On ℓ-open and ℓ-closed C^*-algebras and the Construction of C^*-diagonals in C^*-algebras

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Abstract

This thesis broadly focuses on certain lifting problems related to the stability of relations and the existence of a particular rich abelian subalgebra of an inductive limit C^*-algebra of 1-dimensional noncommutative CW-complexes. We consider the newly introduced notions of ℓ-open and ℓ-closed C^*-algebras by Blackadar. These C^*-algebras derive their definitions and properties from the space of *-homomorphisms from the algebra to another C^*-algebra, equipped with the point-norm topology. We characterize ℓ-open and ℓ-closed C^*-algebras and use these characterizations to resolve some questions posed by Blackadar. Additionally, we explore the relationships of these notions with other C^*-algebraic concepts, such as extension theory and the homotopy lifting property, which is the dual concept of the classical homotopy extension property.

A 1-dimensional noncommutative CW-complex exemplifies an ℓ-open C^*-algebra which serves as a building block for many important examples of stably finite classifiable C^*-algebras. A C^*-diagonal is an abelian C^*-subalgebra with the unique extension property and certain regularity condition. This study investigates the unique extension property of an abelian C^*-subalgebra within a 1-dimensional NCCW complex, along with the limitations and implications of approximating *-homomorphisms between two such complexes. Furthermore, leveraging Leonel's classification result, which extends beyond simple C^*-algebras, we establish the existence of C^*-diagonals in the inductive limits of some 1-dimensional NCCW complexes.

Dedications

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Preface

C^*-algebras are self-adjoint subalgebras of the bounded operators on a complex Hilbert space closed under the operator norm topology. From its origins in the mathematical foundations of quantum mechanics, the study of C^*-algebras is an active research area that has created a bridge between functional analysis and areas such as geometric group theory, dynamical systems, mathematical physics, and algebraic topology. C^*-algebras provide the framework for noncommutative topology through the Gelfand-Naimark theorem [53, Theorem 2.1.10], which shows that every abelian C^*-algebra arises as the algebra $C(X)$ of continuous functions on a compact Hausdorff space X (which can be realized as a family of bounded multiplication operators on $L^2(X)$). Through the viewpoint of Gelfand duality, we can formulate algebraic and analytic properties of $C(X)$ that are equivalent to topological properties of X, and develop abstract noncommutative versions of these properties for arbitrary C^*-algebras.

One particularly prominent example is the success of operator algebraic K-theory [53]; the non-commutative extension of Atiyah–Hirzebruch's topological K-theory [7]. The application of the theory in the Elliott classification program, which aims to classify nuclear C^*-algebras using K-theoretic invariants and traces, stands as a monumental breakthrough in the field which still generates a lot of interest [37, 33]. The use of the theory extends into theoretical physics, offering a rigorous framework for analyzing quantum systems in quantum field theory and statistical mechanics [9].

Topological shape theory provides tools for understanding the global properties of a space by gleaning information from its local properties. One way to do this is through approximating a space by means of more tractable spaces, such as Borsuk's absolute neighborhood retracts (ANRs). ANRs are certain tractable spaces with nice homological and topological properties, such as the homotopy extension property, used, for example, by Milnor and Spanier, to prove certain fiber homotopy-type results [52]. The noncommutative analog of ANR is given by *semiprojectivity* for C^*-algebras. This notion is fundamental in studying C^*-algebraic stability problems (if a property holds approximately, to what extent can it be made to hold exactly nearby?) [50] and plays a crucial role in the classification program of amenable C^*-algebras and the study of noncommutative dynamical systems [37, 55].

The Borsuk Homotopy Extension Theorem is a foundational result in algebraic topology for ANRs that ensures the extension of maps and homotopies from a subspace to a larger space. In the course of extending the theorem to the noncommutative case, Blackadar[15] introduced two new classes called ℓ-open and ℓ-closed C^*-algebras in terms of the properties of the space of *-homomorphism $\mathrm{Hom}(\cdot,\cdot)$ equipped with the point-norm topology. The point-norm topology on $\mathrm{Hom}(\cdot,\cdot)$ is considered natural due to its extensive application in various contexts, including approximation properties such as exactness, nuclearity, and quasidiagonality. Moreover, it is widely utilized in the classification of C^*-algebras.

In Chapter 2, we explore in greater detail the recently introduced C^*-algebras known as ℓ-open and ℓ-closed C^*-algebras, along with the homotopy lifting property. We provide various characterizations, permanence properties, and examples, that answers some questions raised by Blackadar. In particular, we prove:

Theorem A (see Theorem 2.2.16)**.** *Let A be a C^*-algebra. The following are equivalent:*

(i) *A is ℓ-open.*

(ii) *For every C^*-algebra B and ideal $I \subseteq B$, the natural map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/I)$ is open.*

(iii) *A satisfies the Homotopy Lifting Theorem (a noncommutative analog of the Borsuk Homotopy Extension Theorem), and $\mathrm{Hom}(A, B)$ is locally path-connected for every C^*-algebra B.*

Theorem B (see Theorem 2.3.1)**.** *Let A be a separable C^*-algebra. Then A is ℓ-closed if and only if for every C^*-algebra B and ideal $I \subseteq B$, the natural map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/I)$ is uniformly relatively open.*

These characterizations help us establish relationships between ℓ-open and ℓ-closed C^*-algebras and confirm a conjecture of Blackadar. We also prove that for a unital commutative C^*-algebra, semiprojectivity coincides with ℓ-openness, confirming another conjecture from [15, Page 299]. Additionally, we explore the connection between ℓ-closed C^*-algebras, quasidiagonality, and the Ext-groups.

Notable examples of semiprojective C^*-algebras, and thus ℓ-open C^*-algebras, include finite-dimensional C^*-algebras and 1-dimensional noncommutative CW-complexes (NCCW-complexes), which are a generalization of 1-dimensional CW-complexes to the noncommutative setting. The second focus of this thesis is to investigate the existence of commutative C^*-algebras within an inductive limit of 1-dimensional NCCW complexes that exhibit characteristics similar to the subalgebra of diagonal matrices in a matrix algebra. Renault and Kumjian[43, 62] introduced the concept of C^*-diagonals within C^*-algebras, following the foundational work of Feldman and Moore [36] on Cartan subalgebras of von Neumann algebras. They showed that C^*-algebras with C^*-diagonals are realizable as C^*-algebras of some well-behaved topological groupoid.

A significant amount of work has been devoted to identifying C^*-diagonals within C^*-algebras. For instance, An Huef et al. [2] proved that the unique extension property of an abelian C^*-subalgebra of a Fell algebra, and in particular C^*-algebras with continuous trace, is both a necessary and sufficient condition for the subalgebra to be a C^*-diagonal. Additionally, Li [45] showed that every classifiable simple C^*-algebra contains a C^*-diagonal subalgebra. Furthermore, Li and Raad [46] constructed C^*-diagonals in AH-algebras.

In Chapter 3, we begin by explaining why the techniques employed by Li and Raad [46] cannot be directly applied to the inductive limits of 1-dimensional NCCW complexes. Our approach is similar to the method used to construct the C^*-diagonal of AI-algebras [61], but considerable effort is required to write an inductive limit of 1-dimensional NCCW complexes such that the building blocks contain C^*-diagonals, and the connecting maps are suitably compatible with them. We impose no simplicity restriction on the C^*-algebras under consideration. We rely on Robert's classification results [63], which utilize a functor based on the Cuntz semigroup, and techniques of Li [45] to establish:

Theorem C. Every unital inductive limit of 1-dimensional NCCW complexes with trivial K_1-group and unital injective connecting maps has a C^*-diagonal.

Chapter 1

Background

1.1 C^*-algebra

This section briefly outlines some basic notions, notations, and constructions of C^*-algebras in the literature [13, 53] that will be useful throughout this thesis.

Definition 1.1.1. A C^*-algebra $\mathcal{D}$ is a Banach algebra over $\mathbb{C}$ endowed with an involution $w \to w^*$ satisfying $\|w^*w\| = \|w\|^2$ for every $w \in \mathcal{D}$.

We say $\mathcal{D}$ is a *unital* C^*-algebra if it has a multiplicative identity element. It is *σ-unital* if it contains an increasing sequence $(w_n)_{n=1}^{\infty}$ of positive elements in the closed unit ball of $\mathcal{D}$ such that $w = \lim_{n\to\infty} w_n w = \lim_{n\to\infty} w w_n$ for all $w \in \mathcal{D}$. An element $w \in \mathcal{D}$ is (i) *self-adjoint* if $w^* = w$, (ii) a *projection* if $w = w^2 = w^*$, (iii) *normal* if $w^*w = ww^*$, (iv) *unitary* if $ww^* = w^*w = 1$. Throughout this thesis, we assume the ideals are closed, two-sided ideals, and $\mathcal{U}(\mathcal{D})$ is the set of unitaries of $\mathcal{D}$ unless otherwise stated.

Some classical examples of C^*-algebras are the algebra $B(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ and the space of continuous functions on a locally compact Hausdorff space Δ, $C_0(\Delta)$. The involution map for $B(\mathcal{H})$ is adjoint $T \mapsto T^*$, and the involution map for $C_0(\Delta)$ is the map $f \mapsto \overline{f}$. These two examples are fundamental to the study of C^*-algebras since we can realize every abstract C^*-algebra as a norm-closed *-subalgebra of $B(\mathcal{H})$ via the GNS construction [53, Theorem 3.4.1] and every commutative C^*-algebra as $C_0(\Delta)$ for some locally compact Hausdorff space Δ via the Gelfand duality [53, Theorem 2.1.10].

Definition 1.1.2. Let $\mathcal{D}$ and $\mathcal{C}$ be C^*-algebras. A linear map $E : \mathcal{D} \to \mathcal{C}$ is:

(i) a projection if $\mathcal{C}$ is a C^*-subalgebra of $\mathcal{D}$ and $E(w) = w$ for all $w \in \mathcal{C}$.

(ii) a completely positive map (c.p.) if E is positive ($E(c) \geq 0$ whenever $c \geq 0$) and the matrix amplification $\mathrm{id} \otimes E : M_n(\mathcal{D}) \to M_n(\mathcal{C})$ is positive for all $n \in \mathbb{N}$.

(iii) a conditional expectation if it is a contractive and completely positive projection satisfying $E(w_1 z w_2) = w_1 E(z) w_2$ for all $w_1, w_2 \in \mathcal{C}$ and $z \in \mathcal{D}$.

The following theorem establishes a necessary and sufficient condition for a projection $E : \mathcal{D} \to \mathcal{C}$ to be a conditional expectation.

Theorem 1.1.3 ([21, Theorem 1.5.10])**.** Let $\mathcal{C} \subseteq \mathcal{D}$ be C^*-algebras, and $E : \mathcal{D} \to \mathcal{C}$ be a projection. Then, the following are equivalent:

(i) E is a conditional expectation.

(ii) E is a contractive and completely positive map.

(iii) E is contractive.

Constructing New C^*-algebras from Algebras

1.1.0.1 Enveloping C^*-algebra

Consider a pair $(\mathcal{D}, p)$, where $\mathcal{D}$ is a *-algebra and $p : \mathcal{D} \to \mathbb{R}$ is a C^*-seminorm[1]. Then, $I = p^{-1}\{0\}$ is a self-adjoint ideal of $\mathcal{D}$ and $\|w + I\| = p(w)$ is a C^*-norm on the quotient *-algebra $\mathcal{D}/I$. The enveloping C^*-algebra $\mathcal{B}$ of $(\mathcal{D}, p)$ is the Banach space completion of $\mathcal{D}/I$ with respect to the norm $\| \cdot \|$. The canonical map from $\mathcal{D}$ to $\mathcal{B}$ is the map $i : \mathcal{D} \to \mathcal{B}$ that maps w to $w + I$.

1.1.0.2 Multiplier Algebra

Let $\mathcal{D}$ be a C^*-algebra and $M(\mathcal{D})$ be the set of double centralizers of $\mathcal{D}$, that is, a pair (L, R) of bounded linear maps on $\mathcal{D}$ such that

$$L(wz) = L(w)z, \quad R(wz) = wR(z), \quad \text{and} \quad R(w)z = wL(z)$$

for all $w, z \in \mathcal{D}$. For double centralizers (L_1, R_1) and (L_2, R_2) of $\mathcal{D}$,

(i) $\|L_1\| = \|R_1\|$ and $\|(L_1, R_1)\| = \|L_1\| = \|R_1\|$ is a norm on $M(\mathcal{D})$,

(ii) $(L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1)$ is a product operation on $M(\mathcal{D})$,

(iii) The operation $c_1(L_1, R_1) + c_2(L_2, R_2) = (c_1 L_1 + c_2 L_2, c_1 R_1 + c_2 R_2)$ makes $M(\mathcal{D})$ a vector space, where $c_1, c_2 \in \mathbb{C}$,

(iv) For a linear map $L : \mathcal{D} \to \mathcal{D}$, define $L^* : \mathcal{D} \to \mathcal{D}$ by $L^*(w) = (L(w^*))^*$. If (L_1, R_1) is a double centralizer, then $(L_1, R_1)^* = (R_1^*, L_1^*)$ is a double centralizer and $(L_1, R_1) \mapsto (R_1^*, L_1^*)$ is an involution on $M(\mathcal{D})$.

[1] p is a C^*-seminorm means it is a seminorm on $\mathcal{D}$ satisfying $p(wz) \leq p(w)p(z)$, $p(w^*) = p(w)$, and $p(w^*w) = p(w)^2$ for all $w, z \in \mathcal{D}$

$M(\mathcal{D})$ is a unital C^*-algebra under the algebraic operations, involution, and norm defined above. $\mathcal{D}$ can be identified as an ideal of $M(\mathcal{D})$ via an isometric *-homomorphism that maps $w \in \mathcal{D}$ to $(L_w, R_w) \in M(\mathcal{D})$, where L_w and R_w are linear maps on $\mathcal{D}$ defined by $L_w(z) = wz$ and $R_w(z) = zw$, respectively. Note that $\mathcal{D} = M(\mathcal{D})$ if and only if $\mathcal{D}$ is unital.

Theorem 1.1.4 ([13, Theorem II.7.3.1])**.** $M(\mathcal{D})$ has the universal property that whenever $\mathcal{D}$ is an ideal in a C^*-algebra $\mathcal{B}$, the identity map on $\mathcal{D}$ can be uniquely extended to a *-homomorphism from $\mathcal{B}$ to $M(\mathcal{D})$.

1.1.0.3 Direct Sums and Direct Products

Let $\{\mathcal{D}_i : i \in \Omega\}$ be an infinite set of C^*-algebras. Denote by

$$\prod_{i\in\Omega} \mathcal{D}_i := \{(w_i) : w_i \in \mathcal{D}_i \text{ and } \|(w_i)\| := \sup_{i\in\Omega} \|w_i\| < \infty\}$$

and by

$$\bigoplus_{i\in\Omega} \mathcal{D}_i := \{(w_i) : w_i \in \mathcal{D}_i \text{ and for any } \delta > 0,\ \|w_i\| \geq \delta \text{ for finitely many } i\}.$$

Then $\prod_{i\in\Omega} \mathcal{D}_i$ is called the direct product of $\mathcal{D}_i$ and it is a C^*-algebra under the entry-wise algebraic operations and the norm $\|(w_i)\| = \sup_{i\in\Omega} \|w_i\|$. Additionally, $\bigoplus_{i\in\Omega} \mathcal{D}_i$ is called the direct sum of $\mathcal{D}_i$ and it is an ideal of $\prod_{i\in\Omega} \mathcal{D}_i$.

1.1.0.4 Inductive Limits of C^*-algebras

An inductive sequence $(\mathcal{D}_n, \phi_n)$ of C^*-algebras is a sequence $(\mathcal{D}_n)_{n=1}^\infty$ of C^*-algebras and *-homomorphisms $\phi_n : \mathcal{D}_n \to \mathcal{D}_{n+1}$.

Let $\mathcal{D}'$ be the set of $w = (w_k) \in \prod \mathcal{D}_k$ having the property that there exists $N \in \mathbb{N}$ such that $w_{k+1} = \phi_k(w_k)$ for all $k \geq N$. Then, $\mathcal{D}'$ is a *-subalgebra of $\prod \mathcal{D}_k$ and $p(w) = \lim_{k\to\infty} \|w_k\|$ defines a C^*-seminorm since each ϕ_n is norm-decreasing. The inductive limit of the sequence $(\mathcal{D}_n, \phi_n)$, represented by $\varinjlim(\mathcal{D}_n, \phi_n)$ or simply $\varinjlim \mathcal{D}_n$ if there is no ambiguity, is the enveloping C^*-algebra of $(\mathcal{D}', p)$. If $i : \mathcal{D}' \to \varinjlim \mathcal{D}_n$ is the canonical map, then the map

$$\phi_{n,\infty} : \mathcal{D}_n \to \varinjlim \mathcal{D}_n, \quad w \mapsto i(0, 0, \ldots, 0, w, \phi_{n,n+1}(w), \phi_{n,n+2}(w), \ldots)$$

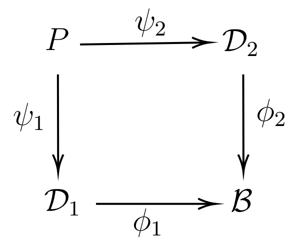
is a natural *-homomorphism from $\mathcal{D}_n$ to $\varinjlim \mathcal{D}_n$. Consequently, we can consider $\varinjlim \mathcal{D}_n$ as the closure of the increasing sequence of its C^*-subalgebras $(\phi_{n,\infty}(\mathcal{D}_n))$.

Example 1.1.5. (i) Let $(\mathcal{D}_n)_{n=1}^\infty$ be an increasing sequence of C^*-subalgebras of a C^*-algebra $\mathcal{C}$ and $\mathcal{D} = \overline{\bigcup_{n=1}^\infty \mathcal{D}_n}$. For each n, let $i_n : \mathcal{D}_n \to \mathcal{D}_{n+1}$ be the inclusion map. Then $\mathcal{D} = \varinjlim(\mathcal{D}_n, i_n)$.

(ii) The C^*-algebra $\mathcal{K}$ of compact operators on a separable, infinite-dimensional Hilbert space $\mathcal{H}$ is isomorphic to the inductive limit $\varinjlim(\mathcal{D}_n, \phi_n)$, where $\mathcal{D}_n = M_n(\mathbb{C})$ and $\phi_n : \mathcal{D}_n \to \mathcal{D}_{n+1}$ maps $w \in \mathcal{D}_n$ to $\begin{bmatrix} w & 0 \\ 0 & 0 \end{bmatrix}$ for each $n \in \mathbb{N}$.

1.1.0.5 Pullbacks of C^*-algebras

Let $\mathcal{D}_1, \mathcal{D}_2, \mathcal{B}$ be C^*-algebras, and $\phi_i : \mathcal{D}_i \to \mathcal{B}$ be *-homomorphisms, $i = 1, 2$. A C^*-algebra P is called a pullback of $\mathcal{D}_1$ and $\mathcal{D}_2$ along ϕ_1 and ϕ_2 if there exist *-homomorphisms $\psi_i : P \to \mathcal{D}_i$ that make the following diagram commute



and satisfy the following universal property: given any C^*-algebra $\mathcal{C}$ and *-homomorphisms $\delta_i : \mathcal{C} \to \mathcal{D}_i$ with $\varphi_1 \circ \delta_1 = \varphi_2 \circ \delta_2$, then there is a unique *-homomorphism $\theta : \mathcal{C} \to P$ such that $\delta_i = \psi_i \circ \theta$.

P is unique up to isomorphism. Concretely, we can consider

$$P = \{(a_1, a_2) \in \mathcal{D}_1 \oplus \mathcal{D}_2 : \phi_1(a_1) = \phi_2(a_2)\}$$

and $\psi_i : P \to D_i$ as projections of P onto its i-th components [57, Page 247].

1.1.0.6 Tensor Products of C^*-algebras

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be C^*-algebras. Let $\mathcal{D}_1 \odot \mathcal{D}_2$ denote the algebraic tensor product of $\mathcal{D}_1$ and $\mathcal{D}_2$, and let $\|\cdot\|_\alpha$ be a C^**-norm*, that is, a norm on $\mathcal{D}_1 \odot \mathcal{D}_2$ satisfying $\|c^*\|_\alpha = \|c\|_\alpha$, $\|c \otimes d\|_\alpha \leq \|c\|_\alpha \|d\|_\alpha$, and $\|c^*c\|_\alpha = \|c\|_\alpha^2$ for all $c, d \in \mathcal{D}_1 \odot \mathcal{D}_2$. The completion of $\mathcal{D}_1 \odot \mathcal{D}_2$ with respect to $\|\cdot\|_\alpha$ is a C^*-algebra, which has $\mathcal{D}_1 \odot \mathcal{D}_2$ as a dense *-subalgebra. We represent this completion by $\mathcal{D}_1 \otimes_\alpha \mathcal{D}_2$. A C^*-norm on the algebraic tensor product of C^*-algebras always exists and may not be unique [53, Page 190]. $\mathcal{D}_1$ is called a *nuclear* C^**-algebra* if for every C^*-algebra $\mathcal{D}_2$, there is a unique C^*-norm on $\mathcal{D}_1 \odot \mathcal{D}_2$.

1.1.0.7 Discrete Cross Product C^*-algebras

Let H be a discrete group, $\mathcal{D}$ be a C^*-algebra, and $Aut(\mathcal{D})$ be the group of automorphisms of $\mathcal{D}$. An *action* of H on $\mathcal{D}$ is a group homomorphism $\alpha : H \to Aut(\mathcal{D})$ defined by $h \mapsto \alpha_h$. A C^*-dynamical system is the triple $(\mathcal{D}, H, \alpha)$.

For a C^*-dynamical system $(\mathcal{D}, H, \alpha)$, let $C_c(H, \mathcal{D})$ be the set of functions $a : H \to \mathcal{D}$ such that $a(h) \neq 0$ for finitely many $h \in H$. $C_c(H, \mathcal{D})$ is a *-algebra under the following operations: For $a, b \in C_c(H, \mathcal{D})$,

$$(a^*)(t) = \alpha_t(a(t^{-1}))^* \quad \text{and} \quad (ab)(t) = \sum_{h \in H} a(h)\alpha_h(b(h^{-1}t)).$$

Let $\mathcal{D}$ be a unital C^*-algebra, and w_h be given by $w_h(t) := \begin{cases} 0 & \text{if } t \neq h, \\ 1 & \text{if } t = h. \end{cases}$ Then $w_h \in C_c(H, \mathcal{D})$ and every $a \in C_c(H, \mathcal{D})$ can be written as $a = \sum_{h \in H} a_h w_h$, where only finitely many a_h are non-zero. Moreover, $w_h a w_h^* = \alpha_h(a)$ for all $h \in H$ and $a \in \mathcal{D}$. This relation allows us to realize the operations above in a natural way. For $a = \sum_{h \in H} a_h w_h$ and $b = \sum_{t \in H} a_t w_t$,

$$a^* = \sum_{h \in H} w_h^* a_h^* = \sum_{h \in H} \alpha_{h^{-1}}(a_h^*) w_{h^{-1}} = \sum_{h \in H} \alpha_h(a_{h^{-1}}^*) w_h.$$

$$ab = \sum_{h,t \in H} a_h w_h a_t w_t = \sum_{h,t \in H} a_h \alpha_h(a_t) w_{ht} = \sum_{h \in H} \left(\sum_{t \in H} a_h \alpha_h(a_{t^{-1}h}) \right) w_h.$$

Definition 1.1.6. Let $(\mathcal{D}, H, \alpha)$ be a C^*-dynamical system, $u : H \to B(\mathcal{H})$ a unitary representation[2] of $\mathcal{H}$, and $\pi : \mathcal{D} \to B(\mathcal{H})$ a representation of $\mathcal{D}$. $(\pi, u, \mathcal{H})$ is a covariant representation of $(\mathcal{D}, H, \alpha)$ if $u(t)\pi(a)u(t)^* = \pi(\alpha_t(a))$ for every $t \in H$ and $a \in \mathcal{D}$.

Any covariant representation $(\pi, u, \mathcal{H})$ induces a representation $\pi \times u : C_c(H, \mathcal{D}) \to \mathcal{H}$ defined by $(\pi \times u)(a) = (\pi \times u) \left(\sum_{h \in H} a_h w_h \right) = \sum_{h \in H} \pi(a_h) u(h)$. The norm $\|\cdot\| = \sup\{\|\pi \times u(\cdot)\|\}$ is a *full C^*-norm* on $C_c(H, \mathcal{D})$, where the supremum ranges over all covariant representations of $(\mathcal{D}, H, \alpha)$.

Definition 1.1.7. The full C^*-crossed product $\mathcal{D} \rtimes H$ is the completion of $C_c(H, \mathcal{D})$ with respect to the full C^*-norm.

Let $\pi : \mathcal{D} \to B(\mathcal{H})$ be any faithful representation of $\mathcal{D}$ on a Hilbert space $\mathcal{H}$ and $\mathbf{H} = l_2(H, \mathcal{H})$. $(\widehat{\pi}, \lambda, \mathbf{H})$ defined below is a covariant representation of $(\mathcal{D}, H, \alpha)$:

$$\widehat{\pi}(a)\delta_{t,\xi} = \delta_{t,\pi(\alpha_t^*(a))\xi}, \quad \lambda(s)\delta_{t,\xi} = \delta_{st,\xi}, \quad a \in \mathcal{D},\ s, t \in H,\ \xi \in \mathcal{H},$$

where the Kronecker delta $\delta_{t,\xi} \in \mathbf{H}$ maps $s \in H$ to $\delta_{t,s}\xi \in \mathcal{H}$. The representation $\widehat{\pi} \times \lambda : C_c(H, \mathcal{D}) \to B(\mathbf{H})$ induced by $(\widehat{\pi}, \lambda, \mathbf{H})$ is called a *regular representation* of $(\mathcal{D}, H, \alpha)$ associated with π. The *reduced C^*-norm* on $C_c(H, \mathcal{D})$ is given by

$$\| \cdot \|_r = \|\widehat{\pi} \times \lambda(\cdot)\|,$$

where π is any faithful representation of $\mathcal{D}$. $\widehat{\pi} \times \lambda$ is a faithful representation, and $\| \cdot \|_r$ is independent of the chosen faithful representation π [21, Theorem 4.1.5].

[2]A continuous homomorphism such that $u(h)$ is a unitary operator on $\mathcal{H}$ for each $h \in H$.

Definition 1.1.8. The reduced C^*-crossed product $\mathcal{D} \rtimes_r H$ is the completion of $C_c(H, \mathcal{D})$ with respect to the reduced C^*-norm.

Lemma 1.1.9 ([21])**.** Let $(\mathcal{D}, H, \alpha)$ be a C^*-dynamical system.

(i) Let $M(\mathcal{D} \rtimes_r H)$ be the multiplier algebra of $\mathcal{D} \rtimes_r H$. Then, there is an inclusion of H into $\mathcal{U}(M(\mathcal{D} \rtimes_r H))$. Moreover, we can consider w_h as an element of $\mathcal{U}(M(\mathcal{D} \rtimes_r H))$ for all $h \in H$.

(ii) The map $a \mapsto aw_e$ defines a canonical inclusion of $\mathcal{D}$ in $\mathcal{D} \rtimes_r H$.

(iii) Let $E : C_c(H, \mathcal{D}) \to \mathcal{D}$ be a projection of norm one defined by

$$E\left(\sum_{h \in H} a_h w_h\right) = a_e.$$

Then, E extends continuously to a faithful conditional expectation $E : \mathcal{D} \rtimes_r H \to \mathcal{D}$.

(iv) For each $h \in H$, the map $E_h : C_c(H, \mathcal{D}) \to \mathcal{D}$ given by $E_h(a) = a(h)$ extends to a norm-one linear map E_h of $\mathcal{D} \rtimes_r H$ onto $\mathcal{D}$. Moreover, $E_h(a) = E(aw_{h^{-1}})$ for all $a \in \mathcal{D} \rtimes_r H$ and $h \in H$.

Next, we consider the case where $\mathcal{D}$ is a commutative C^*-algebra

Definition 1.1.10. We say that a discrete group H with unit e acts on the left of a set Δ if there is a mapping $H \times \Delta \to \Delta$ defined by $(s, w) \mapsto s \cdot w$ such that:

(i) $e \cdot w = w$,

(ii) $s \cdot (t \cdot w) = (st) \cdot w$.

The pair (H, Δ) is called a *transformation group* if Δ is a locally compact Hausdorff space and the mapping $(s, w) \mapsto s \cdot w$ is continuous.

Let (H, Δ) be a transformation group. For each $s \in H$, the map $w \mapsto s \cdot w$ is a homeomorphism of Δ and thus induces an automorphism $\alpha_s : C_0(\Delta) \to C_0(\Delta)$ defined by $\alpha_s(f)(w) = f(s^{-1} \cdot w)$. Hence, $\alpha : H \to \mathrm{Aut}(C_0(\Delta))$ is a group homomorphism and $(C_0(\Delta), H, \alpha)$ is a C^*-dynamical system.

Conversely, if Δ is a locally compact Hausdorff space, H is a discrete group, and $(C_0(\Delta), H, \alpha)$ is a C^*-dynamical system, we can find a homomorphism $H \to \mathrm{Homeo}(\Delta) : s \mapsto h_s$ such that $(s \cdot w) \mapsto h_{s^{-1}}(w)$ defines a group action making (H, Δ) a transformation group [80, Proposition 2.7].

States and Pure States

Let $\mathcal{D}$ be a C^*-algebra and $S(\mathcal{D})$ the set of states of $\mathcal{D}$, that is, the set of positive linear functionals on $\mathcal{D}$ of norm 1. For $\phi \in S(\mathcal{D})$, define

$$N_\phi := \{w : \phi(w^*w) = 0\}.$$

Then, there is a GNS-representation $(\mathcal{H}_\phi, \pi_\phi)$ associated to ϕ, where $\mathcal{H}_\phi$ is a Hilbert space and the completion of $\mathcal{D}/N_\phi$ under a well-defined inner product $\langle \cdot, \cdot \rangle$.

Theorem 1.1.11 ([53, Theorem 5.1.1])**.** Let $\mathcal{D}$ be a C^*-algebra and $\phi \in S(\mathcal{D})$. Then, there exists a unique unit vector $w_\phi \in \mathcal{H}_\phi$ such that

$$\phi(w) = \langle \pi_\phi(w) w_\phi, w_\phi \rangle$$

for all $w \in \mathcal{D}$, where π_ϕ is the GNS representation associated with ϕ and $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}_\phi$.

Definition 1.1.12. We say a state ϕ on a C^*-algebra $\mathcal{D}$ is a pure state if whenever τ is a positive linear functional on $\mathcal{D}$ satisfying $\tau \leq \phi$, there exists $c \in [0, 1]$ such that $\tau = c\phi$. We denote the set of pure states on $\mathcal{D}$ by $PS(\mathcal{D})$.

Theorem 1.1.13 ([53, Theorem 5.1.6])**.** Let $\mathcal{D}$ be a C^*-algebra and $\phi \in S(\mathcal{D})$. Then

(i) $(\mathcal{H}_\phi, \pi_\phi)$ is irreducible if and only if ϕ is a pure state.

(ii) If $\mathcal{D}$ is a commutative C^*-algebra, then ϕ is a pure state if and only if ϕ is a character of $\mathcal{D}$.

Cuntz Semigroup

Here, we recall the notion of the Cuntz semigroup which allows us to compare positive elements in C*-algebras (see [4] for more details).

Let $\mathcal{D}$ be a C*-algebra and $\mathcal{D}_+$ the set of positive elements of $\mathcal{D}$. For $a, b \in \mathcal{D}_+$, we say a is *Cuntz subequivalent* to b, denoted by $a \lesssim b$, if there exists a sequence $(x_n)_n$ in $\mathcal{D}$ such that $x_n^* b x_n \xrightarrow{n\to\infty} a$. Moreover, we say a is *Cuntz equivalent* to b, denoted by $a \sim b$, if and only if $a \lesssim b$ and $b \lesssim a$.

Cuntz equivalence $\sim$ is an equivalence relation, and it can be used to define the following set of equivalence classes:

$$\mathrm{Cu}(\mathcal{D}) := (\mathcal{D} \otimes \mathcal{K})_+ / \sim .$$

$\mathrm{Cu}(\mathcal{D})$ is called the *Cuntz semigroup* of $\mathcal{D}$, and the class of $a \in (\mathcal{D} \otimes \mathcal{K})_+$ in $\mathrm{Cu}(\mathcal{D})$ is represented by $[a]$. For any $a, b \in (\mathcal{D} \otimes \mathcal{K})_+$, the relation

$$[a] \leq [b] \text{ if and only if } a \lesssim b$$

defines a well-defined partial order on $\mathrm{Cu}(\mathcal{D})$. Additionally, $\mathrm{Cu}(\mathcal{D})$ is an abelian ordered semigroup under the binary operation

$$[a] + [b] := [a \oplus b],$$

where $a \oplus b := v_1 a v_1^* + v_2 b v_2^*$ and v_1, v_2 are isometries in $M(\mathcal{D} \otimes \mathcal{K})$ satisfying $v_1 v_1^* + v_2 v_2^* = \mathbf{I}_{M(\mathcal{D}\otimes\mathcal{K})}$. For any *-homomorphism $\phi : \mathcal{D} \to \mathcal{B}$, define $Cu(\phi) : \mathrm{Cu}(\mathcal{D}) \to \mathrm{Cu}(\mathcal{B})$ by $Cu(\phi)[a] := [(\phi \otimes \mathrm{id}_{\mathcal{K}})(a)]$.

Definition 1.1.14. Let $(Z, \leq)$ be an ordered semigroup. We say $a \in Z$ is compactly contained in $b \in Z$, denoted $a \ll b$, if whenever $b \leq \sup_n z_n$ for some increasing sequence (z_n) with supremum in Z, there exists n_0 such that $a \leq z_{n_0}$.

Definition 1.1.15. A Cu-semigroup is a positively ordered monoid Z that satisfies the following properties:

(i) Every increasing sequence in Z has a supremum.

(ii) For every $z \in Z$, there exists a sequence $z_1 \ll z_2 \ll z_3 \ll \ldots$ in Z with supremum z.

(iii) If $z' \ll z$ and $t' \ll t$ for $z', z, t', t \in Z$, then $z' + t' \ll z + t$.

(iv) If (z_n) and (t_n) are increasing sequences in Z, then $\sup_n (z_n + t_n) = \sup_n z_n + \sup_n t_n$.

Definition 1.1.16. Let X be a topological space and Z a Cu-semigroup. A map $g : X \to Z$ is said to be lower semicontinuous if the set $\{w \in X : t \ll g(w)\}$ is open in X for any $t \in Z$. The set of lower semicontinuous maps from X to Z is denoted by $Lsc(X, Z)$.

Example 1.1.17. Let $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Then $Cu(M_n(\mathbb{C})) \cong \overline{\mathbb{N}}$ via $[a] \mapsto \mathrm{Rank}(a)$. Moreover, if $\mathcal{D} := \bigoplus_{i=1}^{l} M_{n_i}(\mathbb{C})$, then $Cu(\mathcal{D}) \cong \overline{\mathbb{N}}^l$ via $[(a_1, \ldots, a_l)] \mapsto (\mathrm{Rank}(a_1), \ldots, \mathrm{Rank}(a_l))$.

Theorem 1.1.18 ([4, Corollary 2.7])**.** Let $\mathcal{D}$ be a separable C*-algebra with stable rank one such that $K_1(I) = 0$ for every closed two-sided ideal I of $\mathcal{D}$. Then the map

$$\gamma : Cu(C([0,1], \mathcal{D})) \to Lsc([0,1], Cu(\mathcal{D}))$$

defined by $\gamma([a])(t) := [a(t)]$ is an isomorphism of Cu-semigroups.

1.2 Spaces of *-homomorphisms

Let $\mathcal{D}$ and $\mathcal{B}$ be C^*-algebras. We denote the space of *-homomorphisms from $\mathcal{D}$ to $\mathcal{B}$, endowed with the point-norm topology, by $\mathrm{Hom}(\mathcal{D}, \mathcal{B})$, and the subspace of unital *-homomorphisms by $\mathrm{Hom}_1(\mathcal{D}, \mathcal{B})$ (if $\mathcal{D}$ and $\mathcal{B}$ are unital). For $\phi \in \mathrm{Hom}(\mathcal{D}, \mathcal{B})$, a neighborhood basis of ϕ is made up of sets

$$U_{\mathcal{B}}(\phi; \mathcal{F}, \epsilon) := \{\psi \in \mathrm{Hom}(\mathcal{D}, \mathcal{B}) : \|\psi(d) - \phi(d)\| < \epsilon \text{ for all } d \in \mathcal{F}\}, \tag{1.2.1}$$

where the union ranges over all finite sets $\mathcal{F} \subset \mathcal{D}$ and all positive real numbers $\epsilon > 0$. This gives a uniform structure to $\mathrm{Hom}(\mathcal{D}, \mathcal{B})$. In fact, the sets of this neighborhood basis are parameterized independently of $\mathcal{B}$, giving a uniform structure to all of $\mathrm{Hom}(\mathcal{D}, \mathcal{B})$ at once. (One might want to put a uniform structure on the disjoint union of $\mathrm{Hom}(\mathcal{D}, \mathcal{B})$ over all C^*-algebras $\mathcal{B}$, but this is not a well-defined set. Instead, one can put a uniform structure on $\coprod_{C \in \mathcal{C}} \mathrm{Hom}(\mathcal{D}, C)$, for any set $\mathcal{C}$ of C^*-algebras.)

