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APPLICATION OF THE DIFFERENTIAL QUADRATURE METHOD TO THE BUCKLING ANALYSIS OF CYLINDRICAL SHELLS AND TANKS

by

Pirooz Mirfakhraei

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Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering
University of Ottawa
Ottawa, Ontario, Canada

0-612-46534-9

To My parents
and my wife

Abstract

The newly developed differential quadrature method (DQM) is applied to the problem of linear elastic buckling of circular cylindrical shells and tanks. The Flügge theory serves as the basis of the analysis. For the first time the DQM is applied to the buckling problem of cylindrical shells and also for the first time the 2-dimensional DQM with two different test functions (polynomial and harmonic test functions) is applied to the structural mechanics problems with circumferential continuity. As well for the first time the nonsymmetrical form of the buckling equations is derived based on the Flügge theory and used for analyzing nonsymmetric buckling problem of tanks under quasi static earthquake loading. Both the 1-dimensional and 2-dimensional DQM are used.

In the 1-dimensional part, the displacement fields (U, V, W) are expressed as products of unknown functions along the axial direction and known trigonometric functions along the circumferential direction. In the 2-dimensional part, the displacement fields are represented by unknown functions in two directions. The derivatives in both the governing equations and the boundary conditions are discretized by the DQM. In this process the governing differential equations and boundary conditions are transformed into a set of algebraic equations, the eigenvalues of which are the buckling loads of the shell or the tank. In the 2-dimensional part, shells under single and combined loads are considered. The latter load represents the tanks containing liquid under quasistatic earthquake loading.

The accuracy and efficiency of the DQM is examined by comparing the results with those in the literature. The results of 1-dimensional DQM are in good agreement with the work of other researchers. In the 2-dimensional problems, while an overall convergence is observed, in some cases a complete convergence is not obtained. The size of the matrices in the 2-dimensional DQM is an order of magnitude larger than those in one dimension, yet good results have been found for tanks under earthquake loading in the last part where more accurate equations and boundary conditions are used. The results show that the application of DQM to the tank and shell problems is successful.

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Contents

List of figures

List of tables

Nomenclature

A	Acceleration
$A^{(r)}_{ij}$	Weighting coefficient of r-th derivative in longitudinal direction
$B^{(r)}_{ij}$	Weighting coefficient of r-th derivative in circumferential direction
D	$Eh^3/12(1-\upsilon^2)$
E	Young's modulus
g	Acceleration of gravity
h	Thickness of cylindrical shell
J	$Eh/(1-\upsilon^2)$
L	Length of cylindrical shell
M	No. of sampling points in longitudinal direction
M_x, $M_{x\theta}$, M_θ	Nondimensional moment resultants
N_x, Q_x, $N_{x\theta}$, N_θ	Nondimensional force resultants
N	No. of sampling points in circumferential direction
n	No. of assumed circumferential buckling mode
R	Radius of cylindrical shell
S_x, T_x	Nondimensional effective shear forces
u, v, w	Non-dimensional displacement components
x	Longitudinal position variable
θ	Nondimensional circumferential position variable
κ	$h^2/12R^2$
υ	Poisson's ratio
ρ	Mass density of contained liquid
φ	Nondimensional longitudinal position variable

Chapter 1

Introduction

Liquid storage tanks are very important facilities in today's society and they mostly have a circular cylindrical shape. These tanks are made of a vertical cylindrical shell welded at the bottom to a circular base plate. The tanks may be anchored to the ground or simply rest unanchored on the foundation. They usually have a roof at the top which is either welded to the cylindrical shell or floats on the contained fluid. Cylindrical tanks are made in different sizes and aspect ratios (length to radius). The radius may vary from a few meters to tens of meters and the aspect ratio may range from 0.5 to 5. Circular cylindrical tanks are easy to build and have good load carrying capability, so they are widely used in the oil and petrochemical industry as well as water storage and distribution and winery industries.

The loads applied to cylindrical tanks can be classified as static or dynamic. The basic static loads are the hydrostatic pressure acting in the radial direction, and the weight of the wall and roof acting in the axial direction. The sources of dynamic loads are hydrodynamic pressure and inertia forces generated, respectively, by the acceleration of the fluid and the tank body in earthquakes.

The major source of potential damage to liquid storage tanks is earthquake loading. In earthquakes different combinations of loads may be present and the dynamic and static loads are superimposed. The design of earthquake resistant tanks is a matter of special importance which extends far beyond the pure economic value of the tanks and their contents. The absence of a water supply, uncontrolled fires and spillage or clouds of toxic chemicals subsequent to earthquake may cause much more damage.

The most commonly observed failure of tanks in seismic action is tank-wall buckling. Three different kinds of buckling can occur in earthquakes. They are elastic buckling due to axial load, elastic buckling due to external pressure, and elastic-plastic buckling. The last of these is called "elephant foot buckling" and occurs most often in the uplifting of tanks. Research on the buckling of cylindrical tanks has been and still is a very active field. Due to difficulties in formulating convenient analytical solutions for the buckling of cylindrical shells and tanks mostly numerical solutions have been presented. The Finite Difference Method (FDM), Finite Element Method (FEM) and the Boundary Element Method (BEM) are widely used for this type of analysis.

The objective of this thesis is to apply the newly developed Differential Quadrature Method (DQM) to the buckling problem of circular cylindrical shells and tanks. While in some ways easier, this method has shown better results compared to other numerical methods in some problems of engineering. Since the method is still in the development stage, it is to be applied to simple problems of elastic buckling to reveal its capabilities for use in more complicated applications in the future. In consideration of earthquake effects the equivalent static loads are used to represent dynamic loads. The study is divided into three major parts. In each major part results are found and compared with results obtained by other methods and researchers.

In the first part the DQM method is presented and the governing differential equations of the buckling problem of thin elastic cylindrical shells are discussed. It is also discussed how to find the DQM analogue of differential equations, how to apply the boundary conditions, and how to prepare the matrices of the eigenvalue problem in buckling. The first part is included in chapters 3 and 4.

In the second part the DQM is applied to the problem of the buckling of cylindrical shells under different loads and boundary conditions. In some problems the solution can be found by applying the discretization in one direction only. This simplifies the procedure greatly. In some other cases the 2-dimensional discretization must be applied. In the 1-dimensional DQM the polynomial approximation is used, in the 2-dimensional polynomial and harmonic functions have to be used. The second part is included in chapters 5 and 6.

In the third part the DQM is applied to the buckling problem of cylindrical tanks containing liquid. In this part the effect of combinations of loads on the shell, resembling the effect of fluid's hydrostatic pressure and dynamic loads induced by accelerated fluid, is studied. The nonsymmetrical buckling equations for the combined loading are derived and solved, using the 2-dimensional DQM. This part includes chapter 7. Finally in chapter 8 suggestions for future research are made.

Chapter 2

Literature survey

2.1 Introduction

Because of its wide scope of application, the buckling of cylindrical shells and tanks has been the subject of investigation of many researchers and, as indicated by the recent surveys of Noor (1990) and Teng (1996), still remains a very active area of research. There is also an ongoing need for faster and more efficient numerical methods to solve problems of engineering. The DQM has been shown to have the potential to become an effective numerical method that can exceed in numerical accuracy as well as computational efficiency the well known FDM and FEM (Malik& Civan,1995).

2.2 Cylindrical shells

Cylindrical shells have been in service for a long time and they are one of the first structural elements that have come under investigation. Fairbain's study in 1858 of the buckling of cylinders under external pressure is one of the first experimental investigations of shell stability.

The first theoretical investigations on the buckling of cylindrical shells were performed by Lorenz (1908), Timoshenko (1910), and Southwell (1914) for axially loaded cylinders during the first decade of this century. Later Flügge (1932), Robertson (1929), Lundquist (1933), Wilson and Newmark (1933) performed related experimental studies.

Generally speaking the problem of shells is a problem of three-dimensional elasticity. But it is known that the solution of problems in three dimensional theory of elasticity involves considerable complications which have been overcome only in a few special cases. Thus a group of simplifying assumptions that provide a reasonable description of the behaviour of elastic shells has been adopted. The initial theoretical investigations were based on many of such assumptions, and reduced the mathematical model to a linear eigen-boundary-value problem (classical bifurcation approach). However, theoretical predictions did not coincide with the experimental results which prompted many researchers to look for the reasons (Simitses, 1986). To get better theoretical results, some simplifying assumptions were reevaluated or removed. Most of the investigations in this regard were done in the 1960s. Some nonlinear theories were developed and the effects of different factors were considered. The effect of membrane prebuckling stresses and deformations were studied by Ohiro (1961), Stein (1964), Hoff (1965), Fischer (1965), Hoff (1966) and Gorman and Evan-Iwanowski (1970) among others. Some other investigators tried to find the buckling load by investigating the postbuckling behaviour of cylinders under axial load. They defined the buckling load as the minimum load carrying capacity of the cylindrical shell after buckling. But Hoff et al (1966) showed that the definition is not correct. In another approach the cylinder with initial geometric imperfections was considered. Donnell (1934) employed nonlinear kinematic relations when applying this approach for the first time . Von Karman and Tsien (1941), Koiter (1945), Donnell and Wan (1950), and Budiansky and Hutchinson (1966) made valuable contributions to this approach. In conclusion it was found that the sensitivity of cylindrical shells to initial imperfection is the main reason for discrepancy between theory and practice.

The second kind of loading of cylindrical shells that came under investigation was lateral loading (pressure). The first analysis in this case was done by von Mises in 1914. Simitses and Aswani (1974) compared the results of this kind of loading for different radius to thickness and length to radius ratios and for various load behaviours during the buckling process (true pressure, constant directional pressure, and centrally directed pressure) using different linear buckling theories. In 1933 the buckling analysis of cylindrical shells under torsional loading was carried out by Donnell for the first time. Hess (1961) studied the effect of combined loads for the first time. The monograph by Yamaki (1984) is a complete reference on the effect of three basic loads (uniform axial, lateral and torsional load), and also combined loads. As is evident from this review the study of shell stability has become more and more complicated, until it has become nearly impossible to solve problems analytically.

Due to the availability of powerful computers and the development of sophisticated finite element and other numerical techniques the solution of specific complicated nonlinear buckling problems of perfect shells of revolution is now feasible (Teng,1996). Many engineering commercial codes based on these techniques are now in use (Ravichandran, 1994). These circumstances do not however imply that there is no longer a need for research on shell buckling. The research in the area of shells still has three major fields to follow. The establishment of a reliable procedure to convert a numerically obtained buckling load to the design strength of a shell is one of the most important challenges facing the shell buckling research community (Samuelson, 1991). Another area is the understanding of the buckling of shells for a wider range of problems. A final area, which is followed in this thesis, is the introduction of less expensive and more efficient methods in the solution of shell buckling problems.

2.3 Differential quadrature method (DQM)

The differential quadrature method is a numerical solution technique for boundary and/or initial value problems. The technique was developed in the early 1970s by Bellman and Casti (1971), and since then it has been successfully employed in a variety of problems in engineering and physical sciences. There are over one hundred papers on DQM in the published literature now (Bert & Malik 1996a). Jang (1987) applied the DQM to structural mechanics problem for the first time. Since then the method has been applied to solve static (Jang, 1989), dynamic (Bert et al, 1988) and structural nonlinear problems (Striz et al, 1988). It has been claimed that in some cases the DQM exceeds in numerical accuracy as well as computational efficiency the finite difference and finite element methods (Malik and Civan, 1995).

After introducing the DQM, Bellman et al (1971) presented the idea of using polynomial test functions for determining the weighting functions in the solution of an initial value problem (Bellman et al, 1972). Following these two introductory papers, the DQM was applied to problems in different areas which led to further development of the method. Kashef and Bellman (1974) used the cardinal spline test functions to determine the weighting coefficients in a second order initial value problem. Mingle (1977) introduced the new method of implementing the boundary conditions in which the quadrature analog equations were obtained from the boundary conditions at the end (boundary) points and from differential equations at the interior grid points. Civan and Sliepcevich (1984) extended the application of the DQM to three-dimensional problems for the first time. The other contributions of these same authors are, introducing the concept of domain decomposition to quadrature solution (Civan and Sliepcevich, 1985) and applying the quadrature method to integro-differential equations (Civan and Sliepcevich, 1986).

2.4 DQM in structural mechanics

The major contributions in the development of DQM arise from its application to structural mechanics problems. The first applications of DQM to structural problems by Jang (1987), Bert et al (1988) and Jang et al (1989), showed that the method is capable of handling different kinds of problems in this field. Because of the higher order of differential equations in structural problems (usually fourth or higher), these problems are of higher complexity compared to problems in other fields (which are usually of the first or second order).

One source of complexity in the application of the DQM to structural problems was the accuracy of the weighting coefficients. The accuracy of the DQM solution is completely dependent on the accuracy of the weighting coefficients. The first investigators selected a special distribution for their grid points (Bellman et al, 1972), or a special kind of test functions (Kashef and Bellman, 1974), so they could find an explicit formula for their desired derivative coefficients and thus obtain the weighting coefficients exactly. Some others used the direct linear solvers to determine the weighting coefficients. In the latter method of determination of the weighting coefficients, it is necessary to find the inverse of the Vandermonde matrices. As shown by

Hamming (1973) these matrices are not singular, but are ill-conditioned. A deterioration of results thus occurs as the size of the matrices increases. It has been verified by Bert and Malik (1996a) that for a given number of sampling points, ill-conditioning the of Vandermonde matrices leads to increasingly erroneous weighting coefficients with increasing higher order derivatives. For example, with the direct solution of Vandermonde equations, errors start creeping into the fourth order derivative weighting coefficients with as few as six equally spaced sampling points. In the earlier works the differential equation was second order at most, and the number of sampling points was limited (less than ten), thus the effect of ill-conditioning was not important. But in structural problems of order four or higher, this effect is important (Quan and Chang 1989b).

To solve this problem Quan and Chang (1989a) derived explicit formulae for determining weighting coefficients for polynomial test functions which could be used for any number of sampling points. Shu and Richards (1992) derived the same formulae for the first order derivative weighting coefficients as Quan and Chang (1989a). But for higher order derivative weighting coefficients, they proposed a general recurrence relationship which is easier to use for programming. Both of these explicit formulas are now widely used in the DQM.

The other cause of complexity in applying the DQM to structural problems is in the implementation of the boundary conditions. Since the order of equations is higher than two, there is more than one boundary equation at each end of the domain. As mentioned earlier the boundary conditions are satisfied at boundary points and so more than one boundary point is needed at each end. To overcome this difficulty Jang et al. (1989) proposed the "δ technique". In this technique, points at a small distance (δ) from the boundary points are employed and the boundary conditions are enforced at both the actual boundary points and at the "delta points".

This technique has been used successfully in many problems and helped to solve the problem (Bert and Malik (1996b), Bert et al. (1988)). The optimum value for δ varies from problem to problem and the selection of a suitable value is an important issue. The δ value must not be too large, to resemble the actual boundary condition better, on the other hand it must not be too small as the solution begins to oscillate. In general any magnitude of δ introduces some approximation and reduces the accuracy of the results.

In order to overcome this difficulty in implementing boundary conditions, another method has been proposed and used by different authors (Wang and Bert (1993), Malik and Bert (1996), Bert et al. (1994)). In this method boundary conditions are incorporated in the weighting coefficient matrices, for example to imply the condition of zero displacement, and the corresponding grid point to that boundary are simply ignored in the quadrature analogue of differential equations. Effectively in this method the boundary conditions are built into the weighting coefficients. Wang and Bert (1993) have shown that this technique improves the accuracy of the DQM dramatically. Nevertheless, the technique has some limitations. The major limitation of this method is that the weighting coefficient matrices can be modified to incorporate only the type of boundary conditions having the derivatives with respect to one spatial coordinate (Malik and Bert, 1996). This limitation prevents application of the method to some special kinds of boundary conditions in multi dimensional problems (e.g., free boundary conditions in plates or shells). This method is mostly suitable for clamped or simply supported boundary conditions.

One more problem which arises in applying the boundary conditions of structural problems is at the corner of two intersecting edges. In this case there are two different boundary conditions (from two different edges) to be satisfied at the corner point. This issue has been dealt with by

Chen et al. (1994, 1995). They have proposed a scheme for adjusting the boundary conditions at the corner to get better results.

Circular cylindrical shells are one of the most important elements of structures. To fulfill the continuity and periodicity condition required in the circumferential direction of these shells, harmonic test functions have been introduced. Malik and Bert (1994) employed such test functions in the analysis of compressible lubrication problems of bearings. Striz et al (1995) used harmonic test functions but they applied these to vibration and buckling problems of rectangular plates. They referred to the resulting method as harmonic differential quadrature method (HDQM), and have reported achieving higher accuracy, especially for higher order frequencies and buckling loads.

2.5 Buckling of liquid storage tanks

With each new major seismic event there are further reports of tank failures in the literature (Anon (1989), Benuska (1990), Haroun et al (1991), Manos (1991), Tremblay et al (1995)). Because of damages to storage tanks in earthquakes in previous years, the field of analysis of earthquake loaded liquid storage tanks has become an active area of engineering science research (Rammerstorfer, 1990). Standards for the seismic design of such tanks were formulated in the late 1970's, and have remained virtually unchanged. Extensive research work is continuing, seeking to address the main engineering issues, and to lead to improvements in design codes (Mirfakhraei et al, 1996).

Basically, the design procedure of storage tanks can be divided into two parts, determination of dynamically activated loads and analysis of strength and stability. The study of buckling of tanks includes consideration of the interaction between roof, cylindrical shell, attached pipes, baseplate, liquid, foundation and soil. The actual load applied on the shell should ideally be found by considering all the mentioned interacting elements. Taking account of these elements is however very complicated, and thus some simpler engineering methods must be used. Numerous investigations have been conducted and some important factors in the behaviour of tanks have already been determined. In the current study some of these will be adopted.

2.5.1 Shell liquid interaction

To accurately study the dynamic characteristics of liquid containers, the correct loading should be considered. Different kinds of loads will result in different kinds of responses. Two major categories of loads applied to the tanks are, loads directly applied to the shell, and loads stemming from base excitation. The latter is the load induced in an earthquake and is more important in storage tanks. The base excitation may be a horizontal, vertical, or rocking motion (rotation about a diameter of the base plate). In an actual earthquake a combination of these excitations may occur. Thus different researchers have used different modes of base excitation in their studies to model the effect of an earthquake.

Through application of a base excitation the liquid is brought into motion. Different methods have been considered to model the response of liquid. One of the simplest and most popular methods is that proposed by Housner (1957, 1959). In this procedure the hydrodynamic pressure is divided into two parts. The first one is called the "impulsive" pressure, and is caused by simultaneous movement of the liquid and the rigid shell. The second part is called the "convective" pressure, and is caused by the motion of the free surface of the liquid (sloshing). This method was used in some practical applications, but later earthquakes and damages to storage tanks showed that the concept of rigid tanks does not always hold. Further investigations showed that the flexibility of the tank wall may increase the hydrodynamic forces significantly (Veletsos and Yang (1976, 1977), Fischer (1979), Haroun (1980, 1981). Other researchers added terms to the Housner method to improve its applicability (Rammerstorfer et al., (1988, 1990)). They added terms due to vibrational interaction between shell and liquid and obtained better results.

2.5.2 Methods of analysis

The least expensive feasible method should be used to solve engineering problems. Therefore whenever possible, researchers try to find analytical or semi analytical solutions for the tank buckling problems (Chiba, Tani & Yamaki, 1987). However, because of complications in the problem of buckling of tanks, only limited literatures can be found using analytical approaches.

Numerical methods are widely used in solving the buckling problem of storage tanks. Because of its flexibility, the FEM is the most popular numerical method used. The FEM has been used in determining both the interaction between different tank elements (Mistry et al. (1995), Liu and Uras (1989a), Yi (1992), Wunderlich et al. (1994)), and the instability of tanks (Liu and Uras (1989b), Zhou et al. (1992)). The boundary element method has also been used, especially for studying the behaviour of the fluid (Zhou et al. (1992), Lay (1993)). In a few cases the finite difference method has been used (Peek, 1988). There have been some experimental studies reported on the buckling of tanks mainly serving to confirm theoretical results (Chiba and Tani (1987), Shih and Babcock (1987)). Because of their importance, sometimes the tests are carried out in full scale (Niwa and Clough, 1982), or sometimes real earthquake damaged tanks are studied (Anon (1989), Benuska (1990), Haroun et al (1991), Manos (1991), Tremblay et al. (1995)).

The DQM has not been applied to the buckling problem of shells or tanks as yet. The closest DQM study in this field is the vibration problem of cylindrical and conical shells (Bert & Malik (1996b), Shu (1996)) or the buckling problem of circular arches, plates and beams (Kang et al (1996), Wang (1995), Sherbourne (1991). Introducing the DQM to tank buckling problems can solve some engineering problems of tanks while enhancing the DQM methodology and capabilities.

Chapter 3

Governing equations

3.1 Introduction

The loss of stability is usually associated with large deformations, involving either large rotations or strains. Such deformations require that the governing equations be nonlinear. In small deformations assumption displacements and their derivatives are small and there are linear expressions for both displacement-strain relations as well as strain-stress relations. Also the equilibrium conditions can be described considering the undeformed geometry of the problem.

In some nonlinear large deformation problems the strains may be small but the rotations are large. Then the stress strain relations remain linear, but the displacement-strain relations become nonlinear. The buckling problem of shell is of this kind. For this case the deformed geometry should be considered in developing the equilibrium conditions. The resultant basic equations have nonlinear displacement terms, and belong to the nonlinear theory of elasticity. Resulting from the nonlinearity, the uniqueness of solution and the stability of equilibrium state cannot be guaranteed. Thus different equilibrium configurations (stable or unstable) may exist under a unique loading and boundary conditions. So, the solution of the nonlinear equations may lead to any of these equilibrium configurations.

In practice the structure adopts the stable configuration with the minimum total potential energy. The mathematical development leads to a variational problem. Solving for the general shape in buckling problems is quite difficult. But for cylindrical shells, as well as some other important structural elements, linear bending theories have been established for approximate analysis within the small deformation range, which can be used here.

So, the basic nonlinear equations for the buckling problem can be solved by considering the foregoing small deformation (Yamaki, 1984). This method is used in deriving the Flügge stability equations of circular cylindrical shells in this chapter.

3.2 Related theories

There are three major theories concerning the buckling of the shell. They are Donnell's, Flügge's, and Sander's theories.

Donnell introduced his theory in 1933 in connection with the analysis of torsional buckling of thin-walled tubes. The theory is relatively simple, with good accuracy for shallow shells. The Donnell theory is based on the following assumptions for the shell of thickness of h, length L and radius R:

(1) The shell is sufficiently thin, i.e., $h/R \ll 1$, $h/L \ll 1$.

(2) The strains are sufficiently small, $\varepsilon<<1$, and Hooke's law holds.

(3) Straight lines normal to the undeformed middle surface remain straight and normal to the deformed middle surface with their lengths unchanged.

(4) The normal stress acting in the direction normal to the middle surface may be neglected in comparison with the stresses acting in the direction parallel to the middle surface.

(5) Displacements U and V are infinitesimal, while W is of the same order as the shell thickness, that is, $|U|<<h$, $|V|<<h$ and $|W|=O(h)$.

(6) The derivatives of W are small, but their squares and products are of the same order as the strains.

(7) Curvature changes are small and the influence of U and V on them are negligible, so that they can be represented by linear functions of W only.

The assumptions (5) to (7) correspond to shallow shell approximations.

Donnell's theory is not appropriate for the analysis of the deformations of a cylinderical shell in which the membrane displacements are significant. Flügge's theory (Flügge 1932, 1973) does not have this deficiency, and is applicable to the problem with any buckling configuration. In addition to the first four assumptions of Donnell, the following three assumptions are made in this theory:

(5) The terms up to the order of $(h/R)^2$ are retained in deriving expression for stress resultants.

(6) The rotations are small but the effect of their product and squares on the mid- surface strains will be considered.

(7) The curvature changes are small enough to have linear expressions for the bending moment.

In Sanders theory emphasis is placed on deriving simplified basic equations, rather than exact ones. Hence the strain displacement equations found are not exact either. Sander's equations are much more complex than those of the Donnell theory but simpler than those of Flügge's theory. Sander's equation are between Donnell's and Flügge's equations in terms of generality as well. Contrary to Donnell's equations, Sander's are applicable to non-shallow shells (larger aspect ratios (L/R) in circular cylindrical shells).

3.3 Variational equations

A circular cylindrical shell of radius R, length L and thickness h is considered. Based on the Flügge assumptions and considering figure 3.1 the governing equations for a cylindrical shell can be developed (Yamaki, 1984). Denoting displacement components of the middle plane by U, V and W, the displacements at a distance of z from the middle surface are given by

$$U_1 = U - zW_{,X}\,, \qquad V_1 = \frac{R-z}{R}V - zW_{,Y}\,, \qquad W_1 = W \tag{3-1}$$

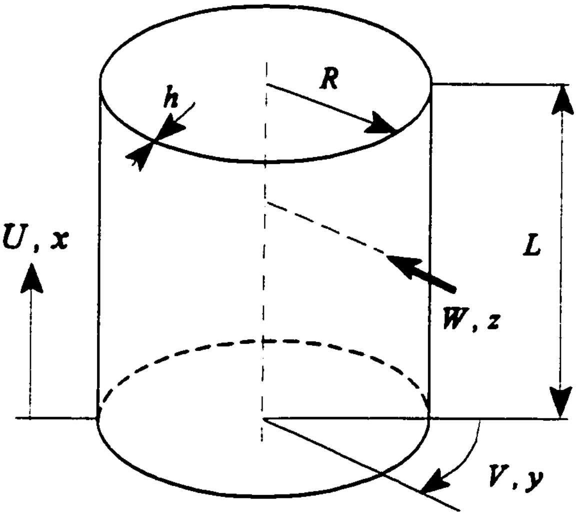

Figure 3.1 Shell geometry and coordinate system

For convenience in this text x and $\varphi = x/R$ and also y and $\theta = y/R$ are used interchangeably to denote the axial and circumferential position of a point.

