{t}\binom{p+q+i+j+2}{i-t}(p+1+t)^{i+j}, \tag{4.3.1}$$

or, symmetrically [28, Corollary 3.4], by

$$A_{p,q}\langle i, j\rangle = \sum_{t=0}^{j}(-1)^{j-t}\binom{p+q+t+1}{t}\binom{p+q+i+j+2}{j-t}(q+1+t)^{i+j}. \tag{4.3.2}$$

4.4 A combinatorial result

Let $v = \langle p, q\rangle$ be a vertex at level $p+q$ and $w = \langle p+i, q+j\rangle$ be a vertex at level $p+q+i+j$. Consider $v_1 = \langle p+1, q\rangle$ and $v_2 = \langle p, q+1\rangle$ the two vertices connected to v at level $p+q+1$; and W_l (respectively W_r), the set of vertices on the left side (respectively on the right side) of w at level $p+q+i+j+1$ (see Figure 4.4). We are going to prove that:

$$\dim(v, w) = \sum_{x\in W_l} \dim(v_1, x) - \dim(v_2, x);$$
$$\dim(v, w) = \sum_{x\in W_r} \dim(v_2, x) - \dim(v_1, x).$$

Formally, we should prove

Theorem 4.4.1. *If* $A_{p,q}\langle i, j\rangle$ *is the number of paths connecting the vertices* $\langle p, q\rangle$ *and* $\langle p+i, q+j\rangle$, *then*

1. $A_{p,q}\langle i, j\rangle = \sum_{l=0}^{j} A_{p+1,q}\langle i+l, j-l\rangle - A_{p,q+1}\langle i+1+l, j-1-l\rangle$

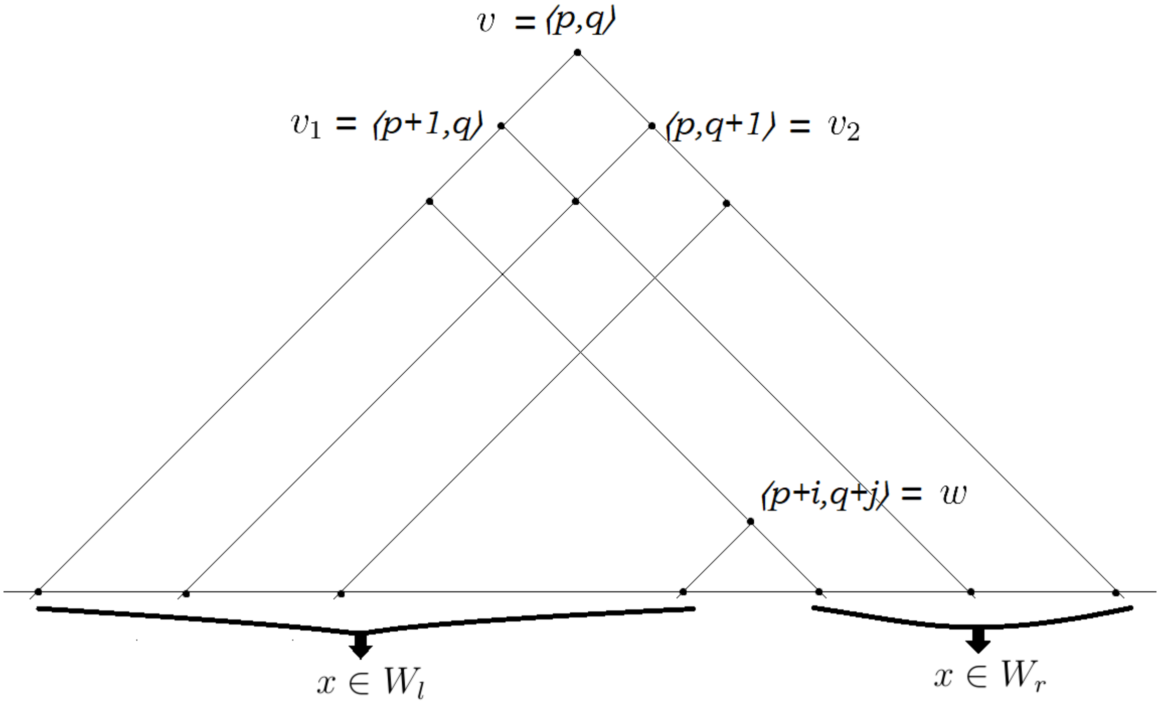

Figure 4.4: Notation for the combinatorial result

2. $A_{p,q}\langle i,j\rangle = \sum_{r=0}^{i} A_{p,q+1}\langle i-r, j+r\rangle - A_{p+1,q}\langle i-1-r, j+1+r\rangle$

To prove the above theorem, we will need some lemmas based on Pascal's rule:

$$\forall n \in \mathbb{N}, 1 \le k \le n, \qquad \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}.$$

We set, if $n < m$, $\binom{n}{m} := 0$.

Lemma 4.4.2. *Fixed $n \in \mathbb{N}$ we have, for all $j \in \mathbb{N}$,*

$$\sum_{k=0}^{j} (-1)^k \binom{n}{k} = (-1)^j \binom{n-1}{j}. \tag{4.4.1}$$

Proof: We will prove this lemma by induction. The case $j = 0$ is trivial, as

$$(-1)^0 \binom{n}{0} = 1 = (-1)^0 \binom{n-1}{0}.$$

Assuming that the lemma is valid for j, let us verify that

$$\sum_{k=0}^{j+1}(-1)^k\binom{n}{k} = (-1)^{j+1}\binom{n-1}{j+1}.$$

Using the induction hypothesis and the Pascal's rule, we have

$$\begin{aligned}
\sum_{k=0}^{j+1}(-1)^k\binom{n}{k} &= \sum_{k=0}^{j}(-1)^k\binom{n}{k} + (-1)^{j+1}\binom{n}{j+1} \\
&= (-1)^j\binom{n-1}{j} + (-1)^{j+1}\binom{n}{j+1} \\
&= (-1)^j\left[\binom{n-1}{j} - \binom{n}{j+1}\right] \\
&= (-1)^j\left[-\binom{n-1}{j+1}\right] = (-1)^{j+1}\binom{n-1}{j+1}.
\end{aligned}$$

∎

Corollary 4.4.3. *The following identity is true:*

$$\sum_{t=0}^{j}(-1)^{j-t}\binom{n}{j-t} = (-1)^j\binom{n-1}{j}$$

.

Proof: This sum is identical to the one in the preceding lemma if we set $t = j-k$. ∎

Lemma 4.4.4. *For $0 \leq k \leq n$ and $j \in \mathbb{N}$, the following inequality holds:*

$$\binom{n+j}{k} + \binom{n-j}{k} \geq 2\binom{n}{k}.$$

Proof: For $j \in \mathbb{N}$, let $a_j := \binom{n+j}{k} + \binom{n-j}{k}$, so

$$a_0 = 2\binom{n}{k}, \quad a_1 = \binom{n+1}{k} + \binom{n-1}{k}, \quad a_2 = \binom{n+2}{k} + \binom{n-2}{k}, \ldots$$

We prove that $a_0 \leq a_1 \leq a_2 \leq \ldots$. In other words, $a_j \geq a_{j-1}$ for every $j \in \mathbb{N}^*$. In fact, using Pascal's rule, we have

$$a_j - a_{j-1} = \binom{n+j}{k} + \binom{n-j}{k} - \binom{n+j-1}{k} - \binom{n-j+1}{k}$$

$$\begin{aligned} &= \binom{n+j}{k} - \binom{n+j-1}{k} + \binom{n-j}{k} - \binom{n-j+1}{k} \\ &= \binom{n+j-1}{k-1} - \binom{n-j}{k-1} \geq 0. \end{aligned}$$

Therefore, $a_j \geq a_{j-1}$ for all $j \in \mathbb{N}^*$. It follows that

$$\binom{n+j}{k} + \binom{n-j}{k} = a_j \geq a_{j-1} \geq \cdots \geq a_1 \geq a_0 = 2\binom{n}{k}.$$

■

The next two lemmas are technical tools that we use in the proof of Theorem 4.4.1. They count the difference between the number of paths connecting two adjacent vertices to another one in a future level.

Lemma 4.4.5. *If $A_{p,q}\langle i,j\rangle$ is the number of paths connecting the vertices $\langle p,q\rangle$ and $\langle p+i,q+j\rangle$, then $A_{p+1,q}\langle i,j\rangle - A_{p,q+1}\langle i+1,j-1\rangle =$*

$$\sum_{t=0}^{j} (-1)^{j-t} \binom{p+q+1+t}{t} \binom{p+q+i+j+3}{j-t} (q+1+t)^{i+j}$$

Proof: Using (4.3.2) we have:

$$\begin{aligned} A_{p+1,q}\langle i,j\rangle &= \sum_{t=0}^{j} (-1)^{j-t} \binom{p+q+t+1}{t} \binom{p+q+i+j+3}{j-t} (q+1+t)^{i+j} \\ A_{p,q+1}\langle i+1,j-1\rangle &= \sum_{t=0}^{j-1} (-1)^{j-1-t} \binom{p+q+t+1}{t} \binom{p+q+i+j+3}{j-1-t} (q+2+t)^{i+j} \end{aligned}$$

By setting $k := j-t$ in $A_{p+1,q}$ and $k := j-1-t$ in $A_{p,q+1}$, we get

$$\begin{aligned} A_{p+1,q}\langle i,j\rangle &= \sum_{k=0}^{j} (-1)^{k} \binom{p+q+j-k+1}{j-k} \binom{p+q+i+j+3}{k} (q+1+j-k)^{i+j} \\ A_{p,q+1}\langle i+1,j-1\rangle &= \sum_{k=0}^{j-1} (-1)^{k} \binom{p+q+j-k}{j-k-1} \binom{p+q+i+j+3}{k} (q+1+j-k)^{i+j} \end{aligned}$$

So $A_{p+1,q}\langle i,j\rangle - A_{p,q+1}\langle i+1,j-1\rangle =$

$$\sum_{k=0}^{j} (-1)^{k} \left[\binom{p+q+j-k+1}{j-k} - \binom{p+q+j-k}{j-k-1} \right] \binom{p+q+i+j+3}{k} (q+1+j-k)^{i+j}$$

Using Pascal's rule, the brackets are equal to $\binom{p+q+j-k}{j-k}$:

$$\sum_{k=0}^{j}(-1)^k\binom{p+q+j-k}{j-k}\binom{p+q+i+j+3}{k}(q+1+j-k)^{i+j}$$

or, reverting back to $t=j-k$ we have $A_{p+1,q}\langle i,j\rangle - A_{p,q+1}\langle i+1,j-1\rangle =$

$$\sum_{t=0}^{j}(-1)^{j-t}\binom{p+q+t}{t}\binom{p+q+i+j+3}{j-t}(q+1+t)^{i+j}$$

■

Lemma 4.4.6. *If* $A_{p,q}\langle i,j\rangle$ *is the number of paths connecting the vertices* $\langle p,q\rangle$ *and* $\langle p+i,q+j\rangle$*, then* $A_{p,q+1}\langle i,j\rangle - A_{p+1,q}\langle i-1,j+1\rangle =$

$$\sum_{t=0}^{i}(-1)^{i-t}\binom{p+q+1+t}{t}\binom{p+q+i+j+3}{i-t}(p+1+t)^{i+j}$$

Proof: Using (4.3.1) we have:

$$\begin{aligned}
A_{p,q+1}\langle i,j\rangle &= \sum_{t=0}^{i}(-1)^{i-t}\binom{p+q+t+1}{t}\binom{p+q+i+j+3}{i-t}(p+1+t)^{i+j}\\
A_{p+1,q}\langle i-1,j+1\rangle &= \sum_{t=0}^{i-1}(-1)^{i-1-t}\binom{p+q+t+1}{t}\binom{p+q+i+j+3}{i-1-t}(p+2+t)^{i+j}
\end{aligned}$$

By setting $k := i-t$ in $A_{p,q+1}$ and $k := i-1-t$ in $A_{p+1,q}$, we get

$$\begin{aligned}
A_{p,q+1}\langle i,j\rangle &= \sum_{k=0}^{i}(-1)^{k}\binom{p+q+i-k+1}{i-k}\binom{p+q+i+j+3}{k}(p+1+i-k)^{i+j}\\
A_{p+1,q}\langle i-1,j+1\rangle &= \sum_{k=0}^{i-1}(-1)^{k}\binom{p+q+i-k}{i-k-1}\binom{p+q+i+j+3}{k}(p+1+i-k)^{i+j}
\end{aligned}$$

So $A_{p,q+1}\langle i,j\rangle - A_{p+1,q}\langle i-1,j+1\rangle =$

$$\sum_{k=0}^{i}(-1)^k\left[\binom{p+q+i-k+1}{i-k}-\binom{p+q+i-k}{i-k-1}\right]\binom{p+q+i+j+3}{k}(p+1+i-k)^{i+j}$$

Using Pascal's rule, the brackets are equal to $\binom{p+q+i-k}{i-k}$:

$$\sum_{k=0}^{i}(-1)^k\binom{p+q+i-k}{i-k}\binom{p+q+i+j+3}{k}(p+1+i-k)^{i+j}$$

or, reverting back to $t=i-k$ we have $A_{p,q+1}\langle i,j\rangle - A_{p+1,q}\langle i-1,j+1\rangle =$

$$\sum_{t=0}^{i}(-1)^{i-t}\binom{p+q+t}{t}\binom{p+q+i+j+3}{i-t}(p+1+t)^{i+j}$$

∎

Now we prove Theorem 4.4.1:

If $A_{p,q}\langle i,j\rangle$ is the number of paths connecting the vertices $\langle p,q\rangle$ and $\langle p+i,q+j\rangle$, then

1. $A_{p,q}\langle i,j\rangle = \sum_{l=0}^{j} A_{p+1,q}\langle i+l,j-l\rangle - A_{p,q+1}\langle i+1+l,j-1-l\rangle$
2. $A_{p,q}\langle i,j\rangle = \sum_{r=0}^{i} A_{p,q+1}\langle i-r,j+r\rangle - A_{p+1,q}\langle i-1-r,j+1+r\rangle$

Proof: First we prove item 1. Replacing i by $i+l$ and j by $j-l$, for $l=0,\ldots,j$, in Lemma 4.4.5, we have $A_{p+1,q}\langle i+l,j-l\rangle - A_{p,q+1}\langle i+1+l,j-1-l\rangle =$

$$\sum_{t=0}^{j-l}(-1)^{j-l-t}\binom{p+q+1+t}{t}\binom{p+q+i+j+3}{j-l-t}(q+1+t)^{i+j} \tag{4.4.2}$$

Thus, $\sum_{l=0}^{j} A_{p+1,q}\langle i+l,j-l\rangle - A_{p,q+1}\langle i+1+l,j-1-l\rangle =$

$$= \sum_{l=0}^{j}\sum_{t=0}^{j-l}(-1)^{j-l-t}\binom{p+q+1+t}{t}\binom{p+q+i+j+3}{j-l-t}(q+1+t)^{i+j}$$

$$= \sum_{l=0}^{j}\sum_{t=0}^{j}(-1)^{j-l-t}\binom{p+q+1+t}{t}\binom{p+q+i+j+3}{j-l-t}(q+1+t)^{i+j}$$

$$= \sum_{t=0}^{j}\sum_{l=0}^{j}(-1)^{j-l-t}\binom{p+q+1+t}{t}\binom{p+q+i+j+3}{j-l-t}(q+1+t)^{i+j}$$

$$= \sum_{t=0}^{j} \sum_{l=0}^{j} (-1)^{j-l-t} \binom{p+q+i+j+3}{j-l-t} \binom{p+q+1+t}{t} (q+1+t)^{i+j}$$

$$= \sum_{t=0}^{j} \left[\sum_{l=0}^{j} (-1)^{j-l-t} \binom{p+q+i+j+3}{j-l-t} \right] \binom{p+q+1+t}{t} (q+1+t)^{i+j}$$

The part inside the brackets can be simplified using Corollary 4.4.3:

$$\sum_{l=0}^{j} (-1)^{j-l-t} \binom{p+q+i+j+3}{j-l-t} = \binom{p+q+i+j+2}{j-t}.$$

