MODELING AND CONTROL OF LARGE SPACE STRUCTURES

By

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ABSTRACT

In this thesis a preliminary formulation of large space structures and their stabilization is considered. The system consists of a (rigid) massive body and flexible configurations which consist of several beams, forming the space structure. The rigid body is located at the center of the space structure and may play the role of experimental modules. A complete dynamics of the system has been developed using Hamilton's principle. Euler-Bernoulli, Rayleigh and Timoshenko beam theories are utilized to derive the dynamic equations governing the vibration of the beams. The equations that govern the motion of the complete system consist of six ordinary differential equations and several partial differential equations together with appropriate boundary conditions. The partial differential equations govern the vibration of flexible components. The ordinary differential equations describe the rotational and translational motion of the central body.

The dynamics indicate very strong interaction among rigid body translation, rigid body rotation and vibrations of flexible members through nonlinear couplings. Hence any rotation of the rigid body induces vibration in the beams and vice-versa. Also any disturbance in the orbit induces vibration in the beams and wobbles in the body rotation and vice-versa. This makes the system performance unsatisfactory for many practical applications. In this thesis stabilization of the above mentioned system subject to external disturbances is considered. The asymptotic stability of the perturbed system by applying several types of stabilizing controls such as proportional controls, deadzone controls or saturation controls is proved using Lyapunov's method.

Numerical simulations are carried out in order to illustrate the impact of dynamic coupling or interaction among several members of the system and the effectiveness of the suggested feedback controls for stabilization. Stability of a spacecraft and a space station under the influence of slew maneuvering is numerically investigated.

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Chapter 1

INTRODUCTION

1.1 Introduction

The advent of a space transportation technology makes it possible to conceive of very large satellite or space stations which could be carried into space and deployed, assembled or constructed in orbit for several purposes such as surveillance, communication, astornomy, space exploration, manufacturing and electric power generation. These large space structure concepts range from large satellite with large flexible appendages to space stations such as the solar power station satellite which would possibly be a structure nearly the size of Manhattan Island[57,91,102,104,106].

Typically suggested space structures consist of rigid main bodies and large flexible structures such as long beams, large solar pannels, antennas and truss configurations etc[64,90,76,55]. For example, the configuration of the SCOLE (Spacecraft COntrol Laboratory Experiment), a new design challenge by NASA and IEEE, shown in Fig. 1.1, consists of the space shuttle orbiter and a large antenna which is connected to the orbiter by a long beam[125]. A schematic diagram of a communication satellite proposed by the Government of Canada is shown in Fig. 1.2. The physical configuration of the U.S. space station (Fig. 1.3) consists of the trusses, the framework to which the other elements are attached, experimental modules which may play the role of rigid bodies and several appendages[101,36].

In recent years considerable attention has been focused on the problem of modelling and control of large space stations whose configurations consist of a massive central body and large flexible structures[49,100,27,7,59,10,63,132,9]. Flexiblity of various components introduces additional complexities in control and stabilization of the space structures. In order to study the effects of structural flexibility on the motion of the system, it is necessary to develop a model which is mathematically rigorous yet simple enough for theoretical as well as numerical investigation. To facilitate design and analysis of control schemes, it is preferable to describe the deformation of the elastic parts with reference to its undeformed configuration. Because large space structures involve high level of mechanical flexibility, it is also essential to design stabilizing controls and to optimally implement the active controllers.

In a most commonly used approach of modeling, the dynamics of the elastic parts is described by modal approximation using finite number of modes. A large number of contributions on this subject can be found in literature[Meirovitch 95, Taylor 125]. However, the number of modes of a continuous system is actually infinite and it is not known how many modes should be included in the approximation in order to provide a given accuracy without really solving the original system with all its complexities. Another crucial question in the modal method is how the modes are defined for complex structures? In designing stabilizing controls, techniques of finite dimensional control theory have been used on the basis of the reduced order model. A well known problem in this finite order model approach is "control spill over" which is the control-law interaction with the unmodelled coordinates. Infinite dimensional formulation of the modeling and control of flexible systems has also been considered by Balakrishnan[13] and Balas [16].

The most natural model for flexible space structures could be given by a hybrid system of equations, i.e. a combination of ordinary differential equations for rigid parts and partial differential equations for flexible members. Hybrid models for some simple structures have been considered by Ahmed and Biswas[3,25,26], Ahmed and Lim[4] and Meirovitch[96]. In [96], the dynamics was developed using a coordinate system whose origin is located at the instantaneous mass center which is time - varying during perturbation. Ahmed and Biswas[3,25,26] eliminated this problem by considering a coordinate system located at unperturbed position of the beam. However, in the model by Ahmed and Biswas the orbital dynamics is not included. In this thesis, **complete† dynamic models** of large flexible spacecraft and space stations have been developed. Based on the dynamic model developed by the author, asymptotic stability of the system subject to external disturbing forces is proved using Lyapunov's method. The effectiveness of the stabilizing controls is numerically illustrated.

† By the terminology, in the sequel, we mean that the dynamics consists of orbital dynamics, attitude dynamics of the system and equations for vibration of flexible members with all the relevant boundary conditions.

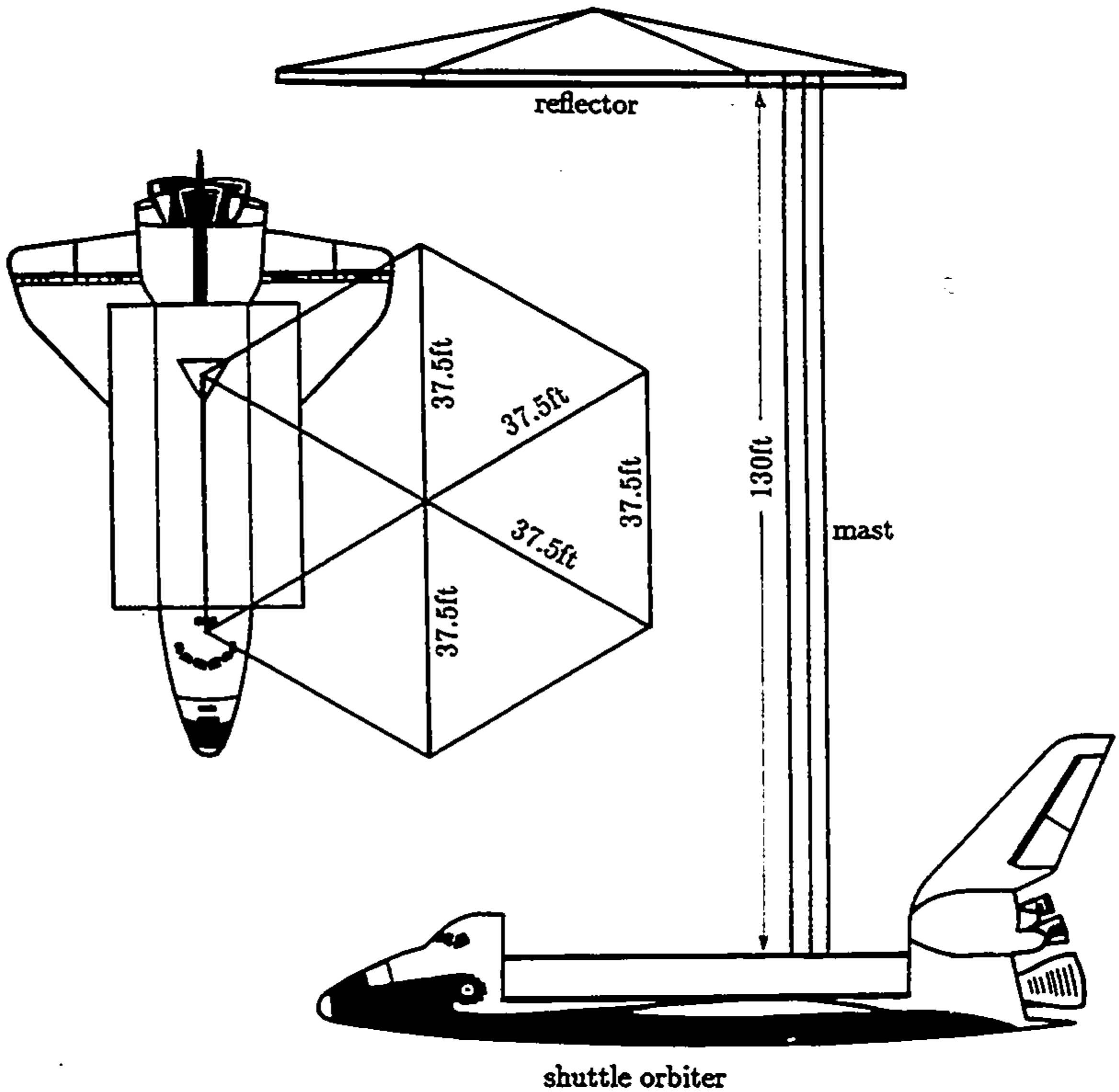


Fig. 1.1 Drawing of the Shuttle/Antenna configuration (NASA).

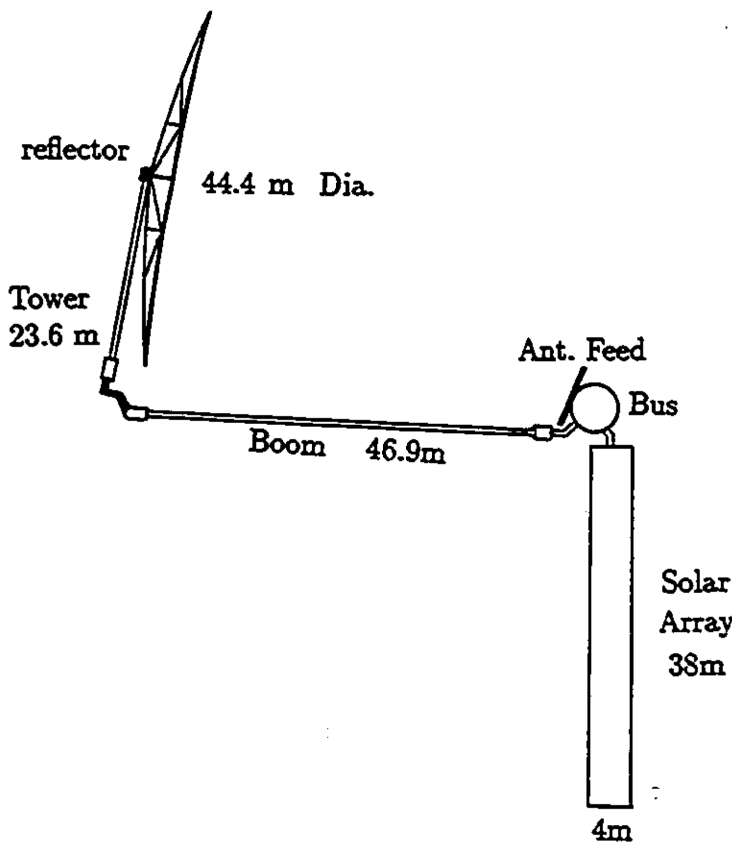


Fig.1.2 Schematic diagram of a flexible spacecraft (Canada).

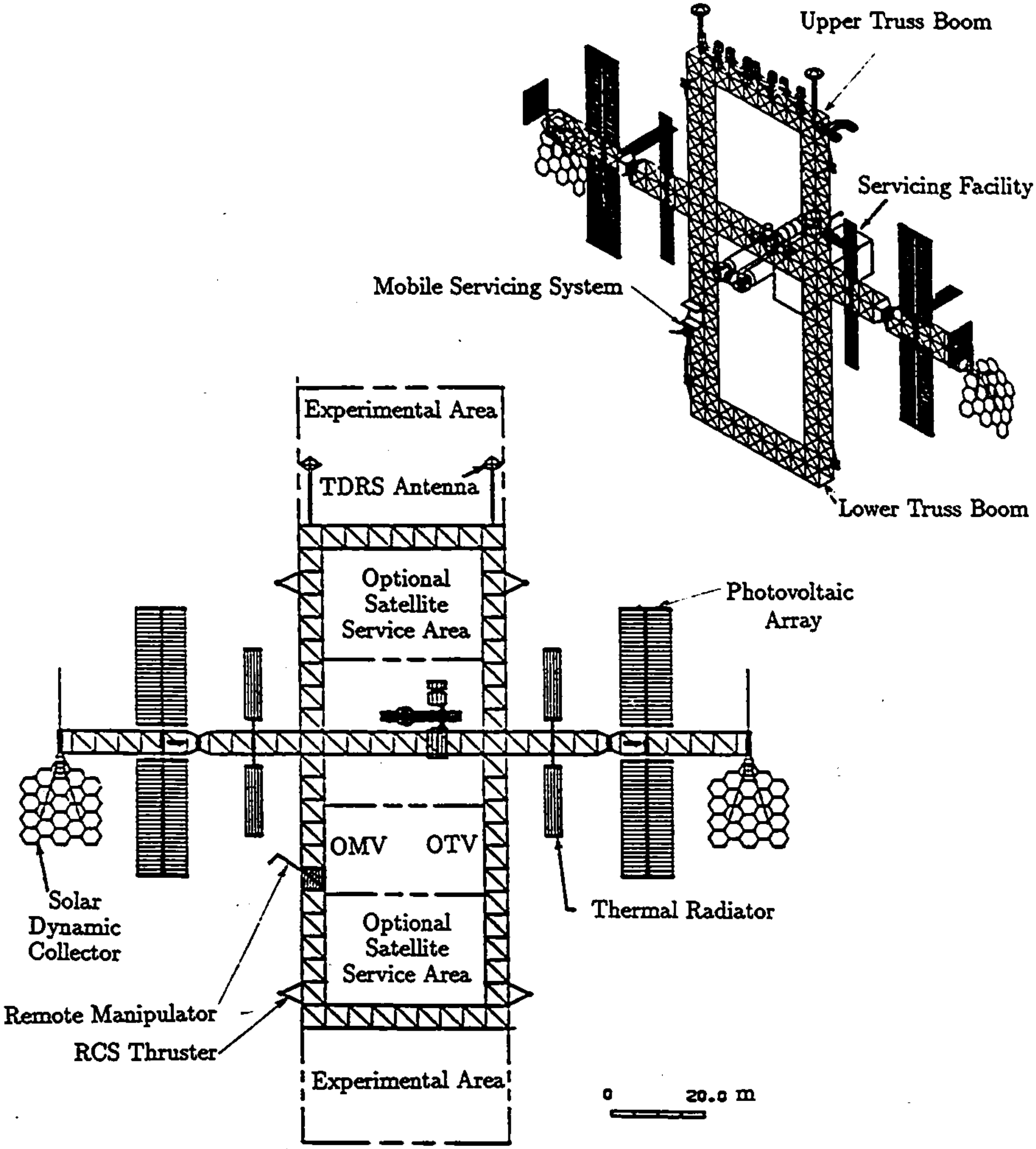


Fig. 1.3 The U.S. Space station.

1.2 A Brief Review of Previous Studies

The U.S. space station program (or European Columbus program) and SCOLE by NASA have opened a new era of space exploration, resulting in a whole new idea of scientific endeavor referred to as modeling and control of large space structures.

In the early stages of space exploration, the elastic deformation was relatively insignificant since spacecraft were small, mechanically simple and essentially inflexible. Hence the assumption of rigidity of satellite was acceptable for all practical purposes for the small satellites. Under the assumption numerous investigations have been carried out in the past[5,56,72,114,128] on the modeling of attitude and orbital dynamics of rigid body satellites and for design of active and passive controllers for attitude control and orbital maneuver. However, in a modern spacecraft carrying lightweight deployable members and very large space structures, which are inherently flexible, the assumption of rigidity is no longer true[28,44,117]. Rather, the structural flexibility may interact with attitude control systems of the spacecraft in a variety of ways. For exmaple, the attitude instabilities of the satellites Explorer I, Explorer XX and Alouette-I were believed to be caused by flexibility of some components of the system. These observation led to an extensive investigations. NASA special report[103] focuses attention on the significance of this problem through a discussion of the anomalous behavior of several spacecraft. The efforts towards modeling and controller design by taking the flexibility of the system into consideration can be summarized in a few categories:

The first category is represented by the SCOLE (Spacecraft COntrol Laboratory Experiment) workshops[125,126] which have been held since 1984, as a new design challenge by NASA/IEEE. The objective of the annual workshop is to provide an example configuration and control objectives which enables direct comparison of diffrent techniques in modeling, system identification and control. A number of contributions could be found in modeling and control of the SCOLE configuration (Fig. 1.1). Taylor and Balakrishnan[126, 13, 14] developed a model for the SCOLE using Newtonian mechanics and transformed the model into a bilinear form in order to establish a stabilization argument and controllability under some simplifying assumptions. In [15], Balakrishnan introduced a nonlinear damping model for flexible flight structures. Taylor and Latimer investigated various kinds of structural damping and compared with experimental results in a recent paper[127]. Kakad[69,70] suggested another model for investigating the slew maneuvers using modal approach. As for the control problem, Lin[87, 88] investigated Bang-Pause-Bang controllers for initial slew maneuvers and vibration control of the SCOLE. Bainum et al.[11, 12] proposed a control law for rapidly orienting the antenna line-of-sight based on the linearized model for the SCOLE configuration. Parameter identification problem using modal data and infinite dimensional approach was considered by many authors[20,22,53, 68,81,112,121].

The next category is to model the spacecraft as assemblage of rigid bodies. In this approach any kinds of complex structures are discretized into a finite number of interconnected rigid bodies and then equations of motion are systematically developed using classical rigid body dynamics. This discrete coordinate formulation is limited only by the available computational capacity[56,115]. Finite dimensional control theories[Ahmed 2] are utilized for the attitude control of these spacecraft.

Another popular approach for modeling of large flexible space structures is the modal method in which the dynamics of the flexible parts are approximated by a finite number of modes. In this method the state variables describing the motion of the elastic continuum are represented by finite sums of the product of space dependent eigenfunctions and time dependent generalized coordinates known as modal coordinates. A number of contributions could be found in the literature on the control of large flexible spacecraft based on the modal equations[45,74,60,66]. A number of papers in the proceedings of AIAA symposium [92] deal with the modal dynamics and control system design using modern control theory. Rank conditions for controllability and observability have been derived by Hughes and Skelton[61] and by Juang and Balas[67]. Techniques of optimal control theory have also been utilized by Biswas and Ahmed[27] and by Singh[119] in designing optimal feedback regulators for stabilization of flexible spacecraft. The modal method has the advantages that it gives rise to a set of linear ordinary differential equations which are readily understood by practicing engineers and which can be treated by using the known results from finite dimensional control theory. However, main drawbacks of the modal method are with the facts that the actual number of modes of a flexible structure is infinite and for a given system the exact number of modes that should be included in the approximation to yield satisfactory results is not known a priori and further the number is different for different structures. Another significant problem that should be resolved is *control spillover* [16,46] which is the control-law interaction with the unmodelled coordinates.

Studies have also been performed in the past towards rigorous modelling of flexible spacecraft in terms of partial differential equations [Balas 16, Wang 133]. In these studies large space structures are described by a set of second order hyperbolic partial differential equations. In the development of the dynamics of flexible parts, three types of beam have been utilized based on the observations that flexible structures can be modelled by a beam with equivalent beam parameters[Dow 42, Juang and Sun

6S, Renton 113, Sun and Kim 122]. This can be summarized as follows. When the cross sectional dimensions are small in comparison with the length of the beam, the motion of a beam can be described by the Euler beam equation:

$$\rho(x)\frac{\partial^2 W}{\partial t^2} + \frac{\partial^2}{\partial x^2}\Big(EI(x)\frac{\partial^2 W}{\partial x^2}\Big) = f(x,t), \tag{1.1}$$

where $W = W(x,t)$ is the displacement of the beam, with the spatial variable x. EI and ρ are the flextural ridiｄity of the beam and mass density of the beam respectively. Further f is the applied force (which is distributed or lumped). If the cross-sectional dimensions are not negligible, the effect of rotary inertia should be considered and then the motion is better described by the Rayleigh beam model:

$$\rho(x)\frac{\partial^2 W}{\partial t^2} + \frac{\partial}{\partial t}\Big(I_\rho(x)\frac{\partial^2 W}{\partial x^2}\Big) + \frac{\partial^2}{\partial x^2}\Big(EI(x)\frac{\partial^2 W}{\partial x^2}\Big) = f(x,t), \tag{1.2}$$

where $I_\rho(x)$ is the rotary inertia and ρ, EI, W, f are the same as defined in equation (1.1). If the deflection due to shear force is also taken into account in addition to the rotary inertia, we arrive at a still more accurate model which is called the Timoshenko beam model:

$$\rho(x)\frac{\partial^2 W}{\partial t^2} - K(x)\frac{\partial^2 W}{\partial x^2} + K(x)\frac{\partial V}{\partial x} = f_1(x,t), \tag{1.3}$$

$$I_\rho(x)\frac{\partial^2 V}{\partial t^2} - EI(x)\frac{\partial^2}{\partial x^2}V + K(x)\Big(V - \frac{\partial W}{\partial x}\Big) = f_2(x,t), \tag{1.4}$$

where K is a constant related to shear modulus and $V(x) \equiv \frac{\partial W}{\partial x} + \beta$, β is shear deformation, f_1, f_2 are (distributed or lumped) forces. These beam equations have been derived through Newtonian mechanics or energy principle by using natural energy of the beam[Meirovitch 93, Russell 116, Timoshenko, Young and Weaver 130, Traill-Nash and Collar 131]. In particular, Timoshenko[130] and Meirovitch[92] discuss various boundary conditions. For one dimensional wave equation with boundary control, Quinn and Russell [111] established the exponential decay of solutions. Later, Chen[33,34] obtained the same result for a wave equation in any space dimension under some geometrical conditions on the domain. A very restrictive part of the geometrical conditions was eliminated by Lagnese[77] with the aid of a new energy estimate. Lagnese[78] also extended the result to linear elastodynamic systems. In contrast to the above works, Lasiecka and Triggiani[80] employed boundary feedback acting in the Dirichlet boundary condition to achieve exponential decay of solutions to the wave equation. Recently, Chen et al. [35] discussed the case of a chain of Euler beams and obtained a similar result. Similar approach was used by Kim and Renardy[73] to prove the uniform stabilization of the Timoshenko beam using the energy method combined with C_0-semigroup theory. Representation of structural vibration (both transverse and torsional) of the spacecraft by partial differential equations is mathematically rigorous as compared to the modal method. These studies would be of practical importance if the elastic body dynamics could be decoupled from the rigid body dynamics. However, in general, the hyperbolic partial differential equation is only a partial representation of the complete dynamics of large flexible spacecraft, because the dynamics of the rigid body and the flexible members are actually nonlinearly coupled. Rigid body rotation and translation are not taken into account in the formulation using partial differential equations.

The most natural model describing the dynamics of the flexible spacecraft would be a coupled set of ordinary differential equations for the rigid body and partial differential equations for the flexible parts. This type of system model of flexible spacecraft has been termed as *Hybrid System* by Biswas and Ahmed[25,26]. A dynamic model of flexible spacecraft in the form of hybrid system has been introduced by Meirovitch [94,95]. Using Lagrangian mechanics, Hamilton's canonical equations in the form of hybrid system have been obtained in these papers. Lyapunov's approach have been utilized in his papers[96,97] to obtain gravity gradient stabilization and spin stabilization of a flexible spacecraft. Although this method provides a rigorous approach towards studying the dynamic behavior of flexible satellites, there are a few criticisms that are worth mentioning. In these papers the deformation of the elastic members is described with respect to a coordinate system located at the mass center of the deformed body. Since this mass center varies with time because of vibrations, calculation of the total deformation of the body would produce computational complexities. In practice, one would, rather, be interested in measuring the deformation with reference to the undeformed state. The proof of stabilization in Meirovitch's work[96,97] requires the assumption that the deflections of the elastic body in the three directions are independent. It also makes critical use of the smallest eigenvalue of the elastic member which is very difficult to determine except for some simple cases. Meirovitch[96,97] considered only passive stabilization of the system. In this regard Ahmed and Biswas [3,25,26] considered the problem of modeling and stabilization for flexible spacecraft consisting of a rigid body and a long beam. Using Euler-Bernoulli beam theory the equations of motion were derived in terms of a hybrid system using Hamilton's principle. In their works[3,25,26] the system is described in terms of variables which can be measured physically and also the deformations of the elastic members were described with respect to coordinate systems which are fixed at their undeformed positions. However, in the works[3,25,26], the satellite was assumed to maintain a circular orbit in an orbital plane. Thus the effects of the orbital disturbance on vibration of the beam and rotation of the rigid body were not taken into consideration.

In view of the above discussion it is clear that, to ensure satisfactory performance, there is a need to take into account the flexibility of the different components of the spacecraft. In order to study the effects of structural flexibility on the satellite motion, it is necessary to develop a model which is mathematically rigorous yet simple enough for theoretical as well as numerical investigation. To facilitate design and analysis of control schemes, it is preferable to describe the deformation of the elastic parts with reference to the undeformed configuration. It is also essential to design active controllers for stabilization of the flexible spacecraft.

Keeping these objectives in mind and considering the flexibility of the large structures as well as taking into account the orbital dynamics, in this thesis complete dynamic models for large space structures have been developed. The space structures consist of a (rigid) massive body, which may play the role of experimental modules located at the center of the space structure and flexible configurations, consisting of several truss bays, forming a space station. Using Euler-Bernoulli, Rayleigh and Timoshenko beam theories three different dynamic models have been derived. The dynamic equations that govern the complete system consist of several partial differential equations with complex boundary conditions describing the vibration of the flexible components and six ordinary differential equations that govern the rotational and translational motion of the central body. The dynamic equations of motion for the system indicate very strong interactions (or couplings) among the translational, rotational and vibrational motions of the space structure.

Structural vibration caused by various sources should be suppressed for mechanical safety of the system and for maintaining the desired line-of-sight for communications. A very useful method of stability analysis was developed by Lyapunov in 1892 and extended later by Lasalle and Lefeschetz[79] and many other contributers. Lasalle's invariance principle[79] and the extension by Hale[50] play a cental role for estimating the domain of asymptotic stability. Recently, Pazy[110] extended the theorem for an abstract evolution equation in Banach space. Stability of sets are treated in Ahmed[2] and Zubov[135]. Extensive contributions in the area of application of stability theories to abstract and practical systems have been carried out [2,9,23,30,31,38,40,41,71,98,110]. Meirovitch investigated the stability of flexible systems using the Lyapunov method[94,95, 96,97] and other method[98]. Ahmed and Biswas[3,25,26] established the asymptotic stability of a rotating rigid body with a flexible beam by use of Lyapunov's approach. Lyapunov stability theory and semigroup theory are combined in Buis' paper[31] for a class of partial differentail equations. Littman, Markus and You[89] used invariance principle and spectral analysis to show strong stabilization of nonrotating Euler beam with a tip mass by applying feedback boundary damping. A stability problem of an extensible beam was considered by Ball[17,18] earlier. By virtue of a Lyapunov functional based on the total energy of the system, in this thesis asymptotic stability of the space structures using state feedback control has been proved. The problem of feedback stabilization of the system subject to external disturbing forces such as shuttle takeoff and docking or crew movements has been also considered. By applying controls such as proportional controls, deadzone controls, and saturation controls, asymptotic stability of the perturbed system is proved using Lyapunov's method. Stability of the slew maneuvering has been numerically investigated.

1.3 Outline of the Thesis and Contributions

The thesis is organized as follows:

Chapter 1 contains the motivation to the problem of modeling and stabilization of large flexible space structures and review of previous studies in this area. Then the problem treated in this thesis is presented.

In Chapter 2 we present a methodology for developing a complete dynamic model for spacecraft consiting of a rigid body and a flexible beam. Complete dynamics of the system is derived using extended Hamilton's principle. In the derivation, Euler-Bernoulli, Rayleigh and Timoshenko beam theories are utilized for expressing the energy of the flexible beam.

Following the same principle and beam theories used in Chapter 2, in Chapter 3 a complete model for flexible space stations (or space structures) consisting of a massive central body and flexible configurations consisting of several flexible beams is developed. The flexible parts introduce very strong coupling effects to the other remaining parts of the system through very complex boundary conditions.

In Chapter 4 stabilization of the space structures is investigated using energy methods. Simple feedback controls are suggested for asymptotic stability of the system.

In Chapter 5 numerical simulations are carried out for illustration of the impact of external perturbations on the performance of the space structures and the effectiveness of the stabilizing controls suggested in Chapter 4. Stability of vibration caused by slew maneuvering of the space structures is also considered in the numerical experiments.

In chpater 6 concluding remarks and suggestions for further research are finally presented.

The original contributions of the thesis include:

(i) Development of a complete dynamic model for flexible spacecraft, which consists of radial dynamics, attitude dynamics and beam equations with boundary conditions, using Euler-Bernoulli, Rayleigh and Timoshenko beam theories: sections

2.4, 2.5, 2.6[82].

(ii) Development of a new dynamic model for flexible space stations, which consists of radial dynamics, attitude dynamics and beam equations with complex boundary conditions, using Euler-Bernoulli, Rayleigh and Timoshenko beam theories: sections 3.4, 3.5, 3.6[84].

(iii) Investigation of the stability of the space structures subject to external perturbations by applying feedback control schemes: sections 3.3 and 3.4[83].

(iv) Numerical simulation of the hybrid system and analysis of the results: sections 5.2, 5.3 and 5.4[82-85].

Chapter 2

MODELING OF FLEXIBLE SPACECRAFT

2.1 Introduction

In this chapter, we consider the modeling of flexible spacecraft consisting of a rigid body (bus) and a long flexible beam rigidly attached to the bus (see Fig.2.1). A flexible spacecraft in an earth-orbit would undergo three types of motion:

(a) rigid body translation perturbing the orbit

(b) rigid body rotation perturbing the orientation of the satellite

(c) vibration of the elastic members causing elastic deformation of the system.

In this study, we shall concern with these three types of motion. In [3,25,26], Ahmed and Biswas used the Hamilton's principle to derive the equations of motion for the motions (b) and (c) under the simplifying assumption that the orbital dynamics has negligible effect on the rotation of the bus and vibration of the beam. The criticizem of the model[3,25,26] is that (i) radial equation (which provides the trajectory of the rigid bus) is not included and (ii) disturbance caused by orbital dynamics is not taken into consideration, although the effect is quite significant. In this chapter, considering the three types of motion mentioned above, a complete dynamic model† is developed for the dynamics of the flexible spacecraft, using three typical beam theories. The equations of motion consist of partial differential equations describing the vibration of the beam and ordinary differential equations governing the translational and rotational dynamics of the rigid body. For the development of dynamics of the spacecraft in an earth orbit, first we introduce several reference coordinate systems

† See, page 3.

in Section 2.2. In the body coordinate system we define the inertia tensor in Section 2.3. In terms of the quantities described in the various coordinate systems in Section 2.4, based on Euler-Bernoulli beam theory, we derive the equations of motion for the flexible spacecraft consisting of a rigid bus and a long flexible beam (Fig. 2.1) using Hamilton's principle. In Section 2.5 we derive the dynamics of the spacecraft based on Rayleigh beam theory, following the same procedure used in Section 2.4. In Section 2.6 we consider modeling of the same system based on Timoshenko beam theory. Finally a summary is presented in the last section.

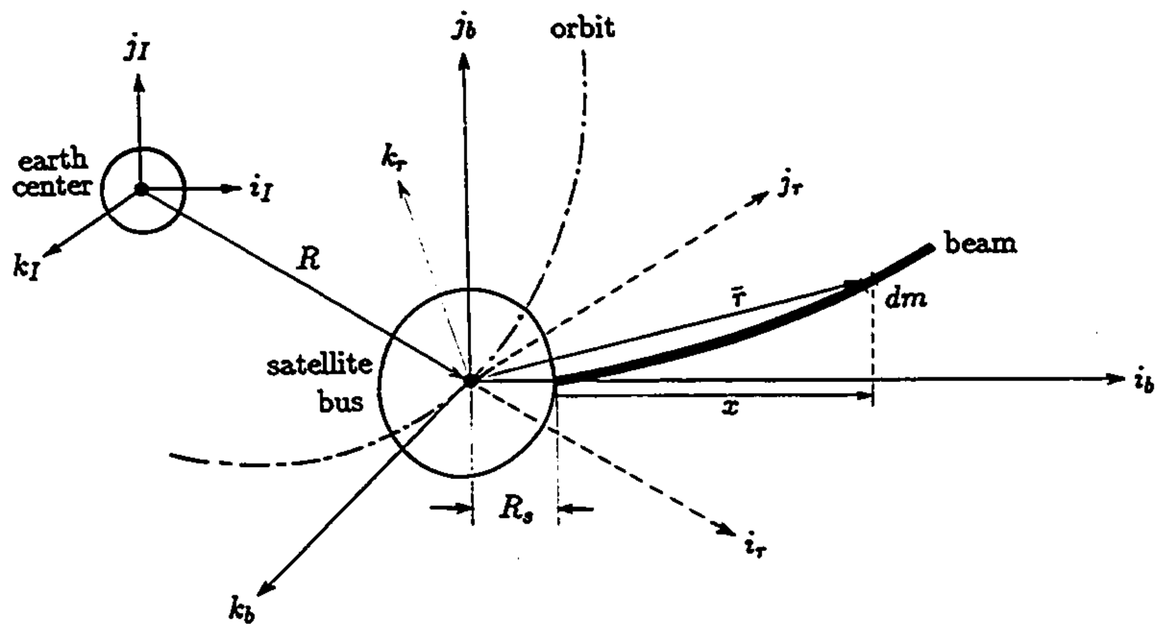


Fig. 2.1 The schematic diagram of a flexible spacecraft and coordinate systems.

2.2 Reference Coordinate Systems

We first define a body coordinate system S_B represented by the right handed vector triad (i_b, j_b, k_b) (Fig.2.1). This coordinate system will be assumed to be fixed to the satellite body with the origin at the mass center of the satellite bus. We suppose that the rest position of the beam is along the i_b axis, with the fixed end of the beam at a distance R_s from the origin of the body frame (see Fig. 2.2). We suppose that the beam is inextensible ,i.e. deflections of the beam could occur in the j_b and k_b directions only. The corresponding deflections, denoted by $y = y(x,t)$ and $z = z(x,t)$ are governed by the beam dynamics as discussed later. Then the position of any point on the perturbed beam can be denoted by the vector

$$\bar{r} = (x + R_s, y, z)' \tag{2.1}$$

where x is the distance measured from the point where the beam is attached and "′" denotes the transpose of a vector (or a matrix).

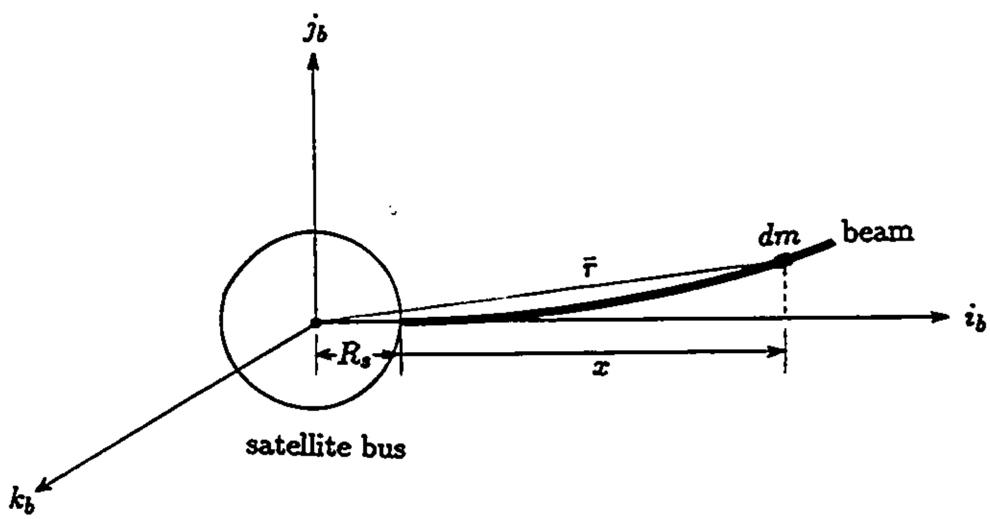


Fig. 2.2 Body coordinate system.

We introduce an orbital reference coordinate system S_r by the right- handed vector triad (i_r, j_r, k_r) with the origin coinciding with the origin of the body frame. The axis k_r is normal to the orbital plane, the axis i_r is along the line connecting the center of the earth and the mass center of the satellite and the axis j_r is such that (i_r, j_r, k_r) makes a right-handed vector triad. The positive direction of the i_r is taken along the radial vector $R \equiv (X, Y, Z)'$.

Let $w_b = (w_x, w_y, w_z)'$ be the angular velocity of the body frame with respect to the orbital reference frame S_r and $w_r = (w_X, w_Y, w_Z)'$ be the angular velocity of the reference frame with respect to the inertial frame $S_I = (i_I, j_I, k_I)$(Fig. 2.1). We shall denote $w = w_b + w_r = (w_1, w_2, w_3)'$.

2.3 Inertia Tensor

Let dm denote the mass of a generic element on the beam. Then the beam inertia tensor with respect to the body frame is given by

$$I_b \equiv \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix} \tag{2.2}$$

where

$$I_{xx} = \int (y^2 + z^2) dm, \tag{2.3.1}$$

$$I_{yy} = \int ((x + R_s)^2 + z^2) dm, \tag{2.3.2}$$

$$I_{zz} = \int ((x + R_s)^2 + y^2) dm, \tag{2.3.3}$$

$$I_{xy} = \int (x + R_s) y dm, \tag{2.3.4}$$

$$I_{xz} = \int (x + R_s) z dm, \tag{2.3.5}$$

$$I_{yz} = \int yz dm. \tag{2.3.6}$$

Unless otherwise specified, the integrals will be assumed to be taken over the whole body of the beam. It is clear from Equations (2.2) and (2.3) that the beam inertia tensor I_b is a functional of the deflections y and z, and consequently an implicit function of time.

The inertia tensor of the satellite main body (bus) is denoted by I_s, which is not necessarily diagonal. However, we assume that I_s is not time varying and is positive definite.

2.4 Dynamic Modeling based on Euler-Bernoulli Beam Theory

In this section we derive the equations of motion for the spacecraft shown in Fig.2.1 based on Euler-Bernoulli beam theory. We shall denote the radial vector by $R = (X, Y, Z)'$ in the inertial frame and the beam deflection by $D = (0, y, z)'$ in the body frame. Then the equations of motion can be derived from the extended Hamilton's principle[Pars 107, Craig 37, Hurty 62]:

$$\delta \int_{t_1}^{t_2} L \, dt + \delta \int_{t_1}^{t_2} W dt = 0 \, , \tag{2.4}$$

subject to the end conditions: at $t = t_1$ and $t = t_2$,

$$\delta\theta = (\delta\theta_1, \delta\theta_2, \delta\theta_3) = (0,0,0)' \, , \tag{2.5.1}$$

$$\delta R = (\delta X, \delta Y, \delta Z)' = (0,0,0)', \tag{2.5.2}$$

$$\delta D = (0, \delta y, \delta z)' = (0,0,0)' \, , \tag{2.5.3}$$

where $\theta_1, \theta_2, \theta_3$ are the Euler angles describing the orientation of the body frame relative to the inertial frame, δD is the virtual dispacement of the beam element dm and δR is the virtual dispacement of the mass center of the satellite . In Equation (2.4), $L \equiv K - P$ is the Lagrangian function, with K the total kinetic energy, P the total potential energy and W is the total work done by external forces. They can be expressed as follows.

$$K = K_s + K_b = \textit{Kinetic Energy of the Bus} + \textit{Kinetic Energy of the Beam} \tag{2.6}$$

$$P = P_g + P_e = \textit{Gravitational Potential Energy} + \textit{Elastic Potential Energy} \tag{2.7}$$

The kinetic energy of the satellite bus K_s is given by[Goldstein 47]

$$K_s \;=\; \frac{1}{2}\int_{bus}\frac{d\tilde{r}}{dt}\circ\frac{d\tilde{r}}{dt}d\tilde{m}\;, \tag{2.8}$$

where "$\circ$" denotes the scalar product of two vectors and

$$\tilde{r} = R + \tilde{\eta} \tag{2.9}$$

denotes the absolute position of the incremental mass element $d\tilde{m}$ of the bus with respect to the inertial frame, with the vector R defining the position of the origin of the body frame with respect to the inertial frame and the vector $\tilde{\eta}$ defining the position of the mass element $d\tilde{m}$ relative to the body frame.

The kinetic energy of the beam is obtained in a similar way and is given by

$$K_b = \frac{1}{2}\int_{beam}\frac{dr}{dt}\circ\frac{dr}{dt}dm, \tag{2.10}$$

with

$$r = R + \bar{r} \;=\; R + D_0 + D, \tag{2.11}$$

where r denotes the absolute position of the incremental mass element dm of the beam relative to the inertial frame, D_0 defines the position of the mass element dm relative to the body frame in the undeformed state and D represents the displacement of dm from the undeformed position of the beam in S_B. Indeed, with reference to Fig. 2.1, one sees that in the body frame,

$$D_0 = (R_s + x, 0, 0)' \;\text{ and }\; D = (0, y, z)'. \tag{2.12}$$

The system potential energy arises from two sourses, namely elastic and gravitational. The elastic potential energy P_e due to deformations of the beam is a function of the partial derivatives of the displacements y, z with respect to the spatial variable x. Under the assumption of small bending deformations neglecting elongation or torsional effects, the elastic potential energy of the beam P_e is given by[Meirovitch 93]

$$P_e = \frac{1}{2}\int_0^l \left\{ EI_y\left(\frac{\partial^2 y}{\partial x^2}\right)^2 + EI_z\left(\frac{\partial^2 z}{\partial x^2}\right)^2 \right\} dx, \tag{2.13}$$

where $EI_y \equiv EI_y(x)$(respectively EI_z) is the flexural rigidity of the beam for deflections in the j_b direction (respectively k_b direction) and l is the length of the beam. The gravitational potential energy P_g is given by the sum of the potential energy of the satellite bus(P_s) and that of the beam(P_b) as follows[Halliday and Resnick 51].

$$\begin{aligned} P_g &= P_s + P_b \\ &= \int_{bus} \frac{-Gm_e}{|\bar{r}|} d\tilde{m} + \int_{beam} \frac{-Gm_e}{|r|} dm, \end{aligned} \tag{2.14}$$

where G = universal gravitational constant, m_e = mass of the earth. Expanding $\frac{1}{|r|}$ and $\frac{1}{|\bar{r}|}$ in the form of power series, we have following expression (see Appendix A):

$$P_g = -\frac{Gm_e m_T}{|R|} - \frac{Gm_e}{2|R|^3}\Big[\int_{bus}\Big\{|\bar{\eta}|^2 - 3(\frac{R}{|R|}\circ\bar{\eta})^2\Big\}d\tilde{m} + \int_{beam}\Big\{|\bar{r}|^2 - 3(\frac{R}{|R|}\circ\bar{r})^2\Big\}dm\Big] + \cdots \tag{2.15}$$

The intergral terms in the above Equation (2.15) contribute to gravity gradient torques and is often used in the design of passive stabilizers [Greensite 48, Santini 118, Frik 43, Nurre 105]. However, in some applications this can be neglected (see Remark 2.1 at the end of this section); for example, for the study of orbital perturbation the quantities $|\bar{\eta}|$ and $|\bar{r}|$ are negligible and hence one can approximate P_g as

$$P_g = -\frac{Gm_e m_T}{|R|}, \tag{2.16}$$

where $m_T(= m_s + m_b)$ is the total mass of the satellite bus and the beam.

Let the time rate of change of any vector A relative to the inertial space be denoted by $\frac{dA}{dt}$ and that relative to the body frame by $\dot{A}$. Then denoting the cross product of two vectors by "×", we have the well known relation[Meriam 99]

$$\frac{dA}{dt} = \dot{A} + \omega \times A. \tag{2.17}$$

In the present problem, since the body frame rotates with respect to the inertial space, one needs some extra care in taking the first variation of the Lagrangian L. The virtual displacement of the elastic body relative to the inertial space is given[Cavin and Dusto 32] by

$$\delta r = \delta R + \delta\theta \times \bar{r} + \delta D, \tag{2.18}$$

where $\delta\theta = (\delta\theta_1, \delta\theta_2, \delta\theta_3)$ is the virtual rigid body rotation of the body frame relative to the reference frame. Further, using Equation (2.17) one obtains (Appendix B)

$$\delta\Big(\frac{dr}{dt}\Big) = \delta\frac{dR}{dt} + \delta\omega \times \bar{r} + \delta\theta \times (\omega \times \bar{r} + \dot{D}) + \frac{d}{dt}(\delta D), \tag{2.19}$$

where $\delta\omega = \frac{d}{dt}(\delta\theta)$ denotes the virtual angular velocity of the body frame.

Assuming that the origin of the body frame coincides with the center of mass of the satellite bus, it follows from Equation (2.8) that[Meriam 99]

$$K_s = \frac{1}{2}m_s\frac{dR}{dt} \circ \frac{dR}{dt} + \frac{1}{2}\omega \circ (I_s \circ \omega), \tag{2.20}$$

where m_s is the mass of the satellite bus and I_s is its inertia tensor with respect to the body frame. Taking the first variation of K_s, one obtains

$$\delta\int_{t_1}^{t_2} K_s dt = \int_{t_1}^{t_2} m_s\Big(\frac{dR}{dt}\Big) \circ \frac{d}{dt}(\delta R)dt + \int_{t_1}^{t_2}(\delta\omega) \circ (I_s \circ \omega)dt\ . \tag{2.21}$$

Substituting Equations (2.18-2.19) into Equation (2.10) and taking the first variation, we have

$$\begin{aligned}\delta\int_{t_1}^{t_2} K_b dt = &\int_{t_1}^{t_2}\int_{beam}\Big(\delta\frac{dR}{dt} \circ \frac{dr}{dt}\Big)dmdt + \int_{t_1}^{t_2}\int_{beam}\Big\{\delta\omega \times \bar{r}\Big\} \circ \frac{dr}{dt}dmdt \\ &+ \int_{t_1}^{t_2}\int_{beam}\Big\{\delta\theta \times (\omega \times \bar{r} + \dot{D})\Big\} \circ \frac{dr}{dt}dmdt \\ &+ \int_{t_1}^{t_2}\int_{beam}\Big\{\frac{d}{dt}(\delta D) \circ \frac{dr}{dt}\Big\}dmdt\ .\end{aligned} \tag{2.22}$$

Using the following vector identity(Appendix C)

$$(A \times B) \circ C = (B \times C) \circ A = (C \times A) \circ B, \tag{2.23}$$

we obtain from Equation (2.22)

$$\delta \int_{t_1}^{t_2} K_b dt = \int_{t_1}^{t_2} \int_{beam} \Big[\delta \frac{dR}{dt} \circ \frac{dr}{dt} + \delta\omega \circ \Big\{ \bar{r} \times \frac{dr}{dt} \Big\}$$

$$+ \delta\theta \circ \Big\{ (\omega \times \bar{r} + \dot{D}) \times \frac{dr}{dt} \Big\} + \frac{dr}{dt} \circ \frac{d}{dt}(\delta D) \Big] dm dt \,. \tag{2.24}$$

Defining

$$EI \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & EI_y & 0 \\ 0 & 0 & EI_z \end{pmatrix}, \tag{2.25}$$

one obtains from Equation (2.13)

$$\delta \int_{t_1}^{t_2} P_e dt = \int_{t_1}^{t_2} \int_0^l EI \Big(\frac{\partial^2 D}{\partial x^2} \Big) \circ \frac{\partial^2}{\partial x^2} (\delta D) dx dt \,. \tag{2.26}$$

Since

$$\int_{t_1}^{t_2} \delta \Big(\frac{1}{|R|} \Big) dt = \int_{t_1}^{t_2} \frac{-1}{|R|^2} (\delta |R|) dt \,, \tag{2.27}$$

it follows from Equation (2.16) that

$$\delta \int_{t_1}^{t_2} P_g dt = \int_{t_1}^{t_2} \frac{G m_e m_T}{|R|^2} (\delta |R|) \; dt, \tag{2.28}$$

or if we use the identity (D.1) (Appendix D)

$$\delta \int_{t_1}^{t_2} P_g dt = \int_{t_1}^{t_2} \frac{G m_e m_T}{|R|^2} 1_R \circ (\delta R) dt, \tag{2.29}$$

where 1_R is the unit vector in the direction of R and given by (see Appendix D)

$$1_R \equiv \frac{R}{|R|} \,. \tag{2.30}$$

Let the force and torque applied to the bus be given by $F_s = (F_1, F_2, F_3)'$ and $T = (T_1, T_2, T_3)'$ respectively, and external force applied to the beam be given by

$\bar{F}_b = (0, F_y, F_z)'$. Then the virtual work[Meriam 99] done by the applied forces is given by

$$\delta W \;=\; F_s \circ \delta R \;+\; T \circ \delta\theta \;+\; \int_0^l \bar{F}_b \circ \delta D dx. \tag{2.31}$$

Substituting Equations (2.21), (2.24), (2.26), (2.29) and (2.31) into Equation (2.4) and integrating by parts with the end conditions in (2.5), we obtain

$$\begin{aligned}
&\delta \int_{t_1}^{t_2} (L+W)dt \equiv \\
&-\int_{t_1}^{t_2} (\delta R) \circ \left[m_s \frac{d^2R}{dt^2} + \frac{d}{dt}\int_{beam}\Big(\frac{dR}{dt} + \omega \times \bar{r} + \dot{D}\Big)dm + \frac{Gm_e m_T R}{|R|^3} - F_s\right] dt \\
&+\int_{t_1}^{t_2} (\delta\theta) \circ \left[T - \frac{d}{dt}\Big\{(I_s \circ \omega) + \int_{beam} \bar{r} \times \frac{dr}{dt} dm\Big\} + \int_{beam}\Big(\omega \times \bar{r} + \dot{D}\Big) \times \Big(\frac{dr}{dt}\Big) dm\right] dt \\
&-\int_{t_1}^{t_2}\int_0^l (\delta D) \circ \Big\{\rho \frac{d^2 r}{dt^2} + \frac{\partial^2}{\partial x^2}\Big(EI\frac{\partial^2 D}{\partial x^2}\Big) - \bar{F}_b\Big\} dx dt \\
&-\int_{t_1}^{t_2}\left[EI\Big(\frac{\partial^2 D}{\partial x^2}\Big) \circ \frac{\partial}{\partial x}(\delta D)\Big|_0^l + \frac{\partial}{\partial x}\Big(EI\frac{\partial^2 D}{\partial x^2}\Big) \circ (\delta D)\Big|_0^l\right] dt = 0 \, .
\end{aligned} \tag{2.32}$$

Since the beam is assumed to be inextensible $\delta D = (0, \delta y, \delta z)'$. Therefore, noting that the variations $\delta\theta, \delta R, \delta D$ are arbitrary, we obtain, from Equation (2.32), a set of coupled differential equations for the dynamics of the flexible spacecraft as follows. From the first part of Equation (2.32) we have

$$(m_s + m_b)\frac{d^2R}{dt^2} + \frac{d}{dt}\int_0^l \rho\Big(\omega \times \bar{r} + \dot{D}\Big)dx + \frac{Gm_e m_T R}{|R|^3} = F_s, \tag{2.33}$$

where $m_b = \int_{beam} dm$ is the mass of the beam and $\rho \equiv \rho(x)$ is the mass per unit length. From the second part of Equation (2.32) we obtain

$$\frac{d}{dt}\{(I_s + I_b) \circ \omega\} + \frac{d}{dt}\int_0^l \rho\bar{r} \times \Big(\frac{dR}{dt} + \dot{D}\Big)dx - \int_0^l \rho\Big\{(\omega \times \bar{r} + \dot{D}) \times \frac{dR}{dt}\Big\}dx = T, \tag{2.34}$$

where $I_b \circ \omega = \int \bar{r} \times (\omega \times \bar{r}) dm$. For notational simplicity we shall denote $F_b \equiv (F_y, F_z)'$ and $I_\tau \equiv I_s + I_b$. Since $\delta D = (0, \delta y, \delta z)'$, we obtain from the third part of Equation (2.32)

$$\rho Q \ddot{D} + Q \frac{\partial^2}{\partial x^2}\Big(EI \frac{\partial^2 D}{\partial x^2}\Big) + \rho Q \Big[\frac{d^2 R}{dt^2} + \dot{\omega} \times \bar{r} + \omega \times \Big\{2\dot{D} + \omega \times \bar{r}\Big\}\Big] = F_b, \qquad (2.35)$$

where

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (2.36)$$

The boundary conditions are obtained from the last part of Equation (2.32) and the cantilever configuration. At the clamped end (x=0) we have

$$y(0,t) = 0, \qquad z(0,t) = 0, \qquad (2.37.1)$$

$$\frac{\partial y}{\partial x}(0,t) = 0, \qquad \frac{\partial z}{\partial x}(0,t) = 0. \qquad (2.37.2)$$

Since the shear force and bending moment are zero at the free end ($x = l$), it is obtained

$$EI_y \frac{\partial^2 y}{\partial x^2}(l,t) = 0, \qquad EI_z \frac{\partial^2 z}{\partial x^2}(l,t) = 0, \qquad (2.37.3)$$

$$\frac{\partial}{\partial x}(EI_y \frac{\partial^2 y}{\partial x^2})(l,t) = 0, \qquad \frac{\partial}{\partial x}(EI_z \frac{\partial^2 z}{\partial x^2})(l,t) = 0. \qquad (2.37.4)$$

Boundary conditions for other configurations can be obtained from standard text-books on structural dynamics (for example, see[37,62,93]). Then the dynamics of the flexibie spacecraft can be rewritten as

$$m_\tau \frac{d^2 R}{dt^2} + \int_0^l \rho \Big\{(\dot{\omega} \times \bar{r}) + \omega \times (2\dot{D} + \omega \times \bar{r}) + \ddot{D}\Big\} dx + \frac{G m_e m_\tau R}{|R|^3} = F_s, \qquad (2.38)$$

$$I_\tau \circ \dot{\omega} + \omega \times (I_\tau \circ \omega) + \dot{I}_b \circ \omega + \int_0^l \rho \Big(\bar{r} \times \frac{d^2 R}{dt^2}\Big) dx$$

$$+ \int_0^l \rho (\bar{r} \times \ddot{D}) dx + \int_0^l \rho \{\omega \times (\bar{r} \times \dot{D})\} dx = T, \qquad (2.39)$$

and

$$\rho\frac{\partial^2}{\partial t^2}\begin{pmatrix} y \\ z \end{pmatrix} + \rho Q\Big(\alpha + \frac{d^2R}{dt^2}\Big) + \frac{\partial^2}{\partial x^2}\left(\begin{pmatrix} EI_y & 0 \\ 0 & EI_z \end{pmatrix}\frac{\partial^2}{\partial x^2}\begin{pmatrix} y \\ z \end{pmatrix}\right) = F_b, \tag{2.40}$$

where $\frac{\partial}{\partial t}$ denotes the partial derivative in time with respect to the body frame and

$$\alpha = \dot{\omega} \times \bar{r} + 2\omega \times \dot{D} + \omega \times (\omega \times \bar{r}). \tag{2.41}$$

The three terms in (2.41) are commonly known as Euler acceleration, Coriolis acceleration and centrifugal acceleration, respectively [Thomson 129].

Let C_B^I denote the coordinate transformation matrix from body to inertial frame (see Appendix E) and $(a_i, a_j, a_k)' \equiv (C_B^I)^{-1}\frac{d^2R}{dt^2}$ where $(C_B^I)^{-1} = C_I^B = (C_B^I)'$ [Greensite 48]. Performing the vector operations, Equations (2.38), (2.39) and (2.40) could be expressed in explicit forms which are suitable for digital computer simulation[Lim and Ahmed 82]. These are given below.

<u>BUS DYNAMICS(TRANSLATION)</u>

$$m_T\frac{d^2}{dt^2}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + C_B^I\begin{pmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \end{pmatrix} + \begin{pmatrix} h_4(t) \\ h_5(t) \\ h_6(t) \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \tag{2.42}$$

where

$$h_1(t) = \int_0^l \rho\Big\{(z\dot{\omega}_2 - y\dot{\omega}_3) + 2(\dot{z}\omega_2 - \dot{y}\omega_3) + \omega_1\omega_2 y - \omega_2^2(R_s + x) - \omega_3^2(R_s + x) + \omega_1\omega_3 z\Big\}dx, \tag{2.43}$$

$$h_2(t) = \int_0^l \rho\Big\{(R_s + x)\dot{\omega}_3 - z\dot{\omega}_1 - 2\dot{z}\omega_1 + \omega_2\omega_3 z - y(\omega_3^2 + \omega_1^2) + \omega_1\omega_2(R_s + x) + \frac{\partial^2 y}{\partial t^2}\Big\}dx, \tag{2.44}$$

$$h_3(t) = \int_0^l \rho\Big\{y\dot{\omega}_1 - (x + R_s)\dot{\omega}_2 + 2\dot{y}\omega_1 + \omega_1\omega_3(x + R_s)$$

$$- z(\omega_1^2 + \omega_2^2) + \omega_2\omega_3 y + \frac{\partial^2 z}{\partial t^2}\Big\}dx, \tag{2.45}$$

$$h_4(t) = \frac{Gm_e m_T}{|R|^3}X, \tag{2.46.1}$$

$$h_5(t) = \frac{Gm_e m_T}{|R|^3}Y, \tag{2.46.2}$$

$$h_6(t) = \frac{Gm_e m_T}{|R|^3}Z, \tag{2.46.3}$$

<u>BUS DYNAMICS(ROTATION)</u>

$$I_r\begin{pmatrix}\dot{\omega_1}\\ \dot{\omega_2}\\ \dot{\omega_3}\end{pmatrix} + \begin{pmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{pmatrix} I_r\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} + \begin{pmatrix}h_7(t)\\ h_8(t)\\ h_9(t)\end{pmatrix} + \begin{pmatrix}h_{10}(t)\\ h_{11}(t)\\ h_{12}(t)\end{pmatrix} = \begin{pmatrix}T_1\\ T_2\\ T_3\end{pmatrix}, \tag{2.47}$$

where

$$h_7(t) = \int_0^l \rho\Big\{y\frac{\partial^2 z}{\partial t^2} - z\frac{\partial^2 y}{\partial t^2} + 2\omega_1 y\frac{\partial y}{\partial t} + 2\omega_1 z\frac{\partial z}{\partial t}\Big\}dx, \tag{2.48}$$

$$h_8(t) = \int_0^l \rho\Big\{-(x+R_s)\frac{\partial^2 z}{\partial t^2} + 2\omega_2 z\frac{\partial z}{\partial t} - 2\omega_3 z\frac{\partial y}{\partial t} - 2\omega_1(x+R_s)\frac{\partial y}{\partial t}\Big\}dx, \tag{2.49}$$

$$h_9(t) = \int_0^l \rho\Big\{(x+R_s)\frac{\partial^2 y}{\partial t^2} - 2\omega_1(x+R_s)\frac{\partial z}{\partial t} - 2\omega_2 y\frac{\partial z}{\partial t} + 2\omega_3 y\frac{\partial y}{\partial t}\Big\}dx, \tag{2.50}$$

$$h_{10}(t) = \int_0^l \rho\{ya_k - za_j\}dx, \tag{2.51}$$

$$h_{11}(t) = \int_0^l \rho\{za_i - (x+R_s)a_k\}dx, \tag{2.52}$$

$$h_{12}(t) = \int_0^l \rho\{(x+R_s)a_j - ya_i\}dx, \tag{2.53}$$

BEAM DYNAMICS(BEAM VIBRATION)

$$\begin{aligned}\rho\frac{\partial^2}{\partial t^2}\begin{pmatrix}y\\ z\end{pmatrix}+\rho\begin{pmatrix}0&-2\omega_1\\ 2\omega_1&0\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}y\\ z\end{pmatrix}+\frac{\partial^2}{\partial x^2}\left(\begin{pmatrix}EI_y&0\\ 0&EI_z\end{pmatrix}\frac{\partial^2}{\partial x^2}\begin{pmatrix}y\\ z\end{pmatrix}\right)\\ +\rho\begin{pmatrix}-\omega_1^2-\omega_3^2&-\dot{\omega_1}+\omega_2\omega_3\\ \dot{\omega_1}+\omega_2\omega_3&-\omega_1^2-\omega_2^2\end{pmatrix}\begin{pmatrix}y\\ z\end{pmatrix}\\ +\rho(x+R_s)\begin{pmatrix}\dot{\omega_3}+\omega_1\omega_2\\ -\dot{\omega_2}+\omega_1\omega_3\end{pmatrix}+\rho QC_I^B\frac{d^2R}{dt^2}=\begin{pmatrix}F_y\\ F_z\end{pmatrix},\end{aligned} \tag{2.54}$$

with the boundary conditions (2.37).

Thus the complete dynamics of the flexible spacecraft consisting of the main rigid body and a flexible beam rigidly attached to it (Fig.2.1) is given by the coupled set of nonlinear ordinary differential equations (2.42),(2.47) and the partial differential equations (2.54) along with the boundary conditions in Eq. (2.37) (it is a so called *Hybrid System*).

We note from Equations (2.38-2.40) or (2.42-2.54) that the three sets of equations are strongly coupled. Thus any rotation of the bus would induce vibration in the beam and vice-versa. Also any disturbance of the orbit would induce vibrations in the beam and rotations in the bus and vice-versa. Further, one can observe the following facts:

(OBS.2.1) The integral in the orbital dynamics (2.38) represents the perturbation force due to vibration of the beam and rotation of the bus. Thus if we bring the integral term to the right hand side, the resulting equation has well-known form of perturbed orbital dynamics (see Eq. (11.1.1) of [Danby 39]).

(OBS.2.2) The integrals, involving the last three terms of the left hand side of Equation(2.39) (or 2.47), represent the contribution of the beam vibrations and orbital disturbance to the total angular momentum of the system. Clearly, in Equation (2.47) the total inertia matrix I_T varies only with the position of the beam, whereas

h_7, h_8 and h_9 depend also on the velocity and acceleration of the beam elements. In the case of an infinitely rigid beam, the relative displacement, velocity and acceleration of the beam elements with respect to the body frame are all zero. Consequently, I_r becomes constant and h_7, h_8, h_9 reduce to zero. In this situation the resulting form of Equation (2.47) becomes the well known attitude dynamics of a rigid body satellite (Eq. (7.1.2) of [Thomson 129]).

(OBS.2.3) The effect of rigid body rotation and translation on the vibration of the beam is represented by the forcing term $\rho Q(\alpha + \frac{d^2R}{dt^2})$ as shown in Equation (2.40). Thus it is clear that without the coupling effect $(\alpha + \frac{d^2R}{dt^2})$, Equation (2.40) becomes Euler-Bernoulli equation of cantilever beam[Timoshenko et al 130]. It is also interesting to note that the vibrations of the beam in the two directions are not independent of each other and rather is given by the coupled system of partial differential equations (2.54).

We have similar equations for the dynamics of the spacecraft as developed by Ahmed and Biswas[3,25,26]. However in their work, the radial dynamics and its effects on the dynamics of bus rotation and beam vibration are not considered while the equations derived in this section include all these effects on the entire system. In other words, the additional equation (2.42) describing the translational motion of the satellite bus and the additional terms $h_{10}(t)$, $h_{11}(t)$, $h_{12}(t)$ in Eq.(2.47) and the term $\rho Q C_I^B \frac{d^2R}{dt^2}$ in Eq.(2.54) are not taken into account in the work[3,25,26]. In fact, $h_{10}(t)$, $h_{11}(t)$, $h_{12}(t)$ and $\rho Q C_I^B \frac{d^2R}{dt^2}$ represent the impact of the radial perturbation on the rotations of the bus and vibrations of the beam, respectively. Hence this model enables us to study the influence of the radial perturbation as well as other disturbing forces on the dynamic performance such as the bus rotation and beam vibtrations. The terms $h_1(t)$, $h_2(t)$, $h_3(t)$ of Eq.(2.42) represent the contribution of the beam vibration and bus rotation to the translational motion of the bus. The influence of the radial perturbation and slew maneuvers on the dynamic performance of the satellite is investigated by numerical simulation in Chapter 5(Section 5.3).

Remark 2.1 (Gravity effect)

In the derivation of equations of motion we have approximated the gravitational potential energy P_g by Eqation (2.16). By using more precise expression for the gravity [Bomford 29, Heiskanen and Mainesz 52, Parvin 109, Swick 123], for example Eq.(2.15) or Eq. (A7), one can also study the effects of the gravity gradient torques on the rotation of the bus and vibration of the beam. However, the main focus of the section is with the development of a complete dynamic equations of motion for radial, rotational motion of the bus and vibration of the beam. Hence the gravity torque effect is not included here. In [134], Yu has shown that long term coupling effects between the planar pitch librational and orbital motions of a *rigid gravitationally* oriented satellite are negligibly small. However, in his work[134] the coupling effects between orbital perturbation and vibration of flexible members are not taken into account. Concerning the coupling effects between rigid rotational and translational modes, equations of motion of an orbiting spacecraft were developed by Santini[118] utilizing modal approach. Although in the work[118] the gravity gradient torques and nonlinear couplings are included, the effects of radial disturbance on flexible members are not considered.

Remark 2.2 (Sources of perturbation)

Flexible spacecraft in space are often subject to external disturbances caused by various sources such as meteorite collisions, variations in solar pressure and the earth's magnetic field due to explosions in the sun, as well as on-board disturbances such as sloshing of liquid fuel, disturbances due to motors and pumps etc. This may cause appreciable perturbation on the dynamic performance of the spacecraft. In Chapter 5, we present some numerical results illustrating the impact of radial perturbation and slew maneuvers on the dynamic behavior of the spacecraft. From the results, one can observe that the impact of the external perturbation on the beam vibration and bus rotation is quite significant and it should be eliminated by application of appropriate stabilizing controls. The stabilization of the spacecraft under the influence of external perturbation is considered in Chapter 4(Section 4.3).

Remark 2.3 (Modeling of more general spacecraft)

Using Euler-Bernoulli beam theory, in this section we have developed a complete dynamic model for the spacecraft consisting of a rigid main body and a flexible beam through the variational formulation. In the formulation of the beam dynamics associated with the spacecraft, we have assumed that the beam vibration is governed by the Euler-Bernoulli equation. The assumption implicit in the theory is that (i) the bending wavelength is several times larger than the cross sectional area of the beam and (ii) the rotary inertia and shear displacement are negligible. The *Euler-Bernoulli beam theory* is fairly accurate in the case of a thin beam. However, more detailed beam equations are given by Rayleigh beam theory or Timoshenko beam equation which include the effects of rotary inertia and shear displacement[Traill-Nash and Collar 131]. In the next two sections, we develop dynamic equations of motion for the spacecraft shown in Fig.2.1 on the basis of *Rayleigh beam theory* and *Timoshenko beam theory*. In the case of space structures consisting of a massive rigid body and several flexible configurations such as space stations, the dynamics of the system could be derived by following the same procedure as used in this chapter. This is done in Chapter 3.

2.5 Dynamic Modeling based on Rayleigh Beam Theory

In this section, using the same coordinate systems in Section 2.2 and following the same procedure as in Section 2.4, we derive the equations of motion for the spacecraft (Fig.2.1), *based on Rayleigh beam theory.* We use the same notations and definitions as in Section 2.4 throuthout this section, unless otherwise stated. The equations of motion derived based on Rayleigh beam theory have different forms compared with those based on Euler-Bernoulli beam theory, because the kinetic energy of the beam has a different representation in Rayleigh beam model.

Assuming that the origin of the body frame coincides with the center of mass of the satellite bus, the kinetic energy of the bus is given by Equation (2.20) and its first variation is given by Equation (2.21).

According to Rayleigh beam theory[Russell 116], the kinetic energy of the beam is given by

$$K_{b_R} = \frac{1}{2}\int_0^l \rho \frac{dr}{dt} \circ \frac{dr}{dt} dx + \frac{1}{2}\int_0^l I_\rho \frac{\partial^2 D}{\partial x \partial t} \circ \frac{\partial^2 D}{\partial x \partial t} dx, \tag{2.55}$$

with

$$r = R + \bar{r} \ = \ R + D_0 + D, \tag{2.56}$$

where r denotes the absolute position of the incremental mass element dm of the beam relative to the inertial frame, D_0 defines the position of the mass element dm relative to the body frame in the undeformed state and given by Equation (2.12). Further, D represents the displacement of dm from the undeformed position of the beam in S_B and given by Equation (2.12). $I_\rho \equiv I_\rho(x)$ is the rotary inertia of the beam element dm.

Under the assumption of small bending deformations neglecting elongation or torsional effects, the elastic potential energy of the beam P_e is given by Equation (2.13) and its first variation is the same as in Equation (2.26). The gravitational potential energy P_g is given by the sum of the potential energy of the satellite bus and that of the beam and it is given by Equation (2.16). Using the variational formula (2.18-2.19) and Equation (2.55), we have

$$\delta \int_{t_1}^{t_2} K_{b_R} dt = \int_{t_1}^{t_2} \int_0^l \rho \Big(\delta \frac{dR}{dt} \circ \frac{dr}{dt} \Big) dx dt + \int_{t_1}^{t_2} \int_0^l \rho \Big\{ \delta\omega \times \bar{r} \Big\} \circ \frac{dr}{dt} dx dt$$

$$+ \int_{t_1}^{t_2} \int_0^l \rho \Big\{ \delta\theta \times (\omega \times \bar{r} + \dot{D}) + \frac{d}{dt}(\delta D) \Big\} \circ \frac{dr}{dt} dx dt$$

$$+\int_{t_1}^{t_2}\int_0^l \delta D \circ \frac{\partial}{\partial x}\Big(I_\rho \frac{d}{dt}\frac{\partial^2 D}{\partial x \partial t}\Big) dx dt - \int_{t_1}^{t_2} \delta D \circ I_\rho \frac{d}{dt}\frac{\partial^2 D}{\partial x \partial t}\Big|_0^l dt. \quad (2.57)$$

Using the vector identity (2.23) we obtain, from Equation (2.57), that

$$\delta \int_{t_1}^{t_2} K_b dt = \int_{t_1}^{t_2}\int_0^l \rho \Big[\delta \frac{dR}{dt} \circ \frac{dr}{dt} + \delta\omega \circ \Big\{\bar{r} \times \frac{dr}{dt}\Big\}$$

$$+ \delta\theta \circ \Big\{(\omega \times \bar{r} + \dot{D}) \times \frac{dr}{dt}\Big\} + \frac{dr}{dt} \circ \frac{d}{dt}(\delta D)\Big] dx dt$$

$$+\int_{t_1}^{t_2}\int_0^l \delta D \circ \frac{\partial}{\partial x}\Big(I_\rho \frac{d}{dt}\frac{\partial^2 D}{\partial x \partial t}\Big) dx dt - \int_{t_1}^{t_2} \delta D \circ I_\rho \frac{d}{dt}\frac{\partial^2 D}{\partial x \partial t}\Big|_0^l dt. \quad (2.58)$$

Let the force and torque applied to the bus be given by $F_s = (F_1, F_2, F_3)'$ and $T = (T_1, T_2, T_3)'$ respectively, and control force applied to the beam be given by $\bar{F}_b = (0, F_y, F_z)'$. Then the virtual work done by the applied forces is given by

$$\delta W \ = \ F_s \circ \delta R \ + \ T \circ \delta\theta \ + \ \int_0^l \bar{F}_b \circ \delta D dx. \quad (2.59)$$

Substituting Equations (2.21), (2.58), (2.26), (2.29) and (2.59) into Equation (2.4) and integrating by parts with the end conditions in (2.5), we obtain

$$\delta \int_{t_1}^{t_2} (L + W) dt \equiv$$

$$- \int_{t_1}^{t_2} (\delta R) \circ \Big[m_s \frac{d^2 R}{dt^2} + \frac{d}{dt}\int_0^l \rho\Big(\frac{dR}{dt} + \omega \times \bar{r} + \dot{D}\Big) dx + \frac{G m_e m_T R}{|R|^3} - F_s\Big] dt$$

$$+ \int_{t_1}^{t_2} (\delta\theta) \circ \Big[T - \frac{d}{dt}\Big\{(I_s \circ \omega) + \int_0^l \rho \bar{r} \times \frac{dr}{dt} dx\Big\} + \int_0^l \rho\Big(\omega \times \bar{r} + \dot{D}\Big) \times \Big(\frac{dr}{dt}\Big) dx\Big] dt$$

$$- \int_{t_1}^{t_2}\int_0^l (\delta D) \circ \Big\{\rho \frac{d^2 r}{dt^2} - \frac{\partial}{\partial x}\Big(I_\rho \frac{d}{dt}\frac{\partial^2 D}{\partial x \partial t}\Big) + \frac{\partial^2}{\partial x^2}\Big(EI \frac{\partial^2 D}{\partial x^2}\Big) - \bar{F}_b\Big\} dx\, dt$$

$$- \int_{t_1}^{t_2} \Big[EI\Big(\frac{\partial^2 D}{\partial x^2}\Big) \circ \frac{\partial}{\partial x}(\delta D)\Big|_0^l + \Big\{I_\rho \frac{d}{dt}\frac{\partial^2 D}{\partial x \partial t} - \frac{\partial}{\partial x}\Big(EI \frac{\partial^2 D}{\partial x^2}\Big)\Big\} \circ (\delta D)\Big|_0^l\Big] dt = 0 \,. \quad (2.60)$$

Since the beam is assumed to be inextensible $\delta D = (0, \delta y, \delta z)'$. Therefore, noting that the variations $\delta\theta, \delta R, \delta D$ are arbitrary, we obtain, from Equation (2.60), a set of coupled differential equations for the dynamics of the flexible spacecraft as follows. From the first integral term with δR in Equation (2.60) we have

$$(m_s + m_b)\frac{d^2R}{dt^2} + \frac{d}{dt}\int_0^l \rho\Big(\omega \times \bar{r} + \dot{D}\Big)dx + \frac{Gm_e m_T R}{|R|^3} = F_s, \tag{2.61}$$

where $m_b = \int_{beam} dm$ is the mass of the beam and $\rho \equiv \rho(x)$ is the mass per unit length. From the second part with $\delta\theta$ in Equation (2.60) we obtain

$$\frac{d}{dt}\{(I_s+I_b)\circ\omega\}+\frac{d}{dt}\int_0^l \rho\bar{r}\times\Big(\frac{dR}{dt}+\dot{D}\Big)dx-\int_0^l \rho\Big\{(\omega\times\bar{r}+\dot{D})\times\frac{dR}{dt}\Big\}dx = T, \tag{2.62}$$

where $I_b\circ\omega = \int_0^l \bar{r}\times(\omega\times\bar{r})dm$. For notational simplicity we shall denote $F_b{\equiv}(F_y, F_z)'$ and $I_T \equiv I_s + I_b$. Since $\delta D = (0, \delta y, \delta z)'$, we obtain from the third part with δD in Equation (2.60) that

$$\begin{aligned}\rho Q\ddot{D} - Q\frac{\partial}{\partial x}\Big(I_\rho\frac{d}{dt}\frac{\partial^2 D}{\partial x\partial t}\Big)+Q\frac{\partial^2}{\partial x^2}\Big(EI\frac{\partial^2 D}{\partial x^2}\Big)&\\ +\rho Q\Big[\frac{d^2R}{dt^2}+\dot{\omega}\times\bar{r}+\omega\times\Big\{2\dot{D}+\omega\times\bar{r}\Big\}\Big]&= F_b,\end{aligned} \tag{2.63}$$

where

$$Q \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.64}$$

The boundary conditions are obtained from the last part of Equation (2.60) and the cantilever configuration. At the clamped end (x=0) we have

$$y(0,t) = 0, \qquad z(0,t) = 0, \tag{2.65}$$

$$\frac{\partial y}{\partial x}(0,t) = 0, \qquad \frac{\partial z}{\partial x}(0,t) = 0. \tag{2.66}$$

Since the shear force and bending moment are zero at the free end ($x = l$), it is obtained

$$EI_y\frac{\partial^2 y}{\partial x^2}(l,t) = 0, \qquad EI_z\frac{\partial^2 z}{\partial x^2}(l,t) = 0, \tag{2.67}$$

$$I_\rho w_1 \frac{\partial^2}{\partial x \partial t} z(l,t) - I_\rho \frac{\partial^3}{\partial x \partial t^2} y(l,t) + \frac{\partial}{\partial x}(EI_y \frac{\partial^2 y}{\partial x^2})(l,t) = 0, \tag{2.68}$$

$$I_\rho w_1 \frac{\partial^2}{\partial x \partial t} y(l,t) + I_\rho \frac{\partial^3}{\partial x \partial t^2} z(l,t) - \frac{\partial}{\partial x}(EI_z \frac{\partial^2 z}{\partial x^2})(l,t) = 0. \tag{2.69}$$

Then the dynamics of the controlled flexible spacecraft can be rewritten as

$$m_T \frac{d^2 R}{dt^2} + \int_0^l \rho\Big\{(\dot{\omega} \times \bar{r}) + \omega \times (2\dot{D} + \omega \times \bar{r}) + \ddot{D}\Big\} dx + \frac{G m_e m_T R}{|R|^3} = F_s, \tag{2.70}$$

$$I_T \circ \dot{\omega} + \omega \times (I_T \circ \omega) + \dot{I}_b \circ \omega + \int_0^l \rho\Big(\bar{r} \times \frac{d^2 R}{dt^2}\Big) dx$$
$$+ \int_0^l \rho(\bar{r} \times \ddot{D}) dx + \int_0^l \rho\{\omega \times (\bar{r} \times \dot{D})\} dx = T, \tag{2.71}$$

and

$$\rho \frac{\partial^2}{\partial t^2}\begin{pmatrix} y \\ z \end{pmatrix} - \frac{\partial}{\partial x}\Big(I_\rho \frac{\partial^3}{\partial x \partial t^2}\begin{pmatrix} y \\ z \end{pmatrix}\Big) - \frac{\partial}{\partial x}\Big(I_\rho \begin{pmatrix} 0 & -w_1 \\ w_1 & 0 \end{pmatrix} \frac{\partial^2}{\partial x \partial t}\begin{pmatrix} y \\ z \end{pmatrix}\Big)$$
$$+ \rho Q\Big(\alpha + \frac{d^2 R}{dt^2}\Big) + \frac{\partial^2}{\partial x^2}\Big(\begin{pmatrix} EI_y & 0 \\ 0 & EI_z \end{pmatrix} \frac{\partial^2}{\partial x^2}\begin{pmatrix} y \\ z \end{pmatrix}\Big) = F_b, \tag{2.72}$$

where

$$\alpha = \dot{\omega} \times \bar{r} + 2\omega \times \dot{D} + \omega \times (\omega \times \bar{r}). \tag{2.73}$$

Equation (2.72) is similar to Equation (2.54). However, because of the second term due to the effect of rotary inertia, the response of the system governed by Eqs.(2.70-2.72) based on Rayleigh beam theory is different from that governed by Eqs.(2.38-2.40) based on Euler-Bernoulli beam theory. Using the same notations C_B^I and $(a_i, a_j, a_k)'$ as in Section 2.4 for the coordinate transformation matrix from body to inertial frame and the accerlation vector, respectively and performing the vector operations with keeping the relation (2.17) in mind, Equations (2.70), (2.71) and (2.72) could be expressed in explicit forms which are suitable for digital computer simulation. These are given below.