Nonlinear terms are considered only for the strain components along the middle surface. Thus for finite deformations, the strains at distance z from middle surface can be written as

$$\begin{aligned}
\varepsilon_x &= U,_x - zW,_{xx} + \varepsilon_{x0} \\
\varepsilon_y &= V,_y - \frac{R}{R-z} z W,_{yy} - \frac{1}{R-z} W + \varepsilon_{y0} \\
\gamma_{xy} &= \frac{R-z}{R} V,_x + \frac{R}{R-z} U,_y - \left(1 + \frac{R}{R-z}\right) z W,_{xy} + \gamma_{xy0}
\end{aligned} \tag{3-2}$$

where the following nonlinear terms will be considered only when $z = 0$

$$\begin{aligned}
\varepsilon_{x0} &= \frac{1}{2}(U,_x^2 + V,_x^2 + W,_x^2) \\
\varepsilon_{y0} &= \frac{1}{2}\left(U,_y^2 + \left(V,_y - \frac{W}{R}\right)^2 + \left(W,_y + \frac{V}{R}\right)^2\right) \\
\gamma_{xy0} &= U,_x U,_y + V,_x\left(V,_y - \frac{w}{R}\right) + W,_x\left(W,_y + \frac{V}{R}\right)
\end{aligned} \tag{3-3}$$

Hooke's law provide the relations

$$\sigma_x = \frac{E}{1-\upsilon^2}(\varepsilon_x + \upsilon\varepsilon_y), \quad \sigma_y = \frac{E}{1-\upsilon^2}(\varepsilon_y + \upsilon\varepsilon_x) \quad \tau_{xy} = \frac{E}{2(1+\upsilon)}\gamma_{xy} \tag{3-4}$$

Using equations (3-3) in (3-4) and by the definition of stress resultants and stress couples given as:

$$N_x=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_x dz,\quad N_y=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_y dz,\quad N_{xy}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy} dz,\quad N_{yx}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{yx} dz,$$

$$M_x=\int_{-\frac{h}{2}}^{\frac{h}{2}}z\sigma_x dz,\quad M_y=\int_{-\frac{h}{2}}^{\frac{h}{2}}z\sigma_y dz,\quad M_{xy}=\int_{-\frac{h}{2}}^{\frac{h}{2}}z\tau_{xy} dz,\quad M_{yx}=\int_{-\frac{h}{2}}^{\frac{h}{2}}z\tau_{yx} dz \tag{3-4b}$$

one can find

$$\begin{aligned}
N_x &= J[U,_x+\upsilon(V,_y-\frac{W}{R})+\varepsilon_{x0}+\upsilon\varepsilon_{y0}]+\frac{D}{R}W,_{xx} \\
N_y &= J(V,_y-\frac{W}{R}+\upsilon U,_x+\varepsilon_{y0}+\upsilon\varepsilon_{x0})-\frac{D}{R}(W,_{yy}+\frac{W}{R^2}) \\
N_{xy} &= \frac{1-\upsilon}{2}[J(U,_y+V,_x+\gamma_{xy0})+\frac{D}{R}(\frac{V,_x}{R}+W,_{xy})] \\
N_{yx} &= \frac{1-\upsilon}{2}[J(U,_y+V,_x+\gamma_{xy0})+\frac{D}{R}(\frac{U,_y}{R}-W,_{xy})] \\
M_x &= -D(W,_{xx}+\upsilon W,_{yy}+\frac{U,_x+\upsilon V,_y}{R}) \\
M_y &= -D(W,_{yy}+\frac{W}{R^2}+\upsilon W,_{xx}) \\
M_{xy} &= -(1-\upsilon)D(W,_{xy}+\frac{V,_x}{R}) \\
M_{yx} &= -(1-\upsilon)D(W,_{xy}+\frac{V,_x-U,_y}{2R})
\end{aligned} \tag{3-5}$$

where

$$J=\frac{Eh}{1-\upsilon^2}\quad,\qquad D=\frac{Eh^3}{12(1-\upsilon^2)} \tag{3-6}$$

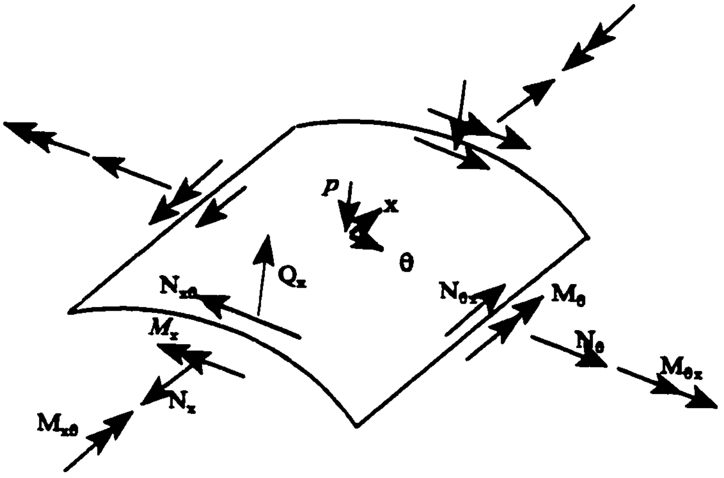

Figure 3.2 Force and moment resultant

The basic equations are found by minimizing the total energy through the variational principle. The elastic strain energy (Yamaki, 1984) is

$$U_e=\frac{1}{2}\int_0^L\int_0^{2\pi R}\int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\tau_{xy}\gamma_{xy})dx\,dy\,dz \tag{3-7}$$

and the potential energy of the external forces is

$$V_e=-\int_0^L\int_0^{2\pi R}(p_xU+p_yV+p_zW)dx\,dy-\int_0^{2\pi R}(p_{xe}U+p_{ye}V+p_{ze}W-M_{xe}W_{,x})\Big|_{x=0}^{x=L}dy \tag{3-8}$$

where p_x, p_y and p_z are distributed forces per unit area of the shell (p_z is the external pressure applied "p"). M_{xe} is the external bending moment and p_{xe}, p_{ye} and p_{ze} are the external applied forces per unit length along the edges. The total potential energy is

$$E_t = U_e + V_e\,.$$

According to the variational principle the variation of the total energy at equilibrium state must be zero, i.e.,

$$\delta E_t = \delta U_e + \delta V_e = 0$$

Expanding, and using Gauss's theorem the following equilibrium equations are obtained

$$\begin{gathered}
[N_x(1+U_{,x})]_{,x} + [N_{yx}(1+U_{,x})]_{,y} + (N_y U_{,y})_{,y} + (N_{xy}U_{,y})_{,x} + p_x - pW_{,x} = 0 \\
[N_{xy}(1+V_{,y}-\frac{W}{R})]_{,x} + [N_y(1+V_{,y}-\frac{W}{R})]_{,y} - \frac{M_{y,y}+M_{xy,x}}{R} + (N_x V_{,x})_{,x} + (N_{yx}V_{,x})_{,y} \\
-\frac{N_{yx}W_{,x}}{R} + p_y - (p+\frac{N_y}{R})(W_{,y}+\frac{V}{R}) = 0 \\
M_{x,xx} + (M_{xy}+M_{yx})_{,xy} + M_{y,yy} + \frac{N_y}{R}(1+V_{,y}-\frac{W}{R}) + [N_x W_{,x} + N_{xy}(W_{,y}+\frac{V}{R})]_{,x} \\
+[N_{yx}W_{,x} + N_y(W_{,y}+\frac{V}{R})]_{,y} + \frac{N_{yx}V_{,x}}{R} + p(1+U_{,x}+V_{,y}-\frac{W}{R}) = 0
\end{gathered} \qquad (3\text{-}9)$$

Equations (3-9) could also be found enforcing equilibrium condition on an infinitesimal element in the axial, circumferential and radial directions respectively. The boundary conditions are

$$\begin{array}{lcl}
N_x(1+U_{,x})+N_{xy}U_{,y} = p_{xe}, & \text{or} & U=U_0 \\
N_{xy}(1+V_{,y}-\frac{W}{R})+N_x V_{,x}-\frac{M_{xy}}{R} = p_{ye} & \text{or} & V=V_0 \\
M_{x,x}+(M_{xy}+M_{yx})_{,y}+N_x W_{,x}+N_{xy}(W_{,y}+\frac{V}{R}) = p_{ze} & \text{or} & W=W_0 \\
M_x = M_{xe}, & \text{or} & W_{,x}=W_{x0}
\end{array} \qquad (3\text{-}10)$$

3.4 Flügge basic equations for buckling

Three basic loads are considered here, and the basic equations of the buckling problem on the basis of Flügge nonlinear theory are derived (Yamaki 1984). The loads are axisymmetric external pressure p, axial force $P=2\pi R\sigma h$, and torque $T=2\pi R^2\tau h$, the two latter loads being applied at the edges of the cylinder (σ and τ are axial normal and shear stresses respectively).

For the prebuckling state the stress resultants and displacements are functions of x only so denoting them by subscript 0, from equations (3-9) we have

$$N_{x0}=-\sigma h-\frac{1}{2}pR\varepsilon,\qquad N_{xy0}=N_{yx0}=\tau h$$

$$N_{y0}=-\upsilon\sigma h-\frac{1}{2}\upsilon pR\varepsilon-\frac{EhW_0}{R}$$

$$DW_{0,xxxx}+(\sigma h+\frac{1}{2}\varepsilon pR)W_{0,xx}+\frac{EhW_0}{R^2}+\frac{\upsilon\sigma h}{R}-p(1-\frac{1}{2}\upsilon\varepsilon)=0 \qquad (3\text{-}11)$$

where $\varepsilon=1$ for a shell closed at the ends, and $\varepsilon=0$ for an open shell. At the buckling state we have

$$(U, V, W) = (U_0, V_0, W_0) + (U_1, V_1, W_1)$$

$$(N_x, N_{xy}, N_{yx}, N_y) = (N_{x0}, N_{xy0}, N_{yx0}, N_{y0}) + (N_{x1}, N_{xy1}, N_{yx1}, N_{y1}) \qquad (3\text{-}12)$$

The subscript 1 corresponds to the infinitesimal increment during buckling. Substituting these expressions into (3-9), considering equations (3-11) and disregarding the nonlinear terms in the incremental displacement, one obtains

$$N_{x1,x}+N_{yx1,y}+N_{x0}U_{1,xx}+2N_{xy0}U_{1,xy}+N_{y0}U_{1,YY}-pW_{1,x}=0$$

$$(N_{xy1}-\frac{m_{xy1}}{R})_{,x}+(N_{y1}-\frac{M_{y1}}{R})_{,y}+N_{x0}V_{1,xx}+2N_{xy0}(V_{1,xy}-\frac{W_{1,x}}{R})+N_{y0}(V_{1,yy}-\frac{V_1}{R^2}-2\frac{W_{1,y}}{R})$$
$$-p(\frac{V_1}{R}+W_{1,y})=0 \qquad (3\text{-}13)$$

$$M_{x1,xx}+(M_{xy1}+M_{yx1})_{,xy}+M_{y1,yy}+\frac{N_{y1}}{R}+W_{0,xx}N_{x1}+N_{x0}W_{1,xx}+2N_{xy0}(W_{1}yy+\frac{V_{1,x}}{R})$$
$$+N_{y0}(2\frac{V_{1,y}}{R}+W_{1,yy}-\frac{W_1}{R^2})+p(U_{1,x}+V_{1,y}-\frac{W_1}{R})=0$$

In a similar procedure, expressions for the incremental stress resultants and stress couples are obtained from equations (3-5) as

$$N_{x1}=J(U_{1,x}+\upsilon(V_{1,y}-\frac{W_1}{R})+W_{0,x}W_{1,x})+\frac{D}{R}W_{1,xx}$$
$$N_{y1}=J(V_{1,y}-\frac{W_1}{R}+\upsilon(U_{1,x}+W_{0,x}W_{1,x}))-\frac{D}{R}(W_{1,yy}+\frac{W_1}{R^2})$$
$$N_{xy1}=\frac{1-\upsilon}{2}(J(U_{1,y}+V_{1,x}+W_{0,x}W_{1,y})+\frac{D}{R}(\frac{V_{1,x}}{R}+W_{1,xy}))$$
$$N_{yx1}=\frac{1-\upsilon}{2}(J(U_{1,y}+V_{1,x}+W_{0,x}W_{1,y})+\frac{D}{R}(\frac{U_{1,y}}{R}-W_{1,xy}))$$
$$M_{x1}=-D(W_{1,xx}+\upsilon W_{1,yy}+\frac{U_{1,x}+\upsilon V_{1,y}}{R})$$
$$M_{y1}=-D(W_{1,yy}+\frac{W_1}{R^2}+\upsilon W_{1,xx})$$
$$M_{xy1}=-(1-\upsilon)D(W_{1,xy}+\frac{V_{1,x}}{R}) \qquad (3\text{-}14)$$
$$M_{yx1}=-(1-\upsilon)D(W_{1,xy}+\frac{V_{1,x}-U_{1,y}}{2R})$$

Where *J* and *D* are defined in (3-6). Now substituting from (3-14) in (3-10), considering homogenous boundary conditions, and using the dimensionless variables of

$$\varphi=\frac{x}{R},\quad \theta=\frac{y}{R},\quad w^0=\frac{w_0}{R},\quad \kappa=\frac{h^2}{12R^2},$$
$$q_p=\frac{pR}{I},\quad q_c=\frac{\sigma h}{I},\quad q_t=\frac{\tau h}{I} \tag{3-15}$$

the expressions for boundary conditions are

$$\begin{aligned}
M_{xl}=0 &: \quad W_{1,\varphi\varphi}+U_{1,\varphi}+\upsilon V_{1,\theta}=0\\
P_{xl}=0 &: \quad (1-q_c)U_{1,\varphi}+q_t U_{1,\theta}+\upsilon V_{1,\theta}+w^0{}_{,\varphi}W_{1,\varphi}+\kappa W_{1,\varphi\varphi}=0\\
P_{yl}=0 &: \quad \frac{1-\upsilon}{2}U_{1,\theta}+[\frac{1-\upsilon}{2}(1+3\kappa)-q_c]V_{1,\varphi}+q_t V_{1,\theta}+1.5(1-\upsilon)\kappa W_{1,\varphi\theta}=0
\end{aligned} \tag{3-16}$$

So the governing equations for the buckling problem, when the effect of prebuckling bending deformation is considered, are

$$\begin{aligned}
&U_{1,\varphi\varphi}+[\frac{1-\upsilon}{2}-(1-\upsilon^2)w^0]U_{1,\theta\theta}+\frac{1+\upsilon}{2}V_{1,\varphi\theta}-(\upsilon-w^0{}_{,\varphi\varphi})W_{1,\varphi}+w^0{}_{,\varphi}W_{1,\varphi\varphi}+\frac{1-\upsilon}{2}w^0{}_{,\varphi}W_{1,\theta\theta}\\
&\qquad +\kappa(\frac{1-\upsilon}{2}U_{1,\theta\theta}+W_{1,\varphi\varphi\varphi}-\frac{1-\upsilon}{2}W_{1,\varphi\theta\theta})-q_p w_{1,\varphi}-q_c(U_{1,\varphi\varphi}+\upsilon U_{1,\theta\theta})+2q_t U_{1,\varphi\theta}=0
\end{aligned}$$

$$\begin{aligned}
&\frac{1+\upsilon}{2}U_{1,\varphi\theta}+\frac{1-\upsilon}{2}V_{1,\varphi\varphi}+V_{1,\theta\theta}-(1-\upsilon^2)w^0(V_{1,\theta\theta}-V_1)+[-1+\frac{1-\upsilon}{2}w^0{}_{,\varphi\varphi}+2(1-\upsilon^2)w^0]W_{1,\theta}\\
&\qquad +\frac{1+\upsilon}{2}w^0{}_{,\varphi}W_{1,\varphi\theta}+\kappa[1.5(1-\upsilon)V_{1,\varphi\varphi}+\frac{3-\upsilon}{2}W_{1,\varphi\varphi\theta}]-q_p(V_1+W_{1,\varphi})\\
&\qquad\qquad -q_c[V_{1,\varphi\varphi}+\upsilon(V_{1,\theta\theta}-V_1-2W_{1,\theta})]+2q_t(V_{1,\varphi\theta}-W_{1,\varphi})=0
\end{aligned}$$

$$\begin{aligned}
&(\upsilon+w^0{}_{,\varphi\varphi})U_{1,\varphi}+[1+\upsilon w^0{}_{,\varphi\varphi}-2(1-\upsilon^2)w^0]V_{1,\theta}-[1+\upsilon w^0{}_{,\varphi\varphi}-(1-\upsilon^2)w^0]W_1\\
&\qquad -\kappa(U_{1,\varphi\varphi\varphi}-\frac{1-\upsilon}{2}U_{1,\varphi\theta\theta}+\frac{3-\upsilon}{2}V_{1,\varphi\varphi\theta}+W_{1,\varphi\varphi\varphi\varphi}+2W_{1,\varphi\varphi\theta\theta}+W_{1,\theta\theta\theta\theta}+2W_{1,\theta\theta}+W_1)\\
&\qquad\qquad +q_p(U_{1,\varphi}+V_{1,\theta}-W_{1,\theta\theta})-q_c[w_{1,\varphi\varphi}+\upsilon(2V_{1,\theta}+W_{1,\theta\theta}-w_1)]+2q_t(V_{1,\varphi}+W_{1,\varphi\theta})=0
\end{aligned} \tag{3-17}$$

These equations are the basic equations of the Flügge linear buckling theory, and are applied in this work for the buckling analysis of cylindrical shells and tanks under axisymmetrical loadings. The equations for nonaxisymmetrical loading are derived in chapter 7.

Chapter 4

Differential Quadrature Method

4.1 Introduction

The methods used for the buckling analysis of cylindrical shells can be classified into two categories. The first category comprises of semi-analytical methods. Included in this category are the Rayleigh-Ritz and Galerkin methods. Yamaki (1984) has applied the Galerkin method to compute the buckling loads and modes for cylindrical shells under a number of different loads and boundary conditions. The problem with the semi analytical methods is that they are usually not easy to apply and are complicated.

The second category comprises of numerical methods. In these methods the domain is discretized and the governing differential equations are transformed into an analogous set of first order equations in terms of the discrete values of the field variables. Included in this category is the FEM. Using the numerical methods, the differential equations are reduced to a set of algebraic equations. The eigenvalues of the resultant homogeneous algebraic equation system provide the buckling loads of the problem.

Most buckling problems can be solved by the FEM but only with the use of a great amount of memory and computational effort. The differential quadrature method offers a promising new approach for the buckling analysis of cylindrical shells. An accurate buckling load can be found using a relatively small amount of memory and computational effort.

4.2 The quadrature rule

The essential basis of the DQM is the approximation of derivatives of a function, f(x) using the quadrature rule. In this method the derivatives are approximated by a linear weighting of the function values at some sampling points of the domain, i.e.,

$$\frac{d^r f}{dx^r}\Big|_{x=x_i} = \sum_{j=1}^{N} A_{ij}^{(r)} f(x_j) \tag{4-1}$$

Here $A_{ij}^{(r)}$ are the unknown weighting coefficients of the "r-th" order derivative at the "i-th" point in the domain and N corresponds to the number of sampling points or stations in the domain. The weighting coefficients can be determined for some appropriately chosen function, usually called a "test function" and for a special set of sampling points. Once the distribution of sampling points and the test function are selected, the $A_{ij}^{(r)}$s are uniquely defined.

For 2-dimensional problems the discretization is done in both directions. Dependent on the sampling points and the test function of a given direction, the weighting coefficients are determined. For example if "f" is a 2-dimensional function, the x-direction derivative is found from equation (4-1) and in the y-direction it is found from

$$\frac{\partial^{s} f}{\partial y^{s}}\Big|_{y=y_i} = \sum_{j=1}^{N} B_{ij}^{(s)} f(x,y_j) \tag{4-2}$$

For mixed derivative one has

$$\frac{\partial^{(r+s)} f}{x^{r}\partial y^{s}}\Big|_{x=x_i,y=y_j} = \frac{\partial^{r}}{\partial x^{r}}\left(\frac{\partial^{s} f}{\partial y^{s}}\right)\Big|_{x=x_i,y=y_j} = \sum_{k=1}^{N} A_{ik}^{(r)} \sum_{l=1}^{M} B_{jl}^{(s)} f(x_k,y_l) \tag{4-3}$$

4.3 Test functions

The set of test functions selected must be complete. Like the interpolation functions in the FEM, the test functions should represent the possible uniform states of the field variables and have differentiability up to the highest order derivative appearing in the governing differential equations (Bert & Malik (1996a), Huebner (1975)).

Among the possible choices for test functions, the polynomial and harmonic test functions are most commonly used.

4.3.1 Polynomial test functions

For polynomials, the test functions are

$$H(x) = x^{(\eta-1)} \quad , \quad \eta = 1,2,3, \ldots , N \tag{4-4}$$

The number of test functions is equal to the number of sampling points (N). For completeness the lowest power of test functions must be equal to the highest order derivative in the corresponding direction, and the number of sampling points must be at least one more than the highest order derivative. For each of the test functions in equation (4-1) there is one equation of the form

$$\sum_{j=1}^{N} x_j^{(k-1)} A_{ij}^{\,r} = \frac{\partial^r}{\partial x^r} x^{(k-1)} \Big|_{x=x_i} = \begin{cases} \dfrac{(k-1)!}{(k-1-r)!} x_i^{(k-1-r)} & \text{when} \quad k-1-r \geq 0 \\ 0 & \text{when} \quad k-1-r < 0 \end{cases} \quad , i, k=1,2,\ldots, N \tag{4-5}$$

From equation (4-5) it is clear that the weighting coefficients, when the test function is specified, are functions of the sampling points (x_i) only.

The relation (4-5) represents a set of N simultaneous linear equations. In the solution of these equations a Vandermonde matrix is set up, which is proved to always be nonsingular (Hamminig, 1973). But this matrix is ill-conditioned and when the rank of the matrix goes beyond the interval of 11 to 13, this equation can lead to inaccurate results. Shu and Richards (1992) and Quan and Cheng (1989) solved this dilemma by introducing explicit formulae. These formulae give accurate answers for virtually any number of sampling points or any rank of the Vandermonde matrix. The explicit formula for the weighting coefficients for the first order derivative is (Shu and Richards, 1992)

$$A_{ij}^{(1)} = \frac{\pi(x_i)}{(x_i - x_j)\pi(x_j)} \qquad i,j = 1,2,\dots,N \quad ,i \neq j \tag{4-6}$$

where

$$\pi(x_i) = \prod_{j=1}^{N} (x_i - x_j) \qquad , \quad i \neq j \tag{4-7}$$

The weighting coefficient for higher order derivatives may be obtained through the following recurrence relationship

$$A_{ij}^{(r)} = r\ [A_{ii}^{(r-1)} A_{ij}^{(1)} - \frac{A_{ij}^{(r-1)}}{x_i - x_j}], \qquad i,j=1,2,\dots,N \quad , i \neq j \quad , 2 \leq r \leq N-1 \tag{4-8}$$

and when $i = j$

$$A_{ij}^{(r)} = A_{ii}^{(r)} = -\sum_{k=1}^{N} A_{ik}^{(r)} \ , \qquad i=1,2,\dots,N \ , \qquad 1 \leq r \leq (N-1) \tag{4-9}$$

4.3.2 Harmonic test functions

Different families of harmonic functions may be used as test functions, but each family is required to be complete. To find buckling loads of rectangular plates under combined bending and compression for N sampling points, Striz et al (1995) used the following set of N test functions.

$w(x)=\{1, \sin\pi x, \cos\pi x, \sin 2\pi x, \ldots, \sin((N-1)/2)\pi x, \cos((N-1)/2)\pi x\}$. Here N is an odd number. As before N sets of N linear algebraic equations are obtained and the A_{ij}s are completely determined. When using harmonic test functions sometimes the DQM is called the harmonic differential quadrature method (HDQ).

Malik and Bert (1994) have used another set of harmonic functions. Following Hamming (1973) they took the set

$$w(x) = \cos[2(k-1)\pi x] \qquad k = 1, 2, 3, \ldots, N/2+1$$

$$w(x) = \sin[2(k-N/2-1)\pi x] \qquad k = N/2+2, N/2+3, \ldots, N$$

where N is an even number. If the sampling points are spaced equally, the inverse of the Vandermonde matrix can explicitly be found as

$$\begin{vmatrix} 0.5 & 1 & \ldots & 1 & 0.5 & 0 & \ldots & 0 \\ 0.5 & \mathrm{Cos}\,\pi/N & \ldots & \mathrm{Cos}\,(N-1)\pi/N & -0.5 & \mathrm{Sin}\,\pi/N & \ldots & \mathrm{Sin}(N-1)\pi/N \\ 0.5 & \mathrm{Cos}2\pi/N & \ldots & \mathrm{Cos}\,(N-1)2\pi/N & 0.5 & \mathrm{Sin}\,2\pi/N & \ldots & \mathrm{Sin}(N-1)2\pi/N \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0.5 & \mathrm{Cos}\,(2N-1)\pi/N & \ldots & \mathrm{Cos}\,(N-1)(2N-1)\pi/N & -0.5 & \mathrm{Sin}(2N-1)\pi/N & \ldots & \mathrm{Sin}(N-1)(2N-1)\pi/N \end{vmatrix}$$

The main application of the harmonic test functions is in cases where the field variable is periodic, eg. in the circumferential direction of shells of revolution.

In some applications different kinds of test functions may be used in different directions. Because there are explicit formulas available, the polynomial test functions are a good choice whenever possible. In cylindrical shells, polynomial test functions are used in the longitudinal direction, but because of the continuity required it is better to use harmonic test functions in the circumferential direction.

4.4 Sampling points

Different choices for the sampling point positions can be considered. The simplest choice for these points is one with equal spacing. But it has been shown that in some cases some unequally spaced points may lead to faster convergence of the solution, and more accurate results.

Another closely related technique to DQM is the collocation method (Quan and Chang, 1989). In the collocation method the collocation points are placed at zeros of orthogonal polynomials to improve the accuracy of the solution. The zeroes of the Legendre polynomials were used by Bellmen et al (1972) in the DQM. Quan and Chang (1989) found that the zeros of the Chebyshev polynomials of the first kind yieldede the most accurate results for initial value problems. But since the zeroes of orthogonal polynomials (e.g., Legendre or Chebyshev polynomials) do not include the end points, with the presence of boundary conditions these grid points will not yield satisfactory results (Quan and Chang, 1989).

Shu and Richards(1992) offered another choice for sampling points that includes the end points. These points are called Chebyshev-Gauss-Lobatto points. The points are found from the equation

$$x(i)=\frac{1-\cos\frac{\pi(i-1)}{N-1}}{2}\times a \qquad 2<i<N-1 \tag{4-10}$$

where x varies as $0 \le x \le a$.

Bert and Malik (1996a) have found these points to be consistently better than the equally spaced, Legendre, or Chebyshev points in many problems. This type of sampling point spacing (4-10) is used in this thesis for sampling points in the longitudinal direction, and equal spacing is used in the circumferential direction.

4.5 Implementing Boundary Conditions

Using the quadrature analogues of differential equations, we actually satisfy the governing equations at each sampling point of the domain, so we have one equation for each point, for each unknown. To satisfy the boundary conditions, at the boundary points, the boundary condition equations are satisfied instead of the governing equations. This procedure is straightforward when there is one boundary condition at each boundary and when we have distributed the sampling points so that there is one point at each boundary. Solving differential equations of orders higher than two requires more than one boundary condition at each boundary.

The problem of buckling of shells involves an eighth order system of differential equations for three unknowns which require four conditions at each boundary. There are three governing equations, but four boundary equations are to be satisfied at each boundary point. To overcome this problem, the δ technique, introduced by Jang et al (1989) can be used. In this technique an additional sampling point is chosen at a small distance from the boundary. In nondimensional coordinates the small distance δ is: $10^{-3} \leq \delta \leq 10^{-5}$

The δ points can all be considered to be located on the boundaries of the domain. Then all the boundary conditions can be satisfied.

Malik and Bert (1996) used another method for applying boundary conditions. In this method the boundary conditions are implemented in the weighting coefficient matrix. This method yields more accurate results. The limitation of this method is that, the weighting coefficient matrices can be modified to incorporate only the type of boundary conditions having the derivatives with respect to one spatial coordinate (Malik & Bert, 1996). This method is mostly suitable for clamped or simply supported boundaries. Due to its lack of generality this latter method is not used in this thesis.

Chapter 5

Application of the 1-dimensional DQM to the buckling of cylindrical shells

5.1. Introduction

In this chapter the DQM is applied to the problem of the linear elastic buckling of circular cylindrical shells. The shell is assumed to be thin, homogeneous and isotropic. The shell has a length L, radius R, and thickness h (Fig. 5.1). The Flügge stability equations serve as the basis of the analysis. The buckling of the shell is investigated under an axial compressive force F, or a uniform lateral pressure p. A membrane prebuckling state for the Flügge theory is considered. A semi-analytical method is used. By assuming an acceptable form for the buckling mode in the circumferential direction, the 2-dimensional shell problem is transformed into a 1-dimensional one, dependent only on the axial coordinate of the shell. This problem is solved by applying the 1-dimensional quadrature method in the longitudinal direction. Thus, the equations are transformed to a set of algebraic equations, the eigenvalues of which are the buckling loads of the shell.

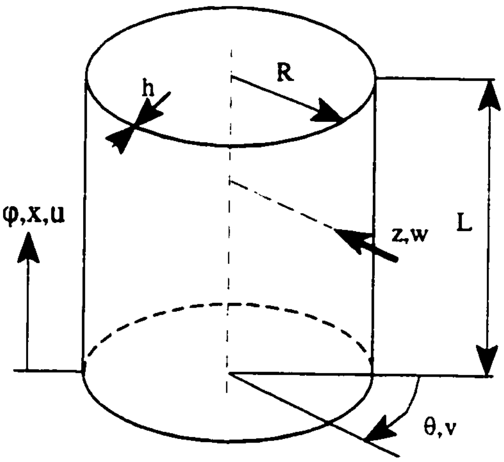

Figure 5.1 Details of the shell

5.2 Formulation

From equation (3-17), assuming the membrane stress state for the prebuckling deformation and ignoring the torsional load (q_s), the stability problem can be formulated as

$$\begin{aligned}
&u,_{\varphi\varphi}+\frac{1-\upsilon}{2}u,_{\theta\theta}+\frac{1+\upsilon}{2}v,_{\varphi\theta}-\upsilon w,_{\varphi}+\kappa\left(\frac{1-\upsilon}{2}u,_{\theta\theta}+w,_{\varphi\varphi\varphi}-\frac{1-\upsilon}{2}w,_{\varphi\theta\theta}\right)=q_c u,_{\varphi\varphi}+q_p(u,_{\theta\theta}+w,_{\varphi})\\
&\frac{1+\upsilon}{2}u,_{\varphi\theta}+\frac{1-\upsilon}{2}v,_{\varphi\varphi}+v,_{\theta\theta}-w,_{\theta}+\kappa\left(\frac{3}{2}(1-\upsilon)v,_{\varphi\varphi}+\frac{3-\upsilon}{2}w,_{\varphi\varphi\theta}\right)=q_c v,_{\varphi\varphi}+q_p(v,_{\theta\theta}-w,_{\theta})\\
&\upsilon u,_{\varphi}+v,_{\theta}-w-\kappa\left(u,_{\varphi\varphi\varphi}-\frac{1-\upsilon}{2}u,_{\varphi\theta\theta}+\frac{3-\upsilon}{2}v,_{\varphi\varphi\theta}+w,_{\varphi\varphi\varphi\varphi}+2w,_{\varphi\varphi\theta\theta}+w,_{\theta\theta\theta\theta}+2w,_{\theta\theta}+w\right)=q_c w,_{\varphi\varphi}\\
&\qquad +q_p(w,_{\theta\theta}-u,_{\theta}-v,_{\theta})
\end{aligned} \tag{5-1}$$

where $u(\varphi,\theta)$, $v(\varphi,\theta)$, and $w(\varphi,\theta)$ are the displacement functions in the axial ($\varphi=x/R$), circumferential (θ), and radial directions respectively. υ is the Poisson's ratio and $\kappa=h^2/(12R^2)$. Finally q_p and q_c are the lateral and axial pressures. In equation (5-1) it is considered that,

when the compression load is applied $\upsilon q_c = -(1-\upsilon^2)w^0$, and

when the lateral pressure is applied $q_p = (1-\upsilon^2)w^0$.