If $\mathcal{D}$ is a separable C^*-algebra, then $\mathrm{Hom}(\mathcal{D}, \mathcal{B})$ is metrizable. Indeed, take any countable dense subset $\{a_1, a_2, \ldots\}$ of the unit ball of $\mathcal{D}$. Then,

$$d(\tau, \sigma) = \sum_{n=1}^{\infty} \frac{\|\tau(a_n) - \sigma(a_n)\|}{2^n}$$

defines a metric on $\mathrm{Hom}(\mathcal{D}, \mathcal{B})$. A net τ_α in $\mathrm{Hom}(\mathcal{D}, \mathcal{B})$ converges to a *-homomorphism τ in the metric d if and only if $\tau_\alpha(a_n)$ converges to $\tau(a_n)$ for all n. Using the density of the set $\{a_1, a_2, \ldots\}$ and the boundedness of the net τ_α, it follows that $\tau_\alpha(a) \to \tau(a)$ for each $a \in \mathcal{D}$.

It is evident that the limit of a net of unital *-homomorphisms is unital. Moreover, it is well-known that a sufficiently close projection to the identity is the identity. Hence, $\mathrm{Hom}_1(\mathcal{D}, \mathcal{B})$ is a clopen subset of $\mathrm{Hom}(\mathcal{D}, \mathcal{B})$. Let Δ and Ω be compact metrizable spaces and Δ^Ω the set of continuous functions from Ω to Δ. Then $\mathrm{Hom}_1(C(\Delta), C(\Omega))$ is homeomorphic to Δ^Ω endowed with the topology of uniform convergence (by Gelfand duality).

Let Δ and Ω be topological spaces. Recall that a map $f : \Delta \to \Omega$ is relatively open if for any given open subset Λ of Δ, $f(\Lambda)$ is an open subset of $f(\Delta)$, that is, $f(\Lambda) = \Xi \cap f(\Delta)$ for some open subset Ξ of Ω. A map $f : \Delta \to \Omega$ is an open map if f is relatively open and $f(\Delta)$ is an open subset of Ω.

Next, we discuss some basic properties of *-homomorphisms between two C^*-algebras and the maps induced by *-homomorphisms. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ be C^*-algebras, and let $\sigma : \mathcal{D} \to \mathcal{B}$ be a *-homomorphism. Then σ induces the maps $\sigma_* : \mathrm{Hom}(\mathcal{A}, \mathcal{D}) \to \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ and $\sigma^* : \mathrm{Hom}(\mathcal{B}, \mathcal{A}) \to \mathrm{Hom}(\mathcal{D}, \mathcal{A})$ defined by $\phi \mapsto \sigma \circ \phi$ and $\psi \mapsto \psi \circ \sigma$, respectively.

Proposition 1.2.1 ([68])**.** Let $\mathcal{D}$ and $\mathcal{B}$ be C^*-algebras and $\sigma : \mathcal{D} \to \mathcal{B}$ a *-homomorphism. Then σ is relatively open.

Proof. It suffices to show that for any $a \in \mathcal{D}$ and any $\epsilon > 0$, $\sigma(B_\epsilon(a))$ is a neighborhood of $\sigma(a)$ in $\sigma(\mathcal{D})$, where $B_\epsilon(a) = \{c \in \mathcal{D} : \|a - c\| < \epsilon\}$.

σ induces a *-isomorphism from $\mathcal{D}/\ker(\sigma)$ to $\mathrm{Im}(\sigma)$ via the map $[a] \mapsto \sigma(a)$. Since $\mathcal{D}/\ker(\sigma)$ and $\mathrm{Im}(\sigma)$ are C^*-algebras, $\|[a]\| = \|\sigma(a)\|$ for all $a \in \mathcal{D}$. Let $\sigma(c) \in B_\delta(\sigma(a))$ for some $\delta < \epsilon$. Then

$$\|[a - c]\| = \|\sigma(a - c)\| = \|\sigma(a) - \sigma(c)\| < \delta.$$

By the definition of the quotient norm on $\mathcal{D}/\ker(\sigma)$, we can find $d \in \ker(\sigma)$ such that $\|a-(c+d)\| < \epsilon$. Hence, $\sigma(c+d) = \sigma(c)$, $\sigma(c) \in \sigma(B_\epsilon(a))$, and $\sigma(a) \in B_\delta(\sigma(a)) \subset \sigma(B_\epsilon(a))$. ∎

Proposition 1.2.2 ([68], Proposition 2.3)**.** Let $\mathcal{D}$ and $\mathcal{B}$ be C^*-algebras and $\sigma : \mathcal{D} \to \mathcal{B}$ an injective *-homomorphism. Then the induced map $\sigma_* : \mathrm{Hom}(\mathcal{A}, \mathcal{D}) \to \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ is relatively open for any C^*-algebra $\mathcal{A}$.

Proof. It suffices to show that for any $\phi \in \text{Hom}(\mathcal{A}, \mathcal{D})$, any finite set $\mathcal{F} \subset \mathcal{A}$, and $\epsilon > 0$,

$$\sigma_*(U_{\mathcal{D}}(\phi; \mathcal{F}, \epsilon)) = U_{\mathcal{B}}(\sigma \circ \phi; \mathcal{F}, \epsilon) \cap Im(\sigma_*).$$

Clearly, $\sigma_*(U_{\mathcal{D}}(\phi; \mathcal{F}, \epsilon)) \subseteq U_{\mathcal{B}}(\sigma \circ \phi; \mathcal{F}, \epsilon) \cap Im(\sigma_*)$. Conversely, let $\psi \in U_{\mathcal{B}}(\sigma \circ \phi; \mathcal{F}, \epsilon) \cap Im(\sigma_*)$. Then, $\psi = \sigma \circ \overline{\psi}$, where $\overline{\psi} \in \text{Hom}(\mathcal{A}, \mathcal{D})$ and

$$\|\psi(a) - \sigma \circ \phi(a)\| = \|\sigma \circ \overline{\psi}(a) - \sigma \circ \phi(a)\| < \epsilon$$

for all $a \in \mathcal{F}$. Injectivity of σ implies $\|\sigma(d)\| = \|d\|$ for all $d \in \mathcal{D}$. Therefore,

$$\|\overline{\psi}(a) - \phi(a)\| = \|\sigma \circ \overline{\psi}(a) - \sigma \circ \phi(a)\| < \epsilon$$

for all $a \in \mathcal{F}$ and $\psi \in \sigma_*(U_{\mathcal{D}}(\phi; \mathcal{F}, \epsilon))$. ∎

Proposition 1.2.3 ([68], Proposition 2.4)**.** Let $\mathcal{D}, \mathcal{B}$ be C^*-algebras and $\sigma : \mathcal{D} \to \mathcal{B}$ a surjective *-homomorphism. Then, the induced map $\sigma^* : \text{Hom}(\mathcal{B}, \mathcal{A}) \to \text{Hom}(\mathcal{D}, \mathcal{A})$ is relatively open for any C^*-algebra $\mathcal{A}$.

Proof. For any $\phi \in \text{Hom}(\mathcal{B}, \mathcal{A})$ and $\epsilon > 0$, we take $\mathcal{F} = \{a\}$. We will prove that

$$\sigma^*(U_{\mathcal{B}}(\phi; \mathcal{F}, \epsilon)) = \bigcup_{b \in \sigma^{-1}(a)} U_{\mathcal{D}}(\phi \circ \sigma; \{b\}, \epsilon) \cap Im(\sigma^*).$$

Let $b \in \sigma^{-1}(a)$ and $\psi \circ \sigma \in \sigma^*(U_{\mathcal{B}}(\phi; \mathcal{F}, \epsilon))$ with $\psi \in U_{\mathcal{B}}(\phi; \mathcal{F}, \epsilon)$. Then

$$\|\psi \circ \sigma(b) - \phi \circ \sigma(b)\| = \|\psi(a) - \phi(a)\| < \epsilon.$$

So,

$$\sigma^*(U_{\mathcal{B}}(\phi; \mathcal{F}, \epsilon)) \subseteq \bigcup_{b \in \sigma^{-1}(a)} U_{\mathcal{D}}(\phi \circ \sigma; \{b\}, \epsilon) \cap Im(\sigma^*).$$

Conversely, for $b \in \sigma^{-1}(a)$ and $\psi \circ \sigma \in U_{\mathcal{D}}(\phi \circ \sigma; \{b\}, \epsilon) \cap Im(\sigma^*)$,

$$\|\phi(a) - \psi(a)\| = \|\phi(\sigma(b)) - \psi(\sigma(b))\| < \epsilon.$$

Hence, $\psi \in U_{\mathcal{B}}(\phi; \mathcal{F}, \epsilon)$ and

$$\sigma^*(U_{\mathcal{B}}(\phi; \mathcal{F}, \epsilon)) = \bigcup_{b \in \sigma^{-1}(a)} U_{\mathcal{D}}(\phi \circ \sigma; \{b\}, \epsilon) \cap Im(\sigma^*),$$

since b is arbitrary. Each $U_{\mathcal{D}}(\phi \circ \sigma; \{b\}, \epsilon) \cap Im(\sigma^*)$ is relatively open in $Im(\sigma^*)$ and therefore their union is relatively open in $Im(\sigma^*)$.

∎

1.3 Extensions of C^*-algebras

Extension theory provides a way of constructing new C^*-algebras from simpler ones. Its relationship to the space of *-homomorphisms, K-theory, and the structure of C^*-algebras often reveals deep structural properties, which will be helpful in our later discussions. We use [12] as the primary source for what follows.

We say a sequence $0 \to \mathcal{B} \xrightarrow{j} \mathcal{E} \xrightarrow{q} \mathcal{D} \to 0$ of *-homomorphisms and C^*-algebras is a short exact sequence if j is injective, q is surjective, and $\ker(q) = \mathrm{Im}(j)$.

Definition 1.3.1. Let $\mathcal{D}$ and $\mathcal{B}$ be C^*-algebras. An extension of $\mathcal{D}$ by $\mathcal{B}$ is a short exact sequence $0 \to \mathcal{B} \xrightarrow{j} \mathcal{E} \xrightarrow{q} \mathcal{D} \to 0$.

Extensions of $\mathcal{D}$ by $\mathcal{B}$ always exist since we can obtain a short exact sequence by setting $\mathcal{E} = \mathcal{B} \oplus \mathcal{D}$, with j as the canonical inclusion of $\mathcal{B}$ in $\mathcal{E}$, and q as the canonical projection of $\mathcal{E}$ onto $\mathcal{D}$.

Given an extension $0 \to \mathcal{B} \xrightarrow{j} \mathcal{E} \xrightarrow{q} \mathcal{D} \to 0$, it follows from the universal property of the multiplier algebra (see Theorem 1.1.4) that there exists a unique *-homomorphism $\rho : \mathcal{E} \to M(\mathcal{B})$ such that $\rho|_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}}$. The composition of ρ with the quotient map $\pi : M(\mathcal{B}) \to M(\mathcal{B})/\mathcal{B}$ induces a *-homomorphism $\tau : \mathcal{D} \to M(\mathcal{B})/\mathcal{B}$, called the Busby invariant of the given extension. The pullback of $\mathcal{D}$ and $M(\mathcal{B})$ along τ and π induces an extension $0 \to \mathcal{B} \to \mathcal{E}(\tau) \to \mathcal{D} \to 0$. Moreover, $\mathcal{E}$ is isomorphic to $\mathcal{E}(\tau)$. Hence, we can uniquely identify every extension $0 \to \mathcal{B} \xrightarrow{j} \mathcal{E} \xrightarrow{q} \mathcal{D} \to 0$ with a *-homomorphism $\tau : \mathcal{D} \to M(\mathcal{B})/\mathcal{B}$ and vice versa [13, II. 8.4.7]. Henceforth, we will use these two notions interchangeably.

The Busby invariant τ is injective if and only if ρ is injective if and only if $\mathcal{B}$ is an essential ideal of $\mathcal{E}$ (i.e., $\mathcal{B} \cap \mathcal{I} \neq \{0\}$ for every nonzero ideal $\mathcal{I}$ of $\mathcal{E}$) [20, Page 67]. Extensions with injective Busby invariants are called essential extensions.

There are different forms of equivalence between two extensions. Here, we consider strong and weak equivalence.

Definition 1.3.2. Let $\mathcal{D}$ and $\mathcal{B}$ be C^*-algebras, and let $\tau_1, \tau_2 : \mathcal{D} \to M(\mathcal{B})/\mathcal{B}$ be extensions.

(i) τ_1 and τ_2 are strongly (unitarily) equivalent if there exists a $v \in \mathcal{U}(M(\mathcal{B}))$ such that $\tau_1(f) = v\tau_2(f)v^*$ for all $f \in \mathcal{D}$.

(ii) τ_1 and τ_2 are weakly (unitarily) equivalent if there exists a $v \in \mathcal{U}(M(\mathcal{B})/\mathcal{B})$ such that $\tau_1(f) = v\tau_2(f)v^*$ for all $f \in \mathcal{D}$.

For any C^*-algebra $\mathcal{B}$, let $\mathcal{Q}(\mathcal{B} \otimes \mathcal{K}) = M(\mathcal{B} \otimes \mathcal{K})/(\mathcal{B} \otimes \mathcal{K})$. We denote the set of strong equivalence classes of extensions $\tau : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$ by $\mathbf{Ext}(\mathcal{D}, \mathcal{B})$, and the set of weak equivalence classes of extensions $\tau : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$ by $\mathbf{Ext}_w(\mathcal{D}, \mathcal{B})$.

Identifying $M(\mathcal{B} \otimes \mathcal{K})$ with $M_2(M(\mathcal{B} \otimes \mathcal{K}))$ and $\mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$ with $M_2(\mathcal{Q}(\mathcal{B} \otimes \mathcal{K}))$ (see [41, Page 98] for details) allows us to define an addition structure on the set of equivalence classes of extensions as follows:

$$[\tau_1] + [\tau_2] = [\tau_1 \oplus \tau_2],$$

where $\tau_1 \oplus \tau_2 : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K}) \oplus \mathcal{Q}(\mathcal{B} \otimes \mathcal{K}) \subseteq M_2(\mathcal{Q}(\mathcal{B} \otimes \mathcal{K})) \cong \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$.

The set $\mathbf{Ext}(\mathcal{D}, \mathcal{B})$ (or $\mathbf{Ext}_w(\mathcal{D}, \mathcal{B})$) is a commutative semigroup under this binary operation.

Definition 1.3.3. Let $\mathcal{D}$ and $\mathcal{B}$ be C^*-algebras. An extension $\tau : \mathcal{D} \to M(\mathcal{B})/\mathcal{B}$ is called a trivial extension if there is a *-homomorphism $\sigma : \mathcal{D} \to M(\mathcal{B})$ such that $\tau = \pi \circ \sigma$, where $\pi : M(\mathcal{B}) \to M(\mathcal{B})/\mathcal{B}$ is the quotient map.

In general, trivial extensions always exist. The zero *-homomorphism $\tau : \mathcal{D} \to M(\mathcal{B})/\mathcal{B}$ is an example of a trivial extension. If $\mathcal{B}$ is unital, then this is the only trivial extension since $M(\mathcal{B})/\mathcal{B} = 0$. However, if $\mathcal{B}$ is non-unital, $M(\mathcal{B})$ is large enough to allow the possibility of many other trivial extensions.

Let S be the set of strong equivalence classes of trivial extensions $\tau : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$. Then S is a subsemigroup of $\mathbf{Ext}(\mathcal{D}, \mathcal{B})$.

Definition 1.3.4. We define $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$ as the quotient semigroup of $\mathbf{Ext}(\mathcal{D}, \mathcal{B})$ by S. For an extension $\tau : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$, we denote the corresponding equivalence class in $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$ by $[[\tau]]$.

If $\tau, \sigma : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$ are extensions, then $[[\tau]] = [[\sigma]]$ if and only if there exist trivial extensions $\tau_1, \sigma_1 : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$ such that $\tau \oplus \tau_1$ is strongly equivalent to $\sigma \oplus \sigma_1$. We can similarly define $\mathrm{Ext}_w(\mathcal{D}, \mathcal{B})$ as above. Since $\mathrm{Ext}(\mathcal{D}, \mathcal{B}) = \mathrm{Ext}_w(\mathcal{D}, \mathcal{B})$ [12, Proposition 15.6.4], we simply refer to $[[\tau]]$ as a stable equivalence class in $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$. We define $\mathrm{Ext}_e(\mathcal{D}, \mathcal{B})$ as the quotient semigroup of the subsemigroup of essential extensions by the subsemigroup of essential trivial extensions. If $\mathcal{D}$ is a separable C^*-algebra, then $\mathrm{Ext}_e(\mathcal{D}, \mathcal{B}) = \mathrm{Ext}(\mathcal{D}, \mathcal{B})$ [12, Proposition 15.6.5], since there exists an essential trivial extension $\psi : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$ [13, II.8.4.7], and for any extension $\phi : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$, we have $[[\phi]] = [[\phi \oplus \psi]]$.

$\mathrm{Ext}(\mathcal{D}, \mathcal{B})$ has a neutral element by construction. An element $[[\tau]] \in \mathrm{Ext}(\mathcal{D}, \mathcal{B})$ is invertible if and only if τ is a semi-split extension, that is, there exists a completely positive contraction $\overline{\tau} : \mathcal{D} \to M(\mathcal{B} \otimes \mathcal{K})$ such that $\tau = \pi \circ \overline{\tau}$ [12, Theorem 15.7.1]. The set of stable equivalence classes of semi-split extensions forms a group, denoted by $\mathrm{ext}(\mathcal{D}, \mathcal{B})$.

One can refer to the work of Arveson [6] to provide a large class of C^*-algebras for which every nonzero extension is invertible.

Theorem 1.3.5 ([12, Corollary 15.8.4])**.** If $\mathcal{D}$ is a separable nuclear C^*-algebra and $\mathcal{B}$ is any C^*-algebra, then $\mathrm{Ext}(\mathcal{D}, \mathcal{B}) = \mathrm{ext}(\mathcal{D}, \mathcal{B})$ is a group.

In some cases, it is possible to identify the quotient semigroup $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$ with a simplified semigroup. Before discussing such an identification for a separable C^*-algebra $\mathcal{D}$ and a σ-unital C^*-algebra $\mathcal{B}$, we recall the following notion of an extension.

Definition 1.3.6. Let $\mathcal{D}$ and $\mathcal{B}$ be C^*-algebras. An extension $\tau : \mathcal{D} \to M(\mathcal{B})/\mathcal{B}$ is called an absorbing extension if for every trivial extension $\sigma : \mathcal{D} \to M(\mathcal{B})/\mathcal{B}$, τ and $\tau \oplus \sigma$ are strongly equivalent.

We denote the set of absorbing extensions $\tau : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$ by $E_a(\mathcal{D}, \mathcal{B})$ and the set of strong equivalence classes of absorbing extensions $\tau : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$ by $\mathbf{Ext}_a(\mathcal{D}, \mathcal{B})$. Note that an absorbing extension $\tau \in \mathbf{Ext}_a(\mathcal{D}, \mathcal{B})$ cannot be a unital extension, and $\sigma \oplus \tau \in \mathbf{Ext}_a(\mathcal{D}, \mathcal{B})$ for any extension σ of $\mathcal{D}$ over $\mathcal{B} \otimes \mathcal{K}$.

A sufficient condition for $\mathbf{Ext}_a(\mathcal{D}, \mathcal{B}) \cong \mathrm{Ext}(\mathcal{D}, \mathcal{B})$ is the existence of an absorbing trivial extension $\tau : \mathcal{D} \to \mathcal{Q}(\mathcal{B})$ [12, Proposition 15.12.2]. Voiculescu [79, Theorem 1.3] proved that if $\mathcal{D}$ is separable, every nonunital essential extension $\tau \in \mathrm{Ext}(\mathcal{D}, \mathbb{C})$ is absorbing, and thus $\mathbf{Ext}_a(\mathcal{D}, \mathbb{C}) \cong \mathrm{Ext}(\mathcal{D}, \mathbb{C})$. Kasparov partly generalized Voiculescu's result to prove the following:

Theorem 1.3.7 ([12, Theorem 15.12.4])**.** Let $\mathcal{D}$ be a separable C^*-algebra and $\mathcal{B}$ a σ-unital C^*-algebra. Assume that either $\mathcal{D}$ or $\mathcal{B}$ is a nuclear C^*-algebra. Then any essential trivial extension $\tau \in \mathrm{Ext}(\mathcal{D}, \mathcal{B})$ regarded as an element of $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$ is an absorbing extension in $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$. Moreover, $\mathbf{Ext}_a(\mathcal{D}, \mathcal{B}) \cong \mathrm{Ext}(\mathcal{D}, \mathcal{B})$ via the map $[\sigma] \mapsto [[\sigma]]$ with an inverse map $[[\sigma]] \mapsto [\sigma \oplus \tau]$, where τ is an absorbing trivial extension.

More generally, Thomsen proved the following result:

Theorem 1.3.8 ([77], Theorem 2.7)**.** Let $\mathcal{D}$ and $\mathcal{B}$ be separable C^*-algebras. Then there exists an absorbing trivial extension of $\mathcal{D}$ by $\mathcal{B} \otimes \mathcal{K}$.

Let $\mathcal{D}$ be a separable C^*-algebra, $\mathcal{B}$ be a C^*-algebra, and τ, σ be *-homomorphisms from $\mathcal{D}$ to $\mathcal{B}$. Recall the metric $d(\tau, \sigma) = \sum_{n=1}^{\infty} \frac{\|\tau(a_n) - \sigma(a_n)\|}{2^n}$ defined on $Hom(\mathcal{D}, \mathcal{B})$, where $\{a_1, a_2, \ldots\}$ is a countable dense subset of the unital ball of $\mathcal{D}$. A natural topology on $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$ is the quotient topology induced by the point-norm topology on the space $Hom(\mathcal{D}, \mathcal{Q}(\mathcal{B} \otimes \mathcal{K}))$, which is referred to as the Brown-Salinas topology. The topology was first introduced by Brown in his work [16], and later, Brown and Ozawa described it further in [21, Section 17.4]. Salinas extended the concept to the case where B is a σ-unital C^*-algebra, as detailed in [66, Remark 3.1]. Schochet [69] showed that the topology coincides with three other topologies on $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$ when $\mathcal{D}$ is a nuclear C^*-algebra. Finally, Marius Dadarlat provided a characterization of the topology in [27, Theorem 6.2].

Using ideas from [65], we explicitly provide the details of the pseudometric that induces the Brown-Salinas topology in the following theorem.

Theorem 1.3.9. Let $\mathcal{D}$ be a separable C^*-algebra and $\mathcal{B}$ a C^*-algebra. Assume there exists an absorbing trivial extension α of $\mathcal{D}$ by $\mathcal{B} \otimes \mathcal{K}$. For a fixed metric d on $Hom(\mathcal{D}, \mathcal{Q})$, define $\widehat{d}$ on $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$ as follows:

$$\widehat{d}([[\tau]], [[\sigma]]) := \inf_{u,v} d(u(\tau \oplus \alpha)u^*, v(\sigma \oplus \alpha)v^*),$$

where u and v are unitaries in $\mathcal{Q}$ that lift to unitaries in $M(\mathcal{B} \otimes \mathcal{K})$.

Then,

(a) $\widehat{d}$ is a pseudometric[3] on $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$.

[3] A function $g : X \times X \to [0, \infty)$ is called a pseudometric on a nonempty X if (i) $g(x, x) = 0$, (ii) $g(x, y) = g(y, x)$, and (iii) $g(x, z) \leq g(x, y) + g(y, z)$ for all $x, y, z \in X$.

(b) The Brown-Salinas topology on $\mathrm{Ext}(\mathcal{D},\mathcal{B})$ coincides with the topology induced by $\widehat{d}$.

(c) A net $([[\sigma_n]])_{n\in\Lambda}$ of essential extensions converges to $[[\sigma]]$ in the Brown-Salinas topology if and only if there exist liftable unitaries u_n such that $\|u_n(\sigma_n(w)\oplus\alpha(w))u_n^* - \sigma(w)\oplus\alpha(w)\|$ converges to zero for each $w \in \mathcal{D}$.

Proof. Let u be a unitary in $\mathcal{Q}(\mathcal{B}\otimes\mathcal{K})$, and let $\sigma_i : \mathcal{D} \to \mathcal{Q}(\mathcal{B}\otimes\mathcal{K})$ be essential extensions of $\mathcal{D}$ by $\mathcal{B}\otimes\mathcal{K}$, where $i = 1, 2, 3, 4$. Then,

$$\begin{aligned} d(u\sigma_1 u^*, \sigma_2) = \sum_{n=1}^{\infty} \frac{\|u\sigma_1(a_n)u^* - \sigma_2(a_n)\|}{2^n} &= \sum_{n=1}^{\infty} \frac{\|\sigma_1(a_n) - u^*\sigma_2(a_n)u\|}{2^n} \\ &= d(\sigma_1, u^*\sigma_2 u). \end{aligned} \tag{1.3.1}$$

Consequently,

$$\begin{aligned} \widehat{d}([[\sigma_1]],[[\sigma_2]]) &= \inf_{u,v} d(u(\sigma_1\oplus\alpha)u^*, v(\sigma_2\oplus\alpha)v^*) = \inf_{u,v} d(v^*u(\sigma_1\oplus\alpha)u^*v, \sigma_2\oplus\alpha) \\ &= \inf_{w} d(w(\sigma_1\oplus\alpha)w^*, \sigma_2\oplus\alpha). \end{aligned} \tag{1.3.2}$$

Note from the definition of $\widehat{d}$ that $\widehat{d}([[\sigma_1]],[[\sigma_1]]) = 0$, $\widehat{d}([[\sigma_1]],[[\sigma_2]]) = \widehat{d}([[\sigma_2]],[[\sigma_1]])$, and

$$d(\sigma_1\oplus\sigma_3, \sigma_2\oplus\sigma_3) = \sum_{n=1}^{\infty} \frac{\|\sigma_1\oplus\sigma_3(a_n) - \sigma_2\oplus\sigma_3(a_n)\|}{2^n} = \sum_{n=1}^{\infty} \frac{\|\sigma_1(a_n) - \sigma_2(a_n)\|}{2^n} = d(\sigma_1,\sigma_2).$$

Next, we prove the triangle inequality to conclude that $\widehat{d}$ is a pseudometric.

$$\begin{aligned} \widehat{d}([[\sigma_1]],[[\sigma_2]]) &= \inf_{u,v} d(u(\sigma_1\oplus\alpha)u^*, v(\sigma_2\oplus\alpha)v^*) \\ &\le \inf_{u,v} [d(u(\sigma_1\oplus\alpha)u^*, \sigma_3\oplus\alpha) + d(\sigma_3\oplus\alpha, v(\sigma_2\oplus\alpha)v^*)] \\ &= \inf_{u} d(u(\sigma_1\oplus\alpha)u^*, \sigma_3\oplus\alpha) + \inf_{v} d(\sigma_3\oplus\alpha, v(\sigma_2\oplus\alpha)v^*) \\ &= \widehat{d}([[\sigma_1]],[[\sigma_3]]) + \widehat{d}([[\sigma_3]],[[\sigma_2]]). \end{aligned} \tag{1.3.3}$$

To prove (b), we first establish that for any open set $U \subseteq \mathrm{Hom}(\mathcal{D}, \mathcal{Q}(\mathcal{B}\otimes\mathcal{K}))$, $R(U)$ is open, where $R(U) = \{\sigma_1 : [[\sigma_1]] = [[\sigma_2]] \text{ for some } \sigma_2 \in U\}$. It is sufficient to show that $R(B_\epsilon(\sigma_1))$ is open for every $\epsilon > 0$ and extension σ_1 of $\mathcal{D}$ over $\mathcal{B}\otimes\mathcal{K}$, where $B_\epsilon(\sigma_1) = \{\sigma_2 : d(\sigma_1,\sigma_2) < \epsilon\}$.

Let $\sigma_2 \in R(B_\epsilon(\sigma_1))$, then $[[\sigma_2]] = [[\sigma_3]]$ for some extension σ_3 satisfying $d(\sigma_3,\sigma_1) < \epsilon$. So, $d(\sigma_3\oplus\alpha, \sigma_1\oplus\alpha) = d(\sigma_3,\sigma_1) < \epsilon$ and $\widehat{d}([[\sigma_3]],[[\sigma_1]]) < \epsilon$. Since $[[\sigma_2]] = [[\sigma_3]]$, $\widehat{d}([[\sigma_2]],[[\sigma_1]]) < \epsilon$. Using (1.3.2), there exists a liftable unitary u such that $d(\sigma_2\oplus\alpha, u(\sigma_1\oplus\alpha)u^*) < \epsilon$. $[u(\sigma_1\oplus\alpha)u^*] = [\sigma_1\oplus\alpha] = [\sigma_1]$ implies $u(\sigma_1\oplus\alpha)u^* = u_1\sigma_1 u_1^*$ for some liftable unitary $u_1 \in \mathcal{Q}(\mathcal{B}\otimes\mathcal{K})$. Similarly, $\sigma_2\oplus\alpha = w\sigma_2 w^*$ for some liftable unitary $w \in \mathcal{Q}(\mathcal{B}\otimes\mathcal{K})$. Hence,

$$d(\sigma_2, w^*u_1\sigma_1 u_1^* w) = d(w\sigma_2 w^*, u_1\sigma_1 u_1^*) = d(\sigma_2\oplus\alpha, u(\sigma_1\oplus\alpha)u^*) < \epsilon \tag{1.3.4}$$

Set

$$\beta = w^*u_1\sigma_1 u_1^* w. \tag{1.3.5}$$

Then $[[\beta]] = [[\sigma_1]]$ and

$$d(\sigma_2, \beta) < \epsilon. \tag{1.3.6}$$

Since $[[\beta]] = [[\sigma_1]]$, we have $R(B_\epsilon(\beta)) = R(B_\epsilon(\sigma_1))$ and

$$\sigma_2 \in B_\epsilon(\beta) \subseteq R(B_\epsilon(\beta)) = R(B_\epsilon(\sigma_1)).$$

Hence $R(B_\epsilon(\sigma_1))$ is open.

Let $\pi : Hom(\mathcal{D}, \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})) \to Ext(\mathcal{D}, \mathcal{B})$ be the quotient map. The equivalence of the two topologies on $Ext(\mathcal{D}, \mathcal{B})$ follows from the equality $B_\epsilon([[\sigma_1]]) = \pi(R(B_\epsilon(\sigma_1)))$, which we show to be true below:

Let $[[\sigma_2]] \in B_\epsilon([[\sigma_1]])$. Then $\widehat{d}([[\sigma_2]], [[\sigma_1]]) < \epsilon$. By (1.3.2), there exists a liftable unitary u such that $d(u(\sigma_2 \oplus \alpha)u^*, \sigma_1 \oplus \alpha) < \epsilon$. Choose an extension β using the same procedure used to obtain (1.3.5). Then $d(\beta, \sigma_1) < \epsilon$ and $[[\beta]] = [[\sigma_2]]$. Therefore, $\beta \in R(B_\epsilon(\sigma_1))$ and $[[\sigma_2]] \in \pi(R(B_\epsilon(\sigma_1)))$. Conversely, let $[[\sigma_2]] \in R(B_\epsilon(\sigma_1))$. Then, $[[\sigma_2]] = [[\sigma_3]]$ and $d(\sigma_3, \sigma_1) < \epsilon$ for some extension σ_3. Consequently, $\widehat{d}([[\sigma_3]], [[\sigma_1]]) < \epsilon$. Thus, $\widehat{d}([[\sigma_2]], [[\sigma_1]]) < \epsilon$ and $[[\sigma_2]] \in B_\epsilon([[\sigma_1]])$.

(c) follows from the definition of infimum and the pseudometric. ∎

Quasidiagonality

Quasidiagonality is an essential concept in the theory of C^*-algebras due to its close connection with nuclearity. Many naturally occurring C^*-algebras, such as commutative C^*-algebras and C^*-algebras arising from certain group actions, are quasidiagonal. In this subsection and subsequent discussions, we are interested in the relationship between quasidiagonality and the space of $*$-homomorphisms. The material covered in this subsection is primarily drawn from [71, 66]. We assume $\mathcal{D}$ is a separable C^*-algebra and $\mathcal{B}$ is a σ-unital C^*-algebra throughout this subsection.

The notion of quasidiagonality originated with bounded linear operators on a separable Hilbert space before it was extended to C^*-algebras. Halmos [38] introduced an operator $S \in L(\mathcal{H})$ as a quasidiagonal operator if there exists an increasing sequence of finite rank projections $(Q_n)_{n=1}^\infty$ that converges to the identity in the strong operator topology (SOT) such that $\lim\limits_{n\to\infty} \|SQ_n - Q_nS\| = 0$.

Let $\mathcal{H}_\mathcal{B}$ be the Hilbert $\mathcal{B}$-module consisting of all sequences $(b_n)_{n=1}^\infty \subset \mathcal{B}$ such that $\sum_{n=1}^\infty b_n b_n^*$ converges in $\mathcal{B}$. Let $\mathcal{K}$ denote the C^*-algebra of compact operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$, and let $\mathcal{L}(\mathcal{H}_\mathcal{B})$ be the set of all bounded $\mathcal{B}$-module maps $T : \mathcal{H}_\mathcal{B} \to \mathcal{H}_\mathcal{B}$ for which there exists a $\mathcal{B}$-module map $T^* : \mathcal{H}_\mathcal{B} \to \mathcal{H}_\mathcal{B}$ satisfying $\langle Tw, z\rangle = \langle w, T^*z\rangle$ for all $w, z \in \mathcal{H}_\mathcal{B}$. Then $\mathcal{L}(\mathcal{H}_\mathcal{B})$ is a C^*-algebra that is $*$-isomorphic to $M(\mathcal{B}\otimes\mathcal{K})$ [42, Corollary 1.14]. Moreover, $\mathcal{H}_\mathcal{B}$ can be identified with $\mathcal{B}\otimes\mathcal{H}$. This identification induces an injective $*$-homomorphism $i : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_\mathcal{B})$. For each $x, y \in \mathcal{H}_\mathcal{B}$, let $\theta_{x,y}$ be defined by $\theta_{x,y}(z) = \langle z, y\rangle x$ for $z \in \mathcal{H}_\mathcal{B}$. Further, let $\mathcal{K}(\mathcal{H}_\mathcal{B})$ denote the norm closure in $\mathcal{L}(\mathcal{H}_\mathcal{B})$ of the set $\{\theta_{x,y} : x, y \in \mathcal{H}_\mathcal{B}\}$. Then $\mathcal{K}(\mathcal{H}_\mathcal{B})$ is $*$-isomorphic to $\mathcal{B} \otimes \mathcal{K}$ [66, Page 98].

Definition 1.3.10 ([26]). Let $\mathcal{H}_i$ be Hilbert $\mathcal{B}$-modules and $\gamma_i : \mathcal{D} \to L(\mathcal{H}_i)$ be representations, $i = 1, 2$. γ_1 is approximately unitarily equivalent to γ_2, denoted $\gamma_1 \approx_u \gamma_2$, if there exists a sequence of unitaries $(v_n)_{n=1}^{\infty} \subseteq \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ satisfying

(i) $\lim_{n\to\infty} \|\gamma_1(w) - v_n\gamma_2(w)v_n^*\| = 0$ for all $w \in \mathcal{D}$,

(ii) $\gamma_1(w) - v_n\gamma_2(w)v_n^* \in \mathcal{K}(\mathcal{H}_1)$ for all $w \in \mathcal{D}$ and n.

Definition 1.3.11 ([26]). A representation $\sigma : \mathcal{D} \to \mathcal{L}(\mathcal{H}_{\mathcal{B}})$ is called an absorbing representation if it is approximately unitarily equivalent to $\sigma \oplus \eta$ for any representation $\eta : \mathcal{D} \to \mathcal{L}(\mathcal{H}_{\mathcal{B}})$.

Definition 1.3.12 ([68]). Let $A \subset \mathcal{L}(\mathcal{H}_{\mathcal{B}})$ be separable. We say A is a $\mathcal{B}$-quasidiagonal set if there exists an increasing sequence of projections $(Q_n)_{n=1}^{\infty}$ in $\mathcal{B}\otimes\mathcal{K}$ such that $\|wQ_n - Q_nw\| \to 0$ for all $w \in A$ and $Q_n \to 1_{\mathcal{H}_{\mathcal{B}}}$ in the SOT as $n \to \infty$.

Definition 1.3.13 ([68]). A C^*-algebra $\mathcal{D}$ is $\mathcal{B}$-quasidiagonal if there exists an absorbing representation $\sigma : \mathcal{D} \to \mathcal{L}(\mathcal{H}_{\mathcal{B}})$ such that $\sigma(\mathcal{D}) \cap (\mathcal{B} \otimes \mathcal{K}) = \{0\}$ and $\sigma(\mathcal{D})$ is a $\mathcal{B}$-quasidiagonal set. We say $\mathcal{D}$ is quasidiagonal if $\mathcal{D}$ is a $\mathbb{C}$-quasidiagonal C^*-algebra.

We highlight some examples and properties of quasidiagonal C^*-algebras below:

Example 1.3.14 ([19]). (a) Commutative C^*-algebras and AF-algebras are quasidiagonal C^*-algebras.

(b) The direct product of quasidiagonal C^*-algebras is a quasidiagonal C^*-algebra.

(c) Subalgebras of quasidiagonal C^*-algebras are quasidiagonal.

(d) Irrational rotation C^*-algebras are quasidiagonal C^*-algebras.

Example 1.3.15 ([66, Lemma 4.3]). If $\mathcal{D}$ is a quasidiagonal C^*-algebra and $\mathcal{B}$ has an increasing sequence of projections $(Q_n)_{n=1}^{\infty}$ such that $\|wQ_n - Q_nw\| \to 0$ for all $w \in \mathcal{B}$ and $Q_n \to 1$ in the SOT as $n \to \infty$, then $\mathcal{D}$ is a $\mathcal{B}$-quasidiagonal C^*-algebra.