BUS DYNAMICS (TRANSLATION)

$$m_T \frac{d^2}{dt^2}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + C_B^I \begin{pmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \end{pmatrix} + \begin{pmatrix} h_4(t) \\ h_5(t) \\ h_6(t) \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \tag{2.74}$$

where

$$h_1(t) = \int_0^l \rho\Big\{(z\dot{\omega}_2 - y\dot{\omega}_3) + 2\Big(\omega_2 \frac{\partial z}{\partial t} - \omega_3 \frac{\partial y}{\partial t}\Big) + \omega_1\omega_2 y - \omega_2^2(R_s + x) - \omega_3^2(R_s + x) + \omega_1\omega_3 z\Big\} dx, \tag{2.75}$$

$$h_2(t) = \int_0^l \rho\Big\{(R_s + x)\dot{\omega}_3 - z\dot{\omega}_1 - 2\omega_1 \frac{\partial z}{\partial t} + \omega_2\omega_3 z - y(\omega_3^2 + \omega_1^2) + \omega_1\omega_2(R_s + x) + \frac{\partial^2 y}{\partial t^2}\Big\} dx, \tag{2.76}$$

$$h_3(t) = \int_0^l \rho\Big\{y\dot{\omega}_1 - (x + R_s)\dot{\omega}_2 + 2\omega_1 \frac{\partial y}{\partial t} + \omega_1\omega_3(x + R_s) - z(\omega_1^2 + \omega_2^2) + \omega_2\omega_3 y + \frac{\partial^2 z}{\partial t^2}\Big\} dx, \tag{2.77}$$

$$h_4(t) = \frac{G m_e m_T}{|R|^3} X, \tag{2.78}$$

$$h_5(t) = \frac{G m_e m_T}{|R|^3} Y, \tag{2.79}$$

$$h_6(t) = \frac{G m_e m_T}{|R|^3} Z, \tag{2.80}$$

BUS DYNAMICS (ROTATION)

$$I_T \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} I_T \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} h_7(t) \\ h_8(t) \\ h_9(t) \end{pmatrix} + \begin{pmatrix} h_{10}(t) \\ h_{11}(t) \\ h_{12}(t) \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}, \tag{2.81}$$

where

$$h_7(t) = \int_0^l \rho\Big\{y\frac{\partial^2 z}{\partial t^2} - z\frac{\partial^2 y}{\partial t^2} + 2\omega_1 y\frac{\partial y}{\partial t} + 2\omega_1 z\frac{\partial z}{\partial t}\Big\}dx, \tag{2.82}$$

$$h_8(t) = \int_0^l \rho\Big\{-(x+R_s)\frac{\partial^2 z}{\partial t^2} + 2\omega_2 z\frac{\partial z}{\partial t} - 2\omega_3 z\frac{\partial y}{\partial t} - 2\omega_1(x+R_s)\frac{\partial y}{\partial t}\Big\}dx, \tag{2.83}$$

$$h_9(t) = \int_0^l \rho\Big\{(x+R_s)\frac{\partial^2 y}{\partial t^2} - 2\omega_1(x+R_s)\frac{\partial z}{\partial t} - 2\omega_2 y\frac{\partial z}{\partial t} + 2\omega_3 y\frac{\partial y}{\partial t}\Big\}dx, \tag{2.84}$$

$$h_{10}(t) = \int_0^l \rho\Big\{ya_k - za_j\Big\}dx, \tag{2.85}$$

$$h_{11}(t) = \int_0^l \rho\Big\{za_i - (x+R_s)a_k\Big\}dx, \tag{2.86}$$

$$h_{12}(t) = \int_0^l \rho\Big\{(x+R_s)a_j - ya_i\Big\}dx, \tag{2.87}$$

<u>BEAM DYNAMICS (BEAM VIBRATION)</u>

$$\begin{aligned}
&\rho\frac{\partial^2}{\partial t^2}\begin{pmatrix} y \\ z \end{pmatrix} + \rho\begin{pmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} y \\ z \end{pmatrix} + \rho(x+R_s)\begin{pmatrix} \dot{\omega_3} + \omega_1\omega_2 \\ -\dot{\omega_2} + \omega_1\omega_3 \end{pmatrix} + \rho Q C_I^B \frac{d^2 R}{dt^2} \\
&\quad + \frac{\partial^2}{\partial x^2}\left(\begin{pmatrix} EI_y & 0 \\ 0 & EI_z \end{pmatrix}\frac{\partial^2}{\partial x^2}\begin{pmatrix} y \\ z \end{pmatrix}\right) + \rho\begin{pmatrix} -\omega_1^2 - \omega_3^2 & -\dot{\omega_1} + \omega_2\omega_3 \\ \dot{\omega_1} + \omega_2\omega_3 & -\omega_1^2 - \omega_2^2 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} \\
&\quad - \frac{\partial}{\partial x}\left(I_\rho\frac{\partial^3}{\partial x\partial t^2}\begin{pmatrix} y \\ z \end{pmatrix}\right) - \frac{\partial}{\partial x}\left(I_\rho\begin{pmatrix} 0 & -w_1 \\ w_1 & 0 \end{pmatrix}\frac{\partial^2}{\partial x\partial t}\begin{pmatrix} y \\ z \end{pmatrix}\right) = \begin{pmatrix} F_y \\ F_z \end{pmatrix}.
\end{aligned} \tag{2.88}$$

<u>BOUNDARY CONDITIONS</u>

$$y(0,t) = 0, \qquad z(0,t) = 0, \tag{2.89}$$

$$\frac{\partial y}{\partial x}(0,t) = 0, \qquad \frac{\partial z}{\partial x}(0,t) = 0. \tag{2.90}$$

$$EI_y \frac{\partial^2 y}{\partial x^2}(l,t) = 0, \qquad EI_z \frac{\partial^2 z}{\partial x^2}(l,t) = 0, \tag{2.91}$$

$$I_\rho w_1 \frac{\partial^2}{\partial x \partial t} z(l,t) - I_\rho \frac{\partial^3}{\partial x \partial t^2} y(l,t) + \frac{\partial}{\partial x}(EI_y \frac{\partial^2 y}{\partial x^2})(l,t) = 0, \tag{2.92}$$

$$I_\rho w_1 \frac{\partial^2}{\partial x \partial t} y(l,t) + I_\rho \frac{\partial^3}{\partial x \partial t^2} z(l,t) - \frac{\partial}{\partial x}(EI_z \frac{\partial^2 z}{\partial x^2})(l,t) = 0. \tag{2.93}$$

In this section we have deveolped the equations of motion for the spacecraft (Fig. 2.1) based on *Rayleigh beam theory*. The complete dynamics of the flexible spacecraft consisting of the main rigid body and a flexible beam rigidly attached to it (Fig.2. 1) is given by a ***hybrid system*** which consists of nonlinear ordinary differential equations (2.74),(2.81) and the partial differential equation (2.88) along with the boundary conditions (2.89)-(2.93).

We note from Equations (2.70-2.72) or (2.74-2.93) that the three sets of equations are strongly coupled. Thus any rotation of the bus would induce vibration in the beam and vice-versa. Also any disturbance of the orbit would induce vibrations in the beam and rotations in the bus and vice-versa. From the equations for rotational and translational motion of the rigid body. we have the same observations as (OBS.2.1) and (OBS.2.2) in Section 2.4. Further, from the beam dynamics we note that

(OBS.2.4) The effect of rigid body rotation and translation on the vibration of the beam is represented by the forcing term $\rho Q(\alpha + \frac{d^2 R}{dt^2})$ as shown in Equation (2.72). Thus it is clear that without the coupling effect $(\alpha + \frac{d^2 R}{dt^2})$, Equation (2.72) becomes Rayleigh beam equation of a cantilever beam[Russell 116].

Remark 2.4

We have similar equations for the dynamics of the spacecraft as developed in Section 2.4 where the equations of motion were derived based on Euler-Bernoulli beam theory. However, note that the second and the third terms of the beam equation (2.72) (or equivalently the 7th and 8th terms of Equation (2.88)) are not taken into account in the Euler-Bernoulli model (Equation (2.40) or (2.54)). In fact, these terms represent the effect of the rotary inertia on the beam dynamics. Hence the beam equation(2.88) enables us to study the influence of the rotary inertia on the dynamic performance such as the bus rotation and beam vibrations. The terms $h_1(t)$, $h_2(t)$, $h_3(t)$ of (2.74) represent the contribution of the beam vibration and bus rotation to the translational motion of the bus. This implies that even though the bus translational and rotational equations(2.74, 2.81) have the same forms as those (2.42,2.47) based on Euler- Bernoulli beam theory, their dynamic response subject to external forces are not the same because the beam euations are different from each other and the three equations are strongly coupled.

Remark 2.5

Based on Rayleigh beam theory in this section we have developed a complete dynamic model for the spacecraft consisting of a rigid main body and a flexible beam(Fig.2.1) through the variational formulation. In our derivation of equations of motion we have approximated the gravitational potential energy P_g by Eqation (2.16) (details are given in Remark 2.1). Flexible structures in space are often subject to external disturbances caused by various sources such as those mentioned in Remark 2.2. The stabilization of the spacecraft under the influence of perturbing forces is considered in Section 4.3.2.

2.6 Dynamic Modeling based on Timoshenko Beam Theory

In this section, using the same coordinate systems as in Section 2.2 and following the same procedure as used in Sections 2.4 and 2.5, we derive the equations of motion for the spacecraft (Fig.2.1), based on Timoshenko beam theory. We use the same notations and definitions as in Section 2.4 throuthout this section, unless otherwise stated. When Timoshenko beam theory[Russell 116] is applied, the equations of motion have different forms compared with those based on Euler-Bernoulli or Rayleigh beam theories, because the kinetic energy of the beam has different representation by Timoshenko beam theory.

Let $I_\rho \equiv I_\rho(x)$ be the rotary inertia of the mass element of the beam and $\beta(x,t)$ be the deformation due to shearing forces. According to Timoshenko beam theory[Russell 116, Traill-Nash and Collar 131], the kinetic energy of the beam is given by

$$K_{b_T} = \frac{1}{2}\int_0^l \rho \frac{dr}{dt} \circ \frac{dr}{dt} dx + \frac{1}{2}\int_0^l I_\rho \frac{\partial \psi}{\partial t} \circ \frac{\partial \psi}{\partial t} dx \tag{2.94}$$

with

$$r = R + \bar{r} \;=\; R + D_0 + D, \tag{2.95}$$

$$\psi = \frac{\partial D}{\partial x} + \beta(x,t). \tag{2.96}$$

where r denotes the absolute position of the incremental mass element dm of the beam relative to the inertial frame, D_0 defines the position of the mass element dm relative to the body frame in the undeformed state and given by Equation (2.12). Further, D and ψ represents the displacement of dm from the undeformed position of the beam in S_B and angular dispacement respectively, whose components are given by

$$D = (0, y, z)', \quad \text{and} \;\; \psi = (0, \psi_y, \psi_z)'. \tag{2.97}$$

The system potential energy arises from two sourses, namely elastic and gravitational. The elastic potential energy P_e due to deformations of the beam is a function of the partial derivatives of the displacements y, z with respect to the spatial variable x. Under the assumption of small bending deformations neglecting elongation or torsional effects, the elastic potential energy of the beam P_{e_T} is given by[Russell 116]

$$P_{e_T} = \frac{1}{2}\int_0^l \Big\{ K\Big(\psi - \frac{\partial D}{\partial x}\Big) \circ \Big(\psi - \frac{\partial D}{\partial x}\Big) + EI\frac{\partial \psi}{\partial x} \circ \frac{\partial \psi}{\partial x} \Big\} dx, \tag{2.98}$$

where $K \equiv kGA$, G is shear modulus, A is cross sectional area of the beam and k is a numerical factor depending on the shape of the section. The gravitational potential energy P_g is given by the sum of the potential energy of the satellite bus and that of the beam as follows[Halliday and Resnick 51].

$$\begin{aligned} P_g &= P_s + P_b \\ &= \int_{bus} \frac{-Gm_e}{|\tilde{r}|} d\tilde{m} + \int_{beam} \frac{-Gm_e}{|r|} dm, \end{aligned} \tag{2.99}$$

where G = universal gravitational constant, m_e = mass of the earth. In Eq. (2.99) P_s and P_b denote the potential energy of the satellite bus and that of the beam respectively. Expanding $\frac{1}{|r|}$ and $\frac{1}{|\tilde{r}|}$ in the form of power series, we have following expression (see Appendix A):

$$\begin{aligned} P_g = -\frac{Gm_e m_T}{|R|} - \frac{Gm_e}{2|R|^3} \Big[& \int_{bus} \Big\{ |\tilde{\eta}|^2 - 3(\frac{R}{|R|} \circ \tilde{\eta})^2 \Big\} d\tilde{m} \\ & + \int_{beam} \Big\{ |\bar{r}|^2 - 3(\frac{R}{|R|} \circ \bar{r})^2 \Big\} dm \Big] + \cdots . \end{aligned} \tag{2.100}$$

The intergral terms in the above equation (2.100) contribute to gravity gradient torques and is often used in the design of passive stabilizers. However, in some applications this can be neglected (see Remark 2.2); for example, for the study of orbital perturbation the quantities $|\tilde{\eta}|$ and $|\bar{r}|$ are negligible and hence one can approximate P_g as in Equation (2.16) and its first variation is given by Equation (2.29). The virtual displacement of the beam relative to the inertial space is given by[Cavin and Dusto 32]:

$$\delta r = \delta R + \delta\theta \times \bar{r} + \delta D, \tag{2.101}$$

where $\delta\theta = (\delta\theta_1, \delta\theta_2, \delta\theta_3)$ is the virtual rigid body rotation of the body frame relative to the reference frame. Further, using (2.17) one obtains from (2.101) that

$$\delta\Big(\frac{dr}{dt}\Big) = \delta\frac{dR}{dt} + \delta\omega \times \bar{r} + \delta\theta \times (\omega \times \bar{r} + \dot{D}) + \frac{d}{dt}(\delta D), \tag{2.102}$$

where $\delta\omega = \frac{d}{dt}(\delta\theta)$ denotes the virtual angular velocity of the body frame.

Assuming that the origin of the body frame coincides with the center of mass of the satellite bus the kinetic energy of the bus is given by Equation (2.20) and its first variation is given by Equation (2.21). Substituting Eqations (2.101-2.102) into Equation (2.94) and taking the first variation, we have

$$\delta \int_{t_1}^{t_2} K_{b_T} dt = \int_{t_1}^{t_2} \int_0^l \rho \Big(\delta \frac{dR}{dt} \circ \frac{dr}{dt} \Big) dx dt + \int_{t_1}^{t_2} \int_0^l \rho \Big\{ \delta\omega \times \bar{r} \Big\} \circ \frac{dr}{dt} dx dt$$

$$+ \int_{t_1}^{t_2} \int_0^l \rho \Big\{ \delta\theta \times (\omega \times \bar{r} + \dot{D}) \Big\} \circ \frac{dr}{dt} dx dt$$

$$+ \int_{t_1}^{t_2} \int_0^l \rho \Big\{ \frac{d}{dt}(\delta D) \circ \frac{dr}{dt} \Big\} dx dt$$

$$+ \int_0^l \delta\psi \circ I_\rho \frac{\partial \psi}{\partial t} \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_0^l (\delta\psi) \circ \frac{d}{dt} \Big(I_\rho \frac{\partial \psi}{\partial t} \Big) dx dt. \tag{2.103}$$

Using the vector identity (2.23) we obtain, from Equation (2.103), that

$$\delta \int_{t_1}^{t_2} K_{b_T} dt = \int_{t_1}^{t_2} \int_0^l \rho \Big[\delta \frac{dR}{dt} \circ \frac{dr}{dt} + \delta\omega \circ \Big\{ \bar{r} \times \frac{dr}{dt} \Big\}$$

$$+ \delta\theta \circ \Big\{ (\omega \times \bar{r} + \dot{D}) \times \frac{dr}{dt} \Big\} + \frac{dr}{dt} \circ \frac{d}{dt}(\delta D) \Big] dx dt$$

$$+ \int_0^l \delta\psi \circ I_\rho \frac{\partial \psi}{\partial t} \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_0^l (\delta\psi) \circ \frac{d}{dt} \Big(I_\rho \frac{\partial \psi}{\partial t} \Big) dx dt. \tag{2.104}$$

Let $EI_y \equiv EI_y(x)$(respectively EI_z) be the flexural rigidity of the beam for deflections in the j_b direction (respectively k_b direction) and define the matrix EI and K by

$$EI \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & EI_y & 0 \\ 0 & 0 & EI_z \end{pmatrix}. \tag{2.105}$$

$$K \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{pmatrix}. \tag{2.106}$$

Then one obtains from Equation (2.98) that

$$\delta \int_{t_1}^{t_2} P_{e_T} dt = \int_{t_1}^{t_2} \int_0^l \Big\{ \delta\Big(\psi - \frac{\partial D}{\partial x}\Big) \circ K\Big(\psi - \frac{\partial D}{\partial x}\Big) + EI\frac{\partial \psi}{\partial x} \circ \Big(\delta\frac{\partial \psi}{\partial x}\Big)\Big\} dxdt.$$

$$= \int_{t_1}^{t_2} \int_0^l \Big[\delta\psi \circ K\Big(\psi - \frac{\partial D}{\partial x}\Big) - \delta\frac{\partial D}{\partial x} \circ K\Big(\psi - \frac{\partial D}{\partial x}\Big) + \delta\frac{\partial \psi}{\partial x} \circ EI\frac{\partial \psi}{\partial x}\Big] dxdt. \tag{2.107}$$

or

$$\delta \int_{t_1}^{t_2} P_{e_T} dt = \int_{t_1}^{t_2} \int_0^l (\delta\psi) \circ \Big\{K\Big(\psi - \frac{\partial D}{\partial x}\Big) - \frac{\partial}{\partial x}\Big(EI\frac{\partial \psi}{\partial x}\Big)\Big\} dxdt$$

$$+ \int_{t_1}^{t_2} \int_0^l (\delta D) \circ \frac{\partial}{\partial x} K\Big(\psi - \frac{\partial D}{\partial x}\Big) dxdt$$

$$+ \int_{t_1}^{t_2} \Big\{ \Big[\delta\psi \circ EI\frac{\partial \psi}{\partial x}\Big]_0^l - \Big[\delta D \circ K\Big(\psi - \frac{\partial D}{\partial x}\Big)\Big]_0^l \Big\} dt. \tag{2.108}$$

Let the force and torque applied to the bus be given by $F_s = (F_1, F_2, F_3)'$ and $T = (T_1, T_2, T_3)'$ respectively, and control force applied to the beam be given by $\bar{F}_b = (0, F_y, F_z)'$, $\bar{F}_{b_T} = (0, F_{yT}, F_{zT})'$. Then the virtual work[Meriam 99] done by the applied forces is given by

$$\delta W \; = \; F_s \circ \delta R \; + \; T \circ \delta\theta \; + \; \int_0^l \bar{F}_b \circ \delta D dx + \int_0^l \bar{F}_{b_T} \circ \delta\psi dx. \tag{2.109}$$

Substituting Equations (2.21), (2.104), (2.107), (2.29) and (2.109) into Equation (2.4) and integrating by parts with the end conditions in (2.5), we obtain

$$\delta \int_{t_1}^{t_2} (L + W) dt \equiv$$

$$-\int_{t_1}^{t_2}(\delta R)\circ\left[m_s\frac{d^2R}{dt^2}+\frac{d}{dt}\int_0^l\rho\Big(\frac{dR}{dt}+\omega\times\bar r+\dot D\Big)dx+\frac{Gm_em_TR}{|R|^3}-F_s\right]dt$$

$$+\int_{t_1}^{t_2}(\delta\theta)\circ\left[T-\frac{d}{dt}\Big\{(I_s\circ\omega)+\int_0^l\rho\bar r\times\frac{dr}{dt}dx\Big\}+\int_0^l\rho\Big(\omega\times\bar r+\dot D\Big)\times\Big(\frac{dr}{dt}\Big)dx\right]dt$$

$$-\int_{t_1}^{t_2}\int_0^l(\delta D)\circ\Big\{\rho\frac{d^2r}{dt^2}+\frac{\partial}{\partial x}(K\psi)-\frac{\partial}{\partial x}\Big(K\frac{\partial D}{\partial x}\Big)-\bar F_b\Big\}dx\ dt$$

$$-\int_{t_1}^{t_2}\int_0^l(\delta\psi)\circ\Big\{\frac{d}{dt}\Big(I_\rho\frac{\partial\psi}{\partial t}\Big)+K\Big(\psi-\frac{\partial D}{\partial x}\Big)-\frac{\partial}{\partial x}\Big(EI\frac{\partial\psi}{\partial x}\Big)-\bar F_{b_T}\Big\}dx\ dt$$

$$-\int_{t_1}^{t_2}\left[EI\Big(\frac{\partial\psi}{\partial x}\Big)\circ(\delta\psi)\Big|_0^l-K\Big(\psi-\frac{\partial D}{\partial x}\Big)\circ(\delta D)\Big|_0^l\right]dt=0\ . \tag{2.110}$$

Noting that the variations $\delta\theta,\delta R,\delta D,\delta\psi$ are arbitrary, we obtain, from Equation (2.110), a set of coupled differential equations for the dynamics of the flexible spacecraft as follows. From the first integral term with δR in Equation (2.110) we have

$$(m_s+m_b)\frac{d^2R}{dt^2}+\frac{d}{dt}\int_0^l\rho\Big(\omega\times\bar r+\dot D\Big)dx+\frac{Gm_em_TR}{|R|^3}=F_s, \tag{2.111}$$

where m_s is the mass of the bus, $m_b\equiv\int_{beam}dm$ the mass of the beam and $\rho\equiv\rho(x)$ is the mass per unit length. From the second part with $\delta\theta$ in Equation (2.110) we obtain

$$\frac{d}{dt}\{(I_s+I_b)\circ\omega\}+\frac{d}{dt}\int_0^l\rho\bar r\times\Big(\frac{dR}{dt}+\dot D\Big)dx-\int_0^l\rho\Big\{(\omega\times\bar r+\dot D)\times\frac{dR}{dt}\Big\}dx=T, \tag{2.112}$$

where $I_b\circ\omega=\int_0^l\bar r\times(\omega\times\bar r)dm$. For notational simplicity we shall denote $F_b\equiv(F_y,F_z)'$, $F_{b_T}\equiv(F_{yT},F_{zT})'$ and $I_T\equiv I_s+I_b$. We obtain from the third part with δD in Equation (2.110) that

$$\rho Q\ddot D+Q\frac{\partial}{\partial x}\Big(K\Big(\psi-\frac{\partial D}{\partial x}\Big)\Big)+\rho Q\Big[\frac{d^2R}{dt^2}+\dot\omega\times\bar r+\omega\times\Big\{2\dot D+\omega\times\bar r\Big\}\Big]=F_b, \tag{2.113}$$

where

$$Q \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.114}$$

From the fourth part of (2.110) we obtain that

$$\frac{d}{dt}\Big(I_\rho \frac{\partial \psi}{\partial t}\Big) + K\Big(\psi - \frac{\partial D}{\partial x}\Big) - \frac{\partial}{\partial x}\Big(EI\frac{\partial \psi}{\partial x}\Big) = \bar{F}_{b_T}. \tag{2.115}$$

The boundary conditions are obtained from the last part of Equation (2.110) and the cantilever configuration. At the clamped end (x=0) we have

$$y(0,t) = 0, \qquad z(0,t) = 0, \tag{2.116}$$

$$\psi_y(0,t) = 0, \qquad \psi_z(0,t). \tag{2.117}$$

Since the shear force and bending moment are zero at the free end (x=l), it is obtained

$$EI_y \frac{\partial}{\partial x}\psi_y(l,t) = 0, \qquad EI_z \frac{\partial}{\partial x}\psi_z(l,t) = 0, \tag{2.118}$$

$$K_y\Big(\psi_y - \frac{\partial}{\partial x}y\Big)(l,t) = 0, \qquad K_z\Big(\psi_z - \frac{\partial}{\partial x}z\Big)(l,t) = 0. \tag{2.119}$$

Therefore the dynamics of the flexible spacecraft can be rewritten as

$$m_T\frac{d^2R}{dt^2} + \int_0^l \rho\Big\{(\dot{\omega}\times\bar{r}) + \omega\times(2\dot{D} + \omega\times\bar{r}) + \ddot{D}\Big\}dx + \frac{Gm_e m_T R}{|R|^3} = F_s, \tag{2.120}$$

$$I_T \circ \dot{\omega} + \omega \times (I_T \circ \omega) + \dot{I}_b \circ \omega + \int_0^l \rho\Big(\bar{r} \times \frac{d^2R}{dt^2}\Big)dx$$

$$+ \int_0^l \rho(\bar{r}\times\ddot{D})dx + \int_0^l \rho\{\omega\times(\bar{r}\times\dot{D})\}dx = T, \tag{2.121}$$

and

$$\rho\frac{\partial^2}{\partial t^2}\begin{pmatrix} y \\ z \end{pmatrix} + \rho Q\Big(\alpha + \frac{d^2R}{dt^2}\Big) + \frac{\partial}{\partial x}\begin{pmatrix} K_y & 0 \\ 0 & K_z \end{pmatrix}\begin{pmatrix} \psi_y - \frac{\partial}{\partial x}y \\ \psi_z - \frac{\partial}{\partial x}z \end{pmatrix} = \begin{pmatrix} F_y \\ F_z \end{pmatrix}, \tag{2.122}$$

$$I_\rho \frac{\partial^2}{\partial t^2}\begin{pmatrix}\psi_y\\ \psi_z\end{pmatrix} + I_\rho \begin{pmatrix}0 & -w_1\\ w_1 & 0\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}\psi_y\\ \psi_z\end{pmatrix} + \begin{pmatrix}K_y & 0\\ 0 & K_z\end{pmatrix}\begin{pmatrix}\psi_y\\ \psi_z\end{pmatrix}$$

$$-\begin{pmatrix}K_y & 0\\ 0 & K_z\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}y\\ z\end{pmatrix} - \frac{\partial}{\partial x}\begin{pmatrix}EI_y & 0\\ 0 & EI_z\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\psi_y\\ \psi_z\end{pmatrix} = \begin{pmatrix}F_{y_T}\\ F_{z_T}\end{pmatrix}, \qquad (2.123)$$

where

$$\alpha = \dot{\omega}\times\bar{r} + 2\omega\times\dot{D} + \omega\times(\omega\times\bar{r}). \qquad (2.124)$$

Using the same notations C_B^I and $(a_i, a_j, a_k)'$ as in Section 2.4 for the coordinate transformation matrix from body to inertial frame and the acceration vector, respectively and performing the vector operations, Equations (2.120-2.122) and (2.123) could be expressed in explicit forms which are suitable for digital computer simulation. These are given below.

BUS DYNAMICS (TRANSLATION)

$$m_T\frac{d^2}{dt^2}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix} + C_B^I\begin{pmatrix}h_1(t)\\ h_2(t)\\ h_3(t)\end{pmatrix} + \begin{pmatrix}h_4(t)\\ h_5(t)\\ h_6(t)\end{pmatrix} = \begin{pmatrix}F_1\\ F_2\\ F_3\end{pmatrix}, \qquad (2.125)$$

where

$$h_1(t) = \int_0^l \rho\Big\{(z\dot{\omega}_2 - y\dot{\omega}_3) + 2\Big(\omega_2\frac{\partial z}{\partial t} - \omega_3\frac{\partial y}{\partial t}\Big) + \omega_1\omega_2 y - \omega_2^2(R_s + x) - \omega_3^2(R_s + x) + \omega_1\omega_3 z\Big\}dx, \qquad (2.126)$$

$$h_2(t) = \int_0^l \rho\Big\{(R_s + x)\dot{\omega}_3 - z\dot{\omega}_1 - 2\omega_1\frac{\partial z}{\partial t} + \omega_2\omega_3 z - y(\omega_3^2 + \omega_1^2) + \omega_1\omega_2(R_s + x) + \frac{\partial^2 y}{\partial t^2}\Big\}dx, \qquad (2.127)$$

$$h_3(t) = \int_0^l \rho\Big\{y\dot{\omega}_1 - (x + R_s)\dot{\omega}_2 + 2\omega_1\frac{\partial y}{\partial t} + \omega_1\omega_3(x + R_s)$$

$$- z(\omega_1^2 + \omega_2^2) + \omega_2\omega_3 y + \frac{\partial^2 z}{\partial t^2}\Big\}dx, \qquad (2.128)$$

$$h_4(t) = \frac{Gm_e m_T}{|R|^3}X, \qquad (2.129)$$

$$h_5(t) = \frac{Gm_e m_T}{|R|^3}Y, \qquad (2.130)$$

$$h_6(t) = \frac{Gm_e m_T}{|R|^3}Z. \qquad (2.131)$$

BUS DYNAMICS (ROTATION)

$$I_\tau \begin{pmatrix} \dot{\omega_1} \\ \dot{\omega_2} \\ \dot{\omega_3} \end{pmatrix} + \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} I_\tau \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} h_7(t) \\ h_8(t) \\ h_9(t) \end{pmatrix} + \begin{pmatrix} h_{10}(t) \\ h_{11}(t) \\ h_{12}(t) \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}, \qquad (2.132)$$

where

$$h_7(t) = \int_0^l \rho\Big\{y\frac{\partial^2 z}{\partial t^2} - z\frac{\partial^2 y}{\partial t^2} + 2\omega_1 y\frac{\partial y}{\partial t} + 2\omega_1 z\frac{\partial z}{\partial t}\Big\}dx, \qquad (2.133)$$

$$h_8(t) = \int_0^l \rho\Big\{-(x+R_s)\frac{\partial^2 z}{\partial t^2} + 2\omega_2 z\frac{\partial z}{\partial t} - 2\omega_3 z\frac{\partial y}{\partial t} - 2\omega_1(x+R_s)\frac{\partial y}{\partial t}\Big\}dx, \qquad (2.134)$$

$$h_9(t) = \int_0^l \rho\Big\{(x+R_s)\frac{\partial^2 y}{\partial t^2} - 2\omega_1(x+R_s)\frac{\partial z}{\partial t} - 2\omega_2 y\frac{\partial z}{\partial t} + 2\omega_3 y\frac{\partial y}{\partial t}\Big\}dx, \qquad (2.135)$$

$$h_{10}(t) = \int_0^l \rho\Big\{ya_k - za_j\Big\}dx, \qquad (2.136)$$

$$h_{11}(t) = \int_0^l \rho\Big\{za_i - (x+R_s)a_k\Big\}dx, \qquad (2.137)$$

$$h_{12}(t) = \int_0^l \rho\Big\{(x+R_s)a_j - ya_i\Big\}dx. \qquad (2.138)$$

BEAM DYNAMICS (BEAM VIBRATION)

$$\rho\frac{\partial^2}{\partial t^2}\begin{pmatrix} y \\ z \end{pmatrix}+\rho\begin{pmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} y \\ z \end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix} K_y & 0 \\ 0 & K_z \end{pmatrix}\begin{pmatrix} \psi_y-\frac{\partial}{\partial x}y \\ \psi_z-\frac{\partial}{\partial x}z \end{pmatrix}$$
$$+\rho\begin{pmatrix} -\omega_1^2-\omega_3^2 & -\dot{\omega_1}+\omega_2\omega_3 \\ \dot{\omega_1}+\omega_2\omega_3 & -\omega_1^2-\omega_2^2 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} \tag{2.139}$$
$$+\rho(x+R_s)\begin{pmatrix} \dot{\omega_3}+\omega_1\omega_2 \\ -\dot{\omega_2}+\omega_1\omega_3 \end{pmatrix}+\rho QC_I^B\frac{d^2R}{dt^2}=\begin{pmatrix} F_y \\ F_z \end{pmatrix},$$

$$I_\rho\frac{\partial^2}{\partial t^2}\begin{pmatrix} \psi_y \\ \psi_z \end{pmatrix}+I_\rho\begin{pmatrix} 0 & -w_1 \\ w_1 & 0 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} \psi_y \\ \psi_z \end{pmatrix}+\begin{pmatrix} K_y & 0 \\ 0 & K_z \end{pmatrix}\begin{pmatrix} \psi_y \\ \psi_z \end{pmatrix}$$
$$-\begin{pmatrix} K_y & 0 \\ 0 & K_z \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} y \\ z \end{pmatrix}-\frac{\partial}{\partial x}\begin{pmatrix} EI_y & 0 \\ 0 & EI_z \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} \psi_y \\ \psi_z \end{pmatrix}=\begin{pmatrix} F_{yT} \\ F_{zT} \end{pmatrix}. \tag{2.140}$$

BOUNDARY CONDITIONS

$$y(0,t)=0, \qquad z(0,t)=0, \tag{2.141}$$
$$\psi_y(0,t)=0, \qquad \psi_z(0,t). \tag{2.142}$$

$$EI_y\frac{\partial}{\partial x}\psi_y(l,t)=0, \qquad EI_z\frac{\partial}{\partial x}\psi_z(l,t)=0, \tag{2.143}$$
$$K_y\Big(\psi_y-\frac{\partial}{\partial x}y\Big)(l,t)=0, \qquad K_z\Big(\psi_z-\frac{\partial}{\partial x}z\Big)(l,t)=0. \tag{2.144}$$

Thus the complete dynamics of the flexible spacecraft consisting of the main rigid body and a flexible beam rigidly attached to it (Fig.2.1) is given by a *hybrid system* which consists of six nonlinear ordinary differential equations (2.125),(2.132) and the partial differential equations (2.139-2.140) along with the boundary conditions in (2.141-2.144).

We note from Equations (2.120-2.123) or (2.125-2.144) that the three sets of equations are strongly coupled. Thus any rotation of the bus would induce vibration in the beam and vice-versa. Also any disturbance of the orbit would induce vibrations in the beam and rotations in the bus and vice-versa. We have similar observations as (OBS.2.1) and (OBS.2.2) in Section 2.4 for the dynamics (2.125,2.132) for translational and rotational motions of the bus. Further from the beam equation (2.139-2.140) we note that

(OBS.2.5) The effect of rigid body rotation and translation on the vibration of the beam is represented by the forcing term $\rho Q(\alpha + \frac{d^2R}{dt^2})$ as shown in Equation (2.122). The effects of the shear displacement are shown in Equations (2.122) and (2.123) through the K_y and K_z terms. Thus it is clear that without the coupling effect $(\alpha + \frac{d^2R}{dt^2})$, Equations (2.122) and (2.123) become Timoshenko equations of a cantilever beam[Traill-Nash and Collar 131].

Remark 2.6

We have similar equations for the dynamics of the spacecraft as developed in the previous two sections where they are derived using Euler-Bernoulli and Rayleigh beam theories respectively. However, note that because of the effects of rotary inertia and shear displacement of an incremental section, the transverse vibration of the beam is governed by Equations (2.122-2.123) (or 2.139-2.140) with boundary conditions (2.141-2.144). In fact, the second term of Equation (2.140) or Equation (2.123) represent the effect of the rotary inertia and all the terms including K_y and K_z reflect the influence of the shear displacement on the transverse vibration of the beam. Hence this improved beam model enables us to study the influence of the beam dynamics including shear displacement on the dynamic performance such as rotation and translation of the rigid bus of the spacecraft.

Remark 2.7

Based on *Timoshenko beam theory* [Traill-Nash and Collar 131] in this section we have developed a complete dynamic model for the spacecraft (Fig.2.1) consisting of a rigid main body and a flexible beam through the variational formulation. In the derivation of equations of motion again we have approximated the gravitational potential energy P_g by Eqation (2.16) (details are given in Remark 2.1). Flexible structures in space are often subject to external disturbances caused by various sources such as those mentioned in Remark 2.2. The stabilization of the spacecraft under the influence of those disturbing forces is considered in Section 4.3.3.

2.7 Summary

In this chapter a *complete* dynamic model for the spacecraft consisting of a rigid body and a long flexible beam has been developed[Lim and Ahmed 82] using the extended Hamilton's principle. The equations of motion is given by a *Hybrid system* which is a combination of ordinary differential equations describing the translational and rotational motions of the rigid body and partial differential equations governing the vibration of the beam along with the boundary conditions. The equations are strongly coupled and thus any perturbation or disturbance in one part of the system will induce vibration or wobbling on the other members of the system. In the development of the dynamics of the beam three typical beam theories i.e., Euler-Bernoulli, Rayleigh and Timoshenko beam equations were utilized and the corresponding dynamic equations of the entire system have been developed. In all the cases, the equations of motion were expressed in the explicit forms which are suitable for computer simulation and system analysis.

Chapter 3

MODELING OF FLEXIBLE SPACE STATIONS

3.1 Introduction

In this chapter, we consider the problem of modeling of a space station consisting of a massive rigid body and flexible configurations. The rigid body may play the role of experimental modules located at the center of the space station. The flexible configurations may consist of several beams, forming the space structure shown in Fig.3.1. Taking into consideration the three types of motion such as translation and rotation of the rigid body and vibration of the flexible members, complete dynamic equations† of motion have been developed using extended Hamilton's principle together with three typical beam theories by the author and Ahmed[84]. The dynamic equations consist of ordinary differential equations describing the translational and rotational dynamics of the rigid body and partial differential equations governing the vibration of the flexible parts. To the knowledge of the author, this type of model for space stations has not been appeared in open literature. Even though the underlying principle is the same as for the case of spacecraft, the dynamic modeling for the space station is much more complex because of the flexible configurations of the system in which each member is strongly coupled with the other members through the geometric boundary conditions. In Section 3.2 the reference coordinate systems are introduced for the space station. In terms of the quantities defined in the body coordinate system, in Section 3.3 inertia tensor is defined. Following the same procedure as used in Section 2.4, equations of motion for the space station(Fig.3.1) based on Euler-Bernoulli beam theory has been developed in Section 3.4. Using Rayleigh

† see, page 3.

and Timoshenko beam theories, different form of dynamics have been developed in Section 3.5 and Section 3.6 respectively. Concluding remarks are presented in Section 3.7. In this chapter, if necessary in order to enhance readability of the thesis, we will redefine some variables and repeat a few notations which were already defined in Chapter 2.

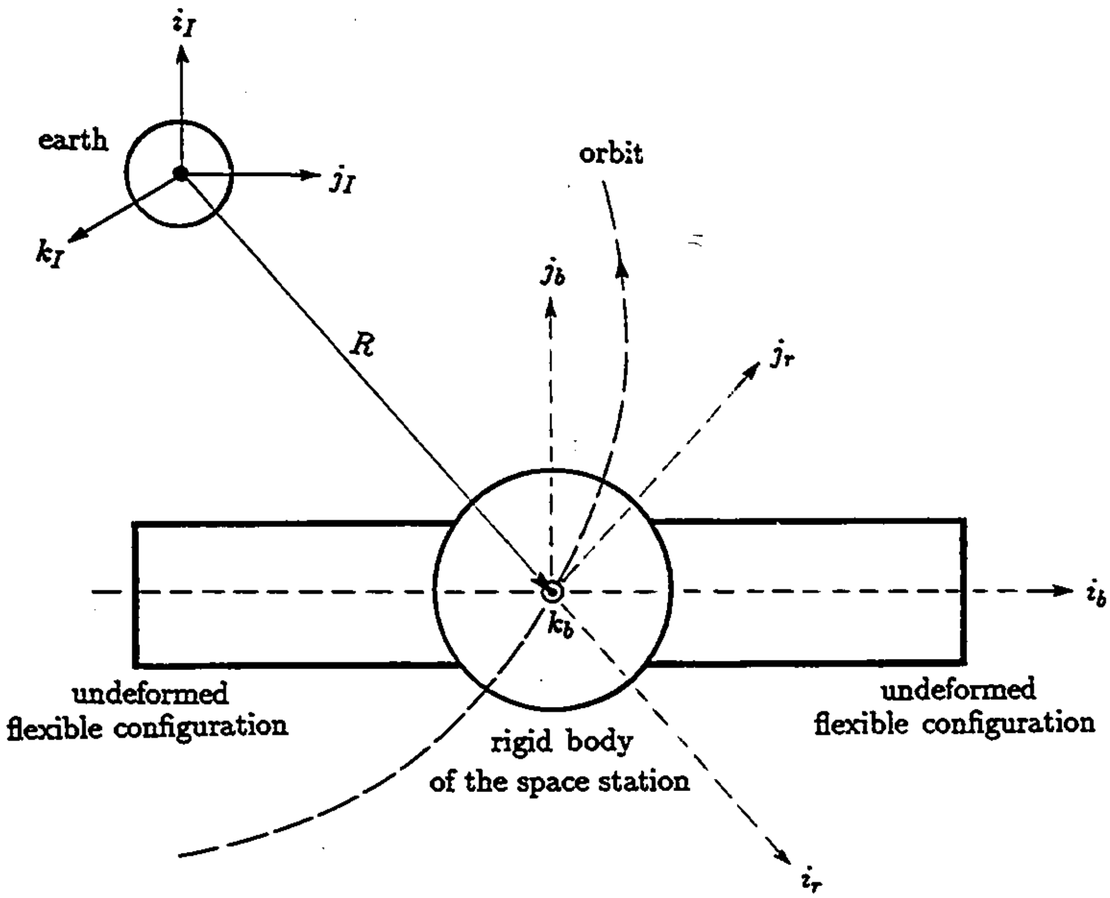


Fig.3.1 Schematic diagram of a space station and reference coordinate systems.

3.2 Reference Coordinate Systems

We introduce an orbital reference coordinate system S_r by the right handed vector triad (i_r, j_r, k_r) with the origin coinciding with the origin of the body frame defined in the below. The axis k_r is normal to the orbital plane, the axis i_r is along the line connecting the center of the earth and the mass center of the satellite and the axis j_r is such that (i_r, j_r, k_r) makes a right- handed vector triad. The positive direction of the i_r is taken along the radial vector R.

Let $w_b = (w_x, w_y, w_z)'$ be the angular velocity of the body frame with respect to the reference frame S_r and $w_r = (w_X, w_Y, w_Z)'$ be the angular velocity of the reference frame with respect to the inertial frame $S_I = (i_I, j_I, k_I)$. We shall denote $w = w_b + w_r = (w_1, w_2, w_3)'$. For simplicity of presentation in the sequel we call each flexible member of the space station as "*beam*" and it is designated by a number as shown in Fig.3.2.

We define a body coordinate system S_b represented by the right handed vector triad (i_b, j_b, k_b). This coordinate system will be assumed to be fixed to the space station body with the origin at the mass center of the rigid bus. We suppose that the rest positions of the beams 1,2,3 and 4 are parallel to the i_b axis while beams 5 and 6 are parallel to the j_b axis in xy plane in S_b. Let D_{i_0}, $i = 1, 2, ..., 6$ denote the displacement of the point $\mathcal{J}_i$ measured from the origin of the body frame in undeformed state and $D_i(\xi_i, t)$, i=1,2,...,6, the displacement (or peturbation) of a mass element on the beam i measured from the undeformed position of the beam in S_b. Let

$$Q_i \equiv \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for i=1,2,3,4,} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for i=5,6.} \end{cases} \tag{3.1}$$

Then the absolute position of the incremental mass element dm_i of the beam i, $i = 1, 2, \cdots, 6$, relative to the inertial frame can be defined by the vector

$$r_i = R + \bar{r}_i, \quad \text{with} \quad \bar{r}_i \equiv (\bar{x}_i, \bar{y}_i, \bar{z}_i)' = D_{i_0} + D_i, \tag{3.2}$$

where $R \equiv (X, Y, Z)'$ represents the position of the origin of S_b with respect to S_I and $D_{i_0} \equiv (x_{i_0}, y_{i_0}, z_{i_0})'$, i=1,2,...,6 defines the position of dm_i relative to S_b in the undeformed state and $D_i \equiv Q_i\ (x_i, y_i, z_i)'$, i=1,2,...,6 represents the displacement of dm_i from the undeformed position of the beam in S_b.

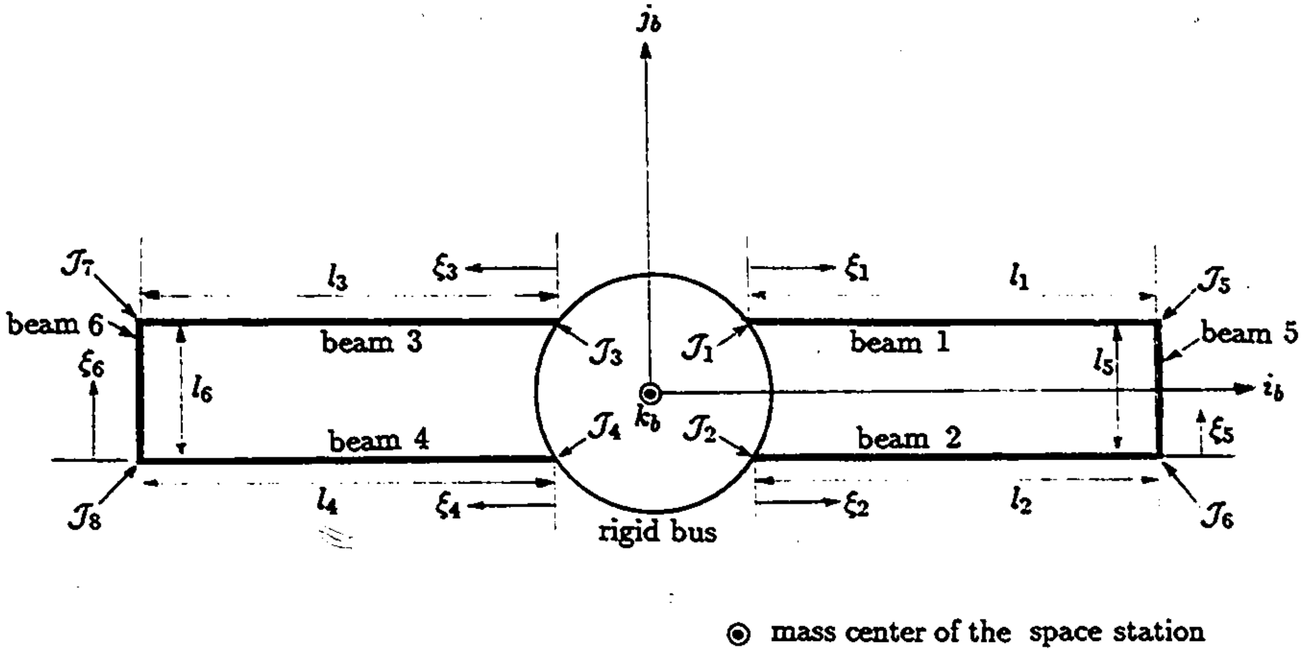


Fig. 3.2 Body coordinate system.

3.3 Inertia Tensor

Let dm_i denote the mass of a generic element on the beam i and given by $dm_i = \rho_i d\xi_i$ and the domain Ω_i of spatial variable ξ_i be defined by $\Omega_i \equiv (0, l_i)$, i=1,2,...,6 for l_i which is the length of the beam i measured from the point of attachment, as shown in Fig.3.2. With respect to Ω_i, D_{i_o} is represented by

$$D_{1_o} = ((s_{1_x} + \xi_1), s_{1_y}, 0)', \quad D_{2_o} = ((s_{2_x} + \xi_2), -s_{2_y}, 0)', \tag{3.3}$$

$$D_{3_o} = (-(s_{3_x} + \xi_3), s_{3_y}, 0)', \quad D_{4_o} = (-(s_{4_x} + \xi_4), -s_{4_y}, 0)', \tag{3.4}$$

$$D_{5_o} = (s_{5_x}, (-s_{5_y} + \xi_5), 0)', \quad D_{6_o} = (-s_{6_x}, (-s_{6_y} + \xi_6), 0)', \tag{3.5}$$

where s_{i_x} (respectively s_{i_y}) is the distance of the joints $\mathcal{J}_i$, in undeformed state of the beams, measured in i_b (respectively j_b) directions and constant. Then the beam inertia tensor with respect to the body frame is given by

$$I_b \equiv \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix} \tag{3.6}$$

where

$$I_{xx} = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i(\bar{y}_i^2 + \bar{z}_i^2) d\xi_i, \tag{3.7}$$

$$I_{yy} = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i(\bar{x}_i^2 + \bar{z}_i^2) d\xi_i, \tag{3.8}$$

$$I_{zz} = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i(\bar{x}_i^2 + \bar{y}_i^2) d\xi_i, \tag{3.9}$$

$$I_{xy} = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \bar{x}_i \bar{y}_i d\xi_i, \tag{3.10}$$

$$I_{xz} = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \bar{x}_i \bar{z}_i d\xi_i, \tag{3.11}$$

$$I_{yz} = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \bar{y}_i \bar{z}_i d\xi_i. \tag{3.12}$$

We suppose that the beams are inextensible, i.e. deflections of the beams 1,2,3 and 4 could occur only in the j_b and k_b directions while for the beams 5 and 6 they are given by in i_b and k_b directions. The corresponding deflections of the beam i=1,2,3 and 4 are denoted by $y_i = y_i(\xi_i, t)$, $z_i = z_i(\xi_i, t)$, $i = 1, 2, 3, 4$ and those of beam j are denoted by $x_j = x_j(\xi_j, t)$, $z_j = z_j(\xi_j, t)$, j=5,6 respectively. The dynamic equations governing x_i, y_i, z_i, $i = 1, 2, ..., 6$ are discussed later. It is clear from (3.6) that the beam inertia tensor I_b is a functional of the deflections x_i,y_i,z_i and consequently, an implicit function of time.

The inertia tensor of the rigid bus of the space station is denoted by I_s, which is not necessarily diagonal. However, for simplicity of presentation, we assume that I_s is not time varying and is positive definite.

3.4 Dynamic Modeling based on Euler-Bernoulli Beam Theory

In order to derive equations of motion, we follow the same procedure used in Section 2.4 with some modifications. The first step is to find a proper expression of the kinetic energy and potential energy of the system in terms of the quantities with reference to the coordinate systems. The kinetic energy of the system is given by Equation (2.6) with the components K_s and K_b, where K_s has the same expression as Equation (2.8) and K_b is given by

$$K_b = \sum_{i=1}^{6} K_{b_i} \tag{3.13}$$

The kinetic energy K_{b_i} of the beam i is given by

$$K_{b_i} = \frac{1}{2}\int_{\Omega_i} \frac{dr_i}{dt} \circ \frac{dr_i}{dt} dm_i. \tag{3.14}$$

where r_i denotes the absolute position of the incremental mass element dm_i of the beam i relative to the inertial frame and is defined in Equation (3.2).

Let $EI_i \equiv EI_i(\xi_i)$ denote the flexural rigidity of the beam i for deflections D_i and be defined by

$$EI_i \equiv \begin{pmatrix} EI_{ix} & 0 & 0 \\ 0 & EI_{iy} & 0 \\ 0 & 0 & EI_{iz} \end{pmatrix}, \quad i = 1, 2, ..., 6. \tag{3.15}$$

The system potential energy arises from two sourses, namely elastic and gravitational as in Equation (2.7). The elastic potential energy P_e due to deformations of the beams are a function of the partial derivatives of the displacements x_i, y_i, z_i with respect to the spatial variable ξ_i. Under the assumption of small bending deformations neglecting elongation or torsional effects, the elastic potential energy of the beams P_e is given by

$$P_e = \sum_{i=1}^{6} P_{e_i}, \tag{3.16}$$

where P_{e_i} is the potential energy of the beam i and given by

$$P_{e_i} = \frac{1}{2}\int_{\Omega_i}\Big\{EI_i\frac{\partial^2 D_i}{\partial \xi_i^{\,2}}\circ\frac{\partial^2 D_i}{\partial \xi_i^{\,2}}\Big\}d\xi_i. \tag{3.17}$$

The gravitational potential energy P_g given by the sum of the potential energy of the rigid bus and that of the beams as follows[Halliday and Resnick 51].

$$\begin{aligned} P_g &= P_s + P_b \\ &= \int_{bus}\frac{-Gm_e}{|\bar{r}|}d\bar{m} + \sum_{i=1}^{6}\int_{\Omega_i}\frac{-Gm_e}{|r_i|}dm_i, \end{aligned} \tag{3.18}$$

where G = universal gravitational constant, m_e = mass of the earth. In Equation (3.18) P_s and P_b denote the potential energy of the rigid bus and that of the beams respectively. P_g given in Equation (3.18) can be approximated by Equation (2.16) as explained in Section 2.4. Taking the first variation of the kinetic energy and the potential energy given by equations(2.6, 2.7) together with the external work W and substituting into Equation (2.4) with the ends conditions (2.5.1), (2.5.2),

$$\delta D_i \equiv Q_i(\delta x_i, \delta y_i, \delta z_i)' = (0,0,0)', \quad i = 1,2,...,6, \tag{3.19}$$

one can derive the dynamic equations of motion for the space station. This is considered in the sequel.

Assuming that the origin of the body frame coincides with the center of mass of the rigid bus, the first variation of K_s is given by Equation (2.21). The virtual displacement and velocity of the elastic body relative to the inertial space are given by[Cavin 32]

$$\delta r_i = \delta R + \delta\theta \times \bar{r}_i + \delta D_i, \tag{3.20}$$

$$\delta\Big(\frac{dr_i}{dt}\Big) = \delta\frac{dR}{dt} + \delta\omega \times \bar{r}_i + \delta\theta \times (\omega \times \bar{r}_i + \dot{D}_i) + \frac{d}{dt}(\delta D_i), \tag{3.21}$$

where $\delta\theta = (\delta\theta_1, \delta\theta_2, \delta\theta_3)'$ is the virtual rigid body rotation of the body frame relative to the reference frame and $\delta\omega = \frac{d}{dt}(\delta\theta)$ denotes the virtual angular velocity of the body frame. Substituting Equation (3.14) into Equation (3.13) and taking the first variation, we have

$$\begin{aligned}\delta \int_{t_1}^{t_2} K_b dt &= \delta \sum_{i=1}^{6} \int_{t_1}^{t_2} K_{b_i} dt \\ &= \sum_{i=1}^{6} \int_{t_1}^{t_2} \int_{\Omega_i} \left(\delta \frac{dR}{dt} \circ \frac{dr_i}{dt}\right) dm_i dt + \sum_{i=1}^{6} \int_{t_1}^{t_2} \int_{\Omega_i} \left\{\delta\omega \times \bar{r}_i\right\} \circ \frac{dr_i}{dt} dm_i dt \\ &+ \sum_{i=1}^{6} \int_{t_1}^{t_2} \int_{\Omega_i} \left\{\delta\theta \times (\omega \times \bar{r}_i + \dot{D}_i)\right\} \circ \frac{dr_i}{dt} dm_i dt \\ &+ \sum_{i=1}^{6} \int_{t_1}^{t_2} \int_{\Omega_i} \left\{\frac{d}{dt}(\delta D_i) \circ \frac{dr_i}{dt}\right\} dm_i dt \,. \end{aligned} \tag{3.22}$$

Using the vector identity (2.23), we obtain from Equation (3.22) that

$$\begin{aligned}\delta \int_{t_1}^{t_2} K_b dt = \sum_{i=1}^{6} \int_{t_1}^{t_2} \int_{\Omega_i} \Bigg[& \delta \frac{dR}{dt} \circ \frac{dr_i}{dt} + \delta\omega \circ \left\{\bar{r}_i \times \frac{dr_i}{dt}\right\} \\ & + \delta\theta \circ \left\{(\omega \times \bar{r}_i + \dot{D}_i) \times \frac{dr_i}{dt}\right\} + \frac{dr_i}{dt} \circ \frac{d}{dt}(\delta D_i)\Bigg] dm_i dt \,. \end{aligned} \tag{3.23}$$

One obtains from Equation (3.17) that

$$\delta \int_{t_1}^{t_2} P_e dt = \sum_{i=1}^{6} \int_{t_1}^{t_2} \int_{\Omega_i} EI_i \left(\frac{\partial^2 D_i}{\partial \xi_i^2}\right) \circ \frac{\partial^2}{\partial \xi_i^2} (\delta D_i) d\xi_i dt \,. \tag{3.24}$$

The first variation of P_g is given by Equation (2.28) or (2.29). Let the external force and torque applied to the bus be given by F_s and T as defined in Section 2.4 respectively, and external forces applied to the beams be given by

$$\bar{F}_{b_i} \equiv \begin{cases} (0, F_{i_y}, F_{i_z})', & \text{for } i=1,2,3,4, \\ (F_{i_x}, 0, F_{i_z})', & \text{for } i=5,6. \end{cases} \tag{3.25}$$

Then the virtual work[Meriam 99] done by the external forces F_s, T and $\bar{F}_{b_i}$ is given by

$$\delta W \;=\; F_s \circ \delta R \;+\; T \circ \delta\theta \;+\; \sum_{i=1}^{6} \int_{\Omega_i} \bar{F}_{b_i} \circ \delta D_i d\xi_i. \tag{3.26}$$

Substituting Equations (2.21), (3.23), (3.24), (2.29) and (3.26) into Equation (2.4) and integrating by parts with the end conditions in Equations (2.5.1), (2.5.2), (3.19) we obtain

$$\delta\int_{t_1}^{t_2}(L+W)dt \equiv -\int_{t_1}^{t_2}\Big\{(\delta R)\circ\mathcal{I}_1-(\delta\theta)\circ\mathcal{I}_2+\sum_{i=1}^{6}\int_{\Omega_i}(\delta D_i)\circ\mathcal{I}_3 d\xi_i+\mathcal{I}_4\Big\}dt=0, \tag{3.27}$$

where

$$\mathcal{I}_1=m_s\frac{d^2R}{dt^2}+\frac{d}{dt}\sum_{i=1}^{6}\int_{\Omega_i}\Big(\frac{dR}{dt}+\omega\times\bar{r}_i+\dot{D}_i\Big)dm_i+\frac{Gm_em_TR}{|R|^3}-F_s, \tag{3.28}$$

$$\mathcal{I}_2=T-\frac{d}{dt}\Big\{(I_s\circ\omega)+\sum_{i=1}^{6}\int_{\Omega_i}\bar{r}_i\times\frac{dr_i}{dt}dm_i\Big\}+\sum_{i=1}^{6}\int_{\Omega_i}\Big(\omega\times\bar{r}_i+\dot{D}_i\Big)\times\Big(\frac{dr_i}{dt}\Big)dm_i, \tag{3.29}$$

$$\mathcal{I}_3=\rho_i\frac{d^2r_i}{dt^2}+\frac{\partial^2}{\partial\xi_i{}^2}\Big(EI_i\frac{\partial^2 D_i}{\partial\xi_i{}^2}\Big)-\bar{F}_{b_i},\ i=1,2,...,6, \tag{3.30}$$

$$\mathcal{I}_4=\sum_{i=1}^{6}\Big\{\Big[EI_i\Big(\frac{\partial^2 D_i}{\partial\xi_i{}^2}\Big)\circ\frac{\partial}{\partial\xi_i}(\delta D_i)\Big]_{\Omega_i}-\Big[\frac{\partial}{\partial\xi_i}\Big(EI_i\frac{\partial^2 D_i}{\partial\xi_i{}^2}\Big)\circ(\delta D_i)\Big]_{\Omega_i}\Big\}, i=1,2,...,6. \tag{3.31}$$

Since the beams are assumed to be inextensible we have

$$\delta D_i\equiv\delta(Q_i(x_i,y_i,z_i)')=\begin{cases}(0,\delta y_i,\delta z_i)', & \text{for } i\text{=1,2,3,4},\\ (\delta x_i,0,\delta z_i)', & \text{for } i\text{=5,6}.\end{cases} \tag{3.32}$$

Therefore, noting that the variations $\delta\theta,\delta R,\delta D_i$, $i=1,2,...,6$ are arbitrary, we obtain, from Equation (3.27), that

$$\mathcal{I}_1=0, \tag{3.33}$$

$$\mathcal{I}_2=0, \tag{3.34}$$

$$\mathcal{I}_3=0, \tag{3.35}$$

$$\mathcal{I}_4=0. \tag{3.36}$$

One obtains from these equations a set of coupled differential equations for the dynamics of the flexible space station as follows. From Equation (3.33) we obtain the dynamics for the translational motion of the rigid bus:

$$(m_s + m_b)\frac{d^2R}{dt^2} + \frac{d}{dt}\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big(\omega\times\bar{r}_i + \dot{D}_i\Big)d\xi_i + \frac{Gm_e m_T R}{|R|^3} = F_s, \tag{3.37}$$

where $m_b = \sum_{i=1}^{6}\int_{\Omega_i} dm_i$ is the total mass of the beams and $\rho_i \equiv \rho_i(\xi_i)$ is the mass per unit length of the beam i. For rotational dynamics (attitude motion) of the bus it is obtained from Equation (3.34) that

$$\frac{d}{dt}\Big\{(I_s + I_b)\circ\omega\Big\} + \frac{d}{dt}\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\bar{r}_i\times\Big(\frac{dR}{dt} + \dot{D}_i\Big)d\xi_i$$
$$-\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{(\omega\times\bar{r}_i + \dot{D}_i)\times\frac{dR}{dt}\Big\}d\xi_i = T, \tag{3.38}$$

where $I_b\circ\omega = \sum_{i=1}^{6}\int\bar{r}_i\times(\omega\times\bar{r}_i)dm_i$. Let us define $I_T \equiv I_s + I_b$ for notational simplicity. Using vector identities we obtain (Appendix F):

$$\sum_{i=1}^{6}\int_{\Omega_i}\Big\{\bar{r}_i\times(2w\times\dot{D}_i)+\bar{r}_i\times(\dot{w}\times\bar{r}_i)\Big\}dm_i = I_b\circ\dot{w}+\dot{I}_b\circ w+\sum_{i=1}^{6}\int_{\Omega_i}w\times(\bar{r}_i\times\dot{D}_i)dm_i, \tag{3.39}$$

$$\sum_{i=1}^{6}\int_{\Omega_i}\bar{r}_i\times(w\times(w\times\bar{r}_i))dm_i = w\times(I_b\circ w). \tag{3.40}$$

Substituting Equations (3.39, 3.40) into Equation (3.38) we obtain

$$I_T\circ\dot{w}+w\times(I_T\circ w)+\dot{I}_b\circ w+\sum_{i=1}^{6}\int_{\Omega_i}\Big\{\bar{r}_i\times\Big(\frac{d^2R}{dt^2}+\ddot{D}_i\Big)+w\times(\bar{r}_i\times\dot{D}_i)\Big\}dm_i = T. \tag{3.41}$$

Since δD_i are given by Equation (3.32), from Equation (3.35) we obtain the dynamics for the beam vibration: for i=1,2,...,6

$$\rho_i\ddot{D}_i + \frac{\partial^2}{\partial\xi_i^2}\Big(EI_i\frac{\partial^2 D_i}{\partial\xi_i^2}\Big)+\rho_i Q_i\Big\{\frac{d^2R}{dt^2} + \dot{\omega}\times\bar{r}_i + \omega\times(2\dot{D}_i + \omega\times\bar{r}_i)\Big\}= \bar{F}_{b_i}. \tag{3.42}$$

Thus we have twelve second order partial differential equations for the dynamics of the flexible members. We need four boundary conditions to solve each beam equation and total number of boundary conditions for the solution of (3.42) is forty eight. The boundary conditions can be obtained from Equation (3.36) and the geometric configuration of the structure [Craig 37, Hertz and Crawley 54, Kounadis 75]. For boundary conditions we suppose that

(i) The structures are in $i_b j_b$ plane in undisturbed state,

(ii) The beams 1,2,3,4 are clamped at the joints $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$(Fig.3.2),

(iii) Displacements of the beams 1,2,...,6 are small,

(iv) The beams 5,6 are rigidly jointed with the other four beams as shown in Fig.3.2.

Let $\frac{\partial}{\partial \xi_i} D_i$, $\frac{\partial^2}{\partial \xi_i^2} D_i$, and $\frac{\partial^3}{\partial \xi_i^3} D_i$ be denoted by D_{ξ_i}, $D_{\xi_i \xi_i}$ and $D_{\xi_i \xi_i \xi_i}$ respectively. Then rewriting Equation (3.36) we obtain for any time t,

$$\mathcal{I}_4 \equiv \delta D_{\xi_1}(l_1) \circ EI_1 D_{\xi_1\xi_1}(l_1) + \delta D_{\xi_5}(l_5) \circ EI_5 D_{\xi_5\xi_5}(l_5) \tag{3.43}$$

$$+\delta D_{\xi_2}(l_2) \circ EI_2 D_{\xi_2\xi_2}(l_2) - \delta D_{\xi_5}(0) \circ EI_5 D_{\xi_5\xi_5}(0) \tag{3.44}$$

$$+\delta D_{\xi_3}(l_3) \circ EI_3 D_{\xi_3\xi_3}(l_3) + \delta D_{\xi_6}(l_6) \circ EI_6 D_{\xi_6\xi_6}(l_6) \tag{3.45}$$

$$+\delta D_{\xi_4}(l_4) \circ EI_4 D_{\xi_4\xi_4}(l_4) - \delta D_{\xi_6}(0) \circ EI_6 D_{\xi_6\xi_6}(0) \tag{3.46}$$

$$-\delta D_{\xi_1}(0) \circ EI_1 D_{\xi_1\xi_1}(0) - \delta D_{\xi_2}(0) \circ EI_2 D_{\xi_2\xi_2}(0) \tag{3.47}$$

$$-\delta D_{\xi_3}(0) \circ EI_3 D_{\xi_3\xi_3}(0) - \delta D_{\xi_4}(0) \circ EI_4 D_{\xi_4\xi_4}(0) \tag{3.48}$$

$$-\delta D_1(l_1) \circ EI_1 D_{\xi_1\xi_1\xi_1}(l_1) - \delta D_5(l_5) \circ EI_5 D_{\xi_5\xi_5\xi_5}(l_5) \tag{3.49}$$

$$-\delta D_2(l_2) \circ EI_2 D_{\xi_2\xi_2\xi_2}(l_2) + \delta D_5(0) \circ EI_5 D_{\xi_5\xi_5\xi_5}(0) \tag{3.50}$$

$$-\delta D_3(l_3) \circ EI_3 D_{\xi_3\xi_3\xi_3}(l_3) - \delta D_6(l_6) \circ EI_6 D_{\xi_6\xi_6\xi_6}(l_6) \tag{3.51}$$

$$-\delta D_4(l_4) \circ EI_4 D_{\xi_4\xi_4\xi_4}(l_4) + \delta D_6(0) \circ EI_6 D_{\xi_6\xi_6\xi_6}(0) \tag{3.52}$$

$$+\delta D_1(0) \circ EI_1 D_{\xi_1\xi_1\xi_1}(0) + \delta D_2(0) \circ EI_2 D_{\xi_2\xi_2\xi_2}(0) \tag{3.53}$$

$$+\delta D_3(0) \circ EI_3 D_{\xi_3\xi_3\xi_3}(0) + \delta D_4(0) \circ EI_4 D_{\xi_4\xi_4\xi_4}(0) = 0. \tag{3.54}$$

Since the displacement and slope vanish at a clamped boundary (i.e. $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$), we obtain the geometric boundary conditions:

$$\text{at } \xi_i = 0, \qquad y_i(0,t) = 0, \qquad z_i(0,t) = 0,$$
$$\frac{\partial y_i}{\partial \xi_i}(0,t) = 0, \qquad \frac{\partial z_i}{\partial \xi_i}(0,t) = 0, \quad \text{for } i = 1,2,3,4. \tag{3.55}$$

Since the displacements and slopes are continuous at the rigid joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$, we obtain:

Continuity of displacements at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$

$$x_5(l_5,t) = 0, \qquad z_1(l_1,t) = z_5(l_5,t), \quad at\ \mathcal{J}_5, \tag{3.56}$$
$$x_5(0,t) = 0, \qquad z_2(l_2,t) = z_5(0,t), \quad at\ \mathcal{J}_6, \tag{3.57}$$
$$x_6(l_6,t) = 0, \qquad z_3(l_3,t) = z_6(l_6,t), \quad at\ \mathcal{J}_7, \tag{3.58}$$
$$x_6(0,t) = 0, \qquad z_4(l_4,t) = z_6(0,t), \quad at\ \mathcal{J}_8, \tag{3.59}$$

Continuity of slopes at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$ (Appendix G):

$$\frac{\partial y_1}{\partial \xi_1}(l_1,t) = -\frac{\partial x_5}{\partial \xi_5}(l_5,t), \ at\ \mathcal{J}_5, \tag{3.60}$$
$$\frac{\partial y_2}{\partial \xi_2}(l_2,t) = -\frac{\partial x_5}{\partial \xi_5}(0,t), \ at\ \mathcal{J}_6, \tag{3.61}$$
$$\frac{\partial y_3}{\partial \xi_3}(l_3,t) = \frac{\partial x_6}{\partial \xi_6}(l_6,t), \ at\ \mathcal{J}_7, \tag{3.62}$$
$$\frac{\partial y_4}{\partial \xi_4}(l_4,t) = \frac{\partial x_6}{\partial \xi_6}(0,t), \ at\ \mathcal{J}_8. \tag{3.63}$$

Applying the moment equation at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$, we have

$$at\ \mathcal{J}_5, \quad EI_{5z}\frac{\partial^2 z_5}{\partial \xi_5^2}(l_5,t) = 0, \quad EI_{1z}\frac{\partial^2 z_1}{\partial \xi_1^2}(l_1,t) = 0, \tag{3.64}$$
$$at\ \mathcal{J}_6, \quad EI_{5z}\frac{\partial^2 z_5}{\partial \xi_5^2}(0,t) = 0, \quad EI_{2z}\frac{\partial^2 z_2}{\partial \xi_2^2}(l_2,t) = 0, \tag{3.65}$$
$$at\ \mathcal{J}_7, \quad EI_{6z}\frac{\partial^2 z_6}{\partial \xi_6^2}(l_6,t) = 0, \quad EI_{3z}\frac{\partial^2 z_3}{\partial \xi_3^2}(l_3,t) = 0, \tag{3.66}$$
$$at\ \mathcal{J}_8, \quad EI_{6z}\frac{\partial^2 z_6}{\partial \xi_6^2}(0,t) = 0, \quad EI_{4z}\frac{\partial^2 z_4}{\partial \xi_4^2}(l_4,t) = 0. \tag{3.67}$$

Using Equations (3.56) and (3.57) into Equation (3.49) and (3.51) respectively, we obtain, from their first components, that

$$EI_{1y}\frac{\partial^3 y_1}{\partial \xi_1^3}(l_1,t)=0, \qquad EI_{3y}\frac{\partial^3 y_3}{\partial \xi_3^3}(l_3,t)=0. \tag{3.68}$$

Similarly using eqs.(3.58) and (3.59) into equations (3.50) and (3.52) respectively we obtain, from their first elements, that

$$EI_{2y}\frac{\partial^3 y_2}{\partial \xi_2^3}(l_2,t)=0, \qquad EI_{4y}\frac{\partial^3 y_4}{\partial \xi_4^3}(l_4,t)=0. \tag{3.69}$$

Substituting Equations (3.56) and (3.57) into Equations (3.49) and (3.52) respectively, we obtain, from their second components, that

$$EI_{1z}\frac{\partial^3 z_1}{\partial \xi_1^3}(l_1,t)+EI_{5z}\frac{\partial^3 z_5}{\partial \xi_5^3}(l_5,t)=0, \tag{3.70}$$

$$EI_{3z}\frac{\partial^3 z_3}{\partial \xi_3^3}(l_3,t)+EI_{6z}\frac{\partial^3 z_6}{\partial \xi_6^3}(l_6,t)=0, \tag{3.71}$$

$$EI_{2z}\frac{\partial^3 z_2}{\partial \xi_2^3}(l_2,t)-EI_{5z}\frac{\partial^3 z_5}{\partial \xi_5^3}(0,t)=0, \tag{3.72}$$

$$EI_{4z}\frac{\partial^3 z_4}{\partial \xi_4^3}(l_4,t)-EI_{6z}\frac{\partial^3 z_6}{\partial \xi_6^3}(0,t)=0. \tag{3.73}$$

It is noted that the above equations (3.68-3.73) can be ontained from the force equations at the joints $\mathcal{J}_5$, $\mathcal{J}_6$, $\mathcal{J}_7$, $\mathcal{J}_8$. Substituting Equations (3.60) and (3.62) into Equations (3.43) and (3.45) respectively, we obtain, from their first elements, that

$$EI_{1y}\frac{\partial^2 y_1}{\partial \xi_1^2}(l_1,t)-EI_{5x}\frac{\partial^2 x_5}{\partial \xi_5^2}(l_5,t)=0, \tag{3.74}$$

$$EI_{3y}\frac{\partial^2 y_3}{\partial \xi_3^2}(l_3,t)+EI_{6x}\frac{\partial^2 x_6}{\partial \xi_6^2}(l_6,t)=0. \tag{3.75}$$

Similarly using Equations (3.61) and (3.63) into Equations (3.44) and (3.46) respectively, we obtain, from their first elements, that

$$EI_{2y}\frac{\partial^2 y_2}{\partial \xi_2^2}(l_2,t)+EI_{5x}\frac{\partial^2 x_5}{\partial \xi_5^2}(0,t)=0, \tag{3.76}$$

$$EI_{4y}\frac{\partial^2 y_4}{\partial \xi_4^2}(l_4,t) - EI_{6x}\frac{\partial^2 x_6}{\partial \xi_6^2}(0,t) = 0. \tag{3.77}$$

We note that the above equations (3.74)–(3.77) also can be obtained from moment equations at the joints $\mathcal{J}_5$, $\mathcal{J}_6$, $\mathcal{J}_7$, $\mathcal{J}_8$.

Let α_i be defined by $\alpha_i = \dot\omega \times \bar r_i + 2\omega \times \dot D_i + \omega \times (\omega \times \bar r_i)$, i=1,2,...,6, and

$$F_{b_i} \equiv \begin{cases} (F_{i_y}, F_{i_z})', & \text{for } i=1,2,3,4, \\ (F_{i_x}, F_{i_z})', & \text{for } i=5,6. \end{cases} \tag{3.78}$$

$$\tilde Q_i \equiv \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for } i=1,2,3,4, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for } i=5,6. \end{cases} \tag{3.79}$$

Then we finally obtain the complete dynamics of the space station subject to external forces in the following[Lim and Ahmed 86].

$$m_T\frac{d^2R}{dt^2} + \sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{(\dot\omega \times \bar r_i) + \omega \times (\omega \times \bar r_i) + \ddot D_i + 2\omega \times \dot D_i\Big\}d\xi_i + \frac{Gm_e m_T R}{|R|^3} = F_s, \tag{3.80}$$

$$I_T \circ \dot\omega + \omega \times (I_T \circ \omega) + \dot I_b \circ \omega + \sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big(\bar r_i \times \frac{d^2R}{dt^2}\Big)d\xi_i + \sum_{i=1}^{6}\int_{\Omega_i}\rho_i(\bar r_i \times \ddot D_i)d\xi_i + \sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{\omega \times (\bar r_i \times \dot D_i)\Big\}d\xi_i = T, \tag{3.81}$$

and for $\xi_i \in \Omega_i$, $\xi_j \in \Omega_j$; i=1,2,3,4, j=5,6,

$$\rho_i\frac{\partial^2}{\partial t^2}\begin{pmatrix} y_i \\ z_i \end{pmatrix} + \rho_i\tilde Q_i\Big(\alpha_i + \frac{d^2R}{dt^2}\Big) + \frac{\partial^2}{\partial {\xi_i}^2}\left(\begin{pmatrix} EI_{iy} & 0 \\ 0 & EI_{iz} \end{pmatrix}\frac{\partial^2}{\partial {\xi_i}^2}\begin{pmatrix} y_i \\ z_i \end{pmatrix}\right) = F_{b_i}, \tag{3.82}$$

$$\rho_j\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_j \\ z_j \end{pmatrix} + \rho_j\tilde Q_j\Big(\alpha_j + \frac{d^2R}{dt^2}\Big) + \frac{\partial^2}{\partial \xi_j^2}\left(\begin{pmatrix} EI_{jx} & 0 \\ 0 & EI_{jz} \end{pmatrix}\frac{\partial^2}{\partial \xi_j^2}\begin{pmatrix} x_j \\ z_j \end{pmatrix}\right) = F_{b_j}, \tag{3.83}$$

with the boundary conditions (3.55-3.77).

Using the same notations C_B^I, $(a_i, a_j, a_k)'$ as in Section 2.4 and performing the vector operations, Equations (3.80) −(3.83) and (3.55-3.77) could be expressed in explicit forms which are suitable for digital computer simulation[Lim and Ahmed 84]. These are given below.

BUS DYNAMICS(TRANSLATION)

$$m_T \frac{d^2}{dt^2}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + C_B^I \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} + \begin{pmatrix} h_4(t) \\ h_5(t) \\ h_6(t) \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \tag{3.84}$$

where

$$f_1(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ (\bar{z}_i \dot{\omega}_2 - \bar{y}_i \dot{\omega}_3) + 2(\omega_2 \frac{\partial z_i}{\partial t} - \omega_3 \frac{\partial y_i}{\partial t})$$
$$+ \omega_1 \omega_2 \bar{y}_i + w_1 w_3 \bar{z}_i - (w_2^2 + w_3^2)\bar{x}_i + \frac{\partial^2 x_i}{\partial t^2} \Big\} d\xi_i \tag{3.85}$$

$$f_2(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{x}_i \dot{w}_3 - \bar{z}_i \dot{w}_1 + 2(w_3 \frac{\partial x_i}{\partial t} - w_1 \frac{\partial z_i}{\partial t})$$
$$+ w_2 w_3 \bar{z}_i + w_2 w_1 \bar{x}_i - (w_3^2 + w_1^2)\bar{y}_i + \frac{\partial^2 y_i}{\partial t^2} \Big\} d\xi_i, \tag{3.86}$$

$$f_3(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{y}_i \dot{w}_1 - \bar{x}_i \dot{w}_2 + 2(w_1 \frac{\partial y_i}{\partial t} - w_2 \frac{\partial x_i}{\partial t})$$
$$+ w_3 w_1 \bar{x}_i + w_3 w_2 \bar{y}_i - (w_1^2 + w_2^2)\bar{z}_i + \frac{\partial^2 z_i}{\partial t^2} \Big\} d\xi_i, \tag{3.87}$$

and $h_4(t)$, $h_5(t)$, $h_6(t)$ are given by Equations (2.46.1), (2.46.2), (2.46.3), respectively.