The derivatives of w^0 are ignored. Since the prebuckling displacements are not considered any more, all displacements are buckling displacements. For convenience the subscript 1 is dropped and lowercase letters are used.

Three sets of boundary conditions are considered in this work

1- Clamped $u = v = w = w_{,x} = 0$

2- Free $N_x = 0$, $T_x = N_{x\theta} + M_{x\theta} = 0$, $S_x = Q_x + M_{x\theta,\theta} = 0$, $M_x = 0$

3- Simple $u = v = w = w_{,xx} = 0$

In these equations N_x, $N_{x\theta}$, Q_x are nondimensional force resultants and M_x, $M_{x\theta}$ are nondimensional moment resultants (Fig. 3.2). S_x and T_x are effective shear forces. Based on the Flügge theory the statement of free end boundary conditions in terms of displacements is

$$N_x = 0 : u_{,\varphi} + \upsilon\, v_{,\theta} - \upsilon\, w + \kappa w_{,\varphi\varphi} = 0$$

$$T_x = 0 : u_{,\theta} + (1+3\kappa)v_{,\varphi} + 3\kappa w_{,\varphi\theta} = 0$$

$$S_x = 0 : u_{,\varphi\varphi} - (1-\upsilon)u_{,\theta\theta}/2 + (3-\upsilon)v_{,\varphi\theta}/2 + w_{,\varphi\varphi\varphi} + (2-\upsilon)\, w_{,\varphi\theta\theta} = 0$$

$$M_x = 0 : u_{,\varphi} + \upsilon\, v_{,\theta} + w_{,\varphi\varphi} + \upsilon\, w_{,\theta\theta} = 0 \qquad (5\text{-}2)$$

To solve this partial differential equation problem, the displacement functions are taken as

$$u(\varphi,\theta)=U(\varphi)\cos n\theta, \qquad v(\varphi,\theta)=V(\varphi)\sin n\theta, \qquad w(\varphi,\theta)=W(\varphi)\cos n\theta$$

Substituting these expansions into the governing equations of the eigenvalue problem gives

$$\begin{gathered}
U_{,\varphi\varphi} - n^2\left(\frac{1-\upsilon}{2}\right)(1+\kappa)U + n\left(\frac{1+\upsilon}{2}\right)V_{,\varphi} + \left(n^2\left(\frac{1-\upsilon}{2}\right)\kappa - \upsilon\right)W_{,\varphi} + \kappa W_{,\varphi\varphi\varphi} = q_c U_{,\varphi\varphi} + q_p(-n^2U + W_{,\varphi}) \\
-\frac{1+\upsilon}{2}nU_{,\varphi} + \left(\frac{1-\upsilon}{2}\right)(1+3\kappa)V_{,\varphi\varphi} - n^2V + nW - \kappa\frac{3-\upsilon}{2}nW_{,\varphi\varphi} = q_c V_{,\varphi\varphi} + q_p(-n^2V + nW) \\
\left(\upsilon - n^2\left(\frac{1-\upsilon}{2}\right)\kappa\right)U_{,\varphi} - \kappa U_{,\varphi\varphi\varphi} + nV - \kappa\frac{3-\upsilon}{2}nV_{,\varphi\varphi} - (1+\kappa n^4 - 2\kappa n^2 + \kappa)W + 2\kappa n^2 W_{,\varphi\varphi} - \kappa W_{,\varphi\varphi\varphi\varphi} = \\
q_c W_{,\varphi\varphi} + q_p(-n^2W - U_{,\varphi} - nV)
\end{gathered} \tag{5-3}$$

The free boundary conditions become

$$N_x = 0 : U_{,\phi} + \upsilon nV - \upsilon W + \kappa W_{,\phi\phi} = 0$$

$$T_x = 0 : -nU + (1+3\kappa)V_{,\phi} - 3\kappa nW_{,\phi} = 0$$

$$S_x = 0 : U_{,\phi\phi} - (1-\upsilon)n^2 U/2 + (3-\upsilon)nV_{,\phi}/2 + W_{,\phi\phi\phi} - (2-\upsilon)n^2 W_{,\phi} = 0$$

$$M_x = 0 : U_{,\phi} + \upsilon n V + W_{,\phi\phi} - \upsilon n^2 W = 0 \tag{5-4}$$

It can be seen that by using the new displacement functions the variables are function of longitudinal coordinate (φ or x) only.

5.3 DQM formulation

To solve the boundary value problem obtained in previous section the quadrature rule is used in the governing equations, which yields

$$\sum_{j=1}^{M} A_{ij}^{(2)}U_j-\frac{(1-\upsilon)}{2}(1+\kappa)n^2U_i+\frac{(1+\upsilon)}{2}n\sum_{j=1}^{M} A_{ij}^{(1)}V_j+\sum_{j=1}^{M}(\kappa A_{ij}^{(3)}+(\kappa\frac{(1-\upsilon)}{2}n^2-\upsilon)A_{ij}^{(1)})W_j=q_p(-n^2U_i+\sum_{j=1}^{M} A_{ij}^{(1)}W_j)$$
$$+q_c\sum_{j=1}^{M} A_{ij}^{(2)}U_j$$
$$-\frac{(1+\upsilon)}{2}n\sum_{j=1}^{M} A_{ij}^{(1)}U_j+\frac{(1-\upsilon)}{2}(1+3\kappa)\sum_{j=1}^{M} A_{ij}^{(2)}V_j-n^2V_i+nW_i-\kappa\frac{(3-\upsilon)}{2}nA_{ij}^{(2)}W_j=q_p(-n^2V_i+nW_i)+q_c(\sum_{j=1}^{M} A_{ij}^{(2)}V_j)$$
$$(\upsilon-\kappa\frac{(1-\upsilon)}{2}n^2)\sum_{j=1}^{M} A_{ij}^{(1)}U_j-\kappa\sum_{j=1}^{M} A_{ij}^{(3)}U_j+nV_i-\kappa\frac{(3-\upsilon)}{2}n\sum_{j=1}^{M} A_{ij}^{(2)}V_j-(1+\kappa n^4-2\kappa n^2+\kappa)W_i+2\kappa n^2\sum_{j=1}^{M} A_{ij}^{(2)}W_j-\kappa\sum_{j=1}^{M} A_{ij}^{(4)}W_j=$$
$$q_p(-n^2W_i-\sum_{j=1}^{M} A_{ij}^{(1)}U_j-nV_i)+q_c(\sum_{j=1}^{M} A_{ij}^{(2)}W_j)$$
$$\tag{5-5}$$

The set (5-5) gives three equations at point “ i ” where $M \geq i \geq 1$.

Applying the DQM to the free boundary equations one obtains

$$N_x=0\ :\quad \sum_{j=1}^{M} A_{ij}^{(1)}U_j+\upsilon nV_i+\kappa\sum_{j=1}^{M} A_{ij}^{(2)}W_j-\upsilon W_i=0$$
$$T_x=0\ :\quad -nU_i+(1+3\kappa)\sum_{j=1}^{M} A_{ij}^{(1)}V_j-3\kappa n\sum_{j=1}^{M} A_{ij}^{(1)}W_j=0$$
$$S_x=0\ :\quad \sum_{j=1}^{M} A_{ij}^{(2)}U_j+\frac{(1-\upsilon)}{2}n^2U_i+\frac{(3-\upsilon)}{2}n\sum_{j=1}^{M} A_{ij}^{(1)}V_j-\sum_{j=1}^{M}((2-\upsilon)n^2A_{ij}^{(1)}-A_{ij}^{(3)})W_j$$
$$M_x=0\ :\quad \sum_{i=1}^{M} A_{ij}^{(1)}U_j+\upsilon nV_i+\sum_{i=1}^{M} A_{ij}^{(2)}W_j-\upsilon n^2W_i=0 \tag{5-6}$$

If the above equations are used for boundary conditions N_x must be applied at U_1 or U_M and correspondingly, T_x at V_1 or V_M, S_x at W_1 or W_M, and M_x at W_2 or W_{M-1}, where 1 and M signify the first and last sampling points and U, V, and W denote the first, second, and third governing equations in (5-5) respectively. In equations (5-5) and (5-6) n is the circumferential mode number and is considered a known value. In practice n is found by trial and error. Different values for n are considered and the one yielding the minimum buckling load is considered as the correct choice.

5.4 Sampling points and algebraic equations

In this problem the discretization is done in the longitudinal direction. With regard to the governing equations, at each end of the cylinder one boundary condition on " u ", one on " v ", and two on " w " are required. Correspondingly two δ points are needed, one at each end to implement one of the " w " boundary conditions. Defining a non-dimensional axial coordinate of $\mathrm{x} = x/l$, the sampling points are taken as

$$\mathrm{x}(1)=0, \qquad \mathrm{x}(2)=0.0001, \qquad \mathrm{x}(M-1)=0.9999, \qquad \mathrm{x}(M)=1$$

$$\mathrm{x}(i)=\frac{1-\cos\frac{\pi(i-2)}{N-3}}{2} \qquad 2<i<M-1 \tag{5-7}$$

where M is the number of sampling points and δ has been taken as 0.0001.

There are M simultaneous equations for each governing equation and there are three governing differential equations. Thus there are 3M algebraic equations to be solved, 8 of which are boundary equations (4 equations at each boundary). Taking the first eight rows of the matrix form of the problem as the collection of boundary equations, the equation (5-5) can be cast into

$$\begin{bmatrix} \underset{8\times 8}{[BB]} & \underset{8\times(3M-8)}{[BD]} \\ \underset{(3M-8)\times 8}{[DB]} & \underset{(3M-8)\times(3M-8)}{[DD]} \end{bmatrix} \begin{bmatrix} [d_b] \\ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \end{bmatrix} = q_p \begin{bmatrix} 0 \quad 0 \quad 0 \\ \underset{(3M-8)\times 8}{[DBG]} \quad \underset{(3M-8)\times(3M-8)}{[DDG]} \end{bmatrix} \begin{bmatrix} [d_b] \\ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \end{bmatrix} \tag{5-8}$$

where the [BB] and [BD] matrices include 8 boundary conditions and [DB] and [DD] include the DQM analogue of the differential equations. The numbers beneath each matrix show the size of the matrix. $[d_b]$ is the column matrix of displacements at the boundary points and $[u\ v\ w]^t$ is the column matrix of displacements in the domain. The elements of these matrices are shown for the case of a cylindrical shell with clamped-free boundary conditions under external pressure in appendix 5.1.

In order to transform the equation (5-8) into a standard eigenvalue equation, the matrix $[d_b]$ must be eliminated. Using the static condensation technique equation (5-8) is cast into the standard form of

$$(-[DBG][BB]^{-1}[BD]+[DDG])^{-1}(-[DB][BB]^{-1}[BD]+[DD])[uvw]^T-q_p[I][uvw]^T=0 \tag{5-9}$$

The eigenvalues of (5-9) are the buckling loads, and the eigenvectors are the buckling modes. Using M sampling points, 3M-8 buckling loads can be found. A computer code labelled tank1d.m performing all the necessary procedures has been written in the Matlab language.

5.5 Results and discussion

Results are presented for three different boundary conditions and two kinds of loading. In each case the loading is either lateral pressure or axial compression. In all cases the Poisson ratio υ is taken as 0.3 and the modulus of elasticity E as 200 GPa.

Table 5.1 shows the DQM results for a clamped-clamped cylindrical shell under lateral pressure. Values for the critical pressure p_{cr} obtained by the method discussed in this chapter, and by Vodenitcharova (1996) using the Fourier series approach are presented. The table indicates that an increase of the L/R ratio, or a decrease of the R/h ratio, leads to an increase in the difference between the two sets of results.

The results of the DQM solution for clamped-clamped cylindrical shells for an axial compression loading are shown on Fig. 5.2 along with the results given by Yamaki (1984). The R/h ratio is fixed at 100 and the L/R is shown as a variable on the horizontal axis. In the graph Γ represents the ratio of the critical axial compression stress σ (obtained by the DQM) to the classical buckling stress σ_{cl}, i.e.

$$\Gamma = \frac{\sigma}{\sigma_{cl}} = [3(1-v^2)]^{\frac{1}{2}} \times \frac{R\sigma}{Eh}$$

Similar comparisons with the works of Vodenitcharova (1996) and Yamaki (1984) are presented in Fig. 5.3 and table 5.2 for the case of simple-simple boundary conditions. For this set of boundary conditions as well as that of clamped-clamped mentioned in the preceding it can be seen that for axial compression load an increase of the L/R ratio leads to a decrease in the difference between the two sets of results in Fig. 5.2 and 5.3.

Results for clamped-free boundary conditions are presented in Fig. 5.4 and table 5.3. It is evident from the results that the free boundary conditions lead consistently to lower buckling loads compared to the clamped or simple boundary conditions. A sample buckling mode for this case is shown in Fig 5.5. This shape is for a circumferential mode number of six, L/R=3 and R/h=300.

Table 5.1 Buckling pressure and mode in clamped-clamped cylindrical shells

L/R	R/h	n	p_{cr} (Pa)	p_{cr} (Pa) in Vodentcharova (1996)	% of difference
0.5	300	17	359300	361790	0.7
	3000	33	1093	1095	0.2
1	300	13	170540	172580	1.2
	500	15	47690	48114	0.9
	1000	18	8480	8536	0.7
	1500	20	3090	3107	0.6
	2000	22	1512	1518	0.4
	3000	24	550	552	0.3
2	300	10	85860	87610	2.0
	3000	17	277	278	0.6
3	300	8	56064	57851	3.1
	3000	14	184	186	1.1
5	300	6	32954	34830	5.4
	3000	11	109	111	2.2

Table 5.2 Buckling pressure and mode in simple-simple cylindrical shells

L/R	R/h	n	p_{cr} (Pa)	p_{cr} (Pa) in Vodentcharova (1996)	% of difference
0.5	300	17	330190	332430	0.7
	3000	33	1081	1083	0.2
1	300	13	166541	168520	1.2
	500	15	46993	47412	0.9
	1000	18	8416	8468	0.6
	1500	20	3074	3090	0.5
	2000	22	1506	1512	0.4
	3000	24	548	550	0.4
2	300	9	84791	86917	2.4
	3000	17	276	278	0.6
3	300	8	55894	57674	3.1
	3000	14	184	186	1.1
5	300	6	32900	34769	5.4
	3000	11	109	111	2.2

Table 5.3 Buckling pressure and mode in clamped-free cylindrical shell

L/R	R/h	n	p_{cr} (Pa)
0.5	300	14	221363.0
	3000	28	723.3
1	300	11	111106.0
	500	12	31353.0
	1000	15	5602.0
	1500	17	2052.9
	2000	18	1000.2
	3000	20	364.8
2	300	8	55390.0
	3000	14	182.4
3	300	6	37286.0
	3000	12	121.5
5	300	5	21078.0
	3000	9	69.7

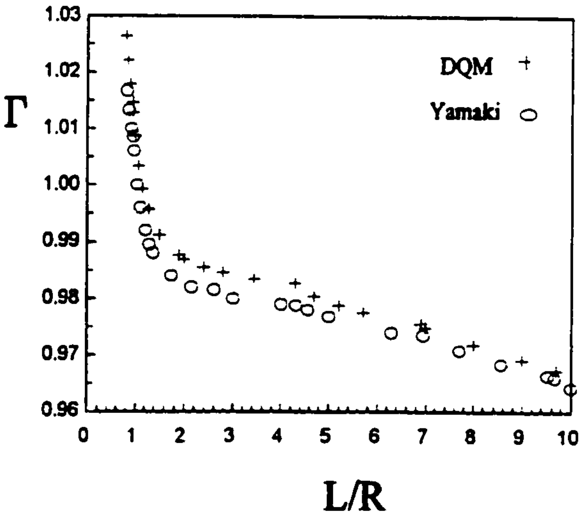

Figure 5.2 Results for clamped-clamped boundary conditions

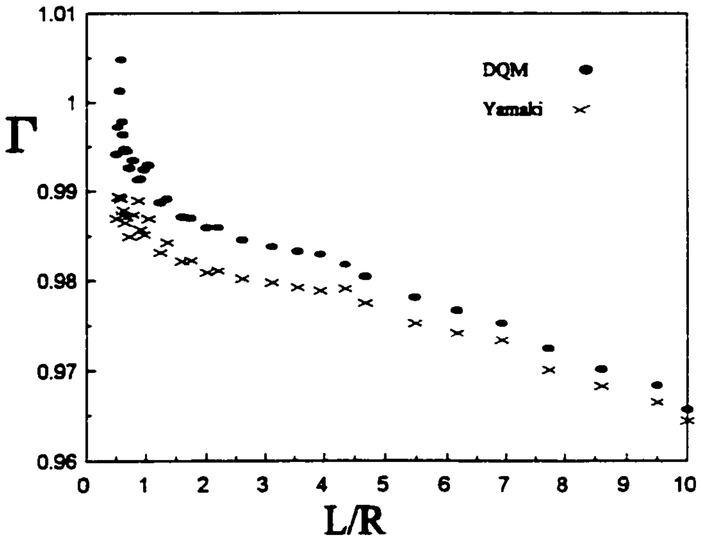

Figure 5.3 Results for simple-simple boundary conditions

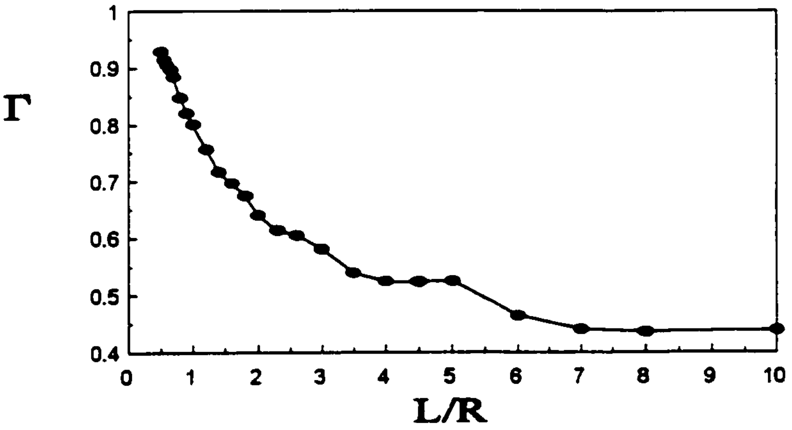

Figure 5.4 Results for clamped-free boundary conditions by DQM

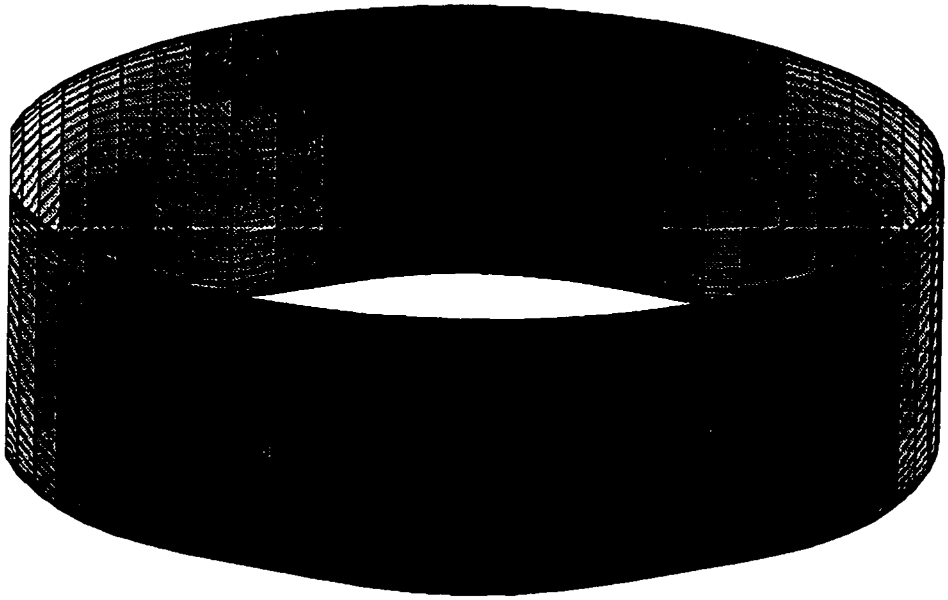

Figure 5.5 Buckling mode for clamped-free boundary conditions. Circumferential mode number n=6, L/R = 3, R/h = 300

5.6 Appendix 5.1

Detail of DQM matrices represented in equation 5-8 for simple-simple boundary conditions.

[BB]

$$\left|\begin{array}{c:c:c:c:c:c:c:c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 & A^{(1)}_{1,1} & A^{(1)}_{1,N} & A^{(1)}_{1,2} & A^{(1)}_{1,N-1} \\ \hdashline
0 & 0 & 0 & 0 & A^{(1)}_{N,1} & A^{(1)}_{N,N} & A^{(1)}_{N,2} & A^{(1)}_{N,N-1}
\end{array}\right|$$

[BD]

$$\left|\begin{array}{c:c:c}
0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 \\ \hdashline
0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 \\ \hdashline
0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 \\ \hdashline
0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 \\ \hdashline
0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 \\ \hdashline
0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 \\ \hdashline
0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 & A^{(1)}_{1,3}\;\; A^{(1)}_{1,4}\;\;\ldots\;\; A^{(1)}_{1,N-2} \\ \hdashline
0\;0\;\ldots\;0\;0 & 0\;0\;\ldots\;0\;0 & A^{(1)}_{N,3}\;\; A^{(1)}_{N,4}\;\;\ldots\;\; A^{(1)}_{N,N-2}
\end{array}\right|$$

[DD]

$AII_{i,j} = -A_{i,j}^{(2)}$ $2 \le i \le N-1$ $2 \le j \le N-1$	$AIM_{i,j} = \dfrac{-(1+\upsilon)\times N\times A_{i,j}^{(1)}}{2}$ $2 \le i \le N-1$ $2 \le j \le N-1$	$AIN_{i,j} = \kappa\times A_{i,j}^{(3)} - (\upsilon-(1-\upsilon)\times\kappa\times\frac{N^2}{2})\times A_{i,j}^{(1)}$ $2 \le i \le N-1$ $3 \le j \le N-2$
$AJI_{i,j} = (1+\upsilon)\times N\times A_{i,j}^{(1)} \div 2$ $2 \le i \le N-1$ $2 \le j \le N-1$	$AJM_{i,j} = -(1-\upsilon)\times(1+3\kappa)\times A_{i,j}^{(2)} \div 2$ $2 \le i \le N-1$ $2 \le j \le N-1$	$AJN_{i,j} = -(3-\upsilon)\times\kappa\times N\times A_{i,j}^{(2)} \div 2$ $2 \le i \le N-1$ $3 \le j \le N-2$
$AKI_{i,j} = -(\kappa\times A_{i,j}^{(3)} - (\upsilon-(1-\upsilon)\times\kappa\times\frac{N^2}{2}))\times A_{i,j}^{(1)}$ $3 \le i \le N-2$ $2 \le j \le N-1$	$AKM_{i,j} = -(3-\upsilon)\times\kappa\times N\times A_{i,j}^{(2)} \div 2$ $3 \le i \le N-2$ $2 \le j \le N-1$	$AKN_{i,j} = \kappa\times A_{i,j}^{(4)} - 2\times N^2\times A_{i,j}^{(2)}$ $3 \le i \le N-2$ $3 \le j \le N-2$

[DB]

AII (2,1) *AII (3,1)* ... ... *AII* *(N-1,1)*	*AII (2,N)* *AII (3,N)* ... ... *AII* *(N-1,N)*	*AIM (2,1)* *AIM (3,1)* ... ... *AIM* *(N-1,1)*	*AIM (2,N)* *AIM (3,N)* ... ... *AIM* *(N-1,N)*	*AIN (2,1)* *AIN (3,1)* ... ... *AIN* *(N-1,1)*	*AIN (2,N)* *AIN (3,N)* ... ... *AIN* *(N-1,N)*	*AIN (2,2)* *AIN (3,2)* ... ... *AIN* *(N-1,2)*	*AIN (2,N-1)* *AIN (3,N-1)* ... ... *AIN* *(N-1,N-1)*
AJI (2,1) *AJI (3,1)* ... *AJI* *(N-1,1)*	*AJI (2,N)* *AJI (3,N)* ... *AJI* *(N-1,N)*	*AJM (2,1)* *AJM (3,1)* ... *AJM* *(N-1,1)*	*AJM (2,N)* *AJM (3,N)* ... *AJM* *(N-1,N)*	*AJN (2,1)* *AJN (3,1)* ... *AJN* *(N-1,1)*	*AJN (2,N)* *AJN (3,N)* ... *AJN* *(N-1,N)*	*AJN (2,2)* *AJN (3,2)* ... *AJN* *(N-1,2)*	*AJN (2,N-1)* *AJN (3,N-1)* ... *AJN* *(N-1, N-1)*
AKI (3,1) *AKI (4,1)* ... *AKI* *(N-2,1)*	*AKI (3,N)* *AKI (4,N)* ... *AKI* *(N-2,N)*	*AKJ (3,1)* *AKJ (4,1)* ... *AKJ* *(N-2,1)*	*AKJ (3,N)* *AKJ (4,N)* ... *AKJ* *(N-2,N)*	*AKN (3,1)* *AKN (4,1)* ... *AKN* *(N-2,1)*	*AKN (3,N)* *AKN (4,N)* ... *AKN* *(N-2,N)*	*AKN (2,2)* *AKN (3,2)* ... *AKN* *(N-2,2)*	*AKN (2,N-1)* *AKN (3,N-1)* ... *AKN* *(N-1, N-1)*

In the above matrices AI* , AJ* and AK* include terms for first, second and third equations of (5-5) respectively. And A*I , A*M and A*N include terms for "u" , "v" and "w" variables respectively.

$[d_b]^T = [u_1, u_N, v_1, v_N, w_1, w_N, w_2, w_{N-1}]$

$[u\ v\ w]^T = [u_2, u_3, \ldots, u_{N-1}, v_2, v_3, \ldots, v_{N-1}, w_3, w_4, \ldots, w_{N-2}]$

[DBG]

$$\left|\begin{array}{c:c:c:c:c:c:c:c}
0 & 0 & 0 & 0 & \begin{matrix}A_{21}^{(1)}\\ \cdots\\ \cdots\\ \cdots\\ A_{n-1,1}^{(1)}\end{matrix} & \begin{matrix}A_{2N}^{(1)}\\ \cdots\\ \cdots\\ \cdots\\ A_{n-1,N}^{(1)}\end{matrix} & \begin{matrix}A_{22}^{(1)}\\ \cdots\\ \cdots\\ \cdots\\ A_{n-1,2}^{(1)}\end{matrix} & \begin{matrix}A_{2,N-1}^{(1)}\\ \cdots\\ \cdots\\ \cdots\\ A_{n-1,N-1}^{(1)}\end{matrix} \\ \hdashline
0 & 0 & 0 & 0 & 0 & 0 & \begin{matrix}N\\ 0\\ ..\\ 0\\ 0\end{matrix} & \begin{matrix}0\\ ..\\ 0\\ 0\\ N\end{matrix} \\ \hdashline
\begin{matrix}-A_{31}^{(1)}\\ \cdots\\ \cdots\\ \cdots\\ -A_{n-2,1}^{(1)}\end{matrix} & \begin{matrix}-A_{3N}^{(1)}\\ \cdots\\ \cdots\\ \cdots\\ -A_{n-2,N}^{(1)}\end{matrix} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right|$$

[DDG]

$$\left|\begin{array}{c:c:c}
\begin{matrix}-N^2 & 0 & \cdots & 0 & 0\\ 0 & -N^2 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & & \\ 0 & 0 & \cdots & -N^2 & 0\\ 0 & 0 & \cdots & 0 & -N^2\end{matrix} & \begin{matrix}0 & 0 & \cdots & 0 & 0\\ & & & & \\ & & \cdots\cdots & & \\ & & & & \\ 0 & 0 & \cdots & 0 & 0\end{matrix} & \begin{matrix}A_{2,3}^{(1)} & \cdots & A_{2,N-2}^{(1)}\\ & \cdots\cdots & \\ A_{N-1,3}^{(1)} & \cdots & A_{N-1,N-2}^{(1)}\end{matrix} \\ \hdashline
\begin{matrix}0 & 0 & \cdots & 0 & 0\\ & & \cdots\cdots & & \\ 0 & 0 & \cdots & 0 & 0\end{matrix} & \begin{matrix}-N^2 & 0 & \cdots & \cdots & 0 & 0\\ 0 & -N^2 & \cdots & \cdots & 0 & 0\\ \cdots & & \cdots & & \cdots & \\ 0 & 0 & \cdots & \cdots & -N^2 & 0\\ 0 & 0 & \cdots & \cdots & 0 & -N^2\end{matrix} & \begin{matrix}0 & 0 & \cdots & \cdots & 0 & 0 & 0\\ N & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & N & 0 & \cdots & 0 & 0 & 0\\ \cdots & \cdots & & \cdots & \cdots & & \cdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & N\\ 0 & 0 & 0 & .. & 0 & 0 & 0\end{matrix} \\ \hdashline
\begin{matrix}-A_{32}^{(1)} & \cdots & -A_{3,N-1}^{(1)}\\ & \cdots\cdots & \\ -A_{N-1,3}^{(1)} & \cdots & -A_{N-1,N-2}^{(1)}\end{matrix} & \begin{matrix}0 & -N & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & -N & 0 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & .. & \\ 0 & 0 & \cdots & & -N & 0 & 0\\ 0 & 0 & \cdots & & 0 & -N & 0\end{matrix} & \begin{matrix}-N^2 & 0 & \cdots & \cdots & 0 & 0\\ 0 & -N^2 & \cdots & \cdots & 0 & 0\\ & \cdots & & \cdots & & \cdots\\ 0 & 0 & \cdots & \cdots & -N^2 & 0\\ 0 & 0 & \cdots & \cdots & 0 & -N^2\end{matrix}
\end{array}\right|$$

Chapter 6

Application of 2-dimensional DQM to buckling of cylindrical shells

6.1 Introduction

In this chapter the DQM is applied to a more general problem of the linear elastic buckling of circular cylindrical shells. The shell has the same characteristics as mentioned in chapter 5 and the Flügge stability equations again serve as the basis of the analysis. The buckling of the shell here however is investigated under uniform torsion load, in addition to axial compression and uniform lateral pressure loads. The 2-dimensional form of DQM is required for the new loading.