It follows that $\sum_{l=0}^{j} A_{p+1,q}\langle i+l, j-l\rangle - A_{p,q+1}\langle i+1+l, j-1-l\rangle =$

$$= \sum_{t=0}^{j} (-1)^{j-t} \binom{p+q+i+j+2}{j-t} \binom{p+q+1+t}{t} (q+1+t)^{i+j} = A_{p,q}\langle i,j\rangle$$

Now we prove item 2 using Lemma 4.4.6. We replace i by $i-r$ and j by $j+r$, for $r = 0, \ldots, i$, and we get $A_{p,q+1}\langle i-r, j+r\rangle - A_{p+1,q}\langle i-1-r, j+1+r\rangle =$

$$\sum_{t=0}^{i-r} (-1)^{i-r-t} \binom{p+q+1+t}{t} \binom{p+q+i+j+3}{i-r-t} (p+1+t)^{i+j}$$

Note that this expression is exactly like (4.4.2) if we replace $(i,r,p) \leftrightarrow (j,l,q)$. Thus $\sum_{r=0}^{i} A_{p,q+1}\langle i-r, j+r\rangle - A_{p+1,q}\langle i-1-r, j+1+r\rangle =$

$$\sum_{t=0}^{i} (-1)^{i-t} \binom{p+q+i+j+2}{i-t} \binom{p+q+1+t}{t} (p+1+t)^{i+j} = A_{p,q}\langle i,j\rangle$$

This proves Theorem 4.4.1. ∎

4.5 Signs of the differences

Let us rewrite Theorem 4.4.1 using the (n,k) notation. We proved that given a vertex (n,k) and a level $p > n$, we have

$$\dim((n,k),(p,j_0)) = \sum_{j \le j_0} \left[\dim((n+1,k),(p+1,j)) - \dim((n+1,k+1),(p+1,j)) \right]$$

$$\dim((n,k),(p,j_0)) = \sum_{j>j_0} \left[\dim((n+1,k+1),(p+1,j)) - \dim((n+1,k),(p+1,j)) \right]$$

Let us rewrite $\dim((n+1,k),(p+1,j)) - \dim((n+1,k+1),(p+1,j))$ as $\mathrm{Diff}_{n,k,p}(j)$, so

$$\dim((n,k),(p,j_0)) = \sum_{j\leq j_0} \mathrm{Diff}_{n,k,p}(j) \quad \text{and} \quad \dim((n,k),(p,j_0)) = \sum_{j>j_0} -\mathrm{Diff}_{n,k,p}(j).$$

In this section we prove (Theorem 4.5.5) that there exists an index J such that

$$\dim((n+1,k),(p+1,j)) \geq \dim((n+1,k+1),(p+1,j)) \qquad \forall j \leq J,$$

$$\dim((n+1,k),(p+1,j)) < \dim((n+1,k+1),(p+1,j)) \qquad \forall j > J.$$

This implies $\mathrm{Diff}_{n,k,p}(j) \geq 0$ for $j \leq J$ and $-\mathrm{Diff}_{n,k,p}(j) \geq 0$ for $j > J$. Therefore,

$$|\mathrm{Diff}_{n,k,p}(j)| = \mathrm{Diff}_{n,k,p}(j) \ \forall j \leq J \quad \text{and} \quad |\mathrm{Diff}_{n,k,p}(j)| = -\mathrm{Diff}_{n,k,p}(j) \geq 0 \ \forall j > J.$$

It follows that

$$\begin{aligned} \sum_{j=1}^{p+1} |\mathrm{Diff}_{n,k,p}(j)| &= \sum_{j\leq J} |\mathrm{Diff}_{n,k,p}(j)| + \sum_{j>J} |\mathrm{Diff}_{n,k,p}(j)| \\ &= \dim((n,k),(p,J)) + \dim((n,k),(p,J)) = 2\dim((n,k),(p,j)). \end{aligned}$$

We can rewrite this result as follows: if v_k, v_{k+1} are adjacent vertices at level n then for $p > n$ there exist $J \in \{1,\ldots,p\}$ such that

$$\sum_{w\in V_{p+1}} |\dim(v_k,w) - \dim(v_{k+1},w)| = 2\dim(v,(p,J)).$$

4.5.1 Ratio between distributions

We now proceed to prove the aforementioned results. Refer to Figure 4.5 in this subsection.

Let $a = \langle p+1, q\rangle$ and $b = \langle p, q+1\rangle$ be two adjacent vertices at level $p+q+1$. For $i,j > 0$, let us denote by $a(i,j)$ (respectively $b(i,j)$) the number of paths from a to $\langle p+1+i, q+j\rangle$ (respectively $\langle p+i, q+1+j\rangle$). Thus $a(i,j) = A_{p+1,q}\langle i,j\rangle$ and $b(i,j) = A_{p,q+1}\langle i,j\rangle$ and, by the recursion property of the Eulerian Bratteli diagram, we have, for $i,j \geq 0$,

$$\begin{aligned} a(i,j) &= a(i,j-1)(i+p+2) + a(i-1,j)(j+q+1) \qquad (4.5.1) \\ b(i,j) &= b(i,j-1)(i+p+1) + b(i-1,j)(j+q+2) \qquad (4.5.2) \end{aligned}$$

where $a(i,j) = 0$ if $i < 0$ or $j < 0$.

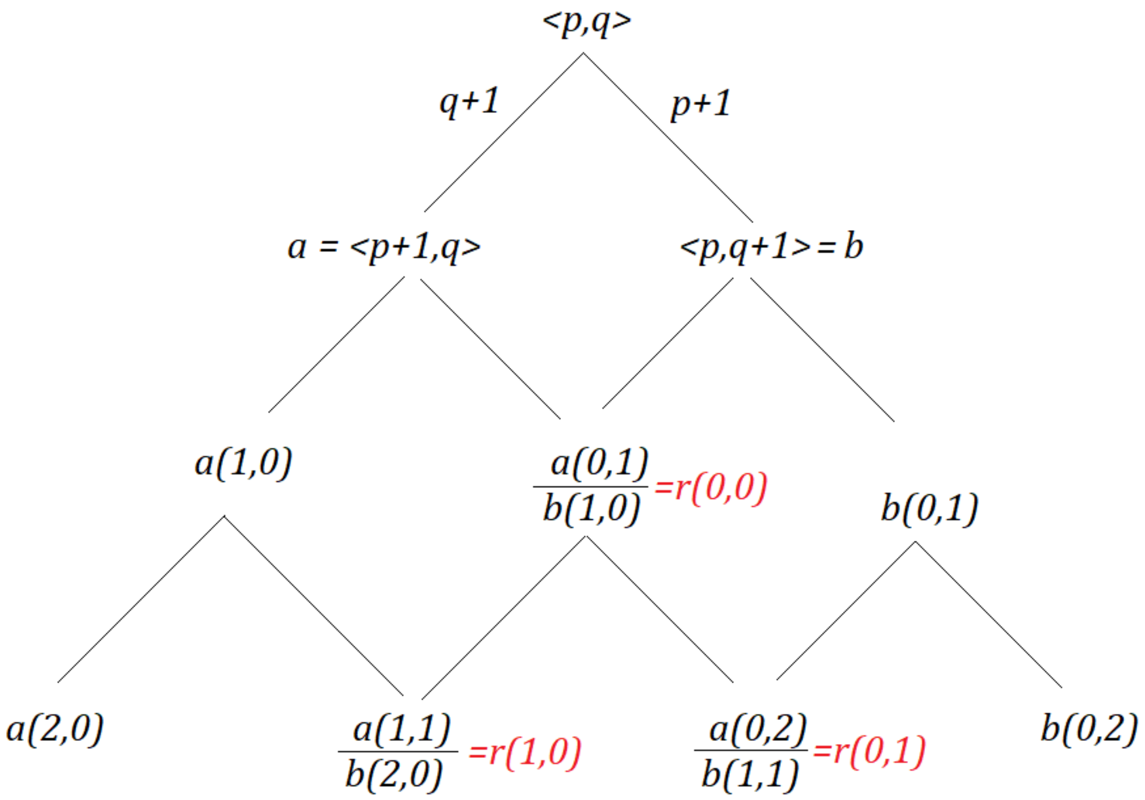

Figure 4.5: Notation for the ratios

For $i, j \geq 0$, let $r(i, j) := \frac{a(i,j+1)}{b(i+1,j)}$ be the ratio of the number of paths connecting a and b to $\langle p+1+i, q+1+j\rangle$. If $r(i, j) > 1$, then there are more paths coming from a than from b (we can simply say a *prevails over* b). Reciprocally, if $r(i, j) < 1$ then b prevails over a.

Example 4.5.1. If $\langle p, q\rangle$ is on the left side of the diagram (i.e. $p > q$) then

$$r(0,0) = \frac{a(0,1)}{b(1,0)} = \frac{(p+2)a(0,0)}{(q+2)b(0,0)} = \frac{p+2}{q+2} > 1. \tag{4.5.3}$$

On the other hand, for $\langle p, q\rangle$ on the right side of the diagram (i.e. $p < q$) then

$$r(0,0) = \frac{a(0,1)}{b(1,0)} = \frac{(p+2)a(0,0)}{(q+2)b(0,0)} = \frac{p+2}{q+2} < 1. \tag{4.5.4}$$

Moreover, if $\langle p, q\rangle$ is on the centre of the diagram (i.e. $p = q$) then

$$r(0,0) = \frac{a(0,1)}{b(1,0)} = \frac{(p+2)a(0,0)}{(q+2)b(0,0)} = \frac{p+2}{q+2} = 1. \tag{4.5.5}$$

In Theorem 4.5.3, we prove that $r(i,j) \leq r(i+1,j)$. This means that when we go "to the left" in the diagram (as i increases), the ratio increases as well. It follows that if a prevails over b at a certain level ($r(i,j) > 1$) then it will continue to do so at all the other vertices "to the left" of that point ($r(i+1,j) > 1$). Respectively, we prove that $r(i,j) \leq r(i,j+1)$. This means that when we go "to the right" in the diagram (as j increases), the ratio decreases. It follows that if b prevails over a at a certain level ($r(i,j) < 1$) then it will continue to do so at all the other vertices "to the right" of that point ($r(i,j+1) < 1$).

Lemma 4.5.2. *For $p,q \geq 0$, assume a collection of positive numbers $\{a(i,j),\ i,j \geq 0,\ (i,j) \neq (0,0)\}$ satisfying (4.5.1) and a collection of positive numbers $\{b(i,j),\ i,j \geq 0,\ (i,j) \neq (0,0)\}$ satisfying (4.5.2). If $r(i,j) = \frac{a(i,j+1)}{b(i+1,j)}$ then for any $i,j \geq 0$ we have*

$$r(i+1,0) \geq r(i,0); \qquad r(0,j) \geq r(0,j+1).$$

Proof: Let $i,j \geq 0$, then

$$\begin{aligned} r(i+1,0) &= \frac{a(i+1,1)}{b(i+2,0)} = \frac{a(i+1,0)(i+p+2) + a(i,1)(q+2)}{b(i+1,0)(q+2)} \\ &= \frac{(i+p+2)}{(q+2)}\frac{a(i+1,0)}{b(i+1,0)} + \frac{a(i,1)}{b(i+1,0)} \\ &= \frac{(i+p+2)}{(q+2)}\frac{a(i+1,0)}{b(i+1,0)} + r(i,0) \geq r(i,0). \end{aligned}$$

By symmetry:

$$r(0,j+1) = \frac{a(0,j+2)}{b(1,j+1)} = \frac{a(0,j+1)(p+2)}{b(1,j)(p+2) + b(0,j+1)(j+q+3)}.$$

If we divide the numerator and the denominator by $b(1,j)(p+2)$ we have:

$$r(0,j+1) = \frac{\frac{a(0,j+1)(p+2)}{b(1,j)(p+2)}}{1 + \frac{b(0,j+1)(j+q+3)}{b(1,j)(p+2)}} = \frac{r(0,j)}{1 + \frac{b(0,j+1)(j+q+3)}{b(1,j)(p+2)}}.$$

Therefore, $r(0,j) = r(0,j+1)\left[1 + \frac{b(0,j+1)}{b(1,j)}\frac{j+q+4}{p+2}\right] \geq r(0,j+1)$. ∎

Below we generalize the above result, instead of starting at $(0,0)$ we will start at a generic (i,j)

Theorem 4.5.3. *For $p,q \geq 0$, assume a collection of positive numbers $\{a(i,j),\ i,j \geq 0,\ (i,j) \neq (0,0)\}$ satisfying (4.5.1) and a collection of positive numbers $\{b(i,j),\ i,j \geq 0,\ (i,j) \neq (0,0)\}$ satisfying (4.5.2). If $r(i,j) = \frac{a(i,j+1)}{b(i+1,j)}$ then for any $i,j \geq 0$ we have*

$$r(i+1,j) \geq r(i,j) \geq r(i,j+1). \tag{4.5.6}$$

Proof: The proof is done by induction. If $(i,j)=(0,0)$ then Lemma 4.5.2 shows that

$$r(i+1,0)\geq r(i,0), \qquad r(1,0)\geq r(0,0)\geq r(0,1) \quad \text{and} \quad r(0,j)\geq r(0,j+1).$$

Now we assume (4.5.6) and prove that

$$r(i+1,j)\geq r(i+1,j+1)\geq r(i,j+1).$$

First we prove $r(i+1,j)\geq r(i+1,j+1)$. From the induction hypothesis, $r(i+1,j)\geq r(i,j+1)$, so

$$\frac{a(i+1,j+1)}{b(i+2,j)}\geq\frac{a(i,j+2)}{b(i+1,j+1)};$$

$$(i+p+3)+\frac{b(i+1,j+1)}{b(i+2,j)}(j+q+3)\geq(i+p+3)+\frac{a(i,j+2)}{a(i+1,j+1)}(j+q+3). \quad (4.5.7)$$

Since

$$\begin{aligned} a(i+1,j+2) &= a(i+1,j+1)(i+p+3)+a(i,j+2)(j+q+3) \\ b(i+2,j+1) &= b(i+2,j)(i+p+3)+b(i+1,j+1)(j+q+3) \end{aligned}$$

then (4.5.7) becomes

$$\frac{b(i+2,j+1)}{b(i+2,j)}\geq\frac{a(i+1,j+2)}{a(i+1,j+1)},$$

$$\frac{a(i+1,j+1)}{b(i+2,j)}\geq\frac{a(i+1,j+2)}{b(i+2,j+1)};$$

which is exactly $r(i+1,j)\geq r(i+1,j+1)$.

The proof of $r(i+1,j+1)\geq r(i,j+1)$ is similar. From the induction hypothesis: $r(i+1,j)\geq r(i,j+1)$, so

$$\frac{a(i+1,j+1)}{b(i+2,j)}\geq\frac{a(i,j+2)}{b(i+1,j+1)};$$

$$\frac{a(i+1,j+1)}{a(i,j+2)}(i+p+3)+(j+q+3)\geq\frac{b(i+2,j)}{b(i+1,j+1)}(i+p+3)+(j+q+3). \quad (4.5.8)$$

Since

$$\begin{aligned} a(i+1,j+2) &= a(i+1,j+1)(i+p+3)+a(i,j+2)(j+q+3) \\ b(i+2,j+1) &= b(i+2,j)(i+p+3)+b(i+1,j+1)(j+q+3) \end{aligned}$$

then (4.5.8) becomes

$$\frac{a(i+1,j+2)}{a(i,j+2)}\geq\frac{b(i+2,j+1)}{b(i+1,j+1)},$$

$$\frac{a(i+1,j+2)}{b(i+2,j+1)} \geq \frac{a(i,j+2)}{b(i+1,j+1)};$$

which is exactly

$$r(i+1,j+1) \geq r(i,j+1)$$

. ▮

4.5.2 Difference between distributions

Now let us rewrite the above results in terms of the notation (n,k). If (n,k) and $(n,k+1)$ are adjacent vertices at level n then

$$r(0,0) = \frac{\dim((n,k),(n+1,k+1))}{\dim((n,k+1),(n+1,k+1))}.$$

A ratio is greater than 1 (respectively less than 1) if the difference between its numerator and denominator is greater than 0 (respectively less than 0). So define $d_{n,k}(p,j) := \dim((n,k),(p,j)) - \dim((n,k+1),(p,j))$. By Theorem 4.5.3, we have

$$d_{n,k}(p+1,j) \geq d_{n,k}(p,j) \geq d_{n,k}(p+1,j+1). \tag{4.5.9}$$

In Example 4.5.1 we proved that:

- $r(0,0) > 1$ if $(n-1,k)$ is on the left side of the diagram;
- $r(0,0) < 1$ if $(n-1,k)$ is on the right side of the diagram;
- $r(0,0) = 1$ if $(n-1,k)$ is on the centre of the diagram.