Definition 1.3.16. Let I be an ideal of $\mathcal{D}$. $\mathcal{D}$ is quasidiagonal relative to I if there exists an increasing sequence of projections $(Q_n)_{n=1}^{\infty}$ of I such that $\|wQ_n - Q_nw\| \to 0$ for all $w \in I$ and $Q_n \to 1$ in the SOT as $n \to \infty$.

The concept of quasidiagonality also applies to the extension of C^*-algebras in the following manner:

Definition 1.3.17. An extension $\tau : \mathcal{D} \to \mathcal{Q}(\mathcal{B} \otimes \mathcal{K})$ is a quasidiagonal extension if the pullback C^*-algebra $\mathcal{E}(\tau)$ of $\mathcal{D}$ and $M(\mathcal{B} \otimes \mathcal{K})$ along τ and π is quasidiagonal relative to $\mathcal{B} \otimes \mathcal{K}$.

Definition 1.3.18. We say that a separable nuclear C^*-algebra $\mathcal{D}$ is quasidiagonal relative to $\mathcal{B}$ if the neutral element of $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$ is quasidiagonal.

Combining Example 1.3.15 with the following result provides one way of establishing quasidiagonality relative to a nuclear C^*-algebra.

Proposition 1.3.19 ([68, Proposition 2.1]). Let $\mathcal{D}$ and $\mathcal{B}$ be separable nuclear C^*-algebras. Then $\mathcal{D}$ is a $\mathcal{B}$-quasidiagonal C^*-algebra if and only if $\mathcal{D}$ is quasidiagonal relative to $\mathcal{B}$.

Quasidiagonality and $Pext^1_{\mathbb{Z}}(G,H)$

There are some relationships between the theory of extensions of groups and the theory of extensions of C^*-algebras.

Given two groups G and H, an extension of G by H is a short exact sequence $0 \to H \to E \to G \to 0$ of groups and group homomorphisms. Another extension $0 \to H \to E_1 \to G \to 0$ is equivalent to the first if there is an isomorphism $\phi : E \to E_1$ that makes the diagram below commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\ & & \Big\| \, \text{id} & & \phi \Big\downarrow & & \Big\| \, \text{id} & & \\ 0 & \longrightarrow & H & \longrightarrow & E_1 & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

We identify $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$ as the set of equivalence classes of extensions of the abelian group G by the abelian group H and note that it has an abelian group structure.

Definition 1.3.20. An extension $0 \to H \to E \to G \to 0$ of abelian groups is called a pure extension if H is a pure subgroup of E, that is, $H \cap nE = nH$ for all positive integers n, where $nE = \{ng : g \in E\}$.

An extension equivalent to a pure extension is pure. Therefore, we can form an equivalence class of pure extensions and use $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ to represent the set of equivalence classes of pure extensions of G by H. $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ is a subgroup of $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$.

We gather some properties of $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ and $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$ that are useful for subsequent discussions (see [70] for a comprehensive review of these properties).

Theorem 1.3.21 ([71])**.** Let G, K, and H be abelian groups.

(a) $\mathrm{Ext}^1_{\mathbb{Z}}(G \oplus K, H) \cong \mathrm{Ext}^1_{\mathbb{Z}}(G,H) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(K,H)$.

(b) $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n, H) \cong H/nH$.

(c) $\mathrm{Ext}^1_{\mathbb{Z}}(G,H) = 0$ if G is a free group.

(d) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$, where $\mathrm{Hom}_{\mathbb{Z}}(G,H)$ is the set of group homomorphisms from G to H.

(e) Let $(G_n)_{n=1}^{\infty}$ be an increasing sequence of finitely generated subgroups of G with $G = \bigcup_{n=1}^{\infty} G_n$. Then there exists a natural short exact sequence

$$0 \to \mathrm{Pext}^1_{\mathbb{Z}}(G,H) \to \mathrm{Ext}^1_{\mathbb{Z}}(G,H) \to \varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(G_n,H) \to 0. \tag{1.3.7}$$

(f) Let $(G_n)_{n=1}^{\infty}$ be an increasing sequence of finitely generated subgroups of a countable abelian group G with $G = \bigcup_{n=1}^{\infty} G_n$. Then[4]

$$\mathrm{Pext}^1_{\mathbb{Z}}(G, H) \cong \varprojlim^1 \mathrm{Hom}_{\mathbb{Z}}(G_n, H). \tag{1.3.8}$$

(g) $\mathrm{Pext}^1_{\mathbb{Z}}(G, H) \cong \mathrm{Ext}^1_{\mathbb{Z}}(G, H)$ if G is a torsion-free abelian group.

(i) $\mathrm{Pext}^1_{\mathbb{Z}}(G, H) = 0$ for all H if G is a finitely generated abelian group.

Salinas [66, Remark 4.2] proved that if τ and σ are essential extensions of $\mathcal{D}$ by $\mathcal{B}\otimes\mathcal{K}$ with τ being a quasidiagonal extension and $[[\tau]] = [[\sigma]]$, then σ is also a quasidiagonal extension. Consequently, we say that $[[\tau]]$ is a quasidiagonal class in $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$ if τ is a semi-split, essential, and quasidiagonal extension. Let $\mathrm{Ext}_{QD}(\mathcal{D}, \mathcal{B})$ be the set of quasidiagonal classes in $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$.

Definition 1.3.22 ([12, Theorem 23.1.1])**.** We say a separable C^*-algebra $\mathcal{D}$ satisfies the universal coefficient theorem (UCT) if

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathcal{D}), K_*(\mathcal{B})) \to KK^*(\mathcal{D}, \mathcal{B}) \to \mathrm{Hom}(K_*(\mathcal{D}), K_*(\mathcal{B})) \to 0$$

is a short exact sequence for every separable C^*-algebra $\mathcal{B}$.

Theorem 1.3.23 ([66, Theorem 4.4], [71, Theorem 2.3])**.** Let $\mathcal{D}$ be a separable nuclear C^*-algebra and $\mathcal{B}$ a σ-unital separable C^*-algebra. Then

(i) $\mathrm{Ext}_{QD}(\mathcal{D}, \mathcal{B})$ is the closure of the neutral element in $\mathrm{Ext}(\mathcal{D}, \mathcal{B})$, that is, the stable equivalence classes of trivial extensions. Consequently, $\mathrm{Ext}_{QD}(\mathcal{D}, \mathcal{B})$ is non-empty if and only if $\mathcal{D}$ is quasidiagonal relative to $\mathcal{B}$.

(ii) Assume $\mathcal{D}$ additionally satisfies the UCT and is quasidiagonal relative to $\mathcal{B}$. Then,

$$\mathrm{Ext}_{QD}(\mathcal{D}, \mathcal{B}) \cong \mathrm{Pext}^1_{\mathbb{Z}}(K_0(\mathcal{D}), K_0(\mathcal{B})) \oplus \mathrm{Pext}^1_{\mathbb{Z}}(K_1(\mathcal{D}), K_1(\mathcal{B}))$$

as a topological group.

(iii) Assume $\mathcal{D}$ is also a quasidiagonal C^*-algebra satisfying the UCT. Then,

$$\mathrm{Ext}_{QD}(\mathcal{D}, \mathbb{C}) \cong \mathrm{Pext}^1_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{Z}).$$

[4]Let (A_j, f_{jk}) be an inverse system of abelian groups, $\varprojlim^1 A_j = \mathrm{Cokernel}(\Psi)$, where $\Psi : \prod_j A_j \to \prod_j A_j$ is defined by $\Psi(a_j) = (a_j - f_{j+1}(a_{j+1}))$.

Unital Extensions by $\mathcal{K}$

Assume $\mathcal{D}$ is a unital separable C^*-algebra, $\mathcal{B} = \mathbb{C}$, and $\mathcal{Q} := B(\mathcal{H})/\mathcal{K}$ the Calkin algebra. We denote the set of weak equivalence classes of essential unital extensions $\tau : \mathcal{D} \to \mathcal{Q}$ by $\mathbf{Ext}_w(\mathcal{D})$ and the set of strong equivalence classes of essential unital extensions $\tau : \mathcal{D} \to \mathcal{Q}$ by $\mathbf{Ext}_s(\mathcal{D})$. A unital injective *-homomorphism $\tau : \mathcal{D} \to \mathcal{Q}$ is a strongly trivial essential extension if there exists an injective *-homomorphism $\overline{\tau} : \mathcal{D} \to B(\mathcal{H})$ such that $\tau = \pi \circ \overline{\tau}$.

The remarkable result of Voiculescu [79] implies that all trivial extensions of $\mathcal{D}$ by $\mathcal{K}$ are strongly equivalent, and the strong (resp. weak) equivalence class of strongly trivial extensions in $\mathbf{Ext}_s(\mathcal{D})$ (resp. $\mathbf{Ext}_w(\mathcal{D})$) is the neutral element for the semigroup. If $\mathcal{D}$ is additionally a nuclear C^*-algebra, it follows from the work of Arveson that $\mathbf{Ext}_w(\mathcal{D})$ (resp. $\mathbf{Ext}_s(\mathcal{D})$) is a group [6, Theorem 8].

Remark 1.3.24. (a) For a unital commutative C^*-algebra $\mathcal{D} := C(\Delta)$, we simply write $\mathbf{Ext}_w(\mathcal{D})$ as $\mathbf{Ext}_w(\Delta)$ and $\mathbf{Ext}_s(\mathcal{D})$ as $\mathbf{Ext}_s(\Delta)$.

(b) In general, $\mathbf{Ext}_w(\mathcal{D})$ does not coincide with $\mathbf{Ext}_s(\mathcal{D})$. For example, $\mathbf{Ext}_w(M_n(\mathbb{C})) = 0$ and $\mathbf{Ext}_s(M_n(\mathbb{C})) = \mathbb{Z}_n$ [12, Example 15.6.6]. However, for a unital commutative C^*-algebra $C(\Delta)$, $\mathbf{Ext}_w(\Delta) = \mathbf{Ext}_s(\Delta)$ [18, Theorem 4.3].

(c) $\mathrm{Ext}(\mathcal{D}, \mathbb{C}) \cong \mathbf{Ext}_w(\mathcal{D})$ [12, Theorem 15.14.2].

1.4 (Weakly) Semiprojective C^*-algebra

In this section, we use [11, 13, 49, 50] as references. We first discuss constructing C^*-algebras from a set of generators satisfying some relations and then the interplay between perturbation of relations and lifting of *-homomorphisms.

Relations and Stability

We assume $\mathrm{G} := \{w_i : i \in \Omega\}$ is a set of variables (to be thought of as generators of a C^*-algebra) and $\mathcal{R}$ is a set of relations of the form

$$\|p(w_{i_1}, w_{i_2}, \ldots, w_{i_n}, w_{i_1}^*, w_{i_2}^*, \ldots, w_{i_n}^*)\| \leq \gamma,$$

where p is a polynomial in $2n$ noncommuting variables with complex coefficients and γ is a non-negative real number.

Definition 1.4.1. By a representation of $(\mathrm{G}|\mathcal{R})$, we mean a set $\{U_i : i \in \Omega\}$ of bounded operators on a Hilbert space $\mathcal{H}$ which satisfy the relations $\mathcal{R}$.

Let $\mathcal{F}(\mathrm{G})$ be the free *-algebra generated by G. A representation of $(\mathrm{G}|\mathcal{R})$ uniquely defines a *-homomorphism from $\mathcal{F}(\mathrm{G})$ to $\mathcal{L}(\mathcal{H})$. Define

$$\|w\| = \sup\{\|\chi(w)\| : \chi \text{ is a representation of } (\mathrm{G}|\mathcal{R})\}$$

for any $w \in \mathcal{F}(\mathrm{G})$. If $\|w\| < \infty$ for all $w \in \mathrm{G}$, then $\|w\| < \infty$ for all $w \in \mathcal{F}(\mathrm{G})$ and the function $\|\cdot\|$ is a seminorm on $\mathcal{F}(\mathrm{G})$.

Definition 1.4.2. The universal C^*-algebra $C^*(\mathrm{G}|\mathcal{R})$ on $(\mathrm{G}|\mathcal{R})$ is the completion of the quotient $\mathcal{F}(\mathrm{G})/\mathcal{I}$ under $\|\cdot\|$, where $\mathcal{I} = \{w \in \mathcal{F}(\mathrm{G}) : \|w\| = 0\}$.

Generally, every C^*-algebra is realizable as a universal C^*-algebra [11, Example 1.3(a) & (b)]. Moreover, any representation of $(\mathrm{G}|\mathcal{R})$ can be uniquely extended to a representation of $C^*(\mathrm{G}|\mathcal{R})$, and the image of G under any representation of $C^*(\mathrm{G}|\mathcal{R})$ is a representation of $(\mathrm{G}|\mathcal{R})$. Here, we recall the universal C^*-algebra formulation of some C^*-algebras of great interest in the later discussions.

Example 1.4.3. (a) If $\mathrm{G} = \{w\}$ and

$$\mathcal{R} = \{w = w^*, \|w\| \leq 1, \|1 - w^2\| \leq 1\},$$

then $C^*(\mathrm{G}|\mathcal{R}) \cong C_0((0,1])$. To see this, first note that the identity function $\mathrm{id} : C_0((0,1]) \to C_0((0,1])$ satisfies the relation $\mathcal{R}$ and generates $C_0((0,1])$. So, there exists a *-homomorphism $\phi : C^*(\mathrm{G}|\mathcal{R}) \to C_0((0,1])$ determined by $w \mapsto \mathrm{id}$. By construction, $C^*(\mathrm{G}|\mathcal{R})$ is a commutative C^*-algebra. Let $\chi : C^*(\mathrm{G}|\mathcal{R}) \to \mathbb{C}$ be an irreducible representation of $C^*(\mathrm{G}|\mathcal{R})$. Then $\chi(w)$ satisfies the relation $\mathcal{R}$ and thus lies in $(0,1]$. Since χ is arbitrary and uniquely determined by $\chi(w)$, the inclusion of the spectrum of $C^*(\mathrm{G}|\mathcal{R})$ in $(0,1]$ induces a surjective *-homomorphism from $C_0((0,1])$ to $C^*(\mathrm{G}|\mathcal{R})$, which acts as an inverse of ϕ.

(b) If $\mathrm{G} = \{w, 1\}$ and

$$\mathcal{R} = \{w = w^*, \|w\| \leq 1, \|1 - w^2\| \leq 1, 1^* = 1 = 1^2, 1w = w1 = w\},$$

then $C^*(\mathrm{G}|\mathcal{R}) \cong C([0,1])$. Using a similar argument as in (a), we obtain an isomorphism via $w \mapsto \mathrm{id}$ and $1 \mapsto \mathbb{1}$, where $\mathbb{1}$ is the constant function on $[0,1]$ with range 1. This example highlights how we can find the unitization of a universal C^*-algebra by expanding its set of generators and relations with the identity element and associated relations.

(c) If $\mathrm{G} = \{w, 1\}$ and

$$\mathcal{R} = \{1 = 1^* = 1^2, 1w = w1 = w, ww^* = w^*w = 1\},$$

then $C^*(\mathrm{G}|\mathcal{R}) \cong C(S^1)$ via a *-isomorphism that identifies w with the identity function $\mathrm{id} : S^1 \to S^1$ and 1 with the constant function on $C(S^1)$ with range 1. Hence, $C(S^1)$ is the universal C^*-algebra generated by a unitary.

(d) If $\mathrm{G} = \{w, 1\}$ and

$$\mathcal{R} = \{1 = 1^* = 1^2, 1w = w1 = w, ww^* = w^*w\},$$

then $C^*(\mathrm{G}|\mathcal{R}) \cong C(\mathbb{D})$ via a *-isomorphism that identifies w with the identity function $\mathrm{id} : \mathbb{D} \to \mathbb{D}$ and 1 with the constant function on $\mathbb{D}$ with range 1. Hence, $C(\mathbb{D})$ is the universal C^*-algebra generated by an identity element and a normal element.

(e) If $\mathrm{G} = \{w\}$ and

$$\mathcal{R} = \{w^*w = 1\},$$

then $C^*(\mathrm{G}|\mathcal{R}) \cong \mathcal{T}$, where $\mathcal{T}$ is the Toeplitz algebra.

-

(f) If $\mathrm{G} = \{r, w\}$, α is a real number, and

$$\mathcal{R} = \{r^*r = rr^* = ww^* = w^*w = 1,\ rw = \exp(2\pi i\alpha)wr\},$$

then $C^*(\mathrm{G}|\mathcal{R}) \cong A_\alpha$, where A_α is called the rotation algebra. We call A_α a rational rotation algebra if α is a rational number and an irrational rotation algebra otherwise.

(g) If $\mathrm{G} = \{e_{ij} : 1 \leq i, j \leq n\}$ and

$$\mathcal{R} = \{e_{ij}^* = e_{ji},\ e_{ij}e_{jk} = \delta_{jk}e_{ik} : 1 \leq i, j, k \leq n\},$$

then $C^*(\mathrm{G}|\mathcal{R}) \cong M_n(\mathbb{C})$, where δ_{ij} is 1 if $i = j$ and 0 otherwise. Indeed, let $E_{ij}, 1 \leq i, j \leq n$ be the usual matrix unit, that is, an $n \times n$ matrix whose non-zero entry is 1 at the i-th row and j-th column. Then $M_n(\mathbb{C})$ is generated by E_{ij}, and E_{ij} satisfies the relation $\mathcal{R}$. The assignment $e_{ij} \mapsto E_{ij}$ uniquely extends to a *-isomorphism $\phi : C^*(\mathrm{G}|\mathcal{R}) \to M_n(\mathbb{C})$. A set $\{h_{ij} : 1 \leq i, j \leq n, h_{ij} \neq 0\}$ in a C^*-algebra $\mathcal{D}$ is called a matrix unit if it satisfies the relation $\mathcal{R}$ above. The C^*-subalgebra of $\mathcal{D}$ generated by these matrix units is isomorphic to $M_n(\mathbb{C})$.

(i) If $\mathrm{G} = \{w_1, w_2, \ldots, w_n\}$ and

$$\mathcal{R} = \{w_i^*w_i = 1, \sum_{j=1}^{n} w_jw_j^* = 1 : 1 \leq i \leq n\},$$

then $C^*(\mathrm{G}|\mathcal{R}) \cong O_n$, the Cuntz algebra. O_n is called the universal C^*-algebra generated by n isometries with mutually orthogonal range projections.

In the sequel, we assume $\mathrm{G} = \{w_1, \ldots, w_n\}$ is a finite set and $\mathcal{R}$ has finitely many relations unless otherwise stated.

Fix $\epsilon > 0$. By changing $\|p(w_1, w_2, \ldots, w_n, w_1^*, w_2^*, \ldots, w_n^*)\| \leq \gamma$ to

$$\|p(w_1, w_2, \ldots, w_n, w_1^*, w_2^*, \ldots, w_n^*)\| \leq \gamma + \epsilon$$

for every *-polynomial p in $\mathcal{R}$, we can form a new relation $(\mathrm{G}|\mathcal{R}_\epsilon)$. The associated universal C^*-algebra is denoted by $C_\epsilon^*(\mathrm{G}|\mathcal{R})$. Consequently, there is an induced surjective *-homomorphism

$$P_\epsilon : C_\epsilon^*(\mathrm{G}|\mathcal{R}) \to C^*(\mathrm{G}|\mathcal{R})$$

which sends x_i to x_i. For $\eta < \epsilon$, we can similarly define

$$P_{\epsilon,\eta} : C_\epsilon^*(\mathrm{G}|\mathcal{R}) \to C_\eta^*(\mathrm{G}|\mathcal{R}).$$

Definition 1.4.4. Let $\mathcal{D}$ be a C^*-algebra and $\epsilon > 0$. We say that a family $\{z_1, \ldots, z_n\}$ of elements of $\mathcal{D}$ is an ϵ-representation of $\mathcal{R}$ in $\mathcal{D}$ if, whenever

$$\|p(w_1, w_2, \ldots, w_n, w_1^*, w_2^*, \ldots, w_n^*)\| \leq \gamma$$

is a relation in $\mathcal{R}$, then

$$\|p(z_1, z_2, \ldots, z_n, z_1^*, z_2^*, \ldots, z_n^*)\| \leq \gamma + \epsilon.$$

Note that the canonical generators of $C^*_\epsilon(\mathrm{G}|\mathcal{R})$ are ϵ-representations of $\mathcal{R}$ in $C^*_\epsilon(\mathrm{G}|\mathcal{R})$. The following result follows from the fact that $\{z_1, \ldots, z_n\}$ is a representation if and only if it is an ϵ_k-representation for all k.

Proposition 1.4.5. Let $\{\epsilon_k\}$ be a decreasing sequence of positive numbers that converges to zero. Then

$$C^*(\mathrm{G}|\mathcal{R}) \cong \varinjlim C^*_{\epsilon_k}(\mathrm{G}|\mathcal{R}).$$

Here, we mention some properties of relations and highlight their relationships.

Definition 1.4.6. Suppose $\mathrm{G} \coloneqq \{w_1, \ldots, w_n\}$ and $\mathcal{R}$ has finitely many relations. We say that $(\mathrm{G}|\mathcal{R})$ is liftable if, given a surjective *-homomorphism $\pi : \mathcal{D} \to \mathcal{B}$ and a representation $\{b_1, \ldots, b_n\}$ of $(\mathrm{G}|\mathcal{R})$ in $\mathcal{B}$, there is a family $\{a_1, \ldots, a_n\}$ in $\mathcal{D}$ satisfying the relation $\mathcal{R}$ with $\pi(a_i) = b_i$.

Definition 1.4.7. Suppose $\mathrm{G} \coloneqq \{w_1, \ldots, w_n\}$ and $\mathcal{R}$ has finitely many relations. We say that $(\mathrm{G}|\mathcal{R})$ is weakly stable if, for every $\epsilon > 0$, there is a $\delta > 0$ such that every δ-representation $\{z_1, z_2, \ldots, z_n\}$ of $(\mathrm{G}|\mathcal{R})$ in a C^*-algebra $\mathcal{D}$ is ϵ-close to some representation $\{c_1, c_2, \ldots, c_n\}$ of $(\mathrm{G}|\mathcal{R})$ in $\mathcal{D}$. That is, there exists a representation $\{c_1, c_2, \ldots, c_n\}$ with $\|z_i - c_i\| < \epsilon$ for all $i = 1, \ldots, n$.

Definition 1.4.8. Suppose $\mathrm{G} \coloneqq \{w_1, \ldots, w_n\}$ and $\mathcal{R}$ has finitely many relations. We say that $(\mathrm{G}|\mathcal{R})$ is stable if, for every $\epsilon > 0$, there is a $\delta > 0$ such that given a surjective *-homomorphism $\pi : \mathcal{D} \to \mathcal{B}$ and $z_1, \ldots, z_n$ in a C^*-algebra $\mathcal{D}$ for which $\{z_1, \ldots, z_n\}$ is a δ-representation of $(\mathrm{G}|\mathcal{R})$ and $\{\pi(z_1), \ldots, \pi(z_n)\}$ is a representation of $(\mathrm{G}|\mathcal{R})$, we can find a representation $\{c_1, \ldots, c_n\}$ of $(\mathrm{G}|\mathcal{R})$ in $\mathcal{D}$ satisfying

$$\|z_j - c_j\| \leq \epsilon \quad \text{and} \quad \pi(z_j) = \pi(c_j) \quad \text{for all } 1 \leq j \leq n.$$

Theorem 1.4.9. Suppose $\mathrm{G} \coloneqq \{w_1, \ldots, w_n\}$ and $\mathcal{R}$ has finitely many relations. Then Liftable relation $\Rightarrow$ Stable relation $\Rightarrow$ Weakly stable relation.

Proof. Suppose $(\mathrm{G}|\mathcal{R})$ is liftable and $\{w_1(m), \ldots, w_n(m)\}$ is a set of the canonical generators of $C^*_{\frac{1}{m}}(\mathrm{G}|\mathcal{R})$. Let

$$A = \prod_{m=1}^{\infty} C^*_{\frac{1}{m}}(\mathrm{G}|\mathcal{R}), \quad I = \bigoplus_{m=1}^{\infty} \mathrm{Ker}(P_{\frac{1}{m}}),$$

and $q : A \to A/I$ be the quotient map. Set $f_i = \Big(w_i(m)\Big)_{m=1}^{\infty}$. Consider a relation $\|p(\cdot)\| \leq \gamma$ in $\mathcal{R}$. Then,

$$\|p(w_1(m), \ldots, w_n(m), w_1^*(m), \ldots, w_n^*(m))\| \leq \frac{1}{m} + \gamma$$

and

$$\limsup_m \|p(w_1(m), \ldots, w_n(m), w_1^*(m), \ldots, w_n^*(m))\| \leq \gamma.$$

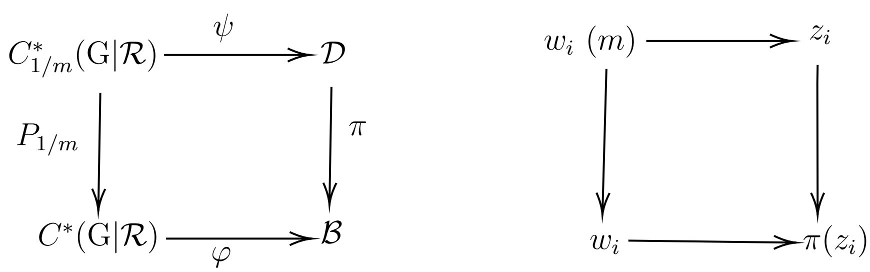
Consequently,

$$\begin{aligned} \|p\Big(q(f_1), \ldots, q(f_n), q(f_1)^*, \ldots, q(f_n)^*\Big)\| &= \|q\bigg(p\Big(w_1(m), \ldots, w_n(m), w_1^*(m), \ldots, w_n^*(m)\Big)_m\bigg)\| \\ &\leq \limsup_m \|p(w_1(m), \ldots, w_n(m), w_1^*(m), \ldots, w_n^*(m))\| \\ &\leq \gamma. \end{aligned}$$

Since p is arbitrary, $\{q(f_1), \ldots, q(f_n)\}$ is a representation of $\mathcal{R}$ in A/I. The assumption implies there exists a representation $\{a_1, a_2, \ldots, a_n\}$ of $\mathcal{R}$ in A such that $q(a_i) = q(f_i)$. Let $a_i = (a_i(m))_{m=1}^{\infty}$. Then $\{a_1(m), \ldots, a_n(m)\}$ is a representation of $\mathcal{R}$ in $C_{\frac{1}{m}}(\mathrm{G}|\mathcal{R})$ for each m,

$$P_{\frac{1}{m}}(a_i(m)) = w_i \text{ for each } m \text{ and } \lim_{m\to\infty} \|w_i(m) - a_i(m)\| = 0.$$

Fix m sufficiently large such that $\|w_i(m) - a_i(m)\| < \epsilon$. Suppose $\pi : \mathcal{D} \to \mathcal{B}$ is a surjective homomorphism and $\{z_1, \ldots, z_n\}$ is a $\frac{1}{m}$-representation with $\{\pi(z_1), \ldots, \pi(z_n)\}$ a representation. We can define a commutative diagram as follows



Let $c_i = \psi(a_i(m))$. Then, $\{c_1, c_2, \ldots, c_n\}$ is a representation in $\mathcal{D}$ and

$$\pi(c_i) = \pi \circ \psi(a_i(m)) = \varphi \circ P_{\frac{1}{m}}(a_i(m)) = \varphi(w_i) = \pi(z_i).$$

Moreover,

$$\|c_i - z_i\| = \|\psi(a_i(m)) - \psi(w_i(m))\| \leq \|a_i(m) - w_i(m)\| < \epsilon.$$

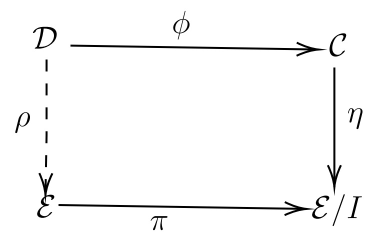
It follows that $(\mathrm{G}|\mathcal{R})$ is stable.

Suppose $(\mathrm{G}|\mathcal{R})$ is stable. For any $\epsilon > 0$, choose δ as in the definition of stability. Let $\{z_1, z_2, \ldots, z_n\}$ be a δ-representation of $\mathcal{R}$ in a C^*-algebra $\mathcal{D}$. The map from $\mathcal{D}$ to $C^*(\mathrm{G}|\mathcal{R})$ sending z_i to w_i is a surjective *-homomorphism. Stability of $(\mathrm{G}|\mathcal{R})$ implies there exists a representation $\{c_1, \ldots, c_n\}$ of $\mathcal{R}$ in $\mathcal{D}$ such that $\|z_i - c_i\| < \epsilon$, and thus $(\mathrm{G}|\mathcal{R})$ is weakly stable. ∎

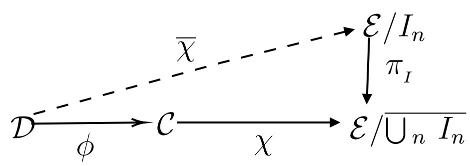
The reverse implication of the stability above does not hold. We will discuss this in the next subsection.

(Weakly) Semiprojective C^*-algebra

Definition 1.4.10. Let $\mathcal{D}$ and $\mathcal{C}$ be C^*-algebras. A *-homomorphism $\phi : \mathcal{D} \to \mathcal{C}$ is *projective* if for any C^*-algebra $\mathcal{E}$, ideal I of $\mathcal{E}$, and *-homomorphism $\eta : \mathcal{C} \to \mathcal{E}/I$, there exists a *-homomorphism $\rho : \mathcal{D} \to \mathcal{E}$ satisfying $\eta \circ \phi = \pi \circ \rho$, where $\pi : \mathcal{E} \to \mathcal{E}/I$ is the quotient map. We say $\mathcal{D}$ is *projective* if the identity map is projective.



Definition 1.4.11. A *-homomorphism $\phi : \mathcal{D} \to \mathcal{C}$ is *semiprojective* if for any C^*-algebra $\mathcal{E}$, any increasing sequence of ideals $I_1 \triangleleft I_2 \triangleleft \cdots$ in $\mathcal{E}$, and any *-homomorphism $\chi : \mathcal{C} \to \mathcal{E}/\overline{\bigcup_n I_n}$, there exists $n \in \mathbb{N}$ and a *-homomorphism $\overline{\chi} : \mathcal{D} \to \mathcal{E}/I_n$ such that $\chi \circ \phi = \pi_{I_n} \circ \overline{\chi}$, where $\pi_{I_n} : \mathcal{E}/I_n \to \mathcal{E}/\overline{\bigcup_n I_n}$ is the quotient map. In the case of *weak semiprojectivity*, we require that for a finite set $F \subset \mathcal{D}$ and $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\|\chi \circ \phi(x) - \pi_{I_n} \circ \overline{\chi}(x)\| < \epsilon$ for all $x \in F$.



We say A is *(weakly) semiprojective* if the identity map is (weakly) semiprojective.

The following result highlights the connection between pertubing relations and lifting of *-homomorphisms.

Theorem 1.4.12 ([50, Theorem 4.1.4 & 14.1.4,],[50, Lemma 10.1.5])**.** Let $\mathcal{D}$ be a finitely presented universal C^*-algebra $C^*(\mathrm{G}|\mathcal{R})$.

(1) $\mathcal{D}$ is projective if and only if $(\mathrm{G}|\mathcal{R})$ is liftable.

(2) $\mathcal{D}$ is (weakly) semiprojective if and only if $(\mathrm{G}|\mathcal{R})$ is (weakly) stable.

Example 1.4.13. All the C^*-algebras in Example 1.4.3 are semiprojective C^*-algebras except (d) and (f) [11].

Remark 1.4.14. It is clear from the definitions that projectivity implies semiprojectivity, and semiprojectivity implies weak semiprojectivity. However, the converse does not hold. Indeed, $C_0((0,1])$ is a projective C^*-algebra, $C(S^1)$ is a semiprojective C^*-algebra which is not projective [11], and $C(X)$, where X is the Cantor set, is a weakly semiprojective C^*-algebra which is not semiprojective [51].

Before stating the characterization of (weak) semiprojectivity for unital commutative C^*-algebras, we recall the following definitions. We assume Δ, Ω and Σ are metrizable spaces in what follows unless otherwise specified. By a map between Δ and Σ, we mean a continuous map between these spaces.

Definition 1.4.15. A metrizable space Δ is called an *absolute retract* (AR) if, for any inclusion map $i : \Omega \to \Sigma$ and any map $g : \Omega \to \Delta$ of metrizable spaces, there exists a map $h : \Sigma \to \Delta$ such that $g = h \circ i$.

Definition 1.4.16. A metrizable space Δ is called an *absolute neighborhood retract* (ANR) if, for any inclusion map $i : \Omega \to \Sigma$ and any map $g : \Omega \to \Delta$ of metrizable spaces, there exists a neighborhood Λ of Ω in Σ and a map $h : \Lambda \to \Delta$ such that $g = h \circ i$.

Definition 1.4.17. A metrizable space Δ is called an *approximate absolute neighborhood retract* (AANR) if, for any inclusion map $i : \Omega \to \Sigma$, any map $g : \Omega \to \Delta$ of metrizable spaces, and any $\epsilon > 0$, there exists a neighborhood Λ of Ω in Σ and a map $h : \Lambda \to \Delta$ such that $d_\Delta(g(w), h \circ i(w)) < \epsilon$ for all $w \in \Omega$.

Example 1.4.18. The interval $[0, 1]$ is an AR, the unit circle S^1 is an ANR but not an AR [78, Page 26], and the Cantor set is an AANR but not an ANR [51].

Though ANR-inspired semiprojectivity provides useful insights, it is insufficient to guarantee the semiprojectivity of a commutative C^*-algebra. For example, S^1 and $S^1 \times S^1$ are ANRs. However, $C(S^1)$ is a semiprojective C^*-algebra, while $C(S^1 \times S^1)$ is not. We now discuss an additional condition that guarantees the semiprojectivity of unital commutative C^*-algebras.

Definition 1.4.19. Let $\mathcal{U} = \{\mathcal{U}_\alpha\}$ be a collection of open sets in a topological space Δ, and let w be a point in Δ. The order of $\mathcal{U}$ at w is the cardinality of $\{\alpha \mid w \in \mathcal{U}_\alpha\}$ and is denoted by $\mathrm{Ord}_w(\mathcal{U})$. Then, the order of $\mathcal{U}$, denoted $\mathrm{Ord}(\mathcal{U})$, is equal to $\sup\{\mathrm{Ord}_w(\mathcal{U}) \mid w \in \Delta\}$.

Note that the order of $\mathcal{U}$ at $w \in \Delta$ can be $+\infty$ if w is contained in infinitely many $\mathcal{U}_\alpha$.

Definition 1.4.20. A topological space Δ has covering dimension $\leq n$ if for any finite open cover $\mathcal{U}$ of Δ, there exists a finite open cover refinement $\mathcal{V}$ of $\mathcal{U}$ with $\mathrm{Ord}(\mathcal{V}) \leq n + 1$. We denote this by $\dim(\Delta) \leq n$.

If every point $w \in \Delta$ has a closed neighborhood Λ such that $\dim(\Lambda) \leq n$, we say Δ has a local covering dimension $\mathrm{locdim}(\Delta) \leq n$ (see [54] for more details).

The combined work of [73, 23, 34] gives the following characterization of (weakly) semiprojective commutative C^*-algebras:

Theorem 1.4.21. Let Δ be a compact metrizable space. Then,

(i) $C(\Delta)$ is a projective C^*-algebra if and only if Δ is an AR and $\dim(\Delta) \leq 1$.

(ii) $C(\Delta)$ is a (weakly) semiprojective C^*-algebra if and only if Δ is an (A)ANR and $\dim(\Delta) \leq 1$.

1.5 Reduced Twisted C^*-algebras and C^*-diagonals

A groupoid $\mathcal{G}$ is a generalization of a group with an additional property: the product of two elements in $\mathcal{G}$ is only defined if they belong to a certain distinguished set. Realizing C^*-algebras as groupoid C^*-algebras allows for approaching problems from different perspectives and with various tools. The materials covered in this subsection are broadly discussed in [72].

We first introduce groupoids and highlight how they generalize groups.

Introduction to Groupoid

Definition 1.5.1. A groupoid is a nonempty set $\mathcal{G}$ endowed with a product map $(w, z) \mapsto wz : \mathcal{G}^{(2)} \to \mathcal{G}$, where $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$ is called the set of composable pairs, and an inverse map $w \mapsto w^{-1} : \mathcal{G} \to \mathcal{G}$ satisfying the following relations:

(i) $(w^{-1})^{-1} = w$ for all $w \in \mathcal{G}$;

(ii) $(w, z), (z, f) \in \mathcal{G}^{(2)}$ implies $(wz, f), (w, zf) \in \mathcal{G}^{(2)}$ and $(wz)f = w(zf)$;

(iii) $(w, w^{-1}) \in \mathcal{G}^{(2)}$ for all $w \in \mathcal{G}$, and if $(w, z) \in \mathcal{G}^{(2)}$, then $w^{-1}(wz) = z$ and $(wz)z^{-1} = w$.

The second axiom above states that associativity holds whenever the products are well-defined. Hence, we can simply write wzf without parentheses, as is done for groups. Unlike groups, groupoids have many identities. We define the set of identities of a groupoid as follows:

Definition 1.5.2. Let $\mathcal{G}$ be a groupoid. The set

$$\mathcal{G}^{(0)} := \{ww^{-1} : w \in \mathcal{G}\} = \{w^{-1}w : w \in \mathcal{G}\}$$

is called the unit space of $\mathcal{G}$. The map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ given by $r(w) = ww^{-1}$ and the map $s : \mathcal{G} \to \mathcal{G}^{(0)}$ given by $s(w) = w^{-1}w$ are called the range map and source map, respectively.