The present work represents the first time that the 2-dimensional DQM is applied to a problem of structural mechanics with circumferential continuity. In this case no functional form is assumed for the circumferential direction and the problem remains 2-dimensional mathematically.

By applying the 2-dimensional quadrature method in the longitudinal and circumferential directions, the equations are again transformed to a set of algebraic equations, the eigenvalues of which are the buckling loads of the shell.

The results of 2-dimensional DQM analyses of the shell under uniform lateral pressure and axial compression are compared with the 1-dimensional ones. The torsion problem can be solved only by 2-dimensional DQM, and the results are compared with values found in the literature. The torsion load is considered because it is closer to the previously solved problems of chapter five. Through experience obtained in this chapter, other more complicated 2-dimensional problems can be handled in later work.

6.2 Formulation

The analysis of the cylindrical shell under lateral pressure and axial compression was described in chapter 5 and thus, the torsion load is considered here. Using the Flügge theory (Yamaki, 1984) the stability problem of cylindrical shell under uniform torsion load can be formulated as

$$\begin{aligned}
&u_{,\varphi\varphi}+\frac{1-\upsilon}{2}u_{,\theta\theta}+\frac{1+\upsilon}{2}v_{,\varphi\theta}-\upsilon w_{,\varphi}+\kappa\left(\frac{1-\upsilon}{2}u_{,\theta\theta}+w_{,\varphi\varphi\varphi}-\frac{1-\upsilon}{2}w_{,\varphi\theta\theta}\right)=-2q_s u_{,\varphi\theta}\\
&\frac{1+\upsilon}{2}u_{,\varphi\theta}+\frac{1-\upsilon}{2}v_{,\varphi\varphi}+v_{,\theta\theta}-w_{,\theta}+\kappa\left(\frac{3}{2}(1-\upsilon)v_{,\varphi\varphi}+\frac{3-\upsilon}{2}w_{,\varphi\varphi\theta}\right)=-2q_s(v_{,\varphi\theta}-w_{,\varphi})\\
&\upsilon u_{,\varphi}+v_{,\theta}-w-\kappa\left(u_{,\varphi\varphi\varphi}-\frac{1-\upsilon}{2}u_{,\varphi\theta\theta}+\frac{3-\upsilon}{2}v_{,\varphi\varphi\theta}+w_{,\varphi\varphi\varphi\varphi}+2w_{,\varphi\varphi\theta\theta}+w_{,\theta\theta\theta\theta}+2w_{,\theta\theta}+w\right)=\\
&\qquad -2q_s(v_{,\varphi}+w_{,\varphi\theta})
\end{aligned}\tag{6-1}$$

where q_s is the nondimensional torsion load, ($q_s=\tau h/J$, $J=Eh/(1-\upsilon^2)$) and other variables are the same as described in chapter 5. In addition to the boundary conditions of chapter 5, another kind of simply support boundary condition (designated S4 in accordance with Yamaki (1984)) is considered for the torsion problem. The S4 boundary condition is defined as:

$$w=0\,,\quad w_{,xx}=0\,,\quad N_x=0$$

as shown in chapter five and

$$N_{x\theta}=0 : u_{,\theta} + (\kappa+1)\, v_{,\varphi} + \kappa\, w_{,\varphi\theta} = 0 \tag{6-2}$$

6.3 DQM formulation

Applying the quadrature rule to the governing equations (6-1) and also considering that

$$u = U(\varphi,\theta)\quad,\quad v = V(\varphi,\theta)\quad,\quad w = W(\varphi,\theta)$$

the DQM analogue of governing equations can be found as:

$$\begin{aligned}
&\sum_{j=1}^{M} A_{ij}^{(2)} U_{jh} + \frac{(1-\upsilon)}{2}(1+\kappa)\sum_{l=1}^{N} B_{hl}^{(2)} U_{il} + \frac{(1+\upsilon)}{2}\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(1)} B_{hl}^{(1)} V_{jl} - [\upsilon\sum_{j=1}^{M} A_{ij}^{(1)} + \kappa\sum_{j=1}^{M} A_{ij}^{(3)}] W_{jh} \\
&\qquad -\kappa\frac{(1-\upsilon)}{2}\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(1)} B_{hl}^{(2)} W_{jl} = -2q_s\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(1)} B_{hl}^{(1)} U_{jl} \\[2ex]
&\frac{(1+\upsilon)}{2}\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(1)} B_{hl}^{(1)} U_{jl} + \sum_{l=1}^{N} B_{hl}^{(2)} V_{il} + \frac{(1-\upsilon)}{2}(1+3\kappa)\sum_{j=1}^{M} A_{ij}^{(2)} V_{jh} + \kappa\frac{(3-\upsilon)}{2}\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(2)} B_{hl}^{(1)} W_{jl} \\
&\qquad -\sum_{l=1}^{N} B_{hl}^{(1)} W_{il} = -2q_s[\sum_{j=1}^{M} A_{ij}^{(1)} W_{jh} - \sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(1)} B_{hl}^{(1)} V_{jl}] \\[2ex]
&\upsilon\sum_{j=1}^{M} A_{ij}^{(1)} U_{jh} + \kappa\frac{(1-\upsilon)}{2}\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(1)} B_{hl}^{(2)} U_{jl} - \kappa\sum_{j=1}^{M} A_{ij}^{(3)} U_{jh} + \sum_{l=1}^{N} B_{hl}^{(1)} V_{il} - \kappa\frac{(3-\upsilon)}{2}\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(2)}) B_{hl}^{(1)} V_{jl} \\
&\qquad - (1+\kappa) W_{ih} - \kappa\sum_{j=1}^{M} A_{ij}^{(4)} W_{jh} - 2\kappa\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(2)} B_{hl}^{(2)} W_{jl} - (k\sum_{l=1}^{N} B_{hl}^{(4)} + 2\kappa\sum_{l=1}^{N} B_{hl}^{(2)}) W_{il} \\
&\qquad = -2q_s(\sum_{j=1}^{M} A_{ij}^{(1)} V_{jh} + \sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(1)} B_{hl}^{(1)} W_{jl})
\end{aligned} \tag{6-3}$$

where $A^{(r)}_{ij}$s and $B^{(r)}_{hl}$s are the weighting coefficients for r-th derivative in the longitudinal and circumferential directions respectively.

The DQM analogue of the free boundary conditions can be found as:

$$
\begin{aligned}
N_x=0 :&\quad \sum_{j=1}^{M} A_{ij}^{(1)}U_{jh}+\upsilon\sum_{l=1}^{N} B_{hl}^{(1)}V_{il}+\kappa\sum_{j=1}^{M} A_{ij}^{(2)}W_{jh}-\upsilon W_{ih}=0\\
T_x=0 :&\quad \sum_{l=1}^{N} B_{hl}^{(1)}U_{il}+(1+3\kappa)\sum_{j=1}^{M} A_{ij}^{(1)}V_{jh}+3\kappa\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(1)}B_{hl}^{(1)}W_{jl}=0\\
S_x=0 :&\quad \sum_{j=1}^{M} A_{ij}^{(2)}U_{jh}-\frac{(1-\upsilon)}{2}\sum_{l=1}^{N} B_{hl}^{(2)}U_{il}+\frac{(3-\upsilon)}{2}\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(1)}B_{hl}^{(1)}V_{jl}\\
&\qquad\qquad+\sum_{j=1}^{M} A_{ij}^{(3)}W_{jh}+(2-\upsilon)\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(1)}B_{hl}^{(2)}W_{jl}=0\\
M_x=0 :&\quad \sum_{j=1}^{M} A_{ij}^{(1)}U_{jh}+\upsilon\sum_{l=1}^{N} B_{hl}^{(1)}V_{il}+\sum_{j=1}^{M} A_{ij}^{(2)}W_{jh}+\upsilon\sum_{l=1}^{N} B_{hl}^{(2)}W_{il}=0
\end{aligned}
\tag{6-4}
$$

The boundary conditions must be satisfied in the same sequence as pointed out in section 5.3. Applying the DQM to the S4 boundary conditions one obtains

$W = 0$, $W_{,xx} = 0$, $N_x = 0$ as shown in equation(6-4),

$$
N_{x\theta}=0 :\quad \sum_{l=1}^{N} B_{hl}^{(1)}U_{il}+(\kappa+1)\sum_{j=1}^{M} A_{ij}^{(1)}V_{jh}+\kappa\sum_{j=1}^{M}\sum_{l=1}^{N} A_{ij}^{(1)}B_{hl}^{(1)}V_{jl}=0
\tag{6-5}
$$

N_x must be applied at U_1 or U_M, $N_{x\theta}$ at V_1 or V_M, W at W_1 or W_M, and $W_{,xx}$ at W_2 or W_{M-1}. For example U_1 and U_M include the first and the last N sampling points at the bottom and at the top of the shell (U_{11} through U_{1N} and U_{M1} through U_{MN}) respectively. The U, V, and W show that the boundary conditions must be satisfied instead of the first, second, and third governing equations respectively.

6.4 Sampling points and algebraic equations

The sampling points distribution in the axial direction is the same as considered in section 5.4, but in the circumferential direction equidistant sampling points are used. Considering "N" sampling points in the circumferential direction, the location of the j-th point will be given by

$$\theta(j) = 2\pi(j-1)/N \quad , \qquad j = 1, 2, \ldots, n.$$

where θ is the rotation angle from the starting point.

Considering "M" sampling points in the longitudinal direction, there are M×N simultaneous equations for each governing equation and there are three governing equations. Thus there will be 3MN algebraic equations to be solved, 8N of which are boundary equations. Using the same procedure as outlined in section 5.4, a matrix equation similar to (5-8) may be obtained. If we follow equation (5-9) to get the standard form of the eigenvalue problem, difficulties arise in finding the inverse of $(-[DBG][BB]^{-1}[BD]+[DDG])$ matrix in torsion and pressure load cases. That is because the determinant of this matrix is a very small value. To avoid this and yet to find the standard form of eigenvalue problem the following procedure is taken. The eigenvalues of the following equation are found

$$(-[DB][BB]^{-1}[BD]+[DD])^{-1}(-[DBG][BB]^{-1}[BD]+[DDG])[u\ v\ w]^T-1/q_p[I][u\ v\ w]^T=0 \qquad (6\text{-}6)$$

The eigenvalues of (6-6) give the inverse of the buckling loads. So the lowest buckling load is found by determining the largest eigenvalue of (6-6) and then taking the inverse. The size of eigenvalue matrix and the number of buckling loads obtained are 3M(N-8). The same procedure as mentioned in section 5.4 may be followed for the compressive load case. Figure (6.1) shows the distribution of sampling points on the cylinder.

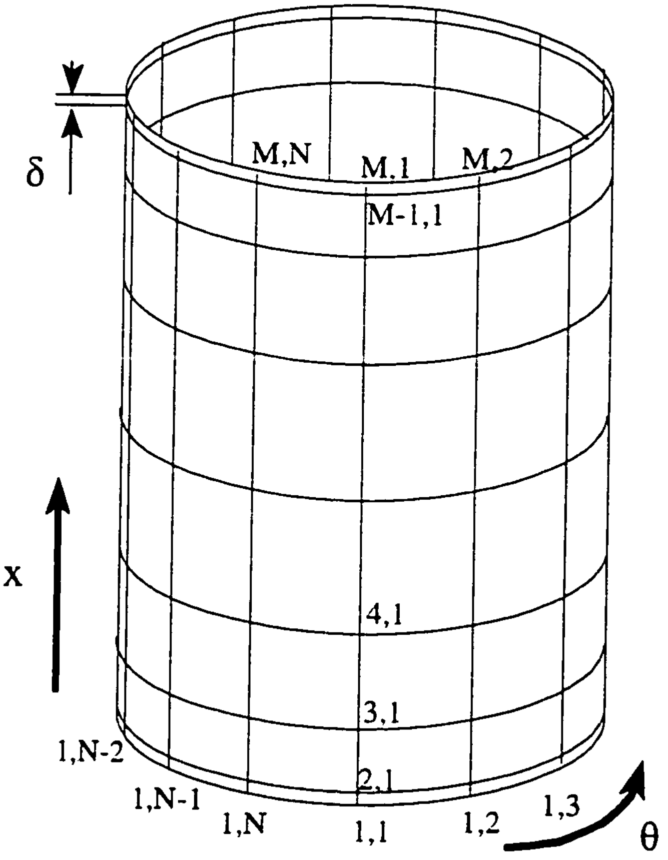

Figure 6.1 Distribution of 2-dimensional sampling points on the cylinder.

6.5 Results and discussions

Results are presented for two different boundary conditions and for three kinds of loading. As in chapter 5 the Poisson's ratio υ is 0.3 and the modulus of elasticity E is 200 GPa.

Figure 6.2 shows the DQM results for a clamped-clamped cylindrical shell under axial compression. Values for the critical axial compression load (q_c) obtained by the 2-dimensional DQM discussed in this chapter, and by the 1-dimensional DQM obtained in chapter 5, and also results found by Yamaki (1984) are presented in this graph. The results of the 2-dimensional DQM for this loading converge to their final values without major oscillation. Figure 6.3 shows the convergence of results for a shell of L/R = 10, R/h =100, as the mesh size varies. The results are found for a large range of mesh sizes. The convergence is clear and quite accurate results are found from an engineering point of view. There are some deviations for specific choices of the number of sampling points. Further study will serve to identify and eliminate the reason for deviations.

The results of the 1 and 2-dimensional DQM solution for clamped-clamped cylindrical shells under lateral pressure are shown in figure 6.4. The results are shown together with the results found by Yamaki (1984). There are some variations from the final value when the number of sampling points changes. The convergence and the accuracy of the results are shown in terms of mesh size in table 6.1. For large aspect ratios the results oscillate around the actual value and converge to different neighbouring buckling loads as the mesh size varies. In table 6.1 when the mesh is fine enough, the highest deviation from the values found in Yamaki (1984) is about 15 percent. The results presented here are found by applying the Hessenberg method installed in Matlab software to the inherent eigenvalue problem. Using the inverse iteration method a smoother convergence with the deviation of less than 15 percent results. The values used in figure 6.4 are the ones that occurred most often (the results oscillate). The oscillation for aspect ratios less than 10 is much lower. e.g. when aspect ratio varies from 1 to 5, the oscillation of results is limited to 2 to 6 percent of the analytical value respectively.

Table 6.1 Convergence of 2-dimensional DQM results, clamped, uniform pressure

Solution	Mesh size M × N	% Error compared to results in Yamaki (1984)			
		L/R = 1	L/R = 3	L/R = 6	L/R = 10
1	6 × 6	9400	3700	1203	654
2	8 × 8	2471	619	90.3	-37.6
3	10 × 10	837	111	-10.2	-14.3
4	12 × 12	322	11.1	-9.8	-0.8
5	14 × 14	128	-1.43	-9.7	-0.8
6	16 × 16	48.9	-1.44	3.5	-14.8
7	18 × 18	16.4	-1.44	42.7	-14.8
8	20 × 20	5.9	-1.44	3.6	-14.7
9	22 × 22	5.9	-	3.6	-0.8

The graph in figure 6.5 shows the results found by 2-dimensional DQM along with those of Yamaki for the clamped-clamped cylinder under torsion load. There is no 1-dimensional DQM solution for this problem. Table 6.2 shows the convergence of results as the mesh size varies. The same pattern as that of pressure load can be seen for convergence of results as mesh size varies, but the oscillation of results is less than that case and the repetition of correct values is more in this case. Here again the accuracy of the results improves as the aspect ratio decreases.

The graph in figure 6.6 shows the buckling results for cylindrical shells with the S4 boundary conditions under torsion load. The 2-dimensional DQM results are compared with those of Yamaki (1984) on this graph. Table 6.3 shows the convergence of results as the mesh size changes. The convergence is not quite smooth here either, although good results are found for fine meshes.

Table 6.2 Convergence of 2-dimensional DQM results, clamped-clamped, torsion

Solution	Mesh size M × N	% Error compared to results in Yamaki (1984)			
		L/R = 4	L/R = 5	L/R = 10	L/R = 12
1	10 ×10	-	-	8.7	31
2	10 × 12	-	-	8.7	116.5
3	12 × 10	-	-	-1.4	-0.5
4	12 × 12	13.5	1.7	-0.9	-0.5
5	12 × 14	13.5	68.3	8.5	22.1
6	14 × 12	13.8	44.5	0	0.5
7	14 × 14	-5.3	-0.6	5.4	11.3
8	14 × 16	13.8	7.7	5.4	0.5
9	16 × 14	13.8	-0.7	5.4	12.5
10	16 × 16	-0.8	-0.7	-0.1	0.2

It can be seen that there is a large difference between the convergence of results under compressive load and those of lateral pressure and torsional load. The problem surfaced when the determinants of matrices (5-8) were small for these two loadings as explained in section 6.4. The problem was bypassed using equations (6-6), but its effects are visible in the results. The formula (6-6) involves more computational effort than (5-9). Inversing of the large matrix in (5-8) introduces round off errors that are added to initial instability of the matrix (the small determinant causes more round off errors, and instability in results).

It can be concluded that the results found for lateral pressure and torsional loads are not completely stable. An overall convergence is observed, but the results are susceptible to variation with the change in the method of finding the eigenvalues or even with the computer used for solving the problem. Round off errors are blamed for inaccuracy of some results, but good results are found in chapter 7 with much larger matrices and more calculations. It is concluded that a combination of round off errors and inaccuracy of equations is the reason for the oscillation of results.

In conclusion it has been shown that it is possible to find buckling loads using 2-dimensional DQM solution for different kinds of loadings. Further investigation is needed to determine the reason for oscillation in 2-dimensional DQM results for uniform lateral pressure and torsion loads.

Table 6.3 Convergence of 2-dimensional DQM results, S4, torsion

Solution	Mesh size	% Error compared to results in Yamaki (1984)			
	M × N	L/R = 3	L/R = 4	L/R = 5	L/R = 10
1	12 ×12	23660	133	29935	1919
2	14 × 12	6.3	21.2	3.5	-
3	12 × 14	23660	133	-	-
4	14 × 14	-1.8	-0.7	570	6383
5	14 × 16	23660	0.53	-	-
6	16 × 14	7.3	0.25	-	-
7	16 × 16	2.3	10	19.4	-0.7
8	18 × 18	2.4	4	10.9	-0.8
9	20 × 20	-	-	0.3	-0.8
10	22 × 22	-	-	6.6	-0.6

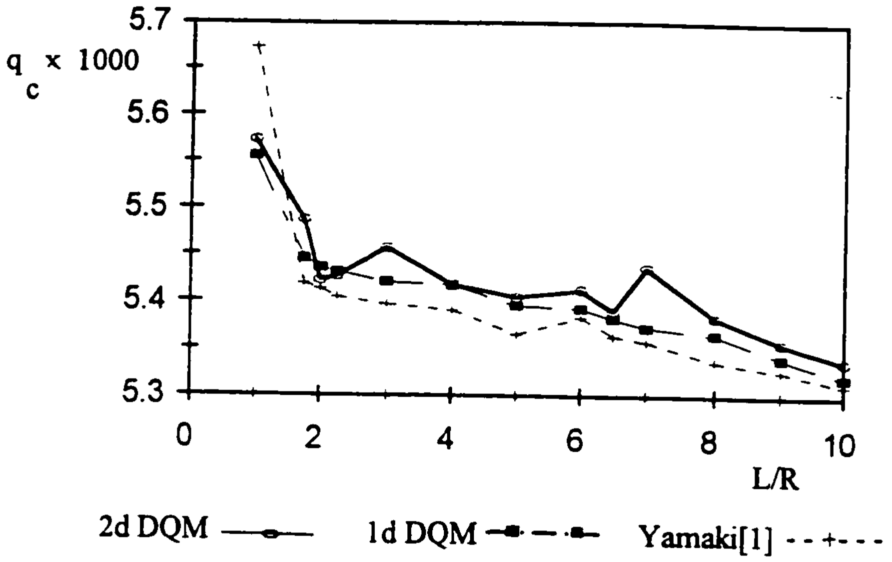

Figure 6.2 Buckling load under axial compression, clamped-clamped, R/h = 100

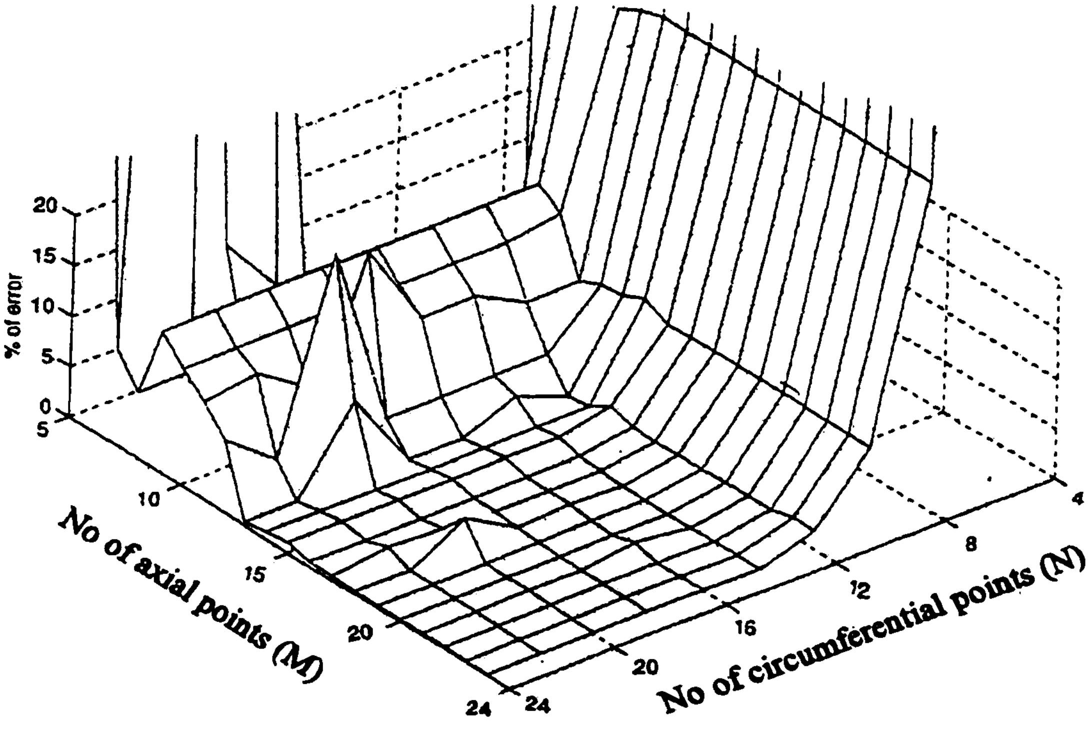

Figure 6.3 Convergence of 2-dimensional DQM results, clamped-clamped, compression load, L/R = 10, R/h = 100

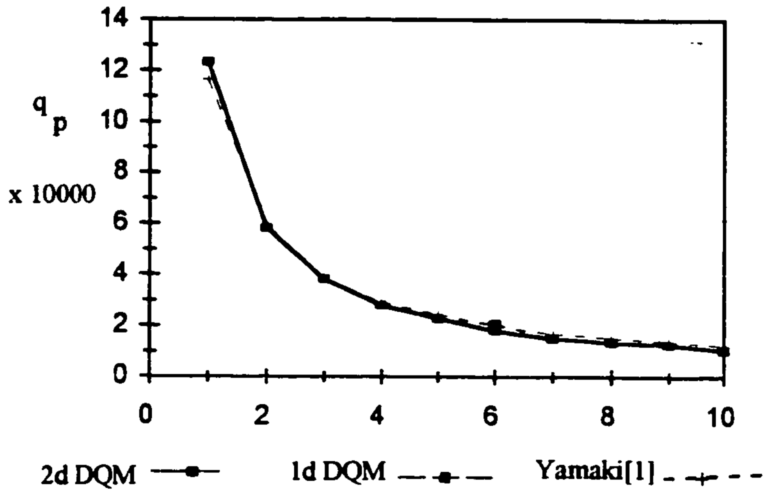

Fig 6.4 Buckling load under lateral pressure, clamped-clamped, R/h = 100.

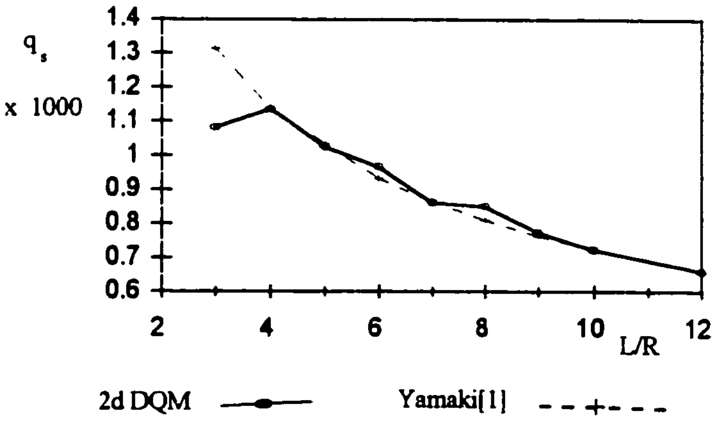

Fig. 6.5 Buckling load under torsion, clamped-clamped, R/h = 100.

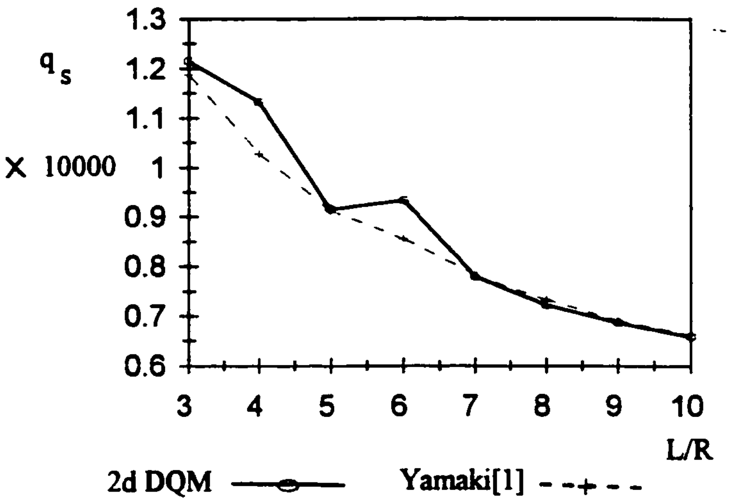

Figure 6.6. Buckling load under torsion, S4, R/h = 100

Chapter 7

Application of DQM to buckling of cylindrical tanks under quasi-static horizontal earthquake loading

7.1 Introduction

In this chapter the DQM is applied to the problem of the linear elastic buckling of circular cylindrical tanks. The tank contains liquid and the loading is the hydrostatic pressure of the contained liquid and/or horizontal forces induced by accelerated liquid inside the tank.