This means that:

Remark 4.5.4. If $d_{n,k}(p,j) = \dim((n,k),(p,j)) - \dim((n,k+1),(p,j))$ then, in the Eulerian Bratteli diagram, we have

- $d_{n,k}(n+1,k+1) > 0$ if $(n-1,k)$ is on the left side of the diagram;
- $d_{n,k}(n+1,k+1) < 0$ if $(n-1,k)$ is on the right side of the diagram;
- $d_{n,k}(n+1,k+1) = 0$ if $(n-1,k)$ is on the centre of the diagram.

Theorem 4.5.5. *Given two adjacent vertices (n,k) and $(n,k+1)$, then for any future level $p > n$, there exists $j_0 \in \{1,\dots,p\}$ such that*

$$\dim((n,k),(p,j)) \geq \dim((n,k+1),(p,j)) \quad \textit{for} \quad j \leq j_0; \tag{4.5.10}$$
$$\dim((n,k),(p,j)) < \dim((n,k+1),(p,j)) \quad \textit{for} \quad j > j_0. \tag{4.5.11}$$

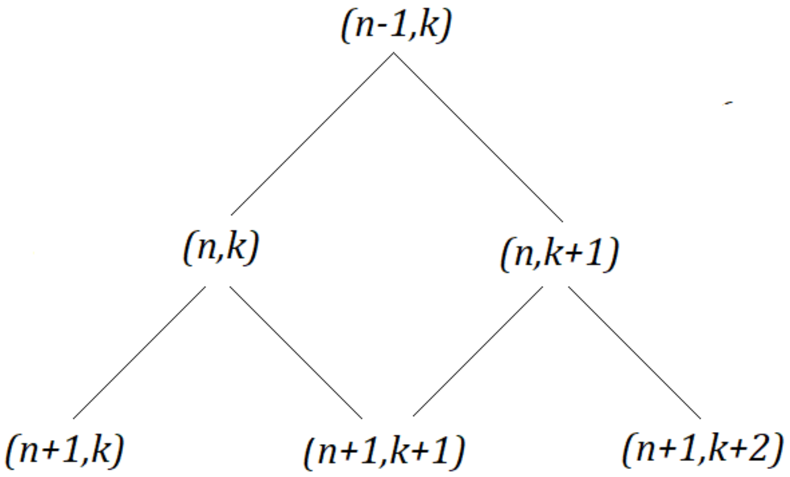

Figure 4.6: Sign of difference at level $n+1$

Proof: By induction on $p > n$, first we prove for $p = n+1$. Refer to Figure 4.6 for this case. Recall that

$$\begin{aligned}
d_{n,k}(n+1,k) &= \dim((n,k),(n+1,k)) - \dim((n,k+1),(n+1,k)),\\
d_{n,k}(n+1,k+1) &= \dim((n,k),(n+1,k+1)) - \dim((n,k+1),(n+1,k+1)),\\
d_{n,k}(n+1,k+2) &= \dim((n,k),(n+1,k+2)) - \dim((n,k+1),(n+1,k+2)).
\end{aligned}$$

Since there is no path connecting $(n,k+1)$ to $(n+1,k)$ and there is no path connecting (n,k) to $(n+1,k+2)$, then $\dim((n,k+1),(n+1,k)) = 0$ and $\dim((n,k),(n+1,k+2)) = 0$. Thus

$$\begin{aligned}
d_{n,k}(n+1,k) &= \dim((n,k),(n+1,k)) > 0,\\
d_{n,k}(n+1,k+2) &= -\dim((n,k+1),(n+1,k+2)) < 0.
\end{aligned}$$

By Remark 4.5.4, the sign of $d_{n,k}(n+1,k+1)$ depends on the side of the diagram where the vertex $(n-1,k)$ is located, i.e.:

- If $(n-1,k)$ is on the left side or on the centre of the diagram then, for $j_0 = k+1$, we have that $d_{n,k}(n+1,j) \geq 0$ for $j \leq j_0$ and $d_{n,k}(n+1,j) < 0$ for $j > j_0$.
- If $(n-1,k)$ is on the right side of the diagram then, for $j_0 = k$, we have that $d_{n,k}(n+1,j) \geq 0$ for $j \leq j_0$ and $d_{n,k}(n+1,j) < 0$ for $j > j_0$.

Therefore, the theorem is valid for $p = n+1$.

Assuming that (4.5.10) and (4.5.11) are valid for an arbitrary $p > n$, we prove that

$$\dim((n,k),(p+1,j)) \geq \dim((n,k+1),(p+1,j)) \quad \text{for} \quad j \leq j_0;$$

$$\dim((n,k),(p+1,j)) < \dim((n,k+1),(p+1,j)) \quad \text{for} \quad j > j_0.$$

If at level p there exists $j_p \in \{1,\dots,p\}$ such that $d_{n,k}(p,j) \geq 0$ for $j \leq j_p$ and $d_{n,k}(p,j) < 0$ for $j > j_p$ then, by (4.5.9), $d_{n,k}(p{+}1,j) \geq 0$ for $j \leq j_p$ and $d_{n,k}(p{+}1,j) < 0$ for $j > j_p + 2$. There are two options for $d_{n,k}(p+1, j_p+1)$:

- If $d_{n,k}(p+1, j_p+1) \geq 0$ then the theorem is valid for $p+1$ using $j_0 = j_p + 1$;
- If $d_{n,k}(p+1, j_p+1) < 0$ then the theorem is valid for $p+1$ using $j_0 = j_p$.

In all cases the induction is complete. ▮

Corollary 4.5.6. *If* a *and* b *are two adjacent vertices at level* n *then, for every* $p > n$*,*

$$\sum_{j=1}^{p+1} |\dim(a,(p+1,j)) - \dim(b,(p+1,j))| \leq 2A(p,c)$$

where $A(p,c)$ *is the central Eulerian number at level* p*.*

Proof: Let $a = (n,k)$ and $b = (n,k+1)$ be two adjacent vertices at level n and let $p > n$. By Theorem 4.5.5, there exist $j_0 \in \{1,\dots,p+1\}$ such that

$$\begin{aligned} \dim(a,(p+1,j)) &\geq \dim(b,(p+1,j)) \quad \text{for} \quad j \leq j_0; \\ \dim(a,(p+1,j)) &< \dim(b,(p+1,j)) \quad \text{for} \quad j > j_0. \end{aligned}$$

By Theorem 4.4.1 using the (n,k) notation (as seen in the introduction of Section 4.5), we have

$$\dim((n-1,k),(p,j_0)) = \sum_{j \leq j_0} \dim(a,(p+1,j)) - \dim(b,(p+1,j))$$

$$\dim((n-1,k),(p,j_0)) = \sum_{j > j_0} \dim(b,(p+1,j)) - \dim(a,(p+1,j))$$

Since $\dim(a,(p+1,j)) \geq \dim(b,(p+1,j))$ for $j \leq j_0$, then

$$\dim(a,(p+1,j)) - \dim(b,(p+1,j)) \geq 0 \qquad \forall j \leq j_0.$$

Thus $|\dim(a,(p+1,j)) - \dim(b,(p+1,j))| = \dim(a,(p+1,j)) - \dim(b,(p+1,j))$ for every $j \leq j_0$.

Since $\dim(a,(p+1,j)) < \dim(b,(p+1,j))$ for $j > j_0$, then

$$\dim(b,(p+1,j)) - \dim(a,(p+1,j)) > 0 \qquad \forall j > j_0.$$

Thus $|\dim(a,(p+1,j)) - \dim(b,(p+1,j))| = \dim(b,(p+1,j)) - \dim(a,(p+1,j))$ for every $j > j_0$.

It follows that $\displaystyle\sum_{j=1}^{p+1} |\dim(a,(p+1,j)) - \dim(b,(p+1,j))| =$

$$\begin{aligned}
&= \sum_{j\le j_0} |\dim(a,(p+1,j)) - \dim(b,(p+1,j))| + \sum_{j>j_0} |\dim(a,(p+1,j)) - \dim(b,(p+1,j))| \\
&= \sum_{j\le j_0} \dim(a,(p+1,j)) - \dim(b,(p+1,j)) + \sum_{j>j_0} \dim(b,(p+1,j)) - \dim(a,(p+1,j)) \\
&= \sum_{j\le j_0} \dim((n-1,k),(p,j_0)) + \sum_{j>j_0} \dim((n-1,k),(p,j_0)) \\
&= 2\dim((n-1,k),(p,j_0))
\end{aligned}$$

Since $\dim((n-1,k),(p,j_0)) < A(p,c)$, we conclude that

$$\sum_{j=1}^{p+1} |\dim(a,(p+1,j)) - \dim(b,(p+1,j))| < 2A(p,c).$$

∎

4.6 Total variation distance for distributions in the Eulerian Bratteli diagram

Lemma 4.6.1. *If v_k and v_r are two vertices at level n (not necessarily adjacent) then there exists $p > n$ such that*

$$\sum_{w\in V_{p+1}} |\dim(v_k,w) - \dim(v_r,w)| < 2nA\left(p,\frac{p}{2}\right)$$

Proof: Let us write $v_k := (n,k) \in V_n$ to denote the vertices at level n. By Corollary 4.5.6, for every two adjacent vertices at level n, say $v_k, v_{k+1} \in V_n$, there exist $p_1 > n$ such that

$$\sum_{w\in V_{p_1+1}} |\dim(v_k,w) - \dim(v_{k+1},w)| < 2A(p_1,c_1).$$

We can assume, without loss of generality, that $c_1 = \frac{p_1}{2}$. Indeed, one can choose the first $p_1 > n$ which is even, in this case the centre of the diagram is the vertex $(p_1, \frac{p_1}{2})$.

So for two nonadjacent vertices at level n we can use the triangle inequality of absolute values. More precisely, if $w \in V_p$ and $1 \leq r \leq n-k$, then

$$|\dim(v_k, w) - \dim(v_{k+r}, w)| \leq \sum_{i=1}^{r} |\dim(v_{k+i-1}, w) - \dim(v_{k+i}, w)|.$$

Since v_{k+i-1} and v_{k+i} are adjacent vertices then there exists p_i such that

$$\sum_{w \in V_{p_i+1}} |\dim(v_{k+i-1}, w) - \dim(v_{k+i}, w)| \leq 2A\left(p_i, \frac{p_i}{2}\right).$$

It follows that for $p > \max\{p_1, \ldots, p_r\}$ we have

$$\begin{aligned}
\sum_{w \in V_{p+1}} |\dim(v_k, w) - \dim(v_{k+r}, w)| &\leq \sum_{w \in V_{p+1}} \sum_{i=1}^{r} |\dim(v_{k+i-1}, w) - \dim(v_{k+i}, w)| \\
&= \sum_{i=1}^{r} \sum_{w \in V_{p+1}} |\dim(v_{k+i-1}, w) - \dim(v_{k+i}, w)| \\
&< \sum_{i=1}^{r} 2A\left(p, \frac{p}{2}\right) = 2rA\left(p, \frac{p}{2}\right).
\end{aligned}$$

Since $r < n$, then $\sum_{w \in V_{p+1}} |\dim(v_k, w) - \dim(v_{k+r}, w)| < 2nA\left(p, \frac{p}{2}\right)$. ∎

The reason we choose even levels is because we want to use a result by Eldar Giladi and Joseph B. Keller [13] that describes the asymptotics of the central Eulerian numbers at even levels. Given two functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that f and g are asymptotic if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. We denote by $f(n) \sim g(n)$ when f is asymptotic to g.

Lemma 4.6.2. *If μ is the symmetric measure defined in Section 2.7, then*

$$\lim_{n\to\infty} \mu(C(2n, n)) = 0.$$

Proof: By [13, equation (7.3)], $A(2n, n) \sim \sqrt{12}(n/e)^n$. By Stirling's formula, $n! \sim \sqrt{2\pi n}(n/e)^n$. Combining these two, we have:

$$\mu\left(C\left(n, \frac{n}{2}\right)\right) = \frac{1}{n!} A\left(n, \frac{n}{2}\right) \sim \frac{\sqrt{12}(n/e)^n}{\sqrt{2\pi n}(n/e)^n} = \sqrt{\frac{6}{\pi n}} \to 0 \qquad \text{as } n \to \infty.$$

∎

For the following theorem, recall that $C(p, j)$ denotes the union of all cylinder sets $C(z)$ such that the path z arrives at vertex (p, j).

Theorem 4.6.3. *If X is the path space of the Eulerian Bratteli diagram then, for all $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $p > n$ such that*

$$\sum_{j=1}^{p} |\mu(C(p,j) \cap C(x)) - \mu(C(p,j) \cap C(y))| < \varepsilon \qquad \forall x, y \in X$$

Proof: Let $n \in \mathbb{N}$ and $\varepsilon > 0$. By the previous lemma, $\lim_{p\to\infty} \mu(C(p, \frac{p}{2})) = 0$. It is also true that $\lim_{p\to\infty} \frac{1}{p!} = 0$; therefore $\lim_{p\to\infty} \frac{n!}{p!} 2nA(p, \frac{p}{2}) = 0$. Let $p_1 > n$ be such that $\frac{n!}{p_1!} 2nA(p_1, \frac{p_1}{2}) < \varepsilon$

Let $x, y \in S_n$ be paths ending, respectively, at vertices $v_1, v_2 \in V_n$. By Lemma 4.6.1, there exist $p_2 > n$ such that

$$\sum_{w \in V_{p_2+1}} |\dim(v_1, w) - \dim(v_2, w)| < 2nA\left(p_2, \frac{p_2}{2}\right)$$

Since $\mu(C(p,j) \cap C(x)) = \frac{n!}{p!} \dim(v_1, (p,j))$ and $\mu(C(p,j) \cap C(y)) = \frac{n!}{p!} \dim(v_2, (p,j))$ then, for $p > \max\{p_1, p_2\}$ we have:

$$\begin{aligned}
\sum_{j=1}^{p+1} |\mu(C(p,j) \cap C(x)) - \mu(C(p,j) \cap C(y))| &= \sum_{j=1}^{p+1} |\frac{n!}{p!} \dim(v_1, (p,j)) - \frac{n!}{p!} \dim(v_2, (p,j))| \\
&= \frac{n!}{p!} \sum_{j=1}^{p+1} |\dim(v_1, (p,j)) - \dim(v_2, (p,j))| \\
&= \frac{n!}{p!} \sum_{w \in V_{p+1}} |\dim(v_1, w) - \dim(v_2, w)| \\
&< \frac{n!}{p!} 2nA\left(p, \frac{p}{2}\right) < \varepsilon.
\end{aligned}$$

∎

Chapter 5

Positivity

In this chapter we characterize the positive elements of the dimension group arising from the Eulerian Bratteli diagram. We do this by showing that every trace on every order ideal of the Eulerian dimension group extends to a trace on the whole dimension group. This allows us to use the fact that strict positivity of an element at all traces is sufficient for positivity of the element (Theorem 5.6.3). This has consequences for comparing coefficients of the images in higher levels of the diagram. The extension property is obtained by a combinatorial argument, analogous to that of Gnedin and Olshanski [18].

5.1 Background

This section is based on [10]. Recall that a *partially ordered group* is a pair (G, G^+) where G is an abelian group and $G^+ \subset G$ (named *positive cone*) satisfies the following conditions:

- $G^+ + G^+ \subseteq G^+$,
- $G^+ - G^+ = G$,
- $G^+ \cap (-G^+) = \{0\}$,

Definition 5.1.1. *Let G be a dimension group. An* order ideal *of G is a subgroup I of G that satisfies:*

- *$I = I^+ - I^+$ (where $I^+ = I \cap G^+$),*
- *If $0 \leq a \leq b \in I$, then $a \in I$.*

Remark 5.1.2. We denote an order ideal I of a dimension group G by $I \lhd G$. The terminology is justified by the fact that the ideals of an AF algebras are in one-to-one correspondence with the ideals of the associated dimension group G [9, Proposition IV.5.1]. We say that G is *simple* if $\{0\}$ and G are its only order ideals. Therefore, there is a one-to-one correspondence between simple dimension groups and simple AF algebras.