The following result highlights how the range and source maps characterize the set of composable pairs, $\mathcal{G}^{(2)}$:

Proposition 1.5.3. For a given groupoid $\mathcal{G}$ with source map s and range map r, $(\alpha, \beta) \in \mathcal{G}^{(2)}$ if and only if $s(\alpha) = r(\beta)$.

Groupoids obey the usual cancellation and inverse of a product rule.

Proposition 1.5.4. Let $\mathcal{G}$ be a groupoid, and let r and s be the range and the source map, respectively.

(i) If $(w, z), (z, f) \in \mathcal{G}^{(2)}$ and $wf = zf$, then $w = z$;

(ii) If $(w, z) \in \mathcal{G}^{(2)}$, then $(z^{-1}, w^{-1}) \in \mathcal{G}^{(2)}$ and $(wz)^{-1} = z^{-1}w^{-1}$.

Example 1.5.5. Every group G is naturally a groupoid if we regard $G^{(2)} = G \times G$. The unit space of G contains its unique identity element.

Example 1.5.6. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids, and define

$$(\mathcal{G} \times \mathcal{H})^{(2)} := \{((w_1, z_1), (w_2, z_2)) : (w_1, w_2) \in \mathcal{G}^{(2)}, (z_1, z_2) \in \mathcal{H}^{(2)}\}.$$

Then, $\mathcal{G} \times \mathcal{H}$ with $(\mathcal{G} \times \mathcal{H})^{(2)}$ and the usual operations $(w_1, z_1)(w_2, z_2) = (w_1 w_2, z_1 z_2)$, $(w_1, z_1)^{-1} = (w_1^{-1}, z_1^{-1})$ for all $((w_1, z_1), (w_2, z_2)) \in (\mathcal{G} \times \mathcal{H})^{(2)}$ is called a product groupoid.

Example 1.5.7. Given a group G acting on a nonempty set Δ, let $\mathcal{G} = G \times \Delta$. Then $\mathcal{G}$ is a groupoid with

$$\mathcal{G}^{(2)} := \{((w, \alpha), (z, \beta)) : w, z \in G, \alpha, \beta \in \Delta,\ \alpha = z \cdot \beta\},$$

a multiplication map $(w, z \cdot \beta)(z, \beta) \mapsto (wz, \beta)$, and an inverse map $(w, \alpha) \mapsto (w^{-1}, w \cdot \alpha)$. The unit space $G^{(0)} = \{e\} \times \Delta$ can be identified with Δ. This groupoid is called the transformation groupoid.

Definition 1.5.8. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A map $\eta : \mathcal{G} \to \mathcal{H}$ is called a groupoid homomorphism if $(\eta(w), \eta(z)) \in \mathcal{H}^{(2)}$ and $\eta(wz) = \eta(w)\eta(z)$ for any $(w, z) \in \mathcal{G}^{(2)}$.

Groupoid homomorphisms have the following natural properties.

Proposition 1.5.9. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids and $\eta : \mathcal{G} \to \mathcal{H}$ a groupoid homomorphism. For any $w \in \mathcal{G}$,

(i) $\eta(\mathcal{G}^{(0)}) \subseteq \mathcal{H}^{(0)}$.

(ii) $\eta(r(w)) = r(\eta(w))$ and $\eta(s(w)) = s(\eta(w))$.

(iii) $\eta(w^{-1}) = \eta(w)^{-1}$.

Isotropy

Let $\mathcal{G}$ be a groupoid. For $w \in \mathcal{G}^{(0)}$, we define

$$\mathcal{G}_w := \{z \in \mathcal{G} : s(z) = w\}, \quad \mathcal{G}^w := \{z \in \mathcal{G} : r(z) = w\}, \quad \text{and} \quad \mathcal{G}_w^w := \mathcal{G}_w \cap \mathcal{G}^w.$$

Definition 1.5.10. The isotropy of a groupoid $\mathcal{G}$ is defined as

$$Iso(\mathcal{G}) := \bigcup_{w \in \mathcal{G}^{(0)}} \mathcal{G}_w^w = \{z \in \mathcal{G} : s(z) = r(z)\}.$$

$\mathcal{G}_w^w$ is called the isotropy at w.

It is evident that $\mathcal{G}^{(0)} \subseteq Iso(\mathcal{G})$.

Example 1.5.11. Let (G, Δ) be as in Example 1.5.7. The range and source maps of $\mathcal{G} = G \times \Delta$ are defined by $r(w, \alpha) = w \cdot \alpha$ and $s(w, \alpha) = \alpha$, respectively. We have $r(w, \alpha) = s(w, \alpha)$ if and only if $w \in \text{Stab}(\alpha) = \{w \in G : w \cdot \alpha = \alpha\}$. Hence,

$$Iso(\mathcal{G}) = \bigcup_{\alpha \in \Delta} \{\alpha\} \times \text{Stab}(\alpha).$$

Étale groupoids

Definition 1.5.12. A topological groupoid is a groupoid $\mathcal{G}$ equipped with a topology such that the inverse map $w \mapsto w^{-1} : \mathcal{G} \to \mathcal{G}$ and the multiplication map $(w, z) \mapsto wz : \mathcal{G}^{(2)} \to \mathcal{G}$ are continuous, where $\mathcal{G}^{(2)}$ has the induced topology from $\mathcal{G} \times \mathcal{G}$.

Next, we discuss some implications of a topology on the range, source, and unit space of a groupoid.

Proposition 1.5.13. Let $\mathcal{G}$ be a topological groupoid. Then the source and range maps are continuous. Moreover, the unit space $\mathcal{G}^{(0)}$ is closed if and only if $\mathcal{G}$ is a Hausdorff topological groupoid.

From now on, we use the term groupoid to mean a Hausdorff topological groupoid.

Example 1.5.14. Let (G, Δ) be as in Example 1.5.7, with G a Hausdorff topological group and Δ a Hausdorff topological space. If the group action of G on Δ is continuous, then $\mathcal{G} = G \times \Delta$ is a topological groupoid under the product topology of G and Δ.

Definition 1.5.15. A topological groupoid $\mathcal{G}$ is étale if the source and range maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ are local homeomorphisms. That is, for every $\alpha \in \mathcal{G}$, there exists an open neighborhood U such that $r(U)$ is an open set in $\mathcal{G}^{(0)}$ and the restriction $r|_U : U \to r(U)$ is a homeomorphism.

Étale groupoids are closely related to the discreteness of groups.

Proposition 1.5.16. Let $\mathcal{G}$ be an étale groupoid. Then the range and source maps are open maps, and the unit space $\mathcal{G}^{(0)}$ is a clopen set. Moreover, $\mathcal{G}_w$ and $\mathcal{G}^w$ are discrete in the relative topology for each $w \in \mathcal{G}^{(0)}$.

Example 1.5.17. Let $\mathcal{G}$ be the transformation groupoid considered in Example 1.5.14. Then $\mathcal{G}$ is an étale topological groupoid if and only if G is a discrete group.

The following notions of a groupoid are needed for later use.

Definition 1.5.18. A groupoid $\mathcal{G}$ is called a principal groupoid if $Iso(\mathcal{G}) = \mathcal{G}^{(0)}$. If $\mathcal{G}$ is a topological groupoid, we say $\mathcal{G}$ is:

- topologically principal if the set
$$\{w \in \mathcal{G}^{(0)} : \mathcal{G}_w^w = \{w\}\}$$
is dense in $\mathcal{G}^{(0)}$;
- effective if the interior of $Iso(\mathcal{G})$ is $\mathcal{G}^{(0)}$.

Proposition 1.5.19 ([62, Proposition 3.6])**.** Let $\mathcal{G}$ be an étale groupoid. Then $\mathcal{G}$ is principal $\Rightarrow$ $\mathcal{G}$ is topologically principal $\Rightarrow$ $\mathcal{G}$ is effective.

Note that the reverse implication of the notions above is not always true. We recall the general notion of a twist over a groupoid introduced by Kumjian.

Definition 1.5.20. A twist $(\mathcal{E}, i, q)$ over a Hausdorff étale groupoid $\mathcal{G}$ is a sequence

$$\mathcal{G}^{(0)} \times \mathbb{T} \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{G},$$

where

(a) $\mathcal{E}$ is a locally compact Hausdorff groupoid with unit space $\mathcal{E}^{(0)} = i(\mathcal{G}^{(0)} \times \{1\})$. Moreover, $\mathcal{E}$ is a locally trivial $\mathcal{G}$-bundle, that is, for each $w \in \mathcal{G}$, there is an open neighborhood $\Lambda_w \subseteq \mathcal{G}$ of w, and a continuous map $P_w : \Lambda_w \to \mathcal{E}$ such that

  (i) $q \circ P_w = \mathrm{id}_{\Lambda_w}$,

  (ii) the map $\chi_{P_w} : \Lambda_w \times \mathbb{T} \to q^{-1}(\Lambda_w)$ that sends (z, α) to $i(r(z), \alpha)P_w(z)$ is a homeomorphism.

(b) $i(\{w\} \times \mathbb{T}) = q^{-1}(w)$ for each $w \in \mathcal{G}^{(0)}$, i is a continuous monomorphism, and q is a continuous epimorphism of groupoids. Moreover, the restrictions of i and q to the unit spaces are homeomorphisms.

(c) $\mathcal{G}^{(0)} \times \mathbb{T}$ is regarded as a trivial group bundle with fibers $\mathbb{T}$. $\mathcal{G}^{(0)} \times \mathbb{T}$, together with the set of composable pairs

$$\{((w, t_1), (w, t_2)) : w \in \mathcal{G}^{(0)}, t_1, t_2 \in \mathbb{T}\},$$

the multiplication map $((w, t_1), (w, t_2)) \mapsto (w, t_1 t_2)$, the inverse map $(w, t)^{-1} \mapsto (w, t^{-1})$, and the product topology is a topological groupoid.

(d) $i(r(\epsilon), t)\epsilon = \epsilon i(s(\epsilon), t)$ for all $\epsilon \in \mathcal{E}$ and $t \in \mathbb{T}$.

We sometimes represent a twist $(\mathcal{E}, i, q)$ over $\mathcal{G}$ simply by $\mathcal{E}$. For $t \in \mathbb{T}$ and $\epsilon \in \mathcal{E}$, we write $t \cdot \epsilon = i(r(\epsilon), t)\epsilon$ and $\epsilon \cdot t = \epsilon i(s(\epsilon), t)$.

Example 1.5.21. Let $\mathcal{G}$ be an étale locally compact groupoid, and let $\mathcal{E} = \mathcal{G} \times \mathbb{T}$ be the product groupoid of $\mathcal{G}$ and $\mathbb{T}$. Let i be the inclusion map of $\mathcal{G}^{(0)} \times \mathbb{T}$ into $\mathcal{E}$, and let q be the usual projection of $\mathcal{E}$ onto $\mathcal{G}$. Then $\mathcal{E}$ is a trivial twist of $\mathcal{G}$.

Reduced Twisted Groupoid C^*-algebra

In this subsection, we highlight the construction of reduced twisted groupoid C^*-algebras, which is essential to the concepts of Cartan subalgebras and C^*-diagonals. Throughout this subsection, we assume $\mathcal{E}$ is a twist over an étale locally compact Hausdorff groupoid $\mathcal{G}$.

Definition 1.5.22. We define

$$C(\mathcal{G},\mathcal{E}) \coloneqq \{f:\mathcal{E}\to\mathbb{C} \mid \text{f is continuous and } f(t.w)=tf(w) \text{ for all } t\in\mathbb{T}, w\in\mathcal{E}\};$$

and the support of a continuous function $f\in C(\mathcal{G},\mathcal{E})$ is defined as

$$\operatorname{supp}(f) \coloneqq \overline{\{w\in\mathcal{E} \mid f(w)\neq 0\}}.$$

Set

$$C_c(\mathcal{G},\mathcal{E}) \coloneqq \{f\in C(\mathcal{G},\mathcal{E}) \mid \operatorname{supp}(f) \text{ is compact}\};$$

and

$$C_c(\mathcal{G}^{(0)}) \coloneqq \{f\in C_c(\mathcal{G},\mathcal{E}) \mid \operatorname{supp}(f)\subseteq\mathcal{G}^{(0)}\}.$$

Then, $C_c(\mathcal{G},\mathcal{E})$ is a *-algebra under the following algebraic operations: for any $f,g\in C_c(\mathcal{G},\mathcal{E})$ and $\gamma\in\mathcal{E}$,

$$f*g(\gamma)=\sum_{\sigma\in\mathcal{G}_{s(\gamma)}} f(\gamma\sigma^{-1})g(\sigma),$$

and

$$f^*(\gamma)=\overline{f(\gamma^{-1})}.$$

Before we introduce the reduced C^*-norm on $C_c(\mathcal{G},\mathcal{E})$, we illustrate how to construct the regular representation of $C_c(\mathcal{G},\mathcal{E})$ associated to $w\in\mathcal{G}^{(0)}$.

Identify $w\in\mathcal{G}^{(0)}$ with $i((w,1))$. The set

$$L^2(\mathcal{G}_w;\mathcal{E}_w) \coloneqq \{T:\mathcal{E}_w\to\mathbb{C} \mid T(s.z)=sT(z) \text{ for all } s\in\mathbb{T}, z\in\mathcal{E}_w, \text{ and } \sum_{\gamma\in\mathcal{G}_w}|T(\gamma)|^2<\infty\}$$

is a Hilbert space under the inner product

$$\langle T,S\rangle=\sum_{\gamma\in\mathcal{G}_w} T(\gamma)\overline{S(\gamma)}.$$

Proposition 1.5.23. For every $w\in\mathcal{G}^{(0)}$, the map $\pi_w: C_c(\mathcal{G},\mathcal{E})\to B(L^2(\mathcal{G}_w;\mathcal{E}_w))$ defined by

$$\pi_w(f)(T)(\gamma)=\sum_{\sigma\in\mathcal{G}_w} f(\gamma\sigma^{-1})T(\sigma)$$

is a *-representation, where $f\in C_c(\mathcal{G},\mathcal{E})$, $T\in L^2(\mathcal{G}_w;\mathcal{E}_w)$, and $\gamma\in\mathcal{E}_w$.

Definition 1.5.24. Given a twist $\mathcal{E}$ over an étale locally compact Hausdorff groupoid $\mathcal{G}$, the completion of $C_c(\mathcal{G},\mathcal{E})$ with respect to the norm $\|\cdot\|_r=\sup\{\|\pi_w(\cdot)\| : w\in\mathcal{G}^{(0)}\}$ is called the reduced twisted C^*-algebra $C^*_r(\mathcal{G},\mathcal{E})$ associated with the twist $\mathcal{E}$.

Let $C_0(\mathcal{G},\mathcal{E})$ be the set of continuous functions in $C(\mathcal{G},\mathcal{E})$ that vanish at infinity. We can identify $C^*_r(\mathcal{G},\mathcal{E})$ with a C^*-subalgebra of $C_0(\mathcal{G},\mathcal{E})$ and consequently consider the completion of $C_c(\mathcal{G}^{(0)})$ with respect to the norm $\|\cdot\|_r$ as $C_0(\mathcal{G}^{(0)})$, the set of elements of $C_0(\mathcal{G},\mathcal{E})$ with open support in $\mathcal{G}^{(0)}$ (see [61] for more details).

Cartan Subalgebras and C^*-diagonals

In this subsection, we introduce Cartan subalgebras, C^*-diagonals and their relationships to reduced twisted groupoid C^*-algebras.

Definition 1.5.25. Let $\mathcal{B}$ be a C^*-subalgebra of a C^*-algebra $\mathcal{D}$. We say $\mathcal{B}$ is a Cartan subalgebra of $\mathcal{D}$ or $(\mathcal{D}, \mathcal{B})$ is a Cartan pair if

(i) $\mathcal{B}$ contains an approximate unit for $\mathcal{D}$,

(ii) $\mathcal{B}$ is a maximal abelian C^*-subalgebra of $\mathcal{D}$,

(iii) $\mathcal{B}$ is regular in $\mathcal{D}$, that is, $\mathcal{D}$ is generated by the set of normalizers

$$N_{\mathcal{D}}(\mathcal{B}) = \{n \in \mathcal{D} : n\mathcal{B}n^*, n^*\mathcal{B}n \subseteq \mathcal{B}\},$$

(iv) There exists a faithful conditional expectation $P : \mathcal{D} \to \mathcal{B}$, that is, $P(a^*a) = 0$ implies $a = 0$.

Let $\mathcal{B} \subseteq \mathcal{D}$ be C^*-algebras. Recall that the extreme points of $S(\mathcal{D})$ coincide with the pure states of $\mathcal{D}$. The set of state extensions of a pure state φ on $\mathcal{B}$ forms a weak*-closed face. Therefore, by the Krein-Milman Theorem, φ can be extended to at least one pure state of $\mathcal{D}$ [58, Lemma 4.1.7]. Some natural questions are when such an extension is unique and what the properties are of subalgebras with a unique pure state extension.

Definition 1.5.26. Let $\mathcal{B}$ be a C^*-subalgebra of a C^*-algebra $\mathcal{D}$. We say $\mathcal{B}$ is a C^*-diagonal of $\mathcal{D}$ if $\mathcal{B}$ is a Cartan subalgebra with the unique extension property; that is, every pure state of $\mathcal{B}$ extends uniquely to a pure state of $\mathcal{D}$.

Let $\mathcal{B} \subseteq \mathcal{D}$ be C^*-algebras and $n \in N_{\mathcal{D}}(\mathcal{B})$. Then, n is a *free normalizer* if $n^2 = 0$. The set of free normalizers is denoted by $N_f(\mathcal{B})$. The next result highlights the equivalence between C^*-diagonals as defined in Definition 1.5.26 and Kumjian's original work [43].

Proposition 1.5.27 ([60, Proposition 2.10])**.** Let $\mathcal{B}$ be an abelian C^*-algebra of a C^*-algebra $\mathcal{D}$. Then $\mathcal{B}$ is a C^*-diagonal of $\mathcal{D}$ if and only if there exists a faithful conditional expectation $P : \mathcal{D} \to \mathcal{B}$ and $\ker(P)$ is the norm-closed linear span of the $N_f(\mathcal{B})$.

Given two Cartan pairs $(\mathcal{D}_1, \mathcal{B}_1)$ and $(\mathcal{D}_2, \mathcal{B}_2)$, we write $(\mathcal{D}_1, \mathcal{B}_1) \cong (\mathcal{D}_2, \mathcal{B}_2)$ if there is an isomorphism from $\mathcal{D}_1$ to $\mathcal{D}_2$ that maps $\mathcal{B}_1$ to $\mathcal{B}_2$.

Example 1.5.28. $(M_n(\mathbb{C}), D_n(\mathbb{C}))$ is a Cartan pair, where $D_n(\mathbb{C})$ consists of all the diagonal matrices. Indeed, $D_n(\mathbb{C})$ is a maximal abelian subalgebra since it has vector space dimension n, and every commutative subalgebra of $M_n(\mathbb{C})$ has dimension at most n. The standard matrix units that generate $M_n(\mathbb{C})$ are normalizers of $D_n(\mathbb{C})$, and the canonical conditional expectation from $M_n(\mathbb{C})$ onto $D_n(\mathbb{C})$ is faithful. It can be further shown that $D_n(\mathbb{C})$ is a C^*-diagonal.

The combined work of Renault and Kumjian gives the following remarkable characterization of Cartan subalgebras and C^*-diagonals.

Theorem 1.5.29 ([62, 43])**.** Let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{D}$. Then $(\mathcal{D}, \mathcal{B})$ is a Cartan pair if and only if there exists an étale locally compact second countable effective Hausdorff groupoid $\mathcal{G}$ and a twist $\mathcal{E}$ over $\mathcal{G}$ such that $(C^*_r(\mathcal{G}, \mathcal{E}), C_0(\mathcal{G}^{(0)})) \cong (\mathcal{D}, \mathcal{B})$. Moreover, $\mathcal{G}$ is principal if and only if $\mathcal{B}$ is a C^*-diagonal of $\mathcal{D}$.

Chapter 2

ℓ-open and ℓ-closed C^*-algebras

Lifting properties of C^*-algebras and their *-homomorphisms have been well-studied for some time, with prominent connections to notions of stability; see [11, 32, 49, 50], for example. They play an important role in modern C^*-algebra theory, including the Elliott classification program ([37, 28, 55], for example). In connection with a non-commutative generalization of Borsuk's homotopy extension theorem, Blackadar [15] defined natural classes of C^*-algebras in terms of lifting properties, called ℓ-open and ℓ-closed C^*-algebras. A C^*-algebra is ℓ-open if the set of liftable maps from the C^*-algebra to any quotient C^*-algebra is a set that is open in the point-norm topology, and ℓ-closedness is defined similarly (precise definitions can be found in Section 2.1).

While these notions were first formalized only recently by Blackadar, their study traces back at least to the celebrated work of Brown, Douglas, and Fillmore: in [17], they seek conditions on a space X that ensure the set of liftable maps from $C(X)$ to the Calkin algebra is closed. More recently, Enders and Shulman further studied when the set of liftable maps from $C(X)$ to the Calkin algebra is closed, including a sufficient condition when $\dim(X) \leq 2$ and a full characterization when $\dim(X) \leq 1$ [35].

The work in this chapter is mostly contained in the preprint [56]. We prove the following characterizations of being ℓ-open and ℓ-closed:

Theorem 2.0.1 (see Theorem 2.2.16)**.** Let A be a C^*-algebra. The following are equivalent:

(i) A is ℓ-open.

(ii) For every C^*-algebra B and ideal $I \subseteq B$, the natural map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/I)$ is open.

(iii) A satisfies the homotopy lifting property (a noncommutative analog of the homotopy extension property), and $\mathrm{Hom}(A, B)$ is locally path-connected for every C^*-algebra B.

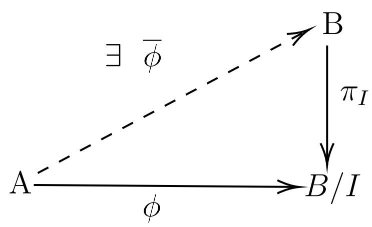
Condition (ii) can be strengthened to uniform openness (see Theorem 2.2.16).

Theorem 2.0.2 (see Theorem 2.3.1)**.** Let A be a separable C^*-algebra. Then A is ℓ-closed if and only if for every C^*-algebra B and ideal $I \subseteq B$, the natural map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/I)$ is uniformly relatively open.

As a consequence, we confirm a conjecture of Blackadar from [15, Page 299] that ℓ-open C^*-algebras are ℓ-closed. Additionally, we prove that a unital commutative C^*-algebra is semiprojective if and only if it is ℓ-open, confirming another conjecture from [15, Page 299].

2.1 Preliminaries

Let A and B be C^*-algebras, and let I be an ideal in B (by which we mean a closed, two-sided ideal). We write $\pi_I : B \to B/I$ for the quotient map, $A^\dagger$ for the minimal unitization of A, and A^+ for the forced unitization of A. Recall that a *-homomorphism $\phi : A \to B/I$ is *liftable* if there exists a *-homomorphism $\overline{\phi} : A \to B$ such that $\phi = \pi_I \circ \overline{\phi}$.



The set of liftable *-homomorphisms $A \to B/I$ is defined as

$$\mathrm{Hom}(A, B, I) \coloneqq \pi_I \circ \mathrm{Hom}(A, B). \tag{2.1.1}$$

The following definition is due to Blackadar [15, Definition 6.1].

Definition 2.1.1. Let A be a C^*-algebra.

(i) A is *ℓ-open* if for any C^*-algebra B and ideal I of B, the set $\mathrm{Hom}(A, B, I)$ is open in $\mathrm{Hom}(A, B/I)$.

(ii) A is *ℓ-closed* if for any C^*-algebra B and ideal I of B, the set $\mathrm{Hom}(A, B, I)$ is closed in $\mathrm{Hom}(A, B/I)$.

Example 2.1.2. [15, Corollary 6.2] All semiprojective C^*-algebras are both ℓ-open and ℓ-closed C^*-algebras.

Recall that if Δ and Ω are compact metrizable spaces and Δ^Ω denotes the set of continuous functions from Ω to Δ, then $\mathrm{Hom}_1(C(\Delta), C(\Omega))$ is homeomorphic to Δ^Ω, endowed with the topology of uniform convergence. Let I be an ideal of $B \coloneqq C(\Omega)$ such that $B/I \cong C(\Gamma)$ for some compact metrizable space Γ. For C^*-algebras $A \coloneqq C(\Delta)$, $B \coloneqq C(\Omega)$, and $B/I \cong C(\Gamma)$, $\mathrm{Hom}(A, B, I)$ is the set of continuous functions $f : \Gamma \to \Delta$ that extend to continuous functions $\overline{f} : \Omega \to \Delta$. In this case, we denote $\mathrm{Hom}(A, B, I)$ by $\Delta^{\Gamma \uparrow \Omega}$.

Blackadar introduced a commutative analog of ℓ-open and ℓ-closed C^*-algebras as follows:

Definition 2.1.3 ([14, Definition 1.1])**.** Let Δ be a compact metrizable space.

(i) Δ is *e-open* if for any compact metrizable space Ω and closed subspace Γ of Ω, the subspace $\Delta^{\Gamma \uparrow \Omega}$ is open in Δ^Γ.

(ii) Δ is *e-closed* if for any compact metrizable space Ω and closed subspace Γ of Ω, the subspace $\Delta^{\Gamma\uparrow\Omega}$ is closed in Δ^Γ.

Example 2.1.4 ([14, Theorems 3.2 & 3.5])**.** All compact ANR spaces are both e-open and e-closed spaces.

Example 2.1.5 ([14, Proposition 2.4])**.** The space $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is an e-closed space which is not e-open.

By slight abuse of notation, if $L \subseteq K \subseteq B$ are ideals, then we also use π_K to denote the quotient map from B/L to B/K.

We recall the following general Chinese remainder theorem for C^*-algebras:

Lemma 2.1.6 ([15], Proposition 2.1)**.** Let B be a C^*-algebra, and let I and J be ideals in B. Then $B/(I \cap J)$ is isomorphic to the fibre product

$$\{(x, y) \in (B/I) \oplus (B/J) \mid \pi_{I+J}(x) = \pi_{I+J}(y)\}$$

via the map $a \mapsto (\pi_I(a), \pi_J(a))$.

2.2 Properties and characterization of ℓ-open C^*-algebras

The following demonstrates that if A is ℓ-open, then the quotient map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/I)$ is always open. In fact, this openness is uniform, as the relationship between $(\mathcal{G}, \delta)$ and $(\mathcal{F}, \epsilon)$ in the statement below does not depend on the C^*-algebra B, the ideal I, or any of the *-homomorphisms under consideration. The conclusion of the following theorem (in the separable case) reformulates the result of [15, Theorem 4.1]. While the proof ideas are similar, additional work is required to accommodate ℓ-openness rather than semiprojectivity as the hypothesis.

Theorem 2.2.1. Let A be an ℓ-open C^*-algebra. Then, for any $\epsilon > 0$ and any finite set $\mathcal{F} \subset A$, there exists a $\delta > 0$ and a finite set $\mathcal{G} \subset A$ such that, for any C^*-algebra B, ideal I of B, and *-homomorphisms γ and φ from A to B/I, if $\|\gamma(u) - \varphi(u)\| < \delta$ for all $u \in \mathcal{G}$ and γ lifts to a *-homomorphism $\overline{\gamma} : A \to B$, then φ also lifts to a *-homomorphism $\overline{\varphi} : A \to B$ with $\|\overline{\gamma}(v) - \overline{\varphi}(v)\| < \epsilon$ for all $v \in \mathcal{F}$. In other words, in the notation of (1.2.1),

$$U_{B/I}(\gamma; \mathcal{G}, \delta) \subseteq \pi_I \circ U_B(\overline{\gamma}; \mathcal{F}, \epsilon). \tag{2.2.1}$$

Proof. Let $(\mathcal{G}_n)_{n\in\Lambda}$ be an increasing net of finite subsets of A whose union is dense in A, and let $(\delta_n)_{n\in\Lambda}$ be a net (over the same index set) of positive numbers such that $\delta_n \to 0$. Suppose that the conclusion of the theorem is false for a fixed $\epsilon > 0$ and finite set $\mathcal{F}$. Then, there exist C^*-algebras B_n, ideals I_n, and *-homomorphisms $\gamma_n, \varphi_n : A \to B_n/I_n$ such that

$$\|\gamma_n(u) - \varphi_n(u)\| < \delta_n \tag{2.2.2}$$

for all $u \in \mathcal{G}_n$, γ_n lifts to $\overline{\gamma}_n : A \to B_n$, but no φ_n lifts to a *-homomorphism $\overline{\varphi}_n : A \to B_n$ with $\|\overline{\gamma}_n(v) - \overline{\varphi}_n(v)\| < \epsilon$ for all $v \in \mathcal{F}$.

Let $B := \prod_{n\in\Lambda} B_n$, $I := \prod_{n\in\Lambda} I_n$, and $J := \{(b_n) \in B \mid \lim_n \|b_n\| = 0\}$. Then $B/I \cong \prod_{n\in\Lambda} B_n/I_n$. Define *-homomorphisms $\overline{\gamma} := (\overline{\gamma}_n)_{n\in\Lambda} : A \to B$ and $\varphi := (\varphi_n)_{n\in\Lambda} : A \to B/I$. Then (2.2.2) implies that $\lim_n \|\gamma_n(x) - \varphi_n(x)\| = 0$ for all $x \in A$, so $\pi_{I+J} \circ \overline{\gamma} = \pi_{I+J} \circ \varphi$.

Using the general Chinese remainder theorem (Lemma 2.1.6), there exists a *-homomorphism $\theta : A \to B/(I \cap J)$ such that

$$\pi_J \circ \overline{\gamma} = \pi_J \circ \theta \quad \text{and} \quad \varphi = \pi_I \circ \theta. \tag{2.2.3}$$

Take a lift $(\theta_n)_{n\in\Lambda} : A \to B$ of θ (which need not be a *-homomorphism), thus defining $\theta_n : A \to B_n$. For $m \in \Lambda$, define $\alpha_m := \pi_{I\cap J} \circ (\alpha_{m,n})_{n\in\Lambda}$, where

$$\alpha_{m,n} := \begin{cases} \theta_n, & \text{if } n \geq m; \\ \overline{\gamma}_n, & \text{otherwise.} \end{cases} \tag{2.2.4}$$

Since θ is a *-homomorphism, $\|\theta_n(xy) - \theta_n(x)\theta_n(y)\| \to 0$ for all $x, y \in A$; from this it follows that α_m is also a *-homomorphism.

The first equation of (2.2.3) implies that $\lim_n \|\overline{\gamma}_n(x) - \theta_n(x)\| = 0$ for all $x \in A$, which in turn implies that

$$\|\alpha_m(x) - \pi_{I\cap J}(\overline{\gamma}(x))\| = \sup_{n\geq m} \|\overline{\gamma}_n(x) - \theta_n(x)\| \to 0 \tag{2.2.5}$$

for all $x \in A$. Thus, $(\alpha_m)_m$ converges in the point-norm topology to the liftable *-homomorphism $\pi_{I\cap J} \circ \overline{\gamma}$, and since A is ℓ-open, it follows that α_m is liftable for some sufficiently large m. Let $\beta = (\beta_n)_{n\in\Lambda} : A \to B$ be a lift of α_m, where $\beta_n : A \to B_n$ is a *-homomorphism for each n. The fact that β is a lift means that

$$(\beta_n(x) - \alpha_{m,n}(x))_{n\in\Lambda} \in I \cap J, \quad \text{for all } x \in A. \tag{2.2.6}$$

This implies that $\lim_n \|\beta_n(x) - \theta_n(x)\| = 0$ for all $x \in A$. Combining this with the first equation in (2.2.3), it follows that

$$\lim_n \|\beta_n(x) - \overline{\gamma}_n(x)\| = 0, \quad \text{for all } x \in A. \tag{2.2.7}$$

From (2.2.6), we also get that $\pi_{I_n} \circ \beta_n = \pi_{I_n} \circ \theta_n$ for all $n \geq m$, and combining this with the second equation in (2.2.3), we have that β_n is a lift of φ_n for $n \geq m$.

In summary, for sufficiently large n, β_n is a lift of φ_n that is point-norm close to $\overline{\gamma}_n$, which contradicts our initial assumption. ∎

We now pick up some consequences, using ideas from Blackadar [15]. We add the proofs for completion. The first tells us that when A is ℓ-open, $\mathrm{Hom}(A, B)$ is locally path-connected in a uniform way.

Corollary 2.2.2 (cf. [15, Corollary 4.2])**.** Let A be an ℓ-open C^*-algebra (or more generally, one that satisfies the conclusion of Theorem 2.2.1). For any $\epsilon > 0$ and any finite set $\mathcal{F} \subset A$, there exists a $\delta > 0$ and a finite set $\mathcal{G} \subset A$ such that for any C^*-algebra B, if φ_0 and φ_1 are *-homomorphisms from A to B/I with $\|\varphi_0(u) - \varphi_1(u)\| < \delta$ for all $u \in \mathcal{G}$, then there exists a point-norm continuous path $(\varphi_t)_{t\in[0,1]}$ of *-homomorphisms from A to B connecting φ_0 and φ_1 with $\|\varphi_0(v) - \varphi_t(v)\| < \epsilon$ for all $v \in \mathcal{F}$ and $t \in [0,1]$. In particular, $\mathrm{Hom}(A, B)$ is locally path-connected for any C^*-algebra B.

Proof.

For any $\epsilon > 0$ and finite set $\mathcal{F} \subset A$, choose $\delta > 0$ and a finite set $\mathcal{G} \subset A$ as in Theorem 2.2.1. Let $D := C([0,1], B)$ and $I := C_0((0,1), B)$. Then $D/I \cong B \oplus B$. Define *-homomorphisms $\gamma, \varphi : A \to D/I$ by $\gamma(x) := (\varphi_0(x), \varphi_0(x))$ and $\varphi(x) := (\varphi_0(x), \varphi_1(x))$. Then γ lifts to a *-homomorphism $\overline{\gamma} := \mathrm{id}_{C([0,1])} \otimes \varphi_0 : A \to D$, and hence these two maps satisfy the hypothesis of Theorem 2.2.1. Therefore, the conclusion of Theorem 2.2.1 holds, and there exists a *-homomorphism $\overline{\varphi} = (\overline{\varphi_t})_{t\in[0,1]} : A \to D$ such that

$$\|\overline{\gamma}(a) - \overline{\varphi}(a)\| < \epsilon \quad \text{for all } a \in \mathcal{F}. \tag{2.2.8}$$

Thus, $\overline{\varphi}$ defines a homotopy of *-homomorphisms from A to B connecting φ_0 to φ_1, and (2.2.8) tells us that $\|\varphi_t(a) - \varphi_0(a)\| < \epsilon$ for all $a \in \mathcal{F}$, as required. ∎

Example 2.2.3. Consider the topologist's sine curve:

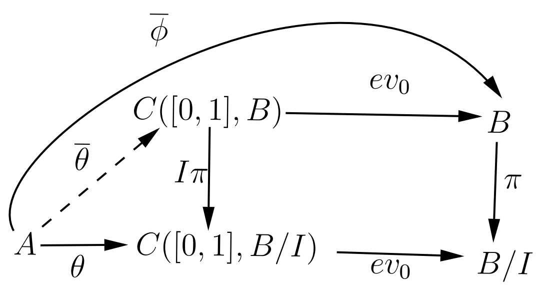
$$X := \left\{(x,y) \mid y = \sin\left(\frac{\pi}{x}\right), 0 < x \le 1\right\} \cup \{(0,y) \mid -1 \le y \le 1\}. \tag{2.2.9}$$

Then $\mathrm{Hom}(C(X), \mathbb{C}) \cong X$, which is not locally path-connected. Therefore, by the above corollary, $C(X)$ is not ℓ-open.

Before we state other properties of ℓ-open C^*-algebras, we discuss the following notion of C^*-algebras.

Definition 2.2.4 ([22])**.** Let (A, π) be a pair consisting of a C^*-algebra A and $\pi : B \to B/I$ a quotient map. (A, π) has the homotopy lifting property (HLP) if, whenever $\varphi : A \to B/I$ is a *-homomorphism that lifts to a *-homomorphism $\overline{\varphi} : A \to B$, any continuous path θ of *-homomorphisms $A \to B/I$ starting at φ lifts to a continuous path $\overline{\theta}$ of *-homomorphisms $A \to B$ starting at $\overline{\varphi}$.

Equivalently, (A, π) has the homotopy lifting property if there exists $\overline{\theta}$ completing the following diagram:



where $I\pi \coloneqq \mathrm{id}_{C([0,1])} \otimes \pi : C([0,1], B) \to C([0,1], B/I)$. We say that a C^*-algebra A satisfies the homotopy lifting property (HLP) if for every quotient map $\pi : B \to B/I$, where B is a C^*-algebra and I is an ideal of B, the pair (A, π) satisfies the homotopy lifting property.

The homotopy lifting property (HLP) of C^*-algebras is the noncommutative analogue of the homotopy extension property in topology. Conway [24, 25] first studied a restricted version of this property, which he called the C^*-covering homotopy property, in an attempt to directly prove the homotopy invariance property of the group $\mathrm{Ext}_s(X)$ and explore the relationship between K-theory and $\mathrm{Ext}(X)$. He considered the case where $A = C(X)$ and B/I is the Calkin algebra. Phillips and Raeburn [59] later studied the HLP for unital AF-algebras.

We now consider some properties and examples of C^*-algebras satisfying the homotopy lifting property.

Example 2.2.5. Every semiprojective C^*-algebra satisfies the homotopy lifting property [15, Theorem 5.1].