The horizontal excitation is the major cause of damage in tanks (Zhou et al, 1992). In horizontally excited tanks, the accelerated liquid applies lateral dynamic forces which are superimposed on the hydrostatic pressure. The combination of these two effects is studied in this chapter. An equivalent static force is used to represent the real lateral dynamic forces.

The prebuckling deformations and internal forces are considered in deriving the buckling equations in this chapter. A new set of buckling equations is derived and solved under the effect of different buckling loads.

7.2 Governing equations

Contrary to earlier work, this chapter deals with nonsymmetrical conditions. The equations found in chapter 3 are all based on an axisymmetric membrane condition before buckling which is not the case here. As the simplified symmetric prebuckling equations are no longer valid new equations must be derived.

7.2.1 Assumptions

The buckling equations which are found by applying the perturbation technique in search for adjacent equilibrium condition contain the prebuckling deformations (u_0, v_0, w_0) and stress resultants (N_{x0}, N_{y0}, N_{xy0}, N_{yx0}, M_{x0}, M_{y0}, M_{xy0}, M_{yx0}). To simplify our equations the small terms are dropped from these equations. As an adjacent equilibrium state is sought, the buckling perturbed values (e.g. u_1, v_1, w_1) are infinitesimally small, so only their linear terms are taken. By application of the method of perturbation a set of equations is found in which three kinds of elements may be distinguished. These three are pure buckling terms, multiplication of linear prebuckling terms with buckling terms, and multiplication of nonlinear prebuckling terms with buckling terms. The first two are kept and the third terms are dropped because, based on Flügge theory they are small and negligible. Yamaki (1984) considered only the effect of N_x, N_θ or $N_{x\theta}$ (depending on the loading) for the second type elements and substituted their values from membrane theory. But all of the elements of second type are considered in this work and their values are found using the linear Flügge equilibrium equations. In this way the equations are more general and can handle nonsymmetric problems.

7.2.2 Loading

Two kinds of load act on the tank simultaneously. The first one is hydrostatic pressure, which is applied internally and varies linearly from zero at the top, to the maximum of $\rho g L$ at the bottom. Here ρ is the mass density of the contained liquid (water), g the acceleration of gravity, and L is the height of the tank (the tank is considered full). The second load is the lateral pressure due to horizontal acceleration. The acceleration is transferred to the contained liquid and causes a unidirectional pressure applied in the opposite direction of acceleration. The pressure varies longitudinally and circumferentially. In the circumferential direction it is a maximum along the direction of acceleration (inward on one side and outward on the other side) and zero at points located 90 degrees from the maximum pressure. Depending on the geometry of the cylindrical shell the pressure will have different distributions in the longitudinal direction. The pressure at each point can be found through the equation (Malhotra & Veletsos, 1994)

$$p = \alpha(x).\rho.R.\cos\theta.A(t) \qquad (7\text{-}7)$$

where p is the pressure, R is again the radius of cylinder, θ is the circumferential position from acceleration direction, A(t) is a function dependent on the amount of acceleration, $\alpha(x)$ is the longitudinal distribution of the pressure, and x is the axial position measured from bottom of the tank. The longitudinal distribution of the pressure is available in the literature for some special

L/R and R/h ratios (h: thickness and L length of the tank). The following functions can be fitted to the graphical distributions given in Malhotra & Veletsos (1994):

For L/R = 0.5 : $\alpha(x) = 0.0532\,x^3 - 0.7129\,x^2 + 0.2812\,x + 0.3799$

For L/R = 1 : $\alpha(x) = -0.886\,x^3 + 0.0421\,x^2 + 0.2009\,x + 0.6516$ (7-8)

For L/R = 3 : $\alpha(x) = -13.4455\,x^5 + 26.2304\,x^4 - 21.0359\,x^3 + 8.108\,x^2 - 0.175x + 0.3239$

Fig. (7.1) shows pressure distributions in the axial and circumferential directions.

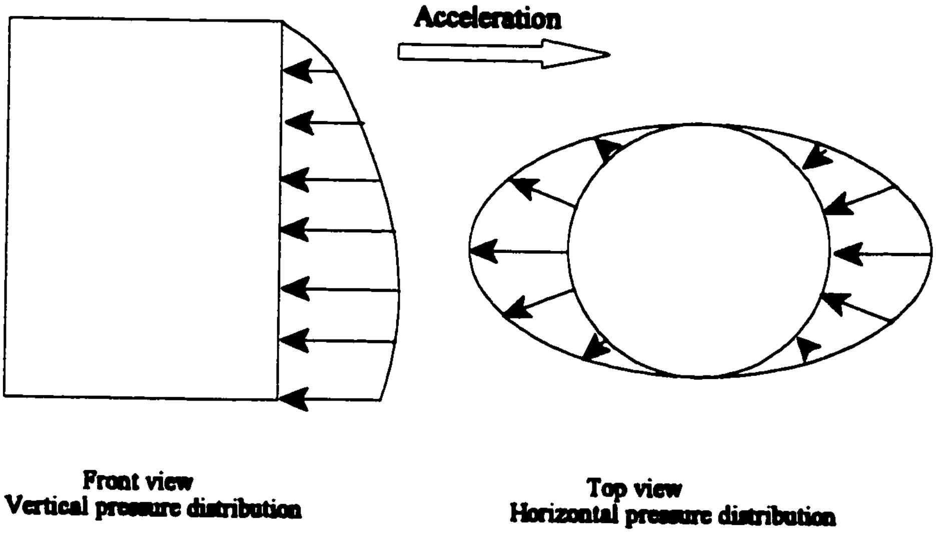

Figure 7.1 Distribution of dynamic loads

7.2.3. Buckling equations

The analysis starts with the nonlinear equilibrium equations (3-9) and employs the perturbation technique. At the buckling state the displacements and resultants can be represented as:

$$(u,v,w) = (u_0, v_0, w_0) + (u_1, v_1, w_1) \tag{7-1}$$

$$(N_x , N_{xy}, N_{yx}, N_y) = (N_{x0}+ N_{xy0}+ N_{yx0}+ N_{y0}) + (N_{x1}+N_{xy1}+ N_{yx1}+ N_{y1})$$

$$(M_x , M_{xy}, M_{yx}, M_y) = (M_{x0}+M_{xy0}+ M_{yx0}+M_{y0}) + (M_{x1}+ M_{xy1}+ M_{yx1}+ M_{y1})$$

Subscripts "0" and "1" correspond to prebuckling and infinitesimal incremental states respectively. Substituting from equations (7-1) in equations (3-9) one obtains:

$$\begin{gathered}
N_{x1,x}(1+u_{0,x})+N_{x1}u_{0,xx}+N_{yx1,y}(1+u_{0,x})+N_{yx1}u_{o,xy}+N_{y1,y}u_{0,y}+N_{y1}u_{0,yy}+N_{xy1,x}u_{0,y} \\
+N_{xy1}u_{0,xy}+(N_{x0,x}+N_{yx0,y})u_{1,x}+N_{x0}u_{1,xx}+(N_{yx0}+N_{xy0})u_{1,xy}+(N_{y0,y}+N_{xy0,x})u_{1,y} \\
+N_{y0}u_{1,yy}-pw_{1,x}=0
\end{gathered}$$

$$\begin{gathered}
N_{xy1,x}(1+v_{0,y}-\frac{w_0}{R})+(N_{xy1}+N_{yx1})(v_{0,xy}-\frac{w_{0,x}}{R})+N_{yx1,y}v_{0,x}+N_{x1,x}v_{0,x}+N_{x1}v_{0,xx} \\
+N_{y1,y}(1+v_{0,y}-\frac{w_0}{R})+N_{y1}(v_{0,yy}-2\frac{w_{0,y}}{R}-\frac{v_0}{R^2})-\frac{M_{y1,y}}{R}-\frac{M_{xy1,x}}{R}+v_{1,x}(N_{x0,x}+N_{yx0,y})+v_{1,xx}N_{x0} \\
+v_{1,y}(N_{xy0,x}+N_{y0,y})+v_{1,yy}N_{y0}+v_{1,xy}(N_{xy0}+N_{yx0})-v_1\frac{N_{y0}}{R^2}-\frac{w_{1,x}}{R}(N_{xy0}+N_{yx0})-2w_{1,y}\frac{N_{y0}}{R} \\
-\frac{w_1}{R}(N_{y0,y}+N_{xy0,x})-p(w_{1,y}+\frac{v_1}{R})=0
\end{gathered}$$

$$\begin{gathered}
M_{x1,xx}+M_{xy1,xy}+M_{yx1,xy}+M_{y1,yy}+N_{x1,x}w_{0,x}+N_{x1}w_{0,xx}+N_{y1}(\frac{1}{R}+2\frac{v_{0,y}}{R}-\frac{w_0}{R^2}+w_{0,yy}) \\
+N_{xy1,x}(w_{0,y}+\frac{v_0}{R})+N_{xy1}(w_{0,xy}+\frac{v_{0,x}}{R})+N_{yx1,y}w_{0,x}+N_{yx1}(w_{0,xy}+\frac{v_{0,x}}{R})+N_{y1,y}(w_{0,y}+\frac{v_0}{R}) \\
+v_{1,x}(\frac{N_{yx0}}{R}+\frac{N_{xy0}}{R})+v_{1,y}(2\frac{N_{y0}}{R})+v_1(\frac{N_{y0,y}+N_{xy0,x}}{R})+w_{1,x}(N_{x0,x}+N_{yx0,y})+w_{1,xx}N_{x0} \\
+w_{1,y}(N_{xy0,x}+N_{y0,y})+N_{y0}w_{1,yy}+w_{1,xy}(N_{xy0}+N_{yx0})-w_1\frac{N_{y0}}{R^2}+p(u_{1,x}+v_{1,y}-\frac{w_1}{R})=0
\end{gathered}$$

(7-2)

The prebuckling resultants in equation (7-2) are the same as those given in equation (3-5) and the incremental resultants may be found by substituting from equations (7-1) in (3-5) and using equations (3-3). Cancelling the equivalent terms from both sides and dropping the nonlinear terms give:

$$
\begin{aligned}
N_{x1} &= J\ [u_{1,x}(1+u_{0,x})+w_{0,x}w_{1,x}+v_{0,x}v_{1,x}+\upsilon v_{1,y}(1+v_{0,y}-\frac{w_0}{R})+\upsilon u_{1,y}u_{0,y} \\
&\quad +\upsilon w_{1,y}(w_{0,y}+\frac{v_0}{R})+\upsilon w_1(-\frac{1}{R}-\frac{v_{0,y}}{R}+\frac{w_0}{R^2})+\upsilon v_1(\frac{w_{0,y}}{R}+\frac{v_0}{R^2})]\ +\ \frac{D}{R}w_{1,xx} \\
N_{y1} &= J\ [v_{1,y}(1+v_{0,y})+w_1(-\frac{1}{R}+\frac{w_0}{R^2}-\frac{v_{0,y}}{R})+u_{1,y}u_{0,y}+v_1\frac{v_0}{R^2}-v_{1,y}\frac{w_0}{R}+w_{1,y}(w_{0,y}+\frac{v_0}{R}) \\
&\quad +v_1\frac{w_{0,y}}{R}++\upsilon u_{1,x}(1+u_{0,x})+\upsilon v_{1,x}v_{0,x}+\upsilon w_{1,x}w_{0,x}]-\frac{D}{R}(w_{1,yy}+\frac{w_1}{R^2}) \\
N_{xy1} &= \frac{1-\upsilon}{2}(\ J\ [u_{1,y}(1+u_{0,x})+v_{1,x}(1+v_{0,y}-\frac{w_0}{R})+u_{1,x}u_{0,y}+v_{1,y}v_{0,x}-w_1\frac{v_{0,x}}{R}+w_{1,y}w_{0,x} \\
&\quad +w_{1,x}(w_{0,y}+\frac{v_0}{R})+v_1\frac{w_{0,x}}{R}\ ]\ +\ \frac{D}{R}(\frac{v_{1,x}}{R}+w_{1,xy})) \\
N_{yx1} &= \frac{1-\upsilon}{2}(\ J\ [u_{1,y}(1+u_{0,x})+v_{1,x}(1+v_{0,y}-\frac{w_0}{R})+u_{1,x}u_{0,y}+v_{1,y}v_{0,x}-w_1\frac{v_{0,x}}{R}+w_{1,y}w_{0,x} \\
&\quad +w_{1,x}(w_{0,y}+\frac{v_0}{R})+v_1\frac{w_{0,x}}{R}\ ]\ +\ \frac{D}{R}(\frac{u_{1,y}}{R}-w_{1,xy})\ )
\end{aligned}
\tag{7-3}
$$

Because of the linearity, all the prebuckling terms are cancelled with their counterparts on the other side, and the incremental moment resultants are the same as those of equation (3-14).

By substituting the stress resultants from equation (7-3) and the moment resultants from equation (3-14) into equation (7-2) the buckling equations are found. Since the incremental stress resultants of (7-3) include some prebuckling terms, there are some nonlinear prebuckling terms in the final equation. The prebuckling terms are considered small so the nonlinear prebuckling terms are dropped in deriving the final equations. The resulting equations for the incremental displacements are:

$$
\begin{gathered}
u_{1,x}\,[J(2u_{0,xx}+\frac{1+\upsilon}{2}u_{0,yy})+Nx_{0,x}+Nyx_{0,y}] \;+\; u_{1,xx}[J(1+2u_{0,x})+Nx_0] \\
+\; u_{1,y}[J(\frac{3-\upsilon}{2}u_{0,xy})+\frac{1-\upsilon}{2}\frac{D}{R^2}u_{0,xy}+Ny_{0,y}+Nxy_{0,x}] \\
+\; u_{1,yy}[J(\frac{1-\upsilon}{2}(1+2u_{0,x}))+\frac{1-\upsilon}{2}\frac{D}{R^2}(1+u_{0,x})+Ny_0] \;+\; u_{1,xy}[J((1+\upsilon)u_{0,y})+Nyx_0+Nxy_0] \\
+\; v_{1,x}[J(v_{0,xx}+\frac{3\upsilon-1}{2}\frac{w_{0,y}}{R}+\upsilon\frac{v_0}{R^2}+\frac{1-\upsilon}{2}v_{0,yy}+(1-\upsilon)u_{0,xy})+\frac{1-\upsilon}{2}\frac{D}{R^2}u_{0,xy}] \\
+\; v_{1,xx}[J(v_{0,x}+\frac{1-\upsilon}{2}u_{0,y})+\frac{1-\upsilon}{2}\frac{D}{R^2}u_{0,y} \;+\; v_{1,y}[J(\frac{1+\upsilon}{2}v_{0,xy}+\frac{1-3\upsilon}{2}\frac{w_{0,x}}{R}+\upsilon u_{0,xx}+u_{0,yy})] \\
+\; v_{1,yy}[J(\frac{1-\upsilon}{2}v_{0,x}+u_{0,y})] \;+\; v_{1,xy}[J(\frac{1+\upsilon}{2}(1+v_{0,y})-\frac{1+\upsilon}{2}\frac{w_0}{R}+\frac{1+\upsilon}{2}u_{0,x})] \\
+\; v_1[J(\frac{1+\upsilon}{2}\frac{w_{0,xy}}{R}+\upsilon\frac{v_{0,x}}{R^2})] \;+\; w_{1,x}[J(w_{o,xx}-\frac{\upsilon}{R}(1+v_{0,y})+\upsilon\frac{w_0}{R^2}+\frac{1-\upsilon}{2}\frac{v_{0,y}}{R}-\upsilon\frac{u_{0,x}}{R}+\frac{1-\upsilon}{2}w_{0,yy})] \\
+\; w_{1,xx}[Jw_{0,x}+\frac{D}{R}u_{0,xx}] \;+\; w_{1,xxx}\frac{D}{R}(1+u_{0,x}) \;+\; w_{1,xxy}\frac{1-\upsilon}{2}\frac{D}{R}u_{0,y} \;-\; w_{1,xyy}\frac{1-\upsilon}{2}\frac{D}{R}(1+u_{0,x}) \\
-\; w_{1,yyy}\frac{D}{R}u_{0,y} \;+\; w_{1,y}[J(\upsilon w_{0,xy}+\frac{3\upsilon-1}{2}\frac{v_{0,x}}{R}+\frac{1-\upsilon}{2}w_{0,xy}-\frac{u_{0,y}}{R})-\frac{D}{R^3}u_{0,y}] \\
+\; w_{1,yy}[J(\frac{1-\upsilon}{2}w_{0,x})-\frac{D}{R}u_{0,yy}] \;+\; w_1[J(-\frac{1+\upsilon}{2}\frac{v_{0,xy}}{R}+\upsilon\frac{w_{0,x}}{R^2}-\upsilon\frac{u_{0,xx}}{R}-\frac{u_{0,yy}}{R})-\frac{D}{R^3}u_{0,yy}] \\
= Pw_{1,x}
\end{gathered}
\qquad (7\text{-}4a)
$$

$$
\begin{aligned}
& u_{1,x}[J(\frac{1+\upsilon}{2}u_{0,xy}+v_{0,x}+\upsilon(v_{0,yy}-2\frac{w_{0,y}}{R}-\frac{v_0}{R^2}))] + u_{1,xx}[J(\frac{1-\upsilon}{2}u_{0,y}+v_{0,x})] \\
& + u_{1,y}[J(1-\upsilon)(v_{0,xy}-\frac{w_{0,x}}{R})+\frac{1-\upsilon}{2}u_{0,xx}+u_{0,yy})+\frac{1-\upsilon}{2}\frac{D}{R^2}(v_{o,xy}-\frac{w_{0,x}}{R})] \\
& + u_{1,yy}[J(\frac{1-\upsilon}{2}v_{0,x}+u_{0,y})+\frac{1-\upsilon}{2}\frac{D}{R^2}v_{0,x}] + u_{1,xy}[J(\frac{1+\upsilon}{2}(1+v_{0,y}-\frac{w_0}{R}+u_{0,x}))] \\
& + v_{1,x}[J(\frac{3-\upsilon}{2}v_{0,xy}-(1-\upsilon)\frac{w_{0,x}}{R})+Nx_{0,x}+Nyx_{0,y}+\frac{1-\upsilon}{2}\frac{D}{R^2}(v_{0,xy}-\frac{w_{0,x}}{R})] \\
& + v_{1,xx}[J(\frac{1-\upsilon}{2}(1+2v_{0,y}-2\frac{w_0}{R}))+\frac{D}{R^2}(3\frac{1-\upsilon}{2}+v_{0,y}-\frac{w_0}{R})+Nx_0] + v_{1,xy}[J(1+\upsilon)v_{0,x}+Nxy_0+Nyx_0] \\
& + v_{1,y}[J(\frac{1+\upsilon}{2}v_{0,xx}+2v_{0,yy}-2\frac{w_{0,y}}{R})+Nxy_{0,x}+Ny_{0,y}] + v_{1,yy}[J(1+2v_{0,y}-2\frac{w_0}{R})+Ny_0] \\
& + v_1[J(\frac{1-\upsilon}{2}\frac{w_{0,xx}}{R}+\frac{w_{0,yy}}{R}+\frac{v_{0,y}}{R^2})-\frac{Ny_0}{R^2}] + w_{1,x}[J(\frac{1+\upsilon}{2}w_{0,xy}-\upsilon\frac{v_{0,x}}{R})-\frac{Nxy_0+Nyx_0}{R}] \\
& + w_{1,xx}[J(\frac{1-\upsilon}{2}(w_{0,y}+\frac{v_0}{R}))+\frac{D}{R}v_{0,xx}] + w_{1,xxx}\frac{D}{R}v_{0,x} + w_{1,xxy}\frac{D}{R}(\frac{3-\upsilon}{2}+\frac{1-\upsilon}{2}(v_{0,y}-\frac{w_0}{R})) \\
& - w_{1,xyy}\frac{1-\upsilon}{2}\frac{D}{R}v_{0,x} + w_{1,y}[J(\frac{1-\upsilon}{2}w_{0,xx}-\frac{1}{R}+w_{0,yy}-\frac{v_{0,y}}{R}+2\frac{w_0}{R^2})-\frac{D}{R^3}(v_{0,y}-\frac{w_0}{R})-2\frac{Ny_0}{R}] \\
& + w_{1,yy}[J(w_{0,y}+\frac{v_0}{R})-\frac{D}{R}(v_{0,yy}-2\frac{w_{0,y}}{R}-\frac{v_0}{R^2})] + w_{1,yyy}\frac{D}{R}(\frac{w_0}{R}-v_{0,y}) \\
& + w_1[J(-\frac{1+\upsilon}{2}\frac{v_{0,xx}}{R}+3\frac{w_{0,y}}{R^2}-2\frac{v_{0,yy}}{R}+\frac{v_0}{R^3})-\frac{D}{R^3}(v_{0,yy}-2\frac{w_{0,y}}{R}-\frac{v_0}{R^2})-\frac{1}{R}(Ny_{0,y}+Nxy_{0,x})] \\
& = P(w_{1,y}+\frac{v_1}{R})
\end{aligned}
$$

(7- 4b)

$$
\begin{aligned}
& u_{1,x}\,[J(w_{0,xx}+\frac{\upsilon}{R}-2\upsilon\frac{v_{0,y}}{R}-\upsilon\frac{w_0}{R^2}+\upsilon w_{0,yy}+\upsilon\frac{u_{0,x}}{R})] \;+\; u_{1,xx}[Jw_{0,x}] \;-\; \frac{D}{R}u_{1,xxx}+\frac{1-\upsilon}{2}\frac{D}{R}u_{1,xyy} \\
& +\; u_{1,y}[J(\frac{u_{0,y}}{R}+(1-\upsilon)(w_{0,xy}+\frac{v_{0,x}}{R}))+\frac{1-\upsilon}{2}\frac{D}{R^2}(w_{0,xy}+\frac{v_{0,x}}{R})] \;+\; u_{1,yy}[J(\frac{1-\upsilon}{2}w_{0,x})+\frac{D}{R^2}\frac{1-\upsilon}{2}w_{0,}x] \\
& +\; u_{1,xy}[J(\upsilon w_{0,y}+\upsilon\frac{v_0}{R})] \;+\; v_{1,x}[J(\frac{v_{0,x}}{R}+(1-\upsilon)w_{0,xy})+\frac{D}{R}\frac{1-\upsilon}{2}(w_{0,xy}+\frac{v_{0,x}}{R})+\frac{1}{R}(Nxy_0+Nyx_0)] \\
& +\; v_{1,xx}[J\frac{1-\upsilon}{2}(w_{0,y}+\frac{v_0}{R})+\frac{1-\upsilon}{2}\frac{D}{R^2}(w_{0,y}+\frac{v_0}{R})] \;+\; v_{1,y}[J(\frac{1}{R}+3\frac{v_{0,y}}{R}-2\frac{w_0}{R^2}+w_{0,yy}+\upsilon w_{0,xx})+2\frac{Ny_0}{R} \\
& +\; v_{1,yy}[J(w_{0,y}+\frac{v_0}{R})] \;-\; v_{1,xxy}\frac{3-\upsilon}{2}\frac{D}{R} \;+\; v_{1,xy}(J\frac{1-\upsilon}{2}w_{0,x}) \;+\; w_{1,xx}(\frac{D}{R}w_{0,xx}+Nx_0) \\
& v_1[J(\frac{w_{0,y}}{R^2}+\frac{v_0}{R^3})+\frac{Ny_{0,y}+Nxy_{0,x}}{R}] \;+\; w_{1,x}[Nx_{0,x}+Nyx_{0,y}] \;+\; w_{1,xxx}\frac{D}{R}w_{0,x} \;-\; Dw_{1\cdot xxxx} \;-\; 2Dw_{1,x} \\
& +\; w_{1,xxy}\frac{D}{R}\frac{1-\upsilon}{2}(w_{0,y}+\frac{v_0}{R}) \;-\; w_{1,xyy}\frac{1-\upsilon}{2}\frac{D}{R}w_{0,x} \;+\; w_{1,y}[Nxy_{0,x}+Ny_{0,y}-\frac{D}{R^3}(w_{0,y}+\frac{v_0}{R})] \;-\; Dw_{1,yy} \\
& +\; w_{1,yy}[Ny_o-\frac{D}{R}(\frac{2+2v_{0,y}}{R}-\frac{w_0}{R^2}+w_{0,yy})] \;-\; w_{1,yyy}\frac{D}{R}(w_{0,y}+\frac{v_0}{R}) \;+\; w_{1,xy}(Nxy_0+Nyx_0) \\
& +\; w_1[J(-\frac{1}{R^2}-3\frac{v_{0,y}}{R^2}+2\frac{w_0}{R^3}-\frac{w_{0,yy}}{R}-\upsilon\frac{w_{0,xx}}{R})-\frac{D}{R^3}(\frac{1}{R}+2\frac{v_{0,y}}{R}-\frac{w_0}{R^2}+w_{0,yy})-\frac{Ny_0}{R^2}] \\
& = \; -P(u_{1,x}+v_{1,y}-\frac{w_1}{R})
\end{aligned}
$$

(7- 4c)

7.2.4 DQM analogue of equations

To solve the equations (7-4) the shell is discretized as shown in figure (6-1) and the quadrature rule is used for both the prebuckling and buckling terms in the governing equations, as well as the boundary conditions. The concept described in sections 5.3 and 6.3, is used again here.

7.3 Boundary conditions

The natural boundary conditions of an upright liquid storage tank can be assumed to be of the clamped-free type. The bottom end is clamped to the foundation and the top end is free to the air. The clamped boundary equations are the same as those given in chapter 5, $u = v = w = w_{,x} = 0$, but the free boundary conditions are different from those of equation (5-2). The difference stems from the fact that the stress resultants are different in this case (they have prebuckling terms here). The free boundary equations are

$$\text{at } u_M : \quad N_x = 0 , \qquad \text{at } v_M : \quad T_x = N_{xy} - M_{xy}/R = 0 \qquad \text{at } w_M : \quad M_x = 0$$

$$\text{at } w_{M-1} : \quad S_x = Q_x + M_{xy,y} = dM_x/dx + d(M_{xy} + M_{yx})/dy = 0 \tag{7-5}$$

In equations (7-5) the subscripts M and M-1 show the position of the set of sampling points at which the boundary condition is satisfied. For example, u_M includes N points at the top of the tank (see figure 6.1)(points (M,1) through (M,N)) and w_{M-1} includes N number of δ points close to the top sampling points ((M-1,1) through (M-1,N)). The u, v, and w determine whether the boundary condition is satisfied instead of the first, second, or third governing equations respectively (equations (7-4a, b, or c)).

Substituting from equations (7-3) in (7-5), the boundary equations can be found in terms of displacements. The displacement statement of the boundary conditions in the u direction is easily satisfied if the first of (7-3) is made equal to zero. The condition at w_M is satisfied when the M_x is put to zero using (3-14). The remaining two equations can be transformed into

$$\begin{aligned} T_x = 0 \rightarrow \quad & J[u_{1,y}(1+u_{0,x})+v_{1,x}(1+v_{0,y}-\frac{w_0}{R})+u_{1,x}u_{0,y}+v_{1,y}v_{0,x}-w_1\frac{v_{0,x}}{R}+w_{1,y}w_{0,x} \\ & +w_{1,x}(w_{0,y}+\frac{v_0}{R})+v_1\frac{w_{0,x}}{R}]+\frac{D}{R}(3w_{1,xy}+3\frac{v_{1,x}}{R}) = 0 \\ S_x = 0 \rightarrow \quad & w_{1,xxx}+(2-\upsilon)w_{1,xyy}+u_{1,xx}-\frac{1-\upsilon}{2}u_{1,yy}+\frac{3_{\upsilon}}{2}v_{1,xy} = 0 \end{aligned} \tag{7-6}$$

7.4 Solution method

Prebuckling displacements and stress resultants are present in both the buckling equations and the boundary conditions. They must be determined as a preliminary step of the buckling analysis. The linear equilibrium equations of Flügge are adopted and solved using the DQM. The weighting coefficients of the DQM are used two times, first for solving the equilibrium equations, and then for solving the buckling problem. The Flügge equilibrium equations used are (Flügge, 1973):

$$\begin{aligned} & u^0{}_{,\varphi\varphi}+\frac{1-\upsilon}{2}u^0{}_{,\theta\theta}+\frac{1+\upsilon}{2}v^0{}_{,\varphi\theta}-\upsilon w^0{}_{,\varphi}+\kappa[\frac{1-\upsilon}{2}u^0{}_{,\theta\theta}+w^0{}_{,\varphi\varphi\varphi}-\frac{1-\upsilon}{2}w^0{}_{,\varphi\theta\theta}] = 0 \\ & \frac{1+\upsilon}{2}u^0{}_{,\varphi\theta}+\frac{1-\upsilon}{2}v^0{}_{,\varphi\varphi}+v^0{}_{,\theta\theta}-w^0{}_{,\theta}+\kappa[3\frac{1-\upsilon}{2}v^0{}_{,\varphi\varphi}+\frac{3-\upsilon}{2}w^0{}_{,\varphi\varphi\theta}] = 0 \\ & \upsilon u^0{}_{,\varphi}+v^0{}_{,\theta}-w^0+\kappa[\frac{1-\upsilon}{2}u^0{}_{,\varphi\theta\theta}-u^0{}_{,\varphi\varphi\varphi}-\frac{3-\upsilon}{2}v^0{}_{,\varphi\varphi\theta}-w^0{}_{,\varphi\varphi\varphi\varphi}-2w^0{}_{,\varphi\varphi\theta\theta} \\ & \qquad -w^0{}_{,\theta\theta\theta\theta}-2w^0{}_{,\theta\theta}-w^0]+q_p=0 \end{aligned} \tag{7-9}$$

The following nondimensional values are used in equation (7-9) and also in programming of the problem.

$$\varphi=\frac{x}{R},\quad \theta=\frac{y}{R},\quad u^0=\frac{u_0}{R},\quad v^0=\frac{v_0}{R},\quad w^0=\frac{w_0}{R},\quad \kappa=\frac{h^2}{12R^2},$$
$$q_p=\frac{pR}{J},\quad Ny^0=\frac{Ny_0}{J}\textit{(same for all stress resultants)},\quad J=\frac{Eh}{1-\upsilon^2} \tag{7-10}$$

The same boundary conditions of (6-3) apply to the free boundary condition for the prebuckling problem. When the displacements are found, the nondimensional stress resultants can be determined using equations (7-3) and (7-10).