Definition 5.1.3. *Let* (G, G^+) *be an ordered group. An* order unit *is an element* $u \in G^+$ *such that for all* $a \in G^+$ *there exists* $n \in \mathbb{N}$ *such that* $nu \geq a$.

Definition 5.1.4. *Let* (G, G^+) *be an ordered group and* $u \in G^+$ *be an order unit. A* trace *(or* state*) is a homomorphism* $\tau : G \to \mathbb{R}$ *such that* $\tau(G^+) \geq 0$ *and* $\tau(u) = 1$.

The collection of all traces, called the *state space*, is denoted by $S_u(G)$ and is a convex compact subset of $\mathbb{R}^G := \{f : G \to \mathbb{R}\}$ endowed with the product topology. We denote by $\partial S_u(G)$ its set of extreme points (also called *pure traces* of $S_u(G)$).

If X is a compact Hausdorff space, we let $C(X, \mathbb{R})$ denote the Banach space of continuous functions $h : X \to \mathbb{R}$ with the sup norm $||h|| = \sup\{|h(x)| \mid x \in X\}$, and consider the *ordinary* and *strict* orderings on $C(X, \mathbb{R})$, given by

$$\begin{aligned} C(X,\mathbb{R})^+ &= \{h \in C(X,\mathbb{R}) \mid h(x) \geq 0 \ \forall x \in X\}, \\ C(X,\mathbb{R})^{++} &= \{h \in C(X,\mathbb{R}) \mid h(x) > 0 \ \forall x \in X\} \cup \{0\}. \end{aligned}$$

If K is a compact convex subset of a locally convex space, we let Aff K be the *affine* functions in $C(K, \mathbb{R})$, i.e., the continuous functions $h \in C(K, \mathbb{R})$ such that

$$h(\alpha p + (1-\alpha)q) = \alpha h(p) + (1-\alpha)h(q) \qquad (p, q \in K; 0 \leq \alpha \leq 1).$$

Aff K is a closed subspace of $C(K, \mathbb{R})$ and we let Aff $K^+ =$ Aff $K \cap C(K, \mathbb{R})^+$, Aff $K^{++} =$ Aff $K \cap C(K, \mathbb{R})^{++}$. In particular if K is a $(d-1)$-simplex, Aff K may be identified with $C(X, \mathbb{R})$ where X is the set of extreme points of K, i.e., the d vertices of K, and we have an ordinary and strict order, norm, and linear isomorphism

$$\text{Aff } K = C(X, \mathbb{R}) \simeq \mathbb{R}^d.$$

If K is the state space $S_u(G)$ of an ordered group G, with order unit u, we define a positive homomorphism $\hat{}: G \to$ Aff K by $\hat{a}(\tau) = \tau(a)$. Since $\tau \in K = S_u(G)$, $\tau(u) = 1$, thus $\hat{u} = 1$.

Recall, from Definition 1.1.2, that if G is a dimension group then $G = \varinjlim(\mathbb{Z}^{r(n)}, \varphi_n)$. Thus a trace $\tau \in S_u(G)$ is determined by a sequence of positive homomorphism $P_n : \mathbb{Z}^{r(n)} \to \mathbb{R}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{Z}^{r(n)} \arrow[r, "\varphi_n"] \arrow[dr, "P_n"] & \mathbb{Z}^{r(n+1)} \arrow[d, "P_{n+1}"] \\ & \mathbb{R} \end{tikzcd} \qquad \forall n \in \mathbb{N}.$$

In other words, $P_n = P_{n+1}\varphi_n$ for every $n \in \mathbb{N}$. Then the map P_n is represented by a row vector $V_n = (V_{n1}, \ldots, V_{n,r(n)})$, satisfying the matrix multiplication:

$$(V_{n1}, \ldots, V_{n,r(n)}) = (V_{n+1,1}, \ldots, V_{n+1,r(n+1)})\varphi_n.$$

5.2 Traces of the Eulerian Bratteli diagram dimension group

From now on, G will represent the Eulerian Bratteli diagram dimension group, i.e., $G = \varinjlim(\mathbb{Z}^n, \varphi_n)$ where $u = [(1), 1] \in G$ is the order unit and

$$\varphi_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ n & 2 & 0 & \cdots & 0 & 0 \\ 0 & n-1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 3 & n-1 & 0 \\ 0 & 0 & \cdots & 0 & 2 & n \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

As seen in the previous section, a trace $\tau : G \to \mathbb{R}$ is determined by a sequence of positive homomorphisms $P_n : \mathbb{Z}^n \to \mathbb{R}$ satisfying $P_n = P_{n+1}\varphi_n$. Writing P_n as a row vector $V_n = (V_{n1}, \ldots, V_{nn})$, we have

$$V_{nk} = kV_{n+1,k} + (n-k+1)V_{n+1,k+1}. \tag{5.2.1}$$

and $V_{11} = 1$ (in order to have $\tau(u) = 1$ for $u = [(1), 1] \in \varinjlim(\mathbb{Z}^n, \varphi_n)$). We call (5.2.1) the *dual* recursion.

Example 5.2.1. If $n = 1$, $V_1 = V_2\varphi_1$ implies $V_{11} = (V_{21}, V_{22}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = V_{21} + V_{22}$. Thus

$$V_{11} = V_{21} + V_{22}.$$

If $n = 2$, $V_2 = V_3\varphi_2$ implies $(V_{21}, V_{22}) = (V_{31}, V_{32}, V_{33}) \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix}$. Thus

$$(V_{21}, V_{22}) = (V_{31} + 2V_{32}, 2V_{32} + V_{33})$$

Therefore, $V_{21} = V_{31} + 2V_{32}$ and $V_{22} = 2V_{32} + V_{33}$.

In [18, Theorem 1], Alexander Gnedin and Grigori Olshanski proved that the dual recursion admits multiple solutions which comprise a convex set. The extreme solutions are called the *boundary* and are parametrized as follows.

Theorem 5.2.2. *Each extreme solution $(W_{nk})_{n,k\geq 1}$ to the dual recursion (5.2.1) is uniquely determined by the parameter W_{21} assuming values in the following set:*

$$\left\{\frac{1}{2} \cdot \frac{z+1}{z} \mid z \in \mathbb{Z}, z < 0\right\} \cup \left\{\frac{1}{2}\right\} \cup \left\{\frac{1}{2} \cdot \frac{z+1}{z} \mid z \in \mathbb{Z}, z > 0\right\}.$$

The solutions are explicitly given by the formulae

$$W_{nk}^{z} = \frac{1}{n!}\prod_{i=1}^{n}\left(1+\frac{i-k}{z}\right) \qquad W_{nk}^{\infty} = \frac{1}{n!}$$

where $z \in \mathbb{Z}^* \cup \{\infty\}$ *(the one-point compactification of* $\mathbb{Z}^*$*).*

Theorem 5.2.3. *[18, Theorem 2] The values* W_{nk}^{z} *can also be written as follows:*

$$\begin{aligned} W_{nk}^{z} &= \frac{1}{z^n}\binom{n-k+z}{n} \quad \text{for} \quad z>0, \\ W_{nk}^{z} &= \frac{1}{|z|^n}\binom{k+1+z}{n} \quad \text{for} \quad z<0. \end{aligned}$$

Example 5.2.4. Below the reader will find arranged the solutions W_{nk}^{z} for $z=\infty$, 1, 2 and 3. The solutions for $z=-1$, -2, -3, etc. are symmetrical with their positive counterparts (as proven in Proposition 5.2.6).

$$\begin{array}{c} W_{11}^{z} \\ W_{21}^{z} \quad W_{22}^{z} \\ W_{31}^{z} \quad W_{32}^{z} \quad W_{33}^{z} \\ W_{41}^{z} \quad W_{42}^{z} \quad W_{43}^{z} \quad W_{44}^{z} \\ W_{51}^{z} \quad W_{52}^{z} \quad W_{53}^{z} \quad W_{54}^{z} \quad W_{55}^{z} \\ W_{61}^{z} \quad W_{62}^{z} \quad W_{63}^{z} \quad W_{64}^{z} \quad W_{65}^{z} \quad W_{66}^{z} \end{array}$$

If $z=\infty$**,** then $W_{nk}^{z}=\frac{1}{n!}$:

$$\begin{array}{c} 1 \\ \frac{1}{2} \quad \frac{1}{2} \\ \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \\ \frac{1}{24} \quad \frac{1}{24} \quad \frac{1}{24} \quad \frac{1}{24} \\ \frac{1}{120} \quad \frac{1}{120} \quad \frac{1}{120} \quad \frac{1}{120} \quad \frac{1}{120} \\ \frac{1}{720} \quad \frac{1}{720} \quad \frac{1}{720} \quad \frac{1}{720} \quad \frac{1}{720} \quad \frac{1}{720} \end{array}$$

If $z = 1$**,** then $W^z_{nk} = \frac{1}{n!}\prod_{i=1}^{n}(1 + i - k)$:

$$\begin{array}{c}
1 \\
1 \quad 0 \\
1 \quad 0 \quad 0 \\
1 \quad 0 \quad 0 \quad 0 \\
1 \quad 0 \quad 0 \quad 0 \quad 0 \\
1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0
\end{array}$$

If $z = 2$**,** then $W^z_{nk} = \frac{1}{n!}\prod_{i=1}^{n}(1 + \frac{i-k}{2})$:

$$\begin{array}{c}
1 \\
\dfrac{3}{4} \quad \dfrac{1}{4} \\
\dfrac{1}{2} \quad \dfrac{1}{8} \quad 0 \\
\dfrac{5}{16} \quad \dfrac{1}{16} \quad 0 \quad 0 \\
\dfrac{3}{16} \quad \dfrac{1}{32} \quad 0 \quad 0 \quad 0 \\
\dfrac{7}{64} \quad \dfrac{1}{64} \quad 0 \quad 0 \quad 0 \quad 0
\end{array}$$

If $z = 3$**,** then $W^z_{nk} = \frac{1}{n!}\prod_{i=1}^{n}(1 + \frac{i-k}{3})$:

$$\begin{array}{c}
1 \\
\dfrac{2}{3} \quad \dfrac{1}{3} \\
\dfrac{10}{27} \quad \dfrac{4}{27} \quad \dfrac{1}{27} \\
\dfrac{5}{27} \quad \dfrac{5}{81} \quad \dfrac{1}{81} \quad 0 \\
\dfrac{7}{81} \quad \dfrac{2}{81} \quad \dfrac{1}{243} \quad 0 \quad 0 \\
\dfrac{28}{729} \quad \dfrac{7}{729} \quad \dfrac{1}{729} \quad 0 \quad 0 \quad 0
\end{array}$$

Now we prove some properties of W^z_{nk} that helps us describe the state space of the Eulerian Bratteli diagram dimension group.

Lemma 5.2.5. *If n, k and z are integers satisfying $n \geq k > z > 0$, then $W_{nk}^{z} = 0$.*

Proof: If $n \geq k > z > 0$, then $1 \leq z < k \leq n$ so that $1 \leq k - z \leq n$. Thus there exists $i \in \{1, \ldots, n\}$ such that $i = k - z$. In this case we have $(1 + \frac{i-k}{z}) = (1 + \frac{-z}{z}) = 0$, therefore $W_{nk}^{z} = 0$ for all $k > z$. ▮

Proposition 5.2.6 (Property of symmetry)**.** *If $n, k \in \mathbb{Z}$ such that $1 \leq k \leq n$, and $z \in \mathbb{Z}^*$; then $W_{nk}^{z} = W_{n,n-k+1}^{-z}$.*

Proof: Indeed,

$$\begin{aligned}
W_{n,n-k+1}^{-z} &= \frac{1}{n!}\prod_{i=1}^{n}\left(1 + \frac{i-(n-k+1)}{-z}\right) \\
&= \frac{1}{n!}\prod_{i=1}^{n}\left(1 + \frac{i-n+k-1}{-z}\right) \\
&= \frac{1}{n!}\prod_{i=1}^{n}\left(1 + \frac{n-k+1-i}{z}\right).
\end{aligned}$$

Performing the change of variables $(n-k+1-i) \leftrightarrow (j-k)$, we have that

$$\prod_{i=1}^{n}\left(1 + \frac{n-k+1-i}{z}\right) = \prod_{j=1}^{n}\left(1 + \frac{j-k}{z}\right)$$

Therefore

$$\begin{aligned}
W_{n,n-k+1}^{-z} &= \frac{1}{n!}\prod_{i=1}^{n}\left(1 + \frac{n-k+1-i}{z}\right) \\
&= \frac{1}{n!}\prod_{j=1}^{n}\left(1 + \frac{j-k}{z}\right) = W_{nk}^{z}.
\end{aligned}$$

▮

Lemma 5.2.7. *If n, k and z are integers satisfying $z < 0$ and $k \leq n + z$, then $W_{nk}^{z} = 0$.*

Proof: If $k \leq n + z$, then $n - k \geq -z > 0$, so $n - k + 1 > -z > 0$. By the symmetric property, $W_{n,n-k+1}^{-z} = W_{nk}^{z}$. By Lemma 5.2.5, $W_{n,n-k+1}^{-z} = 0$, so $W_{nk}^{z} = 0$. ▮

Since, for each $z \in \mathbb{Z}^* \cup \{\infty\}$, the sequence $(W^z_{nk})_{n,k}$ is a solution to the dual recursion (5.2.1), then the map $\tau_z : G \to \mathbb{R}$ given by

$$\tau_z([(v_1, \ldots, v_n), n]) = \sum_{k=1}^{n} v_k W^z_{nk} \tag{5.2.2}$$

defines a trace in the Eulerian Bratteli diagram dimension group $G = \varinjlim(\mathbb{Z}^n, \varphi_n)$.

Example 5.2.8. If $z = \infty$, then

$$\tau_\infty([(v_1, \ldots, v_n), n]) = \frac{1}{n!} \sum_{k=1}^{n} v_k.$$

If an element of the dimension group $a = [(v_1, \ldots, v_n), n]$ is such that $v_k \geq 0$ for all $1 \leq k \leq n$, then $a \in G^+$ and $\tau_\infty(a) = 0$ implies $a = 0$. This means that τ_∞ is a *faithful* trace.

Proposition 5.2.9. *τ_z converges pointwise to τ_∞ as $z \to \pm\infty$.*

Proof: Since

$$\lim_{z\to\pm\infty} W^z_{nk} = \lim_{z\to\pm\infty} \frac{1}{n!} \prod_{i=1}^{n} \left(1 + \frac{i-k}{z}\right) = \frac{1}{n!} \prod_{i=1}^{n} (1) = \frac{1}{n!} = W_{nk}(0),$$

we have, for any $a = [(v_1, \ldots, v_n), n] \in G$,

$$\lim_{z\to\pm\infty} \tau_z(a) = \lim_{z\to\pm\infty} \sum_{k=1}^{n} v_k W^z_{nk} = \sum_{k=1}^{n} v_k W_{nk}(0) = \frac{1}{n!} \sum_{k=1}^{n} v_k = \tau_\infty(a).$$

∎

Combining lemmas 5.2.5 and 5.2.7, we have that, for $a = [(v_1, \ldots, v_n), n] \in G$,

$$z > 0 \quad \Rightarrow \quad \tau_z(a) = \sum_{k=1}^{n} v_k W^z_{nk} = \sum_{k \leq z} v_k W^z_{nk},$$

$$z < 0 \quad \Rightarrow \quad \tau_z(a) = \sum_{k=1}^{n} v_k W^z_{nk} = \sum_{k > n+z} v_k W^z_{nk}.$$

This means that the kernel of τ_z, for $z > 0$, are the elements of the dimension group represented by vectors of $\mathbb{Z}^n$ whose last $n - z$ coordinates are zero. More precisely:

$$z > 0 \Rightarrow \ker \tau_z = \{[(v_1, \ldots, v_n), n] \mid v_k = 0 \ \forall z < k \leq n\}.$$

On the other hand, the kernel of τ_z, for $z < 0$, are the elements of the dimension group represented by vectors of $\mathbb{Z}^n$ whose first z coordinates are zero. More precisely:

$$z < 0 \Rightarrow \ker \tau_z = \{[(v_1, \ldots, v_n), n] \mid v_k = 0 \ \forall 0 \leq k \leq z\}.$$

5.3 The bucket sort algorithm

The extreme solutions (W^z_{nk}) of the dual recursion (5.2.1) given in Theorem 5.2.2 arise from a well-known sorting algorithm called *bucket sort* [18, Section 4]. In this section we describe it.