BUS DYNAMICS(ROTATION)

$$I_r\begin{pmatrix}\dot{\omega_1}\\ \dot{\omega_2}\\ \dot{\omega_3}\end{pmatrix}+\begin{pmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{pmatrix}I_r\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}+\begin{pmatrix}f_4(t)\\ f_5(t)\\ f_6(t)\end{pmatrix}+\begin{pmatrix}f_7(t)\\ f_8(t)\\ f_9(t)\end{pmatrix}=\begin{pmatrix}T_1\\ T_2\\ T_3\end{pmatrix}, \tag{3.88}$$

where

$$f_4(t)=\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{\bar{y}_i\frac{\partial^2 z_i}{\partial t^2}-\bar{z}_i\frac{\partial^2 y_i}{\partial t^2}+2w_1\bar{y}_i\frac{\partial y_i}{\partial t}+2w_1\bar{z}_i\frac{\partial z_i}{\partial t}$$

$$-2w_2\bar{y}_i\frac{\partial x_i}{\partial t}-2w_3\bar{z}_i\frac{\partial x_i}{\partial t}\Big\}d\xi_i, \tag{3.89}$$

$$f_5(t)=\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{\bar{z}_i\frac{\partial^2 x_i}{\partial t^2}-\bar{x}_i\frac{\partial^2 z_i}{\partial t^2}+2w_2\bar{z}_i\frac{\partial z_i}{\partial t}+2w_2\bar{x}_i\frac{\partial x_i}{\partial t}$$

$$-2w_3\bar{z}_i\frac{\partial y_i}{\partial t}-2w_1\bar{x}_i\frac{\partial y_i}{\partial t}\Big\}d\xi_i, \tag{3.90}$$

$$f_6(t)=\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{\bar{x}_i\frac{\partial^2 y_i}{\partial t^2}-\bar{y}_i\frac{\partial^2 x_i}{\partial t^2}+2w_3\bar{x}_i\frac{\partial x_i}{\partial t}+2w_3\bar{y}_i\frac{\partial y_i}{\partial t}$$

$$-2w_1\bar{x}_i\frac{\partial z_i}{\partial t}-2w_2\bar{y}_i\frac{\partial z_i}{\partial t}\Big\}d\xi_i, \tag{3.91}$$

$$f_7(t)=\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{\bar{y}_i a_k-\bar{z}_i a_j\Big\}d\xi_i, \tag{3.92}$$

$$f_8(t)=\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{\bar{z}_i a_i-\bar{x}_i a_k\Big\}d\xi_i, \tag{3.93}$$

$$f_9(t)=\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{\bar{x}_i a_j-\bar{y}_i a_i\Big\}d\xi_i, \tag{3.94}$$

BEAM DYNAMICS(BEAM VIBRATION)

$$\rho_i \frac{\partial^2}{\partial t^2}\begin{pmatrix} y_i \\ z_i \end{pmatrix} + \rho_i \begin{pmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} y_i \\ z_i \end{pmatrix} + \frac{\partial^2}{\partial {\xi_i}^2}\left(\begin{pmatrix} EI_{iy} & 0 \\ 0 & EI_{iz} \end{pmatrix} \frac{\partial^2}{\partial {\xi_i}^2}\begin{pmatrix} y_i \\ z_i \end{pmatrix}\right)$$

$$+ \rho_i \begin{pmatrix} -\omega_1^2 - \omega_3^2 & -\dot{\omega}_1 + \omega_2\omega_3 \\ \dot{\omega}_1 + \omega_2\omega_3 & -\omega_1^2 - \omega_2^2 \end{pmatrix}\begin{pmatrix} \bar{y}_i \\ \bar{z}_i \end{pmatrix} + \rho_i \bar{x}_i \begin{pmatrix} \dot{\omega}_3 + \omega_1\omega_2 \\ -\dot{\omega}_2 + \omega_1\omega_3 \end{pmatrix}$$

$$+ \rho_i \tilde{Q}_i C_I^B \frac{d^2 R}{dt^2} = \begin{pmatrix} F_{i_y} \\ F_{i_z} \end{pmatrix}, \qquad \text{for } \xi_i \in \Omega_i,\ i = 1,2,3,4, \tag{3.95}$$

$$\rho_j \frac{\partial^2}{\partial t^2}\begin{pmatrix} x_j \\ z_j \end{pmatrix} + \rho_j \begin{pmatrix} 0 & 2\omega_2 \\ -2\omega_2 & 0 \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} x_j \\ z_j \end{pmatrix} + \frac{\partial^2}{\partial \xi_j^2}\left(\begin{pmatrix} EI_{jx} & 0 \\ 0 & EI_{jz} \end{pmatrix} \frac{\partial^2}{\partial \xi_j^2}\begin{pmatrix} x_j \\ z_j \end{pmatrix}\right)$$

$$+ \rho_j \begin{pmatrix} -w_2^2 - w_3^2 & \dot{w}_2 + w_3 w_1 \\ -\dot{w}_2 + w_3 w_1 & -w_1^2 - w_2^2 \end{pmatrix}\begin{pmatrix} \bar{x}_j \\ \bar{z}_j \end{pmatrix} + \rho_j \bar{y}_j \begin{pmatrix} -\dot{w}_3 + w_2 w_1 \\ \dot{w}_1 + w_2 w_3 \end{pmatrix}$$

$$+ \rho_j \tilde{Q}_j C_I^B \frac{d^2 R}{dt^2} = \begin{pmatrix} F_{j_x} \\ F_{j_z} \end{pmatrix}, \qquad \text{for } \xi_j \in \Omega_j,\ j = 5,6. \tag{3.96}$$

BOUNDARY CONDITIONS FOR BEAM DYNAMICS

Clamped at $\xi_i = 0$, i=1,2,3,4 for the joints $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$:

$$y_i(0,t) = 0, \qquad z_i(0,t) = 0, \tag{3.97}$$

$$\frac{\partial y_i}{\partial \xi_i}(0,t) = 0, \qquad \frac{\partial z_i}{\partial \xi_i}(0,t) = 0, \tag{3.98}$$

Continuity of displacements at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$:

$$x_5(l_5,t) = 0,\ z_1(l_1,t) = z_5(l_5,t),\ at\ \mathcal{J}_5, \tag{3.99}$$

$$x_5(0,t) = 0,\ z_2(l_2,t) = z_5(0,t),\ at\ \mathcal{J}_6, \tag{3.100}$$

$$x_6(l_6,t) = 0,\ z_3(l_3,t) = z_6(l_6,t),\ at\ \mathcal{J}_7, \tag{3.101}$$

$$x_6(0,t) = 0,\ z_4(l_4,t) = z_6(0,t),\ at\ \mathcal{J}_8, \tag{3.102}$$

Continuity of slopes at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$:

$$\frac{\partial y_1}{\partial \xi_1}(l_1,t) = -\frac{\partial x_5}{\partial \xi_5}(l_5,t), \ at\ \mathcal{J}_5, \tag{3.103}$$

$$\frac{\partial y_2}{\partial \xi_2}(l_2,t) = -\frac{\partial x_5}{\partial \xi_5}(0,t), \ at\ \mathcal{J}_6, \tag{3.104}$$

$$\frac{\partial y_3}{\partial \xi_3}(l_3,t) = \frac{\partial x_6}{\partial \xi_6}(l_6,t), \ at\ \mathcal{J}_7, \tag{3.105}$$

$$\frac{\partial y_4}{\partial \xi_4}(l_4,t) = \frac{\partial x_6}{\partial \xi_6}(0,t), \ at\ \mathcal{J}_8, \tag{3.106}$$

Equilibrium equations:

$$at\ \mathcal{J}_5, \quad EI_{5z}\frac{\partial^2 z_5}{\partial \xi_5^2}(l_5,t) = 0, \quad EI_{1z}\frac{\partial^2 z_1}{\partial \xi_1^2}(l_1,t) = 0, \tag{3.107}$$

$$at\ \mathcal{J}_6, \quad EI_{5z}\frac{\partial^2 z_5}{\partial \xi_5^2}(0,t) = 0, \quad EI_{2z}\frac{\partial^2 z_2}{\partial \xi_2^2}(l_2,t) = 0, \tag{3.108}$$

$$at\ \mathcal{J}_7, \quad EI_{6z}\frac{\partial^2 z_6}{\partial \xi_6^2}(l_6,t) = 0, \quad EI_{3z}\frac{\partial^2 z_3}{\partial \xi_3^2}(l_3,t) = 0, \tag{3.109}$$

$$at\ \mathcal{J}_8, \quad EI_{6z}\frac{\partial^2 z_6}{\partial \xi_6^2}(0,t) = 0, \quad EI_{4z}\frac{\partial^2 z_4}{\partial \xi_4^2}(l_4,t) = 0, \tag{3.110}$$

$$EI_{1y}\frac{\partial^3 y_1}{\partial \xi_1^3}(l_1,t) = 0, \qquad EI_{3y}\frac{\partial^3 y_3}{\partial \xi_3^3}(l_3,t) = 0, \tag{3.111}$$

$$EI_{2y}\frac{\partial^3 y_2}{\partial \xi_2^3}(l_2,t) = 0, \qquad EI_{4y}\frac{\partial^3 y_4}{\partial \xi_4^3}(l_4,t) = 0, \tag{3.112}$$

$$EI_{1z}\frac{\partial^3 z_1}{\partial \xi_1^3}(l_1,t) + EI_{5z}\frac{\partial^3 z_5}{\partial \xi_5^3}(l_5,t) = 0, \tag{3.113}$$

$$EI_{3z}\frac{\partial^3 z_3}{\partial \xi_3^3}(l_3,t) + EI_{6z}\frac{\partial^3 z_6}{\partial \xi_6^3}(l_6,t) = 0, \tag{3.114}$$

$$EI_{2z}\frac{\partial^3 z_2}{\partial \xi_2^3}(l_2,t) - EI_{5z}\frac{\partial^3 z_5}{\partial \xi_5^3}(0,t) = 0, \tag{3.115}$$

$$EI_{4z}\frac{\partial^3 z_4}{\partial \xi_4^3}(l_4,t) - EI_{6z}\frac{\partial^3 z_6}{\partial \xi_6^3}(0,t) = 0, \tag{3.116}$$

$$EI_{1y}\frac{\partial^2 y_1}{\partial \xi_1^2}(l_1,t) - EI_{5x}\frac{\partial^2 x_5}{\partial \xi_5^2}(l_5,t) = 0, \tag{3.117}$$

$$EI_{3y}\frac{\partial^2 y_3}{\partial \xi_3^2}(l_3,t) + EI_{6x}\frac{\partial^2 x_6}{\partial \xi_6^2}(l_6,t) = 0, \tag{3.118}$$

$$EI_{2y}\frac{\partial^2 y_2}{\partial \xi_2^2}(l_2,t) + EI_{5x}\frac{\partial^2 x_5}{\partial \xi_5^2}(0,t) = 0, \tag{3.119}$$

$$EI_{4y}\frac{\partial^2 y_4}{\partial \xi_4^2}(l_4,t) - EI_{6x}\frac{\partial^2 x_6}{\partial \xi_6^2}(0,t) = 0. \tag{3.120}$$

The complete dynamics of the flexible space station consisting of the main rigid body and flexible structures(or beams) rigidly attached to it (Fig.3.1) is given by a *hybrid system* which consists of nonlinear ordinary differential equations (3.84),(3.88) and the partial differential equations (3.95—3.96) along with the boundary conditions in Equations (3.97)— (3.120).

We note from Equations (3.80-3.83) or (3.84-3.96) that the three sets of equations are strongly coupled. Thus any rotation of the bus would induce vibration in the beams and vice-versa. Also any disturbance of the orbit would induce vibrations in the beams and rotations in the bus and vice-versa. Further, one can observe the following facts:

(OBS.3.1) The integrals in the orbital dynamics (3.80) (or f_1, f_2, f_3 in (3.84)) represent the perturbation force due to vibration of the beam and rotation of the bus. Thus if we bring the integral terms to the right hand side, the resulting equation has well-known form of perturbed orbital dynamics (see Eq. (11.1.1) of [Danby 39]).

(OBS.3.2) The integrals, involving the last three terms of the left hand side of Equation (3.81), represent the contribution of the vibration of the beams and orbital disturbance to the total angular momentum of the system. Clearly, in Equation (3.81) (or 3.88) the total inertia matrix I_T varies only with the displacements of the beams, whereas f_4, f_5 and f_6 depend also on the velocity and acceleration of the beam elements. If the beams are all rigid, the relative displacement, velocity and acceleration of the beam elements with respect to the body frame are all zero. Consequently, I_T becomes constant and f_4, f_5, f_6 reduce to zero. In this situation the resulting form of Equation (3.88) becomes the well known attitude dynamics of a rigid body satellite (Eq. (7.1.2) of [Thomson 129]).

(OBS.3.3) The effect of rigid body rotation and translation on the vibration of the beams is represented by the forcing terms $\rho_i\tilde{Q}_i(\alpha_i + \frac{d^2R}{dt^2})$ and $\rho_j\tilde{Q}_j(\alpha_j + \frac{d^2R}{dt^2})$ as shown in Equations (3.82) and (3.83). Thus it is clear that without the coupling effects $(\alpha_i+\frac{d^2R}{dt^2})$ and $(\alpha_j+\frac{d^2R}{dt^2})$, Equations (3.82-3.83) become Euler-Bernoulli equations of simple beams[Timoshenko et al 130].

We have a similar set of equations for the dynamics of the space station (Fig.3.1) as that developed in Section 2.4, except for the beam equations and boundary conditions. However, we note that each of the beam equations (3.95, 3.96) for the flexible structures of the space station is coupled with other beam equations through boundary conditions (3.97-3.120). Further, the boundary conditions (3.97-3.120) for the beams of the space station are very complex and coupled one another while the boundary conditions (2.37) for the beam of the spacecraft(Fig.2.1) are simple and well known. Hence this model enables us to study the coupling effects among the vibrations of the flexible structures, rotation of the central body and translational motion of the central rigid body. The terms $f_1(t)$, $f_2(t)$,$f_3(t)$ of Equation (3.84) represent the contribution of the vibration of the beams and rotation of the main body to the translational motion of the mass center of the rigid body. The influence of the radial perturbation and slew maneuvers on the dynamic performance of the entire system is investigated by numerical simulations given in Section 5.4.

Remark 3.1

Using *Euler-Bernoulli beam theory* in this section we have developed a complete dynamic model for the space station consisting of a rigid main body and flexible structures through the variational formulation. Each member of the flexible structures (assumed to have truss configuration) was treated as a simple beam which can be characterized by equivalent beam parameters of the actual truss or structure[Dow et al 42, Juang and Sun 68, Renton 113, Sun et al 122]. To find the equivalent beam parameters of the actual structure is an identification problem. The parameter identification problem has been treated recently in literature[Banks and Rosen 21, Banks and Wang 22, Hendricks et al 53, Hossain 58, Lee 81, Rajaram and Junkins 112]. The dynamic equations of motion for the space station (Fig.3.1) using *Rayleigh and Timoshenko beam theories* are developed in the next two sections.

Remark 3.2

In our derivation of equations of motion we have approximated the gravitational potential energy P_g by Equation (2.16). By using more precise expression for the gravity, for example Eq.(2.15) or Eq. (A7), one can also study the effects of the gravity gradient torques on the rotation of the bus and vibration of the beams. Details are given in Remark 2.1. Orbiting space stations are often expected to experience several disturbing forces such as those caused by movements of crews or balancing masses, shuttle takeoff and docking as well as the disturbances mentioned in Remark 2.2. This may induce appreciable perturbations to the dynamic performance of space station. In Section 5.4, we present some numerical results illustrating the impact of radial perturbation and slew maneuvers on the motion of the space station. From the results, one can observe that the impact of the external perturbation on the beam vibration and bus rotation is quite significant and it should be eliminated by appropriate stabilizing controls. The stabilization of the space station under the influence of such perturbing forces is considered in Section 4.4.

3.5 Dynamic Modeling based on Rayleigh Beam Theory

In this section, using the coordinate systems defined in Section 3.2 and following the procedure used in Section 3.4, we derive equations of motion for the space station shown in Fig.3.1, based on *Rayleigh beam theory*. We use the same notations and definitions as in Section 3.4 throuthout this section, unless otherwise stated. The equations of motion derived using Rayleigh beam theory have different forms compared with those based on Euler-Bernoulli beam theory, because the kinetic energy of the beam has different representation by Rayleigh beam theory. For the derivation of equations of motion the first step is to find a proper expression of the kinetic energy and potential energy of the system in terms of the quantities with respect to the coordinate systems. The kinetic energy of the system is given by Equation (2.6) with the components K_s and K_{b_R} in place of K_b, where K_s is the same as in Equation (2.8) and K_{b_R} is given by

$$K_{b_R} = \sum_{i=1}^{6} K_{b_{R_i}} \tag{3.121}$$

The kinetic energy $K_{b_{R_i}}$ of the beam i is given by[Russell 116]

$$K_{b_{R_i}} = \frac{1}{2}\int_{\Omega_i} \frac{dr_i}{dt} \circ \frac{dr_i}{dt} dm_i + \frac{1}{2}\int_{\Omega_i} I_{\rho_i} \frac{\partial^2}{\partial \xi_i \partial t} D_i \circ \frac{\partial^2}{\partial \xi_i \partial t} D_i d\xi_i, \tag{3.122}$$

with

$$r_i = R + \bar{r}_i \;=\; R + D_{i_0} + D_i, \tag{3.123}$$

where r_i denotes the absolute position of the incremental mass element dm_i of the beam i relative to the inertial frame, D_{i_0} defines the position of the mass element dm_i relative to the body frame in the undeformed state and given by Equations (3.3-3.5). Further, D_i represents the displacement of dm_i from the undeformed position of the beam in S_B and given by Equation (3.2) and $I_{\rho_i} \equiv I_{\rho_i}(\xi_i)$ is the rotary inertia of the beam element dm_i.

Let $EI_i \equiv EI_i(\xi_i)$ denote the flexural rigidity of the beam i for deflections D_i and be defined by Equation (3.15). The system potential energy arises from two sourses, namely elastic and gravitational as in Equation (2.7). The elastic potential energy P_e due to deformations of the beams are a function of the partial derivatives of the displacements x_i, y_i, z_i with respect to the spatial variable ξ_i. Under the assumption of small bending deformations neglecting elongation or torsional effects, the elastic potential energy of the beams P_e is given by Equations (3.16) and (3.17). The gravitational potential energy P_g is given by the sum of the potential energy of the rigid bus and that of the beams P_g is given by Equation (3.18) and approximated by Equation (2.16). Taking the first variation of the kinetic energy and the potential energy given by Equations(2.6, 2.7) together with the external work W and substituting into Equation (2.4) with the ends conditions (2.5.1), (2.5.2),(3.19) one can derive the dynamic equations of motion for the space station. This is sonsidered in the sequel.

Assuming that the origin of the body frame coincides with the center of mass of the rigid bus the kinetic energy of the bus is given by Equation (2.20) and its first variation by Equation (2.21). Substituting Equation (3.122) into Equation (3.121) and taking its first variation, we have

$$\begin{aligned}
\delta\int_{t_1}^{t_2} K_{b_R} dt &= \delta\sum_{i=1}^{6}\int_{t_1}^{t_2} K_{b_{R_i}} dt \\
&= \sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\left(\delta\frac{dR}{dt}\circ\frac{dr_i}{dt}\right)dm_i dt + \sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\left\{\delta\omega\times\bar{r}_i\right\}\circ\frac{dr_i}{dt}dm_i dt \\
&+ \sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\left\{\delta\theta\times(\omega\times\bar{r}_i+\dot{D}_i)+\frac{d}{dt}(\delta D_i)\right\}\circ\frac{dr_i}{dt}dm_i dt \\
&+ \sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\left\{(\delta D_i)\circ\frac{\partial}{\partial\xi_i}\left(I_{\rho_i}\frac{d}{dt}\frac{\partial^2}{\partial\xi_i\partial t}D_i\right)\right\}d\xi_i dt \\
&- \sum_{i=1}^{6}\int_{t_1}^{t_2}(\delta D_i)\circ I_{\rho_i}\frac{d}{dt}\frac{\partial^2}{\partial\xi_i\partial t}D_i\Big|_{\Omega_i} dt.
\end{aligned}\tag{3.124}$$

Using the vector identity (2.23), we obtain from Equation (3.124) that

$$\begin{aligned}
\delta\int_{t_1}^{t_2} K_{b_R} dt = \sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\Big[&\delta\frac{dR}{dt}\circ\frac{dr_i}{dt}+\delta\omega\circ\left\{\bar{r}_i\times\frac{dr_i}{dt}\right\} \\
&+(\delta\theta)\circ\left\{(w\times\bar{r}_i+\dot{D}_i)\times\frac{dr_i}{dt}\right\}+\frac{dr_i}{dt}\circ\frac{d}{dt}(\delta D_i)\Big]dm_i\, dt
\end{aligned}$$

$$+\sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\Big\{(\delta D_i)\circ\frac{\partial}{\partial\xi_i}\Big(I_{\rho_i}\frac{d}{dt}\frac{\partial^2}{\partial\xi_i\partial t}D_i\Big)\Big\}d\xi_i dt$$

$$-\sum_{i=1}^{6}\int_{t_1}^{t_2}(\delta D_i)\circ I_{\rho_i}\frac{d}{dt}\frac{\partial^2}{\partial\xi_i\partial t}D_i\Big|_{\Omega_i}dt. \tag{3.125}$$

The first variation of P_g is given by Equation (2.28) or Equation (2.29) and that of P_e is given by (3.24). Let the external force and torque applied to the bus be given by F_s and T as defined in Section 2.4 respectively, and external forces applied to the beams be given by

$$\bar{F}_{b_i} \equiv \begin{cases} (0, F_{i_y}, F_{i_z})', & \text{for } i\text{=1,2,3,4,} \\ (F_{i_x}, 0, F_{i_z})', & \text{for } i\text{=5,6.} \end{cases} \tag{3.126}$$

Then the virtual work[Meriam 99] done by the external forces F_s, T and $\bar{F}_{b_i}$ is given by

$$\delta W \;=\; F_s\circ\delta R \;+\; T\circ\delta\theta \;+\; \sum_{i=1}^{6}\int_{\Omega_i}\bar{F}_{b_i}\circ\delta D_i d\xi_i. \tag{3.127}$$

Substituting Equations (2.21), (3.125), (3.24), (2.29) and (3.127) into Equation (2.4) and integrating by parts with the end conditions in Equations (2.5.1), (2.5.2), (3.19) we obtain

$$\delta\int_{t_1}^{t_2}(L+W)dt \equiv -\int_{t_1}^{t_2}\Big\{(\delta R)\circ\mathcal{I}_1-(\delta\theta)\circ\mathcal{I}_2+\sum_{i=1}^{6}\int_{\Omega_i}(\delta D_i)\circ\mathcal{I}_3 d\xi_i+\sum_{i=1}^{6}\mathcal{I}_4\Big\}dt = 0, \tag{3.128}$$

where

$$\mathcal{I}_1 = m_s\frac{d^2R}{dt^2}+\frac{d}{dt}\sum_{i=1}^{6}\int_{\Omega_i}\Big(\frac{dR}{dt}+\omega\times\bar{r}_i+\dot{D}_i\Big)dm_i+\frac{Gm_e m_\tau R}{|R|^3}-F_s, \tag{3.129}$$

$$\mathcal{I}_2 = T-\frac{d}{dt}\Big\{(I_s\circ\omega)+\sum_{i=1}^{6}\int_{\Omega_i}\bar{r}_i\times\frac{dr_i}{dt}dm_i\Big\}+\sum_{i=1}^{6}\int_{\Omega_i}\Big(\omega\times\bar{r}_i+\dot{D}_i\Big)\times\Big(\frac{dr_i}{dt}\Big)dm_i, \tag{3.130}$$

$$\mathcal{I}_3 = \rho_i\frac{d^2r_i}{dt^2}-\frac{\partial}{\partial\xi_i}\Big(I_{\rho_i}\frac{d}{dt}\frac{\partial^2}{\partial\xi_i\partial t}D_i\Big)+\frac{\partial^2}{\partial{\xi_i}^2}\Big(EI_i\frac{\partial^2 D_i}{\partial{\xi_i}^2}\Big)-\bar{F}_{b_i}, \tag{3.131}$$

$$\mathcal{I}_4 = \Big[EI_i\Big(\frac{\partial^2 D_i}{\partial {\xi_i}^2}\Big)\circ\frac{\partial}{\partial \xi_i}(\delta D_i) + \Big\{I_{\rho_i}\frac{d}{dt}\frac{\partial^2}{\partial\xi_i\partial t}D_i - \frac{\partial}{\partial\xi_i}\Big(EI_i\frac{\partial^2 D_i}{\partial{\xi_i}^2}\Big)\Big\}\circ(\delta D_i)\Big]_{\Omega_i} \tag{3.132}$$

where $i = 1, 2, ..., 6$.

Since the beam is assumed to be inextensible we have

$$\delta D_i \equiv \delta(Q_i(x_i, y_i, z_i)' = \begin{cases} (0, \delta y_i, \delta z_i)', & \text{for } i\text{=1,2,3,4,} \\ (\delta x_i, 0, \delta z_i)', & \text{for } i\text{=5,6.} \end{cases} \tag{3.133}$$

Therefore, noting that the variations $\delta\theta, \delta R, \delta D_i$, $i = 1, 2, ..., 6$ are arbitrary, we obtain, from Equation (3.128), that

$$\mathcal{I}_1 = 0, \tag{3.134}$$

$$\mathcal{I}_2 = 0, \tag{3.135}$$

$$\mathcal{I}_3 = 0, \tag{3.136}$$

$$\mathcal{I}_4 = 0. \tag{3.137}$$

One obtains from these equations a set of coupled differential equations for the dynamics of the flexible space station as follows. From Equation (3.134) we obtain the dynamics for the translational motion of the rigid bus:

$$(m_s + m_b)\frac{d^2R}{dt^2} + \frac{d}{dt}\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big(\omega\times\bar{r}_i + \dot{D}_i\Big)d\xi_i + \frac{Gm_e m_T R}{|R|^3} = F_s, \tag{3.138}$$

where $m_b = \sum_{i=1}^{6}\int_{\Omega_i} dm_i$ is the total mass of the beams and $\rho_i \equiv \rho_i(\xi_i)$ is the mass per unit length of the beam i. For rotational dynamics (attitude motion) of the bus it is obtained from Equation (3.135) that

$$\frac{d}{dt}\Big\{(I_s + I_b)\circ\omega\Big\} + \frac{d}{dt}\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\bar{r}_i\times\Big(\frac{dR}{dt} + \dot{D}_i\Big)d\xi_i$$
$$-\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{(\omega\times\bar{r}_i + \dot{D}_i)\times\frac{dR}{dt}\Big\}d\xi_i = T, \tag{3.139}$$

where $I_b \circ \omega = \sum_{i=1}^{6} \int \bar{r}_i \times (\omega \times \bar{r}_i) dm_i$. Let define $I_T \equiv I_s + I_b$ for notational simplicity. Substituting Equations (3.39, 3.40) into Equation (3.139) we obtain

$$I_T \circ \dot{w} + w \times (I_T \circ w) + \dot{I}_{b_i} \circ w + \sum_{i=1}^{6} \int_{\Omega_i} \left\{ \bar{r}_i \times \left(\frac{d^2 R}{dt^2} + \ddot{D}_i \right) + w \times (\bar{r}_i \times \dot{D}_i) \right\} dm_i = T. \quad (3.140)$$

Let

$$F_{b_i} \equiv \begin{cases} (F_{i_y}, F_{i_z})', & \text{for } i{=}1,2,3,4, \\ (F_{i_x}, F_{i_z})', & \text{for } i{=}5,6. \end{cases} \quad (3.141)$$

Since δD_i are given by Equation (3.133), from Equation (3.136) we obtain the dynamics for the beam vibration:

$$\rho_i \ddot{D}_i - \frac{\partial}{\partial \xi_i} \Big(I_{\rho_i} \frac{d}{dt} \frac{\partial^2}{\partial \xi_i \partial t} D_i \Big) + \frac{\partial^2}{\partial {\xi_i}^2} \Big(EI_i \frac{\partial^2 D_i}{\partial {\xi_i}^2} \Big)$$

$$+ \rho_i Q_i \Big[\frac{d^2 R}{dt^2} + \dot{\omega} \times \bar{r}_i + \omega \times \Big\{ 2\dot{D}_i + \omega \times \bar{r}_i \Big\} \Big] = F_{b_i}. \quad (3.142)$$

Thus we have twelve second order partial differential equations for the dynamics of the flexible members. We need four boundary conditions to solve each beam equation and total number of boundary conditions for the solution of Equation (3.142) is forty eight. The boundary conditions can be obtained from Equation (3.137) and the geometric configuration of the structure[Craig 37, Kounadis 75, Russell 116]. For boundary conditions we suppose that

(i) The structures are in $i_b j_b$ plane in undisturbed state,

(ii) The beams 1,2,3,4 are clamped at the joints $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$,

(iii) Displacements of the beams 1,2,...,6 are small,

(iv) The beams 5,6 are rigidly jointed with the other four beams as shown in Fig.3.2.

Let $\frac{\partial}{\partial \xi_i} D_i$, $\frac{\partial^2}{\partial {\xi_i}^2} D_i$, and $\frac{\partial^3}{\partial {\xi_i}^3} D_i$ be denoted by D_{ξ_i}, $D_{\xi_i \xi_i}$ and $D_{\xi_i \xi_i \xi_i}$ respectively. Then rewriting Equation (3.36) we obtain for any time t,

$$\mathcal{I}_4 \equiv \delta D_{\xi_1}(l_1) \circ EI_1 D_{\xi_1 \xi_1}(l_1) + \delta D_{\xi_5}(l_5) \circ EI_5 D_{\xi_5 \xi_5}(l_5) \quad (3.143)$$

$$+\delta D_{\xi_2}(l_2) \circ EI_2 D_{\xi_2\xi_2}(l_2) - \delta D_{\xi_5}(0) \circ EI_5 D_{\xi_5\xi_5}(0) \tag{3.144}$$

$$+\delta D_{\xi_3}(l_3) \circ EI_3 D_{\xi_3\xi_3}(l_3) + \delta D_{\xi_6}(l_6) \circ EI_6 D_{\xi_6\xi_6}(l_6) \tag{3.145}$$

$$+\delta D_{\xi_4}(l_4) \circ EI_4 D_{\xi_4\xi_4}(l_4) - \delta D_{\xi_6}(0) \circ EI_6 D_{\xi_6\xi_6}(0) \tag{3.146}$$

$$-\delta D_{\xi_1}(0) \circ EI_1 D_{\xi_1\xi_1}(0) - \delta D_{\xi_2}(0) \circ EI_2 D_{\xi_2\xi_2}(0) \tag{3.147}$$

$$-\delta D_{\xi_3}(0) \circ EI_3 D_{\xi_3\xi_3}(0) - \delta D_{\xi_4}(0) \circ EI_4 D_{\xi_4\xi_4}(0) \tag{3.148}$$

$$\begin{aligned}+\delta D_1(l_1) \circ &\left[I_{\rho_1}\frac{d}{dt}D_{t\xi_1}(l_1) - EI_1 D_{\xi_1\xi_1\xi_1}(l_1)\right] \\ &+ \delta D_5(l_5) \circ \left[I_{\rho_5}\frac{d}{dt}D_{t\xi_5}(l_5) - EI_5 D_{\xi_5\xi_5\xi_5}(l_5)\right]\end{aligned} \tag{3.149}$$

$$\begin{aligned}+\delta D_2(l_2) \circ &\left[I_{\rho_2}\frac{d}{dt}D_{t\xi_2}(l_2) - EI_2 D_{\xi_2\xi_2\xi_2}(l_2)\right] \\ &- \delta D_5(0) \circ \left[I_{\rho_5}\frac{d}{dt}D_{t\xi_5}(0) - EI_5 D_{\xi_5\xi_5\xi_5}(0)\right]\end{aligned} \tag{3.150}$$

$$\begin{aligned}+\delta D_3(l_3) \circ &\left[I_{\rho_3}\frac{d}{dt}D_{t\xi_3}(l_3) - EI_3 D_{\xi_3\xi_3\xi_3}(l_3)\right] \\ &+ \delta D_6(l_6) \circ \left[I_{\rho_6}\frac{d}{dt}D_{t\xi_6}(l_6) - EI_6 D_{\xi_6\xi_6\xi_6}(l_6)\right]\end{aligned} \tag{3.151}$$

$$\begin{aligned}+\delta D_4(l_4) \circ &\left[I_{\rho_4}\frac{d}{dt}D_{t\xi_4}(l_4) - EI_4 D_{\xi_4\xi_4\xi_4}(l_4)\right] \\ &- \delta D_6(0) \circ \left[I_{\rho_6}\frac{d}{dt}D_{t\xi_6}(0) - EI_6 D_{\xi_6\xi_6\xi_6}(0)\right]\end{aligned} \tag{3.152}$$

$$\begin{aligned}-\delta D_1(0) \circ &\left[I_{\rho_1}\frac{d}{dt}D_{t\xi_1}(0) - EI_1 D_{\xi_1\xi_1\xi_1}(0)\right] \\ &- \delta D_2(0) \circ \left[I_{\rho_2}\frac{d}{dt}D_{t\xi_2}(0) - EI_2 D_{\xi_2\xi_2\xi_2}(0)\right]\end{aligned} \tag{3.153}$$

$$\begin{aligned}-\delta D_3(0) \circ &\left[I_{\rho_3}\frac{d}{dt}D_{t\xi_3}(0) - EI_3 D_{\xi_3\xi_3\xi_3}(0)\right] \\ &- \delta D_4(0) \circ \left[I_{\rho_4}\frac{d}{dt}D_{t\xi_4}(0) - EI_4 D_{\xi_4\xi_4\xi_4}(0)\right] = 0.\end{aligned} \tag{3.154}$$

Since the displacement and slope vanish at a clamped boundary (i.e. $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$), we obtain the geometric boundary conditions:

$$\begin{aligned}\text{at } \xi_i = 0, \qquad & y_i(0,t) = 0, \qquad z_i(0,t) = 0, \\ & \frac{\partial y_i}{\partial \xi_i}(0,t) = 0, \qquad \frac{\partial z_i}{\partial \xi_i}(0,t) = 0, \quad \text{for } i = 1,2,3,4.\end{aligned} \tag{3.155}$$

Since the displacements and slopes are continuous at the rigid joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$, we obtain:

Continuity of displacements at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$:

$$x_5(l_5,t) = 0, \qquad z_1(l_1,t) = z_5(l_5,t), \quad at\ \mathcal{J}_5, \tag{3.156}$$

$$x_5(0,t) = 0, \qquad z_2(l_2,t) = z_5(0,t), \quad at\ \mathcal{J}_6, \tag{3.157}$$

$$x_6(l_6,t) = 0, \qquad z_3(l_3,t) = z_6(l_6,t), \quad at\ \mathcal{J}_7, \tag{3.158}$$

$$x_6(0,t) = 0, \qquad z_4(l_4,t) = z_6(0,t), \quad at\ \mathcal{J}_8, \tag{3.159}$$

Continuity of slopes at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$ (Appendix G):

$$\frac{\partial y_1}{\partial \xi_1}(l_1,t) = -\frac{\partial x_5}{\partial \xi_5}(l_5,t), \ at\ \mathcal{J}_5, \tag{3.160}$$

$$\frac{\partial y_2}{\partial \xi_2}(l_2,t) = -\frac{\partial x_5}{\partial \xi_5}(0,t), \ at\ \mathcal{J}_6, \tag{3.161}$$

$$\frac{\partial y_3}{\partial \xi_3}(l_3,t) = \frac{\partial x_6}{\partial \xi_6}(l_6,t), \ at\ \mathcal{J}_7, \tag{3.162}$$

$$\frac{\partial y_4}{\partial \xi_4}(l_4,t) = \frac{\partial x_6}{\partial \xi_6}(0,t), \ at\ \mathcal{J}_8. \tag{3.163}$$

Applying the moment equation at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$, we have

$$at\ \mathcal{J}_5, \quad EI_{5z}\frac{\partial^2 z_5}{\partial \xi_5^2}(l_5,t) = 0, \quad EI_{1z}\frac{\partial^2 z_1}{\partial \xi_1^2}(l_1,t) = 0, \tag{3.164}$$

$$at\ \mathcal{J}_6, \quad EI_{5z}\frac{\partial^2 z_5}{\partial \xi_5^2}(0,t) = 0, \quad EI_{2z}\frac{\partial^2 z_2}{\partial \xi_2^2}(l_2,t) = 0, \tag{3.165}$$

$$at\ \mathcal{J}_7, \quad EI_{6z}\frac{\partial^2 z_6}{\partial \xi_6^2}(l_6,t) = 0, \quad EI_{3z}\frac{\partial^2 z_3}{\partial \xi_3^2}(l_3,t) = 0, \tag{3.166}$$

$$at\ \mathcal{J}_8, \quad EI_{6z}\frac{\partial^2 z_6}{\partial \xi_6^2}(0,t) = 0, \quad EI_{4z}\frac{\partial^2 z_4}{\partial \xi_4^2}(l_4,t) = 0. \tag{3.167}$$

Using Equations (3.156) and (3.158) into Equations (3.149) and (3.151) respectively, we obtain, from their first components, that

$$I_{\rho_{1y}}\frac{\partial^3 y_1}{\partial \xi_1 \partial t^2}(l_1,t) - I_{\rho_{1y}} w_1 \frac{\partial^2 z_1}{\partial \xi_1 \partial t}(l_1,t) - EI_{1y}\frac{\partial^3 y_1}{\partial \xi_1^3}(l_1,t) = 0. \tag{3.168.1}$$

$$I_{\rho_{3y}}\frac{\partial^3 y_3}{\partial\xi_3\partial t^2}(l_3,t) - I_{\rho_{3y}}w_1\frac{\partial^2 z_3}{\partial\xi_3\partial t}(l_3,t) - EI_{3y}\frac{\partial^3 y_3}{\partial\xi_3^3}(l_3,t) = 0. \tag{3.168.2}$$

Similarly using Equations(3.157) and (3.159) into Equations (3.150) and (3.152) respectively, we obtain, from their first elements, that

$$I_{\rho_{2y}}\frac{\partial^3 y_2}{\partial\xi_2\partial t^2}(l_2,t) - I_{\rho_{2y}}w_1\frac{\partial^2 z_2}{\partial\xi_2\partial t}(l_2,t) - EI_{2y}\frac{\partial^3 y_2}{\partial\xi_2^3}(l_2,t) = 0. \tag{3.169.1}$$

$$I_{\rho_{4y}}\frac{\partial^3 y_4}{\partial\xi_4\partial t^2}(l_4,t) - I_{\rho_{4y}}w_1\frac{\partial^2 z_4}{\partial\xi_4\partial t}(l_4,t) - EI_{4y}\frac{\partial^3 y_4}{\partial\xi_4^3}(l_4,t) = 0. \tag{3.169.2}$$

Substituting Equations (3.156), (3.158), (3.157) and (3.159) into Equations (3.149), (3.151), (3.150) and (3.152) respectively, , we obtain, from their second components, that

$$\begin{aligned} &I_{\rho_{1z}}\frac{\partial^3 z_1}{\partial\xi_1\partial t^2}(l_1,t) + I_{\rho_{1z}}w_1\frac{\partial^2 y_1}{\partial\xi_1\partial t}(l_1,t) - EI_{1z}\frac{\partial^3 z_1}{\partial\xi_1^3}(l_1,t) \\ &+ I_{\rho_{5z}}\frac{\partial^3 z_5}{\partial\xi_5\partial t^2}(l_5,t) - I_{\rho_{5z}}w_2\frac{\partial^2 x_5}{\partial\xi_5\partial t}(l_5,t) - EI_{5z}\frac{\partial^3 z_5}{\partial\xi_5^3}(l_5,t) = 0. \end{aligned} \tag{3.170}$$

$$\begin{aligned} &I_{\rho_{3z}}\frac{\partial^3 z_3}{\partial\xi_3\partial t^2}(l_3,t) + I_{\rho_{3z}}w_1\frac{\partial^2 y_3}{\partial\xi_3\partial t}(l_3,t) - EI_{3z}\frac{\partial^3 z_3}{\partial\xi_3^3}(l_3,t) \\ &+ I_{\rho_{6z}}\frac{\partial^3 z_6}{\partial\xi_6\partial t^2}(l_6,t) - I_{\rho_{6z}}w_2\frac{\partial^2 x_6}{\partial\xi_6\partial t}(l_6,t) - EI_{6z}\frac{\partial^3 z_6}{\partial\xi_6^3}(l_6,t) = 0. \end{aligned} \tag{3.171}$$

$$\begin{aligned} &I_{\rho_{2z}}\frac{\partial^3 z_2}{\partial\xi_2\partial t^2}(l_2,t) + I_{\rho_{2z}}w_1\frac{\partial^2 y_2}{\partial\xi_2\partial t}(l_2,t) - EI_{2z}\frac{\partial^3 z_2}{\partial\xi_2^3}(l_2,t) \\ &- I_{\rho_{5z}}\frac{\partial^3 z_5}{\partial\xi_5\partial t^2}(0,t) + I_{\rho_{5z}}w_2\frac{\partial^2 x_5}{\partial\xi_5\partial t}(0,t) + EI_{5z}\frac{\partial^3 z_5}{\partial\xi_5^3}(0,t) = 0. \end{aligned} \tag{3.172}$$

$$\begin{aligned} &I_{\rho_{4z}}\frac{\partial^3 z_4}{\partial\xi_4\partial t^2}(l_4,t) + I_{\rho_{4z}}w_1\frac{\partial^2 y_4}{\partial\xi_4\partial t}(l_4,t) - EI_{4z}\frac{\partial^3 z_4}{\partial\xi_4^3}(l_4,t) \\ &- I_{\rho_{6z}}\frac{\partial^3 z_6}{\partial\xi_6\partial t^2}(0,t) + I_{\rho_{6z}}w_2\frac{\partial^2 x_6}{\partial\xi_6\partial t}(0,t) + EI_{6z}\frac{\partial^3 z_6}{\partial\xi_6^3}(0,t) = 0. \end{aligned} \tag{3.173}$$

It is noted that the above equations (3.168-3.173) can be ontained from the force equations at the joints $\mathcal{J}_5$, $\mathcal{J}_6$, $\mathcal{J}_7$, $\mathcal{J}_8$. Substituting Equations (3.160) and (3.162) into Equations (3.143) and (3.145) respectively, we have, from their first elements, that

$$EI_{1y}\frac{\partial^2 y_1}{\partial \xi_1^2}(l_1,t) - EI_{5x}\frac{\partial^2 x_5}{\partial \xi_5^2}(l_5,t) = 0, \tag{3.174}$$

$$EI_{3y}\frac{\partial^2 y_3}{\partial \xi_3^2}(l_3,t) + EI_{6x}\frac{\partial^2 x_6}{\partial \xi_6^2}(l_6,t) = 0. \tag{3.175}$$

Similarly using Equations (3.161) and (3.163) into Equations (3.144) and (3.146) respectively we obtain, from their first elements, that

$$EI_{2y}\frac{\partial^2 y_2}{\partial \xi_2^2}(l_2,t) + EI_{5x}\frac{\partial^2 x_5}{\partial \xi_5^2}(0,t) = 0, \tag{3.176}$$

$$EI_{4y}\frac{\partial^2 y_4}{\partial \xi_4^2}(l_4,t) - EI_{6x}\frac{\partial^2 x_6}{\partial \xi_6^2}(0,t) = 0. \tag{3.177}$$

We note that the above equations (3.174)-(3.177) also can be obtained from moment equation at the joints $\mathcal{J}_5$, $\mathcal{J}_6$, $\mathcal{J}_7$, $\mathcal{J}_8$.

Let α_i be defined by $\alpha_i = \dot{\omega} \times \bar{r}_i + 2\omega \times \dot{D}_i + \omega \times (\omega \times \bar{r}_i)$, i=1,2,...,6, and

$$F_{b_i} \equiv \begin{cases} (F_{i_y}, F_{i_z})', & \text{for } i=1,2,3,4, \\ (F_{i_x}, F_{i_z})', & \text{for } i=5,6. \end{cases} \tag{3.178}$$

$$\tilde{Q}_i \equiv \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for i=1,2,3,4,} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for i=5,6.} \end{cases} \tag{3.179}$$

Then we finally obtain the complete dynamics of the space station subject to external forces in the following.

$$m_\tau \frac{d^2 R}{dt^2} + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ (\dot{\omega} \times \bar{r}_i) + \omega \times (\omega \times \bar{r}_i) + \ddot{D}_i + 2\omega \times \dot{D}_i \Big\} d\xi_i + \frac{G m_e m_\tau R}{|R|^3} = F_s, \tag{3.180}$$

$$I_\tau \circ \dot{\omega} + \omega \times (I_\tau \circ \omega) + \dot{I}_b \circ \omega + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big[\bar{r}_i \times \frac{d^2 R}{dt^2} \Big) d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i (\bar{r}_i \times \ddot{D}_i) d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \omega \times (\bar{r}_i \times \dot{D}_i) \Big\} d\xi_i = T, \tag{3.181}$$

and for $\xi_i \in \Omega_i,\ \xi_j \in \Omega_j;\ i$=1,2,3,4, j=5,6,

$$\rho_i \frac{\partial^2}{\partial t^2}\begin{pmatrix} y_i \\ z_i \end{pmatrix} - \frac{\partial}{\partial \xi_i}\left(\begin{pmatrix} I_{\rho_{iy}} & 0 \\ 0 & I_{\rho_{iz}} \end{pmatrix} \frac{\partial^3}{\partial \xi_i \partial t^2}\begin{pmatrix} y_i \\ z_i \end{pmatrix}\right) - \frac{\partial}{\partial \xi_i}\left(I_{\rho_i}\begin{pmatrix} 0 & -w_1 \\ w_1 & 0 \end{pmatrix} \frac{\partial^2}{\partial \xi_i \partial t}\begin{pmatrix} y_i \\ z_i \end{pmatrix}\right)$$
$$+ \rho_i \tilde{Q}_i\left(\alpha_i + \frac{d^2 R}{dt^2}\right) + \frac{\partial^2}{\partial \xi_i{}^2}\left(\begin{pmatrix} EI_{iy} & 0 \\ 0 & EI_{iz} \end{pmatrix} \frac{\partial^2}{\partial \xi_i{}^2}\begin{pmatrix} y_i \\ z_i \end{pmatrix}\right) = F_{b_i}, \qquad (3.182)$$

$$\rho_j \frac{\partial^2}{\partial t^2}\begin{pmatrix} x_j \\ z_j \end{pmatrix} - \frac{\partial}{\partial \xi_j}\left(\begin{pmatrix} I_{\rho_{jx}} & 0 \\ 0 & I_{\rho_{jz}} \end{pmatrix} \frac{\partial^3}{\partial \xi_j \partial t^2}\begin{pmatrix} x_j \\ z_j \end{pmatrix}\right) - \frac{\partial}{\partial \xi_j}\left(I_{\rho_j}\begin{pmatrix} 0 & w_2 \\ -w_2 & 0 \end{pmatrix} \frac{\partial^2}{\partial \xi_j \partial t}\begin{pmatrix} x_j \\ z_j \end{pmatrix}\right)$$
$$+ \rho_j \tilde{Q}_j\left(\alpha_j + \frac{d^2 R}{dt^2}\right) + \frac{\partial^2}{\partial \xi_j^2}\left(\begin{pmatrix} EI_{jx} & 0 \\ 0 & EI_{jz} \end{pmatrix} \frac{\partial^2}{\partial \xi_j^2}\begin{pmatrix} x_j \\ z_j \end{pmatrix}\right) = F_{b_j}, \qquad (3.183)$$

with the boundary conditions (3.155-3.177).

Using the same notations C_B^I, $(a_i, a_j, a_k)'$ as in Section 3.4 and performing the vector operations, Equations (3.180) – (3.183) and (3.155-3.177) could be expressed in explicit forms which are suitable for digital computer simulation. These are given below.

BUS DYNAMICS(TRANSLATION)

$$m_T \frac{d^2}{dt^2}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + C_B^I \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} + \begin{pmatrix} h_4(t) \\ h_5(t) \\ h_6(t) \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \qquad (3.184)$$

where

$$f_1(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ (\bar{z}_i \dot{\omega}_2 - \bar{y}_i \dot{\omega}_3) + 2(\omega_2 \frac{\partial z_i}{\partial t} - \omega_3 \frac{\partial y_i}{\partial t})$$
$$+ \omega_1 \omega_2 \bar{y}_i + w_1 w_3 \bar{z}_i - (w_2^2 + w_3^2)\bar{x}_i + \frac{\partial^2 x_i}{\partial t^2} \Big\} d\xi_i \qquad (3.185)$$

$$f_2(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{x}_i \dot{w}_3 - \bar{z}_i \dot{w}_1 + 2(w_3 \frac{\partial x_i}{\partial t} - w_1 \frac{\partial z_i}{\partial t})$$

$$+ w_2 w_3 \bar{z}_i + w_2 w_1 \bar{x}_i - (w_3^2 + w_1^2)\bar{y}_i + \frac{\partial^2 y_i}{\partial t^2} \Big\} d\xi_i, \qquad (3.186)$$

$$f_3(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{y}_i \dot{w}_1 - \bar{x}_i \dot{w}_2 + 2(w_1 \frac{\partial y_i}{\partial t} - w_2 \frac{\partial x_i}{\partial t})$$

$$+ w_3 w_1 \bar{x}_i + w_3 w_2 \bar{y}_i - (w_1^2 + w_2^2)\bar{z}_i + \frac{\partial^2 z_i}{\partial t^2} \Big\} d\xi_i, \qquad (3.187)$$

and $h_4(t)$,$h_5(t)$, $h_6(t)$ are given by Equations (2.46.1), (2.46.2), (2.46.3), respectively.

BUS DYNAMICS(ROTATION)

$$I_r \begin{pmatrix} \dot{\omega_1} \\ \dot{\omega_2} \\ \dot{\omega_3} \end{pmatrix} + \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} I_r \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} f_4(t) \\ f_4(t) \\ f_6(t) \end{pmatrix} + \begin{pmatrix} f_7(t) \\ f_8(t) \\ f_9(t) \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}, \qquad (3.188)$$

where

$$f_4(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{y}_i \frac{\partial^2 z_i}{\partial t^2} - \bar{z}_i \frac{\partial^2 y_i}{\partial t^2} + 2w_1 \bar{y}_i \frac{\partial y_i}{\partial t} + 2w_1 \bar{z}_i \frac{\partial z_i}{\partial t}$$

$$- 2w_2 \bar{y}_i \frac{\partial x_i}{\partial t} - 2w_3 \bar{z}_i \frac{\partial x_i}{\partial t} \Big\} d\xi_i, \qquad (3.189)$$

$$f_5(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{z}_i \frac{\partial^2 x_i}{\partial t^2} - \bar{x}_i \frac{\partial^2 z_i}{\partial t^2} + 2\omega_2 \bar{z}_i \frac{\partial z_i}{\partial t} + 2w_2 \bar{x}_i \frac{\partial x_i}{\partial t}$$

$$- 2w_3 \bar{z}_i \frac{\partial y_i}{\partial t} - 2w_1 \bar{x}_i \frac{\partial y_i}{\partial t} \Big\} d\xi_i, \qquad (3.190)$$

$$f_6(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{x}_i \frac{\partial^2 y_i}{\partial t^2} - \bar{y}_i \frac{\partial^2 x_i}{\partial t^2} + 2w_3 \bar{x}_i \frac{\partial x_i}{\partial t} + 2w_3 \bar{y}_i \frac{\partial y_i}{\partial t}$$

$$-2w_1\bar{x}_i\frac{\partial z_i}{\partial t}-2w_2\bar{y}_i\frac{\partial z_i}{\partial t}\Big\}d\xi_i, \qquad (3.191)$$

$$f_7(t)=\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{\bar{y}_i a_k-\bar{z}_i a_j\Big\}d\xi_i, \qquad (3.192)$$

$$f_8(t)=\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{\bar{z}_i a_i-\bar{x}_i a_k\Big\}d\xi_i, \qquad (3.193)$$

$$f_9(t)=\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{\bar{x}_i a_j-\bar{y}_i a_i\Big\}d\xi_i, \qquad (3.194)$$

BEAM DYNAMICS(BEAM VIBRATION)

For $\xi_i\in\Omega_i,\ i=1,2,3,4;$ and $\xi_j\in\Omega_j,\ j=5,6,$

$$\rho_i\frac{\partial^2}{\partial t^2}\begin{pmatrix}y_i\\ z_i\end{pmatrix}+\rho_i\begin{pmatrix}0 & -2\omega_1\\ 2\omega_1 & 0\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}y_i\\ z_i\end{pmatrix}+\rho_i\bar{x}_i\begin{pmatrix}\dot{\omega}_3+\omega_1\omega_2\\ -\dot{\omega}_2+\omega_1\omega_3\end{pmatrix}+\rho_i\bar{Q}_iC_I^B\frac{d^2R}{dt^2}$$

$$+\frac{\partial^2}{\partial{\xi_i}^2}\Bigg(\begin{pmatrix}EI_{i_y} & 0\\ 0 & EI_{i_z}\end{pmatrix}\frac{\partial^2}{\partial{\xi_i}^2}\begin{pmatrix}y_i\\ z_i\end{pmatrix}\Bigg)+\rho_i\begin{pmatrix}-\omega_1^2-\omega_3^2 & -\dot{\omega}_1+\omega_2\omega_3\\ \dot{\omega}_1+\omega_2\omega_3 & -\omega_1^2-\omega_2^2\end{pmatrix}\begin{pmatrix}\bar{y}_i\\ \bar{z}_i\end{pmatrix}$$

$$-\frac{\partial}{\partial\xi_i}\Bigg(\begin{pmatrix}I_{\rho_{i_y}} & 0\\ 0 & I_{\rho_{i_z}}\end{pmatrix}\frac{\partial^3}{\partial\xi_i\partial t^2}\begin{pmatrix}y_i\\ z_i\end{pmatrix}\Bigg)+\frac{\partial}{\partial\xi_i}\Bigg[\begin{pmatrix}0 & w_1I_{\rho_{i_y}}\\ -w_1I_{\rho_{i_z}} & 0\end{pmatrix}\frac{\partial^2}{\partial\xi_i\partial t}\begin{pmatrix}y_i\\ z_i\end{pmatrix}\Bigg]=\begin{pmatrix}F_{i_y}\\ F_{i_z}\end{pmatrix}, \qquad (3.195)$$

$$\rho_j\frac{\partial^2}{\partial t^2}\begin{pmatrix}x_j\\ z_j\end{pmatrix}+\rho_j\begin{pmatrix}0 & 2\omega_2\\ -2\omega_2 & 0\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}x_j\\ z_j\end{pmatrix}+\rho_j\bar{y}_j\begin{pmatrix}-\dot{w}_3+w_2w_1\\ \dot{w}_1+w_2w_3\end{pmatrix}+\rho_j\bar{Q}_jC_I^B\frac{d^2R}{dt^2}$$

$$+\frac{\partial^2}{\partial\xi_j^2}\Bigg(\begin{pmatrix}EI_{j_x} & 0\\ 0 & EI_{j_z}\end{pmatrix}\frac{\partial^2}{\partial\xi_j^2}\begin{pmatrix}x_j\\ z_j\end{pmatrix}\Bigg)+\rho_j\begin{pmatrix}-w_2^2-w_3^2 & \dot{w}_2+w_3w_1\\ -\dot{w}_2+w_3w_1 & -w_1^2-w_2^2\end{pmatrix}\begin{pmatrix}\bar{x}_j\\ \bar{z}_j\end{pmatrix}$$

$$-\frac{\partial}{\partial\xi_j}\Bigg(\begin{pmatrix}I_{\rho_{j_x}} & 0\\ 0 & I_{\rho_{j_z}}\end{pmatrix}\frac{\partial^3}{\partial\xi_j\partial t^2}\begin{pmatrix}x_j\\ z_j\end{pmatrix}\Bigg)+\frac{\partial}{\partial\xi_j}\Bigg[\begin{pmatrix}0 & -w_2I_{\rho_{j_x}}\\ w_2I_{\rho_{j_z}} & 0\end{pmatrix}\frac{\partial^2}{\partial\xi_j\partial t}\begin{pmatrix}x_j\\ z_j\end{pmatrix}\Bigg]=\begin{pmatrix}F_{j_x}\\ F_{j_z}\end{pmatrix}. \qquad (3.196)$$

BOUNDARY CONDITIONS FOR BEAM DYNAMICS

Clamped at $\xi_i = 0$, i=1,2,3,4 for the joints $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$:

$$y_i(0,t) = 0, \qquad z_i(0,t) = 0, \tag{3.197}$$

$$\frac{\partial y_i}{\partial \xi_i}(0,t) = 0, \qquad \frac{\partial z_i}{\partial \xi_i}(0,t) = 0, \tag{3.198}$$

Continuity of displacements at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$:

$$x_5(l_5,t) = 0, \;\; z_1(l_1,t) = z_5(l_5,t), \;\; at\ \mathcal{J}_5, \tag{3.199}$$

$$x_5(0,t) = 0, \;\; z_2(l_2,t) = z_5(0,t), \;\; at\ \mathcal{J}_6, \tag{3.200}$$

$$x_6(l_6,t) = 0, \;\; z_3(l_3,t) = z_6(l_6,t), \;\; at\ \mathcal{J}_7, \tag{3.201}$$

$$x_6(0,t) = 0, \;\; z_4(l_4,t) = z_6(0,t), \;\; at\ \mathcal{J}_8, \tag{3.202}$$

Continuity of slopes at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$:

$$\frac{\partial y_1}{\partial \xi_1}(l_1,t) = -\frac{\partial x_5}{\partial \xi_5}(l_5,t), \;\; at\ \mathcal{J}_5, \tag{3.203}$$

$$\frac{\partial y_2}{\partial \xi_2}(l_2,t) = -\frac{\partial x_5}{\partial \xi_5}(0,t), \;\; at\ \mathcal{J}_6, \tag{3.204}$$

$$\frac{\partial y_3}{\partial \xi_3}(l_3,t) = \;\; \frac{\partial x_6}{\partial \xi_6}(l_6,t), \;\; at\ \mathcal{J}_7, \tag{3.205}$$

$$\frac{\partial y_4}{\partial \xi_4}(l_4,t) = \;\; \frac{\partial x_6}{\partial \xi_6}(0,t). \;\; at\ \mathcal{J}_8, \tag{3.206}$$

Equilibrium equations:

$$at\ \mathcal{J}_5, \qquad EI_{5z}\frac{\partial^2 z_5}{\partial \xi_5^2}(l_5,t) = 0, \quad EI_{1z}\frac{\partial^2 z_1}{\partial \xi_1^2}(l_1,t) = 0, \tag{3.207}$$

$$at\ \mathcal{J}_6, \qquad EI_{5z}\frac{\partial^2 z_5}{\partial \xi_5^2}(0,t) = 0, \quad EI_{2z}\frac{\partial^2 z_2}{\partial \xi_2^2}(l_2,t) = 0, \tag{3.208}$$

$$at\ \mathcal{J}_7, \qquad EI_{6z}\frac{\partial^2 z_6}{\partial \xi_6^2}(l_6,t) = 0, \quad EI_{3z}\frac{\partial^2 z_3}{\partial \xi_3^2}(l_3,t) = 0, \tag{3.209}$$

$$at\ \mathcal{J}_8, \qquad EI_{6z}\frac{\partial^2 z_6}{\partial \xi_6^2}(0,t) = 0, \quad EI_{4z}\frac{\partial^2 z_4}{\partial \xi_4^2}(l_4,t) = 0. \tag{3.210}$$

$$I_{\rho_{1_y}}\frac{\partial^3 y_1}{\partial \xi_1 \partial t^2}(l_1,t) - I_{\rho_{1_y}} w_1 \frac{\partial^2 z_1}{\partial \xi_1 \partial t}(l_1,t) - EI_{1y}\frac{\partial^3 y_1}{\partial \xi_1^3}(l_1,t) = 0. \tag{3.211.1}$$

$$I_{\rho_{3y}}\frac{\partial^3 y_3}{\partial\xi_3\partial t^2}(l_3,t) - I_{\rho_{3y}}w_1\frac{\partial^2 z_3}{\partial\xi_3\partial t}(l_3,t) - EI_{3y}\frac{\partial^3 y_3}{\partial\xi_3^3}(l_3,t) = 0. \tag{3.211.2}$$

$$I_{\rho_{2y}}\frac{\partial^3 y_2}{\partial\xi_2\partial t^2}(l_2,t) - I_{\rho_{2y}}w_1\frac{\partial^2 z_2}{\partial\xi_2\partial t}(l_2,t) - EI_{2y}\frac{\partial^3 y_2}{\partial\xi_2^3}(l_2,t) = 0. \tag{3.212.1}$$

$$I_{\rho_{4y}}\frac{\partial^3 y_4}{\partial\xi_4\partial t^2}(l_4,t) - I_{\rho_{4y}}w_1\frac{\partial^2 z_4}{\partial\xi_4\partial t}(l_4,t) - EI_{4y}\frac{\partial^3 y_4}{\partial\xi_4^3}(l_4,t) = 0. \tag{3.212.2}$$

$$\begin{aligned} &I_{\rho_{1z}}\frac{\partial^3 z_1}{\partial\xi_1\partial t^2}(l_1,t) + I_{\rho_{1z}}w_1\frac{\partial^2 y_1}{\partial\xi_1\partial t}(l_1,t) - EI_{1z}\frac{\partial^3 z_1}{\partial\xi_1^3}(l_1,t) \\ &\quad + I_{\rho_{5z}}\frac{\partial^3 z_5}{\partial\xi_5\partial t^2}(l_5,t) - I_{\rho_{5z}}w_2\frac{\partial^2 x_5}{\partial\xi_5\partial t}(l_5,t) - EI_{5z}\frac{\partial^3 z_5}{\partial\xi_5^3}(l_5,t) = 0. \end{aligned} \tag{3.213}$$

$$\begin{aligned} &I_{\rho_{3z}}\frac{\partial^3 z_3}{\partial\xi_3\partial t^2}(l_3,t) + I_{\rho_{3z}}w_1\frac{\partial^2 y_3}{\partial\xi_3\partial t}(l_3,t) - EI_{3z}\frac{\partial^3 z_3}{\partial\xi_3^3}(l_3,t) \\ &\quad + I_{\rho_{6z}}\frac{\partial^3 z_6}{\partial\xi_6\partial t^2}(l_6,t) - I_{\rho_{6z}}w_2\frac{\partial^2 x_6}{\partial\xi_6\partial t}(l_6,t) - EI_{6z}\frac{\partial^3 z_6}{\partial\xi_6^3}(l_6,t) = 0. \end{aligned} \tag{3.214}$$

$$\begin{aligned} &I_{\rho_{2z}}\frac{\partial^3 z_2}{\partial\xi_2\partial t^2}(l_2,t) + I_{\rho_{2z}}w_1\frac{\partial^2 y_2}{\partial\xi_2\partial t}(l_2,t) - EI_{2z}\frac{\partial^3 z_2}{\partial\xi_2^3}(l_2,t) \\ &\quad - I_{\rho_{5z}}\frac{\partial^3 z_5}{\partial\xi_5\partial t^2}(0,t) + I_{\rho_{5z}}w_2\frac{\partial^2 x_5}{\partial\xi_5\partial t}(0,t) + EI_{5z}\frac{\partial^3 z_5}{\partial\xi_5^3}(0,t) = 0. \end{aligned} \tag{3.215}$$

$$\begin{aligned} &I_{\rho_{4z}}\frac{\partial^3 z_4}{\partial\xi_4\partial t^2}(l_4,t) + I_{\rho_{4z}}w_1\frac{\partial^2 y_4}{\partial\xi_4\partial t}(l_4,t) - EI_{4z}\frac{\partial^3 z_4}{\partial\xi_4^3}(l_4,t) \\ &\quad - I_{\rho_{6z}}\frac{\partial^3 z_6}{\partial\xi_6\partial t^2}(0,t) + I_{\rho_{6z}}w_2\frac{\partial^2 x_6}{\partial\xi_6\partial t}(0,t) + EI_{6z}\frac{\partial^3 z_6}{\partial\xi_6^3}(0,t) = 0. \end{aligned} \tag{3.216}$$

$$EI_{1y}\frac{\partial^2 y_1}{\partial\xi_1^2}(l_1,t) - EI_{5x}\frac{\partial^2 x_5}{\partial\xi_5^2}(l_5,t) = 0, \tag{3.217}$$

$$EI_{3y}\frac{\partial^2 y_3}{\partial\xi_3^2}(l_3,t) + EI_{6x}\frac{\partial^2 x_6}{\partial\xi_6^2}(l_6,t) = 0, \tag{3.218}$$

$$EI_{2y}\frac{\partial^2 y_2}{\partial\xi_2^2}(l_2,t) + EI_{5x}\frac{\partial^2 x_5}{\partial\xi_5^2}(0,t) = 0, \tag{3.219}$$

$$EI_{4y}\frac{\partial^2 y_4}{\partial\xi_4^2}(l_4,t) - EI_{6x}\frac{\partial^2 x_6}{\partial\xi_6^2}(0,t) = 0. \tag{3.220}$$

In this section we have deveolped equations of motion for the space station (Fig.3.1) based on *Rayleigh beam theory*. The complete dynamics of the flexible space station consisting of the main rigid body and flexible structures rigidly attached to it (Fig.3.1) is given by a *hybrid system* which consists of nonlinear ordinary differential equations (3.184),(3.188) and the partial differential equations (3.195-3.196) along with the boundary conditions (3.197-3.220).

We note from Equations (3.184-3.220) or (3.180-3.183) that the four sets of equations are strongly coupled. Thus any rotation of the bus would induce vibration in the beam and vice-versa. Also any disturbance of the orbit would induce vibrations in the beam and rotations in the bus and vice-versa. We have the same observations as (OBS.3.1) and (OBS.3.2) in Section 3.4 for the dynamics for translational and rotational motions of the rigid body. Further, from the beam equations (3.195-3.196) (or 3.182-3.183) we note that

(OBS.3.4) The effect of rigid body rotation and translation on the vibration of the beams is represented by the forcing terms $\rho_i\tilde{Q}_i(\alpha_i + \frac{d^2R}{dt^2})$ and $\rho_j\tilde{Q}_j(\alpha_j + \frac{d^2R}{dt^2})$ as shown in Equations (3.182) and (3.183) respectively. Thus it is clear that without the coupling effects $(\alpha_i + \frac{d^2R}{dt^2})$ and $(\alpha_j + \frac{d^2R}{dt^2})$, the beam equations (3.182-3.183) become Rayleigh beam equations of simple beams[Russell 116].

Remark 3.3

We have similar equations for the dynamics of the spacecraft as developed in Section 3.4 where equations of motion were derived based on *Euler-Bernoulli beam theory*. However, note that the second terms and the third terms of the beam equations (3.182-3.183) (or equivalently the seventh and the eighth terms of Eqs.(3.195-3.196)) are not taken into account in the Euler-Bernoulli model (Equation (3.95-3.96) or (3.82- 3.83)). In fact, these terms represent the effect of the rotary moment of inertia on the beam dynamics. Hence the beam equation(2.88) enables us to study the influence of the rotary inertia on the dynamic performance such as the bus rotation and beam vibrations. The terms $f_1(t)$, $f_2(t)$, $f_3(t)$ of (3.184) represent the contribution of the beam vibration and bus rotation to the translational motion of the bus. This implies that even though the bus translational and rotational equations(3.184, 3.188) have the same forms as those based on Euler- Bernoulli beam theory, the dynamic response correponding to external forces are not the same because the beam equations are different from each other and also the four sets of equations (3.184), (3.188), (3.195), (3.196) are strongly coupled.

Remark 3.4

Based on *Rayleigh beam theory*[Russell 116] in this section we have developed a complete dynamic model for the space station consisting of a rigid main body and flexible structures (Fig.3.1) through the variational formulation. In our derivation of equations of motion we have approximated the gravitational potential energy P_g by Equation (2.16) (details are given in Remark 2.1). Orbiting flexible space structures are often subject to external disturbances caused by various sources such as those mentioned in Remark 3.2. The stabilization of the spacecraft under the influence of those perturbing forces is considered in Section 4.4.2.

3.6 Dynamic Modeling based on Timoshenko Beam Theory

In this section, using the coordinate systems defined in Section 3.2 and following the procedure used in Section 3.5, we derive equations of motion for the space station shown in Fig.3.1, based on *Timoshenko beam theory*[Traill-Nash and Collar 131]. We use the same notations and definitions as in Section 3.4 throuthout this section, unless otherwise stated. The equations of motion derived using Timoshenko beam theory have different forms compared with those based on Euler-Bernoulli beam theory, because the kinetic energy of the beam has a different representation by Timoshenko beam theory. The kinetic energy of the system is given by sum of K_s and K_{b_T}, where K_s is given by Equation (2.6) and K_{b_T} is given below.

Let $I_{\rho_i} \equiv I_{\rho_i}(\xi_i)$ be the rotary inertia of the mass element dm_i of the beam i and β_i be the deformation of the element due to shearing force. According to Timoshenko beam theory[Traill-Nash and Collar 131], the kinetic energy of the beams K_{b_T} is given by

$$K_{b_T} = \sum_{i=1}^{6} K_{b_{T_i}}, \tag{3.221}$$

with

$$K_{b_{T_i}} = \frac{1}{2}\int_{\Omega_i} \rho_i \frac{dr_i}{dt} \circ \frac{dr_i}{dt} d\xi_i + \frac{1}{2}\int_{\Omega_i} I_{\rho_i} \frac{\partial \psi_i}{\partial t} \circ \frac{\partial \psi_i}{\partial t} d\xi_i, \tag{3.222}$$

$$r_i = R + \bar{r}_i = R + D_{i_o} + D_i, \tag{3.223}$$

$$\psi_i = \frac{\partial D_i}{\partial \xi_i} + \beta_i(\xi_i, t), \tag{3.224}$$

where r_i denotes the absolute position of the incremental mass element dm_i of the beam i relative to the inertial frame, D_{i_o} defines the postion of the mass element dm_i relative to the body frame in the undeformed state and given by Equations (3.3-3.5). Further, D_i and ψ_i represents the displacement of dm_i from the undeformed position of the beam in S_B and angular dispacement respectively, whose components are given by

$$D_i = Q_i(x_i, y_i, z_i)', \quad \text{and} \quad \psi_i = Q_i(\psi_{i_x}, \psi_{i_y}, \psi_{i_z})'. \tag{3.225}$$

Let $EI_{i_x} \equiv EI_{i_x}(\xi_i)$(respectively EI_{i_y}, EI_{i_z}) be the flexural rigidity of the beam for deflections in the i_b (respectively j_b, k_b)direction and define the matrix EI_i by

$$EI_i \equiv \begin{pmatrix} EI_{i_x} & 0 & 0 \\ 0 & EI_{i_y} & 0 \\ 0 & 0 & EI_{i_z} \end{pmatrix}, \quad i = 1, 2, ..., 6. \tag{3.226}$$

Similarly let $K_{i_x} \equiv K_{i_x}(\xi_i)$(respectively K_{i_y}, K_{i_z}) be the coefficient equal to kGA of the beam i for deflections in the i_b (respectively j_b, k_b)direction, where G is the shear modulus, A is the cross sectional area and k is a numerical factor depending on the shape of the cross section. Let the matrix K_i be defined by

$$K_i \equiv \begin{pmatrix} K_{i_x} & 0 & 0 \\ 0 & K_{i_y} & 0 \\ 0 & 0 & K_{i_z} \end{pmatrix}, \quad i = 1, 2, ..., 6. \tag{3.227}$$

The system potential energy arises from two sourses, namely elastic and gravitational. The elastic potential energy P_e due to deformations of the beam is a function of the partial derivatives of the displacements x_i, y_i, z_i with respect to the spatial variable ξ_i. Under the assumption of small bending deformations neglecting elongation or torsional effects, the elastic potential energy of the beam P_{e_T} is given by[Russell 116]

$$P_{e_T} = \sum_{i=1}^{6} P_{e_{T_i}}, \tag{3.228}$$

where

$$P_{e_{T_i}} = \frac{1}{2}\int_{\Omega_i} \left\{ K_i\left(\psi_i - \frac{\partial D_i}{\partial \xi_i}\right) \circ \left(\psi_i - \frac{\partial D_i}{\partial \xi_i}\right) + EI_i \frac{\partial \psi_i}{\partial \xi_i} \circ \frac{\partial \psi_i}{\partial \xi_i} \right\} d\xi_i. \tag{3.229}$$

The gravitational potential energy P_g is given by the sum of the potential energy of the satellite bus and that of the beam as given in Equation (3.18) and one can approximate P_g as in Equation (2.16) and its first variation is given by Equation (2.29). The virtual displacement of the elastic body relative to the inertial space is given by similar expression as in Equation (2.18):

$$\delta r_i = \delta R + \delta\theta \times \bar{r}_i + \delta D_i, \tag{3.230}$$

where $\delta\theta = (\delta\theta_1, \delta\theta_2, \delta\theta_3)$ is the virtual rigid body rotation of the body frame relative to the reference frame. Further, using Equation (2.17) one obtains[Cavin 32]

$$\delta\left(\frac{dr_i}{dt}\right) = \delta\frac{dR}{dt} + \delta\omega \times \bar{r}_i + \delta\theta \times (\omega \times \bar{r}_i + \dot{D}_i) + \frac{d}{dt}(\delta D_i), \tag{3.231}$$

where $\delta\omega = \frac{d}{dt}(\delta\theta)$ denotes the virtual angular velocity of the body frame.

Assuming that the origin of the body frame coincides with the center of mass of the satellite bus the kinetic energy of the bus is given by Equation (2.20) and its first variation is given by Equation (2.21). Substituting Equations (3.230-3.231) into Equations (3.221-3.224) and taking the first variation, we have

$$\delta\int_{t_1}^{t_2} K_{b_T} dt = \sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\rho_i\Big(\delta\frac{dR}{dt}\circ\frac{dr_i}{dt}\Big)d\xi_i dt + \sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\rho_i\Big\{\delta\omega\times\bar{r}_i\Big\}\circ\frac{dr_i}{dt}d\xi_i dt$$
$$+\sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\rho_i\Big\{\delta\theta\times(w\times\bar{r}_i+\dot{D}_i)\Big\}\circ\frac{dr_i}{dt}d\xi_i dt$$
$$+\sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\rho_i\Big\{\frac{d}{dt}(\delta D_i)\circ\frac{dr_i}{dt}\Big\}d\xi_i dt$$
$$+\sum_{i=1}^{6}\int_{\Omega_i}\delta\psi_i\circ I_{\rho_i}\frac{\partial\psi_i}{\partial t}\Big|_{t_1}^{t_2}d\xi_i - \sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}(\delta\psi_i)\circ\frac{d}{dt}\Big(I_{\rho_i}\frac{\partial\psi_i}{\partial t}\Big)d\xi_i dt. \tag{3.232}$$

Using the vector identity (2.23) we obtain, from Equation (3.232), that

$$\delta\int_{t_1}^{t_2} K_{b_T} dt = \sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\rho_i\Big[\delta\frac{dR}{dt}\circ\frac{dr_i}{dt} + \delta\omega\circ\Big\{\bar{r}_i\times\frac{dr_i}{dt}\Big\}$$
$$+\delta\theta\circ\Big\{(\omega\times\bar{r}_i+\dot{D}_i)\times\frac{dr_i}{dt}\Big\}+\frac{dr_i}{dt}\circ\frac{d}{dt}(\delta D_i)\Big]d\xi_i dt$$
$$+\sum_{i=1}^{6}\int_{\Omega_i}\delta\psi_i\circ I_{\rho_i}\frac{\partial\psi_i}{\partial t}\Big|_{t_1}^{t_2}d\xi_i - \sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}(\delta\psi_i)\circ\frac{d}{dt}\Big(I_{\rho_i}\frac{\partial\psi_i}{\partial t}\Big)d\xi_i dt. \tag{3.233}$$

Then one obtains from Equations (3.228-3.229) that

$$\delta\int_{t_1}^{t_2} P_{e_T} dt = \sum_{i=1}^{6}\int_{t_1}^{t_2}\int_{\Omega_i}\Big\{\delta\Big(\psi_i-\frac{\partial D_i}{\partial\xi_i}\Big)\circ K_i\Big(\psi_i-\frac{\partial D_i}{\partial\xi_i}\Big)+EI_i\frac{\partial\psi_i}{\partial\xi_i}\circ\Big(\delta\frac{\partial\psi_i}{\partial\xi_i}\Big)\Big\}d\xi_i dt$$

$$= \sum_{i=1}^{6} \int_{t_1}^{t_2} \int_{\Omega_i} \Big[\delta\psi_i \circ K_i\Big(\psi_i - \frac{\partial D_i}{\partial \xi_i}\Big) - \delta \frac{\partial D_i}{\partial \xi_i} \circ K_i\Big(\psi_i - \frac{\partial D_i}{\partial \xi_i}\Big) + \delta \frac{\partial \psi_i}{\partial \xi_i} \circ EI_i \frac{\partial \psi_i}{\partial \xi_i}\Big] d\xi_i dt. \tag{3.234}$$

or

$$\delta \int_{t_1}^{t_2} P_{e_T} dt = \sum_{i=1}^{6} \int_{t_1}^{t_2} \int_{\Omega_i} (\delta\psi_i) \circ \Big\{ K_i\Big(\psi_i - \frac{\partial D_i}{\partial \xi_i}\Big) - \frac{\partial}{\partial \xi_i}\Big(EI_i \frac{\partial \psi_i}{\partial \xi_i}\Big) \Big\} d\xi_i dt$$

$$+ \sum_{i=1}^{6} \int_{t_1}^{t_2} \int_{\Omega_i} (\delta D_i) \circ \frac{\partial}{\partial \xi_i} K_i\Big(\psi_i - \frac{\partial D_i}{\partial \xi_i}\Big) d\xi_i dt$$

$$+ \sum_{i=1}^{6} \int_{t_1}^{t_2} \Big\{ \Big[\delta\psi_i \circ EI_i \frac{\partial \psi_i}{\partial \xi_i}\Big]_{\Omega_i} - \Big[\delta D_i \circ K_i\Big(\psi_i - \frac{\partial D_i}{\partial \xi_i}\Big)\Big]_{\Omega_i} \Big\} dt. \tag{3.235}$$

Let the external force and torque applied to the bus be given by F_s and T as defined in Section 2.4 respectively, and exiternal forces $\bar{F}_{b_i}, \bar{F}_{b_{T_i}}$ applied to the beams be given by

$$\bar{F}_{b_i} \equiv \begin{cases} (0, F_{i_y}, F_{i_z})', & \text{for } i\text{=1,2,3,4}, \\ (F_{i_x}, 0, F_{i_z})', & \text{for } i\text{=5,6}. \end{cases} \tag{3.236}$$

$$\bar{F}_{b_{T_i}} \equiv \begin{cases} (0, F_{T_{i_y}}, F_{T_{i_z}})', & \text{for } i\text{=1,2,3,4}, \\ (F_{T_{i_x}}, 0, F_{T_{i_z}})', & \text{for } i\text{=5,6}. \end{cases} \tag{3.237}$$

Then the virtual work [Meriam 99] done by the external forces F_s, T , $\bar{F}_{b_i}$ and $\bar{F}_{b_{T_i}}$ is given by

$$\delta W \;=\; F_s \circ \delta R \;+\; T \circ \delta\theta \;+\; \sum_{i=1}^{6} \int_{\Omega_i} \bar{F}_{b_i} \circ \delta D_i d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \bar{F}_{b_{T_i}} \circ \delta\psi_i d\xi_i. \tag{3.238}$$

Substituting Equations (2.21), (3.233), (3.234), (2.29) and (3.238) into Equation (2.4) and integrating by parts with the end conditions in Equations (2.5.1), (2.5.2), (3.19) we obtain

$$\delta \int_{t_1}^{t_2} (L+W) dt \equiv - \int_{t_1}^{t_2} \Big\{ (\delta R) \circ \mathcal{I}_1 - (\delta\theta) \circ \mathcal{I}_2 + \sum_{i=1}^{6} \int_{\Omega_i} (\delta D_i) \circ \mathcal{I}_3 d\xi_i$$

$$+ \sum_{i=1}^{6} \Big(\int_{\Omega_i} (\delta\psi_i) \circ \mathcal{I}_4 d\xi_i \Big) + \mathcal{I}_5 \Big\} dt = 0, \tag{3.239}$$

where

$$\mathcal{I}_1 = m_s\frac{d^2R}{dt^2} + \frac{d}{dt}\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big(\frac{dR}{dt} + \omega\times\bar{r}_i + \dot{D}_i\Big)d\xi_i + \frac{Gm_em_TR}{|R|^3} - F_s, \qquad (3.240)$$

$$\mathcal{I}_2 = T - \frac{d}{dt}\Big\{(I_s\circ\omega) + \sum_{i=1}^{6}\int_{\Omega_i}\rho_i\bar{r}_i\times\frac{dr_i}{dt}d\xi_i\Big\} + \sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big(\omega\times\bar{r}_i + \dot{D}_i\Big)\times\Big(\frac{dr_i}{dt}\Big)d\xi_i, \qquad (3.241)$$

$$\mathcal{I}_3 = \rho_i\frac{d^2r_i}{dt^2} + \frac{\partial}{\partial\xi_i}(K_i\psi_i) - \frac{\partial}{\partial\xi_i}\Big(K_i\frac{\partial D_i}{\partial\xi_i}\Big) - \bar{F}_{b_i}, \qquad (3.242)$$

$$\mathcal{I}_4 = \frac{d}{dt}\Big(I_{\rho_i}\frac{\partial\psi_i}{\partial t}\Big) + K_i\Big(\psi_i - \frac{\partial D_i}{\partial\xi_i}\Big) - \frac{\partial}{\partial\xi_i}\Big(EI_i\frac{\partial\psi_i}{\partial\xi_i}\Big) - \bar{F}_{b_{T_i}}, \qquad (3.243)$$

$$\mathcal{I}_5 = \sum_{i=1}^{6}\Big[EI_i\Big(\frac{\partial\psi_i}{\partial\xi_i}\Big)\circ(\delta\psi_i) - K_i\Big(\psi_i - \frac{\partial D_i}{\partial\xi_i}\Big)\circ(\delta D_i)\Big]_{\Omega_i}. \qquad (3.244)$$

Since the beam is assumed to be inextensible we have

$$\delta D_i \equiv \delta(Q_i(x_i,y_i,z_i)') = \begin{cases} (0,\delta y_i,\delta z_i)', & \text{for } i=1,2,3,4, \\ (\delta x_i, 0, \delta z_i)', & \text{for } i=5,6. \end{cases} \qquad (3.245)$$

Therefore, noting that the variations $\delta\theta, \delta R, \delta D_i, \delta\psi_i$, $i = 1,2,...,6$ are arbitrary, we obtain, from Equation (3.239), that

$$\mathcal{I}_1 = 0, \qquad (3.246)$$

$$\mathcal{I}_2 = 0, \qquad (3.247)$$

$$\mathcal{I}_3 = 0, \qquad (3.248)$$

$$\mathcal{I}_4 = 0, \qquad (3.249)$$

$$\mathcal{I}_5 = 0. \qquad (3.250)$$

One obtains from these equations a set of coupled differential equations for the dynamics of the flexible space station as follows. From Equation (3.246) we obtain the dynamics for the translational motion of the rigid bus

$$(m_s + m_b)\frac{d^2R}{dt^2} + \frac{d}{dt}\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big(\omega\times\bar{r}_i + \dot{D}_i\Big)d\xi_i + \frac{Gm_em_TR}{|R|^3} = F_s, \qquad (3.251)$$

where $m_b = \sum_{i=1}^{6} \int_{\Omega_i} dm_i$ is the total mass of the beams and $\rho_i \equiv \rho_i(\xi_i)$ is the mass per unit length of the beam i. For rotational dynamics (attitude motion) of the bus it is obtained from Equation(3.247) that

$$\frac{d}{dt}\Big\{(I_s + I_b)\circ\omega\Big\} + \frac{d}{dt}\sum_{i=1}^{6}\int_{\Omega_i} \rho_i \bar{r}_i \times \Big(\frac{dR}{dt} + \dot{D}_i\Big)d\xi_i$$
$$- \sum_{i=1}^{6}\int_{\Omega_i} \rho_i \Big\{(\omega \times \bar{r}_i + \dot{D}_i) \times \frac{dR}{dt}\Big\}d\xi_i = T, \qquad (3.252)$$

where $I_b \circ \omega = \sum_{i=1}^{6} \int \bar{r}_i \times (\omega \times \bar{r}_i) dm_i$. Let define $I_T \equiv I_s + I_b$ for notational simplicity. Using vector identities(Appendix F) we have

$$\sum_{i=1}^{6}\int_{\Omega_i} \bar{r}_i \times (w \times (w \times \bar{r}_i))dm_i = w \times (I_b \circ w), \qquad (3.253.1)$$

$$\sum_{i=1}^{6}\int_{\Omega_i}\Big\{\bar{r}_i \times (2w \times \dot{D}_i) + \bar{r}_i \times (\dot{w} \times \bar{r}_i)\Big\}dm_i = I_b \circ \dot{w} + \dot{I}_b \circ w + \sum_{i=1}^{6}\int_{\Omega_i} w \times (\bar{r}_i \times \dot{D}_i)dm_i. \qquad (3.253.2)$$

Substituting Equation (3.253) into Equation (3.252) we obtain

$$I_T \circ \dot{w} + w \times (I_T \circ w) + \dot{I}_b \circ w + \sum_{i=1}^{6}\int_{\Omega_i}\Big\{\bar{r}_i \times \Big(\frac{d^2R}{dt^2} + \ddot{D}_i\Big) + w \times (\bar{r}_i \times \dot{D}_i)\Big\}dm_i = T. \quad (3.254)$$

Since δD_i are given by Equation (3.245), from Equation (3.248) we obtain the dynamics for the beam vibration:

$$\rho_i \ddot{D}_i + \frac{\partial}{\partial \xi_i}\Big(K_i\Big(\psi_i - \frac{\partial D_i}{\partial \xi_i}\Big)\Big) + \rho_i Q_i\Big[\frac{d^2R}{dt^2} + \dot{\omega} \times \bar{r}_i + \omega \times \Big\{2\dot{D}_i + \omega \times \bar{r}_i\Big\}\Big] = \bar{F}_{b_i}. \quad (3.255)$$

From (3.249) we obtain that

$$\frac{d}{dt}\Big(I_{\rho_i}\frac{\partial \psi_i}{\partial t}\Big) + K_i\Big(\psi_i - \frac{\partial D_i}{\partial \xi_i}\Big) - \frac{\partial}{\partial \xi_i}\Big(EI_i\frac{\partial \psi_i}{\partial \xi_i}\Big) = \bar{F}_{b_{T_i}}. \qquad (3.256)$$

Thus we have twenty four second order partial differential equations for the dynamics of the flexible members. We need two boundary conditions to solve each beam equation and total number of boundary conditions for the solution of Equations (3.255-3.256) is forty eight. The boundary conditions can be obtained form Equation (3.250) and the geometric configuration of the structure[Craig 37, Kounadis 75, Russell 116]. For boundary conditions we suppose that

(i) The structures are in $i_b j_b$ plane in undisturbed state,

(ii) The beams 1,2,3,4 are clamped at the joints $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$,

(iii) Displacements of the beams 1,2,...,6 are small,

(iv) The beams 5,6 are rigidly jointed with the other four beams as shown in Fig.3.2.