Proposition 2.2.6. A C^*-algebra A satisfies the homotopy lifting property if and only if $A^\dagger$ satisfies this property in the category of unital C^*-algebras and unital *-homomorphisms.

Proof. Case 1: A is unital. Suppose $A^\dagger = A$ satisfies the HLP in the category of unital C^*-algebras and unital *-homomorphisms. Let B be a C^*-algebra, I a closed ideal of B, $(\varphi_t)_{t\in[0,1]}$ a point-norm continuous path of *-homomorphisms from A to B/I, and $\overline{\varphi_0} : A \to B$ a lift of φ_0. Set $q_0 \coloneqq \varphi_0(1)$, $q_1 \coloneqq \varphi_1(1)$, and $p_0 \coloneqq \overline{\varphi_0}(1)$. Then q_0 is homotopic to q_1. Since $\mathbb{C}$ is a semiprojective C^*-algebra, it satisfies the HLP, and there exists a continuous path of projections $(p_t)_{t\in[0,1]}$ connecting p_0 and p_1, with $q_1 = \pi_I(p_1)$. Consequently, we can find a continuous path [30, Lemma 3.8] of partial isometries $(v_t)_{t\in[0,1]}$ such that:

$$\begin{aligned} v_0 &= p_0, \\ v_t^* v_t &= p_0 \quad \text{for all } t, \\ v_t v_t^* &= p_t. \end{aligned} \tag{2.2.10}$$

Let $\psi_1 \coloneqq \pi_I(v_1^*)\varphi_1\pi_I(v_1) : A \to q_0(B/I)q_0$. Then $(\pi_I(v_t^*)\varphi_t\pi_I(v_t))_{t\in[0,1]}$ is a point-norm continuous path of unital *-homomorphisms from A to $q_0(B/I)q_0$. Using the homotopy lifting property in the unital category, ψ_1 lifts to a unital *-homomorphism $\overline{\alpha_1} : A \to p_0Bp_0$, and there is a point-norm continuous path $(\overline{\alpha_t})_{t\in[0,1]}$ of unital *-homomorphisms connecting $\overline{\varphi_0}$ to $\overline{\alpha_1}$. Moreover, $\overline{\alpha_t}$ is a lift of $\pi_I(v_t^*)\varphi_t\pi_I(v_t)$ for each $t \in [0,1]$. Set $\overline{\varphi_t} \coloneqq v_t\overline{\alpha_t}v_t^* : A \to B$. Then $(\overline{\varphi_t})_{t\in[0,1]}$ defines a point-norm continuous path of *-homomorphisms from A to B starting at $\overline{\varphi_0}$ such that $\overline{\varphi_t}$ is a lift of φ_t for all $t \in [0,1]$.

The proof of the converse follows directly from the statement and the fact that projections homotopic to an identity element in a C^*-algebra are necessarily identity elements.

Case 2: A is nonunital. Suppose A satisfies the HLP. Let B be a unital C^*-algebra, I a closed ideal of B, and $(\varphi_t)_{t\in[0,1]}$ a point-norm continuous path of unital *-homomorphisms from $A^\dagger$ to B/I. Let $\overline{\varphi_0} : A^\dagger \to B$ be a unital lift of φ_0. Since A satisfies the HLP, the restriction of φ_t to A lifts to a continuous path $(\theta_t)_{t\in[0,1]}$ of *-homomorphisms from A to B, starting at the restriction of $\overline{\varphi_0}$ to A. Each θ_t extends uniquely to $\theta_t : A^\dagger \to B$, and the result follows.

Conversely, suppose $A^\dagger$ satisfies the HLP in the category of unital C^*-algebras and unital *-homomorphisms. Let B be a C^*-algebra, I a closed ideal of B, and $(\varphi_t)_{t\in[0,1]}$ a point-norm continuous path of *-homomorphisms from A to B/I. Let $\theta_0 : A \to B$ be a lift of φ_0. Each φ_t and θ_0 extends to unital homomorphisms $\widehat{\varphi_t} : A^\dagger \to (B/I)^\dagger$ and $\widehat{\theta_0} : A^\dagger \to B^\dagger$, respectively. Then there exists a point-norm continuous path $(\widehat{\theta}_t)_{t\in[0,1]}$ from $A^\dagger$ to B starting at $\widehat{\theta_0}$ which lifts $\widehat{\varphi_t}$. Moreover, $\widehat{\theta}_t(A) \subseteq B$. Therefore, the restriction of $(\widehat{\theta}_t)_{t\in[0,1]}$ to A defines the needed path for lifting $(\varphi_t)_{t\in[0,1]}$. ∎

Lemma 2.2.7 ([30, Lemma 3.9])**.** Let A be a unital C^*-algebra, $\mathcal{D}$ a finite-dimensional C^*-algebra, and $(\varphi_t)_{t\in[0,1]}$ a point-norm continuous path of *-homomorphisms from $\mathcal{D}$ to A. Then there exists a continuous path of unitaries $u_t \in A$ such that $\varphi_t = u_t\varphi_0 u_t^*$ for all $t \in [0,1]$.

Next, we give an alternative proof of [59, Lemma 3.4].

Lemma 2.2.8. Let B be a unital C^*-algebra, I an ideal of B, $\mathcal{D}$ a finite-dimensional C^*-algebra, and $\pi : B \to B/I$ the quotient map. Suppose $(\theta_t)_{t\in[0,1]}$ and $(\gamma_t)_{t\in[0,1]}$ are point-norm continuous paths of unital *-homomorphisms from $\mathcal{D}$ to B satisfying $\pi \circ \theta_t = \pi \circ \gamma_t$ for all $0 \le t \le 1$ and $\theta_0 = \gamma_0$. Then, there is a continuous path of unitaries $w_t \in B$ such that

(a) $\gamma_t(a) = w_t\theta_t(a)w_t^*$ for all $t \in [0,1]$ and $a \in \mathcal{D}$,

(b) $\pi(w_t) = 1$ for all $t \in [0,1]$.

Proof. Using Lemma 2.2.7, there exist continuous paths of unitaries u_t and v_t in B such that

$$\gamma_t = v_t\gamma_0 v_t^* \quad \text{and} \quad \theta_t = u_t\theta_0 u_t^*, \tag{2.2.11}$$

for all $t \in [0,1]$. Since $\gamma_0 = \theta_0$, it follows that $v_t^*\gamma_t v_t = u_t^*\theta_t u_t$ and $\gamma_t = v_t u_t^*\theta_t u_t v_t^*$. Let $w_t := v_t u_t^*$ for all $t \in [0,1]$. Then, $\pi \circ \gamma_t(a) = \pi(w_t)\pi \circ \theta_t(a)\pi(w_t^*)$ for all $a \in \mathcal{D}$, and $\pi(w_t) = \lambda_+ I$ for some $\lambda_+ \in \mathbb{C}$ satisfying $|\lambda_+| = 1$ since $\pi \circ \theta_t = \pi \circ \gamma_t$ and $\mathcal{D}$ is finite-dimensional. By adjusting v_t if necessary, we may assume $\lambda_+ = 1$, and the result follows. ∎

Using the proposition above, we state a broad class of C^*-algebras satisfying the HLP.

Theorem 2.2.9 (cf. [59, Theorem 3.5])**.** AF-algebras satisfy the homotopy lifting property.

Proof. Let A be an AF-algebra. Recall that A is an AF-algebra if and only if $A^\dagger$ is an AF-algebra. Using Proposition 2.2.6, we can reduce our proof to showing that unital AF-algebras satisfy the HLP in the category of unital C^*-algebras and unital *-homomorphisms.

Suppose $A = \overline{\bigcup_{n=1}^\infty A_n}$ is a unital AF-algebra, B is a unital C^*-algebra, I is a closed ideal of B, and $(\varphi_t)_{t\in[0,1]}$ is a point-norm continuous path of unital *-homomorphisms from A to

B/I. Suppose φ_0 lifts to a unital *-homomorphism $\overline{\varphi_0} : A \to B$. For each $n \in \mathbb{N}$ and $t \in [0,1]$, let φ_t^n be the restriction of φ_t to A_n, and $\overline{\varphi_0^n}$ be the restriction of $\overline{\varphi_0}$ to A_n. Since finite-dimensional C^*-algebras are semiprojective, A_n satisfies the HLP for each n. Consequently, $(\varphi_t^n)_{t\in[0,1]}$ lifts to a point-norm continuous path $(\overline{\varphi_t^n})_{t\in[0,1]}$ of unital *-homomorphisms from A_n to B, starting at $\overline{\varphi_0^n}$. Moreover, we have

$$\pi \circ \overline{\varphi_t^n} = \pi \circ \overline{\varphi_t^{n+1}}\Big|_{A_n} \quad \text{and} \quad \overline{\varphi_0^n} = \overline{\varphi_0^{n+1}}\Big|_{A_n}, \tag{2.2.12}$$

where $\pi : B \to B/I$ is the quotient map.

We construct the desired path of *-homomorphisms inductively. Set $\psi_t^1 = \overline{\varphi_t^1}$ for A_1 and suppose we have defined $(\psi_t^n)_{t\in[0,1]}$, a point-norm continuous path of *-homomorphisms from A_n to B, such that $\psi_0^n = \overline{\varphi_0^n}$ and $\pi \circ \psi_t^n = \pi \circ \overline{\varphi_t^n}$ for all $t \in [0,1]$. It follows from (2.2.12) that $\pi \circ \psi_t^n = \pi \circ \overline{\varphi_t^{n+1}}\Big|_{A_n}$. By applying Lemma 2.2.8, there exists a continuous path of unitaries $u_t^{(n)} \in B$ such that $\overline{\psi_t^n}(a) = u_t^{(n)}\overline{\varphi_t^{n+1}}(a)u_t^{(n)*}$ for all $a \in A_n$ and $\pi(u_t^{(n)}) = 1$. Define $\psi_t^{n+1} : A_{n+1} \to B$ by $\psi_t^{n+1} = u_t^{(n)}\overline{\varphi_t^{n+1}}u_t^{(n)*}$. Then, $(\psi_t^{n+1})_{t\in[0,1]}$ is a point-norm continuous path of *-homomorphisms extending $(\psi_t^n)_{t\in[0,1]}$ with $\psi_0^{n+1} = \overline{\varphi_0^{n+1}}$ and $\pi \circ \psi_t^{n+1} = \pi \circ \overline{\varphi_t^{n+1}}$ for all $t \in [0,1]$. For each t, extend ψ_t^n by continuity to $\psi_t : A \to B$ to define the desired point-norm continuous path of *-homomorphisms $(\phi_t)_{t\in[0,1]}$ that lifts $(\varphi_t)_{t\in[0,1]}$. ∎

Next, we generalize [24, Theorem 8] to show that the homotopy lifting property is preserved under taking direct sums of finite C^*-algebras.

Theorem 2.2.10. Let A_1 and A_2 be two C^*-algebras that satisfy the homotopy lifting property. Then, $A_1 \oplus A_2$ also satisfies the homotopy lifting property.

Proof. By Proposition 2.2.6, it is sufficient to assume A_1 and A_2 are unital C^*-algebras and to consider the homotopy lifting property (HLP) in the category of unital C^*-algebras and unital *-homomorphisms. Suppose $\theta : A_1 \oplus A_2 \to C([0,1], B/J)$ defines a point-norm continuous path of unital *-homomorphisms connecting $\theta_0 : A_1 \oplus A_2 \to B/J$ to $\theta_1 : A_1 \oplus A_2 \to B/J$ with $\theta_0 = \pi \circ \overline{\theta}_0$, where $\overline{\theta}_0 : A_1 \oplus A_2 \to B$ is a unital *-homomorphism and $\pi : B \to B/J$ is the quotient map. Then $Q_1 = \theta(1_{A_1})$ and $Q_2 = \theta(1_{A_2})$ are orthogonal projections in $C([0,1], B/J)$ with $Q_1 + Q_2 = 1$. Similarly, $p_1 = \overline{\theta}_0(1_{A_1})$ and $p_2 = \overline{\theta}_0(1_{A_2})$ are orthogonal projections in B.

The *-homomorphism $\psi : \mathbb{C}^2 \to C([0,1], B/J)$ defined by $\psi(1,0) = Q_1$ and $\psi(0,1) = Q_2$ gives a point-norm continuous path of unital *-homomorphisms with a liftable starting point. Using the fact that $\mathbb{C}^2$ satisfies the HLP, we can find orthogonal projections P_1 and $P_2 \in C([0,1], B)$ such that $P_1 + P_2 = 1$ and $\widehat{\pi} \circ P_i = Q_i$ for $i = 1,2$, where $\widehat{\pi}$ is the induced natural map by π from $C([0,1], B)$ to $C([0,1], B/J)$.

For $i = 1,2$, set $D_i = Q_iC([0,1], B/J)Q_i = \widehat{\pi}(P_iC([0,1], B)P_i)$. Then $D_1D_2 = 0$. Moreover, define $\chi_i : A_i \to D_i$ and $\gamma_i : A_i \to B$ by

$$\chi_i(a) = Q_i\theta(a)Q_i,$$
$$\gamma_i(a) = p_i\overline{\theta}_0(a)p_i$$

for all $a \in A_i$. By the hypothesis, χ_i lifts to a unital *-homomorphism $\overline{\chi}_i : A_i \to P_iC([0,1], B)P_i$ with $\overline{\chi}_i(a)(0) = \gamma_i(a)$ for all $a \in A_i$. Consequently, $\overline{\chi}_1 \oplus \overline{\chi}_2 : A_1 \oplus A_2 \to C([0,1], B)$ is a unital lift of θ starting at $\overline{\theta}_0 := \gamma_1 \oplus \gamma_2$. ∎

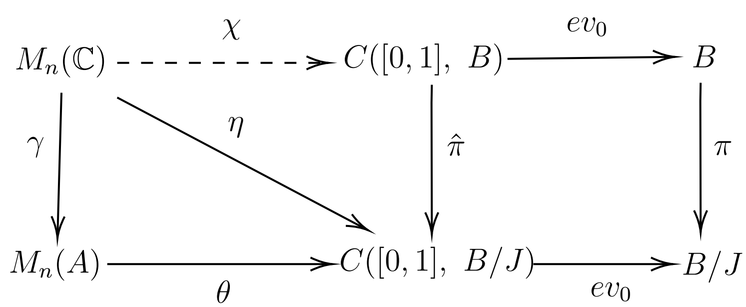
The next result shows that the homotopy lifting property is preserved under taking matrix algebras.

Theorem 2.2.11. If a C^*-algebra A satisfies the homotopy lifting property (HLP), then $M_n(A)$ also satisfies the HLP.

Proof. Without loss of generality, we assume A is unital and use Proposition 2.2.6 to deduce the result. Let $\theta : M_n(A) \to C([0,1], B/J)$ be a point-norm continuous path of unital *-homomorphisms with $\theta_0 = \pi \circ \overline{\theta}_0$, where $\overline{\theta}_0 : M_n(A) \to B$ is a unital *-homomorphism. Define unital *-homomorphisms $\gamma : M_n(\mathbb{C}) \to M_n(A)$ and $\eta : M_n(\mathbb{C}) \to C([0,1], B/J)$ by

$$\gamma(e_{ij}) = 1_A \otimes e_{ij} \ \text{ and } \ \eta = \theta \circ \gamma.$$

Recall that $M_n(\mathbb{C})$ is a semiprojective C^*-algebra and thus satisfies the HLP. The HLP of $M_n(\mathbb{C})$ and the fact that $\eta_0 = \theta_0 \circ \gamma = \pi \circ \overline{\theta}_0 \circ \gamma$ imply that there exists a unital *-homomorphism $\chi : M_n(\mathbb{C}) \to C([0,1], B)$ that completes the following commutative diagram.



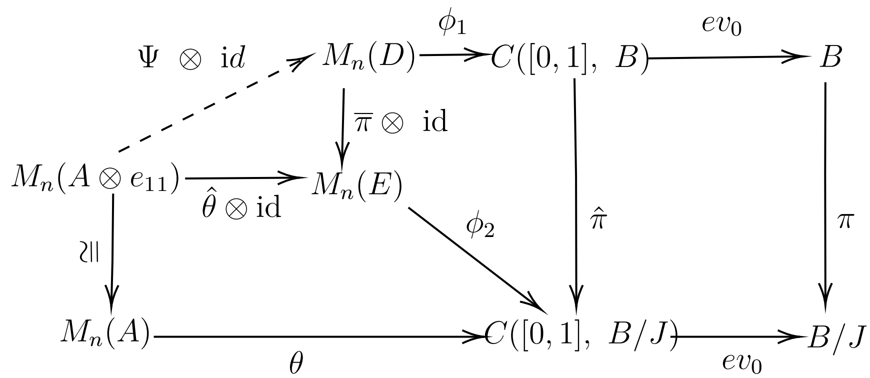
Let

$$D = \chi(e_{11})C([0,1], B)\chi(e_{11}),$$
$$E = \eta(e_{11})C([0,1], B/J)\eta(e_{11}) = \hat{\pi}(D),$$
$$\overline{\pi} = \hat{\pi}|_D \ \text{ and } \ \widehat{\theta} = \theta|_{A\otimes e_{11}}.$$

Note that $(A \otimes e_{11}) \otimes M_n(\mathbb{C}) \cong M_n(A)$ via the map $a \otimes e_{11} \otimes e_{ij} \mapsto a \otimes e_{ij}$. For any $a \in A$, we have

$$\begin{aligned} \theta(a \otimes e_{11}) &= \theta(1_A \otimes e_{11})\theta(a \otimes e_{11})\theta(1_A \otimes e_{11}) \\ &= \theta \circ \gamma(1 \otimes e_{11})\theta(a \otimes e_{11})\theta \circ \gamma(1 \otimes e_{11}) \\ &= \eta(e_{11})\theta(a \otimes e_{11})\eta(e_{11}). \end{aligned} \tag{2.2.13}$$

Hence, $\widehat{\theta}$ defines a point-norm continuous path of unital *-homomorphisms starting at $\widehat{\theta}_0 = \pi \circ \overline{\theta}_0\big|_{A\otimes e_{11}}$ with $\widehat{\theta}(A \otimes e_{11}) \subseteq E$. Using the HLP of $A \cong A \otimes e_{11}$, $\widehat{\theta}$ lifts to a unital *-homomorphism $\Psi : A \otimes e_{11} \to D$ starting at $\overline{\theta}_0\big|_{A\otimes e_{11}}$. The *-homomorphism $\Psi \otimes \mathrm{id}_{M_n} : M_n(A) \to D$ makes the following diagram



commutes, where ϕ_1 and ϕ_2 are *-homomorphisms define by

$$\phi_1(\chi(e_{11})W\chi(e_{11}) \otimes e_{ij}) = \chi(e_{i1})W\chi(e_{1j})$$

and

$$\phi_2(\eta(e_{11})V\chi(e_{11}) \otimes e_{ij}) = \eta(e_{i1})V\eta(e_{1j}).$$

∎

Corollary 2.2.12. Let A be a C^*-algebra satisfying the homotopy lifting property and F a finite dimensional C^*-algebra. Then $A \otimes F$ satisfies the homotopy lifting property.

Proof. The result follows from Theorem 2.2.10 and Theorem 2.2.11. ∎

Proposition 2.2.13. Let A be a unital separable nuclear C^*-algebra satisfying the homotopy lifting property, and let $\mathcal{Q}$ be the Calkin algebra. If two unital injective *-homomorphisms $\phi_0, \phi_1 : A \to \mathcal{Q}$ are homotopic, then ϕ_0 is unitarily equivalent to ϕ_1.

Proof. First consider the case that ϕ_0 is a liftable unital, injective *-homomorphism, that is, an essential trivial extension. The HLP implies ϕ_1 is also an essential trivial extension. By Voiculescu's theorem [79, Theorem 1.4] , all essential trivial extensions are unitarily equivalent and thus ϕ_0 is unitarily equivalent to ϕ_1.

Recall that if A is a unital nuclear C^*-algebra, $Ext_s(A)$ is a group. If ϕ_0 is not a trivial extension, then there exists a unital injective *-homomorphism $\phi : A \to \mathcal{Q}$ such that $\phi_0 \oplus \phi$ is an essential trivial extension, since $Ext_s(A)$ is a group. $\phi_0 \oplus \phi$ is homotopic to $\phi_1 \oplus \phi$. Hence, $\phi_1 \oplus \phi$ is an essential trivial extension by the HLP. By Voiculescu's theorem [79, Theorem 1.4], the maps $\phi_0 \oplus \phi$ and $\phi_1 \oplus \phi$ are unitarily equivalent, as both are essentially trivial extensions. Hence $[\phi_0] + [\phi] = [\phi_1] + [\phi]$. Adding the inverse $[\overline{\phi}]$ of $[\phi]$ to both sides, we get $[\phi_0] = [\phi_0] + [\phi \oplus \overline{\phi}] = [\phi_1] + [\phi \oplus \overline{\phi}] = [\phi_1]$, where the first and the last equality follow from the unital-absorbing property of ϕ_0 and ϕ_1 [12, Theorem 15.12.3]. Therefore, ϕ_0 is unitarily equivalent to ϕ_1. ∎

For any unital *-homomorphism $\phi : A \to B$ between unital separable C^*-algebras, we define $\phi^* : Ext_s(B) \to Ext_s(A)$ by $\phi^*([\tau]) = [\tau \circ \phi \oplus \tau_0]$, where $\tau : B \to \mathcal{Q}$ is an essential extension and τ_0 is any essential trivial extension.

Corollary 2.2.14. Let A be a unital separable nuclear C^*-algebra satisfying the homotopy lifting property and B a unital separable C^*-algebra. If two unital *-homomorphisms $\phi_0, \phi_1 : A \to B$ are homotopic, then $\phi_0^* = \phi_1^*$.

Proof. Let $[\tau] \in Ext_s(B)$. Then $\phi_0^*([\tau]) = [\tau \circ \phi_0 \oplus \tau_0]$ and $\phi_1^*([\tau]) = [\tau \circ \phi_1 \oplus \tau_0]$ for some essential trivial extension τ_0. Since ϕ_0 is homotopic to ϕ_1, $\tau \circ \phi_0 \oplus \tau_0$ is homotopic to $\tau \circ \phi_1 \oplus \tau_0$. By Proposition 2.2.13, this implies $[\tau \circ \phi_0 \oplus \tau_0] = [\tau \circ \phi_1 \oplus \tau_0]$, and thus $\phi_0^* = \phi_1^*$. ∎

Theorem 2.2.15 (Homotopy Lifting Theorem; cf. [15, Theorem 5.1])**.** Let A be an ℓ-open C^*-algebra (or more generally, one that satisfies the conclusion of Theorem 2.2.1). Then A satisfies the homotopy lifting property.

Proof. Let B be a C^*-algebra, I a closed ideal of B, and $(\varphi_t)_{t\in[0,1]}$ a point-norm continuous path of *-homomorphisms from A to B/I. Suppose φ_0 lifts to a *-homomorphism $\overline{\varphi_0} : A \to B$. Take an arbitrary finite set $\mathcal{F} \subseteq A$ and a real number $\epsilon > 0$, and let $\mathcal{G}$ and δ be given by Theorem 2.2.1. We can find a partition $t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1$ such that $\|\varphi_t(a) - \varphi_s(a)\| < \delta$ for all $a \in \mathcal{G}$ whenever $t, s \in [t_{i-1}, t_i]$ for any i.

Let $D \coloneqq C([0, t_1], B)$ and $J \coloneqq C_0((0, t_1], I)$, which is an ideal of D, so that

$$\begin{aligned} D/J &\cong C([0,t_1], B/I) \oplus_{\pi_I} B \\ &= \{(f, b) \in C([0,t_1], B/I) \oplus B \mid f(0) = \pi_I(b)\}. \end{aligned} \tag{2.2.14}$$

Making this identification, define *-homomorphisms $\gamma \coloneqq (\mathrm{id}_{C([0,t_1])} \otimes \varphi_0) \oplus \overline{\varphi_0}$ and $\theta \coloneqq \varphi|_{[0,t_1]} \oplus \overline{\varphi_0} : A \to D/J$ (where $\varphi|_{[0,t_1]}$ denotes the *-homomorphism $A \to C([0,t_1], B/I)$ given by restricting the homotopy (φ_t) to $[0, t_1]$). Then γ lifts to the *-homomorphism $\mathrm{id}_{C([0,t_1])} \otimes \overline{\varphi_0}$, so by Theorem 2.2.1, φ lifts, giving a continuous path of lifts $(\overline{\varphi_t})$ of (φ_t) for $t \in [0, t_1]$. Continuing the same process for successive intervals $[t_1, t_2], \ldots, [t_{n-1}, t_n]$, we get the required continuous path $(\overline{\varphi_t})_{t\in[0,1]}$, such that $\overline{\varphi_t}$ lifts φ_t for all $t \in [0, 1]$. ∎

Combining all the previous theorems and corollaries, we have the following characterization of ℓ-open C^*-algebra.

Theorem 2.2.16. Let A be a C^*-algebra. Then the following are equivalent

(i) A is ℓ-open.

(ii) The system of maps $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/J)$ (over all C^*-algebras B and ideals J) is uniformly open, as in the conclusion of Theorem 2.2.1

(iii) A satisfies the homotopy lifting property (HLP) and $\mathrm{Hom}(A, B)$ is locally path-connected for all C^*-algebras B.

Proof. (i)⇒(ii) is Theorem 2.2.1 and (ii)⇒(iii) is by Corollary 2.2.2 and Theorem 2.2.15.

To prove that (iii) ⇒ (i), let $\phi_n : A \to B/I$ be a net of *-homomorphisms that converges point-norm to a liftable *-homomorphism $\phi : A \to B/I$. Since $\mathrm{Hom}(A, B/I)$ is locally path-connected, ϕ_n is homotopic to ϕ for sufficiently large n. The homotopy lifting property implies that ϕ_n is liftable for these n. This shows that $\mathrm{Hom}(A, B/I)$ is open in $\mathrm{Hom}(A, B/I)$, as required. ∎

Example 2.2.17. Satisfying the homotopy lifting property does not guarantee ℓ-openness of C^*-algebras. M_{2^∞} satisfies the HLP (see Theorem 2.2.9), but it is not an ℓ-open C^*-algebra. To see that M_{2^∞} is not ℓ-open, suppose otherwise.

Given any finite set $\mathcal{F} \subseteq M_{2^\infty}$ and any $\epsilon > 0$, we can obtain $\delta > 0$ and a finite set $\mathcal{G} \subset M_{2^\infty}$ according to Theorem 2.2.1. Without loss of generality, we can assume $\mathcal{G} \subset M_{2^k}$ for some k.

Let $B := B(\mathcal{H})$ and $J := \mathcal{K}$, so that B/J is the Calkin algebra. Let $\phi_1, \phi_2 : A \to B/J$ be unital *-homomorphisms such that ϕ_1 is liftable but ϕ_2 is not (such a ϕ_2 exists by [74]). Define $\varphi_i := \mathrm{id}_{M_{2^k}} \otimes \phi_i : M_{2^k} \otimes M_{2^\infty} \cong M_{2^\infty} \to M_{2^k} \otimes (B/J) \cong (M_{2^k} \otimes B)/(M_{2^k} \otimes J)$.

Then we have $\varphi_1(a) = \varphi_2(a)$ for all $a \in \mathcal{G}$. Hence, Theorem 2.2.1 tells us that if φ_1 is liftable, so is φ_2. The Ext-class of φ_2 is 2^k times the Ext-class of ϕ_2; since $Ext_s(M_{2^\infty})$ is the 2-adic integers (which are torsion-free), it follows that φ_2 is not liftable, leading to a contradiction. Hence, M_{2^∞} is not ℓ-open.

The characterization of ℓ-openness confirms a conjecture of Blackadar [15, Page 299], as follows.

Corollary 2.2.18. Let A be an ℓ-open C^*-algebra. Then A is ℓ-closed.

Proof. Fix $\epsilon > 0$ and a finite set $\mathcal{F} \subset A$, and choose $\delta > 0$ and a finite set $\mathcal{G} \subset A$ as in Theorem 2.2.1. Let $\phi_n : A \to B/I$ be a net of liftable *-homomorphisms that converges point-norm to a *-homomorphism $\phi : A \to B/I$. We can find an index m such that $\|\phi_m(u) - \phi(u)\| < \delta$ for all $u \in \mathcal{G}$. Since ϕ_m is liftable, the conclusion of Theorem 2.2.1 implies that ϕ is also liftable. Hence, A is ℓ-closed. ∎

In what follows, we collect some properties of ℓ-open C^*-algebras.

Lemma 2.2.19 ([50, Theorem 10.1.11])**.** Let A_1 and A_2 be σ-unital C^*-algebras, $\pi : B \to C$ be a surjective *-homomorphism between C^*-algebras, and $\theta : A_1 \oplus A_2 \to C$ be a *-homomorphism. Then there exist strictly positive contractions $h_i \in A_i$ and orthogonal strictly positive contractions $k_i \in B$ such that $\theta(h_i)$ is lifted by k_i, and *-homomorphisms $\theta^i : A_i \to \pi(\overline{k_i B k_i})$ for $i = 1, 2$ such that $\theta = \theta^1 \oplus \theta^2$.

Proposition 2.2.20. Let A_1 and A_2 be σ-unital ℓ-open C^*-algebras. Then $A_1 \oplus A_2$ is ℓ-open.

Proof. Let $\theta_n : A_1 \oplus A_2 \to B/I$ be a net of *-homomorphisms which converges point-norm to a liftable *-homomorphism $\theta : A_1 \oplus A_2 \to B/I$. Each $\theta^i = \theta|_{A_i}$ is liftable, and $\theta^i_n \to \theta^i$ in the point-norm topology. Since A_i is ℓ-open, θ^i_n are liftable *-homomorphisms for sufficiently large n. By Lemma 2.2.19, we can choose orthogonal strictly positive contractions k^n_i and *-homomorphisms $\overline{\theta^i_n} : A_i \to k^n_i B k^n_i$ such that $\overline{\theta_n} = \overline{\theta^1_n} \oplus \overline{\theta^2_n}$ is a lift of θ_n. ∎

Lemma 2.2.21 ([50, Theorem 10.2.3])**.** Let A be a σ-unital C^*-algebra and $\phi : M_n(A) \to B/I$ a *-homomorphism. If $\phi|_A$ is a liftable *-homomorphism, then ϕ is a liftable *-homomorphism.

Proposition 2.2.22. Let A be a σ-unital ℓ-open C^*-algebras. Then $M_n(A)$ is ℓ-open.

Proof. Let $\theta_n : M_n(A) \to B/I$ be a net of *-homomorphisms which converges ponit-norm to a liftable *-homomorphism $\theta : M_n(A) \to B/I$. Then $\theta_n|_A$ converges pointwise to $\theta|_A$. Since $\theta|_A$ is liftable, $\theta_n|_A$ is liftable for sufficiently large n. By Proposition 2.2.21, θ_n is liftable for sufficiently large n. ∎

Recall that a C^*-algebra A homotopically dominates a C^*-algebra B if there exists *-homomorphisms $\theta : A \to B$ and $\gamma : B \to A$ such that $\theta \circ \gamma$ is homotopic to id_B.

Proposition 2.2.23. If a ℓ-open C^*-algebra A homotopically dominates a C^*-algebra B satisfying the homotopy lifting property, then B is ℓ-open.

Proof. Let $\theta : A \to B$ and $\gamma : B \to A$ be *-homomorphisms such that $\theta \circ \gamma \backsim_h \mathrm{id}_B$. Let $\phi_n : B \to C/I$ be a net of *-homomorphisms that converges point-norm to a liftable *-homomorphism $\phi : B \to C/I$. Let $\overline{\phi} : B \to C$ be a *-homomorphism with $\phi = \pi_I \circ \overline{\phi}$. Then, $\phi_n \circ \theta$ converges point-norm to $\phi \circ \theta = \pi_I \circ \overline{\phi} \circ \theta$. Since A is ℓ-open and $\phi \circ \theta$ is a liftable *-homomorphism, $\phi_n \circ \theta$ and $\phi_n \circ \theta \circ \gamma$ are liftable *-homomorphism for sufficiently large n. Note that $\phi_n \backsim_h \phi_n \circ \theta \circ \gamma$. By the homotopy lifting property (HLP) of B, ϕ_n is a liftable *-homomorphism, and therefore B is ℓ-open.

∎

2.3 Characterization of ℓ-closed C^*-algebras

We now characterize ℓ-closed C^*-algebras, showing that the condition is equivalent to the uniform relative openness of the map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/I)$. We require separability for this characterization, and one direction uses a Cauchy sequence argument.

Theorem 2.3.1. Let A be a separable C^*-algebra. Then the following are equivalent:

(i) A is ℓ-closed.

(ii) For any $\epsilon > 0$ and finite set $\mathcal{F} \subset A$, there is a $\delta > 0$ and a finite set $\mathcal{G} \subset A$ such that whenever B is a C^*-algebra, I is a closed ideal of B, ψ and ϕ are *-homomorphisms from A to B with $\|\pi_I \circ \phi(u) - \pi_I \circ \psi(u)\| < \delta$ for all $u \in \mathcal{G}$, then there exists a *-homomorphism $\eta : A \to B$ such that $\|\phi(v) - \eta(v)\| < \epsilon$ for all $v \in \mathcal{F}$ and $\pi_I \circ \psi = \pi_I \circ \eta$.

Proof. (i)⇒(ii). Let $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of A whose union is dense in A. Suppose that (ii) is false for a fixed $\epsilon > 0$ and finite set $\mathcal{F} \subset A$. Then, there are C^*-algebras B_n with ideals I_n, and *-homomorphisms $\phi_n, \psi_n : A \to B_n$ such that

$$\|\pi_{I_n} \circ \phi_n(a) - \pi_{I_n} \circ \psi_n(a)\| < \frac{1}{n} \quad \text{for all } a \in \mathcal{G}_n, \tag{2.3.1}$$

but no *-homomorphism $\eta_n : A \to B_n$ satisfies both $\|\phi_n(a) - \eta_n(a)\| < \epsilon$ for all $a \in \mathcal{F}$ and $\pi_{I_n} \circ \psi_n = \pi_{I_n} \circ \eta_n$.

Let $B := \prod_{n=1}^{\infty} B_n$, $I := \prod_{n=1}^{\infty} I_n$, and $J := \bigoplus_{n=1}^{\infty} B_n$. Define *-homomorphisms $\overline{\phi} := (\phi_1, \phi_2, \dots), \overline{\psi} := (\psi_1, \psi_2, \dots) : A \to B$.

By (2.3.1), it follows that $\pi_{I+J} \circ \overline{\phi} = \pi_{I+J} \circ \overline{\psi}$. Then by the general Chinese remainder theorem (Lemma 2.1.6), there exists a *-homomorphism $\theta : A \to B/(I \cap J)$ such that

$$\pi_J \circ \overline{\phi} = \pi_J \circ \theta \quad \text{and} \ \pi_I \circ \overline{\psi} = \pi_I \circ \theta. \tag{2.3.2}$$

For each $n \in \mathbb{N}$, define the *-homomorphism

$$\overline{\alpha}_n := (\psi_1, \psi_2, \dots, \psi_{n-1}, \phi_n, \phi_{n+1}, \dots) : A \to B. \tag{2.3.3}$$

Then by the definition of J, we have $\pi_J \circ \overline{\alpha}_n = \pi_J \circ \overline{\phi}$. Therefore by (2.3.2), for $x \in A$,

$$\begin{aligned} \|\pi_{I\cap J} \circ \overline{\alpha}_n(x) - \theta(x)\| &= \|\pi_I \circ \overline{\alpha}_n(x) - \pi_I \circ \overline{\psi}(x)\| \\ &= \sup_{m \geq n} \|\pi_{I_m} \circ \phi_m(x) - \pi_{I_m} \circ \psi_m(x)\| \to 0. \end{aligned} \tag{2.3.4}$$

Since A is ℓ-closed, we deduce that θ lifts to a *-homomorphism $\eta = (\eta_1, \eta_2, \dots) : A \to B$. Then (2.3.2) implies that $\pi_{I_n} \circ \psi_n = \pi_{I_n} \circ \eta_n$ and $\lim_{n\to\infty} \|\phi_n(x) - \eta_n(x)\| = 0$ for all $x \in A$. Hence, there is a k such that

$$\|\phi_k(a) - \eta_k(a)\| < \epsilon \tag{2.3.5}$$

for all $a \in \mathcal{F}$. This is a contradiction.

(ii)$\Rightarrow$(i). Suppose $\eta_n : A \to B/I$ is a sequence of liftable *-homomorphisms which converges pointwise to a *-homomorphism $\eta : A \to B/I$. Let $(\mathcal{F}_n)$ be an increasing sequence of finite sets whose union is dense in A. Choose $\delta_n > 0$ and a finite set $\mathcal{G}_n$ such that they satisfy the conditions of (ii) with $\epsilon := \frac{1}{2^n}$ and $\mathcal{F} := \mathcal{F}_n$. By passing to a subsequence, we may assume without loss of generality that

$$\|\eta_n(u) - \eta_{n+1}(u)\| < \delta_n \quad \text{for all } u \in \mathcal{G}_n. \tag{2.3.6}$$

Let $\overline{\eta}_n : A \to B$ be a lift of η_n. Then the choice of $\mathcal{G}_1$ and δ_1 from (ii) implies that there exists a *-homomorphism $\xi_2 : A \to B$ such that $\|\overline{\eta}_1(v) - \xi_2(v)\| < \frac{1}{2}$ for all $v \in \mathcal{G}_1$ and $\pi_I \circ \overline{\eta}_2 = \pi_I \circ \xi_2$. Then we have $\|\pi_I \circ \overline{\eta}_2(u) - \pi_I \circ \overline{\eta}_3(u)\| = \|\pi_1 \circ \xi_2(u) - \pi_I \circ \overline{\eta}_3(u)\| < \delta_2$ for all $u \in \mathcal{G}_2$. Using the choice of $\mathcal{G}_2$ and δ_2 from (ii), we have a *-homomorphism $\xi_3 : A \to B$ such that $\|\xi_2(v) - \xi_3(v)\| < \frac{1}{2^2}$ and $\pi_I \circ \overline{\eta}_3 = \pi_I \circ \xi_3$. Continuing the process and setting $\xi_1 = \overline{\eta}_1$, we get a sequence $(\xi_n : A \to B)$ such that $\|\xi_n(a) - \xi_{n+1}(a)\| < \frac{1}{2^n}$ for all $a \in \mathcal{F}_n$ and $\eta_n = \pi_I \circ \xi_n$. Consequently, the sequence $(\xi_n(a))_{n=1}^{\infty}$ is Cauchy for each $a \in A$, so it converges to some $\xi(a) \in B$. This defines a *-homomorphism $\xi : A \to B$, and for $a \in A$,

$$\pi_I \circ \xi(a) = \lim_n \pi_I \circ \xi_n(a) = \lim_n \eta_n(a) = \eta(a). \tag{2.3.7}$$

Therefore we obtain a lift of η, and this shows that A is ℓ-closed. ∎

Note that condition (iii) of Theorem 2.2.16 strengthens condition (ii) in Theorem 2.3.1, by replacing $\psi : A \to B$ with a map $A \to B/I$ which is (a priori) not liftable. This gives a quick proof of Corollary 2.2.18 in the separable case.