For $z \in \mathbb{N}^*$ fixed, consider z buckets arranged in some fixed order. For each nonzero natural number $1, 2, \ldots \in \mathbb{N}^*$ (henceforth called *digits*), insert it into one of the buckets with equal probabilities $\frac{1}{z}$ and rearrange the digits inside each bucket in decreasing order. For each digit n that we insert into one of the buckets, we obtain a permutation of $\{1, \ldots, n\}$ arising from the buckets when we concatenate them in order.

Example of the bucket sort algorithm for $z = 2$

Step 1: For $z = 2$ buckets, there are 2 possibilities to allocate digit 1 into them, each with probability $\frac{1}{2}$:

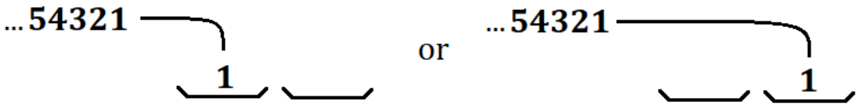

Figure 5.1: Bucket sort, step 1

Step 2: We also have two possibilities to allocate digit 2 in the buckets, each with probability $\frac{1}{2}$. If we insert digit 2 in the same bucket of digit 1, the resulting permutation will be 21 since we must rearrange the digits in the decreasing order. Otherwise, the resulting permutation will be 12.

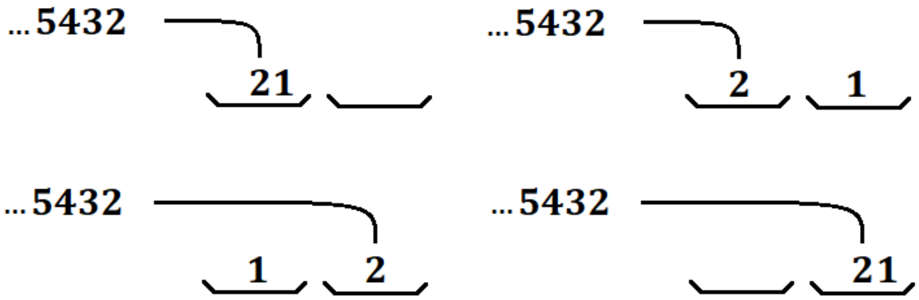

Figure 5.2: Bucket sort, step 2

Note that, out of a sample space of 4 possible outcomes to allocate both digits, 3 of them result in the permutation 21 and 1 of them results in 12 (see Figure 5.2).

Proposition 5.3.1. *Let* x *be a permutation of* $\{1,\dots,n\}$ *with exactly* k *rises (i.e. the set* $\{i \in \{2,\dots,n\} \mid x_{i-1} < x_i\}$ *has cardinality* k*). The number of allocations of digits* $\{1,\dots,n\}$ *into* z *buckets using the bucket sort algorithm whose resulting permutation is* x *is equal to* $\binom{n+z-k-1}{n}$.

Proof: This proof is based on [18, Lemma 9] and it uses the *stars and bars* method. Fix $x \in S_n$ and consider n stars and $z-1$ bars to be rearranged in $n+z-1$ positions (see Figure 5.3).

Figure 5.3: In this example $n = 7$ stars and $z = 4$ buckets

Each arrangement of stars and bars correspond to an allocation of digits $\{1,\dots,n\}$ into z buckets where each digit is represented by a star and each bar separates two adjacent buckets. In order to guarantee that such arrangements result in x, we fix k bars exactly in the position of the rises of x, i.e., if $x_{i-1} < x_i$ then we insert a bar between x_{i-1} and x_i. This way, the number of allocations of $\{1,\dots,n\}$ into z buckets resulting in x will be determined by the remaining $(z-1)-k$ bars. Since the bars are indistinguishable, the number of possibilities equals $\binom{n+z-k-1}{n}$. ∎

Recall that, in the rises and falls representation (Section 2.5), the vertex (n,k) is represented by permutations of $\{1,\dots,n\}$ with exactly $k-1$ rises.

Corollary 5.3.2. *The number of allocations of digits* $\{1,\dots,n\}$ *into* z *buckets using the bucket sort algorithm whose resulting permutation is represented by vertex* (n,k) *is the number* W_{nk}^{z} *for* $z > 0$.

Proof: Let x be a permutation of $\{1,\dots,n\}$ represented by vertex (n,k), then x has exactly $k-1$ rises. By Proposition 5.3.1, the number of allocations of digits $\{1,\dots,n\}$ into z buckets using the bucket sort algorithm whose resulting permutation is x is equal to $\binom{n+z-(k-1)-1}{n} = \binom{n+z-k}{n} = W_{nk}^{z}$ for $z > 0$. ∎

By [18, Lemma 14], the values W_{nk}^{z} resulting from the bucket sorting gives rise to extreme solutions to the dual recursion (5.2.1) which, in turn, gives rise to a pure trace $\tau_z : G \to \mathbb{R}$.

5.4 State space on order ideals

Let $e_{nk} \in \varinjlim(\mathbb{Z}^n, \varphi_n)$ be the element $[(0,\ldots,0,1,0,\ldots,0),n]$ where 1 is at position k, and $u_{nk} := ue_{nk}$ where $u \in \mathbb{Z}^*$. Define $I_{nk} := \{[u_{nk}, n] \mid u \in \mathbb{Z}^*\}$, we refer to it as the *order ideal generated by the vertex* (n,k).

Fix a vertex (n_0,k_0) in the Eulerian Bratteli diagram. In order to describe the traces on the order ideal $I_{n_0k_0}$ we must find solutions to the dual recursion

$$V_{nk} = kV_{n+1,k} + (n-k+1)V_{n+1,k+1} \qquad (n,k) > (n_0,k_0) \tag{5.4.1}$$

subject to the condition $V_{n_0k_0} = 1$. We also use the bucket sort method. An extreme solution to (5.4.1) is a sequence $(W^z_{n_0k_0\to nk})_{n,k}$ where $W^z_{n_0k_0\to nk}$ is the probability to have a permutation represented by (n,k) such that the digits $\{1,\ldots,n_0\}$ is represented by (n_0,k_0). Thus

$$W^z_{n_0k_0\to nk} = \frac{W^z_{nk}}{W^z_{n_0k_0}} = \frac{\binom{n-k+z}{n}}{\binom{n_0-k_0+z}{n_0} z^{n-n_0}}. \tag{5.4.2}$$

1st step: Evaluate the number of possibilities to allocate the numbers $\{1,\ldots,n_0\}$ into z buckets in order to have a permutation represented by (n_0,k_0). By Proposition 5.3.1, this number is $\binom{n_0-k_0+z}{n_0}$.

2nd step: Evaluate the number of possibilities to add the digits $\{n_0+1,\ldots,n\}$ in order to have a permutation of (n,k). By Proposition 5.3.1, this number is $\binom{n-k+z}{n}$.

3rd step: Since there are $\binom{n_0-k_0+z}{n_0}$ starting possibilities and we are adding $n-n_0$ digits, then

$$W^z_{n_0k_0\to nk} = \frac{\binom{n-k+z}{n}}{\binom{n_0-k_0+z}{n_0}\cdot z^{n-n_0}}.$$

The following examples illustrate the algorithm above.

5.4.1 Example: the I_{21} order ideal

Let I_{21} be the order ideal generated by the vertex $(2,1)$, i.e., the element of the dimension group $[(1,0),2]$. The rises and falls representation associated to the vertex $(2,1)$ is the permutation 21. Let us start by describing $\tau_z : I_{21} \to \mathbb{R}$ for $z = 2$. So we set $n_0 = 2$, $k_0 = 1$ and $z = 2$.

There are $\binom{n_0-k_0+z}{n_0} = 3$ ways to allocate the digits 1 and 2 into $z = 2$ buckets in order to form the permutation 21; and they are given by $(21,\emptyset)$, $(2,1)$ and $(\emptyset,21)$. For each of these pairs, there are $z = 2$ ways to allocate the digit 3 in these buckets, and they are:

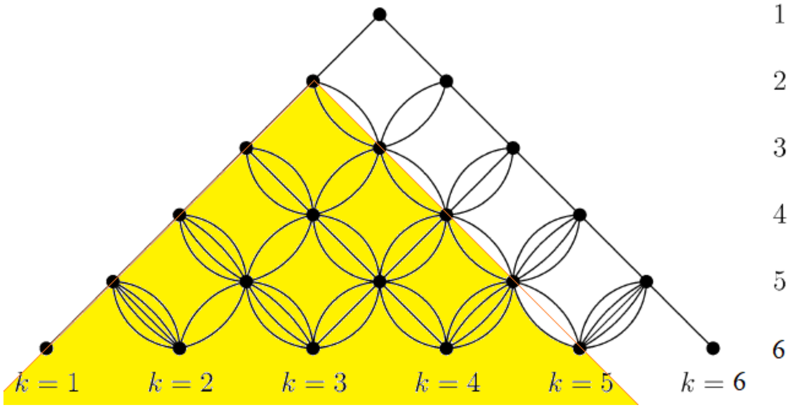

Figure 5.4: The I_{21} order ideal.

$$\begin{array}{cc} (321, \emptyset) & (21, 3) \\ (32, 1) & (2, 31) \\ (3, 21) & (\emptyset, 321) \end{array} \quad \longleftrightarrow \quad \begin{array}{cc} 321 & 213 \\ 321 & 231 \\ 321 & 321 \end{array}$$

Note that the probability to have the sequence 321 (arriving at vertex $(n, k) = (3, 1)$) is $\binom{n-k+z}{n} = 4$ out of a total of $\binom{n_0-k_0+z}{n_0} \cdot z^{n-n_0} = 6$ and the probability to have the sequence 231 or 213 (arriving at vertex $(n, k) = (3, 2)$) is $\binom{n-k+z}{n} = 1$ out of a total of $\binom{n_0-k_0+z}{n_0} \cdot z^{n-n_0} = 6$. Therefore, the array $(W^z_{21 \to nk})$ for $n = 3$ is

$$W^2_{21\to 31} = \frac{4}{6} \qquad W^2_{21\to 32} = \frac{1}{6} \qquad W^2_{21\to 33} = 0$$

Below we write $W^2_{21\to nk}$ up to $n = 6$:

$$\begin{array}{cccccc}
0 \\
1 & 0 \\
\frac{4}{6} & \frac{1}{6} & 0 \\
\frac{5}{12} & \frac{1}{12} & 0 & 0 \\
\frac{6}{24} & \frac{1}{24} & 0 & 0 & 0 \\
\frac{7}{48} & \frac{1}{48} & 0 & 0 & 0 & 0
\end{array}$$

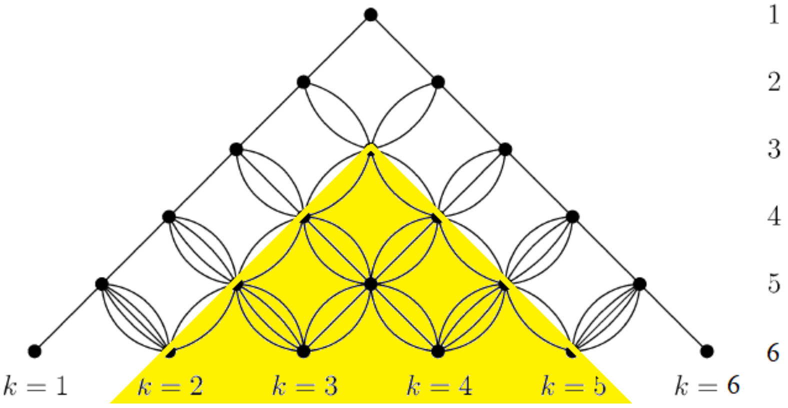

Figure 5.5: The order ideal I_{32}.

5.4.2 Example: The order ideal I_{32}

Now consider the order ideal I_{32}, i.e. set $n_0 = 3$ and $k_0 = 2$. The vertex $(3, 2)$ has four paths arriving to it and they are represented by permutations of $\{1, 2, 3\}$ with 1 rise.

$$\sigma_1 = 231 \qquad \sigma_2 = 213 \qquad \sigma_3 = 312 \qquad \sigma_4 = 132$$

Now let us evaluate $(W^z_{32\to nk})$ for $z = 3$, $n = 4$ and $k = 1, 2, 3, 4$.

Step 1: There are $\binom{n_0-k_0+z}{n_0} = 4$ ways to allocate the digits 1, 2 and $n_0 = 3$ into $z = 3$ buckets in order to have one of the permutations above, e.g. $\sigma_1 = 231$. They are:

$$(2, 31, \emptyset) \qquad (2, 3, 1) \qquad (2, \emptyset, 31) \qquad (\emptyset, 2, 31)$$

Step 2: The number of possibilities to add the digit 4 in order to have one of the permutations associated to $(4, k)$ is $\binom{n-k+z}{n} = \binom{4-k+3}{4}$. See Figure 5.6.

Step 3: For $k = 1, 2, 3, 4$ we compute $W^3_{32\to 4k}$:

$$\begin{aligned}
W^3_{32\to 41} &= 0 \text{ because } k = 1 < 2 = k_0.\\
W^3_{32\to 42} &= \frac{\binom{4-2+3}{4}}{4\cdot 3^1} = \frac{5}{12}\\
W^3_{32\to 43} &= \frac{\binom{4-3+3}{4}}{4\cdot 3^1} = \frac{1}{12}\\
W^3_{32\to 44} &= 0
\end{aligned}$$

Below we write $W^3_{32\to nk}$ up to $n = 6$:

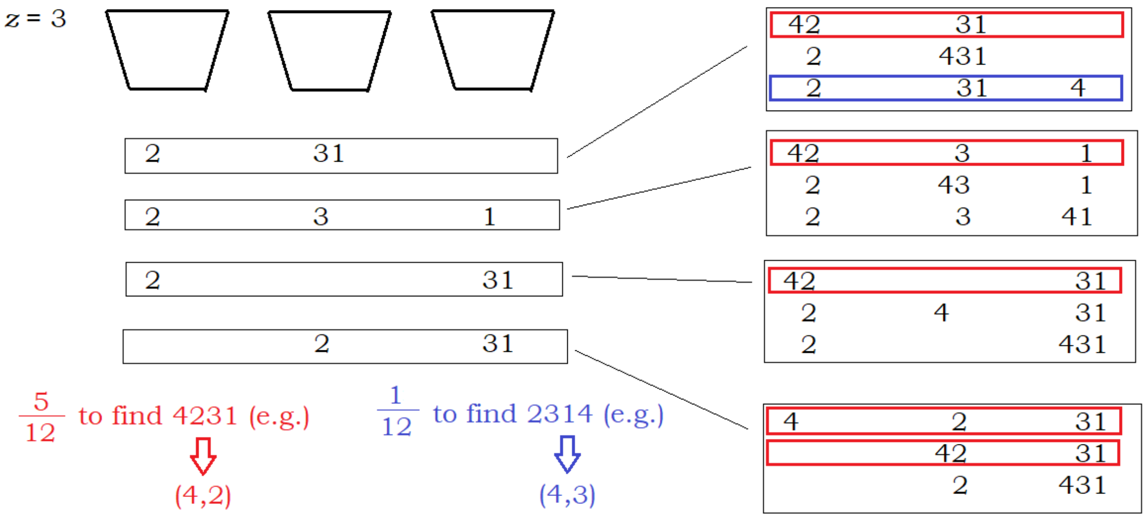

Figure 5.6: Illustration of the bucket sorting method.