The second major step of the analysis is the determination of the buckling loads. The matrix equations of the buckling problem has the form

$$\left[K_b + K_{p1}\right]\begin{bmatrix}(\Delta_b)\\(\Delta_d)\end{bmatrix} = \alpha\left[K_{bg}+K_{p2}\right]\begin{bmatrix}(\Delta_b)\\(\Delta_d)\end{bmatrix} \tag{7-11}$$

The matrix K_b consists of the terms for the DQM analogue of the buckling equations and the boundary conditions. Matrix K_{bg} includes the displacement terms which have a coefficient of the buckling load. Matrix K_{p1} includes the prebuckling terms due to hydrostatic pressure, and K_{p2} the prebuckling terms due to equivalent static load arising from unit acceleration. The resultant matrices on the two sides are full, and thus static condensation, used in the previous chapters, is not possible here. Each matrix is of the size 3MN. The vector Δ_b contains the displacements corresponding to the boundary points, while Δ_d the displacements corresponding to the domain points. The smallest acceleration value (A_{min}) may be found directly using the inverse iteration method with shifting.

Based on the procedure outlined in the preceding a Matlab computer program labelled tankeq.m is developed. The program is given in appendix A. The results given in the following are based on this program.

7.5 Results and discussion

Results are presented here for steel shells again having $\upsilon = 0.3$ and $E = 200Gpa$. The analyses are for different sizes of tanks and shells with the boundary conditions of an upright tank (clamped-free). First the prebuckling problem is solved to use its results in the buckling problem. The analytical solution for a tank under hydrostatic pressure is presented in Flügge (1973). The DQM solution is in good agreement with these results. Table 7.1 shows the convergence of DQM results for the prebuckling displacements under hydrostatic load compared to analytical and FEM results. NE/Nastran software is used for all FEM analyses in this work. 6 D.O.F. rectangular plate elements are used (NE/Nastran, 1996).The results are for the tank of L = 6m, R = 3m, h = 3mm. A complete convergence was observed for the DQM results. Since under

Table 7.1 Convergence of DQM prebuckling results

		Radial deformation at Mid. of tank(at x=L/2) ($\times 10^{-4}$ m)			
Solution	Mesh(M ×N)	Analytical	DQM	FEM	Error %
1	13 × 6	4.410	3.684	-	-12.4
2	18 × 6		4.406	-	-0.1
3	23 × 6		4.344	-	1.5
4	33 × 6		4.410	-	0
5	40 × 30		-	4.528	2.7

symmetric hydrostatic pressure the shape of the tank does not deviate from its initial circular one, six circumferential sampling points are enough to completely represent the shape of the tank. Use of a higher number of circumferential sampling points does not change the results.

In table 7.1, the DQM results are more accurate than the FEM results. The results given are for a point at mid-height of the tank (x = L/2). The same trend is valid for other points on the tank wall. Because of symmetry the displacements are functions of axial variable only and a small number of circumferential points (N = 6) is enough for getting good results.

The accuracy of the new buckling formulation is determined by applying it on cylindrical shells under external uniform lateral loading. The analyses are for tanks of two sizes. Tank 1 has L = 7.5*m*, R = 15*m*, h = 15*mm* and is representative of short tanks. Tank 2 has L = 8*m*, R = 8*m*, h = 8*mm*, and is representative of medium height tanks. The results are compared with those of FEM and results of chapter 5. Table 7.2 shows the buckling loads. While the results of this formulation are close to those of the FEM, they differ from results of chapter 5. The results of chapter 5 are around 30 percent higher than the FEM and DQM.

Despite that less calculation effort and less round off errors are involved in the method used in chapter 5, its results are not accurate enough. Two sources of errors may be considered for inaccuracy of the results in chapter 5. The first source is the simplification which is used in the formulation of equations in chapter 5. A simplified set of membrane prebuckling deformations and stress resultants is considered there (adopted from Yamaki (1984)). In the new formulation there are more accurate prebuckling results in which the effect of boundary conditions and bending moments developed in shell are accounted for. To check the effect of this source of error the new formulation was solved using the simplified membrane prebuckling displacements and stress resultants. The difference in results was less than one percent. So, for the case of symmetrical loading the simplified prebuckling terms considered in Yamaki (1984) are quite accurate.

Table 7.2 Comparison of 1-D, 2-D and FEM buckling loads for uniform lateral pressure

	L/R	R/h	Chapter 5 1-D DQM(Pa)	New 2-D DQM(Pa)	FEM(Pa)
Tank 1	2	300	55390	38339	37528
Tank 2	3	500	31353	20400	22776

The second source of error is in the boundary conditions. In the new formulation a more complete definition is presented for free boundary condition in the buckling problem at top of the tank (equations (7-5) and (7-6)) and it is believed that it is the major reason for inaccuracy of results in chapter 5. Since the other boundary conditions in chapter 5 are easier to define, there is not such a problem with the clamped-clamped and simple-simple supported shells.

The next loading case is the nonsymmetric loading described in section 7-2-2. The analysis is for the same two tanks 1 and 2.

Two load combinations are considered for each tank. The first combination (set 1) consists solely of the equivalent static load. The second combination (set 2) consists of the hydrostatic pressure of the liquid (water) together with the equivalent static load. The hydrostatic pressure causes a prestress in the tank, and the equivalent static load causes buckling. The FEM software used for the verification did not contain an option for determining buckling loads for prestressed tanks. Thus in validation, the results for set 2 loads are not compared with FEM results, but instead with results given in the thesis by Boutros (1997). These latter results are obtained using a series solution in conjunction with the Donnell shell theory.

The numerical results for the minimum horizontal acceleration (A_{min}, m/s^2) to cause buckling are presented in tables 7.3 through 7.5. Table 7.3 shows the convergence of the results in the FEM analysis for tank 1 under the equivalent static loading, while table 7.4 gives similar results for the DQM analysis. Although the DQM results here do not converge as fast as in chapter 5 (for 1-dimensional DQM), there is a steady convergence and there is significantly less oscillation of results than that was observed in chapter 6.

Table 7.3 Convergence of FEM results (tank 1, load set 1)

Solution	Mesh (N × M)	A_{min}
1	60 × 20	3.2509
2	80 × 25	3.0824
3	100 × 40	2.9800

Table 7.5 shows the comparison of the DQM values with FEM results and shell theory results (Boutros, 1997). Indicated in this table are the accelerations required for buckling. Two tanks geometries and 2 load sets are presented. The DQM values are close to FEM and series solution

Table 7.4 Convergence of DQM results for A_{min}(tank 1, load set 1)

		M		
Solution	N	12	14	16
1	30	5.4274	5.4835	5.4919
2	34	3.7383	3.7905	3.7997
3	38	3.1311	3.1765	3.1849
4	40	2.9942	3.0360	3.0439
5	42	2.9130	2.9514	2.9587
6	46	2.8389	2.8721	2.8784

values except for those of tank 2, load set 2. The reason for the difference is that the series solution employed the Donnell shell theory rather than the Flügge theory. The Donnell theory is known to give higher buckling load values, and its accuracy deteriorates with an increase in the L/R aspect ratio.

Considering the results of table 7.5 it is found that a tank filled with liquid has a larger buckling load than an empty one. That's because the hydrostatic pressure causes tensile hoop stresses in shell and also cancels part of external buckling pressure.

Table 7.5 Comparison of DQM results with others for A_{min}

Tank	Load set	DQM	FEM	Series(Boutros, 1997)
1	1	2.88	2.98	-
1	2	9.45	-	10.8
2	1	1.40	1.42	-
2	2	7.50	-	18.64

Figures 7.2 and 7.3 show the buckling modes for the tank 1 and 2 under the load set 1. As expected buckling occurs on the side where the load produces an inward pressure. There are solely elastic displacements on the other side. The buckling deformation for both tanks in the axial direction is a half sine wave. In the circumferential direction tank 1 has a larger number of wavelengths than tank 2 due to its larger radius. Buckling is seen to be due to excessive circumferential stress for both tanks.

Figures 7.4 and 7.5 show the buckling modes for the same tanks under load set 2. Because of the hydrostatic internal pressure the buckling mode is changed. Shear buckling occurs at a point 90 degrees from the point of maximum pressure. These buckling modes have been observed in some experimental cases also (Shih & Babcock (1987), Kokubo et al (1993), Nagashimi et al (1987)).

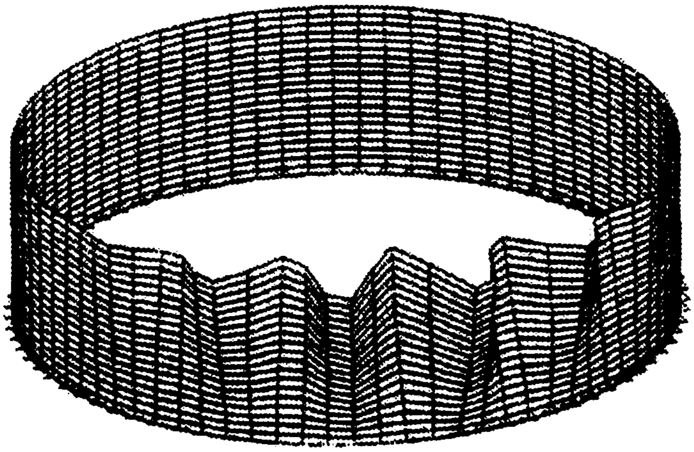

Figure 7.2 Buckling mode of tank 1 under load set 1.

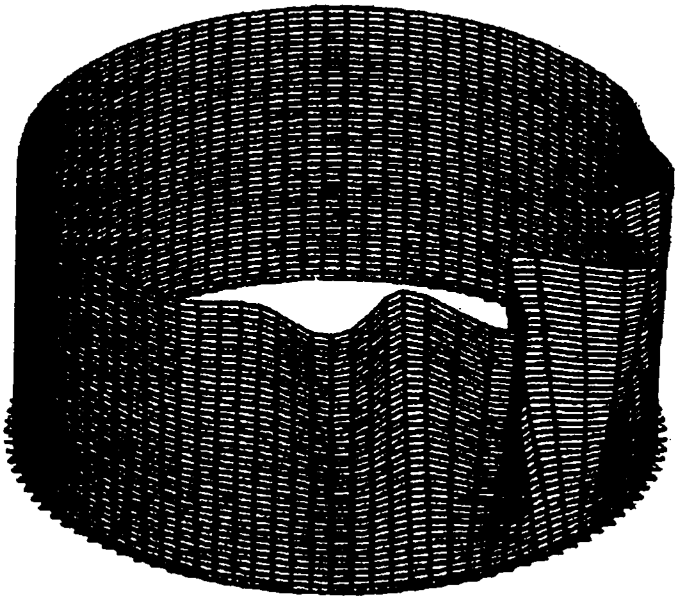

Figure 7.3 Buckling mode of tank 2 under load set 1.

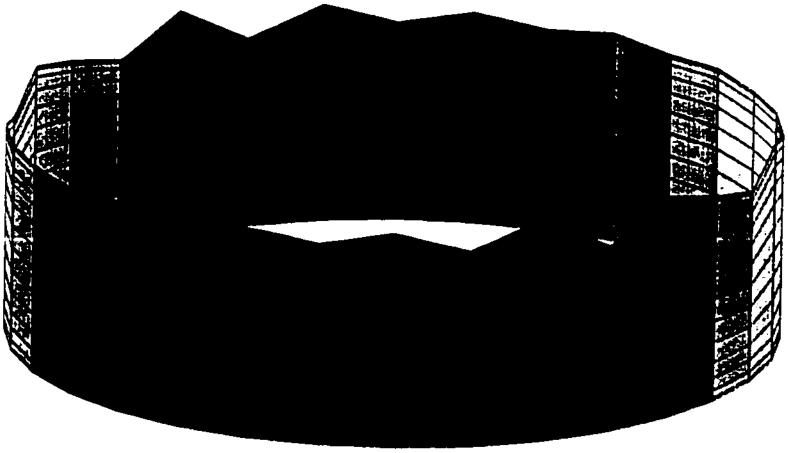

Figure 7.4 Buckling mode of tank 1 under load set 2

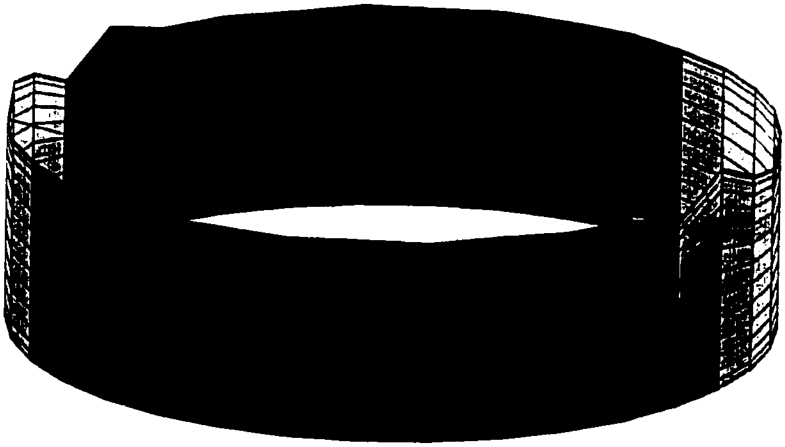

Figure 7.5 Buckling mode of tank 2 under load set 2

Chapter 8

Conclusion

8.1 Achievements

It is believed that the present work is a forward step in furthering the potential of the DQM as an analysis technique in the general area of structural mechanics, especially for problems involving circumferential continuity in geometry. Portions of this study have been presented in some journal and conference papers (Mirfakhraei et al (1996), Mirfakhraei & Redekop (1998), Zhang et al (1998), Mirfakhraei & Redekop (1999a), Mirfakhraei & Redekop (1999b)). This work makes way for further application of this method in the analysis of cylindrical shell structures, pressure vessels and liquid storage tanks.

8.2 Summary

A step by step procedure has been followed in this thesis aimed at the application of DQM analysis to tanks containing liquid. The procedure started with the known problem of buckling of cylindrical shells under uniform lateral pressure and compression and lead to the solution of the untouched problem of buckling of tanks under horizontal acceleration based on Flügge theory using the DQM. The results found are accurate, confirming the correctness of the procedure.

The primary contributions of this thesis are the application of the 1 and 2-dimensional DQM to the problem of buckling of cylindrical shells and tanks. In the 1-dimensional part, application of the DQM to the buckling of cylindrical shells is the main contribution. In the 2-dimensional part, the untouched problem of application of the 2-dimensional DQM to a structure with circumferential continuity is handled. A new formulation for buckling analysis of cylindrical shells under nonsymmetric loading is developed. Using 2-dimensional DQM these equations are solved for a prestressed shell representing a tank filled with liquid under the effect of horizontal acceleration of earthquake. The Flügge theory is used in all parts to use or to derive the related equations. While more accurate than the other theories of shell, because of its complication, the Flügge theory had not been previously used to investigate the buckling problems under combined loading. Using the DQM, the complicated equations can be handled as easily as simple ones.

The results found show the applicability of the DQM to the mentioned problems. Specially good results have been found for the 1-dimensional DQM and for application of the 2-dimensional DQM to buckling of tanks. Oscillation has been observed in the results for some cases in the application of the 2-dimensional DQM to shell buckling problems.

8.3 Recommendations for future work

With the methodology developed a wide variety of shell problems can now be addressed. This thesis is a starting point for analysis of problems of shells of revolution with circumferential continuity and under different combinations of distributed loads. The method has a demonstrated potential in solving practical and industrial problems.

The method needs to be applied to more problems in this field to further show its potential. The same concept used in this thesis may be applied to solve vibration and dynamic problems. It is worth investigating the ways to accelerate the calculation speed of DQM using better programming methods and faster eigenvalue solvers. The instability of some results must also be further studied from a mathematical point of view.

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Appendix A

The following computer program is written in Matlab language and determines the buckling load as the minimum eigenvalue of matrix format of governing equations (7-4) and boundary equations (7-5), using the DQM method. At the end the eigenvector corresponding to the minimum eigenvalue is found and the buckling mode shape is drawn. The main part is called "total" which calls the subroutines one by one. The program is labelled tankeq.m.

TOTAL.m

```
%                    Last revision April 10 1999
% determining the variables of the problem
ainput1;
%
%            ******************
% finding the weighting coefficeint for longitudinal derivatives
baij2;
%
%              ****************
% finding the weighting coefficients for circumferential derivatives
cbij3;
%
%               **************
%    making the DQM analogue of prebuckling differential equations
daii4;
%
%               *******************
%   placing terms in their positions in matrix form
edd5
%
%                ****************
%    Defining prebuckling boundary conditions
fbc6;
%
%                ****************
%   finding the load terms
gload7;
```

```
%
%                   ************
%     solving the linear equations for prebuckling displacements
hsolver8;

%                 ********************
%    preparing the required derivatives for buckling problem
provide9;
%  erasing the unwanted variables
clear dispd disps fqpd fqps ib rr1 rr2 rr3 rr4 stiff
% making the final DQM analogue of L.H.S prebuckling equations
f1aii
% making the final DQM analogue of R.H.S prebuckling equations
f2dii
% making the final L & R H.S matrices
f3dd
% Defining B.C. for buckling(which involves prebuckling displacements)
f4bc
% solving the eigenvalue problem
f5eig
%              **********************
%    Drawing the buckling mode of the cyoinder
f6modshp.m
```

AINPUT1.m

```
% in this part the material and geometrical properties of the
% cylinder are introduced. they are "ar" (aspect ratio=L/R),
% length and radius, "pr" (poisson ratio), E and "rh" (R/thickness).
% all other variables can be found from these three.
% the acceleration is needed for horizontal forces.
rh=1000;
pr=.3;
kp=1/(12*(rh^2));
m=16;
n=12;
delta=0.0001;
% length of cylinder(meter)
lngth=7.5;
% radius of cylinder(meter)
r=15;
ar=lngth/r;
% modulus of elasticity(pascal)
E=200e9;
% horizontal acceleration due to earthquake(m/s^2)
accel=1;
```

BAIJ2.m

```
% this part of program finds the weighting coefficients for longitudinal
direction
% The unequal spacing with one delta point at each end is considered. the
inputs
% for this part are the No. of sampling points "m" and the aspect ratio of the
shell
%(L/R). the outputs are a1(i.j), a2(i,j),a3(i,j),a4(i,j) for 1st to 4th
derivatives.
x(1)=0;
```

```
x(2)=delta;
x(m-1)=1-delta;
x(m)=1;
for i=3:m-2
tt=(pi*(i-2))/(m-3);
x(i)=(1-cos(tt))/2;
end
for i=1:m
pai(i)=1;
for j=1:m
if i~=j
           pai(i)=(x(i)-x(j))*pai(i);
end
end;end
for i=1:m
for j=1:m
if i~=j
    a1(i,j)=pai(i)/((x(i)-x(j))*pai(j));
end
end;end
for i=1:m
a1(i,i)=0;
for j=1:m
if i~=j
     a1(i,i)=a1(i,i)-a1(i,j);
end
end;end
for i=1:m
for j=1:m
if i~=j
  a2(i,j)=2*(a1(i,i)*a1(i,j)-a1(i,j)/(x(i)-x(j)));
end
end;end
for i=1:m
a2(i,i)=0;
for j=1:m
if i~=j
     a2(i,i)=a2(i,i)-a2(i,j);
end
end;end
for i=1:m
for j=1:m
if i~=j
  a3(i,j)=3*(a2(i,i)*a1(i,j)-a2(i,j)/(x(i)-x(j)));
end
end;end
for i=1:m
a3(i,i)=0;
for j=1:m
if i~=j
     a3(i,i)=a3(i,i)-a3(i,j);
end
end;end
for i=1:m
for j=1:m
if i~=j
  a4(i,j)=4*(a3(i,i)*a1(i,j)-a3(i,j)/(x(i)-x(j)));
end
end;end
for i=1:m
a4(i,i)=0;
```

```
 for j=1:m
 if i~=j
     a4(i,i)=a4(i,i)-a4(i,j);
 end
 end;end
 for i=1:m
 for j=1:m
 a1(i,j)=a1(i,j)/ar;
 a2(i,j)=a2(i,j)/ar^2;
 a3(i,j)=a3(i,j)/ar^3;
 a4(i,j)=a4(i,j)/ar^4;
 end;end
```

CBIJ3.m

```
% this part finds the required weighting coefficients in circumferential
% direction (b(i,j)) the sole input into this part is "n" (the number of
% sampling points in
% cifcumferential direction) which must be an even number. sampling points
%  are equally spaced. the outputs are b1(i,j),b2(i,j),b3(i,j),b4(i,j)(for 1st
% to 4th derivations)
n2=n/2;
for i=1:n
tet(i)=2*pi*(i-1)/n;
ib(i,1)=.5;
ib(i,n2+1)=.5;
end
for i=2: 2: n
ib(i,n2+1)=-0.5;
end
for i=2:n2
ib(1,i)=1;
end
for i=n2+2:n
ib(1,i)=0;
end
for i=2:n
for j=n2+2:n
ib(i,j)=sin(2*pi*(i-1)*(j-1-n2)/n);
end
end
for i=2:n
for j=2:n2
ib(i,j)=cos(2*pi*(i-1)*(j-1)/n);
end
end
% the following is a matrix operation
ib=ib/n2;
for i=1:n
for j=1:n
if abs(ib(i,j))<1e-12
ib(i,j)=0;
end
end;end

for i=1:n
for j=1:n2+1
rr1(j)=(1-j)*sin((j-1)*tet(i));
rr2(j)=-1*((j-1)^2)*cos((j-1)*tet(i));
```

```
rr3(j)=((j-1)^3)*sin((j-1)*tet(i));
rr4(j)=((j-1)^4)*cos((j-1)*tet(i));
if abs(rr1(j))<1e-12
rr1(j)=0;
end
if abs(rr2(j))<1e-12
rr2(j)=0;
end
if abs(rr3(j))<1e-12
rr3(j)=0;
end
if abs(rr4(j))<1e-12
rr4(j)=0;
end
end
for j=n2+2:n
rr1(j)=(j-1-n2)*cos((j-1-n2)*tet(i));
rr2(j)=-1*((j-1-n2)^2)*sin((j-1-n2)*tet(i));
rr3(j)=-1*((j-1-n2)^3)*cos((j-1-n2)*tet(i));
rr4(j)=((j-1-n2)^4)*sin((j-1-n2)*tet(i));
if abs(rr1(j))<1e-12
rr1(j)=0;
end
if abs(rr2(j))<1e-12
rr2(j)=0;
end
if abs(rr3(j))<1e-12
rr3(j)=0;
end
if abs(rr4(j))<1e-12
rr4(j)=0;
end
end
b1(i,:)=(ib*rr1')';
b2(i,:)=(ib*rr2')';
b3(i,:)=(ib*rr3')';
b4(i,:)=(ib*rr4')';
end
for i=1:n
for j=1:n
if abs(b1(i,j))<1e-10
b1(i,j)=0;
end
end;end
```

DAII4.m

```
% in this part of program the aii,aim,...,ann matrices which generate the dqm
% analog of the equations are made.the ai* series are for 1st, am* for 2nd
% and an* series are for 3rd eqns respectively. a*i is for U components, a*m
% for V and a*n for W displacements. the inputs to this part are m,n(no of
% sampling points in x and teta dir) and a1,..a4 and b1,..b4 matrices.
% the aii,aim,...,ann are outputs of the program.
l1=(1+kp)*(1-pr)/2;
l2=(1+pr)/2;
l3=kp*(1-pr)/2;
l4=(1-pr)/2+3*kp*(1-pr)/2;
l5=(3-pr)*kp/2;
l6=3*kp*(1-pr)/2;
```

```
 l7=(1-pr)/2;
 aii(:,:)=zeros(m*n);aim(:,:)=zeros(m*n);ain(:,:)=zeros(m*n);
 ami(:,:)=zeros(m*n);amm(:,:)=zeros(m*n);amn(:,:)=zeros(m*n);
 ani(:,:)=zeros(m*n);anm(:,:)=zeros(m*n);ann(:,:)=zeros(m*n);
 % these two i and j loops determine the point (i,j) at which the
 % equation is going to be satisfied.
 for i=1:m
   for j=1:n
   nmbr=(i-1)*n+j;
   % the following amounts are related to terms appearing as pure
   % variable(without any kind of derivation) in the diff. equation
   ann(nmbr,nmbr)=-kp-1;
   % the ii loop is for terms having just "a" matrix
     for ii=1:m
     cntr=(ii-1)*n+j;
     aii(nmbr,cntr)=a2(i,ii);
     ain(nmbr,cntr)=-pr*a1(i,ii)+kp*a3(i,ii)+ain(nmbr,cntr);
     amm(nmbr,cntr)=(l6+l7)*a2(i,ii);
     ani(nmbr,cntr)=pr*a1(i,ii)-kp*a3(i,ii)+ani(nmbr,cntr);
     ann(nmbr,cntr)=-kp*a4(i,ii)+ann(nmbr,cntr);
     % this inner ij loop is for terms of "a*b" matrices
       for ij=1:n
       cntr=(ii-1)*n+ij;
       aim(nmbr,cntr)=l2*a1(i,ii)*b1(j,ij);
       ain(nmbr,cntr)=-l3*a1(i,ii)*b2(j,ij)+ain(nmbr,cntr);
       ami(nmbr,cntr)=l2*a1(i,ii)*b1(j,ij);
       amn(nmbr,cntr)=l5*a2(i,ii)*b1(j,ij);
       ani(nmbr,cntr)=l3*a1(i,ii)*b2(j,ij)+ani(nmbr,cntr);
       anm(nmbr,cntr)=-l5*a2(i,ii)*b1(j,ij);
       ann(nmbr,cntr)=-2*kp*a2(i,ii)*b2(j,ij)+ann(nmbr,cntr);
     end;end
   % the jj loop is for terms having just "b" matrix
   for jj=1:n
   cntr=jj+n*(i-1);
   aii(nmbr,cntr)=l1*b2(j,jj)+aii(nmbr,cntr);
   amm(nmbr,cntr)=b2(j,jj)+amm(nmbr,cntr);
   amn(nmbr,cntr)=-b1(j,jj)+amn(nmbr,cntr);
   anm(nmbr,cntr)=b1(j,jj)+anm(nmbr,cntr);
   ann(nmbr,cntr)=-kp*b4(j,jj)-2*kp*b2(j,jj)+ann(nmbr,cntr);
end;end;end
```