Let $\frac{\partial}{\partial \xi_i} D_i$, $\frac{\partial^2}{\partial \xi_i^2} D_i$, $\frac{\partial^3}{\partial \xi_i^3} D_i$ $\frac{\partial}{\partial \xi_i}\psi_i$ and $\frac{\partial^2}{\partial \xi_i^2}\psi_i$ be denoted by D_{ξ_i}, $D_{\xi_i\xi_i}$, $D_{\xi_i\xi_i\xi_i}$, ψ_{ξ_i} and $\psi_{\xi_i\xi_i}$ respectively. Then rewriting Equation (3.250) we obtain for any time t,

$$\begin{aligned}
\mathcal{I}_5 \equiv\ & \delta\psi_1(l_1) \circ EI_1\psi_{\xi_1}(l_1) + \delta\psi_5(l_5) \circ EI_5\psi_{\xi_5}(l_5) && (3.257)\\
& +\delta\psi_2(l_2) \circ EI_2\psi_{\xi_2}(l_2) - \delta\psi_5(0) \circ EI_5\psi_{\xi_5}(0) && (3.258)\\
& +\delta\psi_3(l_3) \circ EI_3\psi_{\xi_3}(l_3) + \delta\psi_6(l_6) \circ EI_6\psi_{\xi_6}(l_6) && (3.259)\\
& +\delta\psi_4(l_4) \circ EI_4\psi_{\xi_4}(l_4) - \delta\psi_6(0) \circ EI_6\psi_{\xi_6}(0) && (3.260)\\
& -\delta\psi_1(0) \circ EI_1\psi_{\xi_1}(0) - \delta\psi_2(0) \circ EI_2\psi_{\xi_2}(0) && (3.261)\\
& -\delta\psi_3(0) \circ EI_3\psi_{\xi_3}(0) - \delta\psi_4(0) \circ EI_4\psi_{\xi_4}(0) && (3.262)\\
& -\delta D_1(l_1) \circ K_1(\psi_1 - D_{\xi_1})(l_1) - \delta D_5(l_5) \circ K_5(\psi_5 - D_{\xi_5})(l_5) && (3.263)\\
& -\delta D_2(l_2) \circ K_2(\psi_2 - D_{\xi_2})(l_2) + \delta D_5(0) \circ K_5(\psi_5 - D_{\xi_5})(0) && (3.264)\\
& -\delta D_3(l_3) \circ K_3(\psi_3 - D_{\xi_3})(l_3) - \delta D_6(l_6) \circ K_6(\psi_6 - D_{\xi_6})(l_6) && (3.265)\\
& -\delta D_4(l_4) \circ K_4(\psi_4 - D_{\xi_4})(l_4) + \delta D_6(0) \circ K_6(\psi_6 - D_{\xi_6})(0) && (3.266)\\
& +\delta D_1(0) \circ K_1(\psi_1 - D_{\xi_1})(0) + \delta D_2(0) \circ K_2(\psi_2 - D_{\xi_2})(0) && (3.267)\\
& +\delta D_3(0) \circ K_3(\psi_3 - D_{\xi_3})(0) + \delta D_4(0) \circ K_4(\psi_4 - D_{\xi_4})(0) = 0. && (3.268)
\end{aligned}$$

Since the displacement and slope vanish at a clamped boundary (i.e. $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$), we obtain the geometric boundary conditions:

$$\text{at } \xi_i = 0, \qquad y_i(0,t) = 0, \qquad z_i(0,t) = 0, \tag{3.269}$$

$$\psi_{i_y}(0,t) = 0, \qquad \psi_{i_z}(0,t) = 0, \quad \text{for } i = 1,2,3,4. \tag{3.270}$$

Since the displacements and slopes are continuous at the rigid joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$, we obtain:

Continuity of displacements at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$:

$$x_5(l_5,t)=0, \qquad z_1(l_1,t)=z_5(l_5,t), \quad at\ \mathcal{J}_5, \tag{3.271}$$

$$x_5(0,t)=0, \qquad z_2(l_2,t)=z_5(0,t), \quad at\ \mathcal{J}_6, \tag{3.272}$$

$$x_6(l_6,t)=0, \qquad z_3(l_3,t)=z_6(l_6,t), \quad at\ \mathcal{J}_7, \tag{3.273}$$

$$x_6(0,t)=0, \qquad z_4(l_4,t)=z_6(0,t), \quad at\ \mathcal{J}_8, \tag{3.274}$$

Continuity of slopes at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$ (Appendix G):

$$\psi_{1_y}(l_1,t)=-\psi_{5_x}(l_5,t),\ at\ \mathcal{J}_5, \tag{3.275}$$

$$\psi_{2_y}(l_2,t)=-\psi_{5_x}(0,t),\ at\ \mathcal{J}_6, \tag{3.276}$$

$$\psi_{3_y}(l_3,t)=\ \ \psi_{6_x}(l_6,t),\ at\ \mathcal{J}_7, \tag{3.277}$$

$$\psi_{4_y}(l_4,t)=\ \ \psi_{6_x}(0,t),\ at\ \mathcal{J}_8. \tag{3.278}$$

Applying the moment equation at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$, we have

$$at\ \mathcal{J}_5, \quad EI_{5z}\frac{\partial\psi_{5_z}}{\partial\xi_5}(l_5,t)=0, \quad EI_{1z}\frac{\partial\psi_{1_z}}{\partial\xi_1}(l_1,t)=0, \tag{3.279}$$

$$at\ \mathcal{J}_6, \quad EI_{5z}\frac{\partial\psi_{5_z}}{\partial\xi_5}(0,t)=0, \quad EI_{2z}\frac{\partial\psi_{2_z}}{\partial\xi_2}(l_2,t)=0, \tag{3.280}$$

$$at\ \mathcal{J}_7, \quad EI_{6z}\frac{\partial\psi_{6_z}}{\partial\xi_6}(l_6,t)=0, \quad EI_{3z}\frac{\partial\psi_{3_z}}{\partial\xi_3}(l_3,t)=0, \tag{3.281}$$

$$at\ \mathcal{J}_8, \quad EI_{6z}\frac{\partial\psi_{6_z}}{\partial\xi_6}(0,t)=0, \quad EI_{4z}\frac{\partial\psi_{4_z}}{\partial\xi_4}(l_4,t)=0. \tag{3.282}$$

Using Equations (3.271) and (3.273) into Equations (3.263) and (3.265) respectively, we obtain, from their first components, that

$$K_{1y}\Big(\psi_{1_y}-\frac{\partial}{\partial\xi_1}y_1\Big)(l_1,t)=0, \qquad K_{3y}\Big(\psi_{3_y}-\frac{\partial}{\partial\xi_3}y_3\Big)(l_3,t)=0. \tag{3.283}$$

Similarly using Equations (3.272) and (3.274) into Equations (3.264) and (3.266) respectively, we obtain, from their first elements, that

$$K_{2y}\Big(\psi_{2_y}-\frac{\partial}{\partial\xi_2}y_2\Big)(l_2,t)=0, \qquad K_{4y}\Big(\psi_{4_y}-\frac{\partial}{\partial\xi_4}y_4\Big)(l_4,t)=0. \tag{3.284}$$

Substituting Equations (3.271), (3.273), (3.272) and (3.274) into Equations (3.263), (3.265), (3.264) and (3.266) respectively, we obtain, from their second components, that

$$K_{1z}\Big(\psi_{1_x} - \frac{\partial}{\partial \xi_1} z_1\Big)(l_1,t) + K_{5z}\Big(\psi_{5_x} - \frac{\partial}{\partial \xi_5} z_5\Big)(l_5,t) = 0, \tag{3.285}$$

$$K_{3z}\Big(\psi_{3_x} - \frac{\partial}{\partial \xi_3} z_3\Big)(l_3,t) + K_{6z}\Big(\psi_{6_x} - \frac{\partial}{\partial \xi_6} z_6\Big)(l_6,t) = 0, \tag{3.286}$$

$$K_{2z}\Big(\psi_{2_x} - \frac{\partial}{\partial \xi_2} z_2\Big)(l_2,t) - K_{5z}\Big(\psi_{5_x} - \frac{\partial}{\partial \xi_5} z_5\Big)(0,t) = 0, \tag{3.287}$$

$$K_{4z}\Big(\psi_{4_x} - \frac{\partial}{\partial \xi_4} z_4\Big)(l_4,t) - K_{6z}\Big(\psi_{6_x} - \frac{\partial}{\partial \xi_6} z_6\Big)(0,t) = 0. \tag{3.288}$$

It is noted that the above equations (3.283-3.288) can be ontained from the force equations at the joints $\mathcal{J}_5$, $\mathcal{J}_6$, $\mathcal{J}_7$, $\mathcal{J}_8$. Substituting Equations (3.275) and (3.277) into Equations (3.257) and (3.259) respectively, we obtain, from their first elements, that

$$EI_{1y}\frac{\partial \psi_{1_y}}{\partial \xi_1}(l_1,t) - EI_{5x}\frac{\partial \psi_{5_z}}{\partial \xi_5}(l_5,t) = 0, \tag{3.289}$$

$$EI_{3y}\frac{\partial \psi_{3_y}}{\partial \xi_3}(l_3,t) + EI_{6x}\frac{\partial \psi_{6_z}}{\partial \xi_6}(l_6,t) = 0. \tag{3.290}$$

Similarly using Equations (3.276) and (3.278) into Equations (3.258) and (3.260) respectively, we obtain, from their first elements, that

$$EI_{2y}\frac{\partial \psi_{2_y}}{\partial \xi_2}(l_2,t) + EI_{5x}\frac{\partial \psi_{5_z}}{\partial \xi_5}(0,t) = 0, \tag{3.291}$$

$$EI_{4y}\frac{\partial \psi_{4_y}}{\partial \xi_4}(l_4,t) - EI_{6x}\frac{\partial \psi_{6_z}}{\partial \xi_6}(0,t) = 0. \tag{3.292}$$

We note that the above equations (3.289)-(3.292) also can be obtained from moment equations at the joints $\mathcal{J}_5$, $\mathcal{J}_6$, $\mathcal{J}_7$, $\mathcal{J}_8$.

Let α_i be defined by $\alpha_i = \dot{\omega} \times \bar{r}_i + 2\omega \times \dot{D}_i + \omega \times (\omega \times \bar{r}_i)$, i=1,2,...,6, and

$$F_{b_i} \equiv \begin{cases} (F_{i_y}, F_{i_z})', & \text{for } i=1,2,3,4, \\ (F_{i_x}, F_{i_z})', & \text{for } i=5,6. \end{cases} \tag{3.293}$$

$$F_{b_{T_i}} \equiv \begin{cases} (F_{T_{i_y}}, F_{T_{i_z}})', & \text{for } i=1,2,\ 3,4, \\ (F_{T_{i_x}}, F_{T_{i_z}})', & \text{for } i=5,6. \end{cases} \tag{3.294}$$

$$\tilde{Q}_i \equiv \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for } i\text{=1,2,3,4,} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for } i\text{=5,6.} \end{cases} \tag{3.295}$$

Then we finally obtain the complete dynamics of the space station subject to external forces in the following.

$$m_T \frac{d^2R}{dt^2} + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ (\dot{\omega} \times \bar{r}_i) + \omega \times (\omega \times \bar{r}_i) + \ddot{D}_i + 2\omega \times \dot{D}_i \Big\} d\xi_i + \frac{G m_e m_T R}{|R|^3} = F_s, \tag{3.296}$$

$$I_T \circ \dot{\omega} + \omega \times (I_T \circ \omega) + \dot{I}_b \circ \omega + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big(\bar{r}_i \times \frac{d^2R}{dt^2} \Big) d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i (\bar{r}_i \times \ddot{D}_i) d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \omega \times (\bar{r}_i \times \dot{D}_i) \Big\} d\xi_i = T, \tag{3.297}$$

and for $\xi_i \in \Omega_i; \xi_j \in \Omega_j$, i=1,2,3,4, j=5,6,

$$\rho_i \frac{\partial^2}{\partial t^2} \begin{pmatrix} y_i \\ z_i \end{pmatrix} + \rho_i \tilde{Q}_i \Big(\alpha_i + \frac{d^2R}{dt^2} \Big) + \frac{\partial}{\partial \xi_i} \Big[\begin{pmatrix} K_{i_y} & 0 \\ 0 & K_{i_z} \end{pmatrix} \begin{pmatrix} \psi_{i_y} - \frac{\partial}{\partial \xi_i} y_i \\ \psi_{i_z} - \frac{\partial}{\partial \xi_i} z_i \end{pmatrix} \Big] = F_{b_i}, \tag{3.298}$$

$$\rho_j \frac{\partial^2}{\partial t^2} \begin{pmatrix} x_j \\ z_j \end{pmatrix} + \rho_j \tilde{Q}_j \Big(\alpha_j + \frac{d^2R}{dt^2} \Big) + \frac{\partial}{\partial \xi_j} \Big[\begin{pmatrix} K_{j_x} & 0 \\ 0 & K_{j_z} \end{pmatrix} \begin{pmatrix} \psi_{j_x} - \frac{\partial}{\partial \xi_j} x_j \\ \psi_{j_z} - \frac{\partial}{\partial \xi_j} z_j \end{pmatrix} \Big] = F_{b_j}, \tag{3.299}$$

$$I_{\rho_i} \frac{\partial^2}{\partial t^2} \begin{pmatrix} \psi_{i_y} \\ \psi_{i_z} \end{pmatrix} + I_{\rho_i} \begin{pmatrix} 0 & -w_1 \\ w_1 & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \psi_{i_y} \\ \psi_{i_z} \end{pmatrix} + \begin{pmatrix} K_{i_y} & 0 \\ 0 & K_{i_z} \end{pmatrix} \begin{pmatrix} \psi_{i_y} \\ \psi_{i_z} \end{pmatrix} - \begin{pmatrix} K_{i_y} & 0 \\ 0 & K_{i_z} \end{pmatrix} \frac{\partial}{\partial \xi_i} \begin{pmatrix} y_i \\ z_i \end{pmatrix} - \frac{\partial}{\partial \xi_i} \begin{pmatrix} EI_{i_y} & 0 \\ 0 & EI_{i_z} \end{pmatrix} \frac{\partial}{\partial \xi_i} \begin{pmatrix} \psi_{i_y} \\ \psi_{i_z} \end{pmatrix} = F_{b_{T_i}}, \tag{3.300}$$

$$I_{\rho_j} \frac{\partial^2}{\partial t^2} \begin{pmatrix} \psi_{j_x} \\ \psi_{j_z} \end{pmatrix} + I_{\rho_j} \begin{pmatrix} 0 & w_2 \\ -w_2 & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \psi_{j_x} \\ \psi_{j_z} \end{pmatrix} + \begin{pmatrix} K_{j_x} & 0 \\ 0 & K_{j_z} \end{pmatrix} \begin{pmatrix} \psi_{j_x} \\ \psi_{j_z} \end{pmatrix} - \begin{pmatrix} K_{j_x} & 0 \\ 0 & K_{j_z} \end{pmatrix} \frac{\partial}{\partial \xi_j} \begin{pmatrix} x_j \\ z_j \end{pmatrix} - \frac{\partial}{\partial \xi_j} \begin{pmatrix} EI_{j_x} & 0 \\ 0 & EI_{j_z} \end{pmatrix} \frac{\partial}{\partial \xi_j} \begin{pmatrix} \psi_{j_x} \\ \psi_{j_z} \end{pmatrix} = F_{b_{T_j}}, \tag{3.301}$$

with the boundary conditions (3.269-3.292).

Using the same notations C_B^I, $(a_i, a_j, a_k)'$ as in Section 2.4 and performing the vector operations, Equations (3.296) -(3.301) and (3.269-3.292) could be expressed in explicit forms which are suitable for digital computer simulation. These are given below.

BUS DYNAMICS(TRANSLATION)

$$m_T \frac{d^2}{dt^2}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + C_B^I \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} + \begin{pmatrix} h_4(t) \\ h_5(t) \\ h_6(t) \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \tag{3.302}$$

where

$$f_1(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ (\bar{z}_i \dot{\omega}_2 - \bar{y}_i \dot{\omega}_3) + 2(\omega_2 \frac{\partial z_i}{\partial t} - \omega_3 \frac{\partial y_i}{\partial t}) + \omega_1 \omega_2 \bar{y}_i + w_1 w_3 \bar{z}_i - (w_2^2 + w_3^2)\bar{x}_i + \frac{\partial^2 x_i}{\partial t^2} \Big\} d\xi_i \tag{3.303}$$

$$f_2(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{x}_i \dot{w}_3 - \bar{z}_i \dot{w}_1 + 2(w_3 \frac{\partial x_i}{\partial t} - w_1 \frac{\partial z_i}{\partial t}) + w_2 w_3 \bar{z}_i + w_2 w_1 \bar{x}_i - (w_3^2 + w_1^2)\bar{y}_i + \frac{\partial^2 y_i}{\partial t^2} \Big\} d\xi_i, \tag{3.304}$$

$$f_3(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{y}_i \dot{w}_1 - \bar{x}_i \dot{w}_2 + 2(w_1 \frac{\partial y_i}{\partial t} - w_2 \frac{\partial x_i}{\partial t}) + w_3 w_1 \bar{x}_i + w_3 w_2 \bar{y}_i - (w_1^2 + w_2^2)\bar{z}_i + \frac{\partial^2 z_i}{\partial t^2} \Big\} d\xi_i, \tag{3.305}$$

and $h_4(t)$,$h_5(t)$, $h_6(t)$ are given by Equations (2.46.1), (2.46.2), (2.46.3), respectively.

BUS DYNAMICS(ROTATION)

$$I_r \begin{pmatrix} \dot{\omega_1} \\ \dot{\omega_2} \\ \dot{\omega_3} \end{pmatrix} + \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} I_r \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} f_4(t) \\ f_5(t) \\ f_6(t) \end{pmatrix} + \begin{pmatrix} f_7(t) \\ f_8(t) \\ f_9(t) \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}, \tag{3.306}$$

where

$$f_4(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{y}_i \frac{\partial z_i}{\partial t} 2 - \bar{z}_i \frac{\partial y_i}{\partial t} 2 + 2w_1 \bar{y}_i \frac{\partial y_i}{\partial t} + 2w_1 \bar{z}_i \frac{\partial z_i}{\partial t} - 2w_2 \bar{y}_i \frac{\partial x_i}{\partial t} - 2w_3 \bar{z}_i \frac{\partial x_i}{\partial t} \Big\} d\xi_i, \tag{3.307}$$

$$f_5(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{z}_i \frac{\partial^2 x_i}{\partial t^2} - \bar{x}_i \frac{\partial z_i}{\partial t} 2 + 2w_2 \bar{z}_i \frac{\partial z_i}{\partial t} + 2w_2 \bar{x}_i \frac{\partial x_i}{\partial t} - 2w_3 \bar{z}_i \frac{\partial y_i}{\partial t} - 2w_1 \bar{x}_i \frac{\partial y_i}{\partial t} \Big\} d\xi_i, \tag{3.308}$$

$$f_6(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{x}_i \frac{\partial y_i}{\partial t} 2 - \bar{y}_i \frac{\partial^2 x_i}{\partial t^2} + 2w_3 \bar{x}_i \frac{\partial x_i}{\partial t} + 2w_3 \bar{y}_i \frac{\partial y_i}{\partial t} - 2w_1 \bar{x}_i \frac{\partial z_i}{\partial t} - 2w_2 \bar{y}_i \frac{\partial z_i}{\partial t} \Big\} d\xi_i, \tag{3.309}$$

$$f_7(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{y}_i a_k - \bar{z}_i a_j \Big\} d\xi_i, \tag{3.310}$$

$$f_8(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{z}_i a_i - \bar{x}_i a_k \Big\} d\xi_i, \tag{3.311}$$

$$f_9(t) = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \bar{x}_i a_j - \bar{y}_i a_i \Big\} d\xi_i, \tag{3.312}$$

BEAM DYNAMICS(BEAM VIBRATION)

$$\rho_i \frac{\partial^2}{\partial t^2}\begin{pmatrix} y_i \\ z_i \end{pmatrix} + \rho_i \begin{pmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} y_i \\ z_i \end{pmatrix} + \frac{\partial}{\partial \xi_i}\left[\begin{pmatrix} K_{i_y} & 0 \\ 0 & K_{i_z} \end{pmatrix}\begin{pmatrix} \psi_{i_y} - \frac{\partial}{\partial \xi_i} y_i \\ \psi_{i_z} - \frac{\partial}{\partial \xi_i} z_i \end{pmatrix}\right]$$

$$+ \rho_i \begin{pmatrix} -\omega_1^2 - \omega_3^2 & -\dot{\omega}_1 + \omega_2\omega_3 \\ \dot{\omega}_1 + \omega_2\omega_3 & -\omega_1^2 - \omega_2^2 \end{pmatrix}\begin{pmatrix} \bar{y}_i \\ \bar{z}_i \end{pmatrix} + \rho_i \bar{x}_i \begin{pmatrix} \dot{\omega}_3 + \omega_1\omega_2 \\ -\dot{\omega}_2 + \omega_1\omega_3 \end{pmatrix}$$

$$+ \rho_i \tilde{Q}_i C_I^B \frac{d^2 R}{dt^2} = \begin{pmatrix} F_{i_y} \\ F_{i_z} \end{pmatrix}, \qquad \text{for } \xi_i \in \Omega_i,\ i = 1,2,3,4, \tag{3.313}$$

$$\rho_j \frac{\partial^2}{\partial t^2}\begin{pmatrix} x_j \\ z_j \end{pmatrix} + \rho_j \begin{pmatrix} 0 & 2\omega_2 \\ -2\omega_2 & 0 \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} x_j \\ z_j \end{pmatrix} + \frac{\partial}{\partial \xi_j}\left[\begin{pmatrix} K_{j_x} & 0 \\ 0 & K_{j_z} \end{pmatrix}\begin{pmatrix} \psi_{j_x} - \frac{\partial}{\partial \xi_j} x_j \\ \psi_{j_z} - \frac{\partial}{\partial \xi_j} z_j \end{pmatrix}\right]$$

$$+ \rho_j \begin{pmatrix} -w_2^2 - w_3^2 & \dot{w}_2 + w_3 w_1 \\ -\dot{w}_2 + w_3 w_1 & -w_1^2 - w_2^2 \end{pmatrix}\begin{pmatrix} \bar{x}_j \\ \bar{z}_j \end{pmatrix} + \rho_j \bar{y}_j \begin{pmatrix} -\dot{w}_3 + w_2 w_1 \\ \dot{w}_1 + w_2 w_3 \end{pmatrix}$$

$$+ \rho_j \tilde{Q}_j C_I^B \frac{d^2 R}{dt^2} = \begin{pmatrix} F_{j_x} \\ F_{j_z} \end{pmatrix}, \qquad \text{for } \xi_j \in \Omega_j,\ j = 5,6. \tag{3.314}$$

$$I_{\rho_i} \frac{\partial^2}{\partial t^2}\begin{pmatrix} \psi_{i_y} \\ \psi_{i_z} \end{pmatrix} + I_{\rho_i}\begin{pmatrix} 0 & -w_1 \\ w_1 & 0 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} \psi_{i_y} \\ \psi_{i_z} \end{pmatrix} + \begin{pmatrix} K_{i_y} & 0 \\ 0 & K_{i_z} \end{pmatrix}\begin{pmatrix} \psi_{i_y} \\ \psi_{i_z} \end{pmatrix}$$

$$- \begin{pmatrix} K_{i_y} & 0 \\ 0 & K_{i_z} \end{pmatrix}\frac{\partial}{\partial \xi_i}\begin{pmatrix} y_i \\ z_i \end{pmatrix} - \frac{\partial}{\partial \xi_i}\begin{pmatrix} EI_{i_y} & 0 \\ 0 & EI_{i_z} \end{pmatrix}\frac{\partial}{\partial \xi_i}\begin{pmatrix} \psi_{i_y} \\ \psi_{i_z} \end{pmatrix} = \begin{pmatrix} F_{yT_i} \\ F_{zT_i} \end{pmatrix}, \tag{3.315}$$

$$I_{\rho_j} \frac{\partial^2}{\partial t^2}\begin{pmatrix} \psi_{j_x} \\ \psi_{j_z} \end{pmatrix} + I_{\rho_j}\begin{pmatrix} 0 & w_2 \\ -w_2 & 0 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} \psi_{j_x} \\ \psi_{j_z} \end{pmatrix} + \begin{pmatrix} K_{j_x} & 0 \\ 0 & K_{j_z} \end{pmatrix}\begin{pmatrix} \psi_{j_x} \\ \psi_{j_z} \end{pmatrix}$$

$$- \begin{pmatrix} K_{j_x} & 0 \\ 0 & K_{j_z} \end{pmatrix}\frac{\partial}{\partial \xi_j}\begin{pmatrix} x_j \\ z_j \end{pmatrix} - \frac{\partial}{\partial \xi_j}\begin{pmatrix} EI_{j_x} & 0 \\ 0 & EI_{j_z} \end{pmatrix}\frac{\partial}{\partial \xi_j}\begin{pmatrix} \psi_{j_x} \\ \psi_{j_z} \end{pmatrix} = \begin{pmatrix} F_{xT_j} \\ F_{zT_j} \end{pmatrix} \tag{3.316}$$

BOUNDARY CONDITIONS FOR BEAM DYNAMICS

Clamped at $\xi_i = 0$, i=1,2,3,4 for the joints $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$:

$$y_i(0,t) = 0, \qquad z_i(0,t) = 0, \tag{3.317}$$

$$\psi_{i_y}(0,t) = 0, \qquad \psi_{i_z}(0,t) = 0. \tag{3.318}$$

Continuity of displacements at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$:

$$x_5(l_5,t) = 0, \;\; z_1(l_1,t) = z_5(l_5,t), \;\; at \; \mathcal{J}_5, \tag{3.319}$$

$$x_5(0,t) = 0, \;\; z_2(l_2,t) = z_5(0,t), \;\; at \; \mathcal{J}_6, \tag{3.320}$$

$$x_6(l_6,t) = 0, \;\; z_3(l_3,t) = z_6(l_6,t), \;\; at \; \mathcal{J}_7, \tag{3.321}$$

$$x_6(0,t) = 0, \;\; z_4(l_4,t) = z_6(0,t), \;\; at \; \mathcal{J}_8. \tag{3.322}$$

Continuity of slopes at the joints $\mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \mathcal{J}_8$:

$$\psi_{1_y}(l_1,t) = -\psi_{5_x}(l_5,t), \;\; at \; \mathcal{J}_5, \tag{3.323}$$

$$\psi_{2_y}(l_2,t) = -\psi_{5_x}(0,t), \;\; at \; \mathcal{J}_6, \tag{3.324}$$

$$\psi_{3_y}(l_3,t) = \;\; \psi_{6_x}(l_6,t), \;\; at \; \mathcal{J}_7, \tag{3.325}$$

$$\psi_{4_y}(l_4,t) = \;\; \psi_{6_x}(0,t), \;\; at \; \mathcal{J}_8. \tag{3.326}$$

Equilibrium equations:

$$at \; \mathcal{J}_5, \quad EI_{5z}\frac{\partial\psi_{5_z}}{\partial\xi_5}(l_5,t) = 0, \quad EI_{1z}\frac{\partial\psi_{1_z}}{\partial\xi_1}(l_1,t) = 0, \tag{3.327}$$

$$at \; \mathcal{J}_6, \quad EI_{5z}\frac{\partial\psi_{5_z}}{\partial\xi_5}(0,t) = 0, \quad EI_{2z}\frac{\partial\psi_{2_z}}{\partial\xi_2}(l_2,t) = 0, \tag{3.328}$$

$$at \; \mathcal{J}_7, \quad EI_{6z}\frac{\partial\psi_{6_z}}{\partial\xi_6}(l_6,t) = 0, \quad EI_{3z}\frac{\partial\psi_{3_z}}{\partial\xi_3}(l_3,t) = 0, \tag{3.329}$$

$$at \; \mathcal{J}_8, \quad EI_{6z}\frac{\partial\psi_{6_z}}{\partial\xi_6}(0,t) = 0, \quad EI_{4z}\frac{\partial\psi_{4_z}}{\partial\xi_4}(l_4,t) = 0. \tag{3.330}$$

$$K_{1y}\Big(\psi_{1_y} - \frac{\partial}{\partial\xi_1}y_1\Big)(l_1,t) = 0, \qquad K_{3y}\Big(\psi_{3_y} - \frac{\partial}{\partial\xi_3}y_3\Big)(l_3,t) = 0, \tag{3.331}$$

$$K_{2y}\Big(\psi_{2_y} - \frac{\partial}{\partial \xi_2} y_2\Big)(l_2, t) = 0, \qquad K_{4y}\Big(\psi_{4_y} - \frac{\partial}{\partial \xi_4} y_4\Big)(l_4, t) = 0, \tag{3.332}$$

$$K_{1z}\Big(\psi_{1_z} - \frac{\partial}{\partial \xi_1} z_1\Big)(l_1, t) + K_{5z}\Big(\psi_{5_z} - \frac{\partial}{\partial \xi_5} z_5\Big)(l_5, t) = 0, \tag{3.333}$$

$$K_{3z}\Big(\psi_{3_z} - \frac{\partial}{\partial \xi_3} z_3\Big)(l_3, t) + K_{6z}\Big(\psi_{6_z} - \frac{\partial}{\partial \xi_6} z_6\Big)(l_6, t) = 0, \tag{3.334}$$

$$K_{2z}\Big(\psi_{2_z} - \frac{\partial}{\partial \xi_2} z_2\Big)(l_2, t) - K_{5z}\Big(\psi_{5_z} - \frac{\partial}{\partial \xi_5} z_5\Big)(0, t) = 0, \tag{3.335}$$

$$K_{4z}\Big(\psi_{4_z} - \frac{\partial}{\partial \xi_4} z_4\Big)(l_4, t) - K_{6z}\Big(\psi_{6_z} - \frac{\partial}{\partial \xi_6} z_6\Big)(0, t) = 0. \tag{3.336}$$

$$EI_{1y}\frac{\partial \psi_{1_y}}{\partial \xi_1}(l_1, t) - EI_{5x}\frac{\partial \psi_{5_x}}{\partial \xi_5}(l_5, t) = 0, \tag{3.337}$$

$$EI_{3y}\frac{\partial \psi_{3_y}}{\partial \xi_3}(l_3, t) + EI_{6x}\frac{\partial \psi_{6_x}}{\partial \xi_6}(l_6, t) = 0, \tag{3.338}$$

$$EI_{2y}\frac{\partial \psi_{2_y}}{\partial \xi_2}(l_2, t) + EI_{5x}\frac{\partial \psi_{5_x}}{\partial \xi_5}(0, t) = 0, \tag{3.339}$$

$$EI_{4y}\frac{\partial \psi_{4_y}}{\partial \xi_4}(l_4, t) - EI_{6x}\frac{\partial \psi_{6_x}}{\partial \xi_6}(0, t) = 0. \tag{3.340}$$

The complete dynamics of the flexible space station consisting of the main rigid body and flexible structures rigidly attached to it (Fig.3.1) is given by a *hybrid system* which consists of six nonlinear ordinary differential equations (3.302),(3.306) and the partial differential equations (3.313-3.316) along with the boundary conditions in (3.317-3.340).

We note from Equations (3.302-3.340) or (3.296-3.301) that the four sets of equations are strongly coupled. Thus any rotation of the bus would induce vibration in the beam and vice-versa. Also any disturbance of the orbit would induce vibrations in the beam and rotations in the bus and vice-versa. We have similar observations as (OBS.3.1) and (OBS.3.2) in Section 3.4 for the dynamics for translational and rotational motions of the bus. Further from the beam equations (3.313-3.316) we note that

(OBS.3.5) The effect of rigid body rotation and translation on the vibration of the beams is represented by the forcing terms $\rho_i \tilde{Q}_i(\alpha_i + \frac{d^2R}{dt^2})$ and $\rho_j \tilde{Q}_j(\alpha_j + \frac{d^2R}{dt^2})$ as shown in Equations (3.298) and (3.299) respectively. The effects of the shear displacement are shown in Equations (3.300) and (3.301) (or (3.315-3.316) through the K_y and K_z terms. Thus it is clear that without the coupling terms $(\alpha_i + \frac{d^2R}{dt^2})$ and $(\alpha_j + \frac{d^2R}{dt^2})$, Equations (3.298-3.301) (or (3.313-3.316)) become Timoshenko equations[Traill-Nash and Collar 131] of simple beams.

Remark 3.5

We have similar equations for the dynamics of the space station as developed in Section 3.4 and Section 3.5 where they are derived using Euler-Bernoulli and Rayleigh beam theories. However, note that because of the effects of rotary inertia and shear displacement of an incremental section, the transverse vibration of the beams is governed by Equations (3.298-3.301) (or 3.313-3.316) and boundary conditions (3.317-3.340). In fact, the second terms of Equations (3.300-3.301) (or the second terms of Equations (3.315-3.316)) represent the effect of the rotary inertia and all the terms including K_y and K_z reflect the influence of the shear displacement on the transverse vibration of the beams. Hence this improved beam model enables us to study the influence of the beam dynamics including shear displacement on the dynamic performance such as rotation and translation of the rigid bus of the space station.

Remark 3.6

Based on Timoshenko beam theory [Traill-Nash and Collar 131] in this section we have developed a complete model for the space station (Fig.3.1) consisting of a rigid central body and flexible structures through the variational formulation. In the derivation of equations of motion we have approximated the gravitational potential energy P_g by Equation (2.16) (details are given in Remark 2.1). Orbiting space stations are often subject to external disturbances caused by various sources such as those mentioned in Remark 3.2. The stabilization of the space station under the influence of those disturbing forces is considered in Section 4.4.3.

3.7 Summary

In this chapter, using the extended Hamilton's principle a complete dynamic model for the space station has been developed by the author and Ahmed[S4]. The space station consists of a massive central body (which is assumed to be rigid) and flexible structures which are treated as flexible beams with equivalent beam parameters. In the development of equations of motion, *Euler-Bernoulli, Rayleigh* and *Timoshenko beam equations* were utilized for the idealized beams. The dynamics of the space station based on these beam theories are given by hybrid systems which are combinations of ordinary differentail equations describing the translational and rotational motions of the ridid body and partial differentail equations governing the vibration of the flexible structures along with boundary conditions. The equations clearly indicate strong couplings among the state variables such as radial vector, attitude angles (or angular velocities) and displacements of the beams of the system. This complete dynamic model would have many potential applications such as stability analysis and controller design for actual space stations.

Chapter 4

STABILIZATION OF FLEXIBLE SPACE STRUCTURES

4.1 Introduction

In this chapter we consider the problem of stabilization of large flexible space structures subject to disturbing forces. Large orbiting space structures are expected to experience mechanical vibrations arising from several disturbing forces such as those induced by shuttle takeoff or docking and crew movements. For certain applications, for example, on board communication systems, radio telescopes, etc. it is necessary that the system be brought to its rest state (or unperturbed state) following any disturbances. Further, structural vibration caused by various external forces should be suppressed for mechanical safety as well as keeping the desired line-of-sight of antenna for communications. Meirovitch[94-97] used the Lyapunov's method to establish stability of flexible spacecraft. Similarly, Ahmed and Biswas[3,25,26] proved the asymptotic stability of a rigid body with a flexible beam which is rotating without translation. By virtue of a Lyapunov functional based on the total energy of the system, in this chapter asymptotic stability of the large space structures whose dynamics are derived in terms of a hybrid system in the previous two chapters has been proved [82-84] using Lyapunov's approach.

In order to investigate the stability of the system described by the equations of motion in Chapter 2 and Chapter 3, we first introduce nominal trajectory (or orbit) equation in Section 4.2. Then using Lyapunov's approach we consider the feedback stabilization of the spacecraft and space station whose dynamics are given in Chapter 2 and Chapter 3 in Section 4.3 and Section 4.4 respectively. Finally a summary is presented in Section 4.5.

4.2 The Reference Trajectory

In this section we introduce the nominal trajectory equation when no disturbance occurs as a reference trajectory for the radial equation. This is given below.

In order to investigate the influence of the beam vibration and bus rotation on the fluctuations of the radial vector R, we define a vector $\tilde{R}$ by $\tilde{R} = R - R_0$, where R_0 is the nominal trajectory satisfying the following differential equation

$$m_T \frac{d^2 R_0}{dt^2} + \frac{G m_e m_T R_0}{|R_0|^3} = 0 \ . \tag{4.1}$$

We note that the trajectory described by Equation (4.1) depends on the intial conditions $R_0(0)$ and $\frac{dR_0}{dt}(0)$. For circular orbit (in orbital plane), the initial conditions should satisfy the following relation.

$$\frac{dR_0}{dt} \times \left(R_0 \times \frac{dR_0}{dt}\right) = \frac{G m_e R_0}{|R_0|}, \qquad \text{at } \ t = 0. \tag{4.2}$$

The conditions for elliptic, hyperbolic or parabolic orbit could be found in the book by Bate, Mueller and White[24]. Further, $\tilde{R} \equiv (\tilde{X}, \tilde{Y}, \tilde{Z})'$ represents the perturbation of the radial vector R around R_0 (nominal trajectory). Since $|\tilde{R}|$ is negligibly small comapred to $|R_0|$, using Equations (4.1) and (2.33) we obtain the defferential equation for $\tilde{R}$

$$m_T \frac{d^2 \tilde{R}}{dt^2} + \frac{d}{dt}\int_0^l \rho\left(\omega \times \bar{r} + \dot{D}\right) dx + \frac{G m_e m_T \tilde{R}}{|R_0|^3} = F_s, \tag{4.3}$$

which has the same form as Equation (11.4.4) in [Danby 39]. For numerical simulation, in Chapter 5 we use Equations (2.47),(2.54),(4.1) and (4.3).

4.3 Stabilization of the Spacecraft

In this section we consider the problem of stabilization of the spacecraft. In order to use the spacecraft for any application, for example communication, radiotelescopes, etc, it is necessary that the spacecraft be brought to its rest state following any disturbance. Accordingly, it is natural to consider the following state

$$\omega_1 = 0,\ \omega_2 = 0,\ \omega_3 = 0, \tag{4.4}$$

$$\text{and}\ \ y(x,t) = 0,\ z(x,t) = 0,\ \frac{\partial y}{\partial t}(x,t) = 0,\ \frac{\partial z}{\partial t}(x,t) = 0,\ \text{for } x \in [0,l], \tag{4.5}$$

$$\tilde{X}(t) = \tilde{Y}(t) = \tilde{Z}(t) = 0, \quad \tilde{v}_1(t) = \tilde{v}_2(t) = \tilde{v}_3(t) = 0, \tag{4.6}$$

where $\tilde{v} \equiv \frac{d\tilde{R}}{dt} \equiv (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)'$ and $\tilde{R} = (\tilde{X}, \tilde{Y}, \tilde{Z})'$, as the *rest state*. We note that here the trajectory (Equation(4.1)) corresponding to the rest state is the reference trajectory when no disturbance occurs. Now our concern is, if the system is perturbed from the reference trajectory (or the target trajectory) subject to the external forces, can we find a control which will eventually steer the system back to the rest state? This problem of stabilization can be solved following *Lyapunov*'s approach[Ahmed 2,Lasalle and Lefschetz 79]. In the following sections, we show that the spacecraft, using velocity feedback controls, is asymptotically stable with respect to the rest state(4.4-4.6). For notational simplicity we define $(\tilde{A}_y, \tilde{A}_z)' \equiv \frac{-\rho G m_e}{|R_0|^3} Q R_0$ and $(T_x, T_y, T_z)' \equiv \frac{-\rho G m_e}{|R_0|^3} (\int_0^l \bar{r}\, dx) \times R_0$.

4.3.1 Stabilization of the model based on Euler Bernoulli Beam Theory

In the following theorem and its corollories we consider the stabilization of the spacecraft whose dynamics is derived based on Euler Bernoulli beam theory and given by Equations (2.42-2.54) by applying feedback controls. The feedback controls in several different forms such as proportional control, deadzone control, Bang-Bang control and saturation control [Biswas and Ahmed 26, Lim and Ahmed 83] are applied for the vibration suppression of the system arising from external perturbations.

Theorem 4.1 (**Distributed control**)

Consider the perturbed system described by Equations (2.47,2.54,4.3) around the reference trajectory Equation (4.1). Suppose that the controls applied to the system(2.47,2.54,4.3) are given by the feedback law

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)', \quad k_1, k_2, k_3 > 0, \tag{4.7}$$

$$F_b = (\tilde{A}_y - d_1(x)\frac{\partial y}{\partial t},\ \tilde{A}_z - d_2(x)\frac{\partial z}{\partial t})',\ d_1, d_2 > 0 \text{ for } x \in [0, l], \tag{4.8}$$

$$\text{and} \quad F_s = (-e_1\tilde{v}_1,\ -e_2\tilde{v}_2,\ -e_3\tilde{v}_3)', \qquad e_1, e_2, e_3, > 0. \tag{4.9}$$

Then the system is asymptotically stable (in the sense of Lyapunov) with respect to the rest state (4.4-4.6).

Proof

Let $\varphi \equiv (y, \frac{\partial y}{\partial t}, z, \frac{\partial z}{\partial t})'$, $\phi \equiv (y, z)'$ and introduce a scalar functional $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ defined by

$$\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \equiv \frac{1}{2}m_s(\tilde{v} \circ \tilde{v}) + \frac{1}{2}(I_s \circ \omega) \circ \omega + \frac{1}{2}\int_0^l \rho|\tilde{v} + (\dot{D} + \omega \times \bar{r})|^2 dx$$

$$+ \frac{1}{2}\int_0^l EI_y\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^l EI_z\left(\frac{\partial^2 z}{\partial x^2}\right)^2 dx + \frac{1}{2}\frac{Gm_e m_r|\tilde{R}|^2}{|R_0|^3}. \tag{4.10}$$

It is obvious from Equation (4.10) that $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) > 0$ for all $w, \varphi, \tilde{R}, \tilde{v} \neq 0$ and $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) = 0$ for all $w, \varphi, \tilde{R}, \tilde{v} = 0$. Now, in order to complete the proof of asymptotic stability it is enough to show that $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) < 0$ for all $t \geq 0$. In the following we first show that $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \leq 0$ for all $t \geq 0$ and then using additional arguments we conclude that it is indeed negative definite in the following. Scalar multiplying both sides of Eq.(2.47) (or equivalently Eq.(2.39)) by w and noting that $I_T = I_s + I_b$, where I_b varies with time because of vibrations of the beam, we obtain

$$\frac{d}{dt}\Big[\frac{1}{2}(I_T \cdot \omega)\cdot\omega\Big] + \frac{1}{2}(\dot{I}_b \cdot \omega)\cdot\omega + \int_{beam} \{\omega \times (\bar{r} \times \dot{D})\}\cdot\omega dm$$

$$+ \int_{beam} (\bar{r} \times \ddot{D})\cdot\omega dm + \int_{beam} \Big(\bar{r} \times \frac{d\tilde{v}}{dt}\Big)\cdot\omega dm = T\cdot\omega \quad (4.11)$$

It is easy to see that

$$(I_b \cdot \omega)\cdot\omega = \int_{beam} (\bar{r} \times (\omega \times \bar{r}))\cdot\omega dm = \int_{beam} |\omega \times \bar{r}|^2 dm \quad (4.12)$$

$$(\dot{I}_b \cdot \omega)\cdot\omega = \int_{beam} (\dot{D} \times (\omega \times \bar{r}))\cdot\omega dm + \int_{beam} (\bar{r} \times (\omega \times \dot{D}))\cdot\omega dm$$

$$= -2\int_{beam} (\omega \times (\omega \times \bar{r}))\cdot\dot{D} dm \quad (4.13)$$

Substituting Equations (4.12) and (4.13) into (4.11), we obtain

$$\frac{1}{2}\frac{d}{dt}\{(I_s \circ w)\circ w\} + \frac{1}{2}\int_0^l \rho|w \times \bar{r}|^2 dx - \int_0^l \rho(w \times (w \times \bar{r}))\circ\dot{D}dx$$

$$+ \int_0^l \rho\Big(\frac{d^2R_0}{dt^2} + \ddot{D} + \frac{d\tilde{v}}{dt}\Big)\circ(w \times \bar{r})dx = T\circ w. \quad (4.14)$$

Scalar multiplying the Eq.(2.54) (or (2.40)) by $\dot{\phi}$ and integrating by parts over $x \in [0, l]$ and using the boundary conditions (2.37), we obtain

$$\int_0^l \rho Q\Big[\frac{d^2R_0}{dt^2} + \frac{d\tilde{v}}{dt} + \ddot{D} + \dot{w}\times\bar{r} + 2w\times\dot{D} + w\times(w\times\bar{r})\Big]\circ\dot{\phi}dx$$

$$+ \frac{1}{2}\frac{d}{dt}\int_0^l EI\frac{\partial^2\phi}{\partial x^2}\circ\frac{\partial^2\phi}{\partial x^2}dx = \int_0^l F_b\circ\dot{\phi}dx. \quad (4.15)$$

Similarly, scalar multiplying Equation (2.38) by $\tilde{v}$, we get

$$\frac{1}{2}\frac{d}{dt}m_s(\tilde{v}\circ\tilde{v})+\frac{1}{2}\int_0^l\rho\frac{d}{dt}(\tilde{v}\circ\tilde{v})dx+\int_0^l\rho\frac{d^2\bar{r}}{dt^2}\circ\tilde{v}dx+\frac{Gm_em_T\tilde{R}\circ\tilde{v}}{|R_0|^3}=F_s\circ\tilde{v}. \quad (4.16)$$

Using the relation (4.12-4.13), it is not difficult to see that

$$\int_0^l\rho\Big\{|w\times\bar{r}|^2-(w\times(w\times\bar{r}))\circ\dot{D}+\bar{D}\circ(w\times\bar{r})\Big\}dx=\int_0^l\rho\frac{d^2\bar{r}}{dt^2}\circ(w\times\bar{r})dx. \quad (4.17)$$

Substituting this identity into Eq.(4.14) and adding Eqs. (4.14), (4.15) and (4.16) up, we have

$$\frac{1}{2}\frac{d}{dt}m_s(\tilde{v}\circ\tilde{v})+\frac{1}{2}\int_0^l\rho\frac{d}{dt}(\tilde{v}\circ\tilde{v})dx+\int_0^l\rho\frac{d^2\bar{r}}{dt^2}\circ\tilde{v}dx+\frac{Gm_em_T\tilde{R}\circ\tilde{v}}{|R_0|^3}$$

$$+\frac{1}{2}\frac{d}{dt}\Big((I_s\circ w)\circ w\Big)+\int_0^l\rho\frac{d^2\bar{r}}{dt^2}\circ(w\times\bar{r})dx+\int_0^l\rho\frac{d^2R_0}{dt^2}\circ(w\times\bar{r})dx+\int_0^l\rho\frac{d\tilde{v}}{dt}\circ(w\times\bar{r})dx$$

$$+\int_0^l\rho Q\frac{d^2R_0}{dt^2}\circ\dot{\phi}dx+\int_0^l\rho Q\Big(\frac{d\tilde{v}}{dt}+\frac{d^2\bar{r}}{dt^2}\Big)\circ\dot{\phi}dx+\frac{1}{2}\frac{d}{dt}\int_0^l EI\frac{\partial^2\phi}{\partial x^2}\circ\frac{\partial^2\phi}{\partial x^2}dx$$

$$= F_s\circ\tilde{v}+T\circ w+\int_0^l F_b\circ\dot{\phi}dx. \quad (4.18)$$

Then, taking time derivative of Eq. (4.10) (in the inertial frame) one finds that the left-hand-side expression of Eq. (4.18) equals $\frac{d}{dt}\mathrm{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ plus two additional terms as follows.

$$\frac{d}{dt}\mathrm{V}(\omega,\varphi,\tilde{R},\tilde{v};t)+\int_0^l\rho\Big\{\frac{d^2R_0}{dt^2}\circ(w\times\bar{r})+Q\frac{d^2R_0}{dt^2}\circ\dot{\phi}\Big\}dx=F_s\circ\tilde{v}+\int_0^l\Big(F_b\circ\dot{\phi}\Big)\,dx+T\circ\omega\,. \quad (4.19)$$

Substituting the controls given by Eqs.(4.7-4.9) into Eq.(4.19), we have

$$\frac{d}{dt}\mathrm{V}(\omega,\varphi,\tilde{R},\tilde{v};t)=-\Big(k_1\omega_1^2+k_2\omega_2^2+k_3\omega_3^2\Big)-\Big(e_1\tilde{v}_1^2+e_2\tilde{v}_2^2+e_3\tilde{v}_3^2\Big)$$

$$-\int_0^l\Big\{d_1(x)\Big(\frac{\partial y}{\partial t}\Big)^2+d_2(x)\Big(\frac{\partial z}{\partial t}\Big)^2\Big\}\,dx. \quad (4.20)$$

From Eq.(4.20), we see that $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative semidefinite for the given controls. We show that in fact $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative definite in the following. If $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t) \equiv 0$ for any finite interval of time, say $[t_1,t_2]$, the following must be true‡[Ahmed 6, Adams 1]. For all $t \in [t_1,t_2]$,

$$\begin{aligned}
&\text{(i)}\ \ w_1 = w_2 = w_3 = 0,\\
&\text{(ii)}\ \ \frac{\partial y}{\partial t} = \frac{\partial z}{\partial t} = 0, \quad \text{for all } x \in [0,l],\\
&\text{(iii)}\ \ \tilde{v}_1 = \tilde{v}_2 = \tilde{v}_3 = 0 \ \text{ and }\ F_s = 0.
\end{aligned} \tag{4.21}$$

This requires that for all $t \in [t_1,t_2]$, $\frac{d}{dt}\tilde{v} \equiv \frac{d^2}{dt^2}\tilde{R} = 0$, $\dot{w} = 0$ and $\ddot{\vec{\phi}} \equiv (\frac{\partial^2 y}{\partial t^2}, \frac{\partial^2 z}{\partial t^2})' = 0$ for $x \in [0,l]$. Using all these conditions and Eq.(4.21) into Eq.(4.3), it is clear that $\tilde{R} = 0$ for all $t \in [t_1,t_2]$. Then from Eq.(2.37) it follows that

$$\frac{\partial^2}{\partial x^2}\Big(EI\frac{\partial^2\phi}{\partial x^2}\Big) = 0, \qquad \text{for } \ x \in (0,l). \tag{4.22}$$

It is easy to see that the unique solution of the elliptic equation (4.22) along with the boundary conditions (2.37) is $\phi = 0$ for all $x \in [0,l]$. In other words, if for all $t \in [t_1,t_2]$ $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$=0, $w = \tilde{R} = 0$ and $\phi = 0$ for all $x \in [0,l]$. Hence $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative definite. Therefore, the function $\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ defined by Eq.(4.10) is, indeed, a *Lyapunov functional* and consequently, the system (Eqs. 2.47,2.54,4.3) is asymptotically stable with respect to the rest state(4.4-4.6). This completes the proof. ■

Remark 4.2

We note that the first and second terms in Eq.(4.10) represent the kinetic energy of the rigid bus due to orbital perturbation and bus rotation, and the third term gives the kinetic energy of the beam. The two integrals in the fourth and fifth terms

‡ Under the assumption of existence of classical solution, y, z, $\frac{\partial y}{\partial t}$ and $\frac{\partial z}{\partial t}$ are continuous and hence (ii) of Equation (4.21) is true.

represent the strain energy of the beam due to the deformation caused by vibration. Perturbation of the gravitational potential energy due to the orbital perturbation $\tilde{R}$ is represented by the last term. Therefore, the Lyapunov functional defined by Eq.(4.10) is indeed the total energy of the perturbed system (Eqs. 2.47,2.54,4.3).

From Equation (4.20), it is clear that in the absence of any control,

$$\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)=0,$$

so that $\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t) = \mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};0)$ for all t . This implies that the system (without external forces or controls) is conservative. However, interchange of energy between the bus and the beam may take place leading to change in the bus velocity trajectories and in the amplitude of vibration of the beam.

<u>Remark 4.3</u>

The feedback law given by Eqs.(4.7-4.9) can be physically realized through velocity feedback with gains k_1, k_2, k_3, d_1, d_2, e_1, e_2, e_3 and position feedback T_x, T_y, T_z, $\tilde{A}_y$, $\tilde{A}_z$. Since $-\frac{Gm_eR_0}{|R_0|^3}=\frac{d^2}{dt^2}R_0$ (see Eq.(4.1)), the two components $\tilde{A}_y$ and $\tilde{A}_z$ in Eq.(4.8) are the control forces for compensating the force $\rho\frac{d^2}{dt^2}R_0$ induced by the nominal trajectory R_0 which is known. On the other hand, T_x,T_y and T_z in Eq.(4.7) are the control moments for eliminating the torque $m_b r_{cm}\times\frac{d^2}{dt^2}R_0$ due to the nominal trajectory, where r_{cm} denotes the vector for the mass center of the beam with reference to the origin of the body frame. It is noted that under the small deformation of the beam r_{cm} can be treated as time invariant. The choice of feedback gains depends on the requirements of stabilization time and mechanical safety (see Remark 2.5).

The distributed control law in Eq.(4.8) may be difficult and costly for practical applications. In the following corollary, we consider stabilization using localized controls which can be more easily implemented in practice. We note that boundary controls are a special case of localized controls.

Corollary 4.4 (**Localized Control**)

Consider the system described by Equations (2.47,2.54,4.3) and suppose that the controls are given by the localized feedback structure:

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)',\quad k_1,k_2,k_3>0, \tag{4.23}$$

$$F_s = (-e_1\tilde{v}_1, -e_2\tilde{v}_2, -e_3\tilde{v}_3)',\quad e_1,e_2,e_3>0, \tag{4.24}$$

$$F_b = (\tilde{A}_y - d_1(x)\frac{\partial y}{\partial t},\ \tilde{A}_z - d_2(x)\frac{\partial z}{\partial t})',\quad d_1,d_2>0, \tag{4.25}$$

with
$$d_1(x) = \sum_{i=1}^{N} a_i\chi_{\mathcal{A}_i}(x),\quad a_i>0, \tag{4.26}$$

$$d_2(x) = \sum_{i=1}^{M} b_i\chi_{\mathcal{B}_i}(x),\quad b_i>0, \tag{4.27}$$

where χ_E denotes the indicator function of the set E and $\{\mathcal{A}_i, i=1,2,\cdots,N\}$ and $\{\mathcal{B}_i, i=1,2,\cdots,M\}$ are any two families of Lebesgue measurable subsets of $[0,l]$ such that $\frac{\partial y}{\partial t}$ and $\frac{\partial z}{\partial t}$ do not vanish identically on $\cup\mathcal{A}_i$ and $\cup\mathcal{B}_i$ respectively. Then, the system is asymptotically stable with respect to the rest state (4.4-4.6).

Proof The proof is essentially the same as above. ■

In some applications, it is not absolutely essential to stabilize the system with respect to the zero state. It suffices to maintain the system in a sufficiently small neighborhood of the zero state. Further, for the purpose of saving fuel, it is also advisable to allow some deadzone. In this situation, the system is only required to be asymptotically stable with respect to a neighborhood of the origin. This is achieved by use of controls with deadzone.

Corollary 4.5 **(Deadzone Control)**

Consider the system described by Equations (2.47,2.54,4.3). Suppose that the feedback controls are given by

$$T = (T_x - k_1\Upsilon_{\epsilon_1}(\omega_1), T_y - k_2\Upsilon_{\epsilon_2}(\omega_2), T_z - k_3\Upsilon_{\epsilon_3}(\omega_3))', \quad k_1, k_2, k_3 > 0, \tag{4.28}$$

$$F_b = \left(\tilde{A}_y - h_1(x)\Upsilon_{\epsilon_4}\left(\frac{\partial y}{\partial t}\right), \tilde{A}_z - h_2(x)\Upsilon_{\epsilon_5}\left(\frac{\partial z}{\partial t}\right)\right)', \quad h_1, h_2 > 0, \tag{4.29}$$

$$F_s = (-e_1\Upsilon_{\epsilon_6}(\tilde{v}_1), -e_2\Upsilon_{\epsilon_7}(\tilde{v}_2), -e_3\Upsilon_{\epsilon_8}(\tilde{v}_3))', \quad e_1, e_2, e_3 > 0, \tag{4.30}$$

where

$$\Upsilon_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ 0, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \tag{4.31}$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_d of the desired rest state, defined by

$$N_d \equiv \{(\omega, \dot{\phi}, \tilde{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \left|\frac{\partial y}{\partial t}\right| < \epsilon_4, \left|\frac{\partial z}{\partial t}\right| < \epsilon_5, |\tilde{v}_1| < \epsilon_6,$$
$$|\tilde{v}_2| < \epsilon_7, |\tilde{v}_3| < \epsilon_8; \epsilon_i > 0, i = 1, 2, \cdots, 8 \ \}. \tag{4.32}$$

Remark 4.6 **(Bang-Bang Control)**

We note that when $\epsilon_i = 0, i = 1, 2, \cdots, 8$ in Equations (4.28-4.30) we obtain the Bang-Bang controls which can be easily implemented in practical applications, as a special case of deadzone controls.

Corollary 4.7 **(Saturation Control)**

Consider the system described by Equations (2.47,2.54,4.3). Suppose that the feedback controls are given by

$$T = (T_x - k_1 S_{\epsilon_1}(\omega_1), T_y - k_2 S_{\epsilon_2}(\omega_2), T_z - k_3 S_{\epsilon_3}(\omega_3))', \ k_1, k_2, k_3 > 0, \tag{4.33}$$

$$F_b = \left(\tilde{A}_y - h_1(x)S_{\epsilon_4}\left(\frac{\partial y}{\partial t}\right), \tilde{A}_z - h_2(x)S_{\epsilon_5}\left(\frac{\partial z}{\partial t}\right)\right)', \quad h_1, h_2 > 0, \tag{4.34}$$

$$F_s = (-e_1 S_{\epsilon_6}(\tilde{v}_1), -e_2 S_{\epsilon_7}(\tilde{v}_2), -e_3 S_{\epsilon_8}(\tilde{v}_3))', \quad e_1, e_2, e_3 > 0, \tag{4.35}$$

where

$$\mathcal{S}_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ \xi, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \tag{4.36}$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_s of the desired rest state, defined by

$$N_s \equiv \{(\omega, \dot{\phi}, \tilde{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \left|\frac{\partial y}{\partial t}\right| < \epsilon_4, \left|\frac{\partial z}{\partial t}\right| < \epsilon_5, |\tilde{v}_1| < \epsilon_6,$$
$$|\tilde{v}_2| < \epsilon_7, |\tilde{v}_3| < \epsilon_8; \epsilon_i > 0, i = 1, 2, \cdots, 8 \ \}. \tag{4.37}$$

Remark 4.8

We note that the functions defined by Eqs.(4.31) and (4.36) are indeed deadzone and saturation function respectively and they are clearly characterized by following diagrams.

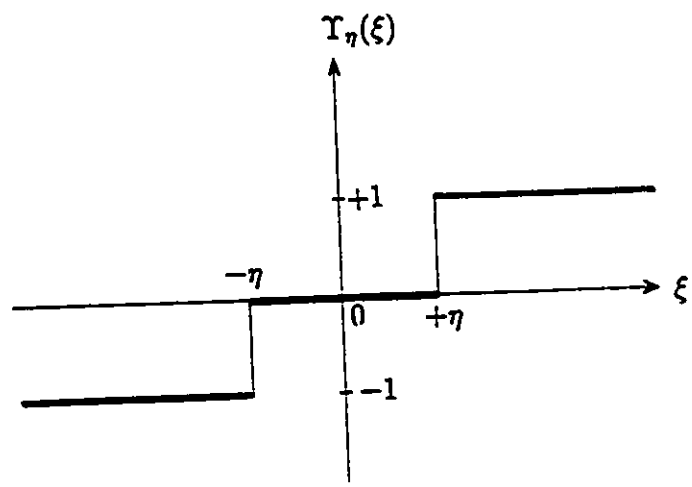


Fig. 4.1 A deadzone function.

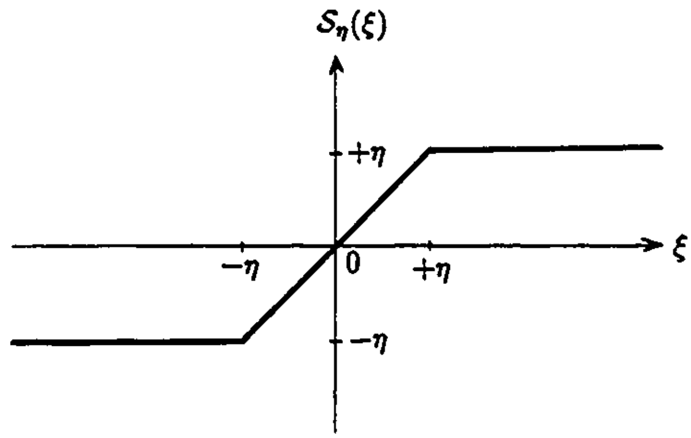


Fig. 4.2 A saturation function.

As a special case of deadzone, we have a Bang-Bang function if we set η=0 in Fig.4.1. It is clear from Fig.4.2 that the saturation curve is in fact a combination of a deadzone with size η and a proportional function (straight line) with slope one.

4.3.2 Stabilization of the model based on Rayleigh Beam Theory

In the following theorem and its corollories we consider the stabilization of the spacecraft whose dynamics is derived based on Rayleigh beam theory and given by equations (2.70-2.72) by applying feedback controls. The feedback controls in several different forms such as proportional control, deadzone control, Bang-Bang control and saturation control are applied for the vibration suppression of the system arising from external perturbations.

Theorem 4.9 **(Distributed control)**

Consider the perturbed system described by Equations (2.70,2.72,4.3) around the reference trajectory equations (4.1). Suppose that the controls applied to the system(2.70,2.72,4.3) are given by the feedback law

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)', \quad k_1, k_2, k_3 > 0, \tag{4.38}$$

$$F_b = (\tilde{A}_y - d_1(x)\frac{\partial y}{\partial t},\ \ \tilde{A}_z - d_2(x)\frac{\partial z}{\partial t})',\ d_1, d_2 > 0 \text{ for } x \in [0, l], \tag{4.39}$$

$$\text{and} \quad F_s = (-e_1\tilde{v}_1,\ -e_2\tilde{v}_2,\ -e_3\tilde{v}_3)', \qquad e_1, e_2, e_3, > 0. \tag{4.40}$$

Then the system is asymptotically stable (in the sense of Lyapunov) with respect to the rest state (4.4-4.6).

Proof

Let $\varphi \equiv (y, \frac{\partial y}{\partial t}, z, \frac{\partial z}{\partial t})'$, $\phi \equiv (y, z)'$ and introduce a scalar functional $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ defined by

$$\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \equiv \frac{1}{2}m_s(\tilde{v} \circ \tilde{v}) + \frac{1}{2}(I_s \circ \omega) \circ \omega + \frac{1}{2}\int_0^l \rho|\tilde{v} + (\dot{D} + \omega \times \bar{r})|^2 dx$$

$$+ \frac{1}{2}\int_0^l I_\rho \left|\frac{\partial^2 D}{\partial t \partial x}\right|^2 dx + \frac{1}{2}\int_0^l \left\{EI_y\left(\frac{\partial^2 y}{\partial x^2}\right)^2 + EI_z\left(\frac{\partial^2 z}{\partial x^2}\right)^2\right\} dx + \frac{1}{2}\frac{Gm_e m_r|\tilde{R}|^2}{|R_0|^3}. \tag{4.41}$$

It is obvious from Equation (4.41) that $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) > 0$ for all $w, \varphi, \tilde{R}, \tilde{v} \neq 0$ and $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) = 0$ for all $w, \varphi, \tilde{R}, \tilde{v} = 0$. Now, in order to complete the proof of asymptotic stability it is enough to show that $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) < 0$ for all $t \geq 0$. In the following we first show that $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \leq 0$ for all $t \geq 0$ and then using additional arguments we conclude that it is indeed negative definite. Scalar multiplying both sides of Eq.(2.71) (or equivalently Eq.(2.81)) by w and using Equations (4.11-4.13), we obtain

$$\frac{1}{2}\frac{d}{dt}\{(I_s \circ w) \circ w\} + \frac{1}{2}\int_0^l \rho|w \times \bar{r}|^2 dx - \int_0^l \rho(w \times (w \times \bar{r})) \circ \dot{D} dx$$

$$+ \int_0^l \rho\left(\frac{d^2 R_0}{dt^2} + \bar{D} + \frac{d\tilde{v}}{dt}\right) \circ (w \times \bar{r}) dx = T \circ w. \tag{4.42}$$

Scalar multiplying the Eq.(2.72) (or (2.88)) by $\dot{\phi}$ and integrating by parts over $x \in [0, l]$ and using the boundary conditions (2.89-2.93), we obtain

$$\int_0^l \rho Q\Big[\frac{d^2 R_0}{dt^2} + \frac{d\tilde{v}}{dt} + \ddot{D} + \dot{w} \times \bar{r} + 2w \times \dot{D} + w \times (w \times \bar{r})\Big] \circ \dot{\phi} dx$$

$$- \int_0^l \dot{\phi} \circ Q\Big(I_\rho \frac{d}{dt}\frac{\partial^2 D}{\partial x \partial t}\Big)dx + \frac{1}{2}\frac{d}{dt}\int_0^l EI\frac{\partial^2 \phi}{\partial x^2} \circ \frac{\partial^2 \phi}{\partial x^2} dx = \int_0^l F_b \circ \dot{\phi} dx. \quad (4.43)$$

Similarly, scalar multiplying Equation (2.70) by $\tilde{v}$, we get

$$\frac{1}{2}\frac{d}{dt}m_s(\tilde{v} \circ \tilde{v}) + \frac{1}{2}\int_0^l \rho\frac{d}{dt}(\tilde{v} \circ \tilde{v})dx + \int_0^l \rho\frac{d^2\bar{r}}{dt^2} \circ \tilde{v} dx + \frac{G m_e m_T \tilde{R} \circ \tilde{v}}{|R_0|^3} = F_s \circ \tilde{v}. \quad (4.44)$$

Using the relations (4.12-4.13), it is not difficult to see that

$$\int_0^l \rho\Big\{|w \times \bar{r}|^2 - (w \times (w \times \bar{r})) \circ \dot{D} + \ddot{D} \circ (w \times \bar{r})\Big\}dx = \int_0^l \rho\frac{d^2\bar{r}}{dt^2} \circ (w \times \bar{r})dx. \quad (4.45)$$

Substituting this identity into Eq.(4.42) and adding Eqs.(4.42), (4.43) and (4.44), we have

$$\frac{1}{2}\frac{d}{dt}m_s(\tilde{v} \circ \tilde{v}) + \frac{1}{2}\int_0^l \rho\frac{d}{dt}(\tilde{v} \circ \tilde{v})dx + \int_0^l \rho\frac{d^2\bar{r}}{dt^2} \circ \tilde{v} dx + \frac{G m_e m_T \tilde{R} \circ \tilde{v}}{|R_0|^3}$$

$$+\frac{1}{2}\frac{d}{dt}\Big((I_s \circ w) \circ w\Big) + \int_0^l \rho\frac{d^2\bar{r}}{dt^2} \circ (w \times \bar{r})dx + \int_0^l \rho\frac{d^2 R_0}{dt^2} \circ (w \times \bar{r})dx + \int_0^l \rho\frac{d\tilde{v}}{dt} \circ (w \times \bar{r})dx$$

$$+ \int_0^l \rho Q\frac{d^2 R_0}{dt^2} \circ \dot{\phi} dx + \int_0^l \rho Q\Big(\frac{d\tilde{v}}{dt} + \frac{d^2\bar{r}}{dt^2}\Big) \circ \dot{\phi} dx + \frac{1}{2}\frac{d}{dt}\int_0^l EI\frac{\partial^2 \phi}{\partial x^2} \circ \frac{\partial^2 \phi}{\partial x^2} dx$$

$$- \int_0^l \dot{\phi} \circ Q\frac{\partial}{\partial x}\Big(I_\rho \frac{d}{dt}\frac{\partial^2 D}{\partial x \partial t}\Big)dx \;=\; F_s \circ \tilde{v} + T \circ w + \int_0^l F_b \circ \dot{\phi} dx. \quad (4.46)$$

Then, taking time derivative of Eq.(4.41) (in the inertial frame) one finds that the left-hand-side expression of Eq.(4.46) equals $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ plus two additional terms as follows.

$$\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) + \int_0^l \rho\Big\{\frac{d^2 R_0}{dt^2} \circ (w \times \bar{r}) + Q\frac{d^2 R_0}{dt^2} \circ \dot{\phi}\Big\}dx = F_s \circ \tilde{v} + \int_0^l \Big(F_b \circ \dot{\phi}\Big)\, dx + T \circ \omega\,. \quad (4.47)$$

Substituting the controls given by Eqs.(4.38-4.40) into Eq.(4.47), we have

$$\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t) = -\left(k_1\omega_1^2 + k_2\omega_2^2 + k_3\omega_3^2\right) - \left(e_1\tilde{v}_1^2 + e_2\tilde{v}_2^2 + e_3\tilde{v}_3^2\right)$$

$$-\int_0^l \left\{ d_1(x)\left(\frac{\partial y}{\partial t}\right)^2 + d_2(x)\left(\frac{\partial z}{\partial t}\right)^2 \right\} dx. \tag{4.48}$$

From Eq.(4.48), we see that $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative semidefinite for the given controls. Using the same arguments as in the proof of theorem 3.1 one can show that in fact $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative definite. Hence $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative definite. Therefore, the function $\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ defined by Equation (4.41) is, indeed, a *Lyapunovfunctional* and consequently, the system described by Equations (2.70,2.72,4.3) is asymptotically stable with respect to the rest state(4.4-4.6). (For details see the proof of theorem 3.1). ■

Remark 4.10

It is interesting to see that the feedback controls have the same form as in theorem 3.1 which is for Euler-Bernoulli beam case, even though the system dynamics are different from those based on Euler-Bernoulli beam thoery. But we note that the Lyapunov functional (4.41) is different from that defined by (4.10).

We note that the Lyapunov functional defined by (4.41) is indeed the total energy of the perturbed system(2.70,2.72,4.3).

Corollary 4.11 **(Localized Control)**

Consider the system (Eqs. 2.70,2.72,4.3) and suppose that the controls are given by the localized feedback structure:

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)', \quad k_1,k_2,k_3 > 0, \tag{4.49}$$

$$F_s = (-e_1\tilde{v}_1, -e_2\tilde{v}_2, -e_3\tilde{v}_3)', \quad e_1,e_2,e_3 > 0, \tag{4.50}$$

$$F_b = (\tilde{A}_y - d_1(x)\frac{\partial y}{\partial t},\ \tilde{A}_z - d_2(x)\frac{\partial z}{\partial t})', \quad d_1,d_2 > 0, \tag{4.51}$$

$$\text{with} \quad d_1(x) = \sum_{i=1}^{N} a_i \chi_{\mathcal{A}_i}(x), \quad a_i > 0, \tag{4.52}$$

$$d_2(x) = \sum_{i=1}^{M} b_i \chi_{\mathcal{B}_i}(x), \quad b_i > 0, \tag{4.53}$$

where χ_E denotes the indicator function of the set E and $\{\mathcal{A}_i, i = 1, 2, \cdots, N\}$ and $\{\mathcal{B}_i, i = 1, 2, \cdots, M\}$ are any two families of Lebesgue measurable subsets of $[0, l]$ such that $\frac{\partial y}{\partial t}$ and $\frac{\partial z}{\partial t}$ do not vanish identically on $\cup \mathcal{A}_i$ and $\cup \mathcal{B}_i$ respectively. Then, the system is asymptotically stable with respect to the rest state (4.4-4.6).

Proof   The proof is essentially the same as above. ■

Corollary 4.12 **(Deadzone Control)**

Consider the system described by Equations (2.70,2.72,4.3). Suppose that the feedback controls are given by

$$T = (T_x - k_1 \Upsilon_{\epsilon_1}(\omega_1), T_y - k_2 \Upsilon_{\epsilon_2}(\omega_2), T_z - k_3 \Upsilon_{\epsilon_3}(\omega_3))', \quad k_1, k_2, k_3 > 0, \tag{4.54}$$

$$F_b = \left(\tilde{A}_y - h_1(x) \Upsilon_{\epsilon_4}\left(\frac{\partial y}{\partial t}\right), \tilde{A}_z - h_2(x) \Upsilon_{\epsilon_5}\left(\frac{\partial z}{\partial t}\right)\right)', \quad h_1, h_2 > 0, \tag{4.55}$$

$$F_s = (-e_1 \Upsilon_{\epsilon_6}(\tilde{v}_1), -e_2 \Upsilon_{\epsilon_7}(\tilde{v}_2), -e_3 \Upsilon_{\epsilon_8}(\tilde{v}_3))', \quad e_1, e_2, e_3 > 0, \tag{4.56}$$

where

$$\Upsilon_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ 0, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \tag{4.57}$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_d of the desired rest state, defined by

$$N_d \equiv \{(\omega, \dot{\phi}, \tilde{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \left|\frac{\partial y}{\partial t}\right| < \epsilon_4, \left|\frac{\partial z}{\partial t}\right| < \epsilon_5, |\tilde{v}_1| < \epsilon_6,$$
$$|\tilde{v}_2| < \epsilon_7, |\tilde{v}_3| < \epsilon_8; \epsilon_i > 0, i = 1, 2, \cdots, 8 \ \}. \tag{4.58}$$

Remark 4.13 **(Bang-Bang Control)**

We note that when $\epsilon_i = 0, i = 1, 2, \cdots, 8$ in Equations (4.54-4.56) we obtain the Bang-Bang controls which can be easily implemented in practical applications, as a special case of deadzone controls.

Corollary 4.14 **(Saturation Control)**

Consider the system described by Equations (2.70,2.72,4.3). Suppose that the feedback controls are given by

$$T = (T_x - k_1 \mathcal{S}_{\epsilon_1}(\omega_1), T_y - k_2 \mathcal{S}_{\epsilon_2}(\omega_2), T_z - k_3 \mathcal{S}_{\epsilon_3}(\omega_3))', \; k_1, k_2, k_3 > 0, \tag{4.59}$$

$$F_b = \left(\tilde{A}_y - h_1(x)\mathcal{S}_{\epsilon_4}\left(\frac{\partial y}{\partial t}\right), \tilde{A}_z - h_2(x)\mathcal{S}_{\epsilon_5}\left(\frac{\partial z}{\partial t}\right)\right)', \; h_1, h_2 > 0, \tag{4.60}$$

$$F_s = (-e_1 \mathcal{S}_{\epsilon_6}(\tilde{v}_1), -e_2 \mathcal{S}_{\epsilon_7}(\tilde{v}_2), -e_3 \mathcal{S}_{\epsilon_8}(\tilde{v}_3))', \; e_1, e_2, e_3 > 0, \tag{4.61}$$

where

$$\mathcal{S}_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ \xi, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \tag{4.62}$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_s of the desired rest state, defined by

$$N_s \equiv \{(\omega, \dot{\phi}, \tilde{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \left|\frac{\partial y}{\partial t}\right| < \epsilon_4, \left|\frac{\partial z}{\partial t}\right| < \epsilon_5, |\tilde{v}_1| < \epsilon_6,$$
$$|\tilde{v}_2| < \epsilon_7, |\tilde{v}_3| < \epsilon_8; \epsilon_i > 0, i = 1, 2, \cdots, 8 \;\}. \tag{4.63}$$

4.3.3 Stabilization of the model based on Timoshenko Beam Theory

In the following theorem and its corollories we consider the stabilization of the spacecraft whose dynamics is derived based on Timoshenko beam theory and given by equations (2.120-2.123) by applying feedback controls. The feedback controls in several different forms such as proportional control, deadzone control, Bang-Bang control and saturation control are applied for the vibration suppression of the system arising from external perturbations.