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & \frac{5}{12} & & \frac{1}{12} & & 0 & & \\
 & 0 & & \frac{6}{36} & & \frac{1}{36} & & 0 & & 0 & \\
0 & & \frac{7}{108} & & \frac{1}{108} & & 0 & & 0 & & 0
\end{array}$$

5.5 Trace extension

As seen in Section 5.2, there exists a correspondence between the solutions to the dual recursion (5.2.1) given by (W^z_{nk}) and the pure traces defined on the Eulerian Bratteli diagram dimension group. In Section 5.4, we used the same algorithm (bucket sort) to find a solution $(W^z_{n_0k_0\to nk})$ to the dual recursion defined on order ideals (5.4.1). In this section we see that one can easily convert a solution on the order ideal, $(W^z_{n_0k_0\to nk})$, into one in the full Bratteli diagram: (W^z_{nk}). This proves that one can extend any trace defined on an order ideal $\tau_z : I_{n_0k_0} \to \mathbb{R}$ into a trace defined on the whole diagram $\overline{\tau}_z : G \to \mathbb{R}$. This extension is not standard (by covering the area not covered by the ideal with zeroes), but instead there is a scaling involved.

Definition 5.5.1. *If $I \subset G$, we say that a trace $\tau : I \to \mathbb{R}$* can be extended after rescaling *to a trace $\overline{\tau} : G \to \mathbb{R}$ if there exists $\lambda \in \mathbb{R}$ such that $\lambda\tau = \overline{\tau} \restriction_J$.*

Proposition 5.5.2. *Given a trace on an order ideal $\tau : I \to \mathbb{R}$, there exists a trace $\overline{\tau} : G \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that*

$$\overline{\tau} \restriction_I = \tau \cdot \lambda.$$

Proof: Let $\tau_z : I_{n_0k_0} \to \mathbb{R}$ be a trace on the order ideal $I_{n_0k_0}$ and let $(W^z_{n_0k_0 \to nk})$ be its corresponding array of solutions of the dual recursion. By (5.4.2),

$$W^z_{n_0k_0 \to nk} \cdot W^z_{n0k0} = W^z_{nk}.$$

Setting $\lambda := W^z_{n_0k_0} \in \mathbb{R}$ and $\overline{\tau_z} : G \to \mathbb{R}$ to be the trace corresponding to the array (W^z_{nk}), we have that $\overline{\tau_z} = \tau_z \cdot \lambda$. ∎

Example 5.5.3. Below we present the array (W^z_{nk}) for $z = 3$ (and $n \leq 6$). This array correspond to the trace $\tau_3 : G \to \mathbb{R}$.

$$\begin{array}{c}
1 \\[1ex]
\frac{2}{3} \quad \frac{1}{3} \\[1ex]
\frac{10}{27} \quad \frac{4}{27} \quad \frac{1}{27} \\[1ex]
\frac{15}{81} \quad \frac{5}{81} \quad \frac{1}{81} \quad 0 \\[1ex]
\frac{21}{243} \quad \frac{6}{243} \quad \frac{1}{243} \quad 0 \quad 0 \\[1ex]
\frac{28}{729} \quad \frac{7}{729} \quad \frac{1}{729} \quad 0 \quad 0 \quad 0
\end{array}$$

Now we present the array $(W^z_{21 \to nk})$ for $z = 3$ (and $n \leq 6$). This correspond to the trace $\sigma_3 : I_{21} \to \mathbb{R}$.

$$\begin{array}{c}
\frac{3}{2} \\[1ex]
1 \quad \frac{1}{2} \\[1ex]
\frac{10}{18} \quad \frac{4}{18} \quad \frac{1}{18} \\[1ex]
\frac{15}{54} \quad \frac{5}{54} \quad \frac{1}{54} \quad 0
\end{array}$$

$$\begin{array}{cccccc} & \frac{21}{162} & \frac{6}{162} & \frac{1}{162} & 0 & 0 \\ \frac{28}{486} & \frac{7}{486} & \frac{1}{486} & 0 & 0 & 0 \end{array}$$

Note that $\sigma_3 \cdot \frac{2}{3} = \tau_3$, thus σ_3 defined on the order ideal I_{21} can be extended up to a rescaling to the trace τ_3 defined on G.

5.6 General criteria for positivity

The following results — from Definition 5.6.1 to Corollary 5.6.6 — are based on [19, Chapter 4].

Definition 5.6.1. *Let G be an ordered group and $u \in G^+ \setminus \{0\}$ be an order unit. An element $a \in G$ is* infinitesimal *if for all natural numbers $N > 0$, $-u \leq Na \leq u$. In other words, if $\epsilon = \frac{p}{q} \in \mathbb{Q}$ then $-pu \leq qa \leq pu$.*

Denote by Inf G the set of infinitesimal elements of G.

Example 5.6.2. If $r_n \geq 2$ for infinitely many $n \in \mathbb{N}$ then $G = \varinjlim(\mathbb{Z}, \times r_n)$ is a simple dimension group without infinitesimals.

Proof: Let I be a non zero order ideal of G, and $[x, n] \in I \setminus \{0\}$. Let us show that $I = G$. Let $[y, m] \in G$. Let $N > \max\{n, m\}$ and consider the elements $[\varphi_{n,N}(x), N]$ and $[\varphi_{m,N}(y), N]$. Since $\varphi_{n,N}(x), \varphi_{m,N}(y) \in \mathbb{Z}$, then there exists $k \in \mathbb{N}$ such that $\varphi_{m,N}(y) \leq k\varphi_{n,N}(x)$. Since I is an order ideal, $[y, m] = [\varphi_{m,N}(y), N] \in I$. Therefore, $I = G$ and G is simple.

Let us prove that Inf $G = \{0\}$. Let $a = [x, n] \in$ Inf G and consider the order unit $u = [1, 1]$, then $\varphi_{1,n}(u) = [y, n]$ with $y \in \mathbb{Q}$. Since a is infinitesimal, for every $N \in \mathbb{N}$, $-y \leq Nx \leq y$. Since $x, y \in \mathbb{Q}$, the inequality is only possible if $x = 0$. Therefore, $a = 0$. ∎

Proposition 5.6.3. *Let G be a dimension group, $u \in G^+ \setminus \{0\}$ be a fixed order unit, and $S_u(G)$ be the set of all traces that map $u \mapsto 1$. If $a \in G \setminus \{0\}$ is such that $\tau(a) > 0$ for all $\tau \in S_u(G)$, then $a \in G^+$*

Proof: If $\tau(a) > 0$ for all $\tau \in S_u(G)$ then there exists $\epsilon > 0$ such that $\tau(a) \geq \epsilon$ for all $\tau \in S_u(G)$. So there exists a rational number α such that $\alpha > \epsilon/2$ and $\alpha u \leq a$. Thus $a \in G^+$. ∎

Proposition 5.6.4. *Let G be a dimension group, $u \in G^+ \setminus \{0\}$ be a fixed order unit, and $S_u(G)$ be the set of all traces that map $u \mapsto 1$. Then a is infinitesimal if and only if for all $\tau \in S_u(G)$, $\tau(a) = 0$.*

Proof: If a is infinitesimal then for all $\epsilon > 0$, $-\epsilon u \leq a \leq \epsilon u$. Then, for all $\tau \in S_u(G)$, $-\epsilon\tau(u) \leq \tau(a) \leq \epsilon\tau(u)$. Since all traces in $S_u(G)$ maps $u \mapsto 1$, then $-\epsilon \leq \tau(a) \leq \epsilon$ for all $\epsilon > 0$. It follows that $\tau(a) = 0$ for all $\tau \in S_u(G)$.

Conversely, if $\tau(a) = 0$ for all $\tau \in S_u(G)$, then given rational number $\epsilon > 0$, $\tau(a + \epsilon u) = \tau(a) + \epsilon > 0$. This implies that $a + \epsilon u \geq 0$, i.e., $a \geq -\epsilon u$. On the other hand, if $\tau(a) = 0$ for all $\tau \in S_u(G)$, then $\tau(-a) = 0$ for all $\tau \in S_u(G)$. Thus given rational number $\epsilon > 0$, $\tau(-a + \epsilon u) = \tau(-a) + \epsilon > 0$. This implies that $-a + \epsilon u \geq 0$, i.e., $a \leq \epsilon u$. ∎

Lemma 5.6.5. *Let G be a dimension group, $u \in G^+ \setminus \{0\}$ be a fixed order ideal, and $S_u(G)$ be the set of all traces that map $u \mapsto 1$. Let $\hat{}: G \to Aff\, S_u(G)$ given by $\hat{a}(\tau) = \tau(a)$. Then the set $F = \{\tau \in S_u(G) \mid \hat{a}(\tau) = \inf \hat{a}\}$ is a closed face, i.e., every convex combination $\tau = \alpha_1\tau_1 + \alpha_2\tau_2$ with $\tau \in F$, $\tau_1, \tau_2 \in S_u(G)$ and $\alpha_1, \alpha_2 \geq 0$; implies in $\tau_1, \tau_2 \in F$.*

Proof: Let $\tau = \alpha_1\tau_1 + \alpha_2\tau_2$ with $\tau \in F$, $\tau_1, \tau_2 \in S_u(G)$ and $\alpha_1, \alpha_2 \geq 0$; then

$$\hat{a}(\tau) = \alpha_1\hat{a}(\tau_1) + \alpha_2\hat{a}(\tau_2).$$

Assume, by contradiction, that $\tau_1 \notin F$, then $\hat{a}(\tau_1) > \inf \hat{a}$. Thus,

$$\hat{a}(\tau) = \alpha_1\hat{a}(\tau_1) + \alpha_2\hat{a}(\tau_2) > \alpha_1 \inf \hat{a} + \alpha_2\hat{a}(\tau_2) = \alpha_1 \inf \hat{a} + \alpha_2\tau_2(a).$$

Since τ_2 is a trace, $\tau_2(a) \geq 0$. Since $\alpha_1, \alpha_2 \geq 0$, then

$$\hat{a}(\tau) > \alpha_1 \inf \hat{a} + \alpha_2\tau_2(a) \geq \inf \hat{a}.$$

So $\hat{a}(\tau) > \inf \hat{a}$, which is a contradiction with the hypothesis that $\tau \in F$. Therefore $\tau_1 \in F$. Analogously we can prove that $\tau_2 \in F$. Therefore F is a closed face. ∎

Proposition 5.6.6. *Let G be a dimension group, $u \in G^+ \setminus \{0\}$ be a fixed order unit, and $S_u(G)$ be the set of all traces that map $u \mapsto 1$. If $a \notin G^+$ then there exists a pure trace $\tau \in \partial S_u(G)$ such that $\tau(a) \leq 0$.*

Proof: If $a \notin G^+$ then, by contrapositive of Proposition 5.6.3, there exists $\tau_0 \in S_u(G)$ such that $\tau_0(a) \leq 0$. Then $\inf \hat{a} \leq 0$. Since $S_u(G)$ is a compact convex subset of the locally compact topological space $\mathbb{R}^G$, the affine map $\hat{a} : S_u(G) \to \mathbb{R}$ admits

its infimum on a closed face of $S_u(G)$. The set $F = \{\tau \in S_u(G) \mid \hat{a}(\tau) = \inf \hat{a}\}$ is a closed face of $S_u(G)$ by Lemma 5.6.5. Since extreme points of F are extreme points of $S_u(G)$, we have that there exists $\tau \in \partial S_u(G)$ such that $\tau(a) = \inf \hat{a} \leq 0$. ∎

Let G be a dimension group with order unit u. For $\tau \in S_u(G)$, denote by $\ker^+ \tau$ the set $\ker \tau \cap G^+$.

Lemma 5.6.7. *Let G be a dimension group with order unit u. Assume that:*

(i) If I is a nonzero order ideal then every pure trace defined on I can be extended after rescaling to a trace in G,

(ii) All order ideals of G admit an order unit,

(iii) For all order ideal I of G, the set G/I has no nonzero infinitesimals.

Let $a \in G$ and assume:

(a) $\hat{a} \geq 0$,

(b) $J = \cap\{I \lhd G \mid a \in I\}$ is an order ideal.

Then a is an order unit for J (consequently $a \in G^+$).

Proof: If, by contradiction, a is not an order unit for J, then $a \notin J^+$. By Proposition 5.6.6, there exists $\tau \in \partial S_u(J)$ such that $\tau(a) \leq 0$. By hypothesis (i), τ can be extended after rescaling to a trace $\overline{\tau}$ in G, i.e., there exists $\lambda \in \mathbb{R}$ such that $\lambda\tau = \overline{\tau} \restriction_J$.

By hypothesis (a), $\hat{a} \geq 0$, i.e., $\hat{a}(\overline{\tau}) \geq 0$, so $\overline{\tau}(a) \geq 0$. But since $\overline{\tau}(a) = \lambda\tau(a) \leq 0$, we have:

$$0 \leq \overline{\tau}(a) = \lambda\tau(a) \leq 0 \Rightarrow \overline{\tau}(a) = 0. \tag{5.6.1}$$

If $\sigma \in \partial S_u(G)$ has the property $\ker^+ \overline{\tau} \subseteq \ker^+ \sigma$, then $\overline{\tau}(a) = 0 \Rightarrow \sigma(a) = 0$.

Set $I := \ker^+ \overline{\tau}$. Let us see how the pure traces on G/I behave. An element of G/I is a coset denoted by $[x]$ where $x \in G$. If $\sigma \in \partial S_u(G/I)$ then $\sigma(I) = 0$ so $\ker^+ \overline{\tau} \subset \ker^+ \sigma$. Thus, if $\sigma \in \partial S_u(G/I)$ then $\sigma([a]) = 0$ so $[a] \in \operatorname{Inf} G/I$. By hypothesis (*iii*), G/I has no nonzero infinitesimals, so $[a] = 0$, i.e., $a \in \ker^+ \overline{\tau}$.

Since $\ker^+ \overline{\tau}$ is an order ideal that contains a then, by hypothesis (b), $J \subset \ker^+ \overline{\tau}$. Thus $\overline{\tau} \restriction_J = 0$. So $\tau = \lambda^{-1}\overline{\tau} \restriction_J = 0$. Contradiction! ∎

Theorem 5.6.8. *Let G be a dimension group with order unit u. Assume that:*

(i) If J is a nonzero order ideal of G, then every pure trace defined on J can be extended after rescaling to a trace defined on G,

(ii') Every increasing chain of order ideals is finite,

(iii') If $J, M \lhd G$, $J \subset M$ and M/J is simple then M/J has no nonzero infinitesimals.

Then $a \in G^+$ if and only if $\hat{a} \geq 0$.

Proof: First, assume that $a \in G^+$. Then, by definition of traces, $\tau(a) \geq 0$ for all $\tau \in S_u(G)$, thus $\hat{a} \geq 0$.

Now, assume that $\hat{a} \geq 0$ and let us prove that $a \in G^+$. By Lemma 5.6.7, it is sufficient to prove the hypothesis (i), (ii), (iii), (a) and (b). We already have (i) and (a) as hypothesis, and condition (ii') of this theorem implies (ii) and (b) of the previous lemma (since every intersection of finitely many order ideals is an order ideal). Thus, the only hypothesis of the previous lemma we need to prove is (iii), i.e., for all order ideal I of G the set G/I contains no nonzero infinitesimals.

Assume, by contradiction, that G/I contains a nonzero infinitesimal, say, the element $[x] \in \operatorname{Inf} G/I \setminus \{0\}$. Let J be the smallest order ideal of G/I containing $[x]$, i.e., $J := \cap\{I' \lhd G/I \mid [x] \in I'\}$. Let τ be a pure trace defined on J. By hypothesis (i), τ can be extended after rescaling to a trace $\overline{\tau}$ defined on G/I. Since $[x] \in \operatorname{Inf} G/I$, $\overline{\tau}([x]) = 0$. It follows that $\tau([x]) = 0$. Thus, $[x] \in \operatorname{Inf} J$.

Now let M be a maximal order ideal of J. By hypothesis (ii'), such maximal order ideal exists. Since M is maximal, J/M is simple. Thus, by hypothesis (iii'), J/M has no nonzero infinitesimals.

Since $[x] \in \operatorname{Inf} J$, for every $\tau \in S_u(J)$, $\tau([x]) = 0$. In particular, if a trace $\sigma \in S_u(G/I)$ is such that $\sigma(M) = 0$, then, since $J \subset M$, $\sigma([x]) = 0$. Then $\tau([x]) = 0$ for every $\tau \in S_u(J) \cap \{\sigma \in S_u(G/I) \mid \sigma(M) = 0\} \cong S_u(J/M)$. So $[[x]]$ is infinitesimal in J/M (the double brackets represent cosets in J/M). Since J/M has no nonzero infinitesimals, $[[x]] = 0$. This means that $[x] \in M$. Since J is the smallest order ideal containing x, $J \subset M$. Contradiction! ∎

In the particular case of the Eulerian Bratteli diagram dimension group G, we proved that every pure trace defined on order ideals can be extender after rescaling to a trace in G (Proposition 5.5.2), this covers hypothesis (i) of the theorem. It is also true that every increasing chain of order ideals is finite (hypothesis (ii') of the theorem), since $I_{nk} \subset I_{pj} \Leftrightarrow n > p$. Thus, every increasing chain of order ideals ends at $I_{11} = G$. For hypothesis (iii'), consider the following results.