EDD5.m

```
% In this part, the coefficients are placed in their real place in the matrix
% form of the equations. the inputs to this part are m,n,a**.
% the outputs are db, dd matrices
dd=zeros(n*(3*m-8));db=zeros(n*(3*m-8),8*n);
nm1=n*(m-1);nm2=n*(m-2);nm4=n*(m-4);
for i=1:nm2
for j=1:nm2
ni=n+i;nj=n+j;
dd(i,j)=aii(ni,nj);
dd(i,nm2+j)=aim(ni,nj);
dd(nm2+i,j)=ami(ni,nj);
dd(nm2+i,nm2+j)=amm(ni,nj);
end;end
for i=1:nm2
for j=1:nm4
ni=n+i;n2j=2*n+j;
```

```
dd(i,2*nm2+j)=ain(ni,n2j);
dd(nm2+i,2*nm2+j)=amn(ni,n2j);
dd(2*nm2+j,i)=ani(n2j,ni);
dd(2*nm2+j,nm2+i)=anm(n2j,ni);
end;end
for i=1:nm4
for j=1:nm4
dd(2*nm2+i,2*nm2+j)=ann(2*n+i,2*n+j);
end;end
for i=1:nm2
for j=1:n
ni=n+i;
db(i,j)=aii(ni,j);db(i,j+n)=aii(ni,nm1+j);
db(i,j+2*n)=aim(ni,j);db(i,j+3*n)=aim(ni,nm1+j);
db(i,j+4*n)=ain(ni,j);db(i,j+5*n)=ain(ni,nm1+j);
db(i,j+6*n)=ain(ni,j+n);db(i,j+7*n)=ain(ni,nm2+j);
db(i+nm2,j)=ami(ni,j);db(i+nm2,j+n)=ami(ni,nm1+j);
db(i+nm2,j+2*n)=amm(ni,j);db(i+nm2,j+3*n)=amm(ni,nm1+j);
db(i+nm2,j+4*n)=amn(ni,j);db(i+nm2,j+5*n)=amn(ni,nm1+j);
db(i+nm2,j+6*n)=amn(ni,j+n);db(i+nm2,j+7*n)=amn(ni,nm2+j);
end;end
for i=1:nm4
for j=1:n
n2i=2*n+i;
db(i+2*nm2,j)=ani(n2i,j);db(i+2*nm2,n+j)=ani(n2i,nm1+j);
db(i+2*nm2,2*n+j)=anm(n2i,j);db(i+2*nm2,3*n+j)=anm(n2i,nm1+j);
db(i+2*nm2,4*n+j)=ann(n2i,j);db(i+2*nm2,5*n+j)=ann(n2i,nm1+j);
db(i+2*nm2,6*n+j)=ann(n2i,j+n);db(i+2*nm2,7*n+j)=ann(n2i,nm2+j);
end;end
```

FBC6.m

```
% This part determines the boundary condition of the problem.the inputs are
% m,n,a1-a4,b1-b4 and various elements of a** depending
% on the boundary condition.
% the following is for CLAMPED-FREE boundary condition.
bb=zeros(n*8);bd=zeros(n*8,n*(3*m-8));
%   ****   The bb matrix   ****
% u, v, w at (1,*) equal to zero
for i=1:n
bb(i,i)=1;bb(i+2*n,i+2*n)=1;bb(i+4*n,i+4*n)=1;
end
% derivative w.r.t. "x"
for i=1:n
bb(n+i,i)=a1(m,1);               bb(n+i,n+i)=a1(m,m);
bb(n+i,4*n+i)=kp*a2(m,1);        bb(n+i,5*n+i)=-pr+kp*a2(m,m);
bb(n+i,6*n+i)=kp*a2(m,2);        bb(n+i,7*n+i)=kp*a2(m,m-1);
bb(3*n+i,2*n+i)=(1+3*kp)*a1(m,1);bb(3*n+i,3*n+i)=(1+3*kp)*a1(m,m);
bb(5*n+i,i)=a1(m,1);             bb(5*n+i,n+i)=a1(m,m);
bb(5*n+i,4*n+i)=a2(m,1);         bb(5*n+i,5*n+i)=a2(m,m);
bb(5*n+i,6*n+i)=a2(m,2);         bb(5*n+i,7*n+i)=a2(m,m-1);
bb(6*n+i,4*n+i)=a1(1,1);         bb(6*n+i,5*n+i)=a1(1,m);
bb(6*n+i,6*n+i)=a1(1,2);         bb(6*n+i,7*n+i)=a1(1,m-1);
bb(7*n+i,i)=a2(m,1);             bb(7*n+i,n+i)=a2(m,m);
bb(7*n+i,4*n+i)=a3(m,1);         bb(7*n+i,5*n+i)=a3(m,m);
bb(7*n+i,6*n+i)=a3(m,2);         bb(7*n+i,7*n+i)=a3(m,m-1);
end
% terms with derivatives in tet direction
for i=1:n
```

```
 for j=1:n
 bb(n+i,3*n+j)=pr*b1(i,j);          bb(3*n+i,n+j)=b1(i,j);
 bb(5*n+i,3*n+j)=pr*b1(i,j);
 bb(5*n+i,5*n+j)=bb(5*n+i,5*n+j)+pr*b2(i,j);
 bb(7*n+i,n+j)=bb(7*n+i,n+j)-(1-pr)*b2(i,j)/2;
 % terms with derivatives in both directions
 bb(3*n+i,4*n+j)=3*kp*a1(m,1)*b1(i,j);
 bb(3*n+i,5*n+j)=3*kp*a1(m,m)*b1(i,j);
 bb(3*n+i,6*n+j)=3*kp*a1(m,2)*b1(i,j);
 bb(3*n+i,7*n+j)=3*kp*a1(m,m-1)*b1(i,j);
 bb(7*n+i,2*n+j)=(3-pr)*a1(m,1)*b1(i,j)/2;
 bb(7*n+i,3*n+j)=(3-pr)*a1(m,m)*b1(i,j)/2;
 bb(7*n+i,4*n+j)=bb(7*n+i,4*n+j)+(2-pr)*a1(m,1)*b2(i,j);
 bb(7*n+i,5*n+j)=bb(7*n+i,5*n+j)+(2-pr)*a1(m,m)*b2(i,j);
 bb(7*n+i,6*n+j)=bb(7*n+i,6*n+j)+(2-pr)*a1(m,2)*b2(i,j);
 bb(7*n+i,7*n+j)=bb(7*n+i,7*n+j)+(2-pr)*a1(m,m-1)*b2(i,j);
 end;end
 %
 %                 ****   The bd matrix   ****
 %  for the first two series of columns related to "u"and "v"
 for i=2:m-1
 for j=1:n
 bd(n+j,j+(i-2)*n)=a1(m,i);
 bd(3*n+j,j+(i+m-4)*n)=(1+3*kp)*a1(m,i);
 bd(5*n+j,j+(i-2)*n)=a1(m,i);
 bd(7*n+j,j+(i-2)*n)=a2(m,i);
 % this inner loop is for combined derivations
  for ii=1:n
  bd(7*n+j,(m+i-4)*n+ii)=(3-pr)*a1(m,i)*b1(j,ii)/2;
  end
 end;end
 % for the last series of columns related to "w"
 for i=3:m-2
 for j=1:n
 bd(n+j,2*n*(m-2)+j+(i-3)*n)=kp*a2(m,i);
 bd(5*n+j,2*n*(m-2)+j+(i-3)*n)=a2(m,i);
 bd(6*n+j,2*n*(m-2)+j+(i-3)*n)=a1(1,i);
 bd(7*n+j,2*n*(m-2)+j+(i-3)*n)=a3(m,i);
 % this inner loop is for combined derivations
  for ii=1:n
  bd(3*n+j,2*n*(m-2)+(i-3)*n+ii)=3*kp*a1(m,i)*b1(j,ii);
  bd(7*n+j,2*n*(m-2)+(i-3)*n+ii)=bd(7*n+j,2*n*(m-2)+(i-3)*n+ii)+(2-
pr)*a1(m,i)*b2(j,ii);
  end
 end;end
```

GLOAD7.m

```
% this part of program finds the pressure load applied at each point on the
% mesh. The unequal spacing with one delta point at each end is considered in
% longitudinal direction. The equal spacing is considered in teta direction.
% The inputs for this part are the No. of sampling points "m,n" and the
% Rh ratio of the shell, l, R, x(i) ,modulus of elasticity(E), accel
% the outputs are qps(i,j), fqps and qpd(i,j), fqpd(static and dynamic loads).
fqps=zeros(3*m*n,1);   fqpd=zeros(3*m*n,1);
for i=1:m
for j=1:n
% the hydrostatic part
qps(i,j)=(-1)*1000*9.8*(1-x(i))*(1-pr^2)*rh*lngth/E;
```

```
% the hydrodynamic (equivalent static part
londistr=-.886*x(i)^3+.0421*x(i)^2+.2009*x(i)+.6516;
qpd(i,j)=-londistr*cos(tet(j))*1000*(1-pr^2)*rh*r*accel/E;
end;end
% the static force matrix
for i=3:(m-2)
for j=1:n
fqps((2*m+4)*n+j+(i-3)*n)=-1*qps(i,j);
end;end
% the dynamic force matrix
for i=3:(m-2)
for j=1:n
fqpd((2*m+4)*n+j+(i-3)*n)=-1*qpd(i,j);
end;end
```

HSOLVER8.m

```
% In this part the stiffness matrix is made
% and then the linear equations are solved for
% Displacements "u, v, w" for each loading.
for i=1:8*n
for j=1:8*n
stiff(i,j)=bb(i,j);
end;end
for i=1:8*n
for j=1:(3*m-8)*n
stiff(i,8*n+j)=bd(i,j);
stiff(8*n+j,i)=db(j,i);
end;end
for i=1:(3*m-8)*n
for j=1:(3*m-8)*n
stiff(8*n+i,8*n+j)=dd(i,j);
end;end
disps=stiff\fqps;
dispd=stiff\fqpd;
```

PROVIDE9.m

```
% this program finds the derivations of displacements and also finds the
% required stress and moment resultants due to each of prebuckling loadings.
% then they are placed in an m*n matrix from the original vector form.
% boundary displacements
for i=1:n
us(1,i)=disps(i);           us(m,i)=disps(n+i);
vs(1,i)=disps(2*n+i);       vs(m,i)=disps(3*n+i);
ws(1,i)=disps(4*n+i);       ws(m,i)=disps(5*n+i);
ws(2,i)=disps(6*n+i);       ws(m-1,i)=disps(7*n+i);
ud(1,i)=dispd(i);           ud(m,i)=dispd(n+i);
vd(1,i)=dispd(2*n+i);       vd(m,i)=dispd(3*n+i);
wd(1,i)=dispd(4*n+i);       wd(m,i)=dispd(5*n+i);
wd(2,i)=dispd(6*n+i);       wd(m-1,i)=dispd(7*n+i);
end
for i=2:m-1
for j=1:n
us(i,j)=disps((6+i)*n+j); vs(i,j)=disps((4+m+i)*n+j);
ud(i,j)=dispd((6+i)*n+j); vd(i,j)=dispd((4+m+i)*n+j);
end;end
```

```
for i=3:m-2
for j=1:n
ws(i,j)=disps((1+2*m+i)*n+j);
wd(i,j)=dispd((1+2*m+i)*n+j);
end;end
% Derivatives of u displacements
usf1=a1*us; usf2=a2*us; ust1=us*b1'; ust2=us*b2'; usft=a1*(us*b1');
udf1=a1*ud; udf2=a2*ud; udt1=ud*b1'; udt2=ud*b2'; udft=a1*(ud*b1');
% Derivatives of v displacements
vsf1=a1*vs; vsf2=a2*vs; vst1=vs*b1'; vst2=vs*b2'; vsft=a1*(vs*b1');
vdf1=a1*vd; vdf2=a2*vd; vdt1=vd*b1'; vdt2=vd*b2'; vdft=a1*(vd*b1');
% Derivatives of w displacements
wsf1=a1*ws ; wsf2=a2*ws ; wsf3=a3*ws ;
wst1=ws*b1'; wst2=ws*b2'; wst3=ws*b3';
wsft=a1*(ws*b1') ; wsft2=a1*(ws*b2') ; wsf2t=a2*(ws*b1');
wdf1=a1*wd ; wdf2=a2*wd ; wdf3=a3*wd ;
wdt1=wd*b1'; wdt2=wd*b2'; wdt3=wd*b3';
wdft=a1*(wd*b1') ; wdft2=a1*(wd*b2') ; wdf2t=a2*(wd*b1');
% Prebuckling stress resultants
nx0s=usf1+pr*(vst1-ws)+kp*wsf2;                nx0d=udf1+pr*(vdt1-wd)+kp*wdf2;
nx0xs=usf2+pr*(vsft-wsf1)+kp*wsf3;             nx0xd=udf2+pr*(vdft-wdf1)+kp*wdf3;
ny0s=vst1-ws+pr*usf1-kp*(wst2+ws);             ny0d=vdt1-wd+pr*udf1-kp*(wdt2+wd);
ny0ys=vst2-wst1+pr*usft-kp*(wst3+wst1);        ny0yd=vdt2-wdt1+pr*udft-
kp*(wdt3+wdt1);
nxy0s=l7*(ust1+vsf1+kp*(vsf1+wsft));           nxy0d=l7*(udt1+vdf1+kp*(vdf1+wdft));
nxy0xs=l7*(usft+vsf2+kp*(vsf2+wsf2t));
nxy0xd=l7*(udft+vdf2+kp*(vdf2+wdf2t));
nyx0s=l7*(ust1+vsf1+kp*(ust1-wsft));           nyx0d=l7*(udt1+vdf1+kp*(udt1-wdft));
nyx0ys=l7*(ust2+vsft+kp*(ust2-wsft2));         nyx0yd=l7*(udt2+vdft+kp*(udt2-
wdft2));
```

F1AII.m

```
% in this part of program the aii,aim,...,ann matrices which generate the dqm
% analogue of the equations are made. the ai* series are for 1st, am* for 2nd
% and an* series are for 3rd eqns respectively. a*i is for U components, a*m
% for V and a*n for W displacements. the inputs to this part are m,n(no of
% sampling points in x and teta dir) and a1,..a4 and b1,..b4 matrices.
% and also prebuckling displacements and their derivatives.
% the aii,aim,...,ann are outputs of the program.
l1=(1+kp)*(1-pr)/2 ; l2=(1+pr)/2 ; l3=kp*(1-pr)/2;
l4=(1-pr)/2+3*kp*(1-pr)/2  ;  l5=(3-pr)*kp/2;
l6=3*kp*(1-pr)/2 ; l7=(1-pr)/2 ; l8=(3-pr)/2;
l9=l3+l8 ; l10=(1-pr)*(1+kp/2) ; l11=2-pr;
l12=1-pr+kp ; l13=l3+(3-pr)/2; l14=(3*pr-1)/2;
aii(:,:)=zeros(m*n);aim(:,:)=zeros(m*n);ain(:,:)=zeros(m*n);
ami(:,:)=zeros(m*n);amm(:,:)=zeros(m*n);amn(:,:)=zeros(m*n);
ani(:,:)=zeros(m*n);anm(:,:)=zeros(m*n);ann(:,:)=zeros(m*n);
% these two i and j loops determine the point (i,j) at which the
% equation is going to be satisfied.
for i=1:m
  for j=1:n
  nmbr=(i-1)*n+j;
  % the following amounts are related to terms appearing as pure
  % variable(without any kind of derivation) in the diff. equation
  aim(nmbr,nmbr)=aim(nmbr,nmbr)+l2*wsft(i,j)+pr*vsf1(i,j);
  ain(nmbr,nmbr)=ain(nmbr,nmbr)-l2*vsft(i,j)+pr*wsf1(i,j)-pr*usf2(i,j)-
(1+kp)*ust2(i,j);
  amm(nmbr,nmbr)=amm(nmbr,nmbr)+l7*wsf2(i,j)+wst2(i,j)+vst1(i,j)-ny0s(i,j)-
```

```
qps(i,j);
  amn(nmbr,nmbr)=amn(nmbr,nmbr)-(2+kp)*vst2(i,j)-
12*vsf2(i,j)+3*wst1(i,j)+vs(i,j)+kp*(vs(i,j)+2*wst1(i,j))-ny0ys(i,j)-
nxy0xs(i,j);
  anm(nmbr,nmbr)=anm(nmbr,nmbr)+wst1(i,j)+vs(i,j)+ny0ys(i,j)+nxy0xs(i,j);
  ann(nmbr,nmbr)=ann(nmbr,nmbr)-kp-1-3*vst1(i,j)+2*ws(i,j)-wst2(i,j)-
pr*wsf2(i,j)-kp*(2*vst1(i,j)-ws(i,j)+wst2(i,j))-qps(i,j);
  % the ii loop is for terms having just "a" matrix
    for ii=1:m
    cntr=(ii-1)*n+j;

aii(nmbr,cntr)=aii(nmbr,cntr)+a1(i,ii)*(2*usf2(i,j)+12*ust2(i,j)+nx0xs(i,j)+ny
x0ys(i;j));
  aii(nmbr,cntr)=aii(nmbr,cntr)+a2(i,ii)*(1+2*usf1(i,j)+nx0s(i,j));

aim(nmbr,cntr)=aim(nmbr,cntr)+a1(i,ii)*(vsf2(i,j)+114*wst1(i,j)+pr*vs(i,j)+17*
vst2(i,j)+110*usft(i,j));
  aim(nmbr,cntr)=aim(nmbr,cntr)+a2(i,ii)*(vsf1(i,j)+11*ust1(i,j));

ain(nmbr,cntr)=ain(nmbr,cntr)+a3(i,ii)*kp*(1+usf1(i,j))+a2(i,ii)*(wsf1(i,j)+kp
*usf2(i,j));
  ain(nmbr,cntr)=ain(nmbr,cntr)+a1(i,ii)*(wsf2(i,j)-pr*(1+vst1(i,j)-
ws(i,j)+usf1(i,j))+17*(wst2(i,j)+vst1(i,j))-qps(i,j));

ami(nmbr,cntr)=ami(nmbr,cntr)+a1(i,ii)*(12*usft(i,j)+vsf1(i,j)+pr*(vst2(i,j)-
2*wst1(i,j)-vs(i,j)));
  ami(nmbr,cntr)=ami(nmbr,cntr)+a2(i,ii)*(17*ust1(i,j)+vsf1(i,j));
  amm(nmbr,cntr)=amm(nmbr,cntr)+a2(i,ii)*(14+112*(vst1(i,j)-
ws(i,j))+nx0s(i,j));
  amm(nmbr,cntr)=amm(nmbr,cntr)+a1(i,ii)*(113*vsft(i,j)-
110*wsf1(i,j)+nx0xs(i,j)+nyx0ys(i,j));
  amn(nmbr,cntr)=amn(nmbr,cntr)+a1(i,ii)*(12*wsft(i,j)-pr*vsf1(i,j)-
nxy0s(i,j)-nyx0s(i,j));

amn(nmbr,cntr)=amn(nmbr,cntr)+a2(i,ii)*(17*(wst1(i,j)+vs(i,j))+kp*vsf2(i,j))+a
3(i,ii)*kp*vsf1(i,j);
  ani(nmbr,cntr)=ani(nmbr,cntr)-a3(i,ii)*kp+a2(i,ii)*wsf1(i,j);
  ani(nmbr,cntr)=ani(nmbr,cntr)+a1(i,ii)*(pr+wsf2(i,j)+pr*(wst2(i,j)-ws(i,j)-
2*vst1(i,j)+usf1(i,j))+qps(i,j));
  anm(nmbr,cntr)=anm(nmbr,cntr)+a2(i,ii)*11*(wst1(i,j)+vs(i,j));

anm(nmbr,cntr)=anm(nmbr,cntr)+a1(i,ii)*((1+13)*vsf1(i,j)+110*wsft(i,j)+nxy0s(i
,j)+nyx0s(i,j));

ann(nmbr,cntr)=ann(nmbr,cntr)+a3(i,ii)*kp*wsf1(i,j)+a2(i,ii)*(kp*wsf2(i,j)+nx0
s(i,j));
  ann(nmbr,cntr)=ann(nmbr,cntr)+a1(i,ii)*(nx0xs(i,j)+nyx0ys(i,j))-kp*a4(i,ii);
    % this inner ij loop is for terms of "a*b" matrices
      for ij=1:n
      cntr=(ii-1)*n+ij;

aii(nmbr,cntr)=aii(nmbr,cntr)+a1(i,ii)*b1(j,ij)*((1+pr)*ust1(i,j)+nyx0s(i,j)+n
xy0s(i,j));
      aim(nmbr,cntr)=aim(nmbr,cntr)+a1(i,ii)*b1(j,ij)*12*(1-
ws(i,j)+vst1(i,j)+usf1(i,j));
      ain(nmbr,cntr)=ain(nmbr,cntr)+13*(a2(i,ii)*b1(j,ij)*ust1(i,j)-
a1(i,ii)*b2(j,ij)*(1+usf1(i,j)));
      ami(nmbr,cntr)=ami(nmbr,cntr)+a1(i,ii)*b1(j,ij)*12*(1+vst1(i,j)-
ws(i,j)+usf1(i,j));

amm(nmbr,cntr)=amm(nmbr,cntr)+a1(i,ii)*b1(j,ij)*((1+pr)*vsf1(i,j)+nxy0s(i,j)+n
```

```
yx0s(i,j));
        amn(nmbr,cntr)=amn(nmbr,cntr)+a2(i,ii)*b1(j,ij)*(15+13*(vst1(i,j)-
ws(i,j)))-a1(i,ii)*b2(j,ij)*13*vsf1(i,j);

ani(nmbr,cntr)=ani(nmbr,cntr)+a1(i,ii)*b2(j,ij)*13+a1(i,ii)*b1(j,ij)*pr*(wst1(
i,j)+vs(i,j));
        anm(nmbr,cntr)=anm(nmbr,cntr)-
a2(i,ii)*b1(j,ij)*15+a1(i,ii)*b1(j,ij)*17*wsf1(i,j);
        ann(nmbr,cntr)=ann(nmbr,cntr)-
2*kp*a2(i,ii)*b2(j,ij)+13*(a2(i,ii)*b1(j,ij)*(wst1(i,j)+vs(i,j))-
a1(i,ii)*b2(j,ij)*wsf1(i,j));
        ann(nmbr,cntr)=ann(nmbr,cntr)+a1(i,ii)*b1(j,ij)*(nxy0s(i,j)+nyx0s(i,j));
     end;end
   % the jj loop is for terms having just "b" matrix
   for jj=1:n
   cntr=jj+n*(i-1);
   aii(nmbr,cntr)=aii(nmbr,cntr)+b2(j,jj)*(11+110*usf1(i,j)+ny0s(i,j));

aii(nmbr,cntr)=aii(nmbr,cntr)+b1(j,jj)*(113*usft(i,j)+ny0ys(i,j)+nxy0xs(i,j));
   aim(nmbr,cntr)=aim(nmbr,cntr)+b1(j,jj)*(pr*usf2(i,j)+ust2(i,j)+12*vsft(i,j)-
114*ws(i,j));
   aim(nmbr,cntr)=aim(nmbr,cntr)+b2(j,jj)*(17*vsf1(i,j)+ust1(i,j));
   ain(nmbr,cntr)=ain(nmbr,cntr)+b2(j,jj)*(17*wsf1(i,j)-kp*ust2(i,j))-
b3(j,jj)*kp*ust1(i,j);
   ain(nmbr,cntr)=ain(nmbr,cntr)+b1(j,jj)*(12*wsft(i,j)+114*vs(i,j)-
(1+kp)*ust1(i,j));
   ami(nmbr,cntr)=ami(nmbr,cntr)+b1(j,jj)*(110*(vsft(i,j)-
wsf1(i,j))+17*usf2(i,j)+ust2(i,j));
   ami(nmbr,cntr)=ami(nmbr,cntr)+b2(j,jj)*(11*vsf1(i,j)+ust1(i,j));
   amm(nmbr,cntr)=amm(nmbr,cntr)+b2(j,jj)*(1+2*vst1(i,j)-2*ws(i,j)-ny0s(i,j));
   amm(nmbr,cntr)=amm(nmbr,cntr)+b1(j,jj)*(2*vst2(i,j)+12*vsf2(i,j)-
2*wst1(i,j)+nxy0xs(i,j)+ny0ys(i,j));
   amn(nmbr,cntr)=amn(nmbr,cntr)+b1(j,jj)*(-1-qps(i,j)+17*wsf2(i,j)+wst2(i,j)-
(1+kp)*vst1(i,j)+(2+kp)*ws(i,j)-2*ny0s(i,j));
   amn(nmbr,cntr)=amn(nmbr,cntr)+b2(j,jj)*(wst1(i,j)+vs(i,j)-kp*(vst2(i,j)-
2*wst1(i,j)-vs(i,j)))+b3(j,jj)*kp*(ws(i,j)-vst1(i,j));

ani(nmbr,cntr)=ani(nmbr,cntr)+b2(j,jj)*11*wsf1(i,j)+b1(j,jj)*(ust1(i,j)+110*(w
sft(i,j)+vsf1(i,j)));

anm(nmbr,cntr)=anm(nmbr,cntr)+b1(j,jj)*(1+wst2(i,j)+pr*wsf2(i,j)+3*vst1(i,j)-
2*ws(i,j)+qps(i,j)+2*ny0s(i,j))+b2(j,jj)*(wst1(i,j)+vs(i,j));
   ann(nmbr,cntr)=ann(nmbr,cntr)-kp*b4(j,jj)+b2(j,jj)*(ny0s(i,j)-
kp*(2+2*vst1(i,j)+wst2(i,j)-ws(i,j)))-b3(j,jj)*kp*(wst1(i,j)+vs(i,j));
   ann(nmbr,cntr)=ann(nmbr,cntr)+b1(j,jj)*(nxy0xs(i,j)+ny0ys(i,j)-
kp*(wst1(i,j)+vs(i,j)));
end;end;end
```