Theorem 4.15 (**Distributed control**)

Consider the perturbed system described by Equations (2.120, 2.121, 4.3) around the reference trajectory equations (4.1). Suppose that the controls applied to the system(2.120,2.121,4.3) are given by the feedback law

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)', \quad k_1, k_2, k_3 > 0, \tag{4.64}$$

$$F_b = (\tilde{A}_y - d_1(x)\frac{\partial y}{\partial t},\ \ \tilde{A}_z - d_2(x)\frac{\partial z}{\partial t})',\ d_1, d_2 > 0 \text{ for } x \in [0,l], \tag{4.65}$$

$$F_{b_r} = (-d_3(x)\frac{\partial \psi_y}{\partial t},\ \ -d_4(x)\frac{\partial \psi_z}{\partial t})',\ d_3, d_4 > 0 \text{ for } x \in [0,l], \tag{4.66}$$

$$\text{and}\quad F_s = (-e_1\tilde{v}_1,\ -e_2\tilde{v}_2,\ -e_3\tilde{v}_3)', \qquad e_1, e_2, e_3, > 0. \tag{4.67}$$

Then the system is asymptotically stable (in the sense of Lyapunov) with respect to the rest state (4.4-4.6).

Proof

Let $\varphi \equiv (y, \frac{\partial y}{\partial t}, z, \frac{\partial z}{\partial t}, \psi_y, \psi_z, \frac{\partial \psi_y}{\partial t}, \frac{\partial \psi_z}{\partial t})'$, $\phi \equiv (y,z)'$ and introduce a scalar functional $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ defined by

$$\begin{aligned}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \equiv &\frac{1}{2} m_s(\tilde{v} \circ \tilde{v}) + \frac{1}{2}(I_s \circ \omega) \circ \omega + \frac{1}{2}\int_0^l \rho|\tilde{v} + (\dot{D} + \omega \times \bar{r})|^2 dx \\ &+ \frac{1}{2}\int_0^l I_\rho \frac{\partial \psi}{\partial t} \circ \frac{\partial \psi}{\partial t} dx + \frac{1}{2}\int_0^l \left\{K\left(\psi - \frac{\partial D}{\partial x}\right) \circ \left(\psi - \frac{\partial D}{\partial x}\right) + EI\frac{\partial \psi}{\partial x} \circ \frac{\partial \psi}{\partial x}\right\} dx \\ &+ \frac{1}{2}\frac{G m_e m_r |\tilde{R}|^2}{|R_0|^3}.\end{aligned} \tag{4.68}$$

It is obvious from Equation (4.68) that $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) > 0$ for all $w, \varphi, \tilde{R}, \tilde{v} \neq 0$ and $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ =0 for all $w, \varphi, \tilde{R}, \tilde{v} = 0$. Now, in order to complete the proof of asymptotic stability it is enough to show that $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) < 0$ for all $t \geq 0$. In the following we first show that $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \leq 0$ for all $t \geq 0$ and then using additional arguments we conclude that it is indeed negative definite. Scalar multiplying both sides of Eq.(2.121) (or equivalently Eq.(2.132)) by w and using Equations (4.11-4.13), we obtain

$$\frac{1}{2}\frac{d}{dt}\{(I_s \circ w)\circ w\} + \frac{1}{2}\int_0^l \rho|w\times\bar{r}|^2 dx - \int_0^l \rho(w\times(w\times\bar{r}))\circ\dot{D}dx$$

$$+\int_0^l \rho\Big(\frac{d^2R_0}{dt^2} + \ddot{D} + \frac{d\bar{v}}{dt}\Big)\circ(w\times\bar{r})dx = T\circ w. \qquad (4.69)$$

Scalar multiplying the Eq.(2.113) (or (2.122)) by $\dot{\phi}$ and integrating by parts over $x\in[0,l]$ and using the boundary conditions (2.116-2.119), we obtain

$$\int_0^l \rho Q\Big[\frac{d^2R_0}{dt^2} + \frac{d\bar{v}}{dt} + \ddot{D} + \dot{w}\times\bar{r} + 2w\times\dot{D} + w\times(w\times\bar{r})\Big]\circ\dot{\phi}dx$$

$$-\int_0^l QK\Big(\psi - \frac{\partial D}{\partial x}\Big)\circ\frac{\partial^2 D}{\partial x\partial t}dx = \int_0^l F_b\circ\dot{\phi}dx. \qquad (4.70)$$

Similarly, scalar multiplying Equation (2.114) (or 2.123) by $\frac{\partial\psi}{\partial t}$ and integrating by parts, we obtain

$$\int_0^l\Big\{\Big(\frac{\partial}{\partial t}I_\rho\frac{\partial\psi}{\partial t}\Big)\circ\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial t}\circ K\Big(\psi - \frac{\partial D}{\partial x}\Big)\Big\}dx + \int_0^l EI\frac{\partial\psi}{\partial x}\circ\frac{\partial\psi}{\partial x}dx = \int_0^l F_{b_T}\circ\frac{\partial\psi}{\partial t}dx. \qquad (4.71)$$

Scalar multiplying Equation (2.120) by $\bar{v}$, we get

$$\frac{1}{2}\frac{d}{dt}m_s(\bar{v}\circ\bar{v}) + \frac{1}{2}\int_0^l \rho\frac{d}{dt}(\bar{v}\circ\bar{v})dx + \int_0^l \rho\frac{d^2\bar{r}}{dt^2}\circ\bar{v}dx + \frac{Gm_em_T\tilde{R}\circ\bar{v}}{|R_0|^3} = F_s\circ\bar{v}. \quad (4.72)$$

Using the relations (4.12-4.13), it is not difficult to see that

$$\int_0^l \rho\Big\{|w\times\bar{r}|^2 - (w\times(w\times\bar{r}))\circ\dot{D} + \ddot{D}\circ(w\times\bar{r})\Big\}dx = \int_0^l \rho\frac{d^2\bar{r}}{dt^2}\circ(w\times\bar{r})dx. \quad (4.73)$$

Substituting this identity into Eq.(4.69) and adding Eqs.(4.69), (4.70) and (4.72) up, we have

$$\frac{1}{2}\frac{d}{dt}m_s(\bar{v}\circ\bar{v}) + \frac{1}{2}\int_0^l \rho\frac{d}{dt}(\bar{v}\circ\bar{v})dx + \int_0^l \rho\frac{d^2\bar{r}}{dt^2}\circ\bar{v}dx + \frac{Gm_em_T\tilde{R}\circ\bar{v}}{|R_0|^3}$$

$$+\frac{1}{2}\frac{d}{dt}\Big((I_s \circ w)\circ w\Big)+\int_0^l \rho\frac{d^2\bar{r}}{dt^2}\circ(w\times\bar{r})dx+\int_0^l \rho\frac{d^2R_0}{dt^2}\circ(w\times\bar{r})dx+\int_0^l \rho\frac{d\tilde{v}}{dt}\circ(w\times\bar{r})dx$$

$$+\int_0^l \rho Q\frac{d^2R_0}{dt^2}\circ\dot{\phi}dx+\int_0^l \rho Q\Big(\frac{d\tilde{v}}{dt}+\frac{d^2\bar{r}}{dt^2}\Big)\circ\dot{\phi}dx-\int_0^l QK\Big(\psi-\frac{\partial D}{\partial x}\Big)\circ\frac{\partial^2 D}{\partial x\partial t}dx$$

$$+\int_0^l\Big\{\Big(\frac{\partial}{\partial t}I_\rho\frac{\partial\psi}{\partial t}\Big)\circ\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial t}\circ K\Big(\psi-\frac{\partial D}{\partial x}\Big)\Big\}dx+\int_0^l EI\frac{\partial\psi}{\partial x}\circ\frac{\partial\psi}{\partial x}dx$$

$$= F_s\circ\tilde{v}+T\circ w+\int_0^l F_b\circ\dot{\phi}dx+\int_0^l F_{b_T}\circ\frac{\partial\psi}{\partial t}dx. \qquad (4.74)$$

Then, taking time derivative of Eq.(4.68) (in the inertial frame) one finds that the left-hand-side expression of Eq.(4.74) equals $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ plus two additional terms as follows.

$$\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)+\int_0^l\rho\Big\{\frac{d^2R_0}{dt^2}\circ(w\times\bar{r})+Q\frac{d^2R_0}{dt^2}\circ\dot{\phi}\Big\}dx$$
$$=F_s\circ\tilde{v}+\int_0^l\Big(F_b\circ\dot{\phi}\Big)\,dx+T\circ\omega+\int_0^l F_{b_T}\circ\frac{\partial\psi}{\partial t}dx. \quad (4.75)$$

Substituting the controls given by Eqs.(4.64-4.67) into Eq.(4.75), we have

$$\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)=-\Big(k_1\omega_1^2+k_2\omega_2^2+k_3\omega_3^2\Big)-\Big(e_1\tilde{v}_1^2+e_2\tilde{v}_2^2+e_3\tilde{v}_3^2\Big)$$

$$-\int_0^l\Big\{d_1(x)\Big(\frac{\partial y}{\partial t}\Big)^2+d_2(x)\Big(\frac{\partial z}{\partial t}\Big)^2+d_3(x)\Big(\frac{\partial\psi_y}{\partial t}\Big)^2+d_4(x)\Big(\frac{\partial\psi_z}{\partial t}\Big)^2\Big\}\,dx. \quad (4.76)$$

From Eq.(4.76), we see that $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative semidefinite for the given controls. We show that in fact $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative definite in the following. If $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)\equiv 0$ for any finite interval of time, say $[t_1,t_2]$, the following must be true‡[Ahmed 6, Adams 1]. For all $t \in [t_1, t_2]$,

$$\begin{aligned}
&\text{(i)}\ \ w_1 = w_2 = w_3 = 0, \\
&\text{(ii)}\ \ \frac{\partial y}{\partial t} = \frac{\partial z}{\partial t} = 0,\ \ \frac{\partial \psi_y}{\partial t} = \frac{\partial \psi_z}{\partial t} = 0, \quad \text{for all } x \in [0, l], \\
&\text{(iii)}\ \ \tilde{v}_1 = \tilde{v}_2 = \tilde{v}_3 = 0 \ \text{ and } \ F_s = 0.
\end{aligned} \tag{4.77}$$

This requires that for all $t \in [t_1, t_2]$, $\frac{d}{dt}\tilde{v} \equiv \frac{d^2}{dt^2}\tilde{R} = 0$, $\dot{w} = 0$ and $\ddot{\phi} \equiv (\frac{\partial^2 y}{\partial t^2}, \frac{\partial^2 z}{\partial t^2})' = 0$, $\frac{\partial^2 \psi}{\partial t^2}$=0 for $x \in [0, l]$. Using all these conditions and Eq.(4.77) into Eq.(4.3), it is clear that $\tilde{R} = 0$ for all $t \in [t_1, t_2]$. Then from Eqs.(2.113-2.114) it follows that

$$\frac{\partial}{\partial x}\Big(\psi - \frac{\partial D}{\partial x}\Big) = 0, \tag{4.78}$$

$$\frac{\partial}{\partial x}\Big(EI\frac{\partial \psi}{\partial x}\Big) = 0, \qquad \text{for } \ x \in (0, l). \tag{4.79}$$

It is easy to see that the unique solution of Equation (4.79) along with the boundary conditions (2.141-2.144) is $\psi = 0$ for all $x \in [0, l]$. This implies from Eq.(4.78) that ϕ=0. In other words, if for all $t \in [t_1, t_2]$ $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$=0, $w = \tilde{R} = 0$ and $\psi = \phi = 0$ for all $x \in [0, l]$. Hence $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ is negative definite. Therefore, the function $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ defined by Eq.(4.68) is, indeed, a *Lyapunov functional* and consequently, the system (Eqs. 2.120,2.121,4.3) is asymptotically stable with respect to the rest state(4.4-4.6). This completes the proof. ■

Remark 4.16

We note that the first and second terms in Eq.(4.68) represent the kinetic energy of the rigid bus due to orbital perturbation and bus rotation, and the third and fourth terms gives the kinetic energy of the beam. The last integral in the fifth term represent the strain energy of the beam due to the deformation caused by vibration. Perturbation of the gravitational potential energy due to the orbital perturbation

‡ Under the assumption of existence of classical solution, y, z, $\frac{\partial y}{\partial t}$ and $\frac{\partial z}{\partial t}$ are continuous and hence (ii) of (4.77) is true.

$\tilde{R}$ is represented by the last term. Therefore, the Lyapunov functional defined by Eq.(4.68) is indeed the total energy of the perturbed system(2.120,2.121,4.3).

Corollary 4.17 **(Localized Control)**

Consider the system described by Equations (2.120,2.121,4.3) and suppose that the controls are given by the localized feedback structure:

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)', \quad k_1, k_2, k_3 > 0, \tag{4.80}$$

$$F_s = (-e_1\tilde{v}_1, -e_2\tilde{v}_2, -e_3\tilde{v}_3)', \quad e_1, e_2, e_3 > 0, \tag{4.81}$$

$$F_b = (\tilde{A}_y - d_1(x)\frac{\partial y}{\partial t},\ \tilde{A}_z - d_2(x)\frac{\partial z}{\partial t})', \quad d_1, d_2 > 0, \tag{4.82}$$

$$F_{b_T} = (-d_3(x)\frac{\partial \psi_y}{\partial t},\ \ -d_4(x)\frac{\partial \psi_z}{\partial t})', \quad d_3, d_4 > 0, \tag{4.83}$$

with
$$d_1(x) = \sum_{i=1}^{N_1} a_i \chi_{\mathcal{A}_i}(x), \quad a_i > 0, \tag{4.84}$$

$$d_2(x) = \sum_{i=1}^{N_2} b_i \chi_{\mathcal{B}_i}(x), \quad b_i > 0, \tag{4.85}$$

$$d_3(x) = \sum_{i=1}^{N_3} c_i \chi_{\mathcal{C}_i}(x), \quad c_i > 0, \tag{4.86}$$

$$d_4(x) = \sum_{i=1}^{N_4} d_i \chi_{\mathcal{D}_i}(x), \quad d_i > 0, \tag{4.87}$$

where χ_e denotes the indicator function of the set e and $\{\mathcal{A}_i, i = 1, 2, \cdots, N_1\}$, $\{\mathcal{B}_i, i = 1, 2, \cdots, N_2\}$, $\{\mathcal{C}_i, i = 1, 2, \cdots, N_3\}$ and $\{\mathcal{D}_i, i = 1, 2, \cdots, N_4\}$ are any four families of Lebesgue measurable subsets of $[0, l]$ such that $\frac{\partial y}{\partial t}$, $\frac{\partial z}{\partial t}$, $\frac{\partial \psi_y}{\partial t}$, and $\frac{\partial \psi_z}{\partial t}$ do not vanish identically on $\cup\mathcal{A}_i$, $\cup\mathcal{B}_i$, $\cup\mathcal{C}_i$ and $\cup\mathcal{D}_i$ respectively. Then, the system is asymptotically stable with respect to the rest state (4.4-4.6).

Proof The proof is essentially the same as above. ■

Corollary 4.18 **(Deadzone Control)**

Consider the system described by Equations (2.120,2.121,4.3). Suppose that the feedback controls are given by

$$T = (T_x - k_1\Upsilon_{\epsilon_1}(\omega_1), T_y - k_2\Upsilon_{\epsilon_2}(\omega_2), T_z - k_3\Upsilon_{\epsilon_3}(\omega_3))', \quad k_1, k_2, k_3 > 0, \quad (4.88)$$

$$F_b = \left(\tilde{A}_y - h_1(x)\Upsilon_{\epsilon_4}\left(\frac{\partial y}{\partial t}\right), \tilde{A}_z - h_2(x)\Upsilon_{\epsilon_5}\left(\frac{\partial z}{\partial t}\right)\right)', \quad h_1, h_2 > 0, \quad (4.89)$$

$$F_s = (-e_1\Upsilon_{\epsilon_6}(\tilde{v}_1), -e_2\Upsilon_{\epsilon_7}(\tilde{v}_2), -e_3\Upsilon_{\epsilon_8}(\tilde{v}_3))', \quad e_1, e_2, e_3 > 0, \quad (4.90)$$

$$F_{b_T} = \left(-h_3(x)\Upsilon_{\epsilon_9}\left(\frac{\partial \psi_y}{\partial t}\right), -h_4(x)\Upsilon_{\epsilon_{10}}\left(\frac{\partial \psi_z}{\partial t}\right)\right)', \quad h_3, h_4 > 0, \quad (4.91)$$

where

$$\Upsilon_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ 0, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \quad (4.92)$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_d of the desired rest state, defined by

$$N_d \equiv \left\{(\omega, \dot{\phi}, \dot{\psi}, \tilde{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \left|\frac{\partial y}{\partial t}\right| < \epsilon_4, \left|\frac{\partial z}{\partial t}\right| < \epsilon_5, |\tilde{v}_1| < \epsilon_6, \right.$$

$$\left. |\tilde{v}_2| < \epsilon_7, |\tilde{v}_3| < \epsilon_8, \left|\frac{\partial \psi_y}{\partial t}\right| < \epsilon_9, \left|\frac{\partial \psi_z}{\partial t}\right| < \epsilon_{10}; \epsilon_i > 0, i = 1, 2, \cdots, 10 \right\}. \quad (4.93)$$

Remark 4.19 **(Bang-Bang Control)**

We note that when $\epsilon_i = 0, i = 1, 2, \cdots, 10$ in Equations (4.88-4.91) we obtain the Bang-Bang controls which can be easily implemented in practical applications, as a special case of deadzone controls.

Corollary 4.20 **(Saturation Control)**

Consider the system described by Equations (2.120,2.121,4.3). Suppose that the feedback controls are given by

$$T = (T_x - k_1 S_{\epsilon_1}(\omega_1), T_y - k_2 S_{\epsilon_2}(\omega_2), T_z - k_3 S_{\epsilon_3}(\omega_3))', \quad k_1, k_2, k_3 > 0, \quad (4.94)$$

$$F_b = \left(\tilde{A}_y - h_1(x) S_{\epsilon_4}\left(\frac{\partial y}{\partial t}\right), \tilde{A}_z - h_2(x) S_{\epsilon_5}\left(\frac{\partial z}{\partial t}\right)\right)', \quad h_1, h_2 > 0, \quad (4.95)$$

$$F_s = (-e_1 S_{\epsilon_6}(\tilde{v}_1), -e_2 S_{\epsilon_7}(\tilde{v}_2), -e_3 S_{\epsilon_8}(\tilde{v}_3))', \quad e_1, e_2, e_3 > 0, \quad (4.96)$$

$$F_{b_T} = \left(-h_3(x) S_{\epsilon_9}\left(\frac{\partial \psi_y}{\partial t}\right), -h_4(x) S_{\epsilon_{10}}\left(\frac{\partial \psi_z}{\partial t}\right)\right)', \quad h_3, h_4 > 0, \quad (4.97)$$

where

$$\mathcal{S}_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ \xi, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \tag{4.98}$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_s of the desired rest state, defined by

$$N_s \equiv \Big\{ (\omega, \dot{\phi}, \dot{\psi}, \tilde{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \left|\frac{\partial y}{\partial t}\right| < \epsilon_4, \left|\frac{\partial z}{\partial t}\right| < \epsilon_5, |\tilde{v}_1| < \epsilon_6,$$

$$|\tilde{v}_2| < \epsilon_7, |\tilde{v}_3| < \epsilon_8, \left|\frac{\partial \psi_y}{\partial t}\right| < \epsilon_9, \left|\frac{\partial \psi_z}{\partial t}\right| < \epsilon_{10}; \epsilon_i > 0, i = 1, 2, \cdots, 10 \Big\}. \tag{4.99}$$

4.4 Stabilization of the Space Station

In this section we consider the problem of stabilization of the space station. For certain applications, for example, on board communication systems, radiotelescopes, etc, it is necessary that the space station be brought to its rest state following any disturbance possibly caused by crew movements, takeoff or docking of space shuttle. Accordingly, it is natural to consider the following state

$$\begin{gathered} \omega_1 = 0,\ \omega_2 = 0,\ \omega_3 = 0, \\ \tilde{X}(t) = \tilde{Y}(t) = \tilde{Z}(t) = 0, \quad \tilde{v}_1(t) = \tilde{v}_2(t) = \tilde{v}_3(t) = 0, \end{gathered} \tag{4.100}$$

and for $k = 5, 6;\ j = 1, 2, 3, 4;\ i = 1, 2, ..., 6,$

$$x_k(\xi_k, t) = y_j(\xi_j, t) = z_i(\xi_i, t) = 0, \ \ \frac{\partial x_k}{\partial t}(\xi_k, t) = \frac{\partial y_j}{\partial t}(\xi_j, t) = \frac{\partial z_i}{\partial t}(\xi_i, t) = 0, \tag{4.101}$$

where $\tilde{v} \equiv \frac{d\tilde{R}}{dt} \equiv (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)'$ and $\tilde{R} = (\tilde{X}, \tilde{Y}, \tilde{Z})'$, as the *rest state*. We note that here the trajectory (Eq.(4.1)) corresponding to the rest state is the reference trajectory when no disturbance occurs. Now our concern is, if the orbit is perturbed from the reference trajetory (or the target trajectory) subject to the external forces, can we find a control which will eventually steer the system back to the rest state? This problem of stabilization can be solved following *Lyapunov*'s approach. In the following, we show that the space station, using feedback controls, is asymptotically stable with respect to the rest state(4.100-4.101).

Throughout this section we use the trajectory equation (4.1) as the nominal trajectory (or reference trajectory) for R_0. That is, we denote the fluctuation of the radial vector by $\tilde{R}$ as in Section 4.2 and define the nominal trajectory by Equation (4.1). Then the perturbed trajectory $\tilde{R}$ for the space station is given by the solution of following differential equation:

$$m_T \frac{d^2 \tilde{R}}{dt^2} + \sum_{i=1}^{6} \frac{d}{dt} \int_{\Omega_i} \rho_i \Big(w \times \bar{r}_i + \dot{D}_i \Big) d\xi_i + \frac{G m_e m_T \tilde{R}}{|R_0|^3} = F_s, \tag{4.102}$$

where nominal trajectory R_0 is given by the solution of Equation (4.1).

For notational simplicity we define $(\tilde{A}_{i_x}, \tilde{A}_{i_y}, \tilde{A}_{i_z})' \equiv -\rho_i \frac{Gm_e}{|R_0|^3} R_0$ and $(T_x, T_y, T_z)'$ $\equiv -\frac{Gm_e}{|R_0|^3} (\sum_{i=1}^{6} \int_{\Omega_i} \rho_i \bar{r}_i d\xi_i) \times R_0$.

4.4.1 Stabilization of the Space Station (Euler Bernoulli Beam Theory)

In the following theorem and its corollories we consider the stabilization of the space station whose dynamics was derived based on Euler Bernoulli beam theory and given by Equations (3.80-3.83) by applying feedback controls. The feedback controls in several different forms such as proportional control, deadzone controls, Bang-Bang controls and saturation controls are applied for the vibration suppression of the system to achieve the stabilization.

Theorem 4.21 **(Distributed Control)**

Consider the perturbed system described by Equations (3.81-3.83) and (4.102) around the reference trajectory (4.1). Suppose that the controls applied to the system(3.81-3.83, 4.102) are given by the feedback law: for $\xi_i \in [0, l_i]$, i=1,2,...,6,

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)', \quad k_1, k_2, k_3 > 0, \tag{4.103}$$

$$F_{b_i} = Q_i\Big(\tilde{A}_{i_x} - d_{i_1}\frac{\partial x_i}{\partial t}, \tilde{A}_{i_y} - d_{i_2}\frac{\partial y_i}{\partial t},\ \ \tilde{A}_{i_z} - d_{i_3}\frac{\partial z_i}{\partial t}\Big)', \quad d_{i_1}, d_{i_2}, d_{i_3} > 0, \tag{4.104}$$

$$F_s = (-e_1\tilde{v}_1,\ -e_2\tilde{v}_2,\ -e_3\tilde{v}_3)', \qquad e_1, e_2, e_3 > 0. \tag{4.105}$$

Then the system is asymptotically stable (in the sense of Lyapunov) with respect to the rest state (4.100-4.101).

<u>Proof</u>

Let $\varphi \equiv \{x_k, \frac{\partial x_k}{\partial t}, y_j, \frac{\partial y_j}{\partial t}, z_i, \frac{\partial z_i}{\partial t}; k = 5,6; j = 1,2,3,4; i = 1,2,...,6\}$ and introduce a scalar functional $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ defined by

$$\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \equiv \frac{1}{2}m_s(\tilde{v} \circ \tilde{v}) + \frac{1}{2}(I_s \circ \omega) \circ \omega + \frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i} \rho_i|\tilde{v} + (\dot{D}_i + \omega \times \bar{r}_i)|^2 d\xi_i$$
$$+ \frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i} EI_i\frac{\partial^2 D_i}{\partial \xi_i^{\,2}} \circ \frac{\partial^2 D_i}{\partial \xi_i^{\,2}} d\xi_i + \frac{1}{2}\frac{Gm_e m_T|\tilde{R}|^2}{|R_0|^3}. \tag{4.106}$$

It is obvious from Equation (4.106) that $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) > 0$ for all $w, \varphi, \tilde{R}, \tilde{v} \neq 0$ and $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$=0 for all $w, \varphi, \tilde{R}, \tilde{v} = 0$. Now, in order to complete the proof of asymptotic stability it is enough to show that $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) < 0$ for all $t \geq 0$. In the following we first show that $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \leq 0$ for all $t \geq 0$ and then using additional arguments we conclude that it is indeed negative definite. Scalar multiplying both sides of Eq.(3.81) (or equivalently Eq.(3.88)) by w and noting that $I_T = I_s + I_b$, where I_b varies with time because of vibrations of the beam, we obtain

$$\frac{d}{dt}\Big[\frac{1}{2}(I_T \cdot \omega) \cdot \omega\Big] + \frac{1}{2}(\dot{I}_b \cdot \omega) \cdot \omega + \sum_{i=1}^{6}\int_{\Omega_i} \rho_i\{\omega \times (\bar{r}_i \times \dot{D}_i)\} \cdot \omega d\xi_i$$
$$+ \sum_{i=1}^{6}\int_{\Omega_i} \rho_i(\bar{r}_i \times \ddot{D}_i) \cdot \omega d\xi_i + \sum_{i=1}^{6}\int_{\Omega_i} \rho_i\Big(\bar{r}_i \times \frac{d\tilde{v}}{dt}\Big) \cdot w d\xi_i = T \cdot w \tag{4.107}$$

It is easy to see that

$$(I_b \cdot \omega) \cdot \omega = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i (\bar{r}_i \times (\omega \times \bar{r}_i)) \cdot \omega d\xi_i = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i |\omega \times \bar{r}_i|^2 d\xi_i \qquad (4.108)$$

$$(\dot{I}_b \cdot \omega) \cdot \omega = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i (\dot{D}_i \times (\omega \times \bar{r}_i)) \cdot \omega d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i (\bar{r}_i \times (\omega \times \dot{D}_i)) \cdot \omega d\xi_i$$

$$= -2 \sum_{i=1}^{6} \int_{\Omega_i} \rho_i (\omega \times (\omega \times \bar{r}_i)) \cdot \dot{D}_i d\xi_i \qquad (4.109)$$

$$\frac{1}{2}\frac{d}{dt}\{(I_s \circ w) \circ w\} + \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{6} \int_{\Omega_i} \rho_i |w \times \bar{r}_i|^2 d\xi_i - \sum_{i=1}^{6} \int_{\Omega_i} \rho_i (w \times (w \times \bar{r}_i)) \circ \dot{D}_i d\xi_i$$

$$+ \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big(\frac{d^2 R_0}{dt^2} + \ddot{D}_i + \frac{d\bar{v}}{dt}\Big) \circ (w \times \bar{r}_i) d\xi_i = T \circ w. \qquad (4.110)$$

Scalar multiplying Eq.(3.42) (or (3.82-3.83)) by $\dot{D}_i$ and integrating by parts over $\xi_i \in [0, l_i]$ and using the boundary conditions (3.97-3.120), we obtain

$$\sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big[\frac{d^2 R_0}{dt^2} + \frac{d\bar{v}}{dt} + \ddot{D}_i + \dot{w} \times \bar{r}_i + 2w \times \dot{D}_i + w \times (w \times \bar{r}_i)\Big] \circ \dot{D}_i d\xi_i$$

$$+ \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{6} \int_{\Omega_i} EI_i \frac{\partial^2 D_i}{\partial \xi_i^2} \circ \frac{\partial^2 D_i}{\partial \xi_i^2} d\xi_i = \sum_{i=1}^{6} \int_{\Omega_i} F_{b_i} \circ \dot{D}_i d\xi_i. \quad (4.111)$$

Similarly, scalar multiplying Equation (4.102) by $\bar{v}$, we get

$$\frac{1}{2}\frac{d}{dt} m_s (\bar{v} \circ \bar{v}) + \frac{1}{2}\sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d}{dt}(\bar{v} \circ \bar{v}) d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d^2 \bar{r}_i}{dt^2} \circ \bar{v} d\xi_i + \frac{G m_e m_T \bar{R} \circ \bar{v}}{|R_0|^3} = F_s \circ \bar{v}. \qquad (4.112)$$

Using the relations (4.108-4.109), it is not difficult to see that

$$\sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{|w \times \bar{r}_i|^2 - (w \times (w \times \bar{r}_i)) \circ \dot{D}_i + \ddot{D}_i \circ (w \times \bar{r}_i)\Big\} d\xi_i = \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d^2 \bar{r}_i}{dt^2} \circ (w \times \bar{r}_i) d\xi_i. \qquad (4.113)$$

Substituting this identity into Eq.(4.110) and adding Eqs.(4.110), (4.111) and (4.112) up, we have

$$\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 = F_s \circ \tilde{v} + T \circ w + \sum_{i=1}^{6} \int_{\Omega_i} F_{b_i} \circ \dot{D}_i d\xi_i, \tag{4.114}$$

where

$$\mathcal{I}_1 \equiv \frac{1}{2}\frac{d}{dt} m_s(\tilde{v} \circ \tilde{v}) + \frac{1}{2}\sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d}{dt}(\tilde{v} \circ \tilde{v}) d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d^2 \bar{r}_i}{dt^2} \circ \tilde{v} d\xi_i + \frac{G m_e m_r \tilde{R} \circ \tilde{v}}{|R_0|^3} \tag{4.115}$$

$$\mathcal{I}_2 \equiv \frac{1}{2}\frac{d}{dt}\Big((I_s \circ w) \circ w\Big) + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d^2 \bar{r}_i}{dt^2} \circ (w \times \bar{r}_i) d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d^2 R_0}{dt^2} \circ (w \times \bar{r}_i) d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d\tilde{v}}{dt} \circ (w \times \bar{r}_i) d\xi_i \tag{4.116}$$

$$\mathcal{I}_3 \equiv \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d^2 R_0}{dt^2} \circ \dot{D}_i d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big(\frac{d\tilde{v}}{dt} + \frac{d^2 \bar{r}_i}{dt^2}\Big) \circ \dot{D}_i d\xi_i + \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{6} \int_{\Omega_i} EI_i \frac{\partial^2 D_i}{\partial \xi_i^2} \circ \frac{\partial^2 D_i}{\partial \xi_i^2} d\xi_i \tag{4.117}$$

Then, taking time derivative of Eq.(4.106) (in the inertial frame) one finds that the left-hand-side expression of Eq.(4.114) equals $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ plus two additional terms as follows.

$$\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{ \frac{d^2 R_0}{dt^2} \circ (w \times \bar{r}_i) + \frac{d^2 R_0}{dt^2} \circ \dot{D}_i \Big\} d\xi_i = F_s \circ \tilde{v} + \sum_{i=1}^{6} \int_{\Omega_i} \Big(F_{b_i} \circ \dot{D}_i \Big) \, d\xi_i + T \circ \omega \, . \tag{4.118}$$

Substituting the controls given by Eqs.(4.103-4.105) into Eq.(4.118), we obtain

$$\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) = -\Big(k_1\omega_1^2 + k_2\omega_2^2 + k_3\omega_3^2\Big) - \Big(e_1\tilde{v}_1^2 + e_2\tilde{v}_2^2 + e_3\tilde{v}_3^2\Big) - \int_{\Omega_i} \Big\{ \sum_{i=5}^{6} d_{i_1}(\xi_i) \Big(\frac{\partial x_i}{\partial t}\Big)^2 + \sum_{i=1}^{4} d_{i_2}(\xi_i) \Big(\frac{\partial y_i}{\partial t}\Big)^2 + \sum_{i=1}^{6} d_{i_3}(\xi_i) \Big(\frac{\partial z_i}{\partial t}\Big)^2 \Big\} \, d\xi_i. \tag{4.119}$$

From Eq.(4.106), we see that $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ is negative semidefinite for the given controls. We show that in fact $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ is negative definite in the following. If $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \equiv 0$ for any finite interval of time, say $[t_1, t_2]$, the following must be true‡[Ahmed 6, Adams 1]. For all $t \in [t_1, t_2]$,

$$
\begin{aligned}
&\text{(i)} \quad w_1 = w_2 = w_3 = 0, \\
&\text{(ii)} \quad Q_i \frac{\partial D_i}{\partial t} = 0, \ \text{for all } \xi_i \in [0, l_i], \\
&\text{(iii)} \quad \tilde{v}_1 = \tilde{v}_2 = \tilde{v}_3 = 0 \ \text{ and } \ F_s = 0.
\end{aligned}
\tag{4.120}
$$

This requires that for all $t \in [t_1, t_2]$, $\frac{d}{dt}\tilde{v} \equiv \frac{d^2}{dt^2}\tilde{R} = 0$, $\dot{w} = 0$ and $\ddot{D}_i \equiv Q_i \left(\frac{\partial^2 x_i}{\partial t^2}, \frac{\partial^2 y_i}{\partial t^2}, \frac{\partial^2 z_i}{\partial t^2}\right)' = 0$ for $\xi_i \in [0, l_i]$. Using all these conditions and Eq.(4.120) into Eq.(4.3), it is clear that $\tilde{R} = 0$ for all $t \in [t_1, t_2]$. Then from Eq.(3.42) it follows that

$$
\frac{\partial^2}{\partial \xi_i{}^2}\left(EI_i \frac{\partial^2 D_i}{\partial \xi_i^2}\right) = 0, \qquad \text{for } \ \xi_i \in (0, l_i). \tag{4.121}
$$

It is easy to see that the unique solution of the elliptic equation (4.121) along with the boundary conditions (3.96-3.120) is $D_i = 0$ for all $\xi_i \in [0, l_i]$. In other words, if $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) = 0$ for all $t \in [t_1, t_2]$, $w = \tilde{R} = 0$ and $D_i = 0$ for all $\xi_i \in [0, l_i]$. Hence $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ is negative definite. Therefore, the function $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ defined by Eq.(4.106) is, indeed, a *Lyapunov functional* and consequently, the system (3.81-3.83,4.102) is asymptotically stable with respect to the rest state(4.100-4.101). This completes the proof. ■

Remark 4.22

We note that the first and second terms in Eq.(4.106) represent the kinetic energy of the rigid bus due to orbital perturbation and bus rotation, and the third term gives the kinetic energy of the beam. The integral in the fourth term represents the strain

‡ Under the assumption of existence of classical solution, x_i, y_i, z_i, $\frac{\partial x_i}{\partial t}$, $\frac{\partial y_i}{\partial t}$ and $\frac{\partial z_i}{\partial t}$ are continuous and hence (ii) of Equation (4.120) is true.

energy of the beam due to the deformation caused by vibration. Perturbation of the gravitational potential energy due to the orbital perturbation $\tilde{R}$ is represented by the last term. Therefore, the Lyapunov functional defined by Eq.(4.106) is indeed the total energy of the perturbed system (Eqs. 3.81-3.83,4.102).

From Equation (4.119), it is clear that in the absence of any control,

$$\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)=0,$$

so that $\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t) = \mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};0)$ for all t . This implies that the system (without external forces or controls) is conservative. However, interchange of energy between the bus and the beam may take place leading to change in the bus velocity trajectories and in the amplitude of vibration of the beam.

Remark 4.23

The feedback law given by Eqs.(4.103-4.105) can be physically realized through velocity feedback with gains k_1, k_2, k_3, $d_{i_1}, d_{i_2}, d_{i_3}$, e_1, e_2, e_3 and position feedback T_x, T_y, T_z, $\tilde{A}_{i_x}$, $\tilde{A}_{i_y}$,$\tilde{A}_{i_z}$. Since $-\frac{Gm_eR_0}{|R_0|^3}=\frac{d^2}{dt^2}R_0$ (see Eq.(4.1)), the components $\tilde{A}_{i_x}$,$\tilde{A}_{i_y}$ and $\tilde{A}_{i_z}$ in (4.104) are the control forces for compensating the force $\rho_i\frac{d^2}{dt^2}R_0$ induced by the nominal trajectory R_0 which is known. On the other hand, T_x,T_y and T_z in Eq. (4.103) are the control moments for eliminating the torque $m_b r_{cm} \times \frac{d^2}{dt^2}R_0$ due to the nominal trajectory, where r_{cm} denotes the vector for the mass center of the beam with reference to the origin of the body frame. It is noted that under the small deformation of the beam r_{cm} can be treated as time invariant. The choice of feedback gains depends on the requirements of stabilization time and mechanical safety (see Remark 2.5).

The distributed control law in Eqs.(4.103-4.105) may be difficult and costly for practical applications. In the following corollary, we consider stabilization using localized controls which can be more easily implemented in practice. We note that boundary controls are a special case of localized controls.

Corollary 4.24 (**Localized Control**)

Consider the system given by Eqs.(3.81-3.83,4.102) and suppose that the controls are given by the localized feedback structure: for $\xi_i \in [0, l_i]$, i=1,2,...,6,

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)', \quad k_1, k_2, k_3 > 0, \tag{4.122}$$

$$F_{b_i} = Q_i(\tilde{A}_{i_x} - d_{i_1}\frac{\partial x_i}{\partial t}, \tilde{A}_{i_y} - d_{i_2}\frac{\partial y_i}{\partial t},\ \ \tilde{A}_{i_z} - d_{i_3}\frac{\partial z_i}{\partial t})', \quad d_{i_1}, d_{i_2}, d_{i_3} > 0, \tag{4.123}$$

$$F_s = (-e_1\tilde{v}_1,\ -e_2\tilde{v}_2,\ -e_3\tilde{v}_3)', \qquad e_1, e_2, e_3 > 0. \tag{4.124}$$

$$\text{with} \qquad d_{i_1}(\eta) = \sum_{j=1}^{N_1} A_{1_j}\chi_{\mathcal{A}_j}(\eta), \quad A_{1_j} > 0, \tag{4.125}$$

$$d_{i_2}(\eta) = \sum_{j=1}^{N_2} A_{2_j}\chi_{\mathcal{B}_j}(\eta), \quad A_{2_j} > 0, \tag{4.126}$$

$$d_{i_3}(\eta) = \sum_{j=1}^{N_3} A_{3_j}\chi_{\mathcal{C}_j}(\eta), \quad A_{3_j} > 0, \tag{4.127}$$

where χ_e denotes the indicator function of the set e and $\{\mathcal{A}_j, j = 1, 2, \cdots, N_1\}$, $\{\mathcal{B}_j, j = 1, 2, \cdots, N_2\}$ and $\{\mathcal{C}_j, j = 1, 2, \cdots, N_3\}$ are any three families of Lebesgue measurable subsets of $[0, l_j]$ such that $\frac{\partial x_i}{\partial t}$, $\frac{\partial y_i}{\partial t}$ and $\frac{\partial z_i}{\partial t}$ do not vanish identically on $\cup\mathcal{A}_j$, $\cup\mathcal{B}_j$ and $\cup\mathcal{C}_j$ respectively. Then, the system is asymptotically stable with respect to the rest state (4.100-4.101).

Proof The proof is essentially the same as above. ■

Corollary 4.25 (**Deadzone Control**)

Consider the system described by Equations (3.81-3.83,4.102). Suppose that the feedback controls are given by for all positive $k_j, h_{i_x}, h_{i_y}, h_{i_z}, e_j$, i=1,2,...,6; j=1,2,3,

$$T = (T_x - k_1\Upsilon_{\epsilon_1}(\omega_1), T_y - k_2\Upsilon_{\epsilon_2}(\omega_2), T_z - k_3\Upsilon_{\epsilon_3}(\omega_3))', \tag{4.128}$$

$$F_{b_i} = Q_i\Big(\tilde{A}_{i_x} - h_{i_x}\Upsilon_{\epsilon_4}\frac{\partial x_i}{\partial t},\ \tilde{A}_{i_y} - h_{i_y}\Upsilon_{\epsilon_5}\frac{\partial y_i}{\partial t},\ \tilde{A}_{i_z} - h_{i_z}\Upsilon_{\epsilon_6}\frac{\partial z_i}{\partial t}\Big)', \tag{4.129}$$

$$F_s = (-e_1\Upsilon_{\epsilon_7}(\tilde{v}_1), -e_2\Upsilon_{\epsilon_8}(\tilde{v}_2), -e_3\Upsilon_{\epsilon_9}(\tilde{v}_3))', \tag{4.130}$$

where

$$\Upsilon_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ 0, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \tag{4.131}$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_d of the desired rest state, defined by

$$N_d \equiv \Big\{(\omega, \dot{D}_i, \tilde{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \Big|\frac{\partial x_i}{\partial t}\Big| < \epsilon_4, \Big|\frac{\partial y_i}{\partial t}\Big| < \epsilon_5, \Big|\frac{\partial z_i}{\partial t}\Big| < \epsilon_6,$$
$$|\tilde{v}_1| < \epsilon_7, |\tilde{v}_2| < \epsilon_8, |\tilde{v}_3| < \epsilon_9; \quad \epsilon_j > 0, j = 1,2,\cdots,9;\ i = 1,2,...,6\Big\}. \tag{4.132}$$

<u>Remark 4.26</u> **(Bang-Bang Control)**

We note that when $\epsilon_j = 0, j = 1, 2, \cdots, 9$ in Equations (4.128-4.130) we obtain the Bang-Bang controls which can be easily implemented in practical applications, as a special case of deadzone controls.

<u>Corollary 4.27</u> **(Saturation Control)**

Consider the system described by Equations (3.81-3.83,4.102). Suppose that the feedback controls are given by for all positive $k_j, h_{i_x}, h_{i_y}, h_{i_z}, e_j$, i=1,2,...,6; j=1,2,3,

$$T = (T_x - k_1 S_{\epsilon_1}(\omega_1), T_y - k_2 S_{\epsilon_2}(\omega_2), T_z - k_3 S_{\epsilon_3}(\omega_3))', \tag{4.133}$$

$$F_{b_i} = Q_i\Big(\tilde{A}_{i_x} - h_{i_x} S_{\epsilon_4}\frac{\partial x_i}{\partial t},\ \tilde{A}_{i_y} - h_{i_y} S_{\epsilon_5}\frac{\partial y_i}{\partial t},\ \tilde{A}_{i_z} - h_{i_z} S_{\epsilon_6}\frac{\partial z_i}{\partial t}\Big)', \tag{4.134}$$

$$F_s = (-e_1 S_{\epsilon_7}(\tilde{v}_1), -e_2 S_{\epsilon_8}(\tilde{v}_2), -e_3 S_{\epsilon_9}(\tilde{v}_3))', \tag{4.135}$$

where

$$S_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ \xi, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \tag{4.136}$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_s of the desired rest state, defined by

$$N_s \equiv \Big\{(\omega, \dot{D}_i, \tilde{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \Big|\frac{\partial x_i}{\partial t}\Big| < \epsilon_4, \Big|\frac{\partial y_i}{\partial t}\Big| < \epsilon_5, \Big|\frac{\partial z_i}{\partial t}\Big| < \epsilon_6,$$
$$|\tilde{v}_1| < \epsilon_7, |\tilde{v}_2| < \epsilon_8, |\tilde{v}_3| < \epsilon_9; \quad \epsilon_j > 0, j = 1,2,\cdots,9;\ i = 1,2,...,6\Big\}. \tag{4.137}$$

4.4.2 Stabilization of the Space Station (Rayleigh Beam Theory)

In the following theorem and its corollories we consider the stabilization of the space station whose dynamics was derived based on Rayleigh beam theory and given by Equations (3.180-3.183) by applying feedback controls. The feedback controls in several different forms such as proportional control, deadzone controls, Bang-Bang controls and saturation controls are applied for the vibration suppression of the system to achieve the stabilization.

Theorem 4.28 **(Distributed Control)**

Consider the perturbed system described by Equations (3.181-3.183,4.102) around the reference trajectory (4.1). Suppose that the controls applied to the system(3.181-3.183,4.102) are given by the feedback law: for $\xi_i \in [0, l_i]$, i=1,2,...,6,

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)', \quad k_1, k_2, k_3 > 0, \tag{4.138}$$

$$F_{b_i} = Q_i\Big(\tilde{A}_{i_x} - d_{i_1}\frac{\partial x_i}{\partial t}, \tilde{A}_{i_y} - d_{i_2}\frac{\partial y_i}{\partial t},\ \ \tilde{A}_{i_z} - d_{i_3}\frac{\partial z_i}{\partial t}\Big)', \quad d_{i_1}, d_{i_2}, d_{i_3} > 0, \tag{4.139}$$

$$F_s = (-e_1\tilde{v}_1,\ -e_2\tilde{v}_2,\ -e_3\tilde{v}_3)', \qquad e_1, e_2, e_3 > 0. \tag{4.140}$$

Then the system is asymptotically stable (in the sense of Lyapunov) with respect to the rest state (4.100-4.101).

Proof

Let $\varphi \equiv \{x_k, \frac{\partial x_k}{\partial t}, y_j, \frac{\partial y_j}{\partial t}, z_i, \frac{\partial z_i}{\partial t}; k = 5,6; j = 1,2,3,4; i = 1,2,...,6\}$ and introduce a scalar functional $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ defined by

$$\begin{aligned}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \equiv \frac{1}{2}m_s(\tilde{v} \circ \tilde{v}) + \frac{1}{2}(I_s \circ \omega) \circ \omega + \frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i} \rho_i|\tilde{v} + (\dot{D}_i + \omega \times \bar{r}_i)|^2 d\xi_i \\ + \frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i} I_{\rho_i}\Big|\frac{\partial^2 D_i}{\partial t \partial \xi_i}\Big|^2 d\xi_i + \frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i} EI_i \frac{\partial^2 D_i}{\partial {\xi_i}^2} \circ \frac{\partial^2 D_i}{\partial {\xi_i}^2} d\xi_i \\ + \frac{1}{2}\frac{Gm_e m_T|\tilde{R}|^2}{|R_0|^3}.\end{aligned} \tag{4.141}$$

It is obvious from Equation (4.141) that $\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t) > 0$ for all $w,\varphi,\tilde{R},\tilde{v} \neq 0$ and $\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$=0 for all $w,\varphi,\tilde{R},\tilde{v} = 0$. Now, in order to complete the proof of asymptotic stability it is enough to show that $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t) < 0$ for all $t \geq 0$. In the following we first show that $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t) \leq 0$ for all $t \geq 0$ and then using additional arguments we conclude that it is indeed negative definite. Scalar multiplying both sides of Eq.(4.140) (or equivalently Eq.(4.188)) by w and using Equations (4.108-4.109), we obtain

$$\frac{1}{2}\frac{d}{dt}\{(I_s \circ w)\circ w\} + \frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i}\rho_i|w\times\bar{r}_i|^2 d\xi_i - \sum_{i=1}^{6}\int_{\Omega_i}\rho_i(w\times(w\times\bar{r}_i))\circ\dot{D}_i d\xi_i$$

$$+\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big(\frac{d^2R_0}{dt^2}+\ddot{D}_i+\frac{d\tilde{v}}{dt}\Big)\circ(w\times\bar{r}_i)d\xi_i = T\circ w. \qquad (4.142)$$

Scalar multiplying Eq.(3.142) (or (3.182-3.183)) by $\dot{D}_i$ and integrating by parts over $\xi_i \in [0,l_i]$ and using the boundary conditions (3.197-3.220), we obtain

$$\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big[\frac{d^2R_0}{dt^2}+\frac{d\tilde{v}}{dt}+\ddot{D}_i+\dot{w}\times\bar{r}_i+2w\times\dot{D}_i+w\times(w\times\bar{r}_i)\Big]\circ\dot{D}_i d\xi_i$$

$$-\sum_{i=1}^{6}\int_{\Omega_i}\dot{D}_i\circ\frac{\partial}{\partial\xi_i}\Big(I_{\rho_i}\frac{d}{dt}\frac{\partial^2 D_i}{\partial\xi_i\partial t}\Big)d\xi_i+\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{6}\int_{\Omega_i}EI_i\frac{\partial^2 D_i}{\partial\xi_i^2}\circ\frac{\partial^2 D_i}{\partial\xi_i^2}d\xi_i$$

$$=\sum_{i=1}^{6}\int_{\Omega_i}F_{b_i}\circ\dot{D}_i d\xi_i. \qquad (4.143)$$

Similarly, scalar multiplying Equation (4.102) by $\tilde{v}$, we get

$$\frac{1}{2}\frac{d}{dt}m_s(\tilde{v}\circ\tilde{v})+\frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d}{dt}(\tilde{v}\circ\tilde{v})d\xi_i+\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d^2\bar{r}_i}{dt^2}\circ\tilde{v}d\xi_i+\frac{Gm_em_\tau\tilde{R}\circ\tilde{v}}{|R_0|^3}=F_s\circ\tilde{v}. \qquad (4.144)$$

Using the relations (4.108-4.109), it is not difficult to see that

$$\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{|w\times\bar{r}_i|^2-(w\times(w\times\bar{r}_i))\circ\dot{D}_i+\ddot{D}_i\circ(w\times\bar{r}_i)\Big\}d\xi_i=\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d^2\bar{r}_i}{dt^2}\circ(w\times\bar{r}_i)d\xi_i. \qquad (4.145)$$

Substituting this identity into Eq.(4.142) and adding Eqs.(4.142), (4.143) and (4.144) up, we have

$$\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 = F_s \circ \tilde{v} + T \circ w + \sum_{i=1}^{6} \int_{\Omega_i} F_{b_i} \circ \dot{D}_i d\xi_i, \tag{4.146}$$

where

$$\mathcal{I}_1 \equiv \frac{1}{2}\frac{d}{dt} m_s(\tilde{v} \circ \tilde{v}) + \frac{1}{2}\sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d}{dt}(\tilde{v} \circ \tilde{v}) d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d^2 \bar{r}_i}{dt^2} \circ \tilde{v} d\xi_i + \frac{G m_e m_T \tilde{R} \circ \tilde{v}}{|R_0|^3}, \tag{4.147}$$

$$\mathcal{I}_2 \equiv \frac{1}{2}\frac{d}{dt}\Big((I_s \circ w) \circ w\Big) + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d^2 \bar{r}_i}{dt^2} \circ (w \times \bar{r}_i) d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d^2 R_0}{dt^2} \circ (w \times \bar{r}_i) d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d\tilde{v}}{dt} \circ (w \times \bar{r}_i) d\xi_i, \tag{4.148}$$

$$\mathcal{I}_3 \equiv \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \frac{d^2 R_0}{dt^2} \circ \dot{D}_i d\xi_i + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big(\frac{d\tilde{v}}{dt} + \frac{d^2 \bar{r}_i}{dt^2}\Big) \circ \dot{D}_i d\xi_i - \sum_{i=1}^{6} \int_{\Omega_i} \dot{D}_i \circ \frac{\partial}{\partial \xi_i}\Big(I_{\rho_i} \frac{d}{dt} \frac{\partial^2 D_i}{\partial \xi_i \partial t}\Big) d\xi_i + \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{6} \int_{\Omega_i} E I_i \frac{\partial^2 D_i}{\partial \xi_i^2} \circ \frac{\partial^2 D_i}{\partial \xi_i^2} d\xi_i \, . \tag{4.149}$$

Then, taking time derivative of Eq.(4.141) (in the inertial frame) one finds that the left-hand-side expression of Eq.(4.149) equals $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ plus two additional terms as follows.

$$\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) + \sum_{i=1}^{6} \int_{\Omega_i} \rho_i \Big\{\frac{d^2 R_0}{dt^2} \circ (w \times \bar{r}_i) + \frac{d^2 R_0}{dt^2} \circ \dot{D}_i\Big\} d\xi_i = F_s \circ \tilde{v} + \sum_{i=1}^{6} \int_{\Omega_i} \Big(F_{b_i} \circ \dot{D}_i\Big) \, d\xi_i + T \circ \omega \, . \tag{4.150}$$

Substituting the controls given by Eqs.(4.138-4.140) into Eq.(4.150), we obtain

$$\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) = -\Big(k_1\omega_1^2 + k_2\omega_2^2 + k_3\omega_3^2\Big) - \Big(e_1\tilde{v}_1^2 + e_2\tilde{v}_2^2 + e_3\tilde{v}_3^2\Big) - \int_{\Omega_i} \Big\{\sum_{i=5}^{6} d_{i_1}(\xi_i)\Big(\frac{\partial x_i}{\partial t}\Big)^2 + \sum_{i=1}^{4} d_{i_2}(\xi_i)\Big(\frac{\partial y_i}{\partial t}\Big)^2 + \sum_{i=1}^{6} d_{i_3}(\xi_i)\Big(\frac{\partial z_i}{\partial t}\Big)^2\Big\} \, d\xi_i \tag{4.151}$$

From Eq.(4.151), we see that $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative semidefinite for the given controls. Using the same arguments as in theorem 4.21, one can show that in fact $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative definite. Hence $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative definite. Therefore, the function $\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ defined by Eq.(4.141) is, indeed, a Lyapunov functional and consequently, the system (3.181-3.183,4.102) is asymptotically stable with respect to the rest state(4.100-4.101). (For details see theorem 4.21) ■

Remark 4.29

We note that the first and second terms in Eq.(4.141) represent the kinetic energy of the rigid bus due to orbital perturbation and bus rotation, and the third and fourth terms gives the kinetic energy of the beam. The integral in the fifth term represents the strain energy of the beam due to the deformations caused by vibration. Perturbation of the gravitational potential energy due to the orbital perturbation $\tilde{R}$ is represented by the last term. Therefore, the Lyapunov functional defined by Eq.(4.141) is indeed the total energy of the perturbed system(3.181-3.183,4.102).

Corollary 4.30 **(Localized Control)**

Consider the system given by Eqs.(3.181-3.183,4.102) and suppose that the controls are given by the localized feedback structure: for $\xi_i \in [0,l_i]$, i=1,2,...,6,

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)', \quad k_1,k_2,k_3 > 0, \tag{4.152}$$

$$F_{b_i} = Q_i(\tilde{A}_{i_x} - d_{i_1}\frac{\partial x_i}{\partial t}, \tilde{A}_{i_y} - d_{i_2}\frac{\partial y_i}{\partial t},\ \tilde{A}_{i_z} - d_{i_3}\frac{\partial z_i}{\partial t})', \quad d_{i_1},d_{i_2},d_{i_3} > 0, \tag{4.153}$$

$$F_s = (-e_1\tilde{v}_1,\ -e_2\tilde{v}_2,\ -e_3\tilde{v}_3)', \qquad e_1,e_2,e_3 > 0. \tag{4.154}$$

$$\text{with} \qquad d_{i_1}(\eta) = \sum_{j=1}^{N_1} A_{1_j}\chi_{A_j}(\eta), \quad A_{1_j} > 0, \tag{4.155}$$

$$d_{i_2}(\eta) = \sum_{j=1}^{N_2} A_{2_j}\chi_{B_j}(\eta), \quad A_{2_j} > 0, \tag{4.156}$$

$$d_{i_3}(\eta) = \sum_{j=1}^{N_3} A_{3_j}\chi_{C_j}(\eta), \quad A_{3_j} > 0, \tag{4.157}$$

where χ_e denotes the indicator function of the set e and $\{\mathcal{A}_j, j = 1,2,\cdots,N_1\}$, $\{\mathcal{B}_j, j = 1,2,\cdots,N_2\}$ and $\{\mathcal{C}_j, j = 1,2,\cdots,N_3\}$ are any three families of Lebesgue measurable subsets of $[0,l_j]$ such that $\frac{\partial x_i}{\partial t}$, $\frac{\partial y_i}{\partial t}$ and $\frac{\partial z_i}{\partial t}$ do not vanish identically on $\cup\mathcal{A}_j$, $\cup\mathcal{B}_j$ and $\cup\mathcal{C}_j$ respectively. Then, the system is asymptotically stable with respect to the rest state (4.100-4.101).

Proof The proof is essentially the same as above. ■

Corollary 4.31 **(Deadzone Control)**

Consider the system described by Equations (3.181-3.183,4.102). Suppose that the feedback controls are given by for all positive $k_j, h_{i_x}, h_{i_y}, h_{i_z}, e_j$, i=1,2,...,6; j=1,2,3,

$$T = (T_x - k_1\Upsilon_{\epsilon_1}(\omega_1), T_y - k_2\Upsilon_{\epsilon_2}(\omega_2), T_z - k_3\Upsilon_{\epsilon_3}(\omega_3))', \tag{4.158}$$

$$F_{b_i} = Q_i\Big(\tilde{A}_{i_x} - h_{i_x}\Upsilon_{\epsilon_4}\frac{\partial x_i}{\partial t},\ \tilde{A}_{i_y} - h_{i_y}\Upsilon_{\epsilon_5}\frac{\partial y_i}{\partial t},\ \tilde{A}_{i_z} - h_{i_z}\Upsilon_{\epsilon_6}\frac{\partial z_i}{\partial t}\Big)', \tag{4.159}$$

$$F_s = (-e_1\Upsilon_{\epsilon_7}(\tilde{v}_1), -e_2\Upsilon_{\epsilon_8}(\tilde{v}_2), -e_3\Upsilon_{\epsilon_9}(\tilde{v}_3))', \tag{4.160}$$

where

$$\Upsilon_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ 0, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \tag{4.161}$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_d of the desired rest state, defined by

$$N_d \equiv \Big\{(\omega, \dot{D}_i, \tilde{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \Big|\frac{\partial x_i}{\partial t}\Big| < \epsilon_4, \Big|\frac{\partial y_i}{\partial t}\Big| < \epsilon_5, \Big|\frac{\partial z_i}{\partial t}\Big| < \epsilon_6, |\tilde{v}_1| < \epsilon_7, |\tilde{v}_2| < \epsilon_8, |\tilde{v}_3| < \epsilon_9;\ \ \epsilon_j > 0, j = 1,2,\cdots,9;\ i = 1,2,...,6\Big\}. \tag{4.162}$$

Remark 4.32 **(Bang-Bang Control)**

We note that when $\epsilon_j = 0, j = 1,2,\cdots,9$ in Equations (4.158-4.160) we obtain the Bang-Bang controls which can be easily implemented in practical applications, as a special case of deadzone controls.

Corollary 4.33 (**Saturation Control**)

Consider the system described by Equations (3.181-3.183,4.102). Suppose that the feedback controls are given by for all positive $k_j, h_{i_x}, h_{i_y}, h_{i_z}, e_j$, i=1,2,...,6; j=1,2,3,

$$T = (T_x - k_1 \mathcal{S}_{\epsilon_1}(\omega_1), T_y - k_2 \mathcal{S}_{\epsilon_2}(\omega_2), T_z - k_3 \mathcal{S}_{\epsilon_3}(\omega_3))', \tag{4.163}$$

$$F_{b_i} = Q_i\Big(\tilde{A}_{i_x} - h_{i_x}\mathcal{S}_{\epsilon_4}\frac{\partial x_i}{\partial t},\ \tilde{A}_{i_y} - h_{i_y}\mathcal{S}_{\epsilon_5}\frac{\partial y_i}{\partial t},\ \tilde{A}_{i_z} - h_{i_z}\mathcal{S}_{\epsilon_6}\frac{\partial z_i}{\partial t}\Big)', \tag{4.164}$$

$$F_s = (-e_1\mathcal{S}_{\epsilon_7}(\tilde{v}_1), -e_2\mathcal{S}_{\epsilon_8}(\tilde{v}_2), -e_3\mathcal{S}_{\epsilon_9}(\tilde{v}_3))', \tag{4.165}$$

where

$$\mathcal{S}_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ \xi, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \tag{4.166}$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_s of the desired rest state, defined by

$$N_s \equiv \Big\{(\omega, \dot{D}_i, \tilde{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \Big|\frac{\partial x_i}{\partial t}\Big| < \epsilon_4, \Big|\frac{\partial y_i}{\partial t}\Big| < \epsilon_5, \Big|\frac{\partial z_i}{\partial t}\Big| < \epsilon_6,$$
$$|\tilde{v}_1| < \epsilon_7, |\tilde{v}_2| < \epsilon_8, |\tilde{v}_3| < \epsilon_9;\ \ \epsilon_j > 0, j = 1, 2, \cdots, 9;\ i = 1, 2, ..., 6\Big\}. \tag{4.167}$$

4.4.3 Stabilization of the Space Station (Timoshenko Beam Theory)

In the following theorem and its corollories we consider the stabilization of the space station whose dynamics was derived based on Timoshenko beam theory and given by Equations (3.296-3.301) by applying feedback controls. The feedback controls in several different forms such as proportional control, deadzone controls, Bang-Bang controls and saturation controls are applied for the vibration suppression of the system to achieve the stabilization.

Theorem 4.34 **(Distributed Control)**

Consider the perturbed system described by Equations (3.297-3.301,4.102) around the reference trajectory (4.1). Suppose that the controls applied to the system(3.297-3.301,4.102) are given by the feedback law: for $\xi_i \in [0, l_i]$, i=1,2,...,6,

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)', \quad k_1, k_2, k_3 > 0, \tag{4.168}$$

$$F_{b_i} = Q_i\Big(\tilde{A}_{i_x} - d_{i_1}\frac{\partial x_i}{\partial t}, \tilde{A}_{i_y} - d_{i_2}\frac{\partial y_i}{\partial t},\ \ \tilde{A}_{i_z} - d_{i_3}\frac{\partial z_i}{\partial t}\Big)', \quad d_{i_1}, d_{i_2}, d_{i_3} > 0, \tag{4.169}$$

$$F_{bT_i} = Q_i\Big(-d_{i_4}\frac{\partial \psi_{i_x}}{\partial t},\ \ -d_{i_5}\frac{\partial \psi_{i_y}}{\partial t},\ \ -d_{i_6}\frac{\partial \psi_{i_z}}{\partial t}\Big)', \quad d_{i_4}, d_{i_5}, d_{i_6} > 0, \tag{4.170}$$

$$F_s = (-e_1\tilde{v}_1,\ -e_2\tilde{v}_2,\ -e_3\tilde{v}_3)', \qquad e_1, e_2, e_3 > 0. \tag{4.171}$$

Then the system is asymptotically stable (in the sense of Lyapunov) with respect to the rest state (4.100-4.101).

Proof

Let $\varphi \equiv \{x_k, \frac{\partial x_k}{\partial t}, y_j, \frac{\partial y_j}{\partial t}, z_i, \frac{\partial z_i}{\partial t}, \psi_{k_x}, \psi_{j_y}, \psi_{i_z}, \frac{\partial}{\partial t}\psi_{k_x}, \frac{\partial}{\partial t}\psi_{j_y}, \frac{\partial}{\partial t}\psi_{i_z}$; k=5,6; j=1,2,3, 4; i=1,2,...,6 $\}$ and introduce a scalar functional $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$ defined by

$$\begin{aligned}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \equiv\ & \frac{1}{2}m_s(\tilde{v} \circ \tilde{v}) + \frac{1}{2}(I_s \circ \omega) \circ \omega + \frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i} \rho_i|\tilde{v} + (\dot{D}_i + \omega \times \bar{r}_i)|^2 d\xi_i \\ & + \frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i}\Big\{K_i\Big(\psi_i - \frac{\partial D_i}{\partial \xi_i}\Big) \circ \Big(\psi_i - \frac{\partial D_i}{\partial \xi_i}\Big) + EI_i\frac{\partial \psi_i}{\partial \xi_i} \circ \frac{\partial \psi_i}{\partial \xi_i}\Big\}d\xi_i \\ & + \frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i} I_{\rho_i}\frac{\partial \psi_i}{\partial t} \circ \frac{\partial \psi_i}{\partial t}d\xi_i + \frac{1}{2}\frac{Gm_e m_T|\tilde{R}|^2}{|R_0|^3}.\end{aligned} \tag{4.172}$$

It is obvious from Equation (4.172) that $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) > 0$ for all $w, \varphi, \tilde{R}, \tilde{v} \neq 0$ and $\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t)$=0 for all $w, \varphi, \tilde{R}, \tilde{v} = 0$. Now, in order to complete the proof of asymptotic stability it is enough to show that $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) < 0$ for all $t \geq 0$. In the following we first show that $\frac{d}{dt}\mathbf{V}(\omega, \varphi, \tilde{R}, \tilde{v}; t) \leq 0$ for all $t \geq 0$ and then using additional arguments we conclude that it is indeed negative definite. Scalar multiplying both sides of Eq.(3.297) (or equivalently Eq.(3.306)) by w and using Equations (4.108-4.109), we obtain

$$\frac{1}{2}\frac{d}{dt}\{(I_s \circ w)\circ w\} + \frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i}\rho_i|w\times\bar{r}_i|^2 d\xi_i - \sum_{i=1}^{6}\int_{\Omega_i}\rho_i(w\times(w\times\bar{r}_i))\circ\dot{D}_i d\xi_i$$
$$+\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big(\frac{d^2R_0}{dt^2}+\ddot{D}_i+\frac{d\tilde{v}}{dt}\Big)\circ(w\times\bar{r}_i)d\xi_i = T\circ w. \quad (4.173)$$

Scalar multiplying Eq.(3.255) (or (3.298-3.299)) by $\dot{D}_i$ and integrating by parts over $\xi_i \in [0, l_i]$ and using the boundary conditions (3.317-3.340), we obtain

$$\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big[\frac{d^2R_0}{dt^2}+\frac{d\tilde{v}}{dt}+\ddot{D}_i+\dot{w}\times\bar{r}_i+2w\times\dot{D}_i+w\times(w\times\bar{r}_i)\Big]\circ\dot{D}_i d\xi_i$$
$$-\sum_{i=1}^{6}\int_{\Omega_i}K_i\Big(\psi_i-\frac{\partial D_i}{\partial\xi_i}\Big)\circ\frac{\partial^2 D_i}{\partial\xi_i\partial t}d\xi_i = \sum_{i=1}^{6}\int_{\Omega_i}F_{b_i}\circ\dot{D}_i d\xi_i. (4.174)$$

Similarly, scalar multiplying Equation (3.256) (or 3.300-3.301) by $\frac{\partial\psi_i}{\partial t}$ and integrating by parts, we obtain

$$\sum_{i=1}^{6}\int_{\Omega_i}\Big\{\Big(\frac{\partial}{\partial t}I_{\rho_i}\frac{\partial\psi_i}{\partial t}\Big)\circ\frac{\partial\psi_i}{\partial t}+\frac{\partial\psi_i}{\partial t}\circ K_i\Big(\psi_i-\frac{\partial D_i}{\partial\xi_i}\Big)\Big\}d\xi_i$$
$$+\sum_{i=1}^{6}\int_{\Omega_i}EI_i\frac{\partial\psi_i}{\partial\xi_i}\circ\frac{\partial\psi_i}{\partial\xi_i}d\xi_i = \sum_{i=1}^{6}\int_{\Omega_i}F_{b_{T_i}}\circ\frac{\partial\psi_i}{\partial t}d\xi_i\ . \quad (4.175)$$

Similarly, scalar multiplying Equation (4.102) by $\tilde{v}$, we get

$$\frac{1}{2}\frac{d}{dt}m_s(\tilde{v}\circ\tilde{v})+\frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d}{dt}(\tilde{v}\circ\tilde{v})d\xi_i+\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d^2\bar{r}_i}{dt^2}\circ\tilde{v}d\xi_i+\frac{Gm_e m_T\bar{R}\circ\tilde{v}}{|R_0|^3} = F_s\circ\tilde{v}. \quad (4.176)$$

Using the relations (4.108-4.109), it is not difficult to see that

$$\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{|w\times\bar{r}_i|^2-(w\times(w\times\bar{r}_i))\circ\dot{D}_i+\ddot{D}_i\circ(w\times\bar{r}_i)\Big\}d\xi_i = \sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d^2\bar{r}_i}{dt^2}\circ(w\times\bar{r}_i)d\xi_i. \quad (4.177)$$

Substituting this identity into Eq.(4.173) and adding Eqs.(4.173), (4.174) and (4.176) up, we have

$$\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4 = F_s\circ\tilde{v}+T\circ w+\sum_{i=1}^{6}\int_{\Omega_i} F_{b_i}\circ\dot{D}_i d\xi_i+\sum_{i=1}^{6}\int_{\Omega_i} F_{bT_i}\circ\frac{\partial\psi_i}{\partial t}d\xi_i. \quad (4.178)$$

where

$$\mathcal{I}_1 \equiv \frac{1}{2}\frac{d}{dt}m_s(\tilde{v}\circ\tilde{v})+\frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d}{dt}(\tilde{v}\circ\tilde{v})d\xi_i+\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d^2\bar{r}_i}{dt^2}\circ\tilde{v}d\xi_i + \frac{Gm_e m_T\tilde{R}\circ\tilde{v}}{|R_0|^3}, \quad (4.179)$$

$$\mathcal{I}_2 \equiv \frac{1}{2}\frac{d}{dt}\Big((I_s\circ w)\circ w\Big)+\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d^2\bar{r}_i}{dt^2}\circ(w\times\bar{r}_i)d\xi_i + \sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d^2R_0}{dt^2}\circ(w\times\bar{r}_i)d\xi_i+\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d\tilde{v}}{dt}\circ(w\times\bar{r}_i)d\xi_i \quad (4.180)$$

$$\mathcal{I}_3 \equiv \sum_{i=1}^{6}\int_{\Omega_i}\rho_i\frac{d^2R_0}{dt^2}\circ\dot{D}_i d\xi_i+\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big(\frac{d\tilde{v}}{dt}+\frac{d^2\bar{r}_i}{dt^2}\Big)\circ\dot{D}_i d\xi_i - \frac{1}{2}\sum_{i=1}^{6}\int_{\Omega_i}K_i\Big(\psi_i-\frac{\partial D_i}{\partial\xi_i}\Big)\circ\frac{\partial^2 D_i}{\partial\xi_i\partial t}d\xi_i, \quad (4.181)$$

$$\mathcal{I}_4 \equiv \sum_{i=1}^{6}\int_{\Omega_i}\Big\{\Big(\frac{\partial}{\partial t}I_{\rho_i}\frac{\partial\psi_i}{\partial t}\Big)\circ\frac{\partial\psi_i}{\partial t}+\frac{\partial\psi_i}{\partial t}\circ K_i\Big(\psi_i-\frac{\partial D_i}{\partial\xi_i}\Big)\Big\}d\xi_i + \sum_{i=1}^{6}\int_{\Omega_i}EI_i\frac{\partial\psi_i}{\partial\xi_i}\circ\frac{\partial\psi_i}{\partial\xi_i}d\xi_i. \quad (4.182)$$

Then, taking time derivative of Eq.(4.172) (in the inertial frame) one finds that the left-hand-side expression of Eq.(4.178) equals $\frac{d}{dt}\mathrm{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ plus two additional terms as follows.

$$\frac{d}{dt}\mathrm{V}(\omega,\varphi,\tilde{R},\tilde{v};t)+\sum_{i=1}^{6}\int_{\Omega_i}\rho_i\Big\{\frac{d^2R_0}{dt^2}\circ(w\times\bar{r}_i)+\frac{d^2R_0}{dt^2}\circ\dot{D}_i\Big\}d\xi_i = F_s\circ\tilde{v}+\sum_{i=1}^{6}\int_{\Omega_i}\Big(F_{b_i}\circ\dot{D}_i\Big)\,d\xi_i+T\circ\omega+\sum_{i=1}^{6}\int_{\Omega_i}F_{bT_i}\circ\frac{\partial\psi_i}{\partial t}d\xi_i. \quad (4.183)$$

Substituting the controls given by Eqs.(4.168-4.171) into Eq.(4.183), we obtain

$$\begin{aligned}\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t) &= -\Big(k_1\omega_1^2+k_2\omega_2^2+k_3\omega_3^2\Big)-\Big(e_1\tilde{v}_1^2+e_2\tilde{v}_2^2+e_3\tilde{v}_3^2\Big)\\ &-\int_{\Omega_i}\Big\{\sum_{i=5}^{6}d_{i_1}(\xi_i)\Big(\frac{\partial x_i}{\partial t}\Big)^2+\sum_{i=1}^{4}d_{i_2}(\xi_i)\Big(\frac{\partial y_i}{\partial t}\Big)^2+\sum_{i=1}^{6}d_{i_3}(\xi_i)\Big(\frac{\partial z_i}{\partial t}\Big)^2\Big\}\,d\xi_i\\ &-\int_{\Omega_i}\Big\{\sum_{i=5}^{6}d_{i_4}(\xi_i)\Big(\frac{\partial\psi_{i_x}}{\partial t}\Big)^2+\sum_{i=1}^{4}d_{i_5}(\xi_i)\Big(\frac{\partial\psi_{i_y}}{\partial t}\Big)^2+\sum_{i=1}^{6}d_{i_6}(\xi_i)\Big(\frac{\partial\psi_{i_z}}{\partial t}\Big)^2\Big\}\,d\xi_i.\end{aligned} \tag{4.184}$$

From Eq.(4.184), we see that $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative semidefinite for the given controls. We show that in fact $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative definite in the following. If $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)\equiv 0$ for any finite interval of time, say $[t_1,t_2]$, the following must be true‡[Ahmed 6, Adams 1]. For all $t\in[t_1,t_2]$,

$$\begin{aligned}&\text{(i)}\;\; w_1=w_2=w_3=0,\\ &\text{(ii)}\;\; Q_i\frac{\partial}{\partial t}D_i=0,\; Q_i\frac{\partial}{\partial t}\psi_i=0,\;\text{ for all } \xi_i\in[0,l_i],\\ &\text{(iii)}\;\; \tilde{v}_1=\tilde{v}_2=\tilde{v}_3=0\;\text{ and }\; F_s=0.\end{aligned} \tag{4.185}$$

This requires that for all $t\in[t_1,t_2]$, $\frac{d}{dt}\tilde{v}\equiv\frac{d^2}{dt^2}\tilde{R}=0$, $\dot{w}=0$ and $\bar{D}_i\equiv Q_i\;(\frac{\partial^2 x_i}{\partial t^2},\frac{\partial^2 y_i}{\partial t^2},\frac{\partial^2 z_i}{\partial t^2})'$=0, $Q_i\frac{\partial^2}{\partial t^2}\psi_i=0$ for $\xi_i\in[0,l_i]$. Using all these conditions and Eq.(4.185) into Eq.(4.102), it is clear that $\tilde{R}=0$ for all $t\in[t_1,t_2]$. Then from Eqs.(3.255-3.256) it follows that

$$\frac{\partial}{\partial\xi_i}\Big(K_i\Big(\psi_i-\frac{\partial D_i}{\partial\xi_i}\Big)\Big)=0, \tag{4.186}$$

$$\frac{\partial}{\partial\xi_i}\Big(EI_i\frac{\partial}{\partial\xi_i}D_i\Big)=0,\qquad \text{for }\; \xi_i\in(0,l_i). \tag{4.187}$$

It is easy to see that the unique solution of Equation (4.187) along with the boundary conditions (3.317-3.340) is $D_i=0$ for all $\xi_i\in[0,l_i]$. This implies that ψ_i=0. In

‡ Under the assumption of existence of classical solution, $x_i, y_i, z_i, \psi_{i_x}, \psi_{i_y}, \psi_{i_z}$; $\frac{\partial x_i}{\partial t}, \frac{\partial y_i}{\partial t}, \frac{\partial z_i}{\partial t}, \frac{\partial\psi_{i_x}}{\partial t}, \frac{\partial\psi_{i_y}}{\partial t}, \frac{\partial\psi_{i_z}}{\partial t}$ are continuous and hence (ii) of Equation (4.185) is true.

other words, if $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ =0 for all $t \in [t_1,t_2]$, $w = \tilde{R} = 0$ and $\psi_i = D_i = 0$ for all $\xi_i \in [0,l_i]$. Hence $\frac{d}{dt}\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ is negative definite. Therefore, the function $\mathbf{V}(\omega,\varphi,\tilde{R},\tilde{v};t)$ defined by (4.172) is, indeed, a *Lyapunov functional* and consequently, the system (3.297-3.301,4.102) is asymptotically stable with respect to the rest state(4.100-4.101). This completes the proof. ■

Corollary 4.35 **(Localized Control)**

Consider the system given by Eqs.(3.297-3.301,4.102) and suppose that the controls are given by the localized feedback structure: for $\xi_i \in [0,l_i]$, i=1,2,...,6,

$$T = (T_x - k_1\omega_1,\ T_y - k_2\omega_2,\ T_z - k_3\omega_3)', \quad k_1,k_2,k_3 > 0, \tag{4.188}$$

$$F_{b_i} = Q_i\Big(\tilde{A}_{i_x} - d_{i_1}\frac{\partial x_i}{\partial t}, \tilde{A}_{i_y} - d_{i_2}\frac{\partial y_i}{\partial t},\ \ \tilde{A}_{i_z} - d_{i_3}\frac{\partial z_i}{\partial t}\Big)', \quad d_{i_1},d_{i_2},d_{i_3} > 0, \tag{4.189}$$

$$F_{b_{T_i}} = Q_i\Big(-d_{i_4}\frac{\partial \psi_{i_x}}{\partial t},\ \ -d_{i_5}\frac{\partial \psi_{i_y}}{\partial t},\ \ -d_{i_6}\frac{\partial \psi_{i_z}}{\partial t}\Big)', \quad d_{i_4},d_{i_5},d_{i_6} > 0, \tag{4.190}$$

$$F_s = (-e_1\tilde{v}_1,\ -e_2\tilde{v}_2,\ -e_3\tilde{v}_3)', \qquad e_1,e_2,e_3 > 0. \tag{4.191}$$

with
$$d_{i_1}(\eta) = \sum_{j=1}^{N_1} A_{1_j}\chi_{\mathcal{A}_j}(\eta), \quad A_{1_j} > 0, \tag{4.192}$$

$$d_{i_2}(\eta) = \sum_{j=1}^{N_2} A_{2_j}\chi_{\mathcal{B}_j}(\eta), \quad A_{2_j} > 0, \tag{4.193}$$

$$d_{i_3}(\eta) = \sum_{j=1}^{N_3} A_{3_j}\chi_{\mathcal{C}_j}(\eta), \quad A_{3_j} > 0, \tag{4.194}$$

$$d_{i_4}(\eta) = \sum_{j=1}^{N_4} A_{4_j}\chi_{\mathcal{D}_j}(\eta), \quad A_{4_j} > 0, \tag{4.195}$$

$$d_{i_5}(\eta) = \sum_{j=1}^{N_5} A_{5_j}\chi_{\mathcal{E}_j}(\eta), \quad A_{5_j} > 0, \tag{4.196}$$

$$d_{i_6}(\eta) = \sum_{j=1}^{N_6} A_{6_j}\chi_{\mathcal{F}_j}(\eta), \quad A_{6_j} > 0, \tag{4.197}$$

where χ_e denotes the indicator function of the set e and $\{\mathcal{A}_j, j = 1,2,\cdots,N_1\}$, $\{\mathcal{B}_j, j = 1,2,\cdots,N_2\}$, $\cdots$, $\{\mathcal{F}_j, j = 1,2,\cdots,N_6\}$ are any six families of Lebesgue measurable subsets of $[0, l_j]$ such that $\frac{\partial x_i}{\partial t}$, $\frac{\partial y_i}{\partial t}$, $\frac{\partial z_i}{\partial t}$, $\frac{\partial \psi_{i_x}}{\partial t}$, $\frac{\partial \psi_{i_y}}{\partial t}$ and $\frac{\partial \psi_{i_z}}{\partial t}$ do not vanish identically on $\cup\mathcal{A}_j$, $\cup\mathcal{B}_j$, $\cdots$, $\cup\mathcal{F}_j$ respectively. Then, the system is asymptotically stable with respect to the rest state (4.100-4.101).