Lemma 5.6.9. *For $i = 1, 2$, let $(G_i, +_i)$ be abelian groups and H_i be subgroups of G_i. Suppose that $\varphi : G_1 \to G_2$ is a homomorphism satisfying $\varphi(H_1) \subset H_2$. Then there exists a homomorphism $\hat{\varphi} : G_1/H_1 \to G_2/H_2$ given by $\hat{\varphi}(g +_1 H_1) = \varphi(g) +_2 H_2$.*

We prove that $\hat{\varphi}$ is well-defined, i.e., it does not depend on the choice of representative. In other words, $g, g' \in G_1/H_1$ are such that $g +_1 H_1 = g' +_1 H_1$, then $\hat{\varphi}(g) = \hat{\varphi}(g')$.

Proof: Let $g, g' \in G_1/H_1$ such that $g +_1 H_1 = g' +_1 H_1$, then $g +_1 (-g') \in H_1$. Since $\varphi(H_1) \subset H_2$, $\varphi(g +_1 (-g')) \in H_2$. Since φ is a homomorphism, $\varphi(g) +_2 (-\varphi(g')) \in H_2$. Thus $\varphi(g) \in \varphi(g') +_2 H_2$ and $\varphi(g') \in \varphi(g) +_2 H_2$, i.e., $\varphi(g) +_2 H_2 = \varphi(g') +_2 H_1$. ∎

The equation $\hat{\varphi}(g +_1 H_1) = \varphi(g) +_2 H_2$ can be written as $\hat{\varphi}\pi_1(g) = \pi_2\varphi(g)$ where $\pi_i : G_i \to G_i/H_i$ is the canonical projection on the quotient defined as $\pi_i(g) = g +_i H_i$ $(i = 1, 2)$. This means that the diagram below commutes:

$$\begin{tikzcd}
G_1 \arrow[r, "\varphi"] \arrow[d, "\pi_1"] & G_2 \arrow[d, "\pi_2"] \\
G_1/H_1 \arrow[r, "\hat{\varphi}"] & G_2/H_2
\end{tikzcd}$$

Lemma 5.6.10. *Let $\{G_n\}_{n\geq 1}$ be abelian groups and $\{H_n\}_{n\geq 1}$ be subgroups $H_n \subset G_n$. Suppose that $\varphi_n : G_n \to G_{n+1}$ is a homomorphism satisfying $\varphi_n(H_n) \subset H_{n+1}$. Consider $\hat{\varphi}_n : G_n/H_n \to G_{n+1}/H_{n+1}$ the homomorphisms given by Lemma 5.6.9. If $H = \varinjlim(H_n, \varphi_n \restriction_{H_n})$, then $G/H \simeq \varinjlim(G_n/H_n, \hat{\varphi}_n)$.*

Proof: Define $\Phi : G \to \varinjlim(G_n/H_n, \hat{\varphi}_n)$ by $\Phi(g_n) = (g_n + H_n)$ and we prove that

1. $\ker \Phi = H$, and
2. $\Phi(G) = \varinjlim(G_n/H_n, \hat{\varphi}_n)$.

Proof of 1. We prove that $g \in \ker \Phi \Leftrightarrow g \in H$:

$$\begin{aligned}
g \in \ker \Phi &\Leftrightarrow \exists N \forall n > N,\ \Phi(g_n) \in H_n \\
&\Leftrightarrow \exists N \forall n > N,\ (g_n + H_n) \in H_n \\
&\Leftrightarrow \exists N \forall n > N,\ (g_n) \in H_n \\
&\Leftrightarrow \exists N \forall n > N,\ \varphi_n \restriction_{H_n} (g_n) = \varphi_n(g_n) \in H_{n+1} \\
&\Leftrightarrow g \in \varinjlim(H_n, \varphi_n \restriction_{H_n}) = H.
\end{aligned}$$

Proof of 2. We have that $x \in \Phi(G)$ if and only if there exists $(g_n) \in G$ such that $x = \Phi(g_n)$. Since $\Phi(g_n) = (g_n + H_n) \in G_n/H_n$ and $\hat{\varphi}(G_n/H_n) \subset G_{n+1}/H_{n+1}$, then $x \in \varinjlim(G_n/H_n/\hat{\varphi}_n)$.

By the first isomorphism theorem, $G/\ker \Phi \simeq \Phi(G)$. Therefore,

$$G/H \simeq \varinjlim(G_n/H_n, \hat{\varphi}_n).$$

∎

In the particular case of the generalized Eulerian Bratteli given in Section 1.2.2, $G_n = \mathbb{Z}^n$ and φ_n is given by the generalized transition matrix. In order to have G_n/H_n simple, we consider subgroups of the form $H_n = \operatorname{span}\{e_1, \ldots, e_{n-1}\}$ and $H'_n = \operatorname{span}\{e_2, \ldots, e_n\}$, which are both isomorphic to $\mathbb{Z}^{n-1}$.

In the quotient $G_n/H_n \simeq \mathbb{Z}$, we have that the representatives are given by $\operatorname{span}\{e_n\}$ and these elements are affected only by the last row of the transition matrix. Since the transition matrix is of the form $\varphi_n = \begin{pmatrix} A & B \\ 0 & r_n \end{pmatrix}$ where A is a $n \times (n-1)$ matrix, B is a $n \times 1$ matrix, and $r_n \in \mathbb{N}^*$; we have that $\hat{\varphi}_n(g + H_n) = r_n g + H_{n+1}$. It follows that $\varinjlim(G_n/H_n, \hat{\varphi}_n) = \varinjlim(\mathbb{Z}, \times r_n)$, therefore, $G/H = \varinjlim(\mathbb{Z}, \times r_n)$. This dimension group is simple and contains no nonzero infinitesimals (Example 5.6.2).

Analogously, in the quotient $G_n/H'_n \simeq \mathbb{Z}$, we have that the representatives are given by $\operatorname{span}\{e_1\}$ and these elements are affected only by the first row of the transition matrix. Since the transition matrix is of the form $\varphi_n = \begin{pmatrix} l_n & 0 \\ A & B \end{pmatrix}$ where A is a $n \times (n-1)$ matrix, B is a $n \times 1$ matrix, and $r_n \in \mathbb{N}^*$, then $\hat{\varphi}_n(g + H_n) = l_n g + H_{n+1}$. It follows that $\varinjlim(G_n/H_n, \hat{\varphi}_n) = \varinjlim(\mathbb{Z}, \times l_n)$, therefore, $G/H = \varinjlim(\mathbb{Z}, \times l_n)$, which is also simple and contains no nonzero infinitesimals (Example 5.6.2).

This means that condition (iii') of Theorem 5.6.8 applies for the Eulerian Bratteli diagram. Therefore we can state the following theorem:

Theorem 5.6.11. *Let $G = \varinjlim(\mathbb{Z}^n, \varphi_n)$ be the dimension group associated to the Eulerian Bratteli diagram, i.e., given by the transition matrices*

$$\varphi_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ n & 2 & 0 & \cdots & 0 & 0 \\ 0 & n-1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 3 & n-1 & 0 \\ 0 & 0 & \cdots & 0 & 2 & n \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Then $a \in G^+$ if and only if $\hat{a} \geq 0$.

For the following examples, recall that e_{nk} is the element of the Eulerian Bratteli diagram dimension group given by $[(0, \ldots, 0, 1, 0, \ldots, 0), n]$ where 1 at position k. Recall also, from Theorem 5.2.2, that

$$W^z_{nk} = \frac{1}{n!} \prod_{i=1}^{n} \left(1 + \frac{i-k}{z}\right), \qquad W^\infty_{nk} = \frac{1}{n!},$$

and, in (5.2.2), we set $\tau_z([(v_1, \ldots, v_n), n]) = \sum_{k=1}^{n} v_k W^z_{nk}$.

Example 5.6.12. Let a be the element of the Eulerian Bratteli diagram dimension group G given by $a = [a_n, n]$ where $a_n = e_{n,k-1} + a_{nk}e_{nk} + e_{n,k+1}$, $a_{nk} \in \mathbb{Z}$, $n \geq 3$, $1 < k < n$. We will prove that $\hat{a} \geq 0$ if and only if $a_{nk} \geq -2$, in other words, we prove:

Claim: $(\forall z \in \mathbb{Z}^* \cup \{\infty\},\ \tau_z(a) \geq 0) \Leftrightarrow a_{nk} \geq -2.$
The proof consists of the following steps:

Step 1: $\tau_\infty(a) \geq 0 \Leftrightarrow a_{nk} \geq -2.$

This proves one side ($\Rightarrow$) of the claim because, assuming $\tau_z(a) \geq 0\ \forall z \in \mathbb{Z}^* \cup \{\infty\}$ then, in particular, for $z = \infty$ we have $\tau_\infty(a) \geq 0$. This also proves part of the other side ($\Leftarrow$), the only thing left to prove is that for $z \neq \infty$, $\tau_z(a) \geq 0$.

Step 2: $a_{nk} \geq -2 \Rightarrow (\forall z \in \mathbb{Z}^*,\ \tau_z(a) \geq 0).$

In order to prove Step 2, for $a \in G$ fixed, we write $\tau_z(a) = f(z)g(z)$, for $z \in \mathbb{Z}^*$, and show:

Part 1: $f(z) \geq 0$, and there exist an interval I such that $f(z) = 0$ for all $z \in I \cap \mathbb{Z}^*$.

Part 2: if $z \notin I$, then $g(z) \geq 0$.

Proof of step 1. Note that

$$\tau_\infty(a) \geq 0 \Leftrightarrow \frac{1}{n!}(1 + a_{nk} + 1) \geq 0 \Leftrightarrow a_{nk} \geq -2.$$

Proof of step 2. Note that $\tau_z(a) = W^z_{n,k-1} + a_{nk}W^z_{nk} + W^z_{n,k+1}$ where

$$W^z_{n,k-1} = \frac{1}{n!}\prod_{i=1}^{n}\left(1 + \frac{i-k+1}{z}\right) = \frac{1}{n!}\prod_{i=2}^{n+1}\left(1 + \frac{i-k}{z}\right) = \frac{1}{n!z^n}\prod_{i=2}^{n+1}(z-k+i),$$

$$W^z_{nk} = \frac{1}{n!}\prod_{i=1}^{n}\left(1 + \frac{i-k}{z}\right) = \frac{1}{n!z^n}\prod_{i=1}^{n}(z-k+i), \text{ and}$$

$$W^z_{n,k+1} = \frac{1}{n!}\prod_{i=1}^{n}\left(1 + \frac{i-k-1}{z}\right) = \frac{1}{n!}\prod_{i=0}^{n-1}\left(1 + \frac{i-k}{z}\right) = \frac{1}{n!z^n}\prod_{i=0}^{n-1}(z-k+i).$$

Thus, $\tau_z(a) = \frac{1}{n!z^n}\prod_{i=2}^{n+1}(z-k+i) + a_{nk}\frac{1}{n!z^n}\prod_{i=1}^{n}(z-k+i) + \frac{1}{n!z^n}\prod_{i=0}^{n-1}(z-k+i) =$

$$\left[\frac{1}{n!z^n}\prod_{i=2}^{n-1}(z-k+i)\right][(z-k+n)(z-k+n+1)+a_{nk}(z-k+1)(z-k+n)+(z-k)(z-k+1)].$$

Set

$$f(z) := \frac{1}{n!z^n}\prod_{i=2}^{n-1}(z-k+i)$$

and

$$g(z) = [(z-k+n)(z-k+n+1) + a_{nk}(z-k+1)(z-k+n) + (z-k)(z-k+1)].$$

To simplify the writing of the proof of Parts 1 and 2, we set $d := z-k$ and denote, for $d \neq -k$, $\tilde{f}(d) = f(d+k) = f(z)$ and $\tilde{g}(d) = g(d+k) = g(z)$. Hence we have, for $d \neq -k$,

$$\tilde{f}(d) = \frac{1}{n!(d+k)^n}\prod_{i=2}^{n-1}(d+i),$$

$$\tilde{g}(d) = [(d+n)(d+n+1) + a_{nk}(d+1)(d+n) + d(d+1)],$$

and $\tau_z(a) = \tilde{f}(d)\tilde{g}(d)$. We shall prove that:

Part 1: $\tilde{f}(d) \geq 0$ and $\tilde{f}(d) = 0$ for $d \in I = [-(n-1), -2] \cap \mathbb{Z} \setminus \{-k\}$;

Part 2: if $d \notin I$, then $\tilde{g}(d) \geq 0$.

For this part, we consider three cases for d.

Case $-(n-1) \leq d \leq -2$. In this case, $d+i = 0$ for some $2 \leq i \leq n-1$, so $\tilde{f}(d) = 0$.

Case $d > -2$. In this case, $d+i > 0$ for all $2 \leq i \leq n-1$. Thus the product $\prod_2^{n-1}(d+i)$ is positive, and so is $(d+k)^n > 0$. Therefore $\tilde{f}(d) > 0$.

Case $d < -(n-1)$. In this case, $d+i < 0$ for every $2 \leq i \leq n-1$. Since the product $\prod_{i=2}^{n-1}(d+i)$ has $n-2$ factors, its sign will be positive if and only if n is even. Similarly, $(d+k)^n$ is positive if and only if n is even. Thus, for any $n \geq 3$, $\tilde{f}(d) > 0$. This proves Part 1.

We now aim to prove Part 2. Assume that $d \notin I$, i.e., $d \geq -1$ or $d \leq -n$.

- If $d = -1$, then $\tilde{g}(d) = (d+n)(d+n+1) + a_{nk}(d+1)(d+n) + d(d+1) = (n-1)n = n^2 - n \geq 0$.
- If $d = -n$, then $\tilde{g}(d) = (d+n)(d+n+1) + a_{nk}(d+1)(d+n) + d(d+1) = (-n)(1-n) = n^2 - n \geq 0$.

- If $d > -1$ or $d < -n$, then suppose, by contradiction, that $\tilde{g}(d) < 0$, i.e.,

  $$(d+n)(d+n+1) + a_{nk}(d+1)(d+n) + d(d+1) < 0.$$

  We can write this inequality as:

  $$a_{nk} < -\frac{(d+n)(d+n+1) + d(d+1)}{(d+1)(d+n)}.$$

  It is easy to prove that

  $$-\frac{(d+n)(d+n+1) + d(d+1)}{(d+1)(d+n)} \leq -2.$$

  Indeed, $-\frac{d(d+1)+(d+n)(d+n+1)}{(d+1)(d+n)} \leq -2$ if and only if

  $$d(d+1) + (d+n)(d+n+1) \geq 2(d+1)(d+n).$$

  Expanding and simplifying this last inequality, we have that $n^2 \geq n$, which is true for any natural n. Therefore, $a_{nk} < -2$, which contradicts our initial hypothesis. So $\tilde{g}(d) \geq 0$ for every $-1 \leq d \leq -n$. This proves Part 2.

Example 5.6.13. Let a be the element of the Eulerian Bratteli diagram dimension group G given by $a = [a_n, n]$ where $a_n = a_{n,k-1}e_{n,k-1} + a_{nk}e_{nk} + a_{n,k+1}e_{n,k+1}$, $a_{n,k-1}, a_{nk}, a_{n,k+1} \in \mathbb{Z}$, $n \geq 3$, $1 < k < n$. If $a_{n,k-1} + a_{n,k+1} > 2a_{nk}$, then $a \in G^+$. This is a consequence of Example 5.6.12 by including the weights $a_{n,k-1}$ and $a_{n,k+1}$ in the factors $W_{n,k-1}$ and $W_{n,k+1}$, respectively.