Theorem 2.3.1 may be reformulated as follows.

Theorem 2.3.2. Let A be a separable C^*-algebra and S a generating set of A. Then the following are equivalent:

(i) A is ℓ-closed.

(ii) For any $\epsilon > 0$ and finite set $\mathcal{F} \subset S$, there is a $\delta > 0$ and a finite set $\mathcal{G} \subset S$ such that whenever B is a C^*-algebra, I is a closed ideal of B, ψ and ϕ are *-homomorphisms from A to B with $\|\pi_I \circ \phi(u) - \pi_I \circ \psi(u)\| < \delta$ for all $u \in \mathcal{G}$, then there exists a *-homomorphism $\eta : A \to B$ such that $\|\phi(v) - \eta(v)\| < \epsilon$ for all $v \in \mathcal{F}$ and $\pi_I \circ \psi = \pi_I \circ \eta$.

Proof. $(i) \Rightarrow (ii)$ Let $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of S whose union is dense in S. Suppose that (ii) is false for a finite set $\mathcal{F} \subset S$. Then follow the corresponding proof of Theorem 2.3.1.

$(ii) \Rightarrow$(i). Let $(\mathcal{F}_n)$ be an increasing sequence of finite sets whose union is dense in S. Then follow the corresponding proof of Theorem 2.3.1. ∎

In [15, Example 6.4], Blackadar asks whether $C^*(\mathbb{F}_\infty)$, the universal C^*-algebra generated by a sequence of unitaries, is ℓ-closed. We now show that it is.

Example 2.3.3. $C^*(\mathbb{F}_\infty)$ is ℓ-closed. To see this, consider $\epsilon > 0$, a finite set $\mathcal{F} \subset \{u_1, u_2, \ldots\}$, an ideal I of B, and *-homomorphisms $\phi, \psi : C^*(\mathbb{F}_\infty) \to B$. Without loss of generality, we may assume $\mathcal{F} = \{u_1, u_2, \ldots, u_n\}$ for some n. Since $C^*(\mathcal{F}) \cong C^*(\mathbb{F}_n)$ is semiprojective (this is well-known; see [11, Corollary 2.22 and Proposition 2.31] for example) and so ℓ-closed by [15, Corollary 6.2]. Choose $\delta > 0$ and a finite set $\mathcal{G} \subset \mathcal{F}$ as in Theorem 2.3.2 (applied to $C^*(\mathcal{F})$). Then, $\|\pi_I \circ \phi(u) - \pi_I \circ \psi(u)\| < \delta$ for all $u \in \mathcal{G}$ implies there exists a *-homomorphism $\xi : C^*(\mathbb{F}_n) \to B$ such that $\|\phi(v) - \xi(v)\| < \epsilon$ for all $v \in \mathcal{F}$ and $\pi_I \circ \xi = \pi_I \circ \psi|_{C^*(\mathbb{F}_n)}$. Define $\eta : C^*(\mathbb{F}_\infty) \to B$ by

$$\eta(u_m) := \begin{cases} \xi(u_m), & \text{for } m \le n, \\ \psi(u_m), & \text{for } m > n. \end{cases} \tag{2.3.8}$$

Then η is a *-homomorphism satisfying $\|\phi(v) - \eta(v)\| < \epsilon$ for all $v \in \mathcal{F}$ and $\pi_I \circ \eta = \pi_I \circ \psi$, as required.

We state some properties of ℓ-closed C^*-algebras.

Proposition 2.3.4. Let A be a ℓ-closed C^*-algebra. Then, any path-connected component of $Hom(A, B)$ is closed for any C^*-algebra B.

Proof. Let U be a path-connected component of $Hom(A, B)$, $D := C([0, 1], B)$ and $I := C_0((0, 1), B)$. Then $D/I \cong B \oplus B$. Suppose $(\phi_n)_{n \in \Lambda} \subseteq U$ is a net of *-homomorphisms $\phi_n : A \to B$ converging point-norm to a *-homomorphism $\phi : A \to B$. Fix $n_0 \in \Lambda$. Then $\phi_n \backsim_h \phi_{n_0}$ for all $n \in \Lambda$.

Define *-homomorphisms $\gamma_n, \gamma : A \to D/I$ by

$$\gamma_n(x) := (\phi_n(x), \phi_{n_0}(x))$$

and

$$\gamma(x) := (\phi(x), \phi_{n_0}(x)).$$

Since γ_n is a liftable *-homomorphism for each n and γ_n converges point-norm to γ, we use the ℓ-closedness of A to conclude that γ is liftable. So, $\phi \backsim_h \phi_{n_0}$ and $\phi \in U$. ∎

The next result is a generalization of [14, Proposition 2.1] to C^*-algebra..

Corollary 2.3.5. Let $C(X)$ be a ℓ-closed C^*-algebra. Then, X is an e-closed space with closed path-connected components.

Proof. X is an e-closed space by definition of ℓ-closed C^*-algebra and $Hom(C(X), \mathbb{C}) \cong X$ has closed path-connected components by Proposition 2.3.4. ∎

Example 2.3.6. The topologist's sine curve:

$$X := \{(x, y) : y = \sin(\frac{\pi}{x}), 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\} \tag{2.3.9}$$

has an open path connected component. Thus $C(X)$ is not ℓ-closed. This example illustrates that weakly semiprojective C^*-algebras are not ℓ-open or ℓ-closed in general.

Enders and Shulman [35] proved that e-closedness of X is not sufficient for $C(X)$ to be ℓ-closed and gave the following necessary condition for ℓ-closedness of $C(X)$:

Proposition 2.3.7 ([35, Corollary 5.16])**.** Let X be a CW-complex. If $C(X)$ is a ℓ-closed C^*-algebra, then $dim(X) \leq 3$.

Proposition 2.3.8. Let A_1 and A_2 be ℓ-closed σ-unital C^*-algebras. Then $A_1 \oplus A_2$ and $M_n(A_1)$ are ℓ-closed C^*-algebras.

Proof. The result follows using similar arguments as in Propositions 2.2.22 and 2.2.20. ∎

We can see that ℓ-closedness is often very restrictive, just by looking at the case $B := \mathcal{B}(\ell^2)$ and $I := \mathcal{K}$. Here, $\mathrm{Hom}(A, B/I)$ corresponds to extensions of A by the compact operators, with $\mathrm{Hom}(A, B, I)$ corresponding to the subset of trivial extensions. Therefore, if A is ℓ-closed then the set of trivial extensions is closed in the set of all extensions (using the corresponding topology). We first state a general result before narrowing it down to extensions by the compact operators.

Theorem 2.3.9. Let A be a nuclear separable C^*-algebra satisfying the UCT and quasidiagonal relative to a σ-unital separable C^*-algebra B. If A is additionally ℓ-closed, then all essential quasidiagonal extensions τ of A over B are trivial in $Ext(A, B)$. Equivalently,

$$Pext^1_{\mathbb{Z}}(K_0(A), K_0(B)) = Pext^1_{\mathbb{Z}}(K_1(A), K_1(B)) = 0.$$

Proof. By Theorem 1.3.23, there exists a sequence $(\tau_n)_{n\in\Lambda}$ of essential trivial extensions such that $[[\tau_n]]$ converges to $[[\tau]]$. By the Brown-Salinas topology on $Ext(A, B)$, we can find an absorbing trivial extension η and liftable unitaries u_n such that

$$\|u_n(\tau_n(a) \oplus \eta(a))u_n^* - \tau(a) \oplus \eta(a)\| \to 0$$

for each $a \in A$. From the ℓ-closedness of A, we conclude that $\tau \oplus \eta$ is a liftable *-homomorphism. Hence, $[[\tau]] = [[\tau \oplus \eta]] = 0$. We also obtain from Theorem 1.3.23 that

$$Pext^1_{\mathbb{Z}}(K_0(A), K_0(B)) = Pext^1_{\mathbb{Z}}(K_1(A), K_1(B)) = 0.$$

∎

Corollary 2.3.10. Suppose that A is a nuclear separable C^*-algebra satisfying the UCT and that A is quasidiagonal. If A is additionally ℓ-closed, then all essential quasidiagonal extensions τ of A over $\mathcal{K}$ is trivial in $Ext(A, \mathbb{C})$. Equivalently,

$$Pext^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) = 0.$$

Proof. By Example 1.3.15 and Proposition 1.3.19, A is quasidiagonal relative to $\mathbb{C}$. Hence, the result follows from Theorem 2.3.9. ∎

Example 2.3.11. Any infinite dimensional noncommutative UHF algebra A is not ℓ-closed, since for these C^*-algebras, all extensions are quasidiagonal, but they are not all trivial in $Ext(A, \mathbb{C})$ (see [64, Remark 2.13] and Remark 1.3.24).To see why quasidiagonal extensions of A are not trivial in $Ext(A, \mathbb{C})$, we first discuss the quasidiagonal extensions of AF-algebras over $\mathcal{K}$. Recall that for any AF-algebra $\mathcal{D} = \varinjlim D_n$, $Ext(\mathcal{D}, \mathbb{C}) \cong Ext^1_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{Z})$ (See [39, Corollary II.10]) and $K_0(\mathcal{D})$ is the union of an increasing sequence of finitely generated abelian group $K_0(\mathcal{D}_n)$ with $K_0(\mathcal{D}_n) \cong \mathbb{Z}^{m_n}$ for some $m_n \in \mathbb{N}$. By Theorem 1.3.21((c) & (d)) and Theorem 1.3.23,

$$Ext(\mathcal{D}, \mathbb{C}) \cong Ext^1_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{Z}) = PExt^1_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{Z}) \cong Ext_{QD}(\mathcal{D}, \mathbb{C}).$$

In other words, all essential extensions of $\mathcal{D}$ over $\mathcal{K}$ are quasidiagonal.

For an infinite dimensional noncommutative UHF algebra $A = \varinjlim M_n(\mathbb{C})$, (1.3.8) implies

$$PExt^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) = \varprojlim{}^1 Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}).$$

Consequently, $Ext(A, \mathbb{C}) = Ext_{QD}(A, \mathbb{C}) = Pext^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \neq 0$ (see [70, Example 3.6]).

Example 2.3.12. Let Y be the triadic solenoid, that is, the inverse limit of circles $\mathbb{T}_n = \mathbb{T}$ with $f_n : \mathbb{T}_{n+1} \to \mathbb{T}_n$ defined by $f_n(z) = z^3$. Let X be the suspension of Y. Then, X is a two dimensional compact space with $Ext_{QD}(X) \neq 0$ by [29, Page 297]. Therefore, $Hom(C(X), B(\ell^2), \mathcal{K})$ is not closed in $Hom(C(X), \mathcal{Q})$, and $C(X)$ is not an ℓ-closed C^*-algebra.

2.4 Commutative unital ℓ-open C^*-algebras

In this section, we show that commutative unital separable ℓ-open C^*-algebras coincide with commutative unital separable semiprojective C^*-algebras. We begin with the following which may be of independent interest.

Proposition 2.4.1. Let A be an ℓ-open C^*-algebra and $\psi : A \to B$ a weakly semiprojective *-homomorphism. Then ψ is a semiprojective *-homomorphism.

Proof. Fix $\epsilon > 0$ and a finite set $\mathcal{F}$ in A, and let $\delta > 0$ and $\mathcal{G} \subset A$ be given by Theorem 2.2.1. Given any *-homomorphism $\varphi : B \to C/\overline{\bigcup_n J_n}$ with $J_1 \triangleleft J_2 \triangleleft \cdots \triangleleft C$ an increasing sequence of closed ideals of a C^*-algebra C, by weak semiprojectivity we can find some n and a *-homomorphism $\phi : A \to C/J_n$ such that

$$\|\varphi \circ \psi(u) - \pi \circ \phi(u)\| < \delta \tag{2.4.1}$$

for all $u \in \mathcal{G}$. It follows from Theorem 2.2.1 that there exists a *-homomorphism $\rho : A \to C/J_n$ such that $\varphi \circ \psi = \pi \circ \rho$. Hence ψ is a semiprojective *-homomorphism. ∎

Lemma 2.4.2 ([23], Proposition 3.1)**.** Let X be a compact, connected, and locally connected metric space of covering dimension greater than 1. Then X contains a topological copy of the circle S^1.

Recall that metrizable space X is an absolute neighbourhood retract (ANR) if, for any metrizable space Y and closed subspace Z of Y, there exists a neighbourhood V of Z such that any continuous map $\eta : Z \to X$ extends to a continuous map $\theta : V \to X$.

Theorem 2.4.3. Let X be a compact metric space. Then the following are equivalent

(i) $C(X)$ is a semiprojective C^*-algebra.

(ii) $C(X)$ is an ℓ-open C^*-algebra.

(iii) X is an ANR and $\dim(X) \leq 1$.

Proof. (i)⇒(ii) follows from [15, Corollary 6.2] and (iii)⇒(i) follows from [73, Theorem 1.2]. We prove that (ii)⇒(iii), along the lines of Sørensen and Thiel's proof of [73, Proposition 3.1].

Suppose $C(X)$ is ℓ-open. Then X is e-open by definition of ℓ-openness of $C(X)$. Blackadar showed that X is locally contractible [14, Corollary 4.3]. The Homotopy Lifting Theorem (Theorem 2.2.15) implies the homotopy extension theorem for X; since X is also locally contractible, we have that X is an ANR by [40, Theorem IV.2.4].

Suppose by contradiction that $\dim(X) \geq 2$. Since X is compact, we have that $\operatorname{locdim}(X) = \dim(X) \geq 2$, which implies that there is an $x_0 \in X$ such that $\dim(D) \geq 2$ for every closed neighbourhood D of x_0 (see [54] for details on $\operatorname{locdim}(X)$). Let $D_1, D_2, \ldots$ be a decreasing sequence of closed neighbourhoods of x_0 with $\dim(D_k) \geq 2$ for all k. Using Lemma 2.4.2, there exists a topological embedding $\psi_k : S^1 \hookrightarrow D_k \subset X$ for each k. Let

$$Y := (0,0) \cup \bigcup_{k\geq 1} S((\frac{1}{2^k}, 0), \frac{1}{4\cdot 2^k}) \subset \mathbb{R}^2, \tag{2.4.2}$$

where $S(x, r)$ is the circle centered at x of radius r. Then $C(Y)$ is weakly semiprojective ([73]]). Define $\psi : Y \to X$ to send $(0,0)$ to x_0 and to be ψ_k on the circle $S((\frac{1}{2^k}, 0), \frac{1}{4.2^k})$. Then ψ induces a *-homomorphism $\psi^* : C(X) \to C(Y)$, which is weakly semiprojective since $C(Y)$ is.

Let $\mathcal{T}$ be the Toeplitz algebra and let $\mathcal{K}$ be the ideal of compact operators. Set

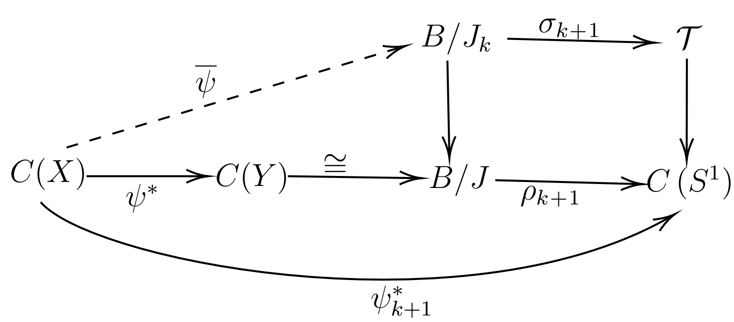
$$\begin{aligned} B &:= (\bigoplus_{k\geq 1} \mathcal{T})^+ \\ &= \{(t_1, t_2, \ldots,) \in \prod_{k\geq 1} \mathcal{T} : (t_k)_k \text{ converges to a scalar multiple of } 1_{\mathcal{T}}\} \end{aligned} \tag{2.4.3}$$

and $J_k := \underbrace{\mathcal{K} \oplus \mathcal{K} \oplus \cdots \mathcal{K}}_{k\ times} \oplus 0 \oplus 0 \cdots$. Then $J_k \subset J_{k+1}$, $J = \overline{\bigcup_k J_k} = \bigoplus_{k\geq 1} \mathcal{K}$,

$$B/J_k = \underbrace{C(S^1) \oplus C(S^1) \oplus \cdots \oplus C(S^1)}_{k\ times} \oplus (\bigoplus_{l\geq k+1} \mathcal{T})^+, \tag{2.4.4}$$

and $B/J = (\bigoplus_{k\geq 1}(C(S^1)))^+ \cong C(Y)$.

Proposition 2.4.1 implies ψ^* is a semiprojective *-homomorphism, so ψ^* lifts to some $\overline{\psi} : C(X) \to B/J_k$.



Let $\sigma_{k+1} : B/J_k \to \mathcal{T}$ be the projection of B/J_k onto the (k+1)-th coordinate and $\rho_{k+1} : B/J \to C(S^1)$ be the projection of B/J onto the (k+1)-th coordinate. Note that $\rho_{k+1} \circ \psi^* : C(X) \to C(S^1)$ coincide with the *-homomorphism induced by $\psi_{k+1} : S^1 \hookrightarrow D_{k+1} \subset X$ and it is surjective since ψ_{k+1} is an inclusion. The generating unitary of $C(S^1)$ lifts to a normal element in $C(X)$ under ψ^*_{k+1}, but it does not lift to a normal element in $\mathcal{T}$, which is a contradiction. Hence, $\dim(X) \leq 1$.

∎

Chapter 3

C^*-diagonals of Inductive Limits of 1-dimensional Noncommutative CW-complexes

1-dimensional Noncommutative CW-complexes (NCCW complexes) are a crucial class of C^*-algebras that extend 1-dimensional classical CW-complexes into the non-commutative world. Key characteristics of these C^*-algebras include being finitely generated [31, Lemma 2.3] and semiprojective [31, Lemma 2.3], which simplifies their study and analysis. Due to their well-understood structure, they serve as useful building blocks in the classification of more complex C^*-algebras.

Recent breakthroughs of Li [45] established the existence of C^*-diagonals for classifiable C^*-algebras, that is, separable simple nuclear $\mathcal{Z}$-stable C*-algebras satisfying the UCT. Since many classifiable C^*-algebras are inductive limits of 1-dimensional NCCW-complexes, it is natural to ask which non-simple inductive limits of 1-dimensional NCCW-complexes have a C^*-diagonal. We rely on the classification result of Robert [63, Corollary 5.2.3] to prove the following:

Theorem 3.0.1. Every unital inductive limit of 1-dimensional NCCW complexes with trivial K_1-group and unital injective connecting maps has a C^*-diagonal.

While classifying real rank zero C^*-algebras, Elliott [33, Theorem 4.4] showed that maximally homogeneous *-homomorphisms between matrix algebras over $C(S^1)$ are dense in the space of unital * homomorphism between the C^*-algebras when it is equipped with the point-norm topology. Li [44, Theorem 2.1.6] extended the result to the case of matrix algebras over 1-dimensional connected CW-complexes. Li and Raad [46] relied on these results to prove that unital AH-algebras whose building blocks have base spaces of dimension at most 1 have C^*-diagonals. We examine the extension of maximal homogeneity to *-homomorphisms between 1-dimensional NCCW complexes and highlight the limited supply of such maps and the limitation of its use for establishing the existence of C^*-diagonals.

3.1 1-dimensional Noncommutative CW-complexes

Based on [31], we define a 1-dimensional NCCW complex as follows:

Definition 3.1.1. Let $\beta_0, \beta_1 : E \to F$ be *-homomorphisms between two finite-dimensional C^*-algebras E and F. A C^*-algebra $A := A(E, F, \beta_0, \beta_1)$ is called a 1-dimensional noncommutative CW-complex (or simply 1-NCCW complex) if it is a pullback of $C([0,1], F)$ and E along $\beta := \beta_0 \oplus \beta_1 : E \to F \oplus F$ and $ev_0 \oplus ev_1 : C([0,1], F) \to F \oplus F$.

$$\begin{array}{ccc} A & \xrightarrow{\ \pi_1\ } & C([0,1],F) \\ {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle ev_0 \oplus ev_1} \\ E & \xrightarrow[\ \beta\ ]{} & F \oplus F \end{array}$$

We can express $A(E, F, \beta_0, \beta_1)$ explicitly as

$$\{(f, a) \in C([0,1], F) \oplus E : f(0) = \beta_0(a), f(1) = \beta_1(a)\} \text{ (see Subsection 1.1.0.5).} \tag{3.1.1}$$

Note that $A(E, F, \beta_0, \beta_1)$ is a unital C^*-algebra if β_0 and β_1 are unital *-homomorphisms. We denote the set of all unital 1-dimensional NCCW complexes and finite-dimensional C^*-algebras by 1-NCCW$_1$ (up to isomorphism).

For $A := A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$, we write E and F as follows:

$$E := \bigoplus_{i=1}^{l} M_{e_i}(\mathbb{C}) \quad \text{and} \quad F := \bigoplus_{j=1}^{k} M_{f_j}(\mathbb{C}), \tag{3.1.2}$$

where e_i and f_j are natural numbers. Setting the following conditions on β will be useful in later discussions:

(A1) $\beta := \beta_0 \oplus \beta_1 : E \to F \oplus F$ is injective.

If β is not injective, take $E' = E/\ker(\beta)$ and $\beta_0', \beta_1' : E' \to F$ as *-homomorphisms induced by β_0 and β_1. Then $A' := A(E', F, \beta_0', \beta_1') \in$ 1-NCCW$_1$ with injective *-homomorphism $\beta' : E' \to F \oplus F$ induced by β. Moreover, $A \cong A' \oplus \ker(\beta)$. Note that the injectivity of either β_0 or β_1 implies the injectivity of β. However, the converse is not true in general. Using Condition (A1), we can identify A with the C^*-subalgebra

$$\{f \in C([0,1], F) : (f(0), f(1)) \in \operatorname{Im}(\beta)\} \tag{3.1.3}$$

of $C([0,1], F)$. We will often identify A in this manner without explicitly stating it whenever we assume (A1) holds.

(A2) $\beta_0 : E \to F$ and $\beta_1 : E \to F$ are of the following form:

$$\beta_0(a_1, a_2, \ldots, a_l) = u \left(\bigoplus_{i=1}^{l} (a_i \otimes \mathbf{I}_{r_i}) \right) u^*, \tag{3.1.4}$$

and

$$\beta_1(a_1, a_2, \ldots, a_l) = v \left(\bigoplus_{i=1}^{l} (a_i \otimes \mathbf{I}_{s_i}) \right) v^*, \tag{3.1.5}$$

where u and v are permutation matrices in F, r_i and s_i are nonnegative integers, $\mathbf{I}_{r_i}$ is the identity matrix of size r_i, and $\sum_{i=1}^{l} e_i r_i = \sum_{j=1}^{k} f_j = \sum_{i=1}^{l} e_i s_i$.

Up to isomorphism of C^*-algebras, every β_0 and β_1 of $A \in$ 1-NCCW$_1$ satisfy condition (A2) with u and v both equal to the identity matrix. To see this, we first note that there exist unitaries $U, V \in F$ such that

$$\beta_0(a) = U \left(\bigoplus_{i=1}^{l} (a_i \otimes \mathbf{I}_{r_i}) \right) U^* \quad \text{and} \quad \beta_1(a) = V \left(\bigoplus_{i=1}^{l} (a_i \otimes \mathbf{I}_{s_i}) \right) V^* \tag{3.1.6}$$

for $a = (a_1, a_2, \ldots, a_l) \in E$, since β_0 and β_1 are unital.Let $w \in C([0,1], \mathcal{U}(F))$ be a continuous path of unitaries satisfying $w(0) = U^*$ and $w(1) = V^*$. Define $B := A(E, F, \beta_0', \beta_1')$, where $\beta_0' = U^*\beta_0 U$ and $\beta_1' = V^*\beta_1 V$. Then, β_0' and β_1' of $B \in$ 1-NCCW$_1$ satisfy condition (A2), and $A \cong B$ via $(f, a) \mapsto (wfw^*, a)$.

Example 3.1.2. $C([0,1], M_n(\mathbb{C}))$ is a 1-dimensional NCCW complex as it can be identified with $A(E, F, \beta_0, \beta_1)$ with $E := M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$, $F := M_n(\mathbb{C})$, β_0 defined by $\beta_0(a, b) = a$ and β_1 defined by $\beta_1(a, b) = b$. The isomorphism is given by the map $f \mapsto (f, (f(0), f(1)))$.

Example 3.1.3. Let X be a 1-dimensional connected finite CW-complex constructed by taking the disjoint union of a finite discrete space X^0 with k copies of the unit interval. Let γ_k be the continuous map that attaches the endpoints of the k-th copy of the unit interval to elements of X^0, and let γ be the disjoint union of these attaching maps. By considering the dual of these maps, we see that $C(X)$ is a 1-dimensional NCCW complex with $E := C(X^0)$, $F := \mathbb{C}^k$, and $\beta := \gamma^*$. For instance, consider X as the unit circle S^1 formed by a single point and a unit interval with both endpoints of the interval attached to the point. Then $C(S^1)$ is a 1-dimensional NCCW complex with $E := \mathbb{C}$, $F := \mathbb{C}$, and $\beta_0 = \beta_1 := \mathrm{id}$.

Example 3.1.4. The splitting interval algebra

$$\{f \in C([0,1], M_m(\mathbb{C})) : f(0) \in \bigoplus_{i=1}^{l} M_{e_i}(\mathbb{C}), f(1) \in \bigoplus_{i=1}^{k} M_{f_i}(\mathbb{C})\},$$

where e_i, f_i are natural numbers satisfying $\sum_{i=1}^{l} e_i = \sum_{i=1}^{k} f_i = m$, is a 1-dimensional NCCW complex with $E := \bigoplus_{i=1}^{l} M_{e_i}(\mathbb{C}) \oplus \bigoplus_{i=1}^{k} M_{f_i}(\mathbb{C})$, $F := M_m(\mathbb{C})$, $\beta_0(a, b) = a$, and $\beta_1(a, b) = b$.

Next, we provide short proofs of some results that illustrate how to derive new 1-dimensional NCCW complexes from known examples.

Proposition 3.1.5 ([57, Theorem 3.9])**.** Let $A \coloneqq A(E, F, \beta_0, \beta_1)$ be a 1-dimensional NCCW complex. Then $M_n(A)$ is a 1-dimensional NCCW complex for any n.

Proof. Take $E' \coloneqq M_n(E)$, $F' \coloneqq M_n(F)$, $\beta_0' \coloneqq \mathrm{id} \otimes \beta_0$, and $\beta_1' \coloneqq \mathrm{id} \otimes \beta_1$. Then $M_n(A)$ is isomorphic to $B \coloneqq B(E', F', \beta_0', \beta_1')$ via $\big((f_{ij}, a_{ij})\big) \mapsto \big((f_{ij}), (a_{ij})\big)$, where $(f_{ij}, a_{ij}) \in A$. ∎

Proposition 3.1.6. Let $A_1 \coloneqq A_1(E_1, F_1, \beta_0, \beta_1)$ and $A_2 \coloneqq A_2(E_2, F_2, \alpha_0, \alpha_1)$ be 1-dimensional NCCW complexes. Then $A_1 \oplus A_2$ is a 1-dimensional NCCW complex.

Proof. $A_1 \oplus A_2$ is isomorphic to $A_3 \coloneqq A_3(E_3, F_3, \gamma_0, \gamma_1)$, where $E_3 \coloneqq E_1 \oplus E_2$, $F_3 \coloneqq F_1 \oplus F_2$, $\gamma_0 \coloneqq \beta_0 \oplus \alpha_0$, and $\gamma_1 \coloneqq \beta_1 \oplus \alpha_1$. ∎

A finite multiset, denoted by $\{a_1^{\sim k_1}, \dots, a_m^{\sim k_m}\}$, extends the concept of a standard finite set by allowing elements to appear multiple times, with $a_i^{\sim k_i}$ representing $\underbrace{a_i, \dots, a_i}_{k_i \text{ times}}$. Suppose $A \coloneqq A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$, with E and F defined as in (3.1.2). We represent the irreducible representation of E that projects E onto its i-th component $M_{e_i}(\mathbb{C})$ by δ_i for $1 \le i \le l$, and denote the set of unitary equivalence classes of all the δ_i by $Sp(E)$. Similarly, we use the notation (t, i), where $0 \le t \le 1$ and $1 \le i \le k$, to denote an irreducible representation of $C([0,1], F)$ that sends f to the i-th component of $f(t)$. Therefore, the set of unitary equivalence classes of all irreducible representations of $C([0,1], F)$ is given by

$$Sp(C([0,1], F)) \coloneqq \coprod_{i=1}^{k} \{(t, i) : 0 \le t \le 1\}.$$

For $(f, a) \in A$, the condition $f(0) = \beta_0(a)$ and the form of β_0 as in (3.1.6) indicate that $(0, i)$ may not correspond to an irreducible representation of A. Instead, it can be identified with $\{\delta_1^{\sim r_{i1}}, \delta_2^{\sim r_{i2}}, \dots, \delta_l^{\sim r_{il}}\} \subset Sp(E)$. Similarly, $(1, i)$ can be identified with $\{\delta_1^{\sim s_{i1}}, \delta_2^{\sim s_{i2}}, \dots, \delta_l^{\sim s_{il}}\} \subset Sp(E)$. So, the set of unitary equivalence classes of irreducible representations of A is given by

$$Sp(A) \coloneqq Sp(E) \cup \coprod_{i=1}^{k} \{(t, i) : 0 < t < 1\} \coloneqq Sp(E) \cup \coprod_{i=1}^{k} (0, 1)_i.$$

We write $sp(A) \coloneqq Sp(E) \cup \coprod_{i=1}^{k} [0, 1]_i$, and further adopt the following notations and definitions from [48]:

$C([0,1], F) \ni f(t) \coloneqq (f(t, 1), f(t, 2), \dots, f(t, l))$ for all $t \in [0, 1]$, where $f(t, i)$ is the i-th component of $f(t)$. $E \ni a \coloneqq (a_1, a_2, \dots, a_l) \coloneqq (a(\delta_1), a(\delta_2), \dots, a(\delta_l))$. $\pi_t : A \to F$ and $\pi_{t_i} : A \to M_{f_i}(\mathbb{C})$ are defined by $\pi_t(f, a) = f(t)$ and $\pi_{t_i}(f, a) = f(t, i)$, respectively, for all $t \in [0, 1]$. Note that we identify π_{t_i} with (t, i) in the preceding paragraph. $\pi_e : A \to E$ and $q : A \to F$ are defined by $\pi_e(f, a) = a$ and $q(f, a) = f$. $(\alpha_{ij})_{k \times l}$ and $(\beta_{ij})_{k \times l}$ are matrices representing $(\beta_0)^* : K_0(E) = \mathbb{Z}^l \to K_0(F) = \mathbb{Z}^k$ and $(\beta_1)^* : K_0(E) = \mathbb{Z}^l \to K_0(F) = \mathbb{Z}^k$, respectively. We use $\bullet$ or $\bullet\bullet$ to represent any nonnegative integer.

For a given $0 < \epsilon < 1$, the topology base on $Sp(A)$ at each δ_j is given by

$$\{\delta_j\} \cup \coprod_{\{i|\alpha_{ij}\neq 0\}} (0,\epsilon)_i \cup \coprod_{\{i|\beta_{ij}\neq 0\}} (1-\epsilon,1)_i.$$

This gives a topology that is non-Hausdorff in general.

In what follows, we recall the construction of test functions useful for relating properties of two *-homomorphisms from a 1-dimensional NCCW complex to a matrix algebra.

For each $\eta = \frac{1}{m}$, where m is a positive integer, consider a partition $0 = w_0 < w_1 < \cdots < w_m = 1$ of $[0,1]$ into m subintervals, each of equal length $\frac{1}{m}$. Let $H(\eta)$ be a finite subset of A that consists of the following two types of positive elements of A:

(a) For any nonnegative integers a_i and b_i, $1 \leq i \leq k$, satisfying $0 \leq a_i < a_i + 2 \leq b_i \leq m$, a test function (f,a) of type 1 corresponding to $\{\delta_j\} \cup \coprod_{\{i|\alpha_{ij}\neq 0\}} [0, a_i\eta]_i \cup \coprod_{\{i|\beta_{ij}\neq 0\}} [b_i\eta, 1]_i$ is defined by

$$a(\delta_s) := \begin{cases} 0_{e_s} & \text{if } s \neq j \\ \mathbf{I}_{e_j} & \text{if } s = j \end{cases}, \tag{3.1.7}$$

where 0_{e_s} is the zero matrix and $\mathbf{I}_{e_j}$ is the identity matrix, and

$$f(t,i) := \begin{cases} \beta_0^i(a)\frac{\eta - \mathrm{dist}(t,[0,a_i\eta]_i)}{\eta} & \text{if } 0 \leq t \leq (a_i+1)\eta \\ 0_{f_i} & \text{if } (a_i+1)\eta \leq t \leq (b_i-1)\eta \\ \beta_1^i(a)\frac{\eta - \mathrm{dist}(t,[b_i\eta,1]_i)}{\eta} & \text{if } (b_i-1)\eta \leq t \leq 1 \end{cases} \tag{3.1.8}$$

for each $t \in [0,1]$ and $i \in \{1,2,\ldots,k\}$, where $\beta_s^i(a)$ is the i-th component of $\beta_s(a)$ and $s = 0,1$.

(b) A test function (f,a) of type 2 corresponding to $X := \bigcup_s [w_{r_s}, w_{r_{s+1}}]_i \subset [\eta, 1-\eta]_i$ is defined by

$$a := 0 \tag{3.1.9}$$

and

$$f(t,r) := \begin{cases} 0_{f_r} & \text{if } r \neq i \\ (1 - \frac{\mathrm{dist}(t,X)}{\eta})\mathbf{I}_{f_i} & \text{if } \mathrm{dist}(t,X) < \eta \text{ and } r = i \\ 0_{f_i} & \text{if } \mathrm{dist}(t,X) \geq \eta \text{ and } r = i \end{cases}. \tag{3.1.10}$$

Let $\{e^i_{mm'} : 1 \leq m,m' \leq e_i,\ 1 \leq i \leq l\}$ and $\{f^i_{mm'} : 1 \leq m,m' \leq f_i,\ 1 \leq i \leq k\}$ be the standard matrix units of E and F, respectively. We modify elements of $H(\eta)$ to define a new finite set $\widetilde{H}(\eta) \subset A_+$. $\widetilde{H}(\eta)$ consists of elements of the form:

- (f,a) constructed like test functions of type 1 with $\mathbf{I}_{e_j}$ in (3.1.7) replaced by $e^j_{mm'}$ for any m,m'.

- (f, a) constructed like test functions of type 2 with $(1 - \frac{\mathrm{dist}(t,X)}{\eta})\mathbf{I}_{f_i}$ in (3.1.10) replaced by $(1 - \frac{\mathrm{dist}(t,X)}{\eta})e^i_{mm'}$ for any m, m'.

Next, we recall a calculation of the Cuntz semigroup of a 1-dimensional NCCW complex (see 1.1.0.7 for a brief overview of the Cuntz semigroup).

Theorem 3.1.7 ([4, Theorem 3.1])**.** Let $A \coloneqq A(E, F, \beta_0, \beta_1)$ be a 1-dimensional NCCW complex. Define $\gamma_0 \coloneqq Cu(\beta_0) : Cu(E) \to Cu(F)$ and $\gamma_1 \coloneqq Cu(\beta_1) : Cu(E) \to Cu(F)$. Then,

$$Cu(A) \cong \{(f, b) \in Lsc([0,1], \overline{\mathbb{N}}^k) \oplus \overline{\mathbb{N}}^l : f(0) = \gamma_0(b), f(1) = \gamma_1(b)\}, \tag{3.1.11}$$

where $\overline{\mathbb{N}} \coloneqq \mathbb{N} \cup \{\infty\}$.

3.2 C^*-diagonals of 1-dimensional NCCW complexes

In this section, we explore the unique extension property and C^*-diagonals of unital 1-dimensional NCCW complexes. We assume $A \coloneqq A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$ satisfies conditions (A1) and (A2) throughout this section unless otherwise stated.

For a subset $S \subset [0,1]$, $s \in [0,1]$, $i \in \{1, 2, \ldots, k\}$, and a C^*-subalgebra $B \subset A$, we adopt the following notations: $A_S \coloneqq \{f|_S : f \in A\} \subset C(S, E)$, $B(s,i) \coloneqq \{f(s,i) : f \in B\}$, and $B_S \coloneqq \{f|_S : f \in B\} \subset A_S$. We denote the set of diagonal matrices in $M_n(\mathbb{C})$ by $D_n(\mathbb{C})$.