Proof The proof is essentially the same as above. ∎

Corollary 4.36 **(Deadzone Control)**

Consider the system described by Equations (3.297-3.301,4.102). Suppose that the feedback controls are given by for all positive $k_j, h_{i_x}, h_{i_y}, h_{i_z}, e_j$, i=1,2,...,6; j=1,2,3,

$$T = (T_x - k_1\Upsilon_{\epsilon_1}(\omega_1), T_y - k_2\Upsilon_{\epsilon_2}(\omega_2), T_z - k_3\Upsilon_{\epsilon_3}(\omega_3))', \tag{4.198}$$

$$F_{b_i} = Q_i\Big(\tilde{A}_{i_x} - h_{i_x}\Upsilon_{\epsilon_4}\frac{\partial x_i}{\partial t},\ \tilde{A}_{i_y} - h_{i_y}\Upsilon_{\epsilon_5}\frac{\partial y_i}{\partial t},\ \tilde{A}_{i_r} - h_{i_z}\Upsilon_{\epsilon_6}\frac{\partial z_i}{\partial t}\Big)', \tag{4.199}$$

$$F_s = (-e_1\Upsilon_{\epsilon_7}(\bar{v}_1), -e_2\Upsilon_{\epsilon_8}(\bar{v}_2), -e_3\Upsilon_{\epsilon_9}(\bar{v}_3)\Big)' \tag{4.200}$$

$$F_{b_{T_i}} = Q_i\Big(-g_{i_x}\Upsilon_{\epsilon_{10}}\frac{\partial \psi_{i_x}}{\partial t},\ -g_{i_y}\Upsilon_{\epsilon_{11}}\frac{\partial \psi_{i_y}}{\partial t},\ -g_{i_z}\Upsilon_{\epsilon_{12}}\frac{\partial \psi_{i_z}}{\partial t}\Big)', \tag{4.201}$$

where

$$\Upsilon_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ 0, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \tag{4.202}$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_d of the desired rest state, defined by

$$\begin{aligned} N_d \equiv \Big\{(\omega, \dot{D}_i, \dot{\psi}_i, \bar{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \left|\frac{\partial x_i}{\partial t}\right| < \epsilon_4, \left|\frac{\partial y_i}{\partial t}\right| < \epsilon_5, \left|\frac{\partial z_i}{\partial t}\right| < \epsilon_6, \\ |\bar{v}_1| < \epsilon_7, |\bar{v}_2| < \epsilon_8, |\bar{v}_3| < \epsilon_9, \left|\frac{\partial \psi_{i_x}}{\partial t}\right| < \epsilon_{10}, \left|\frac{\partial \psi_{i_y}}{\partial t}\right| < \epsilon_{11}, \left|\frac{\partial \psi_{i_z}}{\partial t}\right| < \epsilon_{12}; \\ \epsilon_j > 0, j = 1, 2, \cdots, 12;\ i = 1, 2, ..., 6\Big\}. \end{aligned} \tag{4.203}$$

Remark 4.37 **(Bang-Bang Control)**

We note that when $\epsilon_j = 0, j = 1, 2, \cdots, 12$ in Equations (4.198-4.201) we obtain the Bang-Bang controls which can be easily implemented in practical applications, as a special case of deadzone controls.

Corollary 4.38 **(Saturation Control)**

Consider the system described by Equations (3.297-3.301,4.102). Suppose that the feedback controls are given by for all positive $k_j, h_{i_x}, h_{i_y}, h_{i_z}, e_j$, i=1,2,...,6; j=1,2,3,

$$T = (T_x - k_1 S_{\epsilon_1}(\omega_1), T_y - k_2 S_{\epsilon_2}(\omega_2), T_z - k_3 S_{\epsilon_3}(\omega_3))', \tag{4.204}$$

$$F_{b_i} = Q_i\Big(\tilde{A}_{i_x} - h_{i_x} S_{\epsilon_4}\frac{\partial x_i}{\partial t},\ \tilde{A}_{i_y} - h_{i_y} S_{\epsilon_5}\frac{\partial y_i}{\partial t},\ \tilde{A}_{i_z} - h_{i_z} S_{\epsilon_6}\frac{\partial z_i}{\partial t}\Big)', \tag{4.205}$$

$$F_s = (-e_1 S_{\epsilon_7}(\tilde{v}_1), -e_2 S_{\epsilon_8}(\tilde{v}_2), -e_3 S_{\epsilon_9}(\tilde{v}_3))', \tag{4.206}$$

$$F_{b_{T_i}} = Q_i\Big(-g_{i_x} S_{\epsilon_{10}}\frac{\partial \psi_{i_x}}{\partial t},\ -g_{i_y} S_{\epsilon_{11}}\frac{\partial \psi_{i_y}}{\partial t},\ -g_{i_z} S_{\epsilon_{12}}\frac{\partial \psi_{i_z}}{\partial t}\Big)', \tag{4.207}$$

where

$$S_\eta(\xi) \equiv \begin{cases} +1, & \text{if } \xi > \eta, \\ \xi, & \text{if } -\eta \le \xi \le +\eta, \\ -1, & \text{if } \xi < -\eta. \end{cases} \tag{4.208}$$

Then the system is (approximately) asymptotically stable with respect to the ε-neighborhood N_s of the desired rest state, defined by

$$\begin{aligned} N_s \equiv \Big\{(\omega, \dot{D}_i, \dot{\psi}_i, \tilde{v}) : |\omega_1| < \epsilon_1, |\omega_2| < \epsilon_2, |\omega_3| < \epsilon_3, \left|\frac{\partial x_i}{\partial t}\right| < \epsilon_4, \left|\frac{\partial y_i}{\partial t}\right| < \epsilon_5, \left|\frac{\partial z_i}{\partial t}\right| < \epsilon_6, \\ |\tilde{v}_1| < \epsilon_7, |\tilde{v}_2| < \epsilon_8, |\tilde{v}_3| < \epsilon_9, \left|\frac{\partial \psi_{i_x}}{\partial t}\right| < \epsilon_{10}, \left|\frac{\partial \psi_{i_y}}{\partial t}\right| < \epsilon_{11}, \left|\frac{\partial \psi_{i_z}}{\partial t}\right| < \epsilon_{12}; \\ \epsilon_j > 0, j = 1, 2, \cdots, 12;\ i = 1, 2, ..., 6\Big\}. \end{aligned} \tag{4.209}$$

4.5 Summary

In this chapter, stability of the larege flexible space structures governed by the equations of motion given in Chapter 2 and Chapter 3 has been investigated. By applying feedback controls the asymptotic stability of the perturbed system subject to external disturbances has been proved using Lyapunov method by the author and Ahmed[82]. Several forms of controls such as proportional feedback control, dead-zone control, bang-bang and saturation controls were utilized to stabilize the perturbed system[Lim and Ahmed 83]. The effectiveness of the controls for vibration suppression of the perturbed system is illustrated by numerical simulations in Chapter 5.

Chapter 5

NUMERICAL SIMULATION

5.1 Introduction

In this chapter we present the results of numerical simulation for the space structures whose dynamics were derived in Chapter 2 and Chapter 3, in order to illustrate the effectiveness of the stabilizing controls and the impact of the dynamic coupling on system stability. As mentioned earlier in Chapter 2 and Chapter 3, the dynamics of the spacecraft and space stations are described by the hybrid system of equations which is a combination of ordinary differential equations governing the dynamics of the rigid parts of the system and partial differential equations representing the vibration of the flexible members. The hybrid system could be solved simultaneously using semidiscretization algorithm that is a combination of Runge-Kutta method and finite difference method applied to the partial differential equations with respect to the spatial variable[8,65,108]. In Section 5.2, the semidiscretization method is considered. On the basis of the algorithm, in Section 5.3 we present the simulation results of the spacecraft governed by the dynamics consisting of radial equations (2.42), rotational equations (2.47) and beam equations (2.54) with the boundary conditions (2.37). In Section 5.3 several cases corresponding to different stabilizing controls such as localized control, deadzone control and stabilization during slew maneuvers are investigated. Similarly in Section 5.4 the results for the space station (shown in Fig.3.1) governed by the dynamics (3.84-3.120) are presented. In Section 5.5 concluding remarks are presented.

5.2 A Numerical Algorithm

In this section we consider an algorithm for numerical solution of the hybrid dynamics (2.42-2.54) (or 3.84-3.120), which is a combination of ordinary differential equations and partial differential equations. The algorithm is a combination of Runge-Kutta method[65] for ordinary differential equations and finite difference method[8] for partial differential equations. For simplicity of presentation in this section we develop an algorithm for solving the dynamics of the spacecraft in Section 2.4. However, note that it is valid also for all the dynamics considered in Chapter 2 and Chapter 3.

Defining

$$\phi_1 = (y, z)' \,, \tag{5.1}$$

$$\phi_2 = \frac{\partial \phi_1}{\partial t} \,, \tag{5.2}$$

$$\phi_3 = \frac{\partial^2 \phi_1}{\partial x^2} \,, \tag{5.3}$$

equations (2.54) could be written in three first order equations as

$$\frac{\partial \phi_1}{\partial t} = \phi_2 \,, \tag{5.4}$$

$$\frac{\partial \phi_2}{\partial t} = F_b - A_1\phi_2 - A_2\frac{\partial^2 \phi_3}{\partial x^2} - A_3\phi_1 - A_4(x + R_s) - QC_I^B\frac{\partial^2 R}{\partial t^2}, \tag{5.5}$$

$$\frac{\partial \phi_3}{\partial t} = \frac{\partial^2 \phi_2}{\partial x^2} \,, \tag{5.6}$$

where

$$A_1 = \begin{pmatrix} 0 & -2w_1 \\ 2w_1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} EI_y/\rho & 0 \\ 0 & EI_z/\rho \end{pmatrix}, \tag{5.7}$$

$$A_3 = \begin{pmatrix} -w_1^2 - w_3^2 & -\dot{w}_1 + w_2w_3 \\ \dot{w}_1 + w_2w_3 & -w_1^2 - w_2^2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} \dot{w}_3 + w_1w_3 \\ -\dot{w}_2 + w_1w_3 \end{pmatrix}. \tag{5.8}$$

Note that in terms of the notations used previously, $\phi_1=(y,z)'$ is the displacement vector. Partial derivatives can be approximated by finite differences in many ways. Development of the Taylor series for $f(x+\Delta x,t)$ about (x,t) gives

$$f(x+\Delta x,t) = f(x,t) + \frac{\partial f}{\partial x}(x,t)\Delta x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,t)(\Delta x)^2 + \cdots \tag{5.9}$$

Similarly we have

$$f(x-\Delta x,t) = f(x,t) - \frac{\partial f}{\partial x}(x,t)\Delta x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,t)(\Delta x)^2 - \cdots \tag{5.10}$$

Adding (5.9) and (5.10) we obtain the central difference approximation *

$$f(x+\Delta x,t) + f(x-\Delta x,t) = 2f(x,t) + \frac{\partial^2 f}{\partial x^2}(x,t)(\Delta x)^2 + \mathbf{O}((\Delta x)^4)$$

or

$$\frac{\partial^2 f}{\partial x^2}(x,t) \cong \frac{f(x+\Delta x,t) - 2f(x,t) + f(x-\Delta x,t)}{(\Delta x)^2}. \tag{5.11}$$

Substracting (5.10) from (5.9) we have

$$f(x+\Delta x,t) - f(x-\Delta x,t) = 2\frac{\partial f}{\partial x}(x,t)\Delta x + \mathbf{O}((\Delta x)^3)$$

or

$$\frac{\partial f}{\partial x}(x,t) \cong \frac{f(x+\Delta x,t) - f(x-\Delta x,t)}{2(\Delta x)}. \tag{5.12}$$

Then using (5.11) and (5.12), a finite difference approximations of (5.4-5.6) *with respect to x* are given by

$$\frac{\partial \phi_1}{\partial t}(k,t) = \phi_2(k,t), \tag{5.13}$$

$$\frac{\partial \phi_2}{\partial t}(k,t) = F_b(k,t) - A_2\Big\{\frac{\phi_3(k+1,t) - 2\phi_3(k,t) + \phi_3(k-1,t)}{(\Delta x)^2}\Big\}$$
$$- A_1(t)\phi_2(k,t) - A_3(t)\phi_1(k,t) - A_4(k+R_s) - QC_I^B\frac{\partial^2 R}{\partial t^2}(t), \tag{5.14}$$

$$\frac{\partial \phi_3}{\partial t}(k,t) = \frac{\phi_2(k+1,t) - 2\phi_2(k,t) + \phi_2(k-1,t)}{(\Delta x)^2}, \tag{5.15}$$

* The notation O(h) represent big 'oh ' (for details, see Ames[8]).

where k denotes the discretization points in spatial variable x. Thus the hybrid dynamics (2.42-2.54) is written as a complete system of ordinary differential equations of the form

$$\dot{\Phi} = F(\Phi), \qquad (5.16)$$
$$\Phi(0) = \Phi_0,$$

where $\Phi \equiv \{R, w, (\phi_1(k), \phi_2(k), \phi_3(k))', k = 1, 2, ..., n\}$. Then Runge-Kutta method is used to solve the nonlinear ordinary differential equations (5.16).

For the numerical solution of the dynamics for the space station one can use the same method mentioned above. Following the semi- discretization method, we obtain, from the dynamics of the space station (equations 3.84-3.120), a system of nonlinear ordinary differential equations of the form

$$\dot{\Psi} = G(\Psi), \qquad (5.17)$$
$$\Psi(0) = \Psi_0,$$

where $\Psi \equiv \{R, w, (\psi_{i_1}(k), \psi_{i_2}(k), \psi_{i_3}(k))';\ k = 1, 2, ..., n_i;\ i = 1, 2, ..., 6\}$ for $\psi_i = (y_i, z_i)'$, $i = 1, 2, 3, 4$; $\psi_j = (x_j, z_j)'$, $j = 5, 6$. The equations (5.17) can be solved using Runge-Kutta method. In the following two sections numerical results, which were computed using the *semi- discretization method*, are presented for illustration of the effects of the stabilizing controls discussed in Chapter 4.

5.3 Numerical Results for the Spacecraft

In this section, we present the results of numerical simulation of the flexible spacecraft described by equations (2.42-2.54). For the purpose of simulation, we assume that the satellite body inertia tensor is diagonal and that the beam is uniform with circular cross section. The following parameters were used in the simulation:

Satellite Parameters

$$I_1 = 635\ slug\ ft^2$$
$$I_2 = 100\ slug\ ft^2$$
$$I_3 = 669\ slug\ ft^2$$
$$m_s = 29.362\ slug$$

Beam Parameters

Length $l = 18.8 ft$

Flexural rigidity $EI_y = EI_z = 3550.8\ lb\ ft^2$

Mass density $\rho = 2.86 \times 10^{-2}\ slug/ft$

Distance $R_s = 3\ ft$

I_1, I_2, I_3 are the diagonal elements of the satellite bus inertia I_s.

The *initial conditions* were taken as: $\omega_1(0) = 0.03$ rad/sec, $\omega_2(0)$=0.02 rad/sec, $\omega_3(0)$=0.01 rad/sec and for $x \in [0,l], \frac{\partial y}{\partial t}(x,0) = \frac{\partial z}{\partial t}(x,0) = 0$, $y(x,0) = z(x,0) = \phi_0(x)$, where ϕ_0 satisfies the boundary conditions of the beam and given by

$$\begin{aligned}\phi_0(x) = \gamma\{&(cosh(\lambda l) + cos(\lambda l))(sinh(\lambda x) - sin(\lambda x)) \\ &-(sinh(\lambda l) + sin(\lambda l))(cosh(\lambda x) - cos(\lambda x))\}\end{aligned} \tag{5.18}$$

where $\gamma = -0.001071$ and λ satisfies

$$cos(\lambda l)cosh(\lambda l) + 1 = 0. \tag{5.19}$$

The initial conditions for the radial vector are taken as $R_0(0) = (1.3864 \times 10^8, 0, 0)'\ ft$ (synchronous orbit) and $\frac{dR_0}{dt}(0) = (0, 12338.1, 0)'\ ft/\text{sec}$. *In order to asses the impact of radial perturbation on the beam and body motions,* the external disturbance acting on the orbit equation (i.e. vector $\tilde{R}(t)$) was taken as $F_1(t) = -3000.0\ lb_f$, $F_2(t)$=$F_3(t) = 0\ lb_f$ for the period [2.5,2.6] sec.

In order to study the effect of the controls in Section 4.3 on the dynamic behavior of the spacecraft, the simulation results for the following situations have been compared. For the convenience of presentation, the different schemes of control are designated as:

CASE A: No controls are applied after above mentioned disturbance occurs.

CASE B: Localized feedback controls (Corollary 4.4) are applied.

CASE C: Deadzone controls (Corollary 4.5) are applied.

In Section 5.3.1 the results corresponding to CASES A and B are discussed in detail. In Section 5.3.2 we present the results corresponding to CASES A and C. Stability of the spacecraft under the influence of slewing maneuvers is discussed in Section 5.3.3.

5.3.1 Stabilization of the Spacecraft by Proportional Controls

In this section the results corresponding to CASES A and B are compared[S2,S3]. The feedback controls are taken as in equations (4.23-4.27)), with k_1=600, k_2=100, k_3=600 and d_1=d_2=3 $\chi_{[17.9,18.8]}$, where χ_f represents the characteristic function of the set f, and e_1 = 3000, e_2 = 300, e_3 = 30. Figs.5.1-5.12 show the impact of the orbital disturbance on the beam vibration and bus rotation and the effectiveness of the chosen controls in stabilizing the system.

In the absence of controls or external disturbances, the system energy E, given by the sum of rotational energy of the bus (with respect to the inertial frame) E_B and the beam energy E_b, is conservative (see the curves for the time period 0 to 2.5 sec, Fig.5.1). However, under the influence of orbital disturbance during the period [2.5,2.6] the system gains energy and stays at a higher level. It is observed from Fig.5.1 that interchange of energy between the bus and the beam may take place during the perturbations leading to vibrations of the beam and bus rotation. Further orbital disturbance induces large fluctuations in the beam energy and bus rotational energy. With application of the suggested feedback controls the bus angular motion and the beam vibrations decay to zero.

In Figs.5.3 and 5.5, we show the deflections of the beam at the free end, i.e. $y(l,t)$ and $z(l,t)$. The corresponding velocities are plotted in Figs.5.4 and 5.6 respectively.

From these figures, it is clear that in the absence of controls (case A, Figs.5.3-5.6), immediately after the orbital disturbance lasting over the period [2.5,2.6] sec, the beam vibrations grow. However, with application of the feedback controls, these oscillations are effectively eliminated (case B, Figs.5.3-5.6).

The fact that the beam vibrations are reduced throughout its length can be observed from the *relative beam energy* curve of Fig.5.2. We define the *relative beam energy* function as a measure of vibration of the beam with respect to the body axes by

$$E_r(t) \equiv \int_0^l \left\{ \rho(\frac{\partial y}{\partial t})^2 + \rho(\frac{\partial z}{\partial t})^2 + EI_y(\frac{\partial^2 y}{\partial x^2})^2 + EI_z(\frac{\partial^2 z}{\partial x^2})^2 \right\} dx. \tag{5.20}$$

We note that $E_r(t)$ represents the total energy of the beam relative to the body axes. Clearly $E_r(t)$=0 implies that all the points of the beam are at their rest positions. From Fig.5.2, we observe that for the uncontrolled system the relative beam energy increases with time following the orbital disturbance, indicating growing oscillations of the beam with respect to the body axes (case A, Fig.5.2). Use of the suggested feedback controls effectively reduces the beam vibrations to zero as shown in Fig.5.2 (case B).

The impact of the orbital disturbance on the bus angular velocities is shown in Figs.5.7-5.9. It is observed, from Figs.5.7-5.9, that the angular velocities ω_2 and ω_3 experience larger fluctuations as compared to ω_1. This is based on the fact that the beam vibrations in the j_b and k_b directions would contribute to the total angular momentum about k_b and j_b directions respectively. The orbital disturbance during the period [2.5,2.6]sec introduces additional oscillations in the satellite bus rotation (case A, Figs.5.7-5.9), indicating growing fluctuations in the angular velocities of the bus. Stabilization of the angular velocities by application of the feedback controls can be clearly observed from case B of Figs.5.7-5.9.

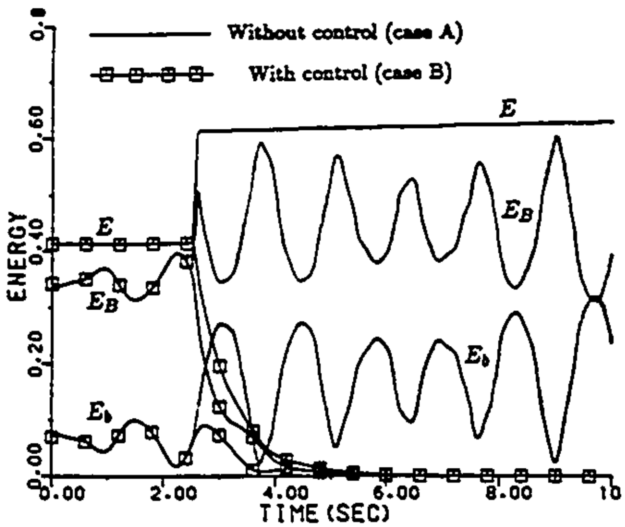


Fig.5.1 Energy of the bus and the beam.

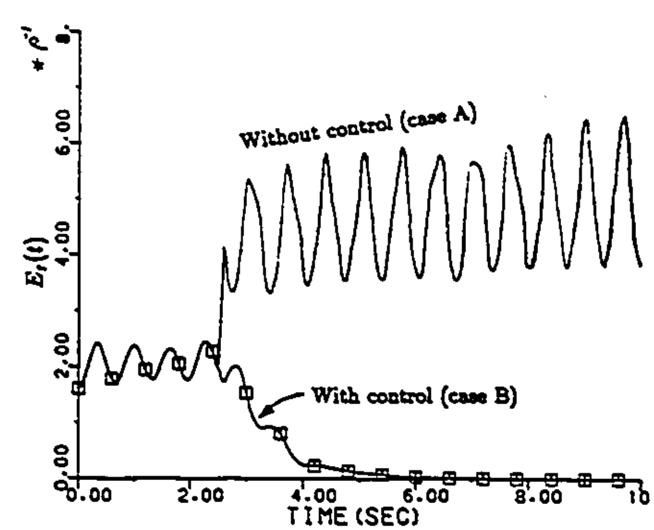


Fig.5.2 Relative beam energy.

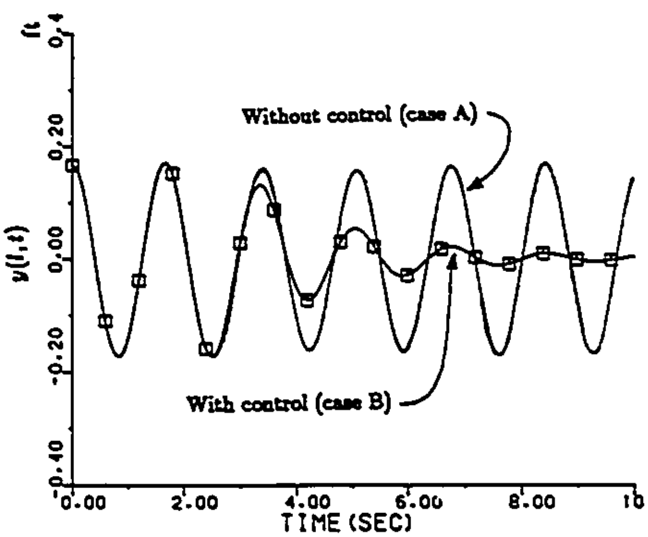


Fig.5.3 Beam displacement $y(l,t)$.

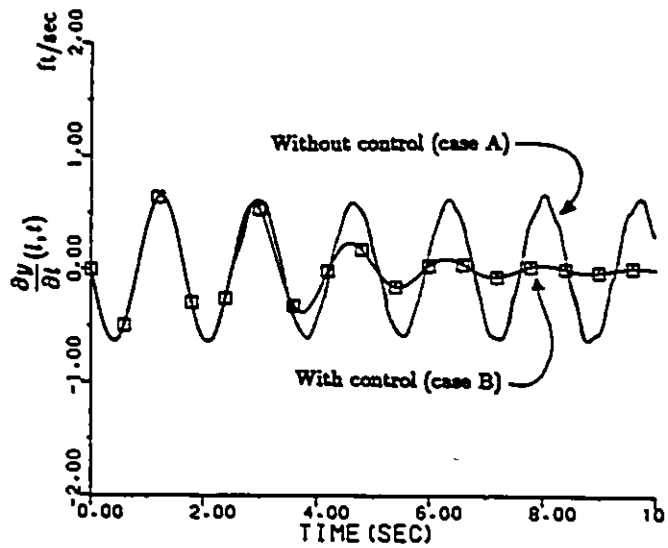


Fig.5.4 Beam velocity $\frac{\partial y}{\partial t}(l,t)$.

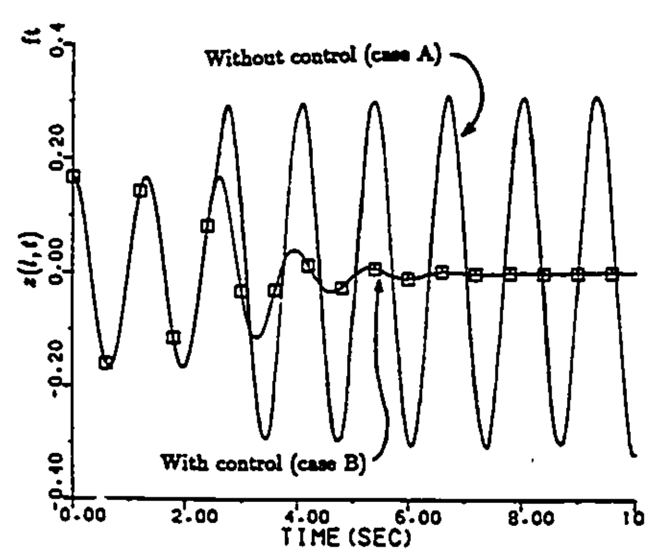


Fig.5.5 Beam displacement $z(l,t)$.

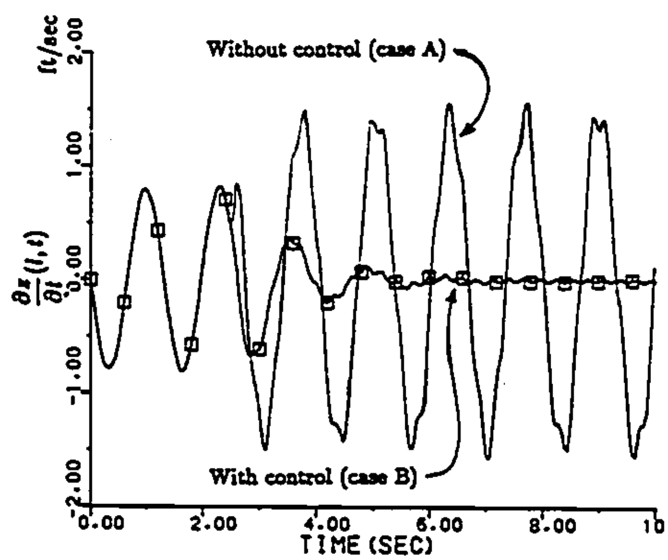


Fig.5.6 Beam velocity $\frac{\partial z}{\partial t}(l,t)$.

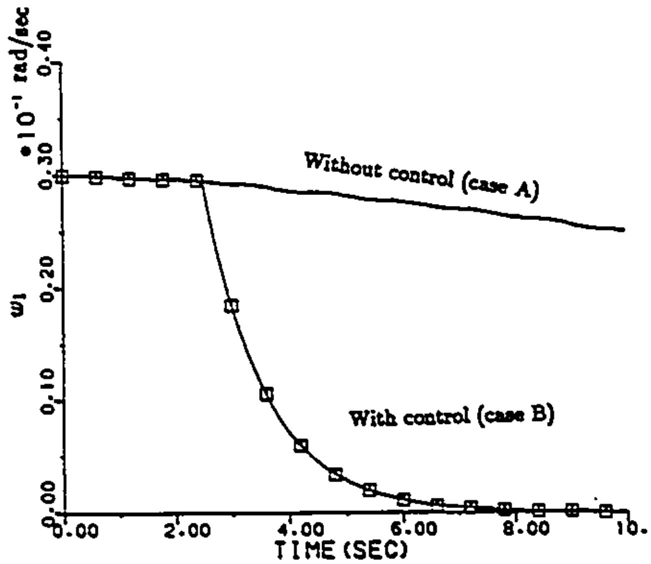


Fig.5.7 Bus angular velocity w_1.

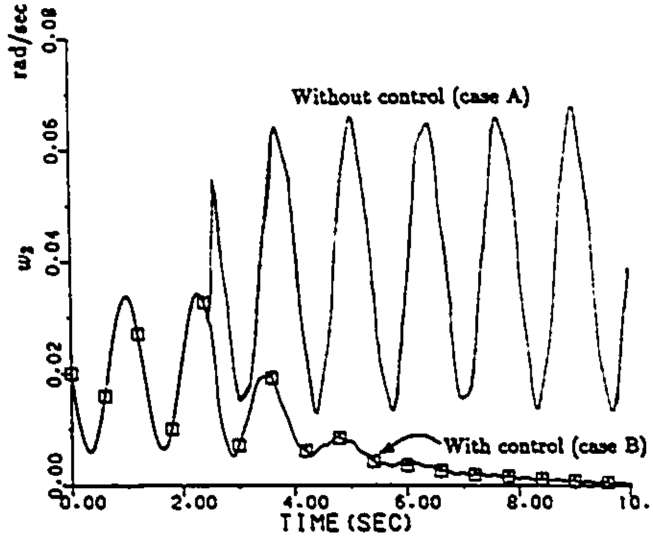


Fig.5.8 Bus angular velocity w_2.

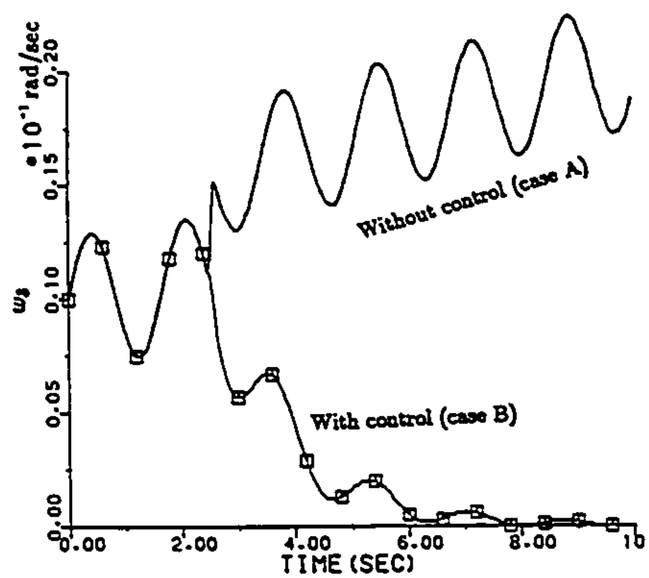


Fig.5.9 Bus angular velocity w_3.

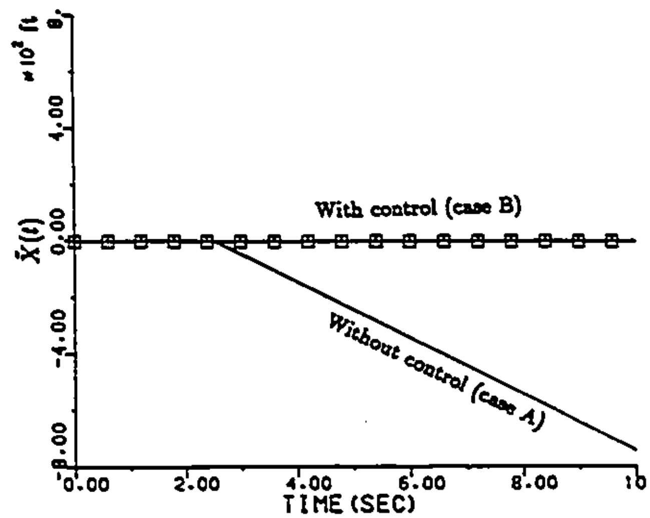


Fig.5.10 Radial perturbation $\tilde{X}$.

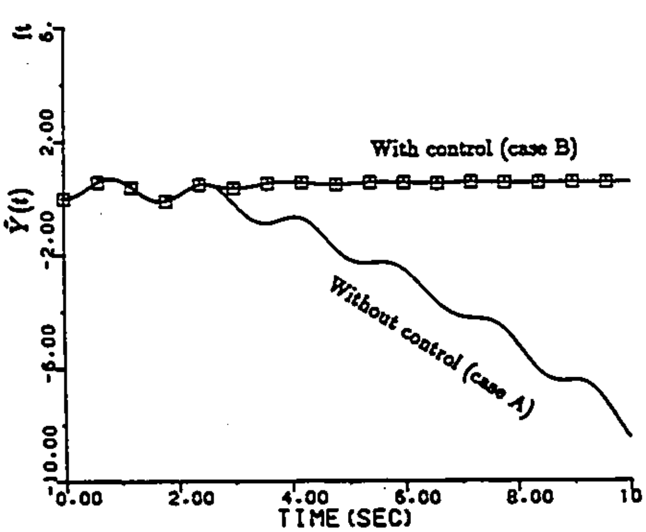


Fig.5.11 Radial perturbation $\tilde{Y}$.

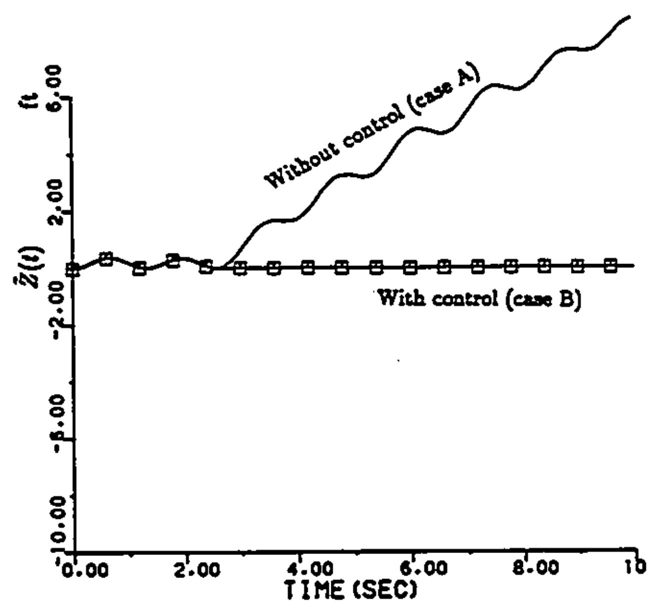


Fig.5.12 Radial perturbation $\tilde{Z}$.

As mentioned earlier in Section 2.4, the satellite translation, rotation and beam vibration are strongly coupled. Thus one can expect some effects of the beam vibration and bus rotation on the radial vector $\tilde{R}(t)$. These are shown in Figs.5.10-5.12. It is clear from these figures that in the absence of the radial disturbance (from 0 to 2.5 sec), the vibration of beam and fluctuations in the bus rotation induce small oscillations in the orbit, i.e. $\tilde{R}(t)$. However, immediately following the external disturbance, the fluctuations in the $\tilde{R}(t)$ grow with time and this also induces vibrations in the beam and oscillations in the bus angular motion as expected (see equation (4.3)). However, by application of the controls, the fluctuations are eliminated. We note that $\tilde{X}$ and $\tilde{Z}$ stabilize to the zero state while $\tilde{Y}$ stabilizes to a neighborhood of the zero state. This error is due to use of the (approximate) linearized equation (4.3) instead of the true nonlinear equation (2.38).

5.3.2 Stabilization of the Spacecraft by Deadzone Controls

In this section the results corresponding to CASES A and C are compared[83]. The deadzone controls(Corollary 4.5) are taken as in equations (4.28-4.32), with $k_1 = 4.5$, $k_2 = 1.0$, $k_3 = 4.5$; $h_1 = h_2 = 0.05\chi_{[17.9,18.8]}$; $e_1 = 3000$, $e_2 = 1000$, $e_3 = 200$; the deadzone parameters: $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.001$; $\epsilon_4 = \epsilon_5 = 0.001$ and $\epsilon_6 = 0.01$, $\epsilon_7 = 0.1$, $\epsilon_8 = 0.01$. Figs.5.13-5.24 show the effectiveness of the chosen deadzone controls in stabilizing the flexible spacecraft under the influence of external disturbances mentioned above. A comparison of Figs.5.13-5.24 with Figs.5.1-5.12 indicates that the overall response of the system under the action of the control law (4.28-4.32) is quite similar to those corresponding to the proportional controls (4.23-4.27) and thus the detailed discussions are omitted. However, it is observed that there are some basic differences in

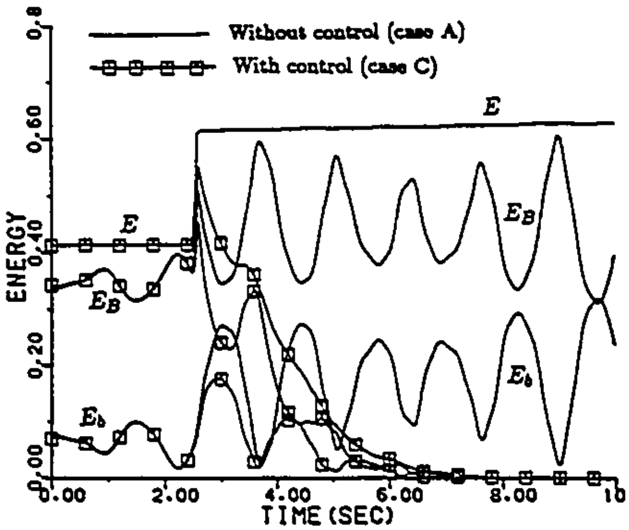


Fig.5.13 Energy of the bus and the beam.

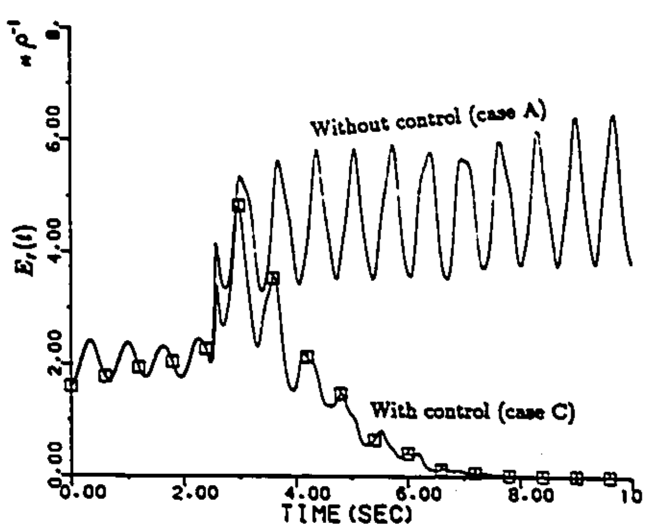


Fig.5.14 Relative beam energy.

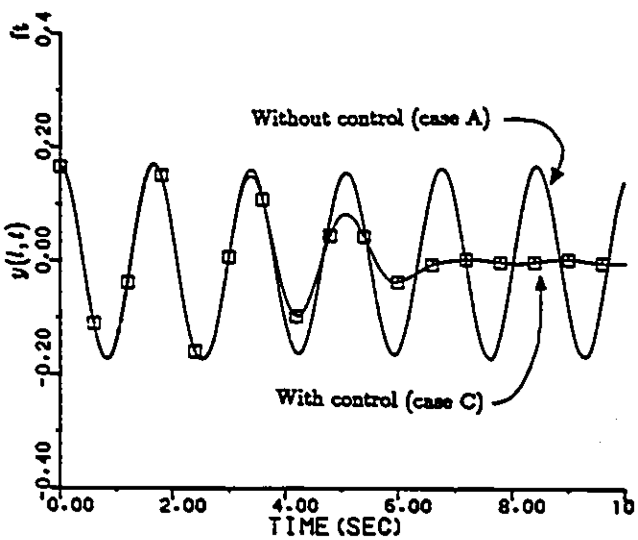


Fig.5.15 Beam displacement $y(l,t)$.

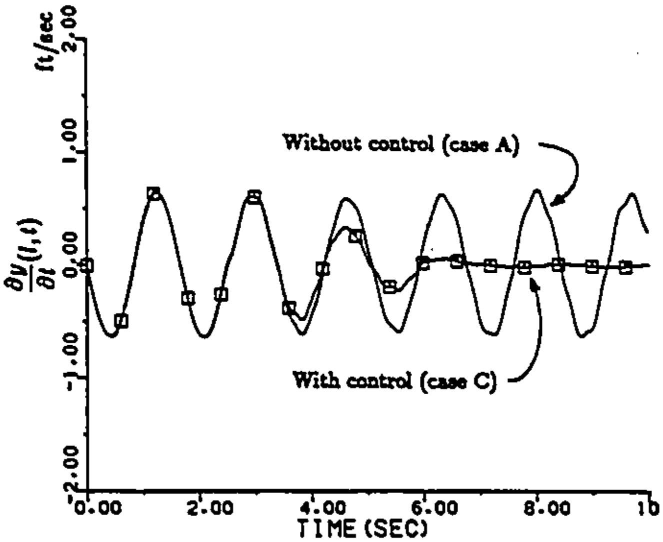


Fig.5.16 Beam velocity $\frac{\partial y}{\partial t}(l,t)$.

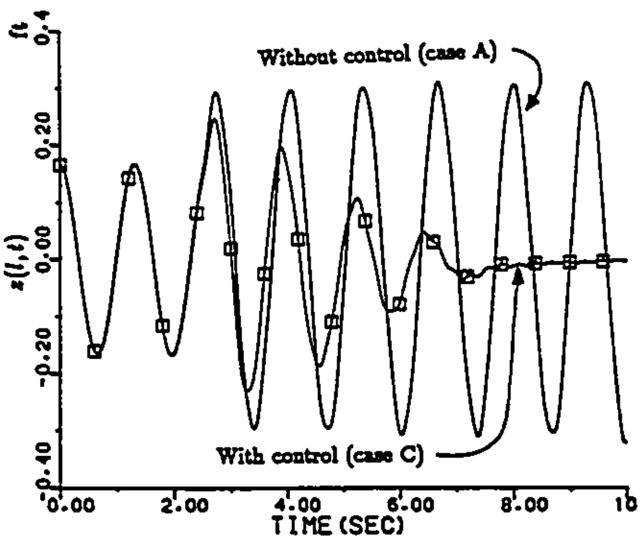


Fig.5.17 Beam displacement $z(l,t)$.

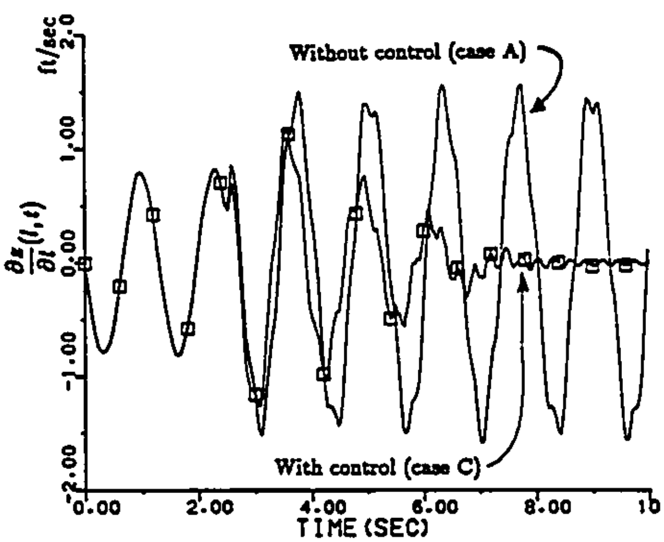


Fig.5.18 Beam velocity $\frac{\partial z}{\partial t}(l,t)$.

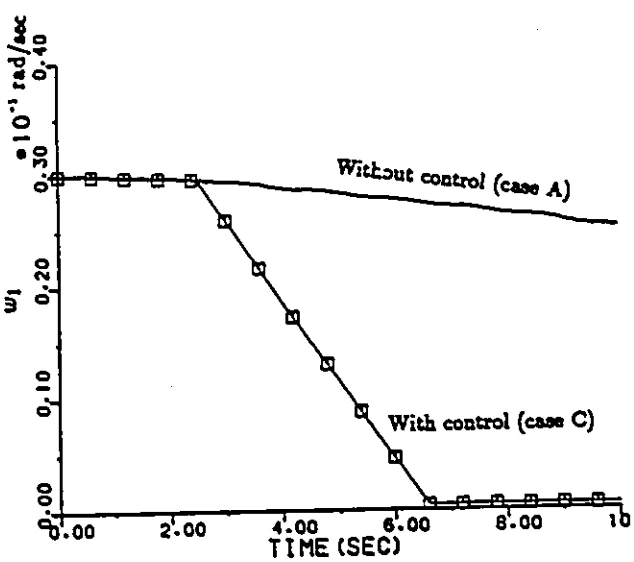


Fig.5.19 Bus angular velocity w_1.

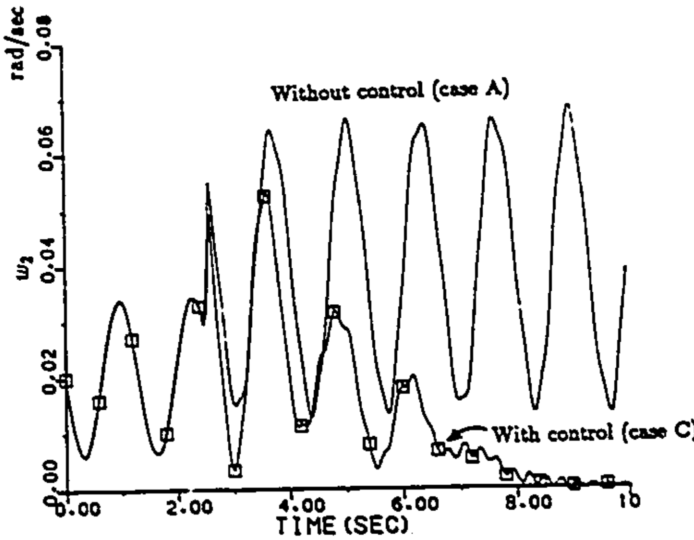


Fig.5.20 Bus angular velocity w_2.

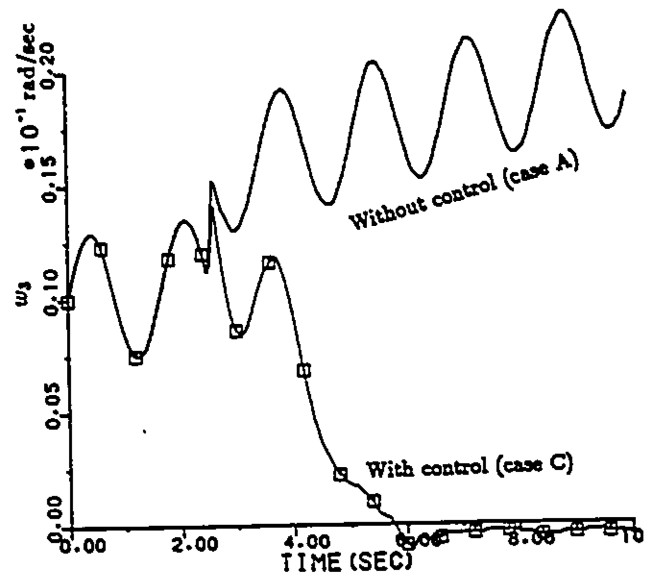


Fig.5.21 Bus angular velocity w_3.

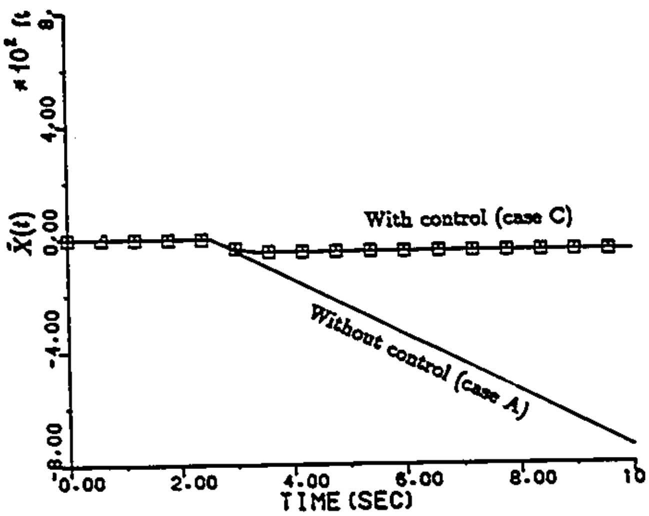


Fig.5.22 Radial perturbation $\tilde{X}$.

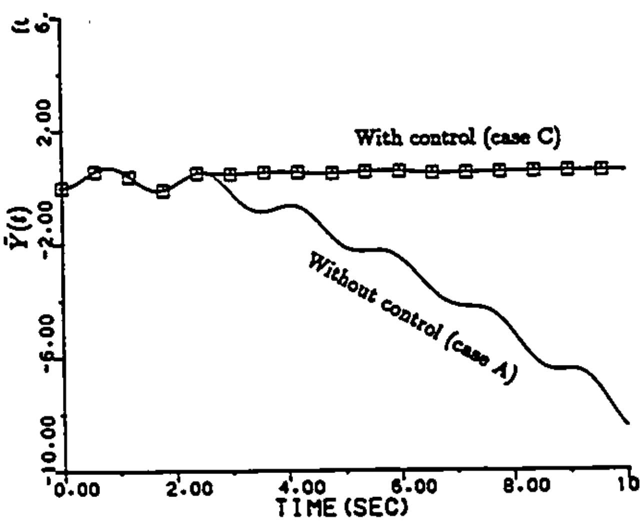


Fig.5.23 Radial perturbation $\tilde{Y}$.

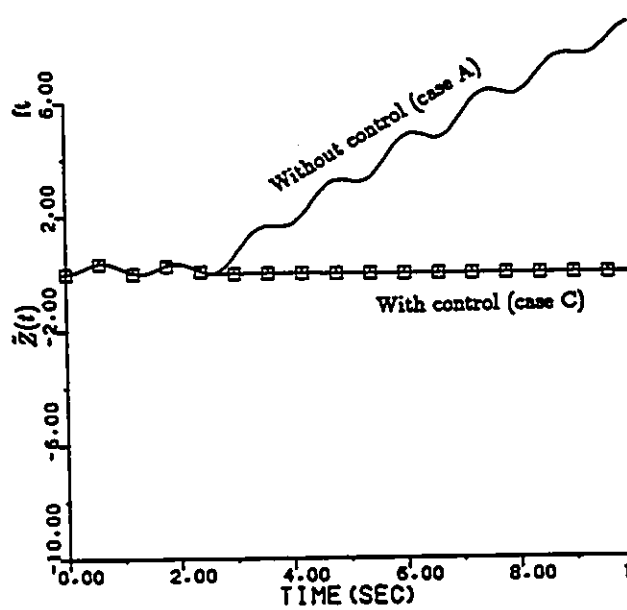


Fig.5.24 Radial perturbation $\tilde{Z}$.

the two sets of results:

(i) *the nature of decay of various states of the system*; in CASE B of Figs.5.1-5.12, the decay rates become smaller as the magnitude of the various states becomes smaller, whereas in CASE C of Figs.5.13-5.24, they remain constants. Indeed, this is quite natural since the control action of the proportional controls is proportional to the errors to be corrected while that of deadzone controls is constant.

(ii) *the limit of the decay*; in CASE B of Figs.5.1-5.12, the various states are reduced to the zero state, whereas in CASE C of Figs.5.13-5.24, they are reduced to the small neighborhood (as pemissible by the size of the deadzone) of the zero state. In Figs.5.13-5.24, the region of the zero state obtained from the simulation is of the order of 0.0001, which is not visible in the figures, while for 0.0004 it is visible as shown in Figures 5.19-5.21. In prctical applications, the size of deadzone is determined by the stabilization requirements of the particular system being considered.

5.3.3 Stabilization of Slew-Maneuvers of the Spacecraft

As mentioned in Remark 2.2, flexible spacecraft are often expected to experience vibrations caused by several disturbing forces. An important disturbance could be due to slewing maneuvers for obtaining or initializing the desired orientation of the spacecraft after it is put into orbit. In this section the results of the stabilization under the influence of slewing maneuvers are presented[85]. To slew a spacecraft for a given angle in a prespecified time, there are many ways to command the slew actuators on board. The one that is easy to implement is bang-bang control. That is, a constant force at its allowable maximum is applied in one direction half of the time and then in the opposite direction the other half. Such control is the most convenient and common with reaction jet thrusters, and most spacecraft including spaceshuttle use thrusters[Lin 87]. For simulation the following *bang-pause bang* control scheme shown in Fig.5.25 is assumed for slew the spacecraft for the desired 0.3rad. (17.2°) under the specified limits on control moments in 2.5 seconds.

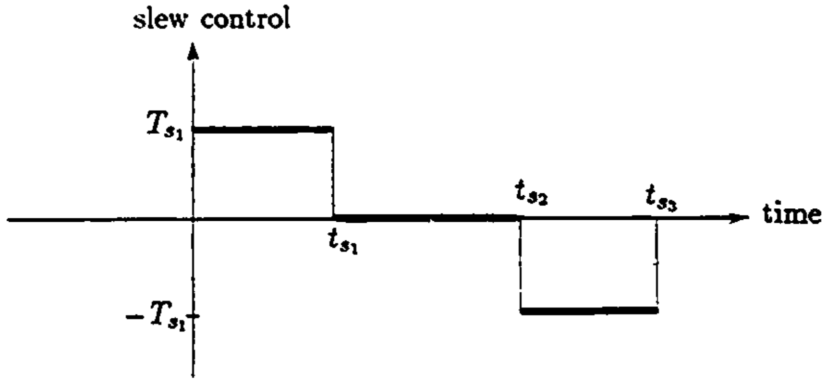


Fig. 5.25 Bang-Pause-Bang control.

For the vibration suppression induced by the slew maneuvers, one can use one of the several control schemes discussed in Chapter 4. For the present problem, we used the proportional control given by equations (4.23-4.27) with the parameters: k_1=600, k_2=100, k_3=600 and d_1=d_2=3 $\chi_{[17.9,18.8]}$, where χ_f represents the characteristic function of the set f, and e_1 = 3000, e_2 = 300, e_3 = 300. $F_1(t)$ = $F_2(t)$= $F_3(t)$ = 0 are assumed and for *slew maneuver* the Bang-Pause-Bang control (Fig.5.25) were used with T_{s_1}= 200.7 *lb-ft*, t_{s_1}=0.5sec, t_{s_2}=2.0sec, t_{s_3}=2.5sec, in the simulation.

Detailed numerical results showing stabilization of the various state trajectories were obtained from the simulation and presented in Figs. 5.26-5.37. Figs. 5.26 and 5.27 represent the slew angle and the angular rate of the spacecraft corresponding to the slew control (Fig. 5.25). We note from the Fig.5.26 that slew maneuver was

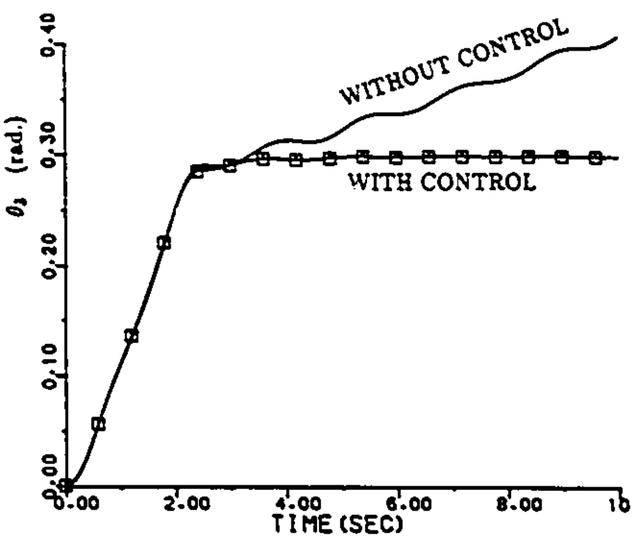


Fig.5.26 The slew angle in k_3 direction.

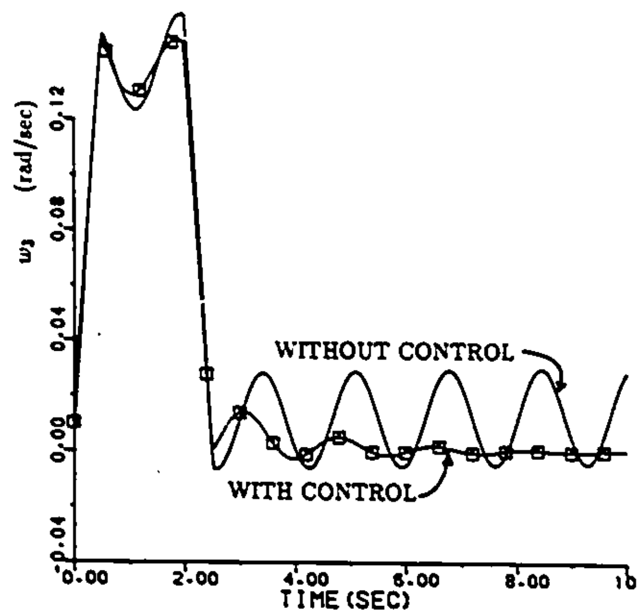


Fig.5.27 Bus angular velocity w_3.

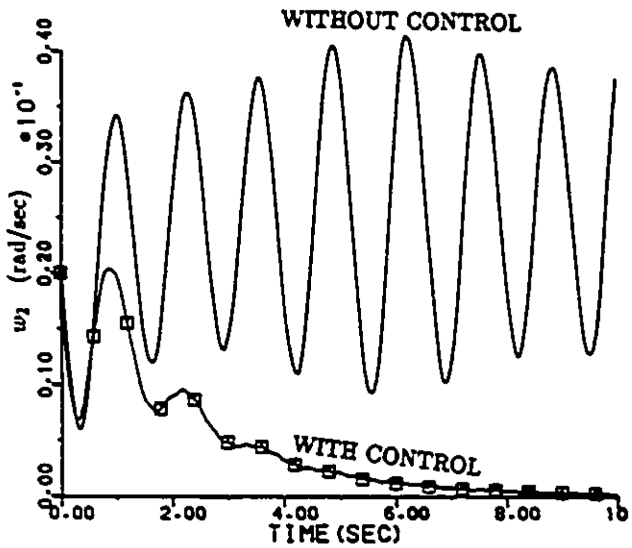


Fig.5.28 Bus angular velocity w_2.

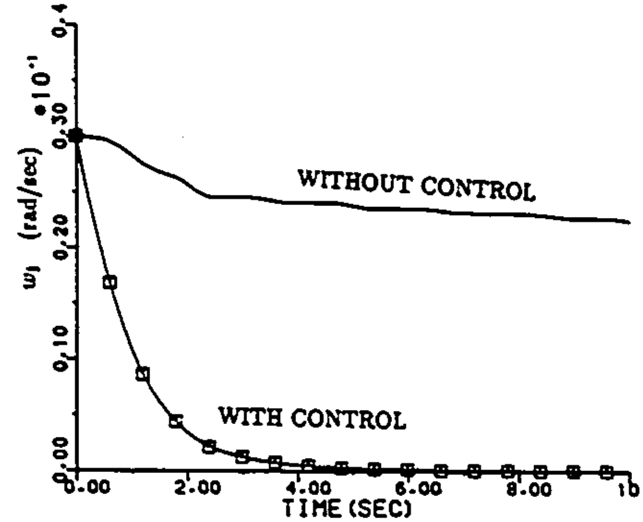


Fig.5.29 Bus angular velocity w_1.

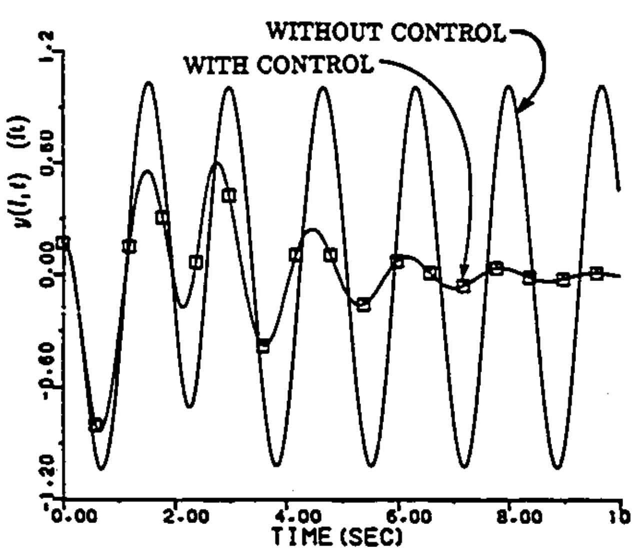


Fig.5.30 Beam displacement $y(l,t)$.

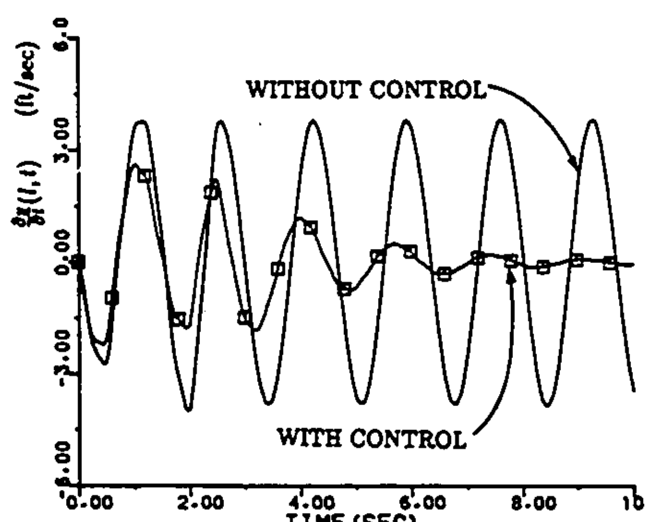


Fig.5.31 Beam velocity $\frac{\partial y}{\partial t}(l,t)$.

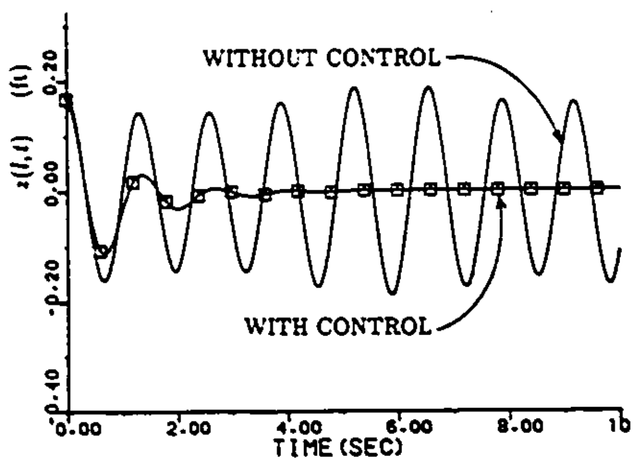


Fig.5.32 Beam displacement $z(l,t)$.

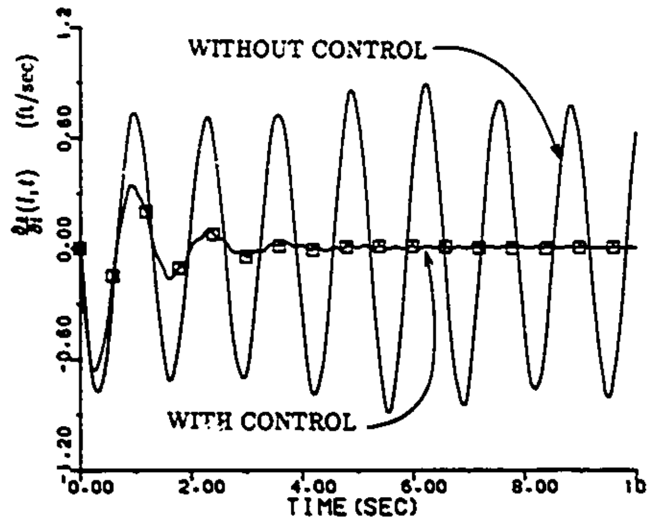


Fig.5.33 Beam velocity $\frac{\partial z}{\partial t}(l,t)$.

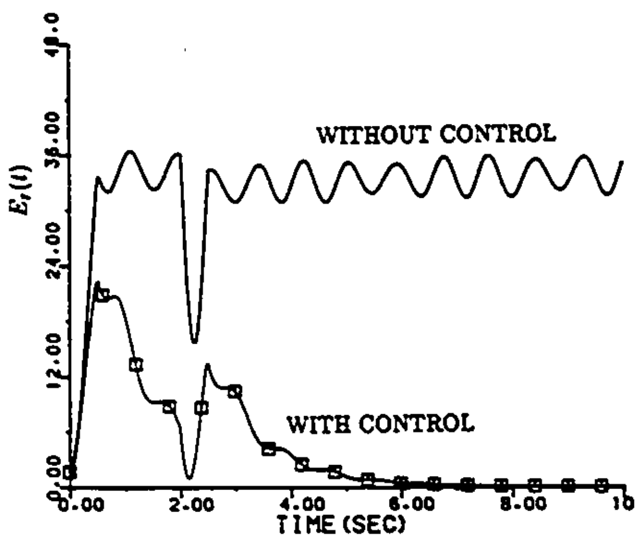


Fig.5.34 Relative beam energy.

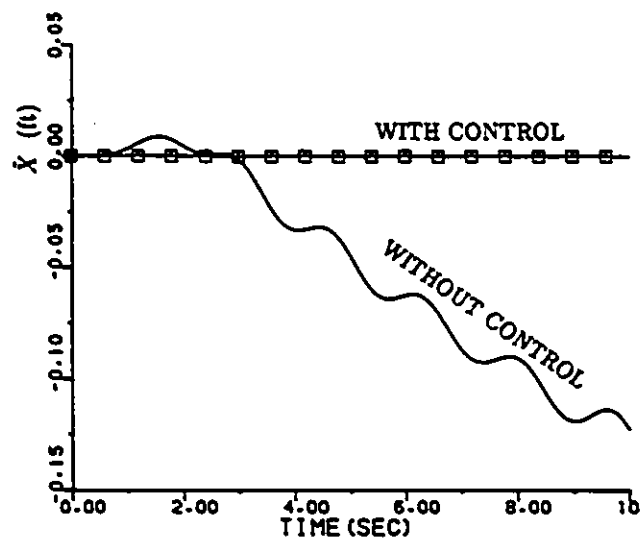


Fig.5.35 Radial perturbation $\tilde{X}$.

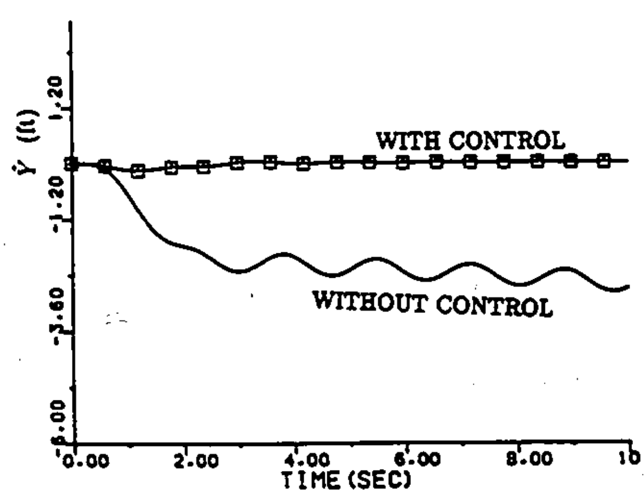


Fig.5.36 Radial perturbation $\tilde{Y}$.

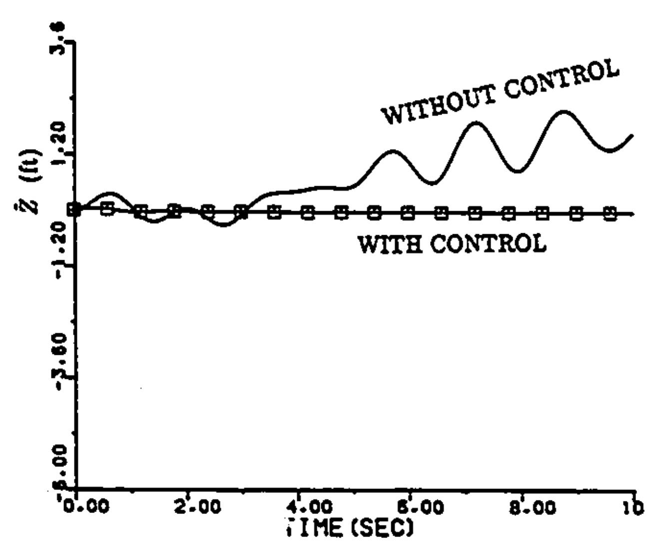


Fig.5.37 Radial perturbation $\tilde{Z}$.

successfully achieved by the slew control commands. As mentioned earlier in Section 2.4, the bus translation, bus rotation and vibration of beams are strongly coupled. This implies that any perturbation in one part of the system will induce disturbances in the other members of the system. Hence the slew maneuver induced significant perturbations leading to vibrations of the beam and bus rotation. This is clearly observed from the case without controls for vibration supression. In Figs. 5.27-5.29, Figs. 5.30-5.33, Figs. 5.35-5.37 and Fig. 5.34, we show the angular velocities of the bus, deflections and velocities of the beam, the components of vector $\tilde{R}$ and the relative beam energy. From these figures it is clear that without stabilizing control the beam vibrations and oscillation of the bus and perturbation in the radial vector persist or grow. However, with application of the feedback controls, these oscillations that were induced by the slew maneuvering are effectively eliminated throughout the entire system.

5.4 Numerical Results for the Space Station

In this section, we present numerical results for the space station whose dynamics are given by eqs.(3.84-3.120). For the purpose of simulation, we assume that (i) the bus inertia tensor is diagonal, (ii) the flexible members consist of three beams (i.e., beams 1,2 and 5 in Fig.3.2) and each beam is uniform. We used the parameters: I_s=diag(I_1, I_2, I_3) = diag(31750, 5000, 33450) $slug\ \ ft^2$ is diagonal matrix of the bus, the length of the beams: $l_1 = l_2$ =187.7 ft, l_5 =66 ft, $EI_{5x} = EI_{5z}$ =$10^5 lb\ ft^2$, $EI_{iy} = EI_{iz}$ =3565.5 $lb\ ft^2$, for i=1,2,5, Mass density: ρ_i =8.25×10^{-2} $slug/ft$,D_{1_o} =(3+ ξ_1, 33, 0)$'$, D_{2_o} =(3+ ξ_2, -33,0)$'$, D_{5_o} =(190.7, -33+ ξ_5, 0)$'ft$, m_s =300 $slug$. The *initial conditions* were taken as ω_1 (0)=0.03, $\omega_2(0)$ =0.02, $\omega_3(0)$ = 0.01 rad/sec and for $\xi_i \in$[0,l_i] and i=1,2, 5, $\frac{\partial x_i}{\partial t}(\xi_i, 0) = \frac{\partial y_i}{\partial t}(\xi_i, 0) = \frac{\partial z_i}{\partial t}(\xi_i, 0)$ =0, $x_i(\xi_i$,0)= $y_i(\xi_i$,0) =$z_i(\xi_i$,0) =ϕ_o (ξ_i), where ϕ_o satisfies the boundary conditions (equations 3.97-3.120).

The initial conditions for the radial vector are taken as $R_0(0)=(1.3864\times 10^8,0,0)'$ ft and $\frac{dR_0}{dt}(0)=(0,12338.1,0)'$ ft/sec. The external disturbance acting on the orbit equation (i.e. vector $\tilde{R}(t)$) was taken as $F_1(t)$ =-30000.0 lb_f, $F_2(t)=F_3(t)=0$ lb_f for $t\in[0.001,0.101]$.

In order to study the effect of the controls in Section 4.4 on the dynamic behavior of the space station, the simulation results for the following situations have been compared. For the convenience of presentation, the different schemes of control are designated as:

CASE D: No controls are applied after above mentioned disturbance occurs.

CASE E: Proportional feedback controls (Theorem 4.21) are applied.

CASE F: Deadzone controls (Corollary 4.25) are applied.

In Section 5.4.1 the results corresponding to CASES D and E are discussed in detail. In Section 5.4.2 we present the results corresponding to CASES D and F. Stability of the space station under the influence of slewing maneuvers is discussed in Section 5.4.3.

5.4.1 Stabilization of the Space Station by Proportional Controls

In this section the results corresponding to CASES D and E are compared[S4]. The *feedback controls* are taken as in equations (4.103-4.105) with k_1=9000, k_2=3000, k_3=7000; e_1=6000, e_2=600, e_3=600 and $d_{i_1}=d_{i_2}=d_{i_3}=0.04$; $i=1,2,5$. Figs.5.38-5.63 show the impact of the orbital disturbance on the beam vibration and bus rotation and the effectiveness of the chosen controls in stabilizing the system.

In the absence of controls or external disturbances, the system energy E, given by the sum of total energy of the bus (with respect to the inertial frame) E_B and the beam energy E_b, is conservative. This is clearly observed from the Fig.5.38 that the energy E remains constant (A slight variation in E of the uncontrolled system could be attributed to the truncation and discretization errors associated with numerical computation). Since the beams are rigidly attached to the main body, it is observed from Fig.5.38 that interchange of energy between the bus and the beam may take place during the perturbations leading to the changes in magnitudes of vibrations of the beam and bus rotation. Further orbital disturbance induces large fluctuations in the beam energy and bus rotational energy. With application of the suggested feedback controls the bus angular motion and the beam vibrations decay to zero.

In Figs. 5.40, 5.44 (respectively 5.42, 5.46), we show the deflections of the beam 1 (beam 2) at the joint 5 (joint 6) , i.e. $y_1(l_1,t)$, $z_1(l_1,t)$ ($y_2(l_2,t)$, $z_2(l_2,t)$). Their corresponding velocities are plotted in Figs.5.41, 5.43, 5.45 and 5.47 respectively. Figs. 5.48 and 5.50 represent the deflections of the beam 5 at the midpoint i.e. $x_5(l_5/2,t)$ and $z_5(l_5/2,t)$. Their corresponding velocities are plotted in Figs. 5.49 and 5.51. From these figures, it is clear that in the absence of controls (CASE D, Figs.5.40-5.51), the beam vibrations persist or grow. However, with application of the feedback controls, these oscillations are effectively eliminated (CASE E, Figs.5.40-5.51).

The fact that the beam vibrations are reduced throughout its length can be observed from the *relative beam energy* curve of Fig.5.39. We define the *relative beam energy* function as a measure of vibration of the beam with respect to the body axes by

$$\begin{aligned} E_r(t) \equiv & \int_{\Omega_1} \left\{ \rho_1 \left(\frac{\partial y_1}{\partial t}\right)^2 + \rho_1 \left(\frac{\partial z_1}{\partial t}\right)^2 + EI_{1_y} \left(\frac{\partial^2 y_1}{\partial {\xi_1}^2}\right)^2 + EI_{1_z} \left(\frac{\partial^2 z_1}{\partial {\xi_1}^2}\right)^2 \right\} d\xi_1 \\ & + \int_{\Omega_2} \left\{ \rho_2 \left(\frac{\partial y_2}{\partial t}\right)^2 + \rho_2 \left(\frac{\partial z_2}{\partial t}\right)^2 + EI_{2_y} \left(\frac{\partial^2 y_2}{\partial {\xi_2}^2}\right)^2 + EI_{2_z} \left(\frac{\partial^2 z_2}{\partial {\xi_2}^2}\right)^2 \right\} d\xi_2 \\ & + \int_{\Omega_5} \left\{ \rho_5 \left(\frac{\partial x_5}{\partial t}\right)^2 + \rho_5 \left(\frac{\partial z_5}{\partial t}\right)^2 + EI_{5_x} \left(\frac{\partial^2 x_5}{\partial {\xi_5}^2}\right)^2 + EI_{5_z} \left(\frac{\partial^2 z_5}{\partial {\xi_5}^2}\right)^2 \right\} d\xi_5. \quad (5.21) \end{aligned}$$

We note that $E_r(t)$ represents the total energy of the beam relative to the body axes. Thus $E_r(t)$=0 implies that all the points of the beam are at their rest positions. From

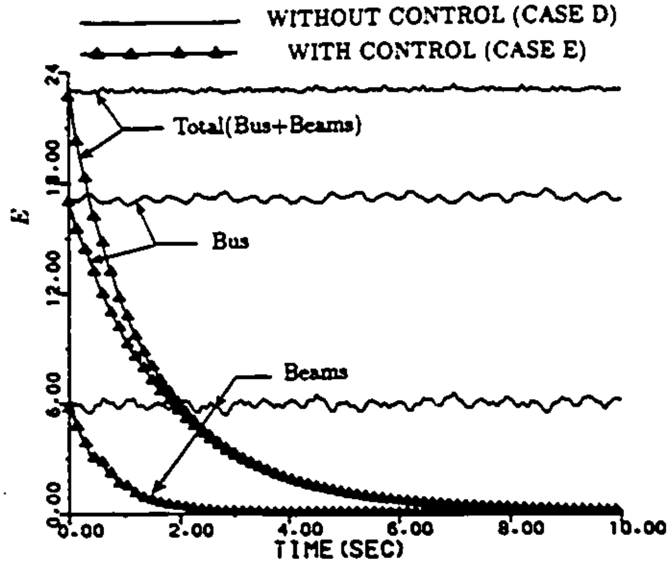


Fig.5.38 Absolute energy.

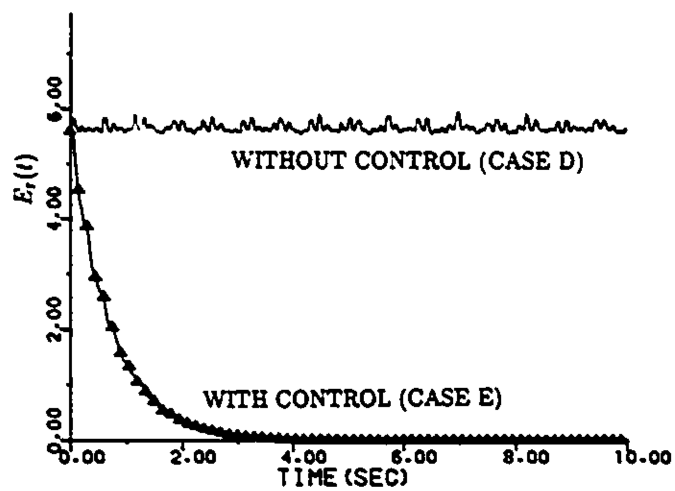


Fig.5.39 Relative beam energy.