Example 5.6.14. Let a be the element of the Eulerian Bratteli diagram dimension group G given by $a = [a_n, n]$ where $a_n = e_{n,k-j} + a_{nk}e_{nk} + e_{n,k+j}$, $a_{nk} \in \mathbb{Z}$, $n \geq 3$, $1 < k < n$ and $0 < j < \min\{k, n-k\}$. Note that this generalizes Example 5.6.12 (in that case, $j = 1$). We show that $\hat{a} \geq 0$ if and only if $a_{nk} \geq -2$, in other words,

Claim: $(\forall z \in \mathbb{Z}^* \cup \{\infty\},\ \tau_z(a) \geq 0) \Leftrightarrow a_{nk} \geq -2.$

The strategy of the proof is identical to Example 5.6.12.

Proof of step 1. Note that

$$\tau_\infty(a) \geq 0 \Leftrightarrow \frac{1}{n!}(1 + a_{nk} + 1) \geq 0 \Leftrightarrow a_{nk} \geq -2.$$

Proof of step 2. Note that $\tau_z(a) = W^z_{n,k-j} + a_{nk}W^z_{nk} + W^z_{n,k+j}$ where

$$W^z_{n,k-j} = \frac{1}{n!}\prod_{i=1}^{n}\left(1+\frac{i-(k-j)}{z}\right) = \frac{1}{n!}\prod_{i=1+j}^{n+j}\left(1+\frac{i-k}{z}\right) = \frac{1}{n!z^n}\prod_{i=1+j}^{n+j}(z-k+i),$$

$$W^z_{nk} = \frac{1}{n!}\prod_{i=1}^{n}\left(1+\frac{i-k}{z}\right) = \frac{1}{n!z^n}\prod_{i=1}^{n}(z-k+i), \text{ and}$$

$$W^z_{n,k+j} = \frac{1}{n!}\prod_{i=1}^{n}\left(1+\frac{i-(k+j)}{z}\right) = \frac{1}{n!}\prod_{i=1-j}^{n-j}\left(1+\frac{i-k}{z}\right) = \frac{1}{n!z^n}\prod_{i=1-j}^{n-j}(z-k+i).$$

Thus, $\tau_z(a) = \frac{1}{n!z^n}\prod_{i=1+j}^{n+j}(z-k+i) + a_{nk}\frac{1}{n!z^n}\prod_{i=1}^{n}(z-k+i) + \frac{1}{n!z^n}\prod_{i=1-j}^{n-j}(z-k+i) =$

$$\left[\frac{1}{n!z^n}\prod_{i=1+j}^{n-j}(z-k+i)\right]\Bigg[\prod_{i=n-j+1}^{n+j}(z-k+i) \;+\; a_{nk}\prod_{i=1}^{j}(z-k+i)\prod_{i=n-j+1}^{n}(z-k+i)$$
$$+ \prod_{i=1-j}^{j}(z-k+i)\Bigg].$$

Set

$$f(z) := \frac{1}{n!z^n}\prod_{i=1+j}^{n-j}(z-k+i)$$

and

$$g(z) = \prod_{i=n-j+1}^{n+j}(z-k+i) + a_{nk}\prod_{i=1}^{j}(z-k+i)\prod_{i=n-j+1}^{n}(z-k+i) + \prod_{i=1-j}^{j}(z-k+i).$$

To simplify the writing of the proof of Parts 1 and 2, we set $d := z-k$ and denote, for $d \neq -k$, $\tilde{f}(d) = f(d+k) = f(z)$ and $\tilde{g}(d) = g(d+k) = g(z)$. Hence we have, for $d \neq -k$,

$$\tilde{f}(d) = \frac{1}{n!(d+k)^n}\prod_{i=1+j}^{n-j}(d+i),$$

$$\tilde{g}(d) := \prod_{i=n-j+1}^{n+j}(d+i) + a_{nk}\prod_{i=1}^{j}(d+i)\prod_{i=n-j+1}^{n}(d+i) + \prod_{i=1-j}^{j}(d+i),$$

and $\tau_z(a) = \tilde{f}(d)\tilde{g}(d)$. We shall prove that,

Part 1: $\tilde{f}(d) \geq 0$ and $\tilde{f}(d) = 0$ for $d \in I = [-(n-j), -(1+j)] \cap \mathbb{Z} \setminus \{-k\}$;

Part 2: if $d \notin I$, then $\tilde{g}(d) \geq 0$.

For this part, we consider three cases for d.

Case $-(n-j) \leq d \leq -(1+j)$. In this case, $d+i=0$ for some $1+j \leq i \leq n-j$, so $\tilde{f}(d)=0$.

Case $d > -(1+j)$. In this case, $d+i>0$ for all $1+j \leq i \leq n-j$. Thus, the product $\prod_{1+j}^{n-j}(d+i)$ is positive, and so is $(d+k)^n > 0$. Therefore $\tilde{f}(d) > 0$.

Case $d < -(n-j)$. In this case, $d+i<0$ for every $1+j \leq i \leq n-j$. Since the product $\prod_{i=1+j}^{n-j}(d+i)$ has $n-2j$ factors, its sign will be positive if and only if n is even. Similarly, $(d+k)^n$ is positive if and only if n is even. Thus, for any $n \geq 3$, $\tilde{f}(d) > 0$. This proves Part 1.

We now aim to prove Part 2. Assume that $d \notin I$, i.e., $d \geq -j$ or $d \leq -(n-j+1)$.

- If $-j \leq d \leq -1$, then $(d+i)=0$ for $1 \leq i \leq j$, this means that $\prod_i (d+i)=0$ if one of the factors i is on the interval $1 \leq i \leq j$. In other words,

$$\begin{aligned} \tilde{g}(d) &= \prod_{i=n-j+1}^{n+j}(d+i) + a_{nk}\prod_{i=1}^{j}(d+i)\prod_{i=n-j+1}^{n}(d+i) + \prod_{i=1-j}^{j}(d+i) \\ &= \prod_{i=n-j+1}^{n+j}(d+i) \geq 0. \end{aligned}$$

  The latter product is non-negative because, for $i \geq n-j+1$, we have that

$$\begin{aligned} d+i &\geq d+n-j+1 \\ &\geq n-2j+1 \quad (\text{since } d \geq -j) \\ &\geq 1 \quad (\text{because } j < \min\{k, n-k\} \text{ and } 1<k<n, \text{ so } j < \frac{n}{2}). \end{aligned}$$

  Since $d+i \geq 1$ for every $i \geq n-j+1$, each factor of the product is non-negative.

- If $-n \leq d \leq -(n-j+1)$, then $(d+i)=0$ for $n-j+1 \leq i \leq n$, this means that $\prod_i (d+i)=0$ if one of the factors i is on the above interval. In other words,

$$\begin{aligned} \tilde{g}(d) &= \prod_{i=n-j+1}^{n+j}(d+i) + a_{nk}\prod_{i=1}^{j}(d+i)\prod_{i=n-j+1}^{n}(d+i) + \prod_{i=1-j}^{j}(d+i) \\ &= \prod_{i=1-j}^{j}(d+i) \geq 0. \end{aligned}$$

  The latter product is non-negative because it consists of an even number of negative factors. Indeed, there are $2j$ factors in the product and each of them is equal to $d+i$, for $i \leq j$, moreover:

$$d+i \leq d+j$$

$$\begin{aligned} &\leq -(n-j+1)+j \quad (\text{since } d \leq -(n-j+1)) \\ &= -n+2j-1 \\ &\leq -1 \quad (\text{because from } j < \frac{n}{2}, \text{ we have } 2j \leq n). \end{aligned}$$

- If $d > -1$ or $d < -n$, then suppose, by contradiction, that $\tilde{g}(d) < 0$, i.e.,

$$\prod_{i=n-j+1}^{n+j}(d+i) + a_{nk}\prod_{i=1}^{j}(d+i)\prod_{i=n-j+1}^{n}(d+i) + \prod_{i=1-j}^{j}(d+i) < 0$$

Since $d > -1$ or $d < -n$, the products $\prod_{i=1}^{j}(d+i)$ and $\prod_{i=n-j+1}^{n}(d+i)$ are non zero, thus we can write the inequality as

$$a_{nk} < -\frac{\prod_{i=n-j+1}^{n+j}(d+i) + \prod_{i=1-j}^{j}(d+i)}{\prod_{i=1}^{j}(d+i)\prod_{i=n-j+1}^{n}(d+i)}.$$

We can expand the fraction above as

$$a_{nk} < -\left[\frac{\prod_{i=n-j+1}^{n+j}(d+i)}{\prod_{i=1}^{j}(d+i)\prod_{i=n-j+1}^{n}(d+i)} + \frac{\prod_{i=1-j}^{j}(d+i)}{\prod_{i=1}^{j}(d+i)\prod_{i=n-j+1}^{n}(d+i)}\right],$$

which, in turn, can be simplified to

$$a_{nk} < -\left[\frac{\prod_{i=n+1}^{n+j}(d+i)}{\prod_{i=1}^{j}(d+i)} + \frac{\prod_{i=1-j}^{0}(d+i)}{\prod_{i=n-j+1}^{n}(d+i)}\right].$$

Performing change of variables on the indices of the products, we have

$$a_{nk} < -\left[\frac{\prod_{i=1}^{j}(d+n+i)}{\prod_{i=1}^{j}(d+i)} + \frac{\prod_{i=1}^{j}(d-i+1)}{\prod_{i=1}^{j}(d+n-i+1)}\right],$$

which can be written as

$$a_{nk} < -\left[\prod_{i=1}^{j}\frac{(d+n+i)}{(d+i)} + \prod_{i=1}^{j}\frac{(d-i+1)}{(d+n-i+1)}\right].$$

Expanding the products using factorials and rewriting in terms of binomial coefficients, we have:

$$a_{nk} < -\left[\binom{d+n+j}{n}\binom{d+n}{n}^{-1} + \binom{d+n-j}{n}\binom{d+n}{n}^{-1}\right].$$

Thus

$$a_{nk} < -\binom{d+n}{n}^{-1}\left[\binom{d+n+j}{n} + \binom{d+n-j}{n}\right].$$

But, by Lemma 4.4.4,

$$\binom{d+n+j}{n} + \binom{d+n-j}{n} \geq 2\binom{d+n}{n}.$$

This means that

$$-\binom{d+n}{n}^{-1}\left[\binom{d+n+j}{n} + \binom{d+n-j}{n}\right] \leq -2.$$

So $a_{nk} < -2$, which contradicts the initial hypothesis. This proves Part 2.

Therefore, $\tau_z(a) \geq 0$ if and only if $a_{nk} \geq -2$.

Bibliography

[1] Scot Adams, George A. Elliott, and Thierry Giordano. Amenable actions of groups. *Trans. Amer. Math. Soc.*, 344(2):803–822, 1994.

[2] Sarah Bailey, Michael Keane, Karl Petersen, and Ibrahim A. Salama. Ergodicity of the adic transformation on the Euler graph. *Math. Proc. Cambridge Philos. Soc.*, 141(2):231–238, 2006.

[3] B. Mitchell Baker, Thierry Giordano, and Radu B. Munteanu. Approximate transitivity of the ergodic action of the group of finite permutations of $\mathbb{N}$ on $\{0,1\}^{\mathbb{N}}$. *Ergodic Theory Dynam. Systems*, 39(11):2881–2895, 2019.

[4] Miklós Bóna and Richard Ehrenborg. A combinatorial proof of the log-concavity of the numbers of permutations with k runs. *J. Combin. Theory Ser. A*, 90(2):293–303, 2000.

[5] Ola Bratteli. Inductive limits of finite dimensional C^*-algebras. *Trans. Amer. Math. Soc.*, 171:195–234, 1972.

[6] Francesco Brenti. Unimodal, log-concave and Pólya frequency sequences in combinatorics. *Mem. Amer. Math. Soc.*, 81(413):viii+106, 1989.

[7] Louis Comtet. *Advanced combinatorics*. D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974. The art of finite and infinite expansions.

[8] J. N. Darroch. On the distribution of the number of successes in independent trials. *Ann. Math. Statist.*, 35:1317–1321, 1964.

[9] Kenneth R. Davidson. *C^*-algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.

[10] Edward G. Effros. *Dimensions and C^*-algebras*, volume 46 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, D.C., 1981.

[11] George A. Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *J. Algebra*, 38(1):29–44, 1976.

[12] Sarah Bailey Frick and Karl Petersen. Random permutations and unique fully supported ergodicity for the Euler adic transformation. *Ann. Inst. Henri Poincaré Probab. Stat.*, 44(5):876–885, 2008.

[13] Eldar Giladi and Joseph B. Keller. Eulerian number asymptotics. *Proc. Roy. Soc. London Ser. A*, 445(1924):291–303, 1994.

[14] Thierry Giordano and David Handelman. Matrix-valued random walks and variations on property AT. *Münster J. Math.*, 1:15–72, 2008.

[15] Thierry Giordano, David Kerr, N. Christopher Phillips, and Andrew Toms. *Crossed products of C^*-algebras, topological dynamics, and classification.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer, Cham, 2018. Lecture notes based on the course held at the Centre de Recerca Matemàtica (CRM) Barcelona, June 14–23, 2011, Edited by Francesc Perera.

[16] Thierry Giordano, Ian F. Putnam, and Christian F. Skau. Topological orbit equivalence and C^*-crossed products. *J. Reine Angew. Math.*, 469:51–111, 1995.

[17] Thierry Giordano, Ian F. Putnam, and Christian F. Skau. The orbit structure of Cantor minimal $\mathbb{Z}^2$-systems. In *Operator Algebras: The Abel Symposium 2004*, volume 1 of *Abel Symp.*, pages 145–160. Springer, Berlin, 2006.

[18] Alexander Gnedin and Grigori Olshanski. The boundary of the Eulerian number triangle. *Mosc. Math. J.*, 6(3):461–475, 587, 2006.

[19] Kenneth R. Goodearl and David Handelman. Rank functions and K_O of regular rings. *J. Pure Appl. Algebra*, 7(2):195–216, 1976.

[20] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete mathematics.* Addison-Wesley Publishing Company, Reading, MA, second edition, 1994. A foundation for computer science.

[21] Andreas Greven, Gerhard Keller, and Gerald Warnecke, editors. *Entropy.* Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2003.

[22] Arshag Hajian, Yuji Ito, and Shizuo Kakutani. Invariant measures and orbits of dissipative transformations. *Advances in Math.*, 9:52–65, 1972.

[23] Richard H. Herman, Ian F. Putnam, and Christian F. Skau. Ordered Bratteli diagrams, dimension groups and topological dynamics. *Internat. J. Math.*, 3(6):827–864, 1992.

[24] Hsien-Kuei Hwang, Hua-Huai Chern, and Guan-Huei Duh. An asymptotic distribution theory for Eulerian recurrences with applications. *Adv. in Appl. Math.*, 112:101960, 125, 2020.

[25] Kathryn Lindsey and Rodrigo Treviño. Infinite type flat surface models of ergodic systems. *Discrete Contin. Dyn. Syst.*, 36(10):5509–5553, 2016.

[26] Xavier Méla. A class of nonstationary adic transformations. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(1):103–123, 2006.

[27] Xavier Méla and Karl Petersen. Dynamical properties of the Pascal adic transformation. *Ergodic Theory Dynam. Systems*, 25(1):227–256, 2005.

[28] Karl Petersen and Alexander Varchenko. The Euler adic dynamical system and path counts in the Euler graph. *Tokyo J. Math.*, 33(2):327–340, 2010.

[29] T. Kyle Petersen. *Eulerian numbers*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, 2015. With a foreword by Richard Stanley.

[30] Jim Pitman. Probabilistic bounds on the coefficients of polynomials with only real zeros. *J. Combin. Theory Ser. A*, 77(2):279–303, 1997.

[31] Richard P. Stanley. Log-concave and unimodal sequences in algebra, combinatorics, and geometry. In *Graph theory and its applications: East and West (Jinan, 1986)*, volume 576 of *Ann. New York Acad. Sci.*, pages 500–535. New York Acad. Sci., New York, 1989.

[32] Anatoly M. Vershik. A description of invariant measures for actions of certain infinite-dimensional groups. *Dokl. Akad. Nauk SSSR*, 218:749–752, 1974.

[33] Anatoly M. Vershik and Sergei V. Kerov. Asymptotic theory of the characters of a symmetric group. *Funktsional. Anal. i Prilozhen.*, 15(4):15–27, 96, 1981.