Here, we present a consequence of [5, Theorem 3.4] that is useful for establishing the unique extension property of a 1-dimensional NCCW complex.

Proposition 3.2.1. Let $A \coloneqq A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$ and let $B \subset A$ be an abelian C^*-subalgebra. Then B has the unique extension property if and only if the vector space dimension $\dim(B_{\{0,1\}}) = \sum\limits_{i=1}^{l} e_i$ and $\dim(B_{\{s\}}) = \sum\limits_{j=1}^{k} f_j$ for all $s \in (0,1)$.

Proof. Suppose B has the unique extension property. Note that $B_{\{s\}} \subset F = A_{\{s\}}$ for all $s \in [0,1]$. Suppose $\dim(B_{\{s\}}) < \sum\limits_{j=1}^{k} f_j$ for some $s \in (0,1)$. Then, there are at least two pure states τ_1 and τ_2 of F that restrict to a pure state τ of $B_{\{s\}}$. Consequently, $\tau_1 \circ \pi_{s_j}$ and $\tau_2 \circ \pi_{s_j}$ are pure states of A that restrict to a pure state $\tau \circ \pi_{s_j}$ of B. This is impossible by the unique extension property of B. Hence, $\dim(B_{\{s\}}) = \sum\limits_{j=1}^{k} f_j$ for all $s \in (0,1)$.

By condition (A1), $A_{\{0,1\}} \cong E$. If $\dim(B_{\{0,1\}}) < \sum\limits_{i=1}^{l} e_i$, then there exists a pure state σ of $\pi_e(B) \subset E$ that extends to at least two pure states σ_1 and σ_2 of E. Consequently, $\sigma \circ \pi_e$ is a pure state of B that extends to two pure states $\sigma_1 \circ \pi_e$ and $\sigma_2 \circ \pi_e$ of A, which is a contradiction. So, $\dim(B_{\{0,1\}}) = \sum\limits_{i=1}^{l} e_i$.

Conversely, suppose $\dim(B_{\{0,1\}}) = \sum_{i=1}^{l} e_i$ and $\dim(B_{\{s\}}) = \sum_{j=1}^{k} f_j$ for all $s \in (0,1)$. Note that $\delta_r(B)$ is abelian for each $r \in \{1, 2, \ldots, l\}$ since B is abelian and $B_{\{0,1\}} \subset \bigoplus_{r=1}^{l} \delta_r(B) \subset E$. The condition $\dim(B_{\{0,1\}}) = \sum_{i=1}^{l} e_i$ implies $B_{\{0,1\}}$ is a maximal abelian subalgebra of E. So, $B_{\{0,1\}} = \bigoplus_{r=1}^{l} \delta_r(B)$ and the dimension of each $\delta_r(B)$ must be e_r. Also, the condition $\dim(B_{\{s\}}) = \sum_{j=1}^{k} f_j$ for all $s \in (0,1)$ implies $B(s,j) = f_j$ for all $j \in \{1, 2, \ldots, k\}$ and $s \in (0,1)$.

Next, we show that B is a maximal abelian subalgebra of A. Let (g, b) be in the relative commutant B' of B. Then $(g, b)(f, a) = (f, a)(g, b)$ for all $(f, a) \in B$. Consequently, $b \in B_{\{0,1\}}$ since $B_{\{0,1\}}$ is a masa in E. Moreover, $g(s,j)f(s,j) = f(s,j)g(s,j)$ for all $s \in (0,1)$ and $j \in \{1, 2, \ldots, l\}$. The maximal dimension of $B(s,j)$ ensures $g(s,j) \in B(s,j)$. Therefore, $(g,b) \in B$ and B is a masa. Since B is a masa and $\pi(B)$ is a masa of $\pi(A)$ for every $\pi \in Sp(A)$, we conclude, using [5, Theorem 3.4], that B has the unique extension property. ∎

The following example demonstrates that an abelian subalgebra of A having a maximal dimension at each fiber does not ensure the unique extension property of the subalgebra.

Example 3.2.2. Let $A := A(E, F, \beta_0, \beta_1)$ be a 1-dimensional NCCW complex with $E := \mathbb{C}^3$, $F := M_2(\mathbb{C})$, $\beta_0 : E \to F$ defined by $\beta_0(a_1, a_2, a_3) := \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, and $\beta_1 : E \to F$ defined by $\beta_1(a_1, a_2, a_3) := \begin{pmatrix} a_2 & 0 \\ 0 & a_3 \end{pmatrix}$. Define

$$B := \{(f, (a_1, a_2, a_3)) \in C([0,1], M_2(\mathbb{C})) \oplus E : a_1 = a_3\} \subset A.$$

We have that $\dim(B(t)) = 2$ for all $t \in [0,1]$, B is abelian, and $\pi(B)$ is a masa for every $\pi \in Sp(A)$. However, B does not exhibit the unique extension property because it is not a masa of A. Note that $\dim(B_{\{0,1\}}) = 2$ and B includes the center of A.

This example further emphasizes the necessity of the maximal abelian condition of B in [5, Theorem 3.4] for 1-dimensional NCCW complexes. Particularly, it illustrates that this condition cannot be replaced by B containing the center of A, as observed in the case of trivial homogeneous C^*-algebras [76, Lemma 1.4].

If an abelian C^*-subalgebra B of a C^*-algebra A has the unique extension property, then there exists a unique conditional expectation $P : A \to B$ [3, Theorem 3.4]. As a special case of [2, Lemma 5.11], we have the following:

Lemma 3.2.3 ([2, Lemma 5.11])**.** Let B be an abelian C^*-subalgebra of a 1-dimensional NCCW complex $A := A(E, F, \beta_0, \beta_1)$ that possesses the unique extension property. Then there exists a unique faithful conditional expectation $P : A \to B$.

Having satisfied all criteria of a C^*-diagonal except the regularity condition, an important question arises: is an abelian C^*-subalgebra of a 1-dimensional NCCW complex $A :=$

$A(E, F, \beta_0, \beta_1)$ satisfying the unique extension property a C^*-diagonal? The following example provides a negative answer to this question.

Example 3.2.4 ([2, Example 5.15])**.** Let $C := \{f \in C([0,1], M_2(\mathbb{C})) \mid f(0) \in \mathbb{C}\mathbf{I}_2\}$ and $D := \{f \in C([0,1], D_2(\mathbb{C})) \mid f(0) \in \mathbb{C}\mathbf{I}_2\}$. C is a 1-dimensional NCCW complex with $E := \mathbb{C} \oplus M_2(\mathbb{C})$, $F := M_2(\mathbb{C})$, $\beta_0 : E \to F$ defined by $\beta_0(z, a) = \mathrm{diag}(z, z)$, and $\beta_1 : E \to F$ defined by $\beta_1(z, a) = a$. Notice that D is abelian, $\dim(D(t)) = 2$ for all $t \in (0, 1)$, and $\dim(D_{\{0,1\}}) = 3$. By Proposition 3.2.1, D satisfies the unique extension property.

For $s > 0$, let $w_s = \begin{pmatrix} \cos(\frac{1}{s}) & \sin(\frac{1}{s}) \\ -\sin(\frac{1}{s}) & \cos(\frac{1}{s}) \end{pmatrix} \in M_2(\mathbb{C})$, where the trigonometric functions are in radians. Define $\theta : C \to C$ by

$$\theta_s(f) = \begin{cases} w_s f(s) w_s^* & \text{if } s \in (0, 1], \\ f(0) & \text{if } s = 0. \end{cases}$$

Let $f \in C$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(t) - f(0)| < \epsilon$ for all $|t| < \delta$.

$$\begin{aligned} \|\theta_t(f) - \theta_0(f)\| = \|w_t f(t) w_t^* - f(0)\| &= \|w_t f(t) w_t^* - w_t f(0) w_t^*\| \\ &= \|w_t\| \|f(t) - f(0)\| \|w_t^*\| < \epsilon \end{aligned}$$

for all $|t| < \delta$. Hence, $\theta(f)$ is continuous and $\theta \in Aut(C)$. Let

$$A := M_2(C) \quad \text{and} \quad B := \left\{ \begin{pmatrix} d_1 & 0 \\ 0 & \theta(d_2) \end{pmatrix} : d_1, d_2 \in D \right\}.$$

By Proposition 3.1.5, A is a 1-dimensional NCCW complex. B is an abelian C^*-algebra of A with the unique extension property since each of D and $\theta(D)$ has the unique extension property with respect to C. The unique faithful conditional expectation $P : A \to B$ is given by

$$P = \begin{pmatrix} \phi & 0 \\ 0 & \theta\phi\theta^{-1} \end{pmatrix},$$

where ϕ is the canonical expectation from C onto D: $\phi(c)(t) = \begin{pmatrix} c_{1,1}(t) & 0 \\ 0 & c_{2,2}(t) \end{pmatrix}$. To show that B is not a C^*-diagonal of A, it is sufficient to find $f \in Ker(P)$ which is not in closed span of $N_f(B)$ by Proposition 1.5.27.

First, we describe the free normalizers of D in C. Let $g \in N_f(D)$. Then $g(s)$ is a normalizer of $M_2(\mathbb{C})$ and thus a generalized permutation matrix for each $s > 0$. Since $g(0) \in \mathbb{C}\mathbf{I}_2$, it follows by continuity that the off-diagonal entries of $g(s)$ tends to 0 as $s \to 0$. $g(0)^2 = 0$ implies $g(0) = 0$.

Similarly, we show that if $h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \in N_f(B)$, then $h(0) = 0$.

Given that h is a normalizer of B, a straightforward computation reveals the following: For $s > 0$,

(i) $h_{11}(s)$ and $w_s^* h_{22}(s) w_s$ are free normalizers of $D(s)$. Consequently, $h_{11}(0) = h_{22}(0) = 0$ by the property of the free normalizer of D.

(ii) Both $h_{12}(s)w_s$ and $w_s^* h_{21}(s)$ are normalizers of $D(s)$ within $C(s)$.

Suppose $h(0) \neq 0$ and $h_{1,2}(0) \neq 0$. By continuity of h_{12}, there exists $\delta > 0$ such that $\|h_{12}(s)\| - \|h_{12}(0)\| \leq \|h_{12}(0) - h_{12}(s)\| \leq \frac{\|h_{12}(0)\|}{4}$ for $|s| < \delta$. So, $\|h_{12}(s)\| > \frac{\|h_{12}(0)\|}{\sqrt{2}}$ and $|(h_{12}(s))_{i,j}| < \frac{\|h_{12}(0)\|}{2}$ for $i \neq j$ and $|s| < \delta$ since $h_{12}(0) \in \mathbb{C}\mathbf{I}_2$. Choose $0 < s_0 < \delta$ such that $w_{s_0} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Since $h_{12}(s_0)w_{s_0}$ is a normaliser of $D(s_0)$, it must be a generalized permutation matrix. Given that $0 < s_0 < \delta$, there is at least one entry of $h_{12}(s_0)w_{s_0}$ with a modulus of at least $\frac{\|h_{12}(0)\|}{\sqrt{2}}$. Consequently, due to the choice of w_{s_0}, we have $h_{12}(s_0) = (h_{12}(s_0)w_{s_0})w_{s_0}^*$ has at least one off-diagonal entry of modulus greater than $\frac{\|h_{12}(0)\|}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{\|h_{12}(0)\|}{2}$. This is contradiction. Hence $h(0) = 0$. Similarly, if $h(0) \neq 0$ and $h_{2,1}(0) \neq 0$ lead to a contradiction.

The function $g \in A$ defined by $g(s) \coloneqq \begin{pmatrix} 0 & \mathbf{I}_2 \\ 0 & 0 \end{pmatrix}$ lies in $\mathrm{Ker}(P)$ and satisfies $g(0) \neq 0$. Therefore, it cannot be included in the closed span of the free normalisers of B in A.

Following the regularity idea in [8, Proposition 5.1], we prove the following:

Proposition 3.2.5. Let $A \coloneqq A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$ and $B \coloneqq \{(f, a) \in A \mid f(t) \in \bigoplus_{i=1}^{k} D_{f_i}(\mathbb{C})$ for all $t \in [0, 1]\}$. Then B is a C^*-diagonal of A.

Proof. Note that Condition (A2) ensures that if $(f, a) \in B$, then a is a diagonal matrix. We have that B is abelian, $\dim(B_{\{s\}}) = \sum_{j=1}^{k} f_j$ for all $s \in (0, 1)$, and $\dim(B_{\{0,1\}}) = \sum_{i=1}^{l} e_i$. Hence, by Proposition 3.2.1, B has the unique extension property. Consequently, it is a maximal abelian C^*-subalgebra of A with a unique faithful conditional expectation $P : A \to B$. Let $\{F^i_{mm'} \in C([0, 1], M_{f_i}(\mathbb{C})) : 1 \leq m, m' \leq f_i, 1 \leq i \leq k\}$ be such that $F^i_{m,m'}(t)$ is the matrix unit $f^i_{mm'}$ of $M_{f_i}(\mathbb{C})$ for all $t \in [0, 1]$. Then, the conditional expectation $P : A \to B$ is given by the restriction of the canonical expectation of $C([0, 1], \bigoplus_{i=1}^{k} M_{f_i}(\mathbb{C}))$ onto $C([0, 1], \bigoplus_{i=1}^{k} D_{f_i}(\mathbb{C}))$:

$$P(h) \coloneqq \sum_{i=1}^{k} \sum_{m=1}^{f_i} F^i_{mm} h F^i_{mm} \quad \text{for all } h \in A.$$

To see that $P(h) \in B$ for $h \coloneqq (f, a) \in A$, note that by Condition (A2) we get the following:

$$\begin{aligned} F^j_{mm}(0)h(0)F^j_{mm} &= F^j_{mm}(0)\beta_0(a)F^j_{mm}(0) \\ &= F^j_{mm}(0)\left(u\left(\bigoplus_{i=1}^{l}(a_i \otimes \mathbf{I}_{r_i})\right)u^*\right)F^j_{mm}(0) \end{aligned}$$

$$= uF^j_{\sigma(m)\sigma(m)}(0)\left(\bigoplus_{i=1}^{l}(a_i\otimes \mathbf{I}_{r_i})\right)F^j_{\sigma(m)\sigma(m)}(0)u^*,$$

for some permutation σ, and

$$P(h)(0)=\sum_{j=1}^{k}\sum_{m=1}^{f_j} uF^j_{\sigma(m)\sigma(m)}(0)\left(\bigoplus_{i=1}^{l}(a_i\otimes \mathbf{I}_{r_i})\right)F^j_{\sigma(m)\sigma(m)}(0)u^* = u\left(\bigoplus_{i=1}^{l}(E_i(a_i)\otimes \mathbf{I}_{r_i})\right)u^*,$$

where $E_i : M_{e_i}(\mathbb{C}) \to D_{e_i}(\mathbb{C})$ is the canonical conditional expectation. Similarly,

$$P(h)(1) = v\left(\bigoplus_{i=1}^{l}(E_i(a_i)\otimes \mathbf{I}_{r_i})\right)v^*.$$

Next, we show the regularity of B in A. Let $f \in A$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(t) - f(s)\| < \epsilon$ whenever $|t-s| < \delta$. Choose $0 < \delta_0 < \delta$ and $1-\delta < \delta_1 < 1$. Then $A_{[\delta_0,\delta_1]} \cong C([\delta_0,\delta_1],F)$ and $B_{[\delta_0,\delta_1]} \cong C([\delta_0,\delta_1],DF)$, where $DF := \bigoplus_{i=1}^{k} D_{f_i}(\mathbb{C})$. Hence, there exists $g_i \in N_{A_{[\delta_0,\delta_1]}}(B_{[\delta_0,\delta_1]})$ and non-zero scalars λ_i such that

$$\| f|_{[\delta_0,\delta_1]} - \sum_i \lambda_i g_i\| < \epsilon. \tag{3.2.1}$$

By the definition of A, $f(0) = \beta_0(a)$ and $f(1) = \beta_1(a)$ for some $a \in E$. Then, there exist $c_i \in N_E(DE)$ and nonnegative integers γ_i such that $\|a - \sum_i \gamma_i c_i\| < \epsilon$. By replacing ϵ with a smaller number if necessary, we can assume that the cardinalities of the λ_i's and γ_i's coincide. Since normalizers of DE in E are generalized matrices, $\beta_0(c_i)$ and $\beta_1(c_i)$ are normalizers of DF in F for each i by their definitions. Moreover, for each $\tau = 0,1$,

$$\|f(\tau) - \sum_i \gamma_i\beta_\tau(c_i)\| = \|\beta_\tau(a) - \sum_i \gamma_i\beta_\tau(c_i)\| < \epsilon. \tag{3.2.2}$$

Define $\overline{g}_i : [0,1] \to F$ by

$$\overline{g}_i(t) := \begin{cases} \frac{t}{\delta_0}g_i(\delta_0) + \left(1-\frac{t}{\delta_0}\right)\frac{\gamma_i}{\lambda_i}\beta_0(c_i) & \text{if } t\in[0,\delta_0],\\ g_i(t) & \text{if } t \in [\delta_0,\delta_1],\\ \frac{1-t}{1-\delta_1}g_i(\delta_1) + \left(1-\frac{1-t}{1-\delta_1}\right)\frac{\gamma_i}{\lambda_i}\beta_1(c_i) & \text{if } t\in[\delta_1,1]. \end{cases} \tag{3.2.3}$$

Then $\overline{g}_i(0) = \beta_0\left(\frac{\gamma_i}{\lambda_i}c_i\right)$ and $\overline{g}_i(1) = \beta_1\left(\frac{\gamma_i}{\lambda_i}c_i\right)$. Therefore, $\overline{g}_i \in A$ and it is a sum of two normalizers of B in A. For $t \in [0,\delta_0]$,

$$\begin{aligned}\|f(t) - \sum_i \lambda_i\overline{g}_i(t)\| &= \|f(t) - \sum_i \lambda_i\left(\frac{t}{\delta_0}g_i(\delta_0) + \left(1-\frac{t}{\delta_0}\right)\frac{\gamma_i}{\lambda_i}\beta_0(c_i)\right)\|\\ &\le \|f(t) - \sum_i \lambda_i g_i(\delta_0)\| + \|f(t) - \sum_i \gamma_i\beta_0(c_i)\|\\ &\le \|f(t) - f(\delta_0)\| + \|f(\delta_0) - \sum_i \lambda_i g_i(\delta_0)\| + \|f(t) - f(0)\|\\ &\quad + \|f(0) - \sum_i \gamma_i\beta_0(c_i)\|\\ &< 5\epsilon.\end{aligned}$$

Similarly, we can show the above approximation for all cases to conclude that $\|f - \sum_i \lambda_i\overline{g}_i\| < 5\epsilon$, and hence B is regular in A. ∎

3.3 C^*-diagonals of Inductive Limits of 1-dimensional NCCW complexes

Throughout this section, we assume $A := A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$ satisfies conditions (A1) and (A2). We begin by generalizing the notion of maximally homogeneous *-homomorphisms between matrix algebras over 1-dimensional CW-complexes, as introduced by [33], to 1-dimensional NCCW complexes. We then examine some limitations associated with this type of morphism for 1-dimensional NCCW complexes. Finally, we establish the existence of C^*-diagonals in certain inductive limits of 1-dimensional NCCW complexes.

Let $A := A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$ and let $\phi : A \to M_n(\mathbb{C})$ be a *-homomorphism. Then there exists a unitary $u \in M_n(\mathbb{C})$ such that

$$\phi(f, a) = u \operatorname{diag}(a(\delta_1) \otimes \mathbf{I}_{s_1}, \dots, a(\delta_l) \otimes \mathbf{I}_{s_l}, f(w_1), f(w_2), \dots, f(w_r), 0_\bullet)\, u^*, \tag{3.3.1}$$

where $w_1, \dots, w_r \in \coprod_{i=1}^{k} (0,1)_i$. We define the spectrum of ϕ by

$$\operatorname{Sp}(\phi) := \{\delta_1^{\sim s_1}, \delta_2^{\sim s_2}, \dots, \delta_l^{\sim s_l}, w_1, w_2, \dots, w_r\}. \tag{3.3.2}$$

Let (X, d) be a metric space. We say two finite sets $X_1 = \{w_1, w_2, \dots, w_n\} \subset X$ and $X_1' = \{w_1', w_2', \dots, w_m'\}$ can be bijectively paired within $\eta > 0$ if $m = n$ and there exists a permutation σ such that $d(w_i, w'_{\sigma(i)}) < \eta$ for all $1 \le i \le n$.

While it is not always possible to pair the spectra of any two *-homomorphisms from $A := A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$ to $M_n(\mathbb{C})$ within a given $\eta = \frac{1}{m}$, where m is a positive integer, we can pair some subsets of their spectra under some mild constraints.

Lemma 3.3.1 ([47, Lemma 2.3])**.** Let $A := A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$, m be any positive integer, ϵ be any positive real number less than 1, and let $\phi, \psi : A \to M_n(\mathbb{C})$ be *-homomorphisms. If $\operatorname{Eig}(\phi(h))$ [1] and $\operatorname{Eig}(\psi(h))$ can be bijectively paired within ϵ for all $h \in H(\eta)$, where $\eta := \frac{1}{m}$, then for each $i \in \{1, 2, \dots, k\}$, there exist $X_i \subset \operatorname{Sp}(\phi) \cap (0,1)_i$ and $X_i' \subset \operatorname{Sp}(\psi) \cap (0,1)_i$ satisfying the following conditions:

(i) X_i and X_i' can be bijectively paired within 2η,

(ii) $\operatorname{Sp}(\phi) \cap [\eta, 1-\eta]_i \subset X_i$ and $\operatorname{Sp}(\psi) \cap [\eta, 1-\eta]_i \subset X_i'$.

By the Weyl spectral variation inequality [10], $\|\phi(h) - \psi(h)\| < \epsilon$ for all $h \in H(\eta)$ is sufficient to establish the bijective pairing of $\operatorname{Eig}(\phi(h))$ and $\operatorname{Eig}(\psi(h))$ within ϵ.

Following [46, Definition 2.2], we extend the concept of maximally homogeneous *-homomorphisms to maps between two C^*-algebras $A, B \in$ 1-NCCW$_1$.

Definition 3.3.2. Let $A, B \in$ 1-NCCW$_1$. A unital *-homomorphism $\phi : A \to B$ is called maximally homogeneous if, for any $\mu \in \operatorname{Sp}(B)$, every element of $\operatorname{Sp}(\phi_\mu)$ has multiplicity at most one, where $\phi_\mu := \mu \circ \phi$.

[1] Set of eigenvalues of $\phi(h)$

Example 3.3.3. Let $A := A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$ with $E := \mathbb{C}^3$, $F := M_2(\mathbb{C})$, and

$$\beta_0(a_1, a_2, a_3) := \begin{pmatrix} a_1 & 0 \\ 0 & a_3 \end{pmatrix} \quad \text{and} \quad \beta_1(a_1, a_2, a_3) := \begin{pmatrix} a_2 & 0 \\ 0 & a_3 \end{pmatrix}. \tag{3.3.3}$$

The *-homomorphism $\phi : A \to C([0,1], M_3(\mathbb{C}))$ defined by

$$\phi_t(f, (a_1, a_2, a_3)) := \begin{cases} \begin{pmatrix} a_1 & 0 \\ 0 & f\left(t + \frac{1}{2}\right) \end{pmatrix} & \text{if } t \in [0, \frac{1}{2}] \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f\left(t - \frac{1}{2}\right) & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \tag{3.3.4}$$

is a maximally homogeneous *-homomorphism.

Let $A := A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$ and $\phi : A \to M_n(\mathbb{C})$ be a unital *-homomorphism. By grouping the δ_i's and using (3.1.4) and (3.1.5), we can reformulate (3.3.1) to have a unitary $v \in M_n(\mathbb{C})$ and

$$\phi(f, a) = v \operatorname{diag}\left(a(\delta_1) \otimes \mathbf{I}_{s_1'}, \ldots, a(\delta_l) \otimes \mathbf{I}_{s_l'}, f(w_1'), f(w_2'), \ldots, f(w_{r'}')\right) v^* \tag{3.3.5}$$

with $w_1', \ldots, w_{r'}') \in \coprod_{i=1}^{k} [0,1]_i$ and $0 \leq s_i' \leq s_i$. In this case, we denote

$$sp(\phi) := \{\delta_1^{\sim s_1'}, \delta_2^{\sim s_2'}, \ldots, \delta_l^{\sim s_l'}, w_1', w_2', \ldots, w_{r'}'\}. \tag{3.3.6}$$

We use an idea of [48, Lemma 3.5] to prove the following:

Proposition 3.3.4. Let $A := A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$, $\phi : A \to M_n(\mathbb{C})$ be a unital *-homomorphism, and $sp(\phi)$ as in (3.3.6). Then ϕ is a limit of maximally homogeneous *-homomorphisms if and only if $s_i' = 0$ or 1 for all $1 \leq i \leq l$.

Proof. Suppose ϕ is a limit of maximally homogeneous *-homomorphisms. Choose a positive integer m such that $w_1', \ldots, w_{r'}' \in \coprod_{i=1}^{k} \left[\frac{1}{m}, 1 - \frac{1}{m}\right]_i$, where $w_i' \in sp(\phi)$. Set $\eta := \frac{1}{8mn}$. Then, there exists a maximally homogeneous *-homomorphism $\psi : A \to M_n(\mathbb{C})$ such that $\|\phi(h) - \psi(h)\| < 1$ for all $h \in H(\eta)$. By Lemma 3.3.1, there exists $[\eta, 1-\eta]_i \subset X_i \subset Sp(\psi) \cap (0,1)_i$ for each $i \in \{1, 2, \ldots, k\}$ such that $Sp(\phi) \cap (0,1)_i$ and X_i can be bijectively paired within 2η. Let $\varpi_i : X_i \to Sp(\phi) \cap (0,1)_i$ be the map that defines the bijective pairing. Define a new *-homomorphism $\psi_1 : A \to M_n(\mathbb{C})$ from ψ by changing all $w \in ((0,\eta)_i \cap Sp(\psi)) \setminus X_i$ to 0_i, all $w \in ((1-\eta, 1)_i \cap Sp(\psi)) \setminus X_i$ to 1_i, and all $w \in X_i$ to $\varpi_i(w)$. Then

$$Sp(\phi) \cap (0,1)_i = Sp(\psi_1) \cap (0,1)_i \tag{3.3.7}$$

for each $i = 1, 2, \ldots, l$. For each $[0,1]_i$, choose positive integers a_i, b_i such that $1 < a_i < a_i + 2 \leq b_i < 8mn$ and

$$Sp(\phi) \cap (a_i\eta, b_i\eta)_i = Sp(\psi) \cap (a_i\eta, b_i\eta)_i = \emptyset.$$

For a test function h_j of type 1 corresponding to $\{\delta_j\} \cup \coprod_{\{i|\alpha_{ij}\neq 0\}} [0, a_i\eta]_i \cup \coprod_{\{i|\beta_{ij}\neq 0\}} [b_i\eta, 1]_i$, $\phi(h_j)$, $\psi(h_j)$ are projections, and

$$\psi(h_j) = \psi_1(h_j) \quad \text{and} \quad \|\phi(h_j) - \psi(h_j)\| < 1 \tag{3.3.8}$$

for each $j = 1, 2, \ldots, k$. (3.3.7) and the inequality in (3.3.8) ensure $Sp(\phi) \cap Sp(E) = Sp(\psi_1) \cap Sp(E)$. By construction of ψ_1, the groupings of $Sp(\phi) \cap Sp(E)$ yield 0_i's, 1_i's $(1 \leq i \leq k)$ and elements of $Sp(E) \cap Sp(\psi)$. Hence, the result holds.

Conversely, consider ϕ in the form of (3.3.5) and suppose $s_i' = 0 \,\text{or}\, 1$ for all $0 \leq i \leq l$. Given a finite set $\mathcal{F}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $\text{dist}(x, y) < \delta$ implies $\|f(x) - f(y)\| < \epsilon$ for all $(f, a) \in \mathcal{F}$. A *-homomorphism $\psi : A \to M_n(\mathbb{C})$ can be defined by substituting points in $\text{Sp}(\phi) \cap \coprod_{i=1}^{k} [0, 1]_i$ with distinct non-zero points in $\coprod_{i=1}^{k} (0, 1)_i$ within δ. This ψ is maximally homogeneous and satisfies $\|\phi(h) - \psi(h)\| < \epsilon$ for all $h \in \mathcal{F}$. ∎

Example 3.3.5. Let A be as in Example 3.3.3. The *-homomorphism $\phi_0 : A \to M_3(\mathbb{C})$ defined by $\phi_0(f, (a_1, a_2, a_3)) := \text{diag}(a_1, f(1))$ is homotopic to the *-homomorphism $\phi_1 : A \to M_3(\mathbb{C})$ defined by $\phi_1(f, (a_1, a_2, a_3)) := \text{diag}(a_1, f(0))$. Notice that ϕ_0 is maximally homogeneous, while ϕ_1 is not. Therefore, homotopy to a maximally homogeneous *-homomorphism does not guarantee maximal homogeneity.

Unlike unital *-homomorphisms between matrix algebras over 1-dimensional CW-complexes, not every unital *-homomorphism between two C^*-algebras $A, B \in$ 1-NCCW$_1$ can be approximated by a maximally homogeneous *-homomorphism. In fact, there may not exist any maximally homogeneous *-homomorphism between two 1-dimensional NCCW complexes, as the next example highlights.

Example 3.3.6. Recall the dimension drop algebra

$$\mathcal{Z}_{p,q} := \{f \in C([0, 1], M_{pq}(\mathbb{C})) : f(0) \in M_p \otimes \mathbf{I}_q \text{ and } f(1) \in \mathbf{I}_p \otimes M_q\} \tag{3.3.9}$$

is a 1-dimensional NCCW complex with $E := M_p \oplus M_q$, $F := M_{pq}$, $\beta_0(a, b) := a \otimes \mathbf{I}_q$, and $\beta_1(a, b) := b \otimes \mathbf{I}_p$.

$\text{Hom}(\mathcal{Z}_{2,3}, \mathcal{Z}_{2,5})$ is non-empty since it contains a *-homomorphism $\phi : \mathcal{Z}_{2,3} \to \mathcal{Z}_{2,5}$ defined by

$$\phi_t(f, (a, b)) := u_t \cdot \text{diag}(a \otimes \mathbf{I}_2, f(t)) \cdot u_t^*, \tag{3.3.10}$$

where $u_t \in C([0, 1], M_{10})$ defines a continuous path of unitaries from the identity matrix to the permutation matrix

$$\begin{pmatrix} \mathbf{I}_2 & 0_{2\times2} & 0_{2\times3} & 0_{2\times3} \\ 0_{3\times2} & 0_{3\times2} & \mathbf{I}_3 & 0_{3\times3} \\ 0_{2\times2} & \mathbf{I}_2 & 0_{2\times3} & 0_{2\times3} \\ 0_{3\times2} & 0_{3\times2} & 0_{3\times3} & \mathbf{I}_3 \end{pmatrix}.$$

However, there is no maximally homogeneous *-homomorphism from $\mathcal{Z}_{2,3}$ to $\mathcal{Z}_{2,5}$ due to the boundary conditions of $\mathcal{Z}_{2,3}$ and $\mathcal{Z}_{2,5}$.

The following example demonstrates that the approximate maximal homogeneity of each ϕ_μ for a unital *-homomorphism ϕ between 1-dimensional NCCW complexes does not ensure the approximate maximal homogeneity of the *-homomorphism itself.

Example 3.3.7. Let $A := A(E, F, \beta_0, \beta_1)$ with $E := \mathbb{C}^3$, $F := M_3(\mathbb{C})$, and

$$\beta_0(a_1, a_2, a_3) := \operatorname{diag}(a_1, a_2, a_3), \quad \beta_1(a_1, a_2, a_3) := \operatorname{diag}(a_1, a_1, a_2). \tag{3.3.11}$$

Define $\phi : A \to C([0, 1], M_4(\mathbb{C}))$ by

$$\phi_t(f, (a_1, a_2, a_3)) := \begin{cases} \begin{pmatrix} a_1 & 0 \\ 0 & f\left(\frac{1}{2} - t\right) \end{pmatrix} & \text{if } t \in [0, \frac{1}{2}], \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_3 & 0 \\ 0 & f\left(\frac{3}{2} - t\right) \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \tag{3.3.12}$$

Then every ϕ_t is a maximally homogeneous *-homomorphism except for $\phi_{\frac{1}{2}}$, which is only approximately maximally homogeneous. Suppose ϕ is the limit of maximally homogeneous *-homomorphisms. Choose a fixed finite set $\mathcal{F}$ and $\delta > 0$. Then, there exists a maximally homogeneous *-homomorphism $\psi : A \to C([0, 1], M_4(\mathbb{C}))$ satisfying $\|\phi(h) - \psi(h)\| < \delta$ for all $h \in \mathcal{F}$.

It follows that

$$\|\phi_0(h) - \psi_0(h)\| < \delta \quad \text{and} \quad \|\phi_1(h) - \psi_1(h)\| < \delta \tag{3.3.13}$$

for all $h \in \mathcal{F}$.

By choosing δ sufficiently small if necessary, we can assume ψ_0 and ψ_1 (using (3.3.13)) are of the same form as ϕ_0 and ϕ_1 with variations in the evaluation points of f. Moreover, the maximal homogeneity of ψ implies that for each $t \in (0, 1)$,

$$\psi_t\left(f, (a_1, a_2, a_3)\right) = U \begin{pmatrix} a_i & 0 \\ 0 & f(w) \end{pmatrix} U^*$$

for some unitary matrix U and some $w \in (0, 1)$. Set $g := (\operatorname{diag}(0, 0, 1 - t), (0, 0, 1)) \in A$.

Define

$$S = \{\psi_t(g) : t \in [0, 1]\},$$
$$S_1 = \{\psi_t(g) : t \in [0, 1]\} \cap \{V \operatorname{diag}(1, 0, 0, 1 - t)V^* : V \in \mathcal{U}(M_4(\mathbb{C})), 0 < t < 1\},$$

and

$$S_2 := \{\psi_t(g) : t \in [0, 1]\} \cap \{V \operatorname{diag}(0, 0, 0, 1 - t)V^* : V \in \mathcal{U}(M_4(\mathbb{C})), 0 < t < 1\}.$$

Then, $S = S_1 \cup S_2$. Notice that $S_1 \cap S_2 = \emptyset$ since matrices in S_1 always have eigenvalue 1 while matrices in S_2 never have such an eigenvalue. Therefore, $S = \{\psi_t(g) : t \in [0, 1]\}$ is not connected, and ψ does not exist. Hence, ϕ is not a limit of maximally homogeneous *-homomorphisms.

Given the aforementioned limitations of maximally homogeneous *-homomorphisms between 1-dimensional NCCW complexes, it is evident that these maps cannot play the same role as maximally homogeneous *-homomorphisms between matrix algebras over 1-dimensional CW complexes in the classification of inductive limit C^*-algebras [44, 33] and inductive limit - diagonals [46]. Thomsen [75] showed that maximally homogeneous *-homomorphisms between matrix algebras over 1-dimensional CW complexes are standard maps. We will now present a variation of these standard maps for 1-dimensional NCCW complexes. This variation differs slightly from the one discussed in [1], generalizes the one considered in [75] and is the same as the one implied in the work of [48].

Definition 3.3.8. Let $A \coloneqq A(E,F,\beta_0,\beta_1) \in$ 1-NCCW$_1$ and $\phi : A \to C([0,1], M_q(\mathbb{C}))$ be a *-homomorphism. We call ϕ an n-standard map if there exists an n-partition $I_1 = [z_0, z_1], I_2 = [z_1, z_2], \dots, I_n = [z_{n-1}, z_n]$ of $[0,1]$ with $z_0 = 0$ and $z_n = 1$, and unitaries $u^{(m)} \in C(I_m, M_q(\mathbb{C}))$ such that for all $h \in A$ and $t \in I_m$,

$$\phi_t(h) \coloneqq u^{(m)}(t) \begin{pmatrix} h \circ \xi_1^{(m)}(t) & & & & \\ & h \circ \xi_2^{(m)}(t) & & & \\ & & \ddots & & \\ & & & h \circ \xi_{k(m)}^{(m)}(t) & \\ & & & & 0_\bullet \end{pmatrix} u^{(m)*}(t), \tag{3.3.14}$$

where $\xi_i^{(m)} \in C(I_m, \mathrm{sp}(A))$ and $m = 1, 2, \dots, n$.

Remark 3.3.9. (a) Recall that if $h \coloneqq (f,a) \in A$, then $h(\delta_i) \coloneqq a(\delta_i)$ and $h(t,i) \coloneqq f(t,i)$.

(b) The number and nature of the eigenfunctions $\xi_i^{(m)}$ depend on the subinterval I_m and may not be the same for all $m = 1, 2, \dots, n$.

(c) Let $U : [0,1] \to M_q(\mathbb{C})$ be defined by

$$U(t) \coloneqq \begin{cases} u^{(m)}(t) & \text{if } t \in [z_{m-1}, z_m) \text{ for } m = 1, 2, \dots, n-1, \\ u^{(n)}(t) & \text{if } t \in [z_{n-1}, z_n]. \end{cases} \tag{3.3.15}$$

Then U is a piecewise continuous function of unitaries.

Definition 3.3.10. Let $A \coloneqq A(E,F,\beta_0,\beta_1)$ and $B \coloneqq B(E',F',\alpha_0,\alpha_1) \in$ 1-NCCW$_1$. A *-homomorphism $\phi : A \to B \subset C([0,1], F')$ is an n-standard map if for each $i = 1, 2, \dots, k'$, the map

$$q_i \circ \phi : A \to C([0,1], M_{f'_i}(\mathbb{C})) \tag{3.3.16}$$

is an n-standard map, where

$$E' \coloneqq \bigoplus_{i=1}^{l'} M_{e'_i}(\mathbb{C}), \quad F' \coloneqq \bigoplus_{j=1}^{k'} M_{f'_j}(\mathbb{C}), \tag{3.3.17}$$

and $q_i : B \to C([0,1], M_{f'_i}(\mathbb{C}))$ is the natural projection map.

Example 3.3.11. The *-homomorphism ϕ in Example 3.3.3 is a 2-standard map with $U : [0,1] \to M_3(\mathbb{C})$ defined by

$$U(t) := \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } t \in [0, \frac{1}{2}], \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

The next lemma shows the existence of a 3-standard map connecting two close unital *-homomorphisms.

Lemma 3.3.12 ([48, Theorem 3.5])**.** Let $A := A(E, F, \beta_0, \beta_1) \in$ 1-NCCW$_1$, $\mathcal{F} \subset A$ be a finite set, and $\epsilon > 0$. There exist η, η_1, and $\delta > 0$ such that if $\phi_0, \phi_1 : A \to M_n(\mathbb{C})$ are unital *-homomorphisms satisfying:

(i) $\|\phi_0(h) - \phi_1(h)\| < 1$ for all $h \in H(\eta_1)$,

(ii) $\|\phi_0(h) - \phi_1(h)\| < \frac{\delta}{8}$ for all $h \in H(\eta) \cup \widetilde{H}(\eta)$.

Then there exists a unital 3-standard map $\phi : A \to C([0,1], M_n(\mathbb{C}))$ connecting ϕ_0 to ϕ_1 such that

$$\|\phi_t(f) - \phi_0(f)\| < \epsilon \quad \forall f \in \mathcal{F},\, t \in [0,1]. \tag{3.3.18}$$

Moreover, for each $\mu \in (\mathrm{Sp}(\phi_0) \cup \mathrm{Sp}(\phi_1)) \cap \coprod_{i=1}^{k} (0,1)_i$, we have

$$\overline{B_{4\eta_1}(\mu)} \subset \bigcup_{t \in [0,1]} \mathrm{Sp}(\phi_t),$$

where $\overline{B_{4\eta_1}(\mu)} := \{\upsilon \in \coprod_{i=1}^{k} [0,1]_i : \mathrm{dist}(\upsilon, \mu) \le 4\eta_1\}$.

Proof. The full details are in [48, Theorem 3.5]. Here, we state the form of the 3-standard map.

Take δ, η, η_1, and $\mathrm{Sp}(\phi_0) \cap [\eta, 1-\eta_1]_i \subset X_i \subset \mathrm{Sp}(\phi_0) \cap (0,1)_i$, $\mathrm{Sp}(\phi_1) \cap [\eta_1, 1-\eta_1]_i \subset X_i' \subset \mathrm{Sp}(\phi_1) \cap (0,1)_i$ such that X_i and X_i' can be bijectively paired within $2\eta_1$ for $i = 1, 2, \ldots, k$ (see [48, Theorem 3.5]). Denote the one-to-one correspondence by $\Lambda_i : X_i \to X_i'$.

There exist unitaries $U_0, U_1 \in M_n(\mathbb{C})$ such that

$$\phi_0(f,a) = U_0 \,\mathrm{diag}\left(a(\delta_1) \otimes \mathbf{I}_{s_1}, \ldots, a(\delta_l) \otimes \mathbf{I}_{s_l}, f(w_1), f(w_2), \ldots, f(w_r)\right) U_0^*, \tag{3.3.19}$$

and

$$\phi_1(f,a) = U_1 \cdot \mathrm{diag}(a(\delta_1) \otimes \mathbf{I}_{t_1}, \ldots, a(\delta_l) \otimes \mathbf{I}_{t_l}, f(c_1), f(c_2), \ldots, f(c_q)) U_1^*, \tag{3.3.20}$$

with $w_m, c_j \in \coprod_{i=1}^{k} (0,1)_i$.

For each $w_m \in \mathrm{Sp}(\phi_0) \cap (0,1)_i$, define a continuous map $\xi_m^{(1)} : [0, \frac{1}{3}] \to \coprod_{i=1}^{k} (0,1)_i$ satisfying:

(i) $\xi_m^{(1)}(0) := w_m$,

(ii) $\xi_m^{(1)}\left(\frac{1}{3}\right) := \begin{cases} 0_i & \text{if } w_m \in (0,\eta_1)_i \setminus X_i, \\ \Lambda(w_m) & \text{if } w_m \in X_i, \\ 1_i & \text{if } w_m \in (1-\eta_1, 1)_i \setminus X_i, \end{cases}$

(iii) $\mathrm{Im}(\xi_m^{(1)}) = \overline{B_{4\eta_1}(w_m)} := \{w \in \coprod_{i=1}^{k} [0,1]_i \mid \mathrm{dist}(w, w_m) < 4\eta_1\}$.

Similarly, for each $c_j \in \mathrm{Sp}(\phi_0) \cap (0,1)_i$, define a continuous map $\xi_j^{(3)} : [\frac{2}{3}, 1] \to \coprod_{i=1}^{k} (0,1)_i$ satisfying:

(i) $\xi_j^{(3)}(0) := c_j$,

(ii) $\xi_j^{(3)}\left(\frac{2}{3}\right) := \begin{cases} 0_i & \text{if } c_j \in (0,\eta_1)_i \setminus X_i, \\ c_j & \text{if } c_j \in X_i, \\ 1_i & \text{if } c_j \in (1-\eta_1, 1)_i \setminus X_i, \end{cases}$

(iii) $\mathrm{Im}(\xi_j^{(3)}) = \overline{B_{4\eta_1}(c_j)} := \{c \in \coprod_{i=1}^{k} [0,1]_i \mid \mathrm{dist}(c, c_j) < 4\eta_1\}$.

Define $\xi_m^{(2)} : [\frac{1}{3}, \frac{2}{3}] \to \coprod_{i=1}^{k} (0,1)_i$ by $\xi_m^{(2)}(t) := \xi_m^{(1)}(\frac{1}{3})$ for $m = 1, 2, \ldots, r$. Finally, ϕ is of the form:

$$\phi_t(f,a) := \begin{cases} u^{(1)}(t)\ \mathrm{diag}\Big(a(\delta_1) \otimes \mathbf{I}_{s_1}, \ldots, a(\delta_l) \otimes \mathbf{I}_{s_l}, f(\xi_1^{(1)}(t)), \ldots, f(\xi_r^{(1)}(t))\Big)\, u^{(1)}(t)^* & \text{if } t \in \left[0, \frac{1}{3}\right], \\ u^{(2)}(t)\ \mathrm{diag}\Big(a(\delta_1) \otimes \mathbf{I}_{s_1}, \ldots, a(\delta_l) \otimes \mathbf{I}_{s_l}, f(\xi_1^{(2)}(\frac{1}{3})), \ldots, f(\xi_r^{(2)}(\frac{1}{3}))\Big)\, u^{(2)}(t)^* & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ u^{(3)}(t)\ \mathrm{diag}\Big(a(\delta_1) \otimes \mathbf{I}_{t_1}, \ldots, a(\delta_l) \otimes \mathbf{I}_{t_l}, f(\xi_1^{(3)}(t)), \ldots, f(\xi_q^{(3)}(t))\Big)\, u^{(3)}(t)^* & \text{if } t \in \left[\frac{2}{3}, 1\right], \end{cases}$$

where $u^{(1)} \in C([0, \frac{1}{3}], M_n(\mathbb{C}))$ and $u^{(3)} \in C([\frac{2}{3}, 1], M_n(\mathbb{C}))$ are constant functions defined by $u^{(1)}(t) = U_0$ for $t \in [0, \frac{1}{3}]$ and $u^{(3)}(t) = U_1$ for $t \in [\frac{2}{3}, 1]$. The function $u^{(2)} \in C([\frac{1}{3}, \frac{2}{3}], M_n(\mathbb{C}))$ defines a continuous path of unitaries from U_0 to $U_1 P$ for some permutation matrix P. We define the associated piecewise continuous function $U : [0,1] \to M_n(\mathbb{C})$ by

$$U(t) := \begin{cases} u^{(1)}(t) & \text{if } t \in [0, \frac{1}{3}) \\ u^{(2)}(t) & \text{if } t \in [\frac{1}{3}, \frac{2}{3}) \\ u^{(3)}(t) & \text{if } t \in [\frac{2}{3}, 1]. \end{cases} \tag{3.3.21}$$

Note that the two forms of $\phi_{\frac{2}{3}}$ above coincide, and there exists a permutation matrix $Q \in M_n(\mathbb{C})$ such that $\mathrm{diag}\Big(a(\delta_1)\otimes\mathbf{I}_{s_1},\dots,a(\delta_l)\otimes\mathbf{I}_{s_l},f(\xi_1^{(2)}(\tfrac{1}{3})),\dots,f(\xi_r^{(2)}(\tfrac{1}{3}))\Big) = Q\ \mathrm{diag}\Big(a(\delta_1)\otimes\mathbf{I}_{t_1},\dots,a(\delta_l)\otimes\mathbf{I}_{t_l},f(\xi_1^{(3)}(\tfrac{2}{3})),\dots,f(\xi_q^{(3)}(\tfrac{2}{3}))\Big)\,Q^*$ by construction. ∎

Remark 3.3.13. Let $V:[0,1]\to M_n(\mathbb{C})$ be a piecewise continuous function defined by

$$V(t) := \begin{cases} \mathbf{I}_n & \text{if } t\in\left[0,\frac{1}{3}\right),\\ \mathbf{I}_n & \text{if } t\in\left[\frac{1}{3},\frac{2}{3}\right),\\ Q & \text{if } t\in\left[\frac{2}{3},1\right],\end{cases}$$

where Q is the permutation matrix referenced in Lemma 3.3.12, and U is defined as in (3.3.21). Let $W:[0,1]\to M_n(\mathbb{C})$ be defined by $W(t):=V(t)U(t)^*$.

Then, from the construction of the 3-standard map in Lemma 3.3.12, we obtain a *-homomorphism $\theta: A\to C([0,1],M_n(\mathbb{C}))$ defined by

$$\theta_t(f,a) := \begin{cases} \mathrm{diag}\Big(a(\delta_1)\otimes\mathbf{I}_{s_1},\dots,a(\delta_l)\otimes\mathbf{I}_{s_l},f(\xi_1^{(1)}(t)),\dots,f(\xi_r^{(1)}(t))\Big) & \text{if } t\in\left[0,\frac{1}{3}\right),\\ \mathrm{diag}\Big(a(\delta_1)\otimes\mathbf{I}_{s_1},\dots,a(\delta_l)\otimes\mathbf{I}_{s_l},f(\xi_1^{(2)}(\tfrac{1}{3})),\dots,f(\xi_r^{(2)}(\tfrac{1}{3}))\Big) & \text{if } t\in\left[\frac{1}{3},\frac{2}{3}\right),\\ Q\ \mathrm{diag}\Big(a(\delta_1)\otimes\mathbf{I}_{t_1},\dots,a(\delta_l)\otimes\mathbf{I}_{t_l},f(\xi_1^{(3)}(t)),\dots,f(\xi_q^{(3)}(t))\Big)\,Q^* & \text{if } t\in\left[\frac{2}{3},1\right]\end{cases}$$

and satisfying $\phi_t = W(t)\theta_t W(t)^*$ for all $t\in[0,1]$.

Following some of the arguments presented in [48, Theorem 3.6], we prove the following result.

Theorem 3.3.14. Let $A := A(E,F,\beta_0,\beta_1)$, $B := B(E',F',\alpha_0,\alpha_1)\in 1\text{-NCCW}_1$, $\mathcal{F}$ be a finite set, and $\epsilon>0$. Let $\phi: A\to B$ be a unital *-homomorphism. Then there exists a unital n-standard map $\psi: A\to B$ such that $\|\phi(h)-\psi(h)\|<\epsilon$ for all $h\in\mathcal{F}$. If ϕ is injective, then ψ is also injective.

Proof. Set $d := \sum_{i=1}^{k'} f_i'$ and consider $\phi: A\to B\subset C([0,1],M_d(\mathbb{C}))$. Without loss of generality, we assume $0<\epsilon<1$. Then apply Lemma 3.3.12 for A, $\frac{\epsilon}{2}$, $M_d(\mathbb{C})$, and the finite set $\mathcal{F}$ to obtain η_1, η, and δ. Since $H(\eta)\cup\widetilde{H}(\eta)\cup H(\eta_1)$ is finite, there exists $\delta'>0$ such that for any i and any $w,w'\in[0,1]_i$ with $\mathrm{dist}(w,w')<\delta'$, we have

$$\|\phi_w(h)-\phi_{w'}(h)\| < \min\left\{\frac{\epsilon}{2},\frac{\delta}{8}\right\}\ \forall\ h\in H(\eta)\cup\widetilde{H}(\eta)\cup H(\eta_1). \tag{3.3.22}$$

For each i, consider an m-partition $0=z_0<z_1<\ldots<z_m=1$ of $[0,1]_i$ with $\mathrm{dist}(z_{j-1},z_j)<\delta'$ for $j=1,2,\dots,m$. Denote the dividing interval $I_j^i := [z_{j-1},z_j]_i$ for $j=1,2,\dots,m$ and $i=1,2,\dots,k'$.

For a fixed $i \in \{1, \ldots, k'\}$, set $w_{j-1} \coloneqq (z_{j-1}, i)$ for each $j = 1, 2, \ldots, m$. Let ϕ_{w_0} be as in (3.3.19) and ϕ_{w_1} be as in (3.3.20). By (3.3.22) and the proof of Lemma 3.3.12, we obtain a 3-standard map $\psi^{i1} \coloneqq \psi|_{I_1^i} : A \to C(I_1^i, M_{e_i'}(\mathbb{C}))$ satisfying

$$\|\psi_z^{i1}(f) - \phi_{w_0}(f)\| < \frac{\epsilon}{2} \;\; \forall f \in \mathcal{F}, \;\; z \in I_1^i, \tag{3.3.23}$$

with $\psi_{z_0}^{i1} \coloneqq \phi_{w_0}$, $\psi_{z_1}^{i1} \coloneqq \phi_{w_1}$, and for each $\mu \in (Sp(\psi_{z_0}^{i1}) \cup Sp(\psi_{z_1}^{i1})) \cap \coprod_{q=1}^{k} (0,1)_q$, we have

$$\overline{B_{4\eta_1}(\mu)} \subset \bigcup_{t \in I_1^i} Sp(\psi_t^{i1}). \tag{3.3.24}$$

By repeating the process for each $j = 2, \ldots, m$, we similarly obtain, from the proof of Lemma 3.3.12, 3-standard maps $\psi^{ij} \coloneqq \psi|_{I_j^i} : A \to C(I_j^i, M_{e_i'}(\mathbb{C}))$ satisfying

(i) $\psi_{z_{j-1}}^{ij} = \phi_{w_{j-1}}$ and $\psi_{z_j}^{ij} = \phi_{w_j}$ for each $j = 2, \ldots, m$,

(ii) For each $j = 2, \ldots, m$, we have that

$$\|\psi_z^{ij}(f) - \phi_{w_{j-1}}(f)\| < \frac{\epsilon}{2} \; \forall f \in \mathcal{F}, \;\; z \in I_j^i. \tag{3.3.25}$$

(iii) For each $j = 2, \ldots, m$ and each $\mu \in (Sp(\psi_{z_{j-1}}^{ij}) \cup Sp(\psi_{z_j}^{ij})) \cap \coprod_{q=1}^{k} (0,1)_q$, we have

$$\overline{B_{4\eta_1}(\mu)} \subset \bigcup_{t \in I_j^i} Sp(\psi_t^{ij}). \tag{3.3.26}$$

It follows from (3.3.22), (3.3.23), and (3.3.25) that

$$\| \, \psi|_{I_j^i}(f) - \phi|_{I_j^i}(f)\| < \epsilon \tag{3.3.27}$$

for all $f \in \mathcal{F}$ and all $j = 1, 2, \ldots, m$. We can fit all the $\psi^{ij} \coloneqq \psi|_{I_j^i}$ together to define a $3m$-standard map $\psi^i : A \to C([0,1], M_{e_i'}(\mathbb{C}))$. For each $\delta_r \in Sp(E')$ and $h \in A$, define

$$\psi_{\delta_r}(h) \coloneqq \phi_{\delta_r}(h) \;\text{ for all } h \in A, \tag{3.3.28}$$

where $\phi_{\delta_r} \coloneqq \delta_r \circ \phi$. Using (3.3.28) and combining all the ψ^i, we define a $3m$-standard map $\psi : A \to B$, since $\phi_\tau(h) = \psi_\tau(h)$ for all $\tau = 0, 1$ and $h \in A$. We obtain from (3.3.27) that

$$\|\psi(f) - \phi(f)\| < \epsilon \tag{3.3.29}$$

for all $f \in \mathcal{F}$.

Next, we prove that if ϕ is injective, meaning that $\mathrm{Sp}(\phi) := \bigcup_{\mu\in \mathrm{Sp}(B)} \mathrm{Sp}(\phi_\mu) = \mathrm{Sp}(A)$, then the map ψ defined above is also injective.Let $\upsilon_0 \in \coprod_{i=1}^{k} (0,1)_i$ and set $\mathcal{V} := \{\upsilon \in \coprod_{i=1}^{k} (0,1)_i \mid \mathrm{dist}(\upsilon, \upsilon_0) \le \eta_1\}$ and $Y := \coprod_{i=1}^{k'} (0,1)_i$. The injectivity of ϕ implies that

$$\mathcal{V} \subset \mathrm{Sp}(\phi) := \bigcup_{\mu\in Y} \mathrm{Sp}(\phi_\mu) \cup \bigcup_{\mu\in Sp(E')} \mathrm{Sp}(\phi_\mu).$$

Hence, $\mathcal{V} \cap \bigcup_{\mu\in Y} \mathrm{Sp}(\phi_\mu)$ contains infinitely many points. Suppose $\upsilon_0 \in (0,1)_r$ for some r and choose $\upsilon_1 \in \mathcal{V} \cap \bigcup_{\mu\in Y} \mathrm{Sp}(\phi_\mu) \cap [\eta_1, 1-\eta_1]_r$. Then $\upsilon_1 \in \mathrm{Sp}(\phi_z)$ for some $z \in I_j^i$. By (3.3.22), $\|\phi_z(h) - \phi_{w_{j-1}}(h)\| < 1$ for all $h \in H(\eta_1)$. It follows from Lemma 3.3.1 that there exists $\upsilon_2 \in \mathrm{Sp}(\phi_{w_{j-1}})$ such that $\mathrm{dist}(\upsilon_1, \upsilon_2) < 2\eta_1$. Consequently, we get that

$$d(\upsilon_0, \upsilon_2) \le d(\upsilon_0, \upsilon_1) + d(\upsilon_1, \upsilon_2) < \eta_1 + 2\eta_1 = 3\eta_1,$$

and $\upsilon_0 \in \overline{B_{4\eta_1}(\upsilon_2)} \subset \bigcup_{t\in I_j^i} \mathrm{Sp}(\psi_t)$ by (3.3.26). Since υ_0 is arbitrary, $\coprod_{i=1}^{k} (0,1)_i \subset \mathrm{Sp}(\psi)$.

For $\upsilon_0 \in (0,\eta_1)_i$, we have

$$d(0_i, \upsilon_0) \le d(0_i, \upsilon_0) + d(\upsilon_0, \upsilon_2) \le 4\eta_1.$$

Similarly, for $\upsilon_0 \in (1-\eta_1, 1)_i$,

$$d(1_i, \upsilon_0) \le d(1_i, \upsilon_0) + d(\upsilon_0, \upsilon_2) \le 4\eta_1.$$

Hence, 0_i and 1_i are in $\overline{B_{4\eta_1}(\upsilon_2)} \subset \bigcup_{t\in I_j^i} \mathrm{Sp}(\psi_t)$, and thus 0_i and 1_i are in $\mathrm{Sp}(\psi)$ for all $i = 1, 2, \ldots, k$. We conclude that $\mathrm{Sp}(E) \subset \mathrm{Sp}(\psi)$ and $\mathrm{Sp}(\psi) = \mathrm{Sp}(A)$. Therefore, ψ is injective. ∎

Let $A := A(E, F, \beta_0, \beta_1)$ and $B := B(E', F', \alpha_0, \alpha_1) \in$ 1-NCCW$_1$, and let $\phi : A \to B$ be a unital n-standard map. As illustrated in Remark 3.3.13, there exists a piecewise continuous function $W : [0,1] \to \mathcal{U}(F')$ of unitaries and a *-homomorphism $\theta : A \to C([0,1], F')$ such that $\phi_t = W(t)\theta_t W(t)^*$ for all $t \in [0,1]$. Moreover,

$$\theta_t(h) := Q \begin{pmatrix} h\circ\xi_1^{(m)}(t) & & & \\ & h\circ\xi_2^{(m)}(t) & & \\ & & \ddots & \\ & & & h\circ\xi_{k(m)}^{(m)}(t) \end{pmatrix} Q^* \tag{3.3.30}$$

for all $h \in A$ and all t in some dividing interval I_m, where $\xi_i^{(m)} \in C(I_m, \mathrm{Sp}(A))$ and Q is a permutation matrix in F'. We call θ the $\mathcal{D}$-map of the n-standard map ϕ.

We denote the set of piecewise continuous function from $[0,1]$ to a finite dimensional $C*$-algebra F by $PC([0,] \to F)$. For a piecewise continuous function $W : [0,1] \to \mathcal{U}(F')$, we define a piecewise continuous function $W^* : [0,1] \to \mathcal{U}(F')$ by $W^*(t) = W(t)^*$.

Lemma 3.3.15. Let A and B be 1-dimensional NCCW complexes with $K_1(A) = 0$, and let $\phi, \psi : A \to B$ be *-homomorphisms. If $\pi \circ \phi$ and $\pi \circ \psi$ are unitarily equivalent for every $\pi \in \mathrm{Sp}(B)$, then ϕ and ψ are approximately unitarily equivalent.

Proof. By [63, Theorems 1.0.1 & 3.2.2], it suffices to show that $Cu(\phi) = Cu(\psi)$. Let $B \coloneqq B(E, F, \beta_0, \beta_1)$. For any $h \in A_+$ and $\pi \in \mathrm{Sp}(B)$, $\pi \circ \phi(h)$ and $\pi \circ \psi(h)$ have the same rank since $\pi \circ \phi$ is unitarily equivalent to $\pi \circ \psi$. Hence, by a slight abuse of notation, we can write the Cuntz semigroup elements as $[\pi_t \circ \phi(h)] = [\pi_t \circ \psi(h)]$ for all $t \in [0, 1]$, and $[\bigoplus_{\pi\in \mathrm{Sp}(E)} \pi \circ \phi(h)] = [\bigoplus_{\pi\in \mathrm{Sp}(E)} \pi \circ \psi(h)]$. Therefore, by the definition of $Cu(B)$ (see Theorem 3.1.7), we obtain that $Cu(\phi) = Cu(\psi)$, and the result follows ∎

Next, we extend [67, Theorem 3] using a similar line of reasoning.

Theorem 3.3.16. Let $A \coloneqq A(E, F, \beta_0, \beta_1)$ and $B \coloneqq B(E', F', \alpha_0, \alpha_1)$ be elements of 1-NCCW$_1$ with trivial K_1-groups, and let $\phi : A \to B$ be a unital n-standard map. Assume W and $\theta : A \to C([0, 1], F')$ are the piecewise continuous function and the $\mathcal{D}$-map associated with ϕ, respectively. Then for any $\epsilon > 0$ and finite set $\mathcal{F} \subset A$, there exists a unitary $H \in B$ and a unital n-standard map $\psi : A \to B$ such that:

(i)
$$\psi_t \coloneqq \mathcal{W}(t)\theta_t\mathcal{W}(t)^*, \tag{3.3.31}$$
where $\mathcal{W} \in C([0, 1], F')$ such that $\mathcal{W}(0)$ and $\mathcal{W}(1)$ are permutation matrices.

(ii)
$$\|\phi(f) - H\psi(f)H^*\| < \epsilon \quad \text{for all } f \in \mathcal{F}. \tag{3.3.32}$$

Proof. Let $\delta'_i \in \mathrm{Sp}(B) \cap \mathrm{Sp}(E')$ and $(f, a) \in A$ be arbitrary. Then there exist unitaries $V_i \in M_{e'_i}(\mathbb{C})$ such that

$$\delta'_i \circ \phi(f, a) = V_i \operatorname{diag}\left(a(\delta_1) \otimes \mathbf{I}_{t^i_1}, \dots, a(\delta_l) \otimes \mathbf{I}_{t^i_l}, f(x^i_1), f(x^i_2), \dots, f(x^i_{r^i})\right) V_i^*. \tag{3.3.33}$$

Using (3.1.3), we can identify $\phi(f, a)$ with $\left(\phi(f, a), \bigoplus_{i=1}^{l'} c_i\right)$, where $c_i \coloneqq \delta'_i \circ \phi(f, a)$. Note that

$$\begin{aligned} \phi_0(f, a) &= \mathcal{W}(0)\theta_0(f, a)\mathcal{W}^*(0), & (3.3.34)\\ &= \alpha_0\left(\bigoplus_{i=1}^{l'} c_i\right), & (3.3.35)\\ &= p\left(\bigoplus_{i=1}^{l'} (c_i \otimes \mathbf{I}_{r_i})\right) p^* \quad \text{by (3.1.4)}, & (3.3.36) \end{aligned}$$

where p is a permutation matrix. Substitute the form of θ_0 from (3.3.30) and c_i from (3.3.33) into (3.3.34) and (3.3.36), respectively. Since the spectrum of ϕ_0 in (3.3.34) and (3.3.36) must coincide, there exists a permutation matrix $S_0 \in F'$ such that

$$S_0\theta_0(f, a)S_0^* = p\left(\bigoplus_{i=1}^{l'} (V_i^* c_i V_i \otimes \mathbf{I}_{r_i})\right) p^*. \tag{3.3.37}$$

Similarly, there exists a permutation matrix $S_1 \in F'$ such that

$$S_1\theta_1(f,a)S_1^* = q\left(\bigoplus_{i=1}^{l'}(V_i^*c_iV_i \otimes \mathbf{I}_{s_i})\right)q^*, \tag{3.3.38}$$

where q is a permutation matrix associated with α_1 as in (3.1.5). Let $\mathcal{W} \in C([0,1],F')$ define a continuous path of unitaries in F' connecting S_0 to S_1. The *-homomorphisms $A \to C([0,1],F')$ that map $h \in A$ to $\mathcal{W}\theta(h)\mathcal{W}^*$ and $A \to E'$ that map $h \in A$ to $\bigoplus_{i=1}^{l'} V_i^*(\delta_i' \circ \phi(h))V_i$ define a map $\psi : A \to B$ satisfying

$$\psi_t \coloneqq \mathcal{W}(t)\theta_t\mathcal{W}(t)^* \ \forall t \in [0,1]. \tag{3.3.39}$$

$\psi(h) \in B$, and ψ is a well-defined *-homomorphism since

$$\alpha_0\left(\bigoplus_{i=1}^{l'}V_i^*c_iV_i\right) = S_0\theta_0(h)S_0^* \quad \text{and} \quad \alpha_1\left(\bigoplus_{i=1}^{l'}V_i^*c_iV_i\right) = S_1\theta_1(h)S_1^*$$

for all $h \in A$. By the definitions of ϕ and ψ, $\pi \circ \psi$ is unitarily equivalent to $\pi \circ \phi$ for all $\pi \in \mathrm{Sp}(B)$. The second part of the theorem follows from Lemma 3.3.15. ∎

An easy application of Elliott's intertwining argument gives the following:

Lemma 3.3.17 ([61, 2.3.46])**.** Let $A \coloneqq \varinjlim(A_n,\phi_n)$ and $B \coloneqq \varinjlim(A_n,\psi_n)$ be inductive limit C^*-algebras, where each A_n is a unital C^*-algebra and the connecting maps ϕ_n and ψ_n are unital *-homomorphisms. If each A_n is finitely generated, and ϕ_n is approximately unitarily equivalent to ψ_n for every n, then $A \cong B$.

Theorem 3.3.18 ([45, Theorem 1.10])**.** For each n, let B_n be a C^*-diagonal of a C^*-algebra A_n, $N_n \coloneqq N_{A_n}(B_n)$ the normalizer of B_n in A_n, and $P_n : A_n \to B_n$ the faithful conditional expectation of A_n onto B_n. If $\phi_n : A_n \to A_{n+1}$ is an injective *-homomorphism satisfying $\phi_n(B_n) \subset B_{n+1}$, $\phi_n(N_n) \subset N_{n+1}$, and $P_{n+1} \circ \phi_n = \phi_n \circ P_n$ for all n, then $\varinjlim(B_n,\phi_n)$ is a C^*-diagonal of $\varinjlim(A_n,\phi_n)$.

We are now ready to prove the main theorem of this chapter.

Theorem 3.3.19. Given a unital, separable C^*-algebra $A \coloneqq \varinjlim(A_n,\phi_n)$, where each A_n is an element of 1-NCCW$_1$ with trivial K_1-groups, if the connecting maps ϕ_n are unital and injective, then A has a C^*-diagonal.

Proof. Let $A_n \coloneqq A_n(E_n,F_n,\beta_0^n,\beta_1^n)$. For any $\epsilon > 0$, n, and finite set $\mathcal{F} \subset A_n$, it follows from Theorem 3.3.14 that there exists an injective unital m_n-standard map $\zeta_n : A_n \to A_{n+1}$ such that

$$\|\phi_n(a) - \zeta_n(a)\| < \frac{\epsilon}{2}, \ \text{ for all } \ a \in \mathcal{F}. \tag{3.3.40}$$

Let θ_n be the $\mathcal{D}$-map of ζ_n. By Theorem 3.3.16, there exists an injective unital m_n-standard map $\psi_n : A_n \to A_{n+1}$ described by $\mathcal{W}_{n+1}\theta_n\mathcal{W}_{n+1}^* : A_n \to C([0,1],F_{n+1})$ and $\gamma_n \coloneqq \pi_e \circ \psi_n :$

$A_n \to E_{n+1}$, where $\mathcal{W}_{n+1} \in C([0,1], F_{n+1})$ with $\mathcal{W}_{n+1}(0)$ and $\mathcal{W}_{n+1}(1)$ being permutation matrices. Moreover,

$$\|\zeta_n(a) - H_n\psi_n(a)H_n^*\| < \frac{\epsilon}{2} \tag{3.3.41}$$

for some unitaries $H_n \in A_{n+1}$ and all $a \in \mathcal{F}$. Combining (3.3.40) and (3.3.41), we conclude that $\|\phi_n(a) - H_n\psi_n(a)H_n^*\| < \epsilon$ for all $a \in \mathcal{F}$. Hence, ϕ_n is approximately unitarily equivalent to an m_n-standard map ψ_n. Since 1-dimensional NCCW complexes are finitely generated [31, Lemma 2.3], we have that $A \cong \varinjlim(A_n, \psi_n)$ by Lemma 3.3.17.

We adopt a slight modification of the technique described in [45, Remark 4.1] to further transform A into an inductive limit C^*-algebra with connecting *-homomorphisms that preserve the C^*-diagonals of each building block.

Regard $\theta_n : C([0,1], F_n) \oplus E_n \to PC([0,1], F_{n+1})$ as an extension of $\theta_n : A_n \to C([0,1], F_{n+1})$. For each n and $s \in [0,1]$, let

(i) $\Omega_n(s) \subset (0,1)$ be the finite set $\mathrm{Sp}(\pi_s \circ \psi_n) \cap \coprod_{i=1}^{k} (0,1)_i$, ignoring multiplicity and index.

(ii) $\Sigma_n \subset (0,1)$ be the finite set $\mathrm{Sp}(\pi_e \circ \psi_n) \cap \coprod_{i=1}^{k} (0,1)_i$, ignoring multiplicity and index.

We proceed by selecting unitary $W_n \in \mathcal{U}(C([0,1], F_n))$, unitary $Z_{n+1} \in \mathcal{U}(C([0,1], F_{n+1}))$, and piecewise continuous function of permutation matrices $C_{n+1} \in PC([0,1], F_{n+1})$ such that, for all n, $W_n(s) = 1$ for all $s \in (\Omega_n(0) \cup \Omega_n(1) \cup \Sigma_n)$, $W_n(t)$ is a permutation matrix for each $t \in \{0,1\}$, $Z_{n+1}(t) = \mathcal{W}_{n+1}(t)$ for $t \in \{0,1\}$, and $W_{n+1} := C_{n+1}\theta_n(W_n, \mathbf{I}_n)Z_{n+1}^*$.

Define $W_1 := 1$. If W_n and Z_n have been chosen, then $\theta_n(W_n, \mathbf{I}_n)$ may not be in $\mathcal{U}(C([0,1], F_{n+1}))$. However, $W_n(t)$ being a permutation matrix for each $t = 0, 1$ ensures the existence of a piecewise continuous function of permutation matrices $C_{n+1} \in PC([0,1], F_{n+1})$ such that $C_{n+1}\theta_n(W_n, \mathbf{I}_n)$ is unitary. Pick $Z_{n+1} \in \mathcal{U}(C([0,1], F_{n+1}))$ such that $Z_{n+1}(t) := \mathcal{W}_{n+1}(t)$ for $t \in \{0,1\}$, and

$$Z_{n+1}(s) := C_{n+1}(s)\theta_n(W_n, \mathbf{I}_n)(s) \quad \forall\, s \in (\Omega_n(0) \cup \Omega_n(1) \cup \Sigma_n). \tag{3.3.42}$$

Moreover, set

$$W_{n+1} := C_{n+1}\theta_n(W_n, \mathbf{I}_n)Z_{n+1}^* \tag{3.3.43}$$

and

$$\overline{A}_n := A(E_n, F_n, \alpha_0^n, \alpha_1^n), \tag{3.3.44}$$

where $\alpha_t^n := W_n(t)\beta_t^n W_n^*(t)$, and $t = \{0,1\}$. Let $\overline{\psi}_n : A_n \to A_{n+1}$ be a *-homomorphism defined by *-homomorphisms $Z_{n+1}\theta_n Z_{n+1}^* : A_n \to C([0,1], F_{n+1})$ and γ_n.

Similarly, let $\widehat{\psi}_n : \overline{A}_n \to \overline{A}_{n+1}$ be a *-homomorphism defined by the map $\overline{\theta}_n := C_{n+1}\theta_n C_{n+1}^* : \overline{A}_n \to C([0,1], F_{n+1})$ and $\overline{\gamma}_n : \overline{A}_n \to E_{n+1}$, where $\overline{\gamma}_n(f, a) = \gamma_n(W_n^* f W_n, a)$. $\overline{\theta}_n$ is a well-defined *-homomorphism since, if $(f, a) \in \overline{A}_n$, then

(i) $W_n(t)^* f(t) W_n(t) = \beta_t^n(a)$ for $t = 0, 1$, and so $(W_n^* f W_n, a) \in A_n$.

(ii) $\overline{\theta}_n(f, a) = C_{n+1}\theta_n(W_n, \mathbf{I}_n)\theta_n(W_n^* f W_n, a)\theta_n(W_n^*, \mathbf{I}_n)C_{n+1}^*$.

(iii) $\overline{\theta}_n(f, a) \in C([0, 1], F_{n+1})$ since $C_{n+1}\theta_n(W_n, \mathbf{I}_n), \theta_n(W_n^* f W_n, a)$, and $\theta_n(W_n^*, \mathbf{I}_n)C_{n+1}^*$ are in $C([0, 1], F_{n+1})$.

Using (3.3.43), we obtain the following commutative diagram:

$$\begin{array}{ccc} A_n & \xrightarrow{\quad \overline{\psi}_n \quad} & A_{n+1} \\ {\scriptstyle Ad(W_n) \oplus \mathrm{id}}\Big\downarrow & & \Big\downarrow{\scriptstyle Ad(W_{n+1}) \oplus \mathrm{id}} \\ \overline{A}_n & \xrightarrow[\quad \hat{\psi}_n \quad]{} & \overline{A}_{n+1} \end{array}$$

Notice that $\overline{A}_n$ satisfies conditions (A1) and (A2). Using a similar argument as in the beginning of the proof and the commutative diagram above, we conclude that

$$A \cong \varinjlim(A_n, \psi_n) \cong \varinjlim(A_n, \overline{\psi}_n) \cong \varinjlim(\overline{A}_n, \widehat{\psi}_n).$$

Now, we show the existence of a C^*-diagonal of A. Let

$$\widehat{B}_n := \{(f, a) \in \widehat{A}_n : f(t) \in DF_n \ \forall \ t \in [0, 1]\},$$

where DF_n is the set of diagonal matrices of F_n. Recall that $\widehat{B}_n$ is a C^*-diagonal of $\overline{A}_n$ for all n by Proposition 3.2.5. Using the definition of $\widehat{\psi}_n$ and θ_n (see (3.3.30)), it is evident that

- $\widehat{\psi}_n(\widehat{B}_n) \subset \widehat{B}_{n+1}$ for all n.
- $\widehat{\psi}_n(N_n) \subset N_{n+1}$ since the normalizers of $\widehat{B}_n$ are of the form (f, a), where $f(t)$ is a generalized permutation matrix.
- $P_{n+1} \circ \widehat{\psi}_n = \widehat{\psi}_n \circ P_n$, where P_n is the canonical conditional expectation described in Proposition 3.2.5.

By Theorem 3.3.18, we conclude that $\varinjlim(\widehat{B}_n, \widehat{\psi}_n)$ is a C^*-diagonal of A.

∎

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