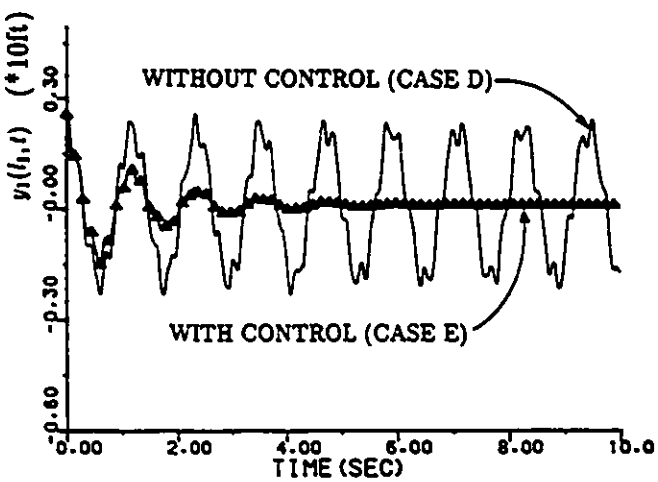


Fig.5.40 Beam displacement $y_1(l_1,t)$.

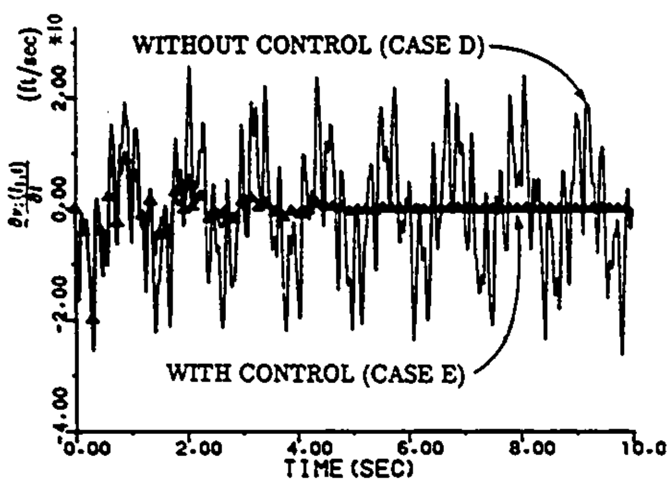


Fig.5.41 Beam velocity $\frac{\partial y_1(l_1,t)}{\partial t}$.

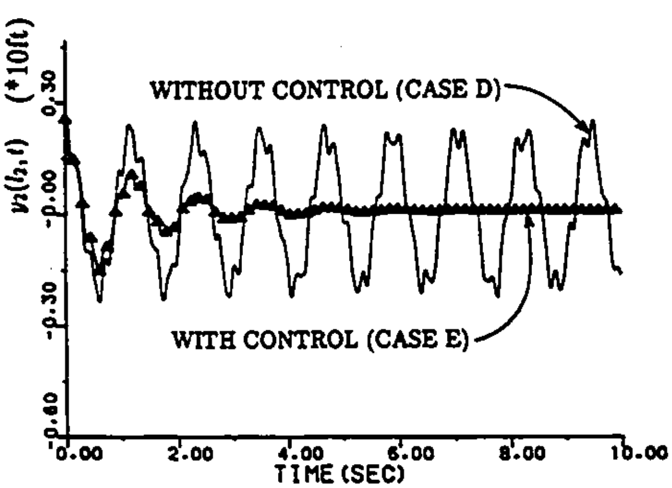


Fig.5.42 Beam displacement $y_2(l_2,t)$.

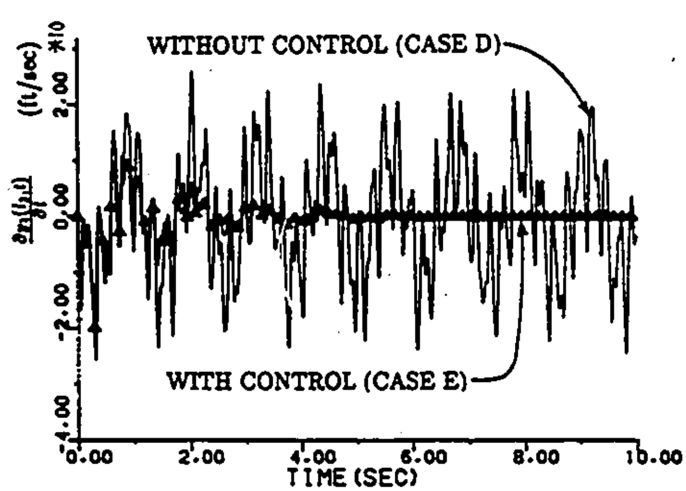


Fig.5.43 Beam velocity $\frac{\partial y_2(l_2,t)}{\partial t}$.

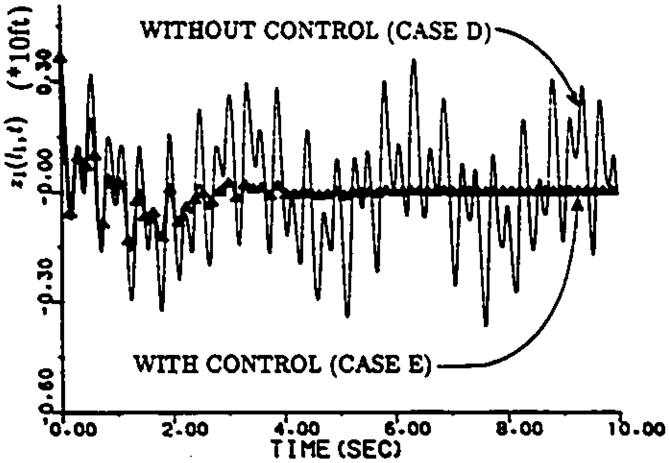


Fig.5.44 Beam displacement $z_1(l_1,t)$.

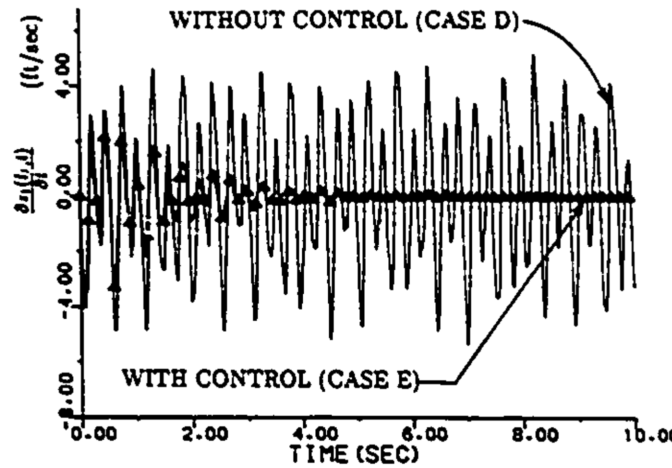


Fig.5.45 Beam velocity $\frac{\partial z_1(l_1,t)}{\partial t}$.

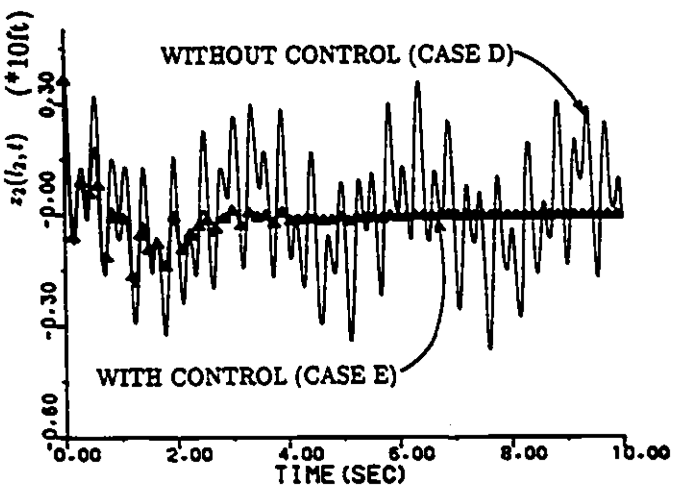


Fig.5.46 Beam displacement $z_2(l_2,t)$.

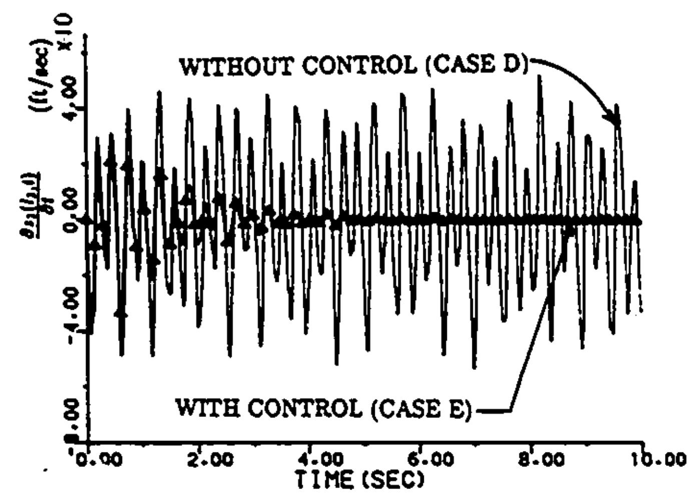


Fig.5.47 Beam velocity $\frac{\partial z_2(l_2,t)}{\partial t}$.

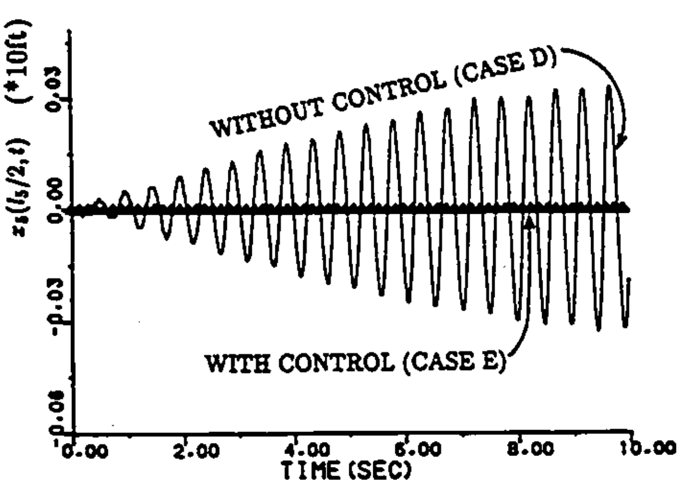


Fig.5.48 Beam displacement $z_5(l_5/2,t)$.

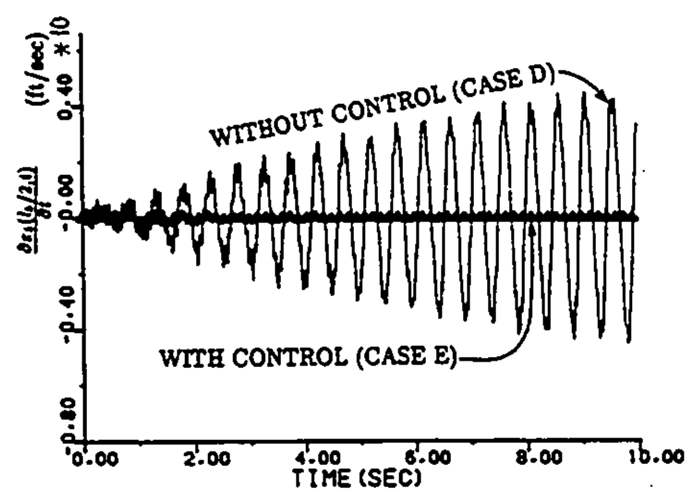


Fig.5.49 Beam velocity $\frac{\partial z_5(l_5/2,t)}{\partial t}$.

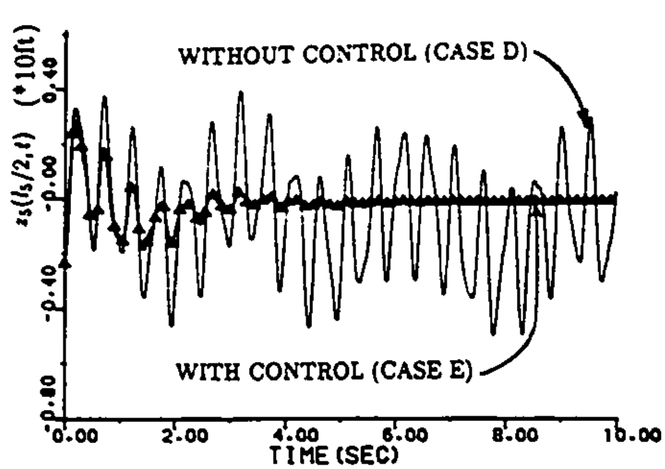


Fig.5.50 Beam displacement $z_3(l_5/2,t)$.

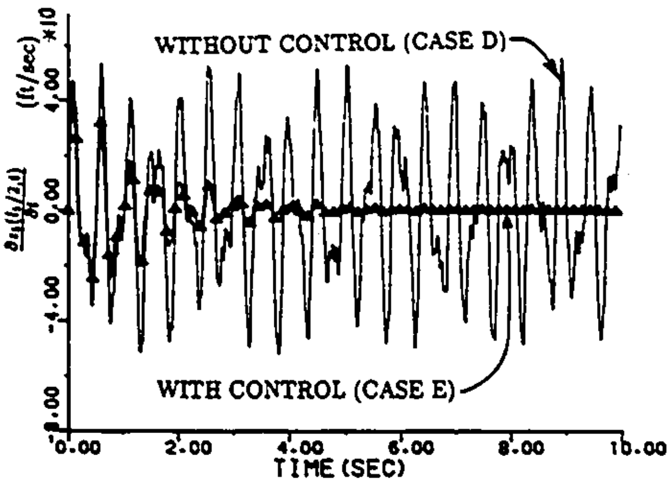


Fig.5.51 Beam velocity $\frac{\partial z_3(l_5/2,t)}{\partial t}$.

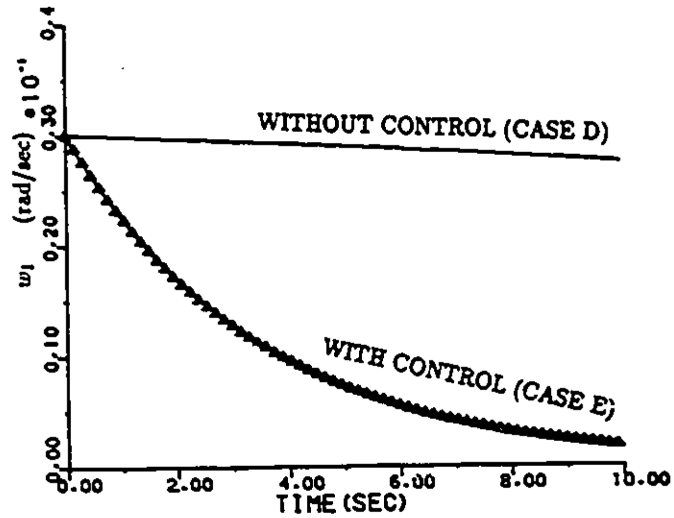


Fig.5.52 Bus angular velocity w_1.

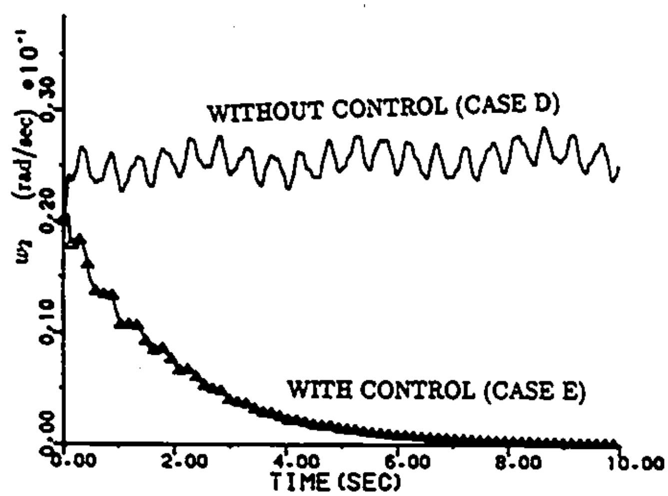


Fig.5.53 Bus angular velocity w_2.

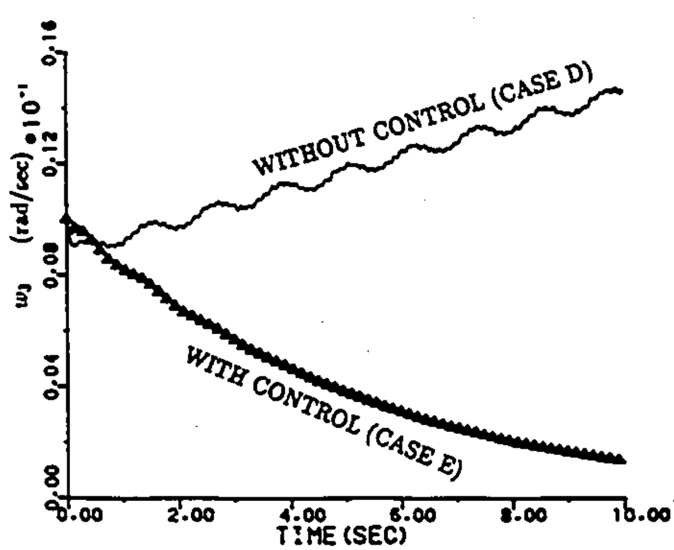


Fig.5.54 Bus angular velocity w_3.

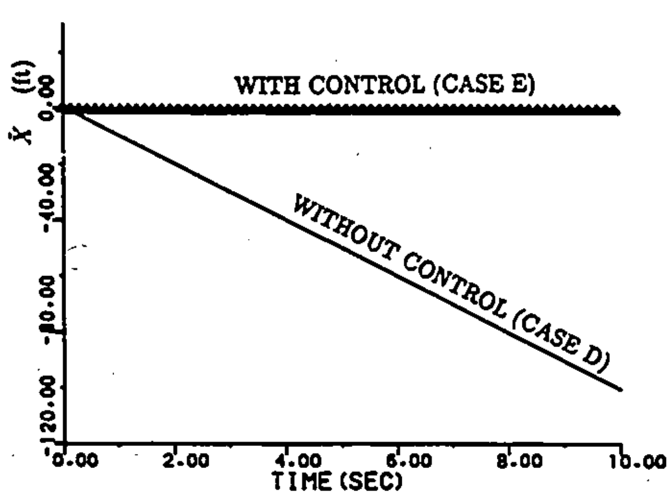


Fig.5.55 Radial perturbation $\bar{X}$.

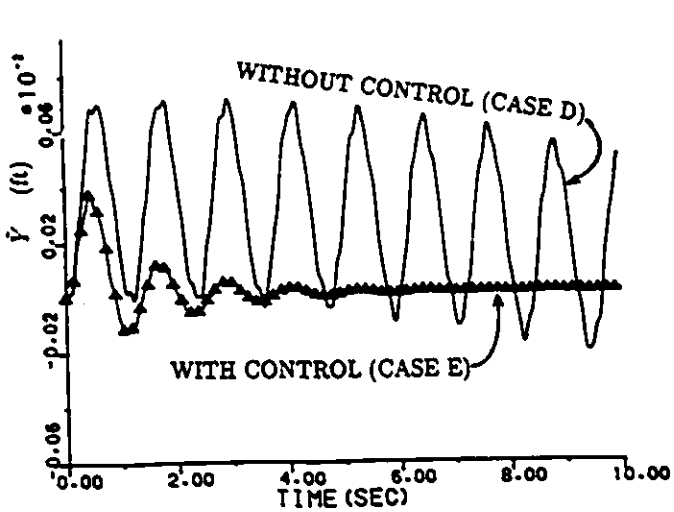


Fig.5.56 Radial perturbation $\tilde{Y}$.

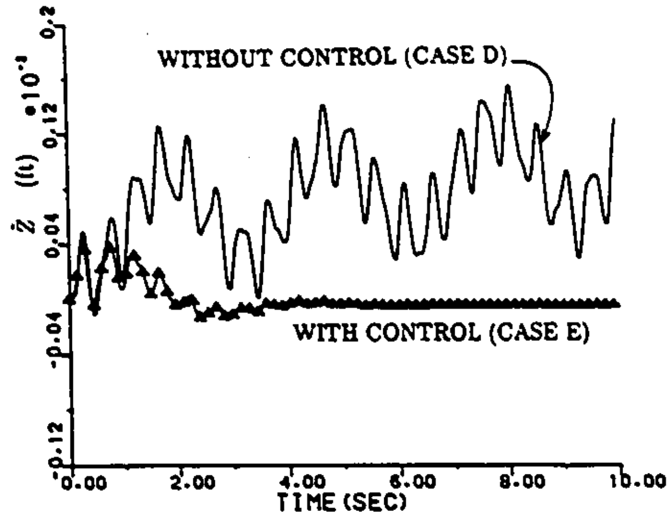


Fig.5.57 Radial perturbation $\tilde{Z}$.

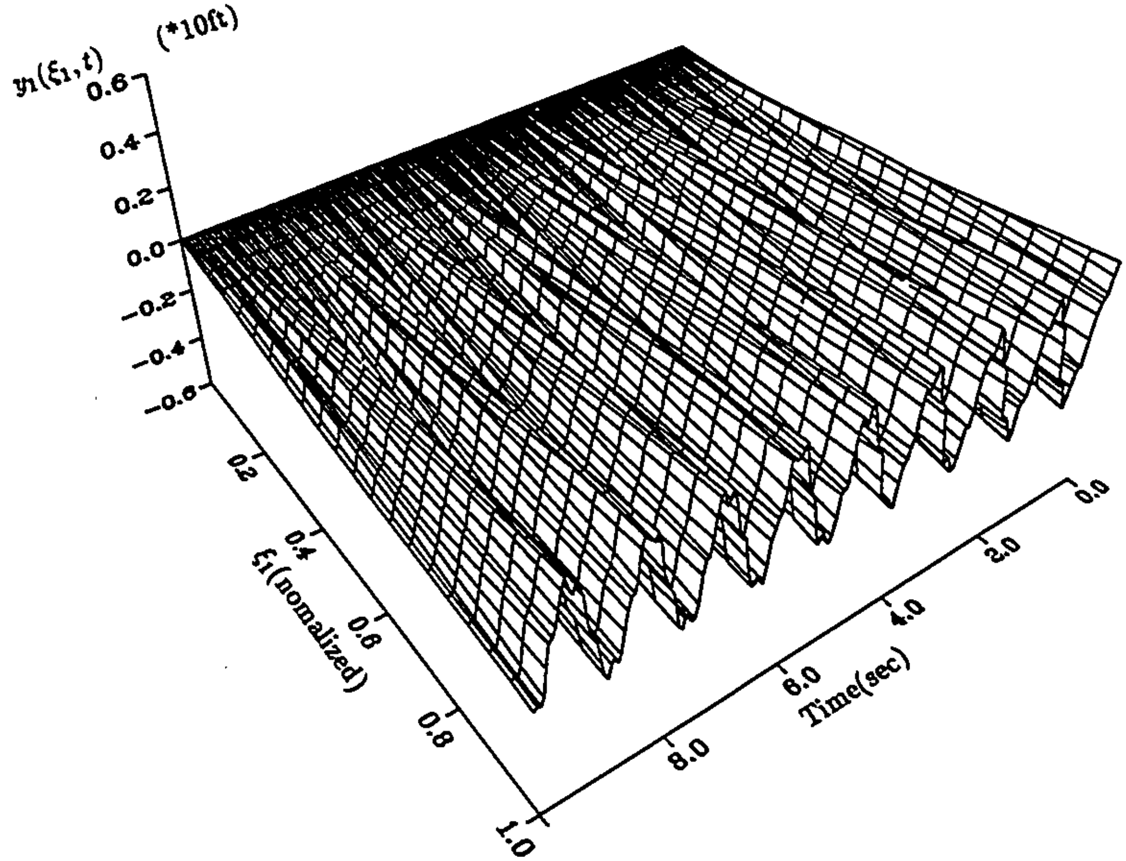


Fig.5.58(a) Uncontrolled beam displacement with time:$y_1(\xi_1, t)$

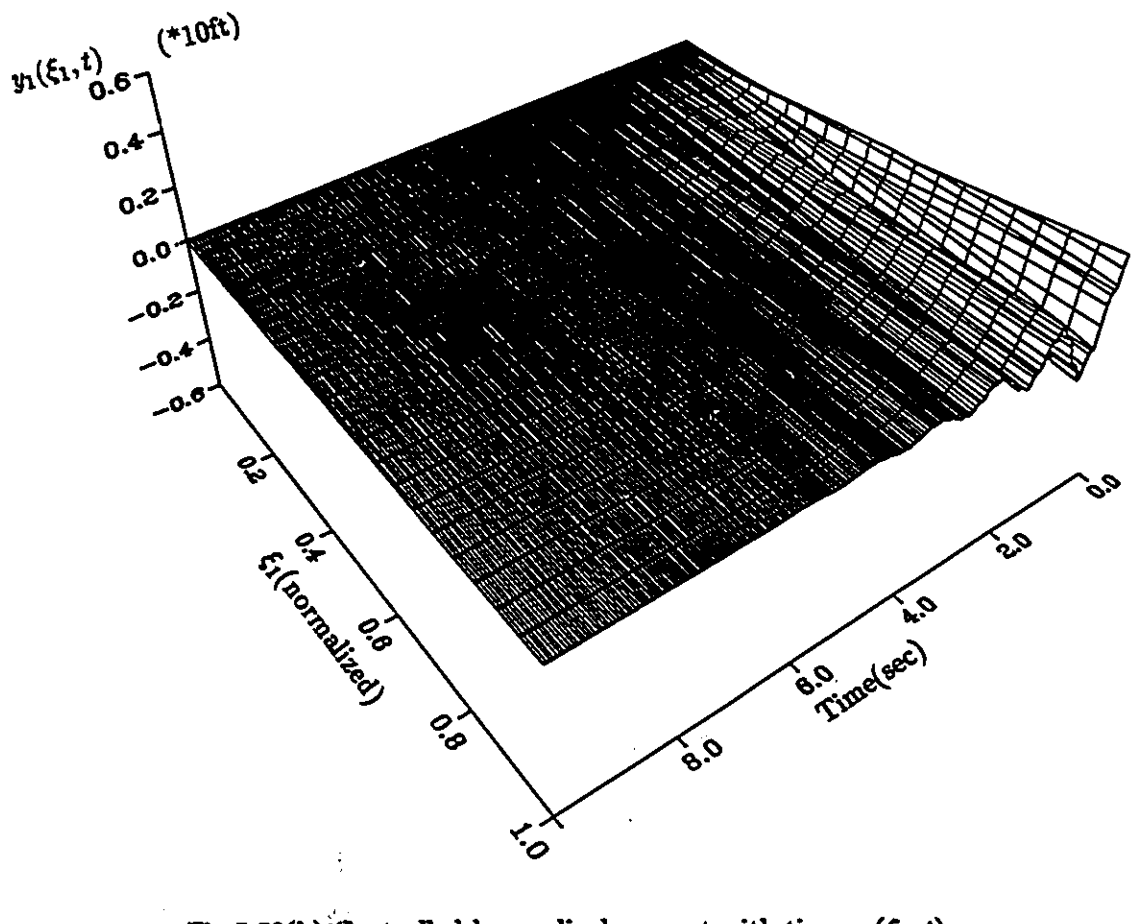


Fig.5.58(b) Controlled beam displacement with time:$y_1(\xi_1, t)$

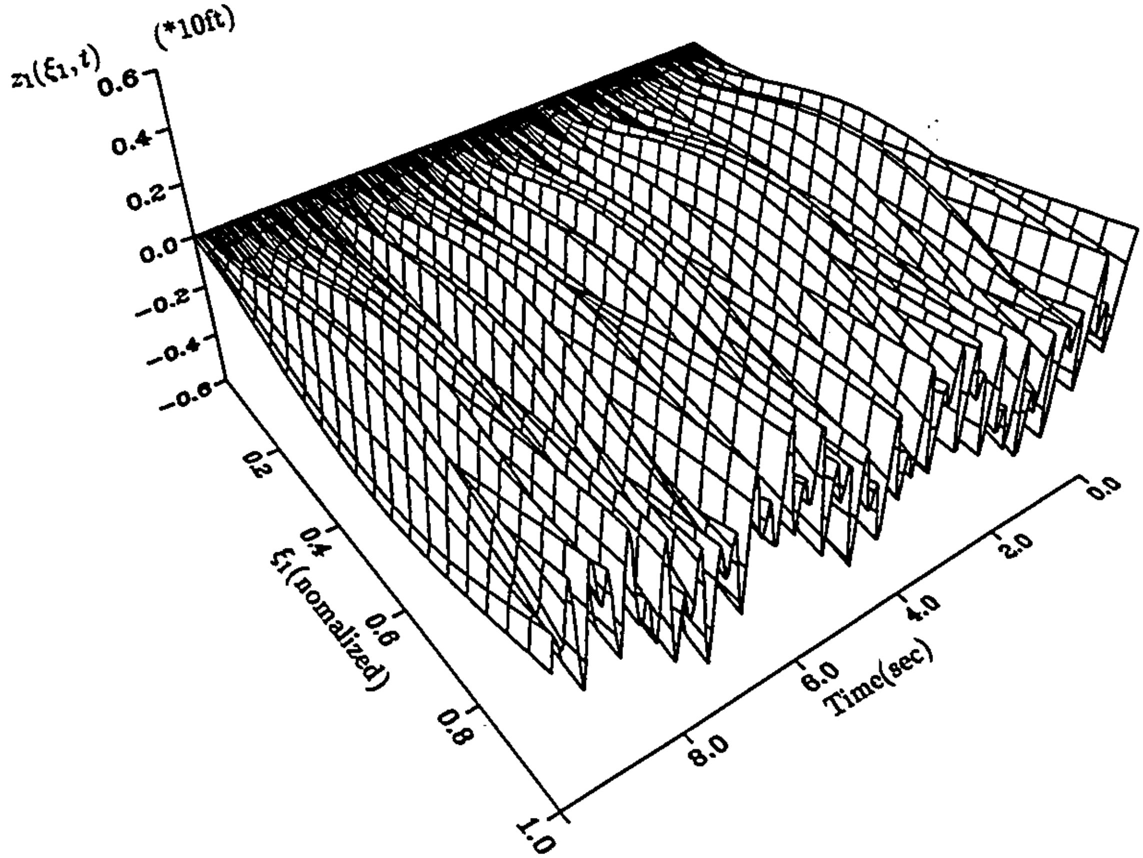


Fig.5.59(a) Uncontrolled beam displacement with time:$z_1(\xi_1,t)$

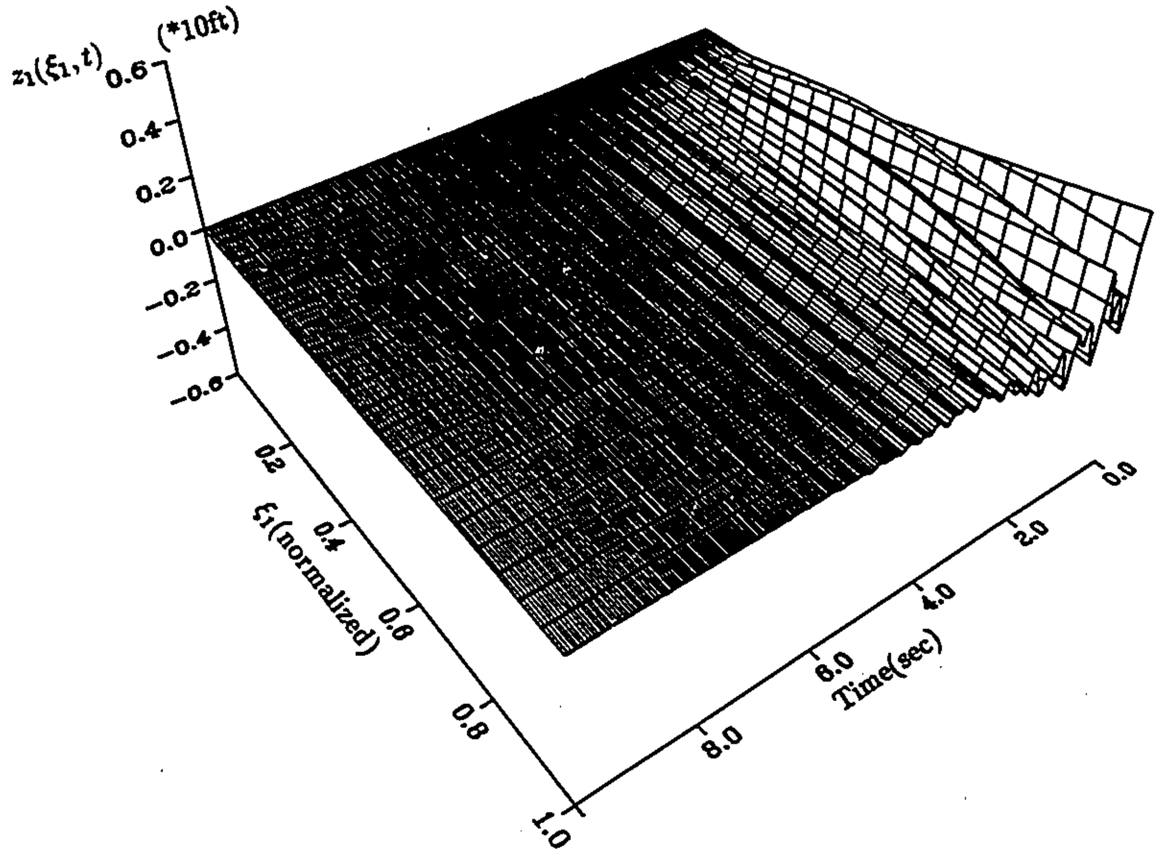


Fig.5.59(b) Controlled beam displacement with time:$z_1(\xi_1,t)$

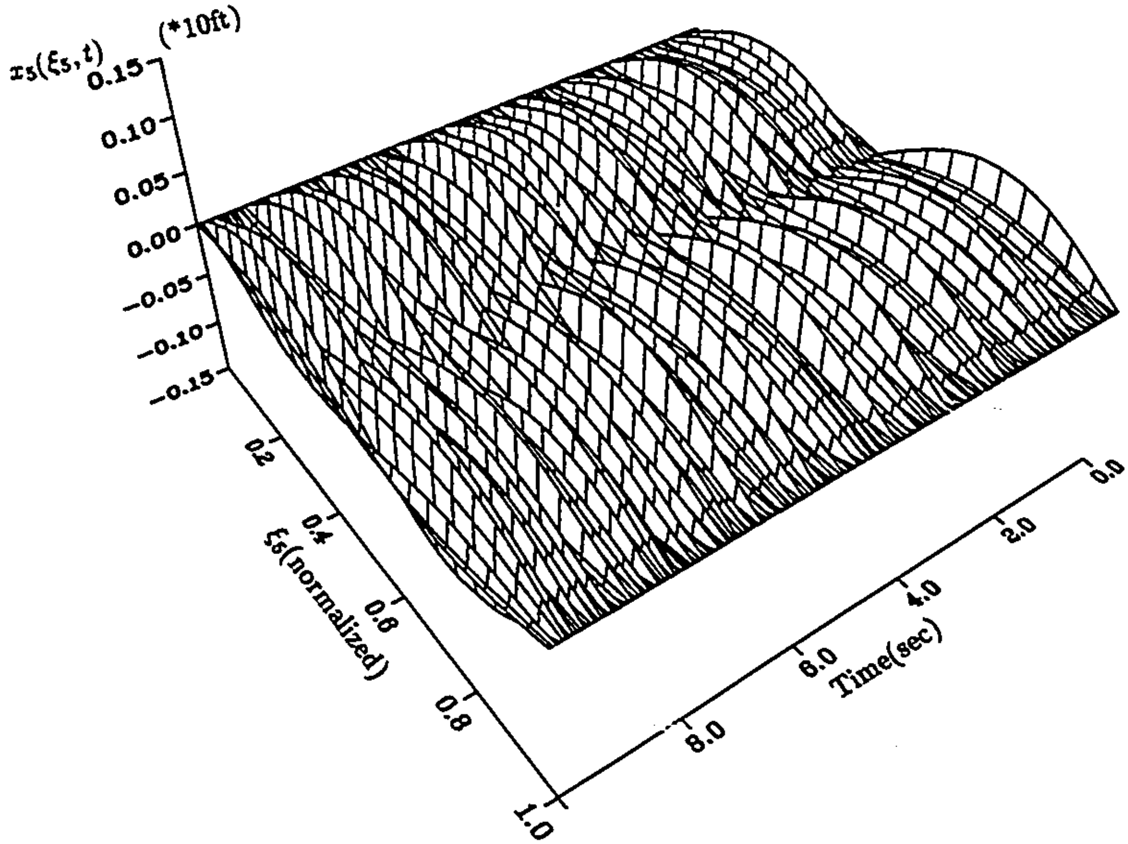


Fig.5.60(a) Uncontrolled beam displacement with time:$x_5(\xi_5, t)$

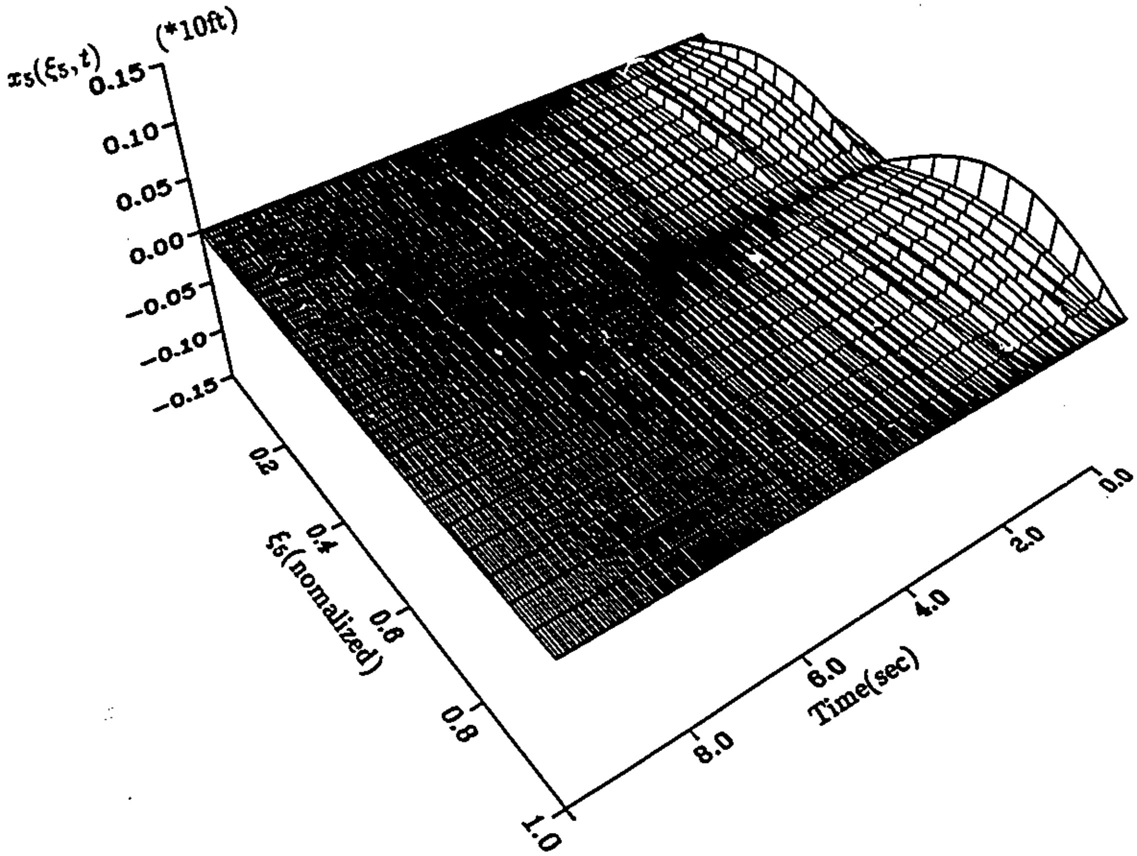


Fig.5.60(b) Controlled beam displacement with time:$x_5(\xi_5, t)$

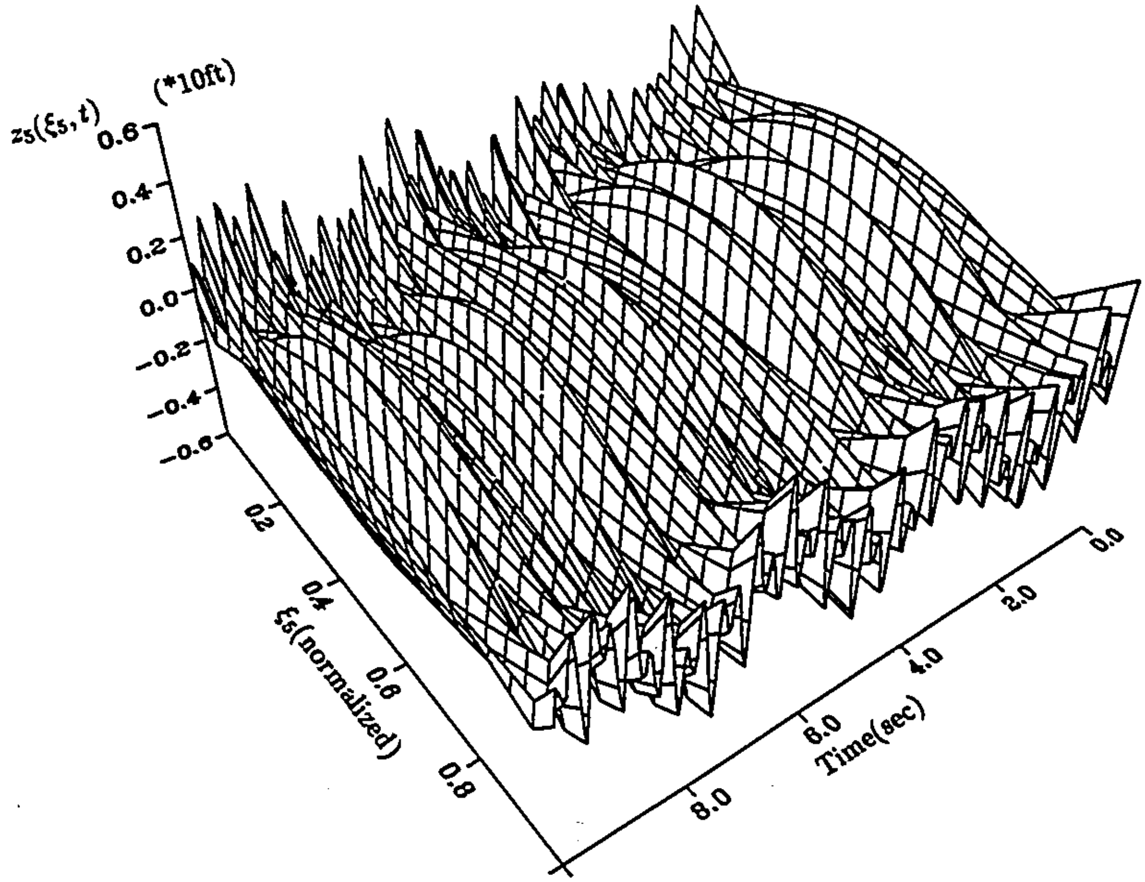


Fig.5.61(a) Uncontrolled beam displacement with time:$z_5(\xi_5, t)$

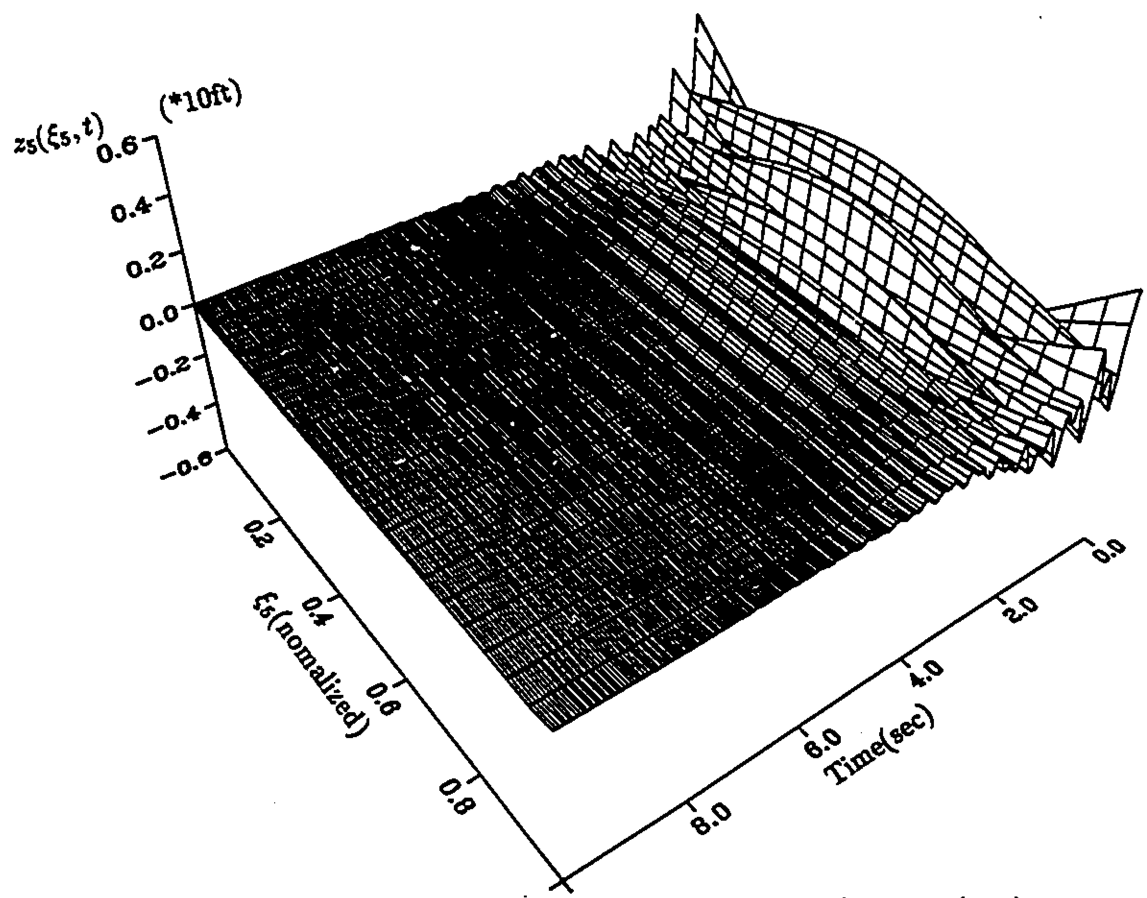


Fig.5.61(b) Controlled beam displacement with time:$z_5(\xi_5, t)$

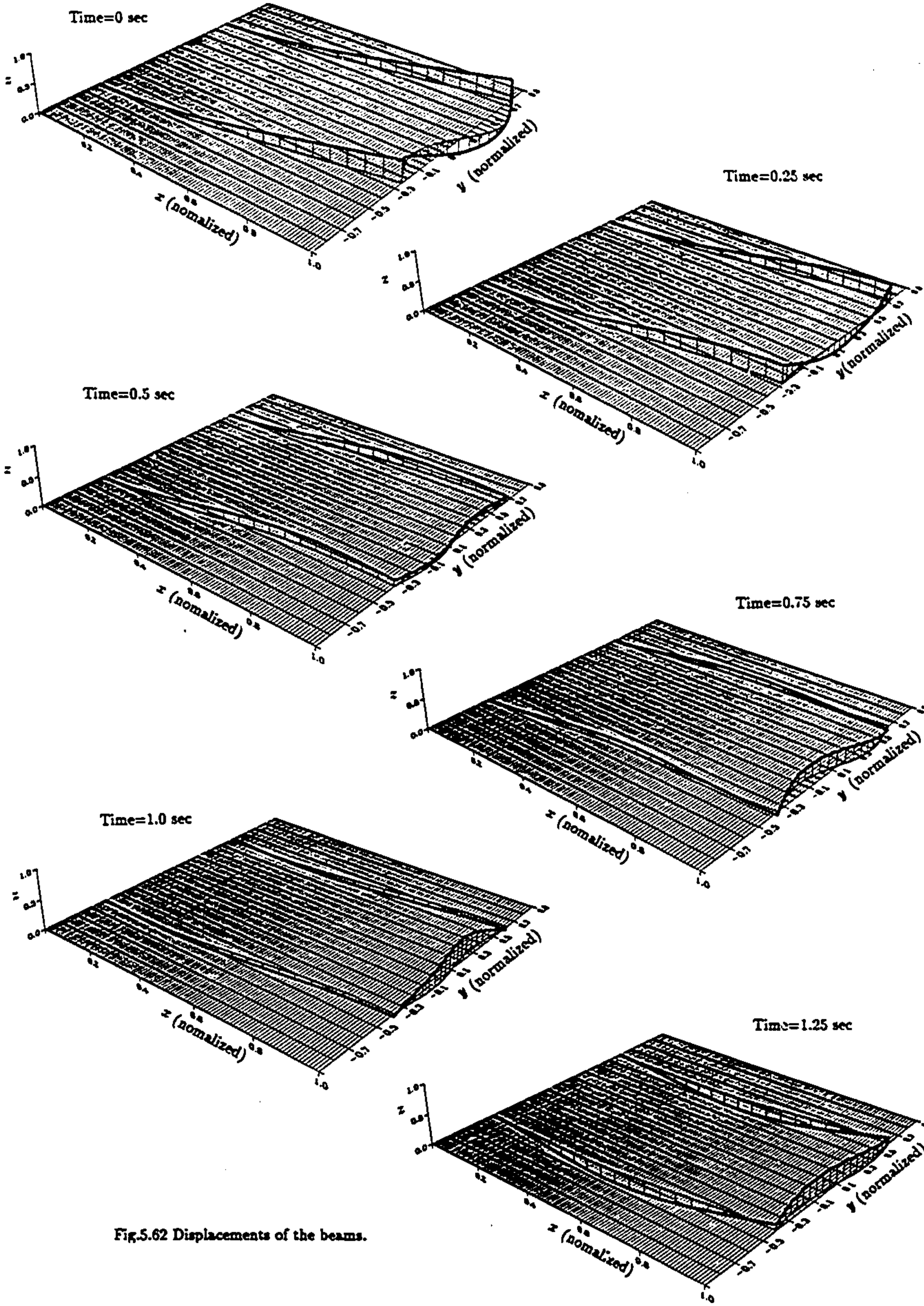
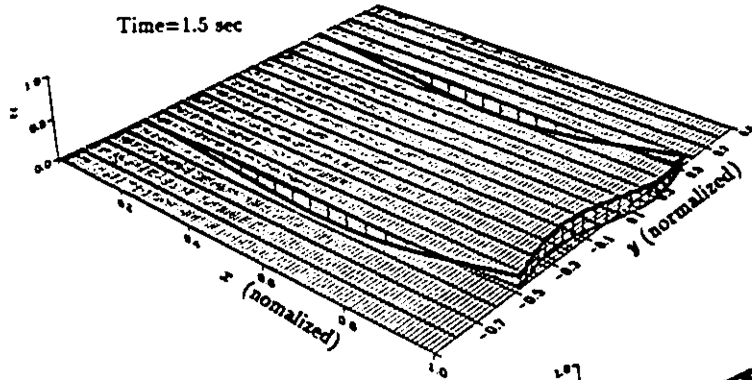
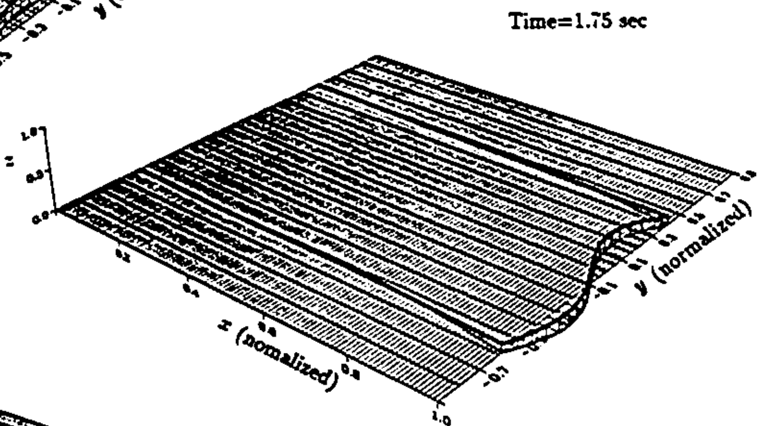
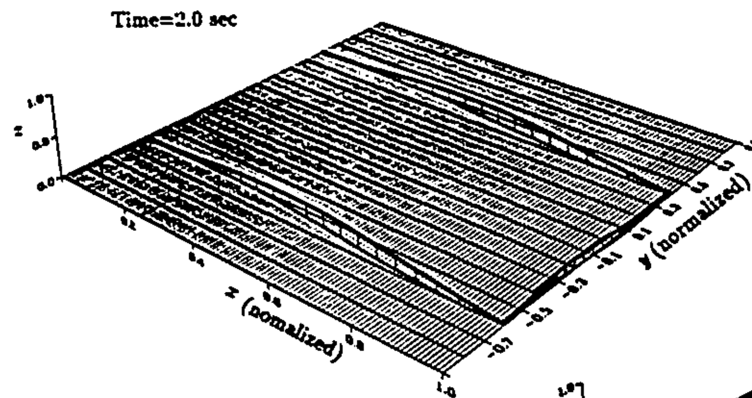
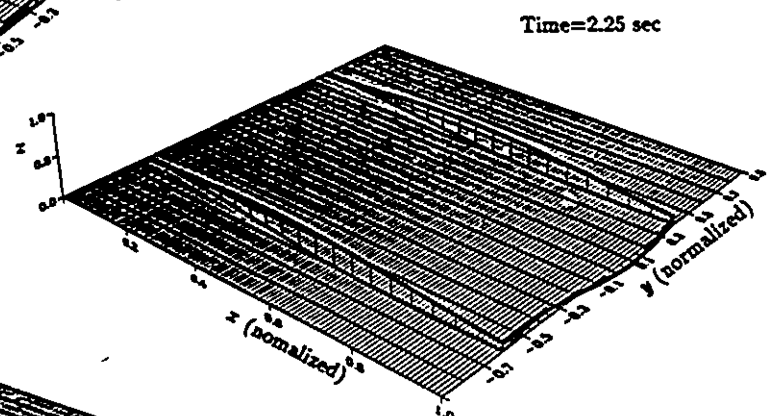


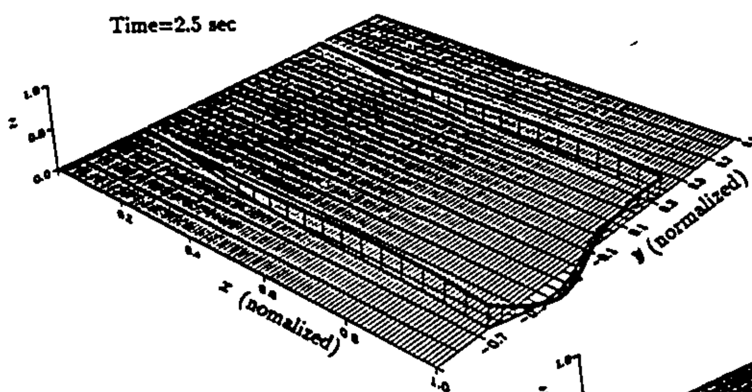
Fig.5.62 Displacements of the beams.











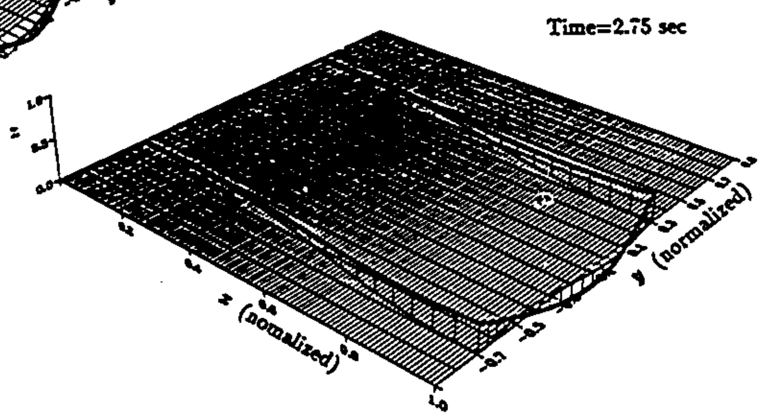


Fig.5.63 Displacements of the beams.

Fig.5.39, we observe that for the uncontrolled system the relative beam energy persist, indicating continued oscillations of the beam with respect to the body axes (CASE D, Fig.5.39). Use of the suggested feedback controls effectively reduces the beam vibrations to zero as shown in Fig.5.39 (case E). In fact, the effects of the control is clearly illustrated from Figs. 5.58-5.61.

The impact of the orbital disturbance on the bus angular velocities is shown in Figs.5.52-5.54. It is observed, from Figs.5.52-5.54, that the angular velocities ω_2 and ω_3 experience larger fluctuations as compared to ω_1. This is based on the fact that the beam vibrations in the j_b and k_b directions would contribute to the total angular momentum about k_b and j_b directions respectively. The orbital disturbance and beam vibrations introduce additional oscillations in the satellite bus rotation (CASE D, Figs.5.52-5.54), indicating growing fluctuations in the angular velocities of the bus. Stabilization of the angular velocities by application of the feedback controls can be clearly observed from Figs.5.52-5.54 (case E).

As mentioned earlier in Section 3.4, the satellite translation, rotation and beam vibration are strongly coupled. Thus one can expect some effects of the beam vibrations and bus rotation on the radial vector $\tilde{R}(t)$. These are shown in Figs.5.55-5.57. It is clear from these figures that the vibration of beams and fluctuations in the bus rotation induce small oscillations in the orbit, i.e. $\tilde{R}(t)$ as expected (see equation 4.102). However, the fluctuations in the $\tilde{R}(t)$ grow with time and this also induces vibration of the beams and oscillations in the bus angular motion. However, by application of the controls, the fluctuations are eliminated. We note that $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ stabilize to a neighborhood of the zero state. This error is due to use of the (approximate) linearized equation (4.102) instead of the true nonlinear equation (3.80). In the absence of control, the structural vibrations are presented schematically in

Figs. 5.62-5.63. In these figures the exact scales are not used for the sake of clear presentation of the overall shape of the vibrations with time.

5.4.2 Stabilization of the Space Station by Deadzone Controls

In this section the results corresponding to CASES D and F are compared. The deadzone controls(Corollary 4.25) are taken as in equations (4.128-4.130), with the parameters: $k_1 = 200$, $k_2 = 50$, $k_3 = 80$; $h_{i_1}=h_{i_2}=0.001$, $h_{i_3}=0.002$ for i=1,2,5; e_1 =5000, e_2 =5, e_3 =10; $\epsilon_1 = \epsilon_2 = 0.0005$, $\epsilon_3=0.0001$; $\epsilon_4=\epsilon_5=\epsilon_6=0.001$, $\epsilon_7=0.01$, $\epsilon_8=0.00005$, $\epsilon_9=0.0002$. Figs.5.64-5.83 show the effectiveness of the chosen deadzone controls in stabilizing the flexible space station under the influence of external disturbances mentioned above. A comparison of Figs.5.64-5.83 with Figs.5.38-5.63 indicates that the overall response of the system under the action of the control law (4.128-4.130) is quite similar to those corresponding to the proportional controls (4.103-4.105) and thus the detailed discussions are omitted. However, it is observed that there are some basic differences in the two sets of results:

(i) *the nature of decay of various states of the system*; in CASE D of Figs.5.38-5.63, the decay rates become smaller as the magnitude of the various states becomes smaller, whereas in CASE F of Figs.5.64-5.83, they remain constants. Indeed, this is quite natural since the control action of the proportional controls is proportional to the errors to be corrected while that of deadzone controls is constant.

(ii) *the limit of the decay*; in CASE D of Figs.5.38-5.63, the various states are reduced to the zero state, whereas in CASE F of Figs.5.64-5.83, they are reduced to the small neighborhood (as pemissible by the size of the deadzone) of the zero state.

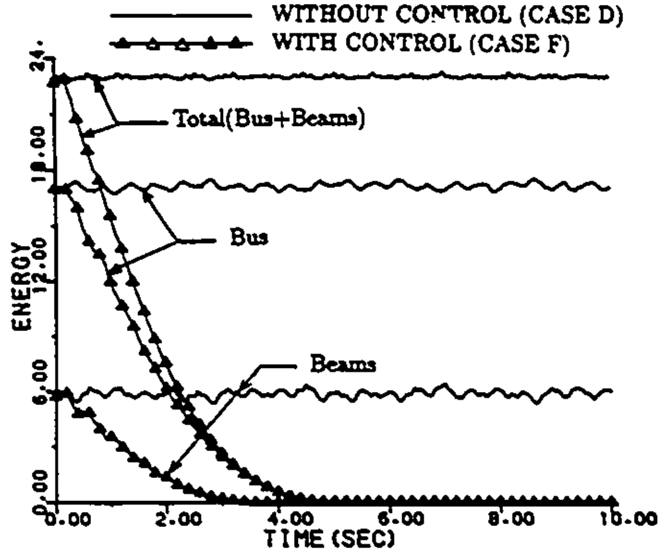


Fig.5.64 Absolute energy.

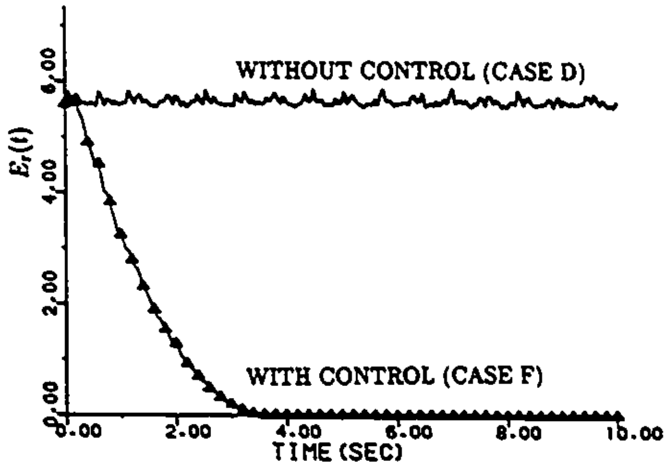


Fig.5.65 Relative beam energy.

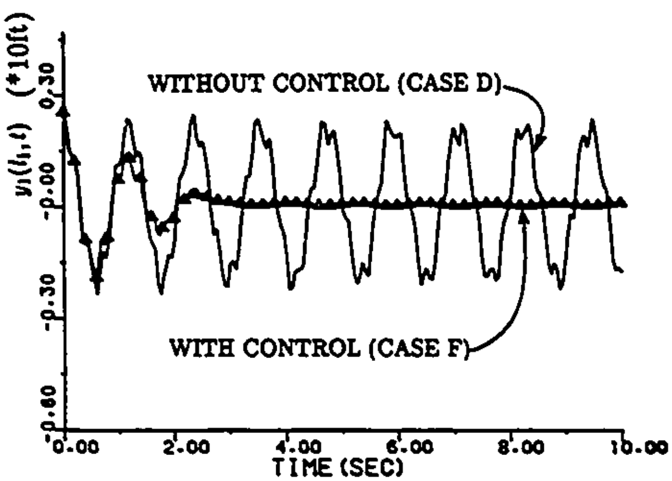


Fig.5.66 Beam displacement $y_1(l_1,t)$

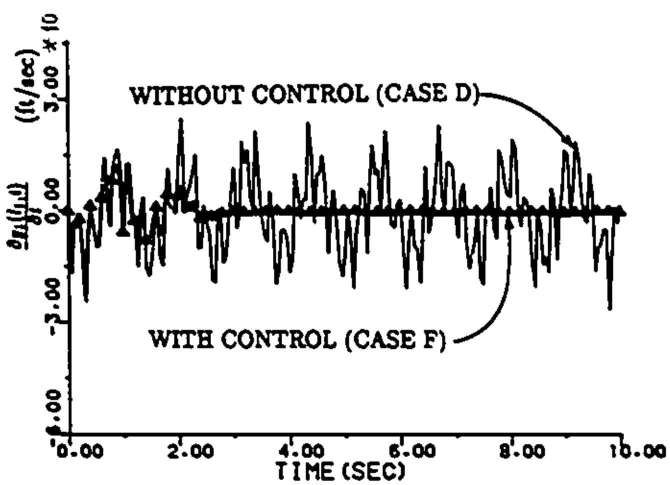


Fig.5.67 Beam velocity $\frac{\partial y_1(l_1,t)}{\partial t}$.

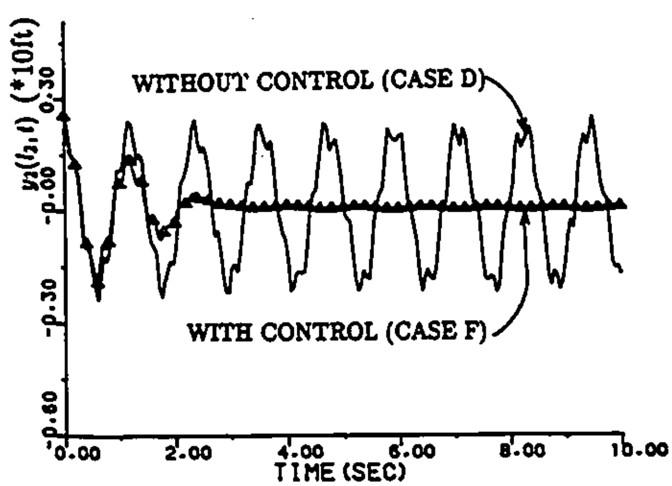


Fig.5.68 Beam displacement $y_2(l_2,t)$.

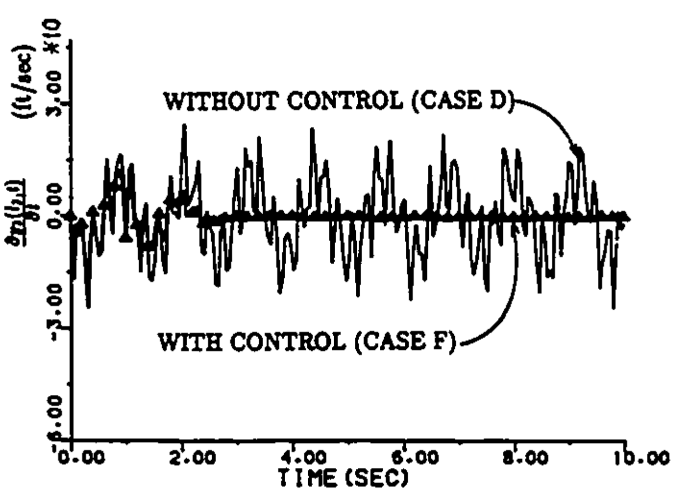


Fig.5.69 Beam velocity $\frac{\partial y_2(l_2,t)}{\partial t}$.

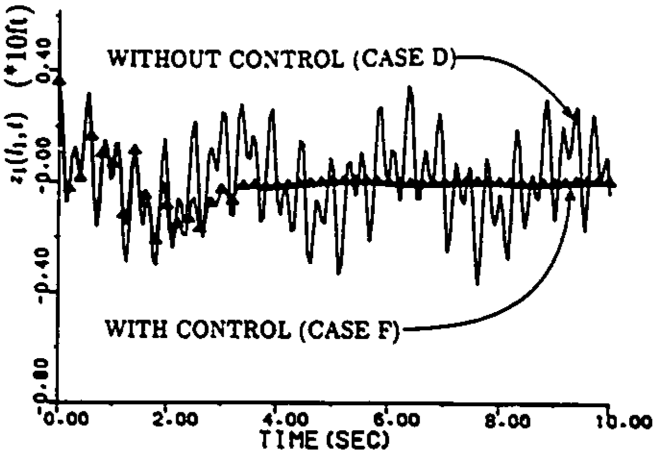


Fig.5.70 Beam displacement $z_1(l_1,t)$.

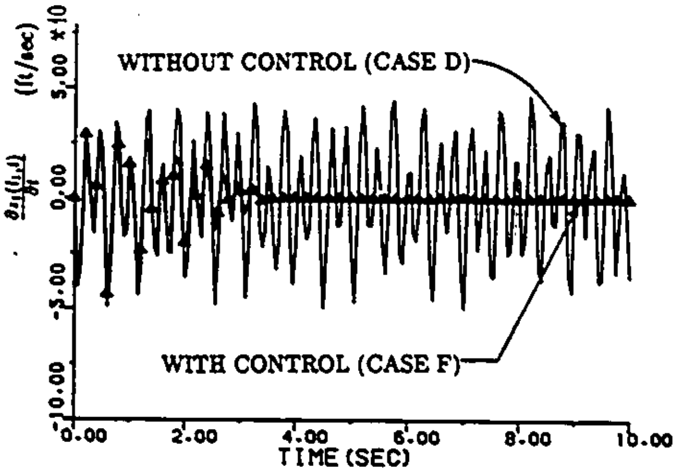


Fig.5.71 Beam velocity $\frac{\partial z_1(l_1,t)}{\partial t}$.

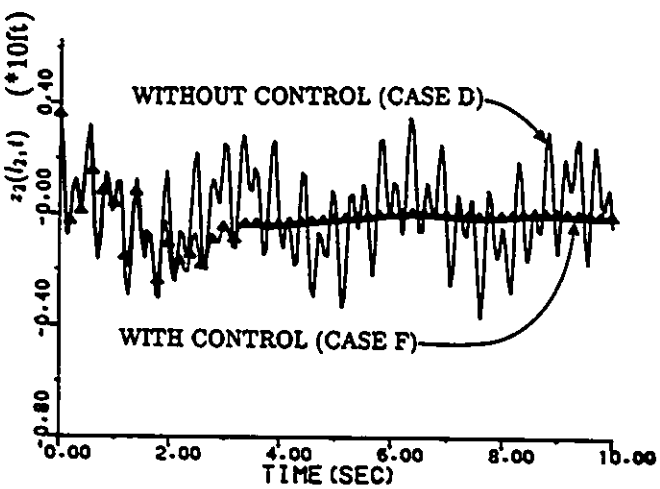


Fig.5.72 Beam displacement $z_2(l_2,t)$.

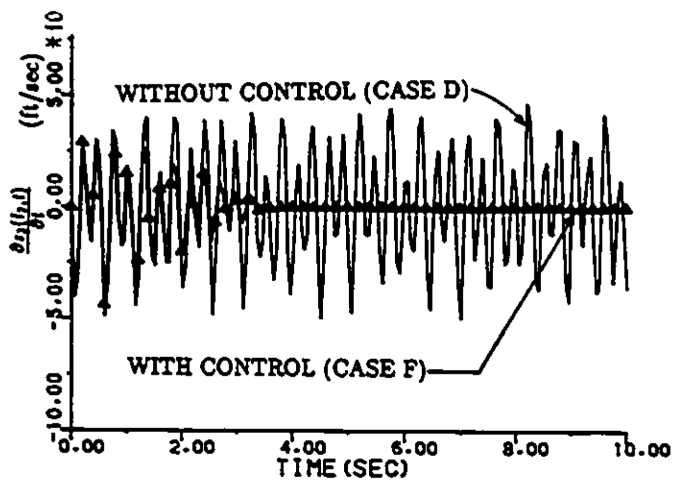


Fig.5.73 Beam velocity $\frac{\partial z_2(l_2,t)}{\partial t}$.

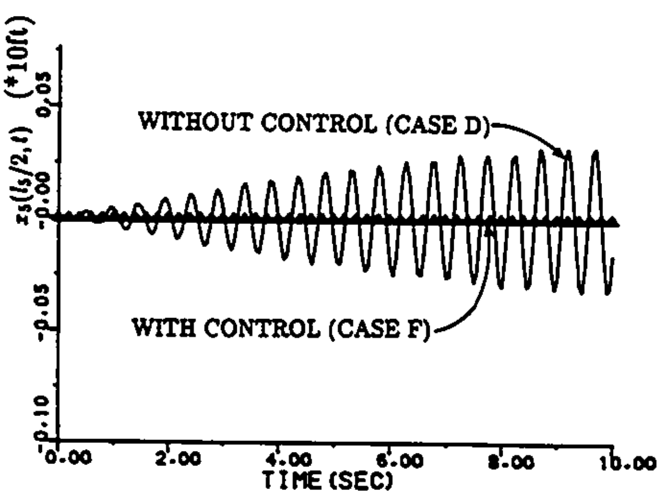


Fig.5.74 Beam displacement $z_5(l_5/2,t)$.

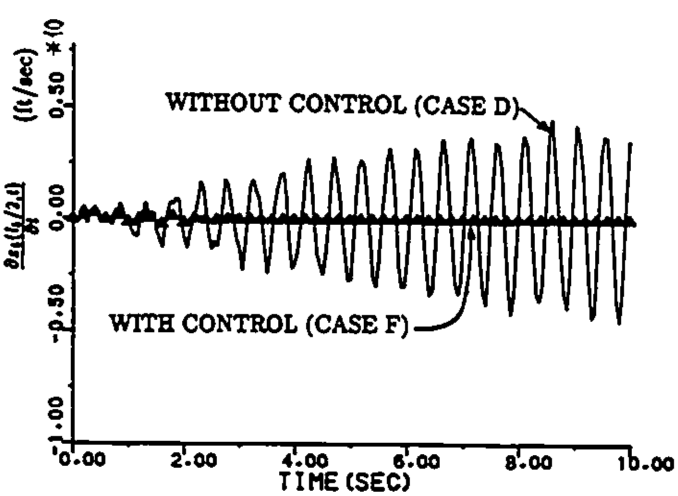


Fig.5.75 Beam velocity $\frac{\partial z_5(l_5/2,t)}{\partial t}$.

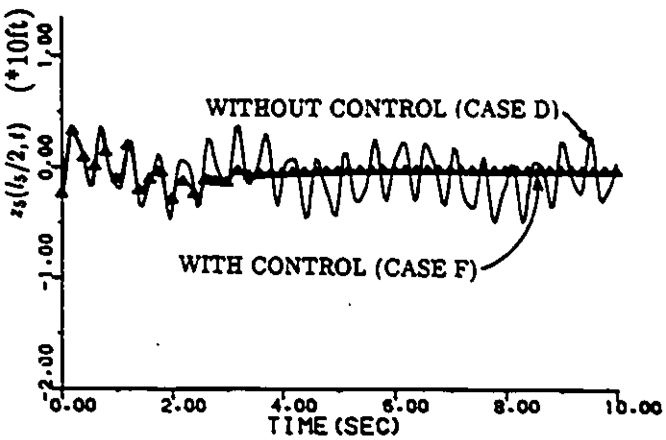


Fig.5.76 Beam displacement $z_3(l_5/2,t)$.

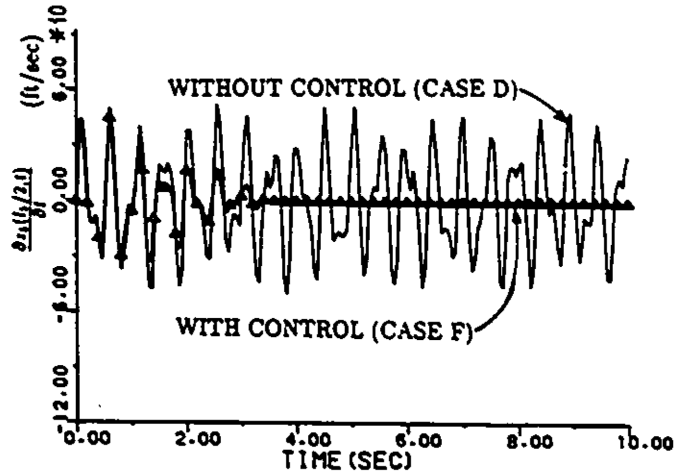


Fig.5.77 Beam velocity $\frac{\partial z_3(l_5/2,t)}{\partial t}$.

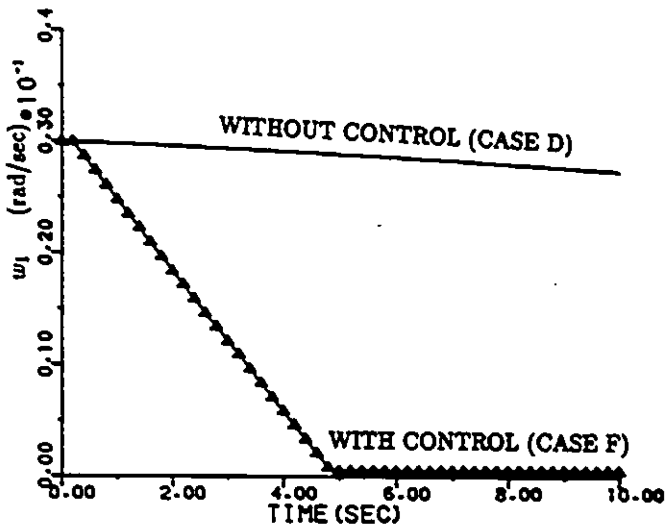


Fig.5.78 Bus angular velocity w_1.

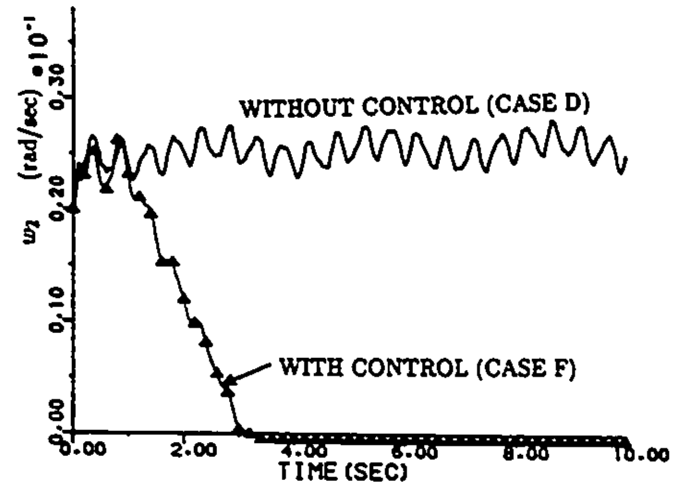


Fig.5.79 Bus angular velocity w_2.

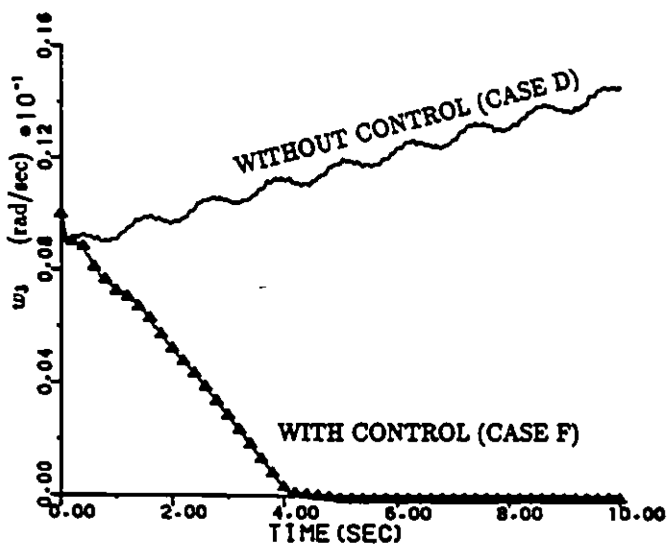


Fig.5.80 Bus angular velocity w_3.

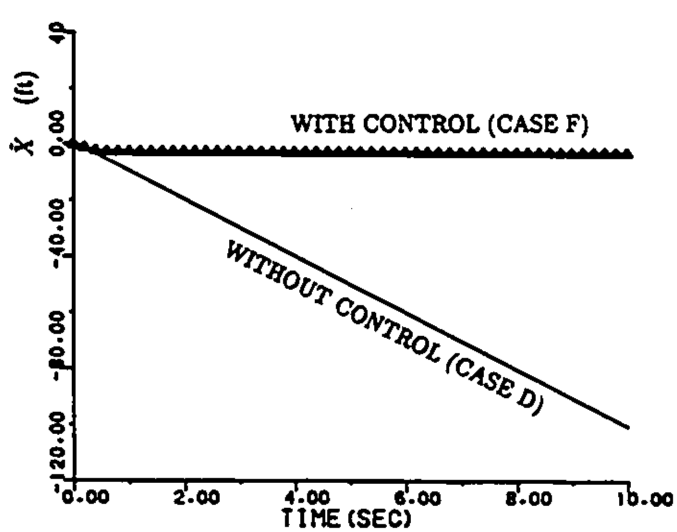


Fig.5.81 Radial perturbation $\tilde{X}$.