F2DII.m

```
% in this part of program the dii,dim,...,dnn matrices which generate the
R.H.S of dqm
% analogue of the equations are made.the di* series are for 1st, dm* for 2nd
% and dn* series are for 3rd eqns respectively. d*i is for U components, d*m
% for V and d*n for W displacements. the inputs to this part are m,n(no of
% sampling points in x and teta dir) and a1,..a4 and b1,..b4 matrices.
% and also prebuckling displacements and their derivatives.
% the dii,dim,...,dnn are outputs of the program.
11=(1+kp)*(1-pr)/2 ; 12=(1+pr)/2 ; 13=kp*(1-pr)/2;
```

```
 l4=(1-pr)/2+3*kp*(1-pr)/2  ;  l5=(3-pr)*kp/2;
 l6=3*kp*(1-pr)/2 ; l7=(1-pr)/2 ; l8=(3-pr)/2;
 l9=l3+l8 ; l10=(1-pr)*(1+kp/2) ; l11=2-pr;
 l12=1-pr+kp ; l13=l3+(3-pr)/2; l14=(3*pr-1)/2;
 dii(:,:)=zeros(m*n);dim(:,:)=zeros(m*n);din(:,:)=zeros(m*n);
 dmi(:,:)=zeros(m*n);dmm(:,:)=zeros(m*n);dmn(:,:)=zeros(m*n);
 dni(:,:)=zeros(m*n);dnm(:,:)=zeros(m*n);dnn(:,:)=zeros(m*n);
 % these two i and j loops determine the point (i,j) at which the
 % equation is going to be satisfied.
 for i=1:m
   for j=1:n
   nmbr=(i-1)*n+j;
   % the following amounts are related to terms appearing as pure
   % variable(without any kind of derivation) in the diff. equation
   dim(nmbr,nmbr)=dim(nmbr,nmbr)-l2*wdft(i,j)-pr*vdf1(i,j);
   din(nmbr,nmbr)=din(nmbr,nmbr)+l2*vdft(i,j)-
pr*wdf1(i,j)+pr*udf2(i,j)+(1+kp)*udt2(i,j);
   dmm(nmbr,nmbr)=dmm(nmbr,nmbr)-l7*wdf2(i,j)-wdt2(i,j)-
vdt1(i,j)+ny0d(i,j)+qpd(i,j);
   dmn(nmbr,nmbr)=dmn(nmbr,nmbr)+(2+kp)*vdt2(i,j)+l2*vdf2(i,j)-3*wdt1(i,j)-
vd(i,j)-kp*(vd(i,j)+2*wdt1(i,j))+ny0yd(i,j)+nxy0xd(i,j);
   dnm(nmbr,nmbr)=dnm(nmbr,nmbr)-wdt1(i,j)-vd(i,j)-ny0yd(i,j)-nxy0xd(i,j);
   dnn(nmbr,nmbr)=dnn(nmbr,nmbr)+3*vdt1(i,j)-
2*wd(i,j)+wdt2(i,j)+pr*wdf2(i,j)+kp*(2*vdt1(i,j)-wd(i,j)+wdt2(i,j))+qpd(i,j);
   % the ii loop is for terms having just "a" matrix
     for ii=1:m
     cntr=(ii-1)*n+j;
   dii(nmbr,cntr)=dii(nmbr,cntr)-
a1(i,ii)*(2*udf2(i,j)+l2*udt2(i,j)+nx0xd(i,j)+nyx0yd(i,j));
   dii(nmbr,cntr)=dii(nmbr,cntr)-a2(i,ii)*(2*udf1(i,j)+nx0d(i,j));
   dim(nmbr,cntr)=dim(nmbr,cntr)-
a1(i,ii)*(vdf2(i,j)+l14*wdt1(i,j)+pr*vd(i,j)+l7*vdt2(i,j)+l10*udft(i,j));
   dim(nmbr,cntr)=dim(nmbr,cntr)-a2(i,ii)*(vdf1(i,j)+l1*udt1(i,j));
   din(nmbr,cntr)=din(nmbr,cntr)-a3(i,ii)*kp*(udf1(i,j))-
a2(i,ii)*(wdf1(i,j)+kp*udf2(i,j));
   din(nmbr,cntr)=din(nmbr,cntr)-a1(i,ii)*(wdf2(i,j)-pr*(vdt1(i,j)-
wd(i,j)+udf1(i,j))+l7*(wdt2(i,j)+vdt1(i,j))-qpd(i,j));
   dmi(nmbr,cntr)=dmi(nmbr,cntr)-
a1(i,ii)*(l2*udft(i,j)+vdf1(i,j)+pr*(vdt2(i,j)-2*wdt1(i,j)-vd(i,j)));
   dmi(nmbr,cntr)=dmi(nmbr,cntr)-a2(i,ii)*(l7*udt1(i,j)+vdf1(i,j));
   dmm(nmbr,cntr)=dmm(nmbr,cntr)-a2(i,ii)*(l12*(vdt1(i,j)-wd(i,j))+nx0d(i,j));
   dmm(nmbr,cntr)=dmm(nmbr,cntr)-a1(i,ii)*(l13*vdft(i,j)-
l10*wdf1(i,j)+nx0xd(i,j)+nyx0yd(i,j));
   dmn(nmbr,cntr)=dmn(nmbr,cntr)-a1(i,ii)*(l2*wdft(i,j)-pr*vdf1(i,j)-
nxy0d(i,j)-nyx0d(i,j));
   dmn(nmbr,cntr)=dmn(nmbr,cntr)-
a2(i,ii)*(l7*(wdt1(i,j)+vd(i,j))+kp*vdf2(i,j))-a3(i,ii)*kp*vdf1(i,j);
   dni(nmbr,cntr)=dni(nmbr,cntr)-a2(i,ii)*wdf1(i,j);
   dni(nmbr,cntr)=dni(nmbr,cntr)-a1(i,ii)*(wdf2(i,j)+pr*(wdt2(i,j)-wd(i,j)-
2*vdt1(i,j)+udf1(i,j))+qpd(i,j));
   dnm(nmbr,cntr)=dnm(nmbr,cntr)-a2(i,ii)*l1*(wdt1(i,j)+vd(i,j));
   dnm(nmbr,cntr)=dnm(nmbr,cntr)-
a1(i,ii)*((1+l3)*vdf1(i,j)+l10*wdft(i,j)+nxy0d(i,j)+nyx0d(i,j));
   dnn(nmbr,cntr)=dnn(nmbr,cntr)-a3(i,ii)*kp*wdf1(i,j)-
a2(i,ii)*(kp*wdf2(i,j)+nx0d(i,j));
   dnn(nmbr,cntr)=dnn(nmbr,cntr)-a1(i,ii)*(nx0xd(i,j)+nyx0yd(i,j));
     % this inner ij loop is for terms of "a*b" matrices
       for ij=1:n
       cntr=(ii-1)*n+ij;
       dii(nmbr,cntr)=dii(nmbr,cntr)-
a1(i,ii)*b1(j,ij)*((1+pr)*udt1(i,j)+nyx0d(i,j)+nxy0d(i,j));
```

```
      dim(nmbr,cntr)=dim(nmbr,cntr)-a1(i,ii)*b1(j,ij)*12*(vdt1(i,j)-
wd(i,j)+udf1(i,j));
      din(nmbr,cntr)=din(nmbr,cntr)-13*(a2(i,ii)*b1(j,ij)*udt1(i,j)-
a1(i,ii)*b2(j,ij)*udf1(i,j));
      dmi(nmbr,cntr)=dmi(nmbr,cntr)-a1(i,ii)*b1(j,ij)*12*(vdt1(i,j)-
wd(i,j)+udf1(i,j));
      dmm(nmbr,cntr)=dmm(nmbr,cntr)-
a1(i,ii)*b1(j,ij)*((1+pr)*vdf1(i,j)+nxy0d(i,j)+nyx0d(i,j));
      dmn(nmbr,cntr)=dmn(nmbr,cntr)-a2(i,ii)*b1(j,ij)*13*(vdt1(i,j)-
wd(i,j))+a1(i,ii)*b2(j,ij)*13*vdf1(i,j);
      dni(nmbr,cntr)=dni(nmbr,cntr)-a1(i,ii)*b1(j,ij)*pr*(wdt1(i,j)+vd(i,j));
      dnm(nmbr,cntr)=dnm(nmbr,cntr)-a1(i,ii)*b1(j,ij)*17*wdf1(i,j);
      dnn(nmbr,cntr)=dnn(nmbr,cntr)-13*(a2(i,ii)*b1(j,ij)*(wdt1(i,j)+vd(i,j))-
a1(i,ii)*b2(j,ij)*wdf1(i,j));
      dnn(nmbr,cntr)=dnn(nmbr,cntr)-a1(i,ii)*b1(j,ij)*(nxy0d(i,j)+nyx0d(i,j));
    end;end
   % the jj loop is for terms having just "b" matrix
   for jj=1:n
   cntr=jj+n*(i-1);
   dii(nmbr,cntr)=dii(nmbr,cntr)-b2(j,jj)*(110*udf1(i,j)+ny0d(i,j));
   dii(nmbr,cntr)=dii(nmbr,cntr)-
b1(j,jj)*(113*udft(i,j)+ny0yd(i,j)+nxy0xd(i,j));
   dim(nmbr,cntr)=dim(nmbr,cntr)-b1(j,jj)*(pr*udf2(i,j)+udt2(i,j)+12*vdft(i,j)-
114*wd(i,j));
   dim(nmbr,cntr)=dim(nmbr,cntr)-b2(j,jj)*(17*vdf1(i,j)+udt1(i,j));
   din(nmbr,cntr)=din(nmbr,cntr)-b2(j,jj)*(17*wdf1(i,j)-
kp*udt2(i,j))+b3(j,jj)*kp*udt1(i,j);
   din(nmbr,cntr)=din(nmbr,cntr)-b1(j,jj)*(12*wdft(i,j)+114*vd(i,j)-
(1+kp)*udt1(i,j));
   dmi(nmbr,cntr)=dmi(nmbr,cntr)-b1(j,jj)*(110*(vdft(i,j)-
wdf1(i,j))+17*udf2(i,j)+udt2(i,j));
   dmi(nmbr,cntr)=dmi(nmbr,cntr)-b2(j,jj)*(11*vdf1(i,j)+udt1(i,j));
   dmm(nmbr,cntr)=dmm(nmbr,cntr)-b2(j,jj)*(2*vdt1(i,j)-2*wd(i,j)-ny0d(i,j));
   dmm(nmbr,cntr)=dmm(nmbr,cntr)-b1(j,jj)*(2*vdt2(i,j)+12*vdf2(i,j)-
2*wdt1(i,j)+nxy0xd(i,j)+ny0yd(i,j));
   dmn(nmbr,cntr)=dmn(nmbr,cntr)-b1(j,jj)*(-qpd(i,j)+17*wdf2(i,j)+wdt2(i,j)-
(1+kp)*vdt1(i,j)+(2+kp)*wd(i,j)-2*ny0d(i,j));
   dmn(nmbr,cntr)=dmn(nmbr,cntr)-b2(j,jj)*(wdt1(i,j)+vd(i,j)-kp*(vdt2(i,j)-
2*wdt1(i,j)-vd(i,j)))-b3(j,jj)*kp*(wd(i,j)-vdt1(i,j));
   dni(nmbr,cntr)=dni(nmbr,cntr)-b2(j,jj)*11*wdf1(i,j)-
b1(j,jj)*(udt1(i,j)+110*(wdft(i,j)+vdf1(i,j)));
   dnm(nmbr,cntr)=dnm(nmbr,cntr)-b1(j,jj)*(wdt2(i,j)+pr*wdf2(i,j)+3*vdt1(i,j)-
2*wd(i,j)+qpd(i,j)+2*ny0d(i,j))-b2(j,jj)*(wdt1(i,j)+vd(i,j));
   dnn(nmbr,cntr)=dnn(nmbr,cntr)-b2(j,jj)*(ny0d(i,j)-kp*(2*vdt1(i,j)+wdt2(i,j)-
wd(i,j)))+b3(j,jj)*kp*(wdt1(i,j)+vd(i,j));
   dnn(nmbr,cntr)=dnn(nmbr,cntr)-b1(j,jj)*(nxy0xd(i,j)+ny0yd(i,j)-
kp*(wdt1(i,j)+vd(i,j)));
end;end;end
```

F3DD.m

```
% In this part, the coefficient are placed in their real place in the matrix
% form of the equations. the inputs to this part are m,n,a**,d**.
% the outputs are db, dd, dbg, ddg matrices
dd=zeros(n*(3*m-8));db=zeros(n*(3*m-8),8*n);
ddg=zeros(n*(3*m-8));dbg=zeros(n*(3*m-8),8*n);
nm1=n*(m-1);nm2=n*(m-2);nm4=n*(m-4);
for i=1:nm2
for j=1:nm2
```

```
ni=n+i;nj=n+j;
dd(i,j)=aii(ni,nj);                        ddg(i,j)=dii(ni,nj);
dd(i,nm2+j)=aim(ni,nj);                    ddg(i,nm2+j)=dim(ni,nj);
dd(nm2+i,j)=ami(ni,nj);                    ddg(nm2+i,j)=dmi(ni,nj);
dd(nm2+i,nm2+j)=amm(ni,nj);                ddg(nm2+i,nm2+j)=dmm(ni,nj);
end;end
for i=1:nm2
for j=1:nm4
ni=n+i;n2j=2*n+j;
dd(i,2*nm2+j)=ain(ni,n2j);                 ddg(i,2*nm2+j)=din(ni,n2j);
dd(nm2+i,2*nm2+j)=amn(ni,n2j);             ddg(nm2+i,2*nm2+j)=dmn(ni,n2j);
dd(2*nm2+j,i)=ani(n2j,ni);                 ddg(2*nm2+j,i)=dni(n2j,ni);
dd(2*nm2+j,nm2+i)=anm(n2j,ni);             ddg(2*nm2+j,nm2+i)=dnm(n2j,ni);
end;end
for i=1:nm4
for j=1:nm4
dd(2*nm2+i,2*nm2+j)=ann(2*n+i,2*n+j); ddg(2*nm2+i,2*nm2+j)=dnn(2*n+i,2*n+j);
end;end
for i=1:nm2
for j=1:n
ni=n+i;
db(i,j)=aii(ni,j);                db(i,j+n)=aii(ni,nm1+j);
db(i,j+2*n)=aim(ni,j);            db(i,j+3*n)=aim(ni,nm1+j);
db(i,j+4*n)=ain(ni,j);            db(i,j+5*n)=ain(ni,nm1+j);
db(i,j+6*n)=ain(ni,j+n);          db(i,j+7*n)=ain(ni,nm2+j);
db(i+nm2,j)=ami(ni,j);            db(i+nm2,j+n)=ami(ni,nm1+j);
db(i+nm2,j+2*n)=amm(ni,j);        db(i+nm2,j+3*n)=amm(ni,nm1+j);
db(i+nm2,j+4*n)=amn(ni,j);        db(i+nm2,j+5*n)=amn(ni,nm1+j);
db(i+nm2,j+6*n)=amn(ni,j+n);      db(i+nm2,j+7*n)=amn(ni,nm2+j);
dbg(i,j)=dii(ni,j);               dbg(i,j+n)=dii(ni,nm1+j);
dbg(i,j+2*n)=dim(ni,j);           dbg(i,j+3*n)=dim(ni,nm1+j);
dbg(i,j+4*n)=din(ni,j);           dbg(i,j+5*n)=din(ni,nm1+j);
dbg(i,j+6*n)=din(ni,j+n);         dbg(i,j+7*n)=din(ni,nm2+j);
dbg(i+nm2,j)=dmi(ni,j);           dbg(i+nm2,j+n)=dmi(ni,nm1+j);
dbg(i+nm2,j+2*n)=dmm(ni,j);       dbg(i+nm2,j+3*n)=dmm(ni,nm1+j);
dbg(i+nm2,j+4*n)=dmn(ni,j);       dbg(i+nm2,j+5*n)=dmn(ni,nm1+j);
dbg(i+nm2,j+6*n)=dmn(ni,j+n);dbg(i+nm2,j+7*n)=dmn(ni,nm2+j);
end;end
for i=1:nm4
for j=1:n
n2i=2*n+i;
db(i+2*nm2,j)=ani(n2i,j);          db(i+2*nm2,n+j)=ani(n2i,nm1+j);
db(i+2*nm2,2*n+j)=anm(n2i,j);      db(i+2*nm2,3*n+j)=anm(n2i,nm1+j);
db(i+2*nm2,4*n+j)=ann(n2i,j);      db(i+2*nm2,5*n+j)=ann(n2i,nm1+j);
db(i+2*nm2,6*n+j)=ann(n2i,j+n);db(i+2*nm2,7*n+j)=ann(n2i,nm2+j);
dbg(i+2*nm2,j)=dni(n2i,j);         dbg(i+2*nm2,n+j)=dni(n2i,nm1+j);
dbg(i+2*nm2,2*n+j)=dnm(n2i,j);     dbg(i+2*nm2,3*n+j)=dnm(n2i,nm1+j);
dbg(i+2*nm2,4*n+j)=dnn(n2i,j);     dbg(i+2*nm2,5*n+j)=dnn(n2i,nm1+j);
dbg(i+2*nm2,6*n+j)=dnn(n2i,j+n);dbg(i+2*nm2,7*n+j)=dnn(n2i,nm2+j);
end;end
```

F4BC.m

```
% This part determines the boundary condition of the problem.
% the inputs are m,n, and various elements of a** depending
% on the boundary condition.
% the following is for CLAMPED-FREE boundary condition.
bb=zeros(n*8);bd=zeros(n*8,n*(3*m-8));
```

```
bbg=zeros(n*8);bdg=zeros(n*8,n*(3*m-8));
%    ****   The bb matrix   ****
% u, v, w at (1,*) equal to zero
for i=1:n
bb(i,i)=1;bb(i+2*n,i+2*n)=1;bb(i+4*n,i+4*n)=1;
end
% derivative w.r.t. "x"
for i=1:n
bb(n+i,i)=a1(m,1)*(1+usf1(m,i)); bb(n+i,n+i)=a1(m,m)*(1+usf1(m,i));
bb(n+i,2*n+i)=a1(m,1)*vsf1(m,i);
bb(n+i,3*n+i)=a1(m,m)*vsf1(m,i)+pr*(wst1(m,i)+vs(m,i));
bb(n+i,4*n+i)=kp*a2(m,1)+a1(m,1)*wsf1(m,i);
bb(n+i,5*n+i)=-pr*(1+vst1(m,i)-ws(m,i))+kp*a2(m,m)+a1(m,m)*wsf1(m,i);
bb(n+i,6*n+i)=kp*a2(m,2)+a1(m,2)*wsf1(m,i);
bb(n+i,7*n+i)=kp*a2(m,m-1)+a1(m,m-1)*wsf1(m,i);
bb(3*n+i,i)=a1(m,1)*ust1(m,i);    bb(3*n+i,n+i)=a1(m,m)*ust1(m,i);
bb(3*n+i,2*n+i)=(1+vst1(m,i)-ws(m,i)+3*kp)*a1(m,1);
bb(3*n+i,3*n+i)=(1+vst1(m,i)-ws(m,i)+3*kp)*a1(m,m)+wsf1(m,i);
bb(3*n+i,4*n+i)=a1(m,1)*(wst1(m,i)+vs(m,i));
bb(3*n+i,5*n+i)=a1(m,m)*(wst1(m,i)+vs(m,i))-vsf1(m,i);
bb(3*n+i,6*n+i)=a1(m,2)*(wst1(m,i)+vs(m,i));
bb(3*n+i,7*n+i)=a1(m,m-1)*(wst1(m,i)+vs(m,i));
bb(5*n+i,i)=a1(m,1);                bb(5*n+i,n+i)=a1(m,m);
bb(5*n+i,4*n+i)=a2(m,1);            bb(5*n+i,5*n+i)=a2(m,m);
bb(5*n+i,6*n+i)=a2(m,2);            bb(5*n+i,7*n+i)=a2(m,m-1);
bb(6*n+i,4*n+i)=a1(1,1);            bb(6*n+i,5*n+i)=a1(1,m);
bb(6*n+i,6*n+i)=a1(1,2);            bb(6*n+i,7*n+i)=a1(1,m-1);
bb(7*n+i,i)=a2(m,1);                bb(7*n+i,n+i)=a2(m,m);
bb(7*n+i,4*n+i)=a3(m,1);            bb(7*n+i,5*n+i)=a3(m,m);
bb(7*n+i,6*n+i)=a3(m,2);            bb(7*n+i,7*n+i)=a3(m,m-1);
% ***** Right hand side terms(in matrix form equations) ******
bbg(n+i,i)=-a1(m,1)*udf1(m,i); bbg(n+i,n+i)=-a1(m,m)*udf1(m,i);
bbg(n+i,2*n+i)=-a1(m,1)*vdf1(m,i);
bbg(n+i,3*n+i)=-a1(m,m)*vdf1(m,i)-pr*(wdt1(m,i)+vd(m,i));
bbg(n+i,4*n+i)=-a1(m,1)*wdf1(m,i);
bbg(n+i,5*n+i)=pr*(vdt1(m,i)-wd(m,i))-a1(m,m)*wdf1(m,i);
bbg(n+i,6*n+i)=-a1(m,2)*wdf1(m,i);
bbg(n+i,7*n+i)=-a1(m,m-1)*wdf1(m,i);
bbg(3*n+i,i)=-a1(m,1)*udt1(m,i);    bbg(3*n+i,n+i)=-a1(m,m)*udt1(m,i);
bbg(3*n+i,2*n+i)=(-1)*(vdt1(m,i)-wd(m,i))*a1(m,1);
bbg(3*n+i,3*n+i)=(-vdt1(m,i)+wd(m,i))*a1(m,m)-wdf1(m,i);
bbg(3*n+i,4*n+i)=-a1(m,1)*(wdt1(m,i)+vd(m,i));
bbg(3*n+i,5*n+i)=-a1(m,m)*(wdt1(m,i)+vd(m,i))+vdf1(m,i);
bbg(3*n+i,6*n+i)=-a1(m,2)*(wdt1(m,i)+vd(m,i));
bbg(3*n+i,7*n+i)=-a1(m,m-1)*(wdt1(m,i)+vd(m,i));
end
% terms with derivatives in tet direction
for i=1:n
for j=1:n
%      *****   Left hand side terms   **************
bb(n+i,n+j)=bb(n+i,n+j)+pr*b1(i,j)*ust1(m,i);
bb(n+i,3*n+j)=bb(n+i,3*n+j)+pr*b1(i,j)*(1+vst1(m,i)-ws(m,i));
bb(n+i,5*n+j)=bb(n+i,5*n+j)+pr*b1(i,j)*(wst1(m,i)+vs(m,i));
bb(3*n+i,n+j)=bb(3*n+i,n+j)+b1(i,j)*(1+usf1(m,i));
bb(3*n+i,3*n+j)=bb(3*n+i,3*n+j)+b1(i,j)*vsf1(m,i);
bb(3*n+i,5*n+j)=bb(3*n+i,5*n+j)+b1(i,j)*wsf1(m,i);
bb(5*n+i,3*n+j)=bb(5*n+i,3*n+j)+pr*b1(i,j);
bb(5*n+i,5*n+j)=bb(5*n+i,5*n+j)+pr*b2(i,j);
bb(7*n+i,n+j)=bb(7*n+i,n+j)-(1-pr)*b2(i,j)/2;
%     ******     Right hand side terms   **************
bbg(n+i,n+j)=bbg(n+i,n+j)-pr*b1(i,j)*udt1(m,i);
```

```
bbg(n+i,3*n+j)=bbg(n+i,3*n+j)-pr*b1(i,j)*(vdt1(m,i)-wd(m,i));
bbg(n+i,5*n+j)=bbg(n+i,5*n+j)-pr*b1(i,j)*(wdt1(m,i)+vd(m,i));
bbg(3*n+i,n+j)=bbg(3*n+i,n+j)-b1(i,j)*udf1(m,i);
bbg(3*n+i,3*n+j)=bbg(3*n+i,3*n+j)-b1(i,j)*vdf1(m,i);
bbg(3*n+i,5*n+j)=bbg(3*n+i,5*n+j)-b1(i,j)*wdf1(m,i);
% terms with derivatives in both directions
%      *****   Left hand side terms  **************
bb(3*n+i,4*n+j)=bb(3*n+i,4*n+j)+3*kp*a1(m,1)*b1(i,j);
bb(3*n+i,5*n+j)=bb(3*n+i,5*n+j)+3*kp*a1(m,m)*b1(i,j);
bb(3*n+i,6*n+j)=bb(3*n+i,6*n+j)+3*kp*a1(m,2)*b1(i,j);
bb(3*n+i,7*n+j)=bb(3*n+i,7*n+j)+3*kp*a1(m,m-1)*b1(i,j);
bb(7*n+i,2*n+j)=bb(7*n+i,2*n+j)+(3-pr)*a1(m,1)*b1(i,j)/2;
bb(7*n+i,3*n+j)=bb(7*n+i,3*n+j)+(3-pr)*a1(m,m)*b1(i,j)/2;
bb(7*n+i,4*n+j)=bb(7*n+i,4*n+j)+(2-pr)*a1(m,1)*b2(i,j);
bb(7*n+i,5*n+j)=bb(7*n+i,5*n+j)+(2-pr)*a1(m,m)*b2(i,j);
bb(7*n+i,6*n+j)=bb(7*n+i,6*n+j)+(2-pr)*a1(m,2)*b2(i,j);
bb(7*n+i,7*n+j)=bb(7*n+i,7*n+j)+(2-pr)*a1(m,m-1)*b2(i,j);
end;end
%
%                ****   The bd matrix   ****
%  for the first two series of columns related to "u"and "v"
for i=2:m-1
for j=1:n
bd(n+j,j+(i-2)*n)=a1(m,i)*(1+usf1(m,j));
bd(n+j,j+(i+m-4)*n)=a1(m,i)*vsf1(m,j);
bd(3*n+j,j+(i-2)*n)=a1(m,i)*ust1(m,j);
bd(3*n+j,j+(i+m-4)*n)=(1+3*kp+vst1(m,j)-ws(m,j))*a1(m,i);
bd(5*n+j,j+(i-2)*n)=a1(m,i);
bd(7*n+j,j+(i-2)*n)=a2(m,i);
%      *****      Right hand side terms  *********
bdg(n+j,j+(i-2)*n)=-a1(m,i)*udf1(m,j);
bdg(n+j,j+(i+m-4)*n)=-a1(m,i)*vdf1(m,j);
bdg(3*n+j,j+(i-2)*n)=-a1(m,i)*udt1(m,j);
bdg(3*n+j,j+(i+m-4)*n)=-(vdt1(m,j)-wd(m,j))*a1(m,i);
% this inner loop is for combined derivations
 for ii=1:n
 bd(7*n+j,(m+i-4)*n+ii)=(3-pr)*a1(m,i)*b1(j,ii)/2;
 end
end;end
% for the last series of columns related to "w"
for i=3:m-2
for j=1:n
%      *****   Left hand side terms  **************
bd(n+j,2*n*(m-2)+j+(i-3)*n)=kp*a2(m,i)+a1(m,i)*wsf1(m,j);
bd(3*n+j,2*n*(m-2)+j+(i-3)*n)=a1(m,i)*(wst1(m,j)+vs(m,j));
bd(5*n+j,2*n*(m-2)+j+(i-3)*n)=a2(m,i);
bd(6*n+j,2*n*(m-2)+j+(i-3)*n)=a1(1,i);
bd(7*n+j,2*n*(m-2)+j+(i-3)*n)=a3(m,i);
%      *****   Right hand side terms  **************
bdg(n+j,2*n*(m-2)+j+(i-3)*n)=-a1(m,i)*wdf1(m,j);
bdg(3*n+j,2*n*(m-2)+j+(i-3)*n)=-a1(m,i)*(wdt1(m,j)+vd(m,j));
% this inner loop is for combined derivations
 for ii=1:n
 bd(3*n+j,2*n*(m-2)+(i-3)*n+ii)=3*kp*a1(m,i)*b1(j,ii);
 bd(7*n+j,2*n*(m-2)+(i-3)*n+ii)=bd(7*n+j,2*n*(m-2)+(i-3)*n+ii)+(2-
pr)*a1(m,i)*b2(j,ii);
 end
end;end
```

F5EIG.m

```
%this part of program finds the eigenvalues and then picks up
% the smallest eigenvalue.
% erasing unwanted terms
clear aii aim ain ami amm amn ani anm ann;
clear dii dim din dmi dmm dmn dni dnm dnn;
clear a1 a2 a3 a4 b1 b2 b3 b4 ib rr1 rr2 rr3 rr4;
clear ud udf1 udf2 udft udt1 udt2 us usf1 usf2 usft ust1 ust2
clear vd vdf1 vdf2 vdft vdt1 vdt2 vs vsf1 vsf2 vsft vst1 vst2
clear wd wdf1 wdf2 wdf3 wdft wdf2t wdft2 wdt1 wdt2 wdt3 ws
clear wsf1 wsf2 wsf3 wsft wsf2t wsft2 wst1 wst2 wst3 qpd qps
clear nx0d nx0s nx0xd nx0xs nxy0d nxy0s nxy0xd nxy0xs ny0d
clear ny0s ny0yd ny0ys nyx0d nyx0s nyx0yd nyx0ys
clear x pai tet;
ll=[bb bd;db dd];
clear bb bd db dd;
rr=[bbg bdg;dbg ddg];
clear bbg bdg dbg ddg;
rslt=inv(ll)*rr;clear rr ll;
e=eig(rslt);
% this part of program finds the eigenvalues and then picks up
% the smallest eigenvalue
mm=0;
for i=1:3*m*n;
if imag(e(i)) == 0
if e(i)> 0
   mm=mm+1;
   l(mm)=e(i);
   end
   end
end
mig=1/max(l)
```

F6MODSHP.m

```
% this part of program finds the eigenvectors and then extracts the
% eigenvector corresponding to the lowest eigenvalue and finally draws the
% buckling mode shape. k is th scaling factor for buckling displacements.
k=3;
% finding eigenvector of the Max eigenvalue
[evc,evl]=eig(rslt);
evlmx=0;
for i=1:3*m*n
   if imag(evl(i,i))==0
      if evl(i,i)>0
         if evl(i,i)>=evlmx
            evlmx=evl(i,i);
            mm=i;
         end
      end
   end
end
vctr=evc(:,mm);
%taking the separate displacements from eigenvector
% u: axial, v circumferential, w radial displacements.
for i=1:n
   u(1,i)=vctr(i);
   u(m,i)=vctr(i+n);
   v(1,i)=vctr(i+2*n);
   v(m,i)=vctr(i+3*n);
```

```
    w(1,i)=vctr(i+4*n);
    w(m,i)=vctr(i+5*n);
    w(2,i)=vctr(i+6*n);
    w(m-1,i)=vctr(i+7*n);
 end
 for i=2:m-1
    for j=1:n
       u(i,j)=vctr(8*n+(i-2)*n+j);
       v(i,j)=vctr(8*n+(m+i-4)*n+j);
    end
 end
 for i=3:m-2
    for j=1:n
       w(i,j)=vctr(8*n+(2*m+i-7)*n+j);
    end
 end
 for i=1:m
    u(i,n+1)=u(i,1);
    v(i,n+1)=v(i,1);
    w(i,n+1)=w(i,1);
 end
 % determining the coordinates of undisplaced points
 x(1)=0;
 x(2)=delta;
 x(m-1)=1-delta;
 x(m)=1;
 for i=3:m-2
 tt=(pi*(i-2))/(m-3);
 x(i)=(1-cos(tt))/2;
 end
 for i=1:m
    for j=1:n+1
       xx(i,j)=cos(2*pi*(j-1)/n);
       yy(i,j)=sin(2*pi*(j-1)/n);
       zz(i,j)=x(i);
    end
 end
 % determining the coordinates of displaced points
 for i=1:m
 for j=1:n+1
 tet=(360/n)*(j-1)*pi/180;
 xdisp(i,j)=xx(i,j)+k*(-w(i,j)*cos(tet)-v(i,j)*sin(tet));
 ydisp(i,j)=yy(i,j)+k*(-w(i,j)*sin(tet)+v(i,j)*cos(tet));
 zdisp(i,j)=zz(i,j)+k*u(i,j);
 end
 end
 % drawing the cylinder
 for i=1:m
    xp(i)=zdisp(i,1);
 end
 for i=1:n/2+1
    yp(i)=xdisp(1,i);
 end
 for i=1:m
    for j=1:n/2+1
       zp(j,i)=ydisp(i,j);
       zp1(j,i)=ydisp(i,j+n/2);
    end
 end
 % to draw a line form point (1,1) at bottom to (m,n/2+1) at top
 lix=[zz(1,1) zz(m,n/2+1)];
```

```
liy=[xx(1,1) xx(m,n/2+1)];
liz=[yy(1,1) yy(m,n/2+1)];
hold on
surf(xp,yp,zp);axis('off');plot3(lix,liy,liz);
surf(xp,yp,zp1);view(270,-40)
hold off
%
%                   ***********************************************
```