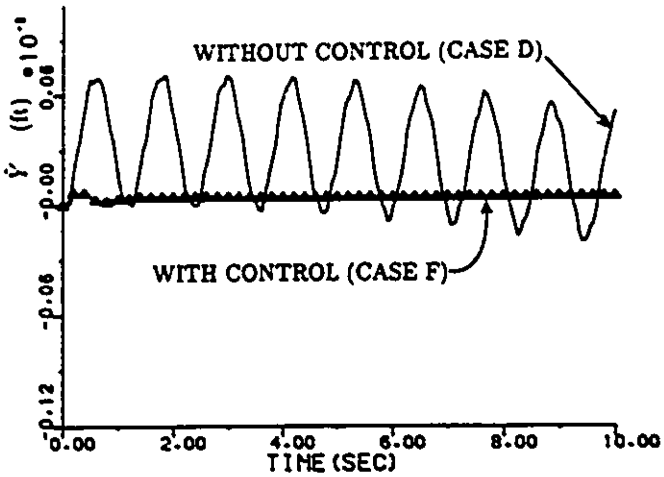


Fig.5.82 Radial perturbation $\tilde{Y}$.

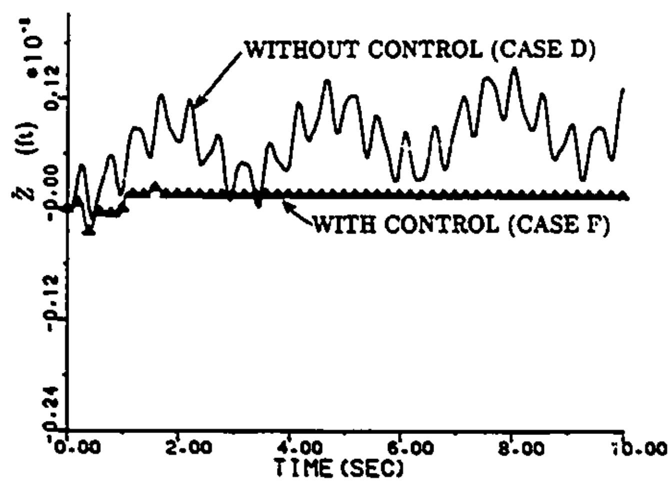


Fig.5.83 Radial perturbation $\tilde{Z}$.

In Figs.5.64-5.83, the region of the zero state obtained from the simulation is of the order of 0.0001, which is not visible in the figures, while for 0.0005 it is visible as shown in Figures 5.78-5.80. In prctical applications, the size of deadzone is determined by the stabilization requirements of the particular system being considered.

5.4.3 Stabilization of Slew-Maneuvers of the Space Station

As mentioned in Section 3.4, flexible space station are often expected to experience vibrations caused by several disturbing forces. An important disturbance could be due to slewing maneuvers for obtaining or initializing the desired orientation of the space station after it is put into orbit. In this section the results of the stabilization of the space station under the influence of slewing maneuvers are presented. To slew a space station for a given angle in a prespecified time, there are many ways to command the slew actuators on board[Lin 87]. The one that is easy to implement is bang-bang control. For simulation the Bang-Pause Bang control scheme shown in Fig. 5.25. is assumed for slew the space station for the desired 0.3rad.(17.2°) under the specified limits on control moments in 2.5 seconds (for details, see Section 5.3.3).

For the vibration suppression induced by the slew maneuvers, one can use one of the several control schemes discussed in Section 4.4. For the present problem, we used the proportional control given by equations (4.103 -4.105) with the parameters: k_1 =30000, k_2 =10000, k_3 =100 for $t \in[0,2.5]$ and k_3 =50000 for $t \in[2.5,10]$; $d_{i_1} = d_{i_2} = d_{i_3}$ =0.05 for i= 1,2,5; e_1=9000, e_2=3000, e_3=6000. $F_1(t) = F_2(t) = F_3(t)$ =0 and for *slew maneuver* the Bang -Pause -Bang control (Fig.5.25) were used with T_{s_1}=10035 $lb\text{-}ft$, t_{s_1}=0.5sec, t_{s_2}=2.0sec, t_{s_3}=2.5sec, in the simulation.

Detailed numerical results showing stabilization of the various state trajectories were obtained from the simulation and presented in Figs. 5.84-5.103. Figs. 5.84 and

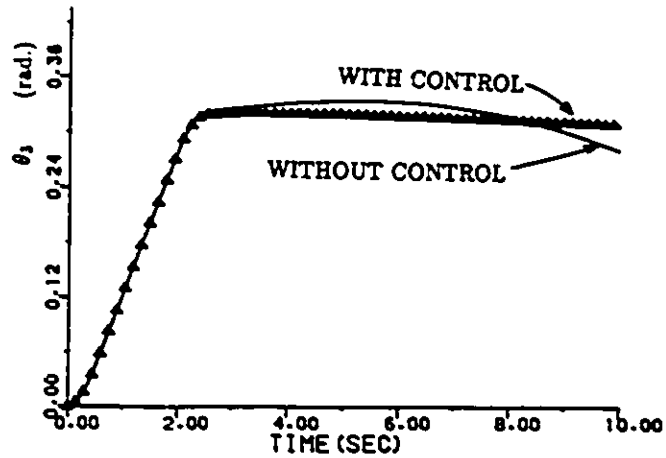


Fig.5.84 Slew angle θ_3 in k_3 direction.

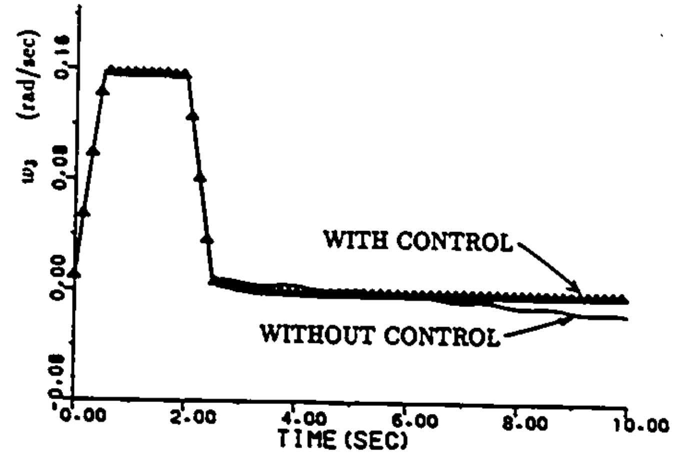


Fig.5.85 Bus angular velocity w_3.

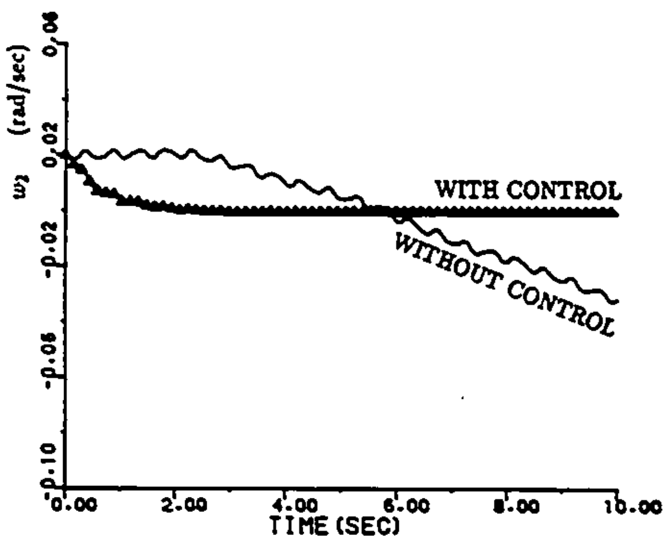


Fig.5.86 Bus angular velocity w_2.

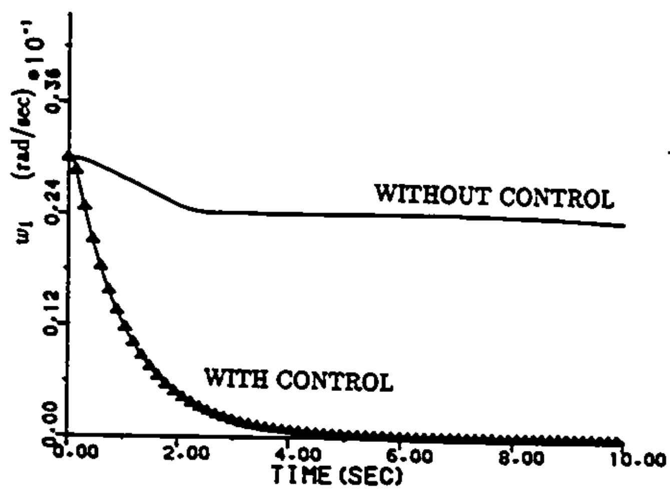


Fig.5.87 Bus angular velocity w_1.

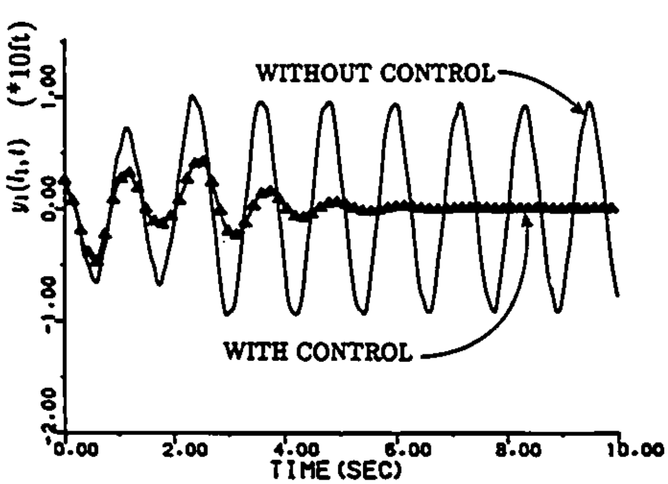


Fig.5.88 Beam displacement $y_1(l_1, t)$.

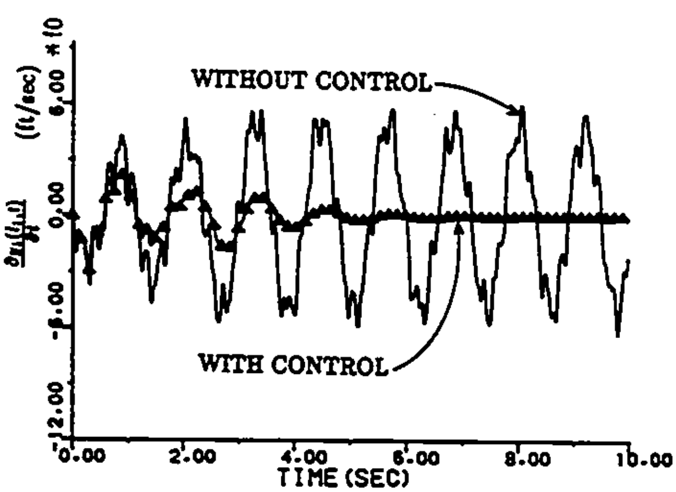


Fig.5.89 Beam velocity $\frac{\partial y_1(l_1,t)}{\partial t}$.

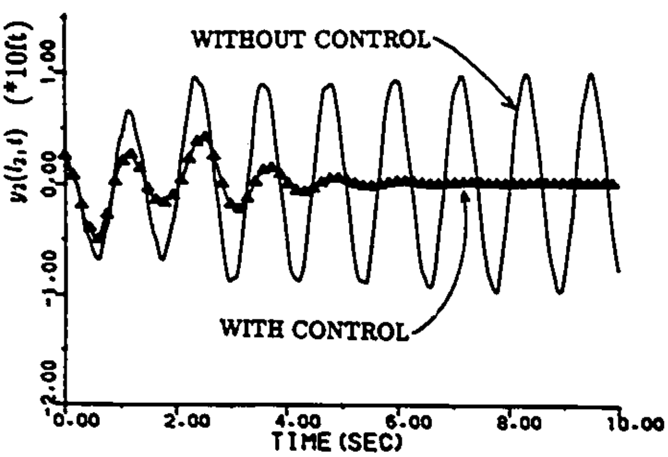


Fig.5.90 Beam displacement $y_2(l_2,t)$.

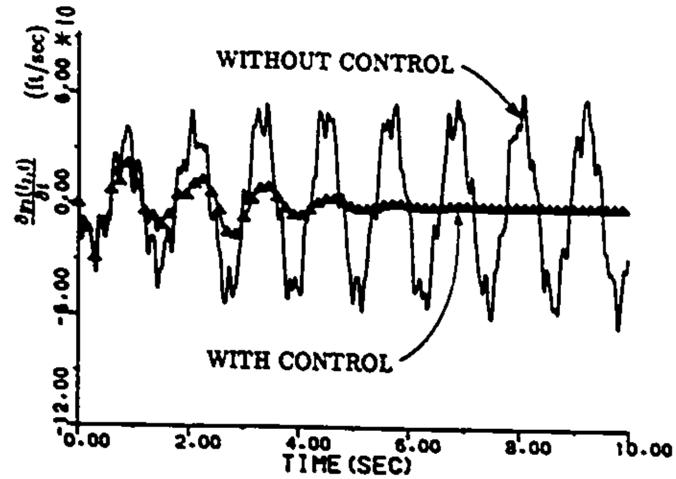


Fig.5.91 Beam velocity $\frac{\partial y_2(l_2,t)}{\partial t}$.

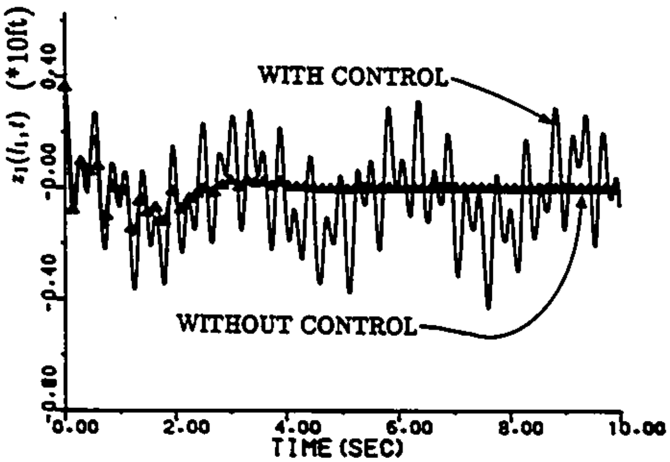


Fig.5.92 Beam displacement $z_1(l_1,t)$.

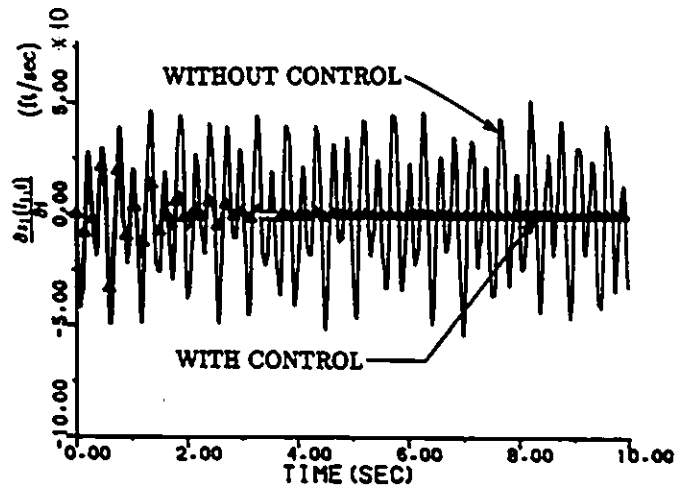


Fig.5.93 Beam velocity $\frac{\partial z_1(l_1,t)}{\partial t}$.

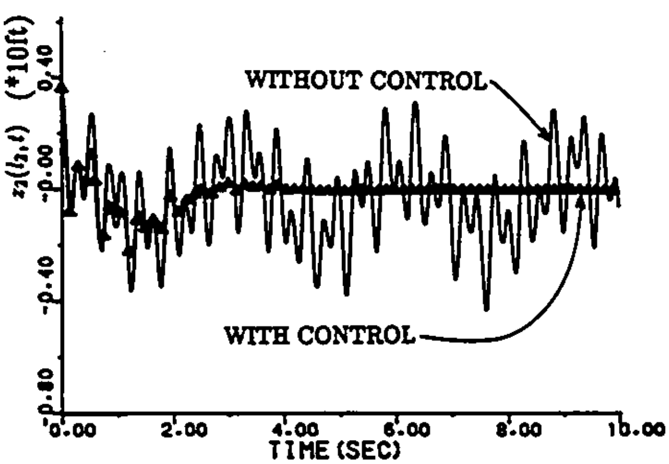


Fig.5.94 Beam displacement $z_2(l_2,t)$.

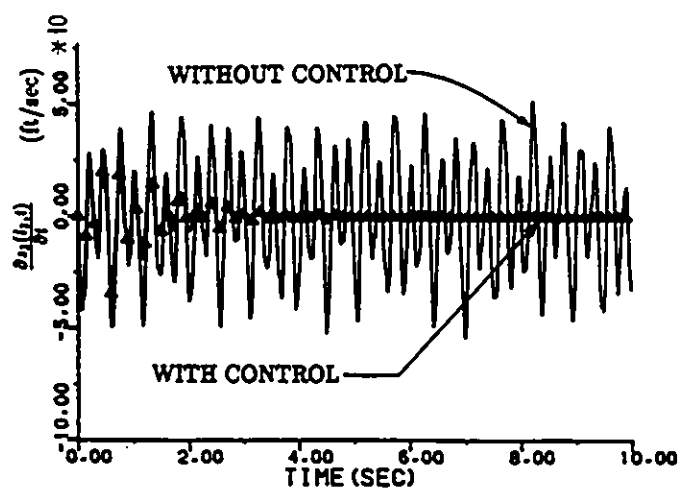


Fig.5.95 Beam velocity $\frac{\partial z_2(l_2,t)}{\partial t}$.

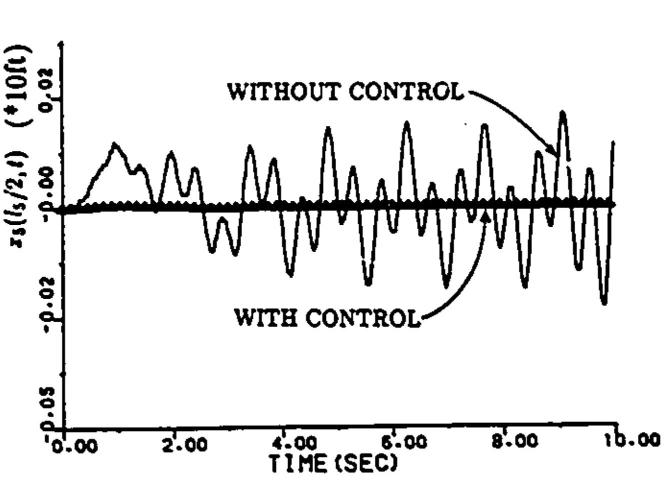


Fig.5.96 Beam displacement $x_5(l_5/2,t)$.

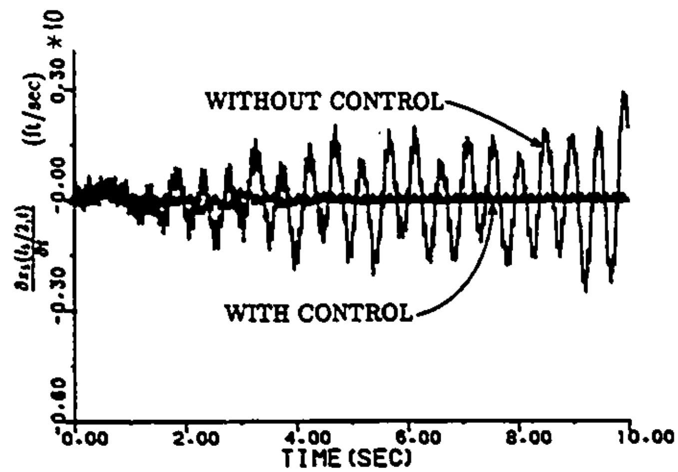


Fig.5.97 Beam velocity $\frac{\partial x_5(l_5/2,t)}{\partial t}$.

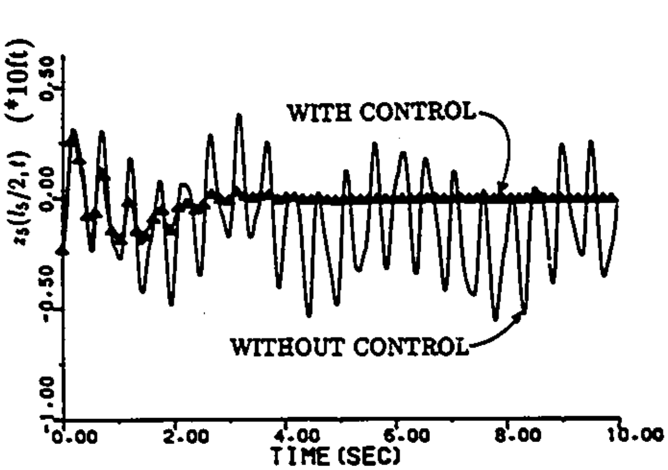


Fig.5.98 Beam displacement $x_5(l_5/2,t)$.

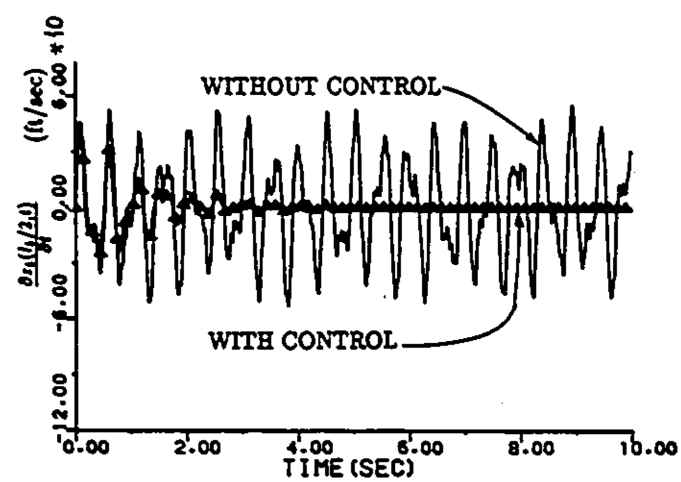


Fig.5.99 Beam velocity $\frac{\partial x_5(l_5/2,t)}{\partial t}$.

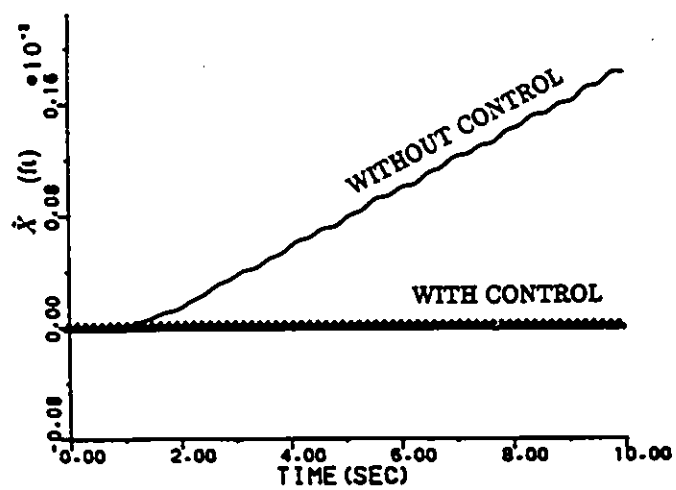


Fig.5.100 Radial perturbation $\tilde{X}$.

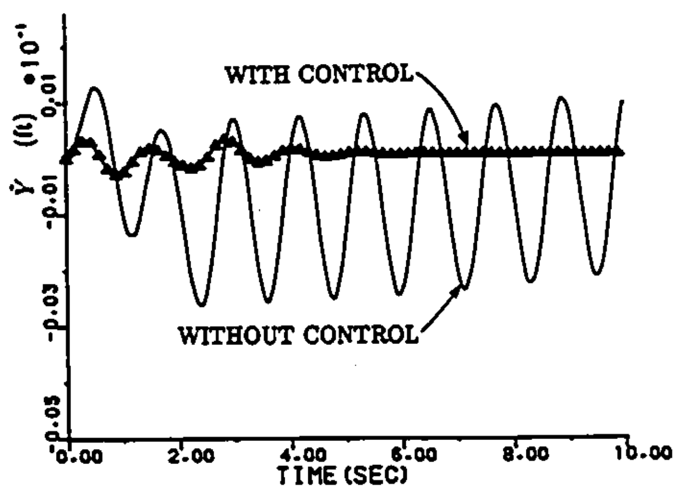


Fig.5.101 Radial perturbation $\tilde{Y}$.

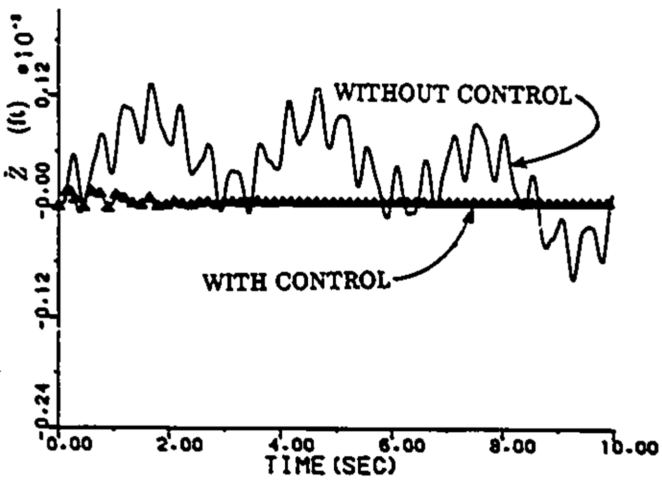


Fig.5.102 Radial perturbation $\tilde{Z}$.

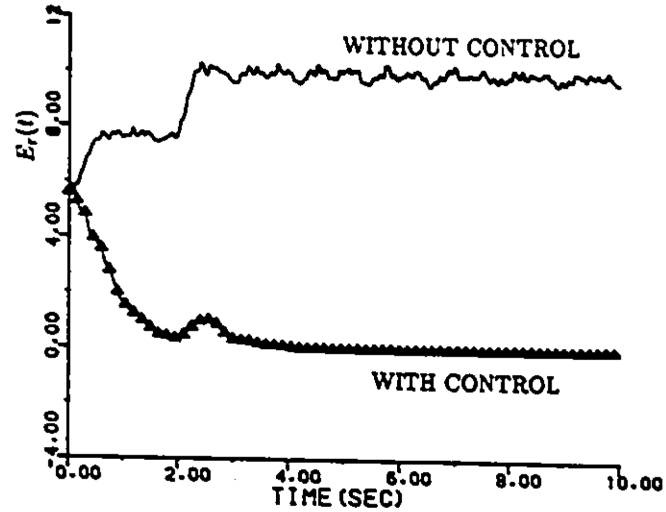


Fig.5.103 Relative beam energy.

5.85 represent the slew angle and the angular velocity of the spacecraft corresponding to the slew control (Fig. 5.25). We note from the Fig. 5.84 that slew maneuver was successfully achieved by the slew control commands. As mentioned earlier in Section 3.4, the bus translation, bus rotation and vibration of beams are strongly coupled. This implies that any perturbation in one part of the system will induce disturbances in the other members of the system. Hence the slew maneuver induced significant perturbations leading to vibrations of the beam and bus rotation. This is clearly observed from the case without controls for vibration supression as expected. In Figs. 5.88-5.99, Figs. 5.85-5.87, Figs. 5.100-5.102 and Fig. 5.103, we show the deflections of the beams, angular velocities of the body, the components of radial vector ($\tilde{R}$) and relative beam energy respectively. From these figures it is clear that without stabilizing controls the beam vibrations, oscillation of the bus and perturbation in the radial vector persist or grow. However, with application of the feedback controls, these oscillations are effectively eliminated throughout the entire system.

5.5 Concluding Remarks

In this chapter several numerical results illustrating the impact of orbital perturbation on the motion of the spacecraft and the space station and stabilization of slew maneuvers were presented. Two external perturbing forces (radial and rotational) were applied to the system and their corresponding stabilizations have been numerically demonstrated through six case studies. From the results it is clearly observed that

(i) The impact of the radial perturbation on the beam vibration and bus rotation is quite significant and it has been eliminated by application of appropriate stabilizing controls as suggested in Chapter 4.

(ii) During the slew maneuver, vibration in the beams and oscillations in the angular velocities of the bus are induced and it has been shown that the stabilizing control can effectively eliminate the oscillations and stabilize the system.

(iii) If the disturbances following external perturbing forces persist, then in the absence of proper controls, these small motions may build up leading to instability of the entire system.

We conclude this chapter with the following remaks.

Remark 5.1

Numerical experiments indicate that the distributed proportional control and Bang -Bang control provide faster stabilization of the system as compared with the localized proportional control and deadzone control respectively. In case of Bang -Bang control, all the errors in the system states eventually decay to zero and one obtains strict asymptotic stabilization. Further, it is noted that the rate of decay of beam vibration and bus angular velocities depends on the magnitude of the coefficients in the feedback law. Larger feedback gains would produce a faster stabilization of the spacecraft. The choice of the feedback gains depends on the requirements of stabilization time and the mechanical safety. One may choose the proper feedback gains in order to stabilize the system in desired time keeping in view the structural safety limits. It was also observed in the process of numerical simulation that (i) the stabilizing controls are *robust* since the stabilizing effect is not affected by the size of the disturbances imposed to the system[5,82,83], (ii) as the disturbance increases, the overall coupling effect becomes more severe and the oscillation of the bus and vibration of the beams grow rapidly as expected.

Remark 5.2

In the numerical solution of the ordinary differential equations (5.16-5.17) Runge-Kutta method was used after discretizing the partial differential equations for beam dynamics by a finite difference scheme discussed in Section 5.2. In this study no attention was given to optimize or minimize the computation time for the solution of the problem, yet the computation time was found to be fairly small. A typical run for the solution of the system equations (5.16) and (5.17) took 36 sec and 370 sec respectively (in CPU time) on a AMDHAL 5860 (VM/HPO Release 4.2) computer. It is expected that the computation time could be further reduced by use of more efficient program.

Remark 5.3

In this numerical experiments, the dynamic equations which were developed based on the Euler-Bernoulli beam theory were used. The other dynamic models developed in Sections 2.5, 2.6, 3.5 and 3.6 based on Rayleigh and Timoshenko beam theories could be solved using the semi-discretization algorithm given in Section 5.2.

Chapter 6

CONCLUSIONS

6.1 Conclusions

For satisfactory performance of the large space structure in space applications such as communications, spacetelescopes, etc, the interactions of the flexibility of elastic members with the other parts should be carefully taken into account in modeling and controller design. In this thesis the important problem of modeling and vibration control of the future space structures which are expected to be very large and flexible was considered. In Chapter 2 and Chapter 3, complete dynamic models for large space structures consisting of a massive rigid body and large flexible configurations were developed using extended Hamilton's principle. In the development, the systems such as (i) a spacecraft consisting of a rigid body and a long flexible beam and (ii) a space station consisting of a massive central body and large flexible configurations were considered. The equations of motion consist of radial equations describing the translational motion of the rigid body, attitude equations governing the rotational motion of the system and partial differential equations representing the vibrational dynamics of the flexible members along with very complex boundary conditions. In the development of the dynamic model, Euler-Bernoulli, Rayleigh and Timoshenko beam equations were utilized to obtain the dynamics of the flexible components of the system. The important aspects of the model developed in this thesis are with that

(i) it includes the trajectory information and attitude dynamics as well as beam equation. This is extremely important in the analysis of the behavior of the entire system and the design of stabilizing controls.

(ii) the dynamics is described in terms of state variables which are physically measurable. Further the displacements of the flexible members are defined with respect to the rest (underformed) position of the structure so that the variable can be easily computed and readily understood.

(iii) The dynamic equations of motion clearly show the strong couplings among the radial dynamics, attitude dynamics and vibrational motion of the flexible parts. Hence with this model one can study the effects of external disturbances in one member of the system on the others. For example, the influence of the orbital perturbation on the rotation of the body and vibration of the beams could be assessed as shown in chapter 5 through numerical simulation.

Large flexible space structures in earth orbit are expected to experience mechanical vibrations subject to several disturbing forces such as orbital corrections, slew maneuvers, shuttle take-off and docking or crew movements etc. In this thesis, the problem of stabilization or vibration suppression of the space structures caused by these disturbances was considered. In Chapter 4, by application of feedback controls such as distributed(or localized) proportional controls, dead zone controls, Bang-Bang or saturation controls, asymptotic stability of the system subject to external perturbations was proved using Lyapunov's method. These controls can be implemented in practice. The robustness property of the stabilizing controls was numerically demonstrated through several case studies in Chapter 5.

In order to illustrate the effectiveness of the stabilizing controls and strong interactions among the vibration of the flexible members, rotational motion and translational motion of the rigid body, in Chapter 5 six case studies were performed through numerical simulations. Although the system dynamics appears to be somewhat complicated, it can be simulated on a digital computer for numerical analysis with reasonable expenditure of computer time. For numerical investigation, a semi-discretization algorithm was developed for the simultaneous solution of the hybrid dynamics. The numerical results indicated that the impact of the external disturbance such as orbital perturbation and slew naneuvers on the beam vibration and bus rotation was quite significant. Hence for satisfactory performance of the space structures the impact should be eliminated by application of appropriate stabilizing controls. The effectiveness of the feedback control as a stabilizing controller was clearly demonstrated from the six case studies. That is, the vibrations or perturbations of each member of the system caused by external forces were effectively eliminated by application of the controls suggested in chapter 4.

6.2 Suggestions for Further Research

As a continuation of this thesis, further research could be conducted in several directions as follows.

(i) As pointed out in Chapter 4, in practice the distributed control may be difficult or costly for implementation. Thus it is essential to consider stabilization of the flexible space structures using a finite number of controllers. Also, the location of sensors and actuators becomes a critically significant degree of freedom in control design of large space structures. Hence, the selection of sensors and actuators which have nontrivial dynamics of their own and their optimal placements would be an interesting and challenging problem.

(ii) In the development of the dynamic model in this thesis, a simplified expression for gravitational potential energy was used. Using more accurate expression as mentioned in Remark 2.1, one can include the effect of gravitational torque on the dynamic behavior of the system. For the small satellites the gravity torques has been utilized in design of passive stabilizing controllers. Development of more detailed dynamics using precise gravitational force will be an interesting problem.

(iii) The dynamic equations of motion for the space structures, which were developed in Chapters 2 and 3, were described in terms of a hybrid system as the combination of three second order ordinary differential equations for translational motion, three first order ordinary differential equations for angular motion of the rigid body and several second order partial differential equations for the dynamics of elastic beams. These equations are strongly coupled one another and expressed in the implicit form. So far as no mathematical theories are applicable to this type of equations. Transformation of the equation into the explicit form or simplification into abstract equations so that known control theories such as semigroup theory can be applied for system analysis is a new outstanding topic for further studies.

(iv) In modeling of flexible structures, the flexibility of the elastic members was modelled by an equivalent beam equation based on certain assumptions. This approach is meaningful only when the equivalent parameters are known. Thus the essential aspect of modeling any flexible system is identification of equvalent beam parameters. Development of a systematic technique for identification of parameters arising in the modeling of large flexible space structures poses a significant problem in space applications.

Appendix A

GRAVITATIONAL POTENTIAL ENERGY

From Equation (2.11) we have $r = R + \bar{r}$. Since $|r|^2 = (R + \bar{r}) \circ (R + \bar{r})$,

$$|r|^2 = |R|^2\Big(1 + \frac{2R \circ \bar{r}}{|R|^2} + \Big(\frac{|\bar{r}|}{|R|}\Big)^2\Big). \tag{A.1}$$

This implies

$$|r|^{-1} = \frac{1}{|R|}\Big(1 + \frac{2R \circ \bar{r}}{|R|^2} + \frac{|\bar{r}|^2}{|R|^2}\Big)^{-\frac{1}{2}}. \tag{A.2}$$

Expanding by power series, since $\left|\frac{2R\circ\bar{r}}{|R|^2} + \frac{|\bar{r}|^2}{|R|^2}\right| < 1$,

$$\begin{aligned} |r|^{-1} &= \frac{1}{|R|}\Big[1 - \frac{1}{2}\Big(\frac{2R \circ \bar{r}}{|R|^2} + \frac{|\bar{r}|^2}{|R|^2}\Big) + \frac{3}{8}\Big(\frac{2R \circ \bar{r}}{|R|^2} + \frac{|\bar{r}|^2}{|R|^2}\Big)^2 - \frac{5}{16}\Big(\frac{2R \circ \bar{r}}{|R|^2} + \frac{|\bar{r}|^2}{|R|^2}\Big)^3 + \cdots\Big] \\ &= \frac{1}{|R|} - \frac{1}{2|R|^3}\Big[2R \circ \bar{r} + |\bar{r}|^2 - 3\Big(\frac{R}{|R|} \circ \bar{r}\Big)^2 - \cdots\Big]. \end{aligned} \tag{A.3}$$

Similarly we have from Equation (2.9) and $|\tilde{r}|^2 = (R + \tilde{\eta}) \circ (R + \tilde{\eta})$ that

$$|\tilde{r}|^{-1} = \frac{1}{|R|} - \frac{1}{2|R|^3}\Big[2R \circ \tilde{\eta} + |\tilde{\eta}|^2 - 3\Big(\frac{R}{|R|} \circ \tilde{\eta}\Big)^2 + \cdots\Big]. \tag{A.4}$$

Introducing Equations (A.3) and (A.4) into Equation (2.14) and integrating over the entire system, we obtain the following expression:

$$\begin{aligned} P_g = \frac{-Gm_e m_T}{|R|} &- \frac{Gm_e}{2|R|^3}\int_{beam}\Big\{2R \circ \bar{r} + |\bar{r}|^2 - 3\Big(\frac{R}{|R|} \circ \bar{r}\Big)^2\Big\} dm \\ &- \frac{Gm_e}{2|R|^3}\int_{bus}\Big\{2R \circ \tilde{\eta} + |\tilde{\eta}|^2 - 3\Big(\frac{R}{|R|} \circ \tilde{\eta}\Big)^2\Big\} d\tilde{m}. \end{aligned} \tag{A.5}$$

Assuming that the origin of the body frame is at the mass center of the system, the integrals of the terms $2R \circ \bar{r}$ and $2R \circ \tilde{\eta}$ in Equation (A.5) are zero. We note that higher order terms have been ignored in Equation (A.5) since in practice they are negligibly small. Thus we have

$$P_g = \frac{-Gm_e m_T}{|R|} - \frac{Gm_e}{2|R|^3} \int_{beam} \left\{ |\bar{r}|^2 - 3\left(\frac{R}{|R|} \circ \bar{r}\right)^2 \right\} dm$$
$$- \frac{Gm_e}{2|R|^3} \int_{bus} \left\{ |\tilde{\eta}|^2 - 3\left(\frac{R}{|R|} \circ \tilde{\eta}\right)^2 \right\} d\tilde{m}. \qquad (A.6)$$

This can be expressed in terms of direction cosines as follows.

$$P_g = \frac{-Gm_e m_T}{|R|} - \frac{Gm_e}{2|R|^3} \Big(J_{11} + J_{22} + J_{33} - 3\alpha^2 J_{11} - 3\beta^2 J_{22} - 3\gamma^2 J_{33}$$
$$+ 6\alpha\beta J_{12} + 6\beta\gamma J_{23} + 6\gamma\alpha J_{31} \Big), \qquad (A.7)$$

where α, β, γ are the direction cosines between R and axes i_b, j_b, k_b respectively and J_{mn}, (where $m, n = 1, 2, 3$) are elements of the inertia tensor of the system with respect to the body frame.

Appendix B

A VARIATIONAL FORMULA

From the definition of the vector r (Equation (2.11)) and Equation (2.17), we have

$$\frac{dr}{dt} = \frac{d}{dt}(R + \bar{r}) = \frac{d}{dt}R + \dot{\bar{r}} + w \times \bar{r}. \tag{B.1}$$

Thus we obtain the variational formula, from (2.18), that

$$\delta\Big(\frac{dr}{dt}\Big) = \delta\Big(\frac{dR}{dt} + \dot{\bar{r}} + w \times \bar{r}\Big) \tag{B.2}$$

$$= \delta\Big(\frac{dR}{dt}\Big) + \delta\Big(\dot{\bar{r}} + w \times \bar{r}\Big) \tag{B.3}$$

$$= \delta\Big(\frac{dR}{dt}\Big) + \delta_b(\dot{\bar{r}} + w \times \bar{r}) + (\delta\theta) \times (\dot{\bar{r}} + w \times \bar{r}) \tag{B.4}$$

$$= \delta\Big(\frac{dR}{dt}\Big) + \delta_b\dot{\bar{r}} + (\delta_b w) \times \bar{r} + w \times (\delta_b \bar{r}) + (\delta\theta) \times (\dot{\bar{r}} + w \times \bar{r}) \tag{B.5}$$

In the above derivation we have used the notation δ_b for the the first variation with respect to the body frame S_b. For details, it is referred to [Cavin and Dusto 32].

Appendix C

SOME IDENTITIES FROM VECTOR ALGEBRA

Some vector relations frequently used in Chapters 2–3 are given below[120]. Let A, B, C be three vectors and T be a tensor in the rectangular coordinate system (i, j, k), definde by

$$A = iA_i + jA_j + kA_k \tag{C.1}$$

$$B = iB_i + jB_j + kB_k \tag{C.2}$$

$$C = iC_i + jC_j + kC_k \tag{C.3}$$

$$T = \begin{pmatrix} i & j & k \end{pmatrix} \begin{pmatrix} T_{ii} & T_{ij} & T_{ik} \\ T_{ji} & T_{jj} & T_{jk} \\ T_{ki} & T_{kj} & T_{kk} \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix} \tag{C.4}$$

Vector Relations

$$A \circ (B \times C) = B \circ (C \times A) = C \circ (A \times B) \tag{C.5}$$

$$A \times (B \times C) + (B \times A) \times C = B \times (A \times C) \tag{C.6}$$

$$A \times (B \times C) = (A \circ C)B - (A \circ B)C \tag{C.7}$$

Vector and Tensor Operations

$$A \circ B = A_iB_i + A_jB_j + A_kB_k \tag{C.8}$$

$$A \times B = i(A_jB_k - A_kB_j) + j(A_kB_i - A_iB_k) + k(A_iB_j - A_jB_i) \tag{C.9}$$

$$\begin{aligned} T \circ A = {} & i(T_{ii}A_i + T_{ij}A_j + T_{ik}A_k) \\ & + j(T_{ji}A_i + T_{jj}A_j + T_{jk}A_k) \\ & + k(T_{ki}A_i + T_{kj}A_j + T_{kk}A_k) \end{aligned} \tag{C.10}$$

Appendix D

PROOF OF AN IDENTITY

We show the relation

$$\delta|R| = (\delta R) \circ 1_R. \tag{D.1}$$

Proof:

We use the relation

$$|R|^2 = R \circ R. \tag{D.2}$$

From the left hand side of Equation (D.2) we obtain that

$$\delta\{|R|^2\} = 2|R|(\delta|R|). \tag{D.3}$$

Taking the first variation of the right hand side of Equation (D.2) we have

$$\begin{aligned} \delta(R \circ R) &= 2(\delta R) \circ R \\ &= 2(\delta R) \circ \left\{|R|\frac{R}{|R|}\right\}. \end{aligned} \tag{D.4}$$

Substituting Equation (D.3) and Equation (D.4) into Equation (D.2), we obtain

$$2|R|(\delta|R|) = 2|R|(\delta R) \circ \left(\frac{R}{|R|}\right), \tag{D.5}$$

or

$$\delta|R| = (\delta R) \circ \left(\frac{R}{|R|}\right). \tag{D.6}$$

Denoting $\frac{R}{|R|}$ by 1_R, we obtain the identity (D.1). Q.E.D.

Appendix E

EULER ANGLES AND THE MATRIX C_I^B

The orientation of the body frame S_B may be related to the inertial frame S_I by a sequence of rotations $\theta_3, \theta_2, \theta_1$ called *Euler Angles*. For details of the sequence of rotations, it is refered to the section 6.1 of [Greensite 48]. For notational simplicity, let S_B denote either the body frame or some vector in the body frame S_B (Similarly for S_I). Then we have

$$S_B = C_I^B S_I, \tag{E.1}$$

where the coordinate transformation matrix from inertial to body frame C_I^B is given by

$$\begin{aligned} C_I^B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta_1 & sin\theta_1 \\ 0 & -sin\theta_1 & cos\theta_1 \end{pmatrix} \begin{pmatrix} cos\theta_2 & 0 & -sin\theta_2 \\ 0 & 1 & 0 \\ sin\theta_2 & 0 & cos\theta_2 \end{pmatrix} \begin{pmatrix} cos\theta_3 & sin\theta_3 & 0 \\ -sin\theta_3 & cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} cos\theta_2 cos\theta_3 & cos\theta_2 sin\theta_3 & -sin\theta_2 \\ sin\theta_1 sin\theta_2 cos\theta_3 - cos\theta_1 sin\theta_3 & sin\theta_1 sin\theta_2 sin\theta_3 + cos\theta_1 cos\theta_3 & sin\theta_1 cos\theta_2 \\ cos\theta_1 sin\theta_2 cos\theta_3 + sin\theta_1 sin\theta_3 & cos\theta_1 sin\theta_2 sin\theta_3 - sin\theta_1 cos\theta_3 & cos\theta_1 cos\theta_2 \end{pmatrix}. \end{aligned} \tag{E.2}$$

Note that the matrix C_I^B is orthogonal, i.e.

$$C_B^I = \left[C_I^B\right]^{-1} = \left[C_I^B\right]'. \tag{E.3}$$

By direct resolution of vectors we find the components of angular velocity in the S_B frame:

$$\begin{aligned} \omega_1 &= \dot\theta_1 - \dot\theta_3 sin\theta_2 \\ \omega_2 &= \dot\theta_2 cos\theta_1 + \dot\theta_3 cos\theta_2 sin\theta_1 \\ \omega_3 &= \dot\theta_3 cos\theta_2 cos\theta_1 - \dot\theta_2 sin\theta_1 \end{aligned} \tag{E.4}$$

These equations can be solved for $\dot\theta_1, \dot\theta_2, \dot\theta_3$ as given below.

$$\begin{aligned}
\dot{\theta}_1 &= w_1 + (w_2 sin\theta_1 + w_3 cos\theta_1) tan\theta_2 \\
\dot{\theta}_2 &= w_2 cos\theta_1 - w_3 sin\theta_1 \\
\dot{\theta}_3 &= (w_2 sin\theta_1 + w_3 cos\theta_1) sec\theta_2
\end{aligned} \tag{E.5}$$

From given angular velocities w_1, w_2, w_3, we obtain the Euler Angles θ_1, θ_2, θ_3 by solving Equation (E.5).

Appendix F

Proof of Equations (3.39) and (3.40)

Since $I_b \circ w \equiv \sum_{i=1}^{6} \int_{\Omega_i} \bar{r}_i \times (w \times \bar{r}_i) dm_i$, we have

$$I_b \circ \dot{w} + \dot{I}_b \circ w = \sum_{i=1}^{6} \int_{\Omega_i} \{\dot{\bar{r}}_i \times (w \times \bar{r}_i) + \bar{r}_i \times (\dot{w} \times \bar{r}_i) + \bar{r}_i \times (w \times \dot{\bar{r}}_i)\} dm_i. \quad (F.1)$$

Thus, the left hand side of Equation (3.39) can be developed as follows:

$$\sum_{i=1}^{6} \int_{\Omega_i} \{\bar{r}_i \times (\dot{w} \times \bar{r}_i) + \bar{r}_i \times (2w \times \dot{D}_i)\} dm_i = \sum_{i=1}^{6} \int_{\Omega_i} \{\bar{r}_i \times (\dot{w} \times \bar{r}_i)$$

$$+ \bar{r}_i \times (w \times \dot{\bar{r}}_i) + \dot{\bar{r}}_i \times (w \times \bar{r}_i) - \dot{\bar{r}}_i \times (w \times \bar{r}_i)\} dm_i + \sum_{i=1}^{6} \int_{\Omega_i} \bar{r}_i \times (w \times \dot{\bar{r}}_i) dm_i$$

$$= I_b \circ \dot{w} + \dot{I}_b \circ w - \sum_{i=1}^{6} \int_{\Omega_i} \dot{\bar{r}}_i \times (w \times \bar{r}_i) dm_i + \sum_{i=1}^{6} \int_{\Omega_i} \bar{r}_i \times (w \times \dot{\bar{r}}_i) dm_i$$

$$= I_b \circ \dot{w} + \dot{I}_b \circ w - \sum_{i=1}^{6} \int_{\Omega_i} \Big(\{w(\dot{\bar{r}}_i \circ \bar{r}_i - \bar{r}_i(\dot{\bar{r}}_i \circ w)\} - \{w(\dot{\bar{r}}_i \circ \bar{r}_i) - \dot{\bar{r}}_i(\bar{r}_i \circ w)\} \Big) dm_i$$

$$= I_b \circ \dot{w} + \dot{I}_b \circ w + \sum_{i=1}^{6} \int_{\Omega_i} \{\bar{r}_i(\dot{\bar{r}}_i \circ w) - \dot{\bar{r}}_i(\bar{r}_i \circ w)\} dm_i$$

$$= I_b \circ \dot{w} + \dot{I}_b \circ w + \sum_{i=1}^{6} \int_{\Omega_i} w \times (\bar{r}_i \times \dot{\bar{r}}_i) dm_i. \quad (F.2)$$

The last expression of Equation (F.2) is the same as the right hand side of Equation (3.39). Q.E.D.

Using the vector identities (C.5) and (C.7), we can show that the left hand side of Equation (3.40) can be developed as follows:

$$\sum_{i=1}^{6} \int_{\Omega_i} \bar{r}_i \times \{w \times (w \times \bar{r}_i)\} dm_i$$

$$= \sum_{i=1}^{6} \int_{\Omega_i} \{w(\bar{r}_i \circ (w \times \bar{r}_i)) - (w \times \bar{r}_i)(\bar{r}_i \circ w)\} dm_i$$

$$= \sum_{i=1}^{6} \int_{\Omega_i} \{w(w \circ (\bar{r}_i \times \bar{r}_i)) - (w \times \bar{r}_i)(\bar{r}_i \circ w)\} dm_i$$

$$= \sum_{i=1}^{6} \int_{\Omega_i} \{\bar{r}_i(w \circ (w \times \bar{r}_i)) - (w \times \bar{r}_i)(\bar{r}_i \circ w)\} dm_i$$

$$= \sum_{i=1}^{6} \int_{\Omega_i} w \times (\bar{r}_i \times (w \times \bar{r}_i)) dm_i = w \times (I_b \circ w). \tag{F.3}$$

The last expression of Equation (F.3) is equivalent to the right hand side of Equation (3.40). Q.E.D.

Appendix G

Angular Boundary Conditions

Here, we derive the boundary conditions for the angular displacements at the joints $\mathcal{J}_5$, $\mathcal{J}_6$, $\mathcal{J}_7$ and $\mathcal{J}_8$. For illustration, only the boundary conditions at the joint $\mathcal{J}_5$ will be considered in the following, since the other joints can be treated in the same way. We have, from the trigonometry, the addition formula[120]

$$tan(\theta_3 - \theta_1) = \frac{tan\theta_3 - tan\theta_1}{1 + tan\theta_3 tan\theta_1}. \qquad (G.1)$$

By the assumption that the two beams are rigidly jointed at the joint $\mathcal{J}_5$, the angle $90°$ is maintained, i.e.

$$\theta_3 - \theta_1 = 90°. \qquad (G.2)$$

Hence $tan(\theta_3 - \theta_1) = \infty$ and this requires from the formula (G.1) that

$$tan\theta_3 tan\theta_1 = -1. \qquad (G.3)$$

From the geometry in Fig.A.1, we know that $\theta_3 = 90° + (-\theta_2)$. This implies that

$$tan\theta_3 = tan(90° - \theta_2) = (tan\theta_2)^{-1}. \qquad (G.4)$$

Also from Fig.A.1, we know that at the joint $\mathcal{J}_5$

$$\theta_1 = tan^{-1}\Big(\frac{\partial y_1}{\partial \xi_1}\Big), \qquad \theta_2 = tan^{-1}\Big(\frac{\partial x_5}{\partial \xi_5}\Big). \qquad (G.5)$$

Substituting Equation (G.4) into Equation (G.3), we obtain

$$tan\theta_1 = -tan\theta_2. \qquad (G.6)$$

Substituting Equation (G.5) into Equation (G.6), the boundary condition for the joint $\mathcal{J}_5$ is obtained:

$$\frac{\partial y_1}{\partial \xi_1} = -\frac{\partial x_5}{\partial \xi_5} \quad . \qquad (G.7)$$

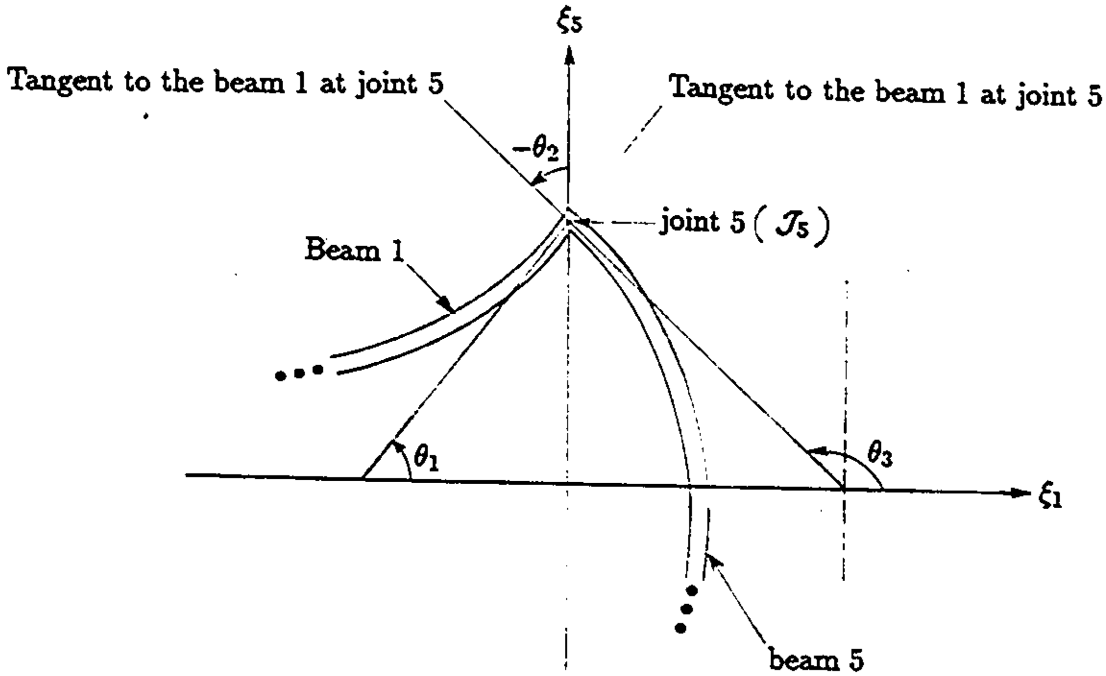


Fig. A.1 Angular relations at the joint $\mathcal{J}_5$(in $i_b j_b$ plane).

REFERENCES

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] N.U. Ahmed, Elements of Finite Dimensional Systems and Control Theory, Longman Science & Technical, England, 1988.

[3] N.U. Ahmed and S.K. Biswas, "Mathematical Modelling and Control of Large Space Structures with Multiple Appendages", Mathematical and Computer Modelling, Vol.10, No.12, pp.891-900, 1988.

[4] N.U. Ahmed and S.S. Lim,"Modelling and Stabilization of Flexible Spacecraft Under the Influence of Orbital Perturbation", Proc. of the 26th Conf. on Decision and Control, Los Angeles, CA, pp.2331-2336, 1987.

[5] N.U. Ahmed and A.K. Sen, "Control of Synchronous Orbit Using Lyapunov Stability Theory", IEEE Trans. on Aerospace and Electronic Systems, Vol. AES-17, pp. 48-53, 1981.

[6] N.U. Ahmed and K.L. Teo, Optimal Control of Distributed Parameter Systems, North Holland, New York, Oxford, 1981.

[7] AIAA, "Large Space Structure Control: Early Experiments", J. of Guid. Cont. and Dynamics, Vol.7, pp.513-562, 1984.

[8] W.A. Ames, Numerical Methods for Partial Differential Equations, Academic Press, New York, 1977.

[9] R. Araya, "A Controllability - Stabilizability Result for th NASA-IEEE Spacecraft Control Laboratory Experiment (SCOLE) Configuration", to appear.

[10] J. Baillieul and M. Levi, "The Rotational Mechanics of a Simple Elastic Structure", IFAC Control of Distributed Parameter Systems, Los Angels, California, pp. 155-160, 1986.

[11] P.M. Bainum and F. Li, "Optimal Torque Control of SCOLE Slewing Maneuvers", 3rd Annual SCOLE Workshop, Nov.17, 1986, NASA Langley Research Center, Hampton, VA.

[12] P.M. Bainum and A.S.S.R. Reddy, "The Dynamics and Control of the In-Orbit SCOLE Configuration", NASA TM-101503, pp.145-180, 1988.

[13] A.V. Balakrishnan, "A Mathematical Formulation of the SCOLE Control Problem", SCOLE Workshop 1984, NASA Langley Research Center, Hampton, Virginia, Dec.6-7, 1984, pp.87-103.

[14] A.V. Balakrishnan, "A Mathematical Formulation of the SCOLE Control Problem, Part II: Optimal Compensator Design", NASA CR-181720, Dec. 1988.

[15] A.V. Balakrishnan, "Some Nonlinear Damping models in Flexible Structures", NASA TM 101503, Oct. 1988, pp. 54-66.

[16] M.J. Balas, "Trends in Large Space Structure Control Theory : Fondest Hopes, Wildest Dreams", IEEE Trans. on Automatic Control, Vol. AC 27, pp. 522-535, 1982.

[17] J.M. Ball, "Initial-Boundary Value Problems for an Extensible Beam", J. of Mathematical Analysis and Applications, Vol.42, 1973, pp. 61-93.

[18] J.M. Ball, "Stability Theory for an Extensible Beam", J. of Differential Equations, Vol.14, 1973, pp.399-418.

[19] H.T. Banks, "Algorithm for Estimation in Distributed Models with Application to Large Space Structures", Proc. of the Workshop on Application of Distributed System Theory to the Control of Large Space Structures, JPL, Pasadena, CA, July 14-16, 1982, pp.505-509.

[20] H.T. Banks, R.H. Fabiano, Y. Wang, D.J. Inman and H. Cudney, Jr., "Spatial Versus Time Hysterisis in Damping Mechanisms", Proc. of the 27th Conf. on Decision and Control, Austin, Tx, Dec.1988, pp.1674-1677.

[21] H.T. Banks and I.G. Rosen, "Approximation Techniques for Parameter Estimation and Feedback Control of Distributed Models of Large Flexible structures", Proc. of NASA/ACC Workshop on Identification and Control of Flexible Space Structures, San Diego, CA, june 4-6, 1984.

[22] H.T. Banks, Y. Wang, D.J. Inman and H. Cudney, Jr., "Parameter Identification Techniques for the Estimation of Damping in Flexible Structure Experiments", Proc. of the 26th Conf. on Decision and Control, Los Angeles, CA, Dec. 1987, pp.1392-1395.

[23] F. J. Barbera, "Liapunov Stability Analysis of Spinning Elastic Spacecraft", AIAA Journal, Vol.11, No.4, 1973, pp.457-466.

[24] R.R. Bate, D.D. Mueller, J.E. White, Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

[25] S.K. Biswas and N.U. Ahmed,"Modelling of Flexible Spacecraft and Their Stabilization", International Journal of Systems Science, Vol. 16, pp. 535-551, 1985.

[26] S.K. Biswas and N.U. Ahmed, "Stabilization of A Class of Hybrid System Arising in Flexible Spacecraft", Journal of Optimization Theory and Applications, Vol. 50, No. 15, pp.83-103, 1986.

[27] S.K. Biswas and N.U. Ahmed, "Optimal Control of Large Space Structures Governed by a Coupled System of Ordinary and Partial Differential Equations", Mathematics of Control, Signal and Systems, Vol.2, 1989, pp.1-18.

[28] C.S. Bodley and A.C. Park, "The Influence of Structural Flexibility on the Dynamic Response in Spinning Spacecraft", AIAA Paper 72-348, San Antonio, Texas, 1972.

[29] B.G. Bomford, Geodesy, 1st Ed., p.333, The Clarendon Press, N.Y., 1952.

[30] D.P. Brown, "Stability of a Spinning Body Containing an Elastic Membrane via Liapunov's Direct Method", AIAA Jounal, Vol.10, No.10, 1972, pp.1286-1290.

[31] G.R. Buis, "Lyapunov Stability for Partial Differential Equations: Part I, Lyapunov Stability Theorem and Stability of Solution to Partial Differential Equations", NASA CR-1100, NASA Langley, 1968.

[32] R.K. Cavin III and A.R. Dusto, "Hamilton's Principle: Finite Element Methods and Flexible Body Dynamics", AIAA Journal, Vol. 15, pp.1684-1690, 1977.

[33] G. Chen,"Energy decay and Exact Boundary Value Controllability for the Wave Equation in a Bounded Domain", J. of Math. Pures Appl., Vol. 58, 1979, pp. 249-273.

[34] G. Chen, "A Note on the Boundary Stabilization of the Wave Equation", SIAM J. Control and Optimization, Vol. 25, No. 6, Nov. 1987, pp.1417-1429.

[35] G. Chen, M.C. Delfour, A.M. Krall and G. Payre, "Modeling, Stabilization and Control of serially Connected Beams", SIAM J. Control and Optimization, Vol. 25, 1987, pp. 527-546.

[36] M.K. Craig, "Definition Status of the U.S. Space Station", Acta Astronautica, Vol.16, 1987-C, pp.27-32.

[37] R.R. Craig, Jr., Structural Dynamics: An Introduction to Computer Methods, John Wiley & Sons Inc., New York, 1981.

[38] C.M. Dafermos and M. Slemrod, "Asymptotic Behavior of Nonlinear Contraction Semigroups", J. of Functional Analysis, Vol.12, 1973, pp.97-106.

[39] J.M. Danby, Fundamentals of Celestial Mechanics, The Macmillan Company, New York, 1962.

[40] R. Datko, "An Extention of a Theorem of A. M. Lyapunov to Semigroups of Operators", J. of math. analy. & appls., Vol.24, 1968, pp.290-295.

[41] W.N. Dong, "Stability of a Spinning Satellite with Flexible Antennas", AIAA Journal, Vol.12, No.12, 1974, pp.1737-1739.

[42] J.O. Dow, Z. W. Su and C.C. Feng, "Equivalent Continuum representation of Structures Composed of Repeated Elements", AIAA Jounal, Vol.23, No.10, 1985, pp.1564-1569.

[43] M.A. Frik, "Attitude stability of Satellite Subject to Gravity Gradient and Aerodynamic Torques", AIAA J., Vol.8, No.10, 1970, pp.1780-1785.

[44] A.H. Gale and P.W. Linkins, "Influence of Flexible Appendages on Dual-Spin Spacecraft Dynamics and Control", Journal of Spacecraft and Rockets, Vol. 7, pp. 1049-1056, 1970.

[45] J.S. Gibson, "Convergence and Stability in Linear Modal Regulation of Flexible Structures", Proc. AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, Virginia, 1979.

[46] C.J. Goh and T.K. Caughey, "On the Stability problem Caused by Finite Actuator Dynamics in the Collocated Control of Large Space Structures", Int. J. of Control, Vol.41, No.3, 1985, pp.787-802.

[47] H. Goldstein, Classical Mechanics, Reading, Addison-Wesley, 2nd ed., 1981.

[48] A.L. Greensite, Analysis and Design of Space Vehicle Flight Control Systems , Spartan Books, New York, 1970.

[49] P.B. Grote, J.C. McMunn and R. Gluck, "Equation of Motion of Flexible Spacecraft", Journal of Spacecraft and Rockets, Vol. 8, pp. 561-567, 1971.

[50] J. Hale, Asymptotic Behavior of Dissipative Systems, American Mathematical Society, providence, Rhode Island, 1988.

[51] D. Halliday and R. Resnick, Fundamentals of Physics, 2nd Ed., John Wiley & Sons, New York, 1981.

[52] W. Heiskanen and V. Mainesz, The Earth and Its Gravitational Field, McGraw-Hill, N.Y., 1958.

[53] S.L. Hendricks, S. Rajaram, M.P. Kamat and J.L. Junkins, "Identification of Large Flexible Structures Mass/Stiffness and Damping from On-Orbit Experiments", Proc. of the Workshop on Application of Distributed System Theory to the Control of Lareg Space Structures, JPL, Pasadena, CA, July 14-16, 1982, pp.511-520.

[54] T.J. Hertz and E.F. Crawley, "Displacement Dependent Friction in Space Structural Joints", AIAA Journal, Vol.23, No.12, 1985, pp.1998-2000.

[55] J.D. Hodge and W.P. Raney, "The Space Station Overview", Acta Astronautica, Vol.16, 1987-E, pp.55-62.

[56] W.W. Hooker and G. Margulies, "The Dynamical Attitude Equations for an n-Body Satellite", The Journal of the Astronautical Sciences, Vol. 12, No. 4, Winter 1965, pp.123-128.

[57] R.M. Hord, CRC Handbook of Space Technology: Status and Projections, CRC Press, Boca Raton, FL, 1985.

[58] S.A. Hossain, "System Identification of Flexible Structures Using Distributed Parameter Models", Ph.D. dissertation, Department of Electrical Engineering, Univ. of Houston, Houston, Tx, May 1986.

[59] J.M. Housner, "Structural Dynamic Model and Response of the Deployable Reference Configuration Space Station", NASA TM-86386, May 1985.

[60] P.C. Hughes, "Space Structure Vibration Modes: How Many Exist? Which Ones Are Important?", IEEE Control System Magazine, Feb. 1987, pp.22-28.

[61] P. Hughes and R. Skelton, "Controllability and Observability for Flexible Spacecraft", Journal of Guidance and Control, Vol. 3, pp. 452-459, 1980.

[62] W.C. Hurty, Dynamics of Structures, Englewood Cliffs, NJ, Prentice-Hall, 1964.

[63] A.M. Ibrahim and V.J. Modi, "A Formulation for Studying Dynamics of N Connected Flexible Deployable Members", Acta Astronautica, Vol.16, 1987, pp.151-164.

[64] C.H.C. Ih, D.S. Bayard and S.J. Wang, "Space Station Adaptive payload articulation Control", Proc. of the 4th IFAC Symposium, Los Angeles, California, 1986, pp.253-261.

[65] M.K. Jain, Numerical Solutions of Differential equations, John Wiley & Sons, New York, 1979.

[66] S.M. Joshi, "A Modal Model for SCOLE Structural Dynamics", First SCOLE Workshop, NASA Langley Research Center, Hampton, VA., Dec. 4-7, 1984.

[67] J. Juang and M.J. Balas, "Dynamics and Control of Large Spinning Spacecraft, Vol. 28, pp. 31-48, 1980.

[68] J.N. Juang and C.T. Sun, "System Identification of Large Flexible Structures by Using Simple Continuum Models", The J. of the Astronautical Sciences, Vol.31, No.1, 1983, pp.77-98.

[69] Y.P. Kakad, "Combined Problem of Slew Maneuver Control and Vibration Suppression", NASA TM-101503, Oct. 1988, pp. 309-320.

[70] Y.P. Kakad, "Dynamics of Spacecraft Control Laboratory Experiment (SCOLE) Slew Maneuvers", NASA CR 4098, Oct. 1987.

[71] R.E. Kalman and J.E. Bertram, "Control System Analysis and Design via the Second Method of Lyapunov", Trans. of the ASME, Journal of Basic Engineering, June 1960, pp.371-393.

[72] J.E. Keat, "Dynamical Equations of Nonrigid satellite", AIAA Journal, Vol.8, No.7, 1970, pp.1344-1345.

[73] J.U. Kim and Y. Renardy, "Boundary Control of the Timoshenko Beam", SIAM J. Control and Optimization, Vol. 25, 1987, pp.1417-1429.

[74] N.F. Knight Jr., "Nonlinear Structural Dynamic Analysis Using a modified Modal method", AIAA Journal, Vol.23, No.10, 1985, pp.1594-1601.

[75] A.N. Kounadis, "Bending Eigenfrequencies of a Two-Bar Frame including the Effect of Axial Inertia", AIAA Journal, Vol.23, No.12, 1985, pp.2000-2002.

[76] A. Kutzer, "The European COLUMBUS Program", IEEE Aerospace and Electronic Systems Magazine, Vol. 2, No.1, 1987, pp.2-12.

[77] J. Lagnese,"Decay of Solutions of Wave Equations in a bounded Region with Boundary Dissipation", J. of Differential Equations, Vol. 50, 1983, pp.163-182.

[78] J. Lagnese, "Boundary Stabilization of Linear Elastodynamic Systems", SIAM J. Control and Optimization, Vol. 21, 1983, pp.968-984.

[79] J.P. Lasalle and S. Lefschetz, Stability by Lyapunov's Direct Method with Applications, Academic Press, New York, London, 1961.

[80] I. Lasiecka and R. Triggiani, "Exponential Uniform Stabilization of the Wave Equation with $L_2(0,\infty;L_2(\Gamma))$ Boundary Feedback Acting in the Dirichlet Boundary Conditions", Proc. of the 24th IEEE Conference on Decision and Control, Fort Lauderdale, FL., 1985.

[81] K.Y. Lee, "Thecnique for the Identification of Distributed Systems Using Finite Element Approximation", Advances in Control and Dynamic Systems, C.T. Leondes (ed.), Vol.24, Academic Press, New York, 1986.

[82] S.S. Lim and N.U. Ahmed, "Mathematical Modeling and Stabilization of Large Flexible Spacecraft Subject to Orbital Perturbation", Mathematical and Computer Modelling, (to appear).

[83] S.S. Lim and N.U. Ahmed, "Stabilizing Controls for Flexible Spacecraft under the Influence of Orbital Perturbation", Int. J. of Science and Technology, Vol.1, No.2, 1988, pp.5-20.

[84] S.S. Lim and N.U. Ahmed, "Towards Modeling and Control of Large Space Stations", IEEE Trans. on Automatic Control, (submitted).

[85] S.S. Lim and N.U. Ahmed, "Vibration Control of Slew Maneuvers for Large Spacecraft", Canadian Conf. on Electrical and Computer Engineering, Montreal, Quebec, Sept.17-20, pp. 650-653, 1989.

[86] S.S.Lim and N.U. Ahmed, "Modeling and Stabilization of Large Flexible Space Stations", Presented at NASA Workshop on Computational Aspects in the Control of Flexible Systems, Williamsburg, VA, July12-14, 1988.

[87] J.G. Lin, "Control Design Challenges of Large Space Systems ans SCOLE", NASA CR-178392, Oct. 1987.

[88] J.G. Lin, "Rapid Torque-Limited Line-of-Sight Pointing of Spacecraft Control Laboratory Experiment Configuration", AIAA/AAS Astrodynamics Conference, Williamsburg, VA, Aug. 1986, AIAA Paper 86-1991, pp. 106-114.

[89] W. Littman, L. Markus and Y.C. You, A Note on Stabilization and Controllability of a Hybrid Elastic System with Boundary Control, Univ. of Minnesota, Math. report #87-103, 1987.

[90] F. Longhurst, J. Graf, G. Bolton, J. Majus and W. Wienss, "The Columbus System - Objectives and Design", Acta Astronautica, Vol.16, 1987-C, pp.33-45.

[91] J.H. McElroy, (Ed.), Space Sciences and Applications, IEEE Press, New York, 1986.

[92] L. Meirovitch, (Ed.), Proc. of AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, Virginia, 1981.

[93] L. Meirovitch, Analytical Methods in vibrations, McGraw-Hill, New York, 1967.

[94] L. Meirovitch, "Liapunov stability Analysis of Hybrid Dynamical Systems in the Neighborhood of Nontrivial Equilibrium", AIAA Journal, Vol.12, No.7, 1974, pp.889-898.

[95] L. Meirovitch, "Liapunov Stability of Hybrid Dynamical Systems with Multi-Elastic Domains", Int. J. of Nonlinear Mechanics, Vol.7, 1972, pp.425-443.

[96] L. Meirovitch, "Stability of a Spinning Body Containing Elastic Parts via Lyapunov's Direct method", AIAA Journal, Vol. 8, pp. 1193-1200, 1970.

[97] L. Meirovitch, "A Method of Lyapunov Stability Analysis of Force Free Dynamical Systems", AIAA Journal, Vol. 9, pp. 1695-1701,1971.

[98] L. Meirovitch, "A Comparative Study of Stability Methods for Flexible Satellites", AIAA Journal, Vol.11, No.1, 1973, pp.91-98.

[99] J.L. Meriam, Dynamics, 2nd Ed., John Wiley & Sons Inc., New York, 1975.

[100] V.J. Modi, "Attitude Dynamics of Satellites with Flexible Appendages: A Brief Review", J. of Spacecraft, Vol.11, 1974, pp.743-751.

[101] NASA, Engineering and Configurations of Space Stations and Platforms, NASA Lyndon B. Johnson Space Center, Systems Engineering and Integration, Space Station Program Office, Houston, Texas, 1985.

[102] NASA, Outlook for Space, NASA Report-386, Jan. 1976.

[103] P.B. Noll, J. Zvara and J.J. Deyst, "Effects of Structural Flexiblity on Spacecraft Control Systems", NASA SP-8016, April 1969.

[104] N.R.C., Canada and the Space Station: Gateway to the 21st Century, National Research Council of Canada, Ottawa, Canada, Aug. 1988.

[105] G.S. Nurre, "Effects of Aerodynamic Torque on an Asymmetric Gravity- Stabilized Satellite", J. of Spacecraft and Rockets, Vol.5, No.9, 1968, pp.1046-1050.

[106] G.K. O'Neill and B. O'Leary (Eds.), Space-Based Manufacturing from Nonterrestrial Materials, Progress in Astronautics and Aeronautics, Vol.57, AIAA, 1977.

[107] L.A. Pars, A Treatise on Analytical Dynamics, Heinemann, London, 1965.

[108] S.V. Parter (Ed.), Numerical methods for Partial Differential Equations, Academic Press, New York, 1979.

[109] R.H. Parvin, Principles of Guided Missile Design: Inertial Navigation, D. Van Nostrand, New York, 1963.

[110] A. Pazy, "The Lyapunov method for semigroups of Nonlinear Contraction in Banach Spaces", J. D'alnalyse Mathematique, Vol.40, 1981, pp.239-262.

[111] J.P. Quinn and D.L. Russell, "Asymptotic Stability and Energy Decay Rates for Solutions of hyperbolic Equations with Boundary Damping", Proc. of Royal Society of Edinburgh: section A, Vol. 77, 1977, pp.97-127.

[112] S. Rajaram and J.L. Junkins, "Identification of Vibrating Flexible Structures", AIAA J. of Guidance, Control and Dynamics, Vol. 87, 1985, pp.463-470.

[113] J.D. Renton, "The Beam-like Behavior of Space Trusses", AIAA Journal, Vol.22, No.2, 1984, pp. 273-280.

[114] R.E. Roberson, "Attitude Control of Satellite and Space Vehicles", in Advances in Space Sciences, Vol. 2, Academic Press, New York, 1960.

[115] R.E. Robertson and J. Wittenburg, "A Dynamical Formalism for an Arbitrary Number of Interconnected Rigid Bodies, with Referenced to the Problem of Satellite Attitude Control", Proc. of the Third Congress of the International Federation of Automatic Control, Butterworth, London, 1967.

[116] D.L. Russell, "Mathematical Models for the Elastic Beams and Their Control-Theoretic Implications", Semigroups, Theory and Applications: Part II, Vol .152, Longman Scientific and Technical, New York, 1986, pp.177-216.

[117] J.C. Samin and P.Y. Willems, "On the Attitude dynamics Spinning Deformable Systems", Vol.13, No.6, 1975, pp.812-817.

[118] P. Santini, "Stability of Flexible Spacecrafts", Acta Astronautica, Vol. 3, PP.685-713, 1976.

[119] S.N. Singh, "Controller Design for Asymptotic Stability of Flexible Spacecraft", Proc. IEEE, Conf on Decision and Control, Sandiego, California, pp. 961-966, 1981.

[120] M.R. Spiegel, Mathematical Handbook of Formulas and Tables, Schaum's Outline Series in Mathematics, McGraw-Hill, New York, 1968.

[121] B. Sridhar, J.-N. Aubrun and K.R. Lorell, "Identification Experiment for Control of Flexible Structures", IEEE Control Systems Magazine, Vol.5, 1985, pp.29-35.

[122] C.T. Sun, B.J. Kim and J.A. Bogdanoff, "On the Derivation of Equivalent Simple Models for Beam- and Plate-like Structures in Dynamic Analysis", AIAA 22nd Structures, Structural Dynamics and Material Conference, Atlanta, Georgia, 1981, pp.523-532.

[123] C. H. Swick, Pendulum Gravity Measurements and Isostatic Reductions, Special Pub. 232, p.61, U.S. Coast and Geodetic Survey, Washington, 1942.

[124] L. Taylor,"SCOLE Workshop 1984", Proc. of a workshop for NASA-IEEE Design Challenge, NASA Langley Research Center, Hampton, Virginia, Dec., 1984.

[125] L. Taylor,"4th Annual SCOLE Workshop, NASA TM-101503, NASA Langley Research Center, Hampton, Virginia, Oct., 1988.

[126] L.W. Taylor and A.V. Balakrishnan, "A Mathematical Problem and Spacecraft Control Laboratory experiment (SCOLE) Used to Evaluate Control Laws for

Flexible spacecraft", First SCOLE Workshop, NASA Langley research Center, Hampton, VA, Dec. 4-7, 1984.

[127] L. Taylor nd K. Latimer, "Nonlinearities in Spacecraft Structural Dynamics", NASA TM 101503, Oct. 1988, pp.67-99.

[128] W.T. Thomson, "Passive Attitude Control of Satellite Vehicles", in Guidance and Control of Aerospace Vehicles, Leondes, Ed., New York, 1963.

[129] W.T. Thomson, Introduction to Space Dynamics, John Wieley, New York, 1964.

[130] S. Timoshenko, D.H. Young, W. Weaver Jr., Vibration Problems in Engineering, 4th ed., John Wiley & Sons, New York, 1974.

[131] R.W. Traill-Nash and A.R. Collar, "The effects of Shear Flexibility and Rotary Inertia on the Bending Vibrations of Beams", J. Mech. Appl. Math., Vol. 6, 1953, pp. 186-222.

[132] K. Tsuchiya and H. Saito, "Dynamics of a Spin-Stabilized Satellite Having Flexible Appendages", AIAA Journal, Vol.12, No.4, 1974, pp.490-495.

[133] P.K.C. Wang, E.C. Hong and J.S. Sarina, "Control of Antenna-Feed Attitude and Reflector Vibrations in Large Spaceborne Antenna by Mechanical Decoupling and Movable Dampers", Jet Propulsion Laboratory Report, California Institute of Technology, Pasadena, California, 1983.

[134] E.Y. Yu,"Long Term Coupling Effects between Librational and Orbital Motions of a Satellite", AIAA Journal, Vol.2, pp.54.53-4.555,1964 .

[135] V.I. Zubov, Methods of A. M. Lyapunov and Their Application, Noordhoff, Groningen, Netherlands, 1964